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Estimating the boundary of the region of attraction of Lotka–Volterra system with time delays

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ABSTRACT

This paper considers the local stability problem and estimates the region of attraction (RA) of the positive equilibrium of Lotka-Volterra (L-V) competitive system with time-delays. Based on the stability theory and the quadratic system theory, by choosing some less conservative integral inequalities and appropriate Lyapunov-Krasovskii (L-K) functional, a local stability condition is obtained by means of linear matrix inequalities (LMI) and the estimate of RA of the positive equilibrium is first discussed. Furthermore, the corresponding optimization problem for the estimate of RA is given. Numerical simulations show that the proposed stability condition in this paper is less conservative as compared with the most existing ones and the method of estimating RA is effective.

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Lotka-Volterra system; delay; quadratic systems; stability region

1. Introduction

In this paper, the following two-species competitive Lotka-Volterra (L-V) type system with discrete delays is considered:

$$\begin{cases}\dot{x}(t) = x(t)[b_1 - a_{11}x(t-\tau_{11}) - a_{12}y(t-\tau_{12})],\\ \dot{y}(t) = y(t)[b_2 - a_{21}x(t-\tau_{21}) - a_{22}y(t-\tau_{22})],\end{cases} \quad (1)$$

where $x(t)$ and $y(t)$ are densities of population at time t, respectively; b_i, $a_{ij}(i,j=1,2)$ are positive constants.

The initial condition of system (1) is given as

$$\begin{cases}x(s) = \phi'_1(s), -\tau \le s \le 0, \phi'_1(s) > 0,\\ y(s) = \phi'_2(s), -\tau \le s \le 0, \phi'_2(s) > 0,\end{cases} \quad (2)$$

where $\tau = \max_{1\le i,j\le 2}\{\tau_{ij}\}$, and $\phi'_1(s)$, $\phi'_2(s)$ are assumed to be continuous.

If $a_{11}/a_{21} > b_1/b_2 > a_{12}/a_{22}$, by denoting $z^*=(x^*,y^*)$ with $x^* = \frac{b_1a_{22}-b_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}}$, $y^* = \frac{b_2a_{11}-b_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$, it is seen that $x^*, y^* \in (0,1)$. The four equilibrium points of system (1) are$(0,0)$, $(0,b_2/a_{22})$, $(b_1/a_{11},0)$, and z^* respectively. Define $z(t) = (x(t),y(t))$ as all the positive solutions of system (1). The point z^* is the only positive equilibrium point. The equilibrium point z^* indicates the coexistence of the two species.

It is well known that the foundation of two-species competitive L-V type system (1) was laid by Lotka and Volterra sixty years ago (Lotka, 1956; Volterra, 1931). During the past several decades, L-V type systems have received much attention. The most basic questions arising from this system are competitive species' persistence, attractivity, extinctions, global (or local) asymptotic behaviour and other dynamical behaviours (Ahmad & Stamova, 2015; Chen et al., 2012; Jiang & Liang, 2020; Kuang, 1996; Lai & Fang, 2020; Ma et al., 2019; Park, 2005; Sun & Meng, 2007; Teng & Yu, 2000; Wang et al., 1995; Zhao et al., 2014; Zhen & Ma, 2002; Zhu & Liu, 2017).

There are considerable works on the study of the global and local asymptotical behaviours of L-V type systems. However, few studies have discussed the estimate of the region of attraction (RA) or the edge of chaotic attractors in L-V systems (Lai & Fang, 2020). In Lai and Fang (2020), a topological approach for plotting the boundary of the region of asymptotic stability of L-V predator-prey system was proposed. But the systems considered by Lai and Fang (2020) don't contain time delay. To the best of our knowledge, the estimate of the RA in L-V system with time delays has not been considered due to probably the mathematical complexity.

Two-species competitive L-V type system is a quadratic system. Quadratic systems play an important role in the modelling of a wide class of nonlinear processes (electrical, robotic, biological, etc.). For such systems, it is of mandatory importance not only to determine whether the origin of the state space is locally asymptotically stable but also to ensure that the operative range is included into the convergence region of the equilibrium. Over the years, several papers have focused on the estimate of

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the RA of the zero-equilibrium point of quadratic system (Caldeira et al., 2018; Chen et al., 2013; Chesi et al., 2005; Chiang et al., 1988; Genesio & Vicino, 1984; Li-Ya et al., 2019; Merola et al., 2017; Tesi et al., 1996; Xu et al., 2020). In Genesio and Vicino (1984), a Lyapunov-based procedure is proposed to compute an ellipsoidal estimate of the RA of a quadratic system. Considering this method is computationally heavy, an estimate of the RA based on topological considerations was provided in Chiang et al. (1988) or based on linear matrix inequalities (LMIs) feasibility problem (Chesi et al., 2005; Tesi et al., 1996).

The estimate of the RA of time-delay systems is tackled by means of a two-steps procedure: (i) choice of a Lyapunov–Krasovskii (L-K) functional, which proves local asymptotic stability of the equilibrium of system (Qian, Li, Chen, et al., 2020; Qian, Xing, and Fei, 2020); (ii) computation of the estimate of the RA associated to that particular L-K functional. The integral terms are common in the derivative of L-K functionals, and the approximation methods are used to replace the integral terms with some more effective expressions (Dong et al., 2019). So, the choice of the optimal L-K functional and the less conservative inequality are important and may severely affect the conservativeness of the estimate.

In this paper, our main contributions are given as follows: (1) we study local stability and estimate the RA of the positive equilibrium of L-V competitive system with time-delays based on quadratic system theory; (2) By choosing an appropriate L-K functionals and using less conservative inequalities, we develop LMI to ensure the local asymptotic stability of the positive equilibrium point with a guaranteed region of stability inside some polytopic region of the state-space.

The paper is organized as follows: In Section 2, the problem we deal with is precisely stated and some preliminary notation is provided. Section 3 proposes some preliminaries. In Section 4, we derive the local stability conditions and estimate the RA of the positive equilibrium z^*. Finally, simulations are given to illustrate the effectiveness of the obtained results.

Notation. The superscript "T" is the transpose of a matrix. The matrix $P > 0 (P \geq 0)$ denotes that P is positive definite (positive semi-definite). $||\cdot||$ denote the 2-norm of a vector. $\lambda(R)_M$ is the maximum eigenvalue value of matrix R. The symmetric terms in a matrix are denoted by $*$.

2. Problem formulation

Letting $u(t) \triangleq x(t) - x^*, v(t) \triangleq y(t) - y^*$, system (1) can be written as

$$\begin{cases} \dot{u}(t) = [u(t) + x^*][-a_{11}u(t-\tau_{11}) - a_{12}v(t-\tau_{12})], \\ \dot{v}(t) = [v(t) + y^*][-a_{21}u(t-\tau_{21}) - a_{22}v(t-\tau_{22})]. \end{cases} \tag{3}$$

Denote $e(t) \triangleq (u^T(t), v^T(t))^T$ and rewrite (3) in the following matrix form:

$$\dot{e}(t) = -\sum_{i,j=1}^{2} [A_{ij} + B_{ij}(e)]e(t-\tau_{ij}), \tag{4}$$

where

$$A_{11} = \begin{pmatrix} a_{11}x^* & 0 \\ 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & a_{12}x^* \\ 0 & 0 \end{pmatrix},$$
$$A_{21} = \begin{pmatrix} 0 & 0 \\ a_{21}y^* & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 0 \\ 0 & a_{22}y^* \end{pmatrix},$$

$$B_{1j}(e) = \begin{pmatrix} e^T A_{1j}/x^* \\ e^T B \end{pmatrix}$$
$$B_{2j}(e) = \begin{pmatrix} e^T B \\ e^T A_{2j}/y^* \end{pmatrix} \quad j = 1, 2$$

with $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The initial condition $e(s) = [\phi_1^T(s), \phi_2^T(s)]^T, s \in [-\tau, 0]$ associated with the system (4) is defined by initial condition (2).

In this paper, the initial condition $e(s) (s \in [-\tau, 0))$ of the system (4) is assumed to belong to a set of the form

$$X_\rho = \left\{ e(s) \in C^1[-\tau, 0] : \max_{s \in [-\tau, 0]} ||e(s)|| \leq \rho_1, \max_{s \in [-\tau, 0]} ||\dot{e}(s)|| \leq \rho_2 \right\}. \tag{5}$$

where $\rho_1 > 0$, $\rho_2 > 0$are scalars to be maximized.

