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Fuzzy Linear Codes

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ABSTRACT

In this paper, we define the notion of fuzzy linear code, fuzzy cyclic code over a Galois ring $\mathbb{Z}_{p^k}$, fuzzy Gray map and we use it to construct fuzzy $\mathbb{Z}_{p^k}$-linear codes and fuzzy $\mathbb{Z}_{p^k}$-cyclic codes.

KEYWORDS

Code over Galois ring; fuzzy linear code; fuzzy cyclic code; fuzzy generalised gray map

1. Introduction

When we study a subject, we always encode its information and decode the received information, this is what the classical code theory deals with, and the information which we handle are certain. However, for uncertain information, the classical code theory has less efficient methods. Since fuzzy mathematics has nice applications when dealing with fuzziness, we try to use the methods of fuzzy mathematics to conduct fuzzy information. The notion of fuzzy subsets was first developed by Zadeh [1] in which imprecise knowledge can be used to define an event. The importance of fuzzy sets comes from the fact that it can deal with imprecise and inexact information. The concept of fuzzy modules was introduced by Negoita and Ralescu [2] while the notion of fuzzy submodule was introduced by Maschinchi and Zahedi [3]. Shum and De Gang [4] introduced the concept of fuzzy linear code over finite fields.

If the data from the information channel is uncertain, then the ordinary method of decoding can not deal with it. For instance, let c be an information of the subject that we study. Since the data from the information channel is uncertain, we can not make sure wether or not the subject (to the arrival) has this information again. We only can estimate the degree of which it possesses the information c and assign a corresponding degree in [0,1]. If for every information there is such a number corresponding to it, then we can get a fuzzy subset A of the block code, we call it the *fuzzy code*.

In this paper, we mainly define the fuzzy linear code and fuzzy cyclic code over the ring $\mathbb{Z}_{p^k}$. We also use the fuzzy generalised Gray map to define $\mathbb{Z}_{p^k}$-fuzzy linear codes and we study the basic properties of all these types of codes.

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2. Preliminaries

In this section, we shall formulate the preliminary definitions and results that are required in the sequel (for references see [2,5]).

Definition 2.1: Let S be a non-empty set.

A fuzzy subset A of S is a function of S into the closed interval $[0, 1]$.

Definition 2.2: Let S be a non-empty set with an additive and multiplicative operation, and let A and B be two fuzzy subsets of S. Then:

- $(A \cap B)(x) = \min\{A(x), B(x)\}$, for all $x \in S$.
- $(A \cup B)(x) = \max\{A(x), B(x)\}$, for all $x \in S$.
- $A = B$ if and only if $A(x) = B(x)$, for all $x \in S$.
- $(A + B)(x) = \max\{A(y) \wedge B(z) \mid x = y + z$ with $y, z \in S\}$, for all $x \in S$.
- $(AB)(x) = \max\{A(y) \wedge B(z) \mid x = yz$ with $y, z \in S\}$, for all $x \in S$.
- $A \subseteq B$ if and only if $A(x) \leq B(x)$, for all $x \in S$.

From now on $(R, +, \cdot)$ will denote a commutative unitary ring or simply $\mathbb{Z}_{p^k}$, where p is a prime integer and $k \in \mathbb{N}$, $k \neq 0$.

Definition 2.3: A R-module M consists of an abelian group $(M, \oplus)$ and an operation $* : R \times M \to M$ (called scalar multiplication, usually just written by juxtaposition, i.e. as rx instead of $r * x$ for $r \in R$ and $x \in M$) such that for all $r, s \in R$, $x, y \in M$, we have

(a) $r(x \oplus y) = rx \oplus ry$,
(b) $(r + s)x = rx \oplus sx$,
(c) $(rs)x = r(sx)$,
(d) $1_R x = x$ where 1_R is the multiplicative identity of the ring R.

From [6,7] we recall the following definition in the fuzzy linear space.

Definition 2.4: A fuzzy subset X of a R-module M is called a *fuzzy submodule* of M if for all $x, y \in M$ and $r \in R$, we have.

(a) $X(x \oplus y) \geq \min\{X(x), X(y)\}$.
(b) $X(rx) \geq X(x)$.

Remark 2.5: If X is a fuzzy submodule of M, then from $b)$ in Definition 2.4 follows $(\forall\, x \in M)(X(0) \geq X(x))$.

Definition 2.6: Let A be a fuzzy subset of a nonempty set M. For $t \in [0, 1]$, the sets $A_t = \{x \in M : A(x) \geq t\}$ and $\overline{A_t} = \{x \in M : A(x) \leq t\}$ are called the *upper t-level cut* and *lower t-level cut* of A, respectively.

Proposition 2.7: *Let M be a R-module. A nonempty subset N of M is a submodule of M if and only if the characteristic function of N is a fuzzy submodule.*

Proposition 2.8: *A is a fuzzy submodule of a R-module M if and only if for all* $\alpha, \beta \in R, x, y \in M$, *we have* $A(\alpha x \oplus \beta y) \geq \min\{A(x), A(y)\}$.

Proof: The proof is similar to the one for fields in [4], just change a field by ring. ■

Definition 2.9: A fuzzy subset I of a ring R is called a fuzzy ideal of R if for each $x, y \in R$:

(a) $I(x - y) \geq \min\{I(x), I(y)\}$.
(b) $I(x \cdot y) \geq \max\{I(x), I(y)\}$.

Let G be a group and R a ring. We denote by RG the set of all formal linear combinations of the form $\alpha = \sum_{g\in G} a_g g$ (where $a_g \in R$ and $a_g = 0$ almost everywhere, that is only a finite number of coefficients are different from zero in each of these sums).

Let $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$ in RG. We define their sum in RG componentwise by: $\alpha + \beta = \sum_{g\in G}(a_g + b_g)g$ and their product by: $\alpha\beta = \sum_{g,h\in G} a_g b_h gh$.

With the operations above, RG is a unitary ring, with $1 = \sum_{g\in G} u_g g$ where the coefficient corresponding to the unit element of the group is equal to 1 and $u_g = 0$ for every other element $g \in G$.

We can also define a product of elements in RG by elements $\lambda \in R$ as $\lambda(\sum_{g\in G} a_g g) = \sum_{g\in G}(\lambda a_g)g$.

Definition 2.10: The set RG with the operations defined above is called the *group ring of G over R*, if R is commutative, then RG is called the *group algebra of G over R*.

Definition 2.11: Let A be a fuzzy subset of the group ring (RG) which is the group algebra of $\langle x\rangle$ over the ring $\mathbb{Z}_{p^k}$, note that x is an invertible element of $\mathbb{Z}_{p^k}$. If for all $\alpha, \beta \in$ RG,

(a) $A(\alpha.\beta) \geq \max\{A(\alpha), A(\beta)\}$;
(b) $A(\alpha - \beta) \geq \min\{A(\alpha), A(\beta)\}$,

then A is call a *fuzzy ideal* of RG.

Proposition 2.12: *A is a fuzzy ideal of RG if and only if for all* $t \in [0, 1]$, *if* $A_t \neq \emptyset$, *then* A_t *is an ideal of RG.*

Proof: The proof follows from the transfer principle in [8]. ■

Definition 2.13: Let n be an integer. A *linear code* of length n over $\mathbb{Z}_{p^k}$ is a submodule of $\mathbb{Z}^n_{p^k}$.

