METRICS OF POSITIVE RICCI CURVATURE ON CONNECTED SUMS: PROJECTIVE SPACES, PRODUCTS, AND PLUMBINGS

by

BRADLEY LEWIS BURDICK

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2019

DISSERTATION APPROVAL PAGE

Student: Bradley Lewis Burdick

Title: Metrics of Positive Ricci Curvature on Connected Sums: Projective Spaces, Products, and Plumbings

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Boris Botvinnik	Chair
Nicholas Proudfoot	Core Member
Robert Lipshitz	Core Member
Micah Warren	Core Member
Graham Kribs	Institutional Representative

and

Janet Woodruff-Borden	Vice Provost & Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2019

DISSERTATION ABSTRACT

Bradley Lewis Burdick

Doctor of Philosophy

Department of Mathematics

June 2019

Title: Metrics of Positive Ricci Curvature on Connected Sums: Projective Spaces, Products, and Plumbings

The classification of simply connected manifolds admitting metrics of positive scalar curvature of initiated by Gromov-Lawson, at its core, relies on a careful geometric construction that preserves positive scalar curvature under surgery and, in particular, under connected sum. For simply connected manifolds admitting metrics of positive Ricci curvature, it is conjectured that a similar classification should be possible, and, in particular, there is no suspected obstruction to preserving positive Ricci curvature under connected sum. Yet there is no general construction known to take two Ricci-positive Riemannian manifolds and form a Ricci-positive metric on their connected sums. In this work, we utilize and extend Perelman's construction of Ricci-positive metrics on connected sums of complex projective planes, to give an explicit construction of Ricci-positive metrics on connected sums given that the individual summands admit very specific Ricci-positive metrics, which we call core metrics. Working towards the new goal of constructing core metrics on manifolds known to support metrics of positive Ricci curvature: we show how to generalize Perelman's construction to all projective spaces, we show that the existence of core metrics is preserved under iterated sphere bundles, and we construct core metrics on certain boundaries of plumbing disk bundles over spheres. These constructions come together to give many new examples of Ricci-positive connected sums, in particular on the connected sum of arbitrary products of spheres and on exotic projective spaces.

CURRICULUM VITAE

NAME OF AUTHOR: Bradley Lewis Burdick

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
The Ohio State University, Columbus, OH

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2019, University of Oregon
Masters of Science, Mathematics, 2016, University of Oregon
Bachelors of Science, Mathematics, 2014, The Ohio State University
Bachelors of Arts, English, 2014, The Ohio State University

AREAS OF SPECIAL INTEREST:

Riemannian Geometry, Smooth Topology

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon 2016-2019

Graduate Teaching Fellow, University of Oregon, 2014-2016

Math and Physics Tutor, The Ohio State University NROTC, 2013-2014

Math and Statistics Tutor, The Ohio State University MSLC, 2011-2013

PUBLICATIONS:

Burdick, Bradley Lewis. (2019). Ricci-positive metrics on connected sums of projective spaces. *Differential Geometry and its Applications*, 62, 212–233.

Bajo, Carlos; Burdick, Bradley Lewis; and Chmutov, Sergei (2014). On the Tutte-Krushkal-Renardy polynomial for cell complexes. *Journal of Combinatorial Theory, Series A*, 123.1, 168–201.

Burdick, Bradley Lewis and Jonker, Jonathan (2013). Generic Polynomials for Transitive Permutation Groups of Degree 8 and 9. *Rose-Hulman Institute of Technology Undergraduate Math Journal*, 13.1, 21 pages.

Burdick, Bradley Lewis (2018). Ricci-positive metrics on connected sums of products with arbitrarily many spheres. *ArXiv e-prints*, 32 pages.

Burdick, Bradley Lewis (2014). A simplicial Tutte “5”-flow Conjecture. *ArXiv eprints,* 26 pages.

ACKNOWLEDGEMENTS

I would like to thank my academic brother and friend Demetre Kazaras, for listening to arguments from early versions of my work. And for working with me on a failed project, which nonetheless forced me to compute several of the quantities that ended being indispensable in my arguments here.

I am very grateful for the guidance of my advisor Boris Botvinnik. For all the guidance early on in this project, for all the helpful references, and for the very healthy skepticism you provided throughout.

Thank you to David Wraith, Christine Escher, and Lee Kennard for the encouragement and useful technical comments.

To my family, thank you for your patience and love despite how far physically and mentally this work has taken me away from you.

Alicia, you have been here with me for most of this journey. Thank you for keeping me grounded when I got lost in thought, comforting me when things were not going well, and sharing the load when time was short. I truly would not have succeeded without you by my side. My dear, thank you.

To Perelman.

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

Figure Page

CHAPTER I

INTRODUCTION

Curvature is the complete, local obstruction to the triviality of a Riemannian metric, and therefore occupies a central role in the study of the geometry and topology of Riemannian manifolds. The Ricci curvature is the trace of the curvatures in a given direction, and it plays a central role in the analysis of distance functions on Riemannian manifolds. When the curvature is bounded from below, in particular when it is positive, it severely restricts the topology of the underlying smooth manifold. While a lower bound on Ricci curvature implies a great deal about the geometry of the space, it is unclear how restrictive positive Ricci curvature is on the topology. It is suspected that positive Ricci curvature is very flexible, yet there are relatively few examples and relatively few constructions that actually demonstrate this flexibility.

We will discuss this flexibility with the following very narrow question: under what conditions on two Riemannian manifolds with positive Ricci curvature will the connected sum also support a metric of positive Ricci curvature? It is suspected for simply connected manifolds that it should always be possible to form Ricci-positive connected sums. Our main results, Theorem A, B, and C, show that many of the Ricci-positive metrics constructed on the manifolds in [1], [2], and [3] can be modified so that the connected sum of any combination of these manifolds still has positive Ricci curvature.

1.1. Background

To begin we recall the definition of curvature and its traces, including the Ricci and scalar curvatures, and the significance of positive curvature in the study of Riemannian manifolds. In particular, we are interested in highlighting the relationship between the smooth topology of a given manifold and the existence of metrics whose curvature is positive in some sense. After establishing the significance that positive sectional curvature plays in determining the topology of space, we recall in greater detail the use of surgery to study manifolds that admit metrics of positive scalar curvature. From the discussion of positive scalar curvature, we expect the existence of positive Ricci curvature metrics to be preserved under connected sum. We discuss to what extent surgery is known to preserve positive Ricci curvature, and conclude that the construction of Ricci-positive connected sums must follow an entirely different approach.

1.1.1. Positive Curvature

Riemannian geometry is the study of classical geometric quantities (angles, distances, volume, and curvature) in the setting of an arbitrary smooth manifold rather than Euclidean space. The principal object needed to define these quantities is a Riemannian metric, which plays the role of the Euclidean dot product. A Riemannian metric is a smoothly varying inner product on the tangent spaces of a smooth manifold. Given a Riemannian metric, one can define the angle and length of vector fields and consequently allows one to define distance, volume, and curvature. We will denote by the pair (M^n, g) a Riemannian manifold, a smooth manifold with specified Riemannian metric. By the remarkable Nash Embedding Theorem [4], these intrinsic definitions are equivalent to considering embeddings

$\varphi : \mathrm{M}^n \hookrightarrow \mathbf{R}^{n+m}$ where we can define these geometric quantities extrinsically using the ambient Euclidean dot product. Our primary interest will be curvature, and specifically the interaction between curvature and the global topology of the underlying smooth manifold.

As a motivating example, let us consider an oriented surface $\varphi : \Sigma^2 \hookrightarrow \mathbf{R}^3$. We can find a global unit normal vector field ν consistent with the orientation of Σ^2 and $\mathbf{R}^3$. By translating ν to the origin, we define the Gauss map $G : \Sigma^2 \to \mathrm{S}^2$, and define the Gaussian curvature K of Σ^2 as the Jacobian of G. The remarkable Gauss-Bonnet theorem [5, Theorem 9.7] claims that

$$\frac{1}{2\pi} \int_{\Sigma^2} \mathrm{K}\, dA = \chi(\Sigma^2).$$

Such an identity allows us to make global topological observations from local geometric information and vice versa. For example if $\mathrm{K} > 0$, then $0 < \chi(\Sigma^2) = 2 - 2g = 2$ and hence Σ^2 is diffeomorphic to the sphere. In the other direction, if we know the genus of Σ^2 is $g \geq 1$, then $\mathrm{K} < 0$ at some point in Σ^2.

1.1.1.1. Sectional Curvature

Given an arbitrary Riemannian manifold (M^n, g) we may isometrically embed it into $\mathbf{R}^{n+m}$ via [4]. Rather than a single normal vector there is a rank m normal bundle $N\,\mathrm{M}^n$ which is the compliment with respect to the Euclidean dot product of the tangent bundle $T\,\mathrm{M}^n$. For X and Y two vector fields of M^n, in analogy to the definition of Gaussian curvature, we take the derivative of X with respect to Y (in the usual sense for vector fields in $\mathbf{R}^{n+m}$) and use the dot product to project onto the normal bundle $N\,\mathrm{M}^n$. This defines a symmetric bilinear form

$\mathrm{II} : T\,\mathrm{M}^n \otimes T\,\mathrm{M}^n \to N\,\mathrm{M}^n$, called the second fundamental form, which is analogous to the Gauss map. The sectional curvature is now defined for every oriented set of vectors as

$$\mathrm{K}_g(X,Y) = \frac{g(\mathrm{II}(X,X),\mathrm{II}(Y,Y)) - g(\mathrm{II}(X,Y),\mathrm{II}(Y,X))}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

When M^n is a hypersurface and $N\,\mathrm{M}^n$ is rank 1 we can see that we have exactly taken a 2×2 minor of the Gauss map, which therefore agrees with the Gaussian curvature in dimension 2. There is an entirely intrinsic definition of $\mathrm{K}_g(X,Y)$ in terms of the Levi-Civita connection of g, but by Guass' Theorema Egregium [5, Theorem 8.6] the two definitions agree.

When $\mathrm{M}^n \hookrightarrow (\mathrm{N}^{n+m}, g)$ we similarly can define the second fundamental form II. We define $N\,\mathrm{M}^n$ the compliment of $T\,\mathrm{M}^n$ with respect to g. Given two tangent vectors of M^n, use the ambient Levi-Civita connection to differentiate X with respect to Y and then use g to project onto the normal bundle $N\,\mathrm{M}^n$. This too defines a symmetric bilinear form $\mathrm{II} : T\,\mathrm{M}^n \otimes T\,\mathrm{M}^n \to N\,\mathrm{M}^n$. Note that, in the case the sectional curvature of M^n is defined in terms of the sectional curvatures of N^{n+m} and II in terms of the Gauss equation [6, Theorem 3.2.4], rather than the formula above. When M^n is a hypersurface, by convention, we consider $\mathrm{II} : T\,\mathrm{M}^n \otimes T\,\mathrm{M}^n \to \mathbf{R}$ by pairing with the unique oriented unit normal of M^n. In this sense, II is a real bilinear form on $T\,\mathrm{M}^n$. The eigenvalues of this form (which are all real) are called the *principal curvatures* of $\mathrm{M}^n \hookrightarrow (\mathrm{N}^{n+1}, g)$. In particular, when (N^{n+1}, g) is a Riemmanian manifold with boundary we can refer to the principal curvatures of the boundary without ambiguity.

The most essential observation about sectional curvature is the Hopf-Killing Theorem [7, 8], which claims that if $\mathrm{K}_g = \Lambda$ is constant then the universal cover of (M^n, g) is isometric to hyperbolic space if $\Lambda < 0$, Euclidean space if $\Lambda = 0$, or a round sphere if $\Lambda > 0$. A Riemannian manifold (M^n, g) is said to have positive (or nonnegative) sectional curvature if $\mathrm{K}_g(X, Y) > 0$ (or $\mathrm{K}_g(X, Y) \geq 0$). In analogy to the Gauss-Bonnet Theorem for surfaces, we ask the following question: to what extent does having positive sectional curvature imply that a Riemannian manifold is like a sphere. We know by example that there are manifolds that are not diffeomorphic to S^n that still have $\mathrm{K}_g(X, Y) > 0$, for instance $\mathbf{R}\mathrm{P}^n$, $\mathbf{C}\mathrm{P}^n$, and $\mathbf{H}\mathrm{P}^n$ with the Fubini-Study metrics. Yet if there is a $\Lambda > 0$ such that $\Lambda < \mathrm{K}_g(X, Y) \leq 4\Lambda$, then M^n is diffeomorphic to the sphere [9].

The field of comparison geometry rephrases this question in a geometric way: under the assumption that $\mathrm{K}_g > \frac{1}{R^2}$, the sectional curvature of a round sphere, which other aspects of the geometry behave like the geometry of a round sphere? Take for example Bonnet's Theorem [10, Theorem 1.26], it claims that the diameter of (M^n, g) is bounded above by πR, the diameter of the round sphere. Or for example Topogonov's Comparison Theorem, it claims that the distance between geodesics in (M^n, g) are bounded above by distance between great circles in round spheres.

While these comparison theorems do not directly answer our topological question, they turn out to be the main tools needed to deduce topological information from positive sectional curvature. For instance, by applying Bonnet's Theorem to the universal cover of (M^n, g), we deduce that $\pi_1(\mathrm{M}^n)$ must be finite if $\mathrm{K}_g > 0$. Or for instance, the work of [11] uses Topogonov's Comparison theorem along with a very careful counting argument to study the homology of a manifold with nonnegative sectional curvature. The main theorem of [11] claims that there is a universal constant $c(n)$ depending only on the dimension such that $\sum b_i(\mathrm{M}^n) < c(n)$ if (M^n, g) is any complete, connected, and non-negatively curved Riemannian manifold. This theorem produced the first examples of simply connected manifolds that admit no metrics with positive sectional curvature but do admit metrics of positive scalar curvature, for instance $\#_k (\mathrm{S}^n \times \mathrm{S}^m)$ or $\#_k \mathbf{C}\mathrm{P}^n$ for $k > c(n)$.

1.1.1.2. Ricci Curvature

Positive sectional curvature is a very rigid requirement; for instance, it is expected that $\mathrm{S}^n \times \mathrm{S}^m$ admits no positively curved metric (see [12]). To weaken the notion of positive curvature, we define the Ricci curvature as the trace of sectional curvatures

$$\mathrm{Ric}_g(X, X) = \sum_{Y \neq X} \frac{\mathrm{K}_g(X, Y)}{g(Y, Y)},$$

where the sum is over an orthogonal frame including X. The Ricci curvature can expanded to a symmetric bilinear form $\mathrm{Ric}_g(X, Y)$ using the polarization identity. We say the (M^n, g) has positive (or nonnegative) Ricci curvature if Ric_g is positive (or nonnegative) definite. We still are interested in the question: does the assumption $\mathrm{Ric}_g > 0$ imply that a Riemannian manifold is like a sphere, or does it restrict the underlying topology?

A number of results from comparison geometry still hold under the weakened hypothesis $\mathrm{Ric}_g(X, X) \geq (n-1)\frac{1}{R^2}$, which is the Ricci curvature of the round sphere. The reason that a lower bound on Ricci curvature is sufficient is because of its appearance in the Bochner formula [13, Corollary 8.3]:

$$\Delta u = \nabla^* \nabla u + \mathrm{Ric}_g(\nabla u, \nabla u).$$

A lower bound on Ricci curvature produces a bound on the Laplacian of distance functions via the Bochner formula, which in turn is the basis of a number of results. Take for example Myers' Theorem [14], it claims that Bonnet's Theorems holds under this weaker assumption. In particular, Myers' Theorem implies that $\pi_1(\mathrm{M}^n)$ is finite if (M^n, g) has positive Ricci curvature. Or for example, the Cheeger-Gromoll Splitting Theorem [15], which has as a topological corollary that $b_1(\mathrm{M}^n) \leq n$ if (M^n, g) has nonnegative Ricci curvature.

For manifolds with positive Ricci curvature, other than the fundamental group (see [16] for a survey), very little is known about the underlying topology. For example, the main theorem of [11] does not hold under the weakened hypothesis of positive Ricci curvature. The first examples of manifolds with homology of arbitrarily large rank that admit metrics of positive Ricci curvature were constructed in [17, 18], which construct Ricci-positive metrics respectively on $\mathrm{S}^4 \times \mathrm{S}^3$ blown-up along arbitrarily many S^3 and $\#_k(\mathrm{S}^n \times \mathrm{S}^m)$ for all k and $n, m \geq 2$. These manifolds represent the first example of spaces known to admit metrics of positive Ricci curvature but no metric of positive sectional curvature.

Though there are far fewer topological restrictions known for positive Ricci curvature than positive sectional curvature, we also have relatively few examples of manifolds which admit Ricci-positive metrics. Outside of those manifolds with positive sectional curvature, one of the main ways to construct new examples of manifolds with positive Ricci curvature is taking iterated fiber bundles, which was initially observed in [2]. Another source of examples of positive Ricci curvature metrics is Kähler and Sasakian geometry. In particular, the celebrated Calabi Conjecture (proven in [19]) reduces the existence of a Ricci-positive Kähler metric to the positivity of an algebraic invariant. For examples of Ricci-positive metrics from Sasakian geometry see [20, 21].

1.1.1.3. Scalar Curvature

We may weaken the notion of curvature even further by taking the trace of Ricci curvature to define the scalar curvature

$$\mathrm{R}_g = \sum_X \frac{\mathrm{Ric}_g(X,X)}{g(X,X)},$$

where the sum is taken over an orthogonal frame. We say that a Riemannian manifold has positive scalar curvature if $\mathrm{R}_g > 0$ at each point. Even though the assumption that $\mathrm{R}_g > 0$ is not sufficient for most of comparison geometry, it still enters into analysis on spin manifolds via the Lichnerowicz formula [13, Theorem 8.8]:

$$\partial\!\!\!/^2 s = \nabla^*\nabla s + \frac{1}{4}\,\mathrm{R}_g\, s,$$

where $\partial\!\!\!/$ is the spin Dirac operator. From the Lichnerowicz fomrula one deduces that a metric with positive scalar curvature admits no harmonic spinors [22]. This with the Atiyah-Singer index theorem [13, Theorem 13.2] gives rise to algebro-topological invariants $\alpha(\mathrm{M}^n) \in KO^{-n}(*)$ that obstruct the existence of positive scalar curvature metrics on a spin manifold M^n. The vanishing of these obstructions is one of the few topological implications of admitting positive scalar curvature metrics. It is a remarkable result of [23, 24] that, for simply connected spin manifolds, the vanishing of $\alpha(\mathrm{M}^n)$ is also *sufficient* to deduce the existence of a positive scalar curvature metric.

This last fact illustrates that the existence of positive scalar curvature metrics is very flexible: for simply connected manifolds it depends on the value of a single numerical invariant. And, as discussed, the existence of a positive sectional curvature metric is extremely rigid: it is expected that $\mathrm{S}^n \times \mathrm{S}^m$ is not positively curved. While positive Ricci curvature implies a great deal about the geometry of the Riemannian manifold, it is not clear how flexible the underlying topology is. For example, if we strengthen our main question to ask whether or not it is possible to preserve positive sectional curvature under connected sums, we know that there is no general construction because of [11]. Or for example, if we weaken our main question to ask whether or not it is possible to preserve positive scalar curvature under connected sums, we know it is always possible by [25, 23]. Other than the fundamental group [14], there is no known or expected obstruction to Ricci-positive connected sums.

1.1.2. Positive curvature and Surgery

One of our main motivations for considering Ricci-positive connected sums is the work of [25, 23]. The main theorem of [23, Theorem A] is that the existence of a positive scalar curvature metric is invariant under p-surgery for $p \leq n-3$. Given an embedding $\varphi : \mathrm{S}^p \times \mathrm{D}^q \hookrightarrow \mathrm{M}^n$, we say that the manifold M^n_φ is the result of performing p-surgery on M^n along φ, where

$$\mathrm{M}^n_\varphi := [\mathrm{M}^n \setminus \operatorname{Im}\varphi] \cup_{\mathrm{S}^p \times \mathrm{S}^{q-1}} \left[\mathrm{D}^{p+1} \times \mathrm{S}^{q-1}\right], \tag{1.1}$$

where the identification along the boundary is given by restricting φ. Note that in the case $p = 0$, if the image of $\varphi : \mathrm{S}^0 \times \mathrm{D}^n \to \mathrm{M}^n$ lies in two seperate components

$N_1^n \sqcup N_2^n = M^n$, that $M_\varphi^n = N_1^n \# N_2^n$. Thus connected sum is a particular instance of $p = 0$ surgery. When $n \geq 3$, [23, Theorem A] implies that the existence of a positive scalar curvature metric is invariant under connected sums.

1.1.2.1. Metrically local surgery and positive scalar curvature

Let us briefly describe the proof of [23, Theorem A]. Given (M^n, g) with positive scalar curvature and an embedding $\varphi : S^p \times D^q \hookrightarrow M^n$ with $q \geq 3$, we would like to produce a positive scalar curvature metric on M_φ^n. The main technical lemma claims that it is possible to alter the metric on $M^n \setminus \operatorname{Im}\varphi$ near the boundary so that it still has positive scalar curvature and is isometric to $dt^2 + ds_p^2 + \varepsilon^2 ds_{q-1}^2$ near the boundary. Assuming we have established this lemma, we could equally well apply the lemma to $S^p \times D^q \hookrightarrow S^n$ with respect to the round metric to produce a positive scalar curvature metric on $S^n \setminus (S^p \times D^q) = D^{p+1} \times S^{q-1}$ with the same boundary metric. We may then glue together this metric with the metric on $M^n \setminus \operatorname{Im}\varphi$ which by (1.1) is a positive scalar curvature metric defined on M_φ^n as desired. We emphasize that this surgery is *metrically local* in the sense that the resulting metric agrees with the original metric g on M^n outside of an arbitrarily small neighborhood of $\varphi(S^p)$ with respect to the metric g.

Surgery induces an equivalence relation on smooth manifolds, which by [26, Corollary 2.10, Theorems 3.12 and 3.13] agrees with cobordism. The secondary results of [23, Theorem B and C] are that existence of a positive scalar curvature metric is invariant under oriented bordism between simply connected non-spin manifolds and under spin bordism between simply connected spin manifolds, both of which follow from standard handle cancelation arguments and the main theorem. The obstruction to positive scalar curvature metric $\alpha(M^n)$ of [22] referenced above, is well defined on spin bordism classes. In [24], a homotopy theoretic computation is carried out to find spin manifolds whose spin bordism classes generate the kernel of α, and a positive scalar curvature metric is constructed on each representative. This combined with the secondary results of [23] implies that every simply connected spin manifold with $\alpha(\mathrm{M}^n) = 0$ must admit a metric with positive scalar curvature.

1.1.2.2. Surgery and positive Ricci curvature

It was conjectured in [27], that the Witten genus $\phi_W(\mathrm{M}^n)$, an invariant of string manifolds, is an obstruction to positive Ricci curvature (see [28, Section 4] for motivation and evidence for this conjecture). A rational generating set for the kernel of ϕ_W was computed in [29], which all admit metrics of positive Ricci curvature. In analogy to the classification of simply connected manifolds admitting positive scalar curvature metrics carried out in [22, 23, 24], this motivates the following question: under what conditions is the existence of a positive Ricci curvature metric invariant under surgery? In particular, the conjecture suggests that the existence of a metric with positive Ricci curvature ought to be invariant under connected sum, as the Witten genus, the conjectured obstruction, is additive under connected sum.

One may ask if the construction used to prove [23, Theorem A], which we outlined in Section 1.1.2.1 above, can be used to preserve positive Ricci curvature under metrically local surgery. There is actually a *metric* obstruction to the construction used in [23]. Assume that $\tilde{g}(r)$ is a metric constructed on $\mathrm{M}^n \setminus \varphi(\mathrm{S}^p)$ that agrees with g outside of a normal neighborhood of $\varphi(\mathrm{S}^p)$ of radius r and agrees with $dt^2 + ds_p^2 + \varepsilon^2 ds_{q-1}^2$ near the boundary. If we let T denote the unit normal of $\varphi(\mathrm{S}^p)$ with respect to $\tilde{g}(r)$, then applying the Codazzi-Mainardi equation [6, Theorem 3.2.5] shows that

$$\mathrm{Ric}_{\tilde{g}(r)}(T, T) = -\frac{1}{r^2} + O(1).$$

Thus the construction used in [23, Theorem A] for positive scalar curvature metric necessarily fails for positive Ricci curvature if the normal neighborhood is too small. We conclude that Ricci-positive surgery cannot be metrically local.

Granted that metrically local surgery cannot preserve positive Ricci curvature, we can still ask if there are any sufficient *metric* conditions on an embedding $\varphi : \mathrm{S}^p \times \mathrm{D}^q \hookrightarrow (\mathrm{M}^n, g)$ to construct a Ricci-positive metric on M^n_φ. The idea of *metrically semi-local surgery* was introduced in [17], where it was shown that it is possible to preserve positive Ricci curvature under surgery in the following setting. Given an embedding $\varphi : \mathrm{S}^3 \times \mathrm{D}^4 \hookrightarrow (\mathrm{M}^7, g)$ into a Ricci-positive manifold, assume that $\varphi^* g = \rho^2 ds_3^2 + ds_4^2$, then it is possible to perform surgery on an ε-neighborhood of φ while preserving positive Ricci curvature if $\varepsilon > c(\rho)$ for some constant c depending on ρ. This is semi-local as it requires some global information about g before deducing that M^7_φ has positive Ricci curvature. For example, we need to know that $c(\rho)$ is less than the injectivity radius of (M^n, g).

The semi-local approach to Ricci-positive surgery has been expanded upon in [18] and [3], where similar Ricci-positive semi-local surgery results are proven. Suppose $p \geq 2$ and $q \geq 3$, and we have $\varphi : \mathrm{S}^p \times \mathrm{S}^q \hookrightarrow (\mathrm{M}^n, g)$ where g is Ricci-positive and $\varphi^* g = \rho^2 ds_p^2 + ds_q^2$. Then [18, Lemma 1] and [3, Theorem 0.3] claim that we can perform surgery on an ε-neighborhood of φ while preserving positive Ricci curvature provided that $\varepsilon > c(p, q, \rho)$. The author of the latter paper has successfully applied this principle to construct new examples of positive Ricci curvature metrics on exotic spheres [30], to study the moduli space of positive Ricci curvature metrics on spheres [31], and to show that highly connected manifolds admit metrics with positive Ricci curvature [32].

1.1.2.3. Ricci-positive semi-local Connected Sum

While the results of [18] and [3] suggest that positive Ricci curvature might always be preserved under semi-local surgery, it is important to note that this technique only works for $2 \leq p \leq n-3$, and so it does not work for connected sum. There is a very simple reason why Ricci-positive connected sum cannot be local in any sense. Were semi-local Ricci-positive connected sum possible, then we could find an embedding $\varphi : \mathrm{S}^0 \times \mathrm{D}^n \hookrightarrow \mathrm{S}^n$ and a Ricci-positive metric on $\mathrm{S}^n_\varphi = \mathrm{S}^1 \times \mathrm{S}^{n-1}$, which would contradict Myers' Theorem. Thus Ricci-positive connected sums, while expected to exist, must rely on an entirely different and global construction.

1.2. Ricci-positive connected sums

While Myers' Theorem prevents Ricci-positive connected sums from being achieved via a local or even semi-local deformation of the metric in the style of [23], we still expect that Ricci-positive connected sums should always exist between two simply connected Ricci-positive Riemannian manifolds. To answer our main question we therefore need an entirely different and global construction of Ricci-positive connected sums. We begin by summarizing previous work on Ricci-positive connected sums in [1, 17, 18, 33, 34, 35]. We then elevate the work in [33] as a general strategy for constructing metrics of positive Ricci curvature on connected sums, by constructing *core metrics* on a given Riemannian manifold. We then discuss some elementary topological considerations that immediately extends the work of [33] to include connected sums of products. We conclude by discussing some preliminary facts about core metrics.

1.2.1. Previous Work

One of the first constructions of Ricci-positive connected sums can be found in [1], where metrics of nonnegative sectional curvature are constructed on the connected sum of any two projective spaces, with either orientation. Combining this with the work of [36], these metrics can be deformed to have positive Ricci curvature for the connected sum of any two of $\mathbf{C}\mathrm{P}^n$, $\mathbf{H}\mathrm{P}^n$, and $\mathbf{O}\mathrm{P}^2$ (with either orientation). The author constructs this metric using Riemannian submersion so that, near the boundary, it is isometric to $dt^2 + ds^2_{kn-1}$. This construction is global, it relies on elementary topology of punctured projective spaces, and therefore cannot be used to produce a non-negatively curved metric with this boundary condition after deleting more than one disk.

In some sense, the Ricci-positive semi-local surgery results of [17, 18] and [3, 30] follow a similar logic to the construction of [1]. They use the elementary topology of $\mathrm{D}^n \times \mathrm{S}^m$ and $(\mathbf{C}\mathrm{P}^2 \setminus \mathrm{D}^4) \times \mathrm{S}^3$ to find a Ricci-positive metric that is, near the boundary, isometric to $R^2 ds^2_{n-1} + \rho^2 ds^2_{m+1}$. Rather than being amenable to glue two of these spaces together, this boundary condition is amenable to glue arbitrarily many of these spaces onto a universal docking station. For [17, 18], this universal docking station is $\mathrm{S}^{n-1} \times \mathrm{S}^{m+1}$, while in [3, 30] the universal docking station are nontrivial sphere bundles over spheres. It is then a matter of topology to identify the spaces that result from these surgeries in [18] as $\#_k(\mathrm{S}^n \times \mathrm{S}^m)$. Even though these are examples of Ricci-positive connected sums, the fact that they rely on this artificial topological identification means that they do not give any insight into how to construct Ricci-positive connected sums given two abstract Ricci-positive manifolds (M_i^n, g_i).

1.2.2. The Work of Perelman: Gluing and Docking

The main purpose of [33] was to prove the following.

Theorem 1.2.1. [33] *For all $k > 0$ there is a Ricci-positive metric on $\#_k \mathbf{C}\mathrm{P}^2$.*

While it is not explicitly stated in [33], the approach taken to construct these metrics merges the ideas of [1] with the docking station idea of [17, 18]. A great deal of the effort needed in the constructions of [17, 18] comes from the fact that the metrics being constructed not only needed to be Ricci-positive but also had to agree on a neighborhood of their boundary with the metric of the docking station being considered. Luckily the metrics being considered in [17, 18] on the docking station transition easily to metrics on the pieces because both can be expressed as doubly warped product metrics. If one wants to combine the Riemannian submersions of [1] with the docking station of [17, 18], we run into the even greater difficulty that this family of Riemannian submersions does not easily transition into the doubly warped products. The two metrics are defined in terms of incompatible topological decompositions of the underlying space.

This difficulty motivates the first technical result of [33]. It is a gluing theorem that removes this necessity to construct metrics that already agree smoothly along their boundary. This gluing theorem claims that we can find a Ricci-positive metric on the boundary union of two Ricci-positive manifolds even if the metrics agree only at the boundary provided that the principal curvatures are compatible.

Theorem 1.2.2. [33, Section 4] *Given two Ricci-positive Riemannian manifolds* (M^n_i, g_i) *with boundaries* N^{n-1}_i*, if there is an isometry* $\Phi : \mathrm{N}^{n-1}_1 \to \mathrm{N}^{n-1}_2$ *such that* $\mathrm{II}_1 + \Phi^* \mathrm{II}_2$ *is positive definite on* N^n_1*, then there is a Ricci-positive metric* g *on* $\mathrm{M}^n_1 \cup_\Phi \mathrm{M}^n_2$ *that agrees with the* g_i *outside of an arbitrarily small neighborhood of gluing site.*

Theorem 1.2.2 has been studied by a number of authors. It was used in [37] along with the work on Ricci-flow in [38] to show that the space of Ricci-positive metrics on D^3 with convex boundary is path connected. A detailed proof is provided in [39, Lemma 2.3], and a version for compact families is proven in [40, Theorem 2].

In order to construct Ricci-positive metrics on $\#_k \mathbf{C}\mathrm{P}^2$, the idea is to find Ricci-positive metrics $\mathbf{C}\mathrm{P}^2 \setminus \mathrm{D}^4$ with boundary conditions amenable to glue to a universal docking station using Theorem 1.2.2 that will self-evidently produce a Ricci-positive metric on the connected sum. The natural candidate for docking station is $\mathrm{S}^4 \setminus \left(\bigsqcup_k \mathrm{D}^4\right)$ as $\mathrm{S}^4 \# \left(\#_k \mathbf{C}\mathrm{P}^2\right)$ is diffeomorphic to $\#_k \mathbf{C}\mathrm{P}^2$. If one wants to construct a Ricci-positive metric on $\mathrm{S}^4 \setminus \left(\bigsqcup_k \mathrm{D}^4\right)$ that is amenable to Theorem 1.2.2, one wants the principal curvatures of the boundary to be as large as possible. Sadly if the principal curvatures were nonnegative, then one could apply Proposition 1.2.11 to find a Ricci-positive metric with positive principal curvatures and then apply Theorem 1.2.2 to identify a pair of boundary components, which would produce a Ricci-positive metric on $\mathrm{S}^1 \times \mathrm{S}^3$ contradicting Myers' Theorem. Thus the principal curvatures all have to be negative.

The following proposition represents our perspective on the metrics constructed in [33], it claims that we can make the principal curvatures of the docking station arbitrarily close to 0.

Proposition 1.2.3. [41, Proposition 1.3] *If $n \geq 4$, then for all $k > 0$ and $1 > \nu > 0$ sufficiently small there is a Ricci-positive metric $g_{docking}(\nu)$ on $\mathrm{S}^n \setminus (\bigsqcup_k \mathrm{D}^n)$ such that*

1. *Each of the boundary S^{n-1} are isometric to $(\mathrm{S}^{n-1}, ds^2_{n-1})$,*

2. *The principal curvatures of each boundary component are all greater than $-\nu$.*

This proposition does not explicitly appear in [33], nonetheless is is a direct consequence of the constructions of [33], which we will give a detailed account of in Section 3.1 below.

Having established Theorem 1.2.2 and Proposition 1.2.3, the desired metric on $\mathbf{C}\mathrm{P}^2 \setminus \mathrm{D}^4$ is clear. It motivates the following definition.

Definition 1.2.4. *We say that a metric g on M^n is a* core metric*, if it has positive Ricci curvature and there is an embedded disk for which the boundary of $\mathrm{M}^n \setminus D^n$ is isometric to ds^2_{n-1} and has positive principal curvatures.*

If one can find core metrics on individual Riemannian manifolds, then gluing them to the docking station using Theorem 1.2.2 produces a Ricci-positive metric on the connected sum. Thus the approach taken in [33] to construct Ricci-positive connected sums can be summarized as follows.

Theorem 1.2.5. [41, Theorem B] *For $n \geq 4$, if M^n_i admits a core metric, then $\#_{i=1}^k \mathrm{M}^n_i$ admits a Ricci-positive metric.*

So all that remains to prove Theorem 1.2.1 is to prove the following.

Proposition 1.2.6. *[33, 34]* $\mathbf{C}\mathrm{P}^2$ *admits a core metric.*

We will discuss the proof of Proposition 1.2.6 below in Section 2.2.2.1.

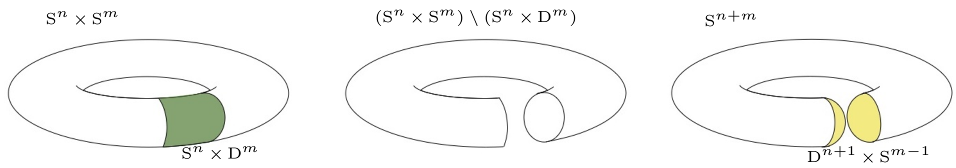

FIGURE 1.1. An illustration of (1.3).

1.2.2.1. Topological considerations

We have already referenced that the surgery result of [18] can be used to construct Ricci-positive metrics on $\#_k\left(S^n \times S^m\right)$ by identifying this space in as iterated surgery on a product of spheres. This idea was carried further in [35] in which Ricci-positive metrics were constructed on $\#_{i=1}^k(S^{n_i} \times S^{m_i})$ where $n_i, m_i \geq 3$ are allowed to vary by identifying this space with alternating surgeries on a product of spheres. Both of these Ricci-positive connected sum results follow from applying serious geometric constructions to elementary topological considerations. In this section we offer one additional elementary topological observation that will allow us to apply the serious work in [33] to construct new Ricci-positive connected sums.

Specifically, we claim the following

$$\#_{i=1}^k(N_i^n \times S^m) = \left[\left(S^{n-1} \times S^{m+1}\right) \setminus \left[\bigsqcup_{k+1}\left(S^{n-1} \times D^{m+1}\right)\right]\right] \cup_\partial \left[(D^n \times S^m) \sqcup \left[\bigsqcup_{i=1}^k((N_i^n \setminus D^n) \times S^m)\right]\right]. \tag{1.2}$$

See [41, Proposition 5.2] for a full proof of this fact. In the special case $N^n = S^n$, (1.2) claims that performing $(n-1)$-surgery $(k+1)$ times on $S^{n-1} \times S^{m+1}$ results in $\#_k(S^n \times S^m)$. While elementary, this identification is not entirely obvious. There are two essential observations that make this identification clear. The first

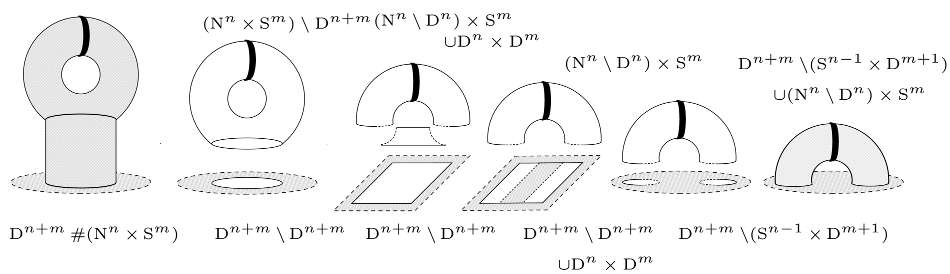

FIGURE 1.2. An illustration of (1.4).

observation is that equation (1.2) is true in the case $k = 0$.

$$\mathrm{S}^{n+m} = \left[\left(\mathrm{S}^{n-1} \times \mathrm{S}^{m+1}\right) \setminus \left(\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}\right)\right] \cup_{\mathrm{S}^{n-1} \times \mathrm{S}^m} \left[\mathrm{D}^n \times \mathrm{S}^m\right]. \tag{1.3}$$

This fact is illustrated in Figure (1.1.).

The second observation is the following general fact. Performing surgery on a trivially embedded $\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}$ is the same as taking the connected sum.

$$\mathrm{D}^{n+m} \#(\mathrm{N}^n \times \mathrm{S}^m) = \left[\mathrm{D}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})\right] \cup_{\mathrm{S}^{n-1} \times \mathrm{S}^m} \left[(\mathrm{N}^n \setminus \mathrm{D}^n) \times \mathrm{S}^m\right]. \tag{1.4}$$

This fact is illustrated in Figure (1.2.). If we apply (1.3) to the righthand side of (1.2), we note that the remaining $\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}$ are trivially embedded within S^{n+m}. After applying (1.4) k times to the righthand side of (1.2) illustrates the equality in (1.2).

While the Ricci-positive surgery result of [18] only allowed for $\mathrm{N}^n = \mathrm{S}^n$ in (1.2), we note that the work of [33] allows us to use $g_{\mathrm{docking}}(\nu)$ of Proposition 1.2.3 as a Ricci-positive metric on the first term on the righthand side of (1.2). Thus with this little bit of topological consideration we are able to state and prove one of our first results.

Proposition 1.2.7. [41, Proposition 5.5] *For $m \geq 3$, if M_i^n admit core metrics then there is a Ricci-positive metric on $\#_{i=1}^k (\mathrm{M}_i^n \times \mathrm{S}^m)$.*

Proof. Let $\mathrm{N}_0^n = \mathrm{D}^n$. Note that a suitable round metric g_0 will be a core metric for D^n, so we may assume all of the N_i^n admit core metrics. The second term on the righthand side of (1.2) is of the form

$$\bigsqcup_{i=0}^{k} (\mathrm{N}_i^n \setminus \mathrm{D}^n) \times \mathrm{S}^m \,. \tag{1.5}$$

Let g_i denote the core metrics on $\mathrm{N}_i^n \setminus \mathrm{D}^n$, and define g_p on (1.5) to agree with the metric $g_i + ds_m^2$ on each component. It is clear that g has positive Ricci curvature, that the boundaries are all isometric to $ds_{n-1}^2 + ds_m^2$, and so that the principal curvatures of the boundary are nonnegative. By Proposition 1.2.11, we can assume instead that g has positive principal curvatures. Let $\nu > 0$ be a lower bound for all of these principal curvatures.

The first term on the righthand side of (1.2) is of the form

$$\mathrm{S}^{n-1} \times \left[\mathrm{S}^{m+1} \setminus \left[\bigsqcup_k \mathrm{D}^{m+1} \right] \right] \tag{1.6}$$

We may take the metric $g_d = ds_{n-1}^2 + g_{\text{docking}}(\nu)$, where this latter metric is the metric constructed in Proposition 1.2.3. It is clear that g_d has positive Ricci curvatures, that the boundaries are all isometric to $ds_{n-1}^2 + ds_m^2$, and that the principal curvatures are all at least $-\nu$.

We may therefore glue together (1.5) and (1.6) along their common boundary using Theorem 1.2.2, which by (1.2) produces a Ricci-positive metric on $\#_{i=1}^k (\mathrm{M}_i^n \times \mathrm{S}^m)$. □

At the moment, we do not know very many core metrics other than those for S^n and $\mathbf{C}P^2$. In this first case Proposition 1.2.7 reduces to the work of [18], but in this second case we have constructed Ricci-positive metrics on $\#_k(\mathbf{C}P^2 \times S^m)$. We note that Proposition 1.2.7 was originally proven alongside Theorem A in [41], and represented one of the major results of the paper. Not so long after we succeeded in proving Theorem B, which subsumes and greatly generalizes all previous results. The reason we include Proposition 1.2.7 at all is that it will provide clear motivation behind the proof of Theorem B in Chapter III below.

We make one final topological observation that allows for further generalization of [33].

Proposition 1.2.8. [41, Corollary 4.5] *For $n \geq 4$, let $G \leq O(2) \oplus O(n-1)$ act freely on S^n with respect to the standard action. If M_i^n admit core metrics, then there is a Ricci-positive metric on*

$$(S^n/G)\# \left(\#_{i=1}^k M_i^n\right).$$

In particular we may take (S^n/G) to be $\mathbf{R}P^n$ or any lens space.

We will discuss the proof of Proposition 1.2.8 in Section 3.1.2.1 after giving a more detailed description of the metric $g_{\text{docking}}(\nu)$ in Proposition 1.2.3. It turns out that lens spaces are the *only* spaces that can be realized as S^n/G in Proposition 1.2.8.

Lemma 1.2.9. *Any finite subgroup of $G \leq O(2) \oplus O(n-1) \leq O(n+1)$ acting freely on S^n is cyclic.*

Proof. Let $\pi_1 : O(2) \oplus O(n-1) \to O(2)$ be the obvious projection. Note that $\pi(G)$ must act freely on S^1, and therefore must contain no reflection. We conclude that $\pi(G) \leq SO(2) = U(1)$. We immediately conclude that G has an irreducible complex representation of dimension 1. By [42, Theorem 7.2.18], this is only possible if G is of Type I, where G is of Type I when

$$G = \langle A, B : A^n = B^m = 1; BAB^{-1} = A^r \rangle,$$

where $r^n \equiv 1 \mod m$, $\gcd(n(r-1), m) = 1$, and r has order d in $(\mathbf{Z}/m\mathbf{Z})^\times$, where d is the dimension of each irreducible complex representation of G. As $d = 1$ in this case, we conclude that $r \equiv 1 \mod m$ (so that the order of r in $(\mathbf{Z}/m\mathbf{Z})^\times$ is 1). As $(r-1) \equiv 0 \mod m$, the requirement that $\gcd(n(r-1), m) = 1$ is only possible if $m = 1$. We conclude that G is cyclic of order n. □

1.2.2.2. Core metrics

Theorem 1.2.5 gives an intrinsic characterization for which Ricci-positive metrics on M_i^n can be used to construct Ricci-positive metrics on the connected sum $\#_i \, \mathrm{M}_i^n$. We emphasize that such core metrics cannot be produced by a local metric deformation for the same reason outlined in Section 1.1.2.3. One might ask if each Ricci-positive Riemannian metric g on M^n admits an embedding $\mathrm{D}^n \hookrightarrow \mathrm{M}^n$ for which g is a core metric. As the boundary of this disk is required to be round, we see that a generic Ricci-positive metric will not admit such an embedding.

Even if we drop the requirement that the boundary be round we make the following observation.

Corollary 1.2.10. [41, Corollary 4.10] *If* $(\mathrm{M}^n \setminus \mathrm{D}^n, g)$ *has positive Ricci curvature and the boundary has positive principal curvatures then* $\pi_1(\mathrm{M}^n) = 0$.

Proof. Assume that $(\mathrm{M}^n \setminus \mathrm{D}^n, g)$ has positive Ricci curvature and positive principal curvatures. We may apply Theorem 1.2.2 to glue this manifold to itself and find a Ricci-positive metric on $\mathrm{M}^n \# \mathrm{M}^n$.

For surfaces, the Gauss-Bonnet Theorem implies that $\mathrm{M}^2 \# \mathrm{M}^2$ is either a sphere or a real projective plane. But both S^2 and $\mathbf{R}\mathrm{P}^2$ are prime, so we conclude that $\mathrm{M}^2 = \mathrm{S}^2$ and hence $\pi_1(\mathrm{M}^2) = 0$.

For $n \geq 3$, the Seifert-van Kampen theorem implies that $\pi_1(\mathrm{M}^n \# \mathrm{M}^n) = \pi_1(\mathrm{M}^n) * \pi_1(\mathrm{M}^n)$. As $\mathrm{M}^n \# \mathrm{M}^n$ is compact manifold with positive Ricci curvature, Myers' Theorem implies that $\pi_1(\mathrm{M}^n \# \mathrm{M}^n) = \pi_1(\mathrm{M}^n) * \pi_1(\mathrm{M}^n)$ is finite. But a free product is only finite if each factor is trivial, i.e. if $\pi_1(\mathrm{M}^n) = 0$. □

In particular, this means that $\mathbf{R}\mathrm{P}^n$ cannot admit a core metric. More generally, if M^n is any Ricci-nonnegative manifold with boundary with positive principal curvatures then $\iota_* : \pi_1(\partial \mathrm{M}^n) \to \pi_1(\mathrm{M}^n)$ is surjective by [43, Proposition 2.8]. This stronger observation implies that the boundary must be connected, which in dimension 2 and 3 implies that $\mathrm{M}^n = \mathrm{D}^n$ [43, Theorem 2.11(b)], though starting in dimension 4 we have the nontrivial example of $\mathbf{C}\mathrm{P}^2$ in Proposition 1.2.6.

It remains an interesting question whether or not a fixed smooth manifold M^n with boundary admits a Ricci-positive (or Ricci-nonnegative) metric so that the boundary has positive principal curvatures. In this paper we are narrowly interested in the case when the boundary is a sphere. We note that Theorem II below further reduces the construction of core metrics, to the construction of Ricci-positive metrics with boundaries with positive principal curvatures so that the boundary metrics are Ricci-positive isotopic to the round metric. One possible approach to this question is to study the space of embeddings $\mathrm{D}^n \hookrightarrow (\mathrm{M}^n, g)$ for a fixed metric g. The perspective we take, however, is to ask which topological constructions can be made to respect such metrics.

We emphasize that it is essential that the metric constructed in Proposition 1.2.3 have principal curvatures arbitrarily close to 0. It is a consequence of [44, Theorem 1], that there is a $\nu > 0$ that depends only on n such that if the principal curvatures of a core metric are greater than ν, then $\mathrm{M}^n = \mathrm{S}^n$. We suspect that the supremum of the principal curvatures over all normalized core metrics is an interesting geometric invariant, which will restrict the possible topology of M^n.

We conclude by noting that if (M^n, g) has positive Ricci curvature and boundary with nonnegative principal curvatures, then one can easily perturb the metric near the boundary so that it has positive Ricci curvature and boundary with positive principal curvatures.

Proposition 1.2.11. *If* (M^n, g) *is a Ricci-positive Riemannian manifold with boundary such that the principal curvatures are nonnegative, then there is a Ricci-positive metric* g_δ *on* M^n *such that the principal curvatures are positive and* $g_\delta = g$ *along the boundary.*

Proof. We will prove the stronger statement that there exists a $\delta > 0$ and a Ricci-positive metric g_δ that agrees with g along the boundary such that

$$\mathrm{II}_{g_\delta} = \mathrm{II}_g + \delta g.$$

Take normal coordinates of the boundary $(a, x) : (-\varepsilon, 0] \times \partial\,\mathrm{M}^n \hookrightarrow \mathrm{M}^n$. In these coordinates the metric splits as $da^2 + k_a$ where k_a is smooth family of metrics on $\partial\,\mathrm{M}^n$, and in these coordinates the second fundamental form can be compute as follows [6, 3.2.1].

$$\mathrm{II}_g = (1/2)\left(\partial_a k_a\right|_{a=0}.$$

Let $\chi_\delta : \mathbf{R} \to \mathbf{R}_+$ be a family of smooth functions such that $\chi_\delta(a) \equiv 1 - \delta$ for $a \in (-\infty, -\varepsilon/2)$, $\chi_\delta(0) = 1$, $\chi'_\delta(0) = \delta$, and that $\chi_\delta(a)$ converges uniformly in all derivatives to 1 as $\delta \to 0$. We can define a family of smooth metrics g_δ on M^n

$$g_\delta = \begin{cases} (1-\delta)^2 g & p \notin (-\varepsilon, 0] \times \partial\,\mathrm{M}^n \\ da^2 + \chi^2_\delta(a) k_a & p \in (-\varepsilon, 0] \times \partial\,\mathrm{M}^n \end{cases}$$

Clearly g_δ restricted to the boundary equals k_0 and therefore agrees with g. Note that g_δ converges uniformly to g in the C^∞ topology as $\delta \to 0$, and so if $\delta > 0$ is sufficiently small g_δ will have positive Ricci curvature. We can again compute the second fundamental form

$$\mathrm{II}_{g_\delta} = (1/2)\left(\partial_a\left(\chi^2_\delta(a) g_a\right)\right|_{a=0} = \chi^2_\delta(0)(1/2)\left(\partial_a g_a\right|_{a=0} + \chi'_\delta(0)\chi_\delta(0) g_0 = \mathrm{II}_g + \delta g.$$

□

Thus to construct a core metric on M^n, it is sufficient to find a Ricci-positive metric on $\mathrm{M}^n \setminus \mathrm{D}^n$ with boundary metric Ricci-positive isotopic to the round metric and with nonnegative principal curvatures.

1.3. Main Results

Our three main theorems, Theorems A, B, and C, all claim that there exist core metrics on certain spaces. Combining these with Theorem 1.2.5 immediately implies the existence of a number of Ricci-positive connected sums. We will give as separate corollaries those implications that are particularly interesting, but we make no attempt to compile a comprehensive list of such examples. Such a project would be impossible as Theorems B and C are both constructions that take as input a core metric on manifold M^n and provide as output a new core metric on a new manifold constructed out of bundles over M^n, so we will instead point out the ways in which these theorems can be combined to produce new examples.

1.3.1. Projective spaces

Our original project was to understand the approach taken in [33, 34] to construct Ricci-positive metrics on $\#_k\,\mathbf{C}\mathrm{P}^2$ and to generalize the construction to connected sums of other projective spaces. While Theorem 1.2.2 and Proposition 1.2.3 generalize immediately to higher dimensions, the construction used to Proposition 1.2.6 relies in an essential way on the dimension. This is because computation of the curvature in [34] relies on the fact that S^3 is a Lie group, which is not the case for higher dimensional spheres. Our first main theorem claims that core metrics exist for all projective spaces.

Theorem A. [41, Theorem C] *For all $n \geq 1$, the spaces $\mathbf{C}\mathrm{P}^n$, $\mathbf{H}\mathrm{P}^n$, and $\mathbf{O}\mathrm{P}^2$ admit core metrics.*

Combining Theorem A with Propositions 1.2.7 and 1.2.8 gives us the following corollary.

Corollary 1.3.1. [41, Theorem A′] *There is a Ricci-positive metric on the connected sum of any of the following spaces: $\mathbf{C}\mathrm{P}^n$, $\mathbf{H}\mathrm{P}^n$, $\mathbf{O}\mathrm{P}^2$, and at most one $\mathbf{R}\mathrm{P}^n$.*

For a fixed $m \geq 3$, there is a Ricci-positive metric on the connected sum of any of the following spaces: $\mathbf{C}\mathrm{P}^n \times \mathrm{S}^m$, $\mathbf{H}\mathrm{P}^n \times \mathrm{S}^m$, and $\mathbf{O}\mathrm{P}^2 \times \mathrm{S}^m$.

The metric constructed to prove Theorem A is a straightforward generalization of the metric used in Proposition 1.2.6. Both rely on the Hopf fibrations to define a sort of doubly warped product metric on the punctured projective spaces. In higher dimensions we need the theory of Riemannian submersions to define these core metrics and to compute their curvatures. In Chapter II we give the necessary background to define these metrics and ultimately prove Theorem A in Section 2.2.2.2 below. We will also give a full account of the core metrics of [33, 34] in Section 2.2.2.1.

1.3.2. Sphere bundles

The work of [17] and [35] combined with Proposition 1.2.7 motivates the following question: is there a Ricci-positive metric on the connected sums of products of three or more spheres? While this is a perfectly natural question, the existing strategies to construct Ricci-positive metrics on such spaces run into the essential difficulty that there is no statement similar to (1.4) for a product of three or more spheres. By Theorem 1.2.5, it would suffice to construct a core metric on a product of three or more spheres. Comparing this with Proposition 1.2.7, we might conjecture that within the metric constructed in Proposition 1.2.7 is an embedded $(\mathrm{N}^n \times \mathrm{S}^m) \setminus \mathrm{D}^{n+m}$ with a round boundary and positive principal curvatures. While this is not the case, it motivates the following theorem.

Theorem B. *If* B^n *admits a core metric and* $\pi : \mathrm{E} \to \mathrm{B}^n$ *is a rank* $m+1 \geq 4$ *vector bundle, then* $S(\mathrm{E})$ *also admits a core metric.*

Note that we may apply Theorem B iteratively, so that any iterated sphere bundle over the space B^n will also admit a core metric, provided that B^n admits a core metric and the fiber spheres all have dimension 3 or more. In particular, we may take trivial bundles in Theorem B and the core metrics from Theorem A to deduce the following.

Corollary 1.3.2. *For $n_i \geq 3$, there is a Ricci-positive metric on the connected sum of any of the following spaces:* $\mathrm{S}^n \times \left[\prod_{i=1}^{k} \mathrm{S}^{n_i}\right]$, $\mathbf{C}\mathrm{P}^n \times \left[\prod_{i=1}^{k} \mathrm{S}^{n_i}\right]$, $\mathbf{H}\mathrm{P}^n \times \left[\prod_{i=1}^{k} \mathrm{S}^{n_i}\right]$, $\mathbf{O}\mathrm{P}^2 \times \left[\prod_{i=1}^{k} \mathrm{S}^{n_i}\right]$, *and at most one lens space* L^n *or* $\mathbf{R}\mathrm{P}^n$

We emphasize that the the values of n and n_i can vary from summand to summand. In particular, if $\mathrm{N}^{2n} = \#_i^k \mathrm{N}_i^{2n}$ where each N_i^{2n} admits core metrics, then Corollary 1.3.2 and Theorem 1.2.5 produces a Ricci-positive metric on the manifold $\mathrm{N}_g^{2n} = \mathrm{N}^{2n} \# (\#_g (\mathrm{S}^n \times \mathrm{S}^n))$. When N^{2n} is spin, we have the following corollary of the main theorem of [45].

Corollary 1.3.3. *For $n \not\equiv 3 \mod 4$ and $n \geq 10$, suppose that $\mathrm{N}^{2n} = \#_i^k \mathrm{N}_i^{2n}$ is spin and each of the N_i^{2n} admit core metrics, then $H^j(\mathcal{R}^{pRc}(\mathrm{N}_g^{2n}); \mathbf{Q})$ is nontrivial for some $1 \leq j \leq 5$ for all g sufficiently large.*

In particular, we may take N_i^{2n} as any spin manifold appearing in Theorems A, B, C. Essentially, one only needs to avoid using $\mathbf{C}\mathrm{P}^{2k}$ in Theorem B.

We note that Corollary 1.3.2 has a fundamental gap in its conclusion: whether or not we may have summands that contain more than one S^2 factor. There is no known obstruction to finding a Ricci-positive metric on $\#_k \left(\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2\right)$, yet our techniques fail to construct such metrics. The one aspect of our construction that fails is the existence of the metric $g_{\mathrm{neck}}(\nu)$ of Proposition 3.1.2, which is an essential ingredient in the proof of Proposition 1.2.3 and hence Theorem 1.2.5. We emphasize that [18] has constructed Ricci-positive metrics on $\#_k \left(\mathrm{S}^2 \times \mathrm{S}^2\right)$, yet when this result was generalized in [35] to $\#_k (\mathrm{S}^{n_i} \times \mathrm{S}^{m_i})$ we must restrict to $n_i, m_i \geq 3$. The difficulties that arise in these constructions is we require further topological decomposition of S^2 into components that will not intrinsically support positive curvature.

While the metric constructed in Proposition 1.2.7 does not directly construct a core metric on $\mathrm{N}^n \times \mathrm{S}^m$ it is still be the basis for the proof of Theorem B. We will discuss in the following section what problems arise when attempting to construct core metrics in this way and how to overcome them. In Section 2.3 we will show how to reduce the construction in Theorem B for all S^m-bundles to the construction for products. The rest of the proof makes up the body of Chapter III, which relies on the Theorems I and II below.

1.3.2.1. The technical constructions

When one attempts to locate an embedded $(\mathrm{N}^n \times \mathrm{S}^m) \setminus \mathrm{D}^{n+m}$ within the topological decomposition (1.2) used in the proof of Proposition 1.2.7, one realizes that the boundary S^{n+m-1} will bound a disk in (1.3). This disk will intersect the two terms on the righthand side of (1.3) which in turn decomposes the disk as

$$\mathrm{D}^{n+m} = (\mathrm{D}^n \times \mathrm{D}^m) \cup_{\mathrm{S}^{n-1} \times \mathrm{D}^m} \left(\mathrm{S}^{n-1} \times \mathrm{B}^{m+1}_+\right). \tag{1.7}$$

This is a decomposition of a smooth manifold with boundary as the union of two manifolds with corners along a common face. We would like to study the resulting metric on the boundary of this disk and its principal curvatures. As the two terms on the righthand side of (1.3) are glued together using Theorem 1.2.2 in the proof of Proposition 1.2.7, we need to understand how this construction interacts with manifolds with corners.

Our first technical theorem is Theorem I, which is a generalization of Theorem 1.2.2 to manifolds with corners. As already observed, we wish to glue together two Ricci-positive Riemannian manifolds with corners along a common face and deduce similarly to Theorem 1.2.2 that there is a smooth Ricci-positive metric on the resulting smooth manifold with boundary. If this is all that one wishes to conclude, then Theorem 1.2.2 would immediately imply this provided we impose the same condition on second fundamental forms of the glued faces. As we are trying to construct core metrics, in addition to preserving Ricci-positivity, we are also interested in preserving the positivity of the principal curvatures of the remaining faces. Theorem I claims it is possible to preserve Ricci-positivity and face convexity under the same hypotheses as Theorem 1.2.2 with the added assumption that the dihedral angles along the corners are not too large.

