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Some Generalised Einstein Hybrid Aggregation Operators and Their Application to Group Decision-making Using Pythagorean Fuzzy Numbers

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ABSTRACT

Pythagorean fuzzy set (PFS) is one of the prosperous extensions of the intuitionistic fuzzy set (IFS) for handling the fuzziness and uncertainties in the data. Under this environment, in this paper, we introduce the notion of two generalised Einstein hybrid operators namely, generalised Pythagorean fuzzy Einstein hybrid averaging (in short GPFEHA) operator and generalised Pythagorean fuzzy Einstein hybrid geometric (in short GPFEHG) operator along with their desirable properties, such as idempotency, boundedness and monotonicity. The main benefit of the proposed operators is that these operators deliver more general, more correct and precise results as compared to their existing methods. Generalised Einstein operators combine Einstein operators with some generalised operators using Pythagorean fuzzy values. Therefore these methods play a vital role in real world problems. Finally, the proposed operators have been applied to decision-making problems to show the validity, practicality and effectiveness of the new attitude.

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Pythagorean fuzzy set; GPFEHA operator; GPFEHG operator; group decision-making

1. Introduction

Multi-criteria group decision-making problems have importance in most kinds of fields such as economics, engineering and management. Generally, it has been assumed that the information which accesses the alternatives in term of criteria and weight are expressed in real numbers. But due to the complexity of the system day-by-day, it is difficult for the decision-makers to make a perfect decision, as most of the preferred value during the decision-making process imbued with uncertainty. In order to handle the uncertainties and fuzziness, intuitionistic fuzzy set [1] theory is one of the prosperous extensions of the fuzzy set theory [2], which is characterised by the degree of membership and degree of non-membership has been presented. Xu [3] developed some basic arithmetic operators, including the IFWA operator, the IFOWA operator, and the IFHA operator. Xu and Yager [4] defined some basic geometric operators, such as the IFWG operator, the IFOWG operator,

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and the IFHG operator. Wang and Liu [5,6] introduced the notion of some Einstein operators, such as the IFEWG operator, the IFEOWG operator, the IFEWA operator, the IFEOWA operator, and applied them to group decision-making. In [7–18], many scholars worked in the field of intuitionistic fuzzy sets and introduced several operators and their applications.

But there are several cases where the decision-maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. For example, suppose a man expresses his preferences towards the alternative in such a way that degree of their satisfaction is 0.6 and degree of rejection is 0.8. Clearly, its sum is greater than one. Therefore, Yager [19] introduced the concept of another set called Pythagorean fuzzy set. Pythagorean fuzzy set is more powerful tool to solve uncertain problems. Like intuitionistic fuzzy operators, Pythagorean fuzzy operators also become an interesting and important area for research, after the advent of Pythagorean fuzzy sets theory. In 2013, Yager and Abbasov [20] introduced the notion of two new operators using the PFVs, such as the PFWA operator and the PFOWA operator. Rahman et al. [21–24] introduced the notion of PFWG operator, PFOWG operator and PFHG operator and applied them on group decision-making problems. Rahman et al. [25–30] introduced the concept of many operators using PFNs and also applied them to group decision-making. Rahman et al. [31,32] introduced the idea of Einstein hybrid aggregation operators using Pythagorean fuzzy numbers and also applied them on group decision-making. Garg [33,34] introduced the notion of several Einstein averaging operators, and Einstein geometric operators such as, PFEWA operator, PFEOWA operator, GPFEWA operator, GPFEOWA operator, PFEWG operator, PFEOWG operator, GPFEWG operator and GPFEOWA operator and applied them to group decision-making. Garg [35] introduced the idea of confidence level and develop some aggregation operators using Pythagorean fuzzy numbers. Garg [36] introduced the idea of logarithmic aggregation operators and their application. Garg [37–43] developed many aggregation operators and applied them on group decision-making. Akram [44] introduced the concept of ELECTRE I Method using Pythagorean fuzzy information. Zang and Xu [45] introduced the notion of TOPSIS for multiple attribute decision-making based on Pythagorean fuzzy information.

This motivation comes from [33,34], in which the author introduced the concept of several Einstein averaging operators, and Einstein geometric operators such as, PFEWA operator, PFEOWA operator, GPFEWA operator, GPFEOWA operator, PFEWG operator, PFEOWG operator, GPFEWG operator and GPFEOWA operator and applied them to group decision-making. Actually, GPFEWA operator and GPFEWG operator weigh only the Pythagorean fuzzy arguments, while GPFEOWA operator and GPFEOWG operator weigh only the ordered positions of the Pythagorean fuzzy arguments instead of weighing the Pythagorean fuzzy arguments themselves. To overcome these limitations, we introduce the concept of GPFEHA operator and GPFEHG operator which weigh both the given Pythagorean fuzzy value and its ordered position. Thus the proposed operators are the generalisation of the existing methods. Therefore we can say that the proposed operators provide more accurate and precise results as compare to the existing methods, because the proposed operators are the generalisation of existing operators. Of course, cursorily, it is more complicated in calculation. However, in real life problems, we need assign the specific parameter ∂, firstly.

The remainder of this paper is arranged as follows. In Section 2, we give some basic definitions which will be used in our later sections. In Section 3, we introduce the notion of the GPFEHA operator, and the GPFEHG operator. In Section 4, we apply the proposed operators to MAGDM using PFNs. In Section 5, we construct numerical example. In Section 6, we compare the proposed operators to others operators. In Section 7, we have conclusion.

2. Preliminaries

Definition 2.1: [19] Let D be a universal set, then Pythagorean fuzzy set can be defined as:

$$P = \{\langle d, \Omega_P(d), \mho_P(d)\rangle \mid d \in D\}, \tag{1}$$

where $\Omega_P(d) : D \to [0,1]$ and $\mho_P(d) : D \to [0,1]$ with condition $0 \le \Omega_P^2(d) + \mho_P^2(d) \le 1$ are called the membership and non-membership function respectively. Let $\pi_P(d) = \sqrt{1 - (\Omega_P^2(d) + \mho_P^2(d))}$ then it is called the degree of indeterminacy. And also $0 \le \pi_P(d) \le 1, \forall d \in D$.

Definition 2.2: [45] Let $\kappa = \langle \Omega_\kappa, \mho_\kappa \rangle$ be a PFN, then score and accuracy function can be defined as: $s(\kappa) = \Omega_\kappa^2 - \mho_\kappa^2$ and $h(\kappa) = \Omega_\kappa^2 + \mho_\kappa^2$ respectively.

