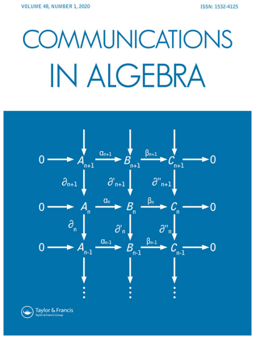

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Rings of congruence preserving functions, II

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ABSTRACT

For several classes of special p-groups G, of exponent p, $p > 2$, we show that the near-ring, $C_0(G)$, of congruence preserving functions on G is a ring if and only if G is a 1-affine complete group.

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1. Introduction

Let $G := \langle G, +, 0\rangle$ be a finite group written additively but not necessarily abelian, with neutral element 0. As usual, we let $M_0(G)$ denote the near-ring of zero-preserving functions on G under the operations of pointwise addition and function composition. We consider subnear-rings $P_0(G)$, the near-ring of polynomial functions on G, and $C_0(G)$, the near-ring of congruence preserving functions on G. We recall $P_0(G)$ is the subnear-ring generated by the inner automorphisms of G while a function $f \in M_0(G)$ is *congruence preserving* if, for each $x, y \in G$ and normal subgroup N of G, if $x - y \in N$, then $f(x) - f(y) \in N$.

We let $\eta(G)$ denote the lattice of normal subgroups of G and recall $\eta(G)$ is lattice isomorphic to the congruence lattice of G. For any subgroup H of G, the normal closure $\overline{H}$ of H is defined by $\overline{H} = \cap\{N \in \eta(G) | N \supseteq H\}$. For $x \in G$ we let $\overline{x} = \overline{\langle x\rangle}$ and thus we have $f \in C_0(G)$ if and only if $f(x) - f(y) \in \overline{x-y}$ for all $x, y \in G$. We have $I(G) = \langle \mathrm{Inn}(G)\rangle = P_0(G) \subseteq C_0(G) \subseteq M_0(G)$.

In this paper we continue the investigation initiated in [11] as to when $C_0(G)$ is a ring. Of course, if $C_0(G)$ is a ring so is $P_0(G)$ and from Chandy ([3]), $P_0(G)$ is a ring if and only if G is a 2-Engel group, i.e., every element of G commutes with all of its conjugates. Since a group G of nilpotency class at most 2 is 2-Engel, in this investigation we restrict to nilpotent groups of class at most 2, and using standard results, can restrict to p-groups of class at most 2. (See [11].) For finite abelian groups, A, $C_0(A)$ is a ring if and only if A is 1-affine complete, and the 1-affine complete finite abelian groups are known ([11]). Recall that a group G is 1-affine complete (1-ac) means $C_0(G) = P_0(G)$. For background material and history see [10, pp 158–160].

Several necessary conditions on finite non-abelian nilpotent p-groups of class 2 for $C_0(G)$ to be a ring were given in [11] (see Theorem II.1 below) and in these cases for $p \neq 2$ all the groups

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G were 1-ac. The first examples of 1-ac non-abelian p-groups were given by Dorda ([6]). These groups were p-groups, nilpotent of class 2, $\exp(G) = p, |G| = p^6, p > 2$.

In light of this example and some GAP examples, we restrict our attention to finite non-abelian p-groups, G, $p \neq 2$, of class 2 and $Z(G) = [G,G]$ and $G/[G,G]$ is elementary abelian, that is a *special* p-group. Recall that a finite group G is special if G is elementary abelian or G is nilpotent of class 2, $Z(G) = [G,G]$ and $G/[G,G]$ is elementary abelian. (The first occurrence we have found of these groups is in Hall and Higman ([9])). From group theory one finds that a non-abelian p-group, G, is special if G is nilpotent of class 2 and $Z(G) = [G,G] = \Phi(G)$ (the Frattini subgroup of G). A special p-group has exponent p or p^2 ([7]). We focus here on non-abelian special p-groups, G, $\exp(G) = p, p \neq 2$.

A further reason for restricting to these special p-groups is that Verardi ([13]) has shown that there exists an injective map $G \mapsto G_p$ from the class of finite groups into the class of special p-groups of exponent p. Thus information about the associated special p-group G_p may be used to obtain information about G.

In the remainder of the paper G will denote a non-abelian special p-group, $p \neq 2$, of exponent p. As usual, $Z(G)$ denotes the center of G, $[G,G] = G'$, the commutator subgroup, and $M_0(G), C_0(G)$ and $P_0(G)$ as defined above.

2. Background results: old and new

As indicated at the end of the previous section, henceforth our groups G will be non-abelian special p-groups of exponent $p \neq 2$. For ease of exposition we denote this by "Let $G \in \text{NAS}(\exp p \neq 2)$".

For $G \in \text{NAS}(\exp p \neq 2)$, let $\eta(G)$ denote the lattice of normal subgroups of G. Let $D, E \in \eta(G), D \subsetneq G, \{0\} \subsetneq E$. The pair (D, E) is called a splitting pair if for each $N \in \eta(G), N \subseteq D$ or $N \supseteq E$. If G contains a splitting pair, we say G *splits* or G *is split*. In the case $D = E$, we say D is a cutting element and G *is cut*.

For $x \in G$, we let $[x, G] = \langle [x,g] | g \in G \rangle = \{[x,g] | g \in G\}$. We have $\bar{x} = \langle x \rangle + [x, G]$ and $\bar{x}$ is abelian ([11, 3.3]).

For use in the sequel we collect some (mostly) known results. We note that some of these hold for any non-abelian p-group of nilpotency class 2.

