International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

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Andreas Bergwall

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
https://doi.org/10.1080/0020739X.2019.1704085

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Andreas Bergwall

School of Science and Technology, Örebro University, Örebro, Sweden

ABSTRACT

Despite the central role of proofs in mathematics, research often shows that school textbooks offer limited support for the teaching and learning of proof-related reasoning. This study contributes to this field of research by studying Swedish and Finnish upper secondary textbooks on logarithms and combinatorics. Justifications in expository sections are analysed and students' tasks are categorized according to the type and nature of reasoning they require. The findings imply that opportunities to learn proof-related reasoning are few, and are more oriented towards deductive reasoning in Finnish textbooks and towards empirical reasoning and conjecturing in Swedish textbooks. The results are discussed in relation to similar studies from both Scandinavian and United States contexts, and address future research and development of the theoretical framing of proof-related reasoning.

ARTICLE HISTORY
Received 21 December 2018

KEYWORDS
Mathematics textbook; upper secondary school; proof-related reasoning

1. Introduction

Proofs are central components of mathematics. They are the agreed-upon way to settle questions about mathematical truth and 'the facet of mathematical activity that characterizes and distinguishes the subject of mathematics' (Baylis, 1983, p. 409). Though few school curricula emphasize proofs, there is a growing consensus that reasoning and proving (in a broad sense) should be part of schooling in all grades and all mathematics topics (e.g. National Council of Teachers of Mathematics, 2000, 2009; Stylianides & Stylianides, 2017). However, it is well documented that students at all levels (including university e.g. Hemmi, 2008; Weber, 2001) have difficulties with proofs, especially in understanding the role of examples, counterexamples, and specific cases (e.g. Almeida, 2001; Harel & Sowder, 2007; Sevimli, 2018; Stylianides, Stylianides & Weber, 2017).

As textbooks are used in classrooms all over the world, researchers' interest has increased in how they treat proofs and proving. Textbooks have been pointed out as one possible factor behind the marginal place of proof in mathematics classrooms (Stylianides, Stylianides & Weber, 2017). The body of research is growing (e.g. Bergwall & Hemmi, 2017; Davis, Smith, Roy, & Bilgic, 2014; Otten, Gilbertson, Males, & Clark, 2014; Stacey & Vincent, 2009; Stylianides, 2009; Thompson, Senk, & Johnson, 2012) but limited, and

CONTACT Andreas Bergwall andreas.bergwall@oru.se

still strives to describe the current state of the art (Stylianides, 2014). General findings are that textbooks offer few opportunities to learn proofs, but there are exceptions and variation within and between countries, textbook series, and mathematical topics. Even when opportunities are few, they differ in character.

Terminology and analytical frameworks[1] differ among researchers, but a common methodological approach is to study textbook explanations and/or reasoning activities in students' tasks. In a study of a reform-oriented middle-school (Grades 6–8) curriculum programme in the United States, Stylianides (2009) found that 40% of the tasks included reasoning-and-proving. Few studies report such high figures. According to Glasnovic Gracin (2018), argumentation and reasoning activities are not present at all in Croatian (Grades 6–8) mathematics textbooks, except in chapters on triangle similarity. Regarding textbook explanations, Stacey and Vincent (2009) found that in Australian Grade 8 textbooks, slightly more than half of the explanations were deductive, with variation among topics from 17% to 100%. Thompson et al. (2012) compared textbook material on exponents, logarithms, and polynomials from 20 United States upper secondary school textbooks (reform-oriented as well as traditional) for three different courses (Algebra 1, Algebra 2, and Precalculus). They found that from Algebra 1 to Precalculus, there was a slight increase in proof-related tasks, tasks requiring reasoning about a general case, and tasks about developing and evaluating arguments. On average, though, only 6% of the textbook tasks involved proof-related reasoning. In expository sections, approximately half of the mathematical properties addressed were justified, half of them with general proofs. In United States textbooks on geometry, Otten et al. (2014) found that the percentage of justified properties was around 75%, with general proofs in approximately 35% of the cases. Twenty-five per cent of the tasks were proof-related.

Bergwall and Hemmi (2017) compared Swedish and Finnish upper secondary school textbooks on primitive functions and definite integrals. In both countries, around 50% of the statements in expository sections were justified. The justifications were almost always general proofs in the Finnish books, whereas the Swedish books mostly based their justifications on specific cases. Also, the mathematical structure was emphasized more in the Finnish books, with clear labelling of statements and proofs. The share of proof-related tasks was low in all textbooks (between 7% and 18%). In the Finnish textbooks, most of them asked the student to prove or show something. Such proving tasks existed in the Swedish books as well, but tasks about investigating the truth of a given statement were as common as those about arguing for a certain statement. Tasks about making a conjecture or evaluating an argument were also more common in the Swedish books. In summary, the Finnish textbooks offered more opportunities for formal deductive reasoning about general cases, while the Swedish textbooks placed more emphasis on conjecturing and inductive reasoning.

Comparisons of different countries highlight and contrast differing educational traditions regarding proofs and proving, and Sweden and Finland provide interesting cases in several respects. They are neighbouring countries with a common history, and Finland has a Swedish-speaking minority. However, several studies indicate that many aspects of mathematics education differ (e.g. Bergwall & Hemmi, 2017; Knutsson, Hemmi, Bergwall, & Ryve (2013); Hemmi & Ryve, 2015; Hemmi, Lepik, & Viholainen, 2013), and Finnish students have outperformed Swedish (and most Western countries') students in many international evaluations (TIMSS and PISA, among others). In the present paper, Swedish and Finnish upper secondary mathematics educations are further investigated through an analysis of textbook material on logarithms and combinatorics. Data on justifications in expository sections, and on the type and nature of reasoning in students' tasks, are combined with similar data from Bergwall and Hemmi (2017) to answer the following research question: What characterizes opportunities to learn proof-related reasoning offered by Swedish and Finnish upper secondary textbooks?

