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# $H_\infty$ control for a hyperchaotic finance system with external disturbance based on the quadratic system theory

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## ABSTRACT

This paper considers the  $H_\infty$  control problem for a hyperchaotic system with energy-bounded disturbance under the delayed feedback controller. Using the quadratic system theory, an augmented Lyapunov functional, some integral inequalities and rigorous mathematical derivations, a sufficient condition is first established by using linear matrix inequalities under which the closed-loop system can achieve some desirable performances including the boundedness, the  $H_\infty$  performance and the asymptotic stability. Subsequently, several convex optimization problems are formulated to obtain the optimal performance indices. Finally, numerical simulations are presented to illustrate the effectiveness of the obtained results.

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$H_\infty$  control; hyperchaotic finance system; external disturbance; delayed feedback controller; quadratic system theory

## 1. Introduction

The economic/finance systems are the special nonlinear systems (Cesare & Sportelli, 2005; Chen & Chen, 2007; Chen & Ma, 2001a, 2001b; Salarieh & Alasty, 2008). It has been recognized that the chaos often occurs in such nonlinear systems (Chen & Chen, 2007; Chen & Ma, 2001a; Salarieh & Alasty, 2008). In fact, the financial crisis is essentially a kind of chaotic behaviour. In the past more than two decades, the economic/finance systems have received wide research attention and the main topics are the dynamic analysis, the feedback control and the synchronization (Chen, 2008; Chen, Liu et al., 2014; Dadras & Momeni, 2010; Jahanshahi et al., 2019; Son & Park, 2011; Tacha et al., 2016; Wang et al., 2012; Xu et al., 2020; Zhao et al., 2011). For example, in Chen (2008), the complex dynamics and the chaos control have been investigated for a finance system under time-delayed feedbacks by numerical simulations, and in Tacha et al. (2016), the problems of dynamic analysis and adaptive control have been addressed for a modified finance system. In Zhao et al. (2011), several control strategies have been employed to consider the synchronization problem for the chaotic finance system. In particular, in our recent work (Xu et al., 2020), the quadratic system theory has been utilized to control the chaos of the finance dynamics under the time-delayed feedback controller.

On the other hand, the finance systems might inevitably be affected by external disturbances stemmed from

environmental interference (Zhao & Wang, 2014). If the external disturbance is ignored in designing the controller, the resultant closed-loop system might have poor performance. In Zhao and Wang (2014), the delayed feedback controller has been designed for a chaotic finance system subject to external disturbance such that the closed-loop system is asymptotically stable with a prescribed  $H_\infty$  performance level. In Xu et al. (2018), the finite-time  $H_\infty$  control problem has been considered for a disturbed chaotic finance system by using the delayed feedback controller. In Harshavarthini et al. (2020), the finite-time resilient fault-tolerant control problem has been studied for a nonlinear finance system.

However, it is worth mentioning that most above references are mainly concerned with the finance models composed of three first-order or fractional-order differential equations. In Yu et al. (2012), by adding an additional state to the model in Chen and Ma (2001a) to represent the average profit margin, a more reasonable finance model has been proposed. It has been shown that such a four-dimensional system displays the more complex hyperchaotic behaviour. In the past several years, the synchronization and control problems have also attracted considerable research attention for various hyperchaotic finance systems (Cai et al., 2012; Hajipour et al., 2018; Vargas et al., 2015; Zheng, 2016). For example, the adaptive algorithm has been proposed in Vargas et al. (2015) to address the synchronization problem

for a hyperchaotic system with unknown parameters. In Zheng (2016), the impulsive control scheme has been utilized to study the stabilization and synchronization of an uncertain hyperchaotic finance system. Nevertheless, it should be pointed out that the external disturbances are not sufficiently incorporated in the considered hyperchaotic finance systems. Moreover, it is observed that the time-delay phenomenon has been ignored in controlling the hyperchaotic finance systems.

Motivated by the above discussions, in this paper, we will be concerned with the  $H_\infty$  control problem for a hyperchaotic system with energy-bounded disturbance via the delayed feedback controller. Using the quadratic system theory (Amato et al., 2007), the augmented Lyapunov functional and some integral inequalities, a sufficient condition is first proposed in the framework of linear matrix inequalities (LMIs) under which the closed-loop dynamics can achieve some desirable performances. Then, several optimization problems are given to handle the different performance requirements. Finally, simulations results are given to illustrate the effectiveness of the obtained results. *The main contributions of this work are as follows: (1) the  $H_\infty$  control problem is addressed, for the first time, for a hyperchaotic finance system under the delayed feedback controller and an LMI-based sufficient condition is established; (2) the quadratic system theory is utilized to investigate a hyperchaotic finance system based on which the performances of the closed-loop dynamics are specifically characterized.*

*Notation.* “ $T$ ” denotes the transpose of a matrix.  $\mathbb{R}^n$  is the  $n$ -dimensional Euclidean space. The real matrix  $P > 0$  ( $P \geq 0$ ) denotes that  $P$  is symmetric and positive definite (semi-definite).  $\|\cdot\|$  is the 2-norm of a vector.  $\lambda(\cdot)_M$  is the maximum eigenvalue value of a matrix.  $I$  is an identity matrix. The symmetric terms in a symmetric matrix are denoted by  $*$ . Matrices are assumed to have compatible dimensions.

## 2. Problem formulation

In Chen and Ma (2001a, 2001b), a chaotic finance system is proposed. Such a finance model contains four sub-blocks (i.e. production, money, stock and labour force) and is formulated by the following three first-order differential equations:

$$\begin{cases} \dot{x}_1(t) = x_3(t) + (x_2(t) - a)x_1(t), \\ \dot{x}_2(t) = 1 - bx_2(t) - x_1^2(t), \\ \dot{x}_3(t) = -x_1(t) - cx_3(t) \end{cases} \quad (1)$$

where the states  $x_1(t)$ ,  $x_2(t)$  and  $x_3(t)$  are, respectively, the interest rate, the investment demand and the price

index;  $a > 0$ ,  $b > 0$  and  $c > 0$  are, respectively, the saving amount, the cost per investment and the demand elasticity of commercial markets.

