International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tmes20

Enrichment in school principals' ways of seeing mathematics

Hanan Innabi & Jonas Emanuelsson

To cite this article: Hanan Innabi & Jonas Emanuelsson (2020): Enrichment in school principals' ways of seeing mathematics, International Journal of Mathematical Education in Science and Technology, DOI: 10.1080/0020739X.2020.1782496

To link to this article: https://doi.org/10.1080/0020739X.2020.1782496

Published online: 29 Jun 2020.

INTERNATIONAL JOURNAL OF MATHEMATICAL EDUCATION IN SCIENCE AND TECHNOLOGY
https://doi.org/10.1080/0020739X.2020.1782496

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS Check for updates

Enrichment in school principals' ways of seeing mathematics

Hanan Innabi and Jonas Emanuelsson

Department of Pedagogical, Curricular and Professional Studies, University of Gothenburg, Gothenburg, Sweden

ABSTRACT

This study presents a case of a problem that did not originate in theory but rather ended in theory. It addresses the issue of the prevalent, traditional public view of the nature of mathematics and that of its teaching and learning that might hamper mathematics reform. In this study, we present a five-day professional development programme that was designed and conducted within the context of practice-oriented research to help elementary school principals to enrich their views of mathematics in order to see mathematics in a different way, more aligned with mathematics reform. Data was generated before, during, and at the end of this programme, and results showed significant enrichment in participants' views of mathematics. By using the Variation Theory of Learning, this study aimed to understand how this programme of short duration can make a difference. The core tenet of Variation Theory is that people learn through experiencing differences and similarities (in this order). The Variation Theory of Learning is used as a lens to see the programme's framework, its activities, and outcomes. The analysis clarified that principals' ways of seeing mathematics were enriched when they experienced patterns of variations and invariance, in particular, the pattern of contrast.

ARTICLE HISTORY
Received 25 November 2019

KEYWORDS
Ways of seeing mathematics; school principals; professional development; elementary school mathematics; variation theory

1. Ways of seeing mathematics

This study refers to the 'traditional view' of the nature of mathematics and the nature of its teaching and learning. This view holds mathematics to be an absolute and unchangeable truth that is independent of human subjectivity. Calls for a wider perspective enriched by more fallibilistic and humanistic views have increased in the last three decades. These calls came as a result of rapidly accelerating human advancements, especially during the last five decades, in science, technology, and communications, as well as in learning theories and education. As a consequence, most countries have adopted mathematics reforms since the 1990s. These reforms have called – and are still calling – for changing traditional practices in teaching and learning mathematics. According to this shift in viewing mathematics, teachers and students have to see mathematics as a subject that can be thought out and that also makes sense. Reformers want mathematics classrooms to function as mathematical communities in which students have opportunities to reason and communicate

CONTACT Hanan Innabi hanan.innabi@gu.se

mathematically and make connections (e.g. National Council of Teachers of Mathematics [NCTM], 2010; NCTM, 2014; National Governors' Association Center & Council of Chief State School Officers, 2010). Throughout this paper, we use the term 'mathematics *reform*' in its general sense to indicate the international orientations of the global mathematics education community over the last three decades. We can read about differences in students' learning that may, in one way or another, mediate the differences between reformist and traditional inclinations. Examples include relational versus instrumental understanding (Skemp, 1976), conceptual and procedural knowledge (Hiebert & Lefevre, 1986), conceptual orientation versus calculation orientation (Thompson et al., 1994), and the discovery learning method as opposed to the 'basic' way of teaching and learning mathematics (Herbel-Eisenmann et al., 2016). The dichotomous view of mathematics as static versus dynamic, in which the former type is rooted in public community and the latter is demanded strongly in mathematics reform, is a challenge facing mathematics education (Latterell & Wilson, 2016; Stephan et al., 2015; White-Fredette, 2010). The traditional view that mathematics learning is a combination of memorizing a series of facts and mastering procedural manipulation is still common around the world. Evidence is presented in support of the existence of these traditional views in the public community, which includes those with and without a formal education in mathematics.

People generally see mathematics as a difficult, mysterious, and abstract subject that is irrelevant and unpleasant; it is viewed as a masculine subject, more suited to elites and those with special abilities than to common people (Darragh, 2018; Ernest, 1996; Forgasz et al., 2014; Latterell & Wilson, 2016). Students see mathematics as a collection of facts to be learned (Bicknell & Hunter, 2009). In addition, students perceive mathematics as a subject that is just for 'smart people' (Markovits & Forgasz, 2017). Frequently, people – including in-service and pre-service teachers – perceive mathematics as a tool for arithmetic computations (Cassel & Vincent, 2011; Latterell & Wilson, 2012; Seaman et al., 2005). Mathematics is seen as static content rather than as a dynamic activity; it is viewed as a collection of fixed theories, facts, rules, and procedures to be mastered (Lim & Ernest, 2000; Nisbet & Warren, 2000; Reeder et al., 2009; Viholainen et al., 2014) or an end product with specific right answers (Cassel & Vincent, 2011). Even individuals whose professions relate to mathematics can see it as a mere collection of formulae and theorems (Peterson, 1996). In addition, the view that mathematical facts should be taught first, before applications and problem solving, is still prevalent (Innabi & El Sheikh, 2007; Oster et al., 1999).

Widespread traditional views on the nature of mathematics and the teaching and learning of the subject not only contradict mathematics reform but they could also hamper it (Bleiler, 2015; Nadelson et al., 2014; Sheffield, 2017; Urick & Bowers, 2014). For instance, if parents think their children are doing well in mathematics because they can do all the computations quickly and precisely or, conversely, if parents think that their children are doing badly in the subject because they cannot do computations, then home support for developing the teaching and learning of mathematics will probably remain minimal (Brez & Allen, 2016; Innabi, 2009; Nanna, 2016; Powell et al., 2012). Ernest (1989) found that teachers', students', and parents' views of mathematics affect the teaching and learning of the subject. Moreover, Goldin et al. (2009) emphasized the importance of 'beliefs' in the mathematics teaching and learning process, not just at student and teacher levels but also in terms of their influence on reform efforts across countries.

The social contexts in which learning takes place are important for pedagogical processes. These contexts are shaped by the beliefs and expectations of students, parents, principals, and policymakers, as well as by the structure of educational systems (Thompson, 1992). Therefore, shifting the educational community's awareness towards a more enriched vision of teaching and learning mathematics is necessary for mathematics reform to succeed; otherwise, all efforts made to change mathematics curricula and train teachers could prove ineffective (Nelson et al., 2007; White-Fredette, 2010). Few studies have explicitly aimed at bringing about change in how people see mathematics. Most of this research was conducted on mathematics teachers (e.g. Gill et al., 2004; Kutaka et al., 2018), in which, the focus was mainly on investigating the effect of changes in instructional pedagogy or classroom environments on pre-service or in-service mathematics teachers' beliefs, images, and perceptions about mathematics learning and teaching (e.g. Anderson & Piazza, 1996; Cerrato, 2019). Sterenberg (2008), for example, succeeded in moving elementary mathematics teachers away from their previous static-image mathematics metaphors towards a view of mathematics as capable of change through communication.

The results of a survey of a wide range of members of the mathematics education community that was aimed at identifying the 'grand challenges' that mathematics educators are currently facing have appeared as commentary in the *Journal for Research on Mathematics Education* (Stephan et al., 2015). These results revealed the following three themes:

- Changing perceptions about what it means to do mathematics: This involves the 'challenge of helping people see that doing mathematics is about problem solving, reasoning, curiosity, and enjoyment, and not about following procedures to get "the answer" or "just about doing well on a test"';
- Changing the public's perception about the role of mathematics in society: This involves asking, 'By what means can we address common thinking such as "I was never good at math"? . . . [There is a] need to see mathematics as something that human beings normally do and that has relevance and beauty'; and
- Achieving equity in mathematics education: It is 'more complex than "just" achievement gaps among different groups of students . . . general beliefs about who can do mathematics and how these beliefs affect teaching and learning' (Stephan et al., 2015, p. 139).

The present research is a response to the above challenges that mathematics education researchers face. It aims to further develop public views on mathematics. It focuses on enriching the *ways of seeing* that apply to the teaching and learning of mathematics for one group of the educational community in particular: school principals.

The educational community is a part of the public, and it is affected by the rooted view of mathematics, but, at the same time, its perspective could affect that of the entire community. For instance, Lim (1999) showed that mathematics teachers and parents could affect students' images of and beliefs about mathematics. Thus, public views of mathematics inside and outside of schools are dependent and interconnected. If we want to make a change, these views have to be addressed and challenged both within the classroom and outside it.

The educational community not only comprises mathematics teachers but also parents, students, principals, administrators, teachers of other subjects, and more. This research is concerned with school principals, who comprise an important group in the educational sector. The next section clarifies the importance of this group to supporting mathematics reform.

2. Principals and mathematics reform

Research confirms the important role school principals play in mathematics reform (Cobb et al., 2018; Darling-Hammond et al., 2009; Johnson, 2013; National Association of Elementary School Principals, 2002; Ontario Principals' Council, 2009; Urick et al., 2018). Despite some unclear findings about how principals influence student achievement or school performance (e.g. Hattie, 2009; Hochbein & Cunningham, 2013), many more studies have shown that principals have an important effect on student achievement (Boberg & Bourgeois, 2016; Chiang et al., 2016; Cotton, 2003; Dempster, 2001; Grissom et al., 2015; Hallinger & Heck, 1998; Hattie, 2015; McCullough et al., 2016; Pelfrey, 2006; Stronge et al., 2008; Tan, 2018). One intermediary explanation for this relationship may be that principals influence teachers; this is valid since many studies have confirmed that principals' instructional leadership affects teacher practices (Lambersky, 2016; Mangin, 2007; Proffitt-White, 2017; Quinn, 2002; Rigby et al., 2017; Supovitz et al., 2010). For example, in a meta-analysis of the literature, Robinson et al. (2008) found that when principals work directly with teachers to plan, coordinate, and assess their instructional practices, student outcomes are significantly higher.

In order to understand principals' influence on mathematics reform, we must consider the numerous challenges that mathematics teachers face, according to this reform. Mathematics teachers must provide opportunities and experiences that allow students to learn mathematics independently (e.g. Hiebert & Grouws, 2007). This requires not only a fuller grasp of mathematical content than that developed by traditional teaching methods (Burrill, 2014) but also more time, better conditions, and more supplies such as teaching materials (McDuffie & Graeber, 2003; Yeliz, 2016). Researchers have found that teachers are likely to feel anxiety and frustration during the process of implementing reform-based approaches (Campbell & White, 1997); teachers commonly have inadequate knowledge or understanding of the new vision of school mathematics (Burrill & Biehler, 2011). In order to overcome these limitations, teachers need support; without it, the challenges they face may tempt them to return to more stable and comfortable traditional practices. However, with continued assistance, the successful implementation of reform goals becomes more likely (McDuffie & Graeber, 2003). Accordingly, it is important that principals provide active pressure and support to establish change (Lamichhane, 2016).