Theorem II.1. *Let $G \in \text{NAS}(\exp p \neq 2)$. If any one of the following holds:*

1. *G is split ([11, 3.1]);*
2. *$|G| < p^6$ ([11, 4.7]);*
3. *G is abelian by cyclic ([11, 4.6]);*
4. *There exists $x \in G$ such that $[x, G]$ is cyclic ([6, Hilfsatz 9]);*
5. *The derived subgroup $G' = [G,G]$ is cyclic ([11, 4.1]);*
6. *G' is 2-generated, that is $G' \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$,*

then $C_0(G)$ is not a ring and thus G is not 1-ac.

Proof. For (4), Dorda ([6]) constructs a function $g \in C_0(G) - P_0(G)$. One finds that $g(id + id) \neq g \cdot id + g \cdot id$ so $C_0(G)$ is not a ring. For (6), we take $G' = \langle c_1, c_2 \rangle \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Let $x \in G - Z(G)$ and note $[x, G] = G'$, otherwise $[x, G]$ is cyclic and the result follows from (4). For $N \in \eta(G)$, if $N \not\subseteq Z(G)$ then for $x \in N - Z(G), G' = [x, G] \subseteq N$. Thus $Z(G)$ cuts G and we use (1). □

We mention two additional cases. In [4], Corsi Tani proved that if G is a finite p-group of nilpotency class 2 having an automorphism $\sigma \neq id$, with $\gcd(|\sigma|, p) = 1$, and such that $\sigma(N) \subseteq N$ for all $N \in \eta(G)$, then $G \in \text{NAS}(\exp p \neq 2)$ and G is cut. Thus these groups are not 1-ac and $C_0(G)$ is not a ring. Gorenstein ([7]) calls $G \in \text{NAS}(\exp p \neq 2)$ *extra special* if $|G'| = p$ and proceeds to discuss the use of extra special groups in the classification problem of finite simple groups. From Theorem II.1, extra special p-groups, G, are not 1-ac and $C_0(G)$ is not a ring.

We know if G is not cut then $\eta(G)$ is a simple lattice ([2, Lemma 6.1]). For $H, K \in \eta(G)$ the interval $I(H, K)$ is said to be a *prime interval* if $|I(H, K)| = 2$ and in this case we write $H \prec K$. From lattice theory, when $\eta(G)$ is simple then any two prime intervals are projective, hence if $H \prec K$ and $A \prec B$, then B/A and K/H are $C_0(G)$ isomorphic (See also [1] and [2]).

Lemma II.2. *Let $G \in \mathrm{NAS}(\exp p \neq 2), |Z(G)| \geq p^2$. Let $f \in C_0(G)$. Then*

1. *1. $f|_{Z(G)} = k \cdot id, k \in \mathbb{Z}_p$.*
2. *If G is not cut, then $h = f - k \cdot id \in C_0(G)$, where $k \in \mathbb{Z}_p$ is given in part 1, and $h(G) \subseteq Z(G), h(Z(G)) = \{0\}$.*

Proof.

1. Let $x, y \in Z(G)$ and $f \in C_0(G)$. Then $f(x) \in \bar{x}$ so $f(x) = k_x \cdot x$ and $f(y) = k_y \cdot y, k_x, k_y \in \mathbb{Z}_p$. Since $f \in C_0(G), f(x) - f(y) \in \overline{x - y}$, that is $k_x \cdot x - k_y \cdot y = k(x - y), k \in \mathbb{Z}_p$. Therefore $(k_x - k)x = (k_y - k)y$. If $\langle x \rangle \neq \langle y \rangle$, then $k_x = k = k_y$. If $\langle x \rangle = \langle y \rangle$, then since $|Z(G)| \geq p^2$, there exists $w \in Z(G), \langle w \rangle \neq \langle x \rangle$ and $\langle w \rangle \neq \langle y \rangle$. But then $k_x = k_y = k_w$ so $f|_{Z(G)} = \ell \cdot id, \ell \in \mathbb{Z}_p$. (This also follows from ([12]) since $Z(G)$ is affine complete.)
2. Let $N, M \in \eta(G), \{0\} \prec N \subseteq Z(G)$ and $M \prec G$. Then $I(\{0\}, N)$ and $I(M, G)$ are projective so G/M and $N/\{0\}$ are $C_0(G)$ isomorphic. For $f \in C_0(G)$ we find from the first part, $f|_{Z(G)} = \ell \cdot id, \ell \in \mathbb{Z}_p$, so for $h = f - \ell \cdot id, h(Z(G)) = (f - \ell \cdot id)(Z(G)) = \{0\}$. Thus $h(G) \subseteq \cap\{M \in \eta(G) \mid M \prec G\} \subseteq Z(G)$. □

From Theorem II.1 we know if G is cut, G is not 1-ac. Thus in the sequel, when G is not cut and when attempting to show that arbitrary $f \in C_0(G)$ is also a polynomial function, without loss of generality we consider $h = f - \ell \cdot id$.

We introduce some further notation and concepts. Let $G \in \mathrm{NAS}(\exp p \neq 2)$. Then $G/Z(G)$ and $Z(G) = G'$ are $\mathbb{Z}_p$-vector spaces, say

$$G/Z(G) = \langle e_1 + Z(G), ..., e_n + Z(G) \rangle \text{ and } Z(G) = \langle c_1, ..., c_s \rangle$$

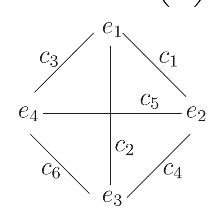
so $G = \langle e_1, ..., e_n, c_1, ..., c_s \rangle$ and $|G| = p^{n+s}, n = \dim_{\mathbb{Z}_p}(G/Z(G)), s = \dim_{\mathbb{Z}_p}(Z(G))$. Note $T := \{[e_i, e_j] | 1 \leq i < j \leq n\}$ is a generating set for $Z(G) = G'$ so without loss of generality we take $B := \{c_1, ..., c_s\} \subseteq T$. Thus each $[e_i, e_j] \in T - B$ is a linear combination of elements from B. See Cortini ([5]) for this representation of $G \in \mathrm{NAS}(\exp p \neq 2)$.