By adding an additional state in the model (1), a more reasonable finance model is proposed in Yu et al. (2012), which is described as follows:

$$\begin{cases} \dot{x}_1(t) = x_3(t) + (x_2(t) - a)x_1(t) + x_4(t), \\ \dot{x}_2(t) = 1 - bx_2(t) - x_1^2(t), \\ \dot{x}_3(t) = -x_1(t) - cx_3(t), \\ \dot{x}_4(t) = -dx_1(t)x_2(t) - ex_4(t) \end{cases} \quad (2)$$

where the state  $x_4(t)$  denotes the average profit margin, and  $d, e$  are positive scalars.

In Yu et al. (2012), it has been identified that the model (2) displays sophisticated hyperchaotic behaviour when the system parameters are selected as  $a = 0.9$ ,  $b = 0.2$ ,  $c = 1.5$ ,  $d = 0.2$  and  $e = 0.17$ . Moreover, it has been verified that, under the assumption  $(abce + be + cd - ce)/(cd - ce) \triangleq \Delta > 0$ , the model (2) has three equilibrium points

$$\left(0, \frac{1}{b}, 0, 0\right), \quad \left(\pm \sqrt{\Delta}, \frac{ace + e}{ce - cd}, \mp \frac{\sqrt{\Delta}}{c}, \frac{\sqrt{\Delta}(ac + 1)d}{cd - ce}\right). \quad (3)$$

Moreover, it has been recognized that the finance systems are unavoidable influenced by external disturbances (Jahanshahi et al., 2019; Zhao & Wang, 2014). Adding the disturbance  $\omega(t) \in \mathbb{R}^l$  and the control input  $u(t) \in \mathbb{R}^m$  to (2) yields that

$$\dot{x}(t) = Ax(t) + f(x(t)) + Bu(t) + D\omega(t) \quad (4)$$

where  $B$  and  $D$  are matrices, and  $x(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]^T$ ,

$$A = \begin{bmatrix} -a & 0 & 1 & 1 \\ 0 & -b & 0 & 0 \\ -1 & 0 & -c & 0 \\ 0 & 0 & 0 & -e \end{bmatrix},$$

$$f(x(t)) = \begin{bmatrix} x_1(t)x_2(t) \\ 1 - x_1^2(t) \\ 0 \\ -dx_1(t)x_2(t) \end{bmatrix}.$$

In this paper, the external disturbance  $\omega(t)$  is assumed to be energy-bounded and satisfies the condition  $\int_0^\infty \omega^T(t) \omega(t) dt \leq \beta$ , where  $\beta$  is a positive scalar.

**Remark 2.1:** The finance systems are inevitably disturbed by external environments, such as the plagues and the wars. For example, due to the impact of the 2019 novel coronavirus (2019-nCoV), the market confidence will be reduced and correspondingly, the lower investment demand and the lower interest rate will occur. In

this case, the impact of 2019-nCoV can be seen as the external disturbance and should be added to the finance systems to reflect the real finance dynamics. In addition, it is worth mentioning that the external disturbances might disappear within the finite time. Therefore, it is reasonable to suppose that the external disturbance is energy-bounded.

As in Zhao and Wang (2014), this paper adopts the delayed feedback controller

$$u(t) = K_1(x(t) - x^*) + K_2(x(t - \tau) - x^*) \quad (5)$$

where  $K_1, K_2$  are the controller gains,  $x^*$  is an unstable equilibrium point, and  $\tau > 0$  is the time delay.

For the given equilibrium point  $x^* = [x_1^* \ x_2^* \ x_3^* \ x_4^*]^T$ , it is clear that

$$Ax^* + f(x^*) = 0. \quad (6)$$

Denoting that  $r(t) \triangleq x(t) - x^*$ , and using (4)–(6), one has the closed-loop system

$$\dot{r}(t) = (A + F + BK_1)r(t) + BK_2r(t - \tau) + \tilde{f}(r(t)) + D\omega(t) \quad (7)$$

where

$$F = \begin{bmatrix} x_2^* & x_1^* & 0 & 0 \\ -2x_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -dx_2^* & -dx_1^* & 0 & 0 \end{bmatrix},$$

$$\tilde{f}(r(t)) = \begin{bmatrix} r_1(t)r_2(t) \\ -r_1^2(t) \\ 0 \\ -dr_1(t)r_2(t) \end{bmatrix}.$$

Note that the nonlinearity  $\tilde{f}(r)$  can be written as follows:

$$\tilde{f}(r) = \begin{bmatrix} r^T G_1 \\ r^T G_2 \\ r^T G_3 \\ r^T G_4 \end{bmatrix} r \triangleq G(r)r \quad (8)$$

where  $G_1 = \text{diag}\{\tilde{G}_1, 0, 0\}$ ,  $G_2 = \text{diag}\{-1, 0, 0, 0\}$ ,  $G_3 = 0_{4 \times 4}$  and  $G_4 = -dG_1$  with  $\tilde{G}_1 = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ . Using (8), the closed-loop dynamics (7) can be further written as

$$\dot{r}(t) = \mathcal{A}(r)r(t) + BK_2r(t - \tau) + D\omega(t) \quad (9)$$

where  $\mathcal{A}(r) = A + F + G(r) + BK_1$ .

The initial condition associated with (9) is denoted by  $r(s) = \phi(s), s \in [-\tau, 0)$ .

The main purpose of our paper is to design the delayed feedback controller (5) such that the closed-loop dynamics (9) has the following properties: (1) all state trajectories are bounded for all admissible initial conditions and

external disturbances; (2) the  $H_\infty$  performance requirement  $\int_0^{+\infty} r^T(s)r(s)ds < \gamma \int_0^{+\infty} \omega^T(s)\omega(s)ds + \gamma V(0)$  is satisfied, where  $\gamma > 0$  is a prespecified scalar and  $V(t)$  is an Lyapunov functional; (3) when  $\omega(t) = 0$ , the asymptotic stability is guaranteed for all admissible initial conditions.