2. Method

2.1. Mathematics topics

In school mathematics, proofs have often been confined to high school geometry. This led Thompson et al. (2012) to investigate other topics: exponents, logarithms and polynomials. Bergwall and Hemmi (2017) followed with a similar study of integral calculus. The present study aims to widen and deepen the results of Thompson et al. (2012) and Bergwall and Hemmi (2017) by focusing on logarithms and combinatorics. Logarithms were chosen for comparison with the results from the United States context provided by Thompson et al. (2012), and combinatorics was chosen to examine whether proof-related reasoning is handled differently depending on the character of the topic. Compared to definite integrals (one of the topics of (Blinded 1) Bergwall & Hemmi, 2017), the theoretical foundation of combinatorics is much simpler: the former requires $\epsilon - \delta$ formalism, the latter only the addition and multiplication principles. Also, combinatorics can easily be related to everyday situations. Taken together, an analysis of textbook sections on logarithms and combinatorics complements earlier studies and has the potential to broaden the picture and provide data for different kinds of comparisons.

2.2. Swedish and Finnish curricula

Swedish and Finnish upper secondary schools are course-based, with national steering documents[2] prescribing the content and learning outcomes in general terms. Here, we briefly describe the courses for students preparing for higher studies in science and technology. For a detailed analysis of proof in Swedish and Finnish steering documents, see Hemmi et al. (2013).

In Sweden there are five mathematics courses. Logarithms are introduced in the second course and combinatorics in the fifth. In the Finnish setting there are 13 courses. Combinatorics is found in the fifth and logarithms are introduced in the eighth course.

The general objectives in the Swedish steering documents state that students should be given opportunities to develop their abilities to follow, conduct and assess mathematical reasoning. Proofs are explicit components of all courses but one, for instance in relation to trigonometry, derivatives, and number theory (Swedish National Agency for Education, 2011). The general objectives in the Finnish steering documents state that students should 'learn to appreciate precision of presentation and clarity of argumentation', 'learn to perceive mathematical knowledge as a logical system', and 'become accustomed to making assumptions, examining their validity, justifying their reasoning and assessing the validity of their arguments and the generalizability of the results' (Finnish National Board of Education, 2004). Proof is however a core content in a non-mandatory specialization course in number theory and logic. Neither country's objectives contain explicit statements about proofs in relation to logarithms and combinatorics.

2.3. Swedish and Finnish textbooks

Textbooks are frequently used in both school systems (Boesen et al., 2014; Joutsenlahti & Vainionpää, 2010), with publishers providing textbooks from a free market without state control or certification. For the present study, textbooks from the series *Matematik 5000* (Alfredsson, Bråting, Erixon, & Heikne, 2011, 2013), *MatematikOrigo* (Szabo, Larson, Viklund, Dufåker, & Marklund, 2012, 2013) and *Ellips* (Kontkanen, Lehtonen, Luosto, & Westermark, 2007; Kontkanen, Lehtonen, Luosto, Savolainen, & Westermark, 2007) were chosen. These series are referred to as Sw1, Sw2 and Fi1, respectively. Sw1 has a dominant market position in Sweden, Sw2 being its primary competitor. Fi1 has been the only Finnish material translated to Swedish for Finland's Swedish-speaking students, while its Finnish version is one of two series that dominate the market for Finnish speakers. Our estimate is that Sw1, Sw2 and Fi1 reflect what around 90% of Swedish- (and many Finnish-) speaking students have used in recent decades in preparation for higher studies.

All textbooks have expository sections, where the authors present new concepts, formulate and justify statements, and present worked examples, mixed with students' exercise sets. Chapters usually end with sets of mixed problems. In the Swedish books the chapters begin with introductory problem-solving activities (whereby students can discover some of the chapter's central ideas) and usually end with spreads offering historical notes, group activities, and sets of discussion questions or true/false questions. Also, the Swedish books frequently have subsections on applications. The Finnish books often contain additional, and more advanced, material in special sections at the end of the books.

Regarding proof, all textbook series contain sections explicitly devoted to proofs and proving in correspondence with the steering documents, but not within chapters on logarithms and combinatorics.

2.4. Data sample

Since logarithms and combinatorics belong to different courses, the data sample consists of textbook material from two books from each textbook series. Sections on logarithms and combinatorics were identified by reading the content pages. All passages and tasks related to these topics and placed within the identified sections were included.

All textbooks introduce logarithms together with exponential equations. Sections on powers and exponents that precede the introduction of logarithms have not been included, and neither have subsections on the derivative of the logarithm function.

Regarding combinatorics, there were subsections in the Swedish books addressing probability, the binomial theorem, and Pascal's triangle. In such subsections, only materials and tasks involving combinatorial reasoning have been included.

The quantities of analysed textbook material are summarized in Table 1.

Table 1. Distribution of analysed textbook material.

Textbook series	Justifications	Worked examples	Students' tasks
Sw1	9	43	342
Sw2	9	41	427
Fi1	15	44	223
Total	33	128	992

2.5. Analytic approach

In the following subsections, we discuss the concept of proof-related reasoning and discuss when a textbook can be said to have provided an opportunity to learn proof-related reasoning. This discussion forms the basis for the analytic frameworks and procedures.

2.5.1. Definitions related to proof-related reasoning

Mathematical proofs vary in form and serve different purposes (e.g. De Villiers, 1990), but always outline how a certain mathematical property is a logical consequence of other known (or assumed) properties. Such processes to settle questions of doubt are conceptualized as deductive proof schemes (Harel & Sowder, 1998). In contrast, the external and empirical proof schemes (Harel & Sowder, 1998, 2007), often seen among mathematics learners, refer to whether conviction about truth comes from external factors or is based on empirical data and specific cases. In the literature, empirical proof schemes are also called empirical responses (Bell, 1976) and pragmatic justifications (Balacheff, 1988).