We introduce the square

$$\chi = [-\bar{e}_1, \bar{e}_1] \times [-\bar{e}_2, \bar{e}_2], \tag{6}$$

which can be equivalently written as follows:

$$\chi = Co\{v_1, v_2, v_3, v_4\} = \{e \in \mathbb{R}^2 : |h_i^T e| \leq \bar{e}_i, i = 1, 2\}, \tag{7}$$

where 'Co' denote the convex hull and the corresponding vertices $v_i (i = 1, 2, 3, 4)$ are given as follows:

$$v_1 = \begin{bmatrix} -\bar{e}_1 & -\bar{e}_2 \end{bmatrix}^T, \quad v_2 = \begin{bmatrix} -\bar{e}_1 & \bar{e}_2 \end{bmatrix}^T,$$
$$v_3 = \begin{bmatrix} \bar{e}_1 & -\bar{e}_2 \end{bmatrix}^T, \quad v_1 = \begin{bmatrix} \bar{e}_1 & \bar{e}_2 \end{bmatrix}^T,$$
$$h_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \quad h_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T.$$

3. Preliminaries

Next, we will introduce the following integral inequalities, which are important in obtaining our main results.

Lemma 3.1: *(**Park et al., 2015**): For a given $n \times n$ matrix $R > 0$, two scalars a and b satisfying $b > a$ and a differentiable vector function $\omega(t) \in \mathbb{R}^n$, the following integral inequalities holds:*

(1) $(b-a)\int_a^b \omega^T(s)R\rho(s)ds \geq \left(\int_a^b \omega(s)ds\right)^T R\left(\int_a^b \rho(s)ds\right) + 3\Omega_1^T R\Omega_2,$

(2) $(b-a)\int_a^b \omega^T(s)R\omega(s)ds \geq \left(\int_a^b \omega(s)ds\right)^T R\left(\int_a^b \omega(s)ds\right) + 3\Omega_1^T R\Omega_1 + 5\Omega_2^T R\Omega_2,$

(3) $\frac{(b-a)^2}{2}\int_a^b\int_\theta^b \omega(s)dsd\theta \geq \left(\int_a^b\int_\theta^b \omega(s)dsd\theta\right)^T R\left(\int_a^b\int_\theta^b \omega(s)dsd\theta\right) + 8\Omega_3^T R\Omega_3,$

where

$$\Omega_1 = \int_a^b \omega(s)ds - \frac{2}{b-a}\int_a^b\int_\theta^b \omega(s)dsd\theta,$$

$$\Omega_2 = \int_a^b \omega(s)ds - \frac{6}{b-a}\int_a^b\int_\theta^b \omega(s)dsd\theta + \frac{12}{(b-a)^2}\int_a^b\int_\lambda^b\int_\theta^b \omega(s)dsd\theta d\lambda,$$

$$\Omega_3 = \int_a^b\int_\theta^b \omega(s)dsd\theta - \frac{3}{b-a}\int_a^b\int_\lambda^b\int_\theta^b \omega(s)dsd\theta d\lambda.$$

Lemma 3.2: *(**Qian, Li, Zhao, et al., 2020**): For a given $n \times n$ matrix $Z > 0$, two scalars a and b satisfying $b > a$ and a differentiable vector function $\omega(t) \in \mathbb{R}^n$, the following two inequalities holds:*

(1) $(b-a)\int_a^b \omega^T(s)Z\omega(s)ds \geq \left(\int_a^b \omega(s)ds\right)^T Z \left(\int_a^b \omega(s)ds\right),$

(2) $\frac{(b^2-a^2)}{2}\int_{-b}^{-a}\int_{t+\theta}^t \omega^T(s)Z\omega(s)dsd\theta\,(b > a \geq 0) \geq \left(\int_{-b}^{-a}\int_{t+\theta}^t \omega(s)dsd\theta\right)^T Z\left(\int_{-b}^{-a}\int_{t+\theta}^t \omega(s)dsd\theta\right).$

4. Main results

In this section, we will first use L-K stability theory and the quadratic systems theory to derive the local asymptotical stability conditions of the equilibrium point z^*. For presentation convenience, we denote that

$$\alpha_{ij}(t) \triangleq \int_{t-\tau_{ij}}^t e(s)ds, \quad \alpha_{ij}^{kl}(t) \triangleq \int_{t-\tau_{ij}}^{t-\tau_{kl}} e(s)ds,$$

$$\beta_{ij}(t) \triangleq \int_{-\tau_{ij}}^0\int_{t+\theta}^t e(s)dsd\theta,$$

$$\beta_{ij}^{kl}(t) \triangleq \int_{-\tau_{ij}}^{-\tau_{kl}}\int_{t+\theta}^t e(s)dsd\theta,$$

$$\tau_{ijkl} \triangleq \tau_{ij} - \tau_{kl}, \quad i,j,k,l = 1,2, \quad 10i + j < 10k + l,$$

$$\bar{e}(t) \triangleq \left[e^T(t-\tau_{11})\, e^T(t-\tau_{12})\, e^T(t-\tau_{21})\, e^T(t-\tau_{22})\right]^T,$$

$$*(t) \triangleq \left[*_{11}{}^T(t) \quad *_{12}{}^T(t) \quad *_{21}{}^T(t) \quad *_{22}{}^T(t)\right]^T,$$

$$\bar{*}(t) \triangleq \left[*_{11}^{12\,T}(t)\, *_{11}^{21\,T}(t)\, *_{11}^{22\,T}(t)\, *_{12}^{21\,T}(t)\, *_{12}^{22\,T}(t)\, *_{21}^{22\,T}(t)\right]^T,$$

$$* = \alpha, \beta,$$

$$\varepsilon(Z,1) \triangleq \{e \in \mathbb{R}^2 : e^T Z e \leq 1, Z > 0\}.$$

Theorem 4.1: *Let the scalars $\bar{e}_1 > 0$, $\bar{e}_2 > 0$ and $\tau_{ij} > 0, i,j = 1,2$ be given. The system (4) is locally asymptotically stable, if there exist 42×42-dimensional symmetric matrix P, and 2×2-dimensional matrices $Q_{ij} > 0, R_{ij} > 0, Q_{ij}^{kl} > 0, R_{ij}^{kl} > 0, k,l = 1,2$ and $10i + j < 10k + l, Z > 0, T_1, T_2$, such that the following LMIs hold:*

$$\begin{bmatrix} \Phi_{11}(v_m) & \Phi_{12} & \Phi_{13}(v_m) \\ * & \Phi_{22} & \Phi_{23} \\ * & * & \Phi_{33} \end{bmatrix} + sym(F_1^T P F_2) < 0, \quad m = 1,2,3,4, \tag{8}$$

$$P + \begin{bmatrix} \bar{\Sigma} & \vartheta \\ * & \Xi \end{bmatrix} \geq 0, \tag{9}$$

$$h_1^T h_1 \leq \bar{e}_1^2 Z, \quad h_2^T h_2 \leq \bar{e}_2^2 Z, \tag{10}$$

where

$$F_1 = \begin{bmatrix} I & 0_{2\times8} & 0_{2\times40} & 0 \\ 0_{40\times2} & 0_{40\times8} & I_{40\times40} & 0_{40\times2} \end{bmatrix},$$

$$\Phi_{12} = \begin{bmatrix} \tilde{R} & 0_{2\times12} \\ \hat{R} & \bar{R} \end{bmatrix},$$

$$\Phi_{13}(v_m) = \begin{bmatrix} \tilde{\Lambda} & 0_{2\times12} & -T_1 \\ \hat{\Lambda} & \bar{\Lambda} & \Pi(v_m) \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0 & 0_{2\times8} & 0_{2\times8} & 0_{2\times12} & 0_{2\times20} & I \\ \Gamma_1 & -I_{8\times8} & 0_{8\times8} & 0_{8\times12} & 0_{8\times20} & 0_{8\times2} \\ 0_{12\times2} & \Gamma_2 & 0_{12\times8} & 0_{12\times12} & 0_{12\times20} & 0_{12\times2} \\ \Gamma_3 & 0_{8\times8} & -I_{8\times8} & 0_{8\times12} & 0_{8\times20} & 0_{8\times2} \\ \Gamma_4 & 0_{12\times8} & 0_{12\times8} & -I_{12\times12} & 0_{12\times20} & 0_{12\times2} \end{bmatrix},$$

$$\Xi = \begin{bmatrix} \Xi^{11} & \Xi^{12} \\ * & \Xi^{22} \end{bmatrix},$$

$$\Phi_{11}(v_m) = \begin{bmatrix} \gamma_1 & \bar{R}_{11}(v_m) & \bar{R}_{12}(v_m) & \bar{R}_{21}(v_m) & \bar{R}_{22}(v_m) \\ * & \gamma_{11} & 3R_{11}^{12} & 3R_{11}^{21} & 3R_{11}^{22} \\ * & * & \gamma_{12} & 3R_{12}^{21} & 3R_{12}^{22} \\ * & * & * & \gamma_{21} & 3R_{21}^{22} \\ * & * & * & * & \gamma_{22} \end{bmatrix},$$

$$\Phi_{22} = -192diag\{\tau_{11}^{-2}R_{11}, \cdots, \tau_{22}^{-2}R_{22}, \tau_{1112}^{-2}R_{11}^{12},$$

