Fuzzy Information and Engineering

ISSN: 1616-8658 (Print) 1616-8666 (Online) Journal homepage: https://www.tandfonline.com/loi/tfie20

Intuitionistic Fuzzy Soft Set Theoretic Approaches to α-ideals in BCI-algebras

Muhammad Touqeer

To cite this article: Muhammad Touqeer (2020): Intuitionistic Fuzzy Soft Set Theoretic Approaches to α-ideals in BCI-algebras, Fuzzy Information and Engineering, DOI: 10.1080/16168658.2020.1751405

To link to this article: https://doi.org/10.1080/16168658.2020.1751405

FUZZY INFORMATION AND ENGINEERING
https://doi.org/10.1080/16168658.2020.1751405

Intuitionistic Fuzzy Soft Set Theoretic Approaches to α-ideals in BCI-algebras

Muhammad Touqeer

Department of Basic Sciences, University of Engineering and Technology, Taxila, Pakistan

ABSTRACT
As an extension of the applications of soft sets in α-ideals of BCI-algebras, we have defined the idea of 'intuitionistic fuzzy soft α-ideals' and proved their basic properties. 'Intuitionistic fuzzy soft BCI-algebras' and 'intuitionistic fuzzy soft ideals' have been described with the help of concrete examples. We proved that any *IFSI* of a BCK-algebra is an 'intuitionistic fuzzy soft BCK-algebra' ($IFS_{BCK}A$). Afterwards we have proceeded towards the detail discussion of $IFS_{\alpha}Is$. The notion of intuitionistic fuzzy soft α-ideals in BCI-algebras is presented, the corresponding properties are proved and concrete applications are given. Connections between various types of intuitionistic fuzzy soft α-ideals and intuitionistic fuzzy soft ideals are described. Intuitionistic fuzzy soft α-ideals are characterised using the idea of soft (δ, η)-level set.

ARTICLE HISTORY
Received 24 September 2019
Revised 28 November 2019
Accepted 2 December 2019

KEYWORDS
Intuitionistic fuzzy soft sets; intuitionistic fuzzy soft BCI-algebra; intuitionistic fuzzy soft ideal; BCI-algebras; intuitionistic fuzzy soft α-ideal; soft BCI-algebra; soft $(\delta; \eta)$-level set

1. Introduction

In the present era, uncertainty is one of the definitive changes in science. The traditional view is that uncertainty is objectionable in science and science should endeavour for certainty through all conceivable means. At the present it is believed that uncertainty is vigorous for science that is not only an inevitable epidemic but also has great effectiveness. The statistical methods particularly the probability theory was the first type of this approach to study the physical process at the molecular level as the existing computational approaches were not able to meet the enormous number of units involved in Newtonian Mechanics.

During the world war II, the development of computer technology assisted quite effectively in overcoming many complicated problems. But later it was realise that complexity can be handled up to a certain limit, that is, there are complications which cannot be overcome by human skills or any computer technology. Then the problem was to deal with such type of complications where no computational power is effective.

Zadeh in 1965 [1] put forward his idea of fuzzy set theory which is considered to be the most suitable tool in overcoming the uncertainties. This theory is considered as a substitute of probability theory and is widely used in solving decision making problems.

CONTACT Muhammad Touqeer touqeer.fareed@uettaxila.edu.pk

Moreover, rough set theory and theory of interval mathematics were also introduced to cope with uncertainties. In daily life, conventional methods are not efficacious for solving difficult problems. Molodtsov [2] pointed out that due to insufficiency of parametrisation tool, the theories like, the probability theory, fuzzy set theory, the theory of interval mathematics are difficult to apply. He solved this problem by presenting the idea of soft set theory. This theory is extensively used in many different fields.

Acar et al. [3] defined soft rings and introduced their basic properties such as soft ideals, soft homomorphisms etc. by using soft set theory. Neong [4] made an attempt to solve a decision problem using imprecise soft sets by considering a hypothetical case study. Ç elik and Yamak [5] elaborated various applications of fuzzy soft set theory in medical diagnosis using fuzzy arithmetic operations. Qin and Hong [6] discussed the algebraic structure of soft sets and constructed the lattice structures of soft sets. They also introduced the concept of soft equality and derived some related properties. Ali et al. [7] presented some new operations like restricted intersection (union, difference), extended intersection, relative complement and verified the De Morgan's Laws using these newly defined operations.

Imai and Iseki [8] introduced the idea of BCK/BCI-algebras by considering the properties of set difference. Masarwah and Ahmad [9–13] discussed the properties of different fuzzy ideals in BCK/BCI-algebras. Senapati et al. [14] applied cubic intuitionistic structures to ideals of BCI-algebras. Senapati and Shum: [15] elaborated the properties of cubic implicative ideals of BCK-algebras. In 2008, Jun [16] applied soft set theory to the theory of BCK/BCI-algebras and presented soft BCK/BCI-algebras, soft subalgebras and deliberated their properties. In [17] Jun et al. presented the notions of fuzzy soft BCK/BCI-algebras, (closed) fuzzy soft ideals and fuzzy soft p-ideals and conferred apposite properties.

Atanassov [18] generalised the fuzzy sets by presenting the idea of intuitionistic fuzzy sets. Hayat et al. [19, 20] applied Intuitionistic fuzzy soft sets to various real life decision making problems. Abu Qamar and Hassan [21] considered an approach towards a Q-neutrosophic soft set and its application in decision making. By considering the idea of intuitionistic fuzzy sets by Atanassov [18] below we extend the study of applications of soft sets in α-ideals of BCI-algebras and introduce the concept of intuitionistic fuzzy soft α-ideals and prove their basic properties. We also describe connections between various types of intuitionistic fuzzy soft α-ideals and intuitionistic fuzzy soft ideals. We explore useful facts on various operations given in [7] and [22] on intuitionistic fuzzy soft α-ideals and characterise intuitionistic fuzzy α-ideals by soft (δ, η)-level sets. Presented examples give applications of our results. A basic literature relevant to the intuitionistic fuzzy soft theory one can find in [23].

2. Preliminaries

BCK/BCI-algebras are important classes of logical algebras introduced by Imai and Iśeki [8] and were comprehensively explored by many researchers.

An algebra $(\Omega, *, 0)$ of type (2, 0) is called a BCI-algebra if it satisfies the following conditions:

(I). $((\imath * \jmath) * (\imath * \ell)) * (\ell * \jmath) = 0$
(II). $(\imath * (\imath * \jmath)) * \jmath = 0$
(III). $\imath * \imath = 0$

(IV). $\imath * \jmath = 0$ *and* $\jmath * \imath = 0$ $\imath = \jmath$ *for all* $\imath, \jmath, \ell \in \Omega$. In a BCI-algebra Ω, we can define a partial ordering '$\leq$' by putting $\imath \leq \jmath$ if and only if $\imath * \jmath = 0$. If a BCI-algebra Ω satisfies the identity:

(V). $0 * \imath = 0$, for all $\imath \in \Omega$, then Ω is called a BCK-algebra. In any BCI-algebra the following hold:

(VI). $(\imath * \jmath) * \ell = (\imath * \ell) * \jmath$

(VII). $\imath * 0 = \imath$

(VIII). $\imath \leq \jmath$ *implies* $\imath * \ell \leq \jmath * \ell$ *and* $\ell * \jmath \leq \ell * \imath$

(IX). $0 * (\imath * \jmath) = (0 * \imath) * (0 * \jmath)$

(X). $\imath * (\imath * (\imath * \jmath)) = (\imath * \jmath)$

(XI). $(\imath * \ell) * (\jmath * \ell) \leq \imath * \jmath$ *for all* $\imath, \jmath, \ell \in \Omega$.

An intuitionistic fuzzy set $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath))|\imath \in \Omega\}$ in a BCI-algebra Ω is called an intuitionistic fuzzy BCI-algebra [24] of Ω if,

$(\alpha_1) \varpi_\Theta(\imath * \jmath) \geq \min\{\varpi_\Theta(\imath), \varpi_\Theta(\jmath)\}$ and $\xi_\Theta(\imath * \jmath) \leq \max\{\xi_\Theta(\imath), \xi_\Theta(\jmath)\}$, for any $\imath, \jmath, \ell \in \Omega$.

An intuitionistic fuzzy set $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath))|\imath \in \Omega\}$ in a BCI-algebra Ω is called an intuitionistic fuzzy ideal [24] of Ω if,

(α_2) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ *and* $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, *for any* $\imath \in \Omega$.

(α_3) $\varpi_\Theta(\imath) \geq \min\{\varpi_\Theta(\imath * \jmath)), \varpi_\Theta(\jmath)\}$ *and* $\xi_\Theta(\imath) \leq \max\{\xi_\Theta(\imath * \jmath)), \xi_\Theta(\jmath)\}$, *for any* $\imath, \jmath, \ell \in \Omega$.

An intuitionistic fuzzy set $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath))|\imath \in \Omega\}$ in a BCI-algebra Ω is called an intuitionistic fuzzy α-ideal [25] of Ω if,

(α_4) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ *and* $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, *for any* $\imath \in \Omega$.

(α_5) $\varpi_\Theta(\jmath * \imath) \geq \min\{\varpi_\Theta((\imath * \ell) * (0 * \jmath)), \varpi_\Theta(\ell)\}$ *and* $\xi_\Theta(\jmath * \imath) \leq \max\{\xi_\Theta((\imath * \ell) * (0 * \jmath)), \xi_\Theta(\ell)\}$ *for any* $\imath, \jmath, \ell \in \Omega$.

3. Intuitionistic Fuzzy Soft set Theoretic Approach to Subalgebras and Ideals in BCI-algebras

In this section, we present the notions of 'intuitionistic fuzzy soft BCI-algebras' and 'intuitionistic fuzzy soft ideals' and elaborate apposite properties. We will also establish relation between them with the help of different examples. In the sequel, *IFS* (*resp. IFSs*), *IFSS* (*resp. IFSSs*) will be 'intuitionistic fuzz set' (resp. 'intuitionistic fuzz sets'), 'intuitionistic fuzzy soft set' (resp. 'intuitionistic fuzzy soft sets') and Ω will be a BCI-algebra.

Definition 3.1: Let (Γ, ς) be an *IFSS* over Ω. If for some $\delta \in \varsigma$, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an 'intuitionistic fuzzy BCI-algebra' (or *IF$_{BCI}$A*) of Ω, then (Γ, ς) is referred as an 'intuitionistic fuzzy soft BCI-algebra' (or *IFS$_{BCI}$A*) over Ω with respect to the parameter δ. If (Γ, ς) is an *IFS$_{BCI}$A* over Ω with respect to all the members of ς (i.e. for all the parameters in ς), then (Γ, ς) is referred as an *IFS$_{BCI}$A* over Ω.

We clarify the above definition by the succeeding example.

Example 3.2: Let the countries; Denmark, Finland, France, Georgia and Hungary constitute a universe set Ω, i.e.

$\Omega = \{$Denmark, Finland, France, Georgia, Hungary$\}$

Suppose that Ⓢ is an operator which acts upon the members of Ω accordingly as:
Denmark Ⓢ $\imath$ = Denmark, for any $\imath \in \Omega$.

$$Finland \circledS \imath = \begin{cases} Denmark, & \text{if } \imath \in \{Finland, Georgia, Hungary\} \\ Finland, & \text{if } \imath \in \{Denmark, France\} \end{cases}$$

$$France \circledS \imath = \begin{cases} Denmark, & \text{if } \imath \in \{France, Hungary\} \\ France, & \text{if } \imath \in \{Denmark, Finland, Georgia\} \end{cases}$$

$$Georgia \circledS \imath = \begin{cases} Georgia, & \text{if } \imath \in \{Denmark, Finland, France\} \\ Denmark, & \text{if } \imath \in \{Georgia, Hungary\} \end{cases}$$

$$Hungary \circledS \imath = \begin{cases} Hungary, & \text{if } \imath \in \{Denmark, Finland\} \\ Georgia, & \text{if } \imath = France \\ France, & \text{if } \imath = Georgia \\ Denmark, & \text{if } \imath = Hungary \end{cases}$$

Then (Ω, Ⓢ, *Denmark*) is a BCK-algebra and hence a BCI-algebra.

Let ς = {*Tourist, Investor, Student*} be a set of types of visas offered by the countries in Ω to the under-developing countries.

Let (Γ, ς) be an *IFSS* over Ω. Then $\Gamma[Tourist]$, $\Gamma[Investor]$ and $\Gamma[Student]$ are *IFSs* in Ω delineated as:

Γ	*Denmark*	*Finland*	*France*	*Georgia*	*Hungary*
Tourist	(0.8, 0.1)	(0.8, 0.2)	(0.8, 0.2)	(0.7, 0.3)	(0.7, 0.3)
Investor	(0.9, 0.1)	(0.8, 0.2)	(0.4, 0.3)	(0.6, 0.4)	(0.4, 0.4)
Student	(0.7, 0.3)	(0.2, 0.4)	(0.6, 0.4)	(0.2, 0.5)	(0.2, 0.5)

Then $\Gamma[Tourist]$, $\Gamma[Investor]$ and $\Gamma[Student]$ are 'intuitionistic fuzzy soft BCI-algebras' ($IFS_{BCI}As$) over Ω based on parameters Tourist, Investor and Student respectively. Hence (Γ, ς) is an $IFS_{BCI}A$ over Ω.

Proposition 3.3: *If (Γ, ς) is an $IFS_{BCI}A$ over Ω, then for any parameter $\delta \in \varsigma$ and $\imath \in \Omega$,*

$$\varpi_{\Gamma[\delta]}(0) \geq \varpi_{\Gamma[\delta]}(\imath) \text{ and } \xi_{\Gamma[\delta]}(0) \leq \xi_{\Gamma[\delta]}(\imath).$$

Proof: By hypothesis, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) | \imath \in \Omega\}$ is an $IF_{BCI}A$ of Ω for any $\delta \in \varsigma$. Thus for any parameter $\delta \in \varsigma$ and $\imath \in \Omega$,

$$\varpi_{\Gamma[\delta]}(0) = \varpi_{\Gamma[\delta]}(\imath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Gamma[\delta]}(\imath)\} = \varpi_{\Gamma[\delta]}(\imath)$$

and

$$\xi_{\Gamma[\delta]}(0) = \xi_{\Gamma[\delta]}(\imath * \imath) \leq \min\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)\} = \xi_{\Gamma[\delta]}(\imath)$$

Hence proved.

