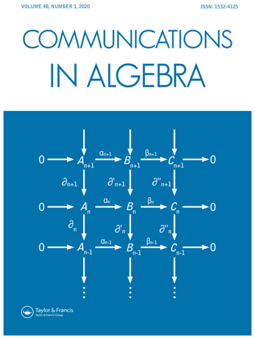

Taylor & Francis
Taylor & Francis Group

Communications in Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

On half-factoriality of transfer Krull monoids

Weidong Gao , Chao Liu , Salvatore Tringali & Qinghai Zhong

To cite this article: Weidong Gao , Chao Liu , Salvatore Tringali & Qinghai Zhong (2020): On half-factoriality of transfer Krull monoids, Communications in Algebra, DOI: 10.1080/00927872.2020.1800720

To link to this article: https://doi.org/10.1080/00927872.2020.1800720

Published online: 08 Aug 2020.

COMMUNICATIONS IN ALGEBRA®
https://doi.org/10.1080/00927872.2020.1800720

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS Check for updates

On half-factoriality of transfer Krull monoids

Weidong Gao[a], Chao Liu[b], Salvatore Tringali[c], and Qinghai Zhong[d]

[a]Center for Combinatorics, LPMC-TJKLC Nankai University, Tianjin, P. R. China; [b]Department of Mathematical Sciences, University of Memphis, Memphis, Tennessee, USA; [c]College of Mathematics and Computer Science, Hebei Normal University, Shijiazhuang, P. R. China; [d]NAWI Graz, Institute for Mathematics and Scientific Computing, University of Graz, Graz, Austria

ABSTRACT

Let H be a transfer Krull monoid over a subset G_0 of an abelian group G with finite exponent. Then every non-unit $a \in H$ can be written as a finite product of atoms, say $a = u_1 \cdot \ldots \cdot u_k$. The set $\mathsf{L}(a)$ of all possible factorization lengths k is called the set of lengths of a, and H is said to be half-factorial if $|\mathsf{L}(a)| = 1$ for all $a \in H$. We show that, if $a \in H$ is a non-unit and $|\mathsf{L}(a^{\lfloor (3 \exp(G) - 3)/2 \rfloor})| = 1$, then the smallest divisor-closed submonoid of H containing a is half-factorial. In addition, we prove that, if G_0 is finite and $|\mathsf{L}(\prod_{g \in G_0} g^{2 \operatorname{ord}(g)})| = 1$, then H is half-factorial.

ARTICLE HISTORY
Received 17 January 2020
Revised 13 July 2020
Communicated by Scott Chapman

KEYWORDS
Half-factorial; sets of lengths; transfer Krull monoids

MATHEMATICS SUBJECT CLASSIFICATION
11B30; 11R27; 13A05; 13F05; 20M13

1. Introduction

Let H be a monoid. If an element $a \in H$ has a factorization $a = u_1 \cdot \ldots \cdot u_k$, where $k \in \mathbb{N}$ and $u_1, \ldots, u_k$ are atoms of H, then k is called a factorization length of a, and the set $\mathsf{L}(a)$ of all possible k is referred to as the set of lengths of a. The monoid H is said to be half-factorial if $|\mathsf{L}(a)| = 1$ for every $a \in H$.

The study of half-factoriality was pioneered by Leonard Carlitz in the setting of algebraic number theory: he proved in [4] that the ring of integers $\mathcal{O}_K$ of a number field K is half-factorial if and only if the cardinality of its class group is either 1 or 2. After this, the concept of half-factoriality seemed to remain dormant for more than a decade until papers by Abraham Zaks [32], Ladislav Skula [28], and Jan Śliwa [29] simultaneously appeared in 1976. In such papers, half-factoriality was studied in the context of Krull domains and c-monoids. Since then half-factoriality has been investigated in different classes of monoids (see [3, 6, 17, 18]) and integral domains (see [5, 7, 13, 19, 20, 23, 33]).

Given $a \in H$, let $[\![a]\!] = \{b \in H \mid b \text{ divides some power of } a\}$ be the smallest divisor-closed submonoid of H containing a. Then $[\![a]\!]$ is half-factorial if and only if $|\mathsf{L}(a^n)| = 1$ for all $n \in \mathbb{N}$, and H is half-factorial if and only if $[\![c]\!]$ is half-factorial for every $c \in H$. It is thus natural to ask:

> Does there exist an integer $N \in \mathbb{N}$ such that, if $a \in H$ and $|\mathsf{L}(a^N)| = 1$, then $[\![a]\!]$ is half-factorial? (Note that, if $[\![a]\!]$ is half-factorial for some $a \in H$, then of course $|\mathsf{L}(a^k)| = 1$ for every $k \in \mathbb{N}$.)

We answer this question affirmatively for transfer Krull monoids over finite abelian groups, and we study the smallest N having the above property (Theorems 1.1 and 1.2).

CONTACT Qinghai Zhong qinghai.zhong@uni-graz.at University of Graz, NAWI Graz, Institute for Mathematics and Scientific Computing, Heinrichstraße 36, 8010 Graz, Austria.

Transfer Krull monoids and transfer Krull domains are a recently introduced class of monoids and domains including, among others, all commutative Krull domains and wide classes of non-commutative Dedekind domains (see Section 2 and, for a survey, see [10]).

Let H be a transfer Krull monoid over a subset G_0 of an abelian group G. Then H is half-factorial if and only if the monoid $\mathcal{B}(G_0)$ of zero-sum sequences over G_0 is half-factorial (in this case, we also say that the set G_0 is half-factorial). It is a standing conjecture that every abelian group has a half-factorial generating set, which implies that every abelian group can be realized as the class group of a half-factorial Dedekind domain [11].

Suppose now that H is a commutative Krull monoid with class group G and that every class contains a prime divisor. It is a classic result that H is half-factorial if and only if $|G| \le 2$, and it turns out that, also for $|G| \ge 3$, half-factorial subsets (and minimal non-half-factorial subsets) of the class group G play a crucial role in a variety of arithmetical questions (see [12, Chapter 6.7], [15]). However, we are far away from a good understanding of half-factorial sets in finite abelian groups (see [25] for a survey, and [21, 22, 26]). To mention one open question, the maximal size of half-factorial subsets is unknown even for finite cyclic groups [22]. Our results open the door to a computational approach to the study of half-factorial sets.

More in detail, denote by $\mathsf{hf}(H)$ the infimum of all $N \in \mathbb{N}$ with the following property:

$$\text{If } a \in H \text{ and } |\mathsf{L}(a^N)| = 1, \text{ then } [\![a]\!] \text{ is half-factorial.}$$

(Here, as usual, we assume $\inf \emptyset = \infty$.) We call $\mathsf{hf}(H)$ the *half-factoriality index* of H. If H is not half-factorial, then $\mathsf{hf}(H)$ is the infimum of all $N \in \mathbb{N}$ with the property that $|\mathsf{L}(a^N)| \ge 2$ for every $a \in H$ such that $[\![a]\!]$ is not half-factorial. In particular, if G is an abelian group with $|G| \ge 3$, then $\mathsf{hf}(\mathcal{B}(G))$ is the infimum of all $N \in \mathbb{N}$ with the property that

$$\textit{For every sequence } S \textit{ over } G, \textit{ if } |\mathsf{L}(S^N)| = 1, \textit{ then } |\mathsf{L}(S^k)| = 1 \textit{ for every } k \ge N.$$

Theorem 1.1. *Let H be a transfer Krull monoid over a finite subset G_0 of an abelian group G with finite exponent. The following are equivalent.*

(a) *H is half-factorial.*
(b) $\mathsf{hf}(H) = 1$.
(c) *G_0 is half-factorial.*
(d) $|\mathsf{L}(\prod_{g \in G_0} g^{2\mathrm{ord}(g)})| = 1$.

