International Journal of Mathematical Education in Science and Technology

ISSN: 0020-739X (Print) 1464-5211 (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

Arc length of function graphs via Taylor's formula

Patrik Nystedt

To cite this article: Patrik Nystedt (2020): Arc length of function graphs via Taylor's formula, International Journal of Mathematical Education in Science and Technology, DOI: 10.1080/0020739X.2020.1751320

To link to this article: https://doi.org/10.1080/0020739X.2020.1751320

Published online: 20 Apr 2020.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
https://doi.org/10.1080/0020739X.2020.1751320

Taylor & Francis
Taylor & Francis Group

CLASSROOM NOTE

OPEN ACCESS Check for updates

Arc length of function graphs via Taylor's formula

Patrik Nystedt

Department of Engineering Science, University West, Trollhättan, Sweden

ABSTRACT

We use Taylor's formula with Lagrange remainder to prove that functions with bounded second derivative are rectifiable in the case when polygonal paths are defined by interval subdivisions which are equally spaced. As a means for generating interesting examples of exact arc length calculations in calculus courses, we recall two large classes of functions f with the property that $\sqrt{1+(f')^2}$ has a primitive, including classical examples by Neile, van Heuraet and Fermat, as well as more recent ones induced by Pythagorean triples of functions. We also discuss potential benefits for our proposed definition of arc length in introductory calculus courses.

ARTICLE HISTORY
Received 27 October 2019

KEYWORDS
Taylor's formula; the fundamental theorem of calculus; Riemann sums; arc length

1. Introduction

One of the first experiences of measurements that we encounter in our lives is that of *length*. Even young children are involved in many everyday activities that concern length measurements. Questions such as 'How *tall* am I?' or 'How *long* can you jump?' or 'How *far* is it to my friends house?' arise naturally from them. In the early years of schooling we are taught how to measure lengths of *straight lines* using a ruler and express our findings in appropriate units. In middle school, we are presented with the problem of measurement of the circumference a the circle and how to relate this to the length of its diameter. For many students the transition from understanding straight line measurements to comprehending length measurement of non-linear curves is not so easily accomplished. Indeed, it is only natural for them to pose questions such as 'How can we measure something curved using a straight ruler?' or 'What do we really mean when we speak of the length of a curve?'. As teachers, we have to treat these questions seriously, because when pondering over this, the students are placed in very good company. Indeed, over the millennia, many of our greatest thinkers failed to provide satisfying answers to such questions. For instance, the Greek philosopher Aristotle (384–322 BC) stated the following concerning comparisons of motions along straight lines and along circles:

> But, once more, if the motions are comparable, we are met by the difficulty aforesaid, namely that we shall have a straight line equal to a circle. But these are not comparable. (Heath, 1970, p. 141)

CONTACT Patrik Nystedt patrik.nystedt@hv.se Department of Engineering Science, University West, SE-461 86 Trollhättan, Sweden

With some exceptions (for instance Archimedes rectification of the circle using a spiral, see e.g. Richesnon, 2013), Aristotle's view on these matters persisted amongst scholars even up to the time of Descartes (1596–1650) who wrote the following in his work *La Géométrie* from 1637:

> ... the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based on such ratios can be accepted as rigorous and exact. (Smith Lantham, 1954, p. 91)

Descartes would only 20 years later be proved wrong on this point by Neile who showed how to rectify the semi-cubical parabola $y^3 = ax^2$. Independently, both van Heuraet and Fermat came to the same conclusion within a few years after Neil's discovery (Traub, 1984). After that, of course, Newton and Leibniz fully developed the calculus machinery including formulas for arc length using integrals (Edwards, 1979, p. 217, 242).

2. Arc length in calculus teaching

The first time students are exposed to arc length calculations of general functions is in introductory calculus courses. In popular calculus books (see e.g. Adams, 2006; Hass et al., 2017; Stewart, 2015) the concept of curve length is typically defined in the following way.

Definition 2.1: Let A and B be two points in the plane and let $|AB|$ denote the distance between A and B. Let C be a curve in the plane joining A and B. Suppose that we choose points $A = P_0, P_1, P_2, \ldots, P_{n-1}$ and $P_n = B$ in order along the curve. The polygonal line $P_0, P_1, P_2, \ldots, P_n$ constructed by joining adjacent pairs of these points with straight lines forms a polygonal approximation to C, having length $L_n = \sum_{i=1}^{n} |P_{i-1}P_1|$. The curve C is said to be rectifiable if the limit L of L_n, as $n \to \infty$ and the maximum segment length $|P_{i-1}P_i| \to 0$, exists. In that case L is called the length of C.