We mention that another computational approach to non-abelian special p-groups of exponent p is given by Grundhöfer and Stroppel ([8]) in their investigations of Heisenberg groups. This approach is used to obtain information about automorphisms of these special p-groups.

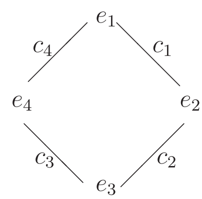
We next introduce a directed graph in which the defining information of our groups is enclosed. Let $G \in \mathrm{NAS}(\exp p \neq 2)$ be given by $G = \langle e_1, ..., e_n, c_1, ..., c_s \rangle$ and the linear combinations for $[e_\ell, e_k] \in T - B, [e_\ell, e_k] = \sum_{i=1}^{s} \alpha_{l,k}^i c_i$. The vertices are the generators $\{e_1, ..., e_n\}$ and the directed edges are $[e_i, e_j], i < j$. For $x, y \in G, x = \sum_{i=1}^{n} a_i e_i + z_1, y = \sum_{j=1}^{n} b_j e_j + z_2, z_1, z_2 \in Z(G), [x, y]$ can be determined from the graph.

Example II.3

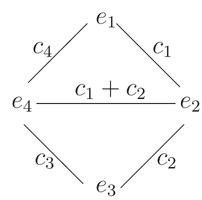
A. G is *full*. G is isomorphic to $\langle e_1, ..., e_n, c_1, ..., c_s \rangle, s = \binom{n}{2}$. In this case $B = T$. For $n = 4$ we have



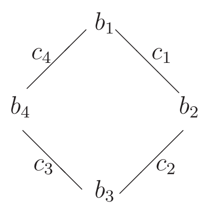
B. G is *circular*. G is isomorphic to $\langle e_1, \ldots, e_n, c_1, \ldots, c_n \rangle, [e_i, e_{i+1}] = c_i$ where we take $e_{n+1} = e_1$, and other $[e_\ell, e_k] = 0$. For $n = 4$ we have



C. Consider G given by



where $[e_i, e_{i+1}] = c_i, 1 \leq i \leq 4$ (with $e_5 = e_1$), and $[e_2, e_4] = c_1 + c_2$. Note $[e_2, e_4] = [e_1, e_2] + [e_2, e_3]$ so $[e_2, e_4 - e_3 + e_1] = 0$. Thus G is isomorphic to



where $b_1 = e_1, b_2 = e_2, b_3 = e_3, b_4 = e_4 - e_3 + e_1$. Therefore G is circular.

In the next section, with the aid of this graphical representation, we determine new classes of non-abelian special p-groups of exponent p which are 1-ac and new classes which are not 1-ac. In these latter classes, $C_0(G)$ is not a ring.

3. Main results

As usual, $G \in \mathrm{NAS}(\exp p \neq 2), |G| = p^{n+s}, n = \dim_{\mathbb{Z}_p}(G/Z(G)), s = \dim_{\mathbb{Z}_p}(Z(G))$.

Theorem III.1. (Full) *Let* $G = \langle e_1, \ldots, e_n, c_1, \ldots, c_s \rangle, n \geq 3, s = \binom{n}{2}$. *Then* G *is 1-ac.*

Proof. Let $f \in C_0(G)$. As we have shown above, we may assume that $f(G) \subseteq Z(G)$, so we let $f(u) = [u, d_u], u, d_u \in G$. We also let $[e_i, e_j] = c_{ij}$ for $1 \leq i < j \leq n$. From $f(e_2) - f(e_1) \in [e_2 - e_1, G]$ we have $f(e_2) - f(e_1) = [e_2 - e_1, x]$ for some $x \in G$. It follows that

$$\begin{aligned} [e_2, d_{e_2} - d_{e_1}] &= [e_2, d_{e_2}] - [e_2, d_{e_1}] - [e_1, d_{e_1}] + [e_1, d_{e_1}] \\ &= f(e_2) - f(e_1) + [e_1, d_{e_1}] - [e_2, d_{e_1}] \\ &= [e_2 - e_1, x] - [e_2 - e_1, d_{e_1}] \\ &\in [e_2 - e_1, G]. \end{aligned}$$

So we have $[e_2, d_{e_2} - d_{e_1}] \in [e_2 - e_1, G] \cap [e_2, G]$. Furthermore,

$$\begin{aligned}[e_2 - e_1, G] &= \langle [e_2 - e_1, e_1], [e_2 - e_1, e_2], [e_2 - e_1, e_3], \ldots, [e_2 - e_1, e_n] \rangle \\ &= \langle c_{12}, c_{23} - c_{13}, \ldots, c_{2n} - c_{1n} \rangle,\end{aligned}$$

while

$$[e_2, G] = \langle [e_2, e_1], [e_2, e_2], [e_2, e_3], \ldots, [e_2, e_n] \rangle = \langle c_{12}, c_{23}, \ldots, c_{2n} \rangle.$$