We observe that in general if H is half-factorial, then $\mathsf{hf}(H) = 1$. But if H is a transfer Krull monoid over a subset of a torsion free group, then $\mathsf{hf}(H) = 1$ does not imply that H is half-factorial (see Example 2.4.1). Furthermore, for every $n \in \mathbb{N}$, there exists a Krull monoid H with finite class group such that $\mathsf{hf}(H) = n$ (see Example 2.4.2).

Theorem 1.2. *Let H be a transfer Krull monoid over an abelian group G.*

1. $\mathsf{hf}(H) < \infty$ *if and only if* $\exp(G) < \infty$.
2. *If* $\exp(G) < \infty$ *and* $|G| \ge 3$, *then* $\exp(G) \le \mathsf{hf}(H) \le \frac{3}{2}(\exp(G) - 1)$.
3. *If G is cyclic or* $\exp(G) \le 6$, *then* $\mathsf{hf}(H) = \exp(G)$.

We postpone the proofs of Theorems 1.1 and 1.2 to Section 3.

2. Preliminaries

Our notation and terminology are consistent with [12]. Let $\mathbb{N}$ be the set of positive integers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\mathbb{Q}$ be the set of rational numbers. For integers $a, b \in \mathbb{Z}$, we denote by $[a, b] = \{x \in \mathbb{Z} | a \le x \le b\}$ the discrete, finite interval between a and b.

Atomic monoids. By a monoid, we mean a semigroup with identity, and if not stated otherwise we use multiplicative notation. Let H be a monoid with identity $1 = 1_H \in H$. The set of invertible elements of H will be denoted by $H^\times$, and we say that H is reduced if $H^\times = \{1\}$. The monoid H is said to be unit-cancellative if for any two elements $a, u \in H$, each of the equations $au = a$ or $ua = a$ implies that $u \in H^\times$. Clearly, every cancellative monoid is unit-cancellative.

Suppose that H is unit-cancellative. An element $u \in H$ is said to be irreducible (or an atom) if $u \notin H^\times$ and for any two elements $a, b \in H$, $u = ab$ implies that $a \in H^\times$ or $b \in H^\times$. Let $\mathcal{A}(H)$ denote the set of atoms of H. We say that H is atomic if every non-unit is a finite product of atoms. If H satisfies the ascending chain condition on principal left ideals and on principal right ideals, then H is atomic [9, Theorem 2.6]. If $a \in H \setminus H^\times$ and $a = u_1 ... u_k$, where $k \in \mathbb{N}$ and $u_1, ..., u_k \in \mathcal{A}(H)$, then k is a factorization length of a, and

$$\mathsf{L}_H(a) = \mathsf{L}(a) = \{k \in \mathbb{N} \mid k \text{ is a factorization length of } a\}$$

denotes the set of lengths of a. It is convenient to set $\mathsf{L}(a) = \{0\}$ for all $a \in H^\times$.

Let H and B be atomic monoids. The homomorphism $\theta : H \to B$ is called a *weak transfer homomorphism* if it satisfies the following two properties.

(T1) $B = B^\times \theta(H) B^\times$ and $\theta^{-1}(B^\times) = H^\times$.

(WT2) If $a \in H$, $n \in \mathbb{N}$, $v_1, ..., v_n \in \mathcal{A}(B)$ and $\theta(a) = v_1 \cdot ... \cdot v_n$, then there exist $u_1, ..., u_n \in \mathcal{A}(H)$ and a permutation $\tau \in \mathfrak{S}_n$ such that $a = u_1 \cdot ... \cdot u_n$ and $\theta(u_i) \in B^\times v_{\tau(i)} B^\times$ for each $i \in [1, n]$.

A *transfer Krull monoid* is a monoid H having a weak transfer homomorphism $\theta : H \to \mathcal{B}(G_0)$, where $\mathcal{B}(G_0)$ is the monoid of zero-sum sequences over a subset G_0 of an abelian group G. If H is a commutative Krull monoid with class group G and $G_0 \subset G$ is the set of classes containing prime divisors, then there is a weak transfer homomorphism $\theta : H \to \mathcal{B}(G_0)$. Beyond that, there are wide classes of non-commutative Dedekind domains having a weak transfer homomorphism to a monoid of zero-sum sequences ([31, Theorem 1.1], [30, Theorem 4.4]). We refer to [10, 16] for surveys on transfer Krull monoids. If $\theta : H \to \mathcal{B}(G_0)$ is a weak transfer homomorphism, then sets of lengths in H and in $\mathcal{B}(G_0)$ coincide [2, Lemma 2.7] and thus the statements of Theorems 1.1 and 1.2 can be proved in the setting of monoids of zero-sum sequences.

Monoids of zero-sum sequences. Let G be an abelian group and let $G_0 \subset G$ be a non-empty subset. Then $\langle G_0 \rangle$ denotes the subgroup generated by G_0. In additive combinatorics, a sequence (over G_0) means a finite unordered sequence of terms from G_0 where repetition is allowed, and (as usual) we consider sequences as elements of the free abelian monoid with basis G_0. Let

$$S = g_1 \cdot ... \cdot g_\ell = \prod_{g \in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0)$$

be a sequence over G_0. We call

$$\begin{aligned} \operatorname{supp}(S) &= \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \text{ the } \textit{support} \text{ of } S, \\ |S| &= \ell = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ the } \textit{length} \text{ of } S, \\ \sigma(S) &= \sum_{i=1}^{\ell} g_i \text{ the } \textit{sum} \text{ of } S, \\ \text{and } \Sigma(S) &= \left\{ \sum_{i \in I} g_i | \emptyset \neq I \subset [1, \ell] \right\} \text{ the } \textit{set of subsequence sums} \text{ of } S. \end{aligned}$$

The sequence S is said to be

- *zero-sum free* if $0 \notin \Sigma(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a nontrivial zero-sum sequence and every proper subsequence is zero-sum free.

The set of zero-sum sequences $\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) | \sigma(S) = 0\} \subset \mathcal{F}(G_0)$ is a submonoid, and the set of minimal zero-sum sequences is the set of atoms of $\mathcal{B}(G_0)$. For any arithmetical invariant $*(H)$ defined for a monoid H, we write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$. In particular, $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ is the set of atoms of $\mathcal{B}(G_0)$ and $\mathsf{hf}(G_0) = \mathsf{hf}(\mathcal{B}(G_0))$.

Let G be an abelian group. We denote by $\exp(G)$ the exponent of G which is the least common multiple of the orders of all elements of G. If there is no least common multiple, the exponent is taken to be infinity. Let $r \in \mathbb{N}$ and let $(e_1, ..., e_r)$ be an r-tuple of elements of G. Then $(e_1, ..., e_r)$ is said to be independent if $e_i \neq 0$ for all $i \in [1, r]$ and if for all $(m_1, ..., m_r) \in \mathbb{Z}^r$ an equation $m_1 e_1 + ... + m_r e_r = 0$ implies that $m_i e_i = 0$ for all $i \in [1, r]$. Suppose G is finite. The r-tuple $(e_1, ..., e_r)$ is said to be a basis of G if it is independent and $G = \langle e_1 \rangle \oplus ... \oplus \langle e_r \rangle$. For every $n \in \mathbb{N}$, we denote by C_n an additive cyclic group of order n. Since $G \cong C_{n_1} \oplus ... \oplus C_{n_r}$, $r = r(G)$ is the rank of G and $n_r = \exp(G)$ is the exponent of G.

Let $G_0 \subset G$ be a non-empty subset. For a sequence $S = g_1 \cdot ... \cdot g_\ell \in \mathcal{F}(G_0)$, we call

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)} \in \mathbb{Q}_{\geq 0} \quad \text{the } \textit{cross number} \text{ of } S, \text{ and}$$

$$\mathsf{K}(G_0) = \max\{\mathsf{k}(S) | S \in \mathcal{A}(G_0)\} \quad \text{the } \textit{cross number} \text{ of } G_0.$$

For the relevance of cross numbers in the theory of non-unique factorizations, see [22, 24, 27] and [12, Chapter 6].