Note that if we use this definition, then we are not calculating a limit of a *sequence*, in the usual sense that the students are used to, but rather the limit of a *net* in the following sense (for the details, see e.g. Bear, 1995 or Olmstead, 1961). A *directed set* is a non-empty set D equipped with a partial ordering $\prec$ satisfying the following three conditions:

(i) $\alpha \prec \alpha$ for all $\alpha \in D$;
(ii) if $\alpha \prec \beta$ and $\beta \prec \gamma$, then $\alpha \prec \gamma$;
(iii) if $\alpha, \beta \in D$, then there is $\gamma \in D$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$.

A net is a function defined on a directed set. Note that a sequence is a type of net, with D being the set of natural numbers directed as usual, namely that $n \prec m$ means $n \leq m$. In Definition 2.1, D is defined to be the set of choices of finite sets of points along the curve C, with the first point being A and the last being B. If α and β are two such choices, then put $\alpha \prec \beta$ if $\alpha \subseteq \beta$, that is if β is a *refinement* of α. In an analogous fashion, Riemann sums can be considered as limits of nets by defining refinements of partitions of intervals. So the approach using nets covers in a natural way many seemingly unrelated mathematical concepts in a beautiful way. It also has the advantage of making different types of limit proofs look like sequence limit proofs. However, it is highly abstract and it is also unsuitable for concrete calculations, for instance using computer simulations. Disregarding these difficulties, the typical calculus book (see e.g. Adams, 2006; Hass et al., 2017; Stewart, 2015) will then state some variant of the following result which is then used in exercises to calculate lengths of function graphs in particular cases.

Theorem 2.2: *If f is a real-valued function defined on $[a, b]$ with the property that its derivative exists and is continuous on $[a, b]$, then f is rectifiable on $[a, b]$ and its length L equals $\int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$. In that case, if G is a primitive function of $\sqrt{1 + (f')^2}$ on $[a, b]$, then $L = G(b) - G(a)$.*

The typical 'proof' of this result runs as follows. For the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$, let P_i be the point $(x_i, f(x_i))$, $0 \le i \le n$. By the mean-value theorem there exists $c_i \in (x_{i-1}, x_i)$ such that $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$. A few lines of calculation now yield that $L_n = \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\,\Delta x_i$ which can be recognized as a Riemann sum for $\int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x$ which ends the proof by invoking the fundamental theorem of calculus (FTC).

The problem with this 'proof' is that it is, in fact, not a proof at all. Why? Well, because it relies on the FTC which is not proved in full detail in any of the popular calculus texts in use today. Sure, parts of it are proved, but the hardest part concerning the convergence of Riemann sums is left out. The reason for skipping this is that a presentation including all details will be long and complicated. For instance, in Tao's book (Tao, 2006) the definition of general Riemann sums and proofs of properties such as these, including the FTC, takes more than 30 pages, excluding an argument for the crucial fact that continuous functions on compact intervals are uniformly continuous, which would make the presentation even longer.

We sympathize with the method of 'cheating' with the theory in calculus courses. To be honest, we can, of course, not prove every statement made in the course. However, we feel that leaving out a valid argument concerning such a central fact as the convergence of Riemann sums should be regarded as cheating at the wrong place.

In a recent article (Nystedt, 2019), we argue that the integral therefore should be defined using equally spaced subdivisions of the interval using only left (or right endpoints). We call the corresponding sums *Euler sums*, inspired by the fact that Euler (1768, Part I, Section I, Chapter 7) proposed such sums for the approximative calculations of integrals.

Definition 2.3: Suppose that f is a real valued function defined on an interval $[a, b]$. For all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$, we put $\Delta x = (b - a)/n$ and $x_k = a + k\Delta x$. We say that $I_n = \sum_{k=0}^{n-1} f(x_k)\Delta x$ is the nth *Euler sum* of f on $[a, b]$ and we say that f is *Euler integrable* on $[a, b]$ if the limit $I = \lim_{n\to\infty} I_n$ exists. In that case, we call I the *integral* of f on $[a, b]$ and we write this symbolically as $\int_a^b f(x)\,\mathrm{d}x = I$.

In (Nystedt, 2019), we show, using an idea of Poisson (see Bressoud, 2011 or Grabiner, 1983), utilizing Taylor's formula with Lagrange remainder, that the following version of the FTC easily can be proved in just a few lines of calculation.

Theorem 2.4: *If F is a real-valued function defined on $[a, b]$ such that its first derivative exists and is continuous on $[a, b]$, and its second derivative exists and is bounded on (a, b), then $f = F'$ is Euler integrable on $[a, b]$ and $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.*

3. Simplified arc length

In this article, we parallel our investigations in Nystedt (2019) and use Euler sums to define length of function graphs (see Definition3.1). We prove (see Theorem 3.4), using our version of the FTC, assuming some regularity conditions, that length of function graphs can be calculated via integrals using the classical formula given in Theorem 2.2.