Theorem I. *If* (X_i^n, g_i) *are Ricci-positive Riemannian manifolds with codimension 2 corners such that the principal curvatures of the faces are all positive and the dihedral angles along the corners are everywhere less than* $\pi/2$. *Suppose that there is an isometry* $\Phi : \mathrm{F}_1^{n-1} \to \mathrm{F}_2^{n-1}$ *between on of their faces such that* $\mathrm{II}_1 + \Phi^* \mathrm{II}_2$ *is positive definite, then there is a smooth Ricci-positive metric on* $\mathrm{X}^n = \mathrm{X}_1^n \cup_\Phi \mathrm{X}_2^n$ *so that the smooth boundary has positive principal curvatures.*

We will discuss Riemannian manifolds with corners in Appendix A where we will rephrase Theorem I in a more precise fashion. We will also introduce the technical tools used in the proof of Theorem 1.2.2 and explain how to adapt them to Riemannian manifolds with corners to prove Theorem I.

Having established Theorem I, we are now able to study the construction of the metric in Proposition 1.2.3 and describe an embedding of the two terms in (1.7) into the corresponding pieces in (1.3) to which a version of Theorem I can be used to construct a Ricci-positive metric on $(\mathrm{N}^n \times \mathrm{S}^m) \setminus \mathrm{D}^{n+m}$ so that the boundary has positive principal curvatures. This is carried out in Sections 3.2.3 and 3.3. This overcomes the first difficulty in proving Theorem B.

The second difficulty in proving Theorem B is precisely that the metric constructed on $(\mathrm{N}^n \times \mathrm{S}^m) \setminus \mathrm{D}^{n+m}$ using Theorem I restricted to the boundary will not be round. We therefore require a technique that would allow us to alter the metric near the boundary. The boundary metric turns out to have positive Ricci curvature, and is connected via a path of positive Ricci curvature metrics to the round metric. This itself is far from obvious, but even if we suppose it were true, this still requires a technique to use Ricci-positive isotopies to deform a collar neighborhood of the boundary. Our second technical theorem claims that the existence of a Ricci-positive isotopy can be used to construct a particular Ricci-positive "neck."

Theorem II. *If g_0 and g_1 are isotopic in the space of Ricci-positive metrics, then g_0 and g_1 are neck-equivalent.*

We give a precise definition of what we mean by neck-equivalent in Appendix B. This definition combined with Theorem 1.2.2 precisely allows us to alter the metric near the boundary in the following way.

Corollary 1.3.4. *If (M^n, g) is a Ricci-positive manifold with boundary so that the principal curvatures of the boundary are positive and the metric g restricted to the boundary is isometric to a Ricci-positive metric g_0 on $\partial\,\mathrm{M}^n$. Suppose that g_0 and g_1 are Ricci-positive isotopic, then there is a Ricci-positive metric $\tilde{g}$ on M^n still has positive principal curvatures along the boundary but now $\tilde{g}$ restricted to the boundary is isometric to g_1.*

The remainder of the proof of Theorem B carried out in Section 3.4 is to show that the metric constructed on the boundary of $(\mathrm{N}^n \times \mathrm{S}^m) \setminus \mathrm{D}^{n+m}$ using Theorem I is Ricci-positive isotopic to the round metric.

1.3.3. Plumbing

One of the main applications of the Ricci-positive surgery theorem of [3] is the construction of Ricci-positive metrics on exotic spheres that bound parallelizable manifolds in [30]. Every such sphere can be built using a construction known as *plumbing*. Suppose we are given a rank n disk bundle over an m dimensional manifold and a rank m disk bundle over an n dimensional manifold, then we can locally glue these bundles together interchanging base and fibers so that the corresponding zero sections are *plumb* with another. This can be repeated with varying bundles and base manifolds, and these particular exotic spheres can be realized as the boundaries of certain plumbings of disk bundles over spheres. The surgery theorem of [3] is applicable because one can equally well describe these boundaries as iterated surgery on a sphere bundle over a spheres. We will discuss these construction in further detail in Chapter IV.

As Theorem B already shows that sphere bundles admit core metrics, and [30, Theorem 2.2] precisely claims that Ricci-positive metrics can be preserved under iterated k surgery, it is reasonable to ask if these exotic spheres admit core metrics as well. These spheres belong to a broader class of manifolds that occur as the boundary of *tree-like plumbings*, i.e. simply connected manifolds constructed by plumbing D^k-bundles over S^k. When plumbing together D^p-bundles over S^q when $p \neq q$, [3, Theorem 2.3] applies to construct a Ricci-positive metric when only plumbing together two disks bundles. Our contribution is that the ideas behind

Theorem B can be combined with the proofs of [30, Theorems 2.2 and 2.3] to conclude that these boundaries of plumbings also admit core metrics.

Theorem C. *For $k \geq 4$, if M^{2k-1} is the boundary of a tree-like plumbing, then M^{2k-1} admits a core metric.*

For $p \geq 4$ and $q \geq 3$, if M^{p+q-1} is the boundary of plumbing a D^p-bundle over S^q with a D^q-bundle over S^p, then M^{p+q-1} admits a core metric.

We give more detailed description of the work of [3, 30] and its application to prove Theorem C below in Chapter IV.

The class of manifolds realized as boundaries of tree-like plumbings is much more than exotic spheres; it contains a sizable portion of all highly connected manifolds [32, Theorem C]. We note that we may use any of the M^n in Theorem C as inputs for Theorem B to produce additional core metrics, which then may be combined with any of the other core metrics constructed using Theorem A and Theorem B using Proposition 1.2.8 to construct further examples of Ricci-positive connected sums.

The exotic spheres that are known to be included in Theorem C are listed in the following corollary.

Corollary 1.3.5. ([46],[47, Satz 12.1]) *If $\Sigma^{2k-1} \in bP_{2k}$, then Σ^{2k-1} admits a core metric. Each $\Sigma^8 \in \Theta_8$, $\Sigma^{16} \in \Theta_{16}$, and $\Sigma^{19} \in \Theta_{19}/bP_{20}$ admit a core metric.*

Thus Proposition 1.2.8 allows us to find Ricci-positive metrics on the connected sum of any the exotic spheres in Corollary 1.3.5 with any of the other manifolds constructed in Theorems A, B, and C. While taking a connected sum with an exotic sphere has no effect on the underlying topological manifold, it can sometimes alter the smooth structure. Thus Corollary 1.3.5 gives us an approach to construct new examples of exotic smooth structures supporting positive Ricci curvature.

By 1.2.8 we may find a Ricci-positive metric on the connected of any of the exotic spheres in Corollary 1.3.5 with $\mathbf{R}\mathrm{P}^n$. In dimension 7, by [48, Corollary 2.11] these connected sums all change the smooth structure.

Corollary 1.3.6. *For all $\Sigma^7 \in bP_8 = \Theta_7$, there is a Ricci-positive metric on $\Sigma^7 \# \mathbf{R}\mathrm{P}^7$, and each $\Sigma^7 \# \mathbf{R}\mathrm{P}^7$ represent distinct smooth structures on $\mathbf{R}\mathrm{P}^7$. Moreover, if M^7 is PL-homeomorphic to $\mathbf{R}\mathrm{P}^7$, then it admits a Ricci-positive metric.*

By Theorem A and Theorem 1.2.5 we can construct a Ricci-positive metric on the connected sums of any of the exotic spheres of Corollary 1.3.5 with any other projective spaces. For $\mathbf{C}\mathrm{P}^n$ in low dimensions $n \leq 8$, by [49, Theorem 1] these connected sums will always change the smooth structures.

Corollary 1.3.7. *Let $n = 4$ or 8. For $\Sigma^{2n} \in \Theta_{2n} \cong \mathbf{Z}/2\mathbf{Z}$, there is a Ricci-positive metric on $\Sigma^{2n} \# \mathbf{C}\mathrm{P}^n$, and each $\Sigma^{2n} \# \mathbf{C}\mathrm{P}^n$ represent distinct smooth structures on $\mathbf{C}\mathrm{P}^n$. Moreover, if M^8 is homeomorphic to $\mathbf{C}\mathrm{P}^4$, then M^8 admits a Ricci-positive metric.*

Where this last sentence is due to [50, Theorem 2.7]. By Theorem B and Theorem 1.2.5 we can construct a Ricci-positive metric on the connected sums of any of the exotic spheres of Corollary 1.3.5 with products of spheres. In dimension 5 or more, by [51, Theorem A] these connected sums will always change the smooth structure.

Corollary 1.3.8. *Let $k \geq 2$ and let d_i be any sequence with $d_1 \geq 2$ and $d_i \geq 3$ which sums to d. Let M^d be the space*

$$\mathrm{M}^d := \prod_{i=1}^{l} \mathrm{S}^{d_i} .$$

Then for each Σ^d listed in Corollary 1.3.5 there is a Ricci-positive metric on $\Sigma^d \# \mathrm{M}^d$, and each $\Sigma^d \# \mathrm{M}^d$ represent distinct smooth structures on M^d.

By Theorem 1.2.5 we can construct a Ricci-positive metric on the connected sum of arbitrarily many manifolds admitting core metrics with the exotic spheres of Corollary 1.3.5. It is likely that many of these connected sums represent distinct smooth structures as well, but very little is known about the effect of connected sum on smooth structures. At the very least, by [52, Corollary 2] we know that taking connected sums with $\mathrm{S}^2 \times \mathrm{S}^{d-2}$ preserves smooth structure.

Corollary 1.3.9. *Let $k \geq 2$ and let d_i be any sequence with $d_i \geq 3$ which sums to d. Let M^d be the space*

$$\mathrm{M}^d := \prod_{i=1}^{l} \mathrm{S}^{d_i}.$$

And let $\mathrm{M}^d_k = M^d \# \left(\#_k \left(\mathrm{S}^2 \times \mathrm{S}^{d-2}\right)\right)$, then for each Σ^d listed in Corollary 1.3.5 there is a Ricci-positive metric on $\Sigma^d \# \mathrm{M}^d_k$, and each $\Sigma^d \# \mathrm{M}^d_k$ represent distinct smooth structures on M^d_k.

By theorem C and Theorem 1.2.5 we can also construct Ricci-positive metrics on the connected sums of any of the exotic spheres of Corollary 1.3.5 with other manifolds belonging to the class of manifolds realized as boundaries of tree-like plumbings. This allows us to strengthen [32, Theorem A] in dimension 19 to the following.

Corollary 1.3.10. *If M^{19} is 8-connected and 9-parallelizable, then there exists a Ricci-positive metric on M^{19}.*

We again emphasize that there are a great deal more examples of Ricci-positive metrics constructed by Theorems A, B, and C and Proposition 1.2.8 then those listed as Corollaries here. These are merely the examples that are known to give distinct smooth structures on the same underlying topological manifolds.

CHAPTER II

RIEMANNIAN SUBMERSIONS

This chapter represents a tour through the theory of Riemannian submersions necessary to include nontrivial sphere bundles in Theorems B and C. Riemannian submersions also arise naturally when considering core metrics on projective spaces, where we recall that projective spaces are built iteratively out of attaching cells to disk bundles. We begin in Section 2.1 providing the definitions, constructions, and preliminary results that will be necessary for our various applications, including the key observation of [41] needed to prove Theorem A, Lemma 2.1.6. Our first application of Riemannian submersions will be to prove Theorem A in Section 2.2 after summarizing the basic topology and geometry of projective spaces. We conclude with our second application of Riemannian submersions in Section 2.3, where we reduce the proof of Theorem B to a technical construction, which occupies the body of our next chapter, Chapter III, as well as set up the proof of Theorem C, which occupies the body of Chapter IV.

2.1. Background

In this section we provide the background to the theory of Riemannian submersions that will be necessary for our two applications. We refer the reader to [53, Chapter 9], which is the standard reference for the subject, and [54] for additional background on the subject. Much of our notation and statements are follow closely to [53, Chapter 9]. We begin with notation, definitions, and constructions, and conclude with our formulas for curvature.

2.1.1. Definitions and constructions

For any smooth surjective submersion between Riemannian manifolds π : $(\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ we may define the *vertical distribution* to be the rank m sub-bundle of $T\,\mathrm{E}^{n+m}$ given by $\ker d\pi$ denoted by V. We may define the *horizontal distribution* to be the rank n sub-bundle to be the compliment of V with respect to g denoted by H, so that $T\,\mathrm{E}^{n+m} = V \oplus H$. In this decomposition we have two orthogonal bundle projections $\mathcal{V} : T\,\mathrm{E}^{n+m} \to V$ and $\mathcal{H} : T\,\mathrm{E}^{n+m} \to H$. The sections of V and H are respectively referred to as *vertical* and emphhorizontal vector fields.

A Riemannian submersion is a submersion where the above data is compatible with the metric $\check{g}$ on the base manifold B^n. We say that a smooth surjective submersion $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ is a *Riemannian submersion* if $d\pi : (H, g) \to (T\,\mathrm{B}^n, \check{g})$ is a fiber-wise isometry of Euclidean vector bundles, i.e. that $g(X, Y) = \check{g}(d\pi X, d\pi Y)$ for horizontal vector fields X, Y. When both E^{n+m} and B^n are compact manifolds, then the fibers $\mathrm{F}_b = \pi^{-1}(b)$ are all diffeomorphic to a compact manifolds F^m [53, Theorem 9.3]. We may define the *fiber metric* $\hat{g}_b$ by restricting g to F_b. In this case, the vertical distribution agrees with the tangent bundle of the fibers $V_x = T_x F_{\pi(x)}$.

The main reason for considering Riemannian submersions is that we would like to relate the geometric properties of the total space and base space. This is similar to the study of immersions $\iota : (\mathrm{N}^n, g_{\mathrm{N}}) \hookrightarrow (\mathrm{M}^{n+m}, g_{\mathrm{M}})$, where we restrict to Riemannian immersions, i.e. $\iota^* g_{\mathrm{M}} = g_{\mathrm{N}}$, in order to compare their geometry. The primary geometric quantity we are interested in is the curvature, which we will discuss in detail in Section 2.1.2 below. The curvature of total space, fibers, and base space are related using two tensorial obstruction defined as follows (where ∇ is the Levi-Civita connection of (E^{n+m}, g)).

$$T_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2, \tag{2.1}$$

$$A_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2. \tag{2.2}$$

Both T and A are tensors, and satisfy a range of identities when restricting the E_i to be horizontal or vertical (see [53, Section 9.C]).

Recall that the curvature of an immersed submanifold is related to the curvature of the ambient manifold in terms of the second fundamental form $\mathrm{II}(X,Y) = \mathcal{N}\nabla_X Y$, where $\mathcal{N} : T\,\mathrm{M}^{n+m} \to N\,\mathrm{N}^n$ is the normal projection and ∇ is the Levi-Civita connection with respect to g_{M}. If we restrict T to vertical vector fields V_i and we restrict A to horizontal vector fields Y_i we have

$$T_{V_1}V_2 = \mathcal{H}\nabla_{V_1}V_2 \qquad A_{Y_1}Y_2 = \mathcal{V}\nabla_{Y_1}Y_2.$$

If we think of $\mathcal{H}$ and $\mathcal{V}$ as the normal projections for V and H respectively, we see that T and A can be thought of as second fundamental forms for V and H respectively. As V agrees with the tangent bundles of the fibers F_b, we see that T actually agrees with the second fundamental form of $(\mathrm{F}_b, \hat{g}_b) \hookrightarrow (\mathrm{E}^{n+m}, g)$. In fact, $T \equiv 0$ if and only if every fiber F_b is a totally geodesic submanifold, and $A \equiv 0$ if and only if H is integrable so that $(\mathrm{E}^{n+m}, g) = (\mathrm{F}^m \times \mathrm{B}^n, \check{g} + \hat{g}_b)$ [53, 9.26].

We can also use Riemannian submersions as a device for constructing new metrics on E^{n+m} out of existing metrics on F^m and B^n. Using the curvature computations in Section 2.1.2 below, we will be able to construct Ricci-positive metrics in this manner. All constructions in this chapter are of the following general form, which will allow us to discuss slightly narrower constructions in Sections 2.1.1.1 and 2.1.1.2.

Proposition 2.1.1. [53, 9.15] *Let* $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$ *be any smooth fiber bundle with fiber* F^n. *Suppose we are given the following data:*

1. *a base metric* $\check{g}$ *on* B^n
2. *a smooth family of fiber metric* $\hat{g}_b$ *on* F^m *parameterized by* $b \in \mathrm{B}^n$
3. *a rank* n *sub-bundle* H *of* $T\,\mathrm{E}^{n+m}$ *that is a vector bundle compliment of* V

Then there is a unique smooth metric g *on* E^{n+m} *such that* $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ *is a Riemannian submersion that restricts to* $\hat{g}_b$ *on the fiber* F_b *and has horizontal distribution given by* H. *Specifically this metric takes the form*

$$g = (\pi^* \check{g}) \circ \mathcal{H} + \hat{g}_b \circ \mathcal{V} \tag{2.3}$$

2.1.1.1. Principal Bundles

A principal G-bundle is a fiber bundle $p : \mathrm{P} \to \mathrm{B}^n$ for which the Lie group G acts freely on P on the right and $p : \mathrm{P} \to \mathrm{P}/G = \mathrm{B}^n$ is the quotient map. A principal connection θ, for us, will be a horizontal distribution of the principal bundle that is invariant under of the action of G, i.e. $R_g^* \theta = \theta$ for each $g \in G$. Given a manifold F^m on which G acts on the left, we can form the associated F^m-bundle

$$\mathrm{P} \times_G \mathrm{F}^m := (\mathrm{P} \times \mathrm{F}^m)/G,$$

where $g \in G$ acts on $(x, y) \in \mathrm{P} \times \mathrm{F}^m$ on the right by $(x, y)g = (xg, g^{-1}y)$. We have that $\pi : \mathrm{P} \times_G \mathrm{F}^m \to \mathrm{B}^n$ is a fiber bundle with fiber F^m, and from a principal connection θ on the principal bundle we have a preferred horizontal distribution of the associated bundle by setting $H = d\pi(\theta \oplus T\,\mathrm{F}^m)$, which again will be invariant under the action of G on the fiber. Fiber bundles formed in this way are said to have structure group G.

In cases where the Riemannian manifold $(\mathrm{F}^m, \hat{g})$ has nontrivial isometry group G, Proposition 2.1.1 can be used to construct metrics on F^m-bundles with structure group G. The following theorem claims that such Riemannian submersions formed in this way also have totally geodesic fibers, i.e. $T \equiv 0$, which is a particularly desirable trait for curvature computations.

Theorem 2.1.2. [53, Theorem 9.59] *Let G be a lie group, $p : \mathrm{P} \to \mathrm{B}^n$ a principal G-bundle, and F^m be any manifold on which G acts. Let $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$ be the associated bundle $\mathrm{E}^{n+m} = \mathrm{P} \times_G \mathrm{F}^m$.*

Given a metric $\check{g}$ on B^n, a G-invariant metric $\hat{g}$ on F^n, and a principal connection θ on P, there exists a unique smooth metric g on E^{n+m} such that $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ is a Riemmanian submersion with totally geodesic fibers isometric to $(\mathrm{F}^n, \hat{g})$ and horizontal distribution associated with θ.

One of the main examples of fiber bundles with designated structure group are vector bundles, which are $\mathbf{R}^m$-bundles with structure group $O(m)$ (though we will only consider oriented vector bundles). In this setting Theorem 2.1.2 takes the following form.

Corollary 2.1.3. *[53, 9.60] Let $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$ be a rank m Euclidean vector with fiber inner product μ and metric connection ∇. Given a metric $\check{g}$ on B^n and an $O(\mathbf{R}^m, \mu)$ invariant metric $\hat{g}$ on $\mathbf{R}^m$ there is a unique metric g on E^{n+m} such that $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ is a Riemannian submersion with totally geodesic fibers isometric to $(\mathbf{R}^m, \hat{g})$ and horizontal distribution H corresponding to ∇.*

Similarly, given a metric $\check{g}$ on B^n and a radius $r > 0$ there is a unique metric g on $S(\mathrm{E})$ such that $\pi : (S(\mathrm{E}), g) \to (\mathrm{B}^n, \check{g})$ is a Riemannian submersion with geodesic fibers isometric to $r^2 ds^2_{m-1}$ and horizontal distribution H corresponding to ∇.

We note that any $O(m)$ invariant metric on $\mathbf{R}^m$ (see [6, Section 4.3.2]) is a warped product metric $\hat{g} = dt^2 + r^2(t) ds^2_{m-1}$ where we think of $\mathbf{R}^n = [0,\infty) \times \mathrm{S}^{m-1} / \left(\{0\} \times \mathrm{S}^{m-1}\right)$ as spherical coordinates and $r(t)$ is a function defined on $[0,\infty)$ that is positive for all values of t other than $t = 0$ where $r^{(\mathrm{even})}(0) = 0$ and $r'(0) = 1$. We again emphasize that the distributions H produced in Theorem 2.1.2 and Corollary 2.1.3 are invariant with respect to the action of the structure group on the fiber.

2.1.1.2. Warped Products and Linear Bundles

In the last section, we have discussed warped product metrics on $\mathbf{R}^n$, in this section we begin by describing the general situation (see [6, Section 4.2.3]). Given any two Riemannian manifolds $(\mathrm{B}^n, \check{g})$ and $(\mathrm{F}^m, \hat{g})$, a *warped product metric* on $\mathrm{B}^n \times \mathrm{F}^m$ is a metric $\check{g} + f^2(b)\hat{g}$, where $\check{g}$ is a metric on B^n, $\hat{g}$ a metric on F^m, and $f(b)$ is a positive smooth function defined on B^n. We call the function $f(b)$ *the warping function.* One can generalize this notion by allowing a product of more than two Riemannian manifolds and allowing warping functions defined on the whole product to scale each metric. Such constructions are quite useful to give examples of certain geometries as the curvature can be computed explicitly in terms of the curvature of the individual Riemannian manifolds and the derivatives of the warping functions. In this paper, when we say *doubly warped products* we will always be referring to a metric on $[0,1] \times \mathrm{B}^n \times \mathrm{F}^m$ defined as $g = dt^2 + h^2(t)\check{g} +$

$f^2(t)\hat{g}$ for two positive functions $f(t)$ and $h(t)$ defined on $t \in [0,1]$. This is not the most general concept of doubly warped product metric, but will be sufficient for our purposes.

If instead we are given a nontrivial fiber bundle $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$ we have an analogous notion of *warped Riemannian submersion metric*. We call a metric g on E^{n+m} a *warped Riemannian submersion metric* if it is constructed via Proposition 2.1.1 using a metric $\check{g}$ and the family of fiber metrics given by $\hat{g}_b = f^2(b)\hat{g}$ for a fixed metric $\hat{g}$ on F^m and a positive smooth function $f(b)$ defined on B^n. Hidden in this definition is the additional data of a horizontal distribution, so one should think of this as taking an existing Riemannian submersion with isometric fibers and allowing the scale of the fiber to change as we move throughout the base. Similarly, a *doubly warped Riemannian submersion metric* is any metric $\tilde{g}$ on $[0,1] \times \mathrm{E}^{n+m}$ such that $\mathrm{id} \times \pi : ([0,1] \times \mathrm{E}^{n+m}, \tilde{g}) \to ([0,1] \times \mathrm{B}^n, dt^2 + h^2(t)\check{g})$ is the Riemannian submersion constructed via Proposition 2.1.1 with family of fiber metrics given by $\hat{g}_b = f^2(t)\hat{g}$ for a fixed metric $\hat{g}$ on F^m and a positive smooth function $f(t)$ defined on $t \in [0,1]$ and the horizontal distribution is specifically flat with respect to t, i.e. is of the form $T[0,1] \oplus H$ where H is any horizontal distribution of $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$. One should think of doubly warped Riemannian submersion metrics as allowing a fixed Riemannian submersion metric to evolve in time by changing the scale of the base and fibers. Note that we have specified all the data necessary to apply Proposition 2.1.1 to construct a Riemannian submersion $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ with fibers isometric to $(\mathrm{F}^m, \hat{g})$. We will say that $\tilde{g}$ is *the doubly warped Riemannian submersion metric associated to the Riemannian submersion* $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ with warping functions $f(t)$ and $h(t)$.

In the special case when $(\mathrm{F}^m, \hat{g}) = (\mathrm{S}^m, ds_m^2)$ a doubly warped product metric can be used to construct metrics on $\mathrm{D}^{m+1} \times \mathrm{B}^n$ and $\mathrm{S}^{m+1} \times \mathrm{B}^n$ in the following way. By allowing the radii of the spheres to decrease to 0 as $t \in [0,1]$ approaches 0 or 1, this has the topological effect of taking the point-wise quotient of the subsets $\mathrm{S}^m \times \{0\} \times \{x\}$ or $\mathrm{S}^m \times \{1\} \times \{x\}$. Doing this at either end produces $\mathrm{D}^{m+1} \times \mathrm{B}^n$ while doing this at both ends produces $\mathrm{S}^{m+1} \times \mathrm{B}^n$. The doubly warped product $g = dt^2 + h^2(t)\check{g} + f^2(t)ds_m^2$ can be used to define a metric on either of these quotients by allowing $f(0) = 0$ or $f(1) = 0$. This metric will have a cusp at $t = 0$ or $t = 1$ if we are not careful about the derivatives of $f(t)$ and $h(t)$ at $t = 0$ or $t = 1$. If we require g to be smooth we must impose the following condition on $f(t)$ [6, Section 1.4.4],

$$f^{(\text{even})}(t) = 0 \text{ and } f'(t) = (-1)^t \text{ for } t \in \{0,1\}. \tag{2.4}$$

The symmetry imposed by allowing $f(t)$ to vanish forces the function $h(t)$ to satisfy the following condition [6, Section 1.4.5],

$$h^{(\text{odd})}(t) = 0 \text{ for } t \in \{0,1\}. \tag{2.5}$$

Thus g defines a smooth metric on $\mathrm{D}^{m+1} \times \mathrm{B}^n$ if $f(t)$ and $h(t)$ satisfy respectively (2.4) and (2.5) at either $t = 0$ or $t = 1$, and defines a smooth metric on $\mathrm{S}^{m+1} \times \mathrm{B}^n$ if $f(t)$ and $h(t)$ satisfy respectively (2.4) and (2.5) at both $t = 0$ and $t = 1$. If, in addition $(\mathrm{B}^n, \check{g}) = (\mathrm{S}^n, ds_n^2)$, then g will define a smooth metric on S^{n+m+1} if $f(t)$ and $h(t)$ satisfy respectively (2.4) and (2.5) at $t = 0$ and respectively (2.5) and (2.4) at $t = 1$.

We would like to extend this idea to use *doubly warped Riemannian submersion metrics* to define smooth metrics on quotients of sphere bundles. Unlike the standard doubly warped product metric, these metrics also depend on a horizontal distribution. If we hope to form a smooth metric on the quotient, these distributions must also be rotationally symmetric. We claim that this is the only additional hypothesis needed to form these quotients.

Proposition 2.1.4. [41, Proposition 2.6] *Let* $\pi : S(\mathrm{E}^{n+m+1}) \to \mathrm{B}^n$ *be the sphere bundle of a rank* $m+1$ *vector bundle with* $O(m+1)$ *invariant horizontal distribution* H*. Let* $\tilde{g}$ *be the doubly warped Riemannian submersion metric associated constructed via Proposition 2.1.1 on* $id \times \pi : [0,1] \times S(\mathrm{E}^{n+m+1}) \to [0,1] \times \mathrm{B}^n$ *with base metric* $dt^2 + h^2(t)\check{g}$*, family of fiber metrics given by* $f^2(t)ds_m^2$*, and horizontal distribution* $T[0,1] \times H$*.*

The metric $\tilde{g}$ *descends to a smooth metric on* $D(\mathrm{E}^{n+m+1})$ *if* $f(t)$ *and* $h(t)$ *satisfy respectively conditions (2.4) and (2.5) at either* $t = 0$ *or* $t = 1$*,* $S(\mathbf{R} \times \mathrm{E}^{n+m+1})$ *if* $f(t)$ *and* $h(t)$ *satisfy respectively conditions (2.4) and (2.5) at both* $t = 0$ *and* $t = 1$*.*

Proof. Using the fact that our horizontal distribution is the product $T[0,1] \oplus H$, the formula (2.3) can be written in this case as

$$\tilde{g} = dt^2 + h^2(t)(\pi^*\check{g}) \circ \mathcal{H} + f^2(t)\hat{g} \circ \mathcal{V}.$$

Restricting the ambient tangent bundle to the submanifold $[0,1] \times F_b \cong [0,1] \times \mathrm{S}^m$, we see that $dt^2 + f^2(T)\hat{g} \circ \mathcal{V} = dt^2 + f^2(t)ds_m^2$, which will descend to a smooth metric on D^{m+1} if $f(t)$ satisfies (2.4) at $t = 0$ or $t = 1$ and will descend to a smooth metric on S^{m+1} if $f(t)$ satisfies (2.4) at both $t = 0$ and $t = 1$. For each horizontal vector field X_i of tangent to H we see that $h^2(t)(\pi^*\check{g})(X_i, X_j)$ will satisfy (2.5) whenever $h(t)$ does. We conclude that $h^2(t)(\pi^*\check{g})(X_i, X_j)$ descends to a smooth function on D^{m+1} or S^{m+1}, and so $h^2(t)(\pi^*\check{g}) \circ \mathcal{H}$ will descend to a smooth tensor on D^{m+1} or S^{m+1}. □

Note that the metric constructed in Proposition 2.1.4 on $D(\mathrm{E}^{n+m+1})$ cannot be constructed by Corollary 2.1.3 as the fibers will not be totally geodesic. That said the second half of Corollary 2.1.3 provides us a natural setting in which rotation invariant horizontal distributions arise, which we will need below Section 2.2 to apply Proposition 2.1.4 to construct metrics on projective spaces. In particular, this means that metrics constructed by 2.1.4 will restrict to metrics on $S(\mathrm{E}^{n+m+1})$ that will agree with those constructed in Corollary 2.1.3 and therefore have totally geodesic fibers. This will be useful when we need to compute the curvature of such metrics in Section 2.1.2.2.

2.1.2. Curvature

In much the same way that the curvatures of an immersed submanifold and the ambient manifold in a Riemannian immersion are related in terms of II via the Gauss [6, Theorem 3.2.4] and Codazzi-Mainardi [6, Theorem 3.2.5] equations, the curvatures of the total space, base space, and fibers in a Riemannian submersion are related in terms of A and T via the O'Neill formulas [53, Theorem 9.28] originally proven in [55].

2.1.2.1. O'Neill formulas

In this section we let $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ be a Riemannian submersion. For the statement of the O'Neill formulas we will let U and V denote any vertical vector fields and X and Y denote any horizontal vector fields of of E^{n+m}. Let X_i with $1 \leq i \leq n$ denote any collection of orthonormal horizontal vector fields. We will not state the O'Neill formulas in their generality, instead we will assume we have a Riemannian submersion with totally geodesic fibers all isometric to $(\mathrm{F}^n, \hat{g})$. The version of the O'Neill formulas for Ricci curvatures are as follows.

Theorem 2.1.5. *If g has totally geodesic fibers, then its Ricci curvatures are as follows*

$$\mathrm{Ric}_g(U,V) = \mathrm{Ric}_{\hat{g}}(U,V) + \sum_{i=1}^{n} g(A_{X_i}U, A_{X_i}U), \tag{2.6}$$

$$\mathrm{Ric}_g(X,U) = \sum_{i=1}^{n} g((\nabla_{X_i}A)_{X_i}X, U), \tag{2.7}$$

$$\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_{\check{g}}(\pi_*X, \pi_*Y) - 2\sum_{i=1}^{n} g(A_X X_i, X_Y X_i). \tag{2.8}$$

2.1.2.2. Curvature of warped products

In this section we let $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check{g})$ denote a Riemannian submersion with horizontal distribution H and totally geodesic fibers isometric to $(\mathrm{F}^m, \hat{g})$. And we let $\tilde{g}$ be the associated doubly warped Riemannian submersion metric on $\mathrm{id} \times \pi : [0,1] \times \mathrm{E}^{n+m} \to [0,1] \times \mathrm{B}^n$. In this setting (2.3) becomes

$$\tilde{g} = dt^2 + h^2(t)(\pi^*\check{g}) \circ \mathcal{H} + f^2(t)\hat{g} \circ \mathcal{V} \tag{2.9}$$

If we denote by II_t the second fundamental form of $\{t\} \times \mathrm{E}^{n+m} \subset [0,1] \times \mathrm{E}^{n+m}$ with respect to $\tilde{g}$ and the unit normal ∂_t, then we can compute II_t readily from

(2.9) and [6, Proposition 3.2.1] as

$$\mathrm{II}_t = h'(t)h(t)(\pi^*\check{g}) \circ \mathcal{H} + f'(t)f(t)\hat{g} \circ \mathcal{V} \tag{2.10}$$

We would like explain how Theorem 2.1.5 is used to compute the Ricci curvature of such metrics. To give a formula for these curvatures we will need the following coordinates. Let Y_i for $1 \le i \le n$ and V_j for $1 \le j \le m$ be horizontal and vertical vector fields of E^{n+m} that comprise a local orthonormal frame with respect to g. If we extend Y_i and V_j to $[0,1] \times \mathrm{E}^{n+m}$ in the obvious way, let $h^2(t)X_i = Y_i$ and $f^2(t)U_j = V_j$. Together with ∂_t, X_i and U_j form a local orthonormal frame of $[0,1] \times \mathrm{E}^{n+m}$ with respect to $\tilde{g}$.

The family of doubly warped Riemannian submersion metrics were previously studied in [56] as part of a broader class of metrics, which we will describe now. Construct a Riemannian submersion metric $\tilde{g}$ on $\mathrm{id} \times \pi : [0,1] \times \mathrm{S}^p \times S(\mathrm{E}^{n+m}) \to [0,1] \times \mathrm{S}^p \times \mathrm{B}^n$ via Proposition 2.1.1 using base metric $dt^2 + \theta^2(t)ds_p^2 + h^2(t)\check{G}$, family of fiber metric $f^2(t)\hat{g}$, and horizontal distribution $T[0,1] \oplus T\,\mathrm{S}^p \oplus H$. It is this class of metrics whose curvatures are computed in [56, Proposition 4.2]. The first step in this computation is to express $\tilde{A}$, the A-tensor of $\tilde{g}$, in terms of the A-tensor of g. This along Theorem 2.1.5 finishes the computation. Our metrics correspond to the special case $\theta(t) \equiv 1$ and $p = 0$. We chose to record the formulas of [56, Proposition 4.2] by rewriting the A-tensor of g in terms of the Ric_g, $\mathrm{Ric}_{\check{g}}$, and $\mathrm{Ric}_{\hat{g}}$ using Theorem 2.1.5.

Lemma 2.1.6. [56, Proposition 4.2] *Let $\tilde{g}$ be a doubly warped Riemannian submersion metric associated to the Riemannian submersion $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^m, \check{g})$ with totally geodesic fibers isometric to $(\mathrm{F}^n, \hat{g})$, then the Ricci curvature of $\tilde{g}$*

are as follows.

$$\mathrm{Ric}_{\tilde g}(\partial_t, \partial_t) = -n\frac{h''}{h} - m\frac{f''}{f}. \tag{2.11}$$

$$\mathrm{Ric}_{\tilde g}(X_i, X_i) = \frac{\mathrm{Ric}_{\check g}(\check Y_i, \check Y_i) - (n-1)h'^2}{h^2} - \frac{h''}{h} - m\frac{f'h'}{fh} + (\mathrm{Ric}_g(Y_i, Y_i) - \mathrm{Ric}_{\check g}(\check Y_i, \check Y_i))\frac{f^2}{h^4}. \tag{2.12}$$

$$\mathrm{Ric}_{\tilde g}(U_j, U_j) = \frac{\mathrm{Ric}_{\hat g}(V_j, V_j) - (m-1)f'^2}{f^2} - \frac{f''}{f} - n\frac{f'h'}{fh} + (\mathrm{Ric}_g(V_j, V_j) - \mathrm{Ric}_{\hat g}(V_j, V_j))\frac{f^2}{h^4}. \tag{2.13}$$

$$\mathrm{Ric}_{\tilde g}(X_i, V_j) = \mathrm{Ric}_g(Y_i, U_j)\frac{f}{h^3} \tag{2.14}$$

$$\mathrm{Ric}_{\tilde g}(X_i, \partial_t) = \mathrm{Ric}_{\tilde g}(U_i, \partial_t) = 0.$$

2.1.2.3. The Canonical Variation

One of the primary difficulties of using Proposition 2.1.1 to construct metrics satisfying certain curvature conditions is precisely the presence of the A tensor terms in Theorem 2.1.5. While we may have good understanding of the curvature of $(\mathrm{B}^n, \check g)$ and $(\mathrm{F}^m, \hat g)$, it is unlikely, unless we have a concrete construction of the fiber bundle E^{n+m}, that we will understand H well enough to compute A. In Lemma 2.1.6 this is avoided by relating the Ricci curvatures of $\tilde g$ to the Ricci curvatures of g. Another approach to avoid this difficulty is the so-called *canonical variation.*

For a given Riemannian submersion $\pi : (\mathrm{E}^{n+m}, g) \to (\mathrm{B}^n, \check g)$ we define the canonical variation to be the family of metrics g_t where the family of fiber metric of $\hat g_b$ of g is replaced by $t^2 \hat g_b$. In other words, we scale the fibers of g by t^2 to construct g_t. Shrinking the fiber spheres causes the corresponding curvatures to blow up, so if the fiber spheres are totally geodesic we may expect this to counteract any negative contribution of the A tensor in the formulas of Corollary 2.1.5.

Proposition 2.1.7. *[53, Proposition 9.70] If g has totally geodesic fibers isometric to $(\mathrm{F}^m, \hat{g})$, then the Ricci curvatures of the canonical variation g_t are as follows*

$$\mathrm{Ric}_{g_t}(U,V) = \mathrm{Ric}_{\hat{g}}(U,V) + t^2 \sum_{i=1}^{n} g(A_{X_i}U, A_{X_i}U), \tag{2.15}$$

$$\mathrm{Ric}_{g_t}(X,U) = t \sum_{i=1}^{n} g((\nabla_{X_i} A)_{X_i} X, U), \tag{2.16}$$

$$\mathrm{Ric}_{g_t}(X,Y) = \mathrm{Ric}_{\check{g}}(\pi_* X, \pi_* Y) - 2t \sum_{i=1}^{n} g(A_X X_i, X_Y X_i). \tag{2.17}$$

We see that Ricci curvatures of g_t on E^{n+m} converge to the corresponding Ricci curvatures of product metric $\check{g} + \hat{g}$ on $\mathrm{B}^n \times \mathrm{F}^m$ as $t \to 0$. Using Theorem 2.1.2 we can construct Riemannian submersions with totally geodesic fibers, and then consider the canonical variation g_t, which by Proposition 2.1.7 will have positive Ricci curvature for t sufficiently small if $(\mathrm{F}^n, \hat{g})$ and $(\mathrm{B}^m, \check{g})$ both have positive Ricci curvature for any horizontal distribution H. This idea was used in [2] to show that metrics of positive Ricci curvature are closed under nontrivial fiber bundles with structure groups acting by isometries on the fibers. This is precisely the construction we will use in Section 2.3 below to reduce the proof of Theorem B to a technical metric construction on an elementary space.

2.2. Projective Spaces

In this section we will use doubly warped Riemannian submersion metrics to construct core metrics for $\mathbf{C}\mathrm{P}^n$, $\mathbf{H}\mathrm{P}^n$, and $\mathbf{O}\mathrm{P}^2$. Recall that the real division algebras are $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, and $\mathbf{O}$. The projective spaces are defined as

$$\mathbf{R}\mathrm{P}^n = (\mathbf{R}^{n+1})^*/\mathbf{R}^* = \mathrm{S}^n \,/\, \mathrm{S}^0$$

$$\mathbf{C}\mathrm{P}^n = (\mathbf{C}^{n+1})^*/\mathbf{C}^* = \mathrm{S}^{2n+1} \,/\, \mathrm{S}^1$$

$$\mathbf{H}\mathrm{P}^n = (\mathbf{H}^{n+1})^*/\mathbf{H}^* = \mathrm{S}^{4n+3} \,/\, \mathrm{S}^3$$

Where, this second equality follows by scaling every vector by $1/|v|$ and restricting the multiplicative action.

We must discuss the octonionic projective plane $\mathbf{O}\mathrm{P}^2$ separately as the multiplicative structure of $\mathbf{O}^*$ restricted to S^7 does not form group. We can define $\mathbf{O}\mathrm{P}^1 := \mathrm{S}^8$ without issue. The action of S^7 by multiplication on $\mathrm{S}^{15} \in (\mathbf{O}^2)^*$ yields a fiber bundle $\eta : \mathrm{S}^{15} \to \mathrm{S}^8$ with fiber S^7, which is the sphere bundle of a disk bundle $D(\gamma^1_{\mathbf{O}})$. This allows us to define $\mathbf{O}\mathrm{P}^2 = \mathrm{D}^{16} \cup_{\mathrm{S}^{15}} D(\gamma^2_{\mathbf{O}})$. The action of S^7 fails to be free on $\mathrm{S}^{8n-1} \in (\mathbf{O}^n)^*$ for $n \geq 3$, and using cohomology operations one can show that there is no CW complex with cohomology ring $\mathbf{Z}[x]/(x^n)$ with $|x| = 8$ and $n \geq 3$ (see [57, 4.L.10]).

2.2.1. The tautological bundle

We have already referenced the existence of a rank 8 vector bundle $\gamma^1_{\mathbf{O}} \to \mathbf{O}\mathrm{P}^1$, for which

$$\mathbf{O}\mathrm{P}^2 = \mathrm{D}^{16} \cup_{\mathrm{S}^{15}} D(\gamma^1_{\mathbf{O}}).$$

We call $\gamma^1_{\mathbf{O}}$ the tautological bundle of $\mathbf{O}\mathrm{P}^1$. In this section we will discuss the tautological bundles $\gamma^n_{\mathbf{R}} \to \mathbf{R}\mathrm{P}^n$, $\gamma^n_{\mathbf{C}} \to \mathbf{C}\mathrm{P}^n$, and $\gamma^n_{\mathbf{H}} \to \mathbf{H}\mathrm{P}^n$. To avoid repetition we will let $\mathbf{K}$ denote the real division algebra of real dimension k, for $k = 1, 2$, or 4, and $\mathbf{K}\mathrm{P}^n$ denote the corresponding projective space.

As points in $\mathbf{K}\mathrm{P}^n$ can be thought of as one-dimensional subalgebras of $\mathbf{K}^{n+1}$, we can define vector bundles $\gamma^n_{\mathbf{K}} \to \mathbf{K}\mathrm{P}^n$ of rank k defined by

$$\gamma^n_{\mathbf{K}} = \{(x, v) \in \mathbf{K}\mathrm{P}^n \times \mathbf{K}^{n+1} : [v] = x\}.$$

The corresponding sphere bundle $S(\gamma^n_{\mathbf{K}})$ is diffeomorphic to $\mathrm{S}^{k(n+1)-1}$. The fiber bundles defined by these sphere bundles, $\eta : \mathrm{S}^{kn+1} \to \mathbf{K}\mathrm{P}^{n-1}$ and $\eta : \mathrm{S}^{15} \to \mathrm{S}^8$ with fibers S^{k-1} and S^7 respectively are called *the generalized hopf fibrations*, while the special case $\eta : \mathrm{S}^3 \to \mathrm{S}^2$ is usually referred to as *the* Hopf fibration. The usual cell structure of $\mathbf{K}\mathrm{P}^n$ uses these hopf fibrations as the attaching maps, which as a smooth manifold can be realized as

$$\mathbf{K}\mathrm{P}^n = \mathrm{D}^{kn} \cup_{\mathrm{S}^{kn-1}} D(\gamma^{n-1}_{\mathbf{K}}). \tag{2.18}$$

2.2.1.1. As a Riemannian submersion

Note that the action of S^{k-1} on $(\mathrm{S}^{kn-1}, ds^2_{kn-1})$ is by isometries as it acts on all of $\mathbf{K}^n$ with the euclidean metric by isometries. There is a unique metric $\check{g}$ on $\mathbf{K}\mathrm{P}^{n-1}$, called the *Fubini-Study metric*, such that $\pi : (\mathrm{S}^{kn-1}, ds^2_{kn-1}) \to (\mathbf{K}\mathrm{P}^{n-1}, \check{g})$ is a Riemannian submersion with totally geodesic fibers isometric to $(\mathrm{S}^{k-1}, ds^2_{k-1})$. If we were to construct a new metric g on S^{kn-1} via Corollary 2.1.3 using base metric $\check{g}$ the Fubini-Study metric, the fiber metric $\hat{g} = ds^2_{k-1}$, and the horizontal connection determined by the round metric, the resulting metric g must equal ds^2_{kn-1} by uniqueness.

One of the key features of the Hopf fibrations as a Riemannian submersion is that it is a rare example where total space, fibers, and base space are all Einstein manifolds. We say that a Riemannian manifold (M^n, g) is an Einstein manifold if $\mathrm{Ric}_g = \Lambda g$ for some constant Λ. The round spheres (S^n, ds_n^2) are Einstein with $\mathrm{Ric}_g = (n-1)g$. We note the Einstein constants for the Fubini-Study metric in the following proposition.

Proposition 2.2.1. [53, Examples 9.81, 9.82, 9.84] *If $\check{g}$ is the Fubini-Study metric on $\mathbf{K}\mathrm{P}^n$, then*

$$\mathrm{Ric}_{\check{g}} = [(n-1)k + 4(k-1)]\check{g}.$$

In the following section we will use doubly warped Riemannian submersion metrics associated to Hopf fibration to construct smooth metrics on $\mathbf{K}\mathrm{P}^n \setminus \mathrm{D}^{kn}$. The fact that the round sphere fibers over projective spaces with totally geodesic fibers allows us to construct these nontrivial core metrics. Sadly we cannot expect to produce any further core metrics in this way, for if $(\mathrm{S}^n, ds_n^2) \to (\mathrm{B}^n, \check{g})$ is a Riemannian submersion with totally geodesic fibers then by [58] the base $(\mathrm{B}^n, \check{g})$ is isometric to a projective space equipped with the Fubini-Study metric. Thus one cannot produce additional core metrics without considering more complicated topological or geometric situations, as we will in the proofs of Theorem B and C.

2.2.2. Core metrics on projective spaces

Having established the basic topology of projective spaces in terms of the tautological bundles as well as the basic geometry of the hopf fibrations, we are prepared to construct core metrics for projective spaces. We start by summarizing the approach taken in [33, 34] to construct core metrics on $\mathbf{C}\mathrm{P}^2$, drawing parallels to the construction of Proposition 2.1.4 and the computation in Lemma 2.1.6 above. Then we apply this construction to general projective spaces and prove Theorem A.

2.2.2.1. The work of Perelman: cores

In [33, Section 1], core metrics are constructed on $\mathbf{C}\mathrm{P}^2$. As we have noted in (2.18), $\mathbf{C}\mathrm{P}^2 \setminus \mathrm{D}^4 = D(\gamma^1_{\mathbf{C}})$. As $S(\gamma^1_{\mathbf{C}}) = \mathrm{S}^3$ and S^3 is a Lie group, there is a left-invariant coframe dx^2, dy^2, and dz^2 dual with respect to ds_3^2 to a left-invariant orthonormal frame X, Y, and Z. The round metric splits as $ds_3^2 = dx^2 + dy^2 + dz^2$ with respect to this coframe. We can choose this coframe so that $dz^2 = d\theta^2$, where $d\theta^2$ is the image of the standard volume form of S^1 under the action of S^1 on S^3. We can define warped product metrics g on $[0,1] \times \mathrm{S}^3$ of the form

$$g = dt^2 + h^2(t)(dx^2 + dy^2) + f^2(t)dz^2 \tag{2.19}$$

Similarly to Proposition 2.1.4, we see that g will descend to a smooth metric on $D(\gamma^1_{\mathbf{C}}) = \mathbf{C}\mathrm{P}^2 \setminus \mathrm{D}^4$ if $f(t)$ and $h(t)$ satisfy respectively (2.4) and (2.5) at $t = 0$. Moreover the boundary will be round if $f(1) = h(1)$ and will have positive principal curvatures if $f'(1)$ and $h'(1)$ are positive.

The Ricci curvatures of such metrics was computed in [34] in terms of the derivatives of the warping functions. This computation relies on the fact that the curvature in a Lie group can be computed in terms of the Lie brackets of the left invariant frame:

$$R(X,Y)Z = \frac{1}{4}[[X,Y],Z].$$

Applying this computation to the case (2.19) yields the following.

Proposition 2.2.2. *[33, Section 2] Let X, Y, and Z are the global left-invariant vector fields of S^3 dual to dx^2, dy^2, and dz^2, then*

$$\begin{aligned}\mathrm{Ric}_g(\partial_t,\partial_t) &= -2\frac{f''}{f} - \frac{h''}{h},\\ \mathrm{Ric}_g(X,X) = \mathrm{Ric}_g(Y,Y) &= 4\frac{h^2-f^2}{h^4} - \frac{h''}{h} - 2\frac{h'^2}{h^2} - \frac{f'h'}{fh} + 2\frac{f^2}{h^4},\\ \mathrm{Ric}_g(Z,Z) &= -\frac{f''}{f} - 2\frac{f'h'}{fh} + 2\frac{f^2}{h^4}.\end{aligned}$$

For a sanity check, we note that Lemma 2.1.6 specialized to the case $m = 1$, $n = 2$, $\mathrm{Ric}_g = 2ds_3^2$, $\mathrm{Ric}_{\hat{g}} = 0$, and $\mathrm{Ric}_{\check{g}} = (1/2)^2 ds_2^2$ agrees with Lemma 2.2.2.

Using Lemma 2.2.2, Perelman concludes his construction of a core metric for $\mathbf{C}\mathrm{P}^2$ by taking the metric g in (2.19) with the choice of $f(t) = \sin t \cos t$ and $h(t) = (1/100)\cosh(t/100)$ restricted to $t \in [0, t_0]$ where $f(t_0) = h(t_0)$. Our construction will directly generalize this construction, using the doubly warped Riemannian submersion metrics of Proposition 2.1.4 to construct core metrics on $\mathbf{K}\mathrm{P}^n$. While defining such metrics is a straightforward generalization, the principal difficulty in showing you have constructed a metric, is ensuring that they will have positive Ricci curvature. Unlike the case considered by Perelman, the spheres we will be considering will not be Lie groups. Thus the use of O'Neill formulas to compute the curvatures in Lemma 2.1.6 represents the main technical contribution needed to prove Theorem A.

2.2.2.2. Generalized cores

We are now ready to construct core metrics on projective spaces. Note that, up until this point, everything we have said about projective spaces has applied equally well to $\mathbf{R}\mathrm{P}^n$. The following construction does not work for $\mathbf{R}\mathrm{P}^n$; we will point out where the construction does not work.

Proof of Theorem A. Note that $\mathbf{K}\mathrm{P}^n \setminus \mathrm{D}^{kn} = D(\gamma_{\mathbf{K}}^{n-1})$ by (2.18). If $f(t)$ and $h(t)$ are smooth functions on $[0,1]$ that satisfy (2.4) and (2.5) respectively at $t = 0$, then by Proposition 2.1.4 the doubly warped Riemannian submersion metric $\tilde{g}$ associated to the Hopf fibration $\eta : (\mathrm{S}^{kn-1}, ds^2_{kn-1}) \to (\mathbf{K}\mathrm{P}^{n-1}, \check{g})$ will descend to a smooth metric on $\mathbf{K}\mathrm{P}^n \setminus \mathrm{D}^{kn}$. As observed, if $f(t) = h(t) = r$ then the metric $\tilde{g}$ restricted to $\{t\} \times \mathrm{S}^{kn-1}$ will agree with $r^2 ds^2_{kn-1}$ by (2.9) and the uniqueness in Proposition 2.1.3. By (2.10) the boundary will have positive principal curvatures provided that $f'(t)$ and $h'(t)$ are positive. It remains to show that we can find such functions $f(t)$ and $h(t)$ for which $\tilde{g}$ will have positive Ricci curvatures.

To begin set $f(t) = \sin(t)$ and $h(t) = \varepsilon$, these clearly satisfy (2.4) and (2.5) at $t = 0$. We would like to apply Lemma 2.1.6 to $\tilde{g}$ in this case. In this case the fiber dimension is $(k-1)$ and the base dimension is $k(n-1)$. We have $\mathrm{Ric}_g = (kn-2)ds^2_{kn-1}$, $\mathrm{Ric}_{\hat{g}} = (k-2)ds^2_{k-1}$, and $\mathrm{Ric}_{\check{g}} = [(n-2)k + 4(k-1)]\check{g}$ by Proposition 2.2.1. Substituting these quantities into Lemma 2.1.6 yields:

$$
\begin{aligned}
\mathrm{Ric}_{\tilde{g}}(\partial_t, \partial_t) &= (k-1),\\
\mathrm{Ric}_{\tilde{g}}(X_i, X_i) &= [(n-2)d + 4(k-1)]\frac{\varepsilon^2 - \sin^2 t}{\varepsilon^4} + (kn-2)\frac{\sin^2 t}{\varepsilon^4},\\
\mathrm{Ric}_{\tilde{g}}(U_j, U_j) &= (k-2)\tan^2 t + 1 + [(nk-2) - (k-2)]\frac{\sin^2 t}{\varepsilon^4}.
\end{aligned}
$$

If we restrict to $t \le t_0$ where $\sin t_0 = \varepsilon$, we see that $\tilde{g}$ has positive Ricci curvature, provided that $k > 1$. This is there reason we must restrict ourselves to $\mathbf{C}\mathrm{P}^n$, $\mathbf{H}\mathrm{P}^n$, and $\mathbf{O}\mathrm{P}^2$.

Clearly if we use $h(t) = \varepsilon$, then $h'(t)$ is not positive, so the boundary will not have positive principal curvatures. We will instead let $h(t)$ be a smooth function satisfying (2.5) at $t = 0$ such that $h'(t) > 0$ for $t > 0$. If we choose $h(t)$ close enough to ε then Ricci curvature will still be positive. We can take, for instance $h(t) = \varepsilon \cosh(t\varepsilon)$. If we again restrict to $t \leq t_0$ where $f(t_0) = h(t_0)$, then $\tilde{g}$ will be a core metric on $\mathbf{K}\mathrm{P}^n \setminus \mathrm{D}^{kn}$. $\square$

2.3. Spherical Fibrations

In this section we will reduce Theorems B and C to two technical constructions that will be carried out in Chapters III and IV respectively. To start we will give a topological decomposition of $\mathrm{E}^{n+m} \setminus \mathrm{D}^{n+m}$. For simplicity, denote $\mathrm{B}^n_* := \mathrm{B}^n \setminus \mathrm{D}^n$. For any fiber bundle $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$ we may fix a trivialization over an embedded $\mathrm{D}^n \hookrightarrow \mathrm{B}^n$, which gives us the following decomposition.

$$\mathrm{E}^{n+m} = \left(\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}\right) \cup_{\mathrm{S}^n \times \mathrm{F}^m} \left(\mathrm{D}^n \times \mathrm{F}^m\right). \tag{2.20}$$

Thus we may delete D^{n+m} from the interior of $\mathrm{D}^n \times \mathrm{F}^m$ on the righthand side of (2.20).

In the case that $\mathrm{F}^m = \mathrm{S}^m$, we note that (1.4) applied to $\mathrm{N}^n = \mathrm{D}^n$ gives us

$$(\mathrm{D}^n \times \mathrm{S}^m) \setminus \mathrm{D}^{n+m} \cong \mathrm{D}^{n+m} \setminus \left(\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}\right).$$

Combining this with (2.20) we can decompose $\mathrm{E}^{n+m} \setminus \mathrm{D}^{n+m}$ as follows

$$\mathrm{E}^{n+m} \setminus \mathrm{D}^{n+m} = \left[\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}\right] \cup_{\mathrm{S}^n \times \mathrm{S}^m} \left[\mathrm{D}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})\right] \tag{2.21}$$

Thus to prove Theorem B it suffices to construct Ricci-positive metrics on each term in (2.21) that can be glued together using Theorem 1.2.2. In Section 2.3.1 we will apply the theory of Riemannian submersions to produce a Ricci-positive metric on $\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}$ that is particularly well adapted to Theorem 1.2.2 that will reduce the proof of Theorem B to the construction of the following metrics on $\mathrm{D}^{n+m} \setminus \left(\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}\right)$.

Theorem 2.3.1. *For $n \geq 2$ and $m \geq 3$ and for all $R > 1$ and $1 > \nu > 0$, there exists a Ricci-positive metric $g_{transition}(R, \nu)$ on $\mathrm{D}^{n+m} \setminus \left(\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}\right)$ such that*

1. *The boundary S^{n+m-1} is isometric to a round sphere,*
2. *The principal curvatures of the boundary S^{n+m-1} are all positive,*
3. *The boundary $\mathrm{S}^{n-1} \times \mathrm{S}^m$ is isometric to $R^2 ds^2_{n-1} + ds^2_m$,*
4. *The principal curvatures of the boundary $\mathrm{S}^{n-1} \times \mathrm{S}^m$ are all greater than $-\nu$.*

If we restrict further to the case $\mathrm{B}^n = \mathrm{S}^n$, then we have a very particular core metric for S^n, the round metric. This allows us in Section 2.3.2 to construct a metric on $S(\mathrm{E}^{n+m+1})|_{\mathrm{D}^n}$ that will both be amenable to Theorem 1.2.2, (2.21), to the Ricci-positive surgery theorem of [3]. This will allow us to reduce Theorem C to Theorem 2.3.1 and a careful application of the work in [30] in Chapter IV below.

2.3.1. Spherical fibrations over cores

In this section we will construct metrics on $\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}$ using Theorem 2.1.2 that can be glued using Theorem 1.2.2 to the metric $g_{\text{transition}}(R, \nu)$. To begin we consider an arbitrary fiber.

Lemma 2.3.2. *Let* $\pi : \mathrm{E}^{n+m} \to \mathrm{B}^n$ *be an* F^m*-bundle with structure group* G *acting by isometries on* $(\mathrm{F}^m, \hat{g})$*. Suppose that* $(\mathrm{F}^m, \hat{g})$ *has positive Ricci curvature and that* $(\mathrm{B}^n_*, \check{g})$ *is a core metric for* B^n*, then for some* $R > 1$ *there is a Ricci-positive metric* g_{piece} *on* $\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}$ *so that the boundary is isometric* $R^2 ds^2_{n-1} + \hat{g}$ *and has positive principal curvatures.*

Proof. Trivialize E^{n+m} over $\mathrm{D}^n \hookrightarrow \mathrm{B}^n$ so that the boundary of $\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}$ is diffeomorphic to $\mathrm{S}^{n-1} \times \mathrm{F}^m$. There is a principal G-bundle $p : \mathrm{P} \to \mathrm{B}^n_*$ so that E^{n+m} is the associated F^m-bundle. Let θ be any principal connection on P that is flat on a neighborhood of the boundary of B^n_*. Let g_t denote the metric on $\mathrm{E}^{n+m}|_{\mathrm{B}^n_*}$ constructed using Theorem 2.1.2 with the base metric $\check{g}$, fiber metric $t^2\hat{g}$, and principal connection θ. Because we chose θ flat on a neighborhood of the boundary, the metrics is isometric to $\check{g} + t^2\hat{g}$ there. By Proposition 2.1.7, we know that g_t has positive Ricci curvature for all t sufficiently small.