For purpose of the subsequent local analysis, we introduce the following box:

$$\mathcal{R} = [-\bar{r}_1, \bar{r}_1] \times [-\bar{r}_2, \bar{r}_2] \times [-\bar{r}_3, \bar{r}_3] \times [-\bar{r}_4, \bar{r}_4] \quad (10)$$

where  $\bar{r}_j > 0 (j = 1, 2, 3, 4)$  are scalars. The above box can be represented as

$$\mathcal{R} = \text{Co}\{v_i, 1 \leq i \leq 16\}$$

$$= \{r : |h_j r| \leq \bar{r}_j^2, j = 1, 2, 3, 4\} \quad (11)$$

where “Co” denote the convex hull and

$$v_1 = [-\bar{r}_1 \ -\bar{r}_2 \ -\bar{r}_3 \ -\bar{r}_4]^T, \quad v_2 = [-\bar{r}_1 \ -\bar{r}_2 \ -\bar{r}_3 \ \bar{r}_4]^T,$$

$$v_3 = [-\bar{r}_1 \ -\bar{r}_2 \ \bar{r}_3 \ -\bar{r}_4]^T, \quad v_4 = [-\bar{r}_1 \ \bar{r}_2 \ -\bar{r}_3 \ -\bar{r}_4]^T,$$

$$v_5 = [\bar{r}_1 \ -\bar{r}_2 \ -\bar{r}_3 \ -\bar{r}_4]^T, \quad v_6 = [-\bar{r}_1 \ -\bar{r}_2 \ \bar{r}_3 \ \bar{r}_4]^T$$

$$v_7 = [-\bar{r}_1 \ \bar{r}_2 \ -\bar{r}_3 \ \bar{r}_4]^T, \quad v_8 = [\bar{r}_1 \ -\bar{r}_2 \ -\bar{r}_3 \ \bar{r}_4]^T,$$

$$v_9 = [-\bar{r}_1 \ \bar{r}_2 \ \bar{r}_3 \ -\bar{r}_4]^T, \quad v_{10} = [\bar{r}_1 \ -\bar{r}_2 \ \bar{r}_3 \ -\bar{r}_4]^T,$$

$$v_{11} = [\bar{r}_1 \ \bar{r}_2 \ -\bar{r}_3 \ -\bar{r}_4]^T, \quad v_{12} = [\bar{r}_1 \ \bar{r}_2 \ \bar{r}_3 \ -\bar{r}_4]^T,$$

$$v_{13} = [\bar{r}_1 \ \bar{r}_2 \ -\bar{r}_3 \ \bar{r}_4]^T, \quad v_{14} = [\bar{r}_1 \ -\bar{r}_2 \ \bar{r}_3 \ \bar{r}_4]^T,$$

$$v_{15} = [-\bar{r}_1 \ \bar{r}_2 \ \bar{r}_3 \ \bar{r}_4]^T, \quad v_{16} = [\bar{r}_1 \ \bar{r}_2 \ \bar{r}_3 \ \bar{r}_4]^T,$$

$$h_1 = [1 \ 0 \ 0 \ 0], \quad h_2 = [0 \ 1 \ 0 \ 0],$$

$$h_3 = [0 \ 0 \ 1 \ 0], \quad h_4 = [0 \ 0 \ 0 \ 1].$$

### 3. Main results

In this section, we will first establish the corresponding sufficient condition by using the following augmented Lyapunov functional (Seuret & Gouaisbaut, 2013):

$$V(t) = \eta^T(t)P\eta(t) + \int_{t-\tau}^t r^T(s)Qr(s)ds$$

$$+ \tau \int_{-\tau}^0 \int_{t+\theta}^t \dot{r}^T(s)Z\dot{r}(s)dsd\theta \quad (12)$$

where  $\eta(t) = [r^T(t) \ \int_{t-\tau}^t r^T(s)ds]^T$ ,  $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ , and  $Q > 0, Z > 0$ .

**Theorem 3.1:** Let the scalars  $\tau > 0, \bar{r}_j (j = 1, 2, 3, 4)$  and  $\epsilon \neq 0$  be given. Assume that there exist symmetric matrices  $\bar{P} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{12}^T & \bar{P}_{22} \end{bmatrix}, \bar{Q} > 0, \bar{Z} > 0, \bar{R} > 0$ , matrices  $X, Y_1, Y_2$ , and scalars  $\alpha > 0, \beta > 0 (\alpha < 1/\beta), \gamma$ , such that the following

LMI hold:

$$\begin{bmatrix} \Sigma_{11}^{v_i} & \Sigma_{12} & \Sigma_{13} & D & \Sigma_{15}^{v_i} & X \\ * & \Sigma_{22} & \Sigma_{23} & 0 & \Sigma_{25} & 0 \\ * & * & -12\bar{Z} & 0 & \tau\bar{P}_{12}^T & 0 \\ * & * & * & -I & \epsilon D^T & 0 \\ * & * & * & * & \Sigma_{55} & 0 \\ * & * & * & * & * & -\gamma I \end{bmatrix} < 0, \quad (13)$$

$$i = 1, 2, \dots, 16, \quad (14)$$

$$\begin{bmatrix} \Pi_{11} & \bar{P}_{12} - 2\bar{Z} \\ * & \Pi_{22} \end{bmatrix} \geq 0, \quad (15)$$

$$\begin{bmatrix} \alpha & h_j X^T \\ * & \bar{r}_j^2 \bar{R} \end{bmatrix} \geq 0 \quad (16)$$

where

$$\begin{aligned} \Sigma_{11}^{v_i} &= \text{Sym}\{[A + F + G(v_i)]X^T + BY_1\} \\ &\quad + \text{Sym}(\bar{P}_{12}) + \bar{Q} - 4\bar{Z}, \\ \Sigma_{12} &= -\bar{P}_{12} + BY_2 - 2\bar{Z}, \quad \Sigma_{13} = \tau\bar{P}_{22} + 6\bar{Z}, \\ \Sigma_{15}^{v_i} &= \bar{P}_{11} - X^T + \epsilon X[A + F + G(v_i)]^T + \epsilon Y_1^T B^T, \\ \Sigma_{22} &= -\bar{Q} - 4\bar{Z}, \quad \Sigma_{23} = -\tau\bar{P}_{22} + 6\bar{Z}, \\ \Sigma_{25} &= \epsilon Y_2^T B^T, \quad \Sigma_{55} = \tau^2\bar{Z} - \epsilon \text{Sym}(X), \\ \Pi_{11} &= \bar{P}_{11} + 2\tau\bar{Z} - \bar{R}, \quad \Pi_{22} = \bar{P}_{22} + \bar{Q}/\tau + 2\bar{Z}/\tau. \end{aligned}$$

Then, there exists the controller (5) with  $K_1 = Y_1 X^{-T}$  and  $K_2 = Y_2 X^{-T}$  such that: (1) all trajectories of the dynamics (9) are bounded for all  $\phi(s)$  satisfying  $V(0) \leq 1/\alpha - \beta$  and all non-zero  $\omega(t)$  satisfying  $\int_0^\infty \omega^T(t)\omega(t)dt \leq \beta$ ; (2)  $H_\infty$  performance constraint  $\int_0^{+\infty} r^T(s)r(s)ds < \gamma \int_0^{+\infty} \omega^T(s)\omega(s)ds + \gamma V(0)$  can be ensured; (3) when  $\omega(t) = 0$ , the asymptotic stability of the dynamics (9) is ensured for all  $\phi(s)$  satisfying  $V(0) \leq 1/\alpha$ .

