Fuzzy Information and Engineering

ISSN: 1616-8658 (Print) 1616-8666 (Online) Journal homepage: https://www.tandfonline.com/loi/tfie20

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To cite this article: Abdus Salam, Wajih Ashraf, Ahsan Mahboob & Noor Mohammad Khan (2020): Characterizations of Ordered Semigroups in Terms of Anti-fuzzy Ideals, Fuzzy Information and Engineering, DOI: 10.1080/16168658.2020.1753496

To link to this article: https://doi.org/10.1080/16168658.2020.1753496

Published online: 15 Sep 2020.

FUZZY INFORMATION AND ENGINEERING
https://doi.org/10.1080/16168658.2020.1753496

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Characterizations of Ordered Semigroups in Terms of Anti-fuzzy Ideals

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ABSTRACT
Adopting the notion of a (k^*, q)-quasi-coincidence of a fuzzy point with a fuzzy set, the idea of an $(\in, \in \vee(k^*, q_k))$-antifuzzy left (right) ideal, $(\in, \in \vee(k^*, q_k))$-antifuzzy ideal and $(\in, \in \vee(k^*, q_k))$-antifuzzy (generalized) bi-ideal in ordered semigroups are proposed, that are the generalization of the idea of an antifuzzy left (right) ideal, antifuzzy ideal and antifuzzy (generalized) bi-ideal in ordered semigroups and a few fascinating characterizations are obtained. In this paper, we tend to focus to suggest a connection between standard generalized bi-ideals and $(\in, \in \vee(k^*, q_k))$-antifuzzy generalized bi-ideals. In addition, different classes of regular ordered semigroups are characterized by the attributes of this new idea. Finally, the (k^*, k)-lower part of an $(\in, \in \vee(k^*, q_k))$-antifuzzy generalized bi-ideal is outlined and a few characterizations are mentioned.

ARTICLE HISTORY
Received 16 November 2019
Revised 25 March 2020
Accepted 29 March 2020

KEYWORDS
Ordered semigroup; fuzzy subset; $(\in, \in \vee(k^*, q_k))$-fuzzy bi-ideal; $(\in, \in \vee(k^*, q_k))$-antifuzzy bi-ideal

1. Introduction

The major advancements in the fascinating world of fuzzy sets started with the work of renowned scientist Zadeh [1] with new directions and ideas. In 1971, Rosenfeld's [2] method of fuzzification of algebraic structures represented a quantum jump in the history of fuzzy sets and related mathematics, and most of the later contributions in this field are the validations of this work. Rosenfeld introduced the concept of fuzzy groups and successfully extended many results from groups to fuzzy groups. The idea of a quasi-coincidence of a fuzzy point with a fuzzy set was initiated by Bhakat and Das [3,4] which played a significant role in generating different types of fuzzy subgroups. Later, they [5, 6] reported the concept of (α, β)-fuzzy subgroups by using 'belongs to' relation $(\in)$ and 'quasi-coincident' with relation (q) between a fuzzy point and a fuzzy set. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup [2]. From the time that fuzzy subgroups gained general acceptance over the decades, it has provided a central trunk to the people to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. Jun et al. [7] had thrown interesting light on the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterization of regular ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. Kehayopulu [8] characterized regular, left regular and right regular ordered semigroups by means of fuzzy left, fuzzy

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right and fuzzy bi-ideals. In another pioneered contribution, Jun [9] generalized the concept of $(\in, \in\vee q)$-fuzzy subalgebra of a BCK/BCI-algebra and introduced a new concept of $(\in, \in\vee q_k)$-fuzzy subalgebras followed by the basic properties of BCK-algebras. In continuation of this idea, Shabir et al. [10] and Shabir and Mahmood [11] reported the concept of generalized forms of (α, β)-fuzzy ideals and defined $(\in, \in\vee q_k)$-fuzzy ideals of semigroups and hemirings comprehensively. Recent developments in fuzzy ideals related to semigroups and hemirings have prompted the formulation of a precise description of numerous classes of semigroups and hemirings and their characterizations (see [11–26]). Moreover, new classifications of ordered semigroups have been investigated by introducing the concept of $(\in, \in\vee(k^*, q_k))$-fuzzy subsystems and $(\in, \in\vee(k^*, q_k))$-fuzzy quasi-ideals by Khan et al. [27] and Mahboob et al. [28], respectively.

Biswas [29] introduced the concept of antifuzzy subgroups of groups and lower level sets of fuzzy subsets to initiate the study of antifuzzy algebraic structures. The concept of lower level sets of fuzzy subsets is one of the mathematical methods for studying antifuzzy algebraic structures and some of the papers [29–32] have used the concept of lower level sets. Modifying and applying Biswas's idea, concepts of different types of antifuzzy algebraic structures had been introduced and studied extensively by many authors [33–39]. For example, Shabir and Nawas [38] in 2009 introduced the concept of an antifuzzy (generalized) bi-ideal of semigroup and characterized antifuzzy (generalized) bi-ideals by using lower level sets. They also characterized semigroups in terms of antifuzzy (generalized) bi-ideals. Khan and Asif [34], in continuation of the work carried out by Shabir and Nawas, introduced antifuzzy interior ideals of a semigroup and characterized semigroups by the properties of antifuzzy (generalized) bi-ideals and antifuzzy interior ideals. Khan et al. [36] obtained relationship between antifuzzy (generalized) bi-ideals and antifuzzy right ideals of semilattices of left groups. Antifuzzy (generalized) bi-ideals and antifuzzy one-sided ideals had been used by Khan et al. [36] to characterize semilattices of left (right) groups. In 2018, Julatha and Siripitukdet [33] characterized antifuzzy subsemigroups, antifuzzy generalized bi-ideals and antifuzzy bi-ideals of semigroups by using certain subsets of S, $[0, 1]$ and $S \times [0, 1]$. Due to such possibilities of applications, semigroups and related structures are studied via antifuzzy generalized bi-ideals and antifuzzy bi-ideals.

Motivated by the above, in Section 3 of the present paper, we introduce, as a generalization of the notion of an antifuzzy bi-ideal defined in an ordered semigroup, the notion of an $(\in, \in\vee(k^*, q_k))$-antifuzzy (generalized) bi-ideal in an ordered semigroup. Then, we obtain different characterizations of ordered semigroups in terms of $(\in, \in\vee(k^*, q_k))$-antifuzzy left ideals, $(\in, \in\vee(k^*, q_k))$-antifuzzy right ideals, $(\in, \in\vee(k^*, q_k))$-antifuzzy generalized bi-ideals and $(\in, \in\vee(k^*, q_k))$-antifuzzy bi-ideals. In Section 4, we present relationship between $(\in, \in\vee(k^*, q_k))$-antifuzzy bi-ideals and similar types of fuzzy left (right) ideals, while the last section offers concluding remarks and some ideas for future work on the topic.

2. Preliminaries

Throughout rest of the paper, S will stand for an ordered semigroup without explicit mention.

A subset $T(\neq \emptyset)$ of S is said to be a subsemigroup of S if for all $x, y \in T, xy \in T$. A subset $K(\neq \emptyset)$ of S is said to be a left (resp. right) ideal of S if $SK \subseteq K (KS \subseteq K)$ and for any $x \in K, y \in S$ such that $y \leq x$, then $y \in K$. A non-empty subset J of S is said to be an ideal of S if J is both a left ideal and a right ideal of S. A subsemigroup B of S is said to be a bi-ideal of S if $BSB \subseteq B$ and for any $x \in B, y \in S$ such that $y \leq x$, then $y \in B$. A subset Q of S is said to be a quasi-ideal of S if $(QS] \cap (SQ] \subseteq Q$ and for any $x \in Q, y \in S$ such that $y \leq x$, then $y \in Q$.

An ordered semigroup S is said to be regular (resp. left regular, right regular) if for each $x \in S, \exists y \in S$ such that $x \leq xyx$(resp.$x \leq yxx, x \leq xxy$); and S is called weakly regular if for each $x \in S, \exists y, z \in S$ such that $x \leq yxzx$.