Note that the second fundamental form of g_t agrees with $\mathrm{II}_{g_t} = \mathrm{II}_{\check{g}} + 0\hat{g}$. Thus g_t has nonnegative principal curvatures. By applying Corollary 1.2.11 and scaling the metric by $R^2 = (1/t)^2$ we have constructed the desired metric g_{piece}. □

Specializing to linear sphere bundles as in Corollary 2.1.3, Lemma 2.3.2 immediately implies the following.

Corollary 2.3.3. *Let* $\pi : \mathrm{E}^{n+m+1} \to \mathrm{B}^n$ *be a rank* $m+1$ *vector bundle over* B^n*. If* $(\mathrm{B}^n_*, \check{g})$ *is a core metric for* B^n*, then for some* $R > 1$ *there is a Ricci-positive metric* g_{piece} *on* $S(\mathrm{E}^{n+m+1})|_{\mathrm{B}^n_*}$ *so that the boundary is is isometric to* $R^2 ds^2_{n-1} + ds^2_m$ *and has positive principal curvatures.*

We now note that we Theorem B has been entirely reduced to the proof of Theorem 2.3.1. Observe that one could expand Theorem B to include arbitrary fibers if we could construct metrics on $(\mathrm{D}^n \times \mathrm{F}^m) \setminus \mathrm{D}^{n+m}$ similar to $g_{\text{transition}}$ of Theorem B. We make no claims to having constructed such metrics, as we rely heavily on the trivial topology of S^m in Section III.

Proof of Theorem B. By Corollary 2.3.3 there is a Ricci-positive metric g_{piece} on $S(\mathrm{E}^{n+m+1})|_{\mathrm{B}^n_*}$ with boundary isometric to $R^2 ds^2_{n-1} + ds^2_m$ and positive principal curvatures. Let $\nu > 0$ be less than smallest principal curvature of g_{piece}, and take $g_{\text{transition}}(R, \nu)$ as in Theorem 2.3.1 on $\mathrm{D}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})$. By Theorem 1.2.2 and (2.21) there is a Ricci-positive metric on $S(\mathrm{E}^{n+m+1}) \setminus \mathrm{D}^{n+m}$ with boundary isometric to a round sphere with positive principal curvatures. □

2.3.2. Spherical fibrations over disks

In this section we revisit the proof of Lemma 2.3.2 in the special case $(\mathrm{F}^m, \hat{g}) = (\mathrm{S}^m, ds^2_m)$ and $(\mathrm{B}^n, \check{g}) = (\mathrm{S}^n, ds^2_j)$, which will allow us to be more specific about the metric and set up our proof of Theorem C.

Lemma 2.3.4. *For $m \geq 3$, let $\pi : \mathrm{E}^{n+m+1} \to \mathrm{S}^n$ be a rank $m+1$ vector bundle over S^n. Then for each $k > 0$, there exists an $N > 0$ and a Ricci-positive metric g_{root} on $S(\mathrm{E}^{n+m+1}) \setminus \mathrm{D}^{n+m}$ so that the boundary is is isometric a round sphere with positive principal curvatures and so that there are k disjoint isometric embeddings of $(\mathrm{D}^n_N \times \mathrm{S}^m, ds^2_{n-1} + \rho^2 ds^2_m)$ for any $\rho > 0$ sufficiently small, where the radius of the embedded D^n is N.*

Proof. Fix a geodesics disk $\mathrm{D}^n_r \hookrightarrow (\mathrm{S}^n, ds^2_n)$ with radius $r < \pi/2$. We can find an N that depends only k, r, and n such that there are k geodesic balls $\mathrm{D}^n_N \hookrightarrow (\mathrm{D}^n_r, ds^2_n)$ of radius N that are pairwise disjoint. Trivialize $S(\mathrm{E}^{n+m+1})|_{\mathrm{D}^n_r}$ near its boundary so that it has boundary diffeomorphic to $\mathrm{S}^{n-1} \times \mathrm{S}^m$. There is a principal $SO(m+1)$-bundle $p : \mathrm{P} \to \mathrm{D}^n_r$ so that $S(\mathrm{E}^{n+m+1})$ is the associated S^m-bundle. Let θ be any principal connection on P that is flat on a neighborhood of the boundary and is also flat over each of the embedded $\mathrm{D}^n_N \hookrightarrow \mathrm{D}^n_r$. Let g_t denote the metric constructed on $S(\mathrm{E}^{n+m+1})|_{\mathrm{D}^n_r}$ using Corollary 2.1.3 with base metric ds^2_n, fiber metric $t^2 ds^2_m$, and principal connection θ. Because we chose θ to be flat on a neighborhood of the boundary, the boundary is isometric to $\sin^2 r ds^2_{n-1} + t^2 ds^2_m$. Because we chose θ to be flat on each of the D^n_N, the metric is isometric to $ds^2_n + t^2 ds^2_m$.

By Proposition 2.1.7, we know g_t will have positive Ricci curvature for all t sufficiently small. The metric g_t again has nonnegative principal curvatures, after applying Corollary 1.2.11 we can glue this metric using Theorem 1.2.2 to the metric $t^2 g_{\text{transition}}((1/t), \nu)$ for ν sufficiently small to produce the desired metric on $S(\mathrm{E}^{n+m+1}) \setminus \mathrm{D}^{n+m}$ by (2.21). $\square$

CHAPTER III

THE TRANSITION METRIC

In the previous chapter we have reduced the proof of Theorem B to the proof of Theorem 2.3.1. As mentioned in Section 1.3, our original inspiration behind stating Theorem B was our proof of Proposition 1.2.7. This same idea will be the approach we take to prove Theorem 2.3.1 in this chapter. Unlike Proposition 1.2.7, to deal with nontrivial sphere bundles in Lemma 2.3.3 we had to assume an asymmetry in the boundary metric. So we now revisit the first half of the construction of Proposition 1.2.7 to incorporate this asymmetry.

Lemma 3.0.1. *For all $R > 0$ there is a $\nu(R) > 0$ such that for all $\nu(R) > \nu > 0$ there is a Ricci-positive metric $g_{sphere}(R, \nu)$ on $\mathrm{S}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})$ such that*

1. *The boundary $\mathrm{S}^{n-1} \times \mathrm{S}^m$ is isometric to $R^2 ds_{n-1}^2 + ds_m^2$,*
2. *The principal curvatures of the boundary are all greater than $-\nu$.*

Proof. For all $R > 0$, consider the metric $4R^2 ds_n^2 + ds_m^2$ on $\mathrm{D}^n \times \mathrm{S}^m$, where the D^n has radius $\pi/6$. The boundary of this metric is isometric to $R^2 ds_{n-1}^2 + ds_m^2$ and has nonnegative principal curvatures. Apply Proposition 1.2.11 to produce a Ricci-positive metric $g_{\text{handle}}(R)$ with isometric boundary and positive principal curvatures bounded below by $\nu(R) > 0$ (we will be more specific below in Section 3.3) .

For all $\nu(R) > \nu > 0$, glue $(\mathrm{S}^{n-1} \times \left[\mathrm{S}^{m+1} \setminus \bigsqcup_2 \mathrm{D}^{m+1}\right], R^2 ds_{n-1}^2 + g_{\text{docking}}(\nu))$ to $(\mathrm{D}^n \times \mathrm{S}^m, g_{\text{handle}}(R))$ using Theorem 1.2.2, which by (1.3) is a Ricci-positive metric on $\mathrm{S}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})$. Call this metric $g_{\text{sphere}}(R, \nu)$. To see why it has these properties near the boundary, note that near the boundary this metric agrees with $R^2 ds_{n-1}^2 + g_{\text{docking}}(\nu)$. □

The first step in our proof of Theorem 2.3.1 will be to construct an embedding $\mathrm{D}^{n+m} \hookrightarrow \mathrm{S}^{n+m}$ respecting (1.7) and (1.3). By restricting $g_{\text{sphere}}(R,\nu)$ to this embedded D^{n+m} and deleting an embedded $\mathrm{S}^{n-1}\times\mathrm{D}^{m+1}$, we have a metric $g_{\text{disk}}(R,\nu)$. We claim that the embedding can be chosen so that the following is true.

Lemma 3.0.2. *Let $n \geq 2$ and $m \geq 3$, for each $R > 1$ and $1 > \nu > 0$ there exists a Ricci-positive metric $g_{disk}(R,\nu)$ on $\mathrm{D}^{n+m}\setminus(\mathrm{S}^{n-1}\times\mathrm{D}^{m+1})$ such that*

1. *The boundary S^{n+m-1} is isometric to $g_0(R,\nu)$,*

2. *The principal curvatures of the boundary S^{n+m-1} are positive,*

3. *The boundary $\mathrm{S}^{n-1}\times\mathrm{S}^m$ is isometric to $R^2ds_{n-1}^2 + ds_m^2$,*

4. *The principal curvatures of the boundary $\mathrm{S}^{n-1}\times\mathrm{S}^m$ are greater than $-\nu$.*

Where we will give a fairly explicit description of the metric of (1) in Lemma 3.3.4 below. We note that conditions (3) and (4) of Lemma 3.0.2 will follow directly from the construction in Lemma 3.0.1. The biggest difficulty we will run into in the proof of Lemma 3.0.2 will to be to describe the geometric properties of the boundary of the embedding $\mathrm{B}_+^{m+1} \hookrightarrow \big(\mathrm{S}^{m+1}\setminus\mathrm{D}^{m+1}, g_{\text{docking}}(\rho)\big)$. This is precisely because the construction of the metric $g_{\text{docking}}(\rho)$ is itself built out of two very technical metrics glued together using Theorem 1.2.2. It is this embedding that will actually require the application of Theorem I (or, more accurately, the version Theorem II″). We will describe the construction of $g_{\text{docking}}(\rho)$ in Section 3.1 below.

The reason we need to apply the more precise version Theorem II″ is precisely to give a qualitative description of the boundary metric $g_0(R,\nu)$ in Lemma 3.0.2. This metric will be a doubly warped product metric, and for ν sufficiently small will have positive Ricci-curvature. Because the Ricci curvatures of a doubly warped product metric are so elementary, we can write down a piecewise linear path in the space of Ricci-positive metrics connecting it to the round metric.

Lemma 3.0.3. *For each $R > 1$, if $\nu > 0$ is sufficiently small the metric $g_0(R, \nu)$ of Lemma 3.0.2 is Ricci-positive isotopic to the round metric.*

Once we have established Lemmas 3.0.2 and 3.0.3, we can prove Theorem 2.3.1 and consequently Theorem B.

Proof of Theorem 2.3.1. Take the Ricci-positive metric $g_{\text{disk}}(R, \nu)$ on $\mathrm{D}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})$ of Lemma 3.0.2. By (3) and (4) of Lemma 3.0.2, the boundary $\mathrm{S}^{n-1} \times \mathrm{S}^m$ already satisfies conditions (3) and (4) of Theorem 2.3.1. By (2) of Lemma 3.0.2, the boundary S^{n+m-1} has positive principal curvatures bounded below by ε. By Lemma 3.0.3 and Corollary 1.3.4 of Theorem II there is a Ricci-positive metric $g_{\text{transition}}(R, \nu)$ on $\mathrm{D}^{n+m} \setminus (\mathrm{S}^{n-1} \times \mathrm{D}^{m+1})$ so that boundary S^{n+m-1} satisfy (1) and (2) of Theorem 2.3.1 and agrees with $g_{\text{disk}}(R, \nu)$ away from the boundary S^{n+m-1}. This metric clearly satisfies all conditions of Theorem 2.3.1. □

3.1. The work of Perelman, again

The metric $g_{\text{docking}}(\rho)$ is constructed out of metrics on the following topological decomposition of the punctured sphere.

$$\mathrm{S}^{m+1} \setminus \bigsqcup_k \mathrm{D}^{m+1} = \left[\mathrm{S}^{m+1} \setminus \bigsqcup_k \mathrm{D}^{m+1} \right] \cup_\partial [[0,1] \times \mathrm{S}^m] \tag{3.1}$$

While topologically this is completely trivial, the metric constructed on $[0,1] \times \mathrm{S}^m$ is the most delicate and technical aspect of [33] and cannot be avoided. The embedding we wish to construct with respect to $g_{\text{docking}}(\rho)$

$$\iota_{\text{docking}} : \mathrm{B}_{+}^{m+1} \hookrightarrow (\mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}, g_{\text{docking}}(\rho)), \tag{3.2}$$

will be decomposed as a union of two further embeddings

$$\begin{aligned} \iota_{\text{ambient}} &: \mathrm{B}_{+}^{m+1} \hookrightarrow (\mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}, g_{\text{ambient}}(\rho)), \\ \iota_{\text{neck}} &: [0,1] \times \mathrm{D}^{m} \hookrightarrow ([0,1] \times \mathrm{S}^{m}, g_{\text{neck}}(\rho)). \end{aligned} \tag{3.3}$$

We will ultimately apply Theorem II″ to the embeddings (3.3) to construct the embedding (3.2).

3.1.1. Perelman's Ambient Space and Neck

Keeping Proposition 1.2.3 in mind, one would look to doubly warped product metrics on S^{n+m} that admits arbitrarily many disjoint geodesic balls with principal curvatures relatively small compared to the radius. The boundary of these geodesic balls will themselves be doubly warped product metrics, which we will need to correct to prove Proposition 1.2.3. This problem can be minimized by considering metrics of the form $dt^2 + h^2(t)dx^2 + f^2(t)ds_{n-2}$ on $[0, \pi/2] \times \mathrm{S}^1 \times \mathrm{S}^{n-2}$, where the boundary of a geodesic ball will be isometric to warped products (rather than doubly warped products). The following proposition claims that, after rescaling, we can find a Ricci-positive metric on $\mathrm{S}^n \setminus \bigsqcup_k \mathrm{D}^n$ with principal curvatures relatively small.

Proposition 3.1.1. [33, Section 3] *For $m \geq 3$ and any $0 < \rho < r$ sufficiently small there exists an $R_0 > 0$, a function $R : [0, \pi/2] \to [0, R_0]$, and $k > 0$ there is a metric*

$$g_{ambient}(\rho) = \cot r(dt^2 + \cos^2(t)dx^2 + R^2(t)ds^2_{m-1}),$$

defined on S^{m+1} with $t \in [0, \pi/2]$ and $x \in [0, 2\pi]$ such that:

1. *is positively curved,*
2. *the geodesic balls of radius r centered along the subspace $t = 0$ for the metric are isometric to $A_1(-b)db^2 + B_1^2 \cos^2(-b)$ for $b \in [-\pi, 0]$,*
3. *the principal curvatures of boundary of such a geodesic ball are all less than 1,*
4. *the sectional curvatures of the boundary of such a geodesic ball are all greater than 1.*

The construction of the function $R(t)$ is elementary, and the verification of the conditions (1)-(4) is straightforward.

The most technical aspect of [33] is the construction of *the neck*. This is a particular metric on the cylinder that transitions from the boundary metric of $g_{\text{ambient}}(\rho)$ to a round metric in a way that allows the principal curvatures to remain relatively large relative to the radius of the boundary. As the construction of this metric is the central to our proof of Theorem 2.3.1 and hence our proof of Theorem B, we take the time to explain the construction in detail in Appendix C. The aspects of this metric that are needed for the proof of Theorem 2.3.1 are summarized as follows.

Proposition 3.1.2. [33, Assertion] *For $m \geq 3$ and any $0 < \rho < r$ sufficiently small there exists a $l > 0$, a function $A : [0, l] \times [0, \pi] \to \mathbf{R}$, and function $B : [0, l] \to \mathbf{R}$*

there is a metric

$$g_{neck} = da^2 + A^2(a,-b)db^2 + B^2(a)\cos^2(-b)ds^2_{m-1},$$

defined on $[0,k] \times \mathrm{S}^m$ *with* $a \in [0,l]$ *and* $b \in [-\pi, 0]$ *such that:*

1. *has positive Ricci curvature,*
2. *at* $a = l$ *has* $A(a,-b) = A_1(-b)$ *and* $B(l) = B_1$*,*
3. *at* $a = 0$ *has* $A(0,-b) = B(0) = \rho/\lambda$*,*
4. *the principal curvatures at* $a = l$ *are all greater than* 1*,*
5. *the principal curvatures at* $a = 0$ *are all* $-\lambda$*,*
6. *the function* $B(a)$ *is increasing and concave down.*

A proof of Proposition 3.1.2 is given in Section C.3.4.1 below.

Note that the construction of $g_{\mathrm{neck}}(\rho)$ in [33], is the one aspect of our construction that cannot be reduced to dimension $m = 2$. This is why we cannot assert Theorem B with $n_i = 2$.

3.1.2. The Docking Station

In order to apply Theorem I to the two embeddings (3.3) to construct (3.2), we must first describe how to apply Theorem 1.2.2 to the two metrics Propositions 3.1.1 and 3.1.2 to construct the metric $g_{\mathrm{docking}}(\nu)$ out of (A.1.2).

Proof of Proposition 1.2.3. For all $r > \rho > 0$ sufficiently small take $g_{\mathrm{ambient}}(\rho)$ on S^{m+1} and delete the k disjoint geodesic balls of radius r to produce a metric on $\mathrm{S}^{m+1} \setminus \bigsqcup_k \mathrm{D}^{m+1}$. Take $g_{\mathrm{neck}}(\rho)$ on each component of $\bigsqcup_k ([0,l] \times \mathrm{S}^m)$. These are both Ricci-positive metrics by (1) and (1) of Propositions 3.1.1 and 3.1.2. The boundaries of $\mathrm{S}^{m+1} \setminus \bigsqcup_k \mathrm{D}^{m+1}$ are isometric to the boundaries of $\bigsqcup_k ([0,l] \times \mathrm{S}^m)$ at $a = l$ by (2) and (2) of Propositions 3.1.1 and 3.1.2. The principal curvatures are compatible to apply Theorem 1.2.2 by (3) and (4) of Propositions 3.1.1 and 3.1.2. We may therefore glue these metrics together using Theorem 1.2.2 and scale the metric by $(\lambda/\rho)^2$, for any $\nu > \rho$ call the resulting metric $g_{\text{docking}}(\nu)$. By (3) of Proposition 3.1.2, the boundary components of $g_{\text{docking}}(\nu)$ are all isometric to ds_m^2. By (5) of Proposition 3.1.2, the principal curvatures of the boundary are all $-\rho > -\nu$. □

3.1.2.1. Lens Spaces as Docking Stations

Having described the construction of $g_{\text{docking}}(\nu)$ we can now describe how to modify this construction to prove Proposition 1.2.8.

Proof of Proposition 1.2.8. Let $G \leq O(2) \oplus O(n-1)$ be finite with $|G| = d$ that acts freely on S^n under the standard action. Take $g_{\text{ambient}}(\nu)$ on S^{m+1} so that there are k disjoint geodesic balls of radius $r > \rho$ centered along $t = 0$. We can assume that we have chosen r small enough so that the orbits of these geodesic balls under G are also disjoint, so that we have a collection of kd geodesic balls that are left fixed by the action of G. We note that G acts by isometries on $(\mathrm{S}^{m+1} \setminus \bigsqcup_{kd} \mathrm{D}^{m+1}, g_{\text{ambient}}(\rho))$, and so this descends to a Ricci-positve metric $g_{\text{lens}}(\rho)$ on $(\mathrm{S}^{m+1}/G) \setminus \bigsqcup_k \mathrm{D}^{m+1}$. Proceeding identically to Proposition 1.2.3 by gluing necks to $g_{\text{lens}}(\rho)$ using Theorem 1.2.2 we produce a Ricci-positive metric $g_{\text{docking}}(\rho)$ on $(\mathrm{S}^{m+1}/G) \setminus \bigsqcup_k \mathrm{D}^{m+1}$ that satisfies the conclusions of Proposition 1.2.3. The proof of Proposition 1.2.8 now follows identically to the proof of Theorem 1.2.5 by scaling and gluing the cores to $(\mathrm{S}^{m+1}/G) \setminus \bigsqcup_k \mathrm{D}^{m+1}$. □

3.2. Embedding in the docking station

In this section we make explicit choices for the embeddings (3.3), and apply Theorem II″ To construct the embedding (3.2). We also take time to describe the metric $g_{\text{docking}}(\nu)$ restricted to the interior face of $\iota_{\text{docking}} : \mathrm{B}_+^{m+1} \hookrightarrow (\mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1})$, which will be necessary in order to describe the metric $g_{\text{sphere}}(R, \nu)$ restricted to the boundary of $\iota : \mathrm{D}^{n+m} \hookrightarrow \mathrm{S}^{n+m}$ in Lemma 3.0.2.

3.2.1. Embedding in the Ambient Space

In this section we will describe the embedding $\iota_{\text{ambient}} : \mathrm{B}_+^{m+1} \hookrightarrow (\mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1})$. In Proposition 3.1.1, the metric $g_{\text{ambient}}(\rho)$ is defined in terms of the coordinates

$$(t, x, \theta) : [0, 1] \times [0, 2\pi] \times \mathrm{S}^{m-1} \to \mathrm{S}^{m+1},$$

where t is given by the distance from a great circle (with respect to the round metric), x is the arclength parameterization of said great circle, and θ surjects onto the level sets of (t, x). In the construction of $g_{\text{docking}}(\nu)$ of Proposition 1.2.3, we delete a geodesic ball of radius r centered at the point $p = (0, 0, \theta)$ in S^{m+1} (when $t = 0$, θ is the point map) with respect to the metric $g_{\text{ambient}}(\rho)$. We want the embedding ι_{ambient} to have nonnegative definite boundary as well as contain a second geodesic ball of radius r centered at a point $q = (0, x_0, \theta)$ in its interior in order to satisfy (3) and (4) of Lemma 3.0.2. Let $(r_1(s), r_2(s))$ be the arclength parameterization of the geodesic sphere of radius r centered at $(0, 0)$ in the (t, x)-plane with respect to the metric $dt^2 + \cos^2(t)dx^2$. By symmetry of the metric, the boundary of a geodesic ball centered at $(0, x_0)$ can be realized as $(r_1(s), x_0 r_2(s), q)$.

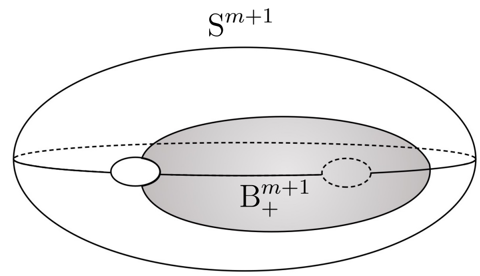

FIGURE 3.1. A schematic of the embedding $\iota_{\text{ambient}} : B_+^{m+1} \hookrightarrow (S^{m+1} \setminus D^{m+1})$ given by the grey region. The small dotted circle indicates a geodesic ball contained in its interior.

Using the terminology of Section A.1, we have a manifold $X_1^{m+1} = B_+^{m+1}$ with faces $Y_1^{m+1} = D^m$ and $\widetilde{Y}_1^{m+1} = D^m$ that intersect in the corner $Z_1^{m-1} = S^{m-1}$. We also have a manifold $M_1^{m+1} = S^{m+1} \setminus D^{m+1}$ with boundary $N_1^m = S^m$. We would like to describe an embedding $\iota_{\text{ambient}} : B_+^{m+1} \hookrightarrow (S^{m+1} \setminus D^{m+1})$ of a manifold with faces into a manifold with boundary relative to the face Y_1^m. To begin we will specify an embedding of the boundary $Y_1 \cup \widetilde{Y}_1 \hookrightarrow (S^{m+1} \setminus D^{m+1})$. Let $(\gamma_1(s), \gamma_2(s))$ be a curve in the (t, x)-plane that starts at the point $(r_1(s_0), r_2(s_0))$ where it meets $(r_1(s), r_2(s))$ perpendicularly, and ends at the point $(0, x_1)$ where it meets the x-axis perpendicularly. Define

$$\begin{aligned} Y_1^m &= \{(t, x, \theta) : (t, x) = (r_1(s), r_2(s)) \text{ for } s \leq s_0\} \cong D^m \\ \widetilde{Y}_1^m &= \{(t, x, \theta) : (t, x) = (\gamma_1(s), \gamma_2(s))\} \cong D^m \,. \end{aligned}$$

Note that Y_1^m lies entirely with the boundary of $S^{m+1} \setminus D^{m+1}$ and that $\widetilde{Y}_1^m$ intersects boundary intersects Y_1^m perpendicularly in the set

$$Z_1^{m-1} = \{(t, x, \theta) \in [0, \pi/2] \times [0, 2\pi] \times S^{m-1} : (t, x) = (r_1(s_0), r_2(s_0))\} \cong S^{m-1} \,.$$

Topologically the union of Y_2 and $\widetilde{\mathrm{Y}}_2$ is an S^m, which bounds a disk in M_1^{m+1}. We may therefore extend this embedding of the boundary to a smooth embedding $\iota_{\mathrm{ambient}} : \mathrm{B}_+^{m+1} \hookrightarrow \mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}$.

As we would like to apply Theorem II$''$, we will need to make a specific choice of the curve $(\gamma_1(s), \gamma_2(s))$ to define ι_{ambient} and record the geometric properties of its faces. We will specifically need to refer to coordinate dependent quantities that utilize the normal coordinates of Section A.1.2.2 below. We note however that the coordinates used to define $g_{\mathrm{ambient}}(\rho)$ do not agree with these normal coordinates.

Lemma 3.2.1. *For any $r_1(s_0) > 0$ and $r_2(s_0) > 0$, one can choose r small enough and a curve $(\gamma_1(s), \gamma_2(s))$ so that the embedding $\iota_{ambient} : \mathrm{B}_+^{m+1} \hookrightarrow \mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}$ defined above in terms of this curve has the following properties.*

1. *The principal curvatures of the face Y_1^m are all greater than -1 with respect to $g_{ambient}(\rho)$,*
2. *The principal curvatures of the face $\widetilde{\mathrm{Y}}_1^m$ are nonnegative, and are positive for all s for which $\gamma_2(s) \neq 0$,*
3. *The image of $\iota_{ambient}$ contains another geodesic ball of $g_{ambient}(\rho)$ of radius r centered at a point $(0, x_1, \theta)$,*
4. *The function $\phi_1(a)$ that defines the face $\widetilde{\mathrm{Y}}_1^m$ in the normal coordinates of Section A.1.2.2 depends only on a and satisfies $\phi_1'(0) = 0$,*
5. *The metric $g_{ambient}(\rho)$ restricted to $\widetilde{\mathrm{Y}}_1^m$ takes the form $ds^2 + R^2(s)ds_{m-1}^2$ where $R''(s) < 0$.*

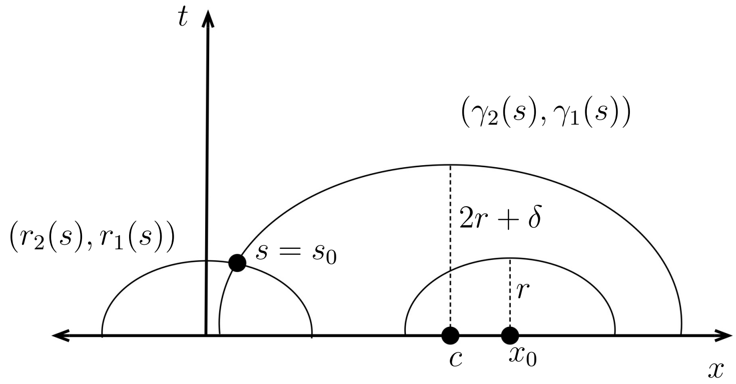

FIGURE 3.2. A schematic for the geodesic balls of Proposition 3.2.1.

Proof. Note that regardless of the chosen curve, that the principal curvatures of Y_1^m will agree with the principal curvatures of N_1^m, which by (3) of Proposition 3.1.1 implies (1).

The tangent space of $\widetilde{\mathrm{Y}}_1^m$ is spanned by a tangent vector $\partial_s = \gamma_1'(s)\partial_t + \gamma_2'(s)\partial_x$ tangent to the (t, x)-plane and by those tangent to $\{\gamma_1(s)\} \times \{\gamma_2(s)\} \times \mathrm{S}^{m-1}$. Because ∂_s is entirely tangent to the (t, x) plane and the metric $dt^2 + \cos^2 t dx^2 = ds_2^2$, we see that that $\mathrm{II}(\partial_s, \partial_s)$ must agree with principal curvatures of a geodesic ball of radius $2r + \delta$ embedded in the unit radius round 2-sphere. This is a standard computation to show that it is $\cot(2r + \delta)$, which is clearly positive provided $r < \pi/4$ and $\delta > 0$ is sufficiently small.

Let σ denote a coordinate vector field of $\{\gamma_1(s)\} \times \{\gamma_2(s)\} \times \mathrm{S}^{m-1}$. We can compute at a given point that the Christoffel symbols are

$$\Gamma^t_{\sigma\sigma} = -R'(t)R(t) \text{ and } \Gamma^x_{\sigma\sigma} = 0.$$

The unit normal vector to the hypersurface $\{\gamma_1(s)\} \times \{\gamma_2(s)\} \times \mathrm{S}^{m-1}$ must lie in the (t,x)-plane and can be written as

$$N = \frac{\cos^2(t)\gamma_2'(s)\partial_t - \gamma_1'(s)\partial_x}{\cos(t)\sqrt{1+\sin^2(t)(\gamma_2'(s))^2}}.$$

We can now compute

$$\mathrm{II}(\partial_\sigma, \partial_\sigma) = -g(\nabla_{\partial_\sigma}\partial_\sigma, N) = -g(\Gamma^t_{\sigma\sigma}\partial_t N) = \frac{\cos(\gamma_1(s))R'(\gamma_1(s))R(\gamma_1(s))\gamma_2'(s)}{\sqrt{1+\sin^2(\gamma_1(s))(\gamma_2'(s))^2}}.$$

As $\gamma_2'(s) \geq 0$ and $R'(t) \geq 0$, this quantity is nonnegative definite. Note that $\gamma_2'(s) = 0$ only when $\gamma_2(s) = 0$.

Clearly if $c - x_1 < \delta$, then this ball contains the geodesic ball of radius r centered at $(0, x_1, \theta)$. By changing δ and c it is possible to intersect the curve $(r_1(s), r_2(s))$ perpendicularly at any point without upsetting the nonnegativity of the principal curvatures. Because $(\gamma_1(s), \gamma_2(s))$ is assumed to be perpendicular to Y_2, this means that $\phi_2'(0) = 0$.

Let $(\gamma_1(s), \gamma_2(s))$ be the unit length parameterization of a geodesic ball of radius $2r + \delta$ centered at $(0, c, \theta)$. The metric $g_{\mathrm{ambient}}(\rho)$ restricted to the face $\widetilde{\mathrm{Y}}_1^m$ clearly takes the following

$$g_{\mathrm{ambient}}(\rho)|_{\widetilde{\mathrm{Y}}_1^m} = ds^2 + R^2(\gamma_1(s))ds^2_{m-1} \tag{3.4}$$

On the other hand the metric $g_{\mathrm{ambient}}(\rho)$ has positive sectional curvature (by (1) of Proposition 3.1.1). It follows that the intrinsic sectional curvature of this boundary is also positive. The sectional curvature of a warped product metric is positive only if $\partial_s^2 R(\gamma_2(s)) < 0$ (see [6, Section 4.2.3]). □

3.2.2. Embedding in the Neck

In this section we will describe the embedding $\iota_{\text{neck}} : [0,l] \times \mathrm{D}^m \hookrightarrow [0,l] \times \mathrm{S}^m$. In Proposition 3.1.2 the metric $g_{\text{neck}}(\rho)$ is defined in the coordinates

$$(a,b,q) : [0,k] \times [-\pi, 0] \times \mathrm{S}^{m-1} \to \times[0,k] \times \mathrm{S}^m,$$

where a is the identity, $-b$ is the distance from the north pole (with respect to the round metric), and q surjects onto the level sets of $-b$.

We have a manifold $\mathrm{X}_2^{m+1} = [0,l] \times \mathrm{D}^m$ with faces $\mathrm{Y}_2^m = \{l\} \times \mathrm{D}^m$, $\widetilde{\mathrm{Y}}_2^m = [0,l] \times \mathrm{S}^m$, and $\check{\mathrm{Y}}_2^m = \{0\} \times \mathrm{D}^m$, and we have a manifold $\mathrm{M}_2^{m+1} = [0,l] \times \mathrm{S}^m$ with boundary $\mathrm{N}_1^m = \{l\} \times \mathrm{S}^m$ and $\check{\mathrm{N}}_2^m = \{0\} \times \mathrm{S}^m$. We would like to specify an embedding $\iota : \mathrm{X}_2^{m+1} \hookrightarrow \mathrm{M}_2^{m+1}$ of a manifold with faces within a manifold with boundary relative to its face Y_2^m. Define the image of ι_{neck} by

$$\iota_{\text{neck}}\left([0,l] \times \mathrm{S}^m\right) = \{(a,b,q) \in [0,l] \times [-\pi,0] \times \mathrm{S}^{m-1} : b \le -b_0\},$$

where $b_0 \in [0, \pi/2]$. These coordinates are consistent with the normal coordinates of Section A.1.2.2, we note that the faces of X_2^{m+1} agree with

$$\begin{aligned}
\mathrm{Y}_2^m &= \{(a,b,q) \in [0,l] \times [-\pi,0] \times \mathrm{S}^{m-1} : b \le -b_0 \text{ and } a = l\} \cong \mathrm{D}^m \\
\widetilde{\mathrm{Y}}_2^m &= \{(a,b,q) \in [0,l] \times [-\pi,0] \times \mathrm{S}^{m-1} : b = -b_0\} \cong [0,l] \times \mathrm{S}^{m-1}, \\
\check{\mathrm{Y}}_2^m &= \{(a,b,q) \in [0,l] \times [-\pi,0] \times \mathrm{S}^{m-1} : b \le -b_0 \text{ and } a = 0\} \cong \mathrm{D}^m\,.
\end{aligned}$$

As we intend to apply Theorem II$''$ to glue together the embeddings (3.3), we need to record some metric properties of ι_{neck}. We note that the coordinates used to define $g_{\text{neck}}(\rho)$ actually do agree with the normal coordinates of Section A.1.2.2 for the corners relative to either face $\{0\} \times \mathrm{D}^m$ or $\{1\} \times \mathrm{D}^m$.

Lemma 3.2.2. *The embedding ι_{neck} satisfies the following*

1. *The principal curvatures of Y_2^m with respect to $g_{neck}(\rho)$ are greater than* 1.

2. *The principal curvatures of $\widetilde{\mathrm{Y}}_2$ with respect to $g_{neck}(\rho)$ are all positive.*

3. *The boundary function that defines $\widetilde{\mathrm{Y}}_2$ as in Section A.1.2.2 is $\phi_2(a) = -b$*

4. *The metric $g_{neck}(\rho)$ restricted to $\widetilde{\mathrm{Y}}_1$ takes the form $ds^2 + B^2(s)$, where $B''(s) < 0$.*

Proof. Note that (1) follows from (4) of Proposition 3.1.2. To compute the principal curvatures of $\widetilde{\mathrm{Y}}_2^m$ we may apply Lemma A.1.12, with $\phi(a) = -b_0$, $\mu(a, b) = A(a, -b)$, and $h(a, b) = B^2(a)\cos^2(-b)ds_{m-1}^2$. One can verify from the formulas in Lemma A.1.12 that (2) is true.

Condition (3) is evident from the definition of ι_{neck}. By Lemma 3.1.2 we see that the metric $g_{\text{neck}}(\rho)$ restricted to $\widetilde{\mathrm{Y}}_2$ takes the form

$$da^2 + B^2(a)\cos^2(-b_0)ds_{m-1}^2.$$

Condition (4) follows from the fact that the second derivative of $B(a)\cos(-b_0)$ is negative by (6) of Lemma 3.1.2. □

3.2.3. Embeddings in the docking station

We have described our two embeddings of (3.3), which we would like to apply Theorem II$''$ to produce the embedding of (3.2). In addition to this, the following

Lemma records the special nature of the metric restricted to the interior face $\widetilde{\mathrm{Y}}^m = \widetilde{\mathrm{Y}}_1^m \cup_{\mathrm{Z}^{m-1}} \widetilde{\mathrm{Y}}_2^m$.

Lemma 3.2.3. *For all $\nu > 0$ sufficiently small, there is an embedding $\iota_{docking}$: $\mathrm{B}_+^{m+1} \hookrightarrow \mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}$ of a manifold with faces into a manifold with boundary with respect to one of its faces, such that the principal curvatures of the interior face $\widetilde{\mathrm{Y}}^m$ are nonnegative and positive near the boundary. Moreover the image of $\iota_{docking}$ contains a geodesic ball of radius r with respect to $g_{ambient}(\rho)$ contained on its interior.*

Restricted to the interior face $\widetilde{\mathrm{Y}}$, the metric $g_{docking}(\nu)$ takes the form

$$ds^2 + k_1^2(s)ds_{m-1}^2,$$

with $s \in [0, P]$. If $\gamma = \cos(b_0)\nu$, then we also have

1. $k_1(0) = \cos(b_0)$,
2. $k_1'(0) = \gamma$,
3. $k_1''(s) < 0$ *for* $s < P$,
4. $k_1^{(even)}(P) = 0$,
5. $k_1'(P) = -1$.

Proof. By (1) and (1) of Lemmas 3.1.1 and 3.1.2 we have two Ricci-positive Riemannian manifolds $(\mathrm{M}_1^{m+1}, g_{\mathrm{ambient}}(\rho))$ and $(\mathrm{M}_2^{m+1}, g_{\mathrm{neck}}(\rho))$, which by (2) and (2) of Lemmas 3.1.1 and 3.1.2 have isometric boundaries $\Phi : \mathrm{N}_1^m \to \mathrm{N}_2^m$. By (3) and (3) of Lemmas 3.2.1 and 3.2.2, we can make sure that this isometry respects the corners $\Phi : \mathrm{Z}_1^{m-1} \to \mathrm{Z}_2^{m-1}$. By (3) and (4) of Lemmas 3.1.1 and 3.1.2 the principal curvatures are such that Theorem 1.2.2 is applicable. By (2) and (5) of Lemma 3.2.1 and (2) and (4) of Lemma 3.2.2 the remaining hypotheses of theorem II″ are true. Thus we may apply Theorem II″ to the embeddings ι_{ambient} and ι_{neck} to define an embedding $\iota_{\text{docking}} : \mathrm{B}_{+}^{m+1} \hookrightarrow (\mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}, g_{\text{docking}}(\nu))$.

We immediately conclude that the principal curvatures of the interior face $\widetilde{\mathrm{Y}}$ are nonnegative and positive near the boundary.

Because both $g_{\text{ambient}}(\rho)$ and $g_{\text{neck}}(\rho)$ are both of the form $da^2 + \mu^2(a,b)db^2 + H^2(a,b)ds_{m-1}^2$, the resulting metric $g_{\text{docking}}(\rho)$ must also take this form. After substituting $b = \phi(a)$ and reparameterizing, we have that the metric restricted to the boundary takes the form $ds^2 + k_1^2(s)ds_m^2$, with $s \in [0, P]$ for some $P > 0$. Conditions (1) and (2) follow from (3) and (5) of Lemmas 3.1.1 and 3.1.2 and the fact that we have scaled by $(\lambda/\rho)^2$. Condition (3) is just the conclusion of Theorem II″. Conditions (4) and (5) follow from the fact that $g_{\text{ambient}}(\rho)$ is a warped product metric on S^{m+1} (see [6, Section 1.4.5]). □

3.3. Assembling the Disk

In this section we will describe an embeddings $\iota_{\text{handle}} : \mathrm{D}^n \times \mathrm{D}^m \hookrightarrow \mathrm{D}^n \times \mathrm{S}^m$ with respect to the metric g_{handle} of Lemma 3.0.1. This will allows us to glue the embedding ι_{handle} to the embedding ι_{docking} of Lemma 3.2.3 to prove Lemma 3.0.2.

3.3.1. Embedding in the handle

We would like to elaborate on the construction of $g_{\text{sphere}}(R,\nu)$ outlined in Lemma 3.0.1, by constructing a specific Ricci-positive metric g_{handle} on $\mathrm{D}^n \times \mathrm{S}^m$.

Definition 3.3.1. *For each $\nu > 0$, let $k_2(a)$ be function defined on $[0, R\pi/3]$ such that*

1. $k_2(0) = 1 - \nu$

2. $k_2(R\pi/3) = 1$

3. $k_2^{(odd)}(0) = 0$

4. $\nu < k_2'(R\pi/3) < 2\nu$

5. $0 \leq k_2''(a) < \nu/(R\pi/3)$

Define the metric $g_{handle}(R,\nu)$ *on* $\mathrm{D}^n \times \mathrm{S}^m$ *to be* $da^2 + (2R)^2 \sin^2(a/2R) ds_{n-1}^2 + k_2^2(a) ds_m^2$.

We claim that $g_{\text{handle}}(R,\nu)$ can be used in Lemma 3.0.1 to construct the metric $g_{\text{sphere}}(R,\nu)$.

Lemma 3.3.2. *For each* R*, if* ν *is sufficiently small, then* $g_{handle}(R,\nu)$ *has positive Ricci curvature. Moreover, it is possible to glue* $g_{handle}(R,\nu)$ *to* $R^2 ds_{n-1}^2 + g_{docking}(\nu)$ *for all* ν *sufficiently small using Theorem 1.2.2. Let* $g_{sphere}(R,\nu)$ *be the corresponding metric which by (1.3) is defined on* S^{n+m}.

Proof. Clearly $g_{\text{handle}}(R,\nu)$ converges to $da^2 + (2R)^2 \sin^2(a/2R) ds_{n-1}^2 + ds_{m-1}^2$ in the C^2-topology as $\nu \to 0$. As this latter metric has positive Ricci curvature, it follows that $g_{\text{handle}}(R,\nu)$ will have positive Ricci curvature for ν sufficiently small.

Using [6, Proposition 3.2.1], we can compute that the principal curvatures of the boundary with respect to $g_{\text{handle}}(R,\nu)$ are $(\sqrt{3}/4)$ and $k'(R\pi/3)$, which are greater than ν for ν sufficiently small. The corresponding principal curvatures of $R^2 ds_{n-1}^2 + g_{\text{docking}}(\nu)$ are 0 and $-\nu$. Clearly the boundary of $g_{\text{handle}}(R,\nu)$ and $g_{\text{docking}}(\nu)$ are both isometric to $R^2 ds_{n-1}^2 + ds_m^2$. Thus Theorem 1.2.2 applies for ν sufficiently small. $\square$

Next we will specify an embedding $\iota_{\text{handle}} : \mathrm{D}^n \times \mathrm{D}^m \hookrightarrow \mathrm{D}^n \times \mathrm{S}^m$. In defining $g_{\text{handle}}(R, \nu)$ we used the coordinates

$$(a, p, b, q) : [0, \pi/2] \times \mathrm{S}^{n-1} \times [-\pi, 0] \times \mathrm{S}^{m-1} \to \mathrm{D}^n \times \mathrm{S}^m,$$

where (a, p) and (b, q) are respectively spherical coordinates on D^n and S^{m+1}. The embedding is given in these coordinates by

$$\iota_{\text{handle}}(\mathrm{D}^n \times \mathrm{D}^m) = \{(a, p, b, q) : b \leq -b_0\}.$$

For some $b_0 \in [0, \pi/2]$. Note that in the notation of Section A.1.2.2 that $\phi_0(a) = -b_0$ and that

$$\begin{aligned}
\mathrm{Y}_0^{n+m-1} &= \{(a, p, b, q) : b \leq -b_0 \text{ and } a = \pi/2\} \cong \mathrm{S}^{n-1} \times \mathrm{D}^m \\
\widetilde{\mathrm{Y}}_0^{n+m-1} &= \{(a, p, b, q) : b = -b_0\} \cong \mathrm{D}^n \times \mathrm{S}^{m-1}
\end{aligned}$$

As we want to glue ι_{handle} to ι_{docking}, we must describe the metric property of the faces of this embedding.

Lemma 3.3.3. *For all $\nu > 0$ there is an embedding $\iota_{handle} : \mathrm{D}^n \times \mathrm{D}^m \hookrightarrow \mathrm{D}^n \times \mathrm{S}^{m+1}$ such that with respect to $g_{handle}(R, \nu)$:*

1. *The principal curvatures of Y_0^{n+m-1} are greater than ρ,*
2. *The principal curvatures of the interior face $\widetilde{\mathrm{Y}}_0^{n+m-1}$ are positive,*
3. *The function that defines the interior face $\widetilde{\mathrm{Y}}_0^{n+m-1}$ in the normal coordinates of Section A.1.2.2 is $\phi_0(a) = -b_0$.*

4. The metric $g_{handle}(\rho)$ restricted to the interior face $\widetilde{\mathrm{Y}}_0^{n+m-1}$ is $ds^2 + (2R)^2\sin^2(a/2R)ds_{n-1}^2 + k_2^2(a)\cos^2(b)ds_m^2$.

Proof. Condition (1) follows from Lemma 3.3.2. For (2), we can compute the principal curvatures of the boundary with respect to $g_{\text{handle}}(R,\nu)$ using Lemma A.1.12 with $\phi_0 = -b_0$, $\mu_0(a,b) = k_1(a)$, and $h_0(a,b) = (2R)^2\sin^2(a/2R)ds_{n-1}^2 + k_1^2(a)\cos^2(b)ds_{m-1}^2$. Condition (3) is evident from the construction of ι_{handle}. Evaluating the metric $g_{\text{handle}}(R,\nu)$ at $b = -b_0$ yields the formula in (4). □

3.3.2. The boundary metric of the Disk

The construction of $g_{\text{sphere}}(R,\nu)$ in Lemma 3.0.1 follows by gluing together $(\mathrm{D}^n \times \mathrm{S}^m, g_{\text{handle}}(R,\nu))$ and $(\mathrm{S}^{n-1} \times \left(\mathrm{S}^{m+1} \setminus \mathrm{D}^{m+1}\right), R^2 ds_{n-1}^2 + g_{\text{docking}}(\nu))$ using Theorem 1.2.2. We note that both the embeddings ι_{handle} and ι_{docking} define interior faces using *the same* boundary function $\phi(a) = -b_0$ in normal coordinates. As these functions already smoothly agree, we conclude that the embeddings ι_{docking} and ι_{handle} glue together, which by (1.7) gives a smooth embedding $\iota_{\text{disk}} : \mathrm{D}^{n+m} \hookrightarrow \mathrm{D}^{n+m}$. We conclude by recording the metric of the boundary of this embedding.

Lemma 3.3.4. *There is a metric $g_{disk}(R,\nu)$ satisfying the claims of Lemma 3.0.2 such that the metric of the boundary S^{n+m-1} takes the form*

$$g_0 = dt^2 + h^2(t)ds_{n-1}^2 + f^2(t)ds_m^2,$$

with $t \in [0,T]$, such that $f(t)$ and $h(t)$ satisfy the boundary conditions of a doubly warped product metric on S^{n+m-1} (see [6, Section 1.4.5]). Moreover there exists $0 < T_0 < T_1 < T$ such that $f(t)$ and $h(t)$ satisfy the following for some ε and δ sufficiently small used in the proof of Theorem 1.2.2.

1. $f''(t) < 0$ *for* $t \geq T_0$

2. $\cos b_0 - 2\gamma R\pi/3 \leq f(t) \leq 2$ *for* $t < T_1$

3. $0 \leq f'(t) \leq 2\gamma$ *for* $t \leq T_1$

4. $f''(t) < 6\gamma/R\pi$ *for* $t \leq T_1$

5. $h(t) \equiv R$ *for* $t \geq T_1$

6. $h''(t) < 0$ *for* $t < T_1$

7. $-\dfrac{h''(t)}{h(t)} > \dfrac{1}{4R}$ *for* $t \leq T_0$

Proof. By (1.7), the two embeddings ι_{handle} and ι_{docking} glue together to define a smooth embedding of $\mathrm{D}^{n+m} \hookrightarrow (\mathrm{S}^{n+m}, g_{\text{sphere}}(R, \nu))$. By (3) of Lemma 3.2.1, there is an embedding of $\mathrm{S}^{n-1} \times \mathrm{D}^{m+1}$ into the interior of this D^{n+m} such that, after deleting we may glue another copy of $([0, l] \times \mathrm{S}^m, g_{\text{neck}}(\rho))$ using Theorem 1.2.2. The boundary is now isometric to $R^2 ds^2_{n-1} + ds^2_m$ with principal curvatures greater than $-\nu$. Call this metric $g_{\text{disk}}(R, \nu)$, which by construction this metric satisfies (3) and (4) of Lemma 3.0.2.

While we know that each part of D^{n+m} in (1.7) had positive principal curvatures near the gluing site by Lemmas 3.2.3 and 3.3.3. When we apply Theorem 1.2.2 this affects the boundary metric as well as the principal curvatures of the respective boundary. Because the boundary function used to define this boundary is identically $-b_0$ near the gluing site, we see that principal curvatures depend only on the ∂_b derivatives of the metric, which by (1) and (1) of Lemmas A.2.1 and A.2.2 will be relatively unchanged by an application of Theorem 1.2.2 (see the proof of Theorem 1.2.2 outlined in Section A.2.2). Thus this metric satisfies (2) of Lemma 3.0.2.

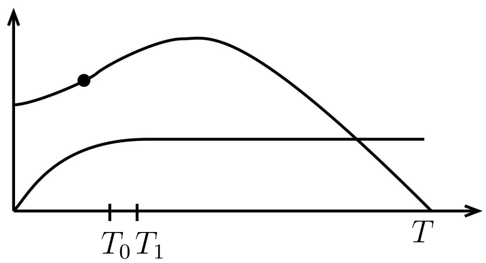

FIGURE 3.3. A graph of the functions $h(t)$ and $f(t)$ satisfying the conditions of Lemma 3.3.4 with the largest inflection point labeled for $f(t)$.

Note that both $g_{\text{handle}}(R,\nu)$ and $g_{\text{docking}}(\nu)$ are of the form

$$da^2 + H^2(a,b)ds_{n-1}^2 + \mu^2(a,b)db^2 + F^2(a,b)ds_{m-1}^2.$$

Clearly the defining equation (A.9) respects this form, so that the resulting metric $g_{\text{sphere}}(R,\nu)$ will also take this form near the gluing site. As the function defining the boundary is $-b_0$, we see that the boundary will take the form $dt^2 + h^2(t)ds_{n-1}^2 + f^2(t)ds_{m-1}^2$ for some smooth functions.

Let $T = R\pi/3 + P$ and $T_1 > R\pi/3 + \varepsilon + \delta$. Note that $R\pi/3 + \varepsilon + \delta$ is largest value of t affected in the smoothing process of Theorem 1.2.2. And therefore $h(t) = R$ so that (5) is true and $f(t) = k_1^2(t)$ for all $t \geq T_1$.

By (3) of Proposition 3.2.3, $f''(t) < 0$ for all $t \geq T_1$. If we consider the value of $f''(t)$ computed in Lemma A.3.1, we see that the terms involving the derivatives of ϕ are all 0, so that $f''(t)$ and $h''(t)$ are both equal to $h_{aa}(a,-b_0)$, where this h is the h in A.1.2.2). We conclude that the effect of applying Theorem 1.2.2 on $f(t)$ and $h(t)$ is dictated by Lemmas A.2.1 and A.2.2 applied directly to the warping functions $f(t)$ and $h(t)$. We will use the same decorations to refer to the two-stage smoothing of $f(t)$ and $h(t)$ as in the proof of Theorem 1.2.2 in Section A.2.2.

When applying (3) of Lemma A.2.1 to $f(t)$, we see that $\overline{f}''(t) < 0$ for $t \geq R\pi/3 - \varepsilon$. When we apply (3) of Lemma A.2.2 to $f(t)$, we see that $\breve{f}''(t) < 0$ for all $t > R\pi/3 - \varepsilon + \delta$. Let $T_0 < R\pi/3 - \varepsilon + \delta$ be the least value for which $f''(t) < 0$ for all $t > T_0$, thus demonstrating (1). Conditions (2), (3), and (4) follow by considering the effect of applying Lemmas A.2.1 and A.2.2 where the boundary values are given by (1), (2), and (3) of Definition 3.3.1 and (1) and (2) of Lemma 3.2.3.

To see why $h(t)$ satisfies (6), one must consult the formula (A.15) for $\breve{h}(t)$ in the specific case where $\overline{h}(t) \equiv R$ for $t \geq R\pi/3 + \varepsilon$ and $\overline{h}''(t) < 0$ for $t < R\pi/3 + \varepsilon$. Condition (7) merely claim that the sectional curvature of $dt^2 + h^2(t)ds^2_{n-1}$ is bounded below uniformly in terms of the sectional curvatures of $dt^2 + (2R)^2 \sin^2(t/2R)ds^2_{n-1}$ for $t \leq T_0$. This is a straightforward conclusion after noting that the denominator in the formulas for sectional curvature (see [6, Section 4.2.4]) is essentially constant during the smoothing process by (1) of Lemma A.2.1 and (1) of Lemma A.2.2, and that the numerator in the formula for the sectional curvature is causing its value to actually increase through the smoothing process by (3) of Lemma A.2.1. □

3.4. The Ricci-positive isotopy

In this section we will demonstrate that the metric $g_0 := g_0(R, \nu)$ is Ricci-positive isotopic to the round metric if ν is sufficiently small. The path is piecewise linear and utilizes the fact that g_0 is a doubly warped product metric. We will first show that g_0 is Ricci-positive isotopic to g_1, where g_1 is a doubly warped product metric with nonnegative sectional curvature. We will conclude by showing any nonnegatively curved doubly warped produce metric is Ricci-positive isotopic to $g_2 = ds^2_{n+m+1}$ (to simplify our subscripts in this section we will reindex our dimensions so that the sphere in question has dimension $n+m+1$). The isotopies will both take the form

$$g_\lambda = dt^2 + ((1-\lambda)h_l(t) + \lambda h_r(t))^2 ds_n^2 + ((1-\lambda)f_l(t) + \lambda f_r(t))^2 ds_m^2.$$

Using an isotopy of doubly warped product metrics will allow us to argue in an elementary fashion using the formulas for curvature found in [6, Section 4.2.4]. In this context, we will let ∂_t, ϕ_i, and θ_j denote an orthonormal frame with respect to g_λ tangent to $[0,T]$, S^n, and S^m respectively.

3.4.1. Path to Non-negatively Curved

Let $f_0(t) = f(t)$ and $h_0(t) = h(t)$ be any smooth functions that satisfy the conditions of Lemma 3.3.4. Let $h_1(t) = h_0(t)$, and let $f_1(t)$ be any function defined on $[0,T]$ that satisfies

1. $f_1^{(\text{odd})}(0) = f_1^{(\text{even})}(T) = 0$
2. $f_1'(T) = -1$
3. $f_1''(t) < 0$ for $t < T$.
4. $f_1(T_1) = f_0(T_1)$
5. $-\gamma < f_1'(t) < 0$ for $t \leq T_1$

The function $f_1(t)$ is concave and crosses over $f_0(t)$ at its largest point of inflection.

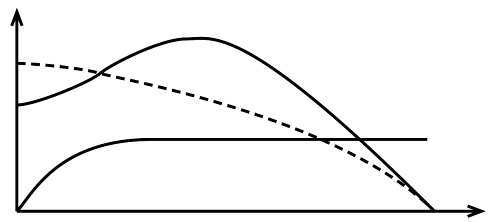

FIGURE 3.4. A graph of $f(t)$ and $h(t)$ with $f_1(t)$ indicated by the dotted line. Compare to (3.3.).

Define functions f_λ and h_λ and metric g_λ as follows.

$$f_\lambda(t) = (1-\lambda)f_0(t) + \lambda f_1(t),$$

$$h_\lambda(t) = (1-\lambda)h_0(t) + \lambda h_1(t) = h(t),$$

$$g_\lambda = dt^2 + h_\lambda^2(t)ds_n^2 + f_\lambda^2(t)ds_m^2.$$

Note that g_λ defines a metric on S^{n+m+1} for all λ as the conditions on the warping functions in [6, Section 1.4.5] are all linear, and we have assumed that f_0, h_0, f_1, and h_1 satisfy these conditions.

Lemma 3.4.1. *If $f(t)$ and $h(t)$ are any functions that satisfy the conditions of Lemma 3.3.4, then for ν sufficiently small the metric g_λ has positive Ricci curvature for all $\lambda \in [0,1]$.*

Proof. First, we will restrict our attention to $t \geq T_1$. Note that $f_0''(t)$ is nonpositive by Lemma 3.2.3, and $f_1''(t)$ is nonpositive by assumption. So $f_\lambda''(t) \leq 0$ for all $\lambda \in [0,1]$. We claim that $-1 \leq f_\lambda'(t) < 1$ for $t \geq T_1$ and each $\lambda \in [0,1]$. Note that $0 < f_0'(T_1) < 2\gamma < 1$ by (2) of Lemma 3.3.4, and that $-1 < -\gamma < f_1'(T_1) < 0$ by assumption. Thus we have $-1 < f_\lambda'(T_1) < 1$ for all λ. We also have that $f_\lambda'(T) = -1$ for all $\lambda \in [0,1]$ by [6, Section 1.4.5]. Because $f_\lambda''(t) \leq 0$ for all $t \geq T_1$, we conclude that $-1 \leq f_\lambda'(t) < 1$.

As g_λ is a doubly warped product for each $\lambda \in [0,1]$, the formulas for its sectional curvature can be found in [6, Section 4.2.4]. By (5) of Proposition 3.3.4, the only sectional curvature that will be nonzero for $t \geq T_1$ will be the following

$$\mathrm{K}_\lambda(\partial_t, \theta) = \frac{-f_\lambda''(t)}{f_\lambda(t)}, \quad \mathrm{K}_\lambda(\theta_i, \theta_j) = \frac{1-(f_\lambda'(t))^2}{f_\lambda^2(t)}, \text{ and } \mathrm{K}_\lambda(\phi_i, \phi_j) = \frac{1}{R^2}.$$

As we have observed $f_\lambda''(t) < 0$ and $-1 < f_\lambda(t) < 1$ for $T_1 \leq t < T$, thus these first two terms are positive for $T_1 \leq t < T$. One can check that the limit of these terms as $t \to T$, is proportional to $f_\lambda'''(t)$, which is necessarily positive. Thus these three sectional curvatures are positive, and so $\mathrm{Ric}_{g_\lambda} > 0$ for all $T_1 \leq t \leq T$.

Next, consider $T_0 \leq t < T_1$. Note that by definition $f_1''(t) < 0$, and by (1) of Lemma 3.2.3 that $f_0''(t) < 0$. Thus $f_\lambda''(t) < 0$ for all $\lambda \in [0,1]$. Similarly we have $h_\lambda''(t) \leq 0$ by (6) of Lemma 3.2.3. Thus $\mathrm{K}_\lambda(\partial_t, \theta) > 0$ and $\mathrm{K}_\lambda(\partial_t, \phi) \geq 0$ from [6, Section 4.2.4], and we conclude that $\mathrm{Ric}_{g_\lambda}(\partial_t, \partial_t) > 0$.