**Proof:** Suppose the LMIs in (13) are feasible, then we have  $\Sigma_{55} < 0$ , which implies that the matrix  $X$  is invertible. Furthermore, one can set  $L \triangleq X^{-1}$  and denote

$$\begin{cases} P \triangleq \tilde{L}\tilde{P}\tilde{L}^T, & (\tilde{L} \triangleq \text{diag}\{L, L\}), \\ Q \triangleq L\tilde{Q}L^T, & Z \triangleq L\tilde{Z}L^T, \\ R \triangleq L\tilde{R}L^T, & K_i \triangleq Y_i L^T, \quad i = 1, 2. \end{cases} \quad (17)$$

Pre- and post-multiplying the LMIs (13) by  $\text{diag}\{L, L, L, I, L, I\}$  and its transpose, and using Schur complement and the notations in (16) yield

$$\begin{bmatrix} \bar{\Sigma}_{11}^{v_i} & \bar{\Sigma}_{12} & \bar{\Sigma}_{13} & LD & \bar{\Sigma}_{15}^{v_i} \\ * & \bar{\Sigma}_{22} & \bar{\Sigma}_{23} & 0 & \bar{\Sigma}_{25} \\ * & * & -12Z & 0 & \tau P_{12}^T \\ * & * & * & -I & \epsilon D^T L^T \\ * & * & * & * & \bar{\Sigma}_{55} \end{bmatrix} < 0,$$

$$i = 1, 2, \dots, 16, \quad (18)$$

where

$$\begin{aligned} \bar{\Sigma}_{11}^{v_i} &= \text{Sym}\{L[A + F + G(v_i) + BK_1]\} \\ &\quad + \text{Sym}(P_{12}) + Q - 4Z + I/\gamma, \\ \bar{\Sigma}_{12} &= LBK_2 - P_{12} - 2Z, \quad \bar{\Sigma}_{13} = \tau P_{22} + 6Z, \\ \bar{\Sigma}_{15}^{v_i} &= P_{11} - L + \epsilon[A + F + G(v_i) + BK_1]^T L^T, \\ \bar{\Sigma}_{22} &= -Q - 4Z, \quad \bar{\Sigma}_{23} = -\tau P_{22} + 6Z, \\ \bar{\Sigma}_{25} &= \epsilon(BK_2)^T L^T, \quad \bar{\Sigma}_{55} = -\epsilon \text{Sym}(L) + \tau^2 Z. \end{aligned}$$

Similarly, from the LMIs (14) and (15), we can obtain the following inequalities:

$$\begin{bmatrix} \bar{\Pi}_{11} & P_{12} - 2Z \\ * & \bar{\Pi}_{22} \end{bmatrix} \triangleq \bar{\Pi} \geq 0, \quad (19)$$

$$(1/\alpha)h_j^T h_j \leq \bar{r}_j^2 R, \quad j = 1, 2, 3, 4 \quad (20)$$

where  $\bar{\Pi}_{11} = P_{11} + 2\tau Z - R$  and  $\bar{\Pi}_{22} = P_{22} + Q/\tau + 2Z/\tau$ .

Next, we will prove that under the conditions (17)–(19), the conclusions of theorem are ensured. By straightforward calculations, it follows from (12) that

$$\begin{aligned} \dot{V}(t) &\leq 2 \left[ r^T(t)P_{11} + \int_{t-\tau}^t r^T(s)ds P_{12}^T \right] \dot{r}(t) \\ &\quad + 2 \left[ r^T(t)P_{12} + \int_{t-\tau}^t r^T(s)ds P_{22} \right] \\ &\quad \times [r(t) - r(t-\tau)] + r^T(t)Qr(t) + \tau^2 \dot{r}^T(t)Z\dot{r}(t) \\ &\quad - r^T(t-\tau)Qr(t-\tau) - \tau \int_{t-\tau}^t \dot{r}^T(s)Z\dot{r}(s)ds \quad (21) \end{aligned}$$

Using the Wirtinger-based inequality (Seuret & Gouaisbaut, 2013), we have

$$\begin{aligned} & -\tau \int_{t-\tau}^t \dot{r}^T(s)Z\dot{r}(s)ds \\ & \leq -\xi^T(t) \begin{bmatrix} 4Z & 2Z & -6Z \\ 2Z & 4Z & -6Z \\ -6Z & -6Z & 12Z \end{bmatrix} \xi(t) \quad (22) \end{aligned}$$

where  $\xi(t) = [r^T(t) \ r^T(t-\tau) \ (1/\tau) \int_{t-\tau}^t r^T(s)ds]^T$ .