In contrast to vector spaces, modules do not admit a basis in general. However modules possess a generating family and therefore a generating matrix, but the decomposition of the elements with respect to this family is not necessarily unique.

Definition 2.14: A *generating matrix* of some linear code over $\mathbb{Z}_{p^k}$ is a matrix in $\mathcal{M}(\mathbb{Z}_{p^k})$, where the lines are the minimal generating family of code.

Definition 2.15: Let C_{p^k} and C'_{p^k} be two linear codes over $\mathbb{Z}_{p^k}$ with generating matrices G and G' respectively. The codes C_{p^k} and C'_{p^k} are *equivalent* if there exists a permutation matrix P, such that $G' = GP$ (the product of the two matrices G and P).

Definition 2.16: Let C_{p^k} be a linear code of length n over $\mathbb{Z}_{p^k}$, the *dual* of the code C_{p^k} which we denote by $C^{\perp}_{p^k}$ is the submodule of $\mathbb{Z}^n_{p^k}$ defined by: $C^{\perp}_{p^k} = \{a|$ for all $b \in C_{p^k}, \langle a, b\rangle = 0\}$, where $\langle , \rangle$ is the inner product.

Remark 2.17: $C^{\perp}_{p^k}$ is also a linear code over $\mathbb{Z}_{p^k}$.

Definition 2.18: A linear code C_{p^k} of length n over $\mathbb{Z}_{p^k}$ is *cyclic* if it is invariant by the shift map s, define by $s((a_0, \ldots, a_{n-1})) = (a_{n-1}, a_0, \ldots, a_{n-2})$. i.e. if $(a_0, \ldots, a_{n-1}) \in C_{p^k}$, then $s((a_0, \ldots, a_{n-1})) \in C_{p^k}$.

3. Fuzzy Linear Code Over $\mathbb{Z}_{p^k}$

In this section are defined and studied fuzzy linear linear code over the Galois ring $\mathbb{Z}_{p^k}$. There are also characterised by using the transfer principle [8].

Definition 3.1: Let $M = \mathbb{Z}^n_{p^k}$ be a $\mathbb{Z}_{p^k}$-module. A fuzzy submodule A of M is called a *fuzzy linear code* of length n over $\mathbb{Z}_{p^k}$.

Example 3.2: Let $\mathbb{Z}_4$ be a ring, then $\mathbb{Z}_4$ is a $\mathbb{Z}_4$-module.

Let $A : \mathbb{Z}_4 \to [0, 1]$ be the map such that $A(0) = A(1) = A(2) = A(3) = t$ ($t \in [0, 1]$), then A is a fuzzy subset of $\mathbb{Z}_4$. It is obvious that A is a fuzzy $\mathbb{Z}_4$-module. Therefore A is a fuzzy linear code over $\mathbb{Z}_4$.

Proposition 3.3: *Let A be a fuzzy subset of $\mathbb{Z}^n_{p^k}$.*

A is a fuzzy linear code of length n over $\mathbb{Z}_{p^k}$ if and only if for any $t \in [0, 1]$, if $A_t \neq \emptyset$, then A_t is a linear code of length n over $\mathbb{Z}_{p^k}$.

Proof: Use the transfer principle in [8]. ■

Corollary 3.4: *Let C be a subset of $\mathbb{Z}^n_{p^k}$.*

C is a linear code of length n over $\mathbb{Z}_{p^k}$ if and only if the characteristic function χ_C of C is a fuzzy linear code over $\mathbb{Z}_{p^k}$.

Proposition 3.5: *Let A be a fuzzy subset of $\mathbb{Z}^n_{p^k}$.*

A is a fuzzy linear code of length n over $\mathbb{Z}_{p^k}$ if and only if the characteristic function of any upper t-level cut $A_t \neq \emptyset$ for $t \in [0, 1]$ is a fuzzy linear code of length n over $\mathbb{Z}_{p^k}$.

Proof: Uses Proposition 3.3 and Corollary 3.4. ■

Remark 3.6: We remark the following:

(i) In the Example 3.2, $A_t = \mathbb{Z}_4$ (for all $t \in [0, 1]$) which is a linear code over $\mathbb{Z}_4$.

(ii) If M is a module over the ring $\mathbb{Z}_{p^k}$ and A a fuzzy linear code A on M such that $\forall\, x \in M$, $A(x) = t$ (where $t \in [0, 1]$), then A is called the *trivial fuzzy linear code* over $\mathbb{Z}_{p^k}$.

Example 3.7: Consider a fuzzy subset A of $\mathbb{Z}_4$ as follows:

$$A : \mathbb{Z}_4 \to [0,1], \quad x \mapsto \begin{cases} 1 & \text{if } x = 0; \\ \frac{1}{3} & \text{if } x = 1; \\ \frac{1}{3} & \text{if } x = 2; \\ \frac{1}{3} & \text{if } x = 3. \end{cases}$$

Then A is a fuzzy submodule of the $\mathbb{Z}_4$-module $\mathbb{Z}_4$, hence A is a fuzzy linear code over $\mathbb{Z}_4$.

Remark 3.8: Let A be a fuzzy linear code of length n over $\mathbb{Z}_{p^k}$. Since $\mathbb{Z}_{p^k}^n$ is a finite set, the image $\mathrm{Im}(A) = \{A(x) \mid x \in \mathbb{Z}_{p^k}^n\}$ is finite as well. Assume that all elements in $\mathrm{Im}(A)$ satisfy: $t_1 > t_2 > \cdots > t_m$ (where $t_i \in [0, 1]$) i.e. $\mathrm{Im}(A)$ has m elements. Since A_{t_i} is a linear code over $\mathbb{Z}_{p^k}$, let G_{t_i} be its generator matrix. Thus A can be determined by m matrixes $G_{t_1}, G_{t_2}, \ldots, G_{t_m}$ (see Theorem 4.7).

Definition 3.9: Let S be a non-empty set.

Let $x_t : S \to [0, 1]$ be a fuzzy subset of S, (where $x \in S$, $t \in [0, 1]$) defined by:

$$x_t(y) = \begin{cases} t & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}, \quad \text{for all } y \in S.$$

x_t is called a *fuzzy singleton or fuzzy point* of S.

Definition 3.10: Let A_1 and A_2 be two fuzzy linear codes over $\mathbb{Z}_{p^k}$ of the same length n. The *residual quotient* of A_1 and A_2 denoted by $(A_1 : A_2)$ is the fuzzy subset of $\mathbb{Z}_{p^k}$ defined by: $(A_1 : A_2)(r) = \sup\{t \in [0, 1] \mid r_t A_2 \subseteq A_1\}$ for all $r \in \mathbb{Z}_{p^k}$.

That is $(A_1 : A_2) = \{r_t \mid r_t A_2 \subseteq A_1, r_t \text{ is fuzzy singleton of } \mathbb{Z}_{p^k}\}$.

