Taylor & Francis
Taylor & Francis Group

Systems Science & Control Engineering
An Open Access Journal

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tssc20

Finite-horizon H_∞ state estimation for time-varying complex networks based on the outputs of partial nodes

Wenhua Zhang , Li Sheng & Ming Gao

To cite this article: Wenhua Zhang , Li Sheng & Ming Gao (2020): Finite-horizon H_∞ state estimation for time-varying complex networks based on the outputs of partial nodes, Systems Science & Control Engineering, DOI: 10.1080/21642583.2020.1837691

To link to this article: https://doi.org/10.1080/21642583.2020.1837691

Published online: 27 Oct 2020.

SYSTEMS SCIENCE & CONTROL ENGINEERING: AN OPEN ACCESS JOURNAL
https://doi.org/10.1080/21642583.2020.1837691

Taylor & Francis
Taylor & Francis Group

OPEN ACCESS Check for updates

Finite-horizon H_∞ state estimation for time-varying complex networks based on the outputs of partial nodes

Wenhua Zhang, Li Sheng and Ming Gao

College of Control Science and Engineering, China University of Petroleum (East China), Qingdao, People's Republic of China

ABSTRACT

In this paper, the partial-nodes-based resilient filtering problem for a class of discrete time-varying complex networks is investigated. In order to reduce the effect of imprecision of filter parameters on estimation performance, a set of resilient filters is proposed. The measurement output from all network nodes may not be available in the actual system, but only from a fraction of nodes. The state estimators are designed for the time-varying complex network based on partial nodes to make the estimation error achieve the H_∞ performance constraint over a finite horizon. By employing the completing-the-square technique and the backward recursive Riccati difference equations, the sufficient conditions for the existence of the estimator are derived. Then the gain of the estimator is calculated. Finally, a numerical example is provided to illustrate the effectiveness of the proposed method.

ARTICLE HISTORY
Received 3 August 2020
Accepted 13 October 2020

KEYWORDS
Time-varying complex network; resilient filter; partial-nodes-based estimation; H_∞ state estimation

1. Introduction

A complex network is composed of a series of nodes with certain dynamic performance, which are connected with each other through a network topology. There are many examples of complex networks in various natural and man-made systems, such as the World Wide Web, genetic networks, power grids and social networks (Albert & Barabási, 2002; Boccaletti et al., 2006; Costa et al., 2011, 2007; Khafaf & Jalili, 2019; Pagani & Aiello, 2013). In the past several decades, the dynamic analysis of complex networks has received extensive attention, such as state estimation (Ding et al., 2012; H. Li, 2013; Sheng et al., 2017; Zou et al., 2017), synchronization (Adu-Gyamfi et al., 2018; Tang et al., 2014; X. Wang et al., 2020) and stability (C. Zhang & Han, 2019). However, owing to the enormous number of nodes and complicated topology of complex networks, it is generally impossible to obtain the information of all nodes directly. Hence, in order to analyse the trajectory of node state change, it is very valuable to use the available measurement output for state estimation.

For decades, filters (Ding et al., 2015; X. M. Li et al., 2020; Liang et al., 2014; Sheng, Niu, Zou, et al., 2018; Z. Wang et al., 2013; Zhao et al., 2018; Zou et al., 2019a, 2019b; Zou, Wang, Hu, et al., 2020; Zou, Wang, & Zhou, 2020) have been widely applied in control, signal processing, target tracking and other engineering fields because they can utilize the available measurement output signals for state estimation. Among the existing filters, H_∞ filter can provide a bound for the worst situation estimation error without statistical information of noise, so it has been a focal point in dynamic analysis. For time-varying systems, compared with moving horizon (Zou et al., 2019a; Zou, Wang, Hu, et al., 2020; Zou, Wang, & Zhou, 2020), it is more meaningful for on-line implementation to study state estimation in finite horizon. In recent years, there have been mainly two methods for finite-horizon filtering, including linear matrix inequality method (X. M. Li et al., 2020; Liang et al., 2014; Zhao et al., 2018) and Riccati difference equations (RDEs) method (Ding et al., 2015; Sheng, Niu, Zou, et al., 2018; Z. Wang et al., 2013). For instance, a periodic neural network over multiple fading channels was considered in X. M. Li et al. (2020) and the parameters of the estimator were calculated by solving the recursive linear matrix inequality (RLMI). In Sheng, Niu, Zou, et al. (2018), a new coupled recursive Riccati difference equations method was proposed to solve the filtering problem of complex networks with multiplicative noise and random coupling strength.

In reality, the filter cannot achieve the effect accurately in some cases. Due to some physical reasons such as digital to analog conversion, rounding error, limited precision or internal noise (D. Zhang et al., 2014), the implementation of digital state estimator will have errors. That is,

CONTACT Li Sheng shengli@upc.edu.cn

its actual gain may fluctuate or vary compared with the expected gain (Yu et al., 2016). Based on this fact, desired filter is insensitive to some gain errors, in other words, the designed filter is resilient (or non fragile). In the past few years, resilient filters have been widely studied in sensor networks (Sheng, Niu, & Gao, 2018; L. Zhang et al., 2017), complex networks (W. Li et al., 2018; F. Wang et al., 2018; Yang et al., 2019) and neural networks (Hou et al., 2016; Sakthivel et al., 2015). For example, in the literature (L. Zhang et al., 2017), the problem of non-fragile H_∞ filtering for large-scale power systems based on sensor networks is studied. In Yang et al. (2019), the resilient state estimation problem for a class of time-delay complex networks with stochastic communication protocol (SCP) is discussed.

Most of the state estimation problems for complex networks mentioned above are based on the assumption that the measurement output signals from all the network nodes are available. However, in reality, because of the harsh physical environment or limited communication resources, not all the measurement outputs of the nodes can be accessed directly. Moreover, due to the large scale of nodes in practical complex networks, some nodes may have similar functions, and the information of such nodes may be redundant for state estimation. Therefore, it is of application significance to estimate the states by measuring outputs from certain network nodes. This is also known as the partial-nodes-based (PNB) estimation problem. Recently, Liu has done pioneering work on the state estimation of delayed complex networks (Liu et al., 2018, 2017). Han extended it to sensor networks and studied the H_∞ consensus filtering algorithm based on partial nodes for time-varying sensor networks (Han et al., 2017). Nevertheless, finite-horizon H_∞ state estimation of complex networks based on partial nodes has not been fully investigated yet, which is the motivation of this paper.

Based on the above discussion, the objective of this paper is to investigate the resilient filtering problem of time-varying complex networks based on the measurement output of partial nodes. The main contributions of this paper are as follows. (1) Considering that the measurement output of complex networks may not be available for all nodes, but come from partial nodes, this can reflect the reality more closely. (2) In view of the harmful influence caused by inaccurate filter parameters, a set of resilient filters is designed to eliminate it. (3) A new parameter variable is designed to facilitate subsequent calculation, which includes disturbance, filter parameter uncertainty and nonlinear uncertainty. By means of the recursive RDEs approach, the estimation error can satisfy finite-horizon H_∞ performance constraint.

$\mathbb{R}$ is the space of all real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ represent the set of all n-dimensional vectors and the set of all $m \times n$ real matrices, respectively. $A > 0$ ($A \geq 0$) means that A is a real symmetric positive definite (positive semi-definite) matrix. A^{T} stands for the transpose of a matrix A, and $A^{\dagger} \in \mathbb{R}^{m\times n}$ respects the Moore–Penrose pseudo inverse of $A \in \mathbb{R}^{n\times m}$. I denotes the unit matrix with appropriate dimension and diag$\{\cdots\}$ represents a diagonal matrix. $\otimes$ is the operation of Kronecker product. $\|x\|$ describes the Euclidean norm of a vector x. $\mathbb{E}\{x\}$ respects the mathematical expectation of the stochastic variable x. For simplicity, sym$\{A\}$ stands for $A + A^{\mathrm{T}}$. $\|A\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix, i.e. $\|A\|_{\mathrm{F}} = (\mathrm{trace}(A^{\mathrm{T}}A))^{\frac{1}{2}}$.

2. Problem formulation and preliminaries

Consider the following time-varying complex network consisting of M coupling nodes:

$$\begin{aligned} x_i(k+1) &= A_i(k)x_i(k) + \sum_{j=1}^{M} w_{ij}\Lambda x_j(k) \\ &\quad + f(k, x_i(k)) + D_i(k)v_i(k), \\ z_i(k) &= E_i(k)x_i(k), \quad i = 1, 2, \ldots, M, \\ x_i(0) &= x_{i0}, \end{aligned} \tag{1}$$

where $x_i(k) \in \mathbb{R}^n, z_i(k) \in \mathbb{R}^{n_z}$ denote the state vector, regulated output of the ith node, respectively. $v_i(k) \in \mathbb{R}^{n_v}$ is the disturbance input which belongs to $l_2([0, N], \in \mathbb{R}^{n_v})$. $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \geq 0$ is an inner-coupling matrix linking the jth state variable if $\lambda_j \neq 0$. $W = [w_{ij}]_{M\times M}$ is the outer coupled matrix satisfying $w_{ij} \geq 0 (i \neq j)$ and $w_{ii}(k) = -\sum_{j=1, j\neq i}^{M} w_{ij}$. x_{i0} is the initial state. $A_i(k)$, $D_i(k)$, and $E_i(k)$ are known real matrices with appropriate dimensions. $f(\cdot) : [0, N] \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector-valued function. Suppose that the nonlinear function $f(\cdot)$ satisfies the sector-bounded condition:

$$\begin{aligned} &\left[f(k,x) - f(k,y) - U_1(k)(x-y)\right]^T \\ &\quad \times \left[f(k,x) - f(k,y) - U_2(k)(x-y)\right] \leq 0, \\ &f(k,0) = 0, \ \forall\, x, y \in \mathbb{R}^n, \end{aligned} \tag{2}$$

where $U_1(k)$ and $U_2(k)$ are real matrices with appropriate dimensions and satisfy $U_1(k) \leq U_2(k)$ for all k.