From the system equation (9), it can be seen that (Qian, Li, Zhao et al., 2020)

$$\begin{aligned} & 2[r^T(t) + \epsilon \dot{r}^T(t)]L[A(r)r(t) + BK_2 r(t-\tau) \\ & \quad + D\omega(t) - \dot{r}(t)] = 0. \quad (23) \end{aligned}$$

Adding the left side of (22) to  $\dot{V}(t)$ , and combining with (20) and (21), one obtains

$$\dot{V}(t) + (1/\gamma)r^T(t)r(t) - \omega^T(t)\omega(t) \leq \zeta^T(t)\Sigma(r)\zeta(t) \quad (23)$$

where  $\zeta(t) = [\xi^T(t) \ \omega^T(t) \ \dot{r}^T(t)]^T$  ( $\xi(t)$  has the same definition as in (21)) and

$$\Sigma(r) = \begin{bmatrix} \bar{\Sigma}_{11}^r & \bar{\Sigma}_{12} & \bar{\Sigma}_{13} & LD & \bar{\Sigma}_{15}^r \\ * & \bar{\Sigma}_{22} & \bar{\Sigma}_{23} & 0 & \bar{\Sigma}_{25} \\ * & * & -12Z & 0 & \tau P_{12}^T \\ * & * & * & -I & \epsilon D^T L^T \\ * & * & * & * & \bar{\Sigma}_{55} \end{bmatrix}.$$

In above matrix  $\Sigma(r)$ ,  $\bar{\Sigma}_{12}$ ,  $\bar{\Sigma}_{13}$ ,  $\bar{\Sigma}_{22}$ ,  $\bar{\Sigma}_{23}$ ,  $\bar{\Sigma}_{25}$ ,  $\bar{\Sigma}_{55}$  are denoted in (17), and

$$\begin{aligned} \bar{\Sigma}_{11}^r &= \text{Sym}\{L[A + F + G(r) + BK_1]\} \\ &\quad + \text{Sym}(P_{12}) + Q - 4Z + I/\gamma, \\ \bar{\Sigma}_{15}^r &= P_{11} - L + \epsilon[A + F + G(r) + BK_1]^T L^T. \end{aligned}$$

Note that  $G(r)$  is affine with respect to  $r_1, r_2, r_3, r_4$ . If the inequalities (17) are true, then we have  $\Sigma(r) < 0$  on the box  $\mathcal{R}$ . Moreover, it follows from (23) that

$$\dot{V}(t) + (1/\gamma)r^T(t)r(t) - \omega^T(t)\omega(t) < 0, \quad r \in \mathcal{R}. \quad (24)$$

Integrating both sides of (24) from 0 to  $t$ , it follows that

$$\begin{aligned} V(t) + \frac{1}{\gamma} \int_0^t r^T(s)r(s)ds \\ \leq V(0) + \int_0^t \omega^T(s)\omega(s)ds \leq V(0) + \beta, \quad r \in \mathcal{R}. \end{aligned} \quad (25)$$

Also, using Jensen inequalities (Qian, Xing et al., 2020), one can obtain from (12) and (18) that (Chen et al., 2017; Qian, Li, Chen et al., 2020)

$$\begin{aligned} V(t) &\geq \eta^T(t)P\eta(t) + \left[ \int_{t-\tau}^t r(s)ds \right]^T Q \left[ \int_{t-\tau}^t r(s)ds \right] \\ &\quad + \frac{2}{\tau} \left[ \int_{-\tau}^0 \int_{t+\theta}^t \dot{r}(s)dsd\theta \right]^T Z \left[ \int_{-\tau}^0 \int_{t+\theta}^t \dot{r}(s)dsd\theta \right] \\ &= \eta^T(t)(\bar{\Pi} + \text{diag}\{R, 0\})\eta(t) \geq r^T(t)Rr(t) > 0 \end{aligned} \quad (26)$$

where  $\bar{\Pi}$  is denoted in (18). In addition, from the condition (19), it is seen that

$$(1/\alpha)r^T(t)h_j^T h_j r(t) \leq \bar{r}_j^2 r^T(t)Rr(t). \quad (27)$$

For all admissible  $\phi(s)$  ( $-\tau \leq s \leq 0$ ) satisfying  $V(0) \leq 1/\alpha - \beta$ , using (25)–(27), it can be verified that all states  $r$  are contained in the box  $\mathcal{R}$ . In particular, it can be seen from (25) and (26) that all trajectories remain in the ellipsoid  $\mathcal{E}(R, 1/\alpha) \triangleq \{r \in \mathbb{R}^4 : r^T R r \leq 1/\alpha\}$ , which implies that all trajectories  $r(t)$  are bounded.

Letting  $t \rightarrow +\infty$  and noting  $V(t) \geq 0$ , it follows from (25) that the  $H_\infty$  performance constraint  $\int_0^t r^T(s)r(s)ds \leq \gamma \int_0^t \omega^T(s)\omega(s)ds + \gamma V(0)$  can be guaranteed in  $\mathcal{E}(R, 1/\alpha)$ .

When  $\omega(t) = 0$ , for all  $\phi(s)$  ( $-\tau \leq s \leq 0$ ) satisfying  $V(0) \leq 1/\alpha$ , using (25)–(27), we can prove that all states  $r$  are still contained in the set  $\mathcal{E}(R, 1/\alpha) \subset \mathcal{R}$ . Moreover, it is seen from (24) that the relation  $\dot{V}(t) < 0$  ( $r \in \mathcal{R}$ ) holds, which means that the closed-loop dynamics (9) is asymptotically stable. The proof is completed. ■

**Remark 3.1:** Recently, the  $H_\infty$  control problem has been addressed in Xu et al. (2018) for a chaotic finance system with external disturbance in the framework of finite time. However, it is noted that the results in Xu et al. (2018) are based on the linearized model and one cannot perform the accurate analysis and design. Very recently, the quadratic system theory has been adopted in Xu et al. (2020) to stabilize the finance system (1). However, the disturbance is ignored in Xu et al. (2020). In fact, when the external disturbance is considered, one has to first determine the admissible initial conditions and disturbances to ensure the boundedness of the state trajectories and then discuss the corresponding  $H_\infty$  performance. Therefore, the proposed Theorem 3.1 in this paper is not the simple extension of the result in Xu et al. (2020).

**Remark 3.2:** In Chen et al. (2013); de Souza and Coutinho (2014), the local stabilization/control problem has been studied for nonlinear quadratic time-delay systems. However, the systems addressed in Chen et al. (2013); de Souza and Coutinho (2014) contain the state delay but not the input delay. Therefore, the results proposed in Chen et al. (2013); de Souza and Coutinho (2014) cannot be applicable for the hyperchaotic finance system subject to the delayed feedback controller. Moreover, it should be pointed out the external disturbance is not considered in de Souza and Coutinho (2014) and the boundedness of system trajectories is not discussed.

For the case of non-delayed feedback, the controller can be denoted as

$$u(t) = K(x(t) - x^*) \quad (28)$$

Correspondingly, the closed-loop system can be written as

$$\begin{aligned} \dot{r}(t) &= [A + F + G(r) + BK]r(t) \\ &\quad + D\omega(t), \quad r(0) = r_0. \end{aligned} \quad (29)$$

Using the Lyapunov function  $\hat{V}(t) = x^T(t)X^{-1}x(t)$ , where  $X > 0$ , the following result can be readily established.