If $\kappa_1 = \langle \Omega_{\kappa_1}, \mho_{\kappa_1} \rangle$ and $\kappa_2 = \langle \Omega_{\kappa_2}, \mho_{\kappa_2} \rangle$, then we having the following some conditions:

(1) If $s(\kappa_1) \prec s(\kappa_2)$, then $\kappa_1 \prec \kappa_2$
(2) If $s(\kappa_1) = s(\kappa_2)$, then there are two cases:
   (i) If $h(\kappa_1) = h(\kappa_2)$, then $\kappa_1 = \kappa_2$
   (ii) If $h(\kappa_1) \prec h(\kappa_2)$, then $\kappa_1 \prec \kappa_2$

Definition 2.3: [33] Let $\kappa_t = \langle \Omega_{\kappa_t}, \mho_{\kappa_t} \rangle (t = 1, 2)$ and $\partial \succ 0$ be any real number, then

$$\kappa_1 \oplus_\varepsilon \kappa_2 = \left\langle \frac{\sqrt{\Omega_{\kappa_1}^2 + \Omega_{\kappa_2}^2}}{\sqrt{1 + \Omega_{\kappa_1}^2 \cdot_\varepsilon \Omega_{\kappa_2}^2}}, \frac{\mho_{\kappa_1} \cdot_\varepsilon \mho_{\kappa_2}}{\sqrt{1 + (1 - \mho_{\kappa_1}^2) \cdot_\varepsilon (1 - \mho_{\kappa_2}^2)}} \right\rangle, \tag{2}$$

$$\kappa_1 \otimes_\varepsilon \kappa_2 = \left\langle \frac{\Omega_{\kappa_1} \cdot_\varepsilon \Omega_{\kappa_2}}{\sqrt{1 + (1 - \Omega_{\kappa_1}^2) \cdot_\varepsilon (1 - \Omega_{\kappa_2}^2)}}, \frac{\sqrt{\mho_{\kappa_1}^2 + \mho_{\kappa_2}^2}}{\sqrt{1 + \mho_{\kappa_1}^2 \cdot_\varepsilon \mho_{\kappa_2}^2}} \right\rangle, \tag{3}$$

$$\kappa^{\wedge_\varepsilon \partial} = \left\langle \frac{\sqrt{2\Omega_\kappa^{2\partial}}}{\sqrt{(2 - \Omega_\kappa^2)^\partial + \Omega_\kappa^{2\partial}}}, \frac{\sqrt{(1 + \mho_\kappa^2)^\partial - (1 - \mho_\kappa^2)^\partial}}{\sqrt{(1 + \mho_\kappa^2)^\partial + (1 - \mho_\kappa^2)^\partial}} \right\rangle, \tag{4}$$

$$\partial \cdot_\varepsilon \kappa = \left\langle \frac{\sqrt{(1 + \Omega_\kappa^2)^\partial - (1 - \Omega_\kappa^2)^\partial}}{\sqrt{(1 + \Omega_\kappa^2)^\partial + (1 - \Omega_\kappa^2)^\partial}}, \frac{\sqrt{2\mho_\kappa^{2\partial}}}{\sqrt{(2 - \mho_\kappa^2)^\partial + (\mho_\kappa^2)^\partial}} \right\rangle, \tag{5}$$

Definition 2.4: [31] PFEHA operator can be defined as:

$$\text{PFEHA}_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \left\langle \frac{\sqrt{\prod_{t=1}^{n} (1+\Omega_{\dot{\kappa}_{\Im(t)}}^2)^{\hbar_t} - \prod_{t=1}^{n} (1-\Omega_{\dot{\kappa}_{\Im(t)}}^2)^{\hbar_t}}}{\sqrt{\prod_{t=1}^{n} (1+\Omega_{\dot{\kappa}_{\Im(t)}}^2)^{\hbar_t} + \prod_{t=1}^{n} (1-\Omega_{\dot{\kappa}_{\Im(t)}}^2)^{\hbar_t}}}, \frac{\sqrt{2\prod_{t=1}^{n} \mho_{\dot{\kappa}_{\Im(t)}}^{2\hbar_t}}}{\sqrt{\prod_{t=1}^{n} (2-\mho_{\dot{\kappa}_{\Im(t)}}^2)^{\hbar_t} + \prod_{t=1}^{n} \mho_{\dot{\kappa}_{\Im(t)}}^{2\hbar_t}}} \right\rangle, \tag{6}$$

where $\dot{\kappa}_{\Im(t)}$ is the largest of the WPFVs $\dot{\kappa}_t(\dot{\kappa}_t = n\hbar_t\kappa_t, t = 1, 2, \ldots, n)$. $\hbar = (\hbar_1, \hbar_2, \ldots, \hbar_n)^T$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)^T$ be the associated and weighted vector, respectively, and both have the same condition, such as both belong to the closed interval and their sum is equal to 1.

Definition 2.5: [32] PFEHG operator can be defined as:

$$\mathrm{PFEHG}_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \left\langle \frac{\sqrt{2\prod_{t=1}^{n} \Omega_{\dot{\kappa}_{\Im(t)}}^{2\hbar_t}}}{\sqrt{\prod_{t=1}^{n}(2-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\hbar_t} + \prod_{t=1}^{n}\Omega_{\dot{\kappa}_{\Im(t)}}^{2\hbar_t}}}, \frac{\sqrt{\prod_{t=1}^{n}(1+\mho^2_{\dot{\kappa}_{\Im(t)}})^{\hbar_t} - \prod_{t=1}^{n}(1-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\hbar_t}}}{\sqrt{\prod_{t=1}^{n}(1+\mho^2_{\dot{\kappa}_{\Im(t)}})^{\hbar_t} + \prod_{t=1}^{n}(1-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\hbar_t}}} \right\rangle, \tag{7}$$

where $\dot{\kappa}_{\Im(t)}$ is the largest of the WPFVs $\dot{\kappa}_t(\dot{\kappa}_t = \kappa_t^{n\hbar_t}, t = 1, 2, \ldots, n)$. $\hbar = (\hbar_1, \hbar_2, \ldots, \hbar_n)^T$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)^T$ be the associated and weighted vector respectively, and both have the same condition, such as both belong to the closed interval and their sum is equal to 1.

3. Some Generalised Pythagorean Fuzzy Einstein Hybrid Operators

In this section, we investigate the generalised Einstein operators such as, generalised Pythagorean fuzzy Einstein hybrid averaging operator and generalised Pythagorean fuzzy Einstein hybrid geometric operator

Definition 3.1: GPFEHG operator can be defined as:

$$\mathrm{GPFEHG}_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n)$$

$$= \left\langle \sqrt{\frac{\left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} - \left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}{\left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} + \left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}}, \frac{\sqrt{2\left\{\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right\}^{\frac{1}{\partial}}}}{\sqrt{\left(\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} + \left(\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}} \right\rangle, \tag{8}$$

where $\dot{\kappa}_{\Im(t)}$ is the largest of the WPFVs $\dot{\kappa}_t(\dot{\kappa}_t = \kappa_t^{n\bar{\lambda}_t}, t = 1, 2, \ldots, n)$. $\hbar = (\hbar_1, \hbar_2, \ldots, \hbar_n)^T$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n)^T$ be the associated and weighted vector, respectively, and both have the same condition, such as both belong to the closed interval and their sum is equal to 1. And the positive number n is called the balancing coefficient, the parameter ∂ is $\partial \succ 0$.