Theorem 3.11 ([2]): *Let A_1 and A_2 be two fuzzy linear codes of length n over $\mathbb{Z}_{p^k}$, then the residual quotient $(A_1 : A_2)$ of A_1 and A_2 is a fuzzy ideal of $\mathbb{Z}_{p^k}$.*

Definition 3.12: Let A and B be two fuzzy submodules of a module $M = \mathbb{Z}_{p^k}^n$ over the ring $\mathbb{Z}_{p^k}$. We say that A is *orthogonal* to B if $Im(B) = \{1 - c \mid c \in Im(A)\}$ and for all $t \in [0, 1]$, $B_{1-t} = (A_t)^{\perp} = \{y \in M \mid \langle x, y\rangle = 0, \text{for all } x \in A_t\}$, where $\langle , \rangle$ is the inner product on M

Let A and B be two fuzzy submodules of a module M. We denote A orthogonal to B by $A \perp B$.

Example 3.13: Consider the fuzzy submodules A and B of $\mathbb{Z}_4$ defined as follows:

$$A : \mathbb{Z}_4 \to [0,1], \quad x \mapsto \begin{cases} \frac{1}{2} & \text{if } x = 0; \\ \frac{1}{4} & \text{if } x = 1; \\ \frac{1}{3} & \text{if } x = 2; \\ \frac{1}{4} & \text{if } x = 3. \end{cases} \quad \text{and} \quad B : \mathbb{Z}_4 \to [0,1], \quad x \mapsto \begin{cases} \frac{3}{4} & \text{if } x = 0; \\ \frac{1}{2} & \text{if } x = 1; \\ \frac{2}{3} & \text{if } x = 2; \\ \frac{1}{2} & \text{if } x = 3. \end{cases}$$

We have:

$$A_{1/2} = \{0\} \quad \text{and} \quad B_{1/2} = \mathbb{Z}_4,$$
$$A_{1/4} = \mathbb{Z}_4 \quad \text{and} \quad B_{3/4} = \{0\},$$
$$A_{1/3} = \{0,2\} \quad \text{and} \quad B_{2/3} = \{0,2\}.$$

Therefore $A \perp B$.

Remark 3.14: Let A be a fuzzy submodule of a module M such that for all $x \in M$, $A(x) = \gamma$ (with $\gamma \in [0,1]$), then it does not exist a fuzzy subset B of M such that $A \perp B$.

The Remark 3.14 shows that the orthogonal of some fuzzy submodule does not always exist, so it is important to see under which conditions the orthogonal of a fuzzy submodule exists. The following theorem shows the existence of the orthogonal of some fuzzy submodule.

Theorem 3.15: *Let A be a fuzzy submodule of a finite module $M = \mathbb{Z}^n_{p^k}$. Then there exists a fuzzy submodule B of M such that $A \perp B$ if and only if $|\mathrm{Im}(A)| > 1$ and for any $\gamma \in \mathrm{Im}(A)$ there exist $\epsilon \in \mathrm{Im}(A)$ such that $A_\gamma = (A_\epsilon)^\perp$.*

Proof: Let A be a fuzzy submodule of M. Assume that $|\mathrm{Im}(A)| = m > 1$ and for any $\gamma \in \mathrm{Im}(A)$ there exists $\epsilon \in \mathrm{Im}(A)$ such that $A_\gamma = (A_\epsilon)^\perp$.

Assume that $\mathrm{Im}(A) = \{t_1 > t_2 > \cdots > t_m\}$. Define the sets $M_i = \{x \in M \mid A(x) = t_i\}$, for $i = 1, \ldots, m$. These sets form a partition of M.

We define a fuzzy set B as follows: $B : M \to [0,1], x \mapsto 1 - t_{m-i+1}$, if $x \in M_i$.

Since $\mathrm{Im}(A) = \{t_1 > t_2 > \cdots > t_m\}$, we have $A_{t_1} \subseteq A_{t_2} \subseteq \cdots \subseteq A_{t_m}$. As for any $\gamma \in \mathrm{Im}(A)$ there exists $\epsilon \in \mathrm{Im}(A)$ such that $A_\gamma = (A_\epsilon)^\perp$, with the properties that we know about the orthogonal over the finite module, we conclude that $A_{t_i} = (A_{t_{m-i+1}})^\perp$. Thus $B_{1-t_{m-i+1}} = \{x \in M \mid B(x) \geq 1 - t_{m-i+1}\} = M_i \cup M_{i-1} \cup \cdots \cup M_1 = A_{t_i} = (A_{t_{m-i+1}})^\perp$. Thus B is the fuzzy submodule we need.

Conversely if there exists a fuzzy submodule B of M such that $A \perp B$, then by Definition 3.12, $|\mathrm{Im}(A)| > 1$. Since $\forall\, t \in [0,1]$, $B_{1-t} = (A_t)^\perp$, then $\forall \gamma \in \mathrm{Im}(A)$, there exist $\epsilon \in \mathrm{Im}(A)$, such that $A_\gamma = (A_\epsilon)^\perp$, because $Im(B) = \{1 - t \mid t \in \mathrm{Im}(A)\}$. ∎

The following result shows the uniqueness of the orthogonal of some fuzzy submodule.

Theorem 3.16: *Let A, B and C be three fuzzy submodules of a module M, such that $A \perp B$ and $A \perp C$, then $B = C$.*

Proof: Assume that $A \perp B$ and $B \perp C$. Let $t \in [0, 1]$, and $y \in B_{1-t}$. Then $\langle x, y \rangle = 0$, for all $x \in A_t$. Thus $y \in C_{1-t}$ and $B_{1-t} \subseteq C_{1-t}$. Therefore $C_t \subseteq B_t$. In the same way, we show that $B_t \subseteq C_t$.
Therefore $B = C$. ■

Corollary 3.17: *Let A be a fuzzy submodule of a finite module M such that there exists a fuzzy set B on M orthogonal to A, then B is a fuzzy submodule of M.*

Corollary 3.18: *Let A be a fuzzy submodule of M. If $A^{\perp}$ exists, then $(A^{\perp})^{\perp} = A$.*

Definition 3.19: Let A and B be two fuzzy linear codes over $\mathbb{Z}_{p^k}$. A and B are *equivalent fuzzy linear codes* over $\mathbb{Z}_{p^k}$ if for all $t \in [0, 1]$, the linear codes A_t and B_t are equivalent.

Example 3.20: (1) All fuzzy linear code are equivalent to itself.

(2) Let C_{G_1} and C_{G_2} be two equivalent linear codes of length n over $\mathbb{Z}_{p^k}$. We define two equivalent fuzzy linear codes as follows:

$$A : \mathbb{Z}_{p^k}^n \to [0, 1], \quad x \mapsto \begin{cases} 1 & \text{if } x \in C_{G_1}; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$B : \mathbb{Z}_{p^k}^n \to [0, 1], \quad x \mapsto \begin{cases} 1 & \text{if } x \in C_{G_2}; \\ 0 & \text{otherwise.} \end{cases}$$

Thus $A_1 = C_{G_1}$ and $B_1 = C_{G_2}$, $A_0 = \mathbb{Z}_{p^k}^n$ and $B_0 = \mathbb{Z}_{p^k}^n$.