The succeeding statement is evident. ■

Theorem 3.4: *Let* (Γ, ς) *be an* $IFS_{BCI}A$ *over* Ω. *If* $\tau \subset \varsigma$, *then* $(\Gamma|_{\tau}, \tau)$ *is an* $IFS_{BCI}A$ *over* Ω.

Now, we demonstrate that there exists an *IFSS* (Γ, ς) over Ω which is not an $IFS_{BCI}A$ over Ω but there exists $\tau \subset \varsigma$ such that $(\Gamma|_{\tau}, \tau)$ is an $IFS_{BCI}A$ over Ω.

Example 3.5: Let (Ω, Ⓢ, *Denmark*) be the BCI-algebra established in Example 3.2 and $\varsigma = \{$*Tourist, Investor, Student, Worker, Athlete*$\}$ be a set of characteristics of members of Ω. Let (Γ, ς) be an *IFSS* over Ω. Then Γ[*Tourist*], Γ[*Investor*], Γ[*Student*], Γ[*Worker*] and Γ[*Athlete*] are *IFSs* in Ω delineated as:

Γ	*Denmark*	*Finland*	*France*	*Georgia*	*Hungary*
Tourist	(0.7, 0.1)	(0.7, 0.2)	(0.7, 0.2)	(0.2, 0.3)	(0.2, 0.4)
Investor	(0.8, 0.2)	(0.7, 0.3)	(0.2, 0.3)	(0.5, 0.4)	(0.2, 0.5)
Student	(0.6, 0.1)	(0.2, 0.2)	(0.4, 0.2)	(0.2, 0.3)	(0.2, 0.3)
Worker	(0.1, 0.1)	(0.2, 0.2)	(0.3, 0.2)	(0.5, 0.3)	(0.6, 0.4)
Athlete	(0.3, 0.2)	(0.2, 0.3)	(0.5, 0.3)	(0.6, 0.4)	(0.2, 0.5)

Then it can be perceived that (Γ, ς) is not an $IFS_{BCI}A$ over Ω, since, Γ[*Worker*] and Γ[*Athlete*] aren't 'intuitionistic fuzzy BCI-algebras' (or $IF_{BCI}As$) in Ω. Whereas, if we contemplate $\tau = \{$*Tourist, Investor, Student*$\} \subset \varsigma$, then $(\Gamma|_{\tau}, \tau)$ is an $IFS_{BCI}A$ over Ω.

Definition 3.6: Let (Γ, ς) be an *IFSS* over Ω. If for some $\delta\ \varsigma$, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an 'intuitionistic fuzzy ideal' (or *IFI*) of Ω, then (Γ, ς) is referred as an 'intuitionistic fuzzy soft ideal' (or *IFSI*) over Ω with respect to the parameter δ. If (Γ, ς) is an *IFSI* over Ω with respect to all the members of ς (i.e. for all the parameters in ς), then (Γ, ς) is referred as an *IFSI* over Ω.

We clarify the above definition by the succeeding example.

Example 3.7: Let $\Omega = \{$*Denmark, Finland, France, Georgia, Hungary*$\}$ be a universe set. Suppose that $\circledcirc$ is an operator which acts upon the members of Ω accordingly as:

Denmark $\circledcirc \imath =$ Denmark, for any $\imath \in \Omega$.

$$Finland \circledcirc \imath = \begin{cases} Denmark, & \text{if } \imath \in \{Finland, Georgia, Hungary\} \\ Finland, & \text{if } \imath \in \{Denmark, France\} \end{cases}$$

$$France \circledcirc \imath = \begin{cases} Denmark, & \text{if } \imath \in \{France, Georgia\} \\ France, & \text{if } \imath \in \{Denmark, Finland, Hungary\} \end{cases}$$

$$Georgia \circledcirc \imath = \begin{cases} Georgia, & \text{if } \imath = Denmark \\ Denmark, & \text{if } \imath = Georgia \\ France, & \text{if } \imath \in \{Finland, Hungary\} \\ Finland, & \text{if } \imath = France \end{cases}$$

$$Hungary \circledcirc \imath = \begin{cases} Denmark, & \text{if } \imath = Hungary \\ Finland, & \text{if } \imath \in \{Finland, Georgia\} \\ Hungary, & \text{if } \imath \in \{Denmark, France\} \end{cases}$$

Then $(\Omega, \circledcirc, Denmark)$ is a BCK-algebra and hence a BCI-algebra.

Let $\varsigma = \{$Tourist, Investor, Student, Worker, Athlete, Artist$\}$ be a set of characteristics of members of Ω.

Let (Γ, ς) be an *IFSS* over Ω. Then $\Gamma[Tourist]$, $\Gamma[Investor]$, $\Gamma[Student]$, $\Gamma[Worker]$, $\Gamma[Athlete]$ and $\Gamma[Artist]$ are *IFSs* in Ω delineated as:

Γ	*Denmark*	*Finland*	*France*	*Georgia*	*Hungary*
Tourist	(0.6, 0.1)	(0.4, 0.2)	(0.4, 0.2)	(0.4, 0.3)	(0.4, 0.4)
Investor	(0.7, 0.2)	(0.5, 0.3)	(0.7, 0.3)	(0.5, 0.4)	(0.5, 0.5)
Student	(0.8, 0.1)	(0.8, 0.2)	(0.3, 0.2)	(0.3, 0.4)	(0.8, 0.2)
Worker	(0.5, 0.3)	(0.3, 0.4)	(0.4, 0.4)	(0.2, 0.5)	(0.2, 0.6)
Athlete	(0.5, 0.1)	(0.5, 0.2)	(0.8, 0.2)	(0.7, 0.3)	(0.4, 0.5)
Artist	(0.6, 0.3)	(0.5, 0.4)	(0.2, 0.4)	(0.2, 0.5)	(0.5, 0.6)

Then (Γ, ς) is an *IFSI* over Ω based on the parameters Tourist, Investor, Student, Worker and Artist. But since,

$$\begin{aligned}\varpi_{\Gamma[Athelet]}(Hungary)\% &= 0.4\\ &< 0.5\\ &= \min\{\varpi_{\Gamma[Athelet]}(Hungary \circledcirc Georgia), \varpi_{\Gamma[Athelet]}(Georgia)\}\end{aligned}$$

and

$$\begin{aligned}\xi_{\Gamma[Athelet]}(Hungary)\% &= 0.5\\ &> 0.3\\ &= \max\{\xi_{\Gamma[Athelet]}(Hungary \circledcirc Georgia), \xi_{\Gamma[Athelet]}(Georgia)\}\end{aligned}$$

i.e. the *IFS* $\Gamma[Athelet] = \{(\varpi_{\Gamma[Athelet]}(\imath), (\xi_{\Gamma[Athelet]}(\imath))|\imath \in \Omega\}$ is not an *IFI* of Ω. Thus (Γ, ς) isn't an *IFSI* over Ω based on the parameter 'Athlete'. Hence (Γ, ς) isn't an *IFSI* over Ω.

Example 3.8: Let $\Omega = \{Denmark, Finland, France, Georgia, Hungary\}$ be a universe set.

Suppose that $\circledast$ is an operator which acts upon the members of Ω accordingly as:

$$Denmark \circledast \imath = \begin{cases} Denmark, & if\ \imath \in \{Denmark, Finland, France\}\\ Hungary, & if\ \imath = Georgia\\ Georgia, & if\ \imath = Hungary \end{cases}$$

$$Finland \circledast \imath = \begin{cases} Denmark, & if\ \imath = Finland\\ Finland, & if\ \imath \in \{Denmark, France\}\\ Hungary, & if\ \imath = Georgia\\ Georgia, & if\ \imath = Hungary \end{cases}$$

$$France \circledast \imath = \begin{cases} Denmark, & if\ \imath = France\\ France, & if\ \imath \in \{Denmark, Finland\}\\ Hungary, & if\ \imath = Georgia\\ Georgia, & if\ \imath = Hungary \end{cases}$$

$$Georgia \circledast \imath = \begin{cases} Denmark, & if\ \imath = Georgia \\ Georgia, & if\ \imath \in \{Denmark, Finland, France\} \\ Hungary, & if\ \imath = Hungary \end{cases}$$

$$Hungary \circledast \imath = \begin{cases} Denmark, & if\ \imath = Hungary \\ Hungary, & if\ \imath \in \{Denmark, Finland, France\} \\ Georgia, & if\ \imath = Georgia \end{cases}$$

Then (Ω, $\circledast$, *Denmark*) is BCI-algebra.

Let $\varsigma = \{$*Tourist, Investor, Student*$\}$ be a set of characteristics of members of Ω.

Let (Γ, ς) be an *IFSS* over Ω. Then Γ[*Tourist*], Γ[*Investor*] and Γ[*Student*] are *IFSs* in Ω delineated as:

Γ	*Denmark*	*Finland*	*France*	*Georgia*	*Hungary*
Tourist	(0.6, 0.1)	(0.4, 0.2)	(0.4, 0.3)	(0.4, 0.3)	(0.6, 0.1)
Investor	(0.3, 0.2)	(0.5, 0.3)	(0.3, 0.4)	(0.3, 0.2)	(0.5, 0.5)
Student	(0.5, 0.3)	(0.6, 0.3)	(0.4, 0.5)	(0.4, 0.4)	(0.6, 0.4)

Then (Γ, ς) is an *IFSI* over Ω.

Any *IFSI* of a BCK-algebra is an 'intuitionistic fuzzy soft BCK-algebra' (or $IFS_{BCK}A$) but the converse isn't valid. To comprehend this we contemplate the succeeding example.

Example 3.9: Let the flowers, Rose, Tulip, Sunflower, Camation and Gerbera constitute a universe set Ω, i.e. $\Omega = \{$*Rose, Tulip, Sunflower, Camation, Gerbera*$\}$. Suppose that $\boxminus$ is an operator which acts upon the members of Ω accordingly as:

Rose $\boxminus \imath$ = Rose, for any $\imath \in \Omega$.

$$Tulip \boxminus \imath = \begin{cases} Rose, & if\ \imath \in \{Tulip, Camation, Gerbera\} \\ Tulip, & if\ \imath \in \{Rose, Sunflower\} \end{cases}$$

$$Sunflower \boxminus \imath = \begin{cases} Rose, & if\ \imath \in \{Sunflower, Camation, Gerbera\} \\ Sunflower, & if\ \imath \in \{Rose, Tulip\} \end{cases}$$

$$Camation \boxminus \imath = \begin{cases} Rose, & if\ \imath \in \{Camation, Gerbera\} \\ Camation, & if\ \imath \in \{Rose, Tulip, Sunflower\} \end{cases}$$

Then (Ω, Ø, *Rose*) is a BCK-algebra.

Let $\varsigma = \{$*white, gold, pink*$\}$ be a set of different colours in which the flowers in Ω exist in nature.

Let (Γ, ς) be an *IFSS* over Ω. Then Γ[*white*], Γ[*gold*] and Γ[*pink*] are *IFSs* in Ω delineated as:

Γ	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Gerbera*
White	(0.8, 0.1)	(0.6, 0.2)	(0.4, 0.2)	(0.4, 0.3)	(0.4, 0.4)
Gold	(0.7, 0.2)	(0.3, 0.3)	(0.3, 0.3)	(0.6, 0.2)	(0.3, 0.5)
Pink	(0.9, 0.1)	(0.8, 0.2)	(0.7, 0.2)	(0.5, 0.3)	(0.5, 0.4)

Then (Γ, ς) is an $IFS_{BCK}A$ over Ω. But since,

$$\varpi_{\Gamma[gold]}(Tulip) = 0.3 < 0.6 = \min\{\varpi_{\Gamma[gold]}(Tulip \boxminus Camation), \varpi_{\Gamma[gold]}(Camation)\}$$

and

$$\xi_{\Gamma[gold]}(Tulip) = 0.3 > 0.2 = \max\{\xi_{\Gamma[gold]}(Tulip \boxminus Camation), \xi_{\Gamma[gold]}(Camation)\}$$

i.e. the *IFS* $\Gamma[gold] = \{(\varpi_{\Gamma[gold]}(\imath), (\xi_{\Gamma[gold]}(\imath))|\imath \in \Omega\}$ is not an *IFI* of Ω. Thus (Γ, ς) isn't an *IFSI* over Ω based on the parameter 'gold'. Hence (Γ, ς) isn't an *IFSI* over Ω.

Theorem 3.10: *Let (Γ, ς) be an IFSS over Ω. If (Γ, ς) satisfies Proposition 3.3 and the implication,*

$$\imath * \jmath \leq \ell \Rightarrow \varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Gamma[\delta]}(\ell)\} \text{ and } \xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Gamma[\delta]}(\ell)\} \quad (1)$$

for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$. Then (Γ, ς) is an IFSI over Ω.

Proof: *Since by axiom (II), $\imath * (\imath * \jmath) \leq \jmath$, for any $\imath, \jmath, \ell \in \Omega$, it follows by the given implication,*

$$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\} \text{ and } \xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\}$$

This along with Proposition 3.3 implies that (Γ, ς) is an *IFSI* over Ω.

Theorem 3.11: *Let (Γ, ς) be an IFSI over Ω. If for any $\imath, \jmath, \ell \in \Omega$, $\imath * \jmath \leq \ell$, then,*

$$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Gamma[\delta]}(\ell)\} \text{ and } \xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Gamma[\delta]}(\ell)\}$$

for any parameter $\delta \in \varsigma$.

Proof: Let $\imath * \jmath \leq \ell$, for any $\imath, \jmath, \ell \in \Omega$. Then $(\imath * \jmath) * \ell = 0$. By given hypothesis, for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$$\varpi_{\Gamma[\delta]}(\imath * \jmath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \jmath) * \ell), \varpi_{\Gamma[\delta]}(\ell)\} = \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}(\ell)\} = \varpi_{\Gamma[\delta]}(\ell)$$

and

$$\xi_{\Gamma[\delta]}(\imath * \jmath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \jmath) * \ell), \xi_{\Gamma[\delta]}(\ell)\} = \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}(\ell)\} = \xi_{\Gamma[\delta]}(\ell)$$

Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\} \geq \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Gamma[\delta]}(\ell)\}$$

and

$$\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\} \leq \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Gamma[\delta]}(\ell)\}.$$

Hence proved. ∎

Theorem 3.12: *Let (Γ, ς) be an IFS$_{BCI}$A over Ω. Then (Γ, ς) is an IFSI of Ω if and only if the implication* (1) *is valid.*

Proof: The necessity is evident by Theorem 3.11. Conversely, let the implication (1) is valid. Since $\imath * (\imath * \jmath) \leq \jmath$ for any $\imath, \jmath, \ell \in \Omega$, thus by (1),
$\varpi_{\Gamma[\delta]}(\imath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\}$ and $\xi_{\Gamma[\delta]}(\imath) \leq \max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\}$ for any $\delta \in \varsigma$. This along with Proposition 3.3 implies that (Γ, ς) is an *IFSI* over Ω. ■

Theorem 3.13: *Any IFSI (Γ, ς) over Ω satisfies,*

$$\varpi_{\Gamma[\delta]}(0 * (0 * \imath)) \geq \varpi(\imath) \text{ and } \xi_{\Gamma[\delta]}(0 * (0 * \imath)) \leq \xi(\imath)$$

for any $\delta \in \varsigma$ and $\imath \in \Omega$.