The set G_0 is called

- *half-factorial* if the monoid $\mathcal{B}(G_0)$ is half-factorial;
- *non-half-factorial* if the monoid $\mathcal{B}(G_0)$ is not half-factorial;
- *minimal non-half-factorial* if G_0 is not half-factorial but all its proper subsets are;
- an *LCN-set* if $\mathsf{k}(A) \geq 1$ for all atoms $A \in \mathcal{A}(G_0)$.

The following simple result [12, Proposition 6.7.3] will be used throughout the article without further mention.

Lemma 2.1. *Let G be a finite abelian group and $G_0 \subset G$ a subset. Then the following statements are equivalent.*

(a) *G_0 is half-factorial.*
(b) *$\mathsf{k}(U) = 1$ for every $U \in \mathcal{A}(G_0)$.*
(c) *$\mathsf{L}(B) = \{\mathsf{k}(B)\}$ for every $B \in \mathcal{B}(G_0)$.*

Lemma 2.2. *Let G be a finite group, let $G_0 \subset G$ be a subset, let S be a zero-sum sequence over G_0, and let A be a minimal zero-sum sequence over G_0.*

(1) *If $\mathsf{k}(A) \neq 1$, then $|\mathsf{L}(A^{\exp(G)})| \geq 2$.*
(2) *If there exists a zero-sum subsequence T of S such that $|\mathsf{L}(T)| \geq 2$, then $|\mathsf{L}(S)| \geq 2$.*
(3) *If $\mathsf{k}(A) < 1$ and $\mathsf{k}(A)$ is minimal over all minimal zero-sum sequences over G_0, then*

$$\left| \mathsf{L}\left(A^{\left\lceil \frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} \right\rceil} \right) \right| \geq 2, \quad \textit{for all } g \in \mathrm{supp}(A).$$

Proof. 1. Suppose $\mathsf{k}(A) \neq 1$ and let $A = g_1 \cdot ... \cdot g_\ell$, where $\ell \in \mathbb{N}$ and $g_1, ..., g_\ell \in G_0$. Then

$$A^{\exp(G)} = \left(g_1^{\mathrm{ord}(g_1)}\right)^{\frac{\exp(G)}{\mathrm{ord}(g_1)}} \cdot ... \cdot \left(g_\ell^{\mathrm{ord}(g_\ell)}\right)^{\frac{\exp(G)}{\mathrm{ord}(g_\ell)}},$$

which implies that

$$\left\{ \exp(G), \sum_{i=1}^{\ell} \frac{\exp(G)}{\mathrm{ord}(g_i)} \right\} = \{\exp(G), \exp(G)\mathsf{k}(A)\} \subset \mathsf{L}\big(A^{\exp(G)}\big).$$

It follows by $\mathsf{k}(A) \neq 1$ that $|\mathsf{L}\big(A^{\exp(G)}\big)| \geq 2$.

2. Suppose T is a zero-sum subsequence of S with $|\mathsf{L}(T)| \geq 2$. It follows by $\mathsf{L}(S) \supset \mathsf{L}(T) + \mathsf{L}(ST^{-1})$ that $|\mathsf{L}(S)| \geq |\mathsf{L}(T)| \geq 2$.

3. Suppose $\mathsf{k}(A) < 1$ and $\mathsf{k}(A)$ is minimal over all minimal zero-sum sequences over G_0. Let $g \in \mathrm{supp}(A)$. Then there exist $s \in \mathbb{N}$ and minimal zero-sum sequences $W_1, \ldots, W_s$ such that

$$A^{\left\lceil \frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} \right\rceil} = g^{\mathrm{ord}(g)} \cdot W_1 \cdot \ldots \cdot W_s.$$

Since

$$k\Big(A^{\left\lceil \frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} \right\rceil}\Big) = \left\lceil \frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} \right\rceil \mathsf{k}(A) = 1 + \sum_{i=1}^{s} \mathsf{k}(W_i) > (1+s)\mathsf{k}(A),$$

we have $\left\lceil \frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} \right\rceil \neq s+1$ and hence $\left|\mathsf{L}\Big(A^{\left\lceil \frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} \right\rceil}\Big)\right| \geq 2$. □

For commutative and finitely generated monoids, we have the following result.

Proposition 2.3. *Let H be a commutative unit-cancellative monoid. If H_{red} is finitely generated, then $\mathsf{hf}(H)$ is finite.*

Proof. We may assume that H is reduced and not half-factorial. Suppose H is finitely generated and suppose $\mathcal{A}(H) = \{u_1, \ldots, u_n\}$, where $n \in \mathbb{N}$. Set $A_0 = \{\prod_{i \in I} u_i \mid \emptyset \neq I \subset [1, n]\}$. Then A_0 is finite and hence there exists $M \in \mathbb{N}$ such that $|\mathsf{L}\big(a_0^M\big)| \geq 2$ for all $a_0 \in A_0$ with $[\![a_0]\!]$ not half-factorial. Let $a \in H \setminus H^\times$ such that $[\![a]\!]$ is not half-factorial. It suffices to show that $|\mathsf{L}(a^M)| \geq 2$. Suppose $a = u_1^{k_1} \cdot \ldots \cdot u_n^{k_n}$, where $k_1, \ldots, k_n \in \mathbb{N}_0$. Set $I_0 = \{i \in [1, n] | k_i \geq 1\}$ and $a_0 = \prod_{i \in I} u_i$. Then a_0 divides a and $[\![a_0]\!] = [\![a]\!]$ is not half-factorial, whence $|\mathsf{L}\big(a_0^M\big)| \geq 2$ and $|\mathsf{L}(a^M)| \geq 2$. □

If G_0 is a finite subset of an abelian group, then $\mathcal{B}(G_0)$ is finitely generated [12, Theorem 3.4.2] and thus $\mathsf{hf}(G_0) < \infty$. We refer to [8, Sections 3.2 and 3.3] and [14] for semigroups of ideals and semigroups of modules that are finitely generated unit-cancellative but not necessarily cancellative.

Example 2.4. The following examples will help us to illustrate some important points.

1. Let (e_1, e_2) be a basis of $\mathbb{Z}^2$ and let $G_0 = \{e_1, -e_1, e_2, -e_2, e_1+e_2, -e_1-e_2\}$. Then $\mathcal{A}(G_0) = \{e_1(-e_1), e_2(-e_2), (e_1+e_2)(-e_1-e_2), e_1e_2(-e_1-e_2), (-e_1)(-e_2)(e_1+e_2)\}$. Since $e_1(-e_1) \cdot e_2(-e_2) \cdot (e_1+e_2)(-e_1-e_2) = e_1e_2(-e_1-e_2) \cdot (-e_1)(-e_2)(e_1+e_2)$, we obtain G_0 is not half-factorial. Furthermore, we have G_1 is half-factorial for every non-empty proper subset $G_1 \subsetneq G_0$. Let $A \in \mathcal{B}(G_0)$. If $\mathrm{supp}(A) = G_0$, then $|\mathsf{L}(A)| \geq 2$ and $[\![A]\!] = \mathcal{B}(G_0)$ is not half-factorial. If $\mathrm{supp}(A) \subsetneq G_0$, then $[\![A]\!] = \mathcal{B}\big(\mathrm{supp}(A)\big)$ is half-factorial and $|\mathsf{L}(A)| = 1$. Therefore $\mathsf{hf}(G_0) = 1$.
2. Let G be a cyclic group with order n and let $g \in G$ with $\mathrm{ord}(g) = n$, where $n \in \mathbb{N}_{\geq 3}$. Set $G_0 = \{g, -g\}$. Then G_0 is not half-factorial. Let $A_0 = g(-g)$. Then $[\![A_0]\!]$ is not half-factorial and $|\mathsf{L}\big(A_0^{n-1}\big)| = 1$, whence $\mathsf{hf}(G_0) \geq n$. Let $A \in \mathcal{B}(G_0)$ such that $[\![A]\!]$ is not half-factorial. Then $\mathrm{supp}(A) = G_0$ and A_0 divides A, whence $|\mathsf{L}(A^n)| \geq 2$. Therefore $\mathsf{hf}(G_0) = n$. Let $G \cong C_2^2$ and let (e_1, e_2) be a basis of G. Set $G_1 = \{e_1, e_2, e_1+e_2\}$. Then G_1 is not half-factorial. Let $A_1 = e_1e_2(e_1+e_2)$. Then $[\![A_1]\!]$ is not half-factorial and $|\mathsf{L}(A_1)| = 1$, whence $\mathsf{hf}(G_1) \geq 2$. Let