**Corollary 3.2:** Let the scalars  $\bar{r}_j$  ( $j = 1, 2, 3, 4$ ) be given. Assume that there exist matrices  $X > 0$ ,  $Y$ , and scalars  $\alpha > 0$ ,  $\beta > 0$  ( $\alpha < 1/\beta$ ),  $\gamma$ , such that the LMIs

$$\begin{bmatrix} \Gamma(v_i) & D & X \\ * & -I & 0 \\ * & * & -\gamma I \end{bmatrix} < 0, \quad (30)$$

$$i = 1, 2, \dots, 16,$$

$$\begin{bmatrix} \alpha & h_j X^T \\ * & \bar{r}_j^2 X \end{bmatrix} \geq 0 \quad (31)$$

are satisfied, where  $\Gamma(v_i) = \text{Sym}\{[A + F + G(v_i)]X^T + BY\}$ . Then, there exists the controller (28) with  $K = YX^{-T}$  such that: (1) all state trajectories of the dynamics (29) are bounded for all  $r_0$  satisfying  $\hat{V}(0) \leq 1/\alpha - \beta$  and all non-zero  $\omega(t)$  satisfying  $\int_0^\infty \omega^T(t)\omega(t)dt \leq \beta$ ; (2)  $H_\infty$  performance constraint can be guaranteed; (3) when  $\omega(t) = 0$ , the dynamics (29) is asymptotically stable for all  $r_0$  satisfying  $\hat{V}(0) \leq 1/\alpha$ .

Finally, we will be concerned with the optimization problems involved in our main results. Before considering the  $H_\infty$  control problem, it is necessary to measure the disturbance tolerance level  $\beta$ . Without loss of generality, it is assumed that  $\phi(s) = 0$  ( $-\tau \leq s \leq 0$ ). In this case, the scalar  $\alpha$  in LMIs (15) and (31) should be substituted by  $1/\beta$ . Correspondingly, the optimization problems concerning the largest disturbance tolerance levels in Theorem 1 and Corollary 2 can be, respectively, described as follows:

**Prob.1.**  $\max_{\bar{P}, \bar{Q}, \bar{Z}, \bar{R}, X, Y_1, Y_2, \beta, \gamma} \beta, \quad \text{s.t.},$

LMIs (13)–(15) hold,

**Prob.1'.**  $\max_{X, Y, \beta, \gamma} \beta, \quad \text{s.t.},$

LMIs (30) and (31) hold.

By solving Prob.1 or Prob.1', we can obtain the largest disturbance tolerance level  $\beta_M$ . For a given scalar  $\beta \leq \beta_M$ , the optimization problems concerning the minimum  $H_\infty$  performance level  $\gamma$  in Theorem 1 and Corollary 2 can be given as follows:

**Prob.2.**  $\max_{\bar{P}, \bar{Q}, \bar{Z}, \bar{R}, X, Y_1, Y_2, \gamma} \gamma, \quad \text{s.t.},$

LMIs (13) – (15) hold,

**Prob.2'.**  $\max_{X, Y, \gamma} \gamma, \quad \text{s.t.},$

LMIs (30) and (31) hold.

When the external disturbance  $\omega(t)$  does not exist, one can maximize the admissible initial condition set in designing the feedback controller (5) or (28). For this case,

the rows and columns related to  $\omega(t)$  in the LMIs (13) and (30) should be deleted. Also, without loss of generality, one can set  $\alpha = 1$  in the LMIs (15) and (31).

As in Xu et al. (2020), we assume that  $\phi(s)$  ( $-\tau \leq s \leq 0$ ) belongs to the set

$$\mathcal{X}_\rho \triangleq \{\phi(s) : \max_{\theta \in [-\tau, 0]} \|\phi(s)\| \leq \rho\} \quad (32)$$

where  $\rho$  is a positive scalar. Let us introduce the following LMI (Chen et al., 2017):

$$\bar{P} \leq \text{diag}\{\bar{P}_1, \bar{P}_2\} \quad (33)$$

where  $\bar{P}_1 > 0$  and  $\bar{P}_2 > 0$ . Using (33) and Jensen inequalities, we have

$$V(0) \leq \delta_1 \max_{s \in [-\tau, 0]} \|\phi(s)\|^2 + \delta_2 \max_{s \in [-\tau, 0]} \|\dot{\phi}(s)\|^2 \quad (34)$$

where

$$\begin{aligned} \delta_1 &= \lambda_M(X^{-1}\bar{P}_1X^{-T}) + \tau^2\lambda_M(X^{-1}\bar{P}_2X^{-T}) \\ &\quad + \tau\lambda_M(X^{-1}\bar{Q}X^{-T}), \\ \delta_2 &= (\tau^3/2)\lambda_M(X^{-1}\bar{Z}X^{-T}). \end{aligned}$$

Moreover, let us set the following matrix inequalities:

$$X^{-1}\bar{P}_1X^{-T} \leq p_1I, \quad (35)$$

$$X^{-1}\bar{P}_2X^{-T} \leq p_2I, \quad (36)$$

$$X^{-1}\bar{Q}X^{-T} \leq qI, \quad (37)$$

$$X^{-1}\bar{Z}X^{-T} \leq zI \quad (38)$$

where  $p_1 > 0$ ,  $p_2 > 0$ ,  $q > 0$  and  $z > 0$ . Note that the inequalities (35)–(38) can be, respectively, ensured by the following LMIs (Chen, Fei et al., 2014):

$$\begin{bmatrix} p_1I & I \\ I & X + X^T - \bar{P}_1 \end{bmatrix} \geq 0, \quad (39)$$

$$\begin{bmatrix} p_2I & I \\ I & X + X^T - \bar{P}_2 \end{bmatrix} \geq 0, \quad (40)$$

$$\begin{bmatrix} qI & I \\ I & X + X^T - \bar{Q} \end{bmatrix} \geq 0, \quad (41)$$

$$\begin{bmatrix} zI & I \\ I & X + X^T - \bar{Z} \end{bmatrix} \geq 0 \quad (42)$$