From the linear independence of the c_{ij}, it follows that $[e_2 - e_1, G] \cap [e_2, G] = \langle c_{12} \rangle$, and hence $d_{e_2} - d_{e_1}$ is forced to have the form $\alpha e_1 + \beta e_2 + c, \alpha, \beta \in \mathbb{Z}_p, c \in Z(G)$. If we let $\widehat{d_1} = d_{e_1} + \alpha e_1 + c$ then $[e_1, \widehat{d_1}] = [e_1, d_{e_1}]$ and $[e_2, d_{e_2}] = [e_2, d_{e_1} + \alpha e_1 + \beta e_2 + c] = [e_2, \widehat{d_1} + \beta e_2] = [e_2, \widehat{d_1}]$. We put $d = \widehat{d_1}$ so that $f(e_1) = [e_1, d]$ and $f(e_2) = [e_2, d]$. For $j \geq 3$, it follows similarly that $f(e_j) - f(e_2) \in [e_j - e_2, G]$ and so $[e_j, d_{e_j} - d] \in [e_j - e_2, G] \cap [e_j, G] = \langle c_{2j} \rangle$. Using $f(e_j) - f(e_1) \in [e_j - e_1, G]$ we get $[e_j, d_{e_j} - d] \in [e_j - e_1, G] \cap [e_1, G] = \langle c_{1j} \rangle$ so $[e_j, d_{e_j} - d] = 0$, and $f(e_i) = [e_i, d], i = 1, 2, \ldots, n$.

For αe_j, let $f(\alpha e_j) = [\alpha e_j, g], g \in G$. Using $f(\alpha e_j) - f(e_i) \in [\alpha e_j - e_i, G], i \neq j$, we get $[\alpha e_j, g - d] = 0$ and $f(\alpha e_j) = [\alpha e_j, d]$.

Thus we find $d \in G, f(\alpha e_i) = [\alpha e_i, d]$ for each $i \in \{1, \ldots, n\}, \alpha \in \mathbb{Z}_p$.

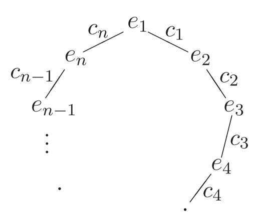
Now let $u = u_i e_i + u_j e_j + c, i \neq j, u_i \neq 0 \neq u_j, \quad c \in Z(G)$. As above we find $[u, d_u - d] \in [e_i, G] \cap [e_j, G] = \langle c_{ij} \rangle$ and for $k \neq i, k \neq j, [u, d_u - d] \in [u + e_k, G] \cap [e_i, G] \cap [e_j, G] = \{0\}$. Hence $f(u) = [u, d]$.

By induction, $f(\sum_{i=1}^n u_i e_i + c) = [\sum_{i=1}^n u_i e_i, d]$ so $f \in P_0(G)$, that is, G is 1-ac.

We may now assume $n \geq 4$. For if $n = 3$, then $s \leq \binom{3}{2} = 3$. If $s = 3$ then from the above theorem, G is 1-ac. If $s = 1$ or $s = 2$, then from Theorem II.1, $C_0(G)$ is not a ring. For $n = 2$, $s = 1$ we are again finished using Theorem II.1. It should be mentioned that the "full" case $n = 3, s = 3$ is the example of Dorda mentioned above.

Theorem III.2. (Circular) *Let G be circular, i.e., G is isomorphic to $\langle e_1, \ldots, e_n, c_1, \ldots, c_n \rangle, [e_i, e_{i+1}] = c_i$ where we take $e_{n+1} = e_1$, and the other $[e_\ell, e_k] = 0$. Then G is 1-ac if and only if $C_0(G)$ is a ring if and only if n is odd.*

Proof. We have



where we take $c_i = [e_i, e_{i+1}]$ and identify $e_{n+1} = e_1, i = 1, \ldots, n$.

Let n be even, $s = n, |G| = p^{2n}$ and define a function $h \in M_0(G)$ by

$$h\left(\sum_{i=1}^n k_i e_i + c\right) = \sum_{i=1}^n (-1)^{i+1} k_i k_{i+1} c_i, \text{ where } k_{n+1} = k_1,$$

and where $c \in Z(G)$ is arbitrary. We show $h \in C_0(G)$ and show $C_0(G)$ is not a ring.

Now

$$h\left(\sum_{i=1}^{n} k_i e_i\right) - h\left(\sum_{i=1}^{n} \ell_i e_i\right) = \sum_{i=1}^{n} (-1)^{i+1}(k_i k_{i+1} - \ell_i \ell_{i+1})c_i, \text{ where } \ell_{n+1} = \ell_1$$
$$= (k_1k_2 - \ell_1\ell_2)c_1 - (k_2k_3 - \ell_2\ell_3)c_2 + (k_3k_4 - \ell_3\ell_4)c_3$$
$$- (k_4k_5 - \ell_4\ell_5)c_4 + - \cdots$$
$$+ (k_{n-1}k_n - \ell_{n-1}\ell_n)c_{n-1} - (k_nk_1 - \ell_n\ell_1)c_n. \ (*)$$

Also,

$$\left[\sum_{i=1}^{n} k_i e_i - \sum_{i=1}^{n} \ell_i e_i, G\right] = \left[\sum_{i=1}^{n} (k_i - \ell_i)e_i, G\right]$$
$$= \langle (k_n - \ell_n)c_n - (k_2 - \ell_2)c_1, (k_1 - \ell_1)c_1 - (k_3 - \ell_3)c_2,$$
$$(k_2 - \ell_2)c_2 - (k_4 - \ell_4)c_3, (k_3 - \ell_3)c_3 - (k_5 - \ell_5)c_4,$$
$$\vdots$$
$$(k_{n-2} - \ell_{n-2})c_{n-2} - (k_n - \ell_n)c_{n-1}, (k_{n-1} - \ell_{n-1})c_{n-1} - (k_1 - \ell_1)c_n\rangle.$$