Any mapping $\eta: S \mapsto [0, 1]$ is called a fuzzy subset of S (or fuzzy set of S). We shall denote, in whatever follows, the fuzzy subset η of S defined by $\eta(u) = 1$ for all $u \in S$ by the symbol S itself.

Let η and ξ be two fuzzy subset of S. Then $\eta \cap \xi, \eta \cup \xi$ and $\eta \circ \xi$ are defined as follows:

$$(\eta \cap \xi)(u) = \min\{\eta(u), \xi(u)\} = \eta(u) \wedge \xi(u),$$

$$(\eta \cup \xi)(u) = \max\{\eta(u), \xi(u)\} = \eta(u) \vee \xi(u)$$

for each $u \in S$ and

$$(\eta \circ \xi)(u) = \begin{cases} \bigvee_{(v,w)\in A_u} \{\eta(v) \wedge \xi(w)\} \text{ if } A_u \neq \emptyset, \\ 0 \text{ if } A_u = \emptyset \end{cases}$$

where $A_u = \{(v, w) \in S \times S | u \leq vw\}$ is a relation on S.

We first collect some notions which are necessary for whatever follows.

Let η be a fuzzy subset of S. Then η is called a fuzzy subsemigroup [8] of S if $\eta(uv) \geq \min\{\eta(u), \eta(v)\}$ for all $u, v \in S$.η is called a fuzzy left (resp. right) ideal [8] of S if: (1) $u \leq v \Rightarrow \eta(u) \geq \eta(v)$ and (2) $\eta(uv) \geq \eta(v)$(resp.$\eta(uv) \geq \eta(u)$) for all $u, v \in S$; and a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subsemigroup η of S is called a fuzzy bi-ideal of S if: (1) $u \leq v \Rightarrow \eta(u) \geq \eta(v)$ and (2) $\eta(uvw) \geq min\{\eta(u), \eta(w)\}$ for all $u, v, w \in S$. If we drop the condition of fuzzy subsemigroup from the definition of a fuzzy bi-ideal, then η is called a fuzzy generalized bi-ideal of S; i.e. a fuzzy subset η of S is called a fuzzy generalized bi-ideal [8] of S if: (1) $u \leq v \Rightarrow \eta(u) \geq \eta(v)$ and (2) $\eta(uvw) \geq\min\{\eta(u), \eta(w)\}$ for all $u, v, w \in S$.

A fuzzy subset η is called an antifuzzy subsemigroup [38] of S if $\eta(uv) \leq\max\{\eta(u), \eta(v)\}$ for all $u, v \in S$. It is called an antifuzzy left (resp. right) ideal of S if: (1) $u \leq v \Rightarrow \eta(u) \leq \eta(v)$ and (2) $\eta(uv) \leq \eta(v)$(resp.$\eta(uv) \leq \eta(u)$) for all $u, v \in S$. η is called an antifuzzy ideal of S if it is both an antifuzzy left and an antifuzzy right ideal of S. Further η is called an antifuzzy generalized bi-ideal [35] of S if: (1) $u \leq v \Rightarrow \eta(u) \leq \eta(v)$ and (2) $\eta(uvw) \leq\max\{\eta(u), \eta(w)\}$; and an antifuzzy bi-ideal [35] of Sif: (1) $u \leq v \Rightarrow \eta(u) \leq \eta(v)$, (2)$\eta(uv) \leq\max\{\eta(u), \eta(v)\}$ for all $u, v \in S$; and (3) $\eta(uvw) \leq\max\{\eta(u), \eta(w)\}$ for all $u, v, w \in S$.

Let $a \in S$ and $u \in (0, 1]$. Then an ordered fuzzy point a_u of S is defined by

$$a_u(x) = \begin{cases} u, \text{ if } x \in (a], \\ 0, \text{ if } x \notin (a]. \end{cases}$$

We shall be denoting, in whatever follows, the classes of: $(\in, \in \vee(k^*, q_k))$- antifuzzy subsemigroups, $(\in, \in \vee(k^*, q_k))$-antifuzzy left ideals, $(\in, \in \vee(k^*, q_k))$-antifuzzy right ideals, $(\in, \in \vee(k^*, q_k))$-antifuzzy ideals, $(\in, \in \vee(k^*, q_k))$-antifuzzy bi-ideals and $(\in, \in \vee(k^*, q_k))$-antifuzzy generalized bi-ideals of S by $AFSSg, AFLI, AFRI, AFI, AFBI$ and $AFGBI$ respectively. Moreover, for any fuzzy subset η of S, we shall write $a_u \in \eta$ for $a_u \subseteq \eta$ in the sequel.

Definition 2.1: [27] An ordered fuzzy point a_u of S is called quasi-coincident with a fuzzy subset η of S, written as $a_u q\eta$, if

$$\eta(a) + u > 1.$$

For any $k^* \in (0, 1]$, a_u is called (k^*, q)-quasi-coincident with a fuzzy subset η of S, denoted by $a_u(k^*, q)\eta$, if

$$\eta(a) + u > k^*.$$

Let $0 \leq k < k^* \leq 1$. For any ordered fuzzy point x_u, we say that

(1) $x_u q_k \eta$ if $\eta(x) + u + k > 1$;
(2) $x_u \in \vee q_k \eta$ if $x_u \in \eta$ or $x_u q_k \eta$;
(3) $x_u(k^*, q_k)\eta$ if $\eta(x) + u + k > k^*$;
(4) $x_u \in \vee(k^*, q_k)\eta$ if $x_u \in \eta$ or $x_u(k^*, q_k)\eta$; and
(5) $x_u \bar{\alpha} \eta$ if $x_u \alpha \eta$ does not hold for $\alpha \in \{q_k, \in \vee q_k, (k^*, q_k), \in \vee(k^*, q_k)\}$.

Khan et al. [27] extended and discussed the generalized form (see Definitions 2.2, 2.4, 2.6, 2.8 to follow) of the notions of $(\in, \in \vee q_k)$-fuzzy subsemigroups, $(\in, \in \vee q_k)$-fuzzy (left, right) ideals, $(\in, \in \vee q_k)$-fuzzy bi-ideals, $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals of an ordered semigroup. Motivated by these notions, we have introduced the notions of $(\in, \in \vee q_k)$-antifuzzy subsemigroups, $(\in, \in \vee q_k)$-antifuzzy (left, right) ideals, $(\in, \in \vee q_k)$-antifuzzy bi-ideals and $(\in, \in \vee q_k)$-antifuzzy generalized bi-ideals in an ordered semigroup (see Definitions 2.3, 2.5, 2.7 and 2.9 below).

Definition 2.2: [12] A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-fuzzy subsemigroup of S if $x_u \in \eta$ and $y_v \in \eta$ imply $(xy)_{\min\{u,v\}} \in \vee q_k \eta$ for all $u, v \in (0, 1]$ and $x, y \in S$.

Definition 2.3: A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-antifuzzy subsemigroup of S if $(xy)_u \in \eta$ imply $x_u \in \vee q_k \eta$ and $y_u \in \vee q_k \eta$ for all $u \in (0, 1]$ and $x, y \in S$.

Definition 2.4: [12] A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S if: (1) $x \leq y, y_u \in \eta \Rightarrow x_u \in \vee q_k \eta$ and (2) $x \in S, y_u \in \eta$ imply $(xy)_u \in \vee q_k \eta$ (resp. $(yx)_u \in \vee q_k \eta$) for all $u \in (0, 1]$ and $x, y \in S$.

A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-fuzzy ideal if it is both an $(\in, \in \vee q_k)$-fuzzy left ideal and an $(\in, \in \vee q_k)$-fuzzy right ideal of S.