Remark 3.21: Let A and B be two equivalent fuzzy linear codes over $\mathbb{Z}_{p^k}$, then $\text{Im}(A) = \text{Im}(B)$.

Let's draw the communication channel as follows:

$$F^k \xrightarrow{\textit{Encoding}} F^n \xrightarrow{\textit{Channel}} \mathbb{R}^n \xrightarrow{\textit{Decoding}} F^k$$

Assume that $F^k = \mathbb{Z}_2^2$ and $F^n = \mathbb{Z}_2^3$, that means that $k = 2$ and $n = 3$. Let $C \subseteq F^3$ be a linear code over F, in the classical case, when we send a codeword $c = (101) \in C$ through a communication channel, the signal receive can be read as $c\prime = (0.98, 0.03, 0.49)$ and modulate to $c\prime\prime = (100)$. Thus to know if $c\prime\prime$ belong to the code C, we use syndrome calculation [9]. Since the modulation have gave a wrong word, we can consider that c' have more information than c'', in the sense that we can estimate a level to which a word 0 is modulate to 1, and a word 1 is modulate to 0. Therefore it is possible to use the idea of fuzzy logic to recover the transmit codeword.

Let a linear code $C \subseteq \mathbb{Z}_2^3$. To each $c \in C$, we find $t \in [0, 1]$ such that t estimate the degree of which the element of $\mathbb{R}^3$, obtain from c through the transmission channel belong to the code C. Thus in $\mathbb{Z}_2^3$ the information that we handle are certain, whereas in $\mathbb{R}^3$ there are uncertain. When we associate to all elements of $\mathbb{Z}_2^3$ the degree of which its correspond element obtain through the transmission channel belong to $\mathbb{Z}_2^3$, then we obtain a fuzzy code. If the fuzzy code are fuzzy linear code, then we can recover the code C just by using the upper t-level cut. Thus we deal directly with the uncertain information to obtain the code C.

The following example illustrate this reconstruction of the code by using uncertain information in the case of fuzzy linear code.

Example 3.22: Let $\mathbb{Z}_2^3 = \{000, 001, 010, 100, 110, 101, 011, 111\}$ and $C = \{000, 001, 110, 111\}$ be a linear code over $\mathbb{Z}_2$.

Assume that after the transmission we obtain respectively $\{000; 0.01, 01; 1.01, 10; 1.001, 1, 0.999\}$. Let

$$A : \mathbb{Z}_2^3 \to [0,1] \quad \text{such that } x \mapsto \begin{cases} \{1\} & \text{if } x = 000; \\ \{0.99\} & \text{if } x = 001; \\ \{0.9\} & \text{if } x = 010; \\ \{0.9\} & \text{if } x = 100; \\ \{0.99\} & \text{if } x = 110; \\ \{0.9\} & \text{if } x = 101; \\ \{0.9\} & \text{if } x = 011; \\ \{0.99\} & \text{if } x = 111. \end{cases}$$

Then by finding a $t \in [0,1]$ such that $A_t = \{x \in \mathbb{Z}_2^3 \mid A(x) \geq t\} = C$, we obtain $t > 0.9$. Thus, for $t = 0.99$, we are sure that the receive codeword is in C.

4. Fuzzy Cyclic Codes Over $\mathbb{Z}_{p^k}$

In this section, we will study the case where the integers n and p are coprime.

Definition 4.1: A fuzzy submodule A of the module $\mathbb{Z}_{p^k}^n$ is called a *fuzzy cyclic code of length n* over $\mathbb{Z}_{p^k}$ if for all $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{p^k}^n$, we have $A((a_{n-1}, a_0, \ldots, a_{n-2})) \geq A((a_0, a_1, \ldots, a_{n-1}))$.

Proposition 4.2: *Let A be a fuzzy submodule A of the module $\mathbb{Z}_{p^k}^n$. A is a fuzzy cyclic code on $\mathbb{Z}_{p^k}^n$ if and only if for all $t \in [0,1]$, if $A_t \neq \emptyset$, then A_t is a cyclic code on $\mathbb{Z}_{p^k}^n$.*

Proof: The proof uses the transfer principle from [8]. ■

As well as the Proposition 3.5 in the linear case, we have the following result in the cyclic case.

Proposition 4.3: *Let A be a fuzzy module A on the module $\mathbb{Z}_{p^k}^n$.*

A is a fuzzy cyclic code on $\mathbb{Z}_{p^k}^n$ if and only if the characteristic of any upper t-level cut $A_t \neq \emptyset$ for $t \in [0,1]$ is a fuzzy cyclic code on $\mathbb{Z}_{p^k}^n$.

Proposition 4.4: *A is a fuzzy cyclic code on $\mathbb{Z}_{p^k}^n$ if and only if for all $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{p^k}^n$, then $A((a_0, a_1, \ldots, a_{n-1})) = A((a_{n-1}, a_0, \ldots, a_{n-2})) = \cdots = A((a_1, a_2, \ldots, a_{n-1}, a_0))$.*

Proof: Assume that A is a fuzzy cyclic code on $\mathbb{Z}_{p^k}^n$. Then

$$A((a_0, a_1, \ldots, a_{n-1})) \leq A((a_{n-1}, a_0, \ldots, a_{n-2})) \leq \cdots \leq A((a_1, a_2, \ldots, a_{n-1}, a_0)) \leq A((a_0, a_1, \ldots, a_{n-1})).$$

Therefore

$$A((a_0, a_1, \ldots, a_{n-1})) = A((a_{n-1}, a_0, \ldots, a_{n-2})) = \cdots = A((a_1, a_2, \ldots, a_{n-1}, a_0)).$$

The converse is straightforward by Definition 4.1. ■

Proposition 4.5: *A is a fuzzy cyclic code of length n over $\mathbb{Z}_{p^k}$ if and only if for all $t \in [0,1]$, if $A_t \neq \emptyset$, then A_t is an ideal of the factor ring $\mathbb{Z}_{p^k}[X]/(X^n - 1)$.*

Proof: Let ϕ be a mapping defined in [5] as follows, $\phi : \mathbb{Z}_{p^k}^n \to \mathbb{Z}_{p^k}[X]/(X^n - 1)$, such that $\underline{c} = (c_0, \ldots, c_{n-1}) \mapsto \phi(\underline{c}) = \sum_{i=0}^{n-1} c_i X^i$. ϕ is an isomorphism of $\mathbb{Z}_{p^k}$-module, which sends a cyclic code over $\mathbb{Z}_{p^k}$ onto the ideals of the factor ring $\mathbb{Z}_{p^k}[X]/(X^n - 1)$.

Let A be a fuzzy subset of $\mathbb{Z}_{p^k}^n$. Assume that A is a fuzzy cyclic code over $\mathbb{Z}_{p^k}$.

Let $t \in [0,1]$ such that $A_t \neq \emptyset$, then A_t is a cyclic code over $\mathbb{Z}_{p^k}$. Therefore, $\forall\, t \in [0,1]$, A_t is an ideal of $\mathbb{Z}_{p^k}[X]/(X^n - 1)$.