We see that $(*)$ can be written as

$$- k_1[(k_n - \ell_n)c_n - (k_2 - \ell_2)c_1] + \ell_2[(k_1 - \ell_1)c_1 - (k_3 - \ell_3)c_2]$$
$$- k_3[(k_2 - \ell_2)c_2 - (k_4 - \ell_4)c_3] + \ell_4[(k_3 - \ell_3)c_3 - (k_5 - \ell_5)c_4]$$
$$\vdots$$
$$- k_{n-1}[(k_{n-2} - \ell_{n-2})c_{n-2} - (k_n - \ell_n)c_{n-1}] + \ell_n[(k_{n-1} - \ell_{n-1})c_{n-1} - (k_1 - \ell_1)c_n],$$

that is,

$$h\left(\sum_{i=1}^{n} k_i e_i\right) - h\left(\sum_{i=1}^{n} \ell_i e_i\right) \in \left[\sum_{i=1}^{n} (k_i - \ell_i)e_i, G\right]$$

for all $k_i, \ell_i \in \mathbb{Z}_p$, which implies $h \in C_0(G)$. However, $h(e_1 + e_2) = c_1 = h(-e_1 - e_2)$, from which it follows that $h\circ(-id) \neq -h$, so $C_0(G)$ is not a ring.

Suppose now n is odd. As above we take $c_i = [e_i, e_{i+1}], 1 \leq i \leq n$, and identify e_{n+1} as e_1. We take $f \in C_0(G)$ and show $f \in P_0(G)$. Recall that we may assume without loss of generality that $f(x) \in [x, G]$ for all $x \in G$.

From $f(e_i) \in [e_i, G], 1 \leq i \leq n$, we have $f(e_i) \in \langle c_{i-1}, c_i\rangle, 1 \leq i \leq n$, (identifying $c_0 = c_n$), say $f(e_i) = \alpha_{i,1}c_{i-1} + \alpha_{i,2}c_i, 1 \leq i \leq n, \alpha_{i,1}, \alpha_{i,2} \in \mathbb{Z}_p$. Next, from $f(e_i) - f(e_{i+2}) \in [e_i - e_{i+2}, G]$ we get

$$\alpha_{i,1}c_{i-1} + \alpha_{i,2}c_i - \alpha_{i+2,1}c_{i+1} - \alpha_{i+2,2}c_{i+2} = \lambda_1(c_i + c_{i+1}) + \lambda_2 c_{i-1} + \lambda_3 c_{i+2}, \lambda_i \in \mathbb{Z}_p,$$

since $[e_i - e_{i+2}, G] = \langle c_i + c_{i+1}, c_{i-1}, c_{i+2}\rangle$. This forces $\alpha_{i,2} = -\alpha_{i+2,1}, 1 \leq i \leq n$, and by putting $\tau_i = \alpha_{i-1,2}, 1 \leq i \leq n$, we find that $f(e_i) = [e_i, d]$ where $d = \sum_{i=1}^{n} \tau_i e_i$.

Take $p \in P_0(G)$ where $p(x) = [x, d], x \in G$ and let $h = f - p \in C_0(G)$. Now $h(e_i) = 0, 1 \leq i \leq n$, and we show $h(x) = 0$ for all $x \in G$, that is $f \in P_0(G)$ and G is 1-ac.

Let $k \in \mathbb{Z}_p, k \notin \{0, 1\}$. Then $h(ke_i) - h(e_{i-2}) \in [ke_i - e_{i-2}, G] = \langle c_{i-3}, kc_{i-1} + c_{i-2}, c_i\rangle$, where we take $c_{-3} = c_{n-3}, c_{-2} = c_{n-2}, c_{-1} = c_{n-1}$ and $h(ke_i) - h(e_{i+2}) \in [ke_i - e_{i+2}, G] = \langle c_{i-1}, kc_i + c_{i+1}, c_{i+2}\rangle$ which implies $h(ke_i) = 0, k \notin \{0, 1\}$. But $h(ke_i) = 0$ for $k \in \{0, 1\}$ so we have $h(ke_i) = 0, 1 \leq i \leq n, k \in \mathbb{Z}_p$.

We also find $h(ke_i + \ell e_j) = 0$ if $|i - j| \geq 2$. In fact, for $|i - j| \geq 2, h(ke_i + \ell e_j) - h(\ell e_j) \in [e_i, G] = \langle c_{i-1}, c_i\rangle$ and $h(ke_i + \ell e_j) - h(ke_i) \in [e_j, G] = \langle c_{j-1}, c_j\rangle$. Since $\langle c_{i-1}, c_i\rangle \cap \langle c_{j-1}, c_j\rangle = \{0\}$ for $|i - j| \geq 2, h(ke_i + \ell e_j) = 0, k, \ell \in \mathbb{Z}_p$.

We next show that $h(ke_i + \ell e_{i+1}) = 0, k, \ell \in \mathbb{Z}_p - \{0\}$. From $h(ke_i + \ell e_{i+1}) - h(\ell e_{i+1}) \in [e_i, G] = \langle c_{i-1}, c_i \rangle$ and $h(ke_i + \ell e_{i+1}) - h(ke_i) \in [e_{i+1}, G] = \langle c_i, c_{i+1} \rangle$, we find that $h(ke_i + \ell e_{i+1}) \in \langle c_i \rangle$, say

$$h(ke_i + \ell e_{i+1}) = \rho_i(k, \ell)c_i, \text{ where } \rho_i(k, l) \in \mathbb{Z}_p.$$