For the remaining Ricci curvatures we will utilize (2) and (3) of Lemma 3.2.3. Note that by assumption $f_1(t)$ satisfies the same bound as (2) in absolute value and the upper bound of (3), so these bounds must remain true for all $f_\lambda(t)$. Applying these to the formulas of [6, Section 4.2.4] gives us the following bounds

$$\mathrm{K}_\lambda(\theta_i, \theta_j) \geq \frac{1-(2\gamma)^2}{\cos^2 b_0} \text{ and } \mathrm{K}_\lambda(\theta, \phi) \geq -\frac{6\gamma h_\lambda'(t)}{R\pi(\cos b_0 - 2\gamma R\pi/3)h_\lambda(t)}.$$

We can take $\gamma = \nu \cos b_0$ to be arbitrarily small, so that

$$\mathrm{K}_\lambda(\theta_i, \theta_j) = \frac{1}{\cos^2 b_0} + O(\nu), \text{ and } \mathrm{K}_\lambda(\theta, \phi) = O(\nu).$$

We immediately have that $\operatorname{Ric}_{g_\lambda}(\theta,\theta) = \mathrm{K}_\lambda(\partial_t,\theta) + (m-2)(1/\cos^2 b_0) + O(\nu)$, which will be positive if ν is taken initially to be small enough.

Because $h'_\lambda(t) = h'_0(t)$, $h'_0(T_1) \equiv 0$, $h'_0(0) = 1$, and $h''_0(t) < 0$ for $t < T_1$, we conclude that $h'_\lambda(t) < 1$ for all $t > 0$. And hence that $\mathrm{K}_\lambda(\phi_i,\phi_j) > 0$. We conclude that $\operatorname{Ric}_{g_\lambda}(\phi,\phi) = \mathrm{K}_\lambda(\partial_t,\phi) + (n-2)\,\mathrm{K}_\lambda(\phi_i,\phi_j) + O(\nu)$ is positive if ν is taken sufficiently small.

Finally, we consider $0 \le t < T_0$. For $\operatorname{Ric}_{g_\lambda}(\theta,\theta)$ and $\operatorname{Ric}_{g_\lambda}(\phi,\phi)$, we argue similarly as for $T_0 \le t < T_1$ taking ν sufficiently small. For $\operatorname{Ric}_{g_\lambda}(\partial_t,\partial_t)$, we no longer have $f''_\lambda(t) < 0$. But by (4) and (7) of Proposition 3.2.3, we have that

$$\mathrm{K}_\lambda(\partial_t,\theta) = O(\nu) \text{ and } \mathrm{K}_\lambda(\partial_t,\phi) > \frac{1}{4R}.$$

We conclude $\operatorname{Ric}_{g_\lambda}(\partial_t,\partial_t) = (n-1)/4R + O(\nu)$ is positive if ν is taken sufficiently small. □

3.4.2. Path to Round

Consider a Ricci-positive doubly warped product metric

$$g_1 = dt^2 + h_1^2(t)ds_{n-1}^2 + f_1^2(t)ds_{m-1}^2,$$

defined on $[0,\pi T/2]$. Note that g_1 has nonnegative sectional curvature if and only if $f''_1(t) \le 0$ and $h''_1(t) \le 0$. We will show that any such metric is Ricci-positive

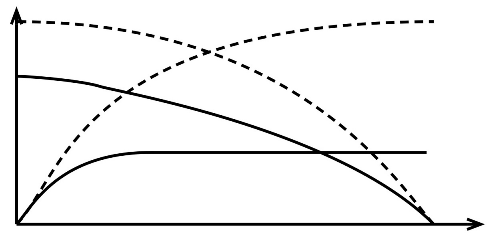

FIGURE 3.5. A graph of $f_1(t)$ and $h_1(t)$ with $T\sin(t/T)$ and $T\cos(t/T)$ indicated by dotted lines.

isotopic to a round metric. For $\lambda \in [1,2]$ we define,

$$h_\lambda(t) = ((2-\lambda)h_1(t) + \lambda T\sin(t/T)),$$
$$f_\lambda(t) = ((1-\lambda)f_1(t) + (\lambda-1)T\cos(t/T)),$$
$$g_\lambda = dt^2 + h_\lambda^2 ds_n^2 + f_\lambda^2 ds_m^2.$$

Note g_λ interpolates between g_1 and $g_2 = T^2 ds_{n+m+1}^2$ the round metric of radius T.

Lemma 3.4.2. *If $f_1''(t) \leq 0$, $h_1''(t) \leq 0$, and g_1 has positive Ricci curvature, then g_λ has positive Ricci curvature for all $\lambda \in [1,2]$.*

Proof. Note that $f_\lambda''(t) < 0$ and $h_\lambda''(t) < 0$ for all $\lambda \in (1,2]$ and $t \in (0, \pi T/2)$. Thus we immediately have by [6, Section 4.2.4] that $\mathrm{K}_\lambda(T,\theta) > 0$ and $\mathrm{K}_\lambda(T,\phi) > 0$ for all $\lambda \in (1,2]$ and $t \in (0,\pi T/2)$. It is straightforward to see that the limit at t approaches the endpoints of these curvatures is proportional to $h_\lambda'''(T)$ or $f_\lambda'''(0)$, which combined with the boundary conditions of [6, Section 1.3.4] implies that these curvatures remain positive for $t = 0, \pi T/2$.

Because $f_\lambda''(t) \leq 0$ and $h_\lambda''(t) \leq 0$, it must be that $f_\lambda'(t)$ and $h_\lambda'(t)$ lie between their extreme values: $0 \leq f_\lambda'(t) \leq 1$ and $-1 \leq h_\lambda'(t) \leq 0$. Thus by [6, Section 4.2.4] we have $\mathrm{K}_\lambda(\theta_i,\theta_j) \geq 0$, $\mathrm{K}_\lambda(\theta,\phi) \geq 0$, and $\mathrm{K}_\lambda(\phi_i,\phi_j) \geq 0$ for all $\lambda \in [1,2]$.

It follows that g_λ has positive Ricci curvature for all $\lambda \in (1, 2]$ and by assumption g_1 has positive Ricci curvature. □

At this point we have finished the proof of Lemma 3.0.3 and consequently our proofs of Proposition 2.3.1 and Theorem B as well.

Proof of Lemma 3.0.3. By Lemma 3.3.4 and Lemma 3.4.1, the boundary metric g_0 of $g_{\text{disk}}(\rho)$ restricted to the boundary S^{n+m-1} can be connected by a path g_λ of Ricci-positive metrics to a doubly warped product metric g_1, where both warping functions have nonpositive second derivative by construction. By Lemma 3.4.2, this nonnegatively curved doubly warped product metric can be connected by a path g_λ of Ricci-positive metrics to the round metric $g_2 = T^2 ds^2_{n+m+1}$. □

CHAPTER IV

PLUMBING DISK BUNDLES OVER SPHERES

In this Chapter we describe precisely what plumbing is, and how it is used to construct smooth manifolds with particular topological characteristics. In [30], Ricci-positive metrics are constructed on a particular class of simply connected manifolds that occur as the boundary of plumbing disk bundles over spheres, which includes a large number of exotic spheres. We claim that the techniques of [30] can be combined with the proof of Theorem B to construct a core metric on the same spaces, proving Theorem C. We begin in Section 4.1 by introducing the basic topological construction related to plumbing and discuss the necessary background of exotic spheres. We conclude in Section 4.2 by discussing the work of [3, 30] and its application to the proof Theorem C.

4.1. Topological Background

We begin in Section 4.1.1 by introducing the basic constructions of plumbing. We will pay special attention to the case of disk bundles over spheres and their relationship to surgery. In Section 4.1.2 we discuss some of the basics of exotic spheres, in particular which exotic spheres can be described as boundaries of plumbings. We conclude in Section 4.1.3 by discussing the relationships between exotic spheres and exotic smooth structures on arbitrary smooth manifolds, which accounts for all of the corollaries to Theorem C in Section 1.3.3.

4.1.1. Plumbing

Given a D^m bundle E_1 over B^n and a D^n bundle E_2 over B^m, fix $\mathrm{D}^n \hookrightarrow B^n$ and $\mathrm{D}^m \hookrightarrow B^m$. Trivializing the E_i over these embedded disks yields embeddings of disk bundles

$$\Phi_1 : \mathrm{E}_1\,|_{\mathrm{D}^n} \to \mathrm{D}^n \times \mathrm{D}^m \;\text{ and }\; \Phi_2 : \mathrm{E}_2\,|_{\mathrm{D}^m} \to \mathrm{D}^m \times \mathrm{D}^n\,.$$

Let $\sigma : \mathrm{D}^n \times \mathrm{D}^m \to \mathrm{D}^m \times \mathrm{D}^n$ be the obvious map. We denote by $\mathrm{E}_1 \Box \mathrm{E}_2$ *the plumbing of* E_1 *and* E_2, which we define as

$$E_1 \Box E_2 = (E_1 \sqcup E_2)\,/\sim \;\text{ where } x \sim \Phi_2^{-1} \circ \sigma \circ \Phi_1(y). \tag{4.1}$$

We consider $E_1 \Box E_2$ as a smooth manifold with boundary by smoothing the inherent corners in the definition (see Lemma A.1.6, for a description of smoothing corners). If B^n and B^m are connected, then the diffeomorphism type of $\mathrm{E}_1 \Box \mathrm{E}_2$ is independent of the choices made in the definition. If E_1 and E_2 are oriented disk bundles then an orientation on $E_1 \Box E_2$ can be chosen to agree with the orientation of E_1, but may disagree with E_2 if n or m is odd. We will assume that $E_1 \Box E_2$ is oriented to agree with an orientation on E_1.

More generally, a *plumbing diagram* is a bipartite graph $G = (V_0, V_1, E)$ with the first vertex set V_0 being labeled by D^m-bundles over n-dimensional manifolds and V_1 being labeled by D^n-bundles over m-dimensional manifolds. If $n = m$, then we will consider arbitrary graphs $G = (V, E)$ with vertices labeled by D^n-bundles over n-dimensional manifolds. In either case, to such a labeled graph G we define a smooth manifold with boundary, denoted $P(G)$, which we call *the plumbing of* *the labeld graph* G. To construct $P(G)$, first take the disjoint union of all of E_i the disk bundles associated to each vertex v_i of G. Next, trivialize the bundle E_i over as many disks as there are edges emanating from v_i. Finally, make the same identification as in (4.1) between any two bundles whose vertices are connected by an edge. In order for the orientation to be well defined we must consider *rooted graphs*, graphs with a designated vertex as *the root*. We will assume the orientation on $P(G)$ agrees with the orientation of the bundle associated to the root. See [59, Section V] for further detail.

Plumbing is a useful way to construct manifolds with certain characteristic numbers and intersection forms. If we restrict to $n = m$, then the intersection form of $P(G)$ can be expressed in terms of the adjacency matrix of G labeled by Euler classes of the various bundles [59, V.1.5]. One can also relate the characteristics classes of $P(G)$ to the characteristics classes of the bundles and the base manifolds in the plumbing diagram. If we restrict to plumbing linear disk bundles over spheres, then the homotopy type of such plumbings is straightforward to understand: it is $\bigvee^i \mathrm{S}^1 \vee \bigvee^j \mathrm{S}^n \vee \bigvee^k \mathrm{S}^m$, where i is the rank of $\pi_1(G)$, $j = |V_0|$, and $k = |V_1|$. As we are interested in constructing Ricci-positive metrics on these plumbings, we will restrict ourselves to simply connected manifolds, so we will only consider plumbing diagrams coming from trees.

The boundary of plumbings are smooth closed manifolds of dimension $n + m - 1$. If we still restrict ourselves to plumbing diagrams with underlying graphs as trees and only disk bundles over spheres, we can describe the boundary of such plumbings in a familiar way. Let $v \in V_0$ be a free vertex (a vertex adjacent to a single edge), then we have

$$\partial(P(G)) = \left[\partial(P(G \setminus \{v\})) \setminus \left(\mathrm{S}^{n-1} \times \mathrm{D}^m\right)\right] \cup_f (\mathrm{D}^n \times \mathrm{S}^{m-1}), \tag{4.2}$$

where f is some diffeomorphism between the respective boundaries. To see this, we note that removing $\mathrm{D}^n \times \mathrm{D}^m$ from a D^m bundle has the effect of removing a $\mathrm{D}^n \times \mathrm{S}^{m-1}$ from the boundary, and that $\partial(E_i) \setminus (\mathrm{D}^n \times \mathrm{S}^{m-1}) \cong \mathrm{D}^n \times \mathrm{S}^{m-1}$. When we compare (4.2) to (1.1), we see that $\partial(P(G))$ can be constructed out of a sequence of alternating $(n-1)$ and $(m-1)$ surgeries on the manifold $\partial(\mathrm{E}_1)$, the sphere bundle of the root.

We will denote by $\partial\mathcal{TP}_k$ the class of tree-like plumbings: all manifolds realized as boundaries of plumbing linear D^k-bundles over S^k where the underlying graphs are trees. It was observed in [32] that this family is closed under connected sum. If ε_k denotes the trivial bundle, then we denote by $G_0 \# G_1$ any graph of the form depicted in Figure (4.1.), then by [32, Proposition 2.6] we have

$$\partial P(G_0 \# G_1) = \partial P(G_0) \# \partial P(G_1). \tag{4.3}$$

It is an elementary exercise in topology to use (4.2) to show that $\partial(E_1 \square E_2)$ has the homotopy type of a sphere if $n < m$. If Θ_d denotes the group of exotic d-dimensional spheres (see Section 4.1.2 for further discussion), then for each $n < m$ and $n + m - 1 = d$ we define the *Milnor plumbing pairing*

$$\sigma_{n,m} : \pi_{n-1}(SO(m)) \times \pi_{m-1}(SO(n)) \to \Theta_d, \text{ where } \sigma_{n,m}(\gamma_1, \gamma_2) = \partial(P(\mathrm{E}_1 \square \mathrm{E}_2)). \tag{4.4}$$

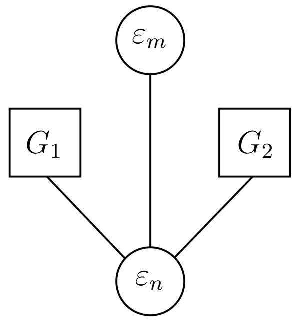

FIGURE 4.1. The graph $G_1 \# G_2$ can be any graph as depicted, where the vertex labeled by ε_n is connected via an edge to any vertex of G_1 and G_2 labeled by a D^m-bundle.

Here, E_i is the disk bundle defined by the clutching function γ_i. Though each set involved in the definition (4.4) is a group, the plumbing pairing $\sigma_{n,m}$ is not bilinear. See [60] for the formula; it is in fact sesquilinear.

The two sets of manifolds $\partial \mathcal{TP}_k$ and $\operatorname{Im} \sigma_{p,q}$ are precisely those for which Ricci-positive metrics are constructed in [30], and are precisely the sets we have claimed admit core metrics in Theorem C. Though $\operatorname{Im} \sigma p, q \subset \Theta_{p+q-1}$, the set $\partial \mathcal{TP}_k$ contains many manifolds that are not homotopy spheres. In particular, we can see from our discussion above, that $H_k(\mathrm{M}^{2k-1}) = \mathbf{Z}^{2j}$, where j is the number of vertices of G. When k is even, this class of manifolds has been described in [32, Theorem C], as those $(k-2)$-connected, $(k-1)$-parallelizable $(2k-1)$-dimensional manifolds (modulo the action of Θ_{2k-1}, which we will discuss in Section 4.1.2). We will discuss in Section 4.1.2 which exotic spheres occur in $\partial \mathcal{TP}_k$.

4.1.2. Exotic Spheres

Let Θ_n denote the set of h-cobordism classes of smooth n-dimensional manifolds that are homotopy equivalent to the sphere. For $n > 4$, the h-cobordism

Theorem implies that Θ_n can be thought of as diffeomorphism classes of smooth homotopy spheres, which can also be described as the set of smooth structures on the topological sphere. Smooth structures which are non-diffeomorphic to the standard smooths structure, are typically called *exotic* smooth structures. This is why the set Θ_n is also called the set of exotic spheres. This set is endowed with a group structure via connected sum, in much the same way as the set of cobordism classes of smooth manifolds. We will now outline the key ideas from the seminal work of [46] on exotic spheres.

Using obstruction theory it can be shown that every exotic sphere has trivial stable normal bundle. This framing is not unique, and the class of all framings can be generated by the action of $\pi_n(SO(n+1))$. This action extends to the group Ω_n^{fr} of framed n-dimensional manifolds by modify the framing on a embedded $\mathrm{D}^n \hookrightarrow \mathrm{M}^n$. This gives rise to a well defined map $\Theta_n \to \Omega_n^{\text{fr}}/\pi_n(SO(n+1))$ sending a homotopy sphere to the class of all possible framings on the underlying sphere. This map has a kernel bP_{n+1}, which is those homotopy spheres which occur as the boundary of a parallelizable $(n+1)$-dimensional smooth manifold. The Pontryagin collapse map $\Omega_n^{\text{fr}} \to \pi_n^S$ is an isomorphism, and the action of $\pi_n(SO(n+1))$ under this isomorphism can be identified with the action defined via the J-homomorphism $J_n : \pi_n(SO(n+1)) \to \pi_{2n+1}(S^{n+1}) \cong \pi_n^S$. This composition gives rise to an injective map $\eta_n : \Theta_n/bP_{n+1} \to \pi_n^S/\operatorname{Im} J_n$. The approach taken to classify Θ_n, initiated in [46], is precisely to study bP_{n+1} and $\pi_n^S/\operatorname{Im} J_n$.

If $n \neq 4k+2$, then η_n is shown to be surjective in [46], and in dimension $n = 4k+2$, the cokernel was shown to either be trivial or order 2. Determining this indeterminacy has since become known as "the Kervaire invariant one problem." This problem, though fairly elementary to state, has only been resolved in the past

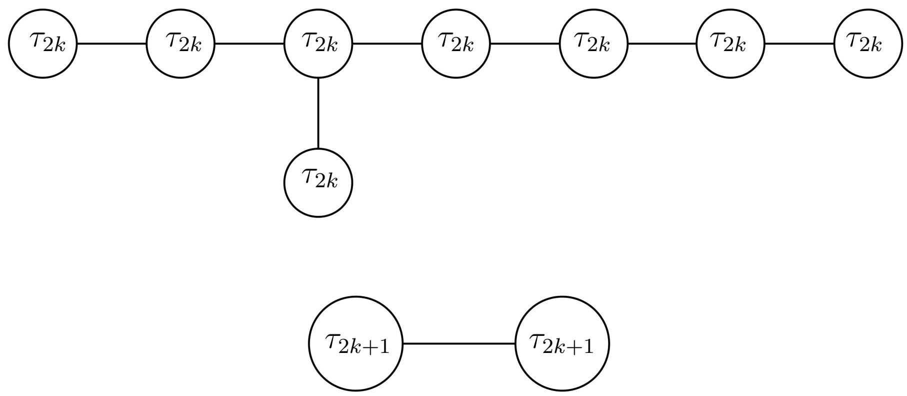

FIGURE 4.2. The E_8 and A_2 graphs, for which $\partial(P(E_8))$ generates bP_{4k} and $\partial(P(A_2))$ generates bP_{4k+2}.

decade in [61]. If $n \equiv 2 \mod 4$, then the cokernel of η_n is trivial other than when $n = 6, 14, 30, 62$, and maybe 126.

The order of the subgroup bP_{n+1} is also determined in [46]. It is shown using surgery that when $n = 2k$, that bP_{n+1} is trivial. When $n = 2k+1$ an explicit generator is constructed using plumbing. If E_8 and A_2 are as in (4.2.), then $\partial(P(E_8))$ generates bP_{4k} and $\partial(P(A_2))$ generates bP_{4k+2}. The order of bP_{4k} is computed exactly, and it is super exponential in k. The order of bP_{4k+2} is shown to either be 1 or 2, and computing the order of bP_{4k+2} is equivalent to the Kervaire invariant one problem: $bP_{4k+2} = \mathbf{Z}/2\mathbf{Z}$ if and only if the cokernel of η_n is trivial. Thus from the above discussion, $bP_{4k+2} = \mathbf{Z}/2\mathbf{Z}$ other than when $n = 6, 14, 30, 62$, and maybe 126. Thus other than dimension $n = 126$ the computation of θ_n has been reduced to computing π_n^S and the extension problem:

$$0 \to bP_{n+1} \to \theta_n \to \pi_n^S / \operatorname{Im} J \to \operatorname{coker} \eta_n \to 0.$$

Which is known to split in most dimensions: when $n \neq 2^j - 3$ [62].

Regardless of their order, we know that bP_{2k} is generated by elements in $\partial\mathcal{TP}_k$. Because the group operation of Θ_n is connected sum and because we know that $\partial\mathcal{TP}_k$ is closed under connected sum by [32, Proposition 2.6], we see that $bP_{2k} \subseteq \partial\mathcal{TP}_k$. We therefore have concrete geometric descriptions of each element in bP_{2k} as either the boundary of a suitable plumbing diagram or, by (4.2), as a sequence of $(k-1)$-surgeries on an S^{k-1}-bundle over S^k. These concrete geometric constructions allow us to construct very concrete metrics on this family of exotic spheres. It is these dual viewpoints that allow [30] to construct Ricci-positive metrics on this family, and will allow us in Section 4.2 to construct core metrics for this family as well.

One can ask if there are any exotic spheres that represent a nontrivial class $\pi_n^S/\operatorname{Im} J_n$ represented in this way. The manifold $P(G)$ will often be stably parallelizable as most of the unstable vector bundles $\pi_{k-1}(SO(k))$ are stably trivializable. It may happen occasionally that a homotopy sphere is built in this way as the boundary of $P(G)$ that is not stably parallelizable, for example, this is always true when considering $\operatorname{Im}\sigma_{p,q}$, if $p < q$, and we have chosen $\gamma_1 \in \pi_{p-1}(SO(q)) \cong \widetilde{KO}^q(\mathrm{S}^p)$ that is not stably trivializable. Even in these cases, it might be that there is an entirely different stably parallelizable manifold that is bounded by the sphere built in this way. While we do not know any general nontriviality statement for $\sigma_{p,q}$, in [47, Satz 12.1] it was noted that the nontrivial elements of $\Theta_8 \cong \Theta_{16} \cong \Theta_{19}/bP_{20} \cong \mathbf{Z}/2\mathbf{Z}$ are in the image of $\sigma_{4,5}$, $\sigma_{8,9}$, and $\sigma_{9,11}$ respectively.

While the image of $\sigma_{p,q}$ in Θ_d/bP_{d+1} is not always trivial, we note that it is not surjective. If $\alpha : \Theta_d \to KO^{-d}(\mathrm{pt})$ is the KO-valued index of the spin Dirac operator defined in [22], then Σ^d admits a positive scalar curvature metric if and only if $\alpha(\Sigma^d) = 0$. There are examples of homotopy spheres $\Sigma^9 \in \Theta_9$ with $\alpha(\Sigma^9) \neq 0$ [26, Theorem 2]. But by Proposition 2.1.7, (4.2), and [23, Theorem A] the image of $\sigma_{p,q}$ will always admit a metric with positive scalar curvature if $d \geq 3$. We conclude that the image of $\sigma_{p,q}$ lies in the kernel of α, and is therefore not surjective.

4.1.3. Inertia Groups

Motivated by Θ_n, for any smooth manifold M^n we define $\mathcal{S}^{\mathrm{Top/O}}(\mathrm{M}^n)$ to be the oriented diffeomorphism classes of smooth manifolds homeomorphic to M^n. This set has a base point given by the identity map. Unlike Θ_n, there will not be a group structure on this set, but $\mathcal{S}^{\mathrm{Top/O}}(\mathrm{M}^n)$ will admit a Θ_n action given by $\widetilde{\mathrm{M}}^n \mapsto \widetilde{\mathrm{M}}^n \# \Sigma^n$. The *inertia group* of M^n, denoted by $\mathcal{I}(\mathrm{M}^n) \leq \Theta_n$, is the stabilizer of the base point, i.e. the subgroup consisting of Σ^n such that $\mathrm{M}^n \# \Sigma^n$ is diffeomorphic to M^n. We have a subset $\Theta_n / \mathcal{I}(\mathrm{M}^n) \subseteq \mathcal{S}^{\mathrm{Top/O}}(\mathrm{M}^n)$. Thus whenever $\mathcal{I}(\mathrm{M}^n) = 0$, we have $\Theta_n \hookrightarrow \mathcal{S}^{\mathrm{Top/O}}(\mathrm{M}^n)$. And $\mathrm{M}^n \# \Sigma^n$ each represent distinct smooth structures on the same underlying topological manifold M^n.

From a computational perspective it is actually more difficult to determine $\mathcal{I}(\mathrm{M}^n)$ than it is to determine $\mathcal{S}^{\mathrm{Top/O}}(\mathrm{M}^n)$. But from a geometric perspective, knowledge that a smooth structure of M^n is realized by $\mathrm{M}^n \# \Sigma^n$ for some $\Sigma^n \in \Theta_n$ gives us a construction of this smooth structure that is as concrete as our understanding of the exotic sphere Σ^n. Thus if $\mathrm{M}^n \# \Sigma^n$ is an exotic smooth structure for $\Sigma^{2k} \in bP_{2k}$ or $\Sigma^d \in \operatorname{Im} \sigma_{p,q}$, we have a good chance to understand the geometric properties of the possible metrics on $\mathrm{M}^n \# \Sigma^n$.

In our case, as we will have successfully constructed core metrics on Σ^n as in Corollary 1.3.5, by Theorem 1.2.5 we will have constructed Ricci-positive metrics on many manifolds of the form $\mathrm{M}^n \# \Sigma^n$, provided that M^n is known to admit a core metric. Corollaries 1.3.6, 1.3.7, 1.3.8, and 1.3.9 as explained in Section 1.3.3 follow from the work of [48], [49], [51], and [52] respectively. These papers respectively prove that $\mathcal{I}(\mathbf{R}\mathrm{P}^7) = 0$; $\mathcal{I}(\mathbf{C}\mathrm{P}^n) = 0$ for $n \leq 8$; $\mathcal{I}(\prod_i \mathrm{S}^{d_i}) = 0$; and $\mathcal{I}(\mathrm{M}^n) = \mathcal{I}(\mathrm{M}^n \# (\#_k(\mathrm{S}^2 \times \mathrm{S}^{n-2}))$ if $n \geq 7$ and M^n is 2-connected. These results allow us to conclude that Ricci-positive metrics constructed on $\mathrm{M}^n \# \Sigma^n$ are all defined on exotic smooth structures on M^n.

4.2. Core metrics on boundary of plumbing

Having established a the relationship between plumbing and surgery, we now explain the work of [3, 30] on Ricci-positive surgery in Section 4.2.1 and how it is applied to construct Ricci-positive metrics on the boundaries of plumbings. In Section 4.2.2 we prove Theorem C by combining the logic of the proofs [30, Theorem 2.2 & Theorem 2.3] with the initial metrics constructed in Lemma 2.3.4 above.

4.2.1. Ricci-positive Surgery

Given a Ricci positive manifold (M^{n+m}, g), suppose we are given an isometric embedding,

$$\varphi : \left(\mathrm{S}^n \times \mathrm{D}^m_R, \rho^2 ds_n^2 + N^2 ds_m^2\right) \hookrightarrow (\mathrm{M}^{n+m}, g). \tag{4.5}$$

Here, by $(\mathrm{D}^m_R, N^2 ds_m^2)$, we mean a geodesic ball of radius R in $(\mathrm{S}^m, N^2 ds_m^2)$. The main technical constructions in [3, 30] were to show that under a suitable compatibility of ρ, N, and R, there is a Ricci-positive metric on M^n_φ.

Lemma 4.2.1. [30, Lemma 2.4] *If $m-1 \geq n \geq 3$, then there is a $\kappa(n, m, \varphi, R/N)$, such that if $\rho/N < \kappa$ we can find a Ricci-positive metric g_φ on the manifold E_φ^{n+m} defined by (1.1) and the embedding (4.5).*

Lemma 4.2.1 is sufficient to construct Ricci-positive metrics and prove [30, Theorem 2.3], and will also be sufficient to construct core metrics on $\operatorname{Im} \sigma_{p,q}$.

One may ask if the metric g_φ of Lemma 4.2.1 can be constructed in such a way as to allow for further applications of Lemma 4.2.1. This is necessary if one hopes to construct Ricci-positive metrics on elements of $\partial\mathcal{TP}_k$.

Lemma 4.2.2. [30, Lemma 2.5] *If $n = m-1 \geq 3$ and $\kappa_1 > 0$, there exists a $\kappa_0(n, \varphi_0, R_0/N_0, \kappa_1) > 0$ such that if $\rho_0/N_i < \kappa_0$ then there is a Ricci-positive metric g_1 on the manifold $\mathrm{M}_{\varphi_0}^{n+m}$ defined by (1.1) and the embedding (4.5). Moreover there is an isometric embedding $\varphi_1 : (\mathrm{S}^n \times \mathrm{D}_{R_1}^{n+1}, \rho_1^2 ds_n^2 + N_1^2 ds_{n+1}^2) \hookrightarrow (\mathrm{M}_{\varphi_0}^{n+m}, g_1)$, where $\rho_1/N_1 < \kappa_1$ and $R_1/N_1 = \Delta(n)$, where $\Delta(n)$ is a dimensional constant.*

Lemma 4.2.2 does not imply that one can continue to perform Ricci-positive surgery indefinitely, but as observed in the proof of [30, Theorem 2.3], in the case of sphere bundles over spheres, we can find a suitable initial metric to perform any fixed number of surgeries. Thus Lemma 4.2.2 will be sufficient to construct core metrics on $\partial\mathcal{TP}_k$.

4.2.2. Core metrics on plumbing pairing

We are now prepared to prove Theorem C. The proof follows the same logic as the proofs of [30, Theorem 2.2 & Theorem 2.3]. The proof begins by taking a suitable Ricci-positive metric on $S(\mathrm{E}_1)$, the sphere bundle of the disk bundle corresponding to the root of G to which Lemmas 4.2.1 and 4.2.2 may be applied.

This is the one aspect of these proofs that we modify: we instead start with a suitable Ricci-positive metric on $S(\mathrm{E}_1) \setminus \mathrm{D}^{n+m}$, g_{root} of Lemma 2.3.4.

Proof of Theorem C. Let $p \geq 4$ and $q \geq 3$, and let $\gamma_1 \in \pi_{p-1}SO(q)$ and $\gamma_2 \in \pi_{q-1}SO(p)$. Let E_1 be the D^q-bundle over S^p corresponding to γ_1. By Lemma 2.3.4 there is a Ricci-positive metric g_{root} on $S(\mathrm{E}_1) \setminus \mathrm{D}^{p+q-1}$ with round boundary that has positive principal curvatures and an isometric embedding $\varphi : (\mathrm{S}^{q-1} \times \mathrm{D}^p_N, ds^2_{q-1} + \rho^2 ds^2_p) \hookrightarrow (S(\mathrm{E}_1) \setminus \mathrm{D}^{p+q-1}, g_{\mathrm{root}})$ for a fixed N and all sufficiently small ρ. By Lemma 4.2.1, if we take ρ small enough so that $\rho/N < \kappa(p, q, \varphi, 1/N)$ we may find a Ricci-positive metric on $S(\mathrm{E}_1^{p+1})_\varphi \setminus \mathrm{D}^{p+q-1}$, with round boundary with positive principal curvatures. But by (4.4) and (4.2),

$$S(\mathrm{E}_1^{p+1})_\varphi \setminus \mathrm{D}^{p+q-1} \cong \partial(\mathrm{E}_1 \,\square\, \mathrm{E}_2) \setminus \mathrm{D}^{p+q-1} = \sigma_{p,q}(\gamma_1, \gamma_2) \setminus \mathrm{D}^{p+q-1}\,.$$

Thus each element of $\mathrm{Im}\,\sigma_{p,q}$ admits a core metric for $q \geq 4$ and $p \geq 3$.

For $P(G) \in \partial\mathcal{TP}_k$, we will argue by induction on the height h of the rooted tree G. Our inductive hypothesis will be much more involved than the conclusion of Theorem C. For all $\kappa_h > 0$, we will assume that for each rooted tree G of height h with leaves v^j_G with $1 \leq j \leq l_G$ each labeled by a natural number d^j with $1 \leq j \leq l_G$ that there is a Ricci-positive metric $g(G, d^j, \kappa_h)$ on $\partial P(G) \setminus \mathrm{D}^{2k-1}$ with round boundary that has positive principal curvatures and that for all $1 \leq i \leq d^j$ embeddings $\varphi^i_{G,j} : (\mathrm{S}^{k-1} \times \mathrm{D}^k_{N^i_{G,j}}, \rho^2 ds^2_{k-1} + (R^i_{G,j})^2 ds^2_k) \hookrightarrow (\partial P(G) \setminus \mathrm{D}^{2k-1}, g)$ whose images lie in the portion corresponding to the leaves v^j_G where $R^i_{G,j}/N^i_{G,j}$ depends only on d^j, G, and the dimension k and ρ can be chosen such that $\rho/N^i_{G,j} < \kappa_h$.

The idea behind this inductive hypothesis, is that for each rooted tree G of height $(h + 1)$ there is a rooted tree G' of height h, so that G can be recovered from

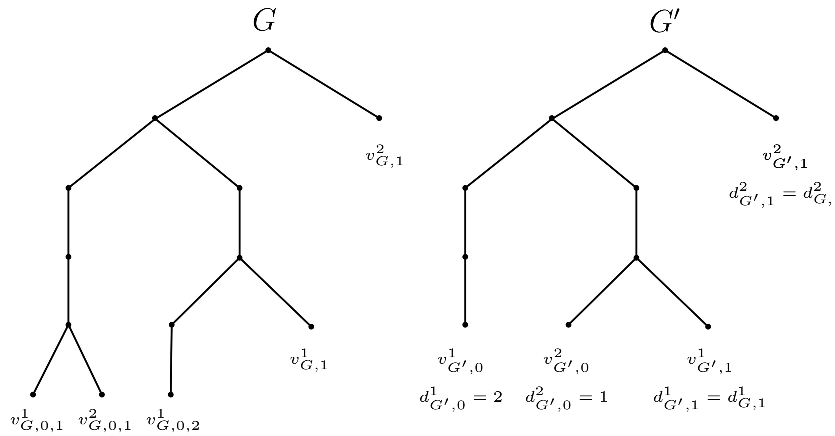

FIGURE 4.3. The labeling and numbering convention for G and G'.

G' by allowing the leaves $v^j_{G'}$ of G' to bud d^j new leaves. Note that the inductive hypothesis is much stronger than the conclusion of Theorem C. Should we be able to prove the inductive hypothesis holds for $(h+1)$ assuming it is true for h, then by disregarding the additional data we will have constructed a Ricci-positive metric on $\partial(P(G)) \setminus \mathrm{D}^{2k-1}$ with round boundary that has positive principal curvatures.

The base case $h = 0$ corresponds to G a single vertex and $\partial P(G) = S(\mathrm{E}_1)$ an S^{k-1}-bundle over S^k. In this case there is only one leaf, and the inductive hypothesis is precisely the conclusion of Lemma 2.3.4.

We now assume our inductive hypothesis holds for graphs of height h, and seek to show it is true for rooted trees of height $(h+1)$. Fix $\kappa_{(h+1)} > 0$ and let G be a rooted graph of height $h+1$. Let us label those leaves whose distance from the root are less than $(h+1)$ by $v^i_{G,1}$ for $1 \leq i \leq l_{G,1}$. If we delete the vertices that are exactly a distance of $(h+1)$ from the root, this produces a rooted tree G' of height h. Some of the leaves of G' have already been labeled as $v^j_{G,1}$, let us also refer to them as $v^j_{G',1}$ for $1 \leq i \leq l_{G',1} = l_{G,1}$. Let us label the remaining vertices of G' by $v^j_{G',0}$ for $1 \leq j \leq l_{G',0}$. Note that each of these leaves may also be considered as vertices of G, where they are adjacent to at least one leave of G. Let $d^j_{G',0}$ denote the number of leaves of G adjacent to $v^j_{G',0}$. We may now finish labeling the leaves of G, by labeling the leaves of G adjacent to $v^j_{G',0}$ as $v^i_{G,0,j}$ for $1 \leq i \leq d^j_{G',0}$. Let us assume with this labeling scheme, that we have assigned natural numbers $d^i_{G,0,j}$ and $d^i_{G,1}$ to each of the similarly labeled leaves of G.

We will apply the inductive hypothesis for height h and Lemma 4.2.2 to show that the inductive hypothesis holds for rooted trees of height $(h+1)$. We will apply the inductive hypothesis to G' for suitably chosen data. We already have labeled the vertices of G', corresponding to these labels we assign natural numbers $d^j_{G',0}$ to the leaves $v^j_{G',0}$ as above the number of leaves of G adjacent to $v^j_{G',0}$ and $d^j_{G',1} = d^j_{G,1}$ to leaves $v^j_{G',1}$ agreeing with those numbers assigned to $v^j_{G,1}$ of G. By the inductive hypothesis, for any $\kappa_h > 0$ we get a Ricci-positive metric g' that depends on G' and the number $d^j_{G',l}$ on $\partial P(G') \setminus \mathrm{D}^{2k-1}$ and isometric embeddings for each $1 \leq i \leq d^j_{G',l}$,

$$\varphi^i_{G',l,j} : (\mathrm{S}^{k-1} \times \mathrm{D}^k_{N^i_{G',l,j}}, \rho^2 ds^2_{k-1} + (R^i_{G,l,j})^2 ds^2_k) \hookrightarrow (\partial P(G') \setminus \mathrm{D}^{2k-1}, g').$$

By Lemma 4.2.2, for any $\kappa_{(h+1)} > 0$ there is a $\kappa^i_{G',l,j} > 0$ that depends on k, $\varphi^i_{G',l,j}$, $N^i_{G',l,j}/R^i_{G',l,j}$, and $\kappa_{(h+1)}$ that guarantees the conclusion of Lemma 4.2.2 under the assumption that $\rho/N^i_{G',l,j} < \kappa^i_{G',l,j}$. We have assumed in the inductive hypothesis that the embedding $\varphi^i_{G',l,j}$ and the quantity $N^i_{G',l,j}/R^i_{G',l,j}$ depend only on the dimension k, the graph G', and the numbers $d^j_{G',l}$. We conclude that $\kappa^i_{G',l,j}$ is independent of κ_h and therefore we may take $\kappa_h < \kappa^i_{G',l,j}$. We will also assume that $\kappa_h < \kappa_{(h+1)}$, which is valid as $\kappa_{(h+1)}$ has been fixed.

We may now apply Lemma 4.2.2 on all of the embeddings $\varphi^i_{G',0,j}$ to produce a Ricci-positive metric g on the resulting space. The smooth manifold that results from performing surgery on all $\varphi^i_{G',0,j}$ in $\partial P(G') \setminus \mathrm{D}^{2k-1}$ is, by (4.2), diffeomorphic to $\partial P(G) \setminus \mathrm{D}^{2k-1}$. As we have not altered the metric outside of the embeddings $\varphi^i_{G',0,j}$, the boundary is still round and has positive principal curvatures with respect to g. We emphasize that we *have not* applied Lemma 4.2.2 to the embeddings $\varphi^i_{G',1,j}$. We instead relabel these embeddings as

$$\varphi^i_{G,1,j} : (\mathrm{S}^{k-1} \times \mathrm{D}^{k+1}_{N^i_{G,1,j}}, \rho^2 ds^2_{k-1} + (R^i_{G,1,j})^2 ds^2_k) \hookrightarrow (\partial P(G) \setminus \mathrm{D}^{2k-1}, g).$$

Their image still lies in the portion of $\partial P(G) \setminus \mathrm{D}^{2k-1}$ corresponding to $v^j_{G',1}$, which we also have relabeled as $v^j_{G,1}$. Lemma 4.2.2 also guarantees the existence of isometric embeddings

$$\varphi_{G,0,j,i} : (\mathrm{S}^{k-1} \times \mathrm{D}^k_{N_{G,0,j,i}}, \rho^2 ds^2_{k-1} + (R_{G,0,j,i})^2 ds^2_k) \hookrightarrow (\partial P(G) \setminus \mathrm{D}^{2k-1}, g).$$

Where the image lies in the portion of $\partial P(G) \setminus \mathrm{D}^{2k-1}$ corresponding to $v^i_{G,0,j}$, and the constant $N_{G,0,j,i}/R_{G,0,j,i}$ depends only on k and $\rho/R_{G,0,j,i} < \kappa_{(h+1)}$.

The only part of the inductive hypothesis that we have not shown for height $(h+1)$, is that we may actually find isometric embeddings $\varphi^l_{G,0,j,i}$ for $1 \leq l \leq d^i_{G,0,j}$. Given $(\mathrm{D}^k_N, R^2 ds^2_k)$, we may find an $N' = N/c(d,k)$ depending only on d and k for which there are d disjoint isometric embeddings $\varphi^i : (\mathrm{D}^k_{N'}, R^2 ds^2_k) \hookrightarrow (\mathrm{D}^k_N, R^2 ds^2_k)$. Applying this observation to the embeddings $\varphi_{G,0,j,i}$, we may find $d^i_{G,0,j}$ many disjoint isometric embeddings

$$\varphi^l : (\mathrm{S}^{k-1} \times \mathrm{D}^k_{N^l_{G,0,j,i}}, \rho^2 ds^2_{k-1} + (R_{G,0,j,i})^2 ds^2_k) \hookrightarrow$$
$$(\mathrm{S}^{k-1} \times \mathrm{D}^k_{N_{G,0,j,i}}, \rho^2 ds^2_{k-1} + (R_{G,0,j,i})^2 ds^2_k).$$

Where $N^l_{G,0,j,i}/R_{G,0,j,i} = (N_{G,0,j,i}/R_{G,0,j,i})\left(1/c(d^i_{G,0,j},k)\right)$ clearly only depends on $d^i_{G,0,j}$ and the dimension k. The desired embeddings are now $\varphi^l_{G,0,j,i} = \varphi^l \circ \varphi_{G,0,j,i}$.

□

APPENDIX A

GLUING AND SMOOTHING FOR RICCI-POSITIVE RIEMANNIAN MANIFOLDS WITH CORNERS

This appendix represents a self contained proof of Theorem I. We begin in Section A.1 by discussing Riemannian manifolds with corners. Much of the terminology and notation introduced in this section has been used above in Chapter III when specifying embeddings of manifolds with corners into manifolds with boundaries. After establishing the needed notation and constructions for manifolds with corners, we then consider an entirely different technical aspect to the proof of Theorem I. In Section A.2, we introduce an explicit family of polynomial splines and study how their derivatives are entirely controlled by their boundary conditions. After completing these two seemingly disparate discussions, we then prove Theorem I in Section A.3 by applying the theory of polynomial splines of Section A.2 to smooth the corners introduced in the constructions of Section A.1.

A.1. Manifolds with Corners

We begin in Section A.1.1 by defining manifolds with corners and outlining the basic smooth topological constructions: collar neighborhoods, smoothing corners, and gluing together two manifolds with corners along a common face. These constructions, while elementary, are not well known. For example, the statement of Theorem I is not precise without discussing exactly what is meant when we say $X_1^n \cup_\Phi X_2^n$, which we discuss in Section A.1.1.2 below. Unlike Riemannian manifolds with boundary, where collar neighborhoods agree with normal coordinates of the boundary, Riemannian manifolds with corners require have competing normal coordinates coming from adjacent faces. Much of the work in Section A.1.2 is to make a choice of normal coordinates for the corners that are amenable to the construction of $\mathrm{X}_1^n \cup_\Phi \mathrm{X}_2^n$ in Theorem I. In Section A.1.2.1 we rephrase Theorem I as the more technical Theorem I′. We then describe the coordinates we will use to prove Theorem I′ in Section A.1.2.2 and compute the relevant curvatures in these coordinates in Section A.1.2.3.

A.1.1. Smooth topology of Manifolds with corners

A smooth manifold with corners is a topological space X^n together with a smooth atlas of charts whose images are $(-\infty, 0]^c \times \mathbf{R}^{n-c}$ for some $0 \leq c \leq n$. We choose the interval $(-\infty, 0]$ as we follow the convention that corners are oriented with respect to outward normal vectors. For such a manifolds we can define the non-continuous function $h(x) = c$ as the maximal value c for which the image of x in $(-\infty, 0]^c \times \mathbf{R}^{n-c}$ lies in the set $\{0\}^c \times \mathbf{R}^{n-c}$. We call the closure of a connected component of the set $h^{-1}(c)$, a codimension c *corner* of X^n, which by definition is a manifold with corners of dimension $(n-c)$. We refer to the codimension 1 corners of X^n as *the faces* of X^n. The boundary of a manifold with corners $\partial\,\mathrm{X}^n$ is the union of its faces along their boundaries, and is not by our convention a manifold with corners.

We say that a manifold with corners is a *manifold with faces* if the codimension c corners are the intersection of c distinct faces. It is with respect to these manifolds that we have stated Theorem I. This is because we have decomposed a manifold into a union of manifolds with boundaries, which in turn decomposes an embedded submanifold with boundary into a union of manifolds with corners. Such intersections will generically result in manifolds with faces.

Proposition A.1.1. *The generic intersection of two codimension 0 manifolds with boundaries is a manifolds with faces with corners of codimension at most* 2*.*

To see this, we note that the corners occur when the two boundaries intersect, which in turn is the intersection of the two faces that correspond to the two boundary components entering the interior of the other manifold.

A.1.1.1. Collar Neighborhoods

We will need a notion of collar neighborhood for the boundary of a manifold with corners. For a given corner Z^{n-c} note that there are $c-k$ distinct inclusion $\varphi_i^k : \mathrm{Z}^{n-c} \hookrightarrow \mathrm{Z}_i^{n-k}$, where Z_i^{n-k} are not necessarily distinct corners. A *collar neighborhood for the boundary of* X^n will be a collection of collar neighborhoods $\Phi : (-\infty, 0]^c \times \mathrm{Z}^{n-c} \hookrightarrow \mathrm{X}^n$ that respect the collar neighborhoods of every embedding $\varphi_i^k : \mathrm{Z}^{n-c} \hookrightarrow \mathrm{Z}_i^{n-k}$. We will not prove the existence of collar neighborhoods in generality, though the argument is much the same as the specific case we are interested in.

Lemma A.1.2. *Let* X^n *be a manifold with corners of codimension at most* 2*. For each corner* Z^{n-2}*, there are two inclusions* $\varphi_i^1 : \mathrm{Z}^{n-2} \hookrightarrow \mathrm{Y}_i^{n-1}$*, where* Y_i^{n-1} *might be the same face. Then there are tubular neighborhoods* $\Phi_i^1 : (-\infty, 0] \times \mathrm{Z}^{n-2} \hookrightarrow \mathrm{Y}_i^{n-1}$*,* $\Phi^0 : (-\infty, 0]^2 \times \mathrm{Z}^{n-1} \hookrightarrow \mathrm{X}^n$*, and* $\Phi_i^0 : (-\infty, 0] \times \mathrm{Y}_i^{n-1} \hookrightarrow \mathrm{X}^n$*, such that*

$$\Phi_1^0(x_0, \Phi_1^1(x_1, z)) = \Phi^0(x_0, x_1, z) = \Phi_0^0(x_1, \Phi_0^1(x_0, z)).$$

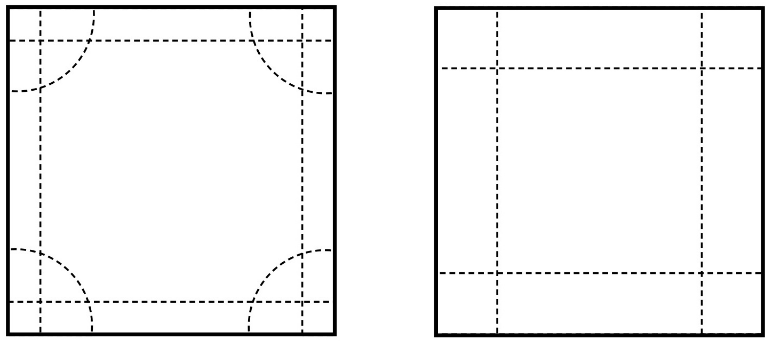

FIGURE A.1. The tubular neighborhood of the corners and faces, and the consistent tubular neighborhoods making up the collar neighborhood.

Proof. We start by noting that each corner $\varphi : \mathrm{Z}^{n-2} \hookrightarrow \mathrm{X}^n$ admits an individual collar neighborhood by [63, Theorem 1]. This means that there is an embedding $\Phi : (-\infty, 0]^2 \times \mathrm{Z}^{n-2} \hookrightarrow \mathrm{X}^n$ of manifolds with corners. Note that there are two embedings $\varphi_i^1 : \mathrm{Z}^{n-2} \hookrightarrow \mathrm{Y}_i^{n-1}$, where Y_i^{n-1} are two, not necessarily distinct, faces. In this coordinates, $\Phi : \{0\} \times (-\infty, 0] \times \mathrm{Z}^{n-2} \hookrightarrow \mathrm{Y}_0^{n-1}$ and $\Phi : (-\infty, 0] \times \{0\} \times \mathrm{Z}^{n-2} \hookrightarrow \mathrm{Y}_1^{n-1}$. Similarly for each face $\varphi_i : \mathrm{Y}_i^{n-1} \hookrightarrow \mathrm{X}^n$, there are collar neighborhoods $\Phi_i : (-\infty, 0] \times \mathrm{Y}^{n-1} \hookrightarrow \mathrm{X}^n$. By compactness, we may shrink the constructed tubular neighborhoods in such a way that the collar neighborhood of Z^{n-2} is disjoint from the neighborhood of any other corner, that the neighborhoods of the faces Y_i^{n-1} only intersect with one another on the interior of the collar neighborhood for a mutual corner, and that the tubular neighborhood of a face only intersect the tubular neighborhood of corners in its boundary.

In order to make these neighborhoods compatible in the desired way, we will need to construct a particular metric on this neighborhood of the boundary we have constructed. On the collar neighborhood of each corner let (x_0, x_1, z) denote the coordinates. Take the metric $g = dx_0^2 + dx_1^2 + h$ with respect to this decomposition, where h is any fixed metric on Z^{n-2}. Note that restriction $x_0 = 0$

corresponds to the Y_1^{n-1} and vice versa, so that the metric restricts to Y_i^{n-1} is $dx_i^2 + h$.

We may extend the function x_0 to the tubular neighborhood of Y_1^{n-1}, by taking a convex combination of this function with the normal coordinate in this tubular neighborhood we get a new coordinate function which we will call x_0 that agrees with x_0 on the collar neighborhood of the corner. We may similarly extend x_1 in this way to Y_0^{n-1}. We may also extend the metric $dx_i^2 + h$ to a metric k_i on Y_i^{n-1}. We define the metric $g_i = dt^2 + k_i$ on the tubular neighborhood of Y_i^{n-1}, which by construction agrees with g in the corner neighborhoods. Taking the g_i together with g gives rise to a smooth metric on a neighborhood of $\partial\,\mathrm{X}^n$.

Taking normal coordinates with respect to this metric gives rise to the desired tubular neighborhoods. By construction the corners and faces are totally geodesic and on the collar neighborhood of the corners the metric restricted to the normal bundle is flat. This first fact means that geodesics emanating from Z^{n-2} in a direction tangent to Y_i^{n-1} will remain inside Y_i^{n-1}. This second fact means that any geodesics emanating from Z^{n-2} in a direction normal to Z^{n-2} commute with one another. These two facts therefore imply the desired compatibility. □

With the existence of collar neighborhoods established we can now describe the elementary smooth constructions that underline the metric constructions we wish to carry out in the proof of Theorem I. The first construction is an embedding theorem for Riemannian manifolds with faces, which allows us to always consider an ambient Riemannian manifold with boundary.

Lemma A.1.3. *Let X^n be a smooth manifold with faces with corners of codimension at most 2. For each face Y^{n-1} of X^n there is a manifold M^n with boundary N^{n-1}, an embedding $\iota : \mathrm{X}^n \hookrightarrow \mathrm{M}^n$ such that $\mathrm{Y}^{n-1} = \mathrm{X}^n \cap \mathrm{N}^{n-1}$ and any*

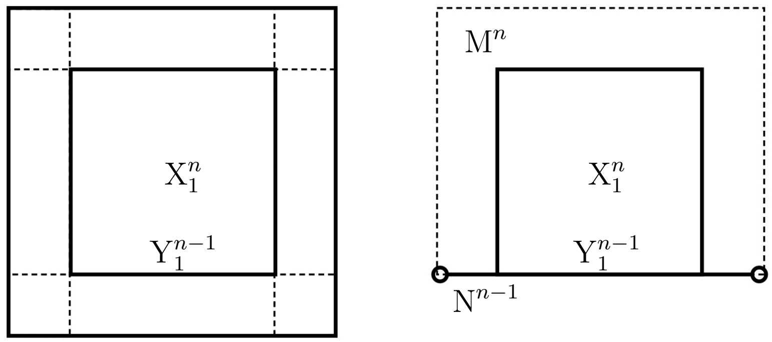

FIGURE A.2. The construction of the embedding $X^n \hookrightarrow M^n$.

other face $\widetilde{Y}^{n-1}$ *intersects* N^{n-1} *transversely, and a diffeomorphisms* $r : M^n \hookrightarrow X^n$ *onto a subspace of* X^n *that carries* N^{n-1} *to the interior of* Y^{n-1}.

Given a metric g *on* X^n *and a choice of an extension of this metric from* Y^{n-1} *to* N^{n-1}*, for all* $\varepsilon > 0$ *there is an extension* $\tilde{g}$ *to* M^n *such that* $r^* g$ *and* $\tilde{g}$ *are* ε *close in the* C^∞ *topology.*

Proof. Take a collar neighborhood of the boundary of X^n by Lemma A.1.2. Define M^n as the subspace of X^n,

$$M^n := X^n \setminus \left[\left(\bigcup_{\widetilde{Y} \neq Y} \widetilde{Y}^{n-1} \right) \cup \left((-1, 0] \times Y^{n-1} \right) \right].$$

We let $r : M^n \hookrightarrow X^n$ denote any map that smoothly sends$(-2, 1] \times Y^{n-1}$ to $(-2, 0] \times Y^{n-1}$. One can similarly choose a diffeomorphism $\iota : X^n \to X_1^n$, where

$$X_1^n := X^n \setminus \left[\bigcup_{Y} (-1, 0] \times Y^{n-1} \right].$$

The rest of the smooth topological claims can be verified from the definition of collar neighborhood.

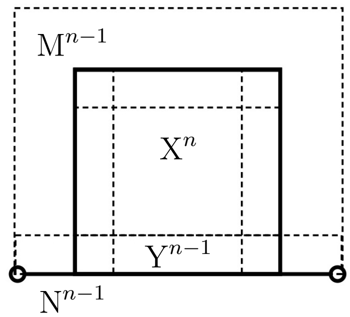

FIGURE A.3. The tubular neighborhood of the boundary and face.

Given a metric g on X^n, we can extend as desired because Y^{n-1} is closed in N^{n-1} and N^{n-1} is closed in M^n. That $\tilde{g}$ and r^*g can be made arbitrarily close follows because we can shrink the initial collar neighborhoods, in such a way that we get a family of metrics $\tilde{g}_\varepsilon$ on M^n that will converges smoothly to g. □

Note that Lemma A.1.3 allows us to talk about any smooth manifold with corners extrinsically. This will be useful in the proof of Theorem I, as we can appeal to existing proofs of Theorem 1.2.2 to immediately conclude the desired metric exists.

Henceforth, we will say that a manifold with faces X^n *is embedded within a manifold with boundary* M^n *relative to a face* Y^{n-1} if the faces of X^n intersect N^{n-1} as in Lemma A.1.3. Note that the proof of Lemma A.1.3 can be modified slightly to prove the following, which claims that we may also choose collar neighborhoods of X^n and M^n that respects the embedding.

Lemma A.1.4. *Given an embedding of a manifold with faces* X^n *into a manifold with boundary* M^n *such that* $Y^{n-1} = X^n \cap N^{n-1}$ *and any other face* $\widetilde{Y}^{n-1}$ *intersects* N^{n-1} *transversely, there is a collar neighborhood of* N^{n-1} *that agrees with the tubular neighborhood of the boundary of* X^n.

A.1.1.2. Gluing and smoothing

The underlying smooth construction behind Theorem 1.2.2 is summarized in the following lemma.

Lemma A.1.5. [64, Lemmas 8.1 and 8.2] *Given two smooth manifolds* M_i^n *with boundaries* N_i^{n-1} *and an orientation reversing diffeomorphism* $\Phi : \mathrm{N}_1^{n-1} \to \mathrm{N}_2^{n-1}$, *there is a smooth structure on the topological manifold* $\mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$ *that agrees with the smooth structure on each* M_i^n. *Moreover, any two smooth structures on* M^n *that respects the given smooth structures on each* M_i^n *are diffeomorphic.*

The construction of the smooth structure takes any collar neighborhoods for the N_i^{n-1} and identifies these collar neighborhoods by $(t,x) \mapsto (1-t,\Phi(x))$. A construction of such a smooth structure can be described similarly using a different choice of collar neighborhoods, or might be described in entirely different terms. The importance of Lemma A.1.5, is that it suffices to find a metric on any preferred smooth structure of M^n. In particular the choice of collar neighborhood could come from normal coordinates of N_i with respect to g_i.

When we have two embedded manifolds with faces X_i^n inside manifolds with boundaries M_i^n, by Lemma A.1.4 we have a collar neighborhood of N_i^{n-1} that respects the collar neighborhood of the boundary of X_i^n. If we assume that Φ restricted to $\Phi : \mathrm{Y}_1^{n-1} \to \mathrm{Y}_2^{n-1}$ is an orientation reversing orientation of manifolds with boundary, then the smooth structure defined on M^n by these collar neighborhoods will naturally give rise to a smooth structure on $\mathrm{X}_1^n \cup_\Phi \mathrm{X}_2^n$ which is a smooth manifold with faces, where the faces of X^n can be described as those faces of X_1^n and X_0^n disjoint from Y_1^{n-1} and Y_2^{n-1} and as $\widetilde{\mathrm{Y}}^{n-1} = \widetilde{\mathrm{Y}}_1^{n-1} \cup_\Phi \widetilde{\mathrm{Y}}_2^{n-1}$ where

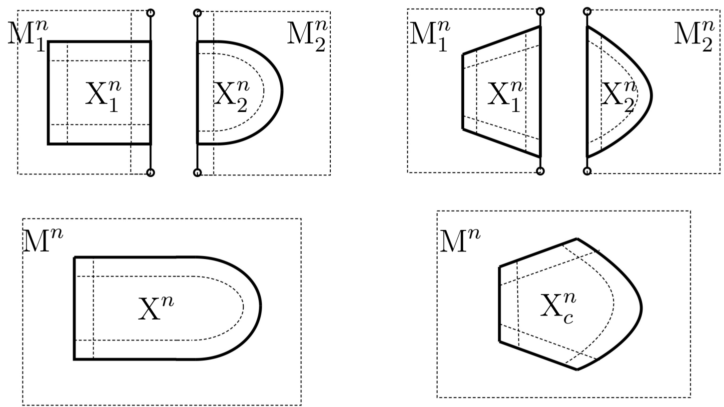

FIGURE A.4. Different choices of gluing result in different submanifolds with faces.

$\widetilde{Y}_i^{n-1}$ is the collection of faces that intersect Y_i^{n-1}. This is what we will mean when we write X^n

If we choose a different tubular neighborhood, we can still form a smooth structure on M^n, but it will not form the same smooth manifold with corners. Instead, it will give rise to a different smooth structure on $X_1^n \cup_\Phi X_2^n$, where the faces can be described as all faces of X_1^n and X_2^n other than the identified faces Y_1^{n-1} and Y_2^{n-1}. This is what we will mean when we write X_c^n. We can recover X^n from X_c^n by the process of *smoothing corners*, which we describe in the following Lemma.