To obtain a larger set  $\mathcal{X}_\rho$ , we can first solve the optimization problem

**Prob.3.**  $\min_{\bar{P}, \bar{P}_1, \bar{P}_2, \bar{Q}, \bar{Z}, \bar{R}, X, Y_1, Y_2, X, p_1, p_2, q, z} \sigma, \quad \text{s.t.},$

LMIs (13)–(15), (33), (39)–(42) hold

where  $\sigma = p_1 + \tau^2 p_2 + \tau q + (\tau^3/2)\kappa z$  ( $\kappa > 0$  is a adjusting scalar)

By solving Prob.3, one can obtain the scalars  $\delta_1$  and  $\delta_2$ . Note that the initial condition  $\phi(s)$  satisfies the relation  $V(0) \leq 1$ , which can be ensured by the inequality

$$\delta_1 \max_{s \in [-\tau, 0]} \|\phi(s)\|^2 + \delta_2 \max_{s \in [-\tau, 0]} \|\dot{\phi}(s)\|^2 \leq 1. \quad (43)$$

From (43), it is seen that  $\|\phi(s)\| \leq \sqrt{1/\delta_1}, s \in [-\tau, 0]$ . As in Xu et al. (2020), we select the scalars  $\hat{r}_1, \hat{r}_2, \hat{r}_3$  and  $\hat{r}_4$  such that the ball  $\mathcal{B}(1/\delta_1) \triangleq \{\phi \in \mathbb{R}^4 : \|\phi\|^2 \leq 1/\delta_1\}$  is contained in the box  $\hat{\mathcal{R}} \triangleq [-\hat{r}_1, \hat{r}_1] \times [-\hat{r}_2, \hat{r}_2] \times [-\hat{r}_3, \hat{r}_3] \times [-\hat{r}_4, \hat{r}_4]$ . Note that the initial condition  $\phi(s)$  ( $s \in [-\tau, 0]$ ) satisfies the equation  $\dot{\phi}(s) = \hat{A}(\phi)\phi(s)$ , where  $\hat{A}(\phi) = A + F + G(\phi)$ . Then, we can choose a scalar  $\mu > 0$  such that the inequality  $\hat{A}^T(\phi)\hat{A}(\phi) \leq \mu I$  holds on  $\hat{\mathcal{R}}$ , which is guaranteed by the following LMIs:

$$\begin{bmatrix} -\mu I & [A + F + G(\hat{v}_i)]^T \\ * & -I \end{bmatrix} \leq 0, \quad i = 1, 2, \dots, 16 \quad (44)$$

where  $\hat{v}_i$  ( $i = 1, 2, \dots, 16$ ) are vertices of the box  $\hat{\mathcal{R}}$ . Corresponding, the scalar  $\rho$  involved in the initial condition set  $\mathcal{X}_\rho$  can be computed by  $\rho = \sqrt{1/(\delta_1 + \mu\delta_2)}$ .

For the non-delayed case, the ellipsoid  $\mathcal{E}(X^{-1}, 1) \triangleq \{r \in \mathbb{R}^4 : r^T X^{-1} r \leq 1\}$  can be seen as the estimate of the domain of attraction. The maximization of the ellipsoid  $\mathcal{E}(X^{-1}, 1)$  can be obtained by solving the optimization problem

**Prob.3'**  $\min_{X, Y, x} \quad \text{s.t.},$   
 LMIs (30), (31) and  $\begin{bmatrix} xI & I \\ I & X \end{bmatrix} \geq 0$  hold.

**Remark 3.3:** The main results of the paper are based on the LMIs. Due to the use of the quadratic system theory, more LMIs are introduced in our obtained results, which will lead to longer computation time in solving optimization problems. The LMIs and decision variables in above optimization problems can be readily calculated. For example, 35 LMIs and  $106 + 8m$  scalar variables are involved in solving Prob.3.

#### 4. Numerical simulation

In the simulation, we choose  $a = 0.9, b = 0.2, c = 1.5, d = 0.2, e = 0.17, \tau = 0.5, B = [0.2 \ 0.2 \ 0.2]^T$  and  $D = I$ . By direct calculations, it is found that the model (2) with above parameters has three unstable equilibrium points, i.e.  $(0, 5, 0, 0)$  and

$$(\vartheta_1, \vartheta_2, -\vartheta_3, \vartheta_4), \quad (-\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4)$$

where  $\vartheta_1 = 1.6660, \vartheta_2 = -8.8778, \vartheta_3 = 1.1107$  and  $\vartheta_4 = 17.4004$ .

First of all, we will consider the local stabilization problem under the delayed controller  $u(t) = K_2(x(t - \tau) - x^*)$  (i.e.  $K_1 = 0$ ), where  $x^* = (\vartheta_1, \vartheta_2, -\vartheta_3, \vartheta_4)^T$ . By solving Problem 3 with  $\bar{r}_1 = 11, \bar{r}_2 = 17, \bar{r}_3 = \bar{r}_4 = 15, \epsilon = 0.02, \kappa = 1000$  and  $Y_1 = 0$ , we have the scalars  $\delta_1 = 0.0124, \delta_2 = 2.9869 \times 10^{-5}$  and the controller gain

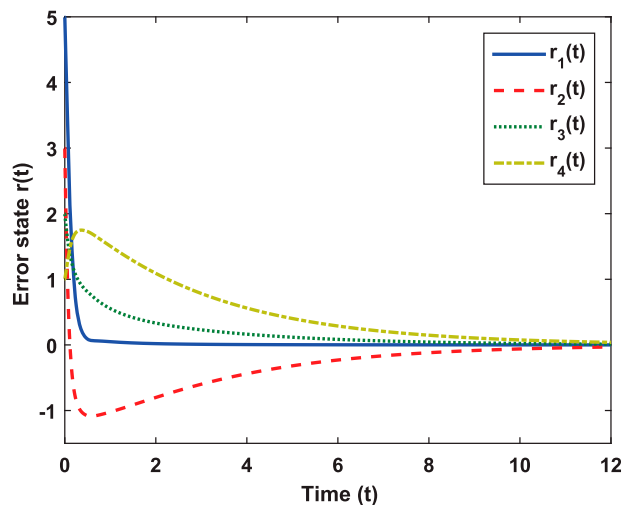
$$K_2 = \begin{bmatrix} 0.0034 & -0.1096 & 0.0354 & -0.0108 \\ 0.0084 & -0.1011 & 0.0413 & 0.0120 \\ 0.0503 & 0.0447 & -0.0232 & 0.3358 \\ -0.0526 & 0.0584 & -0.0528 & -0.3115 \end{bmatrix}.$$