But, for $m \in \mathbb{Z}_p - \{0\}, h(ke_i + \ell e_{i+1}) - h(\ell e_{i+1} + me_{i+2}) \in [ke_i - me_{i+2}, G] = \langle kc_i + mc_{i+1}, c_{i-1}, c_{i+2} \rangle$, that is, $\rho_i(k, \ell)c_i - \rho_{i+1}(\ell, m)c_{i+1} = \lambda(kc_i + mc_{i+1})$ for some $\lambda \in \mathbb{Z}_p$. This implies that

$$\rho_{i+1}(\ell, m) = -mk^{-1}\rho_i(k, \ell) \tag{1}$$

for all $1 \le i \le n, k, \ell, m \in \mathbb{Z}_p - \{0\}$. (Note that $\rho_i(k, \ell) = 0$ if at least one of k and ℓ is zero.) The right-hand side of (1) equals the left-hand side for all $k \in \mathbb{Z}_p - \{0\}$. Hence, $k^{-1}\rho_i(k, \ell) = \rho_i(1, \ell), k \in \mathbb{Z}_p - \{0\}$, so that

$$\rho_i(k, \ell) = k\rho_i(1, \ell), k \in \mathbb{Z}_p - \{0\}. \tag{2}$$

Hence, from (1), $\rho_{i+1}(\ell, m) = -m\rho_i(1, \ell)$. Put $\ell = 1$, and $\rho_{i+1}(1, m) = -m\rho_i(1, 1)$, which implies $m^{-1}\rho_{i+1}(1, m) = -\rho_i(1, 1)$, $m \in \mathbb{Z}_p - \{0\}$. So, $m^{-1}\rho_{i+1}(1, m) = \rho_{i+1}(1, 1)$, giving

$$\rho_{i+1}(1, m) = m\rho_{i+1}(1, 1). \tag{3}$$

From (2) and (3),

$$\rho_i(k, \ell) = k\ell\rho_i(1, 1). \tag{4}$$

Now, put $k = \ell = m = 1$ in (1). Then

$$\rho_1(1, 1) = -\rho_n(1, 1) = \rho_{n-1}(1, 1) = -\rho_{n-2}(1, 1) = \cdots = \rho_2(1, 1) = -\rho_1(1, 1),$$

since n is odd. From $\rho_1(1, 1) = -\rho_1(1, 1)$ we have $\rho_1(1, 1) = 0$, and hence also $-\rho_1(1, 1) = \rho_2(1, 1) = \cdots = -\rho_n(1, 1) = 0$. By (4), $\rho_i(k, \ell) = 0, 1 \le i \le n, k, l \in \mathbb{Z}_p$. This shows that $h(ke_i + \ell e_{i+1}) = 0c_i = 0, 1 \le i \le n, k, \ell \in \mathbb{Z}_p$. So we now have that

$$h(ke_i + \ell e_j) = 0, 1 \le i, j \le n, k, \ell \in \mathbb{Z}_p.$$

We proceed by induction. Let $2 \le t < n$, and assume that $h(k_1e_{i_1} + k_2e_{i_2} + \cdots + k_te_{i_t}) = 0$ for any $\{i_1, i_2, ..., i_t\} \subset \{1, 2, ..., n\}$ and any $k_1, k_2, ..., k_t \in \mathbb{Z}_p$. Then, without loss of generality, put $w = \sum_{i=1}^{t+1} k_ie_i$, with all $k_i \ne 0$. Then $h(w) - h(w - k_1e_1) \in [e_1, G] = \langle c_n, c_1 \rangle$ and $h(w) - h(w - k_3e_3) \in [e_3, G] = \langle c_2, c_3 \rangle$ implies $h(w) \in \langle c_n, c_1 \rangle \cap \langle c_2, c_3 \rangle = \{0\}$. We conclude that

$$h\left(\sum_{i=1}^{n} k_ie_i\right) = 0, k_i \in \mathbb{Z}_p.$$

Thus, $h(x) = 0, x \in G$ and $f \in P_0(G)$. □

In Theorem II.1 several sufficient conditions were stated for a group G to have $C_0(G)$ not a ring. Most of these conditions lead to G being split. In Example 3.2 (2) of [11], GAP was used to find a group H, not split, and a function $f \in C_0(H) - P_0(H)$ with $C_0(H)$ not a ring. In the next theorem we give a construction process for a large collection of groups $G \in \mathrm{NAS}(\exp p \ne 2)$ to construct a function $f \in C_0(G)$ which shows $C_0(G)$ is not a ring.

First some notation. For $a, b, c, d \in G$ we have a 2×2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = [a, d] - [b, c].$$

For $x, y \in G, x = \sum_{i=1}^{n} x_i e_i + \sum_{j=1}^{s} u_j c_j, y = \sum_{i=1}^{n} y_i e_i + \sum_{j=1}^{s} v_j c_j,$

$$[x, y] = \sum_{1 \le i < j \le n} (x_i y_j - x_j y_i)[e_i, e_j] = \sum_{1 \le i < j \le n} \begin{vmatrix} x_i e_i & x_j e_i \\ y_i e_j & y_j e_j \end{vmatrix},$$

similar to the wedge product in multilinear algebra.

Using the above definition of determinant we define a "wedge" product for $x_1, \ldots, x_k, y_1, \ldots, y_k \in G$ by

$$\bigwedge \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \end{pmatrix} = \left(\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \ldots, \begin{vmatrix} x_{k-1} & x_k \\ y_{k-1} & y_k \end{vmatrix} \right),$$

a $\binom{k}{2}$-tuple.

For an abelian subgroup, A, of G, $A \supseteq Z(G)$, we choose a basis $\{e_1 + Z(G), \ldots, e_\ell + Z(G)\}$ for $A/Z(G)$ and extend this to a basis $\{e_1 + Z(G), \ldots, e_\ell + Z(G), e_{\ell+1} + Z(G), \ldots, e_n + Z(G)\}$ of $G/Z(G)$. Thus without loss of generality we have $G = H + A$ where $H = \langle e_{\ell+1}, \ldots, e_n \rangle$ and $A = \langle e_1, \ldots, e_\ell \rangle + Z(G)$.