Lemma A.1.6. *Given a manifold with faces X^n with corners of codimension at most 2. For any corner Z^{n-2} which bounds two faces Y_1^{n-1} and Y_2^{n-1}, there is a smooth structure X_Z^n of a manifold with faces on the underlying topological manifold X^n so that the smooth structure agrees with smooth structure on X^n outside of Z^{n-1}, but so that the faces of X_Z^n agree with the faces of X^n except Y_1^{n-1} and Y_2^{n-1} have been replaced by $Y^{n-1} = Y_1^{n-1} \cup_Z Y_2^{n-1}$. Moreover, any two manifolds with faces satisfying these claims are diffeomorphic as manifolds with faces.*

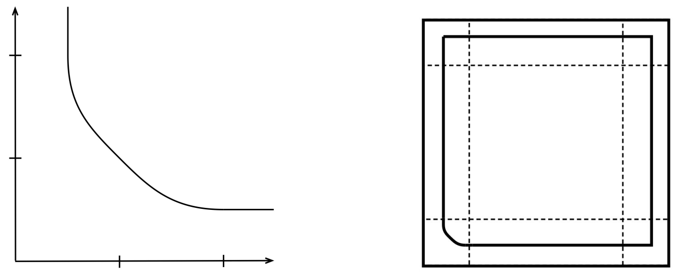

FIGURE A.5. The curve $\gamma(x)$, and $\mathrm{X}^n_{\mathrm{Z}}$ embedded in X^n.

Proof. To construct one such $\mathrm{X}^n_{\mathrm{Z}}$, we take collar neighborhoods as in Lemma A.1.2. Let $(\gamma_0(x), \gamma_1(x)) \subseteq (-\infty, 0]^2$ be a curve that agrees with $\{-1/2\} \times (-\infty, -2]$ and $(-\infty, -2] \times \{-1/2\}$. In the tubular neighborhood of the corner, we define $\mathrm{X}^n_{\mathrm{Z}}$ to be the set $\{(x_0, x_1, z) : \gamma_0(x) < x_0 \text{ and } x_1 = \gamma_1(x)\}$, and in the tubular neighborhood of the faces outside of the corner neighborhoods to be the set $\{(t, z) : t = -1/2\}$.

That the smooth structure on $\mathrm{X}^n_{\mathrm{Z}}$ agrees with X^n outside of Z^{n-2} follows form the fact that the boundary of $\mathrm{X}^n_{\mathrm{Z}}$ is the graph of a smooth function over $\partial\,\mathrm{X}^n$. Uniqueness follows from Lemma A.1.5 applied to the faces Y^{n-1}_1 and Y^{n-1}_2. □

It follows from the uniqueness in Lemma A.1.6 that smoothing the corners of X^n_c introduced along the gluing site is diffeomorphic to X^n. This is precisely the approach we take in Theorem I: we will take coordinates of M^n_i well adapted to the metric to form M^n, and then smooth the corners of X^n_c to produce a smooth embedding $\mathrm{X}^n \hookrightarrow \mathrm{M}^n$.

A.1.2. Riemannian Manifolds with Faces

Suppose that (X^n, g) is a Riemannian manifold with faces with corners of codimension at most 2. In this setting, each face Y^{n-1}_i has a second fundamental form II_i with respect to the metric g defined with respect of the outward unit normal in the same way as a manifold with boundary. Each corner Z^{n-2} bounds two distinct faces Y_0^{n-1} and Y_1^{n-1}. By restring g to Y_i^{n-1} and considering Z^{n-2} as the boundary of Y_i^{n-1} we get two outward unit normals of Z^{n-2}, which we may then measure the angle between using g to get a well defined scalar valued function θ defined on each corner Z^{n-2}, which is called the *dihedral angle* of the corner.

A.1.2.1. Theorem I revisited

Lemma A.1.3 allows us to translate between intrinsically defined manifolds with faces and those defined extrinsically. We can now rephrase Theorem I in this extrinsic setting. In this setting, we will denote by $\widetilde{Y}^{n-1}$ the collection of faces that intersect the designated face Y^{n-1}.

Theorem I′. *Fix two manifolds with faces X_i^n with corners of codimension at most 2, and assume they are embedded within two Riemannian manifolds (M_i^n, g_i) with boundary N_i^{n-1} relative to their face Y_i^{n-1}.*

Assume there is an isometry $\Phi : (N_1^{n-1}, g_1) \to (N_2^{n-1}, g_2)$ that restricts to an isometry $\Phi : (Y_1^{n-1}, g_1) \to (Y_2^{n-1}, g_2)$ of manifolds with boundaries. Assume that $\mathrm{Ric}_{g_i} > 0$, and that the second fundamental forms of N_i^n satisfy $\mathrm{II}_1 + \Phi^* \mathrm{II}_2 > 0$. Assume also that the second fundamental form of $\widetilde{Y}_i^{n-1}$ satisfies $\widetilde{\mathrm{II}}_i > 0$, and that the dihedral angles θ_i along every corner that bounds Y_i^{n-1} satisfies $\theta_1 + \Phi^*\theta_2 < \pi$.

If these assumptions are met, then there exists a smooth Ricci-positive metric g on $M^n = M_1^n \cup_\Phi M_2^n$ and an embedding $X^n \hookrightarrow M^n$ where $X^n = X_1^n \cup_\Phi X_2^n$, so that $\widetilde{\mathrm{II}}$ the second fundamental form of the resulting smooth faces $\widetilde{Y}^{n-1} = \widetilde{Y}_1^{n-1} \cup_\Phi \widetilde{Y}_2^{n-1}$ with respect to g is positive definite. Moreover the metric g restricted to M_i^n agrees with g_i outside of an arbitrarily small neighborhood of the gluing site, and the embedding of $X^n \hookrightarrow M^n$ restricted to $X_i^n \hookrightarrow M_i^n$ agrees with the initial embedding outside of an arbitrarily small neighborhood of the gluing site.

We will now explain how Theorem I′ implies Theorem I. The reason we focus instead on proving Theorem I′, is that it has the benefit that we can we appeal to existing proofs of Theorem 1.2.2 to construct such a metric g, which reduces the proof of Theorem I′ to describing how to smooth the resulting corners on X_c^n while preserving the positivity of $\widetilde{\mathrm{II}}$ with respect to the metric g.

Proof of Theorem I. By Lemma A.1.3 we can find M_i^n and N_i^{n-1} such that there is an embedding $\mathrm{X}_i^n \hookrightarrow \mathrm{M}_i^n$ relative to their face Y_i^{n-1}, where the N_i^{n-1} are diffeomorphic because the Y_i^{n-1} are diffeomorphic by assumption. We can extend the g_i from Y_i^{n-1} to metrics on N_i^{n-1} in such a way that there is still an isometry $\Phi : \mathrm{N}_1^{n-1} \to \mathrm{N}_2^{n-1}$. We may then extend g_i to all of M_i^n by Lemma A.1.3. If we take $\varepsilon > 0$ sufficiently small we will have $\mathrm{Ric}_{g_i} > 0$ on all of M_i^n, and that $\mathrm{II}_1 + \Phi^* \mathrm{II}_2 > 0$ on all of N_1^{n-1}. We may now apply Theorem I′ to construct a metric g on M^n and an embedding $\mathrm{X}^n \hookrightarrow \mathrm{M}^n$ such that the metric restricted to X^n has the desired properties. □

Before we move on to talk about the normal coordinates we will use in the proof of Theorem I′, we take a moment to state a natural corollary of Theorem I′. In our proof of Theorem B, we are constructing a Riemannian metric on a smooth manifold with boundary by building it out of Riemannian manifolds with faces glued together. If we instead think about a manifold with boundary as the smoothing of a manifold with faces (in the sense of Lemma A.1.6), we can ask under what conditions will a metric on the manifold with faces with positive principal curvatures give rise to a metric on the smooth manifold with boundary with positive principal curvatures. The distinction between these two points of views is subtle, and we have chosen to state Theorem I because of the application

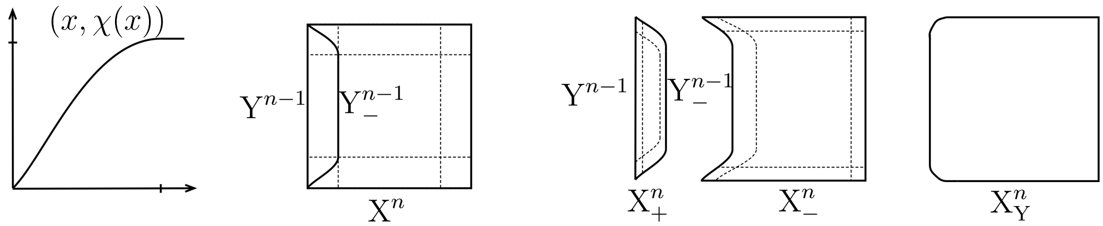

FIGURE A.6. The process of cutting, gluing, and smoothing X^n.

we had in mind. We claim that the following result, which is analogous to Theorem I, follows as a direct corollary of Theorem I′.

Corollary A.1.7. *Given* (X^n, g) *is a Riemannian manifold with faces with corners of codimension at most* 2*, suppose that* $\mathrm{Ric}_g > 0$*, that each face has positive principal curvatures, and that the dihedral angle along each corner never exceeds* π*, then there is a Ricci-positive metric g on the smooth manifold with boundary* M^n *that is the result of smoothing each corner of* X^n*.*

Proof. Take a collar neighborhood of the boundary of X^n as in Lemma A.1.2. Fix a face Y^{n-1} of X^n. We will define a submanifold of X^n, which we will call Y^{n-1}_- as a perturbation of Y^{n-1} relative to its boundary. Let $\chi(x)$ be any smooth function define on $(-\infty, 0]$ such that $\chi(0) = 0$, $\chi'(0) = 1$, $0 \geq \chi(x) \geq -1$, and $\chi(x) \equiv -1$ for all $x \leq -1$. For each corner that bounds Y^{n-1}, in the coordinates $(-\infty, 0]^2 \times Z^{n-2}$ we define $Y^{n-1}_- = \{(x_0, x_1, z) : x_1 \geq \chi(x_0)\}$ (assuming that the points $(x_0, 0, z)$ correspond to Y^{n-1}). On the collar neighborhood $(-\infty, 0] \times Y^{n-1}$ of Y^{n-1} we define $Y^{n-1}_- = \{(t, y) : t \equiv -1\}$. Note that Y^{n-1}_- is diffeomorphic to Y^{n-1}, and is an oriented smooth submanifold of codimension 1 with no relative boundary. It follows that Y^{n-1}_- separates X^n into two submanifolds with faces: X^n_- and X^n_+. Note that X^n_- is diffeomorphic to X^n and that X^n_+ deformation retracts onto Y^{n-1} and has boundary diffeomorphic to the unsmoothed doubling of Y^{n-1}. If were able to apply

Theorem I to X^n_- and X^n_+ along the face Y^{n-1}_-, the resulting manifold with faces X^n_Y can be identified with the manifold X^n with all of the corners bounding Y^{n-1} being smoothed. We claim that with a slight modification of the metric, we can directly apply Theorem I. By cutting, gluing, and smoothing in this fashion for each face of X^n, we will produce the desired metric on M^n.

By construction there is an isometry between the faces Y^{n-1}_- of X^n_- and X^n_+. The sum of dihedral angles $\theta_1 + \Phi^*\theta_2$ is equal to the original dihedral angles of the corners of X^n, which by assumption is less than π. By assumption each manifold has positive Ricci curvature and all faces other than Y^{n-1}_- has positive principal curvatures. The only hypothesis in Theorem I not met is that the sum of principal curvatures along Y^{n-1}_- is not positive, it is 0 because the original metric is smooth. This can be corrected as follows. By Lemma A.1.3, we may embed both X^n_- and X^n_+ into smooth manifolds with boundary relative to their face Y^{n-1}_- in such a way that the faces are still isometric. Apply Proposition 1.2.11 to these manifolds with boundary and then restrict this metric back to X^n_Y and X^n_+. By taking the change small enough we can assume all other hypotheses are preserved, and now may apply Theorem I directly to conclude X^n_Y admits a Ricci-positive metric where all faces have positive principal curvatures. □

A.1.2.2. Normal coordinates for corners

Let g be a Riemannian metric on a smooth manifold with faces X^n with corners of codimension at most 2. We would like to specify normal coordinates for a fixed corner Z^{n-2} with respect to g. As each corner of Z^{n-2} embeds into two faces, it is reasonable to ask if the normal coordinates for Z^{n-2} can be taken to be compatible with the normal coordinates for both of these faces as the tubular neighborhoods in Lemma A.1.2. As explained in the proof of A.1.2, if the normal coordinates were compatible in this way, that would imply one of the sectional curvatures of g is identically 0 along the corner, which will not be true for arbitrary g. Instead, we need to designate one of the two faces that Z^{n-2} bounds, and take normal coordinates of Z^{n-2} *relative to the face* Y^{n-1}. Note that certain geodesics emanating from Z^{n-2} may not exist for any positive time, so we will need to expand our manifold with faces using Lemma A.1.3 to define our normal coordinates.

Lemma A.1.8. *Given a Riemannian manifold with faces* (X^n, g) *with corners of codimension at most* 2*, fix face* Y^{n-1} *and one of its boundary components* Z^{n-2}*. Let* $\widetilde{\mathrm{Y}}^{n-1}$ *denote the other face bounded by* Z^{n-2}*. Take an isometric embedding* $(\mathrm{X}^n, g) \hookrightarrow (\mathrm{M}^n, g)$ *of a manifold with faces into a manifold with boundary relative to its face* Y^{n-1} *as in Lemma A.1.3.*

Then there are normal coordinates $(a, b, z) : (-\infty, 0] \times \mathbf{R} \times \mathrm{Z}^{n-2} \hookrightarrow \mathrm{M}^n$ *of* Z^{n-2} *relative to* Y^{n-1} *such that*

$$g = da^2 + \mu^2(a, b)db^2 + h(a, b). \tag{A.1}$$

Where $\mu(a, b)$ *is a positive function satisfying* $\mu(0, 0) = 1$ *and* $h(a, b)$ *is a two parameter family of metrics on* Z^{n-2}*. In these coordinates* $\mathrm{Z}^{n-2} = \{(a, b, z) : a = b = 0\}$ *and* $\mathrm{Y}^{n-1} = \{(a, b, z) : a = 0$ *and* $b \leq 0\}$*. Moreover, there is a function* $\phi(a, z)$ *satisfying* $\phi(0, z) = 0$ *and such that* $\widetilde{\mathrm{Y}}^{n-1} = \{(a, b, z) : a \leq 0$ *and* $b = \phi(a, z)\}$ *and* $\mathrm{X}^n = \{(a, b, z) : a \leq 0$ *and* $b \leq \phi(a, z)\}$*.*

Proof. Let a be the normal coordinate of $\mathrm{N}^{n-1} \hookrightarrow (\mathrm{M}^n, g)$. Taking the geodesic flowout of N^{n-1} after time a we have a diffeomorphisms $\mathrm{N}^{n-1} \to \{a\} \times \mathrm{N}^{n-1}$ which caries Z^{n-2} to $\{a\} \times \mathrm{Z}^{n-2}$. The metric $g = da^2 + k(a)$, where $k(a)$ is a

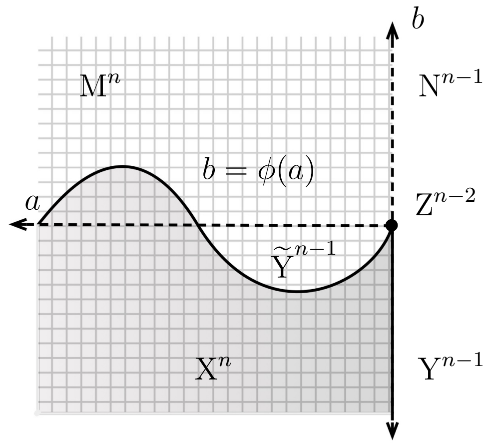

FIGURE A.7. Normal coordinates for Z^{n-2} relative to Y^{n-1}.

metric on $\{a\} \times N^{n-1}$. For each a, let b_a be normal coordinates for $\{a\} \times Z^{n-2} \hookrightarrow (\{a\} \times N^{n-1}, k(a))$. In these coordinates the metric $k(a) = db_a^2 + h(a,b)$. Because solutions to ordinary differential equations vary smoothly as initial data varies smoothly, we have that b_a is smoothly varying with respect to a and in turn $h(a,b)$ is smooth. By constriction the coordinates of $\{a\} \times \{b\} \times Z^{n-2}$, ∂_a, and ∂_b are mutually orthogonal, so the metric must split as $g = da^2 + \mu^2(a,b)db^2 + h(a,b)$ for some smooth function $\mu(a,b)$. We cannot assume $\mu(a,b) = 1$, lest the curvature $K(\partial_a, \partial_b)$ vanish. We may however perform a single reparameterization of b so that $\mu(0,0) = 1$.

By construction, the points $(0,0,z)$ correspond to $Z^{n-2} = \{0\} \times Z^{n-2} \subseteq \{0\} \times N^{n-1} = N^{n-1}$, and the points $\{(0,b,z) : b \leq 0\}$ corresponds to $Y^{n-1} = \{0\} \times Y^{n-1} \subseteq \{0\} \times N^{n-1} = N^{n-1}$. In these coordinates, the face $\widetilde{Y}^{n-1}$ exists as some smooth hypersurface diffeomorphic to $\mathbf{R} \times Z^{n-2}$. By assumption, this hypersurface meets N^{n-1}, the set $(0,b,z)$, transversely. Therefore $\widetilde{Y}^{n-1}$ will be the graph of a smooth function $b = \phi(a,z)$, on some sufficiently small normal neighborhood of N^{n-1}. That $\phi(0,z) = 0$, follows from the fact that $Z^{n-2} = \{(0,0,z)\}$ bounds $\widetilde{Y}^{n-1}$. $\square$

As promised in Section A.1.1.2, these will be the coordinates we use to form the smooth structure on $\mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$. We see that if we have fixed metrics g_i on each X_i^n, then it is not possible to assume we get a smooth structure on X^n unless $\widetilde{\mathrm{Y}}_i^{n-1}$ is totally geodesic near their boundary. Nonetheless, these coordinates will fit together nicely to give us a framework to smooth the metric and the corners simultaneously.

Lemma A.1.9. *Suppose we have two manifolds with faces X_i^n embedded in Riemannian manifolds (M_i^n, g_i) with boundary N_i^{n-1} relative to the face Y_i^{n-1}. Suppose moreover that there is an isometry $\Phi : (\mathrm{N}_1^{n-1}, g_0) \to (\mathrm{N}_2^{n-1}, g_1)$ that restricts to an isometry $\Phi : (\mathrm{Y}_1^{n-1}, g_1) \to (\mathrm{Y}_2^{n-1}, g_2)$. Let $\mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$ and X_c^n be as in Section A.1.1.2.*

Fix a corner Z_i^{n-2} of Y_i^{n-1} and let $\widetilde{\mathrm{Y}}_i^{n-1}$ be the other face bounded by Z^{n-2}. Then there are coordinates $(a, b, z) : \mathbf{R}^2 \times \mathrm{Z}^{n-2} \to \mathrm{M}^n$ such that the C^0 metric $g = g_1 \cup_\Phi g_2$ decomposes as

$$g = da^2 + \mu^2(a, b)db^2 + h(a, b),$$

where $\mu(a, b) = \mu_i((-1)^{i+1}a, b)$ and $h(a, b) = h_i((-1)^{i+1}a, b)$ corresponds to those metric components of the normal coordinates for Z_i^{n-2} in $(\mathrm{M}_i^{n-1}, g_i)$ relative to its face Y_i^{n-1} as in Lemma A.1.8. Moreover, there is a C^0 function $\phi(a, z)$ in these coordinates that agrees with $\phi_i((-1)^{i+1}a, z)$ the function that defines the face $\widetilde{\mathrm{Y}}_i^{n-1}$ in the normal coordinates of Lemma A.1.8.

Proof. Take the normal coordinates of Z_i^{n-2} inside of (M_i^n, g_i) relative to its face Y_i^{n-1} as in Lemma A.1.8. By construction these coordinates agree with normal coordinates of $\mathrm{N}_i^{n-1} \hookrightarrow (\mathrm{M}_i^{n-1}, g_i)$. We may use these normal coordinates to

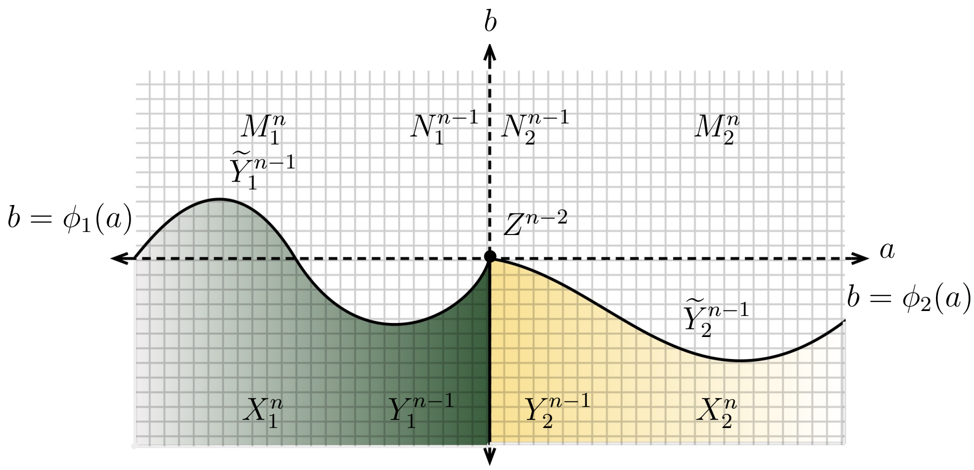

FIGURE A.8. A schematic of the corner charts of $\mathrm{X}_c^n \hookrightarrow \mathrm{M}^n$.

smoothly form $\mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$ as in Lemma A.1.5. This construction glues $(-\infty, 0] \times \mathrm{N}_1^{n-1}$ in an orientation reversing fashion to $(-\infty, 0] \times \mathrm{N}_2^{n-1}$ along the set $\{0\} \times \mathrm{N}^{n-1}$, thus giving us coordinates $\mathbf{R} \times \mathrm{N}^{n-1}$ where for $a \leq 0$ the coordinate a correspond to normal coordinates for N_1^{n-1} with respect to (M_1^n, g_1) and for $a \geq 0$ the coordinate $-a$ correspond to normal coordinates for N_2^{n-1} with respect to (M_2^n, g_2). The rest of the claim follows from Lemma A.1.8. $\square$

As noted at the end of Section A.1.1.2, Lemma A.1.6 implies that smoothing the corners of X_c^n results in X^n. Again by the uniqueness in Lemma A.1.6, we see in Lemma A.1.9 that we may smooth the corners of X_c^n by smoothing the function $\phi(a, z)$ along the set $a = 0$. We will discuss in Section A.2 a particularly concrete method of smoothing that will in turn allow us to prove Theorem I.

A.1.2.3. Curvatures in normal coordinates

In Theorem I there are two curvatures we are interested in: the Ricci curvature and the principal curvatures of the faces. As discussed in the previous section, we will utilize the coordinates of Lemma A.1.9 to prove Theorem I. Thus we will need to actually compute the relevant curvature terms in these coordinates.

Because the coordinates in Lemma A.1.9 agree with the normal coordinates of N_i^{n-1} with respect to g_i. If $(a, x) : (-\infty, 0] \times \mathrm{N}_i^{n-1} \hookrightarrow \mathrm{M}_i^n$ are these normal coordinates, then $g_i = da^2 + k_i(a)$ and we may appeal to the Gauss-Codazzi equations to fully compute the curvature.

Proposition A.1.10. [6, Theorem 3.2.4 & 3.2.5] *Let ∂_a and X_i be a local orthonormal frame of M^n on the normal neighborhood of the boundary N^{n-1} with respect to $g = da^2 + k(a)$, then*

$$\mathrm{Ric}_g(\partial_a, \partial_a) = -\sum_{i=1}^{n-1} \Big((1/2)k''(a)(\mathrm{X}_i, \mathrm{X}_i) + [(1/2)k'(a)(\mathrm{X}_i, \mathrm{X}_i)]^2\Big). \tag{A.2}$$

$$\begin{aligned} \mathrm{Ric}_g(X_i, X_i) = &-(1/2)k''(a)(X_i, X_i) - k'(a)^2(X_i, X_i) + \mathrm{Ric}_{k(a)}(X_i, X_i) \\ &+ \sum_{j \neq i} (1/4)\left[k'(a)^2(X_i, X_j) - k'(a)(X_i, X_i)k'(a)(X_j, X_j)\right]. \end{aligned} \tag{A.3}$$

The remaining off-diagonals of the Ricci tensor can all be expressed in terms of the derivatives of $k'(a)$ and $k(a)$ with respect to X_i.

As mentioned in Section A.1, for each face of X^n there is a corresponding second fundamental form with respect to g. As normal coordinates for Z^{n-2} relative to Y^{n-1} in Lemma A.1.2.2 agree with the normal coordinates of N^{n-1}, we may appeal to existing computation of the second fundamental form.

Proposition A.1.11. [6, Proposition 3.2.1] *Let ∂_a and X_i be a local orthonormal frame of M^n on the normal neighborhood of the boundary N^{n-1} with respect to $g = da^2 + k(a)$, then*

$$\mathrm{II}(X_i, X_j) = (1/2)k'(a)(X_i, X_j). \tag{A.4}$$

Suppose $\widetilde{\mathrm{Y}}^{n-1}$ is the other face that is bounded by Z^{n-2} and let $\widetilde{\mathrm{II}}$ represent its second fundamental form with respect to g. If we take normal coordinates as in Lemma A.1.2.2 for Z^{n-2} relative to its face Y^{n-1}, then $\widetilde{\mathrm{II}}$ will not be as elementary to compute as (A.4) but can still be computed in terms of $\mu(a,b)$, $h(a,b)$, and $\phi(a,z)$. In order to perform this computation we must first pick a local frame for $T\widetilde{\mathrm{Y}}^{n-1}$. Note that there is a unit vector $\tau \in T\widetilde{\mathrm{Y}}^{n-1}$ that lies in the space spanned by ∂_a and ∂_b of Lemma A.1.2.2, and that g gives rise to a splitting $T\widetilde{\mathrm{Y}}^{n-1} = \langle \tau \rangle \oplus T\,\mathrm{Z}^{n-2}$. It is in these coordinates that we compute $\widetilde{\mathrm{II}}$.

Lemma A.1.12. *Let* $(a,b,z) : (-\infty, 0]^2 \times \mathrm{Z}^{n-2} \hookrightarrow \mathrm{M}^n$ *be the normal coordinates for* Z^{n-2} *relative to a face* Y^{n-1}*. If* $\mu(a,b)$*,* $h(a,b)$*, and* $\phi(a,z)$ *be as in Lemma A.1.2.2, then the second fundamental form of* $\widetilde{\mathrm{Y}}^{n-1}$ *can be computed as follows.*

$$\widetilde{\mathrm{II}}(\tau,\tau) = \frac{-\phi_{aa}\mu^2 - \phi_a\mu_a\left((\mu\phi_a)^2 + 2\right) - \phi_a^2\mu_b}{\mu^2\left(1 + (\mu\phi_a))^2\right)^{3/2}},$$
$$\widetilde{\mathrm{II}}(v,w) = \frac{-\phi_a\mu^2 h_a(v,w) + h_b(v,w)}{2\mu\sqrt{1+(\mu\phi_a)^2}}.$$

Where we have suppressed the dependency of $\mu(a,b)$*,* $h(a,b)$*, and* $\phi(a,z)$ *on* (a,b,z) *for legibility.*

Proof. Note that the vector $\partial_a + \dot{\phi}(a)\partial_b$ is tangent to $\widetilde{Y}^{n-1}$, and that $-\dot{\phi}(a)\mu^2(a,b)\partial_a + \partial_b$ is normal to this. Thus τ and its unit normal ν must take the form:

$$\tau = \frac{\partial_a + \phi_a\partial_b}{\sqrt{1+\mu^2\phi_a^2}} \text{ and } \nu = \frac{-\mu^2\phi_a\partial_a + \partial_b}{\mu\sqrt{1+\mu^2\phi_a^2}}. \tag{A.5}$$

We will let τ_a, τ_b, ν_a, and ν_b denote the coefficients of ∂_a and ∂_b that appear in the above expressions of τ and ν (which are all functions of a, b, and z). Note because

$g = da^2 + \mu^2(a,b)db^2 + h(a,b)$, that ν is also normal to $T\,\mathrm{Z}^{n-2}$. Thus ν is in fact the unit normal of the hypersurface $\widetilde{\mathrm{Y}}^{n-1}$ with respect to g.

Next we compute the Christoffel symbols of the metric $da^2 + \mu^2(a,b)db^2 + h(a,b)$ in coordinates. Let $v, w \in T\,\mathrm{Z}^{n-2}$ be any coordinate vector fields. As $\widetilde{\mathrm{II}}(x,y) = -g(\nabla_x y, \nu)$ and $\nu = \nu_a\partial_a + \nu_b\partial_b$, we only need to compute Christoffel symbols of the form Γ^a_{**} and Γ^b_{**}. Up to symmetry, the only nonzero Christoffel symbols of this form are

$$\Gamma^a_{ab} = \frac{\mu_a}{\mu}, \Gamma^a_{bb} = -\mu_a\mu, \Gamma^b_{bb} = \frac{\mu_b}{\mu}, \Gamma^a_{vw} = -\frac{1}{2}h_a(v,w), \text{ and } \Gamma^b_{v,w} = -\frac{1}{2}\frac{h_b(v,w)}{\mu^2}. \tag{A.6}$$

For $\widetilde{\mathrm{II}}(\tau,\tau)$ we have

$$\begin{aligned}\widetilde{\mathrm{II}}(\tau,\tau) &= -g(\nabla_\tau\tau,\nu),\\ \widetilde{\mathrm{II}}(\tau,\tau) &= -\nu_a(\tau_a(\partial_a\tau_a) + \tau_b(\partial_b\tau_a)) - \mu^2\nu_b(\tau_a(\partial_a\tau_b) + \tau_b(\partial_b\tau_b))\\ &\quad - \tau_a\tau_b\mu^2\nu_b(\Gamma^b_{ab} + \Gamma^b_{ba}) - \tau_b^2(\nu_a\Gamma^a_{bb} + \mu^2\nu_b\Gamma^b_{bb}).\end{aligned} \tag{A.7}$$

One can compute that

$$\begin{aligned}\partial_a\tau_a &= -\frac{\mu^2\phi_a\phi_{aa} + \mu\mu_a\phi_a^2}{(1+\mu^2\phi_a^2)^{3/2}}, \quad \partial_b\tau_a = -\frac{\mu\mu_b\phi_a^2}{(1+\mu^2\phi_a^2)^{3/2}},\\ \partial_a\tau_b &= \frac{\phi_{aa} - \mu\mu_a\phi_a^3}{(1+\mu^2\phi_a^2)^{3/2}}, \text{ and } \partial_b\tau_b = -\frac{\mu\mu_b\phi_a^3}{(1+\mu^2\phi_a^2)^{3/2}}.\end{aligned} \tag{A.8}$$

Combining equations (A.5), (A.8), and (A.6) into the equation (A.7) yields the desired equation for $\widetilde{\mathrm{II}}(\tau,\tau)$.

Then noting that $\widetilde{\mathrm{II}}(v,w) = -g(\nabla_v w,\nu) = -\Gamma^a_{vw}\nu_a - \mu^2(a,b)\Gamma^b_{vw}\nu_b$ combined with equations (A.5) and (A.6) yields the desired equation for $\widetilde{\mathrm{II}}|_{T\,\mathrm{Z}^{n-2}}$. □

A.2. Splines

A spline is any piecewise polynomial function. Typically splines are used to interpolate between a sequence of points with specified derivatives or at least a specified degree of differentiability. Splines are particularly well adapted to the field of design and engineering as they produce explicit formulas for models from intuitive user input that can achieve any degree of differentiability while also minimizing computational complexity. It was the insight of [33] that splines are also useful for constructive Riemannian geometry, using them in particular to prove Theorem 1.2.2.

We begin by defining the class of splines we will consider in Section A.2.1 as depending on a continuous family of sections $F(a)$ and a small parameter ε, which represents the scale at which the smoothing occurs. We then study the asymptotic behavior of our splines and their derivatives with respect to $\varepsilon \to 0$ in Sections A.2.1.1 and A.2.1.2. We conclude in Section A.2.2 by repeating the proof of Theorem 1.2.2, which is nearly identically to [40, Proof of Theorem 2], for the sake of completeness in our proof of Theorem I.

A.2.1. Ferguson Splines

We will use a very basic version of a spline in the following general situation. Given a Euclidean vector bundle $\mathrm{E} \to \mathrm{M}^n$, suppose we have $F(a)$, a continuous family of sections of E parameterized by $a \in \mathbf{R}$ that is smooth for $a \neq 0$. Define $P_{k,\varepsilon}(a)$ to be the unique degree $2k+1$ in $\Gamma(\mathrm{E})[a]$ whose coefficients are defined by the following system of $2k+2$ linear equations

$$P_{k,\varepsilon}^{(i)}(\pm\varepsilon) = F^{(i)}(\pm\varepsilon), \text{ for } 0 \leq i \leq k. \tag{A.9}$$

The spline $P_{k,\varepsilon}(a)$ is a natural choice for locally smoothing families of sections that fail to be differentiable at a point. These splines are often called *Ferguson curves* as they were first used by a Boeing engineer in [65] applied to parametric curves and surfaces in $\mathbf{R}^3$.

For each $0 \leq i \leq k$ and $j = 0, 1$, one expects there to be real valued polynomials $H^{ij}_{k,\varepsilon}(a)$ such that

$$P_{k,\varepsilon}(a) = \sum_{i=0}^{k} \left[F^{(i)}(-\varepsilon) H^{i0}_{k,\varepsilon}(a) + F^{(i)}(\varepsilon) H^{i1}_{k,\varepsilon}(a) \right]. \tag{A.10}$$

These are the degree $2k+1$ Hermite polynomials. They are completely independent of the family of sections $F(a)$ and are defined as the unique degree $2k+1$ real-valued polynomials with coefficients defined by the following system of $2k+2$ linear equations

$$(H^{ij}_{k,\varepsilon})^{(l)}((-1)^{m+1}\varepsilon) = \delta_{il}\delta_{jm} \tag{A.11}$$

As $H^{ij}_{k,\varepsilon}(a)$ are polynomials, (A.11) reduces to a system of $2k+2$ linear equations in $2k+2$ variables, and it is not hard to check that this system always has nonzero determinant. Clearly if $H^{ij}_{k,\varepsilon}(a)$ solve (A.11), then $P_{k,\varepsilon}(a)$ defined by (A.10) solves (A.9).

If one expands (A.10) to find the coefficients of a^k, because (A.11) is entirely independent of E, we see that the coefficients of $H^{ij}_{k,\varepsilon}(a)$ define a collection of linear functionals $L^{ij}_{k,\varepsilon} : \mathbf{R}^{k+1} \to \mathbf{R}$ that transform the derivatives of $F(a)$ at $a = \pm\varepsilon$ into the coefficients of a^k in $P_{k,\varepsilon}(a)$ defined by (A.9). If we let $J^k_a F \in (\Gamma\,\mathrm{E})^{\oplus k}$ be the vector whose i-th component is $(J^k_a F)_i = F^{(i)}(a)$, then

$$P_{k,\varepsilon}(a) = \sum_{i=0}^{k} \left(\left[L^{i0}_{k,\varepsilon}(J^k_{-\varepsilon}F) + L^{i1}_{k,\varepsilon}(J^k_{\varepsilon}F) \right] a^k \right), \tag{A.12}$$

where we let linear functionals defined on $\mathbf{R}^n$ also acts as a linear maps $L^{ij}_{k,\varepsilon}$: $V^{\oplus n} \to V$ for any real vector space V using component-wise scalar multiplication.

We will ultimately apply $P_{k,\varepsilon}(a)$ for $k = 1, 2$ to the metric $g = da^2 + \mu^2(a,b)db^2 + h(a,b)$ and the boundary function $\phi(a)$ in the combined normal coordinates described at the end of Section A.1.2.2 to prove Theorem I′. The remainder of this section is dedicated to explicitly computing $P_{1,\varepsilon}(a)$ and $P_{2,\varepsilon}(a)$ in order to summarize the asymptotic behavior as $\varepsilon \to 0$ in Sections A.2.1.1 and A.2.1.2 respectively. We should emphasize that the this approach is motivated by the original proof of Theorem 1.2.2 in [33, Section 4], which claims there exists an interpolation with the desired asymptotics. This proof has been revisited in detail in [40, Section 2], where a specific choice of interpolating functions is made, and these are the functions we also have settled on in (A.9) for our proof of Theorem I′.

A.2.1.1. First order

In this section we will consider the first order spline $P_{1,\varepsilon}(a)$, and study the behavior of its first and second derivatives as $\varepsilon \to 0$.

To begin we solve the differential equations (A.11) in the case $k = 1$ to find:

$$H^{00}_{1,\varepsilon}(a) = \frac{a^3}{4\varepsilon^3} - \frac{3a}{4\varepsilon} + \frac{1}{2} \qquad H^{01}_{1,\varepsilon}(a) = -\frac{a^3}{4\varepsilon^3} + \frac{3a}{4\varepsilon} + \frac{1}{2}$$

$$H^{10}_{1,\varepsilon}(a) = \frac{a^3}{4\varepsilon^2} - \frac{a^2}{4\varepsilon} - \frac{a}{4} + \frac{\varepsilon}{4} \qquad H^{11}_{1,\varepsilon}(a) = \frac{a^3}{4\varepsilon^2} + \frac{a^2}{4\varepsilon} - \frac{a}{4} - \frac{\varepsilon}{4}$$

Let $J^1_{-\varepsilon}F = (l_0, l_1)$ and $J^1_{\varepsilon}F = (r_0, r_1)$, we can expand (A.10) and solve for the linear functionals in (A.12) as follows.

$$
\begin{aligned}
L^{00}_{1,\varepsilon}J^1_{-\varepsilon}F &= \frac{2l_0 + \varepsilon l_1}{4} & \qquad L^{01}_{1,\varepsilon}J^1_{\varepsilon}F &= \frac{2r_0 - \varepsilon r_1}{4} \\
L^{10}_{1,\varepsilon}J^1_{-\varepsilon}F &= -\frac{3l_0 + \varepsilon l_1}{4\varepsilon} & \qquad L^{11}_{1,\varepsilon}J^1_{\varepsilon}F &= \frac{3r_0 - \varepsilon r_1}{4\varepsilon} \\
L^{20}_{1,\varepsilon}J^1_{-\varepsilon}F &= -\frac{l_1}{4\varepsilon} & \qquad L^{21}_{1,\varepsilon}J^1_{\varepsilon}F &= \frac{r_1}{4\varepsilon} \\
L^{30}_{1,\varepsilon}J^1_{-\varepsilon}F &= \frac{l_0 + \varepsilon l_1}{4\varepsilon^3} & \qquad L^{31}_{1,\varepsilon}J^1_{\varepsilon}F &= \frac{-r_0 + \varepsilon r_1}{4\varepsilon^3}
\end{aligned}
$$

Using (A.12) it is now trivial to compute the derivatives of $P_{1,\varepsilon}(a)$. As we intend to apply $P_{1,\varepsilon}(a)$ to g and ϕ of Section A.1.2.2 to prove Theorem I′, we see that we need control on the first and second derivatives of $P_{1,\varepsilon}(a)$ in order to preserve $\mathrm{Ric}_g > 0$ and $\widetilde{\mathrm{II}} > 0$ as in Proposition A.1.10 and Lemma A.1.12. The following Lemma summarizes the aspects of $P_{1,\varepsilon}(a)$ and its first and second derivative that we will need to prove Theorem I′.

Before we state our lemma, we explain big O notation used in this setting. When we write $F(a) = G(a) + O(\varepsilon^k)$ we mean that $||F(a) - G(a)|| = O(\varepsilon^k)$ in the usual sense, where the norm is taken with respect to the bundle metric on E.

Lemma A.2.1. *Let $F(a)$ be a continuous family of sections of a Euclidean vector bundle $\mathrm{E} \to \mathrm{M}^n$ parametrized by $a \in \mathbf{R}$, which is smooth for $a \neq 0$. Let $P_{1,\varepsilon}(a)$ be the cubic polynomial in $\Gamma(\mathrm{E})[a]$ with coefficients defined via equation A.9, then for all $a \in [-\varepsilon, \varepsilon]$ we have*

1. *$DP_{1,\varepsilon}(a) = DF(0) + O(\varepsilon^2)$, where D is any differential operator of* E
2. *$DP'_{k,\varepsilon}(a) = \dfrac{\varepsilon - a}{2\varepsilon}DF'(-\varepsilon) + \dfrac{\varepsilon + a}{2\varepsilon}DF'(\varepsilon) + O(\varepsilon)$*
3. *$P'_{k,\varepsilon}(a) = \dfrac{F'_+(0) - F'_-(0)}{2\varepsilon} + O(1)$.*

Proof. Using (A.12) we have

$$P''_{1,\varepsilon}(a) = 2\left[L^{20}_{1,\varepsilon}J^1_{-\varepsilon}F + L^{21}_{1,\varepsilon}J^1_{\varepsilon}F\right] + 6\left[L^{30}_{1,\varepsilon}J^1_{\varepsilon}F + L^{31}_{1,\varepsilon}J^1_{\varepsilon}F\right]a$$

$$P''_{1,\varepsilon}(a) = \frac{r_1 - l_1}{2\varepsilon} + 6a\left(\frac{l_0 - r_0}{4\varepsilon^3} + \frac{l_1 + r_1}{4\varepsilon^2}\right) \tag{A.13}$$

Note that

$$\frac{l_0 - r_0}{\varepsilon} = -\left(\frac{F(0) - F(-\varepsilon)}{\varepsilon} + \frac{F(\varepsilon) - F(0)}{\varepsilon}\right) = -(l_1 + r_1) + O(\varepsilon). \tag{A.14}$$

Combining (A.14) with the formula $P''_{1,\varepsilon}(a)$ in (A.13) yields (3).

$$P''_{1,\varepsilon}(a) = \frac{r_1 - l_1}{2\varepsilon} + \frac{6a}{4\varepsilon}\frac{O(\varepsilon)}{\varepsilon} = \frac{r_1 - l_1}{2\varepsilon} + O(1).$$

The formulas in (1) and (2) follow by integrating (3) over the interval $[0, a]$ for $a \in [-\varepsilon, \varepsilon]$. □

A.2.1.2. Second order

In this section we will consider the second order spline $P_{2,\varepsilon}(a)$, and study the behavior of its first and second derivatives as $\varepsilon \to 0$.

To begin we solve the differential equations (A.11) in the case $k = 2$ to find:

$$
\begin{aligned}
H^{00}_{2,\varepsilon} &= -\frac{3a^5}{16\varepsilon^5} + \frac{5a^3}{8\varepsilon^3} - \frac{15a}{16\varepsilon} + \frac{1}{2} \\
H^{01}_{2,\varepsilon} &= \frac{3a^5}{16\varepsilon^5} - \frac{5a^3}{8\varepsilon^3} + \frac{15a}{16\varepsilon} + \frac{1}{2} \\
H^{10}_{2,\varepsilon} &= -\frac{3a^5}{16\varepsilon^4} + \frac{a^4}{16\varepsilon^3} + \frac{5a^3}{8\varepsilon^2} - \frac{3a^2}{8\varepsilon} - \frac{7a}{16} + \frac{5\varepsilon}{16} \\
H^{11}_{2,\varepsilon} &= -\frac{3a^5}{16\varepsilon^4} - \frac{a^4}{16\varepsilon^3} + \frac{5a^3}{8\varepsilon^2} + \frac{3a^2}{8\varepsilon} - \frac{7a}{16} - \frac{5\varepsilon}{16} \\
H^{20}_{2,\varepsilon} &= -\frac{a^5}{16\varepsilon^3} + \frac{a^4}{16\varepsilon^2} + \frac{a^3}{8\varepsilon} - \frac{a^2}{8} - \frac{\varepsilon a}{16} + \frac{\varepsilon^2}{16} \\
H^{21}_{2,\varepsilon} &= \frac{a^5}{16\varepsilon^3} + \frac{a^4}{16\varepsilon^2} - \frac{a^3}{8\varepsilon} - \frac{a^2}{8} + \frac{\varepsilon a}{16} + \frac{\varepsilon^2}{16}
\end{aligned}
$$

Let $J^2_{-\varepsilon}F = (l_0, l_1, l_2)$ and $J^2_{\varepsilon}F = (r_0, r_1, r_2)$, we can expand (A.10) and solve for the linear functionals in (A.12) as follows.

$$
\begin{aligned}
L^{00}_{2,\varepsilon}J^2_{-\varepsilon}F &= \frac{\varepsilon^2 l_2 + 5\varepsilon l_1 + 8l_0}{16} & L^{01}_{2,\varepsilon}J^2_{\varepsilon}F &= \frac{\varepsilon^2 r_2 - 5\varepsilon r_1 + 8r_0}{16} \\
L^{10}_{2,\varepsilon}J^2_{-\varepsilon}F &= \frac{-\varepsilon^2 l_2 - 7\varepsilon l_1 - 15l_0}{16\varepsilon} & L^{11}_{2,\varepsilon}J^2_{\varepsilon}F &= \frac{\varepsilon^2 r_2 - 7\varepsilon r_1 + 15\varepsilon r_0}{16\varepsilon} \\
L^{20}_{2,\varepsilon}J^2_{-\varepsilon}F &= \frac{-3l_1 - \varepsilon l_2}{8\varepsilon} & L^{21}_{2,\varepsilon}J^2_{\varepsilon}F &= \frac{3r_1 - \varepsilon r_2}{8\varepsilon} \\
L^{30}_{2,\varepsilon}J^2_{-\varepsilon}F &= \frac{\varepsilon^2 l_2 + 5\varepsilon l_1 + 5l_0}{8\varepsilon^3} & L^{31}_{2,\varepsilon}J^2_{\varepsilon}F &= \frac{-\varepsilon^2 r_2 + 5\varepsilon r_1 - 5r_0}{8\varepsilon^3} \\
L^{40}_{2,\varepsilon}J^2_{-\varepsilon}F &= \frac{\varepsilon l_2 + l_1}{16\varepsilon^3} & L^{41}_{2,\varepsilon}J^2_{\varepsilon}F &= \frac{\varepsilon r_2 - r_1}{16\varepsilon^3} \\
L^{50}_{2,\varepsilon}J^2_{-\varepsilon}F &= \frac{-\varepsilon^2 l_2 - 3\varepsilon l_1 - 3l_0}{16\varepsilon^5} & L^{51}_{2,\varepsilon}J^2_{\varepsilon}F &= \frac{\varepsilon^2 r_2 - 3\varepsilon r_1 + 3r_0}{16\varepsilon^5}
\end{aligned}
$$

Using (A.12) it is now trivial to compute the derivatives of $P_{2,\varepsilon}(a)$. Recall that we intend to use $P_{1,\varepsilon}(a)$ to smooth g and ϕ of Section A.1.2.2 to prove Theorem I′. As $\widetilde{\mathrm{II}}$ and Ric_g depend on the second derivatives of ϕ and g respectively, and $P_{1,\varepsilon}(a)$ is only once differentiable at $a = \pm\varepsilon$, we will need to apply $P_{2,\varepsilon}(a)$ to further smooth the family. Again, we will need to have control over $P_{2,\varepsilon}(a)$ and its derivatives. The following lemma summarizes the aspects of $P_{2,\varepsilon}(a)$ we will need in the proof of Theorem I′.

Lemma A.2.2. *Let $F(a)$ be a once-differentiable family of sections of a Euclidean vector bundle $\mathrm{E} \to \mathrm{M}^n$ parametrized by $a \in \mathbf{R}$, which is smooth for $a \neq 0$. Let $P_{2,\varepsilon}(a)$ be the quintic polynomial in $\Gamma(\mathrm{E})[a]$ with coefficients defined via equation A.9, then for all $a \in [-\varepsilon, \varepsilon]$ we have*

1. $DP_{2,\varepsilon}(a) = DF(0) + O(\varepsilon^3)$
2. $DP'_{2,\varepsilon}(a) = DF'(0) + O(\varepsilon^2)$
3. $P''_{2,\varepsilon}(a) = \dfrac{2-p(a)}{4}F''(-\varepsilon) + \dfrac{2+p(a)}{2}F''(\varepsilon) + O(\varepsilon)$, *for some $p(a) \in [-2, 2]$.*

Proof. Using (A.12) we can easily compute the $P''_{2,\varepsilon}(a)$.

$$\begin{aligned}
P''_{2,\varepsilon}(a) &= 2\left(L^{20}_{2,\varepsilon}J^2_{-\varepsilon}F + L^{21}_{2,\varepsilon}J^2_{\varepsilon}F\right) + 6\left(L^{30}_{2,\varepsilon}J^2_{-\varepsilon}F + L^{31}_{2,\varepsilon}J^2_{\varepsilon}F\right)a \\
&\quad + 12\left(L^{40}_{2,\varepsilon}J^2_{-\varepsilon}F + L^{41}_{2,\varepsilon}J^2_{\varepsilon}F\right)a^2 + 20\left(L^{50}_{2,\varepsilon}J^2_{-\varepsilon}F + L^{51}_{2,\varepsilon}J^2_{\varepsilon}F\right)a^3 \\
P''_{2,\varepsilon}(a) &= \left(-(l_2+r_2) + \frac{3}{\varepsilon}(r_1-l_1)\right)\frac{1}{4} + \left(-(r_2-l_2) + \frac{5}{\varepsilon}(l_1+r_1) - \frac{5}{\varepsilon^2}(r_0-l_0)\right)\frac{3a}{4\varepsilon} \\
&\quad + \left((l_2+r_2) - \frac{1}{\varepsilon}(r_1-l_1)\right)\frac{3a^2}{4\varepsilon^2} + \left((r_2-l_2) - \frac{3}{\varepsilon}(l_1+r_1) + \frac{3}{\varepsilon^2}(r_0-l_0)\right)\frac{5a^3}{4\varepsilon^3}
\end{aligned} \tag{A.15}$$

In order to study $P''_{2,\varepsilon}(a)$ as $\varepsilon \to 0$, there are a number of terms in (A.15) with ε in their denominator that need to be considered. Using Taylor's theorem we can rewrite the terms of the following two forms.

$$\begin{aligned}
\frac{r_0-l_0}{\varepsilon} &= \frac{F(\varepsilon)-F(0)+F(0)-F(-\varepsilon)}{\varepsilon} = r_1+l_1+O(\varepsilon), \\
\frac{r_1-l_1}{\varepsilon} &= \frac{F'(\varepsilon)-F'(0)+F'(0)-F'(-\varepsilon)}{\varepsilon} = r_2+l_2+O(\varepsilon).
\end{aligned} \tag{A.16}$$

Substituting (A.16) into (A.15) we have

$$P''_{2,\varepsilon}(a) = \frac{1}{2}(l_2 + r_2) - \frac{3a}{4\varepsilon}(r_2 - l_2) + \frac{5a^3}{4\varepsilon^3}(r_2 - l_2) + O(\varepsilon).$$

From this, if we set $p(a) = \dfrac{5a^3}{4\varepsilon^3} - \dfrac{3a}{4\varepsilon}$, then we have (A.15) as desired

$$P''_{2,\varepsilon}(a) = \frac{2 - p(a)}{4} l_2 + \frac{2 + p(a)}{2} r_2 + O(\varepsilon).$$

Equations (1) and (2) follow from integrating (3) on the interval $[0, a]$ for $a \in [-\varepsilon, \varepsilon]$. □

A.2.2. Sketch of the proof of Theorem 1.2.2

In this section we will give a sketch of the proof of Theorem 1.2.2 from [40]. This is in part because Theorem I′ would require us to repeat the proof, while simultaneously keeping track of $\widetilde{\mathrm{II}}$. Our main motivation for including this proof is to demonstrate, in a slightly simpler setting, the utility of polynomial splines in preserving curvature conditions while smoothing metrics or corners. Indeed, much of the logic in our proof of Theorem I′ will be similar to the proof of Theorem 1.2.2, we will consider the effect of smoothing g or ϕ of Section A.1.2.2 using the splines $P_{1,\varepsilon}$ and $P_{2,\varepsilon}$ on the formulas for Ric_g and $\widetilde{\mathrm{II}}$ in these coordinates provided in Proposition A.1.10 and Lemma A.1.12 respectively.

Proof of Theorem 1.2.2. Take normal coordinates $(a, x) : (-\infty, 0] \times \mathrm{N}_i^{n-1} \to \mathrm{M}_i^n$ with respect to g_i. Using these coordinates we identify the N_i^{n-1} in an orientation reversing fashion, which gives rise to coordinates $(a, x) : \mathbf{R} \times \mathrm{N}^{n-1} \hookrightarrow \mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$. In these coordinates, the C^0 metric $g = g_1 \cup_\Phi g_2$ splits as $da^2 + k(a)$.

By assumption $\mathrm{Ric}_g > 0$ for all $a \neq 0$, where it is not well defined. Note that $k(a) = k_i((-1)^{i+1}a)$ for $(-1)^i a > 0$, thus $\mathrm{II}_1 + \Phi^* \mathrm{II}_2 = k'(0_-) - k'(0_+)$. The assumption that $\mathrm{II}_1 + \Phi^* \mathrm{II}_2 > 0$ implies that

$$k'(0_-) - k'(0_+) > 0. \tag{A.17}$$

Note that $k(a)$ is a continuous family of sections of $\mathrm{Sym}^2\left(T^* \mathrm{N}^{n-1}\right)$, which is naturally endowed with a bundle metric given by $g_1 = g_2$ on N^{n-1}. We may therefore define, for any $\varepsilon > 0$, a new metric $\overline{g} = da^2 + \overline{k}(a)$, where $\overline{k}(a)$ agrees with $k(a)$ for $a \notin [-\varepsilon, \varepsilon]$ but is replaced by $P_{1,\varepsilon}(a)$ defined by (A.9) using $k(a)$ for $a \in [-\varepsilon, \varepsilon]$. By (1) of Lemma A.2.1, $\overline{g}$ will actually be positive definite for $\varepsilon > 0$ sufficiently small. This new metric $\overline{g}$ is everywhere once-differentiable and is smooth for $a \neq \pm\varepsilon$. We claim that, for ε sufficiently small, the metric $\overline{g}$ will have $\mathrm{Ric}_{\overline{g}} > 0$ for $a \neq \pm\varepsilon$ where it is not well defined. By construction $\overline{g} = g$ for $a \notin [-\varepsilon, \varepsilon]$, and so $\mathrm{Ric}_{\overline{g}} > 0$ by assumption for $a \notin [-\varepsilon, \varepsilon]$. It remains to show that $\mathrm{Ric}_{\overline{g}} > 0$ for $a \in (-\varepsilon, \varepsilon)$.

Take ∂_a and X_i to be a local orthonormal frame of M^n with respect to $\overline{g}$ on $[-\varepsilon, \varepsilon] \times \mathrm{N}^{n-1}$ as in Proposition A.1.10. We note from the formulas in Proposition A.1.10 that the Ricci tensor takes the form

$$\mathrm{Ric}_{\overline{g}} = -(1/2)\overline{k}''(a) + Dk(a). \tag{A.18}$$

Where D is some differential operator for which ∂_a has order at most one. While we have no control over the signature of this second term in (A.18), using (1) and (2) of Lemma A.2.1 we see that this term may be bounded independently of ε.

Applying (3) of Lemma A.2.1 to the first term in (A.18) yields

$$\mathrm{Ric}_{\overline{g}} = -\frac{k'(0_+) - k'(0_-)}{2\varepsilon} + O(1). \tag{A.19}$$

The first term in A.19 is positive by (A.17). We see then that the limit of $\mathrm{Ric}_{\overline{g}}(X, X)$ as $\varepsilon \to 0$ is positive, and so $\mathrm{Ric}_{\overline{g}}$ is positive definite for some ε sufficiently small. Fix such an $\varepsilon > 0$.

Next for $\delta > 0$ we define a new metric $\breve{g} = da^2 + \breve{k}(a)$, where $\breve{k}(a)$ agrees with $\overline{k}(a)$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$ and is replaced by $P_{2,\varepsilon}(a)$ defined by (A.9) using $\overline{k}(a)$ for $a \in [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$. This new metric $\breve{g}$ is everywhere twice-differentiable and is smooth for $a \neq \pm\varepsilon \pm \delta$. We claim that, for δ sufficiently small, the metric $\breve{g}$ will have $\mathrm{Ric}_{\breve{g}} > 0$. By construction $\breve{g} = \overline{g}$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$ and so $\mathrm{Ric}_{\breve{g}} > 0$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$. It remains to show that $\mathrm{Ric}_{\breve{g}} > 0$ for$a \in [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$. After a linear reparameterization of $a \in \mathbf{R}$ we can shift either interval to $a \in [-\delta, \delta]$ to avoid repetition.

Again we take ∂_a and X_i to be a local orthonormal frame of M^n with respect to $\breve{g}$ on $[-\delta, \delta] \times \mathrm{N}^{n-1}$ as in Proposition A.1.10. Note that the diagonals of $\mathrm{Ric}_{\breve{g}}$ in these coordinates, computed in Proposition A.1.10, are linear in $\breve{k}''(a)$, which by (3) of Lemma A.2.2 is arbitrarily close to a convex combination of the values of $\overline{k}''(-\delta)$ and $\overline{k}''(\delta)$. The remaining terms in the diagonals of $\mathrm{Ric}_{\breve{g}}$ depend only on $D\breve{k}'(a)$ and $D\breve{k}(a)$, which by (1) and (2) of Lemma A.2.2 are arbitrarily close to the values $D\overline{k}'(0)$ and $D\overline{k}(0)$. We conclude that

$$\mathrm{Ric}_{\breve{g}} = \frac{2 - p(a)}{4} \mathrm{Ric}_{\overline{g}}|_{\{-\delta\}\times \mathrm{N}^{n-1}} + \frac{2 + p(a)}{4} \mathrm{Ric}_{\overline{g}}|_{\{\delta\}\times \mathrm{N}^{n-1}} + O(\delta), \tag{A.20}$$

where $p(a) \in [-2, 2]$. By construction $\mathrm{Ric}_{\overline{g}}$ is well defined and positive definite for $a = \pm\delta$, and since the space of positive definite 2-tensors is convex we conclude that (A.20) is positive definite for $a \in [-\delta, \delta]$ if δ is sufficiently small.

We have found a twice-differentiable metric $\breve{g}$ on $\mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$ with positive Ricci curvature. It is clear from Proposition A.1.10 that Ric is a second order operator on the metric in normal coordinates. As smooth metrics are dense in the space of twice-differentiable metrics, we can find a smooth metric on M^n arbitrarily close to $\breve{g}$ that will also have positive Ricci curvature. □

A.3. The gluing theorem for manifolds with corners

In this section we will prove Theorem I′ and consequently Theorem I. The proof will proceed parallel to the proof of Theorem 1.2.2 outlined above in Section A.2.2, simultaneously smoothing the boundary functions $\phi(a, z)$ and the metric g using the same splines on the same intervals. This proof makes up the entirety of Section A.3.1.

The remainder of this section is dedicated to a different generalization of Theorem 1.2.2 than Theorem I that will be needed in our proof of Theorem B. We consider a second boundary condition that we would like to be preserved under the smoothing process of Theorem I other than $\widetilde{\mathrm{II}} > 0$. In Section A.3.2 we introduce the notion of intrinsic concavity, and compute the relevant curvatures in the normal coordinates for the corner.

The intrinsic concavity of the boundary is not preserved by the smoothing of Theorem I under the hypothesis that dihedral angles add to less than π. In the borderline case when the dihedral angles are exactly π, the intrinsic concavity can be shown to be preserved under the smoothing of Theorem I. This is Theorem II″, which is the version of Theorem I needed in our proof of Theorem B. We prove it below in Section A.3.3.