Proof: By given hypothesis, for any $\delta \in \varsigma$ and $\imath \in \Omega$,

$$\begin{aligned}\varpi_{\Gamma[\delta]}(0 * (0 * \imath))\% &\geq \min\{\varpi_{\Gamma[\delta]}((0 * (0 * \imath)) * \imath), \varpi_{\Gamma[\delta]}(\imath)\}\\ &= \min\{\varpi_{\Gamma[\delta]}((0 * \imath) * (0 * \imath)), \varpi_{\Gamma[\delta]}(\imath)\}\\ &= \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}(\imath)\}\\ &= \varpi_{\Gamma[\delta]}(\imath)\end{aligned}$$

and

$$\begin{aligned}\xi_{\Gamma[\delta]}(0 * (0 * \imath))\% &\geq \max\{\xi_{\Gamma[\delta]}((0 * (0 * \imath)) * \imath), \xi_{\Gamma[\delta]}(\imath)\}\\ &= \max\{\xi_{\Gamma[\delta]}((0 * \imath) * (0 * \imath)), \xi_{\Gamma[\delta]}(\imath)\}\\ &= \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}(\imath)\}\\ &= \xi_{\Gamma[\delta]}(\imath)\end{aligned}$$

Hence proved. ■

Theorem 3.14: *The AND operation of two intuitionistic fuzzy soft ideals (Γ, ς) and (Υ, τ) over Ω is an IFSI over Ω.*

Proof: The AND operation of (Γ, ς) and (Υ, τ) denoted by, $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau)$, is defined as, $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Delta, \varsigma \times \tau)$, where

$$\begin{aligned}\Delta[\delta, \eta]\% &= \Gamma[\delta] \cap \Upsilon[\eta]\\ &= \{(\varpi_{\Delta[\delta,\eta]}(\imath), \xi_{\Delta[\delta,\eta]}(\imath))|\imath \in \Omega\}\\ &= \{(\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath))|\imath \in \Omega\}\\ &= \{(\min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\}, \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\})|\imath \in \Omega\}\end{aligned}$$

for any $(\delta, \eta) \in \varsigma \times \tau$ and obviously $\delta \in \varsigma, \eta \in \tau$. Thus for any $\imath, \jmath \in \Omega$ and $(\delta, \eta) \in \varsigma \times \tau$,

$$\begin{aligned}\varpi_{\Delta[\delta,\eta]}(\imath)\% &= \varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath)\\ &= \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\}\end{aligned}$$

$$\geq \min\{\min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Gamma[\delta]}(\jmath)\}, \min\{\varpi_{\Upsilon[\eta]}(\imath * \jmath), \varpi_{\Upsilon[\eta]}(\jmath)\}\}$$
$$= \min\{\min\{\varpi_{\Gamma[\delta]}(\imath * \jmath), \varpi_{\Upsilon[\eta]}(\imath * \jmath)\}, \min\{\varpi_{\Gamma[\delta]}(\jmath), \varpi_{\Upsilon[\eta]}(\jmath)\}\}$$
$$= \min\{\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath * \jmath), \varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\jmath)\}$$
$$= \min\{\varpi_{\Delta[\delta,\eta]}(\imath * \jmath), \varpi_{\Delta[\delta,\eta]}(\jmath)\}$$

and

$$\xi_{\Delta[\delta,\eta]}(\imath)\% = \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath)$$
$$= \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\}$$
$$\leq \max\{\max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Gamma[\delta]}(\jmath)\}, \max\{\xi_{\Upsilon[\eta]}(\imath * \jmath), \xi_{\Upsilon[\eta]}(\jmath)\}\}$$
$$= \max\{\max\{\xi_{\Gamma[\delta]}(\imath * \jmath), \xi_{\Upsilon[\eta]}(\imath * \jmath)\}, \max\{\xi_{\Gamma[\delta]}(\jmath), \xi_{\Upsilon[\eta]}(\jmath)\}\}$$
$$= \max\{\xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath * \jmath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\jmath)\}$$
$$= \max\{\xi_{\Delta[\delta,\eta]}(\imath * \jmath), \xi_{\Delta[\delta,\eta]}(\jmath)\}$$

Hence $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Delta, \varsigma \times \tau)$ *is an IFSI over* Ω.

For a BCI-algebra Ω, an *IFSI* over Ω isn't necessarily an *IFS$_{BCI}$A* as can be perceived by the succeeding example. ■

Example 3.15: Let *R* be the set of all non-zero rational numbers. Then it is easy to substantiate that (*R*, ÷, 1) is a BCI-algebra. Delineate an *IFS* $\Gamma[\delta] = \{(\alpha_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) \mid \imath \in \Omega\}$, for any $\delta \in \varsigma$ and $\imath \in R$ as:

$$\varpi_{\Gamma[\delta]}(\imath) = \begin{cases} 0.9 & \textit{if } \imath \in Z' \\ 0.09 & \textit{otherwise} \end{cases}$$

and

$$\xi_{\Gamma[\delta]}(\imath) = \begin{cases} 0.08 & \textit{if } \imath \in Z' \\ 0.8 & \textit{otherwise} \end{cases}$$

Here $Z' =$ set of all non-zero integers. Then (Γ, ς) is an *IFSI* over *R* but since for any $\delta \in \varsigma$,

$$\varpi_{\Gamma[\delta]}(5 \div 4) = 0.09 < 0.9 = \min\{\varpi_{\Gamma[\delta]}(5), \varpi_{\Gamma[\delta]}(4)\}$$

and

$$\xi_{\Gamma[\delta]}(5 \div 4) = 0.8 > 0.08 = \min\{\xi_{\Gamma[\delta]}(5), \xi_{\Gamma[\delta]}(4)\}$$

i.e. $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) | \imath \in \Omega\}$ is not an *IF$_{BCI}$A* of *R* for $\delta \in \varsigma$. Thus (Γ, ς) is not an IFS$_{BCI}$A over R.

An *IFSI* (Γ, ς) over Ω is termed as closed if $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) | \imath \in \Omega\}$ is a 'closed intuitionistic fuzzy ideal' (or *CIFI*) of Ω for any $\delta \in \varsigma$.

Theorem 3.16: *Any closed IFSI over Ω is an IFS$_{BCI}$A over Ω.*

Proof: By given hypothesis, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) | \imath \in \Omega\}$ is a *CIFI* of Ω for any $\delta \in \varsigma$, i.e. $\varpi_{\Gamma[\delta]}(0 * \imath) \geq \varpi_{\Gamma[\delta]}(\imath)$ and $\xi_{\Gamma[\delta]}(0 * \imath) \leq \xi_{\Gamma[\delta]}(\imath)$, for any $\imath \in \Omega$ and $\delta \in \varsigma$. Thus for any $\imath, \jmath \in \Omega$ and $\delta \in \varsigma$,

$$\varpi_{\Gamma[\delta]}(\imath * \jmath)\% \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \jmath) * \imath), \varpi_{\Gamma[\delta]}(\imath)\}$$

$$= \min\{\varpi_{\Gamma[\delta]}((\imath * \imath) * \jmath), \varpi_{\Gamma[\delta]}(\imath)\}$$
$$= \min\{\varpi_{\Gamma[\delta]}(0 * \jmath), \varpi_{\Gamma[\delta]}(\imath)\}$$
$$\geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Gamma[\delta]}(\jmath)\}$$

and

$$\xi_{\Gamma[\delta]}(\imath * \jmath)\% \leq \max\{\xi_{\Gamma[\delta]}((\imath * \jmath) * \imath), \xi_{\Gamma[\delta]}(\imath)\}$$
$$= \max\{\xi_{\Gamma[\delta]}((\imath * \imath) * \jmath), \xi_{\Gamma[\delta]}(\imath)\}$$
$$= \max\{\xi_{\Gamma[\delta]}(0 * \jmath), \xi_{\Gamma[\delta]}(\imath)\}$$
$$\leq \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\jmath)\}$$

Hence proved. ■

Theorem 3.17: *An IFSI (Γ, ς) over Ω is closed $\Longleftrightarrow$ for any $\delta \in \varsigma$ and $\imath, \jmath \in \Omega$ it placates,*

$$\varpi_{\Gamma[\delta]}(\imath * \jmath) \geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Gamma[\delta]}(\jmath)\} \text{ and } \xi_{\Gamma[\delta]}(\imath * \jmath) \leq \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\jmath)\} \quad (2)$$

Proof: The necessity is evident by Theorem 3.16. Conversely, let (2) is valid. Then for any $\delta \in \varsigma$ and $\wp \in \Omega$,

$$\varpi_{\Gamma[\delta]}(0 * \wp) \geq \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}(\wp)\} = \varpi_{\Gamma[\delta]}(\wp)$$

and

$$\xi_{\Gamma[\delta]}(0 * \wp) \leq \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}(\wp)\} = \xi_{\Gamma[\delta]}(\wp)$$

i.e. $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is a CIFI of Ω for any $\delta \in \varsigma$ and $\wp \in \Omega$.

Hence proved. ■

■

4. Intuitionistic Fuzzy Soft set Theoretic Approach to α-Ideals in BCI-algebras

In this section, we present the notions of 'intuitionistic fuzzy soft α-ideals' and elaborate apposite properties. We will also establish their relation with 'intuitionistic fuzzy soft ideals' with the help of different examples. The 'AND' operation, 'extended intersection', 'restricted intersection' and 'union' of 'intuitionistic fuzzy soft α-ideals' will also be conferred. In the sequel, *IFS*$_\alpha$*I* (*resp. IFS*$_\alpha$*Is*) will be 'intuitionistic fuzzy soft α-ideal' (resp. 'intuitionistic fuzzy soft α-ideals') and Ω will be a BCI-algebra.

Definition 4.1: Let (Γ, ς) be an *IFSS* over Ω. If for some $\delta \in \varsigma$, $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an 'intuitionistic fuzzy α-ideal' (or *IF*$_\alpha$*I*) of Ω, then (Γ, ς) is referred as an 'intuitionistic fuzzy soft α-ideal' (or *IFS*$_\alpha$*I*) over Ω with respect to the parameter δ. If

(Γ, ς) is an *IFS*$_\alpha$*I* over Ω with respect to all the members of ς (i.e. for all the parameters in ς), then (Γ, ς) is referred as an *IFS*$_\alpha$*I* over Ω.

We clarify the above definition by the succeeding example.

Example 4.2: Let the four different types of flowers; Rose, Tulip, Sunflower and Camation compose the universe Ω, i.e. $\Omega = \{Rose, Tulip, Sunflower, Camation\}$. Suppose that $\diamondsuit$ is an operator which acts upon the members of Ω accordingly as:

Rose $\diamondsuit \imath = \imath$, for any $\imath \in \Omega$.

$$Tulip \diamondsuit \imath = \begin{cases} Tulip, & \text{if } \imath = Rose \\ Rose, & \text{if } \imath = Tulip \\ Camation, & \text{if } \imath = Sunflower \\ Sunflower, & \text{if } \imath = Camation \end{cases}$$

$$Sunflower \diamondsuit \imath = \begin{cases} Sunflower, & \text{if } \imath = Rose \\ Camation, & \text{if } \imath = Tulip \\ Rose, & \text{if } \imath = Sunflower \\ Tulip, & \text{if } \imath = Camation \end{cases}$$

$$Camation \diamondsuit \imath = \begin{cases} Camation, & \text{if } \imath = Rose \\ Sunflower, & \text{if } \imath = Tulip \\ Tulip, & \text{if } \imath = Sunflower \\ Rose, & \text{if } \imath = Camation \end{cases}$$

Then $(\Omega, \diamondsuit, Rose)$ is a BCI-algebra.

Let $\varsigma = \{Lavender, Red, Orange\}$ be a set of different colours in which the flowers in Ω exist in nature.

Let (Γ, ς) be an *IFSS* Ω. Then $\Gamma[Lavender]$, $\Gamma[Red]$ and $\Gamma[Orange]$ are *IFSs* in Ω delineated as:

Γ	*Rose*	*Tulip*	*Sunflower*	*Camation*
Lavender	(0.8, 0.1)	(0.8, 0.1)	(0.6, 0.2)	(0.6, 0.2)
Red	(0.9, 0)	(0.9, 0)	(0.7, 0.3)	(0.7, 0.3)
Orange	(0.7, 0.2)	(0.7, 0.2)	(0.6, 0.4)	(0.6, 0.4)

Then (Γ, ς) is an $IFS_\alpha I$ over Ω with respect to the parameters Lavender, Red and Orange respectively. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω.

Proposition 4.3: *For any $IFS_\alpha I$ (Γ, ς) over Ω, the succeeding inequalities hold:*

$$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath)) \text{ and } \xi_{\Gamma[\delta]}(\jmath * \imath) \leq \xi_{\Gamma[\delta]}(\imath * (0 * \jmath))$$

for any $\delta \in \varsigma$ and $\imath, \jmath \in \Omega$.