In this paper, any empirical or deductive argument meant to convince the reader of the truth of a mathematical statement will be called a *justification*. When evaluating justifications for, or produced by, school students, one cannot expect justifications to meet the standards of rigour, precision and formalism of mathematicians' proofs; the students' prior knowledge of mathematics and mathematical reasoning must be taken into consideration. To be able to talk about proofs at all levels of schooling, we therefore follow the notions of Stylianides (2007), and consider a justification to be a proof if it is a connected sequence of assertions with the following characteristics: (1) it builds on statements that can be assumed to be accepted by the classroom community, (2) it uses valid forms of reasoning that can be assumed to be known to the classroom community, and (3) it is communicated using forms of expressions appropriate for the classroom community. For upper secondary textbooks, this should be more or less equivalent to what 'any mathematician or mathematics teacher would likely call a proof' (Thompson et al., 2012, p. 259).

Mathematical statements are usually general, in the sense that they ascribe a property to an infinite class of objects. A proof presents an argument valid for all objects in the relevant class at once. The reasoning behind such an argument differs from one based on a specific case. Following Thompson et al. (2012), these two major kinds of reasoning, *general* and *specific*, will be referred to as *types* of reasoning. Of course, an argument with a specific case is enough to prove a statement about that specific case or to refute a false general statement. However, if a specific case is used as conviction for a universal statement, it is an example of an empirical proof scheme.

A specific case offers important insights into why a statement is true, or about how a general proof might appear. It might even contain all relevant aspects of a general argument in such a way that 'one can see the general proof through it because nothing specific to the

[case] enters the proof' (Movshovitz-Hadar, 1988, p. 19). Such generic cases, or *transparent pseudo-proofs,* have great pedagogical value, but we will not refer to such reasoning as general, even though Harel and Sowder (2007) count it as a kind of deductive proof scheme. The distinction between specific and general types of reasoning is, therefore, closer to Balacheff's distinction between pragmatic and conceptual justifications (Balacheff, 1988).

A justification can fail to be a proof for other reasons than not being sufficiently general. For instance, the level of rigour or formalism may be too low, or the argument can be based on vague or intuitive ideas, or on visual impressions from diagrams. Such justifications can also have great pedagogical value and explanatory power (cf. Hanna, 2018). For analytical purposes, we chose to widen the class of specific justifications to include all non-proof justifications instead of defining additional types of reasoning.

Proofs produced by mathematicians are end products. They are the result of a long chain of successive refinements (cf. Lakatos, 2015), and need not bear any trace of the processes that initially led the mathematician to believe in the conclusion. These processes may involve formulating, investigating, and revising conjectures; finding and constructing counterexamples and supporting examples; and developing, evaluating, investigating, finding flaws in, correcting, outlining, or filling in details in arguments. Once again following Thompson et al. (2012), these different elements of reasoning will be grouped in categories of *natures* of reasoning (see later sections for details). Together, they constitute *proof-related reasoning*. In this paper, an underlying assumption is that, to understand proofs and their role in mathematics and to develop an ability to construct proofs, students need to be engaged in reasoning activities that vary in type and nature.

2.5.2. Opportunities to learn proof-related reasoning in textbooks

The written textbook is only one link in the chain between the written curriculum and student learning (Stein, Remillard, & Smith, 2007). The opportunities to learn (Hiebert & Grouws, 2007) that a student is offered depend on how the curriculum is enacted in the classroom. Saying that a textbook offers an opportunity to learn proof-related reasoning is an indication that a potential opportunity exists. We consider such opportunities to occur in the following areas:

- Expository sections when the authors highlight a main result and present some justification for this result, when there are worked examples involving proof-related reasoning, and when the authors discuss the concepts of proof and proving
- Students' exercises and activities when they involve proof-related reasoning

The opportunities to learn proof-related reasoning are characterized by the type and nature of reasoning. Analytic frameworks and procedures are adopted more or less directly from Thompson et al. (2012), with adaptations similar to Bergwall and Hemmi (2017).

2.5.3. Procedure and framework for analysis of expository sections

In the analysis of expository sections, the unit of analysis has been a main result, along with its justification(s). A *main result* refers to a true mathematical statement that the textbook authors highlight, e.g. with a coloured background or a frame, or by labelling it as a theorem, principle, or rule. Statements in the inline text are considered main results if they are accompanied by worked examples illustrating their use.

Table 2. Framework for justifications in expository sections.

Code	Type of justification	Description
G	General proof	The statement is justified with a proof.
S	Specific case or other non-proof justification	The statement is justified using a deductive argument based on a specific case, or that has other flaws that makes it a non-proof justification.
L	Left to the student	A justification of the statement is explicitly left to the student to complete, typically with a problem in the exercises for which a justification of some type is required.
N	No justification	No justification is provided, and no explicit mention is made of leaving the justification to the student.

Table 3. Framework for nature of reasoning in textbook tasks.

Code	Nature of reasoning	Task in which student is asked to ...
M	Make a conjecture	make a conjecture, formulate a true mathematical statement, or find the precise conditions for a certain statement to be true
I	Investigate a conjecture	investigate whether a given conjecture or statement is true or false
D	Develop an argument	justify or explain why a certain statement holds
E	Evaluate an argument	evaluate whether a certain justification or solution is correct
C	Correct or identify a mistake	find and/or correct an error in an argument or a solution
X	Counterexample	find a counterexample to a false mathematical statement
P	Outline a proof	outline an argument without the details of a full proof
O	Other	use some other element of proof-related reasoning
N	Not proof-related	do something not proof-related

When all main results were identified, data on their labelling, logical structure and generality were collected. All the main results' justifications were then classified in terms of the type of reasoning, as shown in Table 2. Notes were also taken about labelling, proof techniques, and whether the justification was placed before or after the statement.

Finally, worked examples in expository sections were analysed and categorized in the same way as students' tasks.

2.5.4. Procedure and framework for analysis of students' tasks

In the analysis of textbook tasks, the unit of analysis has been a task, exercise or activity with its own label, number or name. All such tasks have been categorized as proof-related (PR) or not proof-related. A task is considered proof-related if it involves any of the natures of reasoning specified in Table 3. In addition to nature of reasoning, such tasks are characterized by their type of reasoning, i.e. if they involve reasoning about a specific (S) or a general (G) case.