Definition 3.1: Suppose that f is a real-valued function defined on an interval $[a,b]$. For all $n \in \mathbb{N}$ we put $\Delta x = (b-a)/n$, and for all $k \in \{0,1,\ldots,n-1\}$, we put $x_k = a + k\Delta x$ and $\Delta y_k = f(x_{k+1}) - f(x_k)$. We say that $L_n = \sum_{k=0}^{n-1} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$ is the nth *polygonal length* of f on $[a,b]$ and we say that f is *rectifiable* on $[a,b]$ if the limit $L = \lim_{n\to\infty} L_n$ exists. In that case, we call L the *arc length* of f on $[a,b]$.

The above definition is mathematically crystal clear and the polygonal lengths of this form are easy for students to calculate in particular cases (see Section 5). To prove the main result of the article, we need Taylor's formula with Lagrange remainder, a result which we now state, for the convenience of the reader.

Theorem 3.2: *Let n be a non-negative integer. If f is a real-valued function defined on $[a,b]$ such that its nth derivative exists, is continuous on $[a,b]$, and is differentiable on (a,b), then there exists $c \in (a,b)$ such that*

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b-a)^i + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Proof: For a short proof, see e.g. Hardy (1908), Nystedt (2019) and Olmstead (1961). ■

In the proof of our main result, we also need the following lemma.

Lemma 3.3: *If A, B and C are real numbers, with $A > 0$, then there is a real number D, between 0 and C, such that*

$$\sqrt{A + (B+C)^2} = \sqrt{A + B^2} + \frac{(B+D)C}{\sqrt{A + (B+D)^2}}.$$

Proof: Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \sqrt{A + (B+x)^2}$, for $x \in \mathbb{R}$. Since $A > 0$, the function g is differentiable at all $x \in \mathbb{R}$ with derivative $g'(x) = (B+x)/\sqrt{A + (B+x)^2}$. The claim now follows from Theorem 3.2 with $n = 0$, $a = 0$ and $b = C$ (that is, the mean value theorem). ■

Theorem 3.4: *If f is a real-valued function defined on $[a,b]$ such that its first derivative exists and is continuous on $[a,b]$, its second derivative exists and is bounded on (a,b), then f is rectifiable on $[a,b]$ if and only if the function $\sqrt{1+(f')^2}$ is Euler integrable on $[a,b]$. In that case, the length L of f on $[a,b]$ equals $\int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x$. If, in addition, $\sqrt{1+(f')^2}$ has an antiderivative G on $[a,b]$, then $L = G(b) - G(a)$.*

Proof: We use the notation introduced earlier. From Theorem 3.2 with $n = 1$, we get that

$$\frac{\Delta y_k}{\Delta x} = f'(x_k) + \frac{f''(c_k)\Delta x}{2}$$

for some $c_k \in (x_k, x_{k+1})$, depending on k and Δx, for $k \in \{0, \ldots, n-1\}$. Thus, from Lemma 3.3, it follows that

$$\begin{aligned}\sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2} &= \sqrt{1 + \left(f'(x_k) + \frac{f''(c_k)\Delta x}{2}\right)^2} \\ &= \sqrt{1 + f'(x_k)^2} + \frac{(f'(x_k)^2 + D)f''(c_k)\Delta x}{2\sqrt{1 + (f'(x_k) + D)^2}}\end{aligned}$$

for some real number D between 0 and $f''(c_k)\Delta x/2$. Hence

$$\begin{aligned}L_n &= \sum_{k=0}^{n-1} \sqrt{(\Delta x)^2 + (\Delta y_k)^2} \\ &= \sum_{k=0}^{n-1} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x}\right)^2}\,\Delta x \\ &= \sum_{k=0}^{n-1} \sqrt{1 + f'(x_k)^2}\,\Delta x + \sum_{k=0}^{n-1} \frac{(f'(x_k) + D)f''(c_k)(\Delta x)^2}{2\sqrt{1 + (f'(x_k) + D)^2}}\end{aligned}$$

which proves the claim, since

$$\left|\sum_{k=0}^{n-1} \frac{(f'(x_k) + D)f''(c_k)(\Delta x)^2}{2\sqrt{1 + (f'(x_k) + D)^2}}\right| \leq \frac{(\Delta x)^2}{2} \sum_{k=0}^{n-1} |f''(c_k)| \leq \frac{M(b-a)^2}{2n} \to 0,$$

as $n \to \infty$, for any M satisfying $|f''(x)| \leq M$ when $a < x < b$. The last part follows from Theorem 2.4. ∎

Remark 3.5: From the above proof, we immediately get the error bound

$$|L - L_n| \leq \frac{M(b-a)^2}{2n},$$

for all $n \in \mathbb{N}$, where $M = \sup\{|f''(x)|;\ a < x < b\}$, for the nth polygonal length.