Definition 2.5: A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-antifuzzy left (resp. right) ideal of S if: (1) $x \leq y, x_u \in \eta \Rightarrow y_u \in \vee q_k \eta$ and (2) $x \in S, (xy)_u \in \eta$ imply $y_u \in \vee q_k \eta$ (resp. $y \in S, (xy)_u \in \eta$ imply $x \in \vee q_k \eta$) for all $u \in (0, 1]$ and $x, y \in S$.

A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-antifuzzy ideal if it is both an $(\in, \in \vee q_k)$-antifuzzy left ideal and an $(\in, \in \vee q_k)$-antifuzzy right ideal of S.

Definition 2.6: [10] A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if: (1)$x \leq y, y_u \in \eta \Rightarrow x_u \in \vee q_k \eta$, (2)$x_u \in \eta, y_v \in \eta$ imply $(xy)_{\min\{u,v\}} \in \vee q_k \eta$; and (3) $x_u \in \eta$, $z_v \in \eta$ imply $(xyz)_{\min\{u,v\}} \in \vee q_k \eta$ for all $u, v \in (0, 1]$ and $x, y, z \in S$.

Definition 2.7: A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-antifuzzy bi-ideal of S if

(1) $x \leq y, x_u \in \eta \Rightarrow y_u \in \vee q_k \eta$,

(2) $(xy)_u \in \eta$ imply $x_u \in \vee q_k\eta$ or $y_u \in \vee q_k\eta$; and (3) $(xyz)_u \in \eta$ imply $x_u \in \vee q_k\eta$ or $z_u \in \vee q_k\eta$, for all $u \in (0, 1]$ and $x, y, z \in S$.

It is easy to see that each $(\in, \in \vee q_k)$-antifuzzy bi-ideal of S is an $(\in, \in \vee q_k)$-antifuzzy subsemigroup of S. If we take $k = 0$ in Definition 2.7, then we have the concept of $(\in, \in \vee q)$-antifuzzy bi-ideals of S given in [37]. So the concept of an $(\in, \in \vee q_k)$-antifuzzy bi-ideal is a generalization of $(\in, \in \vee q)$-antifuzzy bi-ideal in S.

Definition 2.8: [10] A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S if: (1) $x \leq y, y_u \in \eta \Rightarrow x_u \in \vee q_k\eta$ and (2) $x_u \in \eta, z_v \in \eta$ imply $(xyz)_{\min\{u,v\}} \in \vee q_k\eta$ for all $u, v \in (0, 1]$ and $x, y, z \in S$.

Definition 2.9: A fuzzy subset η of S is called an $(\in, \in \vee q_k)$-antifuzzy generalized bi-ideal of S if: (1)$x \leq y, x_u \in \eta \Rightarrow y_u \in \vee q_k\eta$ and (2) $(xyz)_u \in \eta$ imply $x_u \in \vee q_k\eta$ or $z_u \in \vee q_k\eta$, for all $u \in (0, 1]$ and $x, y, z \in S$.

In the next three results, Khan et al. [27] investigated the properties of $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) ideal and $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal of an ordered semigroup using the definition of (k^*, q)-quasi-coincident.

Theorem 2.10: [27] *A fuzzy subset η of S is an $(\in, \in \vee(k^*, q_k))$-fuzzy left ideal of $S \Leftrightarrow$ (1)$x \leq y \Rightarrow \eta(x) \geq \min\left\{\eta(y), \frac{k^*-k}{2}\right\}$ and (2)$\eta(xy) \geq \min\left\{\eta(y), \frac{k^*-k}{2}\right\}$ for all $x, y \in S$.*

Theorem 2.11: [27] *A fuzzy subset η of S is an $(\in, \in \vee(k^*, q_k))$-fuzzy right ideal of $S \Leftrightarrow$ (1)$x \leq y \Rightarrow \eta(x) \geq \min\left\{\eta(y), \frac{k^*-k}{2}\right\}$ and (2)$\eta(xy) \geq \min\left\{\eta(x), \frac{k^*-k}{2}\right\}$ for all $x, y \in S$.*

Theorem 2.12: [27] *A fuzzy subset η of S is an $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal of $S \Leftrightarrow$ (1)$x \leq y \Rightarrow \eta(x) \geq \min\left\{\eta(y), \frac{k^*-k}{2}\right\}$ and (2) $\eta(xyz) \geq \min\left\{\eta(x), \eta(z), \frac{k^*-k}{2}\right\}$ for all $x, y, z \in S$.*

In [27], Khan et al. also investigated properties of (k^*, k)-lower parts of $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideals of ordered semigroups (see, Lemmas 2.13 and 2.14).

Lemma 2.13: [27] *Let $L\ (\neq\emptyset)$ be a subset of S. Then the (k^*, k)-lower part $\underline{(\eta_k^{k^*})_L}$ of the characteristic function η_L of L is an $(\in, \in \vee(k^*, q_k))$-fuzzy left (resp. right) ideal of $S \Leftrightarrow L$ is a left (resp. right) ideal of S.*

Lemma 2.14: [27] *Let $B\ (\neq\emptyset)$ be a subset of S. Then the (k^*, k)-lower part $\underline{(\eta_k^{k^*})_B}$ of the characteristic function η_B of B is an $(\in, \in \vee(k^*, q_k))$-fuzzy generalized bi-ideal of $S \Leftrightarrow B$ is a generalized bi-ideal of S.*

We have studied the above results in Sections 3 and 4 in terms of antifuzzy (generalized) bi-ideals.

3. $(\in, \in \vee(k^*, q_k))$-antifuzzy Ideals in Ordered Semigroups

In this section, after introducing the notion of an $(\in, \in \vee(k^*, q_k))$-antifuzzy (generalized) bi-ideal in an ordered semigroup, we have obtained different characterization of ordered semigroups in terms of $(\in, \in \vee(k^*, q_k))$-antifuzzy left ideals, $(\in, \in \vee(k^*, q_k))$-antifuzzy right ideals, $(\in, \in \vee(k^*, q_k))$-antifuzzy generalized bi-ideals and $(\in, \in \vee(k^*, q_k))$-antifuzzy bi-ideals.

Definition 3.1: A fuzzy subset η of S is called an *AFSSg* if $\forall u \in (0,1]$ and $x, y \in S$, $(xy)_u \in \eta$ imply $x_u \in \vee(k^*, q_k)\eta$ and $y_u \in \vee(k^*, q_k)\eta$.

Definition 3.2: A fuzzy subset η of S is called an *AFL*(R)*I* if

(1) $x \leq y, x_u \in \eta \Rightarrow y_u \in \vee(k^*, q_k)\eta$, and

(2) $x \in S$, $(xy)_u \in \eta$ imply $y_u \in \vee(k^*, q_k)\eta$ (resp. $y \in S$, $(xy)_u \in \eta$ imply $x_u \in \vee(k^*, q_k)\eta$)

for all $u \in (0,1]$ and $x, y \in S$.

A fuzzy subset η of S is called an *AFI* if it is both an *AFLI* and an *AFRI*.

If we take $k^* = 1$ and $k = 0$ in Definition 3.2, then we get the idea of $(\in, \in \vee q)$-antifuzzy left (right) ideal. So the idea of *AFI* is a generalization of $(\in, \in \vee q)$-antifuzzy ideals in S.

Definition 3.3: A fuzzy subset η of S is called an *AFBI* if

(1) $x \leq y, x_u \in \eta \Rightarrow y_u \in \vee(k^*, q_k)\eta$,

(2) $(xy)_u \in \eta$ imply $x_u \in \vee(k^*, q_k)\eta$ or $y_u \in \vee(k^*, q_k)\eta$; and

(3) $(xyz)_u \in \eta$ imply $x_u \in \vee(k^*, q_k)\eta$ or $z_u \in \vee(k^*, q_k)\eta$,

for all $u \in (0,1]$ and $x, y, z \in S$.