As mentioned in the introduction, one of the purposes of this paper is to investigate the PNB state estimation of the complex network (1), which the measurement output of the network only comes from a fraction of its nodes. On this basis, we reorder the nodes of system (1). Considering the universality, it is assumed that we can access the measurement output of the first l_0 nodes. The known measurement output expressions of these nodes are as follows:

$$y_i(k) = C_i x_i(k) + H_i(k)v_i(k), \quad 1 \le i \le l_0, \tag{3}$$

where $y_i(k) \in \mathbb{R}^{n_y}(1 \le n_y \le n)$ is the measurement output of the ith node. $C_i(k)$ and $H_i(k)$ are known real matrices with appropriate dimensions.

In order to estimate the state of complex network (1) and consider the imprecision of filter parameters, resilient filters of the ith node of (1) are constructed as follows:

$$\begin{aligned}\hat{x}_i(k+1) &= A_i(k)\hat{x}_i(k) + \sum_{j=1}^{M} w_{ij}\Lambda\hat{x}_j(k) + f(k,\hat{x}_i(k)) \\ &\quad + (K_i(k) + \Delta K_i(k))\left(y_i(k) - C_i(k)\hat{x}_i(k)\right), \\ &\quad i = 1,2,\ldots,l_0,\end{aligned} \tag{4a}$$

$$\begin{aligned}\hat{x}_i(k+1) &= A_i(k)\hat{x}_i(k) + \sum_{j=1}^{M} w_{ij}\Lambda\hat{x}_j(k) + f(k,\hat{x}_i(k)), \\ &\quad i = l_0+1, l_0+2,\ldots,M,\end{aligned} \tag{4b}$$

$$\hat{z}_i(k) = E_i(k)\hat{x}_i(k), \tag{5}$$

where $\hat{x}_i(k) \in \mathbb{R}^n$, $\hat{z}_i(k) \in \mathbb{R}^{n_z}$ are the estimation of state and regulated output on ith node, respectively. $K_i(k) \in \mathbb{R}^{n\times n_y}$ is the filter parameter to be designed. $\Delta K_i(k) \in \mathbb{R}^{n\times n_y}$ represents the uncertainties of filter parameter concerning $K_i(k)$, and it is assumed that it can be described as

$$\Delta K_i(k) = S_i(k)L_i(k)T_i(k), \tag{6}$$

where $S_i(k)$, $T_i(k)$ are known real-valued matrices with proper dimensions, and $L_i(k)$ is unknown matrix with $L_i^T(k)L_i(k) \le I$.

Setting $e_i(k) = x_i(k) - \hat{x}_i(k)$ and $\tilde{z}_i(k) = z_i(k) - \hat{z}_i(k)$, the dynamic estimation error can be derived from (1) and (4) as follows:

$$\begin{aligned}e_i(k+1) &= [A_i(k) - (K_i(k) + \Delta K_i(k))C_i(k)]e_i(k) \\ &\quad + \sum_{j=1}^{M} w_{ij}\Lambda e_j(k) + \tilde{f}(k,x_i(k),\hat{x}_i(k)) \\ &\quad + [D_i(k) - (K_i(k) + \Delta K_i(k))H_i(k)]v_i(k), \\ &\quad i = 1,2,\ldots,l_0,\end{aligned} \tag{7a}$$

$$\begin{aligned}e_i(k+1) &= A_i(k)e_i(k) + \sum_{j=1}^{M} w_{ij}\Lambda e_j(k) + \tilde{f}(k,x_i(k),\hat{x}_i(k)) \\ &\quad + D_i(k)v_i(k), \quad i = l_0+1, l_0+2,\ldots,M,\end{aligned} \tag{7b}$$

$$\tilde{z}_i(k) = E_i(k)e_i(k), \tag{8}$$

where $\tilde{f}(k,x_i(k),\hat{x}_i(k)) = f(k,x_i(k)) - f(k,\hat{x}_i(k))$.

From (2), we have,

$$\begin{aligned}&\left[\tilde{f}(k,x_i(k),\hat{x}_i(k)) - U_1(k)e_i(k)\right]^T \\ &\quad \times \left[\tilde{f}(k,x_i(k),\hat{x}_i(k)) - U_2(k)e_i(k)\right] \le 0,\end{aligned} \tag{9}$$

which can be rewritten as

$$\begin{aligned}&\left[\tilde{f}(k,x_i(k),\hat{x}_i(k)) \right.\\ &\quad \left. - \left(\frac{U_1(k)+U_2(k)}{2} + \frac{U_1(k)-U_2(k)}{2}\right)e_i(k)\right]^T \\ &\quad \times \left[\tilde{f}(k,x_i(k),\hat{x}_i(k)) \right.\\ &\quad \left. - \left(\frac{U_1(k)+U_2(k)}{2} - \frac{U_1(k)-U_2(k)}{2}\right)e_i(k)\right] \le 0.\end{aligned}$$

Denoting $Q(k) = (U_1(k) + U_2(k))/2$, $N(k) = (U_1(k) - U_2(k))/2$, and $\Delta(\tilde{f}(k,x_i(k),\hat{x}_i(k))) = \tilde{f}(k,x_i(k),\hat{x}_i(k)) - Q(k)e_i(k)$, we have

$$\begin{aligned}&\Delta^T(\tilde{f}(k,x_i(k),\hat{x}_i(k)))\Delta(\tilde{f}(k,x_i(k),\hat{x}_i(k))) \\ &\quad \le e_i^T(k)N^T(k)N(k)e_i(k).\end{aligned} \tag{10}$$

It can be inferred that there exists at least a function $Y(k,e_i(k))$ satisfying $\Delta(\tilde{f}(k,x_i(k),\hat{x}_i(k))) = Y(k,e_i(k))e_i(k)$ and $Y^T(k,e_i(k))Y(k,e_i(k)) \le N^T(k)N(k)$. Therefore, the sector-bound condition can be converted to the sector-bound uncertainties expressed as

$$\tilde{f}(k,x_i(k),\hat{x}_i(k)) = Q(k)e_i(k) + \Gamma(k,e_i(k))N(k)e_i(k), \tag{11}$$

with $\Gamma(k,e_i(k)) := Y(k,e_i(k))N^{-1}(k)$ satisfying $\Gamma^T(k,e_i(k))\Gamma(k,e_i(k)) \le I$.

Similarly,

$$f(k,x_i(k)) = Q(k)x_i(k) + \Gamma(k,x_i(k))N(k)x_i(k). \tag{12}$$

By denoting $\eta_i(k) = [x_i^T(k) \quad e_i^T(k)]^T$ and considering (11) and (12), the combination of (1) and (7) generates the following augmented system:

$$\begin{aligned}\eta_i(k+1) &= [\tilde{A}_i(k) + \tilde{I}\tilde{K}_i(k)\tilde{C}_i(k)]\eta_i(k) + \sum_{j=1}^{M} w_{ij}\tilde{\Lambda}\eta_j(k) \\ &\quad + \tilde{I}\Delta\tilde{K}_i(k)\tilde{C}_i(k)\eta_i(k) + \Delta(k) \\ &\quad + [\tilde{D}_i(k) + \tilde{I}\tilde{K}_i(k)\tilde{H}_i(k)]v_i(k) \\ &\quad + \tilde{I}\Delta\tilde{K}_i(k)\tilde{H}_i(k)v_i(k) \\ &= [\tilde{A}_i(k) + \tilde{I}\tilde{K}_i(k)\tilde{C}_i(k)]\eta_i(k) + \sum_{j=1}^{M} w_{ij}\tilde{\Lambda}\eta_j(k) \\ &\quad + B_i(k)\omega_i(k), \quad i = 1,2,\ldots,l_0,\end{aligned} \tag{13a}$$

$$
\begin{aligned}
\eta_i(k+1) &= \tilde{A}_i(k)\eta_i(k) + \sum_{j=1}^{M} w_{ij}\tilde{\Lambda}\eta_j(k) \\
&\quad + \Delta(k) + \tilde{D}_i(k)v_i(k) \\
&= \tilde{A}_i(k)\eta_i(k) + \sum_{j=1}^{M} w_{ij}\tilde{\Lambda}\eta_j(k) + B_i(k)\omega_i(k), \\
&\qquad i = l_0+1, l_0+2, \ldots, M,
\end{aligned} \tag{13b}
$$