4. Primitives of $\sqrt{1 + (f')^2}$

It seems to be a common opinion among mathematics teachers that there are few examples of functions f for which $\sqrt{1 + (f')^2}$ has a primitive function. In this section, we show that this is far from true by recalling two large classes of such functions.

4.1. The examples of Neile, van Heuraet and Fermat

Neile, van Heuraet and Fermat all considered rectification of curves of the type

$$f(x)^n = ax^{n+1}$$

for positive integers n and positive real numbers a (see e.g. Traub, 1984). Here, we will not follow their original approaches, but instead use modern tools from a typical calculus class to investigate this problem. We will also make a particular choice of the constant a to make our calculations simpler.

Proposition 4.1: *If n is a positive integer and $0 \leq x_1 \leq x_2$, then the function*

$$f(x) = \frac{n}{n+1} x^{(n+1)/n}$$

is rectifiable over over $[x_1, x_2]$ with length

$$n2^{-n-1} \int_{t_1}^{t_2} (t^2 - 1)^{n-1}(t^2 + 1)^2 t^{-n-2}\, \mathrm{d}t, \tag{1}$$

where t_1 and t_2 are strictly positive numbers given by $t_1 = x_1^{1/n} + \sqrt{1 + x_1^{2/n}}$ and $t_2 = x_2^{1/n} + \sqrt{1 + x_2^{2/n}}$.

Proof: Since $f'(x) = x^{1/n}$ it follows from Theorem 2.2 that f is rectifiable over $[x_1, x_2]$ with length

$$\int_{x_1}^{x_2} \sqrt{1 + x^{2/n}}\, \mathrm{d}x. \tag{2}$$

If we make the substitution $s = x^{1/n}$, then $\mathrm{d}x/\mathrm{d}s = ns^{n-1}$ and therefore (2) equals

$$\int_{s_1}^{s_2} ns^{n-1}\sqrt{1 + s^2}\, \mathrm{d}s, \tag{3}$$

where $s_1 = x_1^{1/n}$ and $s_2 = x_2^{1/n}$. It is well known that it is always possible to find a primitive function of an expression which is rational in s and $\sqrt{1 + s^2}$ by making the substitution

$$t = s + \sqrt{1 + s^2}.$$

Indeed, from the equality

$$(t - s)^2 = 1 + s^2$$

we get that

$$s = \frac{t^2 - 1}{2t}$$

and thus

$$\sqrt{1 + s^2} = t - s = t - \frac{t^2 - 1}{2t} = \frac{t^2 + 1}{2t}.$$

From the equality

$$s = \frac{t^2 - 1}{2t}$$

we get that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{t^2 + 1}{2t^2}.$$

Therefore (3) equals

$$\int_{t_1}^{t_2} n \cdot \frac{(t^2 - 1)^{n-1}}{(2t)^{n-1}} \cdot \frac{t^2 + 1}{2t} \cdot \frac{t^2 + 1}{2t^2} \,\mathrm{d}t = n2^{-n-1} \int_{t_1}^{t_2} (t^2 - 1)^{n-1}(t^2 + 1)^2 t^{-n-2} \,\mathrm{d}t,$$

where

$$t_1 = s_1 + \sqrt{1 + s_1^2} = x_1^{1/n} + \sqrt{1 + x_1^{2/n}}$$

and

$$t_2 = s_2 + \sqrt{1 + s_2^2} = x_2^{1/n} + \sqrt{1 + x_2^{2/n}}.$$

Note that since

$$t_2 \geq t_1 = x_1^{1/n} + \sqrt{1 + x_1^{2/n}} > x_1^{1/n} + \sqrt{x_1^2} = 2x_1^{1/n} \geq 0,$$

the integrand in (1) is a bounded continuous function and hence is the integral in (1) convergent. ■

To illustrate the above result, we will consider some examples. Note that we will use the same notation as in Proposition 4.1.