Theorem III.3. (Wedge) *Let $G = H + A$ as above with $A = Y + Z(G), Y = \langle e_1, \ldots, e_\ell \rangle$. If there exist $\pi_{\ell+1}, \ldots, \pi_n \in Y$ such that*

i. $\bigwedge \begin{pmatrix} e_{\ell+1} & \cdots & e_n \\ \pi_{\ell+1} & \cdots & \pi_n \end{pmatrix} = 0,$ *and*
ii. $[e_j, \pi_j] \neq 0$ *for at least one* $j \in \{\ell + 1, \ldots, n\}$,

then $C_0(G)$ is not a ring.

Proof. For suitable $\pi_{\ell+1}, \ldots, \pi_n \in Y$, i.e., $\pi_i, \ell + 1 \le i \le n$, satisfying i) and ii), we will show that the function $f \in M_0(G)$, defined by

$$f\left(\sum_{i=1}^{n} x_i e_i\right) = \left[\sum_{i=1}^{n} x_i e_i, \sum_{i=\ell+1}^{n} x_i \pi_i\right]$$
$$= \left[\sum_{i=\ell+1}^{n} x_i e_i, \sum_{i=\ell+1}^{n} x_i \pi_i\right], \text{ since } [e_j, \pi_i] = 0, 1 \le j \le \ell, 1 \le i \le n$$

is in $C_0(G)$, but not in $P_0(G)$. Moreover, we'll show that, for some non-zero $c = \sum_{i=1}^{s} \delta_i c_i \in Z(G)$, $f(\sum_{i=1}^{n} x_i e_i) = f(\sum_{i=1}^{n} (-x_i) e_i) = c$, which shows that $C_0(G)$ contains a non-distributive element, hence $C_0(G)$ is not a ring.

First we show that for arbitrary $x = \sum_{i=1}^{n} x_i e_i, y = \sum_{i=1}^{n} y_i e_i \in G$,

$$f(x) - f(y) = f\left(\sum_{i=1}^{n} x_i e_i\right) - f\left(\sum_{i=1}^{n} y_i e_i\right) \in \left[\sum_{i=1}^{n} (x_i - y_i) e_i, G\right] = [x - y, G].$$

Now, for suitable $\pi_i, (\ell + 1 \le i \le n)$ (those given by i) and ii)):

$$f\left(\sum_{i=1}^{n} x_i e_i\right) - f\left(\sum_{i=1}^{n} y_i e_i\right) = \left[\sum_{i=\ell+1}^{n} x_i e_i, \sum_{i=\ell+1}^{n} x_i \pi_i\right] - \left[\sum_{i=\ell+1}^{n} y_i e_i, \sum_{i=\ell+1}^{n} y_i \pi_i\right]$$
$$= \sum_{j=\ell+1}^{n} \left[x_j e_j, \sum_{i=\ell+1}^{n} (x_i - y_i) \pi_i\right] + \sum_{j=\ell+1}^{n} \left[(x_j - y_j) e_j, \sum_{i=\ell+1}^{n} y_i \pi_i\right]$$
$$= L + R, \text{ where } R \in [x - y, G].$$

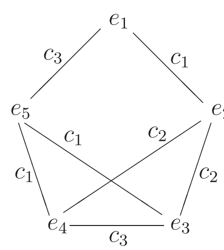
We show that also $L \in [x-y, G]$:

$$\begin{aligned} L &= \sum_{j=\ell+1}^{n}\left[x_je_j, \sum_{i=\ell+1}^{n}(x_i-y_i)\pi_i\right] \\ &= \sum_{j=\ell+1}^{n}\sum_{i=\ell+1}^{n} x_j(x_i-y_i)[e_j,\pi_i] \\ &= \sum_{j=\ell+1}^{n}\sum_{i=\ell+1}^{n} x_j(x_i-y_i)[e_i,\pi_j], \text{ by i)} \\ &= \sum_{j=\ell+1}^{n}\sum_{i=\ell+1}^{n} [(x_i-y_i)e_i, x_j\pi_j] \\ &= \sum_{i=\ell+1}^{n}\left[(x_i-y_i)e_i, \sum_{j=\ell+1}^{n} x_j\pi_j\right] \\ &\in [x-y, G]. \end{aligned}$$

This shows that $f \in C_0(G)$. By ii), there is an $i_0(\ell+1 \le i_0 \le n)$ such that $[e_{i_0}, \pi_{i_0}] \neq 0$. So $f(e_{i_0}) = [e_{i_0}, \pi_{i_0}] = [-e_{i_0}, -\pi_{i_0}] = f(-e_{i_0})$. It follows that $f\circ(-id) \neq -(f\circ id)$, showing that $C_0(G)$ is not a ring. □

Note that condition ii) is necessary here. Otherwise we could have chosen all $\pi_i = 0$ and i) is still satisfied. But in this case f would be the zero function, hence distributive.

Example III.4. *Let G be given by*



Then $G = H + A$ with $H = \langle e_1, e_3, e_4\rangle$ and $A = \langle e_2, e_5\rangle + Z(G)$. We have $\bigwedge\begin{pmatrix} e_1 & e_3 & e_4 \\ e_5 & e_2 & e_2 \end{pmatrix} = 0$ and $[e_1, e_5] \neq 0$. Thus $C_0(G)$ is not a ring.