Conversely, assume that for all $t \in [0,1]$ such that $A_t \neq \emptyset$, A_t is an ideal of factor ring $\mathbb{Z}_{p^k}[X]/(X^n - 1)$. Since A_t is an ideal of factor ring $\mathbb{Z}_{p^k}[X]/(X^n - 1)$, then A_t is a submodule of $\mathbb{Z}_{p^k}$-module $\mathbb{Z}_{p^k}^n$. Hence $A_t \neq \emptyset$, is a linear code over $\mathbb{Z}_{p^k}$, then A is a fuzzy linear code.

Such as ϕ is define, A_t is a cyclic code over $\mathbb{Z}_{p^k}$, for all $t \in [0,1]$. Hence A is a fuzzy cyclic code over $\mathbb{Z}_{p^k}$. ■

Proposition 4.6: *A is a cyclic fuzzy code of length n if and only if A is a fuzzy ideal of the group algebra RG, which is the group algebra of $\langle x \rangle$ over the finite ring $\mathbb{Z}_{p^k}$.*

Proof: Let A be a fuzzy cyclic code. For any $\alpha, \beta \in$ RG, we have $A(\alpha - \beta) \geq min\{A(\alpha), A(\beta)\}$ since A is a fuzzy $\mathbb{Z}_{p^k}$-module on $\mathbb{Z}_{p^k}^n$.

If $\alpha \in$ RG, then $A(x\alpha) \geq A(\alpha)$, $A(x^2\alpha) \geq A(\alpha)$, $\cdots$, $A(x^{n-1}\alpha) \geq A(\alpha)$. So for $\beta \in$ RG, with $\beta = \sum_{i=0}^{n-1} l_i x^i$, $(l_i \in \mathbb{Z}_{p^k})$ we conclude $A(\alpha.\beta) = A(l_0\alpha + l_1 x\alpha + \cdots + l_{n-1}x^{n-1}\alpha) \geq min\{A(l_0\alpha), \ldots, A(l_{n-1}x^{n-1}\alpha)\} \geq A(\alpha)$. Similarly we can also show that $A(\alpha.\beta) \geq A(\beta)$. Hence $A(\alpha.\beta) \geq max\{A(\alpha), A(\beta)\}$.

Conversely, assume that A is a fuzzy ideal of the group algebra RG. (Note that RG is a module that has $G = \langle x \rangle$ as base).

Since A is a fuzzy ideal of RG, also A is a fuzzy submodule of RG.

For any $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}_{p^k}^n$ we associate $\alpha = (a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})$. Then $A(a_{n-1} + a_0 x + \cdots + a_{n-2}x^{n-1}) = A(x(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})) \geq \max\{A(x), A(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})\} \geq A(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1})$. Therefore $A((a_{n-1}, a_0, \ldots, a_{n-2})) \geq A((a_0, \ldots, a_{n-1}))$. Hence A is a fuzzy cyclic code. ■

Since $\mathbb{Z}_{p^k}$ is a finite ring, also $Im(A) = \{A(x) \in [0,1] | x \in \mathbb{Z}_{p^k}^n\}$ is finite. Let $Im(A) = \{t_1 > t_2 > \cdots > t_m\}$, then $A_{t_1} \subseteq A_{t_2} \subseteq \cdots \subseteq A_{t_{m-1}} \subseteq A_{t_m} = \mathbb{Z}_{p^k}^n$. Let $g_i^{(k)}(X) \in \mathbb{Z}_{p^k}[X]$ be the generator polynomial of A_{t_i}. Note that $g_i^{(k)}(X)$ is the Hensel lift of order k of some polynomial

$g_i(X) \in \mathbb{Z}_p[X]$ which divides $X^n - 1$. The cyclic code $\langle g_i^{(k)}(X)\rangle \subset \mathbb{Z}_{p^k}[X]/(X^n-1)$ is called the *lift code* of the cyclic code $\langle g_i(X)\rangle \subset \mathbb{Z}_p[X]/(X^n-1)$. For more information about Hensel lifting see [5].

Since $A_{t_1} \subseteq A_{t_2} \subseteq \cdots \subseteq A_{t_{m-1}} \subseteq A_{t_m} = \mathbb{Z}_{p^k}^n$, it follows that $g_{i+1}^{(k)}(X)|g_i^{(k)}(X)$, $i = 1, \ldots, m-1$.

We can define the polynomial $h_i^{(k)}(X) = (X^n-1)/g_i^{(k)}(X)$ which is called the *check polynomial* of the cyclic code $A_{t_i} = \langle g_i^{(k)}(X)\rangle$, $i = 1, \ldots, m$.

Theorem 4.7: *Let $\mathcal{G} = \{g_1^{(k)}(X), g_2^{(k)}(X), \ldots, g_m^{(k)}(X)\}$ be a set of polynomials in $\mathbb{Z}_{p^k}[X]$, such that $g_i(X)$ divide $X^n - 1$, $i = 1, \ldots, m$. If $g_{i+1}^{(k)}(X) \mid g_i^{(k)}(X)$ for $i = 1, 2, \ldots, m-1$ and $\langle g_m^{(k)}(X)\rangle = \mathbb{Z}_{p^k}^n$, then the set $\mathcal{G}$ determines a fuzzy cyclic code A and $\{\langle g_i^{(k)}(X)\rangle \mid i = 1, \ldots, m\}$ is the family of upper level cut cyclic subcodes of A.*

Proof: Since $g_{i+1}^{(k)}(X) \mid g_i^{(k)}(X)$, we have $\langle g_i^{(k)}(X)\rangle \subseteq \langle g_{i+1}^{(k)}(X)\rangle$ for $i = 1, 2, \ldots, m-1$. Choose $t_i \in [0, 1]$ such that $t_1 > t_2 > \cdots > t_m$.

Let $A_{t_i} = \langle g_i^{(k)}(X)\rangle$ for $i = 1, 2, \ldots, m-1$. We define A as follows.

$$A(\underline{c}) = \begin{cases} t_1, & \text{if } \underline{c} \in \langle g_1^{(k)}(X)\rangle; \\ t_i, & \text{if } \underline{c} \in \langle g_i^{(k)}(X)\rangle \setminus \langle g_{i-1}^{(k)}(X)\rangle, \quad i = 2, \ldots, m. \end{cases}$$

where $\phi : \mathbb{Z}_{p^k}^n \to \mathbb{Z}_{p^k}[X]/(X^n-1)$, $\underline{c} = (c_0, \ldots, c_{n-1}) \mapsto \phi(\underline{c}) = \sum_{i=0}^{n-1} c_i X^i$ is an isomorphism of $\mathbb{Z}_{p^k}$-module.