Example 4.2: Let $n = 1$. This is the problem of the rectification of the parabola $f(x) = x^2/2$. From Proposition 4.1 it follows that the length of f over $[x_1, x_2]$ equals

$$2^{-2} \int_{t_1}^{t_2} (t^2 + 1)^2 t^{-3} \,\mathrm{d}t = \frac{1}{4} \int_{t_1}^{t_2} t + 2t^{-1} + t^{-3} \,\mathrm{d}t = \left[\frac{t^2}{8} + \frac{\ln(t)}{2} - \frac{t^{-2}}{8} \right]_{t_1}^{t_2},$$

where $t_1 = x_1 + \sqrt{1 + x_1^2}$ and $t_2 = x_2 + \sqrt{1 + x_2^2}$. To simplify this, note that

$$t_1^2 = 2x_1^2 + 1 + 2x_1\sqrt{1 + x_1^2}.$$

and

$$(\sqrt{x_1^2 + 1} + x_1)(\sqrt{x_1^2 + 1} - x_1) = 1,$$

so that

$$t_1^{-1} = \sqrt{x_1^2 + 1} - x_1$$

which in turn implies that

$$t_1^{-2} = 2x_1^2 + 1 - 2x_1\sqrt{1 + x_1^2}.$$

Analogous equalities hold for t_2 and x_2. The length of f over $[x_1, x_2]$ hence equals

$$\tfrac{1}{2}[x\sqrt{1 + x^2} + \ln(x + \sqrt{1 + x^2})]_{x_1}^{x_2}.$$

Remark 4.3: Although the general primitive function of t^{-1} is $\ln|t|$, we can remove the absolute value in our examples, thanks to the limits of the integrals being positive, as stated by Proposition 4.1.

Example 4.4: Let $n = 2$. This is the problem of the rectification of the semicubical parabola $f(x) = 2x^{3/2}/3$. From Proposition 4.1 it follows that the length of f over $[x_1, x_2]$ equals

$$2^{-2}\int_{t_1}^{t_2}(t^2-1)(t^2+1)^2t^{-4}\,dt = \frac{1}{4}\int_{t_1}^{t_2} t^2+1-t^{-2}-t^{-4}\,dt$$
$$= \frac{1}{12}[t^3+3t+3t^{-1}+t^{-3}]_{t_1}^{t_2}$$
$$= \frac{1}{12}[(t+t^{-1})^3]_{t_1}^{t_2},$$

where $t_1 = \sqrt{x_1}+\sqrt{1+x_1}$ and $t_2 = \sqrt{x_2}+\sqrt{1+x_2}$. To simplify this, note that

$$(\sqrt{x_1+1}+\sqrt{x_1})(\sqrt{x_1+1}-\sqrt{x_1}) = 1$$

so that

$$t_1^{-1} = \sqrt{x_1+1}-\sqrt{x_1},$$

and analogously for t_2 and x_2. The length of f over $[x_1, x_2]$ hence equals

$$\tfrac{1}{12}[(2\sqrt{x+1})^3]_{x_1}^{x_2} = \tfrac{2}{3}[(x+1)^{3/2}]_{x_1}^{x_2}.$$

Note that the use of the general result in Proposition 4.1 is unnecessary in this case since we see immediately that if $f(x) = 2x^{3/2}/3$, then $f'(x) = \sqrt{x}$ so that the length of f over $[x_1, x_2]$ equals

$$\int_{x_1}^{x_2}\sqrt{1+f'(x)^2}\,dx = \int_{x_1}^{x_2}\sqrt{1+x}\,dx = \frac{2}{3}[(1+x)^{3/2}]_{x_1}^{x_2}.$$

Example 4.5: Let $n = 3$. This is the problem of the rectification of the curve $f(x) = 3x^{4/3}/4$. From Proposition 4.1 it follows that the length of f over $[x_1, x_2]$ equals

$$\frac{3}{16}\int_{t_1}^{t_2}(t^2-1)^2(t^2+1)^2t^{-5}\,dt = \frac{3}{16}\int_{t_1}^{t_2} t^3-2t^{-1}+t^{-5}\,dt$$
$$= \frac{3}{16}\left[\frac{t^4}{4}-2\ln(t)-\frac{t^{-4}}{4}\right]_{t_1}^{t_2},$$

where $t_1 = x_1^{1/3}+\sqrt{1+x_1^{2/3}}$ and $t_2 = x_2^{1/3}+\sqrt{1+x_2^{2/3}}$. To simplify this, note that

$$t_1^4 = (2x_1^{2/3}+1)^2+4x_1^{2/3}(1+x^{2/3})+4x_1^{1/3}(2x_1^{2/3}+1)\sqrt{1+x_1^{2/3}}$$

and

$$(\sqrt{1+x_1^{2/3}}+x_1^{1/3})(\sqrt{1+x_1^{2/3}}-x_1^{1/3}) = 1$$

so that

$$t_1^{-1} = \sqrt{1 + x_1^{2/3}} - x_1^{1/3}$$

which in turn implies that

$$t_1^{-4} = (2x_1^{2/3} + 1)^2 + 4x_1^{2/3}(1 + x^{2/3}) - 4x_1^{1/3}(2x_1^{2/3} + 1)\sqrt{1 + x_1^{2/3}}$$