$A \in \mathcal{B}(G_1)$ such that $[\![A]\!]$ is not half-factorial. Then $\mathrm{supp}(A) = G_1$ and A_1 divides A, whence $|\mathsf{L}(A^2)| \geq 2$. Therefore $\mathsf{hf}(G_1) = 2$.

3. Let H be a bifurcus monoid (i.e., $2 \in \mathsf{L}(a)$ for all $a \in H \setminus (H^\times \cup \mathcal{A}(H))$). For examples, see [1, Examples 2.1 and 2.2]. Since for every $a \in H \setminus H^\times$, we have $\{2, 3\} \subset \mathsf{L}(a^3)$, it follows that $\mathsf{hf}(H) \leq 3$ and $\mathsf{hf}(H)$ is the minimal integer $t \in \mathbb{N}$ such that $|\mathsf{L}(a^t)| \geq 2$ for all $a \in H \setminus H^\times$. Therefore $\mathsf{hf}(H) = 3$ if and only if there exists $a_0 \in \mathcal{A}(H)$ such that $\mathsf{L}(a_0^2) = \{2\}$.
4. Let $H \subset F = F^\times \times [p_1, \ldots, p_s]$ be a non-half-factorial finitely primary monoid of rank s and exponent α (see [12, Definition 2.9.1]). For every $a = \epsilon p_1^{t_1} \ldots p_s^{t_s} \in F$, we define $||a|| = t_1 + \ldots + t_s$, where $t_1, \ldots, t_s \in \mathbb{N}_0$ and $\epsilon \in F^\times$. Let $a \in H \setminus H^\times$. Since H is primary, we have $H = [\![a]\!]$ is not half-factorial. Thus $\mathsf{hf}(H)$ is the minimal integer $t \in \mathbb{N}$ such that $|\mathsf{L}(a^t)| \geq 2$ for all $a \in H \setminus H^\times$. Suppose $a_0 \in H$ with $||a_0|| = \min\{||a|| : a \in H \setminus H^\times\}$. Then $a_0 \in \mathcal{A}(H)$ and $\mathsf{L}(a_0^2) = \{2\}$, whence $\mathsf{hf}(H) \geq 3$.

If $H \setminus H^\times = (p_1 \ldots p_s)^\alpha F$ and $s \geq 2$, then H is bifurcus and hence $\mathsf{hf}(H) = 3$. Suppose $s = 1$ and $H \setminus H^\times = (p_1)^\alpha F$. Let $b = \epsilon p^\beta \in H$. Then $p^{3\alpha}$ divides b^4. It follows by $p^{3\alpha} = (p^\alpha)^3 = p^{\alpha+1} p^{2\alpha-1}$ that $|\mathsf{L}(b^4)| \geq 2$, whence $\mathsf{hf}(H) \leq 4$. If $3\beta \geq 4\alpha$, then $p^{3\alpha}$ divides b^3 and hence $|\mathsf{L}(b^3)| \geq 2$. If (HTML translation failed), then b is an atom and $b^3 = \epsilon^3 p^{2\alpha-1} p^{3\beta-(2\alpha-1)}$, whence $|\mathsf{L}(b^3)| \geq 2$. If $3\beta = 4\alpha - 1$, then $\mathsf{L}(b^3) = \{3\}$. Put all together, if $\alpha \equiv 1 \mod 3$, then $\mathsf{hf}(H) = 4$. Otherwise $\mathsf{hf}(H) = 3$.

3. Proof of main theorem

Proposition 3.1. *Let $G_0 \subset G$ be a non-half-factorial subset and let S be a zero-sum sequence over G_0 with* $\mathrm{supp}(S) = G_0$.

1. *If G_0 is an LCN-set, then* $|\mathsf{L}\left(\prod_{g \in G_0} g^{\mathrm{ord}(g)}\right)| \geq 2$.
2. *If $|G_0| = 2$, then* $|\mathsf{L}\left(\prod_{g \in G_0} g^{\mathrm{ord}(g)}\right)| \geq 2$.
3. *If G_0 is a minimal non-half-factorial subset, then* $|\mathsf{L}(S^{\exp(G)})| \geq 2$.
4. *If $|\{g \in G_0 | \mathrm{ord}(g)/\mathsf{v}_g(S) = \exp(G)\}| \leq 1$, then* $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

Proof. 1. Suppose G_0 is an LCN-set. Since G_0 is not half-factorial, there exists a minimal zero-sum sequence T over G_0 such that $\mathsf{k}(T) > 1$. Note that T is a subsequence of $\prod_{g \in G_0} g^{\mathrm{ord}(g)}$. Then there exist $W_1, \ldots, W_l \in \mathcal{A}(G_0)$ such that

$$\prod_{g \in G_0} g^{\mathrm{ord}(g)} = T \cdot W_1 \cdot \ldots \cdot W_l.$$

Thus $\mathsf{k}\left(\prod_{g \in G_0} g^{\mathrm{ord}(g)}\right) = |G_0| = \mathsf{k}(T) + \sum_{i=1}^{l} \mathsf{k}(W_i) > 1 + l$. The assertion follows by $\{|G_0|, 1 + l\} \subset \mathsf{L}\left(\prod_{g \in G_0} g^{\mathrm{ord}(g)}\right)$.

2. Suppose $|G_0| = 2$ and let $G_0 = \{g_1, g_2\}$. If G_0 is an LCN-set, the assertion follows by 1. Suppose there exists a minimal zero-sum sequence T over G_0 with $\mathsf{k}(T) < 1$. Let $T_0 = g_1^{l_1} \cdot g_2^{l_2}$ be the minimal zero-sum sequence over G_0 such that $\mathsf{k}(T_0)$ is minimal. If $\min\{\frac{\mathrm{ord}(g_1)}{l_1}, \frac{\mathrm{ord}(g_2)}{l_2}\} \leq 2$, say $\frac{\mathrm{ord}(g_1)}{l_1} \leq 2$ then

$$T_0^2 = g_1^{\mathrm{ord}(g_1)} \cdot W, \text{ where } W \text{ is an on-empty zero-sum sequence.}$$

Thus $\mathsf{k}(W) = 2\mathsf{k}(T_0) - 1 < \mathsf{k}(T_0)$, a contradiction to the minimality of $\mathsf{k}(T_0)$. Therefore $\min\{\frac{\mathrm{ord}(g_1)}{l_1}, \frac{\mathrm{ord}(g_2)}{l_2}\} > 2$ and hence

$$g_1^{\mathrm{ord}(g_1)} \cdot g_2^{\mathrm{ord}(g_2)} = T_0^2 \cdot V, \text{ where } V \text{ is non-empty zero-sum sequence.}$$

It follows that $\left|\mathsf{L}\left(g_1^{\mathrm{ord}(g_1)} \cdot g_2^{\mathrm{ord}(g_2)}\right)\right| \geq 2$.