$$\cdots, \tau_{2122}^{-2}R_{21}^{22}\}, \quad \Phi_{23} = \begin{bmatrix} \Theta & 0_{20\times 2} \end{bmatrix},$$

$$\Phi_{33} = -720diag\{\tau_{11}^{-4}R_{11}, \cdots, \tau_{22}^{-4}R_{22}, \tau_{1112}^{-4}R_{11}^{12}, \cdots, \tau_{2122}^{-4}R_{21}^{22}, -\mu/720\},$$

$$\bar{\Sigma} = -Z + \sum_{i,j=1}^{2} 6\tau_{ij}R_{ij},$$

$$\Xi^{11} = diag\{\Xi_{11}, \cdots, \Xi_{22}, \Xi_{11}^{12}, \cdots, \Xi_{21}^{22}\},$$

$$\Xi^{12} = diag\{\Xi_{11}^{1}, \cdots, \Xi_{22}^{1}, \Xi_{11}^{121}, \cdots, \Xi_{21}^{221}\},$$

$$\Xi^{22} = diag\{\Xi_{11}^{2}, \cdots, \Xi_{22}^{2}, \Xi_{11}^{122}, \cdots, \Xi_{21}^{222}\},$$

$$\vartheta = \begin{bmatrix} \Sigma_{11} & \cdots & \Sigma_{22} & \Sigma_{11}^{12} & \cdots & \Sigma_{21}^{22} & \Sigma_{11}^{1} & \cdots & \Sigma_{22}^{1} & \Sigma_{11}^{121} & \cdots & \Sigma_{21}^{221} \end{bmatrix},$$

with

$$\Gamma_1 = \begin{bmatrix} I \\ I \\ I \\ I \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} \tau_{11}I \\ \tau_{12}I \\ \tau_{21}I \\ \tau_{22}I \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} -I & I & 0 & 0 \\ -I & 0 & I & 0 \\ -I & 0 & 0 & I \\ 0 & -I & I & 0 \\ 0 & -I & 0 & I \\ 0 & 0 & -I & I \end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix} \tau_{1112}I \\ \tau_{1121}I \\ \tau_{1122}I \\ \tau_{1221}I \\ \tau_{1222}I \\ \tau_{2122}I \end{bmatrix},$$

$$\gamma_1 = \sum_{i,j=1}^{2} (Q_{ij} - 9R_{ij}),$$

$$\gamma_{11} = -Q_{11} - \sum_{k,l=1}^{2} \tau_{11kl}Q_{11}^{kl} - 9R_{11} - \sum_{k,l=1}^{2} 9R_{11}^{kl},$$

$$\gamma_{22} = -Q_{22} + \sum_{i,j=1}^{2} \tau_{ij22}Q_{ij}^{22} - 9R_{22} - \sum_{i,j=1}^{2} 9R_{ij}^{22},$$

$$\gamma_{\hat{i}\hat{j}} = -Q_{\hat{i}\hat{j}} + \sum_{i,j=1}^{2} \tau_{ij\hat{i}\hat{j}}Q_{ij}^{\hat{i}\hat{j}} - \sum_{k,l=1}^{2} \tau_{\hat{i}\hat{j}kl}Q_{\hat{i}\hat{j}}^{kl} - 9R_{\hat{i}\hat{j}} - \sum_{k,l=1}^{2} 9R_{\hat{i}\hat{j}}^{kl} - \sum_{i,j=1}^{2} 9R_{ij}^{\hat{i}\hat{j}},$$

$$10i + j < 10\hat{i} + \hat{j} < 10k + l, \ (\hat{i}, \hat{j}) = (1, 2) \ or \ (2, 1),$$

$$\bar{R}_{ij}(v_m) = 3R_{ij} - T_1[A_{ij} + B_{ij}(v_m)],$$

$$\tilde{R} = -24\begin{bmatrix} \tau_{11}^{-1}R_{11} & \tau_{12}^{-1}R_{12} & \tau_{21}^{-1}R_{21} & \tau_{22}^{-1}R_{22} \end{bmatrix},$$

$$\hat{R} = 36diag\{\tau_{11}^{-1}R_{11}, \tau_{12}^{-1}R_{12}, \tau_{21}^{-1}R_{21}, \tau_{22}^{-1}R_{22}\},$$

$$\bar{R} = 12\begin{bmatrix} 3\tau_{1112}^{-1}R_{11}^{12} & 3\tau_{1121}^{-1}R_{11}^{21} & 3\tau_{1122}^{-1}R_{11}^{22} & 0 & 0 & 0 \\ 2\tau_{1112}^{-1}R_{11}^{12} & 0 & 0 & 3\tau_{1221}^{-1}R_{12}^{21} & 3\tau_{1222}^{-1}R_{12}^{22} & 0 \\ 0 & 2\tau_{1121}^{-1}R_{11}^{21} & 0 & 0 & 2\tau_{1222}^{-1}R_{12}^{22} & 2\tau_{2122}^{-1}R_{21}^{22} \\ 0 & 0 & 2\tau_{1122}^{-1}R_{11}^{22} & 2\tau_{1221}^{-1}R_{12}^{21} & 0 & 3\tau_{2122}^{-1}R_{21}^{22} \end{bmatrix},$$

$$\tilde{\Lambda} = 60\begin{bmatrix} \tau_{11}^{-2}R_{11} & \tau_{12}^{-2}R_{12} & \tau_{21}^{-2}R_{21} & \tau_{22}^{-2}R_{22} \end{bmatrix},$$

$$\hat{\Lambda} = -60diag\{\tau_{11}^{-2}R_{11}, \tau_{12}^{-2}R_{12}, \tau_{21}^{-2}R_{21}, \tau_{22}^{-2}R_{22}\},$$

$$\Theta = 360diag\{\tau_{11}^{-3}R_{11}, \cdots, \tau_{22}^{-3}R_{22}, \tau_{1112}^{-3}R_{11}^{12}, \cdots, \tau_{2122}^{-3}R_{21}^{22}\},$$

$$\mu = -T_2 - T_2^T + \sum_{i,j=1}^{2} \tau_{ij}^{2}R_{ij} + \sum_{i,j=1}^{2}\sum_{k,l=1}^{2} (\tau_{ij} - \tau_{kl})^2 R_{ij}^{kl}, \quad \Sigma_{ij} = 6R_{ij},$$

$$\Sigma_{ij}^{kl} = 6R_{ij}^{kl}, \quad \Sigma_{ij}^{1} = -24\tau_{ij}^{-1}R_{ij},$$

$$\Sigma_{ij}^{kl1} = -24\tau_{ijkl}^{-1}R_{ij}^{kl}, \quad \Xi_{ij} = (4Q_{ij} + 18R_{ij})\tau_{ij}^{-1},$$

$$\Xi_{ij}^{kl} = 4Q_{ij}^{kl} + 18R_{ij}^{kl}\tau_{ijkl}^{-1},$$

$$\Xi_{ij}^{1} = -6\tau_{ij}^{-2}(Q_{ij} + 8R_{ij}),$$

$$\Xi_{ij}^{kl1} = -6\tau_{ijkl}^{-1}(Q_{ij}^{kl} + 8R_{ij}^{kl}\tau_{ijkl}^{-1}),$$

$$\Xi_{ij}^{2} = 12(Q_{ij} + 12R_{ij})\tau_{ij}^{-3},$$

$$\Xi_{ij}^{kl2} = 12\tau_{ijkl}^{-2}(Q_{ij}^{kl} + 12R_{ij}^{kl}\tau_{ijkl}^{-1}),$$

$$\bar{\Lambda} = 60\begin{bmatrix} \tau_{1112}^{-2}R_{11}^{12} & \tau_{1121}^{-2}R_{11}^{21} & \tau_{1122}^{-2}R_{11}^{22} & 0 & 0 & 0 \\ \tau_{1112}^{-2}R_{11}^{12} & 0 & 0 & \tau_{1221}^{-2}R_{12}^{21} & \tau_{1222}^{-2}R_{12}^{22} & 0 \\ 0 & \tau_{1121}^{-2}R_{11}^{21} & 0 & 0 & \tau_{1222}^{-2}R_{12}^{22} & \tau_{2122}^{-2}R_{21}^{22} \\ 0 & 0 & \tau_{1122}^{-2}R_{11}^{22} & \tau_{1221}^{-2}R_{12}^{21} & 0 & \tau_{2122}^{-2}R_{21}^{22} \end{bmatrix},$$

$$\Pi(v_m) = -\begin{bmatrix} [A_{11} + B_{11}(v_m)]^T \\ [A_{12} + B_{12}(v_m)]^T \\ [A_{21} + B_{21}(v_m)]^T \\ [A_{22} + B_{22}(v_m)]^T \end{bmatrix} T_2^T.$$