Let us select  $\hat{r}_1 = \hat{r}_2 = \hat{r}_3 = \hat{r}_4 = 9.0$  satisfying  $\mathcal{B}(1/\delta_1) \subset \hat{\mathcal{R}}$ , then one can obtain the minimum  $\mu = 387.9917$  such that the LMIs (45) are feasible. Furthermore, we obtain the scalar  $\rho = 6.4550$  involved in the set  $\mathcal{X}_\rho$ . In Figure 1, the state responses of the error dynamics (7) is plotted in the absence of disturbance, where  $\phi(s) = [5 \ 3 \ 2 \ 1]^T \in \mathcal{X}_\rho, s \in [-\tau, 0]$ . From Figure 1, it is seen that our designed delayed controller behaves well.

Next, we will consider the  $H_\infty$  control problem under the delayed controller  $u(t) = K_2(x(t - \tau) - x^*)$ . To this end, we have to estimate the largest disturbance tolerance level  $\beta_M$ . By solving Problem 1 with  $\bar{r}_1 = 0.9, \bar{r}_2 = 2.1, \bar{r}_3 = 3, \bar{r}_4 = 5, \epsilon = 4.3$  and  $Y_1 = 0$ , we have  $\beta_M = 2.0336 \times 10^3$ . Letting  $\beta = 1.8 \times 10^3 < \beta_M$  and solving Problem 2 with the same choosing of the scalars  $\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4$  and  $\epsilon$  as above, one obtains the minimum  $H_\infty$  performance level  $\gamma_m = 0.1467$  and the following controller gain:

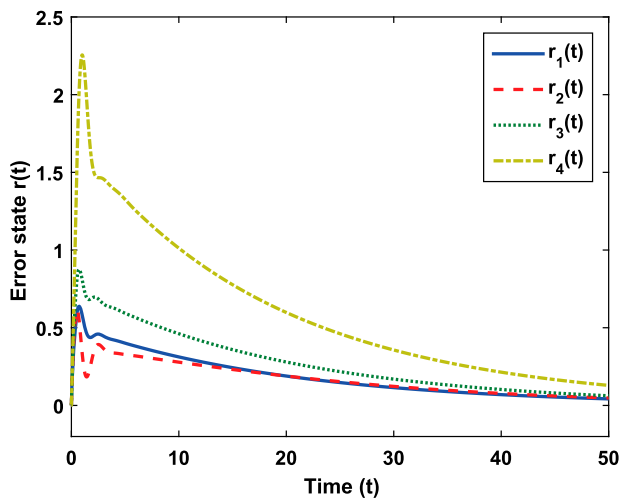
$$K_2 = \begin{bmatrix} -0.0855 & -0.3864 & -0.2265 & -0.3552 \\ -0.4268 & -1.0146 & -0.0480 & 0.0046 \\ -0.0728 & -0.0302 & -1.4135 & 0.0544 \\ -0.0498 & -1.0410 & -0.1823 & -1.4954 \end{bmatrix}.$$

Using the above controller gain, the state responses and the truncated  $H_\infty$  performance level  $\gamma_t$  of the error

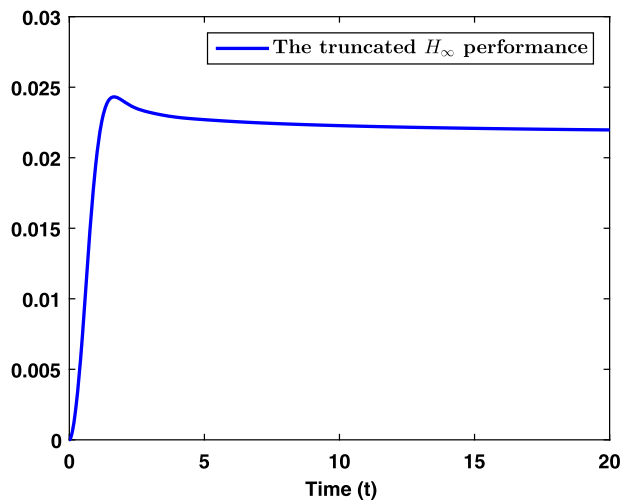


**Figure 1.** State responses of the error dynamics (7).





**Figure 2.** State responses of the error dynamics (7).



**Figure 3.** Truncated  $H_\infty$  performance level  $\gamma_t$ .

dynamics (7) are, respectively, plotted in Figures 2 and 3, where  $\gamma_t = (\int_0^t r^T(s)r(s)ds)/(\int_0^t \omega^T(s)\omega(s)dt)$ . In the simulation, the external disturbance is selected as  $\omega(t) = 13.4 * e^{-0.05t}$  to ensure that  $\int_0^\infty \omega^T(t)\omega(t)dt \leq \beta = 1.8 * 10^3$ . From Figure 2, it is seen that the stability of the error dynamics (7) can be guaranteed when the disturbance  $\omega(t)$  disappears under the proposed control scheme. Moreover, it is clear from Figure 3 that the truncated  $H_\infty$  performance level  $\gamma_t$  is less than  $\gamma_m = 0.1467$ .

## 5. Conclusions

Based on the quadratic system theory, an augmented Lyapunov functional and some integral inequalities, an LMI-based sufficient condition has been obtained in this paper for a hyperchaotic system with energy-bounded disturbance under the delayed feedback controller, which can guarantee that the closed-loop dynamics has

some desirable performances including the boundedness, the  $H_\infty$  performance and the asymptotic stability. Then, several convex optimization problems have been given to handle different system performance requirements. Finally, numerical simulations have been presented to demonstrate the effectiveness of our proposed results.

The existence of the chaotic behaviour in finance systems will result in inherent indefiniteness of the macroeconomic operation. Therefore, it is imperative to propose some effective control schemes to stabilize the chaotic finance dynamics. This paper has attempted to control a hyperchaotic finance system with external disturbance in a more accurate local framework. Our proposed control scheme can be seen as an alternative for the governments in formulating measures to revive the economy.

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