Tasks have been analysed from the perspective of the intended student (e.g. Weinberg & Wiesner, 2011). To accomplish this perspective, it has been assumed that the student has followed the textbook strictly, worked with all the preceding material, and comprehended it. The answer section has been used to understand the textbook authors' intentions and expectations. We have strived to give single codes for nature and type of reasoning, unless a task explicitly asks the student to do things involving different natures or types of reasoning.

The analysis of the data set proceeded iteratively. First, all 'ordinary' tasks (not introductory activities or special tasks at the end of the chapters) were preliminarily classified. After comparison with tasks and analytical decisions made during our work reported in Bergwall and Hemmi (2017), the complete data set was analysed anew. A sample of tasks

Table 4. Examples of tasks in the Swedish textbooks illustrating different natures and types of reasoning

Ex	Task	Code
1	Investigate with your calculator and write down the following values: lg 2, lg 20, lg 200, lg 2000. (a) What pattern can you see? Explain. (b) What should lg 20000 and lg 0.2 be? (Sw1, Book 2c, *p*. 120)	GM/SM
2	(a) Give an example of a positive number which also has a positive ten-logarithm. (b) Give an example of a positive number which has a negative ten-logarithm. (c) Explain when the ten-logarithm of a positive number is positive and when it is negative. (Sw2, Book 2c, *p*. 100)	N/N/GM
3	The number of permutations of a certain sample is always greater than the number of combinations. True or false? Motivate your answer! (Sw1, Book 5, *p*. 57)	GI
4	Gina says that lg 400 must be greater than 2 but less than 3. Is she right? Answer the question without using a calculator, and justify your answer. (Sw1, Book 2c, *p*. 120)	SI
5	Explain why one cannot calculate $\log_2(-5)$. (Sw2, Book 2c, *p*. 113)	SD
6	Using the power laws, show that $\lg A^y = y \cdot \lg A$. (Sw1, Book 2c, *p*. 122)	GD
7	In the EU parliament, there are 754 people from 27 states. Show that at least 28 people are from the same state. (Sw1, Book 5, *p*. 10)	SD
8	Ludvig and Philip are discussing the box principle. 'The box principle says that if I have 13 t-shirts to dye green, blue or red, I will always get at least 4 t-shirts of every colour', Ludvig says. 'No', Philip says, 'the box principle only says that you always get at least one t-shirt of every colour'. Is either of them correct? Justify. (Sw2, Book 5, *p*. 99)	SE
9	What errors do they make? (a) Pierre simplifies $\lg 37 - \lg 8$ and gets $\lg 37/\lg 8$. (b) Fia simplifies $\lg 5x^2$ and gets $2\lg 5x$. (Sw1, Book 2c, *p*. 122)	SC/GC

Table 5. Examples of proof-related tasks in the Swedish textbook series.

Ex	Task	Code
10	Let's say that *m* objects should be placed in *n* boxes. What should be written in the square for the statement to be true? 'If $m > \boxdot$ then at least one of the boxes will contain more than *k* objects'. (Sw2, Book 5, *p*. 63)	GM
11	Per says that a doubling of sound intensity implies an increase in the sound level by 3 dB. Investigate whether this is true. Choose several different intensities. (Sw2, Book 2c, *p*. 114)	GI
12	Is there any number *k* that solves the equation $C(8,5) = P(8,k)$? Justify your answer. (Sw2, Book 5, *p*. 69)	SI
13	Show that if the pH value decreases by 1, then the concentration of hydrogen ions increases by a factor of 10. (Sw2, Book 2c, *p*. 110)	GD
14	Kalle solves the equation $\lg x^2 = 4$ like this: $\lg x^2 = 4$, $2\lg x = 4$, $\lg x = 2$, $x = 10^2$. (a) Kalle finds it strange that the equation does not have two solutions when '*x* is squared'. Show by testing that the equation has the roots $x = -100$ and $x = 100$. (b) Solve the equation in such a way that you get both roots. Explain what is wrong with Kalle's solution. (Sw2, Book 2c, *p*. 123)	SD/GC
15	Lise and Erik solve the equation $3 \cdot 5^x = 12$ correctly, but in two different ways. Lise gets $x = \lg 4/\lg 5$ and Erik gets $x = (\lg 12 - \lg 3)/\lg 5$. Give suggestions for how Lise and Erik might have reached their answers. (Sw2, Book 2c, *p*. 110)	SO
16	At a Nordic conference there were 31 students from Sweden, Norway, Denmark, Finland, and Iceland. (a) Which number is *n* (the number of 'boxes')? (b) Show that some country is represented by at least 7 students. (Sw1, Book 5, *p*. 10)	SO/SD

representing different natures and types of reasoning, as well as tasks found difficult to classify, were selected and discussed with colleagues at a seminar. Based on this discussion, final analytical principles were settled upon, and were applied during a final coding iteration of the complete data set.

Below, we describe the adaptation and operationalization of the framework for nature of reasoning (Table 3), exemplified with proof-related tasks from Sw1 and Sw2 (Table 4). Additional tasks are discussed in the *Results* section (Tables 5–6).

Table 6. Examples of proof-related tasks in the Finnish textbook series.

Ex	Task	Code
17	Show that the value of the expression $(\log_a b/\log_{ac} b) - (\log_a d/\log_c d)$ is independent of the numbers *a*, *b*, *c*, and *d*, which all are greater than one. (Fi1, Book 8, *p.* 126)	GD
18	Prove. $\binom{n}{k} = \binom{n}{n-k}$ (Fi1, Book 6, *p.* 74)	GD

M tasks focus on *what* is true; they are about formulating true mathematical statements, typically based on inductive reasoning and generalization. In the answer section of Example 1 (Table 3), the authors write 'As the number increases by a factor of 10, lg increases by lg 10 = 1', which is a general mathematical statement. The word 'should' in Example 1(b) indicates that the student should make a guess based on the pattern seen in (a). Hence, these tasks are M tasks: the first general, the second specific. The M category also includes tasks involving describing the precise conditions for a statement to be true (Example 2). Specifying conditions is an important aspect of conjecturing. Tasks which only require transformations between different forms of representations are not considered M tasks.