A.3.1. The two step smoothing process

Proof of Theorem I′. As in the proof of Theorem 1.2.2 in Section A.2.2, we take combined normal coordinates for Z^{n-2} inside of $M^n = M_1^n \cup_\Phi M_2^n$ relative to the faces Y_i^{n-1} and the metrics g_i as in Lemma A.1.9.

Because the normal coordinates of the corners agree with the normal coordinates of N_i^{n-1}, we again have that (A.17) is satisfied. The assumption that $\widetilde{\mathrm{II}}_i > 0$ implies that the formulas in Lemma A.1.12 are positive definite for $a \neq 0$ where they are not well defined. The assumption that the dihedral angle $\theta_1 + \Phi^*\theta_2 < \pi$, corresponds to the assumption that

$$\phi_a(0_-, a) - \phi_a(0_+, z) > 0 \tag{A.21}$$

The reason for this is that the metric is Euclidean with respect to ∂_a and ∂_b at $(0, 0, z)$, and one can verify that the angle made by the left and right tangent vectors of $\phi(a, z)$ at $a = 0$ will be less than π if and only if (A.21) is satisfied.

Recall that $X_c^n \hookrightarrow M^n$ is realized in these combined normal coordinates for Z^{n-2} as the set $b \leq \phi(a, z)$. As mentioned, by the uniqueness in Lemma A.1.6, we may construct an embedding $X^n \hookrightarrow M^n$ by smoothing the function $\phi(a, z)$ at $a = 0$. This is the approach we will take in this proof. We will follow along with the proof of Theorem 1.2.2 in Section A.2.2, smoothing the metric g while simultaneously smoothing the function $\phi(a, z)$. We note that because Ric_g independent of the boundary function $\phi(a,z)$, that we can make identical conclusions as in the proof of Theorem 1.2.2.

For $\varepsilon > 0$ we define $\overline{g} = da^2 + \overline{k}(a)$ as in Section A.2.2, where $\overline{k}(a)$ agrees with $k(a)$ for $a \notin [-\varepsilon, \varepsilon]$ but is replaced by $P_{1,\varepsilon}(a)$ defined by (A.9) using $k(a)$ for $a \in [-\varepsilon, \varepsilon]$. Note that because equation (A.9) is linear in F, that $\overline{g} = da^2 + \overline{\mu}^2(a,b)db^2 + \overline{h}(a,b)$ where $\overline{\mu}$ and $\overline{h}$ are themselves similarly defined using $P_{1,\varepsilon}(a)$ defined by (A.9) using $\mu(a,b)$ and $h(a,b)$. We also define $\overline{\phi}(a,z)$ to agree with $\phi(a,z)$ for $a \notin [-\varepsilon, \varepsilon]$ but to be replaced by $P_{1,\varepsilon}(a)$ defined by (A.9) using $\phi(a,z)$ for $a \in [-\varepsilon, \varepsilon]$. We claim that, for $\varepsilon > 0$ sufficiently small, that $\mathrm{Ric}_{\overline{g}} > 0$ and $\widetilde{\mathrm{II}}_{\overline{g}} > 0$. By the proof of Theorem 1.2.2 in Section A.2.2 we only need to show that $\widetilde{\mathrm{II}}_{\overline{g}} > 0$ for ε sufficiently small. By construction, $\overline{g} = g$ and $\overline{\phi}(a,z) = \phi(a,z)$ for $a \notin [-\varepsilon, \varepsilon]$, and so $\widetilde{\mathrm{II}}_{\overline{g}} > 0$ for $a \notin [-\varepsilon, \varepsilon]$. It remains to show that $\widetilde{\mathrm{II}}_{\overline{g}} > 0$ for $a \in [-\varepsilon, \varepsilon]$.

We will show that $\widetilde{\mathrm{II}}_{\overline{g}}(\tau,\tau) > 0$ and $\widetilde{\mathrm{II}}_{\overline{g}}|_{TZ^{n-2}} > 0$ as in Lemma A.1.12 separately to conclude that $\widetilde{\mathrm{II}}_{\overline{g}} > 0$. We will consider first $\widetilde{\mathrm{II}}_{\overline{g}}(\tau,\tau)$. Note that the denominator of the expression for $\widetilde{\mathrm{II}}_{\overline{g}}(\tau,\tau)$ is always positive. We may therefore clear the denominator and to show that the following expression is positive

$$-\phi_{aa} + L(D\overline{\mu}, \overline{\mu}_a, \overline{\phi}, \overline{\phi}_a), \tag{A.22}$$

Where the precise formula for $L(D\overline{\mu}, \overline{\mu}_a, \overline{\phi}, \overline{\phi}_a)$ can be deduced from Lemma A.1.12. By (1) and (2) of Lemma A.2.1 we have that the second term in (A.22) is $O(1)$ with respect to ε. Substituting this observations and (3) of Lemma A.2.1 into (A.22) yields

$$\widetilde{\mathrm{II}}_{\overline{g}}(\tau,\tau) > -\frac{C(\phi_a(0_+,z) - \phi_a(0_-,z))}{\varepsilon} + O(1). \tag{A.23}$$

By (A.21) this first term in (A.23) positive, and so the limit of (A.23) is ∞ as $\varepsilon \to 0$. In particular, $\widetilde{\mathrm{II}}_{\overline{g}}(\tau, \tau)$ will be positive for ε sufficiently small.

Next, we will show that $\widetilde{\mathrm{II}}_{\overline{g}}|_{TZ^{n-2}} > 0$. Note that the denominator in the expression for $\widetilde{\mathrm{II}}_{\overline{g}}|_{TZ^{n-2}}$ in Lemma A.1.12 is always positive. We may therefore clear the denominator, and show the positivity of

$$-\overline{\mu}^2(a,b)\overline{\phi}_a(a,z)\overline{h}_a + \overline{h}_b(a,b) \tag{A.24}$$

By (1) of Lemma A.2.1, $\overline{\mu}^2(a,b) = \mu^2(0,b) + O(\varepsilon)$, similarly $b = \overline{\phi}(a,z) = \phi(0,z) + O(\varepsilon) = O(\varepsilon)$. We conclude that $\overline{\mu}^2(a,b) = 1 + O(\varepsilon)$. Similarly we conclude $\overline{h}_b(a,b) = h_b(0,0) + O(\varepsilon)$. By (2) of Lemma A.2.1 we have that $\overline{h}_a = O(1)$ and $\overline{\phi}_a = O(1)$. Combining these observations, equation (A.24) becomes

$$-\overline{\phi}_a(a,z)\overline{h}_a(a,0) + h_b(0,0) + O(\varepsilon). \tag{A.25}$$

By assumption, $\widetilde{\mathrm{II}}_{\overline{g}}|_{TZ^{n-2}}$ is positive definite at $a = \pm\varepsilon$. In particular, this means that (A.24) is positive definite when $a = \pm\varepsilon$, which in turn implies (A.25) is positive definite when $a = \pm\varepsilon$. Taking the convex combination of (A.25) at the values $a = \pm\varepsilon$ we have

$$h_b(0,0) - \left[\left(\frac{\varepsilon - a}{2\varepsilon}\right)(h_a(-\varepsilon,0)\phi_a(-\varepsilon,z)) + \left(\frac{\varepsilon + a}{2\varepsilon}\right)(h_a(\varepsilon,0)\phi_a(\varepsilon,z))\right] + O(\varepsilon) > 0. \tag{A.26}$$

Where we have used the definition of $\overline{\phi}'(a,z)$ and $\overline{\mu}_a(a,b)$ to rewrite in terms of the $\mu(a,b)$ and $h(a,b)$ at $a = \pm\varepsilon$.

We claim that (A.25) is bounded below by the expression in (A.26) for all $a \in [-\varepsilon, \varepsilon]$. Indeed, note that by (2) of Lemma A.2.1 that both $\overline{h}_a$ and $\overline{\phi}_a$ are approximately linear functions in a with slopes $(\phi_a(\varepsilon,z) - \phi_a(-\varepsilon,z))/2\varepsilon$ and $(h_a(\varepsilon,0) - h_a(-\varepsilon,0))/2\varepsilon$ respectively. By (A.21) and (A.17), these slopes are negative if ε is chosen to be small enough (taking the infimum over all $z \in \mathrm{Z}^{n-2}$). It follows that the product $\overline{\phi}_a(a,z)\overline{h}_a(a,0)$ is approximately the product of decreasing linear functions, which is approximately a concave up quadratic function. It follows from concavity that, for ε sufficiently small that

$$\overline{\phi}_a(a,z)\overline{h}_a(a,0) < \left[\left(\frac{\varepsilon - a}{2\varepsilon}\right)(h_a(-\varepsilon,0)\phi_a(-\varepsilon,z)) + \left(\frac{\varepsilon + a}{2\varepsilon}\right)(h_a(\varepsilon,0)\phi_a(\varepsilon,z))\right] + O(\varepsilon) \tag{A.27}$$

Applying the inequality (A.27) to equation (A.25) shows that (A.25) and consequently (A.24) is bounded below by the expression in (A.26) provided ε is chosen adequately small. We conclude that $\widetilde{\mathrm{II}}_{\overline{g}}|_{T\,\mathrm{Z}^{n-2}} > 0$ for ε sufficiently small.

Fix an $\varepsilon > 0$ so that $\mathrm{Ric}_{\overline{g}} > 0$ and $\widetilde{\mathrm{II}}_{\overline{g}} > 0$. Next, we will choose $\delta > 0$ and define a metric $\breve{g} = da^2 + \breve{k}(a)$, where $\breve{k}(a)$ agrees with $\overline{k}(a)$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$ and is replaced by $P_{2,\varepsilon}(a)$ defined by (A.9) using $\overline{k}(a)$ for $a \in [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$. This new metric $\breve{g}$ is everywhere twice-differentiable and is smooth for $a \neq \pm\varepsilon \pm \delta$. Because (A.9) is linear in $F(a)$, we see that $\breve{g} = da^2 + \breve{\mu}^2(a,b)db^2 + \breve{h}(a,b)$ where $\breve{\mu}$ and $\breve{h}$ are themselves modified by $P_{2,\varepsilon}(a)$ in (A.9) using $\overline{\mu}$ and $\overline{h}$. We may also define a function $\breve{\phi}(a,z)$ that agrees with $\overline{\phi}(a,z)$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$ and is replaced by $P_{2,\varepsilon}(a)$ defined by (A.9) using $\overline{\phi}(a,z)$ on $a \in [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$. We claim that, for δ sufficiently small, that $\mathrm{Ric}_{\breve{g}} > 0$ and $\widetilde{\mathrm{II}}_{\breve{g}} > 0$. By the proof of Theorem 1.2.2 in Section A.2.2 we only need to show that $\widetilde{\mathrm{II}}_{\breve{g}} > 0$ for δ sufficiently small. By construction, $\breve{g} = \overline{g}$ and $\breve{\phi}(a,z) = \overline{\phi}(a,z)$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$, and so $\widetilde{\mathrm{II}}_{\breve{g}} > 0$ for $a \notin [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$. It remains to show that $\widetilde{\mathrm{II}}_{\breve{g}} > 0$ for $a \in [\pm\varepsilon - \delta, \pm\varepsilon + \delta]$.

Ater a linear reparameterization of $a \in \mathbf{R}$ we can shift either interval to $a \in [-\delta, \delta]$ to avoid repetition.

We will again show that $\widetilde{\mathrm{II}}_{\breve{g}}(\tau,\tau) > 0$ and $\widetilde{\mathrm{II}}_{\breve{g}}|_{T\,\mathrm{Z}^{n-2}} > 0$ separately. We begin by considering the formula for $\widetilde{\mathrm{II}}_{\breve{g}}(\tau,\tau)$ in Lemma A.1.12, the numerator of which is of the form

$$\widetilde{\mathrm{II}}_{\breve{g}}(\tau,\tau) = -\breve{\phi}_{aa}(a,z) + \breve{Q}(a,b,z), \tag{A.28}$$

where $\breve{Q}(a,b,z)$ is an expression involving the first derivatives of $\breve{\mu}$ and $\breve{\phi}$. By (1) and (2) of Lemma A.2.2, we see that $\breve{Q}(a,b,z) = \overline{Q}(0,b,z) + O(\delta)$, where $\overline{Q}(a,b,z)$ is the corresponding expression for $\overline{\mu}$ and $\overline{\phi}$. By assumption $\widetilde{\mathrm{II}}_{\breve{g}}(\tau,\tau) > 0$ at $a = \pm\delta$, taking the convex combination of (A.28) at these two endpoints we deduce that

$$\left[\frac{a+\delta}{2\delta}\overline{\phi}_{aa}(-\delta,z) + \frac{a-\delta}{2\delta}\overline{\phi}_{aa}(\delta,z)\right] + \overline{Q}(0,b,z) + \delta > 0. \tag{A.29}$$

But applying (3) of Lemma A.2.2 to $\breve{\phi}_{aa}(a,z)$, we see that the lefthand side of (A.29) is arbitrarily close to $\breve{\phi}_{aa}(a,z)$. From which we deduce that (A.28) is positive and in turn $\widetilde{\mathrm{II}}_{\breve{g}}(\tau,\tau) > 0$ provided δ is chosen small enough.

Finally, we consider $\widetilde{\mathrm{II}}_{\breve{g}}|_{T\,\mathrm{Z}^{n-2}}$. Note from (1) and (2) of Lemma A.2.2 that all the terms in equation for $\widetilde{\mathrm{II}}_{\breve{g}}|_{T\,\mathrm{Z}^{n-2}}$ in Lemma A.1.12 are arbitrarily close to those terms for $\overline{g}$ and $\overline{\phi}$. Because $\widetilde{\mathrm{II}}_{\overline{g}}|_{T\,\mathrm{Z}^{n-2}} > 0$, we immediately conclude that $\widetilde{\mathrm{II}}_{\breve{g}}|_{T\,\mathrm{Z}^{n-2}} > 0$ if $\delta > 0$ is chosen sufficiently small.

We have found a twice-differentiable metric $\breve{g}$ and a twice-differentiable boundary function $\breve{\phi}(a,z)$ on $\mathrm{M}^n = \mathrm{M}_1^n \cup_\Phi \mathrm{M}_2^n$ with positive Ricci curvature and convex faces. It is clear from Proposition A.1.10 and Lemma A.1.12 that Ric and $\widetilde{\mathrm{II}}$ are a second order operator on the metric in normal coordinates. As smooth metrics are dense in the space of twice-differentiable metrics, we can find a smooth metric on M^n arbitrarily close to $\breve{g}$ that will also have positive Ricci curvature and convex faces. $\square$

A.3.2. The intrinsic concavity of the boundary

In normal coordinates for Z^{n-2} relative to Y^{n-1} with respect to g recall that $\widetilde{\mathrm{Y}}$ can describe as the set $b = \phi(a, z)$. In this section we assume that $\phi(a, z) = \phi(a)$ is independent of z. Let $\tilde{g}$ denote the metric g restricted to $\tilde{\mathrm{Y}}$. In this situation we have that

$$\tilde{g} = da^2 + \mu^2(a, \phi(a))(\phi'(a))^2 da^2 + h(a, \phi(a)).$$

We may find an arc-length parameterization $a(s)$ so that $\tilde{g} = ds^2 + h(a(s), b(s)) = ds^2 + h(s)$, where $b(s) = \phi(a(s))$. In this setting, we may discuss the signature of $h''(s)$, which measures the intrinsic concavity of the metric $\tilde{g}$. In particular, we are interested in studying this condition because if $h''(s) < 0$ then necessarily $\mathrm{Ric}_{\tilde{g}}(\partial_s, \partial_s) > 0$ by Proposition A.1.10. In our proof of Theorem B, we will need to show that the boundary metric produced by Theorem I is Ricci-positive isotopic to a round metric, which in particular means we need to show that the boundary will be Ricci-positive. In order to prove such a thing we must consider $h'(s)$ and $h''(s)$ in normal coordinates.

Lemma A.3.1. *Given a Riemannian manifold with faces (X^n, g), in normal coordinates of a corner Z^n relative to a face Y^{n-1} the metric restricted to $\widetilde{\mathrm{Y}}^{n-1}$ can be written as $\tilde{g} = ds^2 + h(a(s), b(s))$. Moreover we have that*

$$\partial_s h(a(s), b(s)) = \frac{h_a + \phi_a h_b}{\sqrt{1 + \mu^2 \phi_a^2}}, \tag{A.30}$$

$$\partial_s^2 h(a(s), b(s)) = \frac{h_{aa}}{(1 + \mu^2 \phi_a^2)^2} + D(h, \mu, \phi), \tag{A.31}$$

where D is some differential operator that has order 1 with respect to ∂_a.

Proof. Clearly we have

$$\partial_s h(a(s), b(s)) = a' h_a + b' h_b \tag{A.32}$$

$$\partial_s^2 h(a(s), b(s)) = a'' h_a + b'' h_b + (a')^2 h_{aa} + 2a'b' h_{ab} + (b')^2 h_{bb}. \tag{A.33}$$

So it suffices to compute the first and second derivatives of $a(s)$ and $b(s)$ in terms of $\mu(a, b)$ and $\phi(a)$. Let $\psi(a)$ denote the arclength of the path $(a, \phi(a))$ with respect to the metric $da^2 + \mu(a, b)db^2$. It is straightforward to compute

$$\begin{aligned} \psi'(a) &= \sqrt{1 + \mu^2 \phi_a^2} \\ \psi''(a) &= \frac{\mu \mu_a \phi_a^2 + \mu \mu_b \phi_a^3 + \mu^2 \phi_a \phi_{aa}}{\sqrt{1 + \mu^2 \phi_a^2}} \end{aligned} \tag{A.34}$$

If s denotes the arclength parameterization, then $a(s) = \psi^{-1}(s)$, and $b(s) = \phi \circ \psi^{-1}(s)$.

$$\begin{aligned} a'(s) &= \frac{1}{\psi'(a)}, \\ a''(s) &= -\frac{\psi''(a)}{(\psi'(a))^3}, \\ b'(s) &= \frac{\phi_a(a)}{\psi'(a)}, \\ b''(s) &= \frac{\phi_{aa}(a)}{(\psi'(a))^2} - \frac{\phi_a(a)\psi''(a)}{(\psi'(a))^3}. \end{aligned} \tag{A.35}$$

Substituting (A.34) and (A.35) into (A.32) yields (A.30), and similarly substituting (A.34) and (A.35) into (A.33) will give a precise formula for D in (A.31). □

The issue we run into when considering smoothing g and ϕ using the splines defined by (A.9), is that (3) of Lemma A.2.1 implies that the signature of $h''(s)$ will be determined by those terms involving h_{aa} and ϕ_{aa} in (A.31) above. If one works out the coefficient of ϕ_{aa} from formulas (A.34), (A.35), and (A.33), we see that it does not have a definitive sign. This is why, in section A.3.3 we must restrict to the borderline case when $\phi_a(0_+) + \phi_a(0_-) = 0$, so that the signature of $h''(s)$ is determined by the signature of h_{aa} when applying the first order spline to g and ϕ.

A.3.3. A borderline case

Assuming that we are in the same situation as Theorem I except we assume that $\phi(a, z) = \phi(a)$ is independent of z and $\phi_a(0_+) + \phi_a(0_-) = 0$. Under these assumptions, an arbitrarily small perturbation of the $\phi_i(a)$ will result in $\phi_a(0_+) + \phi_a(0_-) > 0$ to which we may apply Theorem I. To begin we will show that it is possible to make this perturbation so small, that the second derivatives of the smoothing of $\phi(a)$ are bounded.

Lemma A.3.2. *Assuming everything is the same as in Theorem I′ except that the hypothesis that $\phi_a(0_+) + \phi_a(0_-) > 0$ is replaced by the hypothesis that $\phi_a(0_+) + \phi_a(0_-) = 0$. Then there is a family of boundary functions $\phi_\xi(a)$ and decreasing functions $\varepsilon(\xi)$ and $\delta(\xi)$ such that the conclusions of Theorem I′ holds by smoothing g and $\phi_\xi(a)$ with the first order spline on $[-\varepsilon(\xi), \varepsilon(\xi)]$ and the second order spline on $[\pm\varepsilon(\xi) - \delta(\xi), \pm\varepsilon(\xi) + \delta(\xi)]$. Denote by $\overline{g}$ and $\overline{\phi}_\xi$ this first order smoothing and $\breve{g}$ and $\breve{\phi}_\xi$ this second order smoothing.*

In addition to the conclusions of Theorem I′, we may also bound $|\overline{\phi}''_\xi(a)| < C$ in terms of some C independent of ξ.

Proof. Let $\chi_\xi(a)$ be any family of smooth function defined with $\xi \in [0, 1)$ such that $\chi_\xi(a) \equiv 1$ for $a < -\xi$, $\chi_\xi(0) = 1$ and $\chi'_\xi(0) > 0$, and $\chi_\xi(a)$ uniformly converges to 1 as $\xi \to 0$. Note that the functions $\chi_\xi(a)\phi_1(a)$ and $\phi_2(a)$ satisfy the hypotheses of

Theorem I′ for $\xi > 0$ sufficiently small (so that $\widetilde{\mathrm{II}}_g > 0$). By the proof of Theorem I′, for each ξ, there is a range of possible ε for which $\overline{g}$ and $\overline{\phi}_\xi$ will have $\mathrm{Ric}_{\overline{g}} > 0$ and $\widetilde{\mathrm{II}}_{\overline{g}} > 0$. We claim that we can choose $\varepsilon(\xi)$ for which $\overline{\phi}''_\xi(a) = O(1)$ with respect to ξ.

First note that the equation for $\mathrm{Ric}_{\overline{g}} > 0$ in Proposition A.1.10 is completely independent of $\overline{\phi}_\xi(a)$, thus there exists an ε_0 for which $\mathrm{Ric}_{\overline{g}} > 0$ for $\varepsilon(\xi) < \varepsilon_0$. The equation for $\widetilde{\mathrm{II}}_{\overline{g}}|_{T\,\mathrm{Z}^{n-2}}$ in Lemma A.1.12 depends only on the first derivatives of $\overline{\phi}_\xi$ and $\overline{g}$. In the proof of Theorem I′, it is shown that $\widetilde{\mathrm{II}}_{\overline{g}} > 0$ solely by controlling the error term in (2) of Lemma A.2.1. This error term can be expressed in terms the second derivatives of $\overline{\phi}_\xi(a)$. As $\overline{\phi}_\xi$ converges uniformly to $\overline{\phi}$, this error term can be made arbitrarily small independent of ξ so again there will be a $\varepsilon_0 > 0$ for which $\widetilde{\mathrm{II}}_{\overline{g}}|_{T\,\mathrm{Z}^{n-2}} > 0$ if $\varepsilon(\xi) < \varepsilon_0$.

Finally, we must consider the equation for $\widetilde{\mathrm{II}}_{\overline{g}}(\tau, \tau)$ in Lemma A.2.1. Recall that we needed the fact that $\overline{\chi}''_\xi(a) \to \infty$ as $\varepsilon \to 0$ to dominate the other terms. The other terms are all the first derivatives of $\overline{\phi}_\xi(a)$ and $\overline{g}$, which by (1) and (2) of Lemma A.2.1 can be bounded independently of ξ. Thus there is a $C > 0$ independent of ξ such that $\widetilde{\mathrm{II}}_{\overline{g}}(\tau, \tau) > 0$ provided that $\overline{\phi}''_\xi(a) < -C$ for $a \in [-\varepsilon(\xi), \varepsilon(\xi)]$. We claim that we can choose $\varepsilon(\xi)$ so that $\overline{\phi}''_\xi(a) < -C$ but $|\overline{\phi}''_\xi(a)| < M$ for an M independent of ξ. Considering (3) of Lemma A.2.1, if we can bound the error term in absolute value by D independently of ξ, let $\varepsilon(\xi) < \varepsilon_0$ be a solution to the following:

$$\frac{\overline{\phi}'_\xi(\varepsilon(\xi)) - \overline{\phi}'_\xi(-\varepsilon(\xi))}{2\varepsilon(\xi)} + D < -M.$$

For such a $\varepsilon(\xi)$ we have that $|\overline{\phi}''_\xi(a)| < M + D$ as desired. The error term in (3) of Lemma A.2.1 again depends on the second derivatives of $\overline{\phi}_\xi(a)$, which again can be bounded in terms of the second derivatives of $\overline{\phi}(a)$ independently of ξ.

For each $\xi > 0$ sufficiently small, fix such a $\varepsilon(\xi)$. Then the proof continues identically to the proof of Theorem I′ to produce a $\delta(\xi)$ for which the conclusions hold. □

With Lemma A.3.2 established, we can state the following generalization of Theorem I which in addition claims that intrinsic concavity of the faces is preserved under the smoothing procedure.

Theorem II″. *Assuming everything is the same as in Theorem I′ except that the hypothesis that $\theta_1 + \Phi^*\theta_2 < \pi$ is replaced by the hypothesis that $\theta_1 + \Phi^*\Theta_2 = \pi$. Then the same conclusions as Theorem I′ holds.*

Assume additionally that boundary function of Lemma A.1.9 depends only on a, so that $\phi(a, z) = \phi(a)$, and that the arclength parameterized boundary metrics satisfy $h''(s) < 0$ in normal coordinates for the g_i, then we can conclude the same for the smooth metric g.

Proof. We let ϕ_ξ, $\varepsilon(\xi)$ and $\delta(\xi)$ be as in Lemma A.3.2. We want to in addition consider the effect of these smoothings on the formula for $\partial_s^2 h(a(s), b(s))$ in Lemma A.3.1. Let $\overline{g}$ and $\overline{\phi}_\xi$ be the first order smoothing on the interval $[-\varepsilon(\xi), \varepsilon(\xi)]$. By assumption we have $\partial_s^2 \overline{h}(\overline{a}(s), \overline{b}(s)) < 0$ for $a \notin [-\varepsilon(\xi), \varepsilon(\xi)]$, we claim this remains true for $a \in (-\varepsilon(\xi), \varepsilon(\xi))$.

By (1) and (2) of Lemma A.2.1, all of the arguments of D in the formula for $\partial_s^2 h(a(s), b(s))$ in Lemma A.3.1 other than $\phi''_\xi(a)$ can be bounded in terms of the first derivatives of g and ϕ_ξ, which because ϕ_ξ converges uniformly to ϕ

can be bounded independently of ξ. By Lemma A.3.2, we also can bound $\phi''_\xi(a)$ independently of ξ. Combining this with (3) of Lemma A.2.1 we have

$$\partial_s^2 \overline{h}(\overline{a}(s), \bar{b}(s)) = \frac{h_a(\varepsilon(\xi), b) - h_a(-\varepsilon(\xi), b)}{2\varepsilon(\xi)} + O(1) \tag{A.36}$$

We claim that, for $\varepsilon(\xi)$ sufficiently small, that $(h_a(\varepsilon(\xi), b) - \overline{h}_a(-\varepsilon(\xi), b)) < 0$ by assumption. The limit as $\varepsilon(\xi) \to 0$ is $h_a(0_-, 0) - h_a(0_+, 0) = -2(\mathrm{II}_1 + \Phi^* \mathrm{II}_2)$, which is negative by hypothesis. Thus we see that $\partial_s^2 \overline{h}(\overline{a}(s), \bar{b}(s)) \to -\infty$ for $a \in [-\varepsilon(\xi), \varepsilon(\xi)]$ as $\xi \to 0$. Fix a ξ for which $\partial_s^2 \overline{h}(\overline{a}(s), \bar{b}(s)) < 0$.

Next let $\breve{g}$ and $\breve{\phi}$ denote the second order smoothings of $\overline{g}$ and $\overline{\phi}_\xi$ on $[\pm\varepsilon(\xi) - \delta, \pm\varepsilon(\xi) + \delta]$. By Lemma A.3.2, there is a $\delta(\xi)$ such that $\mathrm{Ric}_{\breve{g}} > 0$ and $\widetilde{\mathrm{II}}_{\breve{g}} > 0$ for all $\delta < \delta(\xi)$. We claim that $\partial_s^2 \breve{h}(\breve{a}(s), \breve{b}(s)) < 0$ for δ sufficiently small. We will again perform a linear reparameterization so that $a \in [-\delta, \delta]$ to avoid repetition.

By inspecting equations (A.33), (A.34), and (A.35), we can see that equation (A.31) is linear in $\breve{h}_{aa}$ and $\breve{\phi}_{aa}$. We conclude that

$$\partial_s^2 \breve{h}(\breve{a}(s), \breve{b}(s)) = \breve{H}(a,b)\breve{h}_{aa} + \breve{F}(a,b)\breve{\phi}_{aa} + \breve{R}(a,b) \tag{A.37}$$

Where $\breve{H}(a,b)$, $\breve{F}(a,b)$, and $\breve{R}(a,b)$ are expressed in terms of the first and second b-derivatives of $\breve{g}$, the mixed derivatives of $\breve{g}$, and the first a-derivatives of $\breve{g}$ and $\breve{\phi}$. By (1) and (2) of Lemma A.2.2 we conclude that these functions are equal to $\overline{H}(0,b) + O(\delta)$, $\overline{F}(0,b) + O(\delta)$, $\overline{R}(0,b) + O(\delta)$, where these are the corresponding expressions evaluated on $\overline{g}$ and $\overline{\phi}$. Substituting these into (A.37) we have

$$\partial_s^2 \breve{h}(\breve{a}(s), \breve{b}(s)) = \overline{H}(0,b)\breve{h}_{aa} + \overline{F}(0,b)\breve{\phi}_{aa} + \overline{R}(0,b) + O(\delta) \tag{A.38}$$

By assumption, (A.38) is negative at $a = \pm\delta$, where it agrees with $\partial_s^2 \overline{h}(\overline{a}(s), \overline{b}(s))$. By (3), equation (A.38) is arbitrarily close to a convex combination of these endpoints and is therefore negative for δ sufficiently small. Fix such a $\delta < \delta(\xi)$ such that $\partial_s^2 \breve{h}(\breve{a}(s), \breve{b}(s)) < 0$.

We have found twice differentiable $\breve{g}$ and $\breve{\phi}$ for which $\mathrm{Ric}_{\breve{g}} > 0$, $\widetilde{\mathrm{II}}_{\breve{g}} > 0$, and $\partial_s^2 \breve{h}(s) < 0$. Note that Ric, $\widetilde{\mathrm{II}}$, and $\partial_s^2 h(s)$ are each second order operators in normal coordinates by Proposition A.1.10, Lemma A.1.12, and Lemma A.3.1 respectively. Taking a smooth metric and smooth boundary function arbitrarily close to $\breve{g}$ and $\breve{\phi}$ proves the claim. $\square$

APPENDIX B

RICCI-POSITIVE ISOTOPY IMPLIES NECK EQUIVALENCE

Fix a smooth manifold M^n. Then the set of all metrics is the subset of the bundle of symmetric 2-tensors $\mathrm{Sym}^2(T^*\mathrm{M}^n) \to \mathrm{M}^n$ consisting of those 2-tensors that are everywhere positive definite. The space $\Gamma(\mathrm{Sym}^2(T^*\mathrm{M}^n))$ is naturally a Frechet manifold by taking the topology of uniform convergence of all derivatives on compact subsets (using some fixed metric g on M^n to define a metric on the bundle $\mathrm{Sym}^2(T^*\mathrm{M}^n)$). The space of all metrics can be endowed with the topology of an open Frechet submanifold by taking the preimage of $C^\infty(\mathrm{M}^n,(0,\infty)^n)$ under the eigenvalue map $\lambda : \mathrm{Sym}^2(T^*\mathrm{M}^n) \to C^\infty(\mathrm{M}^n,\mathbf{R}^n)$. We denote this space of Riemannian metrics on M^n as $\mathcal{R}(\mathrm{M}^n)$.

In this paper, we are interested in a narrower class of metrics, namely those metrics with positive Ricci curvature. The Ricci and scalar curvature give rise to well-defined functions $\mathrm{Ric} : \mathcal{R}(\mathrm{M}^n) \to \mathrm{Sym}^2(T * \mathrm{M}^n)$ and $\mathrm{R} : \mathcal{R}(\mathrm{M}^n) \to C^\infty(\mathrm{M}^n)$. By definition, these maps are related by $\mathrm{tr} \circ \mathrm{Ric} = \mathrm{R}$ where the map $\mathrm{tr} : \mathrm{Sym}^2(T^*\mathrm{M}^n) \to C^\infty(\mathrm{M}^n)$ is the trace of the eigenvalue map. Note that a given metric $g \in \mathcal{R}(\mathrm{M}^n)$ has positive Ricci curvature precisely when $(\lambda \circ \mathrm{Ric})(g) \in C^\infty(\mathrm{M}^n,(0,\infty)^n)$, and has positive scalar curvature precisely when $\mathrm{R}(g) \in C^\infty(\mathrm{M}^n,(0,\infty))$. We see therefore that the set of Ricci-positive metrics and positive scalar curvature metrics are themselves open Frechet submanifold in the space $\mathcal{R}(\mathrm{M}^n)$. Denote these spaces by $\mathcal{R}^{\mathrm{pRc}}(\mathrm{M}^n)$ and $\mathcal{R}^{\mathrm{psc}}(\mathrm{M}^n)$ respectively. Moreover because R factors as $\mathrm{tr} \circ \mathrm{Ric}$ we see that $\mathcal{R}^{\mathrm{pRc}}(\mathrm{M}^n) \subseteq \mathcal{R}^{\mathrm{psc}}(\mathrm{M}^n) \subseteq \mathcal{R}(\mathrm{M}^n)$.

B.1. Isotopy and Concordance of Metrics

Given two $g_0, g_1 \in \mathcal{R}(\mathrm{M}^n)$, we say that they are *isotopic* if there is a path $g_t \in \mathcal{R}(\mathrm{M}^n)$ with $t \in [0,1]$ connecting g_0 to g_1. If $g_0, g_1 \in \mathcal{R}^{\mathrm{psc}}(\mathrm{M}^n)$ or $g_0, g_1 \in \mathcal{R}^{\mathrm{pRc}}(\mathrm{M}^n)$, we similarly say that g_0 and g_1 are *psc isotopic* or *Ricci-positive isotopic* if they are connected by a path within the space $\mathcal{R}^{\mathrm{psc}}(\mathrm{M}^n)$ or $\mathcal{R}^{\mathrm{pRc}}(\mathrm{M}^n)$ respectively. Isotopy defines an equivalence relation, and the set of equivalence classes are the path components of the relevant space of metrics, which is typically denoted by $\pi_0\left(\mathcal{R}(\mathrm{M}^n)\right)$, $\pi_0\left(\mathcal{R}^{\mathrm{psc}}(\mathrm{M}^n)\right)$, or $\pi_0\left(\mathcal{R}^{\mathrm{pRc}}(\mathrm{M}^n)\right)$.

We say that two metrics $g_0, g_1 \in \mathcal{R}(\mathrm{M}^n)$ are *concordant* if there is a metric $G \in \mathcal{R}([0,1] \times \mathrm{M}^n)$ and a $\varepsilon > 0$ such that metric G restricted to $[0, \varepsilon) \times \mathrm{M}^n$ is isometric to $dt^2 + g_0$ and G restricted to $(1-\varepsilon, 1] \times \mathrm{M}^n$ is isometric to $dt^2 + g_1$, and the metric G is referred to as a *concordance*. Concordance is an equivalence relation. For this reason we required the data of product collar neighborhoods, which is needed to show that concordance is transitive. Note that concordance and isotopy define the same equivalence relation on the space of all metrics $\mathcal{R}(\mathrm{M}^n)$. Given an isotopy g_t we define a concordance $G = dt^2 + g_{\lambda(t)}$ for some $\lambda(t) : [0,1] \to [0,1]$ such that $\lambda(t) \equiv 0$ for $t < \varepsilon$ and $\lambda(t) \equiv 1$ for $1-\varepsilon < t$, and given a concordance G we define a path $g_t = G|_{\{t\}\,\mathrm{M}^n}$.

If we restrict to the spaces of positive scalar curvature or positive Ricci curvature, we already have defined what it means to be isotopic. We can define *positive scalar concordance of positive scalar curvature metrics* as a concordance $G \in \mathcal{R}^{\mathrm{psc}}([0,1] \times \mathrm{M}^n)$ that itself has positive scalar curvature. This definition makes sense because the product metric satisfies

$$\mathrm{R}_{(dt^2+g)}(t,x) = \mathrm{R}_g(x).$$

Thus the collar condition of concordance can be satisfied by positive scalar curvature metrics G, and we can similarly check that psc concordance is an equivalence relation. Unlike the space of all metrics, it is no longer clear if the equivalence relations defined by concordance and isotopy are comparable. In Section B.1.1, we discuss how they are related.

If we attempt to define Ricci-positive concordance in a similar manner, we run into the difficulty that

$$\mathrm{Ric}_{dt^2+g}(\partial_t, \partial_t) = 0.$$

And so the collar condition of concordance leaves the space of Ricci-positive metrics. For this reason concordance is not well suited for positive Ricci curvature. We will see in Section B.1.1 that the notion of psc concordance is useful for an array of geometric constructions. Thus even though Ricci-positive concordance is not defined, we would still like to find a Ricci-positive analogue for psc concordance that can be used for similar geometric constructions. This is precisely the idea behind the notion of neck equivalence.

B.1.1. Isotopy implies Concordance for Positive Scalar curvature metrics

Given a psc concordance G, when we restrict G to the sets $\{t\} \times \mathrm{M}^n$, the scalar curvature might be everywhere negative, and there is no clear way how to correct this defect. In dimension 4, [66] used the Seiberg-Witten invariants to show that there are psc concordant psc metrics that are not psc isotopic. It is conjectured in general that there is an obstruction to two concordant positive scalar curvature metrics being positive scalar curvature isotopic.

Given a psc isotopy, we may consider the concordance $G = dt^2 + g_{\lambda(t)}$ as we did for a general isotopy of metrics. When we compute the scalar curvature of this concordance, we see that it depends on the scalar curvature of $R_{g(t)}$ as well as the parametrizing function $\lambda(t)$ and its derivatives. In this case, we have total freedom over our choice $\lambda(t)$, so it becomes plausible that we can, for each path g_t, find an appropriate parametrization $\lambda(t)$ for which G has positive scalar curvature.

Proposition B.1.1 ([67, Lemma II.1]). *If $g_1, g_2 \in \mathcal{R}^{psc}(\mathrm{M}^n)$ are psc isotopic, then they are psc concordant.*

Proof. Let g_s be a positive scalar curvature isotopy between g_1 and g_2. We claim that there is a positive scalar curvature concordance between g_1 and g_2. We will consider metrics of the form $G = dt^2 + g_{\lambda(t)}$, on $[0, t_1] \times \mathrm{M}^n$, where $\lambda : [0, t_1] \to [1, 2]$ is some smooth function such that $\lambda(t) \equiv 1, 2$ for t ε-close to the endpoints of $[0, t_1]$. After reparameterizing the interval, G is a concordance from g_1 to g_2.

It remains to see if we can chose $\lambda(t)$ so that G has positive scalar curvature. By [67, Lemma II.1], the scalar curvature of G takes the following form

$$\mathrm{R}_G(t, x) = \mathrm{R}_{g_{\lambda(t)}}(x) + O(\lambda''(t), \lambda'(t)).$$

From this we see that G will be a psc concordance if $\lambda'(t)$ and $\lambda''(t)$ are sufficiently small. By taking t_1 sufficiently large, we can find such an $\lambda(t)$. □

As mentioned, the set of equivalence classes of psc isotopy is set of path components of the space of psc metrics $\mathcal{R}^{\mathrm{psc}}(\mathrm{M}^n)$. The space of all metrics is in fact convex, and hence path connected. It is unclear whether or not the space of psc metrics is path connected, as the equation $\mathrm{R}_g > 0$ is far from convex. It is a remarkable theorem of [68], that the space $\mathcal{R}^{\mathrm{psc}}(\mathrm{S}^{4k-1})$ has infinitely many path components for all $k > 1$. One of the key geometric constructions needed in the proof of this theorem, is the idea of psc concordance and Proposition B.1.1. Let us briefly describe the construction used in [68] to illustrate the importance of Proposition B.1.1.

Using a plumbing construction, [68] constructs positive scalar curvature metrics g_i on a sequence of spin manifolds M_i^{4k} with boundaries equal to S^{4k-1} so that $g_i = dt^2 + h_i$ near their boundaries. If there was a positive scalar curvature isotopy between h_i and h_j, we may find a positive scalar curvature concordance H_{ij} on $[0,1] \times \mathrm{S}^{4k-1}$. Then by gluing h_i, H, and h_j together, we may find a positive scalar curvature metric on $\mathrm{M}_{ij}^{4k} = \mathrm{M}_i^{4k} \cup_{\mathrm{S}^{4k-1}} \mathrm{M}_j^{4k}$. The α-invariant (see [22]) is computed for these M_{ij}^{4k} and is shown to be nonzero if $i \neq j$. As the existence of a positive scalar curvature metric on M^n implies $\alpha(\mathrm{M}^n) = 0$, this is a contradiction, and we conclude that no psc concordance H_{ij} could exist and hence no psc isotopy between h_i and h_j. Thus the positive scalar curvature metrics h_i each lie in distinct path components of $\mathcal{R}^{\mathrm{psc}}(\mathrm{S}^{4k-1})$, and therefore $\pi_0(\mathcal{R}^{\mathrm{psc}}(\mathrm{S}^{4k-1}))$ is infinite.

B.2. Neck Equivalence

The ability to turn an psc isotopy into a psc concordance, allows us to apply our knowledge of the topological restriction on finite dimensional manifolds supporting psc metrics to deduce topological information about the space of all psc metrics. While Ricci-positive concordances, strictly speaking, do not exist, we would still like to consider Ricci-positive metrics on the cylinder $[0,1] \times \mathrm{M}^n$ that can be used similarly to psc concordance.

As demonstrated by the example of [68], we would like to consider Ricci-positive metrics on $[0,1] \times \mathrm{M}^n$ that can be smoothly glued to a Ricci-positive metric on W^{n+1} with isometric boundary satisfying a certain collar condition. For positive scalar curvature we require that metrics splits as a product, but this requirement is actually equivalent to requiring that the boundary have positive mean curvature by Proposition 1.2.11 and [69]. Considering Theorem 1.2.2 as a technique to glue together Ricci-positive manifolds with isometric boundary, we see that it is reasonable that in place of the collar condition we require a bound on the principal curvatures of the boundary.

If G is a Ricci-positive metric on $[0, 1] \times M^n$, we note that only one of the two boundary components can be convex. Suppose to the contrary that both boundary components were convex, then by Theorem 1.2.2 we may smoothly identify the boundaries and produce a Ricci-positive metric on $S^1 \times M^n$, which contradicts Myers' Theorem. Granted that one end cannot be convex, we must consider metrics on the cylinder with asymmetrical boundary conditions. With this in mind, we now introduce the notion of *convex neck*.

Definition B.2.1. *Given two metrics $g_0, g_1 \in \mathcal{R}(M^n)$, we say that a metric $G \in \mathcal{R}([0, 1] \times M^n)$ is a* convex neck *from g_0 to g_1 with data (ν, r), if it satisfies the following*

(i) G restricted to the boundary $\{0\} \times M^n$ is isometric to g_0,

(ii) the principal curvatures of the boundary $\{0\} \times M^n$ with respect to G are greater than $-\nu$,

(iii) G restricted to the boundary $\{1\} \times M^n$ is isometric to $r^2 g_1$ for some $r > 0$.

(iv) the principal curvatures of the boundary $\{1\} \times M^n$ with respect to G are positive.

Comparing the definition of convex neck with Theorem 1.2.2, we see that Ricci-positive convex necks are a suitable stand in for Ricc-positive concordances in the following sense.

Proposition B.2.2. *If* (W^{n+1}, H) *has positive Ricci curvature with boundary isometric to* (M^n, g_0) *with principal curvatures greater than* $\nu > 0$ *and* G *is a Ricci-positive convex neck from* g_0 *to* g_1 *with data* (ν, r)*, then there is a Ricci positive metric* $\tilde{H}$ *on* W^{n+1} *so that the boundary is isometric to* $(\mathrm{M}^n, r^2 g_1)$ *and has positive principal curvatures.*

While Proposition B.2.2 allows us to perform the same types of constructions as we did with psc concordances, we would still like to use them to define an equivalence relation on the space $\mathcal{R}^{\mathrm{pRc}}(\mathrm{M}^n)$ and be able able to compare that equivalence to Ricci-positive isotopy. As already observed, the notion is inherently not symmetric. But using Theorem 1.2.2 we can discuss to what extent the relation of Ricci-positive convex neck is transitive.

Lemma B.2.3. *Suppose we are given* G_0^1*, a Ricci-positive convex neck from* g_0 *to* g_1 *with data* (ν_0^1, r_1^0) *such that the principal curvatures at* $\{1\} \times \mathrm{M}^n$ *are at least* ν_1^0*, and suppose we are given* G_1^2*, a Ricci-positive convex neck from* g_1 *to* g_2 *with data* (ν_1^2, r_2^1)*. Then there is a Ricci-positive convex neck* G_0^2 *from* g_0 *to* g_2 *with data* $(\nu_0^1, r_1^0 r_2^1)$ *provided that*

$$-\frac{\nu_1^2}{r_1^0} + \nu_1^0 > 0. \tag{B.1}$$

Proof. Scale G_1^2 by $(r_1^0)^2$. The resulting metric $(r_1^0)^2 G_1^2$ is a Ricci-positive convex neck from $(r_1^0)^2 g_1$ to g_2 with data $(\nu_0^1/r_1^0, r_1^0 r_2^1)$. Note that G_0^1 and G_1^2 have boundary $\{1\} \times \mathrm{M}^n$ and $\{0\} \times \mathrm{M}^n$ respectively isometric to $(r_1^0)^2 g_1$, and the principal curvatures there are bounded below respectively by ν_1^0 and ν_1^2/r_1^0. If the sum of these bounds is positive, then by Theorem 1.2.2 we may glue G_0^1 to $(r_1^0)^2 G_1^2$ to produce a smooth Ricci-positive metric G_0^2 on $[0,1] \times \mathrm{M}^n$ with the desired data. Equation (B.1) is exactly this statement. □

From (B.1) we see that the existence of a single Ricci-positive convex neck from g_0 to g_1 is not sufficient to define a transitive relationship. Thus even though transitivity and symmetry fail for the relationship defined by the existence of a single Ricci-positive convex neck, we are ready to define *neck equivalence* in terms of the existence of a family Ricci-positive convex necks in both directions, which forces symmetry by definition and forces transitivity by Lemma B.2.3.

Definition B.2.4. *Two metrics $g_0, g_1 \in \mathcal{R}^{pRc}(\mathrm{M}^n)$ are neck equivalent if there exists $r_i > 1$ such that for all $1 > \nu > 0$ there exists a Ricci-positive convex neck $G_0(\varepsilon)$ from g_0 to g_1 with data (ν, r_1) and a Ricci-positive convex neck $G_1(\varepsilon)$ from g_1 to g_0 with data (ν, r_0).*

While we have called this an equivalence, it is far from obvious why it is reflexive. We will prove this below in Lemma B.2.5. By Proposition B.2.2 we see that if g_0 and g_1 are neck equivalent, then given any Ricci-positive (W^{n+1}, H) with convex boundary isometric to (M^n, g_0) we may find a Ricci-positive metric $\tilde{H}$ on W^{n+1} with convex boundary isometric to $(\mathrm{M}^n, r_1 g_1)$ and vice-versa. The main result of this chapter is Theorem II, which claims that if two metrics are Ricci-positive isotopic then they are Ricci-positive neck equivalent. Thus Proposition B.2.2 and Theorem II together imply Corollary 1.3.4.

Before we prove that neck equivalence is indeed an equivalence, we mention that unlike the case of psc concordance, which is expected to be a coarser equivalence than psc isotopy, we expect that neck equivalence and Ricci-positive isotopy are equivalent. We expect this because we may concatenate infinitely many convex necks to find a Ricci-positive on $[0, \infty) \times \mathrm{M}^n$. One should be able to argue using techniques like those used in [15] to show that the metric splits, giving rise to an isotopy g_t.

Lemma B.2.5. *Ricci-positive neck equivalence is an equivalence relation on* $\mathcal{R}^{pRc}(\mathrm{M}^n)$.

Proof. As symmetry is clear from the definition, it suffices to show that it is transitive and reflexive.

We begin by showing transitivity. Let g_0 and g_1 be Ricci-positive neck equivalent, and let g_1 and g_2 be Ricci-positive neck equivalent. Moreover let $G_0^1(\nu)$, $G_1^0(\nu)$, $G_1^2(\nu)$, and $G_2^1(\nu)$ be the four families of Ricci-positive convex necks guaranteed to exist by Definition B.2.4 with constants r_0^1, r_1^0, r_1^2, and r_2^1 respectively. In order to show that g_0 and g_2 are Ricci-positive neck equivalent we must find two families of Ricci-positive convex necks $G_0^2(\nu)$ and $G_2^0(\nu)$ and constants r_0^2 and r_2^0. We will show how to construct $G_0^2(\nu)$ and r_0^2 in terms of $G_0^1(\nu)$, $G_1^2(\nu)$, r_0^1, and r_1^2. The construction of $G_2^0(\nu)$ and r_2^0 is entirely symmetric.

Fix $\nu > 0$, take $G_0^1(\nu)$. Let $\nu_1^0(\nu) > 0$ be a lower bound on the principal curvatures of $\{1\} \times \mathrm{M}^n$ with respect to $G_0^1(\nu)$. By Lemma B.2.3 we can scale $G_1^2(\varepsilon)$ by $(r_0^1)^2$ and glue it to $G_0^1(\nu)$ provided that

$$\frac{-\varepsilon}{r_0^1} + \nu_1^0(\nu) > 0.$$

We can find ε small enough so that equation (B.1) is satisfied. Define $G_0^2(\nu)$ to be the result of applying Lemma B.2.3 to $G_0^1(\nu)$ and $r_0^1 G_1^2(\varepsilon)$. This is a Ricci-positive convex neck from g_0 to g_2 with data $(\nu, r_1^0 r_2^1)$. We conclude that neck equivalence is transitive.

All that remains is to show that neck equivalence is reflexive. Let g be any Ricci-positive metric on M^n. Fix $r > 1$, to show that Ricci-positive concordance is reflexive, for all $\varepsilon > 0$ we will consider metrics of the form $G(\nu) = dt^2 + R^2(t)g$ where $R(t)$ are smooth functions $R : [0, T(\nu)] \to [1, r]$. We claim that there exists a choice $R(t)$ (that depends on $T(\nu)$) for which $G(\nu)$ will be a Ricci-positive convex neck from g to g with data (ν, r).

To start, we note that $(1/2)\partial_t G(\nu) = R'(t)R(t)g$. We must choose $R(t)$ so that $-\nu < -R'(0)$ and $0 < rR'(T)$. If we decompose $T([0,1] \times \mathrm{M}^n) = T([0,1]) \oplus T\,\mathrm{M}^n$ then the Ricci tensor with respect to G decomposes as

$$\mathrm{Ric}_G = -nR''(t)R(t)dt^2 + \left(R^2(t)\,\mathrm{Ric}_g - \left(R''(t)R(t) + (n-1)(R'(t)R(t))^2\right) g\right). \quad \text{(B.2)}$$

We see from (B.2) that there exists a $c > 0$ depending only on g and r such that Ric_G is positive definite if $R''(t) < 0$ and $|R'(t)| < c$.

We may take, for example, the function $R(t) = (r-1)\sqrt{2}\sin(\pi t/4T(\nu)) + 1$. Clearly $R'(T) > 0$, and as $T(\nu) \to \infty$ we have $R'(t)$ and $R''(t)$ converging to 0 as needed. Thus for $T(\nu)$ adequately large, the metric $G(\nu)$ is a Ricci-positive convex neck with data (ν, r). $\square$

B.3. Ricci-positive Isotopy implies Ricci-positive Neck Equivalent

The goal of this section is to prove Theorem II. Throughout we will assume that g_s with $s \in [0,1]$ is a Ricci-positive isotopy. As isotopy is a symmetric relationship, it suffices to find an r such that for all $\nu > 0$ a Ricci-positive convex neck G from g_0 to g_1 with data (ν, r). The construction of the convex neck combines the constructions used for reflexivity in Lemma B.2.5 and for psc concordance in Proposition B.1.1. We will construct Ricci-positive convex necks G on $[t_0, t_1] \times \mathrm{M}^n$ in the following form

$$G = dt^2 + R^2(t) g_{\lambda(t)}, \tag{B.3}$$

for smooth functions $R : [t_0, t_1] \to [1, r]$ and $\lambda : [t_0, t_1] \to [0, 1]$.

The careful reader will notice that the metrics constructed in this section are very similar to the ones considered in the proof (due to [33]) of Lemma C.0.1 in Appendix C below. The metrics we consider in this section are entirely motivated by the work of [33]. The neck metric and convex necks are both Ricci-positive metrics on a cylinder with particular boundary conditions. In Lemma C.0.1 we require a much stronger condition on the principal curvatures of the boundary than we do for convex necks, and it is for this reason that the Lemma C.0.1 cannot be proven for arbitrary boundary metrics but only a very special class of warped product metrics on S^n. Because we only need positive principal curvatures for convex necks, this allows us to prove Theorem II for general Ricci-positive isotopies.

B.3.1. The curvatures of the convex neck

In this section we will compute the curvatures of metric G of (B.3) in terms of the curvatures of g_s. In order to succinctly record the curvatures of the metric G, we will introduce the intermediate metric $\overline{G}$ on $[0, 1] \times \mathrm{M}^n$ defined as

$$\overline{G} = ds^2 + g_s, \tag{B.4}$$

where we will always assume that $s = \lambda(t)$. Note that the curvatures of $\overline{G}$ are completely independent of the functions $R(t)$, $\lambda(t)$, and the interval $[t_0, t_1]$.

We will let $X_i(s)$ denote normal coordinate frame for $T\,\mathrm{M}^n$ with respect to g_s for a fixed point $p \in \mathrm{M}^n$. We will denote by X_i to be sections of $T([t_0, t_1] \times \mathrm{M}^n)$ defined in the obvious way by the $X_i(\lambda(t))$. Define the vector fields $Y_i = \dfrac{X_i}{R(t)}$. The Y_i together with ∂_t will form a normal coordinate frame for $T([0, 1] \times \mathrm{M}^n)$ with respect to G at the point (t, p) for each t. We will express all of our curvature computations in terms of Y_i, ∂_t, and $X_i(s)$.

To begin we record the extrinsic curvatures of $\{t\} \times \mathrm{M}^n$. Denote by II_G the second fundamental form of $\{t\} \times \mathrm{M}^n$ embedded in $[t_0, t_1] \times \mathrm{M}^n$ with respect to G and the unit normal ∂_t, similarly let $\mathrm{II}_{\overline{G}}$ denote the second fundamental form of $\{s\} \times \mathrm{M}^n$ embedded in $[0, 1] \times \mathrm{M}^n$ with respect to $\overline{G}$ and the unit normal ∂_s.

Proposition B.3.1. $\mathrm{II}_G(Y_i, Y_j) = \dfrac{R'(t)}{R(t)} g_s(X_i, X_j) + \lambda'(t)\, \mathrm{II}_{\overline{G}}(X_i, X_j)$.

Proof. This is a straightforward application of [6, Proposition 3.2.1] to (B.3). □

Note that we have no control over the signature or magnitude of $\mathrm{II}_{\overline{G}}$ in Proposition B.3.1; it is determined entirely by the family g_s. Because $[0, 1] \times \mathrm{M}^n$ is compact we can find a constant N that depends only on g_s for which $|\,\mathrm{II}_{\overline{G}}\,| < N$.

Next we record the sectional curvatures of G involving ∂_t. The actual formula can be derived by applying the Codazzi-Mainardi equation to Proposition B.3.1. The complete formula is very complicated, and will not be necessary to prove Theorem II. Instead we only need to know the specific coefficients of the second order terms.

Proposition B.3.2. *If K_G is sectional curvature of G, then*

$$\mathrm{K}_G(\partial_t, Y) = -\frac{R''(t)}{R(t)} g_s(X, X) - \lambda''(t)\, \mathrm{II}_{\overline{G}}(X, X) + O\left(\frac{R'(t)}{R(t)}, \lambda'(t)\right)^2 .$$

Where the coefficients in this error term depends on g_s and its first and second derivatives.

Proof. This is a straightforward application of [6, Proposition 3.2.11] to Proposition B.3.1. □

The remaining sectional curvatures can all be expressed in terms of the sectional curvatures of g_s and II_G using Gauss' formula. We now record the Ricci tensor of G, again only recording the specific coefficients for the second order terms.

Proposition B.3.3. *At each point in $[t_0, t_1] \times \mathrm{M}^n$ there are local coordinate vector fields ∂_t and X_i tangent to $[t_0, t_1]$ and M^n respectively for which the Ricci tensor of G decomposes as a direct sum:*

$$\begin{aligned}
\mathrm{Ric}_G(\partial_t, \partial_t) &= -n\frac{R''(t)}{R(t)} - \lambda''(t) H_{\overline{G}} + O\left(\frac{R'(t)}{R(t)}, \lambda'(t)\right)^2, \\
\mathrm{Ric}_G(Y_i, Y_j) &= -\frac{R''(t)}{R(t)} g_s(X_i, X_j) - \lambda''(t)\, \mathrm{II}_{\overline{G}}(X_i, X_j) + \frac{1}{R^2(t)} \mathrm{Ric}_{g_s}(X_i, X_j) \\
&\qquad + O\left(\frac{R'(t)}{R(t)}, \lambda'(t)\right)^2, \\
\mathrm{Ric}_G(\partial_t, Y) &= 0.
\end{aligned}$$

Where $H_{\overline{G}}$ is the mean curvature of $\overline{G}$. Where the error term for $\mathrm{Ric}_G(Y_i, Y_j)$ *again depends on g_s and its first and second derivatives.*

Proof. The equation for $\mathrm{Ric}_G(\partial_t, \partial_t)$ follows directly from Proposition B.3.2. The equation for $\mathrm{Ric}_G(Y_i, Y_j)$ follows by applying Gauss' equations (see [6, Theorem

3.2.4]) to Propositions B.3.1 and B.3.2. That $\mathrm{Ric}_G(\partial_t, Y) = 0$ follows applying the Codazzi-Mainardi equation (see [6, Theorem 3.2.5]) to Proposition B.3.1. □

B.3.2. The metric functions

We will define the functions $R(t)$ and $\lambda(t)$ in terms of the following differential equation

$$\alpha(t_0, t_1, r)\frac{R'(t)}{R(t)} = \beta(t_0, t_1)\lambda'(t) = \Gamma(t), \tag{B.5}$$

where α and β are determined by the boundary conditions imposed by requiring bijections $R : [t_0, t_1] \to [1, r]$ and $\lambda : [t_0, t_1] \to [0, 1]$. One can solve explicitly for these functions from (B.5).

$$\alpha(t_0, t_1, r) = \frac{\displaystyle\int_{t_0}^{t_1} \Gamma(t)dt}{\ln r} \text{ and } \beta(t_0, t_1) = \int_{t_0}^{t_1} \Gamma(t)dt \tag{B.6}$$

The idea behind the proof of Theorem II is to investigate the asymptotic behavior of II_G and Ric_G as $t_0 \to \infty$ and $t_1 - t_0 \to \infty$ in terms of the function $\Gamma(t)$, $\Gamma'(t)$, and $\int_{t_0}^{t_1} \Gamma(t)dt$. If we ask that G to be a Ricci-positive convex neck with data (ν, r), this imposes certain conditions on the function $\Gamma(t)$. While it is not immediately obvious what those conditions will be, we emphasize that the following choice of $\Gamma(t)$ is entirely motivated by such a consideration. Fix $t_1 = t_0^2$ and let

$$\Gamma(t) = \frac{1}{t \ln^2 t}. \tag{B.7}$$

B.3.3. Bounding the Curvatures

In this section we bound the curvatures computed in Section B.3.1 in terms of the metric functions chosen in Section B.3.2. Note that because $R(t) : [t_0, t_1] \to [1, r]$, from equation (B.3), G is a convex neck from g_0 and g_1 with data (ν, r) for some ν. It remains to show for any $\nu > 0$ that $|\operatorname{II}_G| < \nu$, $\operatorname{II}_G > 0$, and $\operatorname{Ric}_G > 0$.