$$
\tilde{z}_i(k) = \tilde{E}_i(k)\eta_i(k), \tag{14}
$$

where

$$
\begin{aligned}
&\tilde{A}_i(k) = \mathrm{diag}\{A_i(k)+Q(k), A_i(k)+Q(k)\}, \\
&\tilde{K}_i(k) = \mathrm{diag}\{0, K_i(k)\}, \quad \Delta\tilde{K}_i(k) = \mathrm{diag}\{0, \Delta K_i(k)\}, \\
&\tilde{I} = \mathrm{diag}\{0, -I_n\}, \quad \tilde{\Gamma}_i(k) = \mathrm{diag}\{\Gamma(k, x_i(k)), \Gamma(k, e_i(k))\}, \\
&\tilde{N}(k) = \mathrm{diag}\{N(k), N(k)\}, \\
&\tilde{T}_i(k) = \mathrm{diag}\{0, T_i(k)\}, \quad \tilde{\Lambda} = \mathrm{diag}\{\Lambda, \Lambda\}, \\
&\tilde{C}_i(k) = \mathrm{diag}\{0, C_i(k)\}, \quad \tilde{L}_i(k) = \mathrm{diag}\{0, L_i(k)\}, \\
&\Delta(k) = \begin{bmatrix} \Delta(\tilde{f}(k, x_i(k))) & \Delta(\tilde{f}(k, x_i(k), \hat{x}_i(k))) \end{bmatrix}^T, \\
&\tilde{H}_i(k) = \begin{bmatrix} 0 & H_i^T(k) \end{bmatrix}^T, \quad \tilde{D}_i(k) = \begin{bmatrix} D_i^T(k) & D_i^T(k) \end{bmatrix}^T, \\
&\tilde{E}_i(k) = \begin{bmatrix} 0 & E_i(k) \end{bmatrix}, \\
&B_{i,,i\in[1,l_0]}(k) = \Big[\tilde{D}_i(k) + \tilde{I}\tilde{K}_i(k)\tilde{H}_i(k) \quad \varepsilon_1^{-1}(k)\tilde{I}\tilde{S}_i(k) \\
&\qquad\qquad \varepsilon_1^{-1}(k)\tilde{I}\tilde{S}_i(k) \quad \varepsilon_2^{-1}(k)I\Big], \\
&B_{i,i\in[l_0+1,M]}(k) = \begin{bmatrix} \tilde{D}_i(k) & 0 & 0 & \varepsilon_2^{-1}(k)I \end{bmatrix}, \\
&\omega_{i,i\in[1,l_0]}(k) = \begin{bmatrix} v_i(k) \\ \varepsilon_1(k)\tilde{L}_i(k)\tilde{T}_i(k)\tilde{H}_i(k)v_i(k) \\ \varepsilon_1(k)\tilde{L}_i(k)\tilde{T}_i(k)\tilde{C}_i(k)\eta_i(k) \\ \varepsilon_2(k)\tilde{\Gamma}_i(k)\tilde{N}(k)\eta_i(k) \end{bmatrix}, \\
&\omega_{i,i\in[l_0+1,M]}(k) = \begin{bmatrix} v_i(k) \\ 0 \\ 0 \\ \varepsilon_2(k)\tilde{\Gamma}_i(k)\tilde{N}(k)\eta_i(k) \end{bmatrix}.
\end{aligned}
$$

Denoting $\eta(k) = [\eta_1^T(k)\ \eta_2^T(k)\ \cdots\ \eta_M^T(k)]^T$, $\bar{z}(k) = [\tilde{z}_1^T(k)\ \tilde{z}_2^T(k)\ \cdots\ \tilde{z}_M^T(k)]^T$, and $\omega(k) = [\omega_1^T(k)\ \omega_2^T(k)\ \cdots\ \omega_M^T(k)]^T$, (13)–(14) can be written as following forms:

$$
\begin{aligned}
\eta(k+1) &= \left[\bar{A}(k) + \bar{I}\bar{K}(k)\bar{C}(k)\right]\eta(k) + \bar{B}(k)\omega(k), \\
\bar{z}(k) &= E(k)\eta(k),
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
&\bar{A}(k) = \mathrm{diag}\left\{\tilde{A}_1(k), \ldots, \tilde{A}_M(k)\right\}_M + W \otimes \tilde{\Lambda}, \\
&\bar{K}(k) = \mathrm{diag}\left\{\tilde{K}_1(k), \ldots, \tilde{K}_{l_0}(k), 0, \ldots, 0\right\}_M, \\
&\bar{I} = \mathrm{diag}\left\{\tilde{I}, \ldots, \tilde{I}, 0, \ldots, 0\right\}_M, \\
&\bar{C}(k) = \mathrm{diag}\left\{\tilde{C}_1(k), \ldots, \tilde{C}_{l_0}(k), 0, \ldots, 0\right\}_M, \\
&\bar{B}(k) = \mathrm{diag}\left\{B_1(k), \ldots, B_{l_0}(k), B_{l_0+1}(k), \ldots, B_M(k)\right\}_M, \\
&\bar{E}(k) = \mathrm{diag}\left\{\tilde{E}_1(k), \ldots, \tilde{E}_M(k)\right\}_M.
\end{aligned}
$$

Our objective of this paper is to find the sequence of filter parameter matrices $K_i(k)$, such that the estimation error output $\bar{z}(k)$ satisfies the following H_∞ performance requirement. The main objective is to obtain the filter parameter matrix $\bar{K}(k)$ so that output estimation error $\bar{z}(k)$ can satisfy the H_∞ performance constraint over finite horizon. Given the disturbance attenuation level $\gamma > 0$ and positive definite matrix $\Psi > 0$. If the following inequality holds, then the system satisfies the H_∞ performance requirement,

$$
\begin{aligned}
J = \mathbb{E}\left\{\sum_{k=0}^{N}\left(\|\bar{z}(k)\|^2 - \gamma^2\|v(k)\|^2\right)\right\} \\
- \gamma^2\eta^T(0)\Psi\eta(0) < 0.
\end{aligned} \tag{16}
$$

Redefining H_∞ performance requirement for system (15),

$$
\begin{aligned}
\bar{J} = \mathbb{E}\Bigg\{&\sum_{k=0}^{N}\left(\|\bar{z}(k)\|^2 - \gamma^2\|\omega(k)\|^2\right) \\
&+\gamma^2\left(\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2 + \|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2\right. \\
&\left.+\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\Bigg\} - \gamma^2\eta^T(0)\Psi\eta(0) < 0,
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
&\bar{T}(k) = \mathrm{diag}\left\{\tilde{T}_1(k), \ldots, \tilde{T}_{l_0}(k),\ 0, \ldots, 0\right\}_M, \\
&\bar{H}(k) = \mathrm{diag}\left\{\tilde{H}_1(k), \ldots, \tilde{H}_{l_0}(k),\ 0, \ldots, 0\right\}_M, \\
&\bar{N}(k) = \mathrm{diag}\left\{\tilde{N}_1(k), \ldots, \tilde{N}_M(k)\right\}_M.
\end{aligned}
$$

Lemma 2.1: *Considering the performance requirements of* (16) *and* (17), *it can be inferred that* $J \le \bar{J}$.

Proof: By subtracting (17) from (16), we can get

$$
\begin{aligned}
J - \bar{J} = \mathbb{E}\Bigg\{&\sum_{k=0}^{N}\gamma^2\left(\|\omega(k)\|^2 - \|v(k)\|^2\right. \\
&-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2 - \|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2 \\
&\left.-\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\Bigg\} \\
= \mathbb{E}\Bigg\{&\sum_{k=0}^{N}\gamma^2\left(\|\varepsilon_1(k)\bar{L}(k)\bar{T}(k)\bar{H}v(k)\|^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\|\varepsilon_1(k)\bar{L}(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2\\
&+\|\varepsilon_2(k)\bar{\Lambda}(k)\bar{N}(k)\eta(k)\|^2\\
&-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2-\|\varepsilon_1(k)\bar{T}\bar{C}\eta(k)\|^2\\
&\left.-\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2)\right\}\\
=-\mathbb{E}&\left\{\sum_{k=0}^{N}\gamma^2\left(\|\varepsilon_1(k)(I-\bar{L}^T(k)\bar{L}(k))^{\frac{1}{2}}\bar{T}(k)\bar{H}v(k)\|^2\right.\right.\\
&+\|\varepsilon_1(k)(I-\bar{L}^T(k)\bar{L}(k))^{\frac{1}{2}}\bar{T}(k)\bar{C}(k)\eta(k)\|^2\\
&\left.\left.+\|\varepsilon_2(k)(I-\bar{\Gamma}^T(k)\bar{\Gamma}(k))^{\frac{1}{2}}\bar{N}(k)\eta(k)\|^2\right)\right\}\le 0,
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
\bar{L}(k)&=\operatorname{diag}\left\{\tilde{L}_1(k),\ldots,\tilde{L}_{l_0}(k),0,\ldots,0\right\}_M,\\
\bar{\Gamma}(k)&=\operatorname{diag}\left\{\tilde{\Gamma}_1(k),\ldots,\tilde{\Gamma}_{l_0}(k),0,\ldots,0\right\}_M.
\end{aligned}
$$

Based on $\bar{L}^T(k)\bar{L}(k))\le I$, $\bar{\Gamma}^T(k)\bar{\Gamma}(k)\le I$, we have $J-\bar{J}\le 0$, which means that $J\le\bar{J}$. In this sense, performance index (17) can be used instead of (16) to find the estimator gain. ■

3. Main results

In this section, by utilizing the stochastic analysis method and the complete square approach, the sufficient conditions for system (15) to satisfy the H_∞ performance constraint are obtained, and the estimator gain $K_i(k)$ is solved.