3. Suppose that G_0 is a minimal non-half-factorial set. If S has a minimal zero-sum subsequence A with $\mathsf{k}(A) \neq 1$, then the assertion follows by Lemma 2.2. If G_0 is an LCN-set, then the assertion follows from 1 and Lemma 2.2.2. Therefore we can suppose $\mathsf{L}(S) = \{\mathsf{k}(S)\}$ and suppose there exists a minimal zero-sum sequence T over G_0 with $\mathsf{k}(T) < 1$.

Let $T_0 = \prod_{i=1}^{|G_0|} g_i^{l_i}$ be the minimal zero-sum sequence over G_0 such that $\mathsf{k}(T_0)$ is minimal. The minimality of G_0 implies that $l_i \geq 1$ for all $i \in [1, |G_0|]$. After renumbering if necessary, we let

$$\frac{\mathrm{ord}(g_1)}{l_1} = \min\left\{\frac{\mathrm{ord}(g_i)}{l_i} | i \in [1, |G_0|]\right\}.$$

By Lemma 2.2.3, $\left|\mathsf{L}\left(T_0^{\lceil \frac{\mathrm{ord}(g_1)}{l_1} \rceil}\right)\right| \geq 2$. If $T_0^{\lceil \frac{\mathrm{ord}(g_1)}{l_1} \rceil}$ divides $S^{\exp(G)}$, then the assertion follows by Lemma 2.2.2. Suppose $T_0^{\lceil \frac{\mathrm{ord}(g_1)}{l_1} \rceil} \nmid S^{\exp(G)}$. Let

$$I = \left\{ i \in [1, |G_0|] \middle| \left\lceil \frac{\mathrm{ord}(g_1)}{l_1} \right\rceil l_i > \exp(G)\mathsf{v}_{g_i}(S) \right\}.$$

Thus for each $i \in I$, we have

$$2\mathrm{ord}(g_i) > l_i \left\lceil \frac{\mathrm{ord}(g_i)}{l_i} \right\rceil \geq l_i \left\lceil \frac{\mathrm{ord}(g_1)}{l_1} \right\rceil > \exp(G)\mathsf{v}_{g_i}(S) \geq \exp(G),$$

which implies that $\mathrm{ord}(g_i) = \exp(G)$, $\mathsf{v}_{g_i}(S) = 1$, and $\lceil \frac{\mathrm{ord}(g_1)}{l_1} \rceil > \frac{\mathrm{ord}(g_i)}{l_i} = \frac{\exp(G)}{l_i}$.

Let $i_0 \in I$ such that $l_{i_0} = \max\{l_i | i \in I\}$. Therefore for every $j \in [1, |G_0|] \setminus I$, we have

$$l_j \leq \frac{\exp(G)\mathsf{v}_{g_j}(S)}{\lceil \frac{\mathrm{ord}(g_1)}{l_1} \rceil} \leq \frac{\exp(G)\mathsf{v}_{g_j}(S)}{\frac{\exp(G)}{l_{i_0}}} = l_{i_0}\mathsf{v}_{g_j}(S).$$

Note that for every $i \in I$, we have $l_i \leq l_{i_0} = l_{i_0}\mathsf{v}_{g_i}(S)$. It follows by $\mathsf{v}_{g_{i_0}}(T_0) = l_{i_0} = l_{i_0}\mathsf{v}_{g_{i_0}}(S) = \mathsf{v}_{g_{i_0}}(S^{l_{i_0}})$ that

$$S^{l_{i_0}} = T_0 \cdot W, \text{ where } W \text{ is a zero} - \text{sum sequence over } G_0 \setminus \{g_{i_0}\}.$$

By the minimality of G_0, we have $G_0 \setminus \{g_{i_0}\}$ is half-factorial which implies that $\mathsf{k}(W) \in \mathbb{N}$. Therefore $\mathsf{k}(T_0) = l_{i_0}\mathsf{k}(S) - \mathsf{k}(W)$ is an integer, a contradiction to $\mathsf{k}(T_0) < 1$.

4. Let $G_1 = \{g \in G_0 | \mathrm{ord}(g) = \exp(G)\mathsf{v}_g(S)\}$. Suppose $G_0 \setminus G_1$ is not half-factorial. If $G_0 \setminus G_1$ is an LCN-set, then the assertion follows by Proposition 3.1.1 and Lemma 2.2.2. Otherwise there exists a minimal zero-sum sequence A over $G_0 \setminus G_1$ such that $\mathsf{k}(A) < 1$. We may assume that $\mathsf{k}(A)$ is minimal over all minimal zero-sum sequences over $G_0 \setminus G_1$ and that $\min\{\frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} | g \in \mathrm{supp}(A)\} = \frac{\mathrm{ord}(g_0)}{\mathsf{v}_{g_0}(A)}$ for some $g_0 \in \mathrm{supp}(A) \subset G_0 \setminus G_1$. Thus by Lemma 2.2.3, we have $\left|\mathsf{L}\left(A^{\lceil \frac{\mathrm{ord}(g_0)}{\mathsf{v}_{g_0}(A)} \rceil}\right)\right| \geq 2$. The definition of G_1 implies that

$$A^{\lceil \frac{\mathrm{ord}(g_0)}{\mathsf{v}_{g_0}(A)} \rceil} \quad \text{divides} \quad S^{\exp(G)}$$

and hence the assertion follows.

Suppose $G_0 \setminus G_1$ is half-factorial. Then G_1 is non-empty and hence $G_1 = \{g_0\}$ for some $g_0 \in G_0$. If G_0 is an LCN-set, then the assertion follows by Proposition 3.1.1 and Lemma 2.2.2. Otherwise there exists a minimal zero-sum sequence A over G_0 such that $\mathsf{k}(A) < 1$. We may assume that $\mathsf{k}(A)$ is minimal over all minimal zero-sum sequences over G_0 and that $\min\left\{\frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)} | g \in \mathrm{supp}(A)\right\} = \frac{\mathrm{ord}(g_1)}{\mathsf{v}_{g_1}(A)}$ for some $g_1 \in \mathrm{supp}(A) \subset G_0$. Thus by Lemma 2.2.3, we have $\left|\mathsf{L}\left(A^{\left\lceil\frac{\mathrm{ord}(g_1)}{\mathsf{v}_{g_1}(A)}\right\rceil}\right)\right| \geq 2$. For every $g \in G_0 \setminus G_1$, we obtain

$$\mathsf{v}_g(A)\left\lceil\frac{\mathrm{ord}(g_1)}{\mathsf{v}_{g_1}(A)}\right\rceil \leq \mathsf{v}_g(A)\left\lceil\frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)}\right\rceil < 2\mathrm{ord}(g) \leq \exp(G)\mathsf{v}_g(S).$$

If $\mathsf{v}_{g_0}(A)\left\lceil\frac{\mathrm{ord}(g_1)}{\mathsf{v}_{g_1}(A)}\right\rceil \leq \mathrm{ord}(g_0) = \exp(G)$, then

$$A^{\left\lceil\frac{\mathrm{ord}(g_1)}{\mathsf{v}_{g_1}(A)}\right\rceil} \quad \text{divides} \quad S^{\exp(G)}$$

and hence $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

Otherwise for every $g \in G \setminus G_1$, we have

$$\frac{\exp(G)}{\mathsf{v}_{g_0}(A)} < \left\lceil\frac{\mathrm{ord}(g_1)}{\mathsf{v}_{g_1}(A)}\right\rceil \leq \left\lceil\frac{\mathrm{ord}(g)}{\mathsf{v}_g(A)}\right\rceil \leq \left\lceil\frac{\exp(G)\mathsf{v}_g(S)}{2\mathsf{v}_g(A)}\right\rceil \leq \frac{\exp(G)\mathsf{v}_g(S)}{\mathsf{v}_g(A)}.$$

Therefore $\mathsf{v}_g(A) < \mathsf{v}_{g_0}(A)\mathsf{v}_g(S)$ for all $g \in G_0 \setminus G_1$ which implies that A divides $S^{\mathsf{v}_{g_0}(A)}$. Thus there exists a zero-sum sequence W over $G_0 \setminus G_1$ such that $S^{\mathsf{v}_{g_0}(A)} = A \cdot W$. Since $G_0 \setminus G_1$ is half-factorial, we obtain $k(A) = \mathsf{v}_{g_0}(A)k(S) - k(W)$ is an integer, a contradiction to $k(A) < 1$. □

Proof of Theorem 1.1. By the definition of transfer Krull monoid, it suffices to prove the assertions for $H = \mathcal{B}(G_0)$ and hence H is half-factorial if and only if G_0 is half-factorial. If G_0 is half-factorial, it is easy to see that $\mathsf{hf}(G_0) = 1$ and $\left|\mathsf{L}\left(\prod_{g\in G_0} g^{2\mathrm{ord}(g)}\right)\right| = 1$. Therefore we only need to show that (b) implies (c) and that (d) implies (c).