Studies have shown that principals' views and attitudes influence teachers' experiences (Youngs, 2007; Youngs & Bruce, 2002). Thus, principals can profoundly influence teachers who are working to change how they teach mathematics (Lee & Nie, 2017; Reed et al., 2006). In addition, researchers have found that school principals contribute to higher morale and stronger motivation to succeed among teachers in countries that perform well in mathematics (Abazaoglu & Aztekin, 2016). As Brown and Smith (1997) showed, teachers are concerned about their administrators' opinions; further, they need to know that these administrators understand development approaches (Gokçe, 2009; Nelson & Sassi, 2005).

3. Professional development programmes

In the previous section, we argued for the importance of the school principal's role in ensuring that mathematics reform is successful. This section highlights efforts to help principals meet their needs through professional development programmes so that they can be effective instructional leaders of mathematics.

Mathematics reform that requires a higher quality of teaching and learning has made effective professional development more important than ever (Kelleher, 2003; Peterson, 2002). Sparks (2002) argued that teachers' and principals' professional development is a central factor in determining teaching quality. Principals need support and opportunities that strengthen them so that they, in turn, can provide support for mathematics reform (Donaldson & Papay, 2014; Perry et al., 2015). Despite substantial research on the effect school principals have on mathematics reform, as the previous section showed, far less attention has been given to the principals' own needs in their efforts to support mathematics in their schools and how those needs can be fulfilled. Adequate professional development programmes are needed to help principals confront their own understandings and reflect on their knowledge, as well as learning, curricula, and assessment procedures (Boston et al., 2017; Hussin & Al Abri, 2015; Stein & Spillane, 2005).

As Cobb and Jackson (2011) discussed, researchers have argued about what level of content knowledge of mathematics principals need in order to act as effective instructional leaders. All efforts related to supporting principals in mathematics have focused on improving principals' needed competences. These efforts have always concentrated on knowledge and skills. For example, Lester and Grant (2001) conducted a programme as part of the implementation of National Council of Teachers of Mathematics (NCTM) 2000 standards to help administrators become more effective mathematics supervisors by engaging in mathematical activities and exploring students' thinking and teachers' roles in standards-based classrooms. Another example of these programmes is a training programme for principals that sought to develop their understanding of a more constructivist pedagogy and its application in the classroom, as well as their capacity to use a programme-aligned observation rubric for rating teacher performance (Meyers et al., 2016). As a third example, Boston et al. (2017) investigated how principals can be supported in their development of the knowledge and skills necessary to bolster high-quality teaching and learning in mathematics. In order to enhance principals' capacity to serve as instructional leaders in mathematics, a three-session professional development experience was applied. The results identified changes in the feedback principals gave to mathematics teachers; however, the changes were not sustained in the years that followed. In summary, we can say that none of the school principals' mathematics-related professional development programmes were directly concerned with prevalent traditional views on mathematics among the public.

In examining how to support principals in becoming better instructional leaders, Fox (2018) showed the limited nature of understanding of the design of effective support for principal learning. This is also evident in the processes by which effective support may be designed. As a contribution to supporting principals' learning, which may, in turn, help them to become better instructional leaders, the present study claims that principals need to enrich their rooted views on mathematics that represent the traditional shared views of the entire community. We argue that school principals should be presented with alternatives to the common, traditional understanding of mathematics that was described above.

We want them to envision that mathematics and teaching and learning mathematics can be far more interesting than the mathematics traditionally taught in schools has been thus far. Our argument is that school principals have no opportunities to act as instructional leaders in shaping the direction of mathematics reform if they have never experienced mathematics as a dynamic, humanistic tool for reasoning, communication, and problem solving.

In this study, we present a five-day professional development programme (PDP) that was designed and conducted to help elementary school principals to enrich their view of the meaning of mathematics to see it in a different way – a way that is more aligned with mathematics reform. By describing and explaining the changes that occurred in how participant principals see mathematics and the teaching and learning of it, we try to understand how giving the PDP of a short duration can make a difference. The questions that this study aimed to answer are as follows: What enrichments can a five-day PDP yield that will affect a group of elementary school principals' views about mathematics and teaching and learning mathematics? How can these enrichments be explained?

4. Theoretical framework

In an attempt to bring about a shift in school principals' traditional view of school mathematics in favour of a view that is more in line with reform ideas, we framed the necessary change as a problem of learning and adopted the *Variation Theory of Learning* (VT) as a theoretical framework in this study. VT is an approach that usually serves as a point of departure not only for designing instructional sequences but also for analyzing the process and outcomes of such designs (Lo, 2012). In this study, VT is not used as a point of departure but as a theoretical lens for explaining changes in the participating school principals' views. It is important to bear in mind that it is not uncommon to use phenomenography and VT as tools for the analysis of data materials that were generated for other purposes (e.g. Al-Murani et al., 2019; Baillie et al., 2001; Marton, 2006; Marton & Häggström, 2017). Further, in both Marton and Booth (1997) and Marton (2015) a vast number of the re-analyzed studies where not originally designed with VT-analyses in mind.

Throughout this article, we use the expressions *views* and *ways of seeing* as labels for indicating how people experience, perceive, or understand the nature of mathematics and what it means to learn and teach mathematics. In the literature, terms like *beliefs*, *metaphors*, *images,* and *perceptions* are used to describe fundamental differences in human understanding of learning, teaching, and knowledge. We adopt VT where *ways of seeing*, one of the major terms or ideas, is defined in terms of the *aspects* (attributes) that are *discerned* at a certain point in time (Marton & Tsui, 2004). The following section offers a brief clarification of VT, which is the theoretical framework of this study.

4.1. Variation theory (VT)

VT is a learning theory that explains how learners come to see a phenomenon in certain ways (Marton & Booth, 1997). It represents a theoretical framework that can direct teachers' attention toward what has to be done to provide learners with the necessary learning opportunities (Lo, 2012). VT is developed from phenomenography, which is the study of the different ways people experience a specific phenomenon. According to this approach, learning is conceptualized as a change in the person–world relationship. Nothing is transferred from the world to the individual, but when learning takes place, individuals see the world in new ways. In other words, the learner becomes 'capable of being simultaneously and focally aware of other aspects or more aspects of a phenomenon than was previously the case' (Marton & Booth, 1997, p. 142). Hence, each particular *way of seeing* corresponds to a particular set of *dimensions of variation* (DoV) that are being *simultaneously discerned*. Learning has taken place when seeing is changed from one way to another, more complex way (or what we call in this research *enriched ways of seeing*). VT clarifies, or explains, how such enrichments are possible. In order to see something in a specific way, an individual must discern certain aspects or attributes (or what VT calls *critical aspects*) of that thing. If this is to happen, the individual must experience those aspects, and the only way to experience them is when they vary (Marton, 2015). For instance, in order to experience something as sweet, one must be exposed to variations in taste, such as sourness, bitterness, or saltiness. According to VT, it would not be possible to notice a sweet taste if all tastes were identical. Furthermore, the very concept of taste would be meaningless if there were not different kinds of tastes.

Two concepts are especially important in VT, namely, the *dimension of variation* (DoV) and *values* where at least two values are required to define an aspect or dimension. In the previous example, 'taste' can be seen as a DoV because it can vary between at least two different values (e.g. sweet or bitter). We discern a value when we simultaneously experience another, different value; thus, the meaning of values originates from how they differ from each other. Values and dimensions must be experienced simultaneously because one cannot exist without the other in the learner's awareness (Marton, 2015; Marton & Booth, 1997; Marton & Tsui, 2004).

In VT, it is not enough to tell learners what the critical aspects of a phenomenon are. Rather, learning depends on the individual experiencing variation in the dimensions of the critical aspects. When learning and teaching is examined, the *object of learning*, which is defined as a 'specific insight, skill or capability that the students are expected to develop' (Marton & Pang, 2006, p. 2), should be considered. Learning means simultaneously discerning the critical aspects of the object of learning (Marton, 2015). The differences in how students experience the same object of learning depend on which aspects of this object they discern (Lo, 2012).

According to VT, the discernment of critical aspects occurs during systematic interactions between a learner and a phenomenon (what is to be learned), and variation is the mechanism that generates such an interaction. Thus, variation is a necessary condition for learning. The most effective way to help students learn is by focusing on providing opportunities for them to experience variation in relevant aspects of the concepts they perceive as obvious or understand as undisputable (or what VT expresses as 'taken for granted'). According to Marton and Tsui (2004), the object of learning can be examined from three different perspectives: (a) the *intended object of learning* (what the teacher intended to be learned), (b) the *enacted object of learning* (what is offered in the classroom by the teacher and students), and (c) the *lived object of learning* (what is actually learned).

Another important concept in VT is the *space of learning*. This is related to the enacted object of learning, which describes how the teacher arranges classroom experiences to make it possible for the object of learning and its critical aspects to come to the foreground;

in particular, it is a description of what is varied and what remains invariant in the classroom (Marton & Tsui, 2004). Kullberg and Skodras (2018) clarified that what varies against an invariant background is likely to be noticed. Accordingly, in order to understand what is possible to learn and what is not possible to learn in a learning situation, it is necessary to pay close attention to what varies and what is invariant in the situation (Marton, 2015).

In VT, certain *patterns of variation* may be recognized. One such pattern is *contrast*; in order for a person to experience something, he or she must experience something else that contrasts with it. Thus, comparing different things is a central idea in this learning theory. According to Marton and Tsui (2004),

> [i]n order to understand what 'three' is, for instance, a person must experience something that is not three: 'two' or 'four', for example. This illustrates how a value (three, for instance) is experienced within a certain dimension of variation, which corresponds to an aspect (numerosity or 'manyness'). (Marton & Tsui, 2004, p. 16)

In addition to the contrast pattern of variation, Marton and Tsui (2004) describe three other patterns – generalization, separation, and fusion – that are not used in this study.

In the last two decades, many studies have used VT for teaching different subjects at different educational levels. These studies have examined the patterns of variation inside the classroom and the designed space of learning where variation is used to help students discern critical aspects of objects of learning (for examples, see Fülöp, 2019; Kullberg et al., 2017; Runesson, 1999; To & Pang, 2019). The use of VT in teacher professional development, especially in 'learning studies', is spreading widely. In learning studies, teachers of the same subject examine their practices, usually with the support of a researcher to plan and analyse a specific lesson by using VT (Huang & Li, 2017; Marton & Tsui, 2004). Numerous learning studies have provided evidence that teaching and learning are more effective when learners encounter patterns of variations and invariance (e.g. see Holmqvist Olander & Nyberg, 2014; Kullberg et al., 2020; Pang, 2019; Pang & Ling, 2012).

5. The professional development programme (PDP)

This section presents an intervention project conducted by one of the authors of this study. That project aimed to investigate school principals' views of mathematics and teaching and learning mathematics. Importantly, that project was within the practice-oriented research and not the VT framework when it was designed and implemented. That project formed the basis of the problem addressed in the current study.

The project consisted of three stages. The first stage investigated how school principals view (perceive) the nature of mathematics and the nature of its teaching and learning. The second stage used the results of stage one and created a PDP to enrich this view. The third stage applied the programme and investigated the changes that took place in principals' views by analyzing pre- and post-data in addition to the data generated during the programme. In the following more clarification of these stages is presented.