Theorem 3.1: *For the complex network* (1) *with the estimator* (4), *given a disturbance attenuation level* $\gamma>0$, *two positive scalars* $\varepsilon_1(k)>0$, *and* $\varepsilon_2(k)>0$, *and a positive definite matrix* $\Psi>0$. *The augmented system* (15) *satisfies the* H_∞ *performance constraint defined in* (17) *over a finite-horizon* $[0,N]$, *if there exists a solution* $(P(k),\bar{K}(k))$ *to the following backward RDE:*

$$
\begin{aligned}
P(k)=&\,(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k))^TG(k+1)(\bar{A}(k)\\
&+\bar{I}\bar{K}(k)\bar{C}(k))+\bar{E}^T(k)\bar{E}(k),\\
&+\gamma^2\varepsilon_1^2(k)\bar{C}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{C}(k)\\
&+\gamma^2\varepsilon_2^2(k)\bar{N}^T(k)\bar{N}(k),\\
P(N+1)=&\,0,
\end{aligned}
\tag{19}
$$

subject to

$$
\begin{aligned}
\Phi(k+1)=&-\bar{B}^T(k)P(k+1)\bar{B}(k)\\
&-\gamma^2(\varepsilon_1^2(k)\check{I}^T\bar{H}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{H}(k)\check{I}-I)>0,
\end{aligned}
\tag{20}
$$

$$
P(0)<\gamma^2\Psi, \tag{21}
$$

where

$$
\begin{aligned}
G(k+1)=&\,P(k+1)+P(k+1)\\
&\bar{B}(k)\Phi^{-1}(k+1)\bar{B}^T(k)P(k+1),
\end{aligned}
\tag{22}
$$

$$
\widehat{I}=[I_{n_v}\quad 0\quad 0\quad 0],\quad \check{I}=\operatorname{diag}\{\widehat{I},\ldots,\widehat{I}\}_M, \tag{23}
$$

Proof: Define $J_1(k)=\eta^T(k+1)P(k+1)\eta(k+1)-\eta^T(k)P(k)\eta(k)$. Refer to the expression of augmented system (15) and take the mathematical expectation, we have

$$
\begin{aligned}
&\mathbb{E}\{J_1(k)\}\\
&=\mathbb{E}\left\{\eta^T(k+1)P(k+1)\eta(k+1)-\eta^T(k)P(k)\eta(k)\right\}\\
&=\mathbb{E}\left\{\left[\eta^T(k)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T+\omega^T(k)\bar{B}^T(k)\right]\right.\\
&\qquad P(k+1)\left[\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)\eta(k)+\bar{B}(k)\omega(k)\right]\\
&\quad\left.-\eta^T(k)P(k)\eta(k)\right\}\\
&=\mathbb{E}\left\{\eta^T(k)\left[\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T\right.\right.\\
&\qquad\left.P(k+1)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)-P(k)\right]\eta(k)\\
&\quad+2\eta^T(k)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T\\
&\qquad\left.P(k+1)\bar{B}(k)\omega(k)+\omega^T(k)\bar{B}^T(k)P(k+1)\bar{B}(k)\omega(k)\right\}.
\end{aligned}
\tag{24}
$$

To get the desired conclusion, adding the zero term

$$
\begin{aligned}
&\mathbb{E}\left\{\|\bar{z}(k)\|^2-\gamma^2\|\omega(k)\|^2+\gamma^2\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2\right.\\
&\quad+\gamma^2\|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2+\gamma^2\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\\
&\quad-\left(\|\bar{z}(k)\|^2-\gamma^2\|\omega(k)\|^2+\gamma^2\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2\right.\\
&\quad\left.\left.+\gamma^2\|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2+\gamma^2\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\right\},
\end{aligned}
$$

to the right side of (24) and considering (15), we obtain

$$
\begin{aligned}
&\mathbb{E}\{J_1(k)\}\\
&=\mathbb{E}\left\{\eta^T(k)\left[\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T\right.\right.\\
&\qquad P(k+1)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)+\bar{E}^T(k)\bar{E}(k)\\
&\qquad+\gamma^2\varepsilon_1^2(k)\bar{C}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{C}(k)\\
&\qquad\left.+\gamma^2\varepsilon_2^2(k)\bar{N}^T(k)\bar{N}(k)-P(k)\right]\eta(k)\\
&\qquad+2\eta^T(k)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^TP(k+1)\bar{B}(k)\omega(k)\\
&\qquad+\omega^T(k)\left[\bar{B}^T(k)P(k+1)\bar{B}(k)\right.\\
&\qquad\left.+\gamma^2\left(\varepsilon_1^2(k)\check{I}^T\bar{H}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{H}(k)\check{I}-I\right)\right]\omega(k)\\
&\qquad-\left[\|\bar{z}(k)\|^2-\gamma^2\left(\|\omega(k)\|^2\right.\right.
\end{aligned}
$$

$$-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2 - \|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2 -\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2)\big]\big\}. \quad (25)$$

Drawing support from the completing-the-square technique, we have

$$\begin{aligned}&2\eta^T(k)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T P(k+1)\bar{B}(k)\omega(k)\\&\quad+\omega^T(k)\big[\bar{B}^T(k)P(k+1)\bar{B}(k)\\&\qquad+\gamma^2\left(\varepsilon_1^2(k)\check{I}^T\bar{H}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{H}(k)\check{I}-I\right)\big]\omega(k)\\&\quad=-\left(\omega(k)-\omega^*(k)\right)^T\Phi(k+1)\left(\omega(k)-\omega^*(k)\right)\\&\quad+\left(\omega^*(k)\right)^T\Phi(k+1)\omega^*(k),\end{aligned}\quad (26)$$

where $\Phi(k+1)$ is defined in (20) and

$$\begin{aligned}&\omega^*(k)\\&=\Phi^{-1}(k+1)\bar{B}^T(k)P(k+1)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)\eta(k).\end{aligned}\quad (27)$$

Substituting (26) into (25) yields

$$\begin{aligned}&\mathbb{E}\{J_1(k)\}\\&=\mathbb{E}\Big\{\eta^T(k)\Big[\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T\\&\quad P(k+1)\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)+\bar{E}^T(k)\bar{E}(k)\\&\quad+\gamma^2\varepsilon_1^2(k)\bar{C}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{C}(k)\\&\quad+\gamma^2\varepsilon_2^2(k)\bar{N}^T(k)\bar{N}(k)-P(k)\Big]\eta(k)\\&\quad-\left(\omega(k)-\omega^*(k)\right)^T\Phi(k+1)\left(\omega(k)-\omega^*(k)\right)\\&\quad+\left(\omega^*(k)\right)^T\Phi(k+1)\omega^*(k)\\&\quad-\big[\|\bar{z}(k)\|^2-\gamma^2\left(\|\omega(k)\|^2-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2\right.\\&\quad\left.-\|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2-\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\big]\Big\}.\end{aligned}\quad (28)$$

Substituting (27) into (28) and considering (22) yield

$$\begin{aligned}&\mathbb{E}\{J_1(k)\}\\&=\mathbb{E}\Big\{\eta^T(k)\Big[\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)^T G(k+1)\\&\quad\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)+\bar{E}^T(k)\bar{E}(k)\\&\quad+\gamma^2\varepsilon_1^2(k)\bar{C}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{C}(k)\\&\quad+\gamma^2\varepsilon_2^2(k)\bar{N}^T(k)\bar{N}(k)-P(k)\Big]\eta(k)\\&\quad-\left(\omega(k)-\omega^*(k)\right)^T\Phi(k+1)\left(\omega(k)-\omega^*(k)\right)\\&\quad-\big[\|\bar{z}(k)\|^2-\gamma^2\left(\|\omega(k)\|^2\right.\\&\quad-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2\\&\quad\left.-\|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2-\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\big]\Big\}.\end{aligned}\quad (29)$$

Notice that $J_1(k)=\eta^T(k+1)P(k+1)\eta(k+1)-\eta^T(k)P(k)\eta(k)$. Taking the sum on both sides of (29) from 0 to N and taking (19) into account, we obtain that

$$\begin{aligned}&\mathbb{E}\left\{\eta^T(N+1)P(N+1)\eta(N+1)-\eta^T(0)P(0)\eta(0)\right\}\\&=\mathbb{E}\Big\{-\sum_{k=0}^{N}\left(\omega(k)-\omega^*(k)\right)^T\Phi(k+1)\left(\omega(k)-\omega^*(k)\right)\\&\quad-\sum_{k=0}^{N}\big[\|\bar{z}(k)\|^2-\gamma^2\left(\|\omega(k)\|^2\right.\\&\quad-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2-\|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2\\&\quad\left.-\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\big]\Big\}.\end{aligned}\quad (30)$$

Combined with $\Phi(k+1)>0$, $P(0)<\gamma^2\Psi$, and $P(N+1)=0$, one has

$$\begin{aligned}&\mathbb{E}\Big\{\sum_{k=0}^{N}\big[\|\bar{z}(k)\|^2-\gamma^2\left(\|\omega(k)\|^2-\|\varepsilon_1(k)\bar{T}(k)\bar{H}(k)v(k)\|^2\right.\\&\quad\left.-\|\varepsilon_1(k)\bar{T}(k)\bar{C}(k)\eta(k)\|^2-\|\varepsilon_2(k)\bar{N}(k)\eta(k)\|^2\right)\big]\Big\}\\&\quad-\gamma^2\eta^T(0)\Psi\eta(0)\\&\quad<\mathbb{E}\Big\{-\sum_{k=0}^{N}\left(\omega(k)-\omega^*(k)\right)^T\\&\quad\Phi(k+1)\left(\omega(k)-\omega^*(k)\right)\Big\}\\&\quad\leq 0.\end{aligned}\quad (31)$$