Theorem 3.1: *Let $\kappa_t = \langle \Omega_{\kappa_t}, \mho_{\kappa_t} \rangle$ be a family PFVs, then the following conditions hold:*

(i) $\kappa_1 \oplus_{\varepsilon} \kappa_2 = \kappa_2 \oplus_{\varepsilon} \kappa_1$
(ii) $\kappa_1 \otimes_{\varepsilon} \kappa_2 = \kappa_2 \otimes_{\varepsilon} \kappa_1$
(iii) $(\kappa_1 \oplus_{\varepsilon} \kappa_2) \oplus_{\varepsilon} \kappa_3 = \kappa_1 \oplus_{\varepsilon} (\kappa_2 \oplus_{\varepsilon} \kappa_3)$
(iv) $(\kappa_1 \otimes_{\varepsilon} \kappa_2) \otimes_{\varepsilon} \kappa_3 = \kappa_1 \otimes_{\varepsilon} (\kappa_2 \otimes_{\varepsilon} \kappa_3)$
(v) $\kappa_1^c \oplus_{\varepsilon} \kappa_2^c = (\kappa_1 \otimes_{\varepsilon} \kappa_2)^c$
(vi) $\kappa_1^c \otimes_{\varepsilon} \kappa_2^c = (\kappa_1 \oplus_{\varepsilon} \kappa_2)^c$

Proof: Straightforward. ■

Theorem 3.2: *Let $\kappa_t = \langle \Omega_{\kappa_t}, \mho_{\kappa_t} \rangle$ be a family PFVs, then the following conditions hold:*

(i) $(\kappa_1 \cup \kappa_2) \oplus_{\varepsilon} (\kappa_1 \cap \kappa_2) = \kappa_1 \oplus_{\varepsilon} \kappa_2$
(ii) $(\kappa_1 \cup \kappa_2) \otimes_{\varepsilon} (\kappa_1 \cap \kappa_2) = \kappa_1 \otimes_{\varepsilon} \kappa_2$
(iii) $(\kappa_1 \cup \kappa_2) \oplus_{\varepsilon} (\kappa_1 \cap \kappa_2) = \kappa_1 \oplus_{\varepsilon} \kappa_2$
(iv) $(\kappa_1 \cup \kappa_2) \cap \kappa_3 = (\kappa_1 \cap \kappa_3) \cup (\kappa_2 \cap \kappa_3)$
(v) $(\kappa_1 \cap \kappa_2) \cup \kappa_3 = (\kappa_1 \cup \kappa_3) \cap (\kappa_2 \cup \kappa_3)$
(vi) $(\kappa_1 \cup \kappa_2) \oplus_{\varepsilon} \kappa_3 = (\kappa_1 \oplus_{\varepsilon} \kappa_3) \cup (\kappa_2 \oplus_{\varepsilon} \kappa_3)$
(vii) $(\kappa_1 \cap \kappa_2) \oplus_{\varepsilon} \kappa_3 = (\kappa_1 \oplus_{\varepsilon} \kappa_3) \cap \kappa (\kappa_2 \oplus_{\varepsilon} \kappa_3)$
(viii) $(\kappa_1 \cap \kappa_2) \otimes_{\varepsilon} \kappa_3 = (\kappa_1 \otimes_{\varepsilon} \kappa_3) \cap (\kappa_2 \otimes_{\varepsilon} \kappa_3)$
(ix) $(\kappa_1 \cup \kappa_2) \oplus_{\varepsilon} \kappa_3 = (\kappa_1 \oplus_{\varepsilon} \kappa_3) \cup (\kappa_2 \oplus_{\varepsilon} \kappa_3)$

Proof: Straightforward. ■

Theorem 3.3: *Let $\kappa_t = (\Omega_{\kappa_t}, \mho_{\kappa_t})(t = 1, 2, \ldots, n)$ be a group of PFVs, by using GPFEHG operator then their resulting value is also PFV, and*

$$GPFEHG_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n)$$

$$= \left\langle \sqrt{\frac{\left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} - \left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}{\left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} + \left(\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(1+\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}}, \frac{\sqrt{2\left\{\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right\}^{\frac{1}{\partial}}}}{\sqrt{\left(\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} + \left(\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^2_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}} \right\rangle, \quad (9)$$

Proof: Since

$$\partial \cdot_{\varepsilon} \dot{\kappa} = \left\langle \frac{\sqrt{(1+\Omega^2_{\dot{\kappa}})^{\partial} - (1-\Omega^2_{\dot{\kappa}})^{\partial}}}{\sqrt{(1+\Omega^2_{\dot{\kappa}})^{\partial} + (1-\Omega^2_{\dot{\kappa}})^{\partial}}}, \frac{\sqrt{2\mho^{2\partial}_{\dot{\kappa}}}}{\sqrt{(2-\mho^2_{\dot{\kappa}})^{\partial} + \mho^{2\partial}_{\dot{\kappa}}}} \right\rangle$$ ■

Then

$$\left(\bigotimes_{t=1}^{n}\partial_{t\cdot\varepsilon}(\dot{\kappa}_{\Im(t)})\right)^{\hbar_t}$$

$$=\left\langle \frac{\sqrt{2}\prod_{t=1}^{n}\left(\frac{\sqrt{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}}{\sqrt{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}}\right)^{\hbar_t}}{\sqrt{\prod_{t=1}^{n}\left(2-\frac{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\prod_{t=1}^{n}\left(\frac{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}}, \sqrt{\frac{\prod_{t=1}^{n}\left(1+\frac{2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}-\prod_{t=1}^{n}\left(1-\frac{2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}{\prod_{t=1}^{n}\left(1+\frac{2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\prod_{t=1}^{n}\left(1-\frac{2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}}\right\rangle$$

$$=\left\langle \frac{\left(\frac{\sqrt{2}\prod_{t=1}^{n}\left(\sqrt{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}{\prod_{t=1}^{n}\left(\sqrt{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}\right)}{\sqrt{\prod_{t=1}^{n}\left(\frac{2((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})-(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\left(\frac{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}\right)}}, \sqrt{\frac{\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}-\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}{\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-2(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}}\right\rangle$$

$$=\left\langle \frac{\left(\frac{\sqrt{2}\prod_{t=1}^{n}\left(\sqrt{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}{\prod_{t=1}^{n}\left(\sqrt{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}\right)}{\sqrt{\prod_{t=1}^{n}\left(\frac{2(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+2(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\left(\frac{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}\right)}}, \sqrt{\frac{\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}-\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}{\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}}\right\rangle$$

$$=\left\langle \frac{\left(\frac{\sqrt{2}\left(\sqrt{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})}\right)^{\hbar_t}}{\left(\sqrt{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})}\right)^{\hbar_t}}\right)}{\sqrt{\left(\frac{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}\right)+\left(\frac{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}{\prod_{t=1}^{n}((1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial})^{\hbar_t}}\right)}}, \sqrt{\frac{\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}-\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}{\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}+\prod_{t=1}^{n}\left(\frac{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}}\right)^{\hbar_t}}}\right\rangle$$