I tasks ask *if* something is true; i.e. the student is to determine the truth value of a given statement (Example 3), or whether a statement is correct or not (Example 4). The category does not include tasks with other kinds of choices, like which of two numbers is the greater or whether a certain function is increasing or decreasing.

D tasks ask *why* a statement is true (Example 5). The focus is on deductive reasoning, how things connect to and follow from each other. The result to argue for should be explicit in the task. Proving tasks, i.e. those that ask the student to show or prove a certain statement, fall into this category (Examples 6 and 7). When 'show' refers to describing a procedure, as in 'Show how one uses logarithms to compute (a) 2^3 (b) $5 \cdot 8$' (Sw1, Book 2c, p. 124), the task has not been considered proof-related. However, in cases of doubt as to the textbook authors' intentions, tasks have been coded as D tasks rather than as not proof-related. If students are called to explain their thinking in a task of another nature, as in Examples 1 and 4 above, double codes have not been used. The procedures of simplifying, developing or factorising expressions are seen as algebraic equivalents of calculating and computing. Hence, tasks phrased as 'Simplify the expression … ', without an articulated goal, are considered not proof-related. However, tasks formulated as 'Derive the formula … ', with the formula explicit in the task, are considered D tasks.

E tasks present an argument and ask the student to evaluate its validity. It should not be explicit in the task whether the argument is correct or not. In Example 8, Ludvig's and Philip's words are considered arguments since they describe a situation (number of t-shirts and colours), formulate a conclusion (number of t-shirts of each colour), and present a warrant for their conclusion (Dirichlet's box principle).

C tasks present the student with a false statement or an invalid argument and ask the student to determine exactly what has gone wrong, i.e. give a plausible reason behind an erroneous answer or point out a flaw in an argument (and not simply present a correct answer or argument).

X tasks explicitly ask the student to provide a counterexample to a false universal statement. Other tasks involving proving something is wrong or explaining why something does not work (like Example 7) are considered D tasks.

P tasks are tasks in which a student is asked to outline a proof without providing the level of details necessary to make it a full proof. The opposite situation, i.e. a task that outlines a proof and asks the student to fill in the details, belongs to the D category.

O tasks are tasks that do not fit into any of the categories of natures of reasoning described above, but that involve other elements of reasoning essential in proving. This category was not part of the original framework in Thompson et al. (2012). It was introduced to collect material for a discussion on revisions of the framework itself. All tasks placed in this category are accounted for in the *Results* section.

N tasks, finally, are all the remaining tasks. They are considered 'not proof-related'.

3. Results

Results for the Swedish textbook series are presented first, followed by those for the Finnish series. Examples of proof-related tasks (in addition to those in Table 4) are presented in Tables 5–6. Quantitative data are summarized in Tables 7–9, together with data from Blinded Bergwall and Hemmi (2017).

3.1. Swedish textbooks

3.1.1. Expository sections

The Swedish books introduce logarithms for base 10 and present the logarithm laws for products ($\lg(a \cdot b)$), quotients ($\lg(a/b)$), and powers ($\lg(a^b)$). In subsequent sections, Sw2 also presents the laws for arbitrary bases, while Sw1 only mentions that they hold for any base. In both books, all addressed laws are labelled as such. In Sw2, there is an introductory activity and exercises preceding the presentation of the laws, from which the student has a possibility to conjecture them.

Sw1 justifies one of the logarithm laws with a general proof placed before the statement, but leaves the others unjustified. Sw2 proves all logarithm laws for base 10 with proofs placed after the statement, but also provides specific justifications for two of them before they are stated. Proofs for arbitrary bases turn up as students' exercises. Neither Sw1 nor Sw2 offers proof-related worked examples on logarithms.

Both textbooks' combinatorics chapters start with introductory activities whereby students can get acquainted with, and conjecture, the addition and multiplication principles. However, the basic versions of these principles (with two independent choices) are merely combinatorial representations of addition and multiplication and are hence not classified as main results.

In subsequent sections, the textbooks define and address formulas for permutations and combinations, and present Dirichlet's box principle. In total, Sw1 addresses six main results and justifies five of them, while Sw2 addresses four and justifies all. Neither the results nor the justifications are clearly labelled as such. All justifications are specific. In Sw1 all justifications are placed before the statement, while in Sw2 half of them are. Both textbooks contain examples of specific justifications expressed in ways that give the student a chance to conjecture the general result. Proof-related worked examples are offered in both textbooks, but are related only to Dirichlet's box principle.

3.1.2. Students' exercises

The selected material from Sw1 and Sw2 contains 342 and 427 student tasks, respectively. Forty-four tasks in Sw1 and 65 in Sw2 are proof-related. In both textbook series, the ratio of proof-related tasks is higher in sections on combinatorics than in those on logarithms. All categories of natures of reasoning, except the X and P categories, are represented in both series. More than half of the proof-related tasks are D tasks, which is the dominant category, while E and C tasks are very few. In Sw1 specific tasks dominate, whereas in Sw2 the general tasks are in the majority.

Both textbooks have M tasks on logarithms involving identifying/generalising a pattern (Example 1, Table 4). Sw2 also provides M tasks of a more descriptive character (Example 2(c)). There are no M tasks on combinatorics in Sw1, but Sw2 provides two specific and two general tasks. One task has the character of stating precise conditions (Example 10, Table 5) and another involves identifying/generalising a pattern.

There are specific and general I tasks, on logarithms as well as combinatorics, in both textbooks. The I tasks are frequently formulated as yes/no questions (Example 4) or true/false questions (Example 3). Sw1 has spreads at the end of every chapter with such tasks, often providing connections to definitions and well-known misconceptions. Sw2 also has I tasks with scientific contexts (Example 11), and while some I tasks are conceptual (Example 3) others are more algebraically oriented (Example 12).

Both textbook series contain specific as well as general D tasks on both topics. Specific tasks dominate in Sw1, general tasks in Sw2. Proving tasks always use the imperative 'show'; never 'prove'. Most of them are general (Example 6), but there are also specific proving tasks (Example 7). They also appear in applications (Example 13). D tasks that are not proving tasks often have an explanative character, and are typically formulated as 'Why is ... ' (Example 5) or 'Explain why ... '.