Analogous equalities hold for t_2 and x_2. The length of f over $[x_1, x_2]$ hence equals

$$\tfrac{3}{8}[x^{1/3}(2x^{2/3} + 1)\sqrt{1 + x^{2/3}} - \ln(x^{1/3} + \sqrt{1 + x^{2/3}})]_{x_1}^{x_2}.$$

Remark 4.6: The reader might argue that the 'simplifications' made in Examples 4.2, 4.4 and 4.5 are, in fact, not simplifications at all. Indeed, with the access to a scientific calculator, the student would find it easier to first calculate the values of t_1 and t_2 and then inserting them into the result (1) in terms of t. Nevertheless, we have chosen to include the more complicated formulas depending on x_1 and x_2 since the techniques involved in obtaining them, using products of conjugate expressions, are instructive and might, if an instructor so wished, be used as challenging exercises for the students.

4.2. Pythagorean triples of functions

Suppose that we seek two functions p and q such that $f' = p/q$ and $1 + (f')^2 = g^2$ where g is some function to which we can find a primitive function G. This implies that $1 + p^2/q^2 = g^2$ or equivalently that $(p^2 + q^2)/q^2 = g^2$. One way to accomplish this is if $p^2 + q^2 = r^2$ for some function r of reasonably simple type. This means that (p, q, r) is a Pythagorean triple of functions. It is a classical result in number theory that such triples, consisting of integers, can be parametrized by $p = k(m^2 - n^2)$, $q = k(2mn)$ and $r = k(m^2 + n^2)$, where k, m and n are positive integers with $m > n$, and with m and n coprime and not both odd (see e.g. Long, 1972). In Kubota (1972) Kubota has shown that the same kind of result holds in any unique factorization domain (UFD). In particular, it holds for polynomial rings $\mathbb{R}[X]$, since they are Euclidean domains and hence UFD's. The bottom line is that we can use this kind of parametrization to yield examples of rectifiable curves in the following way.

Proposition 4.7: *Suppose that m and n are positive continuous functions defined over the closed interval $[a, b]$. If f and G are differentiable functions defined over $[a, b]$ satisfying*

$$f' = \frac{m}{2n} - \frac{n}{2m}$$

and

$$G' = \frac{m}{2n} + \frac{n}{2m},$$

then f is rectifiable over $[a, b]$ with length $G(b) - G(a)$.

Proof: Since

$$\begin{aligned}\sqrt{1+(f')^2} &= \sqrt{1+\left(\frac{m}{2n}-\frac{n}{2m}\right)^2}\\ &= \sqrt{\left(\frac{m}{2n}\right)^2+\frac{1}{2}+\left(\frac{n}{2m}\right)^2}\\ &= \sqrt{\left(\frac{m}{2n}+\frac{n}{2m}\right)^2}\\ &= \left|\frac{m}{2n}+\frac{n}{2m}\right|\\ &= \frac{m}{2n}+\frac{n}{2m}\\ &= G'\end{aligned}$$

the claim follows from Theorem 2.2. ■

Let us illustrate Proposition 4.7 in three examples.

Example 4.8: A problem which often comes up in calculus textbooks is to calculate the length of a portion of the hyperbolic cosine function over any interval $[a,b]$. Based on our calculations above, it is easy to see why. Indeed, if we put $f(x)=\cosh(x)$, then $f'(x)=\sinh(x)=m/(2n)-n/(2m)$ where $m=e^x$ and $n=1$ are positive and continuous over $[a,b]$. Since $G(x)=\sinh(x)$ is a primitive function of $m/(2n)+n/(2m)=\cosh(x)$, the length of $f(x)=\cosh(x)$ over $[a,b]$ equals $\sinh(b)-\sinh(a)$. The corresponding task for the students could therefore be:

Problem 4.9: Show that the length of

$$f(x)=\cosh(x)$$

over the interval $[0,1]$ equals

$$\frac{e}{2}-\frac{1}{2e}.$$

Example 4.10: Define functions $m=4x$ and $n=x^2+1$ over any interval $[a,b]$ where a is positive. Then m and n are positive. We need to find f so that

$$f'(x)=\frac{m}{2n}-\frac{n}{2m}=\frac{4x}{2x^2+2}-\frac{x}{8}-\frac{1}{8x}.$$

We choose

$$f(x)=\ln(2x^2+2)-\frac{x^2}{16}-\frac{\ln(x)}{8}.$$

Next, we need to find a primitive G of the function

$$\frac{m}{2n}+\frac{n}{2m}=\frac{4x}{2x^2+2}+\frac{x}{8}+\frac{1}{8x}.$$

We choose

$$G(x) = \ln(2x^2 + 2) + \frac{x^2}{16} + \frac{\ln(x)}{8}.$$

The length of f over $[a, b]$ equals $G(b) - G(a)$. Now we can construct a challenging task for the students:

Problem 4.11: Show that the length of

$$f(x) = \ln(2x^2 + 2) - \frac{x^2}{16} - \frac{\ln(x)}{8}$$

over the interval $[1, 2]$ equals

$$\frac{3}{16} + \ln(5) - \frac{7\ln(2)}{8}.$$

Example 4.12: Define functions $m = (x + 2)^2$ and $n = (x + 1)(x^2 + 1)$ over any interval $[a, b]$ where $a > -2$. We need to find f so that

$$f'(x) = \frac{m}{2n} - \frac{n}{2m} = \frac{(x+2)^2}{2(x+1)(x^2+1)} - \frac{(x+1)(x^2+1)}{2(x+2)^2}.$$

Since

$$\frac{(x+2)^2}{2(x+1)(x^2+1)} = \frac{x}{4(x^2+1)} + \frac{7}{4(x^2+1)} + \frac{1}{4(x+1)}$$

and

$$\frac{(x+1)(x^2+1)}{2(x+2)^2} = \frac{x}{2} - \frac{3}{2} - \frac{5}{2(x+2)^2} + \frac{9}{2(x+2)}$$

we can choose

$$f(x) = \frac{\ln(x^2+1)}{8} + \frac{7\tan^{-1}(x)}{4} + \frac{\ln(x+1)}{4} - \frac{x^2}{4} + \frac{3x}{2} - \frac{5}{2(x+2)} - \frac{9\ln(x+2)}{2}.$$

Next, we need to find a primitve G of the function

$$\frac{m}{2n} + \frac{n}{2m} = \frac{x}{4(x^2+1)} + \frac{7}{4(x^2+1)} + \frac{1}{4(x+1)} + \frac{x}{2} - \frac{3}{2} - \frac{5}{2(x+2)^2} + \frac{9}{2(x+2)}.$$

We choose

$$G(x) = \frac{\ln(x^2+1)}{8} + \frac{7\tan^{-1}(x)}{4} + \frac{\ln(x+1)}{4} + \frac{x^2}{4} - \frac{3x}{2} + \frac{5}{2(x+2)} + \frac{9\ln(x+2)}{2}.$$

The length of f over $[a, b]$ equals $G(b) - G(a)$. Now we can construct a really challenging task for the students:

Problem 4.13: Show that the length of

$$f(x) = \frac{\ln(x^2+1)}{8} + \frac{7\tan^{-1}(x)}{4} + \frac{\ln(x+1)}{4} - \frac{x^2}{4} + \frac{3x}{2} - \frac{5}{2(x+2)} - \frac{9\ln(x+2)}{2}$$

over the interval $[0, 1]$ equals

$$\frac{7\pi}{16} - \frac{5}{3} + \frac{9\ln(3)}{2} - \frac{33\ln(2)}{8}.$$

5. Discussion

In this article, we have presented a simplified definition of arc length as a limit of polygonal sums where the subdivision of the interval is uniform. We feel that such an approach would support the students' learning of calculus for many reasons.

First of all, we have provided a complete proof that the polygonal lengths converge precisely when the associated integral

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{4}$$

exists. In many popular calculus books the proof of this fact is incomplete since convergence of the nets associated with general Riemann sums is not proved.

Secondly and perhaps more importantly, the students can, using a simple computer program, easily calculate approximations of our simplified polygonal lengths, before using (4). For instance, suppose the students are given the task of calculating the arc length of $f(x) = 2x^{3/2}/3$ over the interval $[3, 8]$. For $n \in \mathbb{N}$ we have that $\Delta x = 5/n$ and thus

$$L_n = \sum_{k=0}^{n-1} \sqrt{\frac{25}{n^2} + \left(\frac{2}{3}\left(3 + \frac{5k+5}{n}\right)^{3/2} - \frac{2}{3}\left(3 + \frac{5k}{n}\right)^{3/2}\right)^2}.$$