Proof: Choose the following augmented L-K functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{11}$$

where

$$V_1(t) = \eta^T(t)P\eta(t) + \sum_{i,j=1}^{2}\int_{t-\tau_{ij}}^{t} e^T(s)Q_{ij}e(s)ds + \sum_{i,j=1}^{2}\tau_{ij}\int_{-\tau_{ij}}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)R_{ij}\dot{e}(s)dsd\theta,$$

$$V_2(t) = \sum_{i,j=1}^{2}\sum_{k,l=1}^{2}\tau_{ijkl}\int_{t-\tau_{ij}}^{t-\tau_{kl}} e^T(s)Q_{ij}^{kl}e(s)ds,$$

$$V_3(t) = \sum_{i,j=1}^{2}\sum_{k,l=1}^{2}\tau_{ijkl}\int_{-\tau_{ij}}^{-\tau_{kl}} \times \int_{t+\theta}^{t}\dot{e}^T(s)R_{ij}^{kl}\dot{e}(s)dsd\theta, \quad 10i+j < 10k+l,$$

with

$$\eta(t) = \begin{bmatrix} e^T(t) & \alpha^T(t) & \bar{\alpha}^T(t) & \beta^T(t) & \bar{\beta}^T(t) \end{bmatrix}^T.$$

■

Through some direct calculations, one can obtain that

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \tag{12}$$

where

$$\dot{V}_1(t) = 2\eta^T(t)P\dot{\eta}(t) + e^T(t)\sum_{i,j=1}^{2}Q_{ij}e(t) - \sum_{i,j=1}^{2}e^T(t-\tau_{ij})Q_{ij}e(t-\tau_{ij}) + \dot{e}^T(t)\sum_{i,j=1}^{2}\tau_{ij}^2R_{ij}\dot{e}(t) - \sum_{i,j=1}^{2}\tau_{ij}\int_{t-\tau_{ij}}^{t}\dot{e}^T(s)R_{ij}\dot{e}(s)ds,$$

$$\dot{V}_2(t) = \sum_{i,j=1}^{2}\sum_{k,l=1}^{2}\tau_{ijkl}[e^T(t-\tau_{kl})Q_{ij}^{kl}e(t-\tau_{kl}) - e^T(t-\tau_{ij})Q_{ij}^{kl}e(t-\tau_{ij})],$$

$$\dot{V}_3(t) = \sum_{i,j=1}^{2}\sum_{k,l=1}^{2}\tau_{ijkl}\left[\tau_{ijkl}\dot{e}^T(t)R_{ij}^{kl}\dot{e}(t) - \int_{t-\tau_{ij}}^{t-\tau_{kl}}\dot{e}^T(s)R_{ij}^{kl}\dot{e}(s)ds\right].$$

Using the inequality (2) of Lemma 3.1, it is seen that

$$-\tau_{ij}\int_{t-\tau_{ij}}^{t}\dot{e}^T(s)R_{ij}\dot{e}(s)ds \le -\begin{bmatrix}\xi_{ij}^{(1)}(t)\\ \xi_{ij}^{(2)}(t)\\ \xi_{ij}^{(3)}(t)\end{bmatrix}^T \times \begin{bmatrix} R_{ij} & 0 & 0\\ 0 & 3R_{ij} & 0\\ 0 & 0 & 5R_{ij}\end{bmatrix}\begin{bmatrix}\xi_{ij}^{(1)}(t)\\ \xi_{ij}^{(2)}(t)\\ \xi_{ij}^{(3)}(t)\end{bmatrix}, \tag{13}$$

$$-\tau_{ijkl}\int_{t-\tau_{ij}}^{t-\tau_{kl}}\dot{e}^T(s)R_{ij}^{kl}\dot{e}(s)ds \le -\begin{bmatrix}\xi_{ij}^{kl(1)}(t)\\ \xi_{ij}^{kl(2)}(t)\\ \xi_{ij}^{kl(3)}(t)\end{bmatrix}^T \times \begin{bmatrix} R_{ij}^{kl} & 0 & 0\\ 0 & 3R_{ij}^{kl} & 0\\ 0 & 0 & 5R_{ij}^{kl}\end{bmatrix}\begin{bmatrix}\xi_{ij}^{kl(1)}(t)\\ \xi_{ij}^{kl(2)}(t)\\ \xi_{ij}^{kl(3)}(t)\end{bmatrix}, \tag{14}$$

where

$$\xi_{ij}^{(1)}(t) = e(t) - e(t-\tau_{ij}),$$
$$\xi_{ij}^{(2)}(t) = e(t) + e(t-\tau_{ij}) - 2\tau_{ij}^{-1}\alpha_{ij}(t),$$
$$\xi_{ij}^{(3)}(t) = e(t) - e(t-\tau_{ij}) + 6\tau_{ij}^{-1}\alpha_{ij}(t) - 12\tau_{ij}^{-2}\beta_{ij}(t),$$
$$\xi_{ij}^{kl(1)}(t) = e(t-\tau_{kl}) - e(t-\tau_{ij}),$$
$$\xi_{ij}^{kl(2)}(t) = e(t-\tau_{ij}) + e(t-\tau_{kl}) - 2\tau_{ijkl}^{-1}\alpha_{ij}^{kl}(t),$$
$$\xi_{ij}^{kl(3)}(t) = e(t-\tau_{kl}) - e(t-\tau_{ij}) + 6\tau_{ijkl}^{-1}\alpha_{ij}^{kl}(t) - 12\tau_{ijkl}^{-2}\beta_{ij}^{kl}(t).$$

For any matrix $T_1, T_2 \in \mathbb{R}^{2\times 2}$, it follows that the following equation is true:

$$2[e^T(t)T_1 + \dot{e}^T(t)T_2] \times \left\{-\sum_{i,j=1}^{2}[A_{ij} + B_{ij}(e)]e(t-\tau_{ij}) - \dot{e}(t)\right\} = 0. \tag{15}$$

Adding the left side of (15) to $\dot{V}(t)$ and using (13) and (14) yields

$$\dot{V}(t) \le \zeta^T(t)[\Phi(e) + sym(F_1^TPF_2)]\zeta(t) \tag{16}$$

where

$$\zeta(t) = \left[e^T(t)\ \bar{e}^T(t)\ \alpha^T(t)\ \bar{\alpha}^T(t)\ \beta^T(t)\ \bar{\beta}^T(t)\ \dot{e}^T(t)\right]^T$$

$$\Phi(e) = \begin{bmatrix}\Phi_{11}(e) & \Phi_{12} & \Phi_{13}(e)\\ * & \Phi_{22} & \Phi_{23}\\ * & * & \Phi_{33}\end{bmatrix},$$

$$\Phi_{13}(e) = \begin{bmatrix}\tilde{\Lambda} & 0_{2\times 12} & -T_1\\ \hat{\Lambda} & \bar{\Lambda} & \Pi(e)\end{bmatrix},$$

$$\Phi_{11}(e) = \begin{bmatrix}\gamma_1 & \bar{R}_{11}(e) & \bar{R}_{12}(e) & \bar{R}_{21}(e) & \bar{R}_{22}(e)\\ * & \gamma_{11} & 3R_{11}^{12} & 3R_{11}^{21} & 3R_{11}^{22}\\ * & * & \gamma_{12} & 3R_{12}^{21} & 3R_{12}^{22}\\ * & * & * & \gamma_{21} & 3R_{21}^{22}\\ * & * & * & * & \gamma_{22}\end{bmatrix},$$

with

$$\bar{R}_{ij}(e) = 3R_{ij} - T_1[A_{ij} + B_{ij}(e)],$$

$$\Pi(e) = -\begin{bmatrix}[A_{11} + B_{11}(e)]^T \\ [A_{12} + B_{12}(e)]^T \\ [A_{21} + B_{21}(e)]^T \\ [A_{22} + B_{22}(e)]^T\end{bmatrix} T_2^T.$$