Proof: Let (Γ, ς) be an $IFS_\alpha I$ over Ω. Then $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\delta \in \varsigma$. Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$$

and

$$\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}$$

By substituting $\ell = 0$ we get,

$$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * 0) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(0)\} = \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath))$$

and

$$\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * 0) * (0 * \jmath)), \xi_{\Gamma[\delta]}(0)\} = \xi_{\Gamma[\delta]}(\imath * (0 * \jmath))$$

Hence proved. ■

Theorem 4.4: *Any $IFS_\alpha I$ over Ω is an IFSI over Ω.*

Proof: Let (Γ, ς) be an $IFS_\alpha I$ over Ω. Then $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) | \imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\delta \in \varsigma$. Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$$\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$$

and

$$\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}$$

By substituting $\imath = 0$ we get,

$$\varpi_{\Gamma[\delta]}(\jmath * 0) \geq \min\{\varpi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}$$

and

$$\xi_{\Gamma[\delta]}(\jmath * 0) \leq \max\{\xi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}$$

Since we know that $(0 * \ell) * (0 * \jmath) \leq \jmath * \ell$, therefore,

$$\varpi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)) \geq \varpi_{\Gamma[\delta]}(\jmath * \ell) \text{ and } \xi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)) \leq \xi_{\Gamma[\delta]}(\jmath * \ell)$$

Thus we acquire,

$$\varpi_{\Gamma[\delta]}(\jmath) \geq \min\{\varpi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\} \geq \min\{\varpi_{\Gamma[\delta]}(\jmath * \ell), \varpi_{\Gamma[\delta]}(\ell)\}$$

and

$$\xi_{\Gamma[\delta]}(\jmath) \leq \max\{\xi_{\Gamma[\delta]}((0 * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\} \leq \max\{\xi_{\Gamma[\delta]}(\jmath * \ell), \xi_{\Gamma[\delta]}(\ell)\}$$

i.e. $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath)) | \imath \in \Omega\}$ is an IFI of Ω for any $\delta \in \varsigma$. Hence (Γ, ς) is an *IFSI* over Ω.

The succeeding example proves that an *IFSI* may not be an $IFS_\alpha I$. ■

Example 4.5: Let the five different kinds of flowers; Rose, Tulip, Sunflower, Camation and Lily compose the universe Ω, i.e. $\Omega = \{$*Rose, Tulip, Sunflower, Camation, Lily*$\}$. Suppose that $\oplus$ is an operator which acts upon the members of Ω accordingly as:

$\imath \oplus$ Rose $= \imath$, for any $\imath \in \Omega$.

$$\imath \oplus Tulip = \begin{cases} Rose, & \text{if } \imath \in \{Rose, Tulip\} \\ \imath, & \text{if } \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$\imath \oplus Sunflower = \begin{cases} Lily, & \text{if } \imath \in \{Rose, Tulip\} \\ Rose, & \text{if } \imath = Sunflower \\ Sunflower, & \text{if } \imath = Camation \\ Camation, & \text{if } \imath = Lily \end{cases}$$

$$\imath \oplus Camation = \begin{cases} Camation & \text{if } \imath \in \{Rose, Tulip\} \\ Lily, & \text{if } \imath = Sunflower \\ Rose, & \text{if } \imath = Camation \\ Sunflower, & \text{if } \imath = Lily \end{cases}$$

$$\imath \oplus Lily = \begin{cases} Sunflower, & \text{if } \imath \in \{Rose, Tulip\} \\ Camation, & \text{if } \imath = Sunflower \\ Lily, & \text{if } \imath = Camation \\ Rose, & \text{if } \imath = Lily \end{cases}$$

Then $(\Omega, \oplus, Rose)$ is a BCI-algebra.

Let $\varsigma = \{Lavender, Red, Green\}$ be a set of characteristics of the flowers given in Ω.

Let (Γ, ς) be an *IFSS* over Ω. Then $\Gamma[Lavender]$, $\Gamma[Red]$ and $\Gamma[Green]$ are *IFSs* in Ω delineated as:

Γ	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Lily*
Lavender	(0.9, 0.1)	(0.4, 0.4)	(0.6, 0.3)	(0.8, 0.2)	(0.1, 0.5)
Red	(0.9, 0)	(0.7, 1)	(0.4, 0.4)	(0.5, 0.3)	(0.6, 0.2)
Green	(1, 0)	(0.6, 0.2)	(0.5, 0.3)	(0.3, 0.4)	(0.7, 0.1)

Then (Γ, ς) is an *IFSI* over Ω but since

$$\begin{aligned} \varpi_{\Gamma[Red]}(Tulip \oplus Lily) &= \varpi_{\Gamma[Red]}(Sunflower) \\ &= 0.4 \\ &< 0.6 \\ &= \min\{\varpi_{\Gamma[Red]}((Lily \oplus Rose) \oplus (Rose \oplus Tulip)), \varpi_{\Gamma[Red]}(Rose)\} \end{aligned}$$

and

$$\begin{aligned} \xi_{\Gamma[Red]}(Tulip \oplus Lily) &= \xi_{\Gamma[Red]}(Sunflower) \\ &= 0.4 \\ &> 0.2 \\ &= \max\{\xi_{\Gamma[Red]}((Lily \oplus Rose) \oplus (Rose \oplus Tulip)), \xi_{\Gamma[Red]}(Rose)\} \end{aligned}$$

i.e. $\Gamma[Red] = \{(\varpi_{\Gamma[Red]}(\imath), \xi_{\Gamma[Red]}(\imath))|\imath \in \Omega\}$ isn't an $IF_\alpha I$ of Ω. Therefore (Γ, ς) is not an $IFS_\alpha I$ over Ω with respect to the parameter 'Red'. Hence (Γ, ς) is not an $IFS_\alpha I$ over Ω.

Proposition 4.6: *Let (Γ, ς) be an $IFS_\alpha I$ over Ω. Then for any parameter $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$, $\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)) \geq \varpi_{\Gamma[\delta]}(\imath * (\ell * \jmath))$ and $\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)) \leq \xi_{\Gamma[\delta]}(\imath * (\ell * \jmath))$.*

Proof: Let (Γ, ς) be an $IFS_\alpha I$ over Ω. Since $(\imath * \ell) * (0 * \jmath) = (\imath * \ell) * ((\ell * \jmath) * \ell) \leq \imath * (\ell * \jmath)$. Therefore, $(\imath * \ell) * (0 * \jmath) * (\imath * (\ell * \jmath)) = 0$. By Theorem 4.4, (Γ, ς) is an *IFSI* over Ω. Thus,

$\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an *IFI* of Ω for any $\delta \in \Omega$.

Thus for any $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$$\begin{aligned}\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)) &\geq \min\{\varpi_{\Gamma[\delta]}(((\imath * \ell) * (0 * \jmath)) * (\imath * (\ell * \jmath))), \varpi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\} \\ &= \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\} \\ &= \varpi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\end{aligned}$$

and

$$\begin{aligned}\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)) &\leq \max\{\xi_{\Gamma[\delta]}(((\imath * \ell) * (0 * \jmath)) * (\imath * (\ell * \jmath))), \xi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\} \\ &= \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\} \\ &= \xi_{\Gamma[\delta]}((\imath * (\ell * \jmath)))\end{aligned}$$

Hence proved. ■

Theorem 4.7: *Let (Γ, ς) be an IFSI over Ω. If for any parameter $\delta \in \varsigma$ and $\imath, \jmath \in \Omega$, $\varpi_{\Gamma[\delta]}(\jmath * \imath) \geq \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath))$ and $\xi_{\Gamma[\delta]}(\jmath * \imath) \leq \xi_{\Gamma[\delta]}(\imath * (0 * \jmath))$. Then (Γ, ς) is an $IFS_\alpha I$ over Ω.*

Proof: Since (Γ, ς) is an *IFSI* over Ω, therefore $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an *IFI* of Ω for any $\delta \in \Omega$. Thus for any parameter $\delta \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

$$\begin{aligned}\varpi_{\Gamma[\delta]}(\jmath * \imath) &\geq \varpi_{\Gamma[\delta]}(\imath * (0 * \jmath)) \\ &\geq \min\{\varpi_{\Gamma[\delta]}((\imath * (0 * \jmath)) * \ell), \varpi_{\Gamma[\delta]}\ell)\} \\ &= \min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \Gamma[\delta](\ell)\}\end{aligned}$$

and

$$\begin{aligned}\xi_{\Gamma[\delta]}(\jmath * \imath) &\geq \xi_{\Gamma[\delta]}(\imath * (0 * \jmath)) \\ &\geq \min\{\xi_{\Gamma[\delta]}((\imath * (0 * \jmath)) * \ell), \xi_{\Gamma[\delta]}\ell)\} \\ &= \min\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \Gamma[\delta](\ell)\}\end{aligned}$$

i.e. $\Gamma[\delta] = \{(\varpi_{\Gamma[\delta]}(\imath), \xi_{\Gamma[\delta]}(\imath))|\imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\delta \in \Omega$. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω. ■

Theorem 4.8: *If (Γ, ς) and (Υ, τ) are $IFS_\alpha I$s over Ω, then the 'extended intersection' of (Γ, ς) and (Υ, τ) is an $IFS_\alpha I$ over Ω.*

Proof: We know that the 'extended intersection' of (Γ, ς) and (Υ, τ), denoted by $(\Gamma,\varsigma)\sqcap_E(\Upsilon,\tau)$, can be defined as, $(\Gamma,\varsigma)\sqcap_E(\Upsilon,\tau) = (\Pi,\varrho)$, where $q = \varsigma \cup \tau$ and for any $\wp \in q$,

$$\Pi[\wp] = \begin{cases} \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath))|\imath \in \Omega\} & \text{if } \wp \in \varsigma - \tau \\ \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath))|\imath \in \Omega\} & \text{if } \wp \in \tau - \varsigma \\ \Gamma[\wp] \cap \Upsilon[\wp] = \{(\min\{\varpi_{\Gamma[\wp]}(\imath), \varpi_{\Upsilon[\wp]}(\imath)\}, & \\ \max\{\xi_{\Gamma[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)\})|\imath \in \Omega\} & \text{if } \wp \in \varsigma \cap \tau \end{cases}$$

For any $\wp \in q$ if $\wp \in \varsigma - \tau$, then $\Pi[\wp] = \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath))|\imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω. Similarly, if $\wp \in \tau - \varsigma$, then $\Pi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath))|\imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω. Moreover if $\wp \in q$ such that $\wp \in \varsigma \cap \tau$, then
$\Pi[\wp] = \Gamma[\wp] \cap \Upsilon[\wp] = \{(\min\{\varpi_{\Gamma[\wp]}(\imath), \varpi_{\Upsilon[\wp]}(\imath)\}, \max\{\xi_{\Gamma[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)\})|\imath \in \Omega\}$
which is also an $IF_\alpha I$ of Ω since, the intersection of two $IF_\alpha I$ is an $IF_\alpha I$. Hence $\Pi[\wp]$ is an $IF_\alpha I$ of Ω for any $\wp \in q$. Hence $(\Pi, q) = (\Gamma, \varsigma)$ H_E (Υ, τ) is an $IFS_\alpha I$ over Ω.
The corollaries stated below can be deduced from the above theorem. ■

Corollary 4.9: *If (Γ, ς) and (Υ, ς) are $IFS_\alpha Is$ over Ω, then the 'extended intersection' of (Γ, ς) and (Υ, ς) is an $IFS_\alpha I$ over Ω.*

Corollary 4.10: *The 'restricted intersection' of two $IFS_\alpha Is$ is an $IFS_\alpha I$.*

Theorem 4.11: *Let (Γ, ς) and (Υ, τ) be two $IFS_\alpha Is$ over Ω. If $\varsigma \cap \tau = \varphi$ then the 'union', (Γ, ς) $\tilde{\cup}(\Upsilon, \tau)$ is an $IFS_\alpha I$ over Ω.*

Proof: We know that the 'union' of (Γ, ς) and (Υ, τ), denoted by
$(\Gamma, \varsigma)\,\tilde{\cup}(\Upsilon, \tau)$, can be defined as, $(\Gamma, \varsigma)\,\tilde{\cup}\,(\Upsilon, \tau) = (\Pi, \varrho)$, where $\varrho = \varsigma \cup \tau$ and for any $\wp \in \varrho$,

$$\Pi[\wp] = \begin{cases} \Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\imath), \xi_{\Gamma[\wp]}(\imath))|\imath \in \Omega\} & \text{if } \wp \in \varsigma - \tau \\ \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath))|\imath \in \Omega\} & \text{if } \wp \in \tau - \varsigma \\ \Gamma[\wp] \cup \Upsilon[\wp] = \{(\max\{\varpi_{\Gamma[\wp]}(\imath), \varpi_{\Upsilon[\wp]}(\imath)\}, & \\ \min\{\xi_{\Gamma[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)\})|\imath \in \Omega\} & \text{if } \wp \in \varsigma \cap \tau \end{cases}$$

Since $\varsigma \cap \tau = \varphi$, either $\wp \in \varsigma - \tau$ or $\wp \in \tau - \varsigma$ for all $\wp \in \varrho$.

If $\wp \in \varsigma - \tau$ then $\Pi[\wp] = \Gamma[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath))|\imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω as (Γ, ς) is an $IFS_\alpha I$ over Ω.

If $\wp \in \tau - \varsigma$ then $\Pi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath))|\imath \in \Omega\}$, which is an $IF_\alpha I$ of Ω as (Υ, τ) is an $IFS_\alpha I$ over Ω.

Hence $(\Pi, q) = (\Gamma, \varsigma)\,\tilde{\cup}\,(\Upsilon, \tau)$ is an $IFS_\alpha I$ over Ω.

Below, we consider an example in which we have a non-empty intersection of the sets of parameters (i.e. $\varsigma \cap \tau$ f= φ). ■

Example 4.12: Let $\Omega = \{$*Rose, Tulip, Sunflower, Camation, Lily*$\}$ be a universe set. Suppose that $\otimes$ is an operator which acts upon the members of Ω accordingly as:

$$Rose \otimes \imath = \begin{cases} Rose, & \text{if } \imath \in \{Rose, Tulip\} \\ \imath, & \text{if } \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$Tulip \otimes \imath = \begin{cases} Rose, & \text{if } \imath = Tulip \\ Tulip, & \text{if } \imath = Rose \\ \imath, & \text{if } \imath \in \{Sunflower, Camation, Lily\} \end{cases}$$

$$Sunflower \otimes \imath = \begin{cases} Sunflower, & \text{if } \imath \in \{Rose, Tulip\} \\ Camation, & \text{if } \imath = Lily \\ Rose, & \text{if } \imath = Sunflower \\ Lily, & \text{if } \imath = Camation \end{cases}$$

$$Camation \otimes \imath = \begin{cases} Camation, & \text{if } \imath \in \{Rose, Tulip\} \\ Sunflower, & \text{if } \imath = Lily \\ Lily, & \text{if } \imath = Sunflower \\ Rose, & \text{if } \imath = Camation \end{cases}$$

$$Camation \otimes \imath = \begin{cases} Camation, & \text{if } \imath \in \{Rose, Tulip\} \\ Sunflower, & \text{if } \imath = Lily \\ Lily, & \text{if } \imath = Sunflower \\ Rose, & \text{if } \imath = Camation \end{cases}$$

Then $(\Omega, \otimes, Rose)$ is a BCI-algebra.

Let $\varsigma = \{Lavender, Red, Green, Purple\}$ and $\tau = \{Green, Purple, Blue\}$ be two sets of characteristics of the flowers given in Ω.