Hence

$$\frac{1}{\partial}\left(\bigotimes_{t=1}^{n}\partial\cdot_{\varepsilon}(\dot{\kappa}_{\Im(t)})^{\hbar_t}\right) = \left\langle \sqrt{\frac{\left(1+\frac{2\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}} - \left(1-\frac{2\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}}}{\left(1+\frac{2\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}} + \left(1-\frac{2\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}}}}, \sqrt{\frac{2\left(\frac{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}}}{\left(2-\frac{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}} + \left(\frac{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}}\right)^{\frac{1}{\partial}}}} \right\rangle$$

Thus

$$\mathrm{GPFEHG}_{\bar{\lambda},\hbar}(\kappa_1,\kappa_2,\ldots,\kappa_n) = \left\langle \sqrt{\frac{\left(\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} - \left(\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}{\left(\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} + \left(\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(1+\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(1-\Omega^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}}, \frac{\sqrt{2\left\{\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right\}^{\frac{1}{\partial}}}}{\sqrt{\left(\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}+3\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}} + \left(\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}+3(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}-\prod_{t=1}^{n}\{(2-\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}-(\mho^{2}_{\dot{\kappa}_{\Im(t)}})^{\partial}\}^{\hbar_t}\right)^{\frac{1}{\partial}}}} \right\rangle$$

Theorem 3.4: *Let $\kappa_t = \left\langle \Omega_{\kappa_t}, \mho_{\kappa_t} \right\rangle$ be a group of PFVs, then the following properties hold:*

(i) ***Idempotency:*** *If $\dot{\kappa}_{\Im(t)} = \dot{\kappa}$, then*

$$GPFEHG_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \dot{\kappa}, \tag{10}$$

(ii) ***Boundedness:***

$$\dot{\kappa}_{\min} \leq GPFEHG_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq \dot{\kappa}_{\max}, \tag{11}$$

where $\dot{\kappa}_{\min} = \begin{pmatrix} \Omega_{\dot{\kappa}_{\Im(t)}} & \mho_{\dot{\kappa}_{\Im(t)}} \\ \min_t, & \max_t \end{pmatrix}$ and $\dot{\kappa}_{\max} = \begin{pmatrix} \Omega_{\dot{\kappa}_{\Im(t)}} & \mho_{\dot{\kappa}_{\Im(t)}} \\ \max_t, & \min_t \end{pmatrix}$, which show the minimum and maximum value respectively.

(iii) ***Monotonicity:*** *Let $\kappa_t{}^* = \left\langle \Omega_{\kappa_t{}^*}, \mho_{\kappa_t{}^*} \right\rangle$ $(t = 1, 2, \ldots, n)$ be a family of PFVs with condition, such as $\Omega_{\kappa_t} \leq \Omega^*_{\kappa_t}$ and $\mho_{\kappa_t} \geq \mho^*_{\kappa_t}$ for all j, then*

$$GPFEHG_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq GPFEHG_{\bar{\lambda},\hbar}(\kappa^*_1, \kappa^*_2, \ldots, \kappa^*_n), \tag{12}$$

Proof: (i) **Idempotency:** Since $\dot{\kappa}_{\Im(t)} = \dot{\kappa}$, then

$$\begin{aligned} \text{GPFEHG}_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) &= \frac{1}{\partial}\left(\bigotimes_{t=1}^{n} (\partial \cdot_{\varepsilon} \dot{\kappa}_{\Im(t)})^{\hbar_t}\right) = \frac{1}{\partial}\left(\bigotimes_{t=1}^{n} (\partial \cdot_{\varepsilon} \dot{\kappa})^{\hbar_t}\right) \\ &= \frac{1}{\partial}\left((\partial \cdot_{\varepsilon} \dot{\kappa})^{\sum_{t=1}^{n} \hbar_t} \right) = \dot{\kappa} \end{aligned}$$

(ii) **Boundedness:** We know that $\dot{\kappa}_{\min} = \begin{pmatrix} \Omega_{\dot{\kappa}_{\Im(t)}} & \mho_{\dot{\kappa}_{\Im(t)}} \\ \min_t, & \max_t \end{pmatrix}$ and $\dot{\kappa}_{\max} = \begin{pmatrix} \Omega_{\dot{\kappa}_{\Im(t)}} & \mho_{\dot{\kappa}_{\Im(t)}} \\ \max_t, & \min_t \end{pmatrix}$.

Let GPFEHG $= \dot{\kappa} = (\Omega_{\dot{\kappa}}, \mho_{\dot{\kappa}})$, then by definition (2), we have $(\Omega_{\min}, \mho_{\max}) \leq (\Omega_{\dot{\kappa}}, \mho_{\dot{\kappa}})$ and similarly $(\Omega_{\max}, \mho_{\min}) \geq (\Omega_{\dot{\kappa}}, \mho_{\dot{\kappa}})$. Hence $s(\dot{\kappa}_{\min}) \leq s(\text{GPFEHG})$ and $s(\dot{\kappa}_{\max}) \geq s(\text{GPFEHG})$. Thus we have $\dot{\kappa}_{\min} \leq \text{GPFEHG}_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq \dot{\kappa}_{\max}$.

(iii) **Monotonicity:** As

$$\text{GPFEHG}_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \frac{1}{\partial}\left(\bigotimes_{t=1}^{n} (\partial \cdot_{\varepsilon} \dot{\kappa}_t)^{\hbar_t}\right), \tag{13}$$

and

$$\text{GPFEHG}_{\bar{\lambda},\hbar}(\kappa_1{}^*, \kappa_2{}^*, \ldots, \kappa_n{}^*) = \frac{1}{\partial}\left(\bigotimes_{t=1}^{n} (\partial \cdot_{\varepsilon} \dot{\kappa}^*_t)^{\hbar_t}\right), \tag{14}$$

Since $\Omega_{\kappa_t} \leq \Omega^*_{\kappa_t}$ and $\mho_{\kappa_t} \geq \mho^*_{\kappa_t}$, this means that $\kappa_t \leq \kappa_t{}^*$. Thus $\frac{1}{\partial}\left(\bigotimes_{t=1}^{n} (\partial \cdot_{\varepsilon} \kappa_t)^{\hbar_t}\right) \leq \frac{1}{\partial}\left(\bigotimes_{t=1}^{n} (\partial \cdot_{\varepsilon} \kappa^*_t)^{\hbar_t}\right)$