Since for all $t_i \in [0, 1]$, $A_{t_i} = \langle g_i^{(k)}(X)\rangle$ is a cyclic code as it is an ideal of the principal ring $\mathbb{Z}_p[X]/(X^n-1)$, $i = 1, \ldots, m$, also A is a fuzzy cyclic code and $\{\langle g_i^{(k)}(X)\rangle \mid i = 1, \ldots, m\}$ is the family of upper level cut cyclic subcodes of A. ■

Corollary 4.8: *With the same notations and hypothesis as in Theorem 4.7, if $\langle g_m^{(k)}(X)\rangle \neq \mathbb{Z}_{p^k}^n$, then the set $\mathcal{G}$ determines a fuzzy cyclic code A and $\{\langle g_i^{(k)}(X)\rangle \mid i = 1, \ldots, m\} \cup \mathbb{Z}_{p^k}[X]/(X^n-1)$ is the family of upper level cut cyclic subcodes of A.*

Proof: Take

$$A(\underline{c}) = \begin{cases} t_1, & \text{if } \underline{c} \in \langle g_1^{(k)}(X)\rangle; \\ t_i, & \text{if } \underline{c} \in \langle g_i^{(k)}(X)\rangle \setminus \langle g_{i-1}^{(k)}(X)\rangle, \quad i = 2, \ldots, m; \\ 0, & \text{if } \underline{c} \in \dfrac{\mathbb{Z}_{p^k}[X]}{(X^n-1)} \setminus \langle g_m^{(k)}(X)\rangle. \end{cases}$$

■

Proposition 4.9: *Let A be the fuzzy cyclic code of length n over $\mathbb{Z}_{p^k}$ that can be determined by the set of polynomial $\mathcal{G} = \{g_i^{(k)}(X) \mid i = 1, \ldots, m\}$ as in Theorem 4.7. If for all $g_i^{(k)}(X) \in \mathcal{G}$ there exists $g_j^{(k)}(X) \in \mathcal{G}$ such that $g_i^{(k)}(X).g_j^{(k)}(X) = X^n - 1$, then the set of polynomials $\{h_i^{(k)}(X) = (X^n-1)/g_i^{(k)}(X) \mid i = 1, \ldots, m\}$ determines the orthogonal of A.*

Proof: Under the conditions of Theorem 4.7 we define $A^{\perp}$ as follows.

$$A^{\perp}(\underline{c}) = \begin{cases} 1 - t_1, & \text{if } \underline{c} \in \langle h_m^{(l)}(X) \rangle; \\ 1 - t_i, & \text{if } \underline{c} \in \langle h_{i-1}^{(l)}(X) \rangle \setminus \langle h_i^{(l)}(X) \rangle, \quad i = 2, \ldots, m. \end{cases}$$

Since the upper level set $(A^{\perp})_t$ is a linear code, $A^{\perp}$ is the orthogonal of A. ■

Theorem 4.10: *Let A_1 and A_2 be two fuzzy cyclic codes on $\mathbb{Z}_{p^k}^n$, then:*

(i) *$A_1 \cap A_2$ is a fuzzy cyclic code,*
(ii) *$A_1 + A_2$ is a fuzzy cyclic code,*
(iii) *$A_1 A_2$ is a fuzzy cyclic code.*

Proof: Let A_1 and A_2 be two fuzzy cyclic codes of the $\mathbb{Z}_{p^k}$-module $\mathbb{Z}_{p^k}^n$.

(i) Let $(a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{p^k}^n$.

$$\begin{aligned} & A_1 \cap A_2((a_{n-1}, a_0, \ldots, a_{n-2})) \\ & \quad = \min\{A_1((a_{n-1}, a_0, \ldots, a_{n-2}), A_2((a_{n-1}, a_0, \ldots, a_{n-2}))\} \\ & \quad \geq \min\{A_1(a_0, a_1, \ldots, a_{n-1}), A_2((a_0, a_1, \ldots, a_{n-1}))\} \\ & \quad = A_1((a_0, a_1, \ldots, a_{n-1})) \cap A_2((a_0, a_1, \ldots, a_{n-1})). \end{aligned}$$

Since the intersection of two fuzzy modules is a fuzzy module, we obtain that $A_1 \cap A_2$ is a fuzzy cyclic code over $\mathbb{Z}_{p^k}$.

(ii) For all $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}_{p^k}^n$, we have:

$$\begin{aligned} & (A_1 + A_2)((a_{n-1}, a_0, \ldots, a_{n-2})) \\ & \quad = \max\{A_1((b_{n-1}, b_0, \ldots, b_{n-2})) \wedge A_2((c_{n-1}, c_0, \ldots, c_{n-2})) \mid b_i \\ & \qquad + c_i = a_i,\ i = 0, \ldots, n-1\} \\ & \quad \geq \max\{A_1((b_0, b_1, \ldots, b_{n-1})) \wedge A_2((c_0, c_1, \ldots, c_{n-2})) | b_i + c_i = a_i,\ i = 0, \ldots, n-1\} \\ & \quad = (A_1 + A_2)((a_0, a_1, \ldots, a_{n-1})). \end{aligned}$$

Since $A_1 + A_2$ is a fuzzy module, we conclude as above that $A_1 + A_2$ is a fuzzy cyclic code.

(iii) It is similar to $A_1 + A_2$. ■

5. Fuzzy $\mathbb{Z}_{p^k}$-linear Codes

After having studied the notion of fuzzy linear code over the ring $\mathbb{Z}_{p^k}$ in the previous section, we are now going to construct fuzzy $\mathbb{Z}_{p^k}$-linear codes explicitly.

5.1. Fuzzy Gray Map

Initially, the code of Gray is an order on the binary sequences of a fixed length n, permitting to enumerate all these sequences while modifying only one bit in order to pass from one sequence to the next one. The case that is going to interest us directly is the one of the sequence of length two, for which one has the following Gray code:

$$
\begin{aligned}
0 &\mapsto 00\\
1 &\mapsto 01\\
2 &\mapsto 11\\
3 &\mapsto 10.
\end{aligned}
$$

Let $\psi : \mathbb{Z}_{2^2} \to \mathbb{Z}_2^2$ the Gray map. We are going to define the fuzzy Gray map between two fuzzy spaces by the extension principle [10].

Definition 5.1: Let $\psi : \mathbb{Z}_{2^2} \to \mathbb{Z}_2^2$ be the Gray map, and let $\mathcal{F}(\mathbb{Z}_{2^2})$, $\mathcal{F}(\mathbb{Z}_2^2)$ be the set of all fuzzy subsets of $\mathbb{Z}_{2^2}$ and $\mathbb{Z}_2^2$ respectively. The fuzzy Gray map is the map $\widehat{\psi} : \mathcal{F}(\mathbb{Z}_{2^2}) \to \mathcal{F}(\mathbb{Z}_2^2)$, such that for any $A \in \mathcal{F}(\mathbb{Z}_{2^2})$, $\widehat{\psi}(A)(y) = \sup\{A(x) \mid y = \psi(x)\}$.

Example 5.2: Let

$$
A : \mathbb{Z}_4 \to [0,1], \quad x \mapsto \begin{cases} 1 & \text{if } x = 0;\\ \dfrac{1}{3} & \text{if } x = 1;\\ \dfrac{1}{3} & \text{if } x = 2;\\ \dfrac{1}{3} & \text{if } x = 3. \end{cases}
$$

with the Gray map $\psi(0) = 00, \psi(1) = 01, \psi(2) = 11, \psi(3) = 10$.

By the fuzzy Gray map we have $\widehat{\psi}(A)(00) = 1, \widehat{\psi}(A)(01) = \frac{1}{3}, \widehat{\psi}(A)(11) = \frac{1}{3}, \widehat{\psi}(A)(10) = \frac{1}{3}$, hence $\widehat{\psi}(A)$ is a fuzzy linear code.