E tasks were found only in combinatorics, and there was only one each in Sw1 and Sw2 (Example 8).

Sw1 contained two C tasks on logarithms, one specific and one general (Example 9), both focusing on the identification of an error. Sw2 has one general C task on logarithms (Example 14) and one specific C task on combinatorics. These include correcting an error.

There were no X or P tasks in either of the Swedish textbooks.

Finally, six O tasks were found; i.e. tasks that did not fit the other natures of reasoning but include elements one can argue are proof-related. These were of two kinds. The first includes a call to explain the thinking behind a presented calculation. Such tasks are neither about evaluating an argument nor correcting a mistake, but in the practice of proving and reading proofs, one often faces situations with missing details and must identify the underlying idea. Sw2 presents one such task on logarithms (Example 15) and one on combinatorics. Sw1 also has one with an amendment to evaluate the explained argument's correctness. This task was double-coded as OE.

The second kind occurs once in each textbook. In tasks on Dirichlet's box principle, the student is asked which data correspond to objects and boxes, respectively (Example 16). Matching the given data to preconditions of a mathematical theorem is often the first step in a proving process. When a mathematician realizes that a new situation fulfils the preconditions of a known theorem, it is often of great value.

3.2. Finnish textbook

3.2.1. Expository sections

After an example in which an exponential equation is solved numerically, Fi1 immediately defines logarithms for arbitrary bases. Seven main results are addressed: the logarithm laws and identities, such as $\log_a a = 1$. In the next section, four properties for the logarithm function, including continuity and monotonicity, are addressed (defining properties not counted). All results are labelled as properties. There is no material from which the student can conjecture them.

Five of the addressed results are justified, all of them with general proofs. On three occasions, the proofs are placed before the statement. Some of the unjustified results turn up in students' exercises.

The combinatorics chapter has a short introduction that explains the concept of combinatorics, but no introductory activity. The chapter's first section addresses two main results (general versions of the addition and multiplication principles), and the second section addresses three (formulas for permutations and combinations). The last mentioned are not labelled as results.

One of the addressed combinatorics results is justified with a specific case, three with a general proof, and one with a specific case as well as a general proof. All justifications are placed before the statements. Central concepts are introduced through worked examples before they are defined. In two cases (the general multiplication principle and number of permutations), these are presented so that students can conjecture the general result.

All justifications are unlabelled, but the authors use phrases like 'we prove'. There are no proof-related worked examples, for either logarithms or combinatorics.

3.2.2. Students' exercises

Of 223 student exercises, Fi1 only has six proof-related tasks on logarithms and six on combinatorics. One is an I task, the others are D tasks.

The only I task is a specific true/false question about the number of possible ways to combine pizza toppings.

All the D tasks are general proving tasks, more frequently phrased 'prove that' than 'show that'. Four involve proving unjustified logarithm properties addressed in the expository sections. In two tasks, the student is asked to show that the value of a parametric expression is independent of the parameters (Example 17, Table 6). Regarding combinatorics, most D tasks consist of proving certain identities for combinations (Example 18).

4. Discussion

The aim of the present study was to characterize opportunities to learn proof-related reasoning offered in Swedish and Finnish upper secondary textbooks. Textbook material on logarithms and combinatorics from three textbook series has been analysed, with a

Table 7. Type of reasoning in justifications in expository sections

		No justification (N)	Left to student (L)	Specific case (S)	General proof (G)	Total no. of main results
Sw1	Logarithms & combinatorics	3		5	1	9
	Primitive functions & integrals	13		6	1	20
	Total	16		11	2	29
Sw2	Logarithms & combinatorics	3	1	5	3	9
	Primitive functions & integrals	5		11	2	18
	Total	8	1	16	5	27
Sw1 + Sw2	Logarithms & combinatorics	6	1	10	4	18
	Primitive functions & integrals	18		17	3	38
	Total	24	1	27	7	56
Fi1	Logarithms & combinatorics	6		2	8	15
	Primitive functions & integrals	10	2	3	12	27
	Total	16	2	5	20	42

Remark on double coding: Two logarithm justifications in Sw2 were coded SG, one combinatorics justification in Sw2 was coded LS, and one combinatorics justification in Fi1 was coded SG.

focus on types and natures of reasoning in justifications in expository sections and students' exercises. The textbook series are the ones that most Swedish speaking students in Sweden and Finland have used in recent decades in preparation for higher studies in mathematics. Below is a summary of the main findings, highlighting the similarities and differences between Swedish and Finnish textbooks. These results are related to findings from textbook material on integral calculus obtained from the same textbook series and reported by Bergwall and Hemmi (2017) (see Tables 7–9). The results are also compared with results regarding United States textbooks presented in Thompson et al. (2012) and Otten et al. (2014) (see Tables 10–11). Finally, limitations of the present study, implications for teaching, and suggestions for future research are discussed.

4.1. Comparison between Swedish and Finnish textbooks

4.1.1. Expository sections

In all textbooks, the addressed main results describe general properties (i.e. properties for infinite classes of objects). They are mostly clearly labelled as principles or laws. The Finnish books differ from the Swedish ones in that they provide a more detailed and formal exposition, and address more results. They also head more directly for general results. For instance, logarithms are immediately introduced for arbitrary bases. The Swedish books start with base 10 and view logarithms for arbitrary bases as a generalization, the properties of which the students have to induce themselves. However, the Swedish books usually offer introductory activities and exercises through which students can discover and conjecture general properties.

Both the Swedish and Finnish books usually formulate main results after presenting a justification. Main results are justified about as often in the Swedish books (13 of 19) as in the Finnish ones (10 of 16). Justifications are always unlabelled. The main difference is that in the Finnish books justifications are almost always (9 of 10) general proofs and phrases like 'we prove' are frequently used. Specific arguments dominate in the Swedish books. Proof-related worked examples are, on the other hand, only found in the Swedish books.