The proof is completed. ■

Definition 3.2: GPFEHA operator can be defined as:

$$
\begin{aligned}
&\mathrm{GPFEHA}_{\bar{\lambda},\hbar}(\kappa_1,\kappa_2,\ldots,\kappa_n)\\
&=\left\langle \frac{\sqrt{2\left\{\prod_{t=1}^{n}\left\{\left(2-\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}-\prod_{t=1}^{n}\left\{\left(2-\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right\}^{\frac{1}{\partial}}}}{\sqrt{\begin{array}{l}\left(\prod_{t=1}^{n}\left\{\left(2-\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}+3\prod_{t=1}^{n}\left\{\left(2-\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right)^{\frac{1}{\partial}}\\+\left(\prod_{t=1}^{n}\left\{\left(2-\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}-\prod_{t=1}^{n}\left\{\left(2-\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(\Omega^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right)^{\frac{1}{\partial}}\end{array}}},\right.\\
&\left.\sqrt{\frac{\begin{array}{l}\left(\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}+3\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right)^{\frac{1}{\partial}}\\-\left(\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}-\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right)^{\frac{1}{\partial}}\end{array}}{\begin{array}{l}\left(\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}+3\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right)^{\frac{1}{\partial}}\\+\left(\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}+3\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}-\prod_{t=1}^{n}\left\{\left(1+\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}-\left(1-\mho^2_{\dot{\kappa}_{\Im(t)}}\right)^{\partial}\right\}^{\hbar_t}\right)^{\frac{1}{\partial}}\end{array}}}\right\rangle,
\end{aligned}
\tag{15}
$$

where $\dot{\kappa}_{\Im(t)}$ is the largest of the WPFVs $\dot{\kappa}_t(\dot{\kappa}_t = n\bar{\lambda}_t\kappa_t, t = 1,2,\ldots,n)$. $\hbar = (\hbar_1,\hbar_2,\ldots,\hbar_n)^T$ and $\bar{\lambda} = (\bar{\lambda}_1,\bar{\lambda}_2,\ldots,\bar{\lambda}_n)^T$ be the associated and weighted vector respectively, and both have the same condition, such as both belong to the closed interval and their sum is equal to 1 and n is the balancing coefficient, and the parameter ∂ and is $\partial \succ 0$.

Theorem 3.5: *Let $\kappa_t = (\Omega_{\kappa_t}, \mho_{\kappa_t})(t = 1,2,\ldots,n)$ be a group of PFVs, by using GPFEHA operator then their resulting value is also PFV.*

Proof: For proof see Theorem 3. ■

Theorem 3.6: *Let $\kappa_t = (\Omega_{\kappa_t}, \mho_{\kappa_t})$ be a family of PFVs, then the following properties hold:*

(i) ***Idempotency:*** *If $\dot{\kappa}_{\Im(t)} = \dot{\kappa}$, then*

$$GPFEHA_{\bar{\lambda},\hbar}(\kappa_1,\kappa_2,\ldots,\kappa_n) = \dot{\kappa}, \tag{16}$$

(ii) ***Boundedness:***

$$\dot{\kappa}_{\min} \le GPFEHA_{\bar{\lambda},\hbar}(\kappa_1,\kappa_2,\ldots,\kappa_n) \le \dot{\kappa}_{\max}, \tag{17}$$

where $\dot{\kappa}_{\min} = \left(\min_t \Omega_{\dot{\kappa}_{\Im(t)}}, \max_t \mho_{\dot{\kappa}_{\Im(t)}}\right)$ and $\dot{\kappa}_{\max} = \left(\max_t \Omega_{\dot{\kappa}_{\Im(t)}}, \min_t \mho_{\dot{\kappa}_{\Im(t)}}\right)$, which show the minimum and maximum value respectively.

(iii) ***Monotonicity:*** *Let* $\kappa_t{}^* = \langle \Omega_{\kappa_t{}^*}, \mho_{\kappa_t{}^*} \rangle$ *be a family of PFVs with condition, such as* $\Omega_{\kappa_t} \leq \Omega^*_{\kappa_t}$ *and* $\mho_{\kappa_t} \geq \mho^*_{\kappa_t}$ *for all j, then*

$$GPFEHA_{\bar{\lambda},\hbar}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq GPFEHA_{\bar{\lambda},\hbar}(\kappa^*_1, \kappa^*_2, \ldots, \kappa^*_n), \tag{18}$$

Proof: For proof see Theorem 4. ■

4. An Application of the Proposed Aggregation Operators

This section deals with multiattribute decision-making (MADM) problems based on the above-mentioned operators using PFNs. To show the superiority and practicality of the above-mentioned operators in daily life problems an example is also given.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of m options, $C = \{C_1, C_2, \ldots, C_n\}$ be a set of n qualities, and $D = \{D_1, D_2, \ldots, D_k\}$ be a set of k specialists. Let $\hbar = (\hbar_1, \hbar_2, \ldots, \hbar_m)^T$ be the associated vector of $A_t(t = 1, 2, \ldots, m)$ and $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_k)^T$ be the weighted vector of $D^s(s = 1, 2, \ldots, k)$ both have the same conditions belong to the closed interval and their sum is equal to 1. Let $D = (\kappa_{it}) = (\Omega_{it}, \mho_{it})$, where Ω_{it} shows the degree satisfaction and $\mho_{it}$ shows the degree of non-satisfaction with condition $0 \leq (\Omega_{it})^2 + (\mho_{it})^2 \leq 1$.

Step 1: Construct $D^s = [\kappa^{(s)}_{it}]_{m \times n}(s = 1, 2, \ldots, k)$ for decision.

Step 2: If the criteria have two types, such as benefit criteria and cost criteria, then $D^s = [\kappa^{(s)}_{it}]_{m \times n}$ can be converted into the normalised decision matrices, $R^s = [\kappa^{(s)}_{it}]_{m \times n}$ where

$$r^s_{it} = \begin{cases} \kappa^s_{it}, & \text{for benefit criteria } C_t, \\ (\kappa^c_{it})^s, & \text{for cost criteria } C_t, \end{cases} \begin{pmatrix} t = 1, 2, \ldots, n \\ i = 1, 2, \ldots, m \end{pmatrix}$$

and $(\kappa^c_{it})^s$ is the complement of κ^s_{it}.

Step 3: Utilise the proposed operators to aggregate $R^s = [r^{(s)}_{it}]_{m \times n}$ into $R = [r_{it}]_{m \times n}$.

Step 4: Utilise the $\dot{\kappa}_{it} = n\bar{\lambda}_t\kappa_{it}$.

Step 5: Utilise the proposed operators to derive the overall preference values.

Step 6: Calculate the scores of all values.

Step 7: Select that option which has the highest score function.

5. Numerical Example

Suppose, in Hazara University department of mathematics needs to hire a doctor for department.