During stage one, a national, representative sample of principals of all school levels was selected, and the data were collected using a specially prepared instrument to explore these principals' views of the nature of mathematics, the learning of mathematics, the teaching of mathematics, and the mathematics curriculum. The instrument was based on the mathematics reform. The responses of 244 principals on a 32-item, five-point Likert scale were analyzed. The results that were published (Innabi, 2006) showed that principals held eight specific views that were 'less aligned' with mathematics reform. These views were as follows:

- Mathematics is fixed, unchangeable knowledge that should be accepted;
- To be good at mathematics, one needs talent and an aptitude for the subject;
- Learning mathematics is mainly based on training and practicing;
- The most important indicator of students' learning in mathematics is finding the right answers;
- Problem solving means word problems that usually come as an application exercise at the end of a lesson;
- Sharing by students while they solve problems can have negative effects because errors made by some students could affect others;
- The mathematics curriculum is a collection of procedures, skills, and algorithms;
- It is not suitable to teach statistics and probability to elementary-level students.

At the second stage of the project, based on the above eight views, a PDP to promote elementary school principals' views of mathematics reform was developed to achieve the following two goals: (1) enriching principals' views on the nature of mathematics and teaching and learning mathematics and (2) helping the participants reconsider their role as principals in supporting mathematics reform in their schools.

The framework of the PDP is based on an alternative list of desired views that, to a certain extent, reflect mathematics reform. For each less-aligned view, an alternative, related view was defined, stipulating that school principals would need to support the subject of mathematics in their schools according to the reform. Table 1 shows the eight less-aligned views within each category (Innabi, 2006) and provides brief descriptions of the desired views (i.e. what the PDP intended to achieve). In order to help principals enrich their perceptions of the nature of mathematics and its learning and teaching practices, the eight points listed in Table 1 were challenged by giving the participants opportunities to experience viewpoints that contrasted with their own.

The PDP's activities were prepared and conducted in three steps. First, the principals were asked to express their opinions about situations related to one of the less-aligned views that the PDP attempted to enrich. This step sought to help the participants make their views explicit. Second, several situations that suggested alternative views were presented to the participants. This step aimed to give participants the chance to compare and reflect on their views and, ultimately, enrich them. Finally, discussion and reflection processes were encouraged to promote views that were more aligned with mathematics reform. The PDP was designed to generate maximal dialogue and communication in order to help participants identify common ground and build bridges of understanding among the group of principals. It is important to mention that there was no explicit instruction or direct interference regarding the intended view; alternative situations were merely presented, and then participants were given the chance to discuss their thoughts.

The PDP comprised ten focus group sessions spread over five days. Two sessions discussed the nature of mathematics as a school discipline, while four discussed ways of learning and teaching mathematics. One session was devoted to mathematics curricula,

Table 1. The PDP framework.

Less-aligned views	Category	More-aligned views
Mathematics is fixed, unchangeable knowledge that should be accepted.	Nature of mathematics	Mathematics is a growing body of knowledge. It is not settled and unchangeable. Mathematical knowledge may be changed if contexts and axioms are changed.
To be good at mathematics, one needs talent and an aptitude for the subject. Learning mathematics is mainly based on training and practicing. The most important indicator of students' learning in mathematics is finding the right answers.	Learning mathematics	All students can and should learn mathematics in a meaningful way based on their individual understandings. Learning mathematics should be based on understanding and not only on repeating and training. Finding the right answer is not necessarily the best indicator of student learning.
Problem solving means word problems that usually come as an application at the end of a lesson Students' sharing while solving problems can have negative effects because errors made by some students could affect others.	Teaching mathematics	Teaching mathematics should largely be based on problem solving to promote a situation that challenges students and makes learning happen. Communication is important in teaching mathematics; students have to be encouraged to express and share their thinking verbally and in writing with each other and with their teacher.
The mathematics curriculum is a collection of procedures, skills, and algorithms. It is not suitable to teach statistics and probability to elementary-level students.	Mathematics curricula	A mathematics curriculum is more than calculations and procedures; it is about reasoning, communicating, problem solving, and providing evidence. Data and chance are important and useful for all school levels, starting from kindergarten.

and the final three sessions discussed the role of school principals in supporting the teaching and learning of mathematics. Notably, although every session and sub-session focused on one of the eight views, all the sessions were integrated and built on one another.

Each session combined group and individual activities that included viewing and discussing videotapes of mathematics classes and exploring student thinking by examining students' work. The sessions also incorporated reading and discussing articles that encouraged reasoning about mathematics and teaching and learning mathematics, as well as engaging participants to explore and discuss mathematics. The principals were engaged in dialogues about how the different views could affect their practices as school leaders.

The third stage of the project involved implementing the PDP. In order to select the sample, an invitation was sent to 25 elementary school principals from the original first-stage sample. The invitation stated that a five-day workshop on teaching and learning mathematics would be offered for school principals and that the workshop would be part of a research effort that required data collection. The invitation was accepted by 11 principals, but only eight of them attended all the sessions and completed all the activities including the pre- and post-intervention assessments. Since most (if not all) elementary school principals are female, all the participants were women. These principals came from different educational backgrounds: science, mathematics, social studies, languages, and religion. In addition, each of them holds a degree in education. Their administrative experience as school principals ranges between 5 and 14 years. It is important to mention that these principals are expected to be instructional leaders, attending classes, and assessing teachers' classroom performance.

5.1. Generating data

In order to identify possible enrichments in principals' views of mathematics and teaching and learning mathematics while attending the PDP, data were generated before, during, and at the end of the PDP using two procedures. The first procedure used instruments to generate data by directly asking the participants about their views of teaching and learning mathematics in addition to their opinions about the PDP; three instruments were used. The second procedure involved documentation of PDP activities through participatory observation (Balsiger & Lambelet, 2014) and document analysis (Bowen, 2009).

The first instrument that was used to capture participants' views on the new approach to teaching and learning mathematics was a questionnaire containing eight statements that related directly to the eight less-aligned views on which the PDP was built. This instrument was used twice: once before and once after conducting the PDP. A second instrument was used during focus group interviews. Qualitative responses were generated by asking the participants the following open-ended question: 'What verbs would you use to describe what students are doing in a mathematics class that you are observing?' This group interview was conducted twice: once before and once after the PDP. All the verbs produced by the participants were subjected to frequency analysis. A third instrument was the post-programme questionnaire. Qualitative responses were generated using the following open-ended question: 'To what extent do you think you benefitted from this programme? Explain how'. The qualitative data obtained from the post-programme questionnaire were grouped into categories and labelled. As mentioned above, data were also generated by documenting the PDP activities. Many of these activities were described and analyzed. The following sections present the results of data analysis obtained by these instruments.

6. Enrichments in principals' ways of seeing mathematics

By displaying the results of the data analysis, this section answers the first research question: what enrichments can a five-day PDP yield that will affect a group of elementary school principals' views about mathematics and teaching and learning mathematics?

6.1. Pre- and post-instrument data

Table 2 shows the mean values of the participants' views for each statement of the perceptions instrument before and after PDP. The asterisk in the last column indicates a significant change in perceptions using the Wilcoxon signed-rank test. Notice that the mean ranges between one (*strongly agree*) and five (*strongly disagree*). Higher values indicate perceptions that are more consistent with mathematics reform.

After the PDP, the means of all statements increased; all the changes were statistically significant, and all were in the adjusted disagreement intervals (i.e. none was below the value of 3.4). In about half of the statements, the mean increased to a large degree. Notably, before attending the PDP, the principals' views on all eight points were in the adjusted *agree* or *neutral* intervals (except for one that was just barely in the adjusted *disagree* interval), which suggests that this particular sample of principals embraced these eight less-aligned views.

Table 2. Means of participants' responses on a selection of items from the perceptions instrument before and after PDP ($n = 8$).

	Alignment before	Alignment after
Mathematics is fixed, unchangeable knowledge that should be accepted.	1.9	4.1*
To be good at mathematics, one needs talent and aptitude for this subject.	3.5	4.8*
Learning mathematics is mainly based on training and practicing.	2.9	4.1*
The most important indicator of students' learning in mathematics is getting the right answers.	2.3	4.3*
Problem solving means word problems that come as an application at the end of a lesson.	2	4.6*
Student sharing while solving problems can have negative effects, as errors made by some students could affect the others.	2.9	4.4*
The mathematics curriculum is a collection of procedures, skills, and algorithms.	2.8	4.3*
It is not suitable to teach statistics and probability to elementary-stage students.	4	5*

Note: Responses to the items were given on a five-point Likert scale: 1 = *Strongly agree*, 2 = *Agree*, 3 = *Undecided*, 4 = *Disagree*, and 5 = *Strongly disagree*. This questionnaire was based on the instrument that was used in the first stage of the project, which investigated principals' perceptions using a sample of 244 principals. In order to determine the improvement in principals' perceptions, the instrument was applied twice: once before and once after applying the training programme. Data were analyzed using the means for each statement separately (after considering the direction of the statement, that is, whether it agreed or disagreed with the new view). In order to understand these means, the scale was changed from discontinuous to continuous by dividing the number of intervals (4) by the number of points (5). This provided an interval width of 0.8. Accordingly, the intervals became as follows: *Strongly agree* [1.0–1.8], *Agree* [1.8–2.6], *Undecided* [2.6–3.4], *Disagree* [3.4–4.2], and *Strongly disagree* [4.2–5.0]. In order to investigate the significance of the changes before and after the programme, a non-parametric procedure (the Wilcoxon signed ranks test) was used.

The focus group interviews that were conducted before and after the programme also provided pre- and post-data. The group interviews focused on one question: 'What verbs would you use to describe what students are doing in a mathematics class that you are observing?' Table 3 displays the verbs participants provided before and after PDP. Differences in the variety and quality of these verbs can be seen. For example, before the programme, the verbs were limited to calculations and doing exercises in addition to the regular actions students take in any class, not just mathematics, such as listening and asking questions. In contrast, the post-data represented broader and much richer language for the mathematics classroom. Even though many verbs that participants provided after the programme can also be applied in any class, the results showed that they displayed a more advanced vision of how mathematics class should be.

6.2. *Benefits gained from participants' viewpoints*

Before the end of the PDP, participants were asked to answer the following open-ended question: 'Explain to what extent you benefitted from this programme'. All participants reported that they benefitted from it. Their explanations were classified into the four categories shown in Table 4.

The benefits the participants perceived were consistent with the goals of PDP, namely, to enrich their understanding of the nature of mathematics and of the ways to learn and teach this subject. These benefits corresponded to the categories listed previously in Table 1. In addition, the PDP achieved its goal of clarifying the principals' role as leaders in promoting

Table 3. Principals views of students' acts in mathematics classrooms before and after PDP ($n = 8$).

Pre		Post	
Acts	Frequency	Acts	Frequency
Calculating	8	Representing	3
Doing exercises	7	Connecting	3
Solving problems	1	Justifying	2
Listening	3	Applying	5
Working cooperatively	3	Clarifying	3
Talking	4	Explaining	4
Working individually	4	Deducing	2
Asking questions	2	Estimating	1
		Concluding	1
		Describing	1
		Using	2
		Proving	2
		Discovering	1
		Modelling	1
		Investigating	1
		Solving problems	6
		Formulating	1
		Communicating	4
		Asking questions	3
		Talking with each other	3
		Listening	2

mathematics in their schools. All participants expressed that they benefitted from seeing differences in their roles as school leaders to support mathematics reform in mainly two ways: changing how they evaluate their mathematics teachers and classes, and spreading new ideas about mathematics among teachers, parents, and the community.