Table 8. Labelling and placement of main results and justifications in expository sections.

		Main results				Justifications					
			Labelling				Labelling			Placement	
		Total no.	Theorem	Rule, law, etc.	No/other	Total no. of justified main results	Proof	Check, etc.	No/other	Before	After
Sw1	Logarithms & combinatorics	9		6	3	6			6	6	
	Primitive functions & integrals	20	1	6	13	7			7	7	
	Total	29	1	12	16	13			13	13	
Sw2	Logarithms & Combinatorics	9		7	2	6			6	3	5
	Primitive functions & integrals	18	1	4	13	13			13	13	
	Total	27	1	11	15	19			19	16	5
Sw1 + Sw2	Logarithms & Combinatorics	18		13	5	12			12	9	5
	Primitive functions & integrals	38	2	10	26	20			20	20	
	Total	56	2	23	31	32			32	29	5
Fi1	Logarithms & combinatorics	15		12	3	9			9	7	2
	Primitive functions & integrals	27	6	17	4	15	7		8	7	8
	Total	42	6	29	7	24	7		17	14	10

Remark on double coding: Two logarithm statements in Sw2 had specific justifications placed before and general justifications placed after the main results.

Table 9. Type and nature of reasoning in proof-related students' exercises.

				Type of reasoning		Nature of reasoning							
		Total no. of tasks	PR tasks	S	G	M	I	D	E	C	X	*P*	O
Sw1	Logarithms & Combinatorics	342	44	27	17	2	12	26	1	2			2
	Primitive functions & Integrals	418	74	51	23	11	35	26		2			
	Total	760	118	78	40	13	47	52	1	4			2
Sw2	Logarithms & Combinatorics	427	65	25	40	9	9	41	1	2			4
	Primitive functions & Integrals	449	46	18	28	3	19	21	1	2			
	Total	876	111	43	68	12	28	62	2	4			4
Sw1 + Sw2	Logarithms & Combinatorics	769	109	52	57	11	21	67	2	4			6
	Primitive functions & Integrals	867	120	69	51	14	54	47	1	4			0
	Total	1636	229	121	108	25	75	114	3	8			6
Fi1	Logarithms & Combinatorics	223	12	1	11		1	11					
	Primitive functions & Integrals	524	38	16	26	1	8	29					
	Total	747	50	17	37	1	9	40					

Remark on double coding: One combinatorics task in Sw1 was coded EO and one logarithms task in Sw2 was coded MI.

To conclude, all textbooks convey a picture of mathematics properties as typically being general, but proofs are more visible in the Finnish books. They provide more opportunities for learning formal, deductive reasoning, and that general properties require general arguments, i.e. for developing deductive proof schemes. However, the Swedish textbooks offer better opportunities for the discovery of general properties.

The findings are in line with those reported by Bergwall and Hemmi (2017). In integral calculus, proofs were even more visible in Fi1 since they were frequently labelled as proofs. Also, the proofs were often placed after the theorems, hence emphasizing their verification role.

When it comes to proof-related worked examples, the results differ from that of Bergwall and Hemmi (2017). In integral calculus such examples were only found in the Finnish books, but now only in the Swedish. However, these tasks are few in number and of very specific kinds. The conclusion is therefore that the Swedish and Finnish books alike offer limited opportunities to learn proof-related reasoning from worked examples.

4.1.2. Students' exercises

The percentage of proof-related tasks is low in all textbooks but higher in the Swedish books (around 14%) than in the Finnish ones (5%). In Fi1 only one proof-related task is about a specific case. This is an *investigate a conjecture* task (I task) while all general tasks are *develop an argument* tasks (D tasks). More precisely they are proving tasks. In the Swedish books many proof-related tasks are about a specific case (65% in Sw1, 40% in Sw2). D tasks dominate but are often informal and ask the student to explain or motivate. In the Swedish books, about one-third of the proof-related tasks involve making or investigating conjectures. Though few, there are also tasks in the categories of evaluating and correcting arguments. Finally, there are a few tasks in which the student has to match given data to preconditions of a theorem or suggest plausible thinking behind a presented calculation or derivation.

Our conclusion is that all the textbooks offer few opportunities to learn proof-related reasoning, but that more opportunities exist in the Swedish books. They also offer more variation in natures of proof-related reasoning. As in the case of expository sections, the

Swedish books are more oriented towards conjecturing, evaluation and empirical proof schemes while the Finnish are oriented towards deductive proof-schemes, since almost all proof-related tasks are general proving tasks.

Once again, the findings are in line with those reported by Bergwall and Hemmi (2017). Regarding the number of proof-related tasks in integral calculus and their distribution between general and specific, there were only small differences between Sw2 and Fi1. In integral calculus, the Finnish textbook had a handful of I tasks, all focused on determining whether a certain function is a primitive function to another function, or has a primitive function. Otherwise, D tasks were as dominant in integral calculus as in logarithms and combinatorics, while the Swedish textbooks offered a higher variation. It is worth noting that none of the books included tasks involving providing a counterexample or outlining a proof.

The findings may seem to be in contrast with the fact that proofs are more strongly emphasized in the Swedish national steering documents than in the Finnish ones. One possible explanation for this could be that the Swedish steering documents only emphasize proof within other topics (mainly geometry, trigonometry and number theory).

4.2. Comparison with United States textbooks

Regarding logarithms, Thompson et al. (2012) found that on average, 61% of properties addressed in expository sections in United States textbooks were justified. Just over half of the justifications were general proofs. In the Swedish and Finnish books, half of the main results were justified, but all justifications used general arguments. Regarding students' tasks, 7% of the tasks in the United States textbooks were proof-related, compared to 10% in the Swedish and 4% in the Finnish. While all Finnish proof-related tasks were general D tasks, approximately 75% of the Swedish and 50% of the United States tasks were general, most of them in the D category, but occasionally also in the M, I and E categories. These figures do not suggest definite differences between textbook traditions in the three countries, especially since there was great variation between the analysed United States textbooks. The findings indicate that, compared to an average United States textbook, the Finnish textbooks, and to some extent the Swedish ones, are more oriented towards general justifications. Also, the Swedish and United States textbooks are more alike when it comes to variation in natures of reasoning.