For this resolution, the university constructs a committee of four decision-makers, whose weight vector is $\varpi = (0.10, 0.20, 0.30, 0.40)^T$. After the first selection five doctors, $A_t(t = 1, 2, 3, 4, 5)$are consider for more process. Committee must take a decision according to the following four attributes: C_1: experience and subject knowledge, C_2: teaching skill, C_3: salary and other facilities, C_4: research skill and publications, where C_1, C_3 are cost type criteria and C_2, C_4 are benefiting type criteria, whose weighted vector is $\hbar = (0.40, 0.30, 0.20, 0.10)^T$.

Step 1: Construct the decision matrices (Tables 1–4)

Step 2: Construct the normalised decision matrices (Tables 5–8)

Table 1. Pythagorean fuzzy decision matrix of D^1.

	C_1	C_2	C_3	C_4
A_1	(0.80,0.50)	(0.70,0.40)	(0.70,0.40)	(0.70,0.50)
A_2	(0.80,0.40)	(0.70,0.50)	(0.80,0.50)	(0.80,0.30)
A_3	(0.50,0.60)	(0.60,0.50)	(0.70,0.50)	(0.80,0.30)
A_4	(0.60,0.50)	(0.60,0.40)	(0.60,0.40)	(0.80,0.40)
A_5	(0.60,0.80)	(0.60,0.60)	(0.70,0.30)	(0.60,0.50)

Table 2. Pythagorean fuzzy decision matrix of D^2.

	C_1	C_2	C_3	C_4
A_1	(0.60,0.50)	(0.80,0.40)	(0.60,0.40)	(0.60,0.50)
A_2	(0.70,0.30)	(0.80,0.40)	(0.70,0.50)	(0.70,0.40)
A_3	(0.60,0.60)	(0.60,0.50)	(0.60,0.60)	(0.70,0.40)
A_4	(0.70,0.50)	(0.60,0.60)	(0.70,0.40)	(0.80,0.50)
A_5	(0.60,0.40)	(0.70,0.20)	(0.80,0.40)	(0.80,0.40)

Table 3. Pythagorean fuzzy decision matrix of D^3.

	C_1	C_2	C_3	C_4
A_1	(0.70,0.50)	(0.70,0.40)	(0.60,0.50)	(0.60,0.50)
A_2	(0.80,0.30)	(0.70,0.30)	(0.80,0.30)	(0.90,0.20)
A_3	(0.60,0.50)	(0.60,0.60)	(0.70,0.40)	(0.80,0.30)
A_4	(0.70,0.50)	(0.80,0.50)	(0.90,0.10)	(0.60,0.50)
A_5	(0.70,0.50)	(0.80,0.20)	(0.80,0.20)	(0.70,0.30)

Table 4. Pythagorean Fuzzy Decision Matrix of D^4

	C_1	C_2	C_3	C_4
A_1	(0.80,0.30)	(0.80,0.40)	(0.70,0.40)	(0.70,0.50)
A_2	(0.80,0.30)	(0.80,0.30)	(0.80,0.30)	(0.80,0.20)
A_3	(0.60,0.60)	(0.70,0.60)	(0.70,0.40)	(0.80,0.30)
A_4	(0.70,0.40)	(0.80,0.60)	(0.80,0.20)	(0.70,0.50)
A_5	(0.60,0.60)	(0.80,0.20)	(0.80,0.20)	(0.80,0.30)

Table 5. Normalised decision matrix R^1.

	C_1	C_2	C_3	C_4
A_1	(0.50,0.80)	(0.70,0.40)	(0.40,0.70)	(0.70,0.50)
A_2	(0.40,0.80)	(0.70,0.50)	(0.50,0.80)	(0.80,0.30)
A_3	(0.60,0.50)	(0.60,0.50)	(0.50,0.70)	(0.80,0.30)
A_4	(0.50,0.60)	(0.60,0.40)	(0.40,0.60)	(0.80,0.40)
A_5	(0.80,0.60)	(0.60,0.60)	(0.30,0.70)	(0.60,0.50)

Step 3: Utilise the PFEWA operator, where $\varpi = (0.10, 0.20, 0.30, 0.40)^T$, then (Table 9)
Step 4: Utilise $\dot{\kappa}_{it} = n\bar{\lambda}_t\kappa_t$, where $\bar{\lambda} = (0.40, 0.30, 0.20, 0.10)^T$, and then we have

$$\dot{\kappa}_{11} = (0.542, 0.572);\ \dot{\kappa}_{12} = (0.815, 0.318);\ \dot{\kappa}_{13} = (0.387, 0.718);\ \dot{\kappa}_{14} = (0.424, 0.793)$$

$$\dot{\kappa}_{41} = (0.578, 0.518);\ \dot{\kappa}_{42} = (0.805, 0.470);\ \dot{\kappa}_{43} = (0.232, 0.830);\ \dot{\kappa}_{44} = (0.453, 0.787)$$

$$\dot{\kappa}_{51} = (0.700, 0.439);\ \dot{\kappa}_{52} = (0.818, 0.156);\ \dot{\kappa}_{53} = (0.249, 0.832);\ \dot{\kappa}_{54} = (0.505, 0.699)$$

Table 6. Normalised decision matrix R^2.

	C_1	C_2	C_3	C_4
A_1	(0.50,0.60)	(0.80,0.40)	(0.40,0.60)	(0.60,0.50)
A_2	(0.30,0.70)	(0.80,0.40)	(0.50,0.70)	(0.70,0.40)
A_3	(0.60,0.60)	(0.60,0.50)	(0.60,0.60)	(0.70,0.40)
A_4	(0.50,0.70)	(0.60,0.60)	(0.40,0.70)	(0.80,0.50)
A_5	(0.40,0.60)	(0.70,0.20)	(0.40,0.80)	(0.80,0.40)

Table 7. Normalised decision matrix R^3.

	C_1	C_2	C_3	C_4
A_1	(0.50,0.70)	(0.70,0.40)	(0.50,0.60)	(0.60,0.50)
A_2	(0.30,0.80)	(0.70,0.30)	(0.30,0.80)	(0.90,0.20)
A_3	(0.50,0.60)	(0.60,0.60)	(0.40,0.70)	(0.80,0.30)
A_4	(0.50,0.70)	(0.80,0.50)	(0.10,0.90)	(0.60,0.50)
A_5	(0.50,0.70)	(0.80,0.20)	(0.20,0.80)	(0.70,0.30)

Table 8. Normalised decision matrix R^4.

	C_1	C_2	C_3	C_4
A_1	(0.30,0.80)	(0.80,0.40)	(0.40,0.70)	(0.70,0.50)
A_2	(0.30,0.80)	(0.80,0.30)	(0.30,0.80)	(0.80,0.20)
A_3	(0.60,0.60)	(0.70,0.60)	(0.40,0.70)	(0.80,0.30)
A_4	(0.40,0.70)	(0.80,0.60)	(0.20,0.80)	(0.70,0.50)
A_5	(0.60,0.60)	(0.80,0.20)	(0.20,0.80)	(0.80,0.30)

Table 9. Collective normalised decision matrix R.