Let (Γ, ς) be an *IFSS* over Ω. Then $\Gamma[Lavender]$, $\Gamma[Red]$, $\Gamma[Green]$ and $\Gamma[Purple]$ are *IFSs* in Ω delineated as:

Γ	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Lily*
Lavender	(0.9, 0)	(0.9, 0)	(0.4, 0.3)	(0.4, 0.1)	(0.4, 0.3)
Red	(0.6, 0.2)	(0.6, 0.2)	(0.3, 0.4)	(0.3, 0.4)	(0.5, 0.3)
Green	(0.8, 0.1)	(0.8, 0.1)	(0.2, 0.5)	(0.5, 0.3)	(0.2, 0.5)
Purple	(0.7, 0.2)	(0.7, 0.2)	(0.5, 0.3)	(0.3, 0.5)	(0.3, 0.5)

Then (Γ, ς) is an $IFS_\alpha I$ over Ω with respect to the parameters Lavender, Red, Green and Purple respectively. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω.

Let (Υ, τ) be an *IFSS* over Ω. Then Υ [*Green*], Υ [*Purple*] and Υ [*Blue*] are *IFSs* in Ω defined as follows:

Υ	Rose	*Tulip*	*Sunflower*	*Camation*	*Lily*
Green	(0.7, 0)	(0.7, 0)	(0.5, 0.3)	(0.2, 0.5)	(0.2, 0.5)
Purple	(0.6, 0.2)	(0.6, 0.2)	(0.2, 0.5)	(0.2, 0.5)	(0.4, 0.3)
Blue	(0.9, 0)	(0.9, 0)	(0.4, 0.3)	(0.6, 0.1)	(0.4, 0.3)

Then (Υ, τ) is an $IFS_\alpha I$ over Ω with respect to the parameters creative, comprehensive and perceived respectively. Then (Υ, τ) is an $IFS_\alpha I$ over Ω.

Now, we cogitate the union of (Γ, ς) and (Υ, τ), i.e.
$(\Gamma, \varsigma) \tilde{\cup} (\Upsilon, \tau) = (\Pi, q)$
where $\varrho = \varsigma \cup \tau$. Note that for any parameter $\delta \in \varsigma \cap \tau$,

$$\begin{aligned}
\Pi[\delta] &= \Gamma[\delta] \cup \Upsilon[\delta] \\
&= \{(\varpi_{\Gamma[\delta] \cup \Upsilon[\delta]}, \xi_{\Gamma[\delta] \cup \Upsilon[\delta]}) | \imath \in \Omega\} \\
&= \{(\max\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\delta]}(\imath)\}, \min\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\delta]}(\imath)\}) | \imath \in \Omega\}
\end{aligned}$$

Then

$$\begin{aligned}
& \varpi_{\Pi[Green]}(Camation \otimes Sunflower) \\
&= \varpi_{\Pi[Green]}(Lily) \\
&= \varpi_{(\Gamma[Green] \cup \Upsilon[Green])}(Lily) \\
&= \max\{\varpi_{\Gamma[Green]}(Lily), \varpi_{\Upsilon}[Green](Lily)\} \\
&= \max\{0.2, 0.2\} \\
&= 0.2 \\
&< 0.5 \\
&= \min\{\varpi_{\Pi[Green]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Camation)), \\
&\quad \varpi_{\Pi[Green]}(Sunflower)\} \\
&= \min\{\varpi_{\Pi[Green]}(Camation), \varpi_{\Pi[Green]}(Sunflower)\} \\
&= \min\{\varpi_{\Pi[Green]}(Camation), \varpi_{\Pi[Green]}(Sunflower)\} \\
&= \min\{\varpi_{(\Gamma[Green] \cup \Upsilon[Green])}(Camation), \varpi_{(\Gamma[Green] \cup \Upsilon[Green])}(Sunflower)\} \\
&= \min\{\max\{\varpi_{\Gamma[Green]}(Camation), \varpi\, \Upsilon(Camation)\}, \\
&\quad \max\{\varpi_{\Gamma[Green]}(Sunflower), \varpi_{\Upsilon[Green]}(Sunflower)\}\} \\
&= \min\{\max\{\varpi_{\Gamma[Green]}(Camation), \varpi_{\Upsilon[Green]}(Camation)\}, \\
&\quad \max\{\varpi_{\Gamma[Green]}(Sunflower), \varpi_{\Upsilon[Green]}(Sunflower)\}\} \\
&= \min\{\max\{0.5, 0.2\}, \max\{0.2, 0.5\}\} \\
&= \min\{0.5, 0.5\}
\end{aligned}$$

and

$$\begin{aligned}
& \xi_{\Pi[Green]}(Camation \otimes Sunflower) \\
&= \xi_{\Pi[Green]}(Lily) \\
&= \xi_{(\Gamma[Green] \cup \Upsilon[Green])}(Lily) \\
&= \min\{\xi_{\Gamma[Green]}(Lily), \xi_{\Upsilon}[Green](Lily)\} \\
&= \min\{0.5, 0.5\} \\
&= 0.5 \\
&> 0.3
\end{aligned}$$

$$= \max\{\xi_{\Pi[Green]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Camation)),$$
$$\xi_{\Pi[Green]}(Sunflower)\}$$
$$= \max\{\xi_{\Pi[Green]}(Camation), \xi_{\Pi[Green]}(Sunflower)\}$$
$$= \max\{\xi_{(\Gamma[Green]\cup\Upsilon[Green])}(Camation), \xi_{(\Gamma[Green]\cup\Upsilon[Green])}(Sunflower)\}$$
$$= \max\{\min\{\xi_{\Gamma[Green]}(Camation), \xi_{\Upsilon[Green]}(Camation)\},$$
$$\min\{\xi_{\Gamma[Green]}(Sunflower), \xi_{\Upsilon[Green]}(Sunflower)\}\}$$
$$= \max\{\min\{0.3, 0.5\}, \min\{0.5, 0.3\}\}$$
$$= \max\{0.3, 0.3\}$$

Therefore $\Pi[Green] = \Gamma[Green] \cup \Upsilon[Green] = \{(\varpi_{\Gamma[Green]\cup\Upsilon[Green]}, \xi_{\Gamma[Green]\cup\Upsilon[Green]})|\imath \in \Omega\}$ is not an $IF_\alpha I$ of Ω. Thus $(\Pi, \varrho) = (\Gamma, \varsigma)\tilde{\cup}(\Upsilon, \tau)$ is not an $IFS_\alpha I$ over Ω based on the parameter 'Green'. Hence $(\Pi, \varrho) = (\Gamma, \varsigma)\tilde{\cup}(\Upsilon, \tau)$ is not an $IFS_\alpha I$ over Ω.

Theorem 4.13: *If (Γ, ς) and (Υ, τ) are two $IFS_\alpha Is$ over Ω, then the 'AND' operation, $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau)$ is also an $IFS_\alpha Is$ over Ω.*

Proof: By definition, $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Pi, \varsigma \times \tau)$, where

$$\Pi[\delta, \eta]\% = \Gamma[\delta] \cap \Upsilon[\eta]$$
$$= \{(\varpi_{\Gamma[\delta] \cap \Upsilon[\eta]}(\imath), \xi_{\Gamma[\delta] \cap \Upsilon[\eta]}(\imath))|\imath \in \Omega\}$$
$$= \{(\min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\}, \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\})|\imath \in \Omega\}$$

for all $(\delta, \eta) \in \varsigma \times \tau$. For any $(\delta, \eta) \in \varsigma \times \tau$ (i.e. $\delta \in \varsigma$ and $\eta \in \tau$) and $\imath \in \Omega$,

$$\varpi_{\Pi[\delta,\eta]}(0) = \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(0)$$
$$= \min\{\varpi_{\Gamma[\delta]}(0), \varpi_{\Upsilon[\eta]}(0)\}$$
$$\geq \min\{\varpi_{\Gamma[\delta]}(\imath), \varpi_{\Upsilon[\eta]}(\imath)\}$$
$$= \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\imath)$$
$$= \varpi_{\Pi[\delta,\eta]}(\imath)$$

and

$$\xi_{\Pi[\delta,\eta]}(0) = \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(0)$$
$$= \max\{\xi_{\Gamma[\delta]}(0), \xi_{\Upsilon[\eta]}(0)\}$$
$$\leq \max\{\xi_{\Gamma[\delta]}(\imath), \xi_{\Upsilon[\eta]}(\imath)\}$$
$$= \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\imath)$$
$$= \xi_{\Pi[\delta,\eta]}(\imath)$$

For any $(\delta, \eta) \in \varsigma \times \tau$ (i.e. $\delta \in \varsigma$ and $\eta \in \tau$) and $\imath, \jmath, \ell \in \Omega$,

$$\varpi_{\Pi[\delta,\eta]}(\jmath * \imath) = \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\jmath * \imath)$$
$$= \min\{\varpi_{\Gamma[\delta]}(\jmath * \imath), \varpi_{\Upsilon[\eta]}(\jmath * \imath)\}$$

$$
\begin{aligned}
&\geq \min\{\min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\delta]}(\ell)\}, \\
&\quad \min\{\varpi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Upsilon[\eta]}(\ell)\}\} \\
&= \min\{\min\{\varpi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath))\}, \\
&\quad \min\{\varpi_{\Gamma[\delta]}(\ell), \varpi_{\Upsilon[\eta]}(\ell)\}\} \\
&= \min\{\varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}((\imath * \ell) * (0 * \jmath)), \varpi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\ell)\} \\
&= \min\{\varpi_{\Pi[\delta,\eta]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Pi[\delta,\eta]}(\ell)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_{\Pi[\delta,\eta]}(\jmath * \imath) &= \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\jmath * \imath) \\
&= \max\{\xi_{\Gamma[\delta]}(\jmath * \imath), \xi_{\Upsilon[\eta]}(\jmath * \imath)\} \\
&\leq \max\{\max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\delta]}(\ell)\}, \\
&\quad \max\{\xi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Upsilon[\eta]}(\ell)\}\} \\
&= \max\{\max\{\xi_{\Gamma[\delta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Upsilon[\eta]}((\imath * \ell) * (0 * \jmath))\}, \\
&\quad \max\{\xi_{\Gamma[\delta]}(\ell), \xi_{\Upsilon[\eta]}(\ell)\}\} \\
&= \max\{\xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}((\imath * \ell) * (0 * \jmath)), \xi_{(\Gamma[\delta]\cap\Upsilon[\eta])}(\ell)\} \\
&= \max\{\xi_{\Pi[\delta,\eta]}((\imath * \ell) * (0 * \jmath)), \xi_{\Pi[\delta,\eta]}(\ell)\}
\end{aligned}
$$

Thus $\Pi[\delta,\eta] = \Gamma[\delta] \cap \Upsilon[\eta] = \{(\varpi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath), \xi_{\Gamma[\delta]\cap\Upsilon[\eta]}(\imath))|\imath \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $(\delta, \eta) \in \varsigma \times \tau$. Hence $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Pi, \varsigma \times \tau)$ is an $IFS_\alpha I$ over Ω with respect to the parameter (δ, η). Since (δ, η) is an arbitrary parameter, therefore $(\Gamma, \varsigma)\tilde{\wedge}(\Upsilon, \tau) = (\Pi, \varsigma \times \tau)$ is an $IFS_\alpha I$ over Ω.

Any $IFS_\alpha I$ over a BCK-algebra Ω is an 'intuitionistic fuzzy soft BCK-algebra' (or $IFS_{BCK}A$). We perceive by the succeeding example that the converse isn't valid. ■

Example 4.14: Let $\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$ be a universe set. Suppose that $\otimes$ is an operator which acts upon the members of Ω accordingly as:

Rose $\boxtimes \imath$ = Rose, for all $\imath \in \Omega$.

$$
Tulip \boxplus \imath = \begin{cases} Rose, & if\ \imath \in \{Tulip, Camation, Lily\} \\ Tulip, & if\ \imath \in \{Rose, Sunflower\} \end{cases}
$$

$$
Sunflower \boxplus \imath = \begin{cases} Sunflower, & if\ \imath \in \{Rose, Tulip\} \\ Rose, & if\ \imath \in \{Sunflower, Camation, Lily\} \end{cases}
$$

$$
Camation \boxplus \imath = \begin{cases} Camation, & if\ \imath \in \{Rose, Tulip, Sunflower\} \\ Rose, & if\ \imath \in \{Camation, Lily\} \end{cases}
$$

$$
Lily, \boxplus \imath = \begin{cases} Lily, & if\ \imath \in \{Rose, Tulip, Sunflower, Camation\} \\ Rose, & if\ \imath = Lily \end{cases}
$$

Then $(\Omega, \boxtimes, Rose)$ is a BCK-algebra.

Γ	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Lily*
Lavender	(0.7, 0.1)	(0.5, 0.2)	(0.3, 0.4)	(0.3, 0.4)	(0.3, 0.4)
Red	(0.8, 0.2)	(0.4, 0.5)	(0.4, 0.5)	(0.7, 0.3)	(0.4, 0.5)
Green	(0.8, 0)	(0.7, 0.1)	(0.6, 0.3)	(0.4, 0.4)	(0.4, 0.4)

Let $\varsigma = \{Lavender, Red, Green\}$ be a set of parameters of Ω. Let (Γ, ς) be an *IFSS* over Ω. Then $\Gamma[Lavender]$, $\Gamma[Red]$ and $\Gamma[Green]$ are *IFSs* in Ω delineated as:

Then (Γ, ς) is an $IFS_{BCK}A$ over Ω but since,

$$\begin{aligned} &\varpi_{\Gamma[Red]}(Lily \boxplus Tulip) \\ &= \varpi_{\Gamma[Red]}(Lily) \\ &= 0.4 \\ &< 0.7 \\ &= \min\{\varpi_{\Gamma[Red]}((Tulip \boxplus Camation) \boxplus (Rose \boxplus Lily)), \varpi_{\Gamma[Red]}(Camation)\} \end{aligned}$$

and

$$\begin{aligned} &\xi_{\Gamma[Red]}(Lily \boxplus Tulip) \\ &= \xi_{\Gamma[Red]}(Lily) \\ &= 0.5 \\ &> 0.3 \\ &= \max\{\xi_{\Gamma[Red]}((Tulip \boxplus Camation) \boxplus (Rose \boxplus Lily)), \xi_{\Gamma[Red]}(Camation)\} \end{aligned}$$

i.e. $\Gamma[Red] = \{(\varpi_{\Gamma[Red]}(\imath), \xi_{\Gamma[Red]}(\imath))|\imath \in \Omega\}$ is not an $IF_\alpha I$ of Ω. Therefore (Γ, ς) isn't an $IFS_\alpha I$ over Ω based on the parameter 'Red'. Hence (Γ, ς) isn't an $IFS_\alpha I$ over Ω.