Next, we review the aggregated data on United States textbooks as reported by Thompson et al. (2012) and on Swedish and Finnish textbooks (Sw1, Sw2 and Fi1) as reported here and in Bergwall and Hemmi (2017). In all countries, results addressed in expository sections are justified in approximately 60% of the cases. The Swedish books stand out in having fewer general justifications than their Finnish and United States counterparts (Table 10). The high number of general justifications on logarithms seems to be an exception. However, 14% of the Swedish textbook tasks are proof-related, whereas about 6% of the Finnish and United States textbook tasks are proof-related.

Regarding distribution over types and natures of reasoning, the Swedish and United States books are very similar (Table 11). Here, the Finnish books stand out with a higher percentage of general tasks and D tasks. However, it is worth noting that Thompson et al. (2012) report a shift during upper secondary school towards a higher proportion of general tasks and D tasks. Their results from precalculus textbooks are similar to the overall results

Table 10. Type of reasoning in justifications in expository sections by country.

	No justification	Left to student	Specific case	General proof	Total
Sweden	41%	2%	47%	12%	58
Finland	38%	5%	12%	48%	42
United States	39%	11%	22%	31%	383

Remark: Percentages do not sum to 100% due to occasional double coding and round-off errors.

Table 11. Type and nature of reasoning in proof-related students' exercises by country.

	Total no. of tasks	PR tasks	Type of reasoning		Nature of reasoning				
			S	G	M	I	D	E	C, X, P, O
Sweden	1636	14.0%	53%	47%	11%	33%	50%	1.3%	6,1%
Finland	747	6.7%	34%	74%	2.0%	18%	80%	0%	0%
United States	9742	5.4%	46%	48%	15%	31%	44%	1.9%	11%

Remark: Types and natures are reported in relation to the total number of proof-related tasks. Percentages do not sum to 100% due to occasional double coding, X tasks without type classification, and round-off errors.

obtained from Fi1, which are strongly influenced by integral calculus material from the last mandatory part of the Finnish upper secondary school curriculum.

The study by Thompson et al. (2012) included textbook material on exponents, logarithms and polynomials. In United States textbooks on geometry, Otten et al. (2014) report higher frequencies of justified statements and proof-related tasks. This indicates that proofs continue to be more emphasized in school geometry than in other mathematics topics. Based on formulations in Swedish and Finnish steering documents, it is plausible that the situation is similar in these countries.

4.3. Limitations and future research

We have analysed material from textbooks that most Swedish-speaking (and many Finnish-speaking) students in Sweden and Finland have used when preparing for higher studies in mathematics, science, and technology. Hence, the conclusions set out in this paper only regard what this group of students have typically encountered when studying logarithms and combinatorics (and integral calculus) at the upper secondary level. To generalise further, one would need to extend the data set to other topics and textbook series. To draw general conclusions about Finnish textbook traditions, Finnish textbooks (written in Finnish) must be included. Extensions to geometry would be of special interest, since proofs are often more emphasized in geometry and since there are studies from other countries for comparison (e.g. Otten et al., 2014). However, even when researchers use similar methods, there is no guarantee that analytic frameworks are applied in the same way, which limits the accuracy of such comparisons.

Thompson et al. (2012) report that opportunities to engage in proof-related reasoning differ between mathematics topics and changes during upper secondary schooling. Findings on this issue for the Swedish and Finnish contexts will be reported in a paper in progress. Such studies are important in understanding the transition to tertiary education, where students experience difficulties with the increased focus on proofs and a higher level of formalism.

During the analytical process, situations occurred whereby the framework and its categories did not feel sharp or exhaustive enough to capture all relevant aspects of proof-related reasoning. Revisions and modifications of the frameworks are of theoretical value as well as analytical importance, since they represent a conceptualization of aspects of reasoning that are important in relation to proving. Regarding the nature of reasoning, the counterexample and outline a proof categories are important even if no such tasks were found. For the sake of symmetry, these categories should be supplemented with categories for finding supportive examples and for filling in the details in an outlined proof. Investigations of conditions, and the refinement of conditions to make a statement true, are also important aspects of reasoning that have no evident place in the current framing (though in the present study such tasks were placed in the *investigate* and *make a conjecture* categories, respectively). Also, tasks involving explaining the thinking behind a certain argument or calculation and tasks involving relating a certain situation to conditions of a theorem were found. There is room for refinement regarding the type of reasoning as well. When functions are involved, there is a difference between a situation which can be represented using a finite number of parameters and one which cannot (e.g. Bergwall, 2015). Regarding structure and visibility, it might be equally important to analyse the embedding of statements and justifications in the overall presentation (Bergwall, 2017). All such refinements of the framework need to be tested against data and founded in theoretical considerations.

A revised framework could be a useful tool for the analysis of teaching episodes as well as textbook material. Such a framework could be a starting point for a discussion about design principles for textbooks and learning activities that promote the learning of proof. A natural and necessary next step is to study how opportunities for proof-related reasoning provided in textbooks play out in the classroom context. Such studies could include the perspectives of the teacher as well as the student. One interpretation of the results presented in this study is that textbooks offer teachers limited support for teaching proof. If we think of mathematics as having both 'empirical' elements (experiments, constructing and studying examples and special cases, making guesses and conjectures, using intuition, and making generalizations) and 'deductive' elements (verifications, explaining connections, formalization, making derivations, and proofs), teachers need support for teaching both these aspects and how they are related. The findings reported in this paper indicate that textbooks seldom fill this double role.

Notes

1. Concepts relevant for the present study (*justification*, *proof-related reasoning* etc.) are defined in the subsection on analytical approach.
2. At the time of writing, a revision of the Finnish national curriculum is being implemented. All references in this paper are to the former version, implemented in 2005.

Disclosure statement

No potential conflict of interest was reported by the author.

ORCID

Andreas Bergwall http://orcid.org/0000-0002-8520-8200

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