Next we illustrate an *AFBI* in the following example:

Example 3.4: Let $S = \{m, n, x, y, z\}$. Define a binary operation '$\cdot$' and an order '$\leq$' on S as follows:

·	m	n	x	y	z
m	m	y	m	y	y
n	m	n	m	y	y
x	m	y	x	y	z
y	m	y	m	y	y
z	m	y	x	y	z

$$\leq := \{I, (m,x), (m,y), (m,z), (n,y), (n,z), (y,z)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. Define a fuzzy subset η on S as follows:

$$\eta(\mathrm{a}) = \begin{cases} 0.7 \textit{ if } a = m, \\ 0.5 \textit{ if } a = \{\mathrm{y},\ \mathrm{z}\}, \\ 0.4 \textit{ if } a = x, \\ 0.3 \textit{ if } a = n. \end{cases}$$

Take $k^* = 0.7$ and $k = 0.1$. Clearly η is an *AFBI*.

Definition 3.5: A fuzzy subset η of S is called an *AFGBI* if

(1) $x \leq y, x_u \in \eta \Rightarrow y_u \in \vee(k^*, q_k)\eta$, and

(2) $(xyz)_u \in \eta$ imply $x_u \in \vee(k^*, q_k)\eta$ or $z_u \in \vee(k^*, q_k)\eta$

for all $u \in (0,1]$ and $x, y, z \in S$.

Theorem 3.6: *A fuzzy subset η of S is an AFSSg if and only if*

$$\eta(xy) \wedge \frac{k^* - k}{2} \leq \eta(x) \vee \eta(y)$$

for all $x, y \in S$.

Proof: Let η be an *AFSSg* and $x, y \in S$. On contrary suppose that $(xy) \wedge \frac{k^*-k}{2} > \eta(x) \vee \eta(y)$. Choose $u \in (0,1)$ such that $(xy) \wedge \frac{k^*-k}{2} > u > \eta(x) \vee \eta(y)$. So $(xy)_u \in \eta$, but $x_u \bar{\in} \eta$ and $y_u \bar{\in} \eta$. Now $\eta(x) < u$ and $u < \frac{k^*-k}{2}$. So $\eta(x) + u < \frac{k^*-k}{2} + \frac{k^*-k}{2}$. This implies that $\eta(x) + u + k < k^*$; that is $x_u \overline{\in \vee(k^*, q_k)}\eta$. Similarly $y_u \overline{\in \vee(k^*, q_k)}\eta$. Thus we get a contradiction. Hence $\eta(xy) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(y)\}$.

Conversely assume that $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(x) \vee \eta(y)$ for all $x, y \in S$. Let $(xy)_u \in \eta$ for all $u \in (0,1]$. Then $\eta(xy) \geq u$. So $u \wedge \frac{k^*-k}{2} \leq \eta(x) \vee \eta(y)$.

Case 1. If $u > \frac{k^*-k}{2}$, then $\frac{k^*-k}{2} < u \leq \eta(x) \vee \eta(y)$. So $\eta(x) + u \geq \frac{k^*-k}{2} + \frac{k^*-k}{2}$. This implies that either $\eta(x) + u + k \geq k^*$ or $\eta(y) + u + k \geq k^*$. Therefore, either $x_u \in \eta$ or $y_u \in \eta$. Hence, either $x_u \in \vee(k^*, q_k)\eta$ or $y_u \in \vee(k^*, q_k)\eta$.

Case 2. If $u \leq \frac{k^*-k}{2}$, then $u \leq \eta(x) \vee \eta(y)$. So, either $x_u \in \eta$ or $y_u \in \eta$. Hence, either $x_u \in \vee(k^*, q_k)\eta$ or $y_u \in \vee(k^*, q_k)\eta$. Thus η is an *AFSSg*. ■

Theorem 3.7: *A fuzzy subset η of S is an AFL(R)I if and only if*

(1) $x \leq y \Rightarrow \min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$, and

(2) $\min\left\{\eta(xy), \frac{k^*-k}{2}\right\} \leq \eta(y)$ (resp. $\min\left\{\eta(xy), \frac{k^*-k}{2}\right\} \leq \eta(x)$)

for all $x, y \in S$.

Proof: ($\Rightarrow$) Let η be an *AFLI*. Then, for all $x, y \in S$ such that $x \leq y$ and $u \in (0,1)$, we have $x_u \in \eta \Rightarrow y_u \in \vee(k^*, q_k)\eta$. On the contrary suppose that there exist $x, y \in S$ with $x \leq y$ such that $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} > \eta(y)$. If we choose $u \in (0,1)$ such that $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} > u > \eta(y)$. Then $x_u \in \eta$ but $y_u \bar{\in} \eta$. Moreover, if $u < \frac{k^*-k}{2}$ and $\eta(y) < u$, then $\eta(y) + u < \frac{k^*-k}{2} + \frac{k^*-k}{2}$. This implies that $\eta(y) + u + k < k^*$. Hence $y_u \overline{\in \vee(k^*, q_k)}\eta$ which is a contradiction to our hypothesis. Thus $x \leq y \Rightarrow \min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$.

Now assume that $(xy)_u \in \eta$ implies $y_u \in \vee(k^*, q_k)\eta$ for all $x, y \in S$ and $u \in (0,1)$. Then, by the way of contradiction similar to the above paragraph, we may show that $\min\left\{\eta(xy), \frac{k^*-k}{2}\right\} \leq \eta(y)$ for all $x, y \in S$.

($\Leftarrow$) Assume that (1) and (2) hold. If $x \leq y$ and $x_u \in \eta$, then $\eta(x) \geq u$ and if $\leq \frac{k^*-k}{2}$, then $\eta(y) \geq \eta(x) \wedge \frac{k^*-k}{2} \geq u \wedge \frac{k^*-k}{2} = u$. Therefore $y_u \in \eta$. Again, if $u > \frac{k^*-k}{2}$, then $\eta(y) \geq \eta(x) \wedge \frac{k^*-k}{2} \geq u \wedge \frac{k^*-k}{2} = \frac{k^*-k}{2}$. Therefore $\eta(y) + u > \frac{k^*-k}{2} + \frac{k^*-k}{2}$. This implies that $\eta(y) + u + k > k^*$; that is $y_u \in \vee(k^*, q_k)\eta$.

Next suppose that $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(y)$ and $(xy)_u \in \eta$. Then $\eta(xy) \geq u$. If $u \leq \frac{k^*-k}{2}$, then $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(y)$ implying that $\eta(y) \geq u$. Hence $y_u \in \eta$. If $u > \frac{k^*-k}{2}$, then $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(y)$ which implies $\eta(y) \geq \frac{k^*-k}{2}$. Therefore $\eta(y) + u + k \leq \frac{k^*-k}{2} + \frac{k^*-k}{2} + k = k^*$. Thus $y_u \in \vee(k^*, q_k)\eta$. Hence η is an *AFLI*.

The following theorem may be proved on the lines similar to the lines of the proof of the above theorem. ■

Theorem 3.8: *A fuzzy subset η of S is an AFI if and only if*
(1) $x \leq y \Rightarrow min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$, and
(2) $min\left\{\eta(xy), \frac{k^*-k}{2}\right\} \leq \eta(y)$ and $min\left\{\eta(xy), \frac{k^*-k}{2}\right\} \leq \eta(x)$
for all $x, y \in S$.

Each antifuzzy ideal of S is an *AFI*. However the converse is not true in general, as shown by the following example:

Example 3.9: Let $S = \{w, x, y, z\}$. Define a binary operation '$\cdot$' and an order '$\leq$' on S in the following way:

·	w	x	y	z
w	w	w	w	w
x	w	w	w	w
y	w	w	x	w
z	w	w	x	x

$$\leq := \{(w,w),(x,x),(y,y),(z,z),(x,w),(y,w),(y,x)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. Now define a fuzzy subset η on S as follows:

$$\eta(\mathrm{a}) = \begin{cases} 0.2 \text{ if } a = y, \\ 0.3 \text{ if } a = x, \\ 0.4 \text{ if } a = w, \\ 0.7 \text{ if } a = z. \end{cases}$$

Take $k^* = 0.9$ and $k = 0.5$. Then, by Theorem 3.8, η is an $(\in, \in \vee (0.9, q_{0.5}))$-antifuzzy ideal of S, but

(i) η is not an antifuzzy ideal of S as $\eta(zy) = \eta(x) = 0.3 \nleq \eta(y) = 0.2$.
(ii) η is not an $(\in, \in \vee q)$-antifuzzy ideal of S as $\eta(zy) \wedge 0.5 = 0.3 \nleq \eta(y) = 0.2$.
(iii) η is not an $(\in, \in \vee q_{0.5})$-antifuzzy ideal of S as $\eta(zy) \wedge \frac{1-0.5}{2} = 0.25 \nleq \eta(y) = 0.2$.