The proof of this theorem is complete. ■

Next, the parameter $K_i(k)$ is calculated in the worst situation. Suppose the worst situation is

$$\begin{aligned}\omega(k)=\omega^*(k)=\Phi^{-1}(k+1)\bar{B}^T(k)P(k+1)\\\left(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k)\right)\eta(k).\end{aligned}\quad (32)$$

At this point, the augmented system (15) can be rewritten as

$$\begin{aligned}\eta(k+1)&=\left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)\eta(k)+((\Theta(k)+I)\bar{I}\rho(k),\\\tilde{z}(k)&=E(k)\eta(k),\end{aligned}\quad (33)$$

where

$$\begin{aligned}\Theta(k)&=\bar{B}(k)\Phi^{-1}(k+1)\bar{B}^T(k)P(k+1),\\\rho(k)&=\bar{K}(k)\bar{C}(k)\eta(k).\end{aligned}$$

Moreover, a cost functional is constructed as follows for the sake of the estimator gain:

$$\mathcal{H}(\bar{K}(k),\omega^*(k)) \triangleq \mathbb{E}\left\{\sum_{k=0}^{N}\|\bar{z}(k)\|^2+\sum_{k=0}^{N}\|\rho(k)\|^2\right\}. \quad (34)$$

Theorem 3.2: *For the complex network* (1) *with the estimator* (4)*, given a disturbance attenuation level* $\gamma > 0$*, two positive scalars* $\varepsilon_1(k) > 0$*, and* $\varepsilon_2(k) > 0$*, and a positive definite matrix* $\Psi > 0$*. The augmented system* (15) *satisfies the* H_∞ *performance constraint defined in* (17) *over a finite-horizon* $[0, N]$*, if there exists a solution* $(P(k), R(k), \bar{K}(k))$ *to the following coupled backward RDEs:*

$$\begin{aligned} P(k) &= [(\bar{A}(k)+\bar{I}\bar{K}(k)\bar{C}(k))^T G(k+1)(\bar{A}(k) \\ &\quad +\bar{I}\bar{K}(k)\bar{C}(k))+\bar{E}^T(k)\bar{E}(k) \\ &\quad +\gamma^2\varepsilon_1^2(k)\bar{C}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{C}(k) \\ &\quad +\gamma^2\varepsilon_2^2(k)\bar{N}^T(k)\bar{N}(k), \\ P(N+1) &= 0, \end{aligned} \quad (35)$$

$$\begin{aligned} R(k) &= [(\bar{A}(k)+\Theta(k)\bar{A}(k))^T R(k+1)(\bar{A}(k) \\ &\quad +\Theta(k)\bar{A}(k))+\bar{E}^T(k)\bar{E}(k) \\ &\quad \times \operatorname{sym}\left\{\bar{A}^T(k)\Theta^T(k)R(k+1)(\Theta(k)+I)\bar{I}\bar{K}(k)\bar{C}(k)\right\} \\ &\quad -\bar{A}^T(k)R(k+1)(\Theta(k)+I)\bar{I}\Omega^{-1} \\ &\quad (k+1)\bar{I}^T(\Theta(k)+I)^T R(k+1)\bar{A}(k), \\ R(N+1) &= 0, \end{aligned} \quad (36)$$

subject to

$$\begin{aligned} \Phi(k+1) &= -\bar{B}^T(k)P(k+1)\bar{B}(k) \\ &\quad -\gamma^2(\varepsilon_1^2(k)\breve{I}^T\bar{H}^T(k)\bar{T}^T(k)\bar{T}(k)\bar{H}(k)\breve{I}-I) > 0, \end{aligned} \quad (37)$$

$$\Omega(k+1) = \bar{I}^T(\Theta(k)+I)^T R(k+1)(\Theta(k)+I)\bar{I}+I, \quad (38)$$

$$P(0) < \gamma^2\Psi, \quad (39)$$

$$\begin{aligned} \bar{\mathcal{K}}^*(k) &= \arg\min_{\bar{K}(k)}\|\bar{K}(k)\bar{C}(k) \\ &\quad +\Omega^{-1}(k+1)\bar{I}^T(\Theta(k)+I)^T R(k+1)\bar{A}(k)\|_F. \end{aligned} \quad (40)$$

Proof: On the basis of Theorem 3.1, define $J_2(k) = \eta^T(k+1)R(k+1)\eta(k+1)-\eta^T(k)R(k)\eta(k)$, we obtain

$$\begin{aligned} &\mathbb{E}\{J_2(k)\} \\ &= \mathbb{E}\left\{\eta^T(k+1)R(k+1)\eta(k+1)-\eta^T(k)R(k)\eta(k)\right\} \\ &= \mathbb{E}\Big\{\eta^T(k)\left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)^T R(k+1) \\ &\quad \left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)\eta(k) \\ &\quad +2\eta^T(k)\left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)^T R(k+1)(\Theta(k)+I)^T\bar{I}\rho(k) \\ &\quad +\rho^T(k)\bar{I}^T(\Theta(k)+I)^T R(k+1) \\ &\quad (\Theta(k)+I)\bar{I}\rho(k)-\eta^T(k)R(k)\eta(k)\Big\}. \end{aligned} \quad (41)$$

Adding the zero term

$$\mathbb{E}\left\{\|\bar{z}(k)\|^2+\|\rho(k)\|^2-\|\bar{z}(k)\|^2-\|\rho(k)\|^2\right\}$$

to the right side of (41), one has

$$\begin{aligned} &\mathbb{E}\{J_2(k)\} \\ &= \mathbb{E}\Big\{\eta^T(k)\Big[\left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)^T R(k+1) \\ &\quad \left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)+\bar{E}^T(k)\bar{E}(k) \\ &\quad +\operatorname{sym}\left\{\bar{A}^T(k)\Theta^T(k)R(k+1)(\Theta(k)+I)\bar{I}\bar{K}(k)\bar{C}(k)\right\} \\ &\quad -R(k)\Big]\eta^T(k) \\ &\quad +2\eta^T(k)\left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)^T R(k+1)(\Theta(k)+I)^T\bar{I}\rho(k) \\ &\quad +\rho^T(k)\left[\bar{I}^T(\Theta(k)+I)^T R(k+1)(\Theta(k)+I)\bar{I}+I\right]\rho(k) \\ &\quad -\|\bar{z}(k)\|^2-\|\rho(k)\|^2\Big\}. \end{aligned} \quad (42)$$

Applying the completing-the-square technique, we have

$$\begin{aligned} &2\eta^T(k)\bar{A}^T(k)R(k+1)(\Theta(k)+I)\bar{I}\rho(k) \\ &\quad +\rho^T(k)\left[\bar{I}^T(\Theta(k)+I)^T R(k+1)(\Theta(k)+I)\bar{I}+I\right]\rho(k) \\ &= \left(\rho(k)+\rho^*(k)\right)^T\Omega(k+1)\left(\rho(k)+\rho^*(k)\right) \\ &\quad -\left(\rho^*(k)\right)\Omega(k+1)\rho^*(k), \end{aligned} \quad (43)$$

where

$$\rho^*(k) = \Omega^{-1}(k+1)\bar{I}^T(\Theta(k)+I)^T R(k+1)\bar{A}(k)\eta(k).$$

Furthermore, it can be deduced that

$$\begin{aligned} &\mathbb{E}\{J_2(k)\} \\ &= \mathbb{E}\Big\{\eta^T(k)\Big[\left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)^T R(k+1) \\ &\quad \left(\bar{A}(k)+\Theta(k)\bar{A}(k)\right)+\bar{E}^T(k)\bar{E}(k) \\ &\quad +\operatorname{sym}\left\{\bar{A}^T(k)\Theta^T(k)R(k+1)(\Theta(k)+I)\bar{I}\bar{K}(k)\bar{C}(k)\right\} \\ &\quad -\bar{A}^T(k)R(k+1)(\Theta(k)+I)\bar{I}\Omega^{-1}(k+1)\bar{I}^T \\ &\quad (\Theta(k)+I)^T R(k+1)\bar{A}(k)-R(k)\Big]\eta(k) \\ &\quad +\left(\rho(k)+\rho^*(k)\right)^T\Omega(k+1)\left(\rho(k)+\rho^*(k)\right) \\ &\quad -\|\bar{z}(k)\|^2-\|\rho(k)\|^2\Big\}. \end{aligned} \quad (44)$$

In the light of condition (36), one obtains

$$\begin{aligned} &\mathcal{H}(\bar{K}(k),\omega^*(k)) \\ &= \mathbb{E}\left\{\sum_{k=0}^{N}\|\bar{z}(k)\|^2+\sum_{k=0}^{N}\|\rho(k)\|^2\right\} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left\{\sum_{k=0}^{N} \left(\rho(k)+\rho^*(k)\right)^T \Omega(k+1) \left(\rho(k)+\rho^*(k)\right)\right\} \\
&\quad + \eta^T(0)R(0)\eta(0) \\
&\leq \mathbb{E}\left\{\sum_{k=0}^{N} \|\bar{K}(k)\bar{C}(k) + \Omega^{-1}(k+1)\bar{I}^T \right. \\
&\quad (\Theta(k)+I)^T R(k+1)\bar{A}(k)\|_F^2 \\
&\quad \left. \times \|\Omega(k+1)\|_F \|\eta(k)\|^2 \right\} + \eta^T(0)R(0)\eta(0).
\end{aligned} \tag{45}
$$