We start by bounding II_G provided above by Proposition B.3.1 in terms of $\Gamma(t)$, α, and β of Section B.3.2.

Proposition B.3.4. *If G is defined as in (B.3) with functions $R(t)$ and $\lambda(t)$ defined as in (B.5). There are constants $N, C > 0$ that depend only on g_s such that*

$$\operatorname{II}_G(Y_i, Y_j) > \left(\frac{1}{\alpha} - \frac{N}{\beta}\right)\Gamma(t) g_s(X_i, X_j) \tag{B.8}$$

$$|\operatorname{II}_G(Y_i, Y_j)| < \left(\frac{1}{\alpha} + \frac{N}{\beta}\right)\Gamma(t) C \tag{B.9}$$

Proof. These equations follow directly from Proposition B.3.1 and the definition of $R(t)$ and $\lambda(t)$ in (B.5). The precise values of N and C are determined by $\operatorname{II}_{\overline{G}}$ and g_s respectively. □

Next we bound Ric_G below, again in terms of $\Gamma(t)$, α, and β of Section B.3.2.

Proposition B.3.5. *If G is defined as in (B.3) with functions $R(t)$ and $\lambda(t)$ defined as in (B.5). There is a constant $N > 0$ that depend only on g_s such that*

$$\operatorname{Ric}_g(\partial_t, \partial_t) > -\left(\frac{n}{\alpha} - \frac{N}{\beta}\right)\Gamma'(t) + O\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)^2 \Gamma^2(t) \tag{B.10}$$

$$\operatorname{Ric}_G(Y_i, Y_j) > -\left(\frac{1}{\alpha} - \frac{N}{\beta}\right)\Gamma'(t) + \frac{1}{r^2}\operatorname{Ric}_{g_s}(X_i, X_j) \tag{B.11}$$

$$O\left(\frac{1}{\alpha}, \frac{1}{\beta}\right)^2 \Gamma^2(t), \tag{B.12}$$

where the coefficients in these error terms again depend on g_s and its first and second derivative.

Proof. First we note how $\dfrac{R''(t)}{R(t)}$ can be expressed in terms of $\Gamma(t)$ and α from (B.5).

$$\frac{R''(t)}{R(t)} = \left(\frac{R'(t)}{R(t)}\right)' + \left(\frac{R'(t)}{R(t)}\right)^2 = \frac{\Gamma'(t)}{\alpha} + \frac{\Gamma^2(t)}{\alpha^2}. \tag{B.13}$$

In the expressions for Ric_G in Proposition B.3.3 we rewrite $\dfrac{R'(t)}{R(t)}$, $\lambda'(t)$, and $\lambda''(t)$ in terms of $\Gamma(t)$ using (B.5) and we rewrite $\dfrac{R''(t)}{R(t)}$ using (B.13). We group the term coming from $\dfrac{\Gamma^2(t)}{\alpha^2}$ in (B.13) with the error term.

Taking N to be larger than $|H_{\overline{G}}|$ and $|\operatorname{II}_{\overline{G}}(X, X)|$ these lower bounds are now clear. □

B.3.4. Limiting Behavior

To finish our proof of Theorem II we make the observation that the coefficient of $\Gamma'(t)$ in (B.8) can be made to be negative by choosing a suitable r.

Proposition B.3.6. *Let α and β be as in (C.28). For any constant $N > 0$ depending only on g_s, there is an $r > 0$ for which*

$$\frac{1}{\alpha} - \frac{N}{\beta} > 0.$$

Proof. From (C.28) this is equivalent to $\ln r - N > 0$, which can be solved as N is assumed to be independent of r. □

Proposition B.3.7. *For some $r > 1$ depending only on g_s and all $\nu > 0$, we have $\mathrm{II}_G > 0$ and $|\,\mathrm{II}_G\,| < \nu$ for $t_1 = t_0^2$ and all t_0 sufficiently large.*

Proof. By Proposition B.3.6 we see that the lower bound on II_G in Proposition B.8 is positive if we take r sufficiently large.

We claim that the upper bound for $|\,\mathrm{II}_G\,|$ converges to 0 as $t_0 \to \infty$. We see from (C.28) that it is sufficient to show that

$$\lim_{t_0 \to \infty} \left(\frac{\Gamma(t)}{\displaystyle\int_{t_0}^{t_0^2} \Gamma(t) dt} \right) = 0.$$

Using (C.25), this large parenthetical is equivalent to

$$\frac{2 \ln t_0}{t \ln^2 t}.$$

Clearly this converges to 0 as $t_0 \to \infty$. □

Proposition B.3.8. *For some $r > 1$ depending only on g_s, for $t_1 = t_0^2$, $\mathrm{Ric}_G > 0$ for all t_0 adequately large.*

Proof. Let $C_0, C_1 > 0$ be constants that are independent of t_0 and t_1. Define the function $L(t)$ by

$$L(t) = C_0 \frac{\Gamma'(t)}{\displaystyle\int_{t_0}^{t_1} \Gamma(t)dt} - C_1 \frac{\Gamma^2(t)}{\left(\displaystyle\int_{t_0}^{t_1} \Gamma(t)dt\right)^2} \tag{B.14}$$

We claim that for $t_1 = t_0^2$ that $L(t) < 0$ for all t_0 sufficiently large. Using the definition of $\Gamma(t)$ in (C.25), this is equivalent to

$$-\frac{4C_0}{t_0^2 \ln t_0} + \frac{4(C_1 - C_0)}{t_0^2 \ln^2 t_0} < 0.$$

If we divide the lefthand side by $\frac{1}{t_0^2 \ln t_0}$, we see that resulting quantity converges to $-4C_0$ as $t_0 \to \infty$. As $\frac{1}{t_0^2 \ln t_0} > 0$, we conclude that the lefthand side must be negative for all t_0 sufficiently large.

Considering the lower bound for $\mathrm{Ric}_G(\partial_t, \partial_t)$ in Proposition B.3.5, by Proposition B.3.6, if r is chosen sufficiently large there are constants $C_0, C_1 > 0$ for which $\mathrm{Ric}_G(\partial_t, \partial_t) > -L(t)$, where $L(t)$ is as in (B.14). Thus if we take $t_1 = t_0^2$ and t_0 sufficiently large, $\mathrm{Ric}_G(\partial_t, \partial_t) > 0$.

Because ∂_t is an eigenvector of Ric_G at every point of $[t_0, t_1] \times \mathrm{M}^n$, if E is any other eigenvector of Ric_G at a point (t, p) it must be tangent to $\{t\} \times \mathrm{M}^n$. We may therefore define a family of vector fields $F(s)$ of M^n such that $F(\lambda(t)) = ER^2(t)$. It is possible to extend E and F to belong to the normal coordinate vector fields Y_i and X_i respectively. We can now argue similarly for $\mathrm{Ric}_G(E, E)$ as we did for $\mathrm{Ric}_G(\partial_t, \partial_t)$.

Considering the lower bound in Proposition B.3.5 for $\mathrm{Ric}_G(E, E)$, by Proposition B.3.6, if r is chosen sufficiently large there are constants $C_0, C_1 > 0$

for which

$$\mathrm{Ric}_G(E,E) > -L(t)g_s(F,F) + \frac{1}{r^2}\,\mathrm{Ric}_{g_s}(F,F).$$

By assumption Ric_{g_s} is positive definite, thus if $t_1 = t_0^2$ and t_0 is sufficiently large, the righthand side is positive and consequently $\mathrm{Ric}_G(E,E) > 0$. □

APPENDIX C

CONSTRUCTION OF THE NECK

In this section we will construct the metric $g_{\text{neck}}(\rho)$ promised to exist in Lemma 3.1.2 above. This metric is explicitly claimed to exist in [33], so we will now quote the original statement. The added details in Lemma 3.1.2 above can be deduced from the construction given below.

Lemma C.0.1. *[33, Assertion] Assume that $n \geq 3$, $0 < r < R < 1$, and $g_1 = d\phi^2 + f_1^2(\phi)ds_{n-1}^2$ is a metric on S^n with $\phi \in [0, \pi R]$, $\sup_\phi f_1(\phi) = r$, and $\mathrm{K}_{g_1} > 1$. Then for any any $\rho > 0$ satisfying $r^{(n-1)/n} < \rho < R$, there exists a metric $g = g(\rho)$ defined on $S^n \times [0,1]$ and constant $\lambda > 0$ such that the following are true.*

(i) Ric_g *is positive definite;*

(ii) *the restriction $g|_{t=0}$ coincides with $\rho^2/\lambda^2 ds_n^2$ on $S^n \times \{0\}$;*

(iii) *the restriction $g|_{t=1}$ coincides with g_1 on $S^n \times \{1\}$;*

(iv) *the principal curvatures along the boundary $S^n \times \{0\}$ are equal to $-\lambda$;*

(v) *the principal curvatures along the boundary $S^n \times \{1\}$ are at least* 1.

Lemma C.0.1 was originally proven in [33, Section 2]. The purpose of this appendix is to provide a detailed version of this proof, filling in some of the missing technical details. Indeed, the proof given in [33] is completely accurate, though several assertions go without explanation.

This section is organized into four parts. The first, Section C.1, considers families of warped product metrics and demonstrates the existence of a one

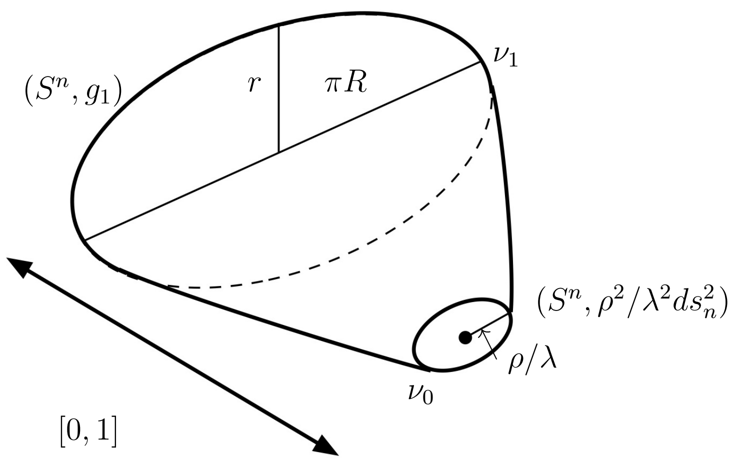

FIGURE C.1. The neck.

parameter family $\tilde{g}(a, b(a))$ connecting the metric g_1 to the round metric of radius ρ, so that the sectional curvatures remain bounded below by 1 in this entire family. The second part, Section C.2, considers metrics on the cylinder $[t_0, t_1] \times S^n$ of the form $g = \kappa^2(dt^2 + t^2\tilde{g}(h(t), k(t)))$, where h and k are essentially parameterizations of the path $(a, b(a))$. The definition of h and k includes three independent parameters that determine a family of metrics $g(t_0, \varepsilon, \delta)$ that will all satisfy the claims of Lemma C.0.1 at $t = t_0$ and $t = t_1$ and will have large sectional curvature in the spherical directions. The third part, Section C.3, is dedicated to showing that the parameters of g can be chosen so that Ric_g is positive definite. The last part, Section C.3.5 is dedicated to computing the relevant curvatures needed for the previous three sections.

The main theme of this proof, which is entirely due to Perelman, is to define for any interval $[t_0, t_1]$ a metric whose ends are isometric to the desired metrics

$\rho^2/\lambda^2 ds_n^2$ and g_1 so that the sectional curvatures and extrinsic curvatures of the ends will exhibit the desired behavior as you take the limit as $t_0 \to \infty$ and $(t_1 - t_0) \to \infty$. While the details needed to make this work are very technical, the key insight in [33] is the existence of the path of metric $\tilde{g}(a, b(a))$, the result of which is that the sectional curvature in the spherical directions are bounded below by $1/t^2$. This essentially reduces the problem to choosing functions h and k that cause the other curvatures decay faster than this.

C.1. A two parameter family of warped product metrics

In this section we begin by defining a two parameter family of warped product metrics $\tilde{g}(a, b)$ on S^n that will connect $\rho^2 ds_n^2$ to g_1. In Section C.1.1 we give some background on warped products metrics. Specifically in Lemma C.1.1 we reparameterize all positively curved warped product metrics so that they are defined on a universal domain $[-\pi/2, \pi/2] \times S^n$. This domain is essential to defining the two parameter family in Section C.1.2. The parameter b is the waist of the warped product metric, and the parameter a is the maximum velocity of the parameterization given in Section C.1.1. The metrics $\tilde{g}(a, b)$ are defined by taking a convex combination of the metric functions for $r^2 ds_n^2$ and g_1 in terms of a and scaling that metric by b. We conclude with Section C.1.3 in which we show that the one parameter family defined by assuming b is proportional to a power of a has sectional curvature larger than 1. Thus showing that $\rho^2 ds_n^2$ and g_1 are in the same path component of the space of all metrics on S^n with $\mathrm{K}_g > 1$. This path will be used explicitly to construct a Ricci positive metric on the cylinder in Section C.2.

C.1.1. Renormalizing warped products

We begin by recalling the definition of warped product metric on S^n, which we have already touched on in Section 2.1.1.2 above. Such metrics can be defined as $d\phi^2 + f^2(\phi)ds^2_{n-1}$ on $[0, D] \times S^{n-1}$ for a function $f : [0, D] \to [0, W]$ such that

$$f^{(\text{even})}(0) = f^{(\text{even})}(D) = 0, \quad f'(0) = 1, \text{ and } f'(D) = 1. \tag{C.1}$$

In this application, we will only be considering $W \leq D$. In this case, we will call W the *waist* and D the *diameter*. Let X and $\{\Sigma_i\}_{i=1}^{n-1}$ denote an orthonormal local frame of S^n tangent to $[0, D]$ and S^{n-1} respectively. The sectional curvatures of the warped product metric in these coordinates is given by

$$K(X, \Sigma_i) = -\frac{f''(\phi)}{f(\phi)} \text{ and } K(\Sigma_i, \Sigma_j) = \frac{1 - (f'(\phi))^2}{f^2(\phi)}. \tag{C.2}$$

By assumption, g_1 is a warped product with $D = \pi R$ and $W = r$. The assumption of Lemma C.0.1, that g_1 is a metric on S^n is equivalent to the fact that f_1 satisfies equation (C.1). The assumption that $\mathrm{K}_{g_1} > 1$ is equivalent to f_1 satisfying certain inequalities determined by equation (C.2). Notice also that round metrics can be realized as warped products on $[0, \pi\rho] \times S^{n-1}$ as follows

$$\rho^2 ds_n^2 = d\phi^2 + \rho^2 \sin^2\left(\frac{\phi}{\rho}\right) ds^2_{n-1}.$$

We wish to define a family of warped product metrics that connect g_1 to $\rho^2 ds_n^2$. Because these metrics are defined for ϕ in intervals of different lengths, one would have to write down a family of functions defined on a family of intervals. To reduce this complexity, we reparameterize any warped product metric with concave warping function to be defined on the universal domain: $[-\pi/2, \pi/2] \times S^{n-1}$.

Lemma C.1.1. *Let $f : [0, D] \to [0, W]$ be a concave function satisfying equation (C.1), let $g_f = d\phi^2 + f^2(\phi) ds_{n-1}^2$. Then there is a parameterization $\phi : [-\pi/2, \pi/2] \to [0, D]$ such that $g_{f\circ\phi} = A^2(x)dx^2 + W^2 \cos^2 x ds_{n-1}^2$. Where $A(x)$ is a positive function that satisfies*

$$A(\pm\pi/2) = W, \quad A'(\pm\pi/2) = 0, \text{ and } \sup_x A(x) \geq \frac{D}{\pi}. \tag{C.3}$$

Proof. As f is concave, there is a unique point at which f has a global maximum. This splits $[0, D]$ into two intervals on which f is bijective. On each interval, we may therefore define

$$\phi(x) = f^{-1}(W \cos x). \tag{C.4}$$

This function is smooth, and by definition $f(\phi(x)) = W \cos x$. It follows then that

$$g_{f\circ\phi} = \left(\frac{d\phi}{dx}\right)^2 dx^2 + W^2 \cos^2 x ds_{n-1}^2.$$

Therefore $A(x) = \phi'(x)$, which by (C.4) can be computed explicitly as

$$A(x) = \frac{-W \sin(x)}{f'(\phi(x))}. \tag{C.5}$$

The statements in equation (C.3) about the values of A and A' at 0 and D follow immediately from equation (C.1).

To see the claim about the supremum of $A(x)$, suppose the the contrary that $\phi'(x) < D/\pi$. It follows that the arc-length of $\phi(x)$ is less than D contradicting the fact that ϕ is a bijection. $\square$

C.1.2. The two parameters

If $K(X, \Sigma_i) > 0$ for a warped product metric, from equation (C.2) we see $f'' < 0$. Thus Lemma C.1.1 applies, in particular to g_1 and $\rho^2 ds_n^2$. Let $A_1(x)$ be the function $A(x)$ determined by g_1. In this case $D = R$ and $W = r$. For $\rho^2 ds_n^2$, we see that the function $A(x)$ is ρ, and $D = W = \rho$. We want use Lemma C.1.1 to define a two parameter family of metrics $\tilde{g}(a, b)$ that will connect g_1 and $\rho^2 ds_{n-1}^2$. To achieve this, we will define functions $A(a, b, x)$ and $B(b, x)$ that interpolate between ρ and $A_1(x)$ and between $\rho \cos x$ and $W \cos x$ respectively. The latter has an obvious candidate, $B(b, x) = b \cos x$ with $b \in [r, \rho]$. As we see that b is already naturally associated with the waist, we now endeavor to find a suitable choice for a.

If we imagine fixing the waist $b = W$, then we want a way of transitioning from $b^2 ds_n^2$ to $(b/r)\, g_1$ (two metrics with waist b). In this case, we need a function $A(a, b, x)$ that interpolates between b and $((b/r)\, A_1(x)$. We can consider the convex combination of these functions with respect to $\tau \in [0, 1]$,

$$A(\tau, b, x) = \tau \frac{b}{r} A_1(x) + (1 - \tau) b. \tag{C.6}$$

We want to replace the parameter τ with one that has geometric meaning, we will take a to be the maximum velocity of $\tilde{g}(a, b)$ of the parameterization determined by Lemma C.1.1. We can define $a(\tau) = \sup_x (1/b)\, A(\tau, b, x)$. Clearly a is linear in τ. Let $a_1 = \sup_x (A_1(x)/r)$. Clearly $A(0, b, x) = (b/r)\, A_1(x)$ and $A(1, b, x) = b$. Thus $a(0) = a_1$ and $a(1) = 1$. Thus $\tau = (a - 1) / (a_1 - 1)$ for $a \in [1, a_1]$, and we may replace τ with this in equation (C.6) to get

$$A(a, b, x) = b \left(\frac{a - 1}{a_1 - 1} \frac{A_1(x)}{r} + \left(1 - \frac{a - 1}{a_1 - 1} \right) \right).$$

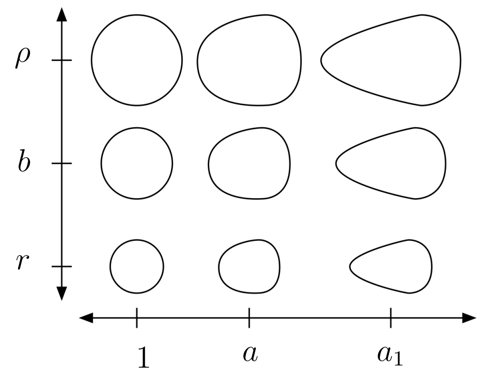

FIGURE C.2. The effect of (a, b) on the metric $\tilde{g}(a, b)$.

Which, following Perelman, we will rewrite in terms of a function $\eta(x)$ as follows

$$A(a, b, x) = b\left(\frac{a-1}{a_1-1}\frac{A_1(x)-r}{r}+1\right) = b((a-1)\eta(x)+1). \tag{C.7}$$

Definition C.1.2. *Let* $a_1 = \sup_x (A_1(x)/r)$ *and* $\eta(x)$ *be as defined in equation (C.7). Define for each* $(a, b) \in [1, a_1] \times [r, \rho]$ *a metric* $\tilde{g}(a, b)$ *on* S^n *as follows.*

$$\tilde{g}(a, b) = A^2(a, b, x)dx^2 + B^2(b, x)ds^2_{n-1}.$$

Where A *and* B *are defined as follows.*

$$A(a, b, x) = b(\eta(x)(a-1)+1),$$
$$B(b, x) = b\cos x.$$

For a fixed a, we see that changing b simply scales the metric. For a fixed b we see that changing a interpolates the velocity of the parameterizations $[-\pi/2, \pi/2] \to [0, b]$ between the constant velocity parameterization and the parameterization determined by $(b/r)\,\phi_1$. This behavior is illustrated in Figure C.2..

C.1.3. A one parameter family with $K > 1$

By definition $\tilde{g}(1,\rho) = \rho^2 ds_n^2$ and $\tilde{g}(a_1, r) = g_1$, so we have succeeded in finding a two parameter family of metrics that connects our two significant metrics. We now wish to study the curvature of these metrics. In particular, both $\rho^2 ds_n^2$ and g_1 have sectional curvature bounded below by 1. This family of metrics $\tilde{g}(a,b)$ remain close to the extreme metrics, so it is reasonable to believe that there is path through this parameter space along which the sectional curvature remains bounded below by 1. If we assume that b is a function of a, then we can consider the one parameter family $\tilde{g}(a) = \tilde{g}(a, b(a))$ and ask which choice of $b(a)$ will have $K_{\tilde{g}(a)} > 1$. A first guess at which function will work, would be to take a linear relationship. In the course of trying to bound the curvature below by one, one needs to bound the slope $(a_1 - 1)\,/\,(r - \rho)$ below by 1, which is false for some choices of g_1.

The next simplest relationship is to assume proportionality. Indeed, if $b(a) = c/\left(a^{1/\tilde{\alpha}}\right)$, then

$$\tilde{\alpha} = \frac{\ln a_1}{\ln \rho - \ln r} \text{ and } c = \rho.$$

To bound the curvature below by 1, one must bound $\tilde{\alpha}$ below by 1. This is implied by the hypotheses of Lemma C.0.1. To see this, recall the fact that $\sup_x A_1(x) \geq R$, by Lemma C.1.1, and the assumption that $R > \rho$ in Lemma C.0.1. Thus $a_1 \geq R/r > \rho/r$, which shows $\tilde{\alpha} > 1$. Note that the proportionality of b to a is equivalent to

$$-\frac{b'(a)}{b(a)} = \frac{1}{\tilde{\alpha} a}. \tag{C.8}$$

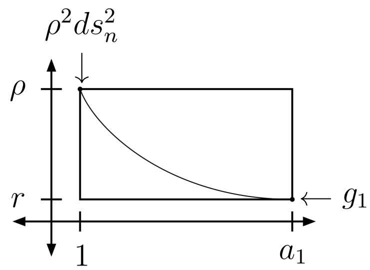

FIGURE C.3. The path $\left(a, \rho/\left(a^{1/\tilde{\alpha}}\right)\right)$ through the parameter space.

This characterization will be convenient in Section C.2 when we want to consider metrics on the cylinder. If we imagine that $b(t)$ and $a(t)$ are functions of interval, then equation (C.8) is equivalent to

$$-\frac{b'(t)}{b(t)} = \frac{a'(t)}{\tilde{\alpha}a(1)}. \tag{C.9}$$

This will allow us to define functions in a way that is easily relatable to the path $(a, b(a))$, but that will also be convenient for prescribing certain asymptotics and boundary conditions.

We define a one parameter family of metrics with respect to $a \in [1, a_1]$ as follows

$$\tilde{g}(a) = \tilde{g}\left(a, \frac{\rho}{a^{1/\tilde{\alpha}}}\right). \tag{C.10}$$

The following Lemma is implicit in [33, Section 2], it claims that this one parameter family satisfies the desired curvature bound.

Lemma C.1.3. *Let $\tilde{g}(a)$ be as in (C.10), then $\mathrm{K}_{\tilde{g}(a)} > 1$ for all $a \in [1, a_1]$.*

Before proving this lemma we provide the formulas for the sectional curvatures of $\tilde{g}(a, b)$. These curvatures can be computed directly from Definition C.1.2 and (i) of Proposition C.3.9.

Corollary C.1.4. *The sectional curvatures of $\tilde{g}(a,b)$ are as follows.*

$$\mathrm{K}_{\tilde{g}(a,b)}(X,\Sigma) = \frac{1}{b^2}\left(\frac{1}{(1+\eta(x)(a-1))^2} - \frac{\eta'(x)\tan x(a-1)}{(1+\eta(x)(a-1))^3}\right)$$
$$\mathrm{K}_{\tilde{g}(a,b)}(\Sigma_i,\Sigma_j) = \frac{1}{b^2}\left(\frac{1}{\cos^2 x} - \frac{\tan^2 x}{(1+\eta(x)(a-1))^2}\right)$$

Before turning to prove Lemma C.1.3, we state a lemma concerning the function $(1+(a-1)\eta(x))$. As this function is key to the definition of $\tilde{g}(a,b)$, it will appear in all but one curvature term. While the following lemma is elementary, it is used repeatedly in the proof of Lemma C.1.3.

Lemma C.1.5. *For all $x \in [-\pi/2, \pi/2]$,*

$$\frac{-1}{a_1 - 1} < \eta(x) \leq 1. \tag{C.11}$$

If $\eta(x) \geq 0$, then $(1+(a-1)\eta(x))$ is nondecreasing with respect to $a \in [1, a_1]$ and

$$1 \leq (1+(a-1)\eta(x)) \leq a_1. \tag{C.12}$$

If $\eta(x) < 0$, then $(1+(a-1)\eta(x))$ is decreasing with respect to $a \in [1, a_1]$ and

$$0 \leq |\sin x| < (1+(a-1)\eta(x)) \leq 1. \tag{C.13}$$

Moreover, there is a constant $h > 0$, depending only on g_1, such that for all a and x we have

$$(1+(a-1)\eta(x)) > ha. \tag{C.14}$$

Proof. By the definition of $\eta(x)$ and a_1 in equation (C.7), we see that $\eta(x) \leq 1$ (with equality necessarily attained). Considering equation (C.5) we see that there is a constant $c > 0$ depending only on g_1, such that $A_1(x) > c$. Applying this to the definition of $\eta(x)$ in equation (C.7) we may assume that $c < r$, and so we see that

$$\eta(x) > -\frac{r-c}{r}\frac{1}{a_1-1}.$$

This proves the bounds in equation (C.11).

Assume now that $\eta(x) \geq 0$. The function is linear in a, so the fact that it is nondecreasing is clear. By (C.11) we have $0 \leq \eta(x) \leq 1$, which yields

$$1 \leq 1 + (a-1)\eta(x) \leq 1 + (a-1).$$

This proves equation (C.12).

Assume next that $\eta(x) < 0$. Again, that the function is decreasing is clear. So for each $\eta(x) < 0$ the function has a maximum at $a = 1$, implying

$$1 + (a-1)\eta(x) \leq 1.$$

For the lower bound, one must use the definition of $A_1(x)$ in equation (C.5) in more detail. By assumption, $\mathrm{K}_{g_1} > 1$. Looking at the equation for $\mathrm{K}_{g_1}(\Sigma_i, \Sigma_j)$ in equation (C.2) yields that $f_1'(\phi(x)) < 1$. The definition of $\phi(x)$ in equation (C.4) for f_1, forces the sign of $-r\sin x$ and $f_1'(\phi(X))$ to agree. Thus we have

$$A_1(x) = \frac{-r\sin x}{f_1'(\phi(x))} > r|\sin x|.$$

Rewriting $1 + (a-1)\eta(x)$ in terms of $A_1(x)$ yields

$$1 + (a-1)\eta(x) = 1 + \frac{a-1}{a_1-1}\left(\frac{A_1(x)}{r} - 1\right) > 1 + \frac{a-1}{a_1-1}\left(|\sin x| - 1\right) \qquad \text{(C.15)}$$

That this function is decreasing means it obtains a minimum value at $a = a_1$. Substituting $a = a_1$ into the righthand side of equation (C.15), yields the lower bound of equation (C.13).

To see why equation (C.14) holds, notice that $1 + (a-1)\eta(x)$ is linear in a with vertical intercept $1 - \eta(x)$ and slope $\eta(x)$. By (C.11), the vertical intercept is always nonnegative. Consider now the value at a_1. By equation (C.13), for all x this has a minimum value equal to $d > 0$. It follows then that

$$1 + (a-1)\eta(x) > \frac{d}{a_1 - 1}a.$$

Because $\eta(x)$ in equation (C.7) is entirely determined by g_1, so is this constant $h > 0$. □

The geometric meaning of Lemma C.1.5 is roughly as follows. The function $1 + (a-1)\eta(x)$, by definition interpolates between 1 and a function $A(x) = r^{-1}A_1(x)$. For a fixed a, the function determines the parameterization of the angle of inclination in terms of the variable $x \in [-\pi/2, \pi/2]$. Necessarily, as a increases this will compress the interval to $[0, \pi R]$. The purpose of Lemma C.1.5 is to keep track of behavior with respect to an individual point $x \in [-\pi/2, \pi/2]$. This is important as the curvature is determined by this variable as given by equation (C.2). That the sign of $\eta(x)$ determines the behavior is clear from the formulas, but also has a clear geometric meaning. If we consider equation (C.7), we can see that the sign of $\eta(x)$ is determined by the difference in slope of $f_1(\phi)$ and $r\cos x$ at the same output values, which by (C.2) is the sign of $\mathrm{K}_{g_1}(\Sigma_i, \Sigma_j) - \mathrm{K}_{r^2 ds_n^2}(\Sigma_i, \Sigma_j)$. If we think carefully about the definition of $\phi(x)$ in Lemma C.1.1, we can see that $\eta(x)$ being positive or negative corresponds to whether $(1 + (a-1)\eta(x))$ is locally stretching or compressing the interval as a increases. Thus one expects distinct behaviors, which is seen directly in Lemma C.1.5.

We now give the proof Lemma C.1.3. The interested reader can see the penultimate line of [33, Page 160] for Perelman's statement of the lemma, and read the following two paragraphs for his proof. Our proof is not substantially different, though we include some elementary arguments that were omitted. For instance, the results of C.1.5 are omitted, which are elementary but not obvious.

Proof of Lemma C.1.3. Define two functions $L_\Sigma(a)$ and $L_X(a)$ as follows

$$L_\Sigma(a) = \ln\left(\mathrm{K}_{\tilde{g}(a)}(\Sigma_i, \Sigma_j)\right) \text{ and } L_X(a) = \ln\left(\mathrm{K}_{\tilde{g}(a)}(X, \Sigma_i)\right).$$

Where we have suppressed the fact that $L_X(a)$ and $L_\Sigma(a)$ also depend on the parameter $x \in [-\pi/2, \pi/2]$. We will consider a fixed x by cases corresponding to Lemma C.1.5. Note that the claim of Lemma C.1.3 is equivalent to showing that $L_X(a) > 0$ and $L_\Sigma(a) > 0$. By construction the metric $\tilde{g}(a)$ interpolates between $\rho^2 ds_n^2$ and g_1 on the interval $a \in [1, a_1]$, which by assumption have sectional curvatures greater than 1. Thus $L_X(a)$ and $L_\Sigma(a)$ are both positive at $a = 1$ and a_1.

We begin by considering $L_\Sigma(a)$. First we compute $L'_\Sigma(a)$ using Corollary C.1.4 and equation (C.9).

$$L'_\Sigma(a) = \frac{2}{\tilde{\alpha} a}\left(1 + \frac{\tilde{\alpha}\eta(x)a}{(1+(a-1)\eta(x))((1+(a-1)\eta(x))^2 - \sin^2 x)}\right). \tag{C.16}$$

Following Lemma C.1.5, we will consider the cases $\eta(x) \geq 0$ and $\eta(x) < 0$ separately.

Assume that x is fixed such that $\eta(x) \geq 0$. We claim that $L'_\Sigma(a) \geq 0$ for such values of x. Were this the case, then $L_\Sigma(a) \geq L_\Sigma(a_1) > 0$ proving the claim. To see this, consider equation (C.12). We see that both factors in the denominator of the large fraction in equation (C.16) are nonnegative. By assumption, the numerator is nonnegative. Since $\tilde{\alpha}a$ is positive, this shows that $L'_\Sigma(a)$ is nonnegative.

Next, fix x so that $\eta(x) < 0$. For such x, we claim that $L_\Sigma(a)$ has no relative minima and hence is bounded below by its boundary values. To see this we claim that $L'_\Sigma(a)$ is decreasing. Note that the large fraction in equation (C.16) is decreasing with respect to a. Indeed, by equation (C.13), both factors in the denominator are positive and decreasing, and by assumption the numerator is negative and decreasing. Applying the quotient and product rules shows directly that the sign of the derivative of this fraction must be negative.

This covers all possible values of x, and we conclude that $L_\Sigma(a) > 0$ for all x and a.

Next consider $L_X(a)$. One sees in Corollary C.1.4 that the sign of $\tan x\eta'(x)$ will influence the behavior of $L_X(a)$. We will therefore consider $\tan x\eta'(x) \geq 0$ and $\tan x\eta'(x) < 0$ separately.

Fix x such that $\tan x\eta'(x) < 0$. We claim that $L_X(a) > -2\left(\ln a + \ln b(a)\right)$. This is obvious from Corollary C.1.4 and equation (C.13). We claim that $-2\left(\ln a + \ln b(a)\right)$ is positive. Note that

$$-2\frac{d}{da}\left(\ln a + \ln b\right) = -2\left(\frac{1}{a} - \frac{1}{\tilde{\alpha}a}\right) = -2\frac{1}{\tilde{\alpha}a}\left(\tilde{\alpha} - 1\right).$$

Because $\tilde{\alpha} > 1$, $-2(\ln a + \ln b(a))$ is decreasing and therefore bounded below by $-2(\ln a_1 + \ln r)$, where $b(a_1) = r$. Notice that when $\eta(x) = 1$ (which is guaranteed to happen), by Corollary C.1.4, $\mathrm{K}_{g_1}(X, \Sigma_i) = 1/(r^2 a_1^2)$, but by assumption $K_{g_1} > 1$. Taking logarithms shows that $-2(\ln a_1 + \ln r) > 0$, therefore proving that $L_X(a) > 0$ in this case.

Next fix x such that $\tan x \eta'(x) \geq 0$. We will again need to consider the log derivative of $\mathrm{K}_{\tilde{g}(a)}(X, \Sigma_i)$, $L'_X(a)$.

$$L'_X(a) = \frac{1}{\tilde{\alpha} a}\left(2 + \frac{\eta(x)\tilde{\alpha} a - \eta'(x)\tan x \tilde{\alpha} a}{(1 + (\eta(x) - \eta'(x)\tan x)(a-1)} - 3\frac{\eta(x)\tilde{\alpha} a}{1 + \eta(x)(a-1)}\right). \quad \text{(C.17)}$$

We will again need to consider the cases $\eta(x) \geq 0$ and $\eta(x) < 0$ seperately.

Fix x such that $\tan x \eta'(x) \geq 0$ and $\eta(x) \geq 0$. Equation (C.17) can be factored as follows.

$$\begin{aligned} L'_X(a) = \frac{1}{\tilde{\alpha} a}\frac{1}{1 + \eta(x)(a-1)} &\left(2(1 - \eta(x)) + 2\eta(x) a (1 - \tilde{\alpha})\right. \\ &\left. - \frac{\eta'(x)\tan x \tilde{\alpha} a}{1 + (\eta(x) - \eta'(x)\tan x)(a-1)}\right). \end{aligned} \quad \text{(C.18)}$$

We claim that $L'_X(a)$ is decreasing, which would demonstrate that $L_X(a) > 0$. The two terms in equation (C.18) outside of the large parentheses are nonnegative by assumption and equation (C.13). We claim that the function inside the large parentheses is decreasing with a. The terms on the first line of equation (C.18) are linear in a with slope $2\eta(x)(1-\tilde{\alpha})$, which by assumption is negative. Thus this term is decreasing. It remains to check that the large fraction is increasing (as it appears with a minus sign). Its derivative is

$$\frac{d}{da}\left(\frac{\eta'(x)\tan x \tilde{\alpha} a}{1 + (\eta(x) - \tan x \eta'(x))(a-1)}\right) = \frac{\tilde{\alpha}\left(\eta'(x)\tan x(1 - \eta(x)) + (\eta'(x)\tan x)^2\right)}{(1 + (\eta(x) - \tan x \eta'(x))(a-1))^2},$$

which is nonnegative by assumption and equation (C.12). We conclude that $L'_X(a)$ is decreasing and hence $L_X(a) > 0$

Finally, fix x such that $\tan x\eta'(x) \geq 0$ and $\eta(x) < 0$. Equation (C.17) can be factored as follows.

$$L'_X(a) = \frac{1}{\tilde{\alpha}a}\left[2 - \frac{\tilde{\alpha}a}{1+\eta(x)(a-1)}\left(2\eta(x) + \frac{\eta'(x)\tan x}{1+(\eta(x)-\tan x\eta'(x))(a-1)}\right)\right]. \tag{C.19}$$

We claim that either $L'_X(a) \neq 0$ or that $L_X(a)$ has no relative minima. In either of these cases we conclude that $L_X(a)$ is bounded by its boundary values and hence positive. If $L'_X(a)$ has constant sign we are done, so let us assume that $L_X(a) = 0$ at some point. Because $\tilde{\alpha}a$ is positive, this is only possible if the large bracketed term is zero and changes sign. Notice that the large bracketed term is zero only if the large parenthetical term is positive. The fraction in the large parenthetical term, is the reciprocal of a linear function of a with negative slope, which is therefore increasing. Consider next the fraction multiplying the parenthetical term. It is the quotient of a positive linear functions of a with positive slope by a positive linear function of a with negative slope. Thus this fraction is positive and increasing. Thus the large bracketed term is decreasing as we are subtracting an increasing function from 2, and so can only change from positive to negative. We conclude that $L_X(a)$ has no relative minima.

This covers all possible cases of x, and we conclude that $L_X(a) > 0$ for all x and a. □

C.2. The metric on the neck

We now wish to use $\tilde{g}(a, b)$ of Definition C.1.2 to define a metric g on the cylinder $[0, 1] \times S^n$. We will define this metric on the cylinder so that the metric restricted to each slice $\{t\} \times S^n$ is conformal to $\tilde{g}(h(t), k(t))$, where $h(t)$ and $k(t)$ are some functions on the interval. This choice of metric allows us to use Lemma C.1.3 to reduce bounding the sectional curvatures in the spherical directions below by bounding the second fundamental form of the slices above. This second fundamental form is proportional to $\partial_t \tilde{g}(h(t), k(t))$. Thus to make this small we want $h'(t)$ and $k'(t)$ to essentially vanish. To achieve this end, we will instead consider defining $g(\kappa, h(t), k(t))$ on $[t_0, t_1] \times S^n$ assuming that $[t_0, t_1]$ is an arbitrarily long interval. After a reparameterization, this will define a metric on $[0, 1] \times S^n$ with the same curvature properties. Thus to prove Lemma C.0.1 it suffices to construct a metric on $[t_0, t_1] \times S^n$.

We begin in Section C.2.1 by defining the family of metrics $g(\kappa, h(t), k(t))$, referred to in the preceding paragraph, on $[t_0, t_1] \times S^n$, where $h(t)$ and $k(t)$ are functions on $[t_0, t_1]$ and $\kappa > 0$ is a scaling factor. We then spend some time discussing the necessary assumptions on $h(t)$, $k(t)$, and κ imposed by assuming that g satisfies (ii) through (v) of Lemma C.0.1, which turns out to be very easily summarized with Proposition C.2.3. Next in section C.2.2, we find a class of functions $h(t)$ and $k(t)$ defined in terms of two new parameters ε and δ that will satisfy the necessary conditions described in Section C.2.1 and that remain close enough to the special path in Lemma C.1.3 for suitable choice of parameters. In section C.2.3 we give a precise description of the control we have over g with our choice of parameters. And finally in Section C.2.4, we explain which choices of parameters will guarantee that g satisfies Proposition C.2.3 and therefore (ii) through (v) of Lemma C.0.1, and we show how the parameters force g to inherit large sectional curvature in the spherical directions from Lemma C.1.3. Going forward we will have produced a three parameter family of metrics $g(t_0, \varepsilon, \delta)$ on $[0,1] \times S^n$ that satisfy (ii) through (v) of Lemma C.0.1 and that has large sectional curvatures which will facilitate in the proof of (i) in Section C.3.

C.2.1. The form of the metric

While $\tilde{g}(a,b)$ of definition C.1.2 was discussed only for $(a,b) \in [1,a_1] \times [r,\rho]$, it is well defined for all $a \geq 1$ and $b > 0$. Thus we can consider metrics of the following form on $[t_0,t_1] \times S^n$.

Definition C.2.1. *Suppose we are given functions $h(t) \geq 1$ and $k(t) > 0$ defined on $[t_0,t_1]$ and a constant $\kappa > 0$. Define the metric $g = g(\kappa, h(t), k(t))$ on $[t_0,t_1] \times S^n$, as follows*

$$\begin{aligned} g(\kappa, h(t), k(t)) &= \kappa^2 \left(dt^2 + t^2 \tilde{g}(h(t), k(t))\right) \\ &= \kappa^2 \left(dt^2 + t^2 k^2(t)(1 + (h(t)-1)\eta(x))^2 dx^2 + t^2 k^2(t) \cos^2 x ds_{n-1}^2\right). \end{aligned} \tag{C.20}$$

We see that the metric restricted to $\{t\} \times S^n$ will be $(t\kappa)^2 \tilde{g}(h(t), k(t))$. Thus the intrinsic curvature of $\{t\} \times S^n$ with respect to g is clearly just $1/\left(t^2\kappa^2\right) \mathrm{K}_{\tilde{g}(h(t),k(t))}$. If $(h(t), k(t))$ is close to the path $\left(a, \rho/\left(a^{1/\tilde{\alpha}}\right)\right)$, then we can immediately conclude from Lemma C.1.3 that the intrinsic curvatures are greater than $1/\left(t^2\kappa^2\right)$. We will explain in a moment why $(h(t), k(t))$ is *not* exactly a parameterization of this path. The extrinsic curvature can be computed directly from (i) of Proposition C.3.9.

Corollary C.2.2. *Let* II_t *denote the second fundamental form of* $(\{t\} \times S^n, g(\kappa, h(t), k(t)))$ *with respect to the unit normal* ∂_t. *The extrinsic curvatures are as follows.*

$$\mathrm{II}_t(X, \Sigma_i) = \mathrm{II}_t(\Sigma_i, \Sigma_j) = 0,$$
$$\mathrm{II}_t(X, X) = \frac{1}{\kappa}\left(\frac{1}{t} + \frac{k'(t)}{k(t)} + \frac{\eta(x)h'(t)}{1 + (h(t) - 1)\eta(x)}\right), \tag{C.21}$$
$$\mathrm{II}_t(\Sigma_i, \Sigma_i) = \frac{1}{\kappa}\left(\frac{1}{t} + \frac{k'(t)}{k(t)}\right). \tag{C.22}$$

We can now explain the motivation for defining $g(\kappa, h(t), k(t))$ in this way. Obviously, we want the slices $\{t\} \times S^n$ to be conformal to $\tilde{g}$, so we can conclude the intrinsic curvatures are large. The purpose of scaling $\tilde{g}$ by t is to correct the following defect. If we considered instead the metric $\kappa^2(dt^2 + \tilde{g}(h(t), k(t)))$, then the second fundamental form of $\{t\} \times S^n$ as in (i) of Proposition C.3.9 gives

$$\mathrm{II}_t(\Sigma_i, \Sigma_i) = \frac{1}{\kappa}\frac{k'(t)}{k(t)}.$$

Since $k(t)$ is decreasing from ρ to r, we see that $\mathrm{II}_t(\Sigma_i, \Sigma_i) < 0$. Thus the metric $\kappa^2(dt^2 + \tilde{g}(h(t), k(t)))$ could not satisfy claims (iv) or (v) of Lemma C.0.1. Considering instead $g(\kappa, h(t), k(t))$ then the second fundamental becomes as in equation (C.22). The limit of this as t increases can be made to be positive if $k'(t)/k(t)$ decays more rapidly than $1/t$. Since we have the freedom to choose $[t_0, t_1]$ and the function $k(t)$, this can be used to make II_t positive definite for all t. It will satisfy (v) if $1/(\kappa t_1) > 1$.

We will consider very carefully which choices of $h(t)$ and $k(t)$ should be made, but regardless, to satisfy (iii) of Lemma C.0.1, we need $g|_{t=t_1} = g_1$. We can see from formula (C.20) that this can only happen if $h(t_1) = a_1$ and that we choose our scale to be $\kappa = r/t_1 k(t_1)$. In order for g to satisfy (ii) we see from equation (C.20) that $h(t_0) = 1$ and therefore the choice of λ must be

$$\lambda = \frac{\rho t_1 k(t_1)}{r t_0 k(t_0)}.$$

In our application, we will assume for ease that $k(t_0) = \rho$, thus $\lambda = (t_1 k(t_1)) / (r t_0)$. Then considering (iv), we see that $\mathrm{II}_t = -\lambda$ if and only if $k'(t_0) = h'(t_0) = 0$.

Already with this short discussion we have seen how metrics of the form $g(\kappa, h(t), k(t))$ satisfy the claims of Lemma C.0.1 at $t = t_0$ and $t = t_1$ under certain assumptions. We summarize this in the following proposition.

Proposition C.2.3. *For any functions $h(t)$ and $k(t)$ on $[t_0, t_1]$ let $\kappa = r/(t_1 k(t_1))$ and let $\lambda = (t_1 k(t_1)) / (t_0 r)$. Then the claims of Lemma C.0.1 for the metric $g(\kappa, h(t), k(t))$ reduce to the following conditions on $h(t)$ and $k(t)$.*

(ii) $h(t_0) = 1$ *and* $k(t_0) = \rho$,

(iii) $h(t_1) = a_1$,

(iv) $h'(t_0) = k'(t_0) = 0$,

(v) $-k'(t_1)/k(t_1) < 1/t_1$ *and* $\kappa < 1/t_1$.

Proposition C.2.3 shows that the one condition of Lemma C.0.1 that requires the most work is (i), that the metric has positive Ricci curvature. The key observation needed to show this has already been established in Lemma C.1.3. We will use the fact that the intrinsic curvatures of time slices are relatively large to dominate all other terms appearing in the Ricci tensor.

C.2.2. The choice of metric functions

We still have the freedom to choose $h(t)$ and $k(t)$. As mentioned, we wish to choose $(h(t), k(t))$ close to the path $\left(a, \rho / \left(a^{1/\tilde{a}}\right)\right)$. Notice that assuming this $(h(t), k(t))$ is a parameterization of this path is equivalent to assuming similar to equation (C.9) that

$$\frac{h'(t)}{\tilde{\alpha} h(t)} = -\frac{k'(t)}{k(t)}. \tag{C.23}$$

We will therefore define $h(t)$ and $k(t)$ in terms of a separable differential equation. We see that this is also amenable to Proposition C.2.3, as we have the freedom to assume initial conditions to satisfy (ii) and (iv). This definition will also allow us to have direct control over the magnitude of $-k'(t_1)/k(t_1)$. However, if we take equation (C.23) as our definition, then $k(t_1) = r$ and so $\kappa = 1/t_1$. Thus $\mathrm{II}_{t_1}(\Sigma_i, \Sigma_i) < 1$, and g will not satisfy (v). To correct this, we will assume that $k(t_1) = \tilde{r} > r$, forcing $\kappa < 1/t_1$. We will assume then that

$$\frac{h'(t)}{\alpha h(t)} = -\frac{k'(t)}{k(t)} = \beta \Gamma(t). \tag{C.24}$$

Where α and β are some constants that will be determined by the assumptions that $h(t_1) = a_1$ and $k(t_1) = \tilde{r}$, and $\Gamma(t)$ is some function with the appropriate asymptotics and initial conditions.

We begin by making a choice for $\Gamma(t)$. According to Proposition C.2.3, in order for g to satisfy (iv), we must have that $\Gamma(t_0) = 0$. And in order to satisfy (v) we must assume something like $\Gamma(t) = O\left(1/\left(t \ln t\right)\right)$. This motivates the following

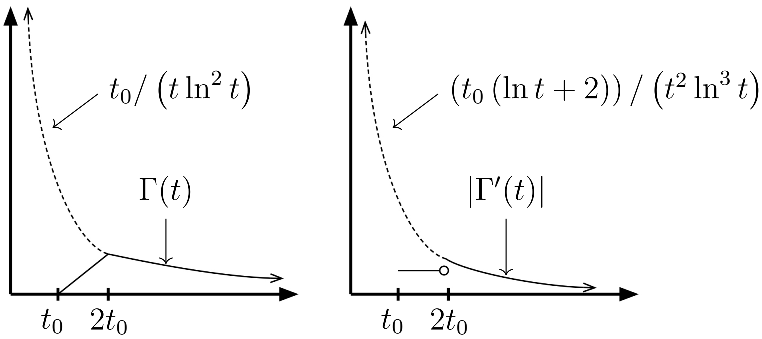

FIGURE C.4. Graph of $\Gamma(t)$ and $|\Gamma'(t)|$, demonstrating the bounds of Lemma C.2.4.

definition of the function $\Gamma(t)$.

$$\Gamma(t) = \begin{cases} \dfrac{t - t_0}{2t_0^2 \ln(2t_0)} & t_0 \leq t \leq 2t_0 \\ \dfrac{\ln(2t_0)}{t \ln^2 t} & 2t_0 < t \end{cases} \tag{C.25}$$

The graphs of this function and its derivative appear in Figure C.4.. One can see that this function is only continuous at $t = 2t_0$, so the functions $h(t)$ and $k(t)$ will only be once-differentiable. This issue has no bearing on the metric at $t = t_0$ and $t = t_1$ nor does it affect the smoothness of K_g is the spherical directions. So we will wait to resolve this issue in Section C.3. The following claim is obvious from Figure C.4. and is easy to verify from the definition (C.25).

Lemma C.2.4. *If $t_0 > 2$, then there exists constants c_Γ and d_Γ such that $\Gamma(t)$ and $\Gamma'(t)$ satisfy the following bounds for all $t \geq t_0$.*

$$|\Gamma(t)| < \frac{c_\Gamma \ln(2t_0)}{t \ln^2 t}$$
$$|\Gamma'(t)| < \frac{d_\Gamma \ln(2t_0)}{t^2 \ln^2 t}$$

For fixed $[t_0, t_1]$ and $\tilde{r}$, then α and β are uniquely determined by equation (C.24). As we are partly trying to illuminate Perelman's proof of Lemma C.0.1, we now introduce two new parameters ε and δ in the following way. Assuming that $h(t_1) = a_1$, then the limit of $h(t)$ as $t \to \infty$ must be greater than a_1. Therefore there must exist a $\delta > 0$ such that

$$\int_{t_0}^{\infty} \frac{h'(t)}{h(t)} dt = (1 + \delta) \ln a_1. \tag{C.26}$$

Assuming that $k(t_1) = \tilde{r} > r$, we may assume that the limit of $k(t)$ as $t \to \infty$ is greater than r. Therefore there must exist an $\varepsilon > 0$ such that

$$\int_{t_0}^{\infty} \frac{k'(t)}{k(t)} dt = (1 - \varepsilon)(\ln r - \ln \rho). \tag{C.27}$$

Conversely if we define $h(t)$ and $k(t)$ by equation (C.24), then $h(t)$ and $k(t)$ satisfy equations (C.26) and (C.27) only if α and β are as follows

$$\beta(t_0, \varepsilon) = \frac{(1 - \varepsilon)(\ln r - \ln \rho)}{\Delta(t_0)} \text{ and } \alpha(t_0, \varepsilon, \delta) = \frac{(1 + \delta) \ln a_1}{\beta \Delta(t_0)}. \tag{C.28}$$

Where $\Delta(t_0) = \int_{t_0}^{\infty} \Gamma(t) dt$. Clearly t_0 and δ determines the value of t_1 by the equation $h(t_1) = a_1$, and then $k(t_1) = \tilde{r}$ is determined additionally by ε. With this we are ready to define our metric in terms of this choice of $h(t)$ and $k(t)$ parameterized by t_0, ε, and δ.

We have finally settled on the following choice of metric.

Definition C.2.5. *For any $t_0 > 2$, $0 < \varepsilon \leq 1/4$, and $0 < \delta \leq 1/4$, let $\alpha = \alpha(t_0, \varepsilon, \delta)$ and $\beta = \beta(t_0, \varepsilon)$ be as defined in equation (C.28), and $h(t)$ and $k(t)$ be as defined by equation (C.24) with initial conditions $h(t_0) = 1$ and $k(t_0) = \rho$. Then define*

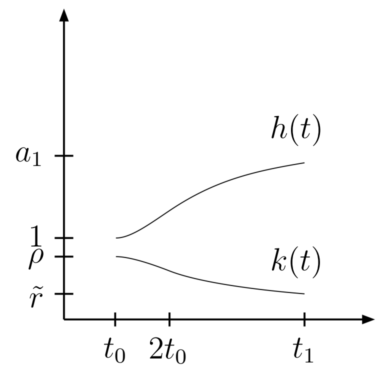

FIGURE C.5. The functions $h(t)$ and $k(t)$.

$g = g(t_0, \varepsilon, \delta) = g\left(r/\left(t_1 k(t_1)\right), h(t), k(t)\right)$, and define t_1 to be the unique number so that $h(t_1) = a_1$.

C.2.3. Control

We claim that the metric $g(t_0, \varepsilon, \delta)$ will satisfy Lemma C.0.1 for suitably chosen parameters. Let us briefly explain how each of these parameters will be used in the proof of Lemma C.0.1. By choosing t_0 larger, one can see from Lemma C.2.4 that this controls the size of the functions $h'(t)/h(t)$ and $k'(t)/k(t)$. This is made precise in Lemma C.2.6. By choosing δ smaller, we make t_1 larger relative to t_0, essentially controlling the length of the interval. By choosing ε smaller, we have control over the lower bound on $\tilde{r}$, which is also determined by t_1. These last two facts are made precise in Lemma C.2.7.

Lemma C.2.6. *Let $h(t)$ and $k(t)$ be as in Definition C.2.5. There are constants $c_1 > 0$ and $c_2 > 0$ that depend only on g_1, such that*

$$\left|\frac{h'(t)}{\alpha h(t)}\right| = \left|\frac{k'(t)}{k(t)}\right| < \frac{c_1 \ln(2t_0)}{t \ln^2 t}, \tag{C.29}$$

$$\left|\frac{\eta(x)h'(t)}{(1+(h(t)-1)\eta(x))}\right| < \frac{c_1 \ln(2t_0)}{t \ln^2 t}, \tag{C.30}$$

$$\left|\frac{\eta(x)h''(t)}{1+(h(t)-1)\eta(x)}\right| < \frac{c_2 \ln(2t_0)}{t^2 \ln^2 t}, \tag{C.31}$$

$$\left|\frac{k''(t)}{k(t)}\right| < \frac{c_2 \ln(2t_0)}{t^2 \ln^2 t}. \tag{C.32}$$

Proof. The constants c_1 and c_2 are partially determined by α and β. It is important to notice that by assuming $t_0 > 2$, $0 < \varepsilon \leq 1/4$, and $0 < \delta \leq 1/4$, we have assumed that α and β are bounded above and below independently of our specific choices t_0, ε, and δ.

Inequality (C.29) follows from Definition C.2.5 and Lemma C.2.4. Using the definition of $k(t)$ to compute $k''(t)/k(t)$ yields.

$$-\frac{k''(t)}{k(t)} = -\beta^2\Gamma^2(t) + \beta\Gamma'(t).$$

By applying Lemma C.2.4, and noting that $\ln(2t_0) < \ln t$ proves equation (C.32).

By equation (C.13) and (C.12) we have

$$|\eta(x)| < \max\left\{1, \frac{1}{a_1 - 1}\right\}.$$

And by equation (C.14), there is an $c > 0$ such that $(1 + (h(t) - 1)\eta(x)) > ch(t)$. Thus there is a constant $d > 0$ such that

$$\left|\frac{\eta(x)}{1 + (h(t) - 1)\eta(x)}\right| < d\left|\frac{1}{h(t)}\right|.$$

Thus we have

$$\left|\frac{\eta(x)h'(t)}{1 + (h(t) - 1)\eta(x)}\right| < d\left|\frac{h'(t)}{h(t)}\right| \text{ and } \left|\frac{\eta(x)h''(t)}{1 + (h(t) - 1)\eta(x)}\right| < d\left|\frac{h''(t)}{h(t)}\right|. \tag{C.33}$$

Applying equations (C.29) and (C.32) to equation (C.33) proves equations (C.30) and (C.31). □

The following lemma is a straightforward computation achieved by explicitly solving the separable differential equation (C.24).

Lemma C.2.7. *Let $h(t)$, $k(t)$, and β be as in Definition C.2.5. The number t_1 such that $h(t_1) = t_1$ and $\tilde{r} = k(t_1)$ are expressed in terms of t_0, ε, and δ as follows.*

$$\begin{aligned} t_1(t_0, \delta) &= \exp\left(\frac{1+\delta}{\delta}\frac{4\ln^2(2t_0)}{1 + 4\ln(2t_0)}\right), \\ \tilde{r}(t_0, \delta, \varepsilon) &= r\left(\frac{\rho}{r}\right)^{\varepsilon}\exp\left(\frac{\beta\ln(2t_0)}{\ln t_1}\right). \end{aligned} \tag{C.34}$$

Moreover, for all $T_1 > 0$, there are $T_0 > 0$ and $\delta_0 > 0$ such that $t_1 > T_1$ either if $t_0 > T_0$ or $\delta < \delta_0$. And, for all $r < r_0 < r_1$, there are $0 < \varepsilon_0 < \varepsilon_1$ and $\delta_1 > 0$ such that $r_0 < \tilde{r} < r_1$ if both $\varepsilon_0 < \varepsilon < \varepsilon_1$ and $\delta < \delta_1$.

C.2.4. Summary of benefits

Now that we understand the precise relationship of t_0, ε, and δ to our metric g, we are ready to summarize the discussion of this section in two corollaries. The first Corollary summarizes the discussion of Section C.2.1. It says that Proposition C.2.3 will apply to our metric g if ε is fixed and δ is chosen small relative to δ. The second summarizes the discussion of Section C.2.2. It says that if ε and δ are small enough, then the path $(h(t), k(t))$ is close enough to $\left(a, \rho/\left(a^{1/\tilde{\alpha}}\right)\right)$, implying that the curvature of g is relatively large.