Note that $\Phi(e)$ is affine with respect to the states $e_1(t)$ and $e_2(t)$. Hence, if the LMIs in (8) are satisfied, the inequality $\Phi(e) < 0$ can be ensured on χ. Then, we have

$$\dot{V}(t) < 0 \tag{17}$$

on the square χ, which implies that

$$V(t) \leq V(0),\ t \geq 0. \tag{18}$$

On the other hand, using the integral inequalities (1) and (3) in Lemma 3.1, one can obtain the inequality

$$\begin{aligned}
V(t) \geq\ & \eta^T(t)P\eta(t) + \sum_{i,j=1}^{2} \tau_{ij}^{-1}\{\alpha_{ij}^T(t)Q_{ij}\alpha_{ij}(t) \\
& + 2[\tau_{ij}e(t) - \alpha_{ij}(t)]^T R_{ij}[\tau_{ij}e(t) - \alpha_{ij}(t)] \\
& + 4[\tau_{ij}e(t) + 2\alpha_{ij}(t) - 6\tau_{ij}^{-1}\beta_{ij}(t)]^T \\
& \times R_{ij}[\tau_{ij}e(t) + 2\alpha_{ij}(t) - 6\tau_{ij}^{-1}\beta_{ij}(t)] \\
& + 3[\alpha_{ij}(t) - 2\tau_{ij}^{-1}\beta_{ij}(t)]^T Q_{ij}[\alpha_{ij}(t) - 2\tau_{ij}^{-1}\beta_{ij}(t)]\} \\
& + \sum_{i,j=1}^{2}\sum_{k,l=1}^{2} \{\alpha_{ij}^{klT}(t)Q_{ij}^{kl}\alpha_{ij}^{kl}(t) \\
& + 4\tau_{ijkl}^{-1}[\tau_{ijkl}e(t) + 2\alpha_{ij}^{kl}(t) - 6\tau_{ijkl}^{-1}\beta_{ij}^{kl}(t)]^T \\
& \times R_{ij}^{kl}[\tau_{ijkl}e(t) + 2\alpha_{ij}^{kl}(t) - 6\tau_{ijkl}^{-1}\beta_{ij}^{kl}(t)] \\
& + 3[\alpha_{ij}^{kl}(t) - 2\tau_{ijkl}^{-1}\beta_{ij}^{kl}(t)]^T Q_{ij}^{kl}[\alpha_{ij}^{kl}(t) - 2\tau_{ijkl}^{-1}\beta_{ij}^{kl}(t)] \\
& + 2\tau_{ijkl}^{-1}(\tau_{ijkl}e(t) - \alpha_{ij}^{kl}(t))^T R_{ij}^{kl}(\tau_{ijkl}e(t) - \alpha_{ij}^{kl}(t))\} \\
=\ & \eta^T(t)\left(P + \begin{bmatrix}\Sigma & \vartheta \\ * & \Xi\end{bmatrix}\right)\eta(t)
\end{aligned} \tag{19}$$

where $\Sigma = \sum_{i,j=1}^{2} 6\tau_{ij}R_{ij}$.

Using (9) and noting (19), it follows that

$$V(t) \geq e^T(t)Ze(t), \tag{20}$$

which shows that the L-K functional $V(t)$ is positive definite.

In addition, it can be inferred from (10) that

$$\varepsilon(Z, 1) \subseteq \chi. \tag{21}$$

For all initial state $e(s) \in X_\rho$ satisfying $V(0) \leq 1$, it is seen from (18) and (20) that all trajectories $e(t)$ are contained in the set $\varepsilon(Z, 1)$. Noting (17) and (21), it is concluded that the system (4) is asymptotically stable for all $e(s)$ satisfying $V(0) \leq 1$. This completes the proof.

Remark 4.1: Let $Q_{ij}^{kl}, R_{ij}^{kl} = 0$, the interconnected terms related to four time delays in the functional (11) have been ignored, then the dimension of matrix P in Theorem 4.1 will be greatly reduced to 18×18. The LMI (8) will be turned into

$$\begin{bmatrix}\Phi_{11}(v_m) & \Phi'_{12} & \Phi'_{13}(v_m) \\ * & \Phi'_{22} & \Phi'_{23} \\ * & * & \Phi'_{33}\end{bmatrix} + sym(F_1'^T P F_2') < 0, \quad m = 1, 2, 3, 4,$$

where

$$F_1' = \begin{bmatrix} I & 0_{2\times8} & 0_{2\times16} & 0 \\ 0_{16\times2} & 0_{16\times8} & I_{16\times16} & 0_{16\times2}\end{bmatrix},$$

$$\Phi'_{12} = \begin{bmatrix}\tilde{R}^T & \hat{R}^T\end{bmatrix}^T, \quad \Phi'_{13}(v_m) = \begin{bmatrix}\tilde{\Lambda} & -T_1 \\ \hat{\Lambda} & \Pi(v_m)\end{bmatrix},$$

$$F_2' = \begin{bmatrix} 0 & 0_{2\times8} & 0_{2\times8} & 0_{2\times8} & I \\ \Gamma_1 & -I_{8\times8} & 0_{8\times8} & 0_{8\times8} & 0_{8\times2} \\ \Gamma_3 & 0_{8\times8} & -I_{8\times8} & 0_{8\times8} & 0_{8\times2}\end{bmatrix},$$

$$\Xi = \begin{bmatrix}\Xi^{11} & \Xi^{12} \\ * & \Xi^{22}\end{bmatrix},$$

$$\Phi'_{22} = -192diag\{\tau_{11}^{-2}R_{11}, \cdots, \tau_{22}^{-2}R_{22}\},$$

$$\Phi'_{23} = \begin{bmatrix}\Theta' & 0_{8\times2}\end{bmatrix},$$

$$\Phi'_{33} = -720diag\{\tau_{11}^{-4}R_{11}, \cdots, \tau_{22}^{-4}R_{22}, -\mu/720\},$$

$$\Theta' = 360diag\{\tau_{11}^{-3}R_{11}, \cdots, \tau_{22}^{-3}R_{22}\}.$$

When the time delays are varying, the system we concern is as follows:

$$\begin{cases}\dot{x}(t) = x(t)[b_1 - a_{11}x(t - \tau_{11}(t)) - a_{12}y(t - \tau_{12}(t))], \\ \dot{y}(t) = y(t)[b_2 - a_{21}x(t - \tau_{21}(t)) - a_{22}y(t - \tau_{22}(t))],\end{cases} \tag{22}$$

where $0 \leq \tau_{ij}(t) \leq d_{ij}, \dot{\tau}_{ij}(t) \leq h_{ij}$ with $d_{ij} > 0, h_{ij} \geq 0 (i, j = 1, 2)$.

The initial condition of system (22) is given as

$$\begin{cases}x(s) = \phi'_1(s), -\tau \leq s \leq 0, \phi'_1(s) > 0, \\ y(s) = \phi'_2(s), -\tau \leq s \leq 0, \phi'_2(s) > 0,\end{cases}$$

where $\tau = \max_{1\leq i,j\leq 2}\{\tau_{ij}(t)\}$, and $\phi_1'(s), \phi_2'(s)$ are assumed to be continuous.

Similar to the previous operation, we can rewrite (22) in the following matrix form:

$$\dot{e}(t) = -\sum_{i,j=1}^{2}[A_{ij} + B_{ij}(e)]e(t - \tau_{ij}(t)), \tag{23}$$

denote $\tilde{\alpha}_{ij}(t) \triangleq \int_{t-\tau_{ij}(t)}^{t} e(s)ds$, then we can obtain the following Corollary 4.1.

Corollary 4.1: *Let the scalars $\bar{e}_1 > 0, \bar{e}_2 > 0$ and $d_{ij} > 0, \tau_{ij} \geq 0, i,j = 1,2$ be given. The system (23) is locally asymptotically stable, if there exist 10×10-dimensional symmetric matrix $\tilde{P}$, and 2×2-dimensional matrices $\tilde{Q}_{ij} > 0$, $\tilde{R}_{ij} > 0$, $i,j = 1,2$, $\tilde{Z} > 0$, $\tilde{T}_1$, $\tilde{T}_2$, such that the following LMIs hold:*

$$\begin{bmatrix} \tilde{\Phi}_{11}(v_m) & \tilde{\Phi}_{12}(v_m) \\ * & \tilde{\Phi}_{22} \end{bmatrix} + sym(\tilde{F}_1^T \tilde{P} \tilde{F}_2) < 0, \ m = 1,2,3,4, \tag{24}$$

$$\tilde{P} + \begin{bmatrix} \tilde{\bar{\Sigma}} & \tilde{\vartheta} \\ * & \tilde{\Xi} \end{bmatrix} \geq 0, \tag{25}$$

$$h_1^T h_1 \leq \bar{e}_1^2 \tilde{Z}, \ h_2^T h_2 \leq \bar{e}_2^2 \tilde{Z}, \tag{26}$$

where

$$\tilde{\Phi}_{11}(v_m) = \begin{bmatrix} \tilde{\gamma}_1 & \hat{R}_{11}(v_m) & \hat{R}_{12}(v_m) & \hat{R}_{21}(v_m) & \hat{R}_{22}(v_m) \\ * & \tilde{\gamma}_{11} & 0 & 0 & 0 \\ * & * & \tilde{\gamma}_{12} & 0 & 0 \\ * & * & * & \tilde{\gamma}_{21} & 0 \\ * & * & * & * & \tilde{\gamma}_{22} \end{bmatrix},$$

$$\tilde{\Phi}_{12}(v_m) = \begin{bmatrix} \widehat{R} & -T_1 \\ \breve{R} & \Pi(v_m) \end{bmatrix},$$

$$\tilde{\Phi}_{22} = -12diag\{d_{11}^{-2}\tilde{R}_{11}, d_{12}^{-2}\tilde{R}_{12}, d_{21}^{-2}\tilde{R}_{21}, \times d_{22}^{-2}\tilde{R}_{22}, -\tilde{\mu}/12\},$$

$$\tilde{\mu} = -T_2 - T_2^T + \sum_{i,j=1}^{2} d_{ij}^2 \tilde{R}_{ij}, \quad \tilde{\gamma}_1 = \sum_{i,j=1}^{2} (\tilde{Q}_{ij} - 4\tilde{R}_{ij}),$$