If the matrix $\bar{K}(k)$ satisfies Equation (40), then the cost of $\mathcal{H}(\bar{K}(k), \omega^*(k))$ can be minimized, which completes the proof. ■

Finally, according to (40), we will determine the estimator parameter $K_i(k)$, $i = 1, 2, \ldots, l_0$.

$$
\begin{aligned}
&\bar{\mathcal{K}}^*(k) \\
&= \arg\min_{\bar{K}(k)} \|\bar{K}(k)\bar{C}(k) \\
&\quad + \Omega^{-1}(k+1)\bar{I}^T(\Theta(k)+I)^T R(k+1)\bar{A}(k)\|_F \\
&= \arg\min_{\bar{K}(k)} \|\bar{K}(k)\bar{C}(k) - u(k)\|_F \\
&= \arg\min_{\bar{K}(k)} \\
&\qquad \left\| \operatorname{diag}\left\{\tilde{K}_i(k)\tilde{C}_1(k), \ldots, \tilde{K}_{l_0}(k)\tilde{C}_{l_0}(k), 0, \ldots, 0\right\} \right. \\
&\quad \left. - \left[(u^1(k))^T, \ldots, (u^{l_0}(k))^T, 0, \ldots, 0\right]^T \right\|_F \\
&= \arg\min_{\bar{K}(k)} \left\| \left[\left(\tilde{K}_1(k)[\tilde{C}_1(k) \quad 0 \quad \cdots \quad 0] - u^1(k)\right)^T \cdots \right. \right. \\
&\qquad \left. \left. \left(\tilde{K}_{l_0}(k)[0 \quad 0 \quad \cdots \quad \tilde{C}_{l_0}(k)] - u^{l_0}(k)\right)^T \right]^T \right\|_F,
\end{aligned} \tag{46}
$$

where

$$
\begin{aligned}
u(k) &= -\Omega^{-1}(k+1)\bar{I}^T(\Theta(k)+I)^T R(k+1)\bar{A}(k) \\
&= \left[(u^1(k))^T \quad (u^2(k))^T \cdots (u^{l_0}(k))^T \right. \\
&\qquad \left. (u^{l_0+1}(k))^T \cdots (u^M(k))^T\right]^T.
\end{aligned} \tag{47}
$$

It can be inferred from the above formula that

$$
\begin{aligned}
\tilde{K}_1(k) &= u^1(k)[\tilde{C}_1(k) \quad 0 \quad \cdots \quad 0]^\dagger, \\
\tilde{K}_2(k) &= u^2(k)[0 \quad \tilde{C}_2(k) \quad 0 \quad \cdots \quad 0]^\dagger, \\
&\vdots \\
\tilde{K}_{l_0}(k) &= u^{l_0}(k)[0 \quad \cdots \quad 0 \quad \tilde{C}_{l_0}(k) \quad 0 \cdots \quad 0]^\dagger.
\end{aligned} \tag{48}
$$

As a result, $K_i(k)$ is the block matrix from row $n+1$ to $2n$ in the block diagonal matrix $\tilde{K}_i(k)$,

$$
K_i(k) = \begin{bmatrix} 0_{n\times n} & I_{n\times n} \end{bmatrix} \tilde{K}_i(k) \begin{bmatrix} 0_{n_y\times n_y} \\ I_{n_y\times n_y} \end{bmatrix}. \tag{49}
$$

On the basis of Theorems 3.1 and 3.2, the method of solving state estimator gain is summarized as Algorithm 1. In this way, online state estimation can be realized by software.

Algorithm 1 The estimator design algorithm.

step 1. *Set $k = N$. Then one has $P(k+1) = 0$ and $R(k+1) = 0$.*
step 2. *According to (??), compute $\Omega(k+1)$ with known $R(k+1)$. If $\Omega(k+1) > 0$, the estimator gain matrices $\bar{K}(k)$ can be derived by (??) and go to the next procedure, else jump to Step 6.*
step 3. *According to (??), compute $\Phi(k+1)$ with known $P(k+1)$ and $\bar{K}(k)$, respectively. If $\Phi(k+1) > 0$, go to the next step, else jump to Step 6.*
step 4. *Solve the backward RDEs (??) and (??) to calculate $P(k)$ and $R(k)$, respectively.*
step 5. *If $k \neq 0$, let $k = k-1$ and jump to Step 2, else go to the next step.*
step 6. *If not all conditions $\Omega(k+1) > 0$, $\Phi(k+1) > 0$ and $P(0) < \gamma^2\Psi$ are fulfilled, this algorithm is infeasible, Stop.*

4. Numerical example

In this section, a numerical example is presented to illustrate the effectiveness of the proposed method. Consider the complex network (1) with 4 nodes, and the parameters are given as follows:

$$
A_1(k) = \begin{bmatrix} 0.42 & 0.35 + 0.2\cos(1.2k) \\ -0.21 & -0.2 \end{bmatrix},
$$

$$
A_2(k) = \begin{bmatrix} 0.42 + 0.2\sin(k) & 0.35 \\ -0.22 & -0.2 + 0.1\cos(k) \end{bmatrix},
$$

$$
A_3(k) = \begin{bmatrix} 0.4 - 0.15\cos(k) & 0.34 \\ -0.22 & -0.2 + 0.05\cos(k) \end{bmatrix},
$$

$$
A_4(k) = \begin{bmatrix} 0.43 & 0.3 \\ -0.2 & -0.2 + 0.1\cos(k) \end{bmatrix},
$$

$$
W = \begin{bmatrix} -0.3 & 0.1 & 0.1 & 0.1 \\ 0.1 & -0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.3 & 0.1 \\ 0.1 & 0.1 & 0.1 & -0.3 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix},
$$

Table 1. Estimator parameters $K_i(k), i = 1, 2$.

k	0	1	2	⋯	48	49	50
$K_1(k)$	$\begin{bmatrix}0.0541\\0.0593\end{bmatrix}$	$\begin{bmatrix}0.0478\\0.0383\end{bmatrix}$	$\begin{bmatrix}0.0297\\0.0198\end{bmatrix}$	⋯	$\begin{bmatrix}0.0634\\0.0489\end{bmatrix}$	$\begin{bmatrix}0.0354\\0.0327\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$
$K_2(k)$	$\begin{bmatrix}0.0852\\0.0570\end{bmatrix}$	$\begin{bmatrix}0.0856\\0.0516\end{bmatrix}$	$\begin{bmatrix}0.0854\\0.0480\end{bmatrix}$	⋯	$\begin{bmatrix}0.0805\\0.0462\end{bmatrix}$	$\begin{bmatrix}0.0571\\0.0326\end{bmatrix}$	$\begin{bmatrix}0\\0\end{bmatrix}$

$$D_1(k) = \begin{bmatrix}0.16 + 0.05\cos(0.3k)\\ 0.18\end{bmatrix},$$

$$D_2(k) = \begin{bmatrix}0.1\\0.2\end{bmatrix}, \quad D_3(k) = \begin{bmatrix}0.1\\ 0.2 + 0.2\sin(0.3k)\end{bmatrix},$$

$$D_4(k) = \begin{bmatrix}0.2 + 0.1\cos(0.3k)\\ 0.18\end{bmatrix},$$

$$E_1(k) = \begin{bmatrix}0.22\\0.2\end{bmatrix}, \quad E_2(k) = \begin{bmatrix}0.2\\0.12\end{bmatrix}, \quad E_3(k) = \begin{bmatrix}0.21\\0.1\end{bmatrix},$$

$$E_4(k) = \begin{bmatrix}0.2\\0.12\end{bmatrix}.$$

The nonlinear function is selected as

$$f(k, x_i(k)) = \begin{bmatrix}0.4x_{i1}(k) - \tanh(0.2x_{i1}(k))\\ 0.3x_{i2}(k) - \tanh(0.2x_{i2}(k))\end{bmatrix},$$

which implies that

$$U_1(k) = \begin{bmatrix}0.4 & 0\\ 0 & 0.3\end{bmatrix}, \quad U_2(k) = \begin{bmatrix}0.2 & 0\\ 0 & 0.1\end{bmatrix}.$$

Thus, it can be concluded that

$$Q(k) = \begin{bmatrix}0.3 & 0\\ 0 & 0.2\end{bmatrix}, \quad N(k) = \begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}.$$

The disturbance input $v_i(k)$ obeys uniform distribution over $[-0.2 \ \ 0.2]$. It is assumed that the measured output of two nodes is known, i.e. $l_0 = 2$. Other parameters are shown below (Table 1)

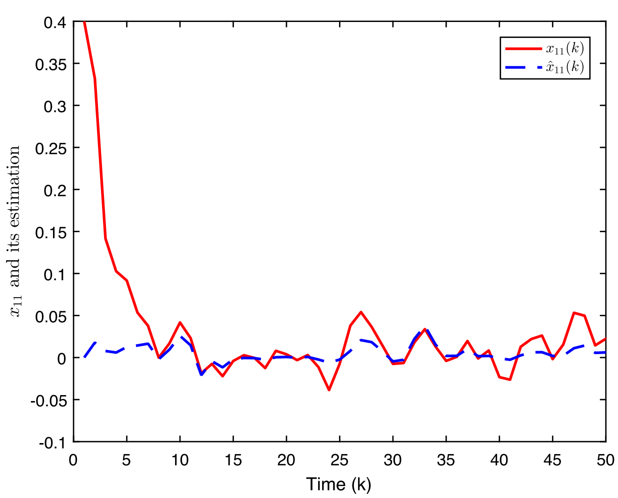

Figure 1. The state evolution $x_{11}(k)$ and its estimate $\hat{x}_{11}(k)$.