5. Intuitionistic Fuzzy Soft set Theoretic Approach to α-Ideals Based on Soft set Theoretic Approach to BCI-Algebras

Now, we will confer 'intuitionistic fuzzy soft α-ideal' of a 'soft BCI-algebra' and discuss its properties. We will elaborate the 'AND' operation, 'extended intersection', 'restricted intersection' and 'union' of 'intuitionistic fuzzy soft α-ideals' of a 'soft BCI-algebra'. Here first we familiarise with *IFI* and $IF_\alpha I$ related to a subalgebra. In the sequel, Ω, as usual will be a BCI-algebra.

Definition 5.1: Let Ξ be a subalgebra of Ω. An *IFS* $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath))|\imath \in \Omega\}$ in Ω is an *IFI* of Ω related to Ξ (or briefly, Ξ-*IFI* of Ω), symbolised as $\Theta\blacktriangle\Xi$ if,

(i) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ and $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, for any $\imath \in \Xi$
(ii) $\varpi_\Theta(\imath) \geq \min\{\varpi_\Theta(\imath * \jmath), \varpi_\Theta(\jmath)\}$ and $\xi_\Theta(\imath) \leq \max\{\xi_\Theta(\imath * \jmath), \xi_\Theta(\jmath)\}$, for any $\imath, \jmath \in \Xi$

Definition 5.2: Let Ξ be a subalgebra of Ω. An *IFS* $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath))|\imath \in \Omega\}$ in Ω is an $IF_\alpha I$ of Ω related to Ξ (or briefly, Ξ-$IF_\alpha I$ of Ω), symbolised as $\Theta\blacktriangle_\alpha\Xi$, if,

(i) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ *and* $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, *for any* $\imath \in \Xi$.
(ii) $\varpi_\Theta(\jmath * \imath) \geq \min\{\varpi_\Theta((\imath * \ell) * (0 * \jmath)), \varpi_\Theta(\ell)\}$ *and* $\xi_\Theta(\jmath * \imath) \leq \max\{\xi_\Theta((\imath * \ell) * (0 * \jmath)), \xi_\Theta(\ell)\}$, *for any* $\imath, \jmath, \ell \in \Xi$

Example 5.3: Cogitate the BCI-algebra (Ω, $\otimes$, *Rose*) defined in Example 4.12.

Let Ξ = {*Rose, Sunflower, Camation, Lily*} be a subset of Ω. Then (Ξ, $\otimes$, *Rose*) is also a BCI-algebra, i.e. Ξ is a subalgebra of Ω. Delineate an *IFS* $\Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath)) | \imath \in \Omega\}$ in Ω as:

$$\varpi_\Theta(Rose) = \varpi_\Theta(Tulip) = \varpi_\Theta(Sunflower) = 0.8,$$

$$\varpi_\Theta(Camation) = \varpi_\Theta(Lily) = 0.2.$$

$$\xi_\Theta(Rose) = \xi_\Theta(Tulip) = \xi_\Theta(Sunflower) = 0.1,$$

$$\xi_\Theta(Camation) = \xi_\Theta(Lily) = 0.6$$

Then it can be observed that,

(i) $\varpi_\Theta(0) \geq \varpi_\Theta(\imath)$ *and* $\xi_\Theta(0) \leq \xi_\Theta(\imath)$, *for any* $\imath \in \Xi$.
(ii) $\varpi_\Theta(\jmath * \imath) \geq \min\{\varpi_\Theta((\imath * \ell) * (0 * \jmath)), \varpi_\Theta(\ell)\}$ and $\xi_\Theta(\jmath * \imath) \leq \max\{\xi_\Theta((\imath * \ell) * (0 * \jmath)), \xi_\Theta(\ell)\}$, for any $\imath, \jmath, \ell \in \Xi$.

Hence $\Theta = \Theta = \{(\varpi_\Theta(\imath), \xi_\Theta(\imath)) | \imath \in \Omega\}$ is an Ξ-$IF_\alpha I$ of Ω.
It is eminent that any Ξ-$IF_\alpha I$ of Ω is an Ξ-*IFI* of Ω.

Definition 5.4: Let (Γ, ς) be a $S_{BCI}A$ over Ω. An *IFSS* (Υ, τ) over Ω is an *IFSI* of (Γ, ς), symbolised as (Υ, τ)$\tilde{\blacktriangle}_\alpha$($\Gamma$, ς), if $\tau \subset \varsigma$ and for any $\delta \in \tau$,
$\Upsilon[\delta] = \{(\varpi_{\Upsilon[\delta]}(\imath), \xi_{\Upsilon[\delta]}(\imath)) | \imath \in \Omega\} \blacktriangle \Gamma[\delta]$

Definition 5.5: Let (Γ, ς) be a $S_{BCI}A$ over Ω. An *IFSS* (Υ, τ) over Ω is an $IFS_\alpha I$ of (Γ, ς), symbolised as (Υ, τ) $\tilde{\blacktriangle}_\alpha$($\Gamma$, ς), if $\tau \subset \varsigma$ and for any $\delta \in \tau$,
$\Upsilon\,[\delta] = \{(\varpi_{\Upsilon[\delta]}(\imath), \xi_{\Upsilon[\delta]}(\imath)) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\delta]$

Below, we discuss an example to explore the above definition.

Example 5.6: Cogitate the BCI-algebra (Ω, $\otimes$, *Rose*) defined in Example 4.12, where Ω = {*Rose, Tulip, Sunflower, Camation, Lily*}. Let ς = {*Lavender, Pink, Golden, Purple*} be a set of characteristics of members of Ω. Let (Γ, ς) be a soft set over Ω. Then

Γ[Lavender] = Γ[Pink] = {Rose, Sunflower, Camation, Lily}
Γ[Golden] = {Rose, Sunflower}
Γ[Purple] = {Rose, Lily}

that are all subalgebras of Ω. Hence (Γ, ς) is a 'soft BCI-algebra' over Ω. Let (Υ, τ) be an *IFSS* over Ω, where τ = {*Lavender, Pink*} $\subset \varsigma$. Then Υ[*Lavender*] and Υ[*inflential*] are *IFSs* in Ω delineated as:

Υ	Rose	*Tulip*	*Sunflower*	*Camation*	*Lily*
Lavender	(0.8, 0.1)	(0.8, 0.1)	(0.8, 0.1)	(0.2, 0.3)	(0.2, 0.3)
Pink	(0.6, 0.2)	(0.6, 0.2)	(0.6, 0.2)	(0.3, 0.4)	(0.3, 0.4)

Then

$$\Upsilon[Lavender] = \{(\varpi_{\Upsilon[Lavender]}(\imath), \xi_{\Upsilon[Lavender]}(\imath))|\imath \in \Omega\}$$

and

$$\Upsilon[Pink] = \{(\varpi_{\Upsilon[Pink]}(\imath), \xi_{\Upsilon[Pink]}(\imath))|\imath \in \Omega\}$$

are $IFS_\alpha Is$ of Ω related to $\Gamma[Lavender]$ and $\Gamma[Pink]$ respectively. Hence $(\Upsilon, \tau)\ \tilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Any $IFS_\alpha I$ (Υ, τ) of a $S_{BCI}A$ (Γ, ς) is an *IFSI* of (Γ, ς) but the converse isn't true, as can be perceived by the succeeding example.

Example 5.7: Assume $\Omega = \{Rose, Tulip, Sunflower, Camation, Lily\}$ is a universe set. Let '$\circledast$' be an operator which operates on the elements of Ω accordingly as;

Rose $\circledast\ \imath$ = Rose, for all $\imath \in \Omega$.

$$Tulip \circledast \imath = \begin{cases} Tulip, & if\ \imath = Rose \\ Rose, & if\ \imath \in \{Tulip, Sunflower, Camation, Lily\} \end{cases}$$

$$Sunflower \circledast \imath = \begin{cases} Sunflower, & if\ \imath \in \{Rose, Tulip, Camation\} \\ Rose, & if\ \imath \in \{Sunflower, Lily\} \end{cases}$$

$$Camation \circledast \imath = \begin{cases} Camation, & if\ \imath \in \{Rose, Tulip, Sunflower\} \\ Rose, & if\ \imath \in \{Camation, Lily\} \end{cases}$$

$$Lily \circledast \imath = \begin{cases} Lily, & if\ \imath \in \{Rose, Tulip\} \\ Rose, & if\ \imath = Lily \\ Camation, & if\ \imath = Sunflower \\ Sunflower, & if\ \imath = Camation \end{cases}$$

Then $(\Omega, \circledast, Rose)$ is a 'BCK-algebra' and thus a 'BCI-algebra'.

Let $\varsigma = \{Lavender, Pink, Gold, Purple, Yellow\}$ be a set of different types of colours in which the flowers in Ω exist in nature.

Let (Γ, ς) be a soft set over Ω. Then $\Gamma[Lavender] = \Omega$,

Γ[Pink] = Γ[Gold] = {Rose, Sunflower, Camation, Lily} and

Γ[Purple] = Γ[Yellow] = {Rose, Sunflower}

that are all subalgebras of Ω. Hence (Γ, ς) is a $S_{BCI}A$ over Ω.

Suppose that (Υ, τ), where $\tau = \{Gold, Purple, Yellow\} \subset \varsigma$ is an *IFSS* over Ω. Then $\Upsilon[Gold]$, $\Gamma[Purple]$ and $\Upsilon\ [Yellow]$ are *IFSs* in Ω delineated as:

Υ	Rose	*Tulip*	*Sunflower*	*Camation*	*Lily*
Gold	(0.8, 0.1)	(0.7, 0.2)	(0.6, 0.3)	(0.4, 0.4)	(0.4, 0.4)
Purple	(0.7, 0)	(0.6, 0.1)	(0.5, 0.2)	(0.3, 0.3)	(0.3, 0.3)
Yellow	(0.6, 0.2)	(0.5, 0.3)	(0.4, 0.4)	(0.2, 0.5)	(0.2, 0.5)

Then (Υ, τ) is an *IFSI* of (Γ, ς) but since

$$\varpi_{\Upsilon[Gold]}(Camation \circledast Sunflower)$$
$$= \varpi_{\Upsilon[Gold]}(Camation)$$

$$
\begin{aligned}
&= 0.4\\
&< 0.6\\
&= \min\{\varpi_{\Upsilon[Gold]}((Sunflower \circledast Rose) \circledast (Rose \circledast Camation)), \varpi_{\Upsilon[Gold]}(Rose)\}\\
&= \xi_{\Upsilon[Gold]}(Camation \circledast Sunflower)\\
&= \xi_{\Upsilon[Gold]}(Camation)\\
&= 0.4\\
&> 0.3\\
&= \max\{\xi_{\Upsilon[Gold]}((Sunflower \circledast Rose) \circledast (Rose \circledast Camation)), \xi_{\Upsilon[Gold]}(Rose)\}
\end{aligned}
$$

i.e. $\Upsilon\,[Gold] = \{(\varpi_{\Upsilon[Gold]}(\imath), \xi_{\Upsilon[Gold]}(\imath)) | \imath \in \Omega\}$ is not an $IF_\alpha I$ of Ω related to $\Gamma[Gold]$. Therefore (Υ, τ) is not an $IFS_\alpha I$ of $S_{BCI}A$ (Γ, ς).

Theorem 5.8: *Let (Γ, ς) be a $S_{BCI}A$ over Ω. If (Υ, τ) and (Π, q) are $IFS_\alpha Is$ of (Γ, ς), then, the 'extended intersection' of (Υ, τ) and (Π, ϱ) is an $IFS_\alpha I$ of (Γ, ς).*

Proof: We know that the 'extended intersection' of (Υ, τ) and (Π, q), denoted by $(\Upsilon, \tau) \sqcap_E (\Pi, \varrho)$, can be defined as, $(\Upsilon, \tau) \sqcap_E (\Pi, \varrho) = (\Xi, \zeta)$, where $\zeta = \tau \cup \varrho \subset \varsigma$ and for any $\wp \in \zeta$,

$$
\Xi[\wp] = \begin{cases}
\Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) | \imath \in \Omega\} & \text{if } \wp \in \tau - \varrho\\
\Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) | \imath \in \Omega\} & \text{if } \wp \in \varrho - \tau\\
\Upsilon[\wp] \cap \Pi[\wp] = \{(\min\{\varpi_{\Upsilon[\wp]}(\imath), \varpi_{\Pi[\wp]}(\imath)\}, & \\
\max\{\xi_{\Upsilon[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)\}) | \imath \in \Omega\} & \text{if } \wp \in \tau \cap \varrho
\end{cases}
$$

For any $\wp \in \zeta$ if $\wp \in \tau - \varrho$, then $\Xi[\wp] = \Upsilon\,[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\wp]$, since $(\Upsilon, \tau)\ \tilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

Similarly, if $\wp \in \varrho - \tau$, then $\Xi[\wp] = \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\wp]$, since $(\Pi, \varrho)\ \tilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

Moreover if $\wp \in \zeta$ such that $\wp \in \tau \cap \varrho$, then, $\Xi[\wp] = \Upsilon[\wp] \cap \Pi[\wp] = \{(\min\{\varpi_{\Upsilon[\wp]}(\imath), \varpi_{\Pi[\wp]}(\imath)\},$

$\max\{\xi_{\Upsilon[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)\}) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\wp]$, since the intersection of two $IF_\alpha Is$ is an $IF_\alpha I$.

Hence $\Xi[\wp] \blacktriangle_\alpha \Gamma[\wp]$ for any $\wp \in \zeta$. Hence $(\Xi, \zeta) = (\Upsilon, \tau) \sqcap_E (\Pi, \varrho)\ \tilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

It is easy to extract the following corollaries from the above result.