(b) ⇒ (c) Suppose $\mathsf{hf}(G_0) = 1$ and assume to the contrary that G_0 is not half-factorial. Then there exists $A \in \mathcal{A}(G_0)$ such that $\mathsf{k}(A) \neq 1$, whence $\mathrm{supp}(A)$ is not half-factorial. Therefore $\mathsf{hf}(\mathrm{supp}(A)) \geq 2$, a contradiction.

(d) ⇒ (c) Suppose $\left|\mathsf{L}\left(\prod_{g\in G_0} g^{2\mathrm{ord}(g)}\right)\right| = 1$ and assume to the contrary that G_0 is not half-factorial. If G_0 is an LCN set, then Proposition 3.1.1 implies that $\left|\mathsf{L}\left(\prod_{g\in G_0} g^{\mathrm{ord}(g)}\right)\right| \geq 2$, a contradiction. Thus there exists an atom $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) < 1$ and we may assume that $\mathsf{k}(A)$ is minimal over all atoms of $\mathcal{B}(G_0)$. Let $g_0 \in \mathrm{supp}(A)$. Then by Lemma 2.2.3, we have $\left|\mathsf{L}\left(A^{\left\lceil\frac{\mathrm{ord}(g_0)}{\mathsf{v}_{g_0}(A)}\right\rceil}\right)\right| \geq 2$, a contradiction to $A^{\left\lceil\frac{\mathrm{ord}(g_0)}{\mathsf{v}_{g_0}(A)}\right\rceil} \mid \prod_{g\in G_0} g^{2\mathrm{ord}(g)}$. □

Proof of Theorem 1.2. By the definition of transfer Krull monoid, it suffices to prove all assertions for $H = \mathcal{B}(G)$.

1. Suppose $\exp(G) < \infty$. If $|G| \geq 3$, then 2 implies that $\mathsf{hf}(G) < \infty$. If $|G| \leq 2$, then $\mathcal{B}(G)$ is half-factorial and hence $\mathsf{hf}(G) = 1$.

Suppose $\exp(G) = \infty$. If there exists an element $g \in G$ with $\mathrm{ord}(g) = \infty$, then $A_n = ((n+1)g)(-ng)(-g)$ is an atom for every $n \in \mathbb{N}$. Since $\{(n+1)g, -ng, -g\}$ is not half-factorial and $|\mathsf{L}(A_n^n)| = 1$ for every $n \geq 2$, we obtain that $\mathsf{hf}(G) \geq n$ for every $n \geq 2$, that is, $\mathsf{hf}(G) =$

∞. Otherwise G is torsion. Then there exists a sequence $(g_i)_{i=1}^{\infty}$ with $g_i \in G$ and $\lim_{i\to\infty} \mathrm{ord}(g_i) = \infty$. It follows by 1 that $\mathsf{hf}(G) \geq \mathsf{hf}(\langle g_i \rangle) \geq \mathrm{ord}(g_i)$ for all $i \in \mathbb{N}$, that is, $\mathsf{hf}(G) = \infty$.

2. If G is an elementary 2-group and e_1, e_2 are two independent elements, then $\{e_1, e_2, e_1 + e_2\}$ is not a half-factorial set and $|\mathsf{L}(e_1 e_2 (e_1 + e_2))| = 1$ which implies that $\mathsf{hf}(G) \geq 2 = \exp(G)$. Otherwise there exists an element $g \in G$ with $\mathrm{ord}(g) = \exp(G) \geq 3$. Since $\{g, -g\}$ is not half-factorial and $\left|\mathsf{L}\left(g^{\mathrm{ord}(g)-1}(-g)^{\mathrm{ord}(g)-1}\right)\right| = 1$, we obtain $\mathsf{hf}(G) \geq \mathrm{ord}(g) = \exp(G)$.

Let S be a zero-sum sequence over G such that $\mathrm{supp}(S)$ is not half-factorial. In order to prove $\mathsf{hf}(G) \leq \lfloor \frac{3\exp(G)-3}{2} \rfloor$, we show that

$$\left|\mathsf{L}\left(S^{\lfloor \frac{3\exp(G)-3}{2} \rfloor}\right)\right| \geq 2.$$

Set $G_0 = \mathrm{supp}(S)$. If G_0 is an LCN-set, the assertion follows by Proposition 3.1.1. Suppose there exists an atom $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) < 1$. Let $A_0 \in \mathcal{A}(\mathrm{supp}(S))$ be such that $\mathsf{k}(A_0)$ is minimal over all minimal zero-sum sequences over G_0 and set $A_0 = g_1^{l_1} \cdot \ldots \cdot g_y^{l_y}$, where $y, l_1, \ldots l_y \in \mathbb{N}$ and $g_1, \ldots, g_y \in \mathrm{supp}(S)$ are pairwise distinct elements. If there exists $j \in [1, y]$ such that $2l_j \geq \mathrm{ord}(g_j)$, then $g_j^{\mathrm{ord}(g_j)}$ divides A_0^2 and hence $A_0^2 = g_j^{\mathrm{ord}(g_j)} \cdot W$ for some non-empty sequence $W \in \mathcal{B}(\mathrm{supp}(S))$. Thus $\mathsf{k}(W) = 2\mathsf{k}(A_0) - 1 < \mathsf{k}(A_0)$, a contradiction to the minimality of $\mathsf{k}(A_0)$. Therefore

$$2l_i \leq \mathrm{ord}(g_i) - 1 \text{ for all } i \in [1, y].$$

After renumbering if necessary, we assume $\frac{\mathrm{ord}(g_1)}{l_1} = \min\left\{\frac{\mathrm{ord}(g_i)}{l_i} | i \in [1, y]\right\}$. Then

$$l_i \left\lceil \frac{\mathrm{ord}(g_1)}{l_1} \right\rceil \leq l_i \left\lceil \frac{\mathrm{ord}(g_i)}{l_i} \right\rceil \leq l_i \frac{\mathrm{ord}(g_i) + l_i - 1}{l_i} \leq \mathrm{ord}(g_i) + \frac{\mathrm{ord}(g_i) - 1}{2} - 1 \leq \frac{3\exp(G) - 3}{2},$$

which implies that $A_0^{\lceil \frac{\mathrm{ord}(g_1)}{l_1} \rceil}$ divides $S^{\lfloor \frac{3\exp(G)-3}{2} \rfloor}$. The assertion follows by Lemma 2.2.3.

3(a). Suppose that G is cyclic and that $g \in G$ with $\mathrm{ord}(g) = |G| \geq 3$. We will show that $\mathsf{hf}(G) = \exp(G)$.