6.3. *Data generated from PDP activities*

Some of the PDP activities are described here alongside the data that showed improvement in participants' views and supported the results found in the previous section (in particular, the benefits gained as illustrated in Table 4). One of these activities was an ongoing activity regarding the principals' roles. Throughout the programme, the principals were asked after each session to write down all actions they could take as school leaders to support the implementation of the new vision of mathematics they had heard described during the sessions. A flip board was set up for this activity to enable principals to add their suggestions. Another activity was conducted at the end of the programme, in which participants were invited to modify or rebuild the assessment tools they were using to observe mathematics instruction and evaluate teacher performance. Participants worked in pairs for one hour and put their work on transparencies to enable them to present their ideas using an overhead projector. Each pair's work was presented and discussed. A third activity came at the end of the programme that required pairs of principals to design a one-semester plan that would help their schools move towards supporting mathematics reform. Each of these three activities (principals' roles, assessment, and school plan) generated qualitative data in the form of statements written by principals. For the first activity, these statements were placed on the board dedicated to this ongoing activity. During the other two activities, statements were written on transparencies. The three sets of data were analyzed

Table 4. Benefits gained from participants' viewpoints ($n = 8$).

Benefit	Frequencies	Examples of participants' responses
Enrichments in seeing the nature of mathematics	8	I learned that mathematics is not fixed but flexible and changeable. My view of mathematics as a subject is different now. My overall perception of mathematics has been changed from fixed calculations to acts and growing structures.
Enrichments in seeing the ways to teach mathematics	8	I now understand how mathematics should be taught. Before this programme, my view of teaching mathematics was different and quite traditional.
Enrichments in seeing the ways mathematics should be learned	7	Any student can understand mathematics if it is taught in the proper way. My view of students' learning has changed from giving the right answer to expressing their thinking strategies. I know now that our students can learn mathematics.
Enrichments in seeing different roles as school leader	8	I have a new understanding of how to evaluate my math teachers. I realize now that my evaluations of my mathematics teachers have been improper and bound by tradition. I see the necessity of transferring this viewpoint to parents to be able to produce a generation that thinks and reflects, moving away from memorization and routines. I learned that there are many ways to make students and the community believe that mathematics is important and that everybody can learn it. I should share my new attitudes toward mathematics with others.

separately, and responses were grouped into categories. Three tables were then generated for the three activities. As all these activities were related to the principals' actions (i.e. acts and roles), many duplications appeared in the three original tables. Accordingly, the three tables were examined together to produce one set of results that was labelled 'principals' views on their role in supporting mathematics reform during and at the end of the programme.

As can be seen in Table 5, three major actions were identified: (1) spreading the mathematics reform vision, which requires the principals to work with the community, parents, teachers, and students; (2) helping mathematics teachers use the reform approach, which can be divided into two subcategories, namely, helping mathematics teachers in their professional development (such as attending workshops and conferences) and encouraging them to teach according to the reform approach (for example, by listening to their needs and fears); and (3) assessing mathematics teachers and classrooms in a different way. Each action was categorized according to one of the four study concerns (the nature of mathematics, the learning of mathematics, the teaching of mathematics, and the mathematics curriculum). Table 5 shows examples of how the principals viewed their roles to support mathematics in their schools within each of the four PDP concerns.

Table 5. Principals' views on their role in promoting mathematics reform ($n = 8$).

	Nature of mathematics	Learning of mathematics	Teaching of mathematics	Mathematics curriculum
Spread new vision of mathematics (parents, teachers, students)	Arrange activities for parents, such as a family mathematics evening and newsletters to discuss the traditional and alternative views of mathematics. Show students that math is fun and attractive through school activities, such as competitions and mathematics days.	Parents need to promote their awareness about what mathematics their kids need to learn (doing mathematics, recognizing patterns, explaining thinking strategies, listening to others, and connecting ideas).	I will highlight the importance of having parents know how to teach and help their kids to learn.	Mathematics curriculum is not limited to a specific textbook; rather, activities and experiences should be part of the curriculum inside and outside the school.
Help math teachers to teach according to new view (professional development, encouragement)	Arrange workshops for teachers to help them to see the real picture of math.	Give workshops on conceptual understanding, recent learning theories.	Professional development for math teachers through training, cooperation with other math teachers, conferences, and higher studies. Meet with math teachers to identify the needs, weaknesses, fears, and challenges they face when applying math reform.	Make sure math teachers are aware of math curriculum standards. Provide instructional tools.
Change assessment of math classroom (teacher's skills, classroom environment)	When assessing mathematics teachers, I will check that she considers problems that do not have one specific answer. Patterns and relationships are important criteria for my assessment.	Lesson plans have to contain high-quality outcomes requiring students to solve problems, communicate, give evidence, explain, and present their work.	Questioning strategies that reflect interesting questions and that ask for clarification and justification; give encouragement to students to think creatively by giving them new methods of finding solutions.	I will assess math teachers according to the new curriculum standards.

During data analysis, it was noted that some of the participants' responses could be seen as general ideas that might already have been suggested in other professional development programmes, such as 'Teachers have to have classroom management skills' or 'content knowledge'. However, the language used reflects mathematics reform, including that all students should and can learn mathematics and students have to 'do', 'reason', and 'communicate' mathematics. In addition, the indicators that principals will watch for in mathematics classes are clearly aligned with the major points that were discussed throughout the PDP, such as problem-solving norms. Thus, the focus points that the programme addressed, including conceptual understanding, doing mathematics, and teaching mathematics effectively, were being considered by the participants by the programme's end. Alongside principals' comments in the post-PDP questionnaire about what they learned from the programme, this finding confirms that the PDP helped the principals enrich their former views regarding their role in promoting mathematics reform.

7. PDP through the lens of VT

In this section, we use VT to explain the enrichments in participants' views on mathematics as described in the previous section. Two interconnected analytical tracks are used in explicating enrichments in the principals' ways of seeing mathematics. First, we clarify the relationship between the rationale for developing the PDP and its different design features; hence, we show parallels with VT assumptions on what it takes to enrich ways of seeing. This part of the result is described in terms of the intended object of learning. Second, analyse the activities within the programme from the viewpoint of VT, thus enabling the enrichments in the principals' ways of seeing to be explained in terms of the lived object of learning.

7.1. Seeing PDP through VT: the intended object of learning

Neither the phenomenographic approach nor its theoretical development in VT was used as a point of departure when the PDP was originally developed. However, the search for an explanation for how the principals' views of mathematics enriched after attendance in this short programme revealed strong similarities between VT and the design principles that were used when developing the PDP. This section discusses these similarities.

The goal of the PDP was to enrich principals' views on school mathematics and align them with mathematics reform. Using VT language, we can say that the intended object of learning of the PDP was to enable principals to view mathematics in a broader and richer way than they had in the past. Principals need to see mathematics as a dynamic rather than as a static body of knowledge; they have to see the learning of mathematics in terms of understanding and meaningfulness, and they need to see mathematics as a reasoning and communication tool rather than solely as a tool of calculation. In order to understand mathematics in this particular way, a set of features (critical aspects) was identified that are necessary to form that vision. These features were determined from the suggested more-aligned views, and they correspond to the less-aligned views that were previously found. Notably, the less-aligned views obtained in the first stage of the project were also found among the specific group of PDP participants as the pre-data analysis has shown (see Table 2).

Table 1 shows the list of pairwise less-aligned and more-aligned views that formed the PDP's theoretical framework; all the activities used in the PDP related directly to this framework. Accordingly, based on VT, the contrast pattern of variation can be seen in the way the PDP was developed not only in the PDP's theoretical framework, as presented in Table 1, but also more clearly in the formation of activities throughout the five-day programme. The PDP activities were created by providing alternatives for the same situation in question.

According to VT, in order to change learners' views, the best teaching design is to first find out what these views are and then uncover individual perspectives on the relevant issues (Lo, 2012). These views should be described in terms of DoVs to be opened so that learners may modify their views (Marton & Booth, 1997). It is worth noting that many other theorists before VT have considered the learner's previous knowledge. One of the most-cited examples is Ausubel (1968) who stated that 'the most important single factor influencing learning is what the learner already knows. Ascertain this and teach him

Table 6. Object of learning, critical aspects, dimensions, and values of the PDP.

'Object of learning' of the PDP: To enrich the principals' views of the nature of mathematics and the nature of its teaching and learning		
Critical aspects	Dimensions	Values
Mathematics is a growing body of knowledge. It is not settled and unchangeable. Mathematical knowledge might be changed if contexts and axioms are changed.	Nature of mathematics	static–dynamic
All students can and should learn mathematics in a meaningful way based on their individual understandings.	Who can learn mathematics (or mathematics for all)	elite–all
Learning mathematics should be based on understanding and not only on repeating and training.	To learn mathematics	practice–understand
Finding the right answer is not necessarily the best indicator of student learning.	Student learning indicators	right answer–not necessarily the right answer
Teaching mathematics should largely be based on problem solving to promote a situation that challenges students and makes learning happen.	Problem Solving	at the end of lesson–not necessarily at the end
Communication is important in teaching mathematics. Students have to be encouraged to express and share their thinking verbally and in writing with each other and with their teacher.	Communication	risky on learning–basic tool for learning
The mathematics curriculum is more than calculations and procedures. It is about reasoning, communicating, problem solving, and providing evidence.	Content of mathematics	calculations–patterns & relationships
Data and chance are important and useful for all levels starting from KG.	Statistics and probability for elementary level	suitable–not suitable

accordingly' (p. vi). However, VT takes this general idea a step further and offered a model of learning in which discerning variation was taken as the basic mechanism.

In retrospect, if the PDP is looked at through the lens of VT, the eight less- and more-aligned views shown in Table 1 can be understood as eight critical aspects, and each one can form a DoV with different values. For example, the perception that '[t]o be good at mathematics, one needs talent and aptitude' can be viewed as a DoV entitled 'who can learn mathematics'. Some of its values include that mathematics is 'for elites' or, alternatively, 'for all'. Table 6, which shows how the information in Table 1 can be seen through VT, presents the object of learning of the PDP, the critical aspects, and their corresponding dimensions and values. Notice that the suggested dimensions and values are defined by their meanings and not by their wordings. For example, the value 'elite' could be replaced by 'smart' or 'talented' because, according to our analysis, their meanings ('not for all') are the same.

7.2. Constructing PDP activities

This section offers clarifications on the strategy used to develop the PDP activities, and describes how this strategy can be seen through the lens of VT. In hindsight and from the perspective of VT, the aim of the PDP can be described as supporting principals to see the nature of mathematics and the process of teaching and learning mathematics in a different way to what they had previously experienced. The three-step strategy that was adopted to create and implement the programme's activities can be seen as a design that was fine-tuned to raise the participants' awareness of the eight DoVs identified for this research.

During the first step of each activity, all participating principals were invited to express their opinions or understandings of a specific situation related to one of the eight dimensions included in the PDP. The reasons for this step (from a VT perspective) were to determine what aspects were critical for the particular group of learners, to determine how this particular group differed from the general group (Innabi, 2006), and to help the participants recognize their own views. Care was taken to give all participants an opportunity to share their understanding. For example, one of the sessions, which showed the participants that mathematics is not fixed but rather changeable knowledge, started by asking the question 'What does mathematics mean to you?' Another session, which demonstrated that student communication while solving problems helps increase learning, started with the statement 'Students talking with each other while solving problems could have negative effects because errors made by some students could affect other students'. The following question was then asked of the participants: 'What do you think about this statement?' All the answers were recorded and categorized in the presence of the participants. One prominent feature of this step is that it created a joint focus among the principals. The principals' understandings became the subject matter of further discussions and enabled them to contextualize their experiences.