$$\hat{R}_{ij}(v_m) = -2\tilde{R}_{ij} - T_1[A_{ij} + B_{ij}(v_m)],$$

$$\tilde{\gamma}_{ij} = (h_{ij} - 1)\tilde{Q}_{ij} - 4\tilde{R}_{ij},$$

$$\widehat{R} = 6\begin{bmatrix} d_{11}^{-1}\tilde{R}_{11} & d_{12}^{-1}\tilde{R}_{12} & d_{21}^{-1}\tilde{R}_{21} & d_{22}^{-1}\tilde{R}_{22} \end{bmatrix},$$

$$\breve{R} = 6diag\begin{bmatrix} d_{11}^{-1}\tilde{R}_{11} & d_{12}^{-1}\tilde{R}_{12} & d_{21}^{-1}\tilde{R}_{21} & d_{22}^{-1}\tilde{R}_{22} \end{bmatrix},$$

$$\tilde{F}_1 = \begin{bmatrix} I & 0_{2\times 8} & 0_{2\times 8} & 0 \\ 0_{8\times 2} & 0_{8\times 8} & I_{8\times 8} & 0_{8\times 2} \end{bmatrix},$$

$$\tilde{F}_2 = \begin{bmatrix} 0 & 0_{2\times 8} & 0_{2\times 8} & I \\ \Gamma_1 & \Gamma_5 & 0_{8\times 8} & 0_{8\times 2} \end{bmatrix},$$

$$\Gamma_5 = diag\{(h_{11}-1)I, (h_{12}-1)I, \times (h_{21}-1)I, (h_{22}-1)I\},$$

$$\tilde{\bar{\Sigma}} = -\tilde{Z} + \sum_{i,j=1}^{2} 2d_{ij}\tilde{R}_{ij},$$

$$\tilde{\vartheta} = -2\begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & \tilde{R}_{21} & \tilde{R}_{22} \end{bmatrix},$$

$$\tilde{\Xi} = diag\{d_{11}^{-1}(\tilde{Q}_{11} + 2\tilde{R}_{11}), d_{12}^{-1}(\tilde{Q}_{12} + 2\tilde{R}_{12}), \times d_{21}^{-1}(\tilde{Q}_{21} + 2\tilde{R}_{21}), d_{22}^{-1}(\tilde{Q}_{22} + 2\tilde{R}_{22})\},$$

Proof: Choose the following augmented L-K functional:

$$V(t) = \tilde{\eta}^T(t)\tilde{P}\tilde{\eta}(t) + \sum_{i,j=1}^{2} \int_{t-\tau_{ij}(t)}^{t} e^T(s)\tilde{Q}_{ij}e(s)ds + \sum_{i,j=1}^{2} d_{ij} \int_{-\tau_{ij}(t)}^{0} \int_{t+\theta}^{t} \dot{e}^T(s)\tilde{R}_{ij}\dot{e}(s)dsd\theta, \tag{27}$$

with

$$\tilde{\eta}(t) = \begin{bmatrix} e^T(t) & \tilde{\alpha}_{11}^T(t) & \tilde{\alpha}_{12}^T(t) & \tilde{\alpha}_{21}^T(t) & \tilde{\alpha}_{22}^T(t) \end{bmatrix}^T.$$

■

Through some direct calculations, one can obtain that

$$\dot{V}(t) \leq 2\tilde{\eta}^T(t)\tilde{P}\dot{\tilde{\eta}}(t) + e^T(t)\sum_{i,j=1}^{2}\tilde{Q}_{ij}e(t) - \sum_{i,j=1}^{2}(1-h_{ij})e^T(t-\tau_{ij}(t))\tilde{Q}_{ij}e(t-\tau_{ij}(t)) + \dot{e}^T(t)\sum_{i,j=1}^{2} d_{ij}^2\tilde{R}_{ij}\dot{e}(t) - \sum_{i,j=1}^{2} d_{ij}\int_{t-\tau_{ij}(t)}^{t}\dot{e}^T(s)\tilde{R}_{ij}\dot{e}(s)ds,$$

Using the inequality (1) of Lemma 3.1, it is seen that

$$-d_{ij}\int_{t-\tau_{ij}(t)}^{t}\dot{e}^T(s)R_{ij}\dot{e}(s)ds \leq -\begin{bmatrix} \xi_{ij}^{(1)}(t) \\ \xi_{ij}^{(2)}(t) \end{bmatrix}^T \begin{bmatrix} \tilde{R}_{ij} & 0 \\ 0 & 3\tilde{R}_{ij} \end{bmatrix} \begin{bmatrix} \xi_{ij}^{(1)}(t) \\ \xi_{ij}^{(2)}(t) \end{bmatrix} \tag{28}$$

where

$$\xi_{ij}^{(1)}(t) = e(t) - e(t-\tau_{ij}(t)),$$

$$\xi_{ij}^{(2)}(t) = e(t) + e(t-\tau_{ij}(t)) - 2d_{ij}^{-1}\tilde{\alpha}_{ij}(t),$$

For any matrix $T_1, T_2 \in \mathbb{R}^{2\times 2}$, it follows that the following equation is true:

$$2[e^T(t)T_1 + \dot{e}^T(t)T_2] \times \left\{-\sum_{i,j=1}^{2}[A_{ij} + B_{ij}(e)]e(t-\tau_{ij}(t)) - \dot{e}(t)\right\} = 0. \tag{29}$$

Adding the left side of (29) to $\dot{V}(t)$ and using (28) yields

$$\dot{V}(t) \leq \tilde{\zeta}^T(t)[\tilde{\Phi}(e) + sym(\tilde{F}_1^T\tilde{P}\tilde{F}_2)]\tilde{\zeta}(t) \tag{30}$$

where

$$\tilde{\zeta}(t) = \left[e^T(t) \quad e^T(t-\tau_{11}(t)) \quad \cdots \quad e^T(t-\tau_{22}(t)) \right.$$
$$\left. [\tilde{\alpha}_{11}(t) \quad \cdots \quad \tilde{\alpha}_{22}(t)\, \dot{e}^T(t)\right]^T$$

$$\tilde{\Phi}(e) = \begin{bmatrix} \tilde{\Phi}_{11}(e) & \tilde{\Phi}_{12}(e) \\ * & \tilde{\Phi}_{22} \end{bmatrix}, \quad \tilde{\Phi}_{12}(e) = \begin{bmatrix} \widehat{R} & -T_1 \\ \breve{R} & \Pi(e) \end{bmatrix}$$

$$\tilde{\Phi}_{11}(e) = \begin{bmatrix} \tilde{\gamma}_1 & \hat{R}_{11}(e) & \hat{R}_{12}(e) & \hat{R}_{21}(e) & \hat{R}_{22}(e) \\ * & \tilde{\gamma}_{11} & 0 & 0 & 0 \\ * & * & \tilde{\gamma}_{12} & 0 & 0 \\ * & * & * & \tilde{\gamma}_{21} & 0 \\ * & * & * & * & \tilde{\gamma}_{22} \end{bmatrix},$$

with

$$\hat{R}_{ij}(e) = -2\tilde{R}_{ij} - T_1[A_{ij} + B_{ij}(e)],$$

Note that $\tilde{\Phi}(e)$ is affine with respect to the states $e_1(t)$ and $e_2(t)$. Hence, if the LMIs in (24) are satisfied, the inequality $\tilde{\Phi}(e) < 0$ can be ensured on χ. Then, we have

$$\dot{V}(t) < 0 \tag{31}$$

on the square χ, which implies that

$$V(t) \le V(0),\ t \ge 0. \tag{32}$$

On the other hand, using the integral inequalities (1) and (2) in Lemma 3.2, one can obtain the inequality

$$\begin{aligned} V(t) \ge& \tilde{\eta}^T(t)\tilde{P}\tilde{\eta}(t) + \sum_{i,j=1}^{2} d_{ij}^{-1}\{\tilde{\alpha}_{ij}^T(t)\tilde{Q}_{ij}\tilde{\alpha}_{ij}(t) + \\ &+ 2d_{ij}^2\tau_{ij}^{-2}(t)[\tau_{ij}(t)e(t) - \tilde{\alpha}_{ij}(t)]^T\tilde{R}_{ij} \\ &\times [\tau_{ij}(t)e(t) - \tilde{\alpha}_{ij}(t)]\} \\ =& \tilde{\eta}^T(t)\left(\tilde{P} + \begin{bmatrix} \tilde{\Sigma} & \tilde{\vartheta} \\ * & \tilde{\Xi} \end{bmatrix}\right)\tilde{\eta}(t) \end{aligned} \tag{33}$$

where $\tilde{\Sigma} = \sum_{i,j=1}^{2} 2d_{ij}\tilde{R}_{ij}$.