Theorem 5.3: *The fuzzy Gray map $\widehat{\psi}$ is a bijection.*

Proof: This follows from the fact that ψ is a one to one function. ∎

As in crisp case, we have the following proposition.

Proposition 5.4: *If A is a fuzzy linear code over $\mathbb{Z}_{2^2}$ and ψ the Gray map, then $\widehat{\psi}(A)$ is not always a fuzzy linear code over the field $\mathbb{Z}_2$*

The Gray map allows to construct nonlinear codes as binary images of the linear codes, that is the case of Kerdock, Preparata, and Goethals codes. For a good understanding, we suggest the reader to examine [11,12]. In fact if $\mathcal{C}$ is a linear code of length n over $\mathbb{Z}_4$, then $C = \psi(\mathcal{C})$ is a nonlinear code of length $2n$ over $\mathbb{Z}_2$ in general [11]. In that way we construct a fuzzy Kerdock code in the following example.

Example 5.5: Let $\mathcal{C}$ be a linear code of length 8 over $\mathbb{Z}_4$ with the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 3 & 2 \end{pmatrix},$$

then its image under the Gray map ψ gives a Kerdock code C (see construction in [9]).

Let

$$A : \mathbb{Z}_4^8 \to [0,1], \quad x \mapsto \begin{cases} 1, & \text{if } x \in \mathcal{C}; \\ 0, & \text{otherwise.} \end{cases}$$

Then A is a fuzzy linear code over $\mathbb{Z}_4$.

Since ψ is a bijection, we construct

$$\widehat{\psi}(A) : \mathbb{Z}_2^{16} \to [0,1], \quad y \mapsto \begin{cases} 1, & y \in E; \\ 0, & \text{otherwise,} \end{cases}$$

where $E = \{y \in \mathbb{Z}_2^{16} \mid y = \psi(x) \text{ and } x \in C\}$

Since E is not a linear code over $\mathbb{Z}_2$, we conclude that $\widehat{\psi}(A)$ is a fuzzy $\mathbb{Z}_2$-linear code but not a fuzzy linear code $\mathbb{Z}_2$. Consequently, $\widehat{\psi}(A)$ is a fuzzy Kerdock code of length 16.

Remark 5.6: A fuzzy $\mathbb{Z}_4$-linear code is not in general a fuzzy linear code over $\mathbb{Z}_2$

If we define the fuzzy binary relation R_ψ on $\mathbb{Z}_{2^2} \times \mathbb{Z}_2^2$ by

$$R_\psi(x,y) = \begin{cases} 1, & \text{if } y = \psi(x); \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see [13] that $\widehat{\psi}(A)(y) = \sup\{A(x) \mid y = \psi(x)\}$ can be represented by $\widehat{\psi}(A)(y) = \sup\{\min\{A(x), R_\psi(x,y)\} \mid x \in \mathbb{Z}_2^2\}$.

As in [5], let $\Psi : \mathbb{Z}_{p^k} \to \mathbb{Z}_p^{p^{k-1}}$ be the generalised Gray map.

Definition 5.7: We call the map $\widehat{\Psi} : \mathcal{F}(\mathbb{Z}_{p^k}) \to \mathcal{F}(\mathbb{Z}_p^{p^{k-1}})$, such that for any $A \in \mathcal{F}(\mathbb{Z}_{p^k})$,

$$\widehat{\Psi}(A)(y) = \begin{cases} \sup\{A(x) \mid y = \Psi(x)\}, & \text{if a such } x \text{ exists;} \\ 0, & \text{otherwise.} \end{cases}$$

The *fuzzy generalised gray map*.

Since $\Psi : \mathbb{Z}_{p^k} \to \mathbb{Z}_p^{p^{k-1}}$ cannot give more than one image for one element, then Definition 5.7 can be simply write

$$\widehat{\Psi}(A)(y) = \begin{cases} A(x), & \text{if } y = \Psi(x); \\ 0, & \text{otherwise.} \end{cases}$$

Remark 5.8: Let $B \in \mathcal{F}(\mathbb{Z}_p^{p^{k-1}})$ such that $B(y) = t \neq 0$ for any $y \in \mathbb{Z}_p^{p^{k-1}}$. There does not exist a fuzzy subset $A \in \mathcal{F}(\mathbb{Z}_{p^k})$ such that $\widehat{\Psi}(A) = B$. Thus $\widehat{\Psi}$ is not a bijection map.

5.2. Fuzzy $\mathbb{Z}_{p^k}$-linear Codes

In the following, we will denote by $\widehat{\Psi}$ the map from $\mathcal{F}(Z_{p^k}^n)$ onto $\mathcal{F}(Z_p^{n.p^{k-1}})$ which spreads the fuzzy generalised Gray map.

Definition 5.9: A fuzzy code A over $\mathbb{Z}_p$ is a fuzzy $\mathbb{Z}_{p^k}$-linear code if it is an image under the fuzzy generalised Gray map of a fuzzy linear code over the ring $\mathbb{Z}_{p^k}$.

Definition 5.10: A fuzzy code A is a fuzzy $\mathbb{Z}_{p^k}$-cyclic code if it is a fuzzy $\mathbb{Z}_{p^k}$-linear code and if it is the image under the generalised Gray map of a cyclic code over the ring $\mathbb{Z}_{p^k}$.

Remark 5.11: A fuzzy $\mathbb{Z}_{p^k}$-linear code is a fuzzy code over the field $\mathbb{Z}_p$.

Example 5.12: (1) Let

$$B : \mathbb{Z}_2^6 \to [0,1], \quad w = (a,b,c,d,e,f) \mapsto \begin{cases} 1, & \text{if } e = f = 0; \\ 0, & \text{otherwise.} \end{cases}$$

B is a fuzzy linear code of length 6 over $\mathbb{Z}_2$.Let

$$A : \mathbb{Z}_4^3 \to [0,1], \quad v = (x,y,z) \mapsto \begin{cases} 1, & \text{if } z = 0; \\ 0, & \text{otherwise.} \end{cases}$$

A is a fuzzy linear code of length 3 over $\mathbb{Z}_4$.
Moreover, if $B = \widehat{\psi}(A)$, then B is a fuzzy $\mathbb{Z}_4$-linear code.