During the second step of the PDP activities, the participants were shown situations (scenes and behaviours) that contrasted with their traditional views in relation to the specific critical aspects that were the focus of the first step. The phenomena were discussed by presenting alternative or contrasting views that reflected the object of learning. This step sought to give each participant the chance to see the situation in question in different ways, to compare and reflect on their views, and to provide possibilities for enrichment. For example, one of the less-aligned perceptions found among school principals and addressed in the PDP was that mathematical knowledge is a settled, unchangeable, and finished product. In order to help principals broaden this view (i.e. open this DoV), the PDP provided several activities that showed that any piece of mathematical knowledge can be changed if contexts and axioms are changed. Additional activities were conducted to demonstrate that mathematics is about seeing patterns and relationships and that it is not a finished product but is instead a continuously evolving field. These activities mainly involved mathematical patterns in nature, such as the Fibonacci series, chaos, and fractal geometry; in real life, mathematics may evolve according to axioms and assumptions, the digital system, and the need to estimate. Another example, which is related to a dimension included in the PDP: 'The most important indicator of students' learning in mathematics is finding the right answer'. Since the goal was to help principals enrich the perspective that arriving at the right answer is not necessarily an indicator that students understand mathematics, an activity was provided to show that many other indicators other than the right answer can signal student learning. More importantly, the principals needed to become aware that, depending on the indicator, arriving at the right answer can sometimes be misleading.

7.3. An example of PDP sessions

According to mathematics reform, problem solving is often discussed as an unfamiliar mathematical situation that can be described as novel to the students and that challenges them. A problem that might be contextualized in terms of a story might also be represented as a symbolic item. In addition, problem solving can come at the end of a lesson as a practical application, but, more importantly, this situation has to be the main tool used to teach what students need to learn (Lester, 2013).

Before attending the PDP, principals saw problem solving as nothing more than a 'story problem' that comes at the end of a mathematics lesson as an application exercise. In order to support principals in seeing problem solving in a richer way, enabling them to recognize its powerful role in learning mathematics, an activity was conducted to reveal how a lesson that starts with a specific problem-solving task can lead the class to achieve the lesson's goal. This activity was a classroom scenario that showed how a fifth-grade teacher engaged her students in problem solving to help them achieve the lesson's objective, which was to have the students think about and develop methods for adding decimals. It was emphasized to the principals that, although students had previously worked with the subject of representing decimals, they had not yet discussed adding them. In this scenario, the teacher presented the students with a problem-solving situation that required adding decimals. This problem was taken from the NCTM 2000 standards (Schifter et al., 1999, pp. 114–201). The scenario showed how the teacher succeeded in helping the students realize the algorithms of adding decimals using a base-ten model without her direct instructions and connect the problem with their knowledge of adding whole numbers. This happened through sharing thinking, asking questions, and explaining and justifying the students' own ideas as well as the ideas of others.

This session about problem solving started with the following introduction (Step 1): 'We will discuss today a lesson about adding decimals, but before we start, let me ask about how you expect mathematics teachers in your school to teach this lesson'. All answers were similar to this scenario: the teacher shows the procedure for adding decimals; the teacher shows examples; the students do more examples; and maybe, at the end of the lesson, a word problem could be given as an application example. After this discussion, Step 2 was given (showing alternatives). Thus, we can see an invariance issue (the decimal lesson) and a variation (the teaching scenario). The principals were offered two different ways to teach the same content (adding decimals): one was the approach described by the principals, and the other was an alternative approach shown to the principals.

8. Explaining enrichments: the perspective of VT

The traditional views of mathematics that principals had before attending the PDP are the result of the deep-rooted, traditional ideas about mathematics held by the general public; these may be related to what Marton and Booth (1997) refer to as 'natural attitudes'. Principals habitually share these traditional views. We claim that this image of mathematics is 'taken for granted' in their awareness of this subject, and thus, we want to support them in enriching this traditional view by enabling them to see mathematics in a richer and more differentiated way. This change in ways of seeing can result from making what is 'taken for granted' an object of reflection and considering alternatives to what is assumed to be the case, thereby raising participants' awareness of the possibility that something may be other than what they originally perceived it to be. According to VT, the change in ways of seeing might happen suddenly as Marton and Booth (1997) explained in using the term 'spontaneous change' (pp. 148–149). Thus, in order for the principals to develop another image of mathematics than the one they took for granted, they had to experience contrasting situations, which is what Marton and Booth (1997) referred to as opening dimensions of variation in awareness. If these individuals could become aware that something was a certain way, they could become aware that it could also be viewed in some other way. On the basis of this approach, we can consider that eight DoVs were included in the PDP's activities; and they were designed to 'open' the participants' views, thus allowing them to 'enrich' their ways of experiencing mathematics and consider those aspects of mathematics that are highlighted in mathematics reform. In order to clarify this, we will highlight in the next section one of the PDP sessions that made it possible for participants to see a 'taken-for-granted' view on learning mathematics differently.

8.1. An example of enriched ways of seeing

In the traditional way of experiencing mathematics education, the main concern is that students get the right answer rather than developing their reasoning and solving strategies and enhancing their capacity to communicate with others. Regarding this view, the PDP sought to help principals see that achieving the right answer is not necessarily the only important indicator of student learning. To achieve this goal, one of the PDP's ten sessions was dedicated to the dimension of student learning indicators.

As can be seen in the results that were displayed earlier (in Tables 2, 4, and 5), many indicators showed that principals experienced enrichment in relation to this view after attending the PDP. One of these indicators came from the measurement instrument that was designed to depict views and was administered to participants before and after the PDP. On a scale ranging from one (*strongly agree*) to five (*strongly disagree*), the mean value of the participants' responses to the statement 'The most important indicator of students' learning in mathematics is finding the right answers' was 2.3 before PDP and 4.3 after PDP. As shown in Table 2, this change was statistically significant. Furthermore, this change is supported by the principals' statements about the benefits they received from the PDP; for example, one principal wrote, 'My view of students' learning has changed from giving the right answer to expressing their thinking strategies'.

In order to describe the broader view of learning indicators that emerged, we need to clarify what happened during the session. At the beginning of the session, the participating principals were asked the following question: 'How do you assess student learning in fourth-grade division lessons?' The participants provided slightly different answers, which were written on a whiteboard, including 'the ability to divide', 'the student's test score', 'when the student divides without mistakes', and 'when the student can get the right answer'. The similarity in meaning among the responses was pointed out, and all the responses were circled and labelled as 'getting the right answer'.

A two-part video clip was then viewed and discussed. The first part showed a fourth-grade student who was asked to solve a division problem (20 divided by 5) using the long division method. Using paper and a pencil, the student solved this division problem correctly and confidently by following the teacher and the textbook's long division method. In the second part, this student was asked to justify what she had just done (i.e. explain her answer) verbally and use a set of cubes to demonstrate the process of division. The student failed to complete these tasks successfully. She kept uttering unrelated and vague sentences and giving wrong representations even after she was provided with some support.

Therefore, we can say that the principals experienced a pattern of variation (different ways of solving the problem) and invariance (the problem was the same in the two situations, and the same student solved the problem in the two situations) before watching the video; the principals seemed to focus only on the correctness of the students' mathematical activities. In accordance with the traditional view of mathematics, they were oriented towards whether or not the student's responses were acceptable. Thus, the group's way of experiencing the phenomenon in question (i.e. assessing student learning) was taken for granted. As described above, we then introduced a contrasting example in which a young student succeeded in finding the right answer to a division problem but failed to demonstrate a corresponding understanding of the method using concrete materials. By opening the DoV of 'student learning indicator', the principals were exposed to a contrasting example that generated discussion. After seeing the video, a discussion started with the question 'What can this example tell you?' The participants came up with a much broader variety of indicators of student learning that included the following statements: 'We need to look also at her ability to explain her answer', 'She has to be able to show us what she did with these cubes', 'She has to put 20 cubes into 5 groups', and 'She has to talk about her method'. Other suggestions were 'She did not understand what she was doing', and 'If she understood, she would have represented the problem as five groups, each with four cubes'. These responses indicate that the discussion allowed the participants to come up with other values of the dimension than just 'correctness'. The participants even mentioned values that had not been demonstrated in the video, such as 'She should be able to apply it in real life'.

We saw two values for the 'student learning indicator' dimension: (1) achieving the right answer and (2) not necessarily achieving the right answer. For the second value, several examples can be represented, such as 'communicating the mathematical strategy', 'representing the mathematical strategy', and 'applying mathematical knowledge'. When the participants watched the video, they experienced something different that opened this dimension to values other than the one the principals had previously taken for granted.

9. Discussion

This paper describes and explains the enrichments that occurred in a group of elementary school principals' ways of seeing mathematics during their participation in a five-day PDP. The goal of this programme was to help principals align their traditional views with reformist views on the nature of mathematics and learning and teaching mathematics. The results of implementing the PDP showed noteworthy enrichments. Moreover, the principals showed awareness of the enrichments of their views about the subject of mathematics.

In addition to describing the enrichments in principals' views, this study aimed to answer the question; how to explain those enrichments. The analytical tool used to answer this question was the VT of learning. The analysis addresses two levels. The first aimed to clarify the relationship between the rationale for developing the PDP and VT. The second (more specific) level involved analyzing the activities of PDP from a VT viewpoint. This analysis revealed patterns that helped explain the enrichments that occurred during PDP.

The VT lens illustrated that the object of learning, which was 'to help principals enrich their views on the mathematics subject', was decomposed into eight critical aspects that were derived from a previous, broader sample of principals as well as the PDP participants. The aim was achieved by determining the less-aligned (with mathematics reform) views and suggesting alternative, more-aligned (with mathematics reform) views. These two sets of views were described in terms of DoVs and varying values.

Before attending the PDP, the participating principals demonstrated a traditional view of mathematics that was 'taken for granted' in their awareness. We showed that enrichments in the principals' views occurred during their participation in the PDP, enabling them to perceive mathematics in a different way. Their 'natural attitudes' were thus broken in the phenomenological sense to which Marton and Booth (1997, p. 148) refer. The principals' ways of seeing mathematics and learning and teaching mathematics expanded, allowing them to see the subject in ways that are more differentiated and more aligned with mathematics reform.

According to VT, this process can take place by making what is 'taken for granted' an object of scrutiny; by offering alternatives to what is assumed, it is possible to raise participants' awareness of the possibility that something may be other than what they thought it was. Thus, in order for the principals to develop a view that is more aligned with mathematics reform than the one they took for granted, they had to experience variations in the eight identified critical aspects, which opened DoVs in their awareness. If the participants could become aware that something was a certain way, they could also become aware that it could be some other way.

On the basis of this approach, and by using a VT lens, we were able to see that the eight DoVs were included in the PDP's activities, which were designed to 'open' the participants' views, allowing them to enrich their ways of experiencing mathematics and consider those aspects of mathematics that are highlighted by mathematics reform.