Let S be a zero-sum sequence over G such that $\mathrm{supp}(G)$ is not half-factorial. It suffices to show that $|\mathsf{L}(S^{\exp(G)})| \geq 2$. If $|\{g \in \mathrm{supp}(S) | \mathrm{ord}(g) = |G| \mathsf{v}_g(S)\}| \leq 1$, then the assertion follows from Proposition 3.1.4. Suppose $|\{g \in \mathrm{supp}(S) | \mathrm{ord}(g) = |G| \mathsf{v}_g(S)\}| \geq 2$. Then there exist distinct $g_1, g_2 \in \mathrm{supp}(S)$ such that $\mathrm{ord}(g_1) = \mathrm{ord}(g_2) = |G|$. We may assume that $g_1 = kg_2$ for some $k \in \mathbb{N}_{\geq 2}$ with $\gcd(k, |G|) = 1$. It follows by $\mathsf{k}\left(g_1^{|G|-k} \cdot g_2\right) < 1$ that $G_0 = \{g_1, g_2\} = \{g_1, kg_1\}$ is not half-factorial. By Proposition 3.1.2, we obtain that $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

3(b). Suppose G is a finite abelian group with $\exp(G) \leq 6$. We need to prove that $\mathsf{hf}(G) = \exp(G)$. Let S be a zero-sum sequence over G such that $\mathrm{supp}(G)$ is not half-factorial. It suffices to show that $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

If $\mathrm{supp}(S)$ is an LCN-set, the assertion follows by Proposition 3.1.1. Thus there is a minimal zero-sum sequence W over $\mathrm{supp}(S)$ such that

$$\mathsf{k}(W) < 1.$$

By Proposition 3.1.2 and Lemma 2.2.2, we have

$$|\mathrm{supp}(W)| \geq 3.$$

Suppose $W \mid S$. Since $\mathsf{k}(W) < 1$, it follows by Lemma 2.2 that $|\mathsf{L}(W^{\exp(G)})| \geq 2$ and hence $|\mathsf{L}(S^{\exp(G)})| \geq 2$. Therefore we may assume that $W \nmid S$, whence $|W| \geq |\mathrm{supp}(W)| + 1 \geq 4$. It follows that $6 \geq \exp(G) \geq \frac{|W|}{\mathsf{k}(W)} > |W| \geq 4$.

We distinguish two cases according to $\exp(G) \in \{5, 6\}$.

Case 1. $\exp(G) = 5$.

Then, $G \cong C_5^r$ and for all $W \in \mathcal{A}(\mathrm{supp}(S))$ with $\mathsf{k}(W) < 1$, we have that

$$W \nmid S, \quad |\mathrm{supp}(W)| = 3, \quad \text{and} \quad |W| = 4.$$

Let W_0 be an atom over $\mathrm{supp}(S)$ with $\mathsf{k}(W_0) < 1$. Then

$$W_0 = g_1^2 g_2 g_3 \quad \text{and} \quad S = T g_1 g_2 g_3,$$

where $g_1, g_2, g_3 \in \mathrm{supp}(S)$ are pairwise distinct and $T \in \mathcal{F}(\mathrm{supp}(S) \setminus \{g_1\})$ with $\sigma(T) = g_1$.

We may assume T is zero-sum free. Otherwise $T = T_0 T'$, where T_0 is a zero-sum sequence and T' is zero-sum free. We can replace S by $T' g_1 g_2 g_3$, since $\left|\mathsf{L}\left((T' g_1 g_2 g_3)^5\right)\right| \geq 2$ implies that $|\mathsf{L}(S^5)| \geq 2$. Therefore S is a product of at most three atoms and every term of S has order 5.

Assume to the contrary that $|\mathsf{L}(S^5)| = 1$, that is, $\mathsf{L}(S^5) = \{|T| + 3\}$. Since $g_1^5 g_2^5 g_3^5 = W_0^2 (g_1 g_2^3 g_3^3)$ is a zero-sum subsequence of S^5, we obtain that $g_1 g_2^3 g_3^3$ is an atom. Note that

$$(g_1^2 g_2 g_3)^2 S = (g_1^4 T)(g_1 g_2^3 g_3^3) \text{ is a zero} - \text{sum subsequence of } S^5.$$

Suppose S is an atom. Then $\mathsf{L}(g_1^4 T) = \{2\}$ and hence $|g_1^4 T| \geq 2 \times 4 = 8$, that is, $|T| \geq 4$. It follows by $\{5\} = \mathsf{L}(S^5) = \{|T| + 3\}$ that $|T| = 2$, a contradiction.

Suppose S is a product of two atoms. Then $\mathsf{L}(g_1^4 T) = \{3\}$ and hence $|g_1^4 T| \geq 3 \times 4 = 12$, that is, $|T| \geq 8$. It follows by $\{10\} = \mathsf{L}(S^5) = \{|T| + 3\}$ that $|T| = 7$, a contradiction.

Suppose S is a product of three atoms. Then $\mathsf{L}(g_1^4 T) = \{4\}$ which implies that $T = T_1 T_2 T_3 T_4$ such that $g_1 T_i$ is zero-sum for all $i \in [1, 4]$. Since $g_1 T_i \mid S$, we obtain $\mathsf{k}(g_1 T_i) \geq 1$ and hence $|g_1 T_i| \geq 5$. Therefore $|g_1^4 T| \geq 4 \times 5 = 20$, that is, $|T| \geq 16$. It follows by $\{15\} = \mathsf{L}(S^5) = \{|T| + 3\}$ that $|T| = 12$, a contradiction.

Case 2. $\exp(G) = 6$.

Let W be an atom over $\mathrm{supp}(S)$ with $\mathsf{k}(W) < 1$. If $|W| = 4$, then $|\mathrm{supp}(W)| = 3$ and hence W must be of the form

$$W = g_1^2 g_2 g_3$$

where $g_1, g_2, g_3 \in \mathrm{supp}(S)$ are pairwise distinct. Since $(g_1^6)(g_2^6)(g_3^6) = W^3 (g_2^3 g_3^3)$, we obtain that $|\mathsf{L}(g_1^6 g_2^6 g_3^6)| \geq 2$. It follows from the fact that $g_1^6 g_2^6 g_3^6$ divides $S^{\exp(G)}$ that $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

If $|W| = 5$, then $|\mathrm{supp}(W)| = 3$ or 4 and hence W can be written in one of the following ways.

i. $W = g_1^3 g_2 g_3$, where $g_1, g_2, g_3 \in \mathrm{supp}(S)$ are pairwise distinct.
ii. $W = g_1^2 g_2^2 g_3$, where $g_1, g_2, g_3 \in \mathrm{supp}(S)$ are pairwise distinct.
iii. $W = g_1^2 g_2 g_3 g_4$, where $g_1, g_2, g_3, g_4 \in \mathrm{supp}(S)$ are pairwise distinct.

Suppose (i) holds. Then $0 = 2\sigma(W) = 6g_1 + 2g_2 + 2g_3 = 2g_2 + 2g_3$. Since $(g_1^6)(g_2^6)(g_3^6) = W^2 (g_2^2 g_3^2)^2$, we obtain that $|\mathsf{L}(g_1^6 g_2^6 g_3^6)| \geq 2$. It follows from the fact that $g_1^6 g_2^6 g_3^6$ divides $S^{\exp(G)}$ that $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

Suppose (ii) holds. Then $0 = 3\sigma(W) = 6g_1 + 6g_2 + 3g_3 = 3g_3$. Thus $\mathrm{ord}(g_3) = 2$ and hence $\mathsf{k}(W) \geq 1/2 + 4/6 > 1$, a contradiction.