Using a computer program, rounding off to four decimal places, we get

$$L_1 \approx 12.6508 \quad L_2 \approx 12.6622 \quad L_3 \approx 12.6646 \quad L_4 \approx 12.6655$$
$$L_5 \approx 12.6659 \quad L_{10} \approx 12.6665 \quad L_{20} \approx 12.6666 \quad L_{100} \approx 12.6666$$

which strongly suggests that $L = 38/3$. After this the students can try to make the exact calculation, which, as we saw before, is the rectification of the semicubical parabola. Namely, since $f'(x)^2 = x$, we get, using Theorem 2.2, that

$$L = \int_3^8 \sqrt{1+x}\,\mathrm{d}x = \left[\frac{2(1+x)^{3/2}}{3}\right]_3^8 = \frac{2 \cdot 9^{3/2}}{3} - \frac{2 \cdot 4^{3/2}}{3} = \frac{38}{3},$$

which confirms what the students guessed. The students could then move on to try to calculate the length of the parabola $f(x) = x^2/2$ over the interval $[0, 1]$. Again, making approximative calculations, we have $\Delta x = 1/n$ and thus

$$L_n = \sum_{k=0}^{n-1} \sqrt{\frac{1}{n^2} + \frac{1}{4}\left(\left(\frac{k+1}{n}\right)^2 - \left(\frac{k}{n}\right)^2\right)^2}.$$

Using a computer program, rounding off to four decimal places, we get

$$L_1 \approx 1.1180 \quad L_2 \approx 1.1404 \quad L_3 \approx 1.1445 \quad L_4 \approx 1.1459 \quad L_5 \approx 1.1466$$
$$L_{10} \approx 1.1475 \quad L_{20} \approx 1.1477 \quad L_{100} \approx 1.1478 \quad L_{200} \approx 1.1478.$$

After this, the students could try to calculate the exact value of the integral. From the discussion in the previous section, this is the length of the parabola, which equals

$$\int_0^1 \sqrt{1+x^2}\,\mathrm{d}x = \frac{\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2}.$$

Finally, the students could try to calculate the length of $f(x) = x^3/3$ over the interval $[0, 1]$. Numerically, they would easily get $L_{100} = 1.0894$, rounding off to four decimal places. However, when considering the exact length calculation, they have to deal with the problem of calculating

$$\int_0^1 \sqrt{1+x^4}\,\mathrm{d}x$$

which is a so-called elliptic integral (see e.g. Hancock, 1958) and is impossible to calculate exactly using the elementary functions. It is our firm belief that students should be subjected to the calculation of such integrals in a typical calculus course, in order for them to appreciate the numerical calculations, which, after all, are crucially important for them in a future work-life as e.g. engineers.

Acknowledgments

The author wishes to thank the anonymous referees for several corrections on the first version of the manuscript.

Disclosure statement

No potential conflict of interest was reported by the author(s).

ORCID

Patrik Nystedt http://orcid.org/0000-0001-6594-7041

References

Adams R. A. (2006). *Calculus a complete course*. Addison Wesley.
Bear H. S. (1995). *A primer of Lebesgue integration*. Academic Press.
Bressoud, D. M. (2011). Historical reflections on teaching the fundamental theorem of integral calculus. *The American Mathematical Monthly*, *118*(2), 99–115. https://doi.org/10.4169/amer.math.monthly.118.02.099.
Edwards Jr. C. H. (1979). *The historical development of the calculus*. Springer-Verlag.
Euler J.L. (1768). *Institutionum calculi integralis*. Note that: http://www.17centurymaths.com/ provides a complete English translation of this work by Ian Bruce.
Grabiner, J. V. (1983). Who gave you the epsilon? The origins of Cauchy's Rigorous Calculus. *The American Mathematical Monthly*, *90*(3), 185–194. https://doi.org/10.1080/00029890.1983.11971185.
Hancock H. (1958). *Elliptic integrals*. Dover Publications.
Hardy G. H. (1908). *A course of pure mathematics*. The English Language Book Society: Cambridge University Press.
Hass J. R., Heil C. E., & Weir M. D. (2017). *Thomas' calculus* (14 ed.). Pearson.
Heath T. (1970). *Mathematics in aristotle*. Oxford University Press.
Kubota, K. (1972). Pythagorean triples in unique factorization domains. *American Mathematical Monthly*, *79*, 503–505. https://doi.org/10.2307/2317570.

Long C. T. (1972). *Elementary introduction to number theory*. D. C. Heath and Company.
Nystedt P. (2019). Poisson's fundamental theorem of calculus via Taylor's formula. Accepted for publication in *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2019.1682699.
Olmstead J. H. (1961). *Advanced calculus*. Appleton-Century Crofts.
Richesnon D. (2013). Circular reasoning: Who first proved that C/d is a constant? arXiv:1303.0904v2.
Smith D. E., & Lantham M. L. (1954). *The geometry of René descartes*. Dover Publications.
Stewart J. (2015). *Calculus: Early transcendentals* (8th ed.). Cengage Learning.
Tao T. (2006). *Analysis I*. Hindustan Book Agency.
Traub G. (1984). *The development of the mathematical analysis of curve length from archimedes to lebesgue* [PhD thesis]. New York University.