A notable mechanism for enrichment is the use of variation in certain aspects by the provision of contrasts in the same aspect. The idea of putting learners in situations in which their conceptions become visible and contrasted with other possibilities is not unique to VT; other views such as conceptual change theory (Kuhn, 1970; Liljedahl, 2011) and the Gestalt theorists (cf. Koffka, 1935/1963), and different types of constructivism, present the same general idea about exposing learners to alternatives. However, variation theorists take this general idea a step further by suggesting a systematic learning mechanism in terms of DoV (e.g. Emanuelsson, 2001; Emanuelsson & Sahlström, 2008; Kullberg et al., 2016; Lo, 2012; Ryve et al., 2013). A more specific way of addressing the desired enrichment on quite a detailed level such as the 'correctness of the answer' or 'problem solving, at the beginning or the end of the class'. In the re-analysis of our PDP, we described the nature of mathematics as well as the nature of learning and teaching mathematics in terms of several dimensions; each of these dimensions was opened through the introduction of a contrasting value.

Using VT, we can describe the detailed systematics that the PDP provided because it revealed details about how the programme was designed that could not have been seen clearly before. We described the object of learning in terms of several DoVs, and in each of these dimensions, we offered the participants an opportunity to experience variation on a micro-level. Hence, we went beyond the general and more overarching idea of using alternatives and contrasts as vehicles for learning. Instead, we opened aspects for systematic variation that were considered critical to mathematics reform.

The PDP comprised ten sessions, each focusing on one aspect to be discerned. These sessions were not separate; rather, they connected and supported each other. Using VT helped us see that the design of the PDP addressed the same DoV from different angles by using several examples. For instance, in the activity that addressed the problem-solving dimension, the session's goal was to convey that problem solving is a powerful tool for teaching and that students can learn through problem solving. The problem-solving session also illustrated that it is not always beneficial for teachers to focus exclusively on whether or not students arrive at the right answer. Instead, the teacher focused on supporting student reasoning in other, better ways. This session can provide a good opportunity for participants to see that if teachers focus on the right answer, they might overlook opportunities to help students learn better.

In summary, an instructional programme has the capacity to make a significant difference by systematically and empirically breaking down complex conceptions into less-complex, interrelated parts (critical aspects and DoV), then challenging participants' rooted ideas by confronting them with patterns of variations and invariance, in particular, the pattern of contrast. Although the focus of this paper was to understand the enrichments that occurred in principals' awareness while they participated in the PDP, this discussion deems it necessary to raise two related issues. The first connects to the rationale behind the PDP itself – that is, the need to change the dominant, traditional public view on school mathematics because it negatively affects or limits students' learning of mathematics. The mathematics education community needs to invest considerable effort in remedying this issue (Stephan et al., 2015). The entire educational community, including teachers, administrators, parents, and students, needs to view mathematics as a subject that can be explained and made understandable. In this study, we included literature that explained how principals' views on mathematics influence teachers' work in the classroom, and, consequently, students' learning possibilities.

If we assume that the public's common view of mathematics adversely affects our children's learning of the subject, we need to challenge the traditional and taken-for granted ways of seeing mathematics. In order to clarify this point, let us examine the related idea of *mathematics for all*, which is in widespread usage (e.g. NCTM Research Committee, 2018). In general, people view mathematics as a subject for just a few – for those with the right inclination and especially for gifted and high-achieving students. Whether such a view is present among principals, teachers, parents, or students, it presents a large obstacle to the reform view that mathematics is *for all*. We believe that a more enriched view is beneficial for the learning of mathematics, especially for those who display the weakest competence. In order to move *mathematics for all* from present political rhetoric into the field of practice, efforts are needed to change people's views. The programme presented by this study can be considered supportive of such an effort since it exemplifies that these views can be enriched if systematically addressed with patterns of variation.

Despite the volume of research describing how people see mathematics (different terms have been used, such as *perceptions*, *beliefs*, *images*, *metaphors*, etc.), very little empirical work has been done on modifying or changing these views. Hence, this study can add to the literature and support the call for grand challenges (Stephan et al., 2015) in a more direct and explicit way.

The other issue that we want to raise is a call for teachers and instructors to research their own practices. Current worldwide discussions about practice-oriented research indicate that well-documented professional development programmes and similar approaches can be considered for potential research with an integrated design that allows for further systematic analyses. Such research should be scrutinized by practitioners and the research community. Our PDP can be presented as an example of a commissioned development programme that has been designed to generate research materials for later analysis. Through this study, we aim to show that it is possible to do research on one's own teaching and instruction in a systematic way. This is an important point because it gives school, college, and university teachers, as well as teacher educators and people working in professional development, evidence that it is possible to approach teaching in systematic ways that allow for solid research that can be documented and published. When the PDP was created and implemented a few years ago, it was considered in the context of practice-oriented research that aims to develop knowledge for the improvement of professional practices. Hermans and Schoeman (2015) clarified that the knowledge created in practice-oriented research contributes directly to professional practice and that research problems do not need to originate in theory. This paper presents a case in which the problem did not originate in theory but rather ended in theory.

10. Significance and limitations

In terms of significance and contributions to the field, we have showed that if a professional development programme is designed with systematic use of variation in critical aspects can enrich traditional views of mathematics. Very little research has aimed at purposefully supporting the changing of views on mathematics. Notice that previous studies mainly addressed teachers of mathematics and not the public or principals.

Our use of VT as an analytical tool for understanding enrichments in a professional development programme is a significant contribution to VT since this theory is usually utilized in normal teaching situations with conventional subjects or courses, such as mathematics, science, and language. Furthermore, we argue that the work represents an important contribution since we show how good instructional practice, if sufficiently documented and evaluated, can generate materials that can be analyzed in a rigorous and theoretically informed way and hence be scrutinized by a wider community of peers where 'peers' are principals and teachers, as well as researchers in the field of mathematics education.

In terms of limitations, and within the constraints of the existing data materials, we were unable to study whether the enrichment that was observed after the participants attended the PDP had an impact on the schools in which the principals worked at the time. Moreover, we did not study how this enrichment can be transferred into other contexts or can affect either teachers' views of mathematics or school policy and practice. Further, we have no claims on the longevity of the observed enrichment. However, we argue that in order for school principals to have an impact on teaching and learning possibilities within their schools, they need to develop their own capacity to see mathematics as a dynamic, humanistic tool for reasoning, communication, and problem solving.

A final point should be mentioned. Although positive changes appeared when the PDP was applied to a limited number of principals, this study does not attempt to prove the effectiveness of this PDP or make any formal generalization (c.f. Eisner, 1991, pp. 103–105). Instead the goal of this study is to describe the enrichments that appeared within a group over a short period of time and explain these enrichments in order to widen our understanding of the possibility of modifying the traditional public view of mathematics and the teaching and learning of mathematics.

Acknowledgements

We thank anonymous reviewers, the research group for studies in Phenomenography and Variation theory at the University of Gothenburg and Professor Ference Marton for their invaluable comments on previous versions of this manuscript. We also want to thank our colleagues in Mathematics Education Professor Yosef Elemam and Maryam Al Daheri for their contributions in developing the pedagogical development program. Finally, we want to mention our appreciation for the school principals in United Arab Emirates that participated, without them this study would not have been possible.

Disclosure statement

No potential conflict of interest was reported by the author.

ORCID

Hanan Innabi http://orcid.org/0000-0001-6258-1486
Jonas Emanuelsson http://orcid.org/0000-0001-9668-0642

References

Abazaoglu, I., & Aztekin, S. (2016). The role of teacher morale and motivation on students' science and math achievement: Findings from Singapore, Japan, Finland and Turkey. *Universal Journal of Educational Research*, *4*(11), 2606–2617. https://doi.org/10.13189/ujer.2016.041114

Al-Murani, T., Kilhamn, C., Morgan, D., & Watson, A. (2019). Opportunities for learning: The use of variation to analyse examples of a paradigm shift in teaching primary mathematics in England. *Research in Mathematics Education*, *21*(1), 6–24. https://doi.org/10.1080/14794802.2018.1511460

Anderson, D. S., & Piazza, J. A. (1996). Changing beliefs: Teaching and learning mathematics in constructivist preservice classrooms. *Action in Teacher Education*, *18*(2), 51–62. https://doi.org/10.1080/01626620.1996.10462833

Ausubel, D. P. (1968). *Educational psychology: A cognitive view*. Holt, Rinehart & Winston.

Baillie, C., Emanuelsson, J., & Marton, F. (2001). Building knowledge about the interface. *Composites Part A: Applied Science and Manufacturing*, *32*(3–4), 305–312. https://doi.org/10.1016/S1359-835X(00)00111-1

Balsiger, P., & Lambelet, A. (2014). Participant observation. In D. della Porta (Ed.), *Methodological practices in social movement research* (pp. 144–172). Oxford University Press.

Bicknell, B., & Hunter, R. (2009). Explorations of year 6 to year 7 transition in numeracy. In Ministry of Education (Ed.), *Findings from the New Zealand numeracy development projects* (pp. 98–109). Learning Media Limited.

Bleiler, S. K. (2015). Increasing awareness of practice through interaction across communities: The lived experiences of a mathematician and mathematics teacher educator. *Journal of Mathematics Teacher Education*, *18*(3), 231–252. https://doi.org/10.1007/s10857-014-9275-6

Boberg, J. E., & Bourgeois, S. J. (2016). The effects of integrated transformational leadership on achievement. *Journal of Educational Administration*, *54*(3), 357–374. https://doi.org/10.1108/JEA-07-2014-0086

Boston, M. D., Henrick, E. C., Gibbons, L. K., Berebitsky, D., & Colby, G. T. (2017). Investigating how to support principals as instructional leaders in mathematics. *Journal of Research on Leadership Education*, *12*(3), 183–214. https://doi.org/10.1177/1942775116640254

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Research Journal*, *9*(2), 27–40. https://doi.org/10.3316/QRJ0902027

Brez, C. C., & Allen, J. J. (2016). Adults' views on mathematics education: A Midwest sample. *European Journal of Science and Mathematics Education*, *4*(2), 155–160.

Brown, C., & Smith, M. (1997). Supporting the development of mathematics pedagogy. *Mathematics Teacher*, *90*(2), 138–143.

Burrill, G. (2014). Preparing future teachers to teach statistics. In K. Makar, B. de Sousa, & R. Gould (Eds.), *Sustainability in statistics education: Proceedings of the ninth international conference on teaching statistics (ICOTS9, July 2014)*. International Statistical Institute.

Burrill, G., & Biehler, R. (2011). Fundamental statistical ideas in the school curriculum and in training teachers. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics—challenges for teaching and teacher education – A joint ICMI/IASE Study: The 18th ICMI study* (pp. 57–69). Springer.

Campbell, P. F., & White, D. (1997). Project IMPACT: Influencing and supporting teacher change in predominantly minority schools. In E. Fennema & B. Nelson (Eds.), *Mathematics teachers in transition* (pp. 309–355). Lawrence Erlbaum Associates.

Cassel, D., & Vincent, D. (2011). Metaphors reveal preservice elementary teachers' views of mathematics and science teaching. *School Science and Mathematics*, *111*(7), 319–324. https://doi.org/10.1111/j.1949-8594.2011.00094.x

Cerrato, E. (2019). Illuminating changes in preservice teachers' perceptions about teaching elementary mathematics in an introductory methods course (Order No. 13858111). Available from ProQuest Dissertations & Theses Global. (2240090025). http://matematikdidaktik.org/wp-content/uploads/2019/01/Madif11_webb-002.pdf

Chiang, H., Lipscomb, S., & Gill, B. (2016). Is school value-added indicative of principal quality? *Education Finance and Policy*, *11*(3), 283–309. https://doi.org/10.1162/EDFP_a_00184

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development*, *13*(1), 6–33.