From Theorem II.1, if G has a maximal abelian normal subgroup A of order $|A| = p^{n-1+s}$ then $C_0(G)$ is not a ring. As an application of the wedge theorem we consider the case where a maximal abelian normal subgroup, A, has order p^{n-2+s} and G/A is not cyclic. As above we choose a basis $\{e_1 + Z(G), ..., e_{n-2} + Z(G)\}$ of $A/Z(G)$ and get $G = \langle e_{n-1}, e_n\rangle + A, A = \langle e_1, ..., e_{n-2}\rangle + Z(G)$.

For $i = 1, 2, ..., n-2, [e_i, G] = \langle [e_i, e_{n-1}], [e_i, e_n]\rangle$. If $[e_i, e_{n-1}] = 0$ or $[e_i, e_n] = 0$, then from Theorem II.1, $C_0(G)$ is not a ring. Thus we take $[e_i, e_{n-1}] \neq 0 \neq [e_i, e_n]$ for $i \in \{1, 2, ..., n-2\}$. Let $A_{n-1} = \langle [e_i, e_{n-1}] | 1 \le i \le n-2\rangle$ and $A_n = \langle [e_i, e_n] | 1 \le i \le n-2\rangle$. If $\{[e_i, e_{n-1}] | 1 \le i \le n-2\}$ is linearly dependent over $\mathbb{Z}_p$, then $\sum_{i=1}^{n-2}\alpha_i[e_i, e_{n-1}] = 0$ and not all $\alpha_i = 0$. So, from $[\sum_{i=1}^{n-2}\alpha_ie_i, e_{n-1}] = 0$, we see that $y = \sum_{i=1}^{n-2}\alpha_ie_i$ is in A and $y \neq 0$. We have $\bigwedge\begin{pmatrix} e_{n-1} & e_n \\ 0 & y \end{pmatrix} = 0$ and $[e_n, y] \neq 0$, otherwise $y \in Z(G)$, a contradiction. From the Wedge Theorem, $C_0(G)$ is not a ring. Thus we now take $\{[e_i, e_{n-1}] | 1 \le i \le n-2\}$ to be linearly independent and similarly $\{[e_i, e_n] | 1 \le i \le n-2\}$ is linearly independent. We have $|A_{n-1}| = p^{n-2} = |A_n|$.

Suppose $A_{n-1} \cap A_n \neq \{0\}$, say $[h, e_{n-1}] = [g, e_n], h, g \in A$, say $g = \sum_{i=1}^{n-2}\beta_ie_i$, not all $\beta_i = 0$. If $[g, e_{n-1}] = 0$ then $0 = \sum_{i=1}^{n-2}\beta_i[e_i, e_{n-1}]$, a contradiction to the linear independence of

$\{[e_i, e_{n-1}] | 1 \leq i \leq n-2\}$. We have $\bigwedge \begin{pmatrix} e_{n-1} & e_n \\ g & h \end{pmatrix} = 0, h, g \in A$ and $[e_{n-1}, g] \neq 0$. so $C_0(G)$ is not a ring. Consequently $|A_{n-1} + A_n| = p^{2n-4}$ or $C_0(G)$ is a ring.

So we have

Theorem III.5. *Let $G \in \mathrm{NAS}(\exp p \neq 2)$ and let $G = H + A, A \in \eta(G)$, A abelian with $Z(G) \subseteq A$ and $|G/A| = p^2$. If $s < 2n-4$ then $C_0(G)$ is not a ring.* □

We use the notation and definitions from the above discussion. When $s \not< 2n-4$, then $s \geq 2n-4$.

If $s = 2n-4$, then $[e_{n-1}, e_n] = 0$ or $[e_{n-1}, e_n] \in A_{n-1} + A_n$. Suppose $[e_{n-1}, e_n] \in A_{n-1} + A_n$, say $[e_{n-1}, e_n] = \sum_{i=1}^{n-2} \alpha_i [e_i, e_{n-1}] + \sum_{i=1}^{n-2} \beta_i [e_i, e_n]$. Let $g = \sum_{i=1}^{n-2} \alpha_i e_i$ and $h = \sum_{i=1}^{n-2} \beta_i e_i$, so $[e_{n-1}, e_n] = [g, e_{n-1}] + [h, e_n]$, hence $[e_{n-1}, e_n + g] = [h, e_n]$. Let $\hat{e}_n = e_n + g$ and note $[e_{n-1}, \hat{e}_n] = [h, e_n] = [h, \hat{e}_n]$ so $[e_{n-1} - h, \hat{e}_n] = 0$ and $[e_i, e_{n-1} - h] = [e_i, e_{n-1}]$ and $[e_i, e_n + g] = [e_i, e_n], i = 1, 2, \ldots, n-2$. By using the basis, $\{e_1, e_2, \ldots, e_{n-2}, e_{n-1} - h, e_n + g, c_1, \ldots, c_s\}$, we have $G = \langle e_{n-1} - g, e_n + h \rangle + A$ with $[e_n - g, e_{n-1} + h] = 0$ so when $s = 2n-4$ we may take $[e_{n-1}, e_n] = 0$. When $n = 4$ we see that G is circular with n even so $C_0(G)$ is not a ring. The case for $n > 4, s = 2n-4$ remains open.

When $s > 2n-4$ and $n = 4$, then $s = 2n-3$, since $s \leq n(n+1)/2 = 6$ and $s = 6$ is the full case. For $n = 4$ and $s = 5$ one finds *via* tedious calculations that G is 1-ac. The case $n > 4$ remains open.

In conclusion, we have identified several further classes of non-abelian p-groups, G, $p \neq 2$, for which $C_0(G)$ is a ring if and only if G is 1-ac. However, the original conjecture as to whether this is true for all finite non-abelian p-groups, $p \neq 2$, remains open.

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