	C_1	C_2	C_3	C_4
A_1	(0.432,0.728)	(0.764,0.400)	(0.432,0.649)	(0.653,0.500)
A_2	(0.311,0.779)	(0.764,0.335)	(0.372,0.779)	(0.653,0.500)
A_3	(0.572,0.589)	(0.643,0.568)	(0.459,0.679)	(0.782,0.317)
A_4	(0.572,0.589)	(0.753,0.546)	(0.259,0.789)	(0.684,0.489)
A_5	(0.568,0.629)	(0.767,0.224)	(0.263,0.789)	(0.757,0.335)

Table 10. Pythagorean fuzzy hybrid decision matrix R.

	C_1	C_2	C_3	C_4
A_1	(0.815,0.318)	(0.542,0.572)	(0.387,0.718)	(0.424,0.793)
A_2	(0.815,0.318)	(0.564,0.627)	(0.393,0.648)	(0.333,0.824)
A_3	(0.704,0.390)	(0.695,0.485)	(0.527,0.687)	(0.411,0.793)
A_4	(0.704,0.390)	(0.578,0.518)	(0.453,0.787)	(0.232,0.830)
A_5	(0.818,0.156)	(0.700,0.439)	(0.505,0.699)	(0.249,0.832)

Now using definition 2, and calculating the score functions, we have (Tables 10 and 11)

$$s(\dot{\kappa}_{53}) = (0.249, 0.832) = -0.630; s(\dot{\kappa}_{54}) = (0.505, 0.699) = -0.233$$

6. Compare with the Other Methods

To show the practicality and effectiveness of the proposed methods and operators, we can compare the proposed methods with some existing methods. First, we compare the

Table 11. Ranking of the alternative at different values of ∂.

∂	Operators	Score function	Ranking
$\partial \to 1$	PFEHAPFEHGGPFEHAGPFEHG	$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
$\partial \to 2$	PFEHAPFEHGGPFEHAGPFEHG	$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_4) \succ s(r_3) \succ s(r_2)$	(5, 1, 4, 3, 2)
		$s(r_5) \succ s(r_1) \succ s(r_4) \succ s(r_3) \succ s(r_2)$	(5, 1, 4, 3, 2)
$\partial \to 5$	PFEHAPFEHGGPFEHAGPFEHG	$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_3) \succ s(r_4) \succ s(r_2)$	(5, 1, 3, 4, 2)
		$s(r_5) \succ s(r_1) \succ s(r_3) \succ s(r_4) \succ s(r_2)$	(5, 1, 3, 4, 2)
$\partial \to 10$	PFEHAPFEHGGPFEHAGPFEHG	$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_2) \succ s(r_3) \succ s(r_4)$	(5, 1, 2, 3, 4)
		$s(r_5) \succ s(r_1) \succ s(r_4) \succ s(r_3) \succ s(r_2)$	(5, 1, 4, 3, 2)
		$s(r_5) \succ s(r_1) \succ s(r_4) \succ s(r_3) \succ s(r_2)$	(5, 1, 4, 3, 2)

propose methods with methods, such as Pythagorean fuzzy hybrid geometric aggregation operator and Pythagorean fuzzy hybrid averaging aggregation operator proposed by Rahman et al. [24,26], are based on algebraic operations, and those in this paper are based on the generalised Einstein operations. Because the generalised Einstein operations for Pythagorean fuzzy numbers are with parameter ∂, the methods proposed in this paper are more general and more flexible. Secondly, we can compare with Einstein operators, such as Pythagorean fuzzy Einstein hybrid averaging aggregation operator and Pythagorean fuzzy Einstein hybrid geometric aggregation operator proposed by Rahman et al. [31,32], they are only the special cases of the proposed operators in this paper. The proposed methods can be comparing the methods proposed by Garg [33,34], in which the author introduced several operators such as GPFEWA operator, GPFEWG operator GPFEOWA operator and GPFEOWG operator. Actually, GPFEWA operator and GPFEWG operator weigh only the Pythagorean fuzzy arguments, while GPFEOWA operator and GPFEOWG operator weigh only the ordered positions of the Pythagorean fuzzy arguments instead of weighing the Pythagorean fuzzy arguments themselves. To overcome these limitations, we introduce the concept of GPFEHA operator and GPFEHG operator which weigh both the given Pythagorean fuzzy value and its ordered position. Thus the proposed operators are the generalisation of the existing methods.

6.1. Benefit of the Proposed Operators

Generalised Einstein operators combine Einstein operators with some generalised operators using Pythagorean fuzzy values. The main benefit of the proposed operators is that these operators deliver more general, more correct and precise results as compared to their existing methods. Therefore these methods play a vital role in real world problems.

7. Conclusion

The objective of this paper is to investigate the generalised Einstein hybrid operators based on PFNs and their application for daily life problems. Firstly, we have investigated two generalised Einstein operators along with their properties, namely the generalised Pythagorean fuzzy Einstein hybrid averaging operator and the generalised Pythagorean fuzzy Einstein hybrid geometric operator by combining the parameter of the decision-making ∂ during the calculation process. Furthermore, we have industrialised a technique for multi-criteria decision-making based on these operators, and the operational procedures have proved in detail. The suggested methodology can be used for any type of selection problem involving any number of selection attributes. We ended the paper with an application of the new approach in a decision-making problem. Garg [33,34] introduced the notion of GPFEWA operator, GPFEOWA operator, GPFEWG operator, GPFEOWG operator. Actually, GPFEWA operator and GPFEWG operator weigh only the Pythagorean fuzzy arguments, GPFEOWA operator and GPFEOWG operator weigh only the ordered positions of the Pythagorean fuzzy arguments instead of weighing the Pythagorean fuzzy arguments themselves. To overcome these limitations, we introduce the concept of GPFEHA operator and GPFEHG operator which weigh both the given Pythagorean fuzzy value and its ordered position. Thus the proposed operators are the generalisation of the existing methods.

In the future, we will extend the proposed approach to the different environment and then will apply to the fields of the pattern recognition, Symmetric operator, Inducing variable, Logarithmic operator, Power operator, Hamacher operator**,** Dombi operator**,** Linguistic terms, Confidence levels, Interval valued etc.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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References

[1] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Sets Syst. 1986;20(1):87–96.
[2] Zadeh LA. Fuzzy sets. Inf Control. 1965;8(3):338–353.
[3] Xu ZS. Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst. 2007;15(6):1179–1187.
[4] Xu ZS, Yager RR. Some geometric aggregation operators based on intuitionistic fuzzy sets. Int J Gen Syst. 2006;35(4):417–433.
[5] Wang W, Liu X. Intuitionistic fuzzy geometric aggregation operators based on Einstein operations. Int J Intell Syst. 2011;26(11):1049–1075.