Corollary 5.9: *If (Υ, τ) and (Π, τ) are $IFS_\alpha Is$ of a $S_{BCI}A$ (Γ, ς), then the 'extended intersection' of (Υ, τ) and (Π, τ) is an $IFS_\alpha I$ of (Γ, ς).*

Corollary 5.10: *The 'restricted intersection' of two $IFS_\alpha Is$ (Υ, τ) and (Π, q) of a $S_{BCI}A$ (Γ, ς) is an $IFS_\alpha I$ of (Γ, ς).*

Theorem 5.11: *Let (Υ, τ) and (Π, q) be two $IFS_\alpha Is$ of a $S_{BCI}A$ (Γ, ς). If $\tau \cap q = \varphi$ then the 'union', $(\Upsilon, \tau)\ \tilde{\cup}\ (\Pi, q)$ is an $IFS_\alpha I$ of (Γ, ς).*

Proof: We know that the 'union' of (Υ, τ) and (Π, q), denoted by $(\Upsilon, \tau)\,\tilde{\cup}\,(\Pi, q)$, can be defined as, $(\Upsilon, \tau)\,\tilde{\cup}\,(\Pi, q) = (\Xi, \zeta)$, where $\zeta = \tau \cup \rho \subset \varsigma$ and for any $\wp \in \zeta$,

$$\Xi[\wp] = \begin{cases} \Upsilon[\wp] = \{(\varpi_{\Upsilon[\wp]}(\imath), \xi_{\Upsilon[\wp]}(\imath)) | \imath \in \Omega\} & \textit{if } \wp \in \tau - \varrho \\ \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) | \imath \in \Omega\} & \textit{if } \wp \in \varrho - \tau \\ \Upsilon[\wp] \cup \Pi[\wp] = \{(\max\{\varpi_{\Upsilon[\wp]}(\imath), \varpi_{\Pi[\wp]}(\imath)\}, & \\ \min\{\xi_{\Upsilon[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)\}) | \imath \in \Omega\} & \textit{if } \wp \in \tau \cap \varrho \end{cases}$$

Since $\tau \cup \rho = \varphi$, either $\wp \in \tau - \varrho$ or $\wp \in \varrho - \tau$ for all $\wp \in \zeta$. If $\wp \in \tau - \varrho$ then

$\Xi[\wp] = \Upsilon[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) | \imath \in \Omega\} \blacktriangle \Gamma[\wp]$

since $(\Upsilon, \tau)\ \tilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$. If $\wp \in \varrho - \tau$ then

$\Xi[\wp] = \Pi[\wp] = \{(\varpi_{\Pi[\wp]}(\imath), \xi_{\Pi[\wp]}(\imath)) | \imath \in \Omega\} \blacktriangle \Gamma[\wp]$

since $(\Pi, \varrho)\ \tilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$. Hence $(\Xi, \zeta) = ((\Upsilon, \tau)\tilde{\cup}(\Pi, \varrho))\tilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Below, we discuss the case when we have a non-empty intersection of the sets of parameters (i.e. $\tau \cap \varrho \neq \varphi$). ■

Example 5.12: Cogitate the BCI-algebra (Ω, *Rose*) defined in Example 4.12. Suppose that ς = {*Lavender, Pink, Purple, Yellow*} is a set of parameters relevant to the universe set Ω. Let (Γ, ς) be a soft set over Ω. Then

Γ[*Lavender*] = Ω

Γ[Pink] = Γ[Purple] = {Rose, Sunflower, Camation, Lily}

Γ[Y ellow] = {Rose, Sunflower}

that are all subalgebras of Ω. Hence (Γ, ς) is a 'soft BCI-algebra' over Ω.

Let τ = {*Lavender, Pink, Purple*} $\subset \varsigma$ and ϱ = {*Purple, Yellow*} $\subset \varsigma$ be two sets of characteristics of the flowers given in Ω. Let (Υ, τ) be an *IFSS* over Ω. Then Υ [*Lavender*], Υ[*Pink*] and Υ [*Purple*] are *IFSs* in Ω delineated as:

Υ	Rose	*Tulip*	*Sunflower*	*Camation*	*Lily*
Lavender	(0.7, 0.1)	(0.7, 0.1)	(0.4, 0.2)	(0.2, 0.4)	(0.2, 0.4)
Pink	(0.8, 0)	(0.8, 0)	(0.2, 0.5)	(0.3, 0.2)	(0.2, 0.5)
Purple	(0.5, 0.1)	(0.5, 0.1)	(0.2, 0.3)	(0.2, 0.3)	(0.4, 0.2)

Then

$$\Upsilon[\textit{Lavender}] = \{(\varpi_{\Upsilon[\textit{Lavender}]}(\imath), \xi_{\Upsilon[\textit{Lavender}]}(\imath)) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\textit{Lavender}],$$

$$\Upsilon[\textit{Pink}] = \{(\varpi_{\Upsilon[\textit{Pink}]}(\imath), \xi_{\Upsilon[\textit{Pink}]}(\imath)) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\textit{Pink}],$$

$$\Upsilon[\textit{Purple}] = \{(\varpi_{\Upsilon[\textit{Purple}]}(\imath), \xi_{\Upsilon[\textit{Purple}]}(\imath)) | \imath \in \Omega\} \blacktriangle_\alpha \Gamma[\textit{Purple}]$$

Hence $(\Upsilon, \tau)\ \tilde{\blacktriangle}_\alpha (\Gamma, \varsigma)$.

Let (Π, ϱ) be an *IFSS* over Ω. Then Π[*Purple*] and Π[*ellow*] are *IFSs* in Ω delineated as:

Π	*Rose*	*Tulip*	*Sunflower*	*Camation*	*Lily*
Purple	(0.8, 0.1)	(0.8, 0.1)	(0.5, 0.2)	(0.3, 0.4)	(0.3, 0.4)
Yellow	(0.7, 0.2)	(0.7, 0.2)	(0.4, 0.5)	(0.4, 0.5)	(0.6, 0.3)

Then

$$\Pi[Purple] = \{(\varpi_{\Pi[Purple]}(\imath), \xi_{\Pi[Purple]}(\imath))|\imath \in \Omega\}\blacktriangle_\alpha\Gamma[Purple]$$
$$\Pi[Yellow] = \{(\varpi_{\Pi[Yellow]}(\imath), \xi_{\Pi[Yellow]}(\imath))|\imath \in \Omega\}\blacktriangle_\alpha\Gamma[Yellow]$$

Thus $(\Pi, \tau)\ \tilde{\blacktriangle}_\alpha(\Gamma, \varsigma)$.

Now we consider the union of (Υ, τ) and (Π, q), i.e. $(\Upsilon, \tau)\ \tilde{\cup}\ (\Pi, q) = (\Xi, \zeta)$, where $\zeta = \tau \cup q$.

Note that for any parameter like, Purple $\in \tau \cap q$,

$$\begin{aligned}\Xi[Purple] &= \Upsilon[Purple] \cup \Pi[Purple]\\ &= \{(\varpi_{\Upsilon[Purple]\cup\Pi[Purple]}(\imath), \xi_{\Upsilon[Purple]\cup\Pi[Purple]}(\imath))|\imath \in \Omega\}\\ &= \{(\max\{\varpi_{\Upsilon[Purple]}(\imath), \varpi_{\Pi[Purple]}(\imath)\}, \min\{\xi_{\Upsilon[Purple]}(\imath), \xi_{\Pi[Purple]}(\imath)\})|\imath \in \Omega\}\end{aligned}$$

Since

$$\begin{aligned}&\varpi_{\Xi[Purple]}(Lily \otimes Sunflower)\\ &= \varpi_{\Xi[Purple]}(Camation)\\ &= \varpi_{(\Upsilon[Purple]\cup\Pi[Purple])}(Camation)\\ &= \max\{\varpi_{\Upsilon[Purple]}(Camation), \varpi_{\Pi[Purple]}(Camation)\}\\ &= \max\{0.2, 0.3\}\\ &= 0.3\\ &< 0.4\\ &= \min\{\varpi_{\Xi[Purple]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Lily)),\\ &\quad \varpi_{\Xi[Purple]}(Sunflower)\}\\ &= \min\{\varpi_{\Xi[Purple]}(Lily), \varpi_{\Xi[Purple]}(Sunflower)\}\\ &= \min\{\varpi_{(\Upsilon[Purple]\cup\Pi[Purple])}(Lily), \varpi_{(\Upsilon[Purple]\cup\Pi[Purple])}(Sunflower)\}\\ &= \min\{\max\{\varpi_{\Upsilon[Purple]}(Lily), \varpi_{\Pi[Purple]}(Lily)\},\\ &\quad \max\{\varpi_{\Upsilon[Purple]}(Sunflower), \varpi_{\Pi[Purple]}(Sunflower)\}\}\\ &= \min\{\max\{0.4, 0.3\}, \max\{0.2, 0.5\}\}\\ &= \min\{0.4, 0.5\}\end{aligned}$$

and

$$\begin{aligned}&\xi_{\Xi[Purple]}(Lily \otimes Sunflower)\\ &= \xi_{\Xi[Purple]}(Camation)\\ &= \xi_{(\Upsilon[Purple]\cup\Pi[Purple])}(Camation)\\ &= \min\{\xi_{\Upsilon[Purple]}(Camation), \xi_{\Pi[Purple]}(Camation)\}\\ &= \min\{0.3, 0.4\}\end{aligned}$$

$= 0.3$

> 0.2

$= \max\{\xi_{\Xi[Purple]}((Sunflower \otimes Sunflower) \otimes (Rose \otimes Lily)),$

$\xi_{\Xi[Purple]}(Sunflower)\}$

$= \max\{\xi_{\Xi[Purple]}(Lily), \xi_{\Xi[Purple]}(Sunflower)\}$

$= \max\{\xi_{(\Upsilon[Purple] \cup \Pi[Purple])}(Lily), \xi_{(\Upsilon[Purple] \cup \Pi[Purple])}(Sunflower)\}$

$= \max\{\min\{\xi_{\Upsilon[Purple]}(Lily), \xi_{\Pi[Purple]}(Lily)\},$

$\min\{\xi_{\Upsilon[Purple]}(Sunflower), \xi_{\Pi[Purple]}(Sunflower)\}\}$

$= \max\{\min\{0.2, 0.4\}$

$= \min\{0.3, 0.2\}\}$

$= \max\{0.2, 0.2\}$

i.e. $\Xi[Purple] = \Upsilon\,[Purple] \cup \Pi[Purple]$ is not an $IF_\alpha I$ of Ω related to $\Gamma[Purple]$. Therefore, $(\Xi, \zeta) = (\Upsilon, \tau)\, \tilde{\cup}\, (\Pi, \varrho)$ is not an $IFS_\alpha I$ of $S_{BCI}A$ (Γ, ς).

Now, we confer the characterisation of an $IFS_\alpha I$ (Γ, ς) over Ω using the idea of a soft (δ, η)-level set, $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\}$, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$.

Theorem 5.13: *An IFSS (Γ, ς) over Ω is an $IFS_\alpha I$ over Ω if and only if soft (δ, η)-level set, $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$ is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$.*

Proof: Let (Γ, ς) be an $IFS_\alpha I$ over Ω. Then $\Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\Im), \xi_{\Gamma[\wp]}(\Im)) | \Im \in \Omega\}$ is an $IF_\alpha I$ of Ω, for any parameter $\wp \in \varsigma$.

Let $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ *and* $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$. Then for any $\imath \in L(\Gamma[\wp]; \delta; \eta)$, $\varpi_{\Gamma[\wp]}(0) \geq \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(0) \leq \xi_{\Gamma[\wp]}(\imath) \leq \eta$, i.e. $0 \in L(\Gamma[\wp]; \delta; \eta)$.

Let $(\imath * \ell) * (0 * \jmath) \in L(\Gamma[\wp]; \delta; \eta)$ and $A \in L(\Gamma[\wp]; \delta; \eta)$, for any $\imath, \jmath, \ell \in \Omega$. Then,

$$\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)) \geq \delta, \varpi_{\Gamma[\wp]}(\ell) \geq \delta,$$

and

$$\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)) \leq \eta, \xi_{\Gamma[\wp]}(\ell) \leq \eta$$

Thus for any $\imath, \jmath, \ell \in \Omega$,

$$\varpi_{\Gamma[\wp]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\} \geq \delta,$$

and

$$\xi_{\Gamma[\wp]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\} \leq \eta$$

i.e. $\jmath * \imath \in L(\Gamma[\wp]; \delta; \eta)$. Hence $L(\Gamma[\wp]; \delta; \eta) \neq \varnothing$ is an 'α-ideal' of Ω for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$.

Conversely assume that $L(\Gamma[\wp]; \delta; \eta)$ is an 'α-ideal' of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$. If for some $\imath_\circ \in \Omega$ and $\wp_\circ \in \varsigma$, $\alpha_{\Gamma[\wp^\circ]}(0) < \alpha_{\Gamma[\wp^\circ]}(\imath_\circ)$ and $\alpha_{\Gamma[\wp^\circ]}(0) > \alpha_{\Gamma[\wp^\circ]}(\imath_\circ)$, then $\Gamma[\wp_\circ](0) < \delta_\circ \leq \Gamma[\wp_\circ](\imath_\circ)$ and $\Gamma[\wp_\circ](0) > \eta_\circ \geq \Gamma[\wp_\circ](\imath_\circ)$, for some $\delta_\circ, \eta_\circ \in [0, 1]$. This implies that $\imath_\circ \in L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$ but $0 \notin L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$, a contradiction. Thus $\varpi_{\Gamma[\wp]}(0) \geq \varpi_{\Gamma[\wp]}(\imath)$ and $\xi_{\Gamma[\wp]}(0) \leq \xi_{\Gamma[\wp]}(\imath)$, for any $\wp \in \varsigma$ and $\imath \in \Omega$.

Moreover if there are elements $\imath_\circ, \jmath_\circ, \ell_\circ \in \Omega$ and $\wp_\circ \in \varsigma$ such that,

$$\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) < \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\},$$

and

$$\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) > \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$$

Then for some $\delta_\circ, \eta_\circ \in [0, 1]$,

$$\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) < \delta_\circ \leq \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\},$$

and

$$\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) > \eta_\circ \geq \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$$

i.e. $(\imath_\circ * \ell_\circ) * (0 * \jmath_\circ) \in L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ)$ and $\ell_\circ \in L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ)$ but $\jmath_\circ * \imath_\circ \notin L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ)$, again contradicts the hypothesis that $L(\Gamma[\wp_\circ]; \delta_\circ, \eta_\circ) \neq \emptyset$ is an 'α-ideal' of Ω. Thus for any $\imath, \jmath, \ell \in \Omega$ and any $\wp \in \varsigma$,

$$\varpi_{\Gamma[\wp]}(\jmath * \imath) \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\},$$

and

$$\xi_{\Gamma[\wp]}(\jmath * \imath) \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\}$$

i.e. $\Gamma[\wp] = \{(\varpi_{\Gamma[\wp]}(\Im), \xi_{\Gamma[\wp]}(\Im)) | \Im \in \Omega\}$ is an $IF_\alpha I$ of Ω for any $\wp \in \varsigma$. Hence (Γ, ς) is an $IFS_\alpha I$ over Ω.