$$C_1(k) = \begin{bmatrix}0.3 & 0.62 + 0.35\cos(0.3k)\end{bmatrix},$$

$$C_2(k) = \begin{bmatrix}0.4 & 0.2 + 0.3\cos(0.3k)\end{bmatrix},$$

$$H_1(k) = 0.9, \quad H_2(k) = 0.85.$$

Concerning the resilient filter, suppose that

$$S(k) = \begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, \quad T(k) = \begin{bmatrix}0.6\\0.5\end{bmatrix}.$$

In this example, set theH_∞ performance level $\gamma = 1.3$, parameter $\varepsilon_1 = 0.4, \varepsilon_2 = 0.6$. The initial value of the state

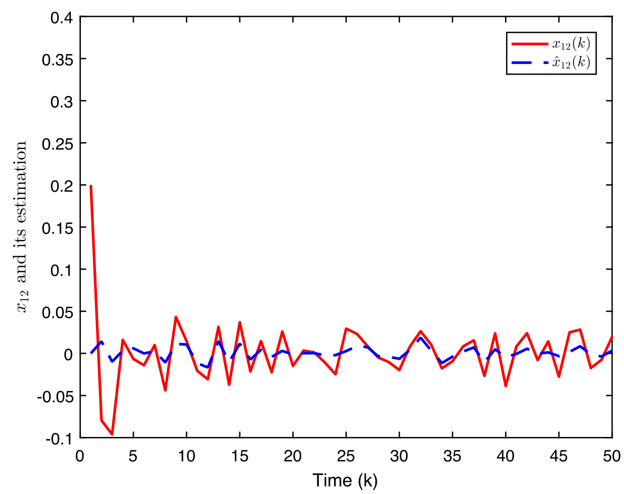

Figure 2. The state evolution $x_{12}(k)$ and its estimate $\hat{x}_{12}(k)$.

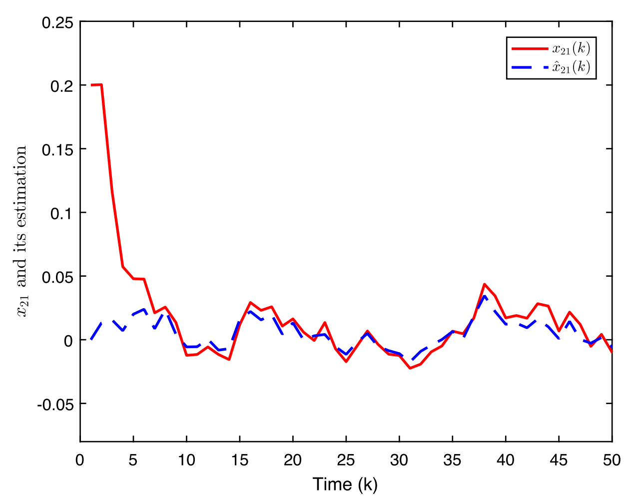

Figure 3. The state evolution $x_{21}(k)$ and its estimate $\hat{x}_{21}(k)$.

is

$$x_1(0) = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}, \quad x_2(0) = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad x_3(0) = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix},$$

$$x_4(0) = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}.$$

The simulation results are shown in Figures 1–5.

Figures 1–4 depict the curves of two states of node 1 and node 2 and their estimations. Figure 5 shows the variation curves of estimation errors, which indicate that the state estimator designed by this algorithm is effective.

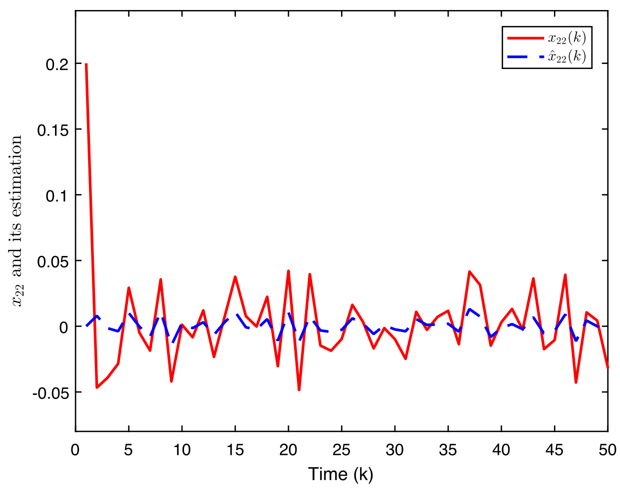

Figure 4. The state evolution $x_{22}(k)$ and its estimate $\hat{x}_{22}(k)$.

From (16), the H_∞ performance constraint can be calculated from $(\mathbb{E}\{\sum_{k=0}^{N} \|\bar{z}(k)\|^2\})^{\frac{1}{2}} / (\mathbb{E}\{\sum_{k=0}^{N} \|v(k)\|^2 + \eta^T(0)\Psi\eta(0)\})^{\frac{1}{2}} = 0.1801 < \gamma = 1.3$.

5. Conclusion

The problem of H_∞ state estimation for time-varying complex networks over finite horizon has been investigated. Because of the possible physical environment, the measurement output may not be measured from all nodes. Thus the measurement outputs for state tracking in this paper come from partial nodes. Moreover, because of the inaccuracy of the parameters in practical application, the estimator parameters may deviate from the expected results, so resilient filters have been designed to eliminate its adverse effects. By using the stochastic analysis method and the complete square method, the sufficient conditions for the estimation error to satisfy the H_∞ performance constraint have been obtained. Then, the filter gain has been calculated by solving backward RDEs. Finally, a numerical simulation example has been given to demonstrate the effectiveness of the proposed state estimation method.

Disclosure statement

No potential conflict of interest was reported by the author(s).

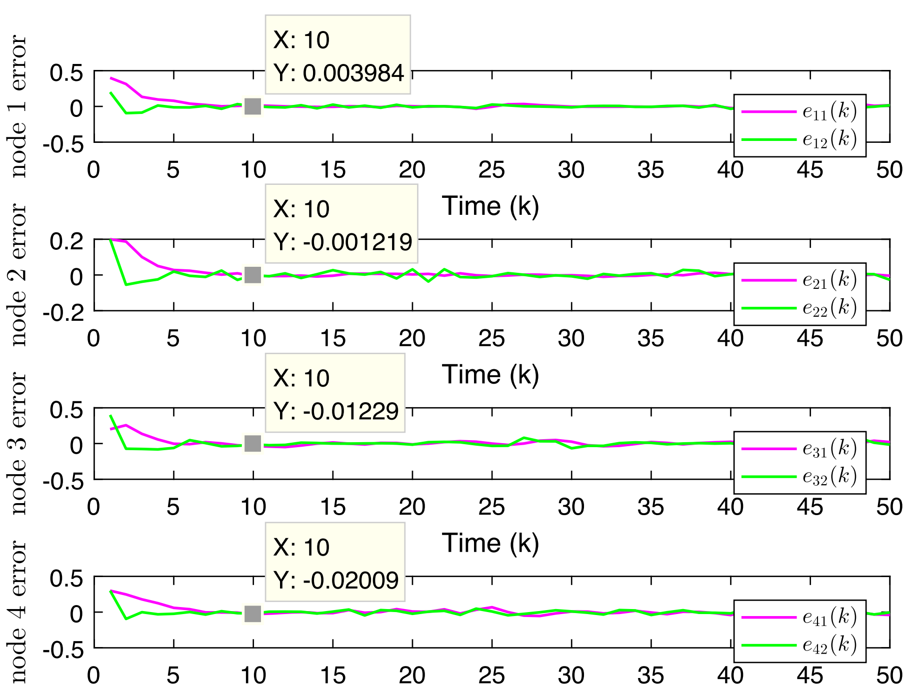

Figure 5. Estimation error of each node.

Funding

This work was supported by the National Natural Science Foundation of China under Grants 61773400, 62073339, 62033008, the Shandong Provincial Key Program of Research and Development under Grant 2019GGX101046, the Fundamental Research Funds for the Central Universities of China under Grants 19CX02044A, 20CX02309A, and the Opening Fund of National Engineering Laboratory of Offshore Geophysical and Exploration Equipment.