Using (25) and noting (33), it follows that

$$V(t) \ge e^T(t)\tilde{Z}e(t), \tag{34}$$

which shows that the L-K functional $V(t)$ is positive definite.

In addition, it can be inferred from (26) that

$$\varepsilon(\tilde{Z}, 1) \subseteq \chi. \tag{35}$$

For all initial state $e(s) \in X_\rho$ satisfying $V(0) \le 1$, it is seen from (32) and (34) that all trajectories $e(t)$ are contained in the set $\varepsilon(\tilde{Z}, 1)$. Noting (31) and (35), it is concluded that the system (22) is asymptotically stable for all $e(s)$ satisfying $V(0) \le 1$. This completes the proof.

Remark 4.2: Because of the difficulties in tackling the varying time-delays, in above proof, we use some simple inequalities to obtain the results. In (28), we use (1) in Lemma in place of (2) in lemma 3.1. In (33), we use Jensen inequalities in place of Wirtinger inequalities.

Now, we will consider the estimate of RA of the positive equilibrium the region of asymptotic. In this end, we introduce the following inequalities:

$$\begin{cases} P \le diag\{\Lambda_0, \Lambda_{11}, \cdots, \Lambda_{22}, \Lambda_{11}^{12}, \cdots, \Lambda_{21}^{22}, \bar{\Lambda}_{11}, \\ \cdots, \bar{\Lambda}_{22}, \bar{\Lambda}_{11}^{12}, \cdots, \bar{\Lambda}_{21}^{22}\}, \\ \Lambda_0 \le p_0 I,\ \Lambda_{ij} \le p_{ij}I,\ \bar{\Lambda}_{ij} \le \bar{p}_{ij}I,\ Q_{ij} \\ \quad \le q_{ij}I,\ R_{ij} \le r_{ij}I,\ i,j = 1,2, \\ \Lambda_{ij}^{kl} \le p_{ij}^{kl}I,\ \bar{\Lambda}_{ij}^{kl} \le \bar{p}_{ij}^{kl}I,\ Q_{ij}^{kl} \le q_{ij}^{kl}I,\ R_{ij}^{kl} \\ \quad \le r_{ij}^{kl}I,\ i,j,k,l = 1,2 \text{ and } 10i + j < 10k + l. \end{cases} \tag{36}$$

where

$$\Lambda_0 > 0, \Lambda_{ij} > 0, \Lambda_{ij}^{kl} > 0,\ \bar{\Lambda}_{ij} > 0,$$
$$\bar{\Lambda}_{ij}^{kl} > 0, p_0 > 0, p_{ij} > 0, p_{ij}^{kl} > 0, \bar{p}_{ij}$$
$$> 0, \bar{p}_{ij}^{kl} > 0, q_{ij} > 0, r_{ij} > 0, q_{ij}^{kl} > 0, r_{ij}^{kl} > 0.$$

Using (36) and the Jensen inequalities, one can obtain

$$\begin{aligned} V(0) &\le \lambda_1 \max_{s\in[-\tau,0]} ||e(s)||^2 + \lambda_2 \max_{s\in[-\tau,0]} ||\dot{e}(s)||^2 \\ &\le \tilde{\lambda}_1 \max_{s\in[-\tau,0]} ||e(s)||^2 + \tilde{\lambda}_2 \max_{s\in[-\tau,0]} ||\dot{e}(s)||^2 \end{aligned} \tag{37}$$

where

$$\begin{aligned} \lambda_1 =& \lambda_M(\Lambda_0) + \sum_{i,j=1}^{2} \tau_{ij}[\tau_{ij}\lambda_M(\Lambda_{ij}) \\ &+ (\tau_{ij}^3/4)\lambda_M(\bar{\Lambda}_{ij}) + \lambda_M(Q_{ij})] + \\ &\sum_{i,j=1}^{2}\sum_{k,l=1}^{2} \tau_{ijkl}^2[\lambda_M(\Lambda_{ij}^{kl}) \\ &+ ((\tau_{ij} + \tau_{kl})^2/4)\lambda_M(\bar{\Lambda}_{ij}^{kl}) + \lambda_M(Q_{ij}^{kl})], \end{aligned}$$

$$\begin{aligned} \lambda_2 =& \sum_{i,j=1}^{2} (\tau_{ij}^3/2)\lambda_M(R_{ij}) \\ &+ \sum_{i,j=1}^{2}\sum_{k,l=1}^{2} [(\tau_{ijkl}^2(\tau_{ij} + \tau_{kl})/2)\lambda_M(R_{ij}^{kl})], \end{aligned}$$

$$\begin{aligned} \tilde{\lambda}_1 =& p_0 + \sum_{i,j=1}^{2} \tau_{ij}[\tau_{ij}p_{ij} + 0.25\tau_{ij}^3\bar{p}_{ij} + q_{ij}] \\ &+ \sum_{i,j=1}^{2}\sum_{k,l=1}^{2} \tau_{ijkl}^2[p_{ij}^{kl} + 0.25(\tau_{ij} + \tau_{kl})^2\bar{p}_{ij}^{kl} + q_{ij}^{kl}] \end{aligned}$$

$$\tilde{\lambda}_2 = \sum_{i,j=1}^{2} 0.5\tau_{ij}^3 r_{ij} + \sum_{i,j=1}^{2}\sum_{k,l=1}^{2} 0.5[\tau_{ijkl}^2(\tau_{ij} + \tau_{kl})r_{ij}^{kl}].$$

Then, the maximization of the estimate of the region of asymptotic stability in Theorem 4.1 can be formulated as

$$\text{OP. 1.} \quad \min_{P,Q_{ij},R_{ij},Q_{ij}^{kl},R_{ij}^{kl},Z,\Lambda_0,\Lambda_{ij},\Lambda_{ij}^{kl},\bar{\Lambda}_{ij},\bar{\Lambda}_{ij}^{kl},p_0,p_{ij},p_{ij}^{kl},\bar{p}_{ij},\bar{p}_{ij}^{kl},q_{ij},r_{ij},q_{ij}^{kl},r_{ij}^{kl},} \tilde{\lambda}_1 + \tilde{\lambda}_2,$$

$$s.t. \qquad \text{LMIs (8)} - \text{(10) and (36) hold.}$$

By solving the optimization problem OP.1, one can obtain the scalars λ_1 and λ_2. Recalling that all initial condition $e(s)$ satisfy $V(0) \leq 1$, from (37), we can require that the initial condition $e(s)$ satisfy the following relationship:

$$\lambda_1 \max_{s\in[-\tau,0]} ||\phi(s)||^2 + \lambda_2 \max_{s\in[-\tau,0]} ||\dot{\phi}(s)||^2 \leq 1. \qquad (38)$$

5. Numerical simulations

In this section, we will consider the numerical example given in Zhen and Ma (2002), Park (2005), Sun and Meng (2007) and Dong et al. (2019) to illustrate the effectiveness and the sharpness of our results. In Zhen and Ma (2002), Park (2005) and Sun and Meng (2007), $b_1 = b_2 = a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0.5$. It is easy to verify that the system (1) has one positive equilibrium point $(2/3,\ 2/3)$.

When $\tau_{ij} = \tau$, the upper bounds τ for local asymptotic stability obtained by the existing results and the Theorem 4.1 in this paper are listed in Table 1.

Table 1 shows that the result in this paper can propose larger delay bound than those in Zhen and Ma (2002), Park (2005), Sun and Meng (2007) and Dong et al. (2019).

For the equilibrium point (0, 0) of the system (4), by solving the optimization problem OP.1 with $\tau_{ij} = 0.6\ (i,j = 1,2)$, $\bar{e}_1 = 0.66$, $\bar{e}_2 = 0.57$, we can obtain$\lambda_1 = 5.669$, $\lambda_2 = 0.8017$. Then, we obtain the local stability condition of system (4) is the initial function $e(s)$ should satisfy the following relationship as shown in Figure 1:

$$5.669||e(s)||^2 + 0.8017||\dot{e}(s)||^2 \leq 1. \qquad (39)$$

For example, if $e(s)$ is constant, then we estimate the RA of the equilibrium point $(0,\ 0)$ of system (4) should satisfy

$$\phi_1^2(s) + \phi_2^2(s) \leq 0.176. \qquad (40)$$

Choosing the initial condition $(\phi_1{}'(s), \phi_2{}'(s))^T = (0.5\cos t, 0.4)^T$, we obtain the state responses of system (1) as Figure 2.

Table 1. Maximum admissible ranges of the time delay τ.