(2) Let $B : \mathbb{Z}_3^9 \to [0,1]$,

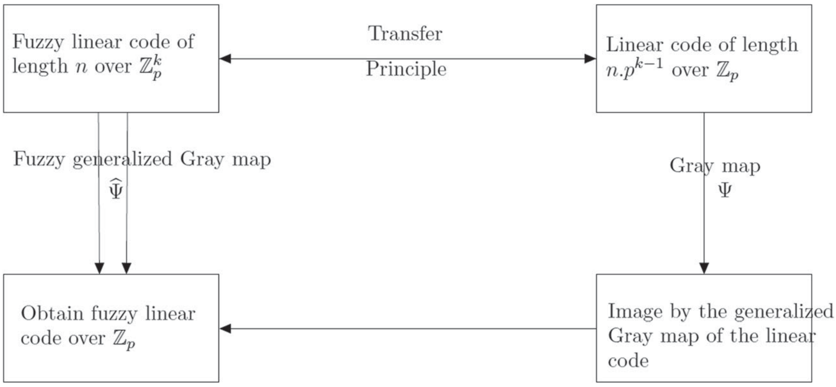
$$v = (a,b,c,d,e,f,g,h,i) \mapsto \begin{cases} \frac{2}{3}, & \text{if } d = e = f = g = h = i = 0 \text{ and } abc \in \\ & \{000, 012, 021, 111, 120, 102, 222, 201, 210\}; \\ \frac{1}{3}, & \text{if } g = h = i = 0, abc \in \{000, 012, 021, 111, \\ & 120, 102, 222, 201, 210\} \text{ and def} \in \{012, 021, \\ & 111, 120, 102, 222, 201, 210\}; \\ 0, & \text{otherwise.} \end{cases}$$

B is a fuzzy linear code over $\mathbb{Z}_3$ and B is a fuzzy $\mathbb{Z}_{3^2}$-linear code.

It is easy to show the next proposition, whose Example 5.5 is a perfect illustration of it.

Proposition 5.13: *Let B be a fuzzy $\mathbb{Z}_{p^k}$-linear code, then B is not always a fuzzy linear code over the field $\mathbb{Z}_p$.*

Since the fuzzy generalised Gray map image of fuzzy linear code is a fuzzy codes over the field $\mathbb{Z}_p$, we can also construct fuzzy $\mathbb{Z}_{p^k}$-linear codes using the following diagram:



Example 5.14: (1) Let $A : \mathbb{Z}_{p^k}^n \to [0, 1]$ be a linear code such that A has three upper t-level cuts $A_{t_3} \subseteq A_{t_2} \subseteq A_{t_1}$. Let $A'_{t_3} = \Psi(A_{t_3})$, $A'_{t_2} = \Psi(A_{t_2})$ and $A'_{t_1} = \Psi(A_{t_1})$, we have $A'_{t_3} = \Psi(A_{t_3}) \subseteq A'_{t_2} = \Psi(A_{t_2}) \subseteq A'_{t_1} = \Psi(A_{t_1})$. We construct $A' = \widehat{\Psi}(A)$ as follow.

$$A' : \mathbb{Z}_p^{n.p^{k-1}} \to [0, 1], \quad y \mapsto \begin{cases} t_3, & \text{if } y \in A'_{t_3}; \\ t_2, & \text{if } y \in A'_{t_2}; \\ t_1, & \text{if } y \in A'_{t_1}; \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let

$$A : \mathbb{Z}_4 \to [0, 1], \quad x \mapsto \begin{cases} \frac{1}{2} & \text{if } x = 0; \\ \frac{1}{3} & \text{if } x = 2; \\ \frac{1}{4} & \text{if } x = 1, 3, \end{cases}$$

be a fuzzy linear code over $\mathbb{Z}_4$. Then $A_{1/2} = \{0\}$, $A_{1/3} = \{0, 2\}$ and $A_{1/4} = \mathbb{Z}_4$.

We construct $A'_{1/2} = \{00\}$, $A'_{1/3} = \{00, 11\}$ and $A'_{1/4} = \mathbb{Z}_2^2$, the Gray map image of $A_{1/2}$, $A_{1/3}$ and $A_{1/4}$ respectively, we define

$$A' : \mathbb{Z}_2^2 \to [0, 1], \quad y \mapsto \begin{cases} \frac{1}{2} & \text{if } x \in A_{1/2}, y = \psi(x); \\ \frac{1}{3} & \text{if } x \in A_{1/3} \setminus A_{1/2}, y = \psi(x); \\ \frac{1}{4} & \text{if } x \in A_{1/4} \setminus A_{1/3}, y = \psi(x). \end{cases}$$

Remark 5.15: A' and $\widehat{\psi}(A)$ are the same codes.

Proposition 5.16: *If for all $t \in [0, 1]$, $A'_t = \Psi(A_t)$ (when $A_t \neq 0$) is a linear code over $\mathbb{Z}_p$, then these two constructions of fuzzy $\mathbb{Z}_p$-linear codes above give the equivalent fuzzy codes.*

Proof: This follows directly from the definition of the fuzzy generalised Gray map and the fact that the image under the generalised Gray map of a linear code is not a linear code in general. ■

6. Conclusion

In this paper where we study fuzzy coding, we define fuzzy linear codes over the finite ring $\mathbb{Z}_{p^k}$, fuzzy generalised Gray map and fuzzy $\mathbb{Z}_{p^k}$-linear codes. We also investigate some of their properties and remark that many of them are similar to the classical form. The codes of Kerdock are permit us to show that fuzzy $\mathbb{Z}_{p^k}$-linear codes is not a fuzzy linear code.

This work allows us to reinforce the hypothesis of Von Kaenel [14], that the theory of fuzzy sets is a natural setting for this study.

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No potential conflict of interest was reported by the authors.

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References

[1] Zadeh LA. Fuzzy sets. Inf Control. 1965;8:338–353.
[2] Negoita CV, Ralescu DA. Applications of fuzzy sets and system analysis. Basel and Stuttgart: Birkhäuser verlag; 1975. 191PP.
[3] Maschinchi M, Zahedi MM. On L-fuzzy primary submodules. Fuzzy Sets Syst. 1992;49:231–236.
[4] Shum KP, De Gang C. Some note on the theory of fuzzy code. BUSEFAL (Bulletin Pour Les Sous Ensembles Flous et Leurs Applications) University of Savoie, France. 2000;81:132–136.
[5] Galand F. Construction de codes $\mathbb{Z}_{p^k}$-linéaires de bonne distance minimale et schémas de dissimulation fondés sur les codes de recouvrement [PhD thesis]. Université de Caen; 2004.
[6] Biswas R. Fuzzy fields and fuzzy linear spaces redefined. Fuzzy Sets Syst. 1989;33:257–259.
[7] Nanda S. Fuzzy fields and fuzzy linear space. Fuzzy Sets Syst. 1986;19:89–94.
[8] Kondo M, Dubek WA. On transfer principle in fuzzy theory. Mathware Soft Comput. 2005;12:41–55.
[9] Carlet C. $\mathbb{Z}_{2^k}$-linear codes. IEEE Trans Inf Theory. 1998;44:1543–1547.
[10] Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning I, II, III. Inf Sci. 1975;8–9:199–249, 301–357, 43–80.
[11] Hammons R, Kumar PV, Calderbank AR, et al. Kerdock, Preparata, Goethals and other codes are linear over $\mathbb{Z}_4$. IEEE Trans Inf Theory. 1994;40:301–319.
[12] Kerdock AM. A class of low-rate nonlinear binary codes. Inf Control. 1972;20:182–187.

[13] Perfilieva I. Fuzzy Function: Theoretical and Practical Point of View. Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology, Aix-les-Bains, France: Atlantis Press; 2011. pp. 480–486. Available from: https://doi.org/10.2991/eusflat.2011.142.
[14] Von Kaenel PA. Fuzzy codes and distance properties. Fuzzy Sets Syst. 1982;8:199–204.