Corollary C.2.8. *There is an $\varepsilon_0 > 0$, and $\delta(\varepsilon_0) > 0$ so that $g(t_0, \varepsilon, \delta)$ satisfies claims (ii), (iii), (iv), and (v) of Lemma C.0.1 with $\lambda = (t_1 k(t_1)) / (t_0 r)$ for all $\varepsilon > \varepsilon_0$ and $\delta < \delta_0(\varepsilon_0)$.*

Proof. By definition $h(t_0) = 1$ and $k(t_0) = \rho$, $h'(t_0) = k'(t_0) = 0$, and $h(t_1) = a_1$, so by Proposition C.2.3, g satisfies (ii) with the specified λ, (iv) with the specified λ, and (iii). It remains to show that we can choose δ_0 and ε_0 such that g also satisfies (v).

By Corollary C.2.2 and equations (C.29) and (C.30) of Lemma C.2.6, there is some $c > 0$ such that

$$|\,\mathrm{II}_{t_1}\,| > \frac{\tilde{r}}{r} - \frac{c}{\ln t_1} - \frac{c}{\ln^2 t_1}. \tag{C.35}$$

If we fix $\varepsilon_0 > 0$, then by Lemma C.2.7 $\tilde{r}/r \geq 1 + \zeta$ for some $\zeta(\varepsilon_0) > 0$. Substituting this into equation (C.35), we see that $|\,\mathrm{II}_{t_1}\,| > 1$ if t_1 is chosen large enough so that the remaining terms are bounded by ζ. Again, by Lemma C.2.7, there exists a $\delta_0(\varepsilon_0) > 0$ depending on ε_0, such that this holds for all $\delta < \delta_0$. Thus for all $\delta < \delta_0$, g satisfies (v). $\square$

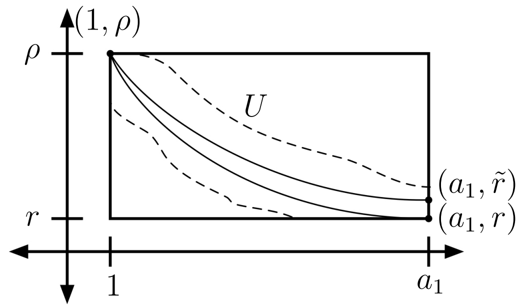

FIGURE C.6. By Lemma C.1.3, the open set $U = \{(a,b) : \mathrm{K}_{\tilde{g}(a,b)} > 1\}$ contains $\left(a, \rho/\left(a^{1/\tilde{\alpha}}\right)\right)$. Clearly, if $\tilde{r}$ is close to r, then $(h(t), k(t))$ is also contained in U.

The above discussion of (v) in the proof of Corollary C.2.8 is the final consideration in Perelman's proof of Lemma C.0.1. In the last paragraph of [33, p. 160], having already fixed ε and before scaling by κ, Perelman says, "it remains to choose $\delta > 0$ so small, and correspondingly t_1 so large, that normal curvatures of $S^n \times \{t_1\}$ are $> (r/\rho)^{\varepsilon} 1/t_1$." We can see from equation (C.34), that $\tilde{r}/r > (\rho/r)^{\varepsilon}$, so after scaling by $\kappa = r/(\tilde{r}t_1)$ we will have normal curvatures greater than 1.

Corollary C.2.9. *Let $g(t_0, \varepsilon, \delta)$ be as in Definition C.2.5, then there exists $\varepsilon_0 > 0$, $\delta_0 > 0$, and $c_s > 0$ such that for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ the ambient sectional curvatures in the spherical of $\{t\} \times S^n$ are as follows.*

$$\mathrm{K}_g(X, \Sigma_i) > \frac{c_s}{t^2}, \text{ and } \mathrm{K}_g(\Sigma_i, \Sigma_j) > \frac{c_s}{t^2}.$$

Proof. Let $a \in [1, a_1]$ and $\tilde{\alpha} = \ln a_1/(\ln \rho - \ln r)$ be as in Section C.1. Define $\tilde{b}(a)$

$$\tilde{b}(a) = \frac{\rho}{a^{1/\tilde{\alpha}}}. \tag{C.36}$$

By Lemma C.1.3, $\mathrm{K}_{\tilde{g}(a,\tilde{b}(a))} > 1$.

Let α be defined as in equation (C.28), then the definition of $h(t)$ and $k(t)$ in Definition C.2.5 is equivalent to

$$k(t) = \frac{\rho}{h^{1/\alpha}(t)}. \tag{C.37}$$

Which we may in turn take to define a function $b(a)$ for $a \in [1, a_1]$. It is clear by comparing equation (C.36) and (C.37) that $||b(a) - \tilde{b}(a)||_{C^2} \to 0$ as $(\alpha - \tilde{\alpha}) \to 0$, where explicitly

$$\alpha - \tilde{\alpha} = \left(\frac{1+\delta}{1-\varepsilon} - 1\right) \frac{\ln a_1}{\ln \rho - \ln r}.$$

Thus $||b(a) - \tilde{b}(a)||_{C^2} \to 0$ as $(\varepsilon, \delta) \to (0, 0)$. This is also clear graphically in Figure C.6., noting that, by Lemma C.2.7, $\tilde{r} \to r$ as $(\varepsilon, \delta) \to (0, 0)$.

Because $\mathrm{K} > 1$ is an open condition on the space of all metrics, the set $U = \{(a, b) : K_{\tilde{g}(a,b)} > 1\}$ is open, and by Lemma C.1.3 $(a, \tilde{b}(a)) \in U$. It follows that there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $(a, b(a)) \in U$ for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. Thus $\mathrm{K}_{\tilde{g}(h(t),k(t))} > d > 1$ for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$.

Still assuming that $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. If we combine this fact with Gauss' formula, Corollary C.2.2, and equations (C.29) and (C.30) of Lemma C.2.6 we have the following.

$$\begin{aligned}
\mathrm{K}_g(X, \Sigma_i) &= \left(\frac{t_1 k(t_1)}{r}\right)^2 \left(\mathrm{K}_{t^2\tilde{g}(h(t),k(t))}(X, \Sigma_i) - \mathrm{II}_t^2(X, \Sigma_i)\right) \\
&> \left(\frac{t_1 k(t_1)}{r}\right)^2 \left(\frac{d}{t^2} - \frac{1}{t^2} - \frac{c}{t^2 \ln t} - \frac{c}{t^2 \ln^3 t}\right). \\
\mathrm{K}_g(\Sigma_i, \Sigma_j) &= \left(\frac{t_1 k(t_1)}{r}\right)^2 \left(\mathrm{K}_{t^2\tilde{g}(h(t),k(t))}(\Sigma_i, \Sigma_j) - \mathrm{II}_t^2(\Sigma_i, \Sigma_j)\right) \\
&> \left(\frac{t_1 k(t_1)}{r}\right)^2 \left(\frac{d}{t^2} - \frac{1}{t^2} - \frac{c}{t^2 \ln t} - \frac{c}{t^2 \ln^3 t}\right).
\end{aligned}$$

There is some $T > 0$ such that the remaining terms $c/\left(t^2 \ln t\right)$ and $c/\left(t^2 \ln^3 t\right)$ bounded above by $(d-1)/t^2$ for all $t > T$. Thus if $t_0 > T$ we have proven the claim. $\square$

In the second full paragraph of [33, p. 160], Perelman suggests "we choose $\varepsilon > 0$ first in such a way that $(\rho/r)^\varepsilon g_1$ still has sectional curvatures > 1." We can see this in our formula (C.34) that $\tilde{r}/r \approx (\rho/r)^\varepsilon$ if δ is chosen very small, and by definition $\tilde{g}(h(t_1), k(t_1)) = (\tilde{r}/r)\, g_1$. The assumption that the sectional curvatures of $(\rho/r)^\varepsilon g_1$ are greater than 1, imply that the proof of Lemma C.1.3 still applies to the path $(a, b) = (h(t), k(t))$. The proof of Corollary C.2.9, is essentially the same observation.

C.3. The curvatures of the neck

By Corollary C.2.8, the metric $g(t_0, \varepsilon, \delta)$ can be chosen to satisfy claims (ii) through (v) of Lemma C.0.1. In this section we consider the validity of (i) for suitable choices of t_0, ε, and δ. That is, we prove that Ric_g is positive definite for suitable choices of parameters. The care that was taken in Section C.2.2 was to ensure that the curvature in spherical directions was large, as recorded in Corollary C.2.9. This one fact will allow us to dominate other sectional curvatures by choosing t_0 even larger.

The first difficulty we face in proving something about Ric_g is that the remaining curvatures are all second order in $h(t)$ and $k(t)$, but $h(t)$ and $k(t)$ are both only once differentiable at $t = 2t_0$. In Section C.3.1, we explain how to apply Perelman's gluing lemma to resolve this issue, and therefore only need to prove positive Ricci curvature for $t \neq 2t_0$. Once this is resolved, in Section 1.3 we see directly that the spherical curvatures dominate $\mathrm{Ric}_g(X, X)$ and $\mathrm{Ric}_g(\Sigma_i, \Sigma_i)$, ensuring their positivity. In Section C.3.3, with a little more care, we show that $\mathrm{Ric}_g(T,T)$ is also positive. As the Ricci tensor is not diagonalized in this frame, this does not prove that it is positive definite. There is exactly one nonzero off-diagonal term: $\mathrm{Ric}_g(T,X)$. Thus positive definiteness of Ric_g reduces to showing that the 2-by-2 sub-matrix spanned by T and X is positive definite. As we will have proven that its trace is positive, we need only show that the determinant is positive. In Section C.3.4, we check that this determinant is dominated by $\mathrm{Ric}_g(X,X)$. We conclude by carefully examining the dependency of choices of t_0, ε, and δ, to ensure that there is a metric g satisfying all of the claims of Lemma C.0.1.

C.3.1. Smoothing the neck

In [33, p. 160], when Perelman defines $h(t)$ and $k(t)$, he recommends that, "b must be smoothed near $t = 2t_0$." Certainly this works, but then Lemma C.2.6 does not follow directly from Lemma C.2.4. Instead of smoothing the metric functions, we claim that if we succeed in constructing a metric that satisfies Lemma 3.1.2 that is C^1 at $t = 2t_0$, then we can smooth the metric while still satisfying Lemma 3.1.2.

Corollary C.3.1. *Assume that there is a $T > 0$ such that, for all $t_0 > T$ the metric $g = g(t_0, \varepsilon, \delta)$ defined on $[t_0, t_1] \times S^n$ satisfies claims (ii) through (v) of Lemma C.0.1 and satisfies claim (i) for all $t \neq 2t_0$. Then for some $t_0 > T$ there exists a smooth metric $\tilde{g}$ on $[t_0, t_1] \times S^n$ that agrees with g outside of an arbitrarily small neighborhood of the set $t = 2t_0$ that satisfies (i) for all t.*

Proof. Note that because g is C^1, the two Riemannian manifolds $([t_0, 2t_0] \times \mathrm{S}^n, g)$ and $([2t_0, t_1] \times \mathrm{S}^n, g)$ have isometric boundaries at $\{2t_0\} \times \mathrm{S}^n$ where the second fundamental forms satisfy $\mathrm{II}_1 + \Phi^* \mathrm{II}_2 = 0$. Apply Lemma 1.2.11 to one of these two manifolds and then apply Theorem 1.2.2 to glue these together to produce a Ricci-positive metric $\tilde{g}$ that agrees with g outside of a neighborhood of the gluing site. □

C.3.2. The Ricci curvature in the spherical directions is large

We have already found suitable lower bound for $\mathrm{K}_g(X, \Sigma_i)$ and $\mathrm{K}_g(\Sigma_i, \Sigma_j)$ in Corollary C.2.9. We must now consider the remaining curvatures of g. The following is a direct corollary of (iii) of Proposition C.3.9 and Lemma C.3.10. Keep in mind that not all of the formulas are defined at $t = 2t_0$.

Corollary C.3.2. *Let g be as in Definition C.2.5. The sectional curvatures in the time direction are as follows.*

$$\mathrm{K}_g(T, \Sigma_i) = -\left(\frac{t_1 k(t_1)}{r}\right)^2 \left(\frac{k''(t)}{k(t)} + \frac{2k'(t)}{tk(t)}\right), \tag{C.38}$$

$$\begin{aligned} \mathrm{K}_g(T, X) = -\left(\frac{t_1 k(t_1)}{r}\right)^2 \Bigg(&\frac{k''(t)}{k(t)} + \frac{2k'(t)}{tk(t)} + \frac{\eta(x)h''(t)}{1 + (h(t)-1)\eta(x)} \\ &+ \frac{2\eta(x)h'(t)}{t(1 + (h(t)-1)\eta(x))} + \frac{k'(t)}{k(t)} \frac{2\eta(x)h'(t)}{1 + (h(t)-1)\eta(x)}\Bigg), \end{aligned} \tag{C.39}$$

$$\mathrm{R}_g(T, \Sigma_i, \Sigma_i, X) = -\left(\frac{t_1 k(t_1)}{r}\right)^2 \left(\frac{\tan(x)\eta(x)h'(t)}{tk(t)(1 - (h(t)-1)\eta(x))^2}\right).$$

All other Riemannian curvatures of the form $\mathrm{R}_g(A, B, B, C)$ vanish.

Investigating Corollary C.3.2 term by term, we see that each is bounded in absolutely value by $(c \ln(2t_0)) \,/\, \left(t^2 \ln^2 t\right)$ as in Lemma C.2.6. Thus the following corollary is immediate.

Corollary C.3.3. *There exists $c_u > 0$ such that the curvature of g in the time directions is bounded in absolute value as follows.*

$$|\operatorname{K}_g(T, \Sigma_i)| < \frac{c_u \ln(2t_0)}{t^2 \ln^2 t},$$
$$|\operatorname{K}_g(T, X)| < \frac{c_u \ln(2t_0)}{t^2 \ln^2 t},$$
$$|R(T, \Sigma_i, \Sigma_i, X)| < \frac{c_u \ln(2t_0)}{t^2 \ln^2 t}.$$

Comparing Corollary C.3.3 with the asymptotics of $\operatorname{K}_g(X, \Sigma_i)$ and $\operatorname{K}_g(\Sigma_i, \Sigma_j)$ in Corollary C.2.9 shows that $\operatorname{Ric}_g(X, X)$ and $\operatorname{Ric}_g(\Sigma_i, \Sigma_i)$ remain as large. In particular, these Ricci curvatures are positive.

Corollary C.3.4. *Let $\varepsilon_0 > 0$ and $\delta_0 > 0$ be as in Corollary C.2.9. There exists $T > 0$ and $c_l > 0$ such that for all $t_0 > T$, $\varepsilon < \varepsilon_0$, $\delta < \delta_0$ we have*

$$\operatorname{Ric}_g(X, X) > \frac{c_l}{t^2},$$
$$\operatorname{Ric}_g(\Sigma_i, \Sigma_i) > \frac{c_l}{t^2}.$$

C.3.3. The Ricci curvature in the time direction is positive

It remains to show that $\operatorname{Ric}_g(T, T)$ is positive and that $\operatorname{Ric}_g(T, X)$ is dominated by $\operatorname{Ric}_g(T, T)$ and $\operatorname{Ric}_g(X, X)$. To achieve both we must bound $\operatorname{Ric}_g(T, T)$ below. Let us begin by considering equation (C.38) and (C.39). For simplicity, let us focus on equation (C.38) and ignore the scaling factor, we want to show that the following is negative

$$\frac{k''(t)}{k(t)} + \frac{2k'(t)}{tk(t)}. \tag{C.40}$$

Note that $k''(t)$ changes from positive to negative at $t = 2t_0$. For small choices of t_0, we can see directly from the definition that this curvature will be negative. It is also not immediately clear from the asymptotics in Lemma C.2.6 that picking t_0 large will resolve this as the two terms in equation (C.38) are both proportional to $(\ln(2t_0)) / \left(t^2 \ln^2 t\right)$. We must therefore return to the definition. In terms of $\Gamma(t)$, equation (C.40) becomes

$$\beta\left(\beta\Gamma^2(t) - \Gamma'(t) - \frac{2}{t}\Gamma(t)\right).$$

By Lemma C.2.4, the $\Gamma^2(t)$ term has smaller asymptotic behavior, so we must show that $\Gamma'(t) + (2/t)\,\Gamma(t)$ is positive for large enough t_0.

Lemma C.3.5. *There exists a $T > 0$ and $c_n > 0$, such that for all $t_0 > T$ we have*

$$\Gamma'(t) + \frac{2}{t}\Gamma(t) > \frac{c_n \ln(2t_0)}{t^2 \ln^2 t}.$$

Proof. For $t < 2t_0$, we have

$$\begin{aligned}\Gamma'(t) + \frac{2}{t}\Gamma(t) &= \frac{1}{2t_0^2 \ln(2t_0)} + \frac{1}{t_0^2 \ln(2t_0)} - \frac{1}{t t_0 \ln(2t_0)} \\ &> \frac{1}{2t_0^2 \ln(2t_0)}.\end{aligned}$$

Because $1/\left(t^2 \ln t\right)$ is decreasing, we have $1/\left(t_0^2 \ln t_0\right) \geq 1/\left(t^2 \ln t\right)$. This proves the claim for $t < 2t_0$.

For $t > 2t_0$, we have

$$\begin{aligned}\Gamma'(t) + \frac{2}{t}\Gamma(t) &= -\frac{\ln(2t_0)}{t^2\ln^2 t} - \frac{2\ln(2t_0)}{t^2\ln^3 t} + \frac{2\ln(2t_0)}{t^2\ln^2 t}\\ &> \frac{\ln(2t_0)}{t^2\ln^2 t} - \frac{2\ln(2t_0)}{t^2\ln^3 t}.\end{aligned}$$

Clearly if t_0 is large enough the negative term may be ignored, thus there exists a $T > 0$ for which the claim is true for $t_0 > T$. □

It follows that $K(T, \Sigma_i) > 0$ for t_0 large enough (so that $\Gamma^2(t)$ is sufficiently small). While the first terms of $K(T, X)$ in equation (C.39) agree with $K(T, \Sigma_i)$, the following two terms that appear have identical asymptotics with opposite sign. One must therefore consider $\mathrm{Ric}(T, T)$ in its entirety, and show that terms from $K(T, \Sigma_i)$ dominate those from $K(T, X)$. Rewriting $h'(t)/h(t)$ in terms of $k'(t)/k(t)$, one sees that the coefficients that need to be compared are determined by n and α. The following lemma compares the exact coefficients of the terms that dominate $\mathrm{Ric}(T, T)$.

Lemma C.3.6. *Let α be as in equation (C.28), where ρ and r satisfy the hypotheses of Lemma C.0.1. There exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that α satisfies the following for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$.*

$$\frac{\alpha h(t)}{1 + (h(t) - 1)\eta(x)} - n < 0. \tag{C.41}$$

Proof. We begin by claiming that $\alpha < n$ for ε and δ small enough. As observed in the proof of Lemma C.1.3, by Corollary C.1.4 and the assumption that $\mathrm{K}_{g_1} > 0$ that $1/(r^2 a_1^2) > 1$. It follows then that $\ln a_1 < -\ln r$. By assumption, $r^{n-1} < \rho^n$. Taking logarithms and solving for $-\ln r$ yields $-\ln r < n(\ln \rho - \ln r)$. Combining these observations yields $\ln a_1 < n(\ln \rho - \ln r)$. Plugging this into the definition of α in equation (C.28) yields

$$\alpha < \frac{1+\delta}{1-\varepsilon} n.$$

Thus $\alpha < n$ if $\varepsilon = \delta = 0$. As this is an open condition, there exists $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that $\alpha < n$ for all $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$.

Now we turn to prove inequality (C.41). If $\eta(x) < 0$, then both terms are negative and the claim is obvious. If $0 \leq \eta(x) \leq 1$, then clearly $\eta(x)h(t) + (1 - \eta(x)) \geq \eta(x)h(t)$. Thus the left-hand side of (C.41) is bounded above by $\alpha - n$, which is negative if $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. □

The first paragraph in the proof of Lemma C.3.6 appears as the penultimate paragraph of [33, p. 160], where Perelman has already assumed that ε is fixed as in the remark after Corollary C.2.9 so that $(\rho/r)^{\varepsilon} g_1$ still has sectional curvatures greater than 1.

We are now ready to prove that $\mathrm{Ric}(T,T)$ is positive for large enough t_0. The proof amounts to showing that the coefficient of $-\beta\left(\Gamma'(t) + (2/t)\,\Gamma(t)\right)$ in $\mathrm{Ric}(T,T)$ is the left-hand side of (C.41).

Lemma C.3.7. *Let $\varepsilon_0 > 0$ and $\delta_0 > 0$ be as in Lemma C.3.6. There exists $T > 2$ such that $\mathrm{Ric}(T,T)$ is positive for all $t_0 > T$, $\varepsilon < \varepsilon_0$, and $\delta < \delta_0$. Moreover, there exists $c_T > 0$ such that $\mathrm{Ric}(T,T)$ satisfies the following.*

$$\mathrm{Ric}(T,T) > \frac{c_T \ln(2t_0)}{t^2 \ln^2 t}.$$

Proof. We can compute $\mathrm{Ric}(T,T)$ by adding together equations (C.39) and (C.38). We then factor this expression for $\mathrm{Ric}(T,T)$ so that the leading term is transparently $-\beta\left(\Gamma'(t) + (2/t)\,\Gamma(t)\right)$, where we notice that

$$\left(\frac{k'(t)}{k(t)}\right)' + \frac{2}{t}\frac{k'(t)}{k(t)} = -\beta\left(\Gamma'(t) + \frac{2}{t}\Gamma(t)\right).$$

We then combine Lemmas C.1.5, C.2.6, C.3.5, and C.3.6 bound $\operatorname{Ric}(T,T)$ below as follows.

$$\begin{aligned}
\operatorname{Ric}(T,T) &= (n-1)K(T,\Sigma_i) + K(T,X),\\
&= \left(\frac{\alpha\eta(x)h(t)}{1+(h(t)-1)\eta(x)} - n\right)\left(\left(\frac{k'(t)}{k(t)}\right)' + \frac{2}{t}\frac{k'(t)}{k(t)}\right)\\
&\qquad - n\left(\frac{k'(t)}{k(t)}\right)^2 - \frac{\eta(x)h(t)}{1+(h(t)-1)\eta(x)}\left(2\frac{k'(t)}{k(t)}\frac{h'(t)}{h(t)} + \left(\frac{h'(t)}{h(t)}\right)^2\right),\\
&> \frac{c\ln(2t_0)}{t^2\ln^2 t} - \frac{c\ln^2(2t_0)}{t^2\ln^4 t}.
\end{aligned}$$

To apply Lemma C.3.6, we must assume that $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$. It is clear that there exists a $T > 2$ for which we may disregard the negative term for all $t_0 > T$. □

C.3.4. The existence of the neck

With Corollary C.3.4, this shows that the diagonals of the Ricci tensor of g are positive. As mentioned in the introduction of Section C.3, in order to prove positive definiteness we must show that the determinant of the 2-by-2 submatrix spanned by T and X is positive, i.e. that $\operatorname{Ric}_g(T,T)\operatorname{Ric}_g(X,X) - \operatorname{Ric}_g^2(T,X)$ is positive. The following claims that this is true for large enough t_0. It is a direct corollary of Corollary C.3.3, Corollary C.3.4, and Lemma C.3.7.

Corollary C.3.8. *There exists a $T > 0$ such that for all $t_0 > T$ we have*

$$\mathrm{Ric}_g(T,T)\,\mathrm{Ric}_g(X,X) > \mathrm{Ric}_g^2(T,X).$$

Proof. Taking the lower bounds for Corollary C.3.4 and Lemma C.3.7 on the left-hand side, and the upper bounds from Corollary C.3.3 for the right-hand side. The claim reduces to the following inequality

$$\frac{c\ln(2t_0)}{t^4\ln^2 t} > \frac{\ln^2(2t_0)}{t^4\ln^4 t}.$$

And clearly, there is a $T > 2$ for which this is true for all $t_0 > T$. □

We have shown therefore that for large t_0 and small ε and δ that the Ricci tensor of g is positive definite. Combining this with Corollaries C.2.8 and C.3.1 we are ready to prove Lemma C.0.1.

Proof of Lemma C.0.1. By Corollary C.2.8, g will satisfy claims (ii), (iii), and (iv) of Lemma C.0.1 for any choice of t_0, ε, and δ.

Consider next the curvature of g. By Corollary C.3.1, it suffices to show that there is a $T > 0$ and specific choices of ε and δ for which $g(t_0, \varepsilon, \delta)$ satisfies (i) for all $t_0 > T$. So we may disregard the fact that g is not smooth at $t = 2t_0$.

The conclusions of Corollary C.3.4, Lemma C.3.7, and Corollary C.3.8 combined tell us that Ric_g is positive definite if t_0 is large. The hypotheses of these claims require that ε and δ be chosen small in the sense of Corollary C.2.9 so that the sectional curvatures in the spherical directions are bigger than c/t^2 and in the sense of Lemma C.3.6 so that $\mathrm{Ric}_g(T,T)$ is positive. Fix ε small enough for the hypotheses of both Corollary C.2.9 and Lemma C.3.6.

Finally, by Corollary C.2.8, g will satisfy (v) if δ is chosen small relative to ε. Fix δ small enough to satisfy Corollary C.2.8 as well as Corollary C.2.9 and Lemma C.3.6. Thus there is a $T > 0$ such that Ric_g is positive definite for all $t_0 > T$. We conclude that there exists a t_0 so that $g(t_0, \varepsilon, \delta)$ will, after smoothing, satisfy all of the claims of Lemma C.0.1. □

C.3.4.1. Describing $g_{\mathbf{neck}}(\rho)$

Proof of Proposition 3.1.2. We have not gone over the construction of $g_{\text{ambient}}(\rho)$ of [33, Section 3], but it directly established that the boundary of $(\mathrm{S}^n \setminus \bigsqcup_k \mathrm{D}^n, g_{\text{docking}}(\rho))$ takes the form $g_1 = A_1^2(x)dx^2 + B_1^2 \cos^2 x ds_{n-1}^2$ with $\mathrm{K}_{g_1} \geq 1$. This allows us to apply Lemma C.0.1 to g_1 to produce the metric $g_{\text{neck}}(\rho)$. Conditions (1)-(5) of Proposition 3.1.2 correspond directly to Lemma C.0.1. The last condition (6), follows from the fact that $B(t,x) = t^2k^2(t)\cos^2 x$. We have already shown that $B_{tt}(t,x) < 0$ in Lemma C.3.5, and similarly that $B_t(T,x) > 0$ in Lemma C.2.6. □

C.3.5. Scaled One-parameter Families of Warped Products

In this section we will use g to denote a metric on $[t_0, t_1] \times [x_0, x_1] \times S^n$ of the form $g = dt^2 + g_t$, and g_t is a one parameter family of warped product metrics on $[x_0, x_1] \times S^n$ of the form $g_t = A(t,x)dx^2 + B(t,x)ds_n^2$. Let T, X, and Σ_i denote a local orthonormal frame for g tangent to $[t_0, t_1]$, $[x_0, x_1]$, and S^n respectively. The purpose of this section is to prove the following for such metrics g.

Proposition C.3.9. *The curvatures of g are as follows.*

(i) *The sectional curvature of $g(t)$ on $\{t\} \times [x_0, x_1] \times S^n$ is as follows.*

$$\mathrm{K}_{g_t}(X, \Sigma_i) = \frac{A_x B_x}{A^3 B} - \frac{B_{xx}}{BA^2}, \text{ and } \mathrm{K}_{g_t}(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{A^2 B^2}.$$

Moreover $\mathrm{Ric}_{g(t)}$ *is diagonalized in this frame.*

(ii) *The second fundamental form II_t of the submanifold $\{t\} \times [x_0, x_1] \times S^n$ inside of $[t_0, t_1] \times [x_0, x_1] \times S^n$ with respect to the normal vector T is as follows.*

$$\mathrm{II}_t(X, X) = \frac{A_t}{A}, \quad \mathrm{II}_t(X, \Sigma_i) = 0, \text{ and } \mathrm{II}_t(\Sigma_i, \Sigma_j) = \frac{B_t}{B}\delta_{ij}.$$

(iii) *Combining (1) and (2), the sectional curvatures of g not involving T are*

$$\mathrm{K}_g(X, \Sigma_j) = -\frac{A_t B_t}{AB} + \frac{A_x B_x}{A^3 B} - \frac{B_{xx}}{BA^2}, \text{ and } \mathrm{K}_g(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{A^2 B^2} - \frac{B_t^2}{B^2}.$$

The remaining sectional curvatures are

$$\mathrm{K}_g(T, X) = -\frac{A_{tt}}{A}, \text{ and } \mathrm{K}_g(T, \Sigma_i) = -\frac{B_{tt}}{B}.$$

And the Ricci tensor has one off-diagonal term

$$\mathrm{Ric}_g(X, T) = n\mathrm{R}_g(X, \Sigma_i, \Sigma_i, T) = n\left(\frac{A_t B_x}{BA^2} - \frac{B_{xt}}{AB}\right).$$

The proof of Proposition C.3.9 is presented in the following sections. In Section C.3.5.1 we compute the second fundamental forms relevant to the computation, in particular proving (ii). Next in Section C.3.5.2, we compute the instrinsic sectional curvatures of the time slice $\{t\} \times [x_0, x_1] \times S^n$ proving part (i).

In Section C.3.5.3 the sectional curvatures of g are computed proving most of (iii). And finally concluding in Section C.3.5.4 by proving the part of (iii) concerning the off-diagonals of the Ricci tensor.

As our main application is the metric of Definition C.2.5, which is of the form $\kappa^2 g$, we quote the following is general fact about scaling metrics.

Lemma C.3.10. [53, Theorem 1.159] *Let A', B', C', and D' be the orthonormal vector fields of $\kappa^2 g$ corresponding to the orthonormal vector fields A, B, C, and D of g. Let II' and II be second fundamental forms of the same embedded hypersurface with respect to $\kappa^2 g$ and g respectively.*

$$\mathrm{R}_{\kappa^2 g}(A', B', C', D') = \frac{1}{\kappa^2}\,\mathrm{R}_g(A, B, C, D) \text{ and } \mathrm{II}'(A', B') = \frac{1}{\kappa}\,\mathrm{II}(A, B).$$

C.3.5.1. Extrinsic Curvature

In this short section we record the second fundamental forms relevant to our curvature computations. Obviously we will consider the second fundamental form of $\{t\} \times [x_0, x_1] \times S^n$ inside of $[t_0, t_1] \times [x_0, x_1] \times S^n$. Notice that g_t is a warped product metric on $[x_0, x_1] \times S^n$, so it will also be necessary to consider the second fundamental form of $\{t\} \times \{x\} \times S^n$ inside of $\{t\} \times [x_0, x_1] \times S^n$ with respect to g_t.

Lemma C.3.11. *Let II_t be the second fundamental form of $\{t\} \times [x_0, x_1] \times S^n$ embedded in $[t_0, t_1] \times [x_0, x_1] \times S^n$ with respect to g and the unit normal T. Then*

$$\mathrm{II}_t = \frac{A_t}{A}(A^2 dx^2) + \frac{B_t}{B}(B^2 ds_n^2). \tag{C.42}$$

Proof. Because the intervals $[t_0, t_1] \times \{x\} \times \{p\}$ are geodesics with respect to g, by [6, Proposition 3.2.1] we have that the second fundamental form of $\{t\}\times[x_0, x_1]\times S^n$

is

$$\mathrm{II}_t = \frac{1}{2}\partial_t g = \frac{A_t}{A}(A^2 dx^2) + \frac{B_t}{B}(B^2 ds_n^2).$$

As dx^2 and ds_n^2 are invariant with respect to T. $\square$

Lemma C.3.12. *Let* II_x *be the second fundamental form of* $\{t\}\times\{x\}\times S^n$ *embedded in* $\{t\}\times[x_0,x_1]\times S^n$ *with respect to* g_t *and the unit normal* X*. Then*

$$\mathrm{II}_x = \frac{A_x}{A^2}(A^2 dx^2) + \frac{B_x}{AB}(B^2 ds_n^2) \tag{C.43}$$

Proof. Note that g restricted to $\{t\}\times[x_0,x_1]\times S^n$ is $A^2(t,x)dx^2 + B^2(t,x)ds_n^2$. Because $X = \partial_x/A$ is a unit vector, applying [6, Proposition 3.2.1] we have

$$\mathrm{II}_x = \frac{1}{2}\mathcal{L}_X g_t = \frac{A_x}{A^2}(A^2 dx^2) + \frac{B_x}{AB} ds_n^2.$$

As dx^2 and ds_n^2 are invariant with respect to X. $\square$

C.3.5.2. The intrinsic curvatures of a time slice

Next we compute the curvatures in the spherical directions, X and Σ. To do this, we consider the restricted metric $g(t)$ on the submanifolds $\{t\}\times[x_0,x_1]\times S^n$, and compute its intrinsic curvature.

Lemma C.3.13. *The curvatures* g_t *involving* X *are as follows.*

$$\mathrm{K}_{g_t}(\Sigma, X_i) = \frac{A_x B_x}{A^3 B} - \frac{B_{xx}}{A^2 B}$$

Proof. By [6, Proposition 3.2.11] we have

$$\mathrm{K}_{g_t}(X, W) = \mathrm{II}_x^2(W,W) - (\mathcal{L}_X \mathrm{II}_x)(W,W). \tag{C.44}$$

Where II_x is as in (C.43). We compute the derivative of II_x as follows

$$\begin{aligned}\mathcal{L}_X \mathrm{II}_x &= \left(\frac{A_{xx}A^2 - 2A_x^2 A}{A^5} + \frac{A_x}{A^4}\right)(A^2 dx^2) \\ &\qquad + \left(\frac{B_{xx}AB - B_x^2 A - B_x A_x B}{A^3 B^2} + \frac{2B_x^2}{A^2B^2}\right)(B^2 ds_n^2) \\ &= \frac{A_{xx}}{A^3}(A^2 dx^2) + \left(\frac{B_{xx}}{A^2 B} + \frac{B_x^2}{A^2B^2} - \frac{A_x B_x}{A^3 B}\right)(B^2 ds_n^2). \end{aligned} \tag{C.45}$$

Taking the square of equation (C.43) and substituting it with (C.45) into (C.44) yields the following

$$\begin{aligned}\mathrm{K}_{g_t}(X,-) &= \frac{A_x^2}{A^4}(A^2dx^2) + \frac{B_x^2}{A^2B^2}(B^2 ds_n^2) - \frac{A_{xx}}{A^3}(A^2dx^2) \\ &\qquad - \left(\frac{B_{xx}}{A^2B} + \frac{B_x^2}{A^2B^2} - \frac{A_xB_x}{A^3B}\right)(B^2ds_n^2) \\ &= \left(\frac{A_x^2}{A^4} - \frac{A_{xx}}{A^3}\right)(A^2dx^2) + \left(-\frac{B_{xx}}{A^2B} + \frac{A_xB_x}{A^3B}\right)(B^2ds_n^2)\end{aligned}$$

The claim follows. □

Finally we must compute those curvatures of g_t in the spherical directions.

Lemma C.3.14. *The curvatures of g_t not involving X are as follows.*

$$\mathrm{K}_{g_t}(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{A^2B^2}.$$

Proof. Let g_x denote the metric g_t restricted to $\{t\} \times \{x\} \times S^n$ inside of $\{t\} \times [x_0, x_1] \times S^n$. Notice that $g_x = B^2(t,x)ds_n^2$ is round with radius B, so $\mathrm{K}_{g_x}(\Sigma_i, \Sigma_j) = 1/B^2$. By Gauss' formula and equation (C.43) we have

$$\mathrm{K}_{g_t}(\Sigma_i,\Sigma_j) = \mathrm{K}_{g_x}(\Sigma_i,\Sigma_j) - \mathrm{II}_x(\Sigma_i,\Sigma_i)\,\mathrm{II}_x(\Sigma_j,\Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{A^2B^2}.$$

□

C.3.5.3. The sectional curvatures

We begin by computing the sectional curvatures involving T.

Lemma C.3.15. *The sectional curvatures of g involving T are as follows.*

$$\mathrm{K}_g(T,X) = -\frac{A_{tt}}{A}, \text{ and } \mathrm{K}_g(T,\Sigma) = -\frac{B_{tt}}{B}.$$

Proof. Because the intervals $[t_0,t_1]\times\{x\}\times\{p\}$ are geodesics of g, the Codazzi-Mainardi equations reduce to

$$K(T,W) = \mathrm{II}_t^2(W,W) - (\partial_t \mathrm{II}_t)(W,W), \tag{C.46}$$

where W is any vector normal to T. Computing the derivative of II_t we have

$$\begin{aligned}
\partial_t \mathrm{II}_t &= \frac{A_{tt}A - A_t^2}{A^2}(A^2dx^2) + \frac{2A_t^2}{A^2}(A^2dx^2) + \frac{B_{tt}B - B_t^2}{B^2}(B^2ds_{n-1}^2) + \frac{2B_t^2}{B^2}(B^2ds_{n-1}^2),\\
&= \frac{A_{tt}A + A_t^2}{A^2}(A^2dx^2) + \frac{B_{tt}B + B_t^2}{B^2}(B^2ds_{n-1}^2).
\end{aligned} \tag{C.47}$$

Taking the square of equation (C.42) and substituting that with (C.47) into (C.46) yields the following.

$$\begin{aligned}
K(T,-) &= \frac{A_t^2}{A^2}(A^2dx^2) + \frac{B_t^2}{B^2}(B^2ds_{n-1}^2) - \frac{A_{tt}A + A_t^2}{A^2}(A^2dx^2) - \frac{B_{tt}B + B_t^2}{B^2}(B^2ds_{n-1}^2)\\
&= -\frac{A_{tt}}{A}(A^2dx^2) - \frac{B_{tt}}{B}(B^2ds_{n-1}^2).
\end{aligned}$$

The claim follows. □

Having determined K_{g_t} in Lemmas C.3.13 and C.3.14 and II_t in equation (C.42), the following is a direct consequence of Gauss' formula.

Corollary C.3.16. *The curvatures of g not involving T are as follows.*

$$\mathrm{K}_g(X, \Sigma_i) = \frac{B_x A_x}{A^3 B} - \frac{B_{xx}}{A^2 B} - \frac{B_t A_t}{AB}, \text{ and } \mathrm{K}_g(\Sigma_i, \Sigma_j) = \frac{1}{B^2} - \frac{B_x^2}{B^2 A^2} - \frac{B_t^2}{B^2}.$$

C.3.5.4. Off-diagonals of the Ricci tensor

In this section we consider those curvatures of the form $\mathrm{R}_g(A, B, B, C)$. Choose local coordinates σ_i for S^n so that $S_i := \partial_{\sigma_i}$ is an orthonormal basis of (S^n, ds_n^2) at one point. We may assume that $S_i/B = \Sigma_i$ at this point. We will use S_i along with ∂_x and ∂_t as a local frame. Because ∂_x and ∂_t are global and g is homogeneous in the S^n factor, it suffices to perform computations in these coordinates.

Lemma C.3.17. *In this frame, the off-diagonals of* Ric_g *are zero except for* $\mathrm{Ric}_g(X, T) = n\,\mathrm{R}_g(X, \Sigma_i, \Sigma_i, T)$*, where*

$$\mathrm{R}_g(X, \Sigma_i, \Sigma_i, T) = \frac{B_{xt}}{AB} - \frac{A_t B_x}{A^2 B}.$$

Proof. As $\mathrm{Ric}(A, B) = \sum_{C_i} R(A, C_i, C_i, B)$, the off-diagonal terms of Ric_g are determined by $R(T, \Sigma_i, \Sigma_i, X)$, $\mathrm{R}_g(T, \Sigma_i, \Sigma_i, \Sigma_j)$, $\mathrm{R}_g(X, \Sigma_i, \Sigma_i, \Sigma_j)$, and $R(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k)$.

First, consider $\mathrm{R}_g(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k)$. By applying Gauss' formula twice using II_x followed by II_t, one sees that the terms involving II_t and II_x vanish, thus reducing to $\mathrm{R}_g(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k) = \mathrm{R}_{g_x}(\Sigma_i, \Sigma_j, \Sigma_j, \Sigma_k)$. But g_x is round, and it is known that this curvature vanishes for the round metric in these coordinates.

Consider next $\mathrm{R}_g(T, \Sigma_i, \Sigma_i, \Sigma_j)$ and $\mathrm{R}_g(X, \Sigma_i, \Sigma_i, \Sigma_j)$. Let II'_x and g'_x be the second fundamental form and metric of $[t_0, t_1] \times \{x\} \times S^n$ with respect to g. By applying Gauss' formula with respect to II_t and II'_x, one can check that $\mathrm{R}_g(T, \Sigma_i, \Sigma_i, \Sigma_j) = \mathrm{R}_{g'_x}(T, \Sigma_i, \Sigma_i, \Sigma_j)$ and $\mathrm{R}_g(X, \Sigma_i, \Sigma_i, \Sigma_j) = \mathrm{R}_{g_t}(X, \Sigma_i, \Sigma_i, \Sigma_j)$. Both the metric g_t and g'_x are warped product metrics. This curvature is known to vanish in these coordinates [6, Section 4.2.3].

We can compute $\mathrm{R}_g(\partial_x, S_i, S_i, T)$ in these coordinates as follows.

$$\begin{aligned}\mathrm{R}_g(\partial_x, S_i, S_i, T) = \frac{1}{2}\left(\partial_{\sigma_i}\partial_{\sigma_i} g_{tx} + \partial_t\partial_x g_{\sigma_i\sigma_i} - \partial_{\sigma_i}\partial_x g_{t\sigma_i} - \partial_t\partial_{\sigma_i} g_{\sigma_i x}\right) \\ + g_{ab}\left(\Gamma^a_{\sigma_i\sigma_i}\Gamma^b_{tx} - \Gamma^a_{\sigma_i x}\Gamma^b_{t\sigma_i}\right).\end{aligned} \tag{C.48}$$

First consider the second derivatives of the metric. Of those metric functions being considered, only $g_{\sigma_i\sigma_i}$ is nonzero. And the desired second derivative is as follows.

$$\partial_t\partial_x g_{\sigma_i\sigma_i} = \partial_t\partial_x(B^2) = 2\partial_t(B_x B) = 2(B_{xt}B + B_x B_t). \tag{C.49}$$

Second, considering the metric functions being summed against in the Christoffel symbol term, the function is nonzero only if $i = j$. Consider the second Christoffel identity.

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}\left(\partial_a g_{bd} - \partial_d g_{ab} + \partial_b g_{ad}\right).$$

If all three indices a, b and c are distinct, then all three metric functions will vanish. Thus, in equation (C.48) the only indices that could possibly give a nonzero summand are $a = b = t$, $a = b = x$, and $a = b = \sigma_i$. Within these, only 9 need to be computed. Of those 9, only 5 are nonzero. These are as follows.

$$\Gamma^t_{\sigma_i\sigma_i} = -B_t B, \quad \Gamma^x_{\sigma_i\sigma_i} = \frac{-B_x B}{A^2}, \quad \Gamma^x_{tx} = \frac{A_t}{A}, \quad \Gamma^{\sigma_i}_{\sigma_i x} = \frac{B_x}{B}, \text{ and } \Gamma^{\sigma_i}_{t\sigma_i} = \frac{B_t}{B}.$$

Finally, when written down, only the cases $a = b = x$ and $a = b = \sigma_i$ are nonzero. These are as follows.

$$g_{xx}\left(\Gamma^{x}_{\sigma_i\sigma_i}\Gamma^{x}_{tx} - 0\right) = A^2\left(\frac{-B_x B}{A^2}\frac{A_t}{A}\right) = \frac{-A_t B_x}{A} \tag{C.50}$$

$$g_{\sigma_i\sigma_i}\left(0 - \Gamma^{\sigma_i}_{\sigma_i x}\Gamma^{\sigma_i}_{t\sigma_i}\right) = -B^2\left(\frac{B_x}{B}\frac{B_t}{B}\right) = -B_t B_x. \tag{C.51}$$

Thus the only nonzero terms of (C.48) are those computed in equations (C.49), (C.50), and (C.51). Combining these yields the following.

$$\mathrm{R}_g(\partial_x, S_i, S_i, T) = \frac{1}{2}(2(B_{xt}B + B_x B_t)) + \frac{-A_t B_x B}{A} - B_t B_x = B_{xt}B - \frac{A_t B_x B}{A}.$$

Finally, using the fact that R_g is a tensor, we see that

$$\mathrm{R}_g(X, \Sigma_i, \Sigma_i, T) = \mathrm{R}_g\left(\frac{\partial_x}{A}, \frac{S_i}{B}, \frac{S_i}{B}, T\right) = \frac{1}{AB^2}\,\mathrm{R}_g(\partial_x, S_i, S_i, T) = \frac{B_{xt}}{AB} - \frac{A_t B_x}{A^2 B}.$$

□

REFERENCES CITED

[1] Jeff Cheeger. Some examples of manifolds of nonnegative curvature. *J. Differential Geometry*, 8:623–628, 1973. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214431964`.

[2] John C. Nash. Positive Ricci curvature on fibre bundles. *J. Differential Geom.*, 14(2):241–254, 1979. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214434973`.

[3] David Wraith. Surgery on Ricci positive manifolds. *J. Reine Angew. Math.*, 501: 99–113, 1998. ISSN 0075-4102. doi: 10.1515/crll.1998.082. URL `https://doi.org/10.1515/crll.1998.082`.

[4] John Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)*, 63:20–63, 1956. ISSN 0003-486X. doi: 10.2307/1969989. URL `https://doi.org/10.2307/1969989`.

[5] John M. Lee. *Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. ISBN 0-387-98271-X. doi: 10.1007/b98852. URL `https://doi.org/10.1007/b98852`. An introduction to curvature.

[6] Peter Petersen. *Riemannian geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer, Cham, third edition, 2016. ISBN 978-3-319-26652-7; 978-3-319-26654-1. doi: 10.1007/978-3-319-26654-1. URL `https://doi.org/10.1007/978-3-319-26654-1`.

[7] Heinz Hopf. Zum Clifford-Kleinschen Raumproblem. *Math. Ann.*, 95(1):313–339, 1926. ISSN 0025-5831. doi: 10.1007/BF01206614. URL `https://doi.org/10.1007/BF01206614`.

[8] Wilhelm Killing. Ueber die Clifford-Klein'schen Raumformen. *Math. Ann.*, 39(2): 257–278, 1891. ISSN 0025-5831. doi: 10.1007/BF01206655. URL `https://doi.org/10.1007/BF01206655`.

[9] Simon Brendle and Richard Schoen. Manifolds with 1/4-pinched curvature are space forms. *J. Amer. Math. Soc.*, 22(1):287–307, 2009. ISSN 0894-0347. doi: 10.1090/S0894-0347-08-00613-9. URL `https://doi.org/10.1090/S0894-0347-08-00613-9`.

[10] Jeff Cheeger and David G. Ebin. *Comparison theorems in Riemannian geometry*. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, 1975. North-Holland Mathematical Library, Vol. 9.

[11] Michael Gromov. Curvature, diameter and Betti numbers. *Comment. Math. Helv.*, 56(2):179–195, 1981. ISSN 0010-2571. doi: 10.1007/BF02566208. URL `https://doi.org/10.1007/BF02566208`.

[12] Wolfgang Ziller. Riemannian manifolds with positive sectional curvature. In *Geometry of manifolds with non-negative sectional curvature*, volume 2110 of *Lecture Notes in Math.*, pages 1–19. Springer, Cham, 2014.

[13] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989. ISBN 0-691-08542-0.

[14] S. B. Myers. Riemannian manifolds with positive mean curvature. *Duke Math. J.*, 8:401–404, 1941. ISSN 0012-7094. URL `http://projecteuclid.org/euclid.dmj/1077492655`.

[15] Jeff Cheeger and Detlef Gromoll. The splitting theorem for manifolds of nonnegative Ricci curvature. *J. Differential Geometry*, 6:119–128, 1971/72. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214430220`.

[16] Stefano Pigola. Ricci curvature and fundamental group of complete manifolds, 2010.

[17] Ji-Ping Sha and DaGang Yang. Examples of manifolds of positive Ricci curvature. *J. Differential Geom.*, 29(1):95–103, 1989. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214442635`.

[18] Ji-Ping Sha and DaGang Yang. Positive Ricci curvature on the connected sums of $S^n \times S^m$. *J. Differential Geom.*, 33(1):127–137, 1991. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214446032`.

[19] Shing Tung Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3): 339–411, 1978. ISSN 0010-3640. doi: 10.1002/cpa.3160310304. URL `https://doi.org/10.1002/cpa.3160310304`.

[20] Charles P. Boyer, Krzysztof Galicki, and Michael Nakamaye. On positive Sasakian geometry. *Geom. Dedicata*, 101:93–102, 2003. ISSN 0046-5755. doi: 10.1023/A:1026363529906. URL `https://doi.org/10.1023/A:1026363529906`.

[21] Charles P. Boyer, Krzysztof Galicki, and Michael Nakamaye. Sasakian geometry, homotopy spheres and positive Ricci curvature. *Topology*, 42(5): 981–1002, 2003. ISSN 0040-9383. doi: 10.1016/S0040-9383(02)00027-7. URL `https://doi.org/10.1016/S0040-9383(02)00027-7`.

[22] Nigel Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974. ISSN 0001-8708. doi: 10.1016/0001-8708(74)90021-8. URL `https://doi.org/10.1016/0001-8708(74)90021-8`.

[23] Mikhael Gromov and H. Blaine Lawson, Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 111(3): 423–434, 1980. ISSN 0003-486X. doi: 10.2307/1971103. URL `https://doi.org/10.2307/1971103`.

[24] Stephan Stolz. Simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 136(3):511–540, 1992. ISSN 0003-486X. doi: 10.2307/2946598. URL `https://doi.org/10.2307/2946598`.

[25] R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979. ISSN 0025-2611. doi: 10.1007/BF01647970. URL `https://doi.org/10.1007/BF01647970`.

[26] John Milnor. *Lectures on the h-cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

[27] Stephan Stolz. A conjecture concerning positive Ricci curvature and the Witten genus. *Math. Ann.*, 304(4):785–800, 1996. ISSN 0025-5831. doi: 10.1007/BF01446319. URL `https://doi.org/10.1007/BF01446319`.

[28] Stephan Stolz. Manifolds of positive scalar curvature. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 661–709. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[29] Anand Dessai. Some geometric properties of the Witten genus. In *Alpine perspectives on algebraic topology*, volume 504 of *Contemp. Math.*, pages 99–115. Amer. Math. Soc., Providence, RI, 2009. doi: 10.1090/conm/504/09877. URL `https://doi.org/10.1090/conm/504/09877`.

[30] David Wraith. Exotic spheres with positive Ricci curvature. *J. Differential Geom.*, 45(3):638–649, 1997. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214459846`.

[31] David J. Wraith. On the moduli space of positive Ricci curvature metrics on homotopy spheres. *Geom. Topol.*, 15(4):1983–2015, 2011. ISSN 1465-3060. doi: 10.2140/gt.2011.15.1983. URL `https://doi.org/10.2140/gt.2011.15.1983`.

[32] Diarmuid Crowley and David J. Wraith. Positive Ricci curvature on highly connected manifolds. *J. Differential Geom.*, 106(2):187–243, 2017. ISSN 0022-040X. doi: 10.4310/jdg/1497405625. URL `https://doi.org/10.4310/jdg/1497405625`.

[33] G. Perelman. Construction of manifolds of positive Ricci curvature with big volume and large Betti numbers. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 157–163. Cambridge Univ. Press, Cambridge, 1997.

[34] G. Perelman. A complete Riemannian manifold of positive Ricci curvature with Euclidean volume growth and nonunique asymptotic cone. In *Comparison geometry (Berkeley, CA, 1993–94)*, volume 30 of *Math. Sci. Res. Inst. Publ.*, pages 165–166. Cambridge Univ. Press, Cambridge, 1997.

[35] David J. Wraith. New connected sums with positive Ricci curvature. *Ann. Global Anal. Geom.*, 32(4):343–360, 2007. ISSN 0232-704X. doi: 10.1007/s10455-007-9066-8.

[36] Paul Ehrlich. Metric deformations of curvature. I. Local convex deformations. *Geometriae Dedicata*, 5(1):1–23, 1976. ISSN 0046-5755. doi: 10.1007/BF00148134. URL `https://doi.org/10.1007/BF00148134`.

[37] Antonio Aché, Davi Maximo, and Haotian Wu. Metrics with nonnegative Ricci curvature on convex three-manifolds. *Geom. Topol.*, 20(5):2905–2922, 2016. ISSN 1465-3060. doi: 10.2140/gt.2016.20.2905. URL `https://doi.org/10.2140/gt.2016.20.2905`.

[38] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1214436922`.

[39] Hui-Hsien Wang. *Boundary convexity of manifolds with nonnegative Ricci curvature*. ProQuest LLC, Ann Arbor, MI, 1997. ISBN 978-0591-55380-2. Thesis (Ph.D.)–Indiana University.

[40] Boris Botvinnik, Mark G. Walsh, and David J. Wraith. Homotopy groups of the observer moduli space of Ricci positive metrics. *arXiv e-prints*, art. arXiv:1712.05937, Dec 2017.

[41] Bradley Lewis Burdick. Ricci-positive metrics on connected sums of projective spaces. *Differential Geom. Appl.*, 62:212–233, 2019. ISSN 0926-2245. doi: 10.1016/j.difgeo.2018.11.005. URL `https://doi.org/10.1016/j.difgeo.2018.11.005`.

[42] Joseph A. Wolf. *Spaces of constant curvature*. AMS Chelsea Publishing, Providence, RI, sixth edition, 2011. ISBN 978-0-8218-5282-8.

[43] Ailana Fraser and Martin Man-chun Li. Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary. *J. Differential Geom.*, 96(2):183–200, 2014. ISSN 0022-040X. URL `http://projecteuclid.org/euclid.jdg/1393424916`.

[44] Hui-Hsien Wang. Boundary convexity on manifolds with nonnegative Ricci curvature. *Pacific J. Math.*, 191(2):393–398, 1999. ISSN 0030-8730. doi: 10.2140/pjm.1999.191.393. URL `https://doi.org/10.2140/pjm.1999.191.393`.

[45] Boris Botvinnik, Johannes Ebert, and David J. Wraith. On the topology of the space of Ricci-positive metrics. *arXiv e-prints*, art. arXiv:1809.11050, Sep 2018.

[46] Michel A. Kervaire and John W. Milnor. Groups of homotopy spheres. I. *Ann. of Math. (2)*, 77:504–537, 1963. ISSN 0003-486X. doi: 10.2307/1970128. URL `https://doi.org/10.2307/1970128`.

[47] Stephan Stolz. *Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder*, volume 1116 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985. ISBN 3-540-15209-1. doi: 10.1007/BFb0101604. URL `https://doi.org/10.1007/BFb0101604`. With an English introduction.

[48] Ramesh Kasilingam. Smooth structures on a fake real projective space. *Topology Appl.*, 206:1–7, 2016. ISSN 0166-8641. doi: 10.1016/j.topol.2016.03.027. URL `https://doi.org/10.1016/j.topol.2016.03.027`.

[49] Katsuo Kawakubo. Inertia groups of low dimensional complex projective spaces and some free differentiable actions on spheres. I. *Proc. Japan Acad.*, 44: 873–875, 1968. ISSN 0021-4280. URL `http://projecteuclid.org/euclid.pja/1195520972`.

[50] Ramesh Kasilingam. Classification of smooth structures on a homotopy complex projective space. *Proc. Indian Acad. Sci. Math. Sci.*, 126(2):277–281, 2016. ISSN 0253-4142. doi: 10.1007/s12044-016-0269-4. URL `https://doi.org/10.1007/s12044-016-0269-4`.

[51] Reinhard Schultz. On the inertia group of a product of spheres. *Trans. Amer. Math. Soc.*, 156:137–153, 1971. ISSN 0002-9947. doi: 10.2307/1995604. URL `https://doi.org/10.2307/1995604`.

[52] Michael Frame. On the inertia groups of h-cobordant manifolds. *Proc. Amer. Math. Soc.*, 90(3):433–439, 1984. ISSN 0002-9939. doi: 10.2307/2044489. URL `https://doi.org/10.2307/2044489`.

[53] Arthur L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008. ISBN 978-3-540-74120-6. Reprint of the 1987 edition.

[54] Maria Falcitelli, Stere Ianus, and Anna Maria Pastore. *Riemannian submersions and related topics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2004. ISBN 981-238-896-6. doi: 10.1142/9789812562333. URL `https://doi.org/10.1142/9789812562333`.

[55] Barrett O'Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966. ISSN 0026-2285. URL `http://projecteuclid.org/euclid.mmj/1028999604`.

[56] David J. Wraith. Bundle stabilisation and positive Ricci curvature. *Differential Geom. Appl.*, 25(5):552–560, 2007. ISSN 0926-2245. doi: 10.1016/j.difgeo.2007.06.005. URL `https://doi.org/10.1016/j.difgeo.2007.06.005`.

[57] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. ISBN 0-521-79160-X; 0-521-79540-0.

[58] Akhil Ranjan. Riemannian submersions of spheres with totally geodesic fibres. *Osaka J. Math.*, 22(2):243–260, 1985. ISSN 0030-6126. URL `http://projecteuclid.org/euclid.ojm/1200778256`.

[59] William Browder. *Surgery on simply-connected manifolds*. Springer-Verlag, New York-Heidelberg, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65.

[60] Jerome Levine. Self-equivalences of $S^n \times S^k$. *Trans. Amer. Math. Soc.*, 143: 523–543, 1969. ISSN 0002-9947. doi: 10.2307/1995262. URL `https://doi.org/10.2307/1995262`.

[61] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016. ISSN 0003-486X. doi: 10.4007/annals.2016.184.1.1. URL `https://doi.org/10.4007/annals.2016.184.1.1`.

[62] G. Brumfiel. The homotopy groups of BPL and PL/O. III. *Michigan Math. J.*, 17:217–224, 1970. ISSN 0026-2285. URL `http://projecteuclid.org/euclid.mmj/1029000468`.

[63] Adrien Douady. Variétés à bord anguleux et voisinages tubulaires. In *Séminaire Henri Cartan, 1961/62, Exp. 1*, page 11. Secrétariat mathématique, Paris, 1961/1962.

[64] John Milnor. *Collected papers of John Milnor. III*, chapter Differentiable manifolds which are homotopy spheres, pages xvi+343. American Mathematical Society, Providence, RI, 2007. ISBN 978-0-8218-4230-0; 0-8218-4230-7. Differential topology.

[65] James Ferguson. Multivariable curve interpolation. *J. Assoc. Comput. Mach.*, 11:221–228, 1964. ISSN 0004-5411. doi: 10.1145/321217.321225. URL `https://doi.org/10.1145/321217.321225`.

[66] Daniel Ruberman. Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants. *Geom. Topol.*, 5:895–924, 2001. ISSN 1465-3060. doi: 10.2140/gt.2001.5.895. URL `https://doi.org/10.2140/gt.2001.5.895`.

[67] Mark Walsh. Metrics of positive scalar curvature and generalised Morse functions, Part I. *Mem. Amer. Math. Soc.*, 209(983):xviii+80, 2011. ISSN 0065-9266. doi: 10.1090/S0065-9266-10-00622-8. URL `https://doi.org/10.1090/S0065-9266-10-00622-8`.

[68] Rodney Carr. Construction of manifolds of positive scalar curvature. *Trans. Amer. Math. Soc.*, 307(1):63–74, 1988. ISSN 0002-9947. doi: 10.2307/2000751. URL `https://doi.org/10.2307/2000751`.

[69] Misha Gromov. Metric inequalities with scalar curvature. *Geom. Funct. Anal.*, 28(3):645–726, 2018. ISSN 1016-443X. doi: 10.1007/s00039-018-0453-z. URL `https://doi.org/10.1007/s00039-018-0453-z`.