Suppose (iii) holds. Then $0 = 3\sigma(W) = 6g_1 + 3g_3 + 3g_3 + 3g_4 = 3g_2 + 3g_3 + 3g_4$. Therefore $W_0 = g_2^3 g_3^3 g_4^3$ is zero-sum. If $W_0 = g_1^6 g_2^6 g_3^6 g_4^6 (W)^{-3}$ is not a minimal zero-sum sequence, then $|\mathsf{L}(g_1^6 g_2^6 g_3^6 g_4^6)| \geq 2$ and hence $|\mathsf{L}(S^{\exp(G)})| \geq 2$. If W_0 is minimal zero-sum, then $g_1^6 g_2^6 g_3^6 g_4^6 = (g_1^6) W_0^2$ implies that $|\mathsf{L}(g_1^6 g_2^6 g_3^6 g_4^6)| \geq 2$ and hence $|\mathsf{L}(S^{\exp(G)})| \geq 2$. □

Acknowledgments

We thank the referee for very careful reading and for providing valuable suggestions.

Funding

This work has been supported in part by the National Science Found of China with grant no. 11671218 and by the Austrian Science Fund with grant no. P33499.

ORCID

Chao Liu http://orcid.org/0000-0001-6759-709X

References

[1] Adams, D., Ardila, R., Hannasch, D., Kosh, A., McCarthy, H., Ponomarenko, V., Rosenbaum, R. (2009). Bifurcus semigroups and rings. *Involve*. 2(3):351–356. DOI: 10.2140/involve.2009.2.351.

[2] Baeth, N. R., Smertnig, D. (2015). Factorization theory: from commutative to noncommutative settings. *J. Algebra* 441:475–551. DOI: 10.1016/j.jalgebra.2015.06.007.

[3] Baginski, P., Kravitz, R. (2010). A new characterization of half-factorial Krull monoids. *J. Algebra Appl.* 09(05):825–837. DOI: 10.1142/S0219498810004269.

[4] Carlitz, L. (1960). A characterization of algebraic number fields with class number two. *Proc. Am. Math. Soc.* 11:291–392.

[5] Chapman, S. T., Coykendall, J. (2000). Half-factorial domains, a survey. In: *Non-Noetherian Commutative Ring Theory, Mathematics and Its Applications*, Vol. 520, Boston, MA: Kluwer Academic Publishers, pp. 97–115.

[6] Chapman, S. T., Krause, U., Oeljeklaus, E. (2000). Monoids determined by a homogeneous linear diophantine equation and the half-factorial property. *J. Pure Appl. Algebra* 151(2):107–133. DOI: 10.1016/S0022-4049(99)00062-6.

[7] Coykendall, J. (2005). Extensions of half-factorial domains: A survey. In: *Arithmetical Properties of Commutative Rings and Monoids, Lect. Notes Pure Appl. Math.*, Vol. 241. Boca Raton, FL: Chapman & Hall/CRC, pp. 46–70.

[8] Fan, Y., Geroldinger, A., Kainrath, F., Tringali, S. (2017). Arithmetic of commutative semigroups with a focus on semigroups of ideals and modules. *J. Algebra Appl.* 16(12):1750234–1750242. DOI: 10.1142/S0219498817502346.

[9] Fan, Y., Tringali, S. (2018). Power monoids: A bridge between factorization theory and arithmetic combinatorics. *J. Algebra* 512:252–294. DOI: 10.1016/j.jalgebra.2018.07.010.

[10] Geroldinger, A. (2016). Sets of lengths. *Amer. Math. Mon.* 123:960–988.

[11] Geroldinger, A., Göbel, R. (2003). Half-factorial subsets in infinite abelian groups. *Houst. J. Math.* 29: 841–858.

[12] Geroldinger, A., Halter-Koch, F. (2006). Non-unique factorizations. In: *Algebraic, Combinatorial and Analytic Theory, Pure and Applied Mathematics*, Vol. 278, Boca Raton, FL: Chapman & Hall/CRC.

[13] Geroldinger, A., Kainrath, F., Reinhart, A. (2015). Arithmetic of seminormal weakly Krull monoids and domains. *J. Algebra* 444:201–245. DOI: 10.1016/j.jalgebra.2015.07.026.

[14] Geroldinger, A., Reinhart, A. (2019). The monotone catenary degree of monoids of ideals. *Int. J. Algebra Comput.* 29(03):419–457. DOI: 10.1142/S0218196719500097.

[15] Geroldinger, A., Zhong, Q. (2016). The set of minimal distances in Krull monoids. *Acta Arith.* 173:1–120. DOI: 10.4064/aa7906-1-2016.

[16] Geroldinger, A., Zhong, Q. (2020). Factorization theory in commutative monoids. *Semigroup Forum.* 100(1):22–51. DOI: 10.1007/s00233-019-10079-0.

[17] Gotti, F. (2020). Geometric and combinatorial aspects of submonoids of a finite-rank free commutative monoid. *Linear Algebra Appl*. 604:146–186. DOI: 10.1016/j.laa.2020.06.009.

[18] Kainrath, F., Lettl, G. (2000). Geometric notes on monoids. *Semigroup Forum*. 61(2):298–302. DOI: 10.1007/PL00006026.

[19] Malcolmson, P., Okoh, F. (2009). Power series extensions of half-factorial domains. *J. Pure Appl. Algebra* 213(4):493–495. DOI: 10.1016/j.jpaa.2008.07.014.

[20] Malcolmson, P., Okoh, F. (2016). Half-factorial subrings of factorial domains. *J. Pure Appl. Algebra* 220(3): 877–891. DOI: 10.1016/j.jpaa.2015.06.011.

[21] Plagne, A., Schmid, W. A. (2005). On large half-factorial sets in elementary p-groups: maximal cardinality and structural characterization. *Isr. J. Math*. 145(1):285–310. DOI: 10.1007/BF02786695.

[22] Plagne, A., Schmid, W. A. (2005). On the maximal cardinality of half-factorial sets in cyclic groups. *Math. Ann*. 333(4):759–785. DOI: 10.1007/s00208-005-0690-y.

[23] Roitman, M. (2011). A quasi-local half-factorial domain with an atomic non-half-factorial integral closure. *J. Commut. Algebra* 3(3):431–438. DOI: 10.1216/JCA-2011-3-3-431.

[24] Schmid, W. A. (2005). Differences in sets of lengths of Krull monoids with finite class group. *J. Théor. Nombres Bordeaux*. 17(1):323–345. DOI: 10.5802/jtnb.493.

[25] Schmid, W. A. (2005). Half-factorial sets in finite abelian groups: a survey. *Grazer Math. Ber*. 348:41–64.

[26] Schmid, W. A. (2006). Half-factorial sets in elementary p-groups. *Far East J. Math. Sci*. 22:75–114.

[27] Schmid, W. A. (2009). Arithmetical characterization of class groups of the form $\mathbb{Z}/n\mathbb{Z}\oplus\mathbb{Z}/n\mathbb{Z}$ via the system of sets of lengths. *Abh. Math. Semin. Univ. Hamb*. 79:25–35.

[28] Skula, L. (1976). On c-semigroups. *Acta Arith*. 31(3):247–257. DOI: 10.4064/aa-31-3-247-257.

[29] Śliwa, J. (1976). Factorizations of distinct lengths in algebraic number fields. *Acta Arith*. 31(4):399–417. DOI: 10.4064/aa-31-4-399-417.

[30] Smertnig, D. (2019). Factorizations in bounded hereditary noetherian prime rings. *Proc. Edinb. Math. Soc*. 62(2):395–442. DOI: 10.1017/S0013091518000305.

[31] Smertnig, D. (2013). Sets of lengths in maximal orders in central simple algebras. *J. Algebra* 390:1–43. DOI: 10.1016/j.jalgebra.2013.05.016.

[32] Zaks, A. (1976). Half factorial domains. *Bull. Amer. Math. Soc*. 82(5):721–723. DOI: 10.1090/S0002-9904-1976-14130-4.

[33] Zaks, A. (1980). Half-factorial-domains. *Isr. J. Math*. 37(4):281–302. DOI: 10.1007/BF02788927.