Theorem 3.10: *A fuzzy subset η of S is an AFGBI if and only if*
(1) $x \leq y \Rightarrow min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$, and
(2) $\eta(xyz) \wedge \frac{k^*-k}{2} \leq max\{\eta(x), \eta(z)\}$
for all $x, y, z \in S$.

Proof: By Theorem 3.7, it is sufficient to show that the following conditions are equivalent:

(i) $(xyz)_u \in \eta$ imply $x_u \in \vee(k^*, q_k)\eta$ or $z_u \in \vee(k^*, q_k)\eta$ for all $x, y, z \in S$.
(ii) $\eta(xyz) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(z)\}$ for all $x, y, z \in S$.

To prove this, suppose (i) holds. If there exist $x, y, z \in S$ such that $\eta(xyz) \wedge \frac{k^*-k}{2} > \max\{\eta(x), \eta(z)\}$, then we can choose $u \in (0, 1)$ such that $\eta(xyz) \wedge \frac{k^*-k}{2} > u > \max\{\eta(x), \eta(z)\}$. So $(xyz)_u \in \eta$, but $x_u \bar{\in} \eta$ and $z_u \bar{\in} \eta$. Therefore $\eta(x) < u$ and, hence, $\eta(x) + u + k <$

$\frac{k^*-k}{2} + \frac{k^*-k}{2} + k = k^*$ implying that $x_u \overline{\in \vee (k^*, q_k)}\eta$. Similarly $z_u \overline{\in \vee (k^*, q_k)}\eta$. Thus we get a contradiction to our hypothesis. Hence $\eta(xyz) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(z)\}$ for all $x, y, z \in S$.

Conversely suppose that $(xyz)_u \in \eta$ for all $x, y, z \in S$. Then $\eta(xyz) \geq u$. If $> \frac{k^*-k}{2}$, then either $\eta(x) \geq u \wedge \frac{k^*-k}{2} = \frac{k^*-k}{2}$ or $\eta(z) \geq u \wedge \frac{k^*-k}{2} = \frac{k^*-k}{2}$. So either $\eta(x) + u + k \geq \frac{k^*-k}{2} + \frac{k^*-k}{2} + k = k^*$ or $\eta(z) + u + k \geq \frac{k^*-k}{2} + \frac{k^*-k}{2} + k = k^*$. Therefore either $x_u(k^*, q_k)\eta$ or $z_u(k^*, q_k)\eta$. Again, if $u \leq \frac{k^*-k}{2}$, then either $\eta(x) \geq u \wedge \frac{k^*-k}{2} = \frac{k^*-k}{2}$ or $\eta(z) \geq u \wedge \frac{k^*-k}{2} = \frac{k^*-k}{2}$. Hence either $x_u \in \eta$ or $z_u \in \eta$. Thus either $x_u \in \vee(k^*, q_k)\eta$ or $z_u \in \vee(k^*, q_k)\eta$ for all $x, y, z \in S$, as required. ■

Theorem 3.11: *A fuzzy subset η of S is an AFBI if and only if*

(1) $x \leq y \Rightarrow min\{\eta(x), \frac{k^*-k}{2}\} \leq \eta(y)$,

(2) $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(x) \vee \eta(y)$; *and*

(3) $\eta(xyz) \wedge \frac{k^*-k}{2} \leq max\{\eta(x), \eta(z)\}$

for all $x, y, z \in S$.

Proof: The proof follows from Theorems 3.6 and 3.10. ■

Each *AFBI* is an *AFGBI*. However the converse is not true in general as shown by the following example:

Example 3.12: Let $S = \{x, y, z\}$. Define a binary operation '$\cdot$' and an order '$\leq$' on S in the following way:

·	x	y	z
x	x	x	x
y	x	x	x
z	x	x	y

$$\leq := \{(x, x), (y, y), (z, z), (x, y), (x, z)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. Define a fuzzy subset η on S by $\eta(x) = 0.10, \eta(y) = 0.45, \eta(z) = 0.30$. Take $k^* = 0.9$ and $k = 0.1$. By routine calculations, it may be easily verified that η is an *AFGBI*. Since $\eta(zz) \wedge \frac{k^*-k}{2} = \eta(y) \wedge \frac{k^*-k}{2} = 0.4 \nleq 0.3 = \eta(z) \vee \eta(z)$, η is not an *AFSSg*. So η is not an *AFBI*.

One may easily observe that each antifuzzy bi-ideal of S is an *AFBI* and each $(\in, \in \vee q_k)$-antifuzzy bi-ideal of S is an *AFBI*, but the converse is not true in general as illustrated by the following example.

Example 3.13: In the Example 3.4, η is clearly an *AFBI*. Since $m \leq x \Rightarrow \eta(m) \wedge \frac{1-k}{2} = 0.45 > \eta(x) = 0.40$, we have that η is not an $(\in, \in \vee q_k)$-antifuzzy bi-ideal of S. Moreover, η is also not an antifuzzy bi-ideal of S.

Next we show (see Theorems 3.14 and 3.15 below) that, in regular and weakly regular ordered semigroups, concepts of *AFGBI* and *AFBI* coincide.

Theorem 3.14: *Each AFGBI of a regular ordered semigroup S is an AFBI.*

Proof: Let η be an *AFGBI*. Since S is a regular ordered semigroup, for each $y \in S$, there exists $x \in S$ such that $y \le yxy$. Then $xy \le x(yxy)$. So $\eta(xy) \wedge \frac{k^*-k}{2} \le \eta(x(yxy)) = \eta(x(yx)y)$. This implies $\min\left\{\eta(xy), \frac{k^*-k}{2}\right\} \le \eta(x(yx)y) \wedge \frac{k^*-k}{2} \le \eta(x) \vee \eta(y)$. Therefore η is an *AFSSg*. Hence η is an *AFBI*. ■

Theorem 3.15: *Each AFGBI of a weakly regular ordered semigroup S is an AFBI.*

Proof: As the proof is similar to that of Theorem 3.14 with a slight modification, we omit it. ■

Theorem 3.16: *A subset $G(\neq \emptyset)$ of S is a generalized bi-ideal of S if and only if the fuzzy subset η of S defined by*

$$\eta(x) = \begin{cases} t, \text{ if } x \notin G \\ r, \text{ if } x \in G \end{cases}$$

is an AFGBI, where $r, t \in [0,1] (r \le t)$ for some fixed members of $[0,1]$.

Proof: The proof is straightforward, so we omit it. ■

Remark 3.17: From the above theorem, we conclude that a subset $G(\neq \emptyset)$ of S is a generalized bi-ideal of S if and only if the characteristic function $\eta_{(A^c)}$ of the complement of A is an *AFGBI*.

Corollary 3.1: *A subset $G(\neq \emptyset)$ of S is a generalized bi-ideal of S if and only if the fuzzy subset η of S defined by*

$$\eta(x) = \begin{cases} t, \text{ if } x \notin G \\ r, \text{ if } x \in G \end{cases}$$

is an antifuzzy generalized bi-ideal of S, where $r, t \in [0,1]$ such that $r \le t$.