From the above statement the following corollary is evident. ■

Corollary 5.14: *An IFSS (Γ, ς) over Ω is an $IFS_\alpha I$ over Ω if and only if the soft (δ,η)-level set et $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$ is an $\alpha -$ ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$.*

Theorem 5.15: *A soft (δ, η)-level set*

$$L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta \text{ and } \xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$$

is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$ if and only if the following conditions are valid: for any $\wp \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$,

(i) $\max\{\varpi_{\Gamma[\wp]}(0), 0.5\} \geq \varpi_{\Gamma[\wp]}(\imath)$ *and* $\max\{\xi_{\Gamma[\wp]}(0), 0.5\} \leq \xi_{\Gamma[\wp]}(\imath)$

(ii) $\max\{\varpi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\}$ *and* $\max\{\xi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\}$

Proof: Let the soft (δ, η)-level set $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ and $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an '$\alpha$-ideal' of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$. If for some $\imath_\circ \in \Omega$ and $\wp_\circ \in \varsigma$.

$$\max\{\varpi_{\Gamma[\wp_\circ]}(0), 0.5\} < \varpi_{\Gamma[\wp_\circ]}(\imath_\circ) \text{ and } \max\{\xi_{\Gamma[\wp_\circ]}(0), 0.5\} > \xi_{\Gamma[\wp_\circ]}(\imath_\circ)$$

Then there are $\delta_\circ, \eta_\circ \in (0.5, 1]$ such that,

$$\max\{\varpi_{\Gamma[\wp_\circ]}(0), 0.5\} < \delta_\circ \leq \varpi_{\Gamma[\wp_\circ]}(\imath_\circ) \text{ and } \max\{\xi_{\Gamma[\wp_\circ]}(0), 0.5\} > \eta_\circ \geq \xi_{\Gamma[\wp_\circ]}(\imath_\circ)$$

This implies

$$\varpi_{\Gamma[\wp_\circ]}(0) < \delta_\circ \leq \varpi_{\Gamma[\wp_\circ]}(\imath_\circ) \text{ and } \xi_{\Gamma[\wp_\circ]}(0) > \eta_\circ \geq \xi_{\Gamma[\wp_\circ]}(\imath_\circ)$$

i.e. $\imath_\circ \in L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$ but $0 \in / L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$, a contradiction. Thus (i) is valid.

Moreover if for some $\imath_\circ, \jmath_\circ, \ell_\circ \in \Omega$ and $\wp_\circ \in \varsigma$,

$$\max\{\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ), 0.5\} < \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$$

and

$$\max\{\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ), 0.5\} > \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$$

Then for some $\delta_\circ, \eta_\circ \in (0.5, 1]$,

$$\max\{\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ), 0.5\} < \delta_\circ \leq \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\},$$

and

$$\max\{\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ), 0.5\} > \eta_\circ \geq \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$$

i.e. $\varpi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) < \delta_\circ \leq \min\{\varpi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \varpi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$,

and

$$\xi_{\Gamma[\wp_\circ]}(\jmath_\circ * \imath_\circ) > \eta_\circ \geq \max\{\xi_{\Gamma[\wp_\circ]}((\imath_\circ * \ell_\circ) * (0 * \jmath_\circ)), \xi_{\Gamma[\wp_\circ]}(\ell_\circ)\}$$

i.e. $(\imath_\circ * \ell_\circ) * (0 * \jmath_\circ) \in L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$ and $\ell_\circ \in L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$ *but* $\jmath_\circ * \imath_\circ \notin L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ)$, which contradicts the hypothesis that $L(\Gamma[\wp_\circ]; \delta_\circ; \eta_\circ) \neq \emptyset$ is an α-ideal of Ω, for any $\wp_\circ \in \varsigma$ and

$\delta_\circ, \eta_\circ \in (0.5, 1]$. Hence (ii) is valid.

Conversely, suppose that (*i*) and (*ii*) are valid. Let $L(\Gamma[\wp]; \delta; \eta) \neq \varnothing$, for any $\wp \in \varsigma$ and δ, $\eta \in (0.5, 1]$. Then for any $\imath \in L(\Gamma[\wp]; \delta; \eta)$,

$$\max\{\varpi_{\Gamma[\wp]}(0), 0.5\} \geq \varpi_{\Gamma[\wp]}(\imath) \geq \delta > 0.5 \text{ and } \max\{\xi_{\Gamma[\wp]}(0), 0.5\} \leq \xi_{\Gamma[\wp]}(\imath) \leq \eta$$

which implies $\varpi_{\Gamma[\wp]}(0) \geq \delta$ and $\xi_{\Gamma[\wp]}(0) \leq \eta$ and thus $0 \in L(\Gamma[\wp]; \delta; \eta)$.

Let $(\imath * \ell) * (0 * \jmath) \in L(\Gamma[\wp]; \delta; \eta))$ and $\ell \in L(\Gamma[\wp]; \delta; \eta)$, for any $\imath, \jmath, \ell \in \Omega$. Then

$$\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)) \geq \delta, \varpi_{\Gamma[\wp]}(\ell) \geq \delta \text{ and } \xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)) \leq \eta, \xi_{\Gamma[\wp]}(\ell) \leq \eta$$

Thus from (*ii*) we get,

$$\max\{\varpi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\} \geq \delta > 0.5$$

and

$$\max\{\xi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\} \leq \eta$$

This implies, $\varpi_{\Gamma[\wp]}(\jmath * \imath) \geq \delta$ *and* $\xi_{\Gamma[\wp]}(\jmath * \imath) \leq \eta$. Thus $\jmath * \imath \in L(\Gamma[\wp]; \delta; \eta)$. Therefore $L(\Gamma[\wp]; \delta; \eta) \neq \varnothing$ is an 'α-ideal' of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in (0.5, 1]$. ■

6. Conclusion

By extending the study of applications of soft sets in α-ideals of BCI-algebras, we have defined the idea of 'intuitionistic fuzzy soft α-ideals' ($IFS_\alpha Is$) and proved their basic properties. In chapter 5, initially the properties of 'intuitionistic fuzzy soft BCI-algebras' ($IFS_{BCI}As$) and 'intuitionistic fuzzy soft ideals' (*IFSIs*) have been described with the help of concrete examples. We proved that any *IFSI* of a BCK-algebra is an 'intuitionistic fuzzy soft BCK-algebra' ($IFS_{BCK}A$). Afterwards we have proceeded towards the detail discussion of $IFS_\alpha Is$. $IFS_\alpha Is$ are related with *IFSIs* and various characterisations are discussed. Useful facts have been explored on various operations on intuitionistic fuzzy soft α-ideals. For instance, it has been proved that the 'AND' operation, extended intersection and restricted intersection of two $IFS_\alpha Is$ is an $IFS_\alpha I$.

The union of two $IFS_\alpha Is$ is an $IFS_\alpha I$ if the intersection of the sets of parameters is empty.

$IFS_\alpha I$ of a 'soft BCI-algebra' ($S_{BCI}A$ has been defined and apposite properties have been explored. We have proved that any $IFS_\alpha I$ (Υ, τ) of a $S_{BCI}A$ (Γ, ς) is an *IFSI* of (Γ, ς). If (Γ, ς) is a $S_{BCI}A$ over Ω and (Υ, τ) and (Π, ϱ) are $IFS_\alpha Is$ of (Γ, ς), then, the extended intersection of (Υ, τ) and (Π, ϱ) is an $IFS_\alpha I$ of (Γ, ς). Also, if (Υ, τ) and (Π, ϱ) be two $IFS_\alpha Is$ of a $S_{BCI}A$ (Γ, ς) and $\tau \cap \varrho = \varphi$, then the union, $(\Upsilon, \tau)\tilde{\cup}(\Pi, \varrho)$ is an $IFS_\alpha I$ of (Γ, ς). Lastly we have characterised $IFS_\alpha Is$ by a soft (δ, η)-level set. It has been proved that an intuitionistic fuzzy soft set (Γ, ς) over a BCI-algebra Ω is an $IFS_\alpha I$ over Ω if and only if the soft (δ, η)-level set, $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ *and* $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an α-ideal of Ω, for any $\wp \in \varsigma$ and $\delta, \eta \in [0, 1]$. Moreover, a soft (δ, η)-level set $L(\Gamma[\wp]; \delta; \eta) = \{\imath \in \Omega | \varpi_{\Gamma[\wp]}(\imath) \geq \delta$ *and* $\xi_{\Gamma[\wp]}(\imath) \leq \eta\} \neq \emptyset$, is an α-ideal of Ω, for any $\wp \in \varsigma$ *and* $\delta, \eta \in (0.5, 1]$ if and only if the following conditions are valid:

(*i*) $\max\{\varpi_{\Gamma[\wp]}(0), 0.5\} \geq \varpi_{\Gamma[\wp]}(\imath)$ *and* $\max\{\xi_{\Gamma[\wp]}(0), 0.5\} \leq \xi_{\Gamma[\wp]}(\imath)$\$.
(*ii*) $\max\{\varpi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \geq \min\{\varpi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \varpi_{\Gamma[\wp]}(\ell)\}$
and $\max\{\xi_{\Gamma[\wp]}(\jmath * \imath), 0.5\} \leq \max\{\xi_{\Gamma[\wp]}((\imath * \ell) * (0 * \jmath)), \xi_{\Gamma[\wp]}(\ell)\}$
for any $\wp \in \varsigma$ and $\imath, \jmath, \ell \in \Omega$.

7. Recommendations for Further Study

This study may further pave the way for applying fuzzy sets and intuitionistic fuzzy sets to soft hyper BCK-ideals, soft hyper *p*-ideals, soft hyper *h*-ideals etc. Also some types of intuitionistic fuzzy α-ideals may also be characterised by $\in$-soft sets. Moreover, fuzzy sets and intuitionistic fuzzy sets may be applied to other soft ideals. After this application, the connections between different fuzzy soft ideals and intuitionistic fuzzy soft ideals may be considered.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributor

Muhammad Touqeer did his PhD from University of the Punjab, Lahore in 2015 and starting working as an Assistant Professor of Mathematics in the Department of Basic Sciences, University of Engineering and Technology, Taxila, Pakistan. His field of intersects include Fuzzy sets, Soft sets, Fuzzy decision making and BCK/BCI-algebras.

References

[1] Zadeh LA. Fuzzy sets. Inf Control. 1965;8:338–353.
[2] Molodtsov D. Soft set theory, first results. Comput Math Appl. 1999;37:19–31.
[3] Acar U, Koyuncu F, Tanay B. Soft sets and soft rings. Comput Math Appl. 2010;59:3458–3463.
[4] Neog TJ. An application of imprecise soft sets in teaching evaluation. Int J Basic Sci Soc Sci (IJBSS). 2012;1(2):25–29.
[5] Çelik Y, Yamak S. Fuzzy soft set theory applied to medical diagnosis using fuzzy arithmetic operations. J Inequalities Appl. 2013;1:1–9. Article no. 82.
[6] Qin K, Hong Z. On soft equality. Comput Math Appl. 2010;234:1347–1355.
[7] Ali MA, Feng F, Liu X, et al. On some new operations in soft set theory. Comp Math Appl. 2009;57:1547–1553.
[8] Imai Y, Iseki K. On axioms of propositional calculi. XIV Proc, Jpn Acad. 1966;42:19–22.
[9] Al-Masarwah A, Ahmad AG. m-Polar fuzzy ideals of BCK/BCI-algebras. J King Saud Univ Sci. 2018. doi:10.1016/j.jksus.2018.10.002.
[10] Al-Masarwah A, Ahmad AG. m-Polar (α, β)-fuzzy ideals in BCK/BCI-algebras. Symmetry (Basel). 2019;11(1):44), doi:10.3390/sym11010044.
[11] Al-Masarwah A, Ahmad AG. On (complete) normality of m-pF subalgebras in BCK/BCI-algebras. AIMS Math. 2019;4(3):740–750.
[12] Al-Masarwah A, Ahmad AG. Novel concepts of doubt bipolar fuzzy H-ideals of BCK/BCI-algebras. Int J Innov Comput Inf Control. 2018;14(6):2025–2041.
[13] Al-Masarwah A, Ahmad AG. On some properties of doubt bipolar fuzzy H-ideals in BCK/BCI-algebras. Eur J Pure Appl Math. 2018;11(3):652–670.
[14] Senapati T, Jun YB, Muhiuddin G, et al. Cubic intuitionistic structures applied to ideals of BCI-algebras. An St Univ Ovidius Constanta. 2019;25(2):213–232.
[15] Senapati T, Shum KP. Cubic implicative ideals of BCK-algebras. Missouri J Math Sci. 2017;29(2):125–138.
[16] Jun YB. Soft BCK/BCI-algebras. Comput Math Appl. 2008;56:1408–1413.
[17] Jun YB, Lee KJ, Park CH. Fuzzy soft set theory applied to BCK/BCI-algebras. Comput Math Appl. 2010;59:3180–3192.
[18] Atanassov KT. Intuitionistic fuzzy sets. Fuzzy Set Syst. 1986;20:87–96.
[19] Hayat K, Liu XC, Cao BY. On intuitionistic fuzzy filters of filteristic Soft BE-algebras. International workshop on Mathematics and Decision Science 2016 Sep 12 (pp. 260-265). Springer, Cham.
[20] Hayat K, Ali M, Cao BY, et al. Another view of aggregation operators on group-based generalized intuitionistic fuzzy soft sets: multi-attribute decision making methods. Symmetry (Basel). 2018 Dec;10(12):1–26. Article no. 753.
[21] Abu Qamar M, Hassan N. An approach toward a Q-neutrosophic soft set and its application in decision making. Symmetry (Basel). 2019;11(2):1–17. Article no. 139.
[22] Maji PK, Biswas AR, Roy AR. Soft set theory. Comput Math Appl. 2003;45:555–562.
[23] Maji PK. More on intuitionistic fuzzy soft sets. In: Sakai H, Chakraborty MK, Hassanien AE, Slezak D, Zhu W, editors. Proceedings of the 12th international conference on rough sets, fuzzy sets, data mining and granular computing (RSFDGrC 2009). Lecture Notes in Computer Science, 5908, Springer; 2009, p. 231240.
[24] Jun YB, Kim KH. Intuitionistic fuzzy ideals of BCK-algebras. Int J Math Math Sci. 2000;24(12): 839–849.
[25] Zhan J, Tan Z. Intuitionistic fuzzy α-ideals in BCI-algebras. Soochow J Math. 2004;30(2):207–2116.