Cobb, P., Jackson, K., Henrick, E., & Smith, T. M. (2018). *Systems for instructional improvement: Creating coherence from the classroom to the district office*. Harvard Education Press.

Cotton, K. (2003). *Principals and student achievement: What the research says*. Association for Supervision & Curriculum Development (ASCD).

Darling-Hammond, L., LaPointe, M., Meyerson, D., & Orr, M. T. (2009). *Preparing principals for a changing world: Lessons from effective school leadership programs*. Jossey-Bass.

Darragh, L. (2018). Loving and loathing: Portrayals of school mathematics in young adult fiction. *Journal for Research in Mathematics Education*, *49*(2), 178–209. https://doi.org/10.5951/jresematheduc.49.2.0178

Dempster, N. (2001). *The professional development of school principals: A fine balance*. Professorial Lecture, Griffith University Public Lecture Series Griffith University, Brisbane, May 24.

Donaldson, M. L., & Papay, J. P. (2014). Teacher evaluation reform: Policy lessons for school principals. *Principal's Research Review*, *9*(5), 1–8. National Association of Secondary School Principals.

Eisner, E. (1991). *The enlightened eye. Qualitative inquiry and the enhancement of educational practice*. McMillan.

Emanuelsson, J. (2001). *En fråga om frågor. Hur lärares frågor i klassrummet gör det möjligt att få reda på elevernas sätt att förstå det som undervisningen behandlar i matematik och naturvetenskap*. Acta Universitatis Gothoburgensis [A question about questions. How teachers' questioning makes it possible to learn about the students' ways of understanding the content taught in mathematics and science; in Swedish with summary in English]. Göteborg Studies in Educational Sciences 168.

Emanuelsson, J., & Sahlström, F. (2008). The price of participation: Teacher control versus student participation in classroom interaction. *Scandinavian Journal of Educational Research*, *52*(2), 205–223. https://doi.org/10.1080/00313830801915853

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–254). Falmer Press.

Ernest, P. (1996). Popularisation: Myths, mass media and modernism. In A. J. Bishop (Ed.), *International handbook of mathematics education* (pp. 785–817). Kluwer Academic Publishers.

Forgasz, H. J., Leder, G. C., & Tan, H. (2014). Public views on the gendering of mathematics and related careers: International comparisons. *Educational Studies in Mathematics*, *87*(3), 369–388. https://doi.org/10.1007/s10649-014-9550-6

Fox, A. (2018). *Becoming an instructional leader for elementary mathematics: Transforming principal learning through a research-practice partnership* [Doctoral dissertation, University of Washington]. University of Washington Digital Library. https://digital.lib.washington.edu/researchworks/handle/1773/42089

Fülöp, É. (2019). *Learning to solve problems that you have not learned to solve: Strategies in mathematical problem solving*. Doctoral Theses from University of Gothenburg.

Gill, M. G., Ashton, P. T., & Algina, J. (2004). Changing preservice teachers' epistemological beliefs about teaching and learning in mathematics: An intervention study. *Contemporary Educational Psychology*, *29*(2), 164–185. https://doi.org/10.1016/j.cedpsych.2004.01.003

Gokçe, F. (2009). Behaviour of Turkish elementary school principals in the change process: An analysis of the perceptions of both teachers and school principals. *Educational Management Administration and Leadership*, *37*(2), 198–215. https://doi.org/10.1177/1741143208100298

Goldin, G., Rösken, B., & Törner, G. (2009). Beliefs—No longer a hidden variable in mathematical and learning processes. In W. Schlöglmann & J. Maasz (Eds.), *Beliefs and attitudes in mathematics education: New research results* (pp. 1–18). Sense Publishers.

Grissom, J. A., Kalogrides, D., & Loeb, S. (2015). Using student test scores to measure principal performance. *Educational Evaluation and Policy Analysis*, *37*(1), 3–28. https://doi.org/10.3102/0162373714523831

Hallinger, P., & Heck, R. H. (1998). Exploring the principal's contribution to school effectiveness: 1980–1995. *School Effectiveness and School Improvement*, *9*(2), 157–191. https://doi.org/10.1080/0924345980090203

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement*. Routledge.

Hattie, J. (2015). High-impact leadership. *Educational Leadership*, *72*(5), 36–40.

Herbel-Eisenmann, B., Sinclair, N., Chval, K. B., Clements, D. H., Civil, M., Pape, S. J., Stephan, M., Wanko, J. J., & Wilkerson, T. L. (2016). Positioning mathematics education researchers to influence storylines. *Journal for Research in Mathematics Education*, *47*(2), 102–117. https://doi.org/10.5951/jresematheduc.47.2.0102

Hermans, C., & Schoeman, W. G. (2015). Practice-oriented research in service of designing interventions. *Acta Theologica*, *22*(3), 26–44. https://doi.org/10.4314/actat.v21i1.4S

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Information Age.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Erlbaum.

Hochbein, C., & Cunningham, B. C. (2013). An exploratory analysis of the longitudinal impact of principal change on elementary school achievement. *Journal of School Leadership*, *23*(1), 64–90. https://doi.org/10.1177/105268461302300103

Holmqvist Olander, M., & Nyberg, E. (2014). Learning study guided by variation theory: Exemplified by children learning to halve and double whole numbers. *Journal of Research in Childhood Education*, *28*(2), 238–260. https://doi.org/10.1080/02568543.2014.884030

Huang, R., & Li, Y. (Eds.). (2017). *Teaching and learning mathematics through variation: Confucian heritage meets Western theories*. Sense Publishers.

Hussin, S., & Al Abri, S. (2015). Professional development needs of school principals in the context of educational reform. *International Journal of Educational Administration and Policy Studies*, *7*(4), 90–97. https://doi.org/10.5897/IJEAPS2015.0391
Innabi, H. (2006). The perceptions of UAE principals regarding the new vision of teaching mathematics. *Studies in Curriculum and Instruction*, *113*, 61–84.
Innabi, H. (2009, September 11–17). A model to develop mathematics education: Modify the public traditional perceptions of mathematics-case of UAE schools' principals. In L. Paditz & A. Rogerson (Eds.), *Proceedings of the tenth international conference on Models in Developing Mathematics Education*, Dresden, Saxony, Germany (pp. 262–266).
Innabi, H., & El Sheikh, O. (2007). The change in mathematics teachers' perceptions of critical thinking after 15 years of educational reform in Jordan. *Educational Studies in Mathematics*, *64*(1), 45–68. https://doi.org/10.1007/s10649-005-9017-x
Johnson, J. (2013). The human factor. *Educational Leadership*, *70*(7), 16–21.
Kelleher, J. (2003). A model for assessment-driven professional development. *Phi Delta Kappan*, *84*(10), 751–756. https://doi.org/10.1177/003172170308401008
Koffka, K. (1935/1963). *Principles of gestalt psychology*. Harcourt, Brace & World.
Kuhn, T. (1970). *The structure of scientific revolutions* (2nd ed., enlarged). University of Chicago Press.
Kullberg, A., Björklund, C., Brkovic, I., & Runesson Kempe, U. (2020). Effects of learning addition and subtraction in preschool by making the first ten numbers and their relations visible with finger patterns. *Educational Studies in Mathematics*, *103*(2), 157–172. https://doi.org/10.1007/s10649-019-09927-1
Kullberg, A., Mårtensson, P., & Runesson, U. (2016). What is to be learned? Teachers' collective inquiry into the object of learning. *Scandinavian Journal of Educational Research*, *60*(3), 309–322. https://doi.org/10.1080/00313831.2015.1119725
Kullberg, A., Runesson Kempe, U., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education*, *49*(4), 559–569. https://doi.org/10.1007/s11858-017-0858-4
Kullberg, A., & Skodras, C. (2018). Systematic variation in examples in mathematics teaching. In C. Osbeck, A. Ingerman, & S. Claesson (Eds.), *Didactic classroom studies: A potential research direction* (pp. 47–67). Nordic Akademic Press, Kapitel.
Kutaka, T. S., Ren, L., Smith, W. M., Beattie, H. L., Edwards, C. P., Green, J. L., Chernyavskiy, P., Stroup, W., Heaton, R. M., & Lewis, W. J. (2018). Examining change in K-3 teachers' mathematical knowledge, attitudes, and beliefs: The case of primarily math. *Journal of Mathematics Teacher Education*, *21*(2), 147–177. https://doi.org/10.1007/s10857-016-9355-x
Lambersky, J. (2016). Understanding the human side of school leadership: Principals' impact on teachers' morale, self-efficacy, stress, and commitment. *Leadership and Policy in Schools*, *15*(4), 379–405. https://doi.org/10.1080/15700763.2016.1181188
Lamichhane, K. (2016). Individuals with visual impairments teaching in Nepal's mainstream schools: A model for inclusion. *International Journal of Inclusive Education*, *20*(1), 16–31. https://doi.org/10.1080/13603116.2015.1073374
Latterell, C. M., & Wilson, J. L. (2012). Students' perceptions of what mathematicians do. *The Mathematics Educator*, *13*(2), 73–84. https://doi.org/10.5901/mjss.2014.v5n3p431
Latterell, C. M., & Wilson, J. L. (2016). Math is like a lion hunting a sleeping gazelle: Preservice elementary teachers' metaphors of mathematics. *European Journal of Science and Mathematics Education*, *4*(3), 283–292.
Lee, A. N., & Nie, Y. (2017). Teachers' perceptions of school leaders' empowering behaviours and psychological empowerment: Evidence from a Singapore sample. *Educational Management Administration & Leadership*, *45*(2), 260–283. https://doi.org/10.1177/1741143215578448
Lester, F. K. Jr. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, *10*(1), 244–278.
Lester, J. B., & Grant, C. M. (2001). Mathematics supervision through a new lens. *Educational Leadership*, *58*(5), 60–63.

Liljedahl, P. (2011). The theory of conceptual change as a theory for changing conceptions. *Nordisk Matematikkdidaktikk*, *16*(1–2), 101–124.

Lim, C. S. (1999). *Public images of mathematics* [Unpublished doctoral dissertation]. University of Exeter.

Lim, C. S., & Ernest, P. (2000). A survey of public images of mathematics. *Research in Mathematics Education Journal*, *2*(1), 193–206. https://doi.org/10.1080/14794800008520076

Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Göteborgs Universitet, Acta Universitatis Gothoburgensis, Göteborg.

Mangin, M. M. (2007). Facilitating elementary principals' support for instructional teacher leadership. *Educational Administration Quarterly*, *43*(3), 319–357. https://doi.org/10.1177/0013161X07299438

Markovits, Z., & Forgasz, H. (2017). Mathematics is like a lion': Elementary students beliefs about mathematics. *Educational Studies in Mathematics*, *96*(1), 49–64. https://doi.org/10.1007/s10649-017-9759-2

Marton, F. (2006). Sameness and difference in transfer. *Journal of the Learning Sciences*, *15*(4), 499–535. https://doi.org/10.1207/s15327809jls1504_3

Marton, F. (2015). *Necessary conditions of learning*. Routledge.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Lawrence Erlbaum Associates.