Theorem 3.18: *Let $\{g_j | j \in J\}$ be a family of AFBI. Then $g = \underset{j\in J}{\cap} g_j$ is an AFBI, where $\left(\underset{j\in J}{\cap} g_j\right)(x) = \underset{j\in J}{\wedge}(g_j(x))$.*

Proof: Take any $x, y, z \in S$. Since each $g_j (j \in J)$ is an *AFBI*, we have

$$g(xy) \wedge \frac{k^*-k}{2} = \bigcap_{j\in J} g_j(xy) \wedge \frac{k^*-k}{2} = \bigwedge_{j\in J}(g_j(x)) \wedge \frac{k^*-k}{2}$$
$$= \underset{j\in J}{\wedge}\left(g_j(xy) \wedge \frac{k^*-k}{2}\right)$$

$$\leq \underset{j\in J}{\wedge}(g_j(x) \vee g_j(y))$$
$$= \Big[\underset{j\in J}{\wedge}(g_j(x))]\vee\Big[\underset{j\in J}{\wedge}(g_j(y))]$$
$$= [(\underset{j\in J}{\cap} g_j)(x)] \vee [(\underset{j\in J}{\cap} g_j)(y)]$$
$$= g(x) \vee g(y)$$

and

$$g(xyz) \wedge \frac{k^* - k}{2} = \left[\bigcap_{j\in J} g_j(xyz)\right] \wedge \frac{k^* - k}{2} = [\bigwedge_{j\in J}(g_j(xy))] \wedge \frac{k^* - k}{2}$$
$$= \underset{j\in J}{\wedge}\left(g_j(xyz) \wedge \frac{k^* - k}{2}\right)$$
$$\leq \underset{j\in J}{\wedge}(g_j(x) \vee g_j(z))$$
$$= \Big[\underset{j\in J}{\wedge}(g_j(x))]\vee\Big[\underset{j\in J}{\wedge}(g_j(z))]$$
$$= [(\underset{j\in J}{\cap} g_j)(x)] \vee [(\underset{j\in J}{\cap} g_j)(z)]$$
$$= g(x) \vee g(z).$$

Now we show that if $x \leq y$, then $g(x) \wedge \frac{k^*-k}{2} \leq g(y)$. Since each $g_j (j \in J)$ is an *AFBI*, we have

$$g(x) \wedge \frac{k^* - k}{2} = \left(\bigcap_{j\in J} g_j\right)(x) \wedge \frac{k^* - k}{2} = [\bigwedge_{j\in J}(g_j(x))] \wedge \frac{k^* - k}{2}$$
$$= \underset{j\in J}{\wedge}\left(g_j(x) \wedge \frac{k^* - k}{2}\right)$$
$$\leq \underset{j\in J}{\wedge}(g_j(y))$$
$$= (\underset{j\in J}{\cap} g_j)(y)$$
$$= g(y).$$

Therefore, by Theorem 3.11, g is an *AFBI*. ■

Theorem 3.19: *Let* $\{g_j | j \in J\}$ *be a family of AFBI. Then* $g = \underset{j\in J}{\cup} g_j$ *is an AFBI, where* $\left(\underset{j\in J}{\cup} g_j\right)(x) = \underset{j\in J}{\vee}(g_j(x))$.

Proof: As the proof is similar to that of Theorem 3.18 with a slight modification, we omit it.

For any fuzzy subset η of an ordered semigroup S and $u \in [0,1]$, let

$$\mathrm{U}(\eta;u) = \{x \in S | \eta(x) \geq u\}, \mathrm{L}(\eta;u) = \{x \in S | \eta(x) \leq u\};$$

$$\mathrm{Q}(\eta;u) = \{x \in S | x_u(k^*, q_k)\eta\}, \tilde{\mathrm{Q}}(\eta;u) = \{x \in S | x_u\overline{(k^*, q_k)}\eta\};$$

$$[\eta]_u = \{x \in S | x_u \in \vee(k^*, q_k)\eta\}, \widetilde{[\eta]}_u = \{x \in S | x_u\overline{\in \vee(k^*, q_k)}\eta\}.$$

Then $[\eta]_u = \mathrm{U}(\eta;u) \cup \mathrm{Q}(\eta;u)$ and $\widetilde{[\eta]}_u = \mathrm{L}(\eta;u) \cup \tilde{\mathrm{Q}}(\eta;u)$. We call $[\eta]_u$ an $(\in, \in \vee(k^*, q_k))$-level bi-ideal of η and $\mathrm{Q}(\eta;u)$ a q_k-level bi-ideal of η. In the following results, above defined subsets of S have been shown to be related with different types of *AFBI*.

Theorem 3.20: *A fuzzy subset η of S is an AFGBI $\Leftrightarrow$ $\mathrm{L}(\eta;u)$ is a generalized bi-ideal of S for all $u \in \left[0, \frac{k^*-k}{2}\right)$.*

Proof: ($\Rightarrow$) Let η be an *AFGBI* and $u \in \left[0, \frac{k^*-k}{2}\right)$ be such that $\mathrm{L}(\eta;u) \neq \emptyset$. Then, by Theorem 3.10(1), $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$ with $x \leq y$. If $y \in \mathrm{L}(\eta;u)$, then $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y) \leq u$. This implies that $\eta(x) \leq u$ (since $u \in \left[0, \frac{k^*-k}{2}\right)$). Hence $x \in \mathrm{L}(\eta;u)$. Next, take any $x, y, z \in S$ with $x, z \in \mathrm{L}(\eta;u)$. Then $\eta(x) \leq u$ and $\eta(z) \leq u$. Therefore, by Theorem 3.10(2), $\eta(xyz) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(z)\} \leq \max\{u, u\} = u$. So $\eta(xyz) \leq u$ (since $u \in \left[0, \frac{k^*-k}{2}\right)$). Thus $xyz \in \mathrm{L}(\eta;u)$.

Hence $\mathrm{L}(\eta;u)$ is a generalized bi-ideal of S for all $u \in \left[0, \frac{k^*-k}{2}\right)$, as required.

($\Leftarrow$) Let $\mathrm{L}(\eta;u)$ be a generalized bi-ideal of S for all $u \in \left[0, \frac{k^*-k}{2}\right)$. On contrary assume that there exist $x, y \in S$ with $x \leq y$ and such that $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} > \eta(y)$. Choose $u \in \left[0, \frac{k^*-k}{2}\right)$ such that $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} > u > \eta(y)$. So $y \in \mathrm{L}(\eta;u)$ but $x \notin \mathrm{L}(\eta;u)$, a contradiction as $x \leq y$. Hence $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$ for $x \leq y$.

Again, for any $x, y, z \in S$, we have to show that $\min\left\{\eta(xyz), \frac{k^*-k}{2}\right\} \leq \eta(x) \vee \eta(z)$. On contrary assume that there exist $x, y, z \in S$ such that $\min\left\{\eta(xyz), \frac{k^*-k}{2}\right\} > \eta(x) \vee \eta(z)$. Choose $u \in \left(0, \frac{k^*-k}{2}\right)$ such that $\min\left\{\eta(xyz), \frac{k^*-k}{2}\right\} > u > \eta(x) \vee \eta(z)$. Then $\eta(x) < u$ and $\eta(z) < u$. This implies $x, z \in \mathrm{L}(\eta;u)$. But $\eta(xyz) > u$ implies that $xyz \notin \mathrm{L}(\eta;u)$, a contradiction. This completes the proof. ■

Lemma 3.21: *Let η be an AFGBI. Then $\eta_{\frac{k^*-k}{2}} = \left\{x \in S | \eta(x) < \frac{k^*-k}{2}\right\}$ is a generalized bi-ideal of S.*

Proof: Let $x, y \in S$ such that $x \leq y$ and $y \in \eta_{\frac{k^*-k}{2}}$. Then $\eta(y) < \frac{k^*-k}{2}$. Since η is an *AFGBI*, by Theorem 3.10(1), we have $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y) < \frac{k^*-k}{2}$. This implies $\eta(x) < \frac{k^*-k}{2}$ (since $\frac{k^*-k}{2} \nless \frac{k^*-k}{2}$). Therefore $x \in \eta_{\frac{k^*-k}{2}}$.