[6] Wang W, Liu X. Intuitionistic fuzzy information aggregation using Einstein operations. IEEE Trans Fuzzy Syst. 2012;20(5):923–938.
[7] Liu W, Liao H. A Bibliometrics analysis of fuzzy decision research during 1970–2015. Int J Fuzzy Syst. 2017;19(1):1–14.
[8] Yu D, Liao H. Visualization and quantitative research on intuitionistic fuzzy studies. J Intell Fuzzy Syst. 2016;30(6):3653–3663.
[9] Liao H, Xu Z. Intuitionistic fuzzy hybrid weighted aggregation operators. Int J Intell Syst. 2014;29(11):971–993.
[10] Xu Z, Da QL. The ordered weighted geometric averaging operators. Int J Intell Syst. 2002;17(7):709–716.
[11] Xu Z, Hu H. Projection models for intuitionistic fuzzy multiple attributes decision making. Int J Inf Technol Decis Mak. 2010;9(2):267–280.
[12] Xu Z, Cai XQ. Recent advances in intuitionistic fuzzy information aggregation. Fuzzy Optim Decis Mak. 2010;9(4):359–381.
[13] Xu Z. Intuitionistic fuzzy aggregation and Clustering. Berlin: Springer Verlag; 2013.
[14] Gou X, Xu Z, Ren P. The properties of continuous Pythagorean fuzzy information. Int J Intell Syst. 2016;31(5):401–424.
[15] Ma ZM, Xu Z. Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multi-criteria decision making problems. Int J Intell Syst. 2016;31(12):1198–1219.
[16] Liang D, Xu Z. Projection model for fusing information in Pythagorean fuzzy multi-criteria group decision making based on geometric Bonferroni mean. Int J Intell Syst. 2017;32(9):966–987.
[17] Ren P, Xu Z, Gou X. Pythagorean fuzzy TODIM approach to multi-criteria decision making. Appl Soft Comput. 2016;42:246–259.
[18] Xu Z. On consistency of the weighted geometric mean complex judgement matrix in AHP. Eur J Oper Res. 2000;126(3):683–687.
[19] Yager RR. Pythagorean fuzzy subsets. In Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada. 2013. p. 57–61.
[20] Yager RR, Abbasov AM. Pythagorean membership grades, complex numbers and decision making. Int J Intell Syst. 2013;28(5):436–452.
[21] Rahman K, Abdullah S, Husain F, et al. Approaches to Pythagorean fuzzy geometric aggregation operators. Int J Comp Sci Inf Secur. 2016;14(9):174–200.
[22] Rahman K, Khan MSA, Ullah M, et al. Multiple attribute group decision making for plant location selection with Pythagorean fuzzy weighted geometric aggregation operator. The Nucleus. 2017;54(1):66–74.
[23] Rahman K, Abdullah S, Husain F, et al. Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making. J Appl Environ Biol Sci. 2017;7(4):67–83.
[24] Rahman K, Abdullah S, Khan MSA, et al. Pythagorean fuzzy hybrid geometric aggregation operator and their applications to multiple attribute decision making. Int J Comput Sci Inf Secur. 2016;14(6):837–854.
[25] Rahman K, Ali A, Shakeel M, et al. Pythagorean fuzzy weighted averaging aggregation operator and its application to decision making theory. The Nucleus. 2017;54(3):190–196.
[26] Rahman K, Husain F, Khan MSA. Pythagorean fuzzy hybrid averaging aggregation operator and its application to multiple attribute decision making. Ital J Pure Appl Math. 2018;40:180–187.
[27] Rahman K, Khan MSA, Abdullah S, et al. Some properties of Pythagorean fuzzy hybrid averaging aggregation operator. J Appl Environ Biol Sci. 2017;7(2):122–133.
[28] Rahman K, Abdullah S, Ali A, et al. Some induced averaging aggregation operators based on Pythagorean fuzzy numbers. Mathematics Letters. 2017;3(4):40–45.
[29] Rahman K, Abdullah S, Ahmed R, et al. Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making. J Intell Fuzzy Syst. 2017;33(1):635–647.
[30] Rahman K, Abdullah S, Khan MSA, et al. Some basic operations on Pythagorean fuzzy sets. J Appl Environ Biol Sci. 2017;7(1):111–119.

[31] Rahman K, Abdullah S, Ali A, et al. Pythagorean fuzzy Einstein hybrid averaging aggregation operator and its application to multiple-attribute group decision making. J Intell Syst. 2018. DOI:10.1515/jisys-2018-0071.

[32] Rahman K, Ali A. New approach to multiple attribute group decision-making based on Pythagorean fuzzy Einstein hybrid geometric operator. Granular Computing. 2019. DOI:10.1007/s41066-019-00166-6.

[33] Garg H. A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. Int J Intell Syst. 2016;31(9):886–920.

[34] Garg H. Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process. Int J Intell Syst. 2017;32(6):597–630.

[35] Garg H. Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process. Comput Math Organ Theory. 2017;23(4):546–571.

[36] Garg H. New logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications. Int J Intell Syst. 2019;34(1):82–106.

[37] Garg H. Hesitant Pythagorean fuzzy Maclaurin symmetric mean operators and its applications to multiattribute decision-making process. Int J Intell Syst. 2019;34(4):601–626.

[38] Garg H. Linguistic Pythagorean fuzzy sets and its applications in multi attribute decision making process. Int J Intell Syst. 2018;33(6):1234–1263.

[39] Garg H. Hesitant Pythagorean fuzzy sets and their aggregation operators in multiple-attribute decision-making. Int J Uncertain Quantif. 2018;8(3):267–289.

[40] Garg H. A new exponential operational laws and their aggregation operators of interval-valued Pythagorean fuzzy information. Int J Intell Syst. 2018;33(3):653–683.

[41] Garg H. Neutrality operations–based Pythagorean fuzzy aggregation operators and its applications to multiple attribute group decision–making process. J Ambient Intell Humaniz Comput. 2019. DOI:10.1007/s12652-019-01448-2.

[42] Garg H. Novel neutrality operation–based Pythagorean fuzzy geometric aggregation operators for multiple attribute group decision analysis. Int J Intell Syst. 2019;34(10):2459–2489.

[43] Garg H. Generalized Pythagorean fuzzy geometric interactive aggregation operators using Einstein operations and their application to decision making. J Exp Theor Artif Intell. 2018;30(6):763–794.

[44] Akram M, Ilyas F, Garg H. Multi-criteria group decision making based on ELECTRE I method in Pythagorean fuzzy information. Soft comput. DOI:10.1007/s00500-019-04105-0.

[45] Zang X, Xu ZS. Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets. Int J Intell Syst. 2014;29(12):1061–1078.