ORCID

Li Sheng http://orcid.org/0000-0003-2940-209X

References

Adu-Gyamfi, F., Cheng, Y., Yin, C., & Zhong, S. (2018). ExponentialH_∞ synchronization of non-fragile sampled-data controlled complex dynamical networks with random coupling and time varying delay. *Systems Science & Control Engineering*, *6*(1), 370–387. https://doi.org/10.1080/21642583.2018.1509396

Albert, R., & Barabási, A. L. (2002). Statistical mechanics of complex networks. *Reviews of Modern Physics*, *74*(1), 47–97. https://doi.org/10.1103/RevModPhys.74.47

Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., & Hwang, D. U. (2006). Complex networks: Structure and dynamics. *Physics Reports*, *424*(4–5), 175–308. https://doi.org/10.1016/j.physrep.2005.10.009

Costa, L. D. F., Oliveira, O. N., Travieso, G., Rodrigues, F. A., P. R. V. Boas, Antiqueira, L., Viana, M. P., & Rocha, L. E. C. (2011). Analyzing and modeling real-world phenomena with complex networks: A survey of applications. *Advances in Physics*, *60*(3), 329–412. https://doi.org/10.1080/00018732.2011.572452

Costa, L. D. F., Rodrigues, F. A., Travieso, G., & Boas, P. R. V. (2007). Characterization of complex networks: A survey of measurements. *Advances in Physics*, *56*(1), 167–242. https://doi.org/10.1080/00018730601170527

Ding, D., Wang, Z., Lam, J., & Shen, B. (2015). Finite-horizon H_∞ control for discrete time-varying systems with randomly occurring nonlinearities and fading measurements. *IEEE Transactions on Automatic Control*, *60*(9), 2488–2493. https://doi.org/10.1109/TAC.2014.2380671

Ding, D., Wang, Z., Shen, B., & Shu, H. (2012). H_∞ state estimation for discrete-time complex networks with randomly occurring sensor saturations and randomly varying sensor delays. *IEEE Transactions on Neural Networks and Learning Systems*, *23*(5), 725–736. https://doi.org/10.1109/TNNLS.2012.2187926

Han, F., Wang, Z., Dong, H., & Liu, H. (2017). Partial-nodes-based scalable H_∞-consensus filtering with censored measurements over sensor networks. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*. doi: 10.1109/TSMC.2019.2907649

Hou, N., Dong, H., Wang, K. Z., Ren, W., & Alsaadi, F. E. (2016). Non-fragile state estimation for discrete markovian jumping neural networks. *Neurocomputing*, *179*, 238–245. https://doi.org/10.1016/j.neucom.2015.11.089

Khafaf, N. A., & Jalili, M. (2019). Optimization of synchronizability in complex spatial networks. *Physica A: Statistical Mechanics and Its Applications*, *514*, 46–55. https://doi.org/10.1016/j.physa.2018.09.030

Li, H. (2013). Cluster synchronization and state estimation for complex dynamical networks with mixed time delays. *Applied Mathematical Modelling*, *37*, 7223–7244. https://doi.org/10.1016/j.apm.2013.02.019

Li, W., Jia, Y., & Du, J. (2018). Resilient filtering for nonlinear complex networks with multiplicative noise. *IEEE Transactions on Automatic Control*, *64*(6), 2522–2528. https://doi.org/10.1109/TAC.9

Li, X. M., Zhang, B., Li, P., Zhou, Q., & Lu, R. (2020). Finite-horizon H_∞ state estimation for periodic neural networks over fading channels. *IEEE Transactions on Neural Networks and Learning Systems*, *31*(5), 1450–1460. https://doi.org/10.1109/TNNLS.5962385

Liang, J., Sun, F., & Liu, X. (2014). Finite-horizon H_∞ filtering for time-varying delay systems with randomly varying nonlinearities and sensor saturations. *Systems Science and Control Engineering*, *2*, 108–118. https://doi.org/10.1080/21642583.2014.883339

Liu, Y., Wang, Z., Yuan, Y., & Alsaadi, F. E. (2018). Partial-nodes-based state estimation for complex networks with unbounded distributed delays. *IEEE Transactions on Neural Networks and Learning Systems*, *29*(8), 3906–3912. https://doi.org/10.1109/TNNLS.2017.2740400

Liu, Y., Wang, Z., Yuan, Y., & Liu, W. (2019). Event-triggered partial-nodes-based state estimation for delayed complex networks with bounded distributed delays. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, *49*(6), 1088–1098. https://doi.org/10.1109/TSMC.6221021

Pagani, G. A., & Aiello, M. (2013). The power grid as a complex network: A survey. *Physica A: Statistical Mechanics and Its Applications*, *392*(11), 2688–2700. https://doi.org/10.1016/j.physa.2013.01.023

Sakthivel, R., Anbuvithya, R., Mathiyalagan, K., & Prakash, P. (2015). Combined H_∞ and passivity state estimation of memristive neural networks with random gain fluctuations. *Neurocomputing*, *168*, 1111–1120. https://doi.org/10.1016/j.neucom.2015.05.012

Sheng, L., Niu, Y., & Gao, M. (2018). Distributed resilient filtering for time-varying systems over sensor networks subject to Round-Robin/Stochastic protocol. *ISA Transactions*, *87*, 55–67. https://doi.org/10.1016/j.isatra.2018.11.012

Sheng, L., Niu, Y., Zou, L., Liu, Y., & Alsaadi, F. E. (2018). Finite-horizon state estimation for time-varying complex networks with random coupling strengths under round-robin protocol. *Journal of the Franklin Institute*, *355*(15), 7417–7442. https://doi.org/10.1016/j.jfranklin.2018.07.026

Sheng, L., Wang, Z., Zou, L., & F. E. Alsaadi (2017). Event-based state estimation for time-varying stochastic dynamical networks with state- and disturbance-dependent noises. *IEEE Transactions on Neural Networks and Learning Systems*, *28*(10), 2382–2394. https://doi.org/10.1109/TNNLS.2016.2580601

Tang, Y., Qian, F., Gao, H., & Kurths, J. (2014). Synchronization in complex networks and its application-a survey of recent advances and challenges. *Annual Reviews in Control*, *38*(2), 184–198. https://doi.org/10.1016/j.arcontrol.2014.09.003

Wang, Z., Dong, H., Shen, B., & Gao, H. (2013). Finite-horizon H_∞ filtering with missing measurements and quantization effects. *IEEE Transactions on Automatic Control*, *58*(7), 1707–1718. https://doi.org/10.1109/TAC.2013.2241492

Wang, F., Liang, J., & Dobaie, A. M. (2018). Resilient filtering for time-varying stochastic coupling networks under the event-triggering scheduling. *International Journal of General*

Systems, *47*(5–6), 491–505. https://doi.org/10.1080/03081079.2018.1455193

Wang, X., Park, J. H., Yang, H., Zhang, X., & Zhong, S. (2020). Delay-dependent fuzzy sampled-data synchronization of T-S fuzzy complex networks with multiple couplings. *IEEE Transactions on Fuzzy Systems*, *28*(1), 178–189. https://doi.org/10.1109/TFUZZ.91

Yang, N., Chen, D., Ji, D., & Wu, Z. (2019). Resilient state estimation for nonlinear complex networks with time-delay under stochastic communication protocol. *Neurocomputing*, *346*, 38–47. https://doi.org/10.1016/j.neucom.2018.07.085

Yu, Y., Dong, H., Huo, F., Kang, C., & Li, J. (2016). Non-fragile state estimation for discrete neural networks. In *International conference on informative and cybernetics for computational social systems*. IEEE. doi:10.1109/ICCSS.2016.7586458

Zhang, D., Cai, W., Xie, L., & Wang, Q. G. (2014). Non-fragile distributed filtering for T-S fuzzy systems in sensor networks. *IEEE Transactions on Fuzzy Systems*, *23*(5), 1883–1890. https://doi.org/10.1109/TFUZZ.2014.2367101

Zhang, C., & Han, B. S. (2020). Stability analysis of stochastic delayed complex networks with multi-weights based on Razumikhin technique and graph theory. *Physica A: Statistical Mechanics and Its Applications*, *538*, 122827. https://doi.org/10.1016/j.physa.2019.122827

Zhang, L., Zhang, H., & Ding, X. (2017). Non-fragile H_∞ filtering for large-scale power systems with sensor networks. *IET Generation, Transmission & Distribution*, *11*(4), 968–977. https://doi.org/10.1049/iet-gtd.2016.0988

Zhao, Z., Wang, Z., Zou, L., & Liu, H. (2018). Finite-horizon H_∞ state estimation for artificial neural networks with component-based distributed delays and stochastic protocol. *Neurocomputing*, *321*, 169–177. https://doi.org/10.1016/j.neucom.2018.08.031

Zou, L., Wang, Z., Gao, H., & Liu, X. (2017). Event-triggered state estimation for complex networks with mixed time delays via sampled data information: The continuous-time case. *IEEE Transactions on Cybernetics*, *45*(12), 2804–2815. https://doi.org/10.1109/TCYB.2014.2386781

Zou, L., Wang, Z., Han, Q., & Zhou, D. (2019a). Moving horizon estimation for networked time-delay systems under Round-Robin protocol. *IEEE Transactions on Automatic Control*, *64*(12), 5191–5198. https://doi.org/10.1109/TAC.9

Zou, L., Wang, Z., Han, Q., & Zhou, D. (2019b). Recursive filtering for time-varying systems with random access protocol. *IEEE Transactions on Automatic Control*, *64*(2), 720–727. https://doi.org/10.1109/TAC.2018.2833154

Zou, L., Wang, Z., Hu, J., & Zhou, D. (2020). Moving horizon estimation with unknown inputs under dynamic quantization effects. *IEEE Transactions on Automatic Control*. https://doi.org/10.1109/TAC.2020.2968975

Zou, L., Wang, Z., & Zhou, D. (2020). Moving horizon estimation with non-uniform sampling under component-based dynamic event-triggered transmission. *Automatica*, *120*, Article ID: 109154, 13 pages. https://doi.org/10.1016/j.automatica.2020.109154