Next, let $x, y, z \in S$ such that $x, z \in \eta_{\frac{k^*-k}{2}}$. Then $\eta(x) < \frac{k^*-k}{2}$ and $\eta(z) < \frac{k^*-k}{2}$. So, by Theorem 3.10(2), $\min\left\{\eta(xyz), \frac{k^*-k}{2}\right\} \leq \eta(x) \vee \eta(z) < \frac{k^*-k}{2}$. This implies $\eta(xyz) < \frac{k^*-k}{2}$; that is $xyz \in \eta_{\frac{k^*-k}{2}}$. Hence $\eta_{\frac{k^*-k}{2}}$ is a generalized bi-ideal of S. ■

Theorem 3.22: *A fuzzy subset η of S is an AFGBI $\Leftrightarrow$ $\tilde{Q}(\eta; u)$ is a generalized bi-ideal of S for all $u \in \left[0, \frac{k^*-k}{2}\right)$.*

Proof: Let η be an *AFGBI* and $u \in \left(0, \frac{k^*-k}{2}\right)$ be such that $\tilde{Q}(\eta; u) \neq \emptyset$. Then, by Theorem 3.10(1), we have $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y)$ with $x \leq y$. If $y \in \tilde{Q}(\eta; u)$, then $\min\left\{\eta(x), \frac{k^*-k}{2}\right\} \leq \eta(y) < k^* - u - k \leq (k^* - k) - \frac{k^*-k}{2} = \frac{k^*-k}{2}$. This implies that $\eta(x) < \frac{k^*-k}{2} \Rightarrow \eta(x) + u + k < \frac{k^*-k}{2} + \frac{k^*-k}{2} + k = k^*$. Hence $x \in \tilde{Q}(\eta; u)$.

Next, let $x, y, z \in S$ such that $x, z \in \tilde{Q}(\eta; u)$. Then $\eta(x) + u + k < k^*$ and $\eta(z) + u + k < k^*$. Therefore, by Theorem 3.10(2), $\eta(xyz) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(z)\} \leq \max\{k^* - k - u, k^* - k - u\} = k^* - k - u$. So $\eta(xyz) \leq k^* - k - u$ (since $k^* - k - u > \frac{k^*-k}{2}$) and, so, $xyz \in \tilde{Q}(\eta; u)$. Hence $\tilde{Q}(\eta; u)$ is generalized bi-ideal of S.

The following result easily follows from Theorems 3.20 and 3.22. ■

Theorem 3.23: *A fuzzy subset η of S is an AFGBI $\Leftrightarrow$ $\widetilde{[\eta]}_u$ is a generalized bi-ideal of S for all $u \in \left[0, \frac{k^*-k}{2}\right)$.*

Theorem 3.24: *$\widetilde{[\eta]}_u$ is a subsemigroup of S $\Leftrightarrow$ $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(x) \vee \eta(y)$ for all $x, y \in S$ and $u \in \left[0, \frac{k^*-k}{2}\right)$.*

Proof: ($\Rightarrow$) Suppose to the contrary that there exist $x, y \in S$ such that $\eta(xy) \wedge \frac{k^*-k}{2} > \max\{\eta(x), \eta(y)\}$. Choose $u \in \left(0, \frac{k^*-k}{2}\right)$ such that $\eta(xy) \wedge \frac{k^*-k}{2} > u > \max\{\eta(x), \eta(y)\}$. Then $x, y \in \mathrm{L}(\eta; u) \subseteq \widetilde{[\eta]}_u$, but $xy \bar{\in} \widetilde{[\eta]}_u$, a contradiction. Thus $\eta(xy) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(y)\}$, as required.

($\Leftarrow$) Take any $x, y \in \widetilde{[\eta]}_u$ and $u \in \left(0, \frac{k^*-k}{2}\right)$. Then $x_u \overline{\in \vee (k^*, q_k)} \eta$ and $y_u \overline{\in \vee (k^*, q_k)} \eta$ which implies that either $\eta(x) \leq u$ or $\eta(x) + u + k \leq k^*$; and either $\eta(y) \leq u$ or $\eta(y) + u + k \leq k^*$. Thus, we have $\eta(xy) \wedge \frac{k^*-k}{2} \leq \max\{\eta(x), \eta(y)\}$.

Case 1. Let $\eta(x) \leq u$ and $\eta(y) \leq u$. Then $\eta(xy) \wedge \frac{k^*-k}{2} \leq \max\{u, u\} = u \leq \frac{k^*-k}{2}$. So $\eta(xy) \leq \frac{k^*-k}{2}$. Therefore $\eta(xy) + u \leq \frac{k^*-k}{2} + \frac{k^*-k}{2}$ implying $\eta(xy) + u + k \leq k^*$. Hence $(xy)_u \overline{\in \vee (k^*, q_k)} \eta$.

Case 2. Let $\eta(x) \leq u$ and $\eta(y) + u + k \leq k^*$. Then $\eta(xy) \wedge \frac{k^*-k}{2} \leq \max\{u, k^* - u - k\} = k^* - u - k$ (since $k^* - u - k > \frac{k^*-k}{2}$). Therefore $\eta(xy) + u + k \leq k^*$. Hence $(xy)_u \overline{\in \vee (k^*, q_k)} \eta$.

Case 3. Let $\eta(x) + u + k \leq k^*$ and $\eta(y) \leq u$. Then, as in Case 2, we may show that

$$(xy)_u \overline{\in \vee (k^*, q_k)} \eta.$$

Case 4. Let $\eta(x) + u + k \leq k^*$ and $\eta(y) + u + k \leq k^*$. Then $\eta(xy) \wedge \frac{k^*-k}{2} \leq \max\{k^* - u - k, k^* - u - k\} = k^* - u - k$. So $\eta(xy) \leq k^* - u - k$ (since $k^* - u - k > \frac{k^*-k}{2}$). Therefore $\eta(xy) + u + k \leq k^*$. Hence $(xy)_u \overline{\in \vee (k^*, q_k)} \eta$.

Thus, in each case, we have $(xy)_u \overline{\in \vee (k^*, q_k)} \eta$ and so $xy \in \widetilde{[\eta]}_u$. Hence $\widetilde{[\eta]}_u$ is a subsemigroup of S. ■

Example 3.25: Let $(S, \cdot, \leq)$ be the ordered semigroup of Example 3.4. Define a fuzzy subset ξ on S as follows:

$$\xi(a) = \begin{cases} 0.5 \text{ if } a \in \{m\} \\ 0 \text{ if } a \in \{\mathrm{n, x, y, z}\}. \end{cases}$$

Then

$$\widetilde{[\xi]}_u = \begin{cases} \emptyset \text{ if } 0 < u < 0.5 \\ \{\mathrm{m}\} \text{ if } 0.5 < u < 0.9 \\ \{\mathrm{m, n, x, y, z}\} \text{ if } 0.9 < u < 1.0. \end{cases}$$

Now it is routine to verify that $\widetilde{[\xi]}_u$ is a subsemigroup of S. ■

4. (k^*, k)-lower Part of $(\in, \in \vee (k^*, q_k))$-antifuzzy Ideals in Ordered Semigroups

In this section, we have investigated properties of (k^*, k)-lower parts of $(\in, \in \vee (k^*, q_k))$-antifuzzy generalized bi-ideals of ordered semigroups and presented relationship between $(\in, \in \vee (k^*, q_k))$-antifuzzy bi-ideals and $(\in, \in \vee (k^*, q_k))$-fuzzy left (right) ideals. We have also shown that if η is an *AFGBI*, then $\underline{\eta_k^{k^*}}$ is an antifuzzy generalized bi-ideal of S.