Zhen and Ma (2002)	Park (2005)	Sun and Meng (2007)	Dong et al. (2019)	Theorem 4.1
$0 < \tau < 0.34$	$0 < \tau < 0.88$	$0 < \tau < 1.4307$	$0 < \tau < 1.5674$	$0 < \tau < 1.5804$

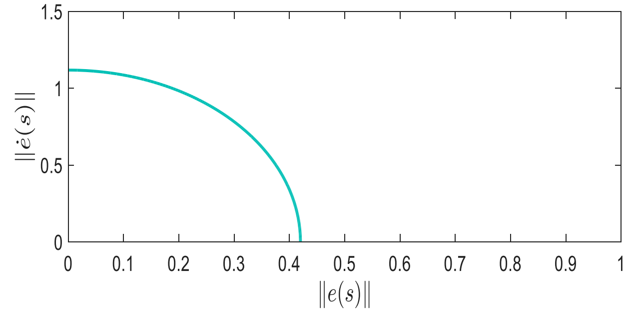

Figure 1. . The boundary of RA of the L-V system (4).

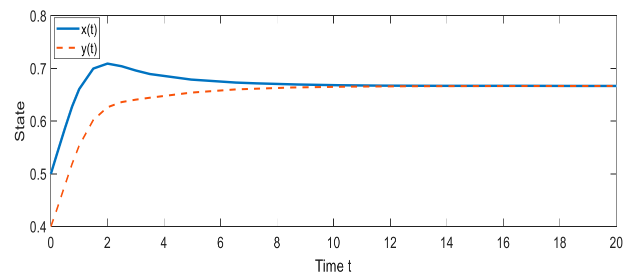

Figure 2. . The trajectories with initial states in the RA.

6. Conclusions

In this paper, local stability analyses and the estimate of the RA of the L-V competitive system with time-delays are conducted firstly based on quadratic system theory. Some less conservative stability conditions have been obtained by constructing an appropriate L-K functional and choosing less conservative inequalities. On the other hand, the boundary of the RA of the system is plotted and maximized by solving the optimization problem OP.1 in the framework of LMIs. Finally, simulation results have been given to demonstrate the effectiveness of the obtained conditions. However, wirtinger-based inequalities are not the most effective, and simulations show that the boundary of the RA estimation in this paper is conservative. This is our future research.

Disclosure statement

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References

Ahmad, S., & Stamova, I. M. (2015). Partial persistence and extinction in N-dimensional competitive systems. *Nonlinear Analysis: Theory, Methods & Applications*, *60*(5), 821–836. https://doi.org/10.1016/j.na.2004.04.013

Caldeira, A. F., Prieur, C., Coutinho, D., & Leite, V. J. S. (2018). Regional stability and stabilization of a class of linear hyperbolic systems with nonlinear quadratic dynamic boundary conditions. *European Journal of Control*, *43*(September), 46–56. https://doi.org/10.1016/j.ejcon.2018.05.003

Chen, F. D., Ma, Z. Z., & Zhang, H. Y. (2012). Global asymptotical stability of the positive equilibrium of the Lotka-Volterra prey-predator model incorporating a constant number of prey refuges. *Nonlinear Analysis-Real World Applications*, *13*(6), 2790–2793. https://doi.org/10.1016/j.nonrwa.2012.04.006

Chen, F., Xu, S., Zou, Y., & Zhang, M. (2013). Finite-time boundedness and stabilization for a class of non-linear quadratic time-delay systems with disturbances. *Control Theory & Applications Iet*, *7*(13), 1683–1688. https://doi.org/10.1049/iet-cta.2012.0631

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2005). LMI-based computation of optimal quadratic Lyapunov functions for odd polynomial systems. *International Journal of Nonlinear and Robust Control*, *15*(1), 35–49. https://doi.org/10.1002/rnc.967

Chiang, H. D., Hirsch, M. W., & Wu, F. F. (1988). Stability regions of nonlinear autonomous dynamical systems. *IEEE Transactions on Automatic Control*, *33*(1), 16–27. https://doi.org/10.1109/9.357

Dong, R., Chen, L., & Chen, Y. (2019). Improved LMI-based stability conditions for a competitive Lotka–Volterra system with time-varying delays. *Journal of Inequalities & Applications*.2019(1),130. https://doi.org/10.1186/s13660-019-2072-0

Genesio, R., & Vicino, A. (1984). Some results on the asymptotic stability of second-order nonlinear systems. *IEEE Transaction on Automatic Control*, *29*(9), 857–861. https://doi.org/10.1109/TAC.1984.1103656

Jiang, J., & Liang, F. (2020). Global dynamics of 3d competitive Lotka-Volterra equations with the identical intrinsic growth rate. *Journal of Differential Equations*, *268*(6), 2551–2586. https://doi.org/10.1016/j.jde.2019.09.039

Kuang, Y. (1996). Global stability in delayed nonautonomous Lotka-Volterra type systems without saturated equilibria. *Differential and Integral Equations*, *9*(3), 557–567.

Lai, Y. H., & Fang, L. (2020). Estimating the boundary of the asymptotic stability region of Lotka-Volterra system by using the trajectory reversing method. *IOP Conference Series Earth and Environmental Ence*, *431*(1), 012047. https://doi.org/10.1088/1755-1315/431/1/012047

Li-Ya, G., Jian, X., & Yi-Cheng, W. (2019). On almost periodic solutions and periodic solutions to a class of quadratic differential systems. *Journal of Biomathematics*, 34(1), 14-26.

Lotka, A. J. (1956). *Elements of mathematical biology*. Dover.

Ma, Z., Yuan, R., Wang, Y., & Wu, X. (2019). Multidimensional stability of planar traveling waves for the delayed nonlocal dispersal competitive Lotka-Volterra system. *Communications on Pure & Applied Analysis*, *18*(4), 2069–2092. https://doi.org/10.3934/cpaa.2019093

Merola, A., Cosentino, C., Colacino, D., & Amato, F. (2017). Optimal control of uncertain nonlinear quadratic systems. *Automatica*, *83*(9), 345–350. https://doi.org/10.1016/j.automatica.2017.05.012

Park, J. H. (2005). Stability for a competitive Lotka–Volterra system with delays: LMI optimization approach. *Applied Mathematics Letters*, *18*(6), 689–694. https://doi.org/10.1016/j.aml.2004.04.014

Park, P. G., Lee, W. I., & Lee, S. Y. (2015). Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems. *Journal of the Franklin Institute*, *352*(4), 1378–1396. https://doi.org/10.1016/j.jfranklin.2015.01.004

Qian, W., Li, Y., Chen, Y., & Liu, W. (2020). L2-L_infinity filtering for stochastic delayed systems with randomly occurring nonlinearities and sensor saturation. *International Journal of Systems Science*. https://doi.org/10.1080/00207721.2020.1794080

Qian, W., Li, Y., Zhao, Y., & Chen, Y. (2020). New optimal method for L2-L_infinity state estimation of delayed neural networks. *Neurocomputing*, *415*, 258–265. https://doi.org/10.1016/j.neucom.2020.06.118

Qian, W., Xing, W., & Fei, S. (2020). H_infinity state estimation for neural networks with general activation function and mixed time-varying delays. *IEEE Transactions on Neural Networks and Learning Systems*. https://doi.org/10.1109/TNNLS.2020.3016120

Sun, Y. G., & Meng, F. W. (2007). LMI approach to stability for a competitive Lotka-Volterra system with time-varying delays. *Applied Mathematics & Computation*, *194*(2), 291–297. https://doi.org/10.1016/j.amc.2007.04.024

Teng, Z. D., & Yu, Y. H. (2000). Some new results of nonautonomous Lotka-Volterra competitive systems with delays. *Mathematical Analysis and Applications*, *241*(2), 254–275. https://doi.org/10.1006/jmaa.1999.6643

Tesi, A., Villoresi, F., & Genesio, R. (1996). On the stability domain estimation via a quadratic Lyapunov function: Convexity and optimality properties for polynomial systems. *IEEE Transactions on Automatic Control*, *41*(11), 1650–1657. https://doi.org/10.1109/9.544002

Volterra, V. (1931). *Lecons sur la theorie mathematique de la lutte pour la vie*. Gauthier-Villars.

Wang, W., Chen, L., & Lu, Z. (1995). Global stability of a competition model with periodic coefficients and time delay. *Canadian Applied Mathematics Quarterly*, *3*(3), 365–373.

Xu, E., Zhang, Y., & Chen, Y. (2020). Time-delayed local feedback control for a chaotic finance system. *Journal of Inequalities and Applications*, *2020*(1), 100. https://doi.org/10.1186/s13660-020-02364-2

Zhao, J. D., Zhang, Z. C., & Ju, J. (2014). Necessary and sufficient conditions for permanence and extinction in a three-dimensional competitive Lotka-Volterra system. *Applied Mathematics and Computation*, *230*, 587–596. https://doi.org/10.1016/j.amc.2013.12.140

Zhen, J., & Ma, Z. (2002). Stability for a competitive Lotka–Volterra system with delays. *Nonlinear Analysis Theory Methods & Applications*, *51*(7), 1131–1142. https://doi.org/10.1016/S0362-546X(01)00881-1

Zhu, Y., & Liu, M. (2017). Permanence and extinction in a stochastic service-resource mutualism model. *Applied Mathematics Letters*, *69*, 1–7. doi:10.1016/j.aml.2017.01.020