Marton, F., & Häggström, J. (2017). Teaching through variation: A European perspective. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 389–406). Sense Publishers.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences*, *15*(2), 193–220. https://doi.org/10.1207/s15327809jls1502_2

Marton, F., & Tsui, A. (2004). *Classroom discourse and the space of learning*. Lawrence Erlbaum.

McCullough, M., Lipscomb, S., Chiang, H., & Gill, B. (2016). *Do principals' professional practice ratings reflect their contributions to student achievement? Evidence from Pennsylvania's framework for leadership* (Working Paper 46). Mathematical Policy Research, Inc., Cambridge.

McDuffie, A. R., & Graeber, A. O. (2003). Institutional norms and policies that influence college mathematics professors in the process of changing to reform-based practices. *School Science and Mathematics*, *103*(7), 331–344. https://doi.org/10.1111/j.1949-8594.2003.tb18210.x

Meyers, C., Molefe, A., Brandt, W. C., Zhou, B., & Dhillon, S. (2016). Impact results of the eMINTS professional development validation study. *Educational Evaluation and Policy Analysis*, *38*(3), 455–476. https://doi.org/10.3102/0162373716638446

Nadelson, L. S., Pluska, H., Moorcroft, S., Jeffrey, A., & Woodard, S. (2014). Educators' perceptions and knowledge of the common core state standards. *Issues in Teacher Education*, *22*(2), 47–66.

Nanna, R. J. (2016). Public conceptions of algorithms and representations in the common core state standards for mathematics. *The Mathematics Educator*, *25*(Special Issue), 29–51.

National Association of Elementary Schools Principals (NAESP). (2002). *What principals need to know about teaching math*.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to action: Ensuring mathematical success for all*.

National Governors' Association (NGA) Center for Best Practices & Council of Chief State School. (2010). *Common core state standards*.

NCTM Research Committee. (2018). Asset-based approaches to equitable mathematics education research and practice. *Journal for Research in Mathematics Education*, *49*(4), 373–389. https://doi.org/10.5951/jresematheduc.49.4.0373

Nelson, B. S., & Sassi, A. (2005). *The effective principal: Instructional readership for high quality learning*. Teachers College Press.

Nelson, B. S., Stimson, V. C., & Jordan, W. J. (2007). *Leadership content knowledge for mathematics of staff engaged in key school leadership functions* [Paper presentation]. University Council of Education Administration Annual Meeting, Washington, DC, USA.

Nisbet, S., & Warren, E. (2000). Primary school teachers' beliefs relating to mathematics, teaching and assessing mathematics and factors that influence these beliefs. *Mathematics Teacher Education and Development*, *2*, 34–47.
Ontario Principals' Council. (2009). *The principal as mathematics leader series: Leading student achievement series*. SAGE Publishing, Corwin.
Oster, E., Graudgenett, N., McGlamery, S., & Topp, N. (1999). How to avoid common problems and misunderstandings of the NCTM standards. *Education*, *120*(2), 397–400.
Pang, M. F. (2019). Enhancing the generative learning of young people in the domain of financial literacy through learning study. *International Journal for Lesson and Learning Studies*, *8*(3), 170–182. https://doi.org/10.1108/IJLLS-09-2018-0065
Pang, M. F., & Ling, L. (2012). Learning study: Helping teachers to use theory, develop professionally and produce new knowledge to be shared. *Instructional Science: An International Journal of the Learning Sciences*, *40*(3), 589–606. https://doi.org/10.1007/s11251-011-9191-4
Pelfrey, R. (2006). *Mathematics program improvement review: A comprehensive evaluation process for K-12 schools*. Association for Supervision and Curriculum Development (ASCD).
Perry, R. R., Finkelstein, N. D., Seago, N., Heredia, A., Sobolew-Shubin, S., & Carroll, C. (2015). *Taking stock of common core math implementation: Supporting teachers to shift instruction: Insights from the math in common 2015 baseline survey of teachers and administrators*. WestEd.
Peterson, I. (1996). Search for new mathematics. *The Mathematics Forum*. http://forum.swarthmore.edu/social/articles/ivars.html
Peterson, K. (2002). The professional development of principals: Innovations and opportunities. *Educational Administration Quarterly*, *38*(2), 213–232. https://doi.org/10.1177/0013161X02382006
Powell, D. R., Son, S. H., File, N., & Froiland, J. M. (2012). Changes in parent involvement across the transition from public school prekindergarten to first grade and children's academic outcomes. *The Elementary School Journal*, *113*(2), 276–300. https://doi.org/10.1086/667726
Proffitt-White, R. (2017). Putting teachers first: Leading change through design—Initiating and sustaining effective teaching of mathematics. *Australian Mathematics Teacher*, *73*(1), 14–22.
Quinn, D. M. (2002). The impact of principal leadership behaviors on instructional practice and student engagement. *Journal of Educational Administration*, *40*(5), 447–467. https://doi.org/10.1108/09578230210440294
Reed, K. E., Goldsmith, L. T., & Nelson, B. S. (2006). *Elementary and middle school teachers' perceptions of their principals' instructional leadership in mathematics: Preliminary results* [Paper presentation]. Research Pre-Session of the Annual Meeting of the National Council of Teachers of Mathematics, St. Louis, MO, USA.
Reeder, S., Utley, J., & Cassel, D. (2009). Using metaphors as a tool for examining preservice elementary teachers' beliefs about mathematics teaching and learning. *School Science and Mathematics*, *109*(5), 290–297. https://doi.org/10.1111/j.1949-8594.2009.tb18093.x
Rigby, J. G., Larbi-Cherif, A., Rosenquist, B. A., Sharpe, C. J., Cobb, P., & Smith, T. (2017). Administrator observation and feedback: Does it lead toward improvement in inquiry-oriented math instruction? *Educational Administration Quarterly*, *53*(3), 475–516. https://doi.org/10.1177/0013161X16687006
Robinson, V. M. J., Lloyd, C. A., & Rowe, K. J. (2008). The impact of leadership on student outcomes: An analysis of the differential effects of leadership types. *Educational Administration Quarterly*, *44*(5), 635–674. https://doi.org/10.1177/0013161X08321509
Runesson, U. (1999, August 24–28). *Teaching as constituting a space of variation* [Paper presentation]. 8th EARLI conference, Göteborg, Sweden.
Ryve, A., Larsson, M., & Nilsson, P. (2013). Analyzing content and participation in classroom discourse: Dimensions of variation, mediating tools, and conceptual accountability. *Scandinavian Journal of Educational Research*, *57*(1), 101–114. https://doi.org/10.1080/00313831.2011.628689
Schifter, D., Russell, S. J., & Bastable, V. (1999). Teaching to the big ideas. In M. Z. Solomon (Ed.), *The diagnostic teacher: Constructing new approaches to professional development* (pp. 114–201). Teachers College Press.

Seaman, C. E., Szydlik, J. E., Szydlik, S. D., & Beam, J. E. (2005). A comparison of preservice elementary teachers' beliefs about mathematics and teaching mathematics: 1968 and 1998. *School Science and Mathematics*, *105*(4), 197–210. https://doi.org/10.1111/j.1949-8594.2005.tb18158.x

Sheffield, L. J. (2017). Dangerous myths about 'gifted' mathematics students. *ZDM: The International Journal on Mathematics Education*, *49*(1), 13–23. https://doi.org/10.1007/s11858-016-0814-8

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20–26.

Sparks, D. (2002). *Designing powerful professional development for teachers and principals*. National Staff Development Council.

Stein, M. K., & Spillane, J. (2005). What can researchers on educational leadership learn from research on teaching? Building a bridge. In W. A. Firestone & C. Riehl (Eds.), *A new agenda for research in educational leadership* (pp. 28–45). New York, NY: Teachers College Press.

Stephan, M. L., Chval, K. B., Wanko, J. J., Civil, M., Fish, M. C., Herbel-Eisenmann, B., Konold, C., & Wilkerson, T. L. (2015). Grand challenges and opportunities in mathematics education research. *Journal for Research in Mathematics Education*, *46*(2), 134–146. https://doi.org/10.5951/jresematheduc.46.2.0134

Sterenberg, G. (2008). Investigating teachers' images of mathematics. *Journal of Mathematics Teacher Education*, *11*(2), 89–105. https://doi.org/10.1007/s10857-007-9062-8

Stronge, J. H., Richard, H. B., & Catano, N. (2008). *Qualities of effective principals*. Association for Supervision & Curriculum Development (ASCD).

Supovitz, J., Sirinides, P., & May, H. (2010). How principals and peers influence teaching and learning. *Educational Administration Quarterly*, *46*(1), 31–56. https://doi.org/10.1177/1094670509353043

Tan, C. Y. (2018). Examining school leadership effects on student achievement: The role of contextual challenges and constraints. *Cambridge Journal of Education*, *48*(1), 21–45. https://doi.org/10.1080/0305764X.2016.1221885

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). National Council of Teachers of Mathematics.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. (1994). Calculational and conceptual orientations in teaching mathematics. In D. B. Aichele & A. F. Coxford (Eds.), *Professional development for teachers of mathematics* (pp. 79–92). NCTM.

To, K. K., & Pang, M. F. (2019). A study of variation theory to enhance students' genre awareness and learning of genre features. *International Journal for Lesson and Learning Studies*, *8*(3), 183–195. https://doi.org/10.1108/IJLLS-10-2018-0070

Urick, A., & Bowers, A. J. (2014). The impact of principal perception on student academic climate and achievement in high school: How does it measure up? *Journal of School Leadership*, *24*(2), 386–414. https://doi.org/10.1177/105268461402400207

Urick, A., Wilson, A. S. P., Ford, T. G., Frick, W. C., & Wronowski, M. L. (2018). Testing a framework of math progress indicators for ESSA: How opportunity to learn and instructional leadership matter. *Educational Administration Quarterly*, *54*(3), 396–438. https://doi.org/10.1177/0013161X18761343

Viholainen, A., Asikainen, M., & Hirvonen, P. E. (2014). Mathematics student teachers' epistemological beliefs about the nature of mathematics and the goals of mathematics teaching and learning in the beginning of their studies. *Eurasia Journal of Mathematics, Science and Technology Education*, *10*(2), 159–171. https://doi.org/10.12973/eurasia.2014.1028a

White-Fredette, K. (2010). Why not philosophy? Problematizing the philosophy of mathematics in a time of curriculum reform. *The Mathematics Educator*, *19*(2), 21–30. http://citeseerx.ist.psu.edu/viewdoc/download?doi = 10.1.1.361.4387&rep = rep1&type = pdf

Yeliz, T. D. (2016). Effective learning environment characteristics as a requirement of constructivist curricula: Teachers' needs and school principals' views. *International Journal of Instruction*, *9*(2), 183–198. https://doi.org/10.12973/iji.2016.9213a

Youngs, P. (2007). How elementary principals' beliefs and actions influence new teachers' experiences. *Educational Administration Quarterly*, *43*(1), 101–137. https://doi.org/10.1177/0013161X06293629

Youngs, P., & Bruce, K. M. (2002). Principal leadership for professional development to build school capacity. *Educational Administration Quarterly*, *38*(5), 643–670. https://doi.org/10.1177/0013161X02239642