Definition 4.1: [27] The (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η is defined as follows:

$$\underline{\eta_k^{k^*}}(x) = \min\left\{\eta(x), \frac{k^* - k}{2}\right\}$$

for all $x \in S$ and $0 \leq k < k^* \leq 1$. Clearly $\underline{\eta_k^{k^*}}$ is a fuzzy subset of S. For a subset $A(\neq \emptyset)$ of S, the (k^*, k)-lower part $\underline{(\eta_A)_k^{k^*}}$ of the characteristic function η_A, will be denoted by $(\underline{\eta_k^{k^*}})_A$ in the sequel.

We now show that if η is an $AFL(R)I$, then the (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η is an antifuzzy left (right) ideal of S.

Proposition 4.2: *If η is an $AFL(R)I$, then the (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η is an antifuzzy left (right) ideal of S.*

Proof: Let η be an *AFRI* and $x, y \in S$. Then $\eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(x)$. So $\underline{\eta_k^{k^*}}(xy) \wedge \frac{k^*-k}{2} = \eta(xy) \wedge \frac{k^*-k}{2} \leq \eta(x) \wedge \frac{k^*-k}{2} = \underline{\eta_k^{k^*}}(x)$. Moreover, if $x \leq y$, then we show that $\underline{\eta_k^{k^*}}(x) \wedge \frac{k^*-k}{2} \leq \underline{\eta_k^{k^*}}(y)$. So assume that $x \leq y$. Since η is an *AFI*, we have $\eta(x) \wedge \frac{k^*-k}{2} \leq \eta(y)$ and, so, $\underline{\eta_k^{k^*}}(x) = \eta(x) \wedge \frac{k^*-k}{2} \leq \eta(y) \wedge \frac{k^*-k}{2} = \underline{\eta_k^{k^*}}(y)$. Therefore $\underline{\eta_k^{k^*}}$ is an antifuzzy right ideal of S. Similary we may show that $\underline{\eta_k^{k^*}}$ is also an antifuzzy left ideal of S.

In Example 3.9, we have illustrated that an *AFI* need not necessarily be an antifuzzy ideal of S. In the following result, we show that if η is an *AFI*, then the (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η is an antifuzzy ideal of S. ■

Proposition 4.3: *If η is an AFI, then the (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η is an antifuzzy ideal of S.*

Proof: The proof follows from Proposition 4.2.

Let $A(\neq \emptyset)$ be a subset of S. Then the (k^*, k)-lower part $(\underline{\eta_k^{k^*}})_{A^C}$ of the characteristic function η_{A^C} of the complement of A is defined as the mapping of S into [0, 1] as follows:

$$(\underline{\eta_k^{k^*}})_{A^C}(x) = \begin{cases} \frac{k^*-k}{2}, \text{ if } x \notin A \\ 0, \text{ if } x \in A \end{cases}$$

■

Theorem 4.4: *Let $B(\neq \emptyset)$ be a subset of S. Then the (k^*, k)-lower part $(\underline{\eta_k^{k^*}})_{B^C}$ of the characteristic function η_{B^C} of the complement of B is an AFGBI $\Leftrightarrow$ B is a generalized bi-ideal of S.*

Proof: $(\Rightarrow)$ Suppose that $(\underline{\eta_k^{k^*}})_{B^C}$ is an *AFGBI*. Let $x, y \leq S$ such that $x \leq y$ with $y \in B$. Then $(\underline{\eta_k^{k^*}})_{B^C}(y) = 0$. Since $(\underline{\eta_k^{k^*}})_{B^C}$ is an *AFGBI* and $x \leq y$, we have $(\underline{\eta_k^{k^*}})_{B^C}(x) \wedge \frac{k^*-k}{2} \leq (\underline{\eta_k^{k^*}})_{B^C}(y)$. Therefore $(\underline{\eta_k^{k^*}})_{B^C}(x) = 0$. So $x \in B$. Next, take any $x, z \in B$ and $y \in S$. Then $(\underline{\eta_k^{k^*}})_{B^C}(x) = 0$ and $(\underline{\eta_k^{k^*}})_{B^C}(z) = 0$. Therefore, by definition of *AFGBI*, we have $(\underline{\eta_k^{k^*}})_{B^C}(xyz) \wedge \frac{k^*-k}{2} \leq (\underline{\eta_k^{k^*}})_{B^C}(x) \vee (\underline{\eta_k^{k^*}})_{B^C}(z) = 0$. Hence $(\underline{\eta_k^{k^*}})_{B^C}(xyz) = 0$ and so $xyz \in B$. Thus B is a generalized bi-ideal of S.

$(\Leftarrow)$ Let B be a generalized bi-ideal of S. Then, by Theorem 3.16 and Corollary 3.17, $(\underline{\eta_k^{k^*}})_{B^C}$ is an *AFGBI*. ■

Similarly we may prove the following:

Theorem 4.5: *Let $L(\neq \emptyset)$ be a subset of S. Then the (k^*, k)-lower part $(\underline{\eta_k^{k^*}})_{L^C}$ of the characteristic function η_{L^C} is an AFL(R)I $\Leftrightarrow$ L is a left (resp. right) ideal of S.*

In the last theorem, we show that if η is an AFGBI, then $\underline{\eta_k^{k^*}}$ is an antifuzzy generalized bi-ideal of S.

Theorem 4.6: *If η is an AFGBI, then $\underline{\eta_k^{k^*}}$ is an antifuzzy generalized bi-ideal of S.*

Proof: Let $x, y \in S$ with $x \leq y$. Since η is an *AFGBI*, we have $\eta(x) \wedge \frac{k^*-k}{2} \leq \eta(y)$, and so $\underline{\eta_k^{k^*}}(x) = \eta(x) \wedge \frac{k^*-k}{2} = \left(\eta(x) \wedge \frac{k^*-k}{2}\right) \wedge \frac{k^*-k}{2} \leq \eta(y) \wedge \frac{k^*-k}{2} = \underline{\eta_k^{k^*}}(y)$. For $x, y, z \in S$, we have $\eta(xyz) \wedge \frac{k^*-k}{2} \leq \eta(x) \vee \eta(z)$ and so $\underline{\eta_k^{k^*}}(xyz) = \eta(xyz) \wedge \frac{k^*-k}{2} = \left(\eta(xyz) \wedge \frac{k^*-k}{2}\right) \wedge \frac{k^*-k}{2} \leq (\eta(x) \vee \eta(z)) \wedge \frac{k^*-k}{2} = \left(\eta(x) \wedge \frac{k^*-k}{2}\right) \vee \left(\eta(z) \wedge \frac{k^*-k}{2}\right) = \underline{\eta_k^{k^*}}(x) \vee \underline{\eta_k^{k^*}}(z)$. Consequently $\underline{\eta_k^{k^*}}$ is an antifuzzy generalized bi-ideal of S. ■

5. Conclusion and Ideas for Future Work

The aim of the present paper is to enhance the understanding of ordered semigroups and regular ordered semigroups by considering the structural influence of *AFBI*. In this aspect, we have obtained several characterization of ordered semigroups in terms of *AFLI*, *AFRI*, *AFBI* and *AFGBI*. In addition, we have also characterized (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η in terms of *AFLI* and *AFRI*. Finally, we have established relationship between *AFGBI* and (k^*, k)-lower part $\underline{\eta_k^{k^*}}$ of η. Following are particular cases of the present paper:

(1) If we put $k^* = 1$, then most of the results of this paper reduce in the setting of $(\in, \in \vee q_k)$-antifuzzy bi-ideals and $(\in, \in \vee q_k)$-antifuzzy generalized bi-ideals.
(2) If we put $k^* = 1$ and $k = 0$, then most of the results of this paper reduce in the setting of $(\in, \in \vee q_k)$-antifuzzy bi-ideals and $(\in, \in \vee q_k)$-antifuzzy generalized bi-ideals.

It is hoped that the properties of *AFLI*, *AFRI*, *AFBI* and *AFGBI* obtained in this paper may prove to be highly instrumental for characterizing different classes of ordered semigroups such as regular ordered semigroups, intra-regular ordered semigroups and semisimple ordered semigroups.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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