METRICS OF POSITIVE SCALAR CURVATURE AND GENERALISED MORSE FUNCTIONS

by

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Title: METRICS OF POSITIVE SCALAR CURVATURE AND GENERALISED MORSE FUNCTIONS

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Dr. Boris Botvinnik

We study the topology of the space of metrics of positive scalar curvature on a compact manifold. The main tool we use for constructing such metrics is the surgery technique of Gromov and Lawson. We extend this technique to construct families of positive scalar curvature cobordisms and concordances which are parametrised by Morse functions and later, by generalised Morse functions. We then use these results to study concordances of positive scalar curvature metrics on simply connected manifolds of dimension at least five. In particular, we describe a subspace of the space of positive scalar curvature concordances, parametrised by generalised Morse functions. We call such concordances Gromov-Lawson concordances. One of the main results is that positive scalar curvature metrics which are Gromov-Lawson concordant are in fact isotopic. This work relies heavily on contemporary Riemannian geometry as well as on differential topology, in particular pseudo-isotopy theory. We make substantial use of the work of Eliashberg and Mishachev on wrinkled maps and of results by Hatcher and Igusa on the space of generalised Morse functions.

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INTRODUCTION

I.1 Why Positive Scalar Curvature?

In the 2-dimensional setting, scalar curvature is a fairly intuitive concept. Round spheres are positively curved, planes and cylinders have no curvature while a saddle surface displays curvature which is negative. Geometrically then, positive scalar curvature can be thought to make a surface close in on itself whereas negative scalar curvature causes it to spread out. The topological consequences of this are evident from the classical theorem of Gauss-Bonnet. This theorem relates the scalar curvature R, of a compact oriented Riemannian 2-manifold M, with its Euler characteristic $\chi(M)$ by the formula

$$\frac{1}{4\pi}\int_M R = \chi(M).$$

It follows that a closed surface with non-positive Euler characteristic, such as a torus, does not admit a metric of strictly positive (or negative) scalar curvature. Similarly, a surface with positive Euler characteristic such as a sphere cannot have scalar curvature which is everywhere non-positive. From the Uniformisation Theorem we know that every closed surface admits a metric of constant scalar curvature. This implies the following classification result: A closed surface admits a metric of positive, zero or negative scalar curvature if and only if its Euler characteristic is respectively positive, zero or negative.

In higher dimensions, the relationship between curvature and topology is much more complicated. The scalar curvature is one of three curvatures which are commonly studied, the others being the Ricci and sectional curvatures. These curvatures vary greatly in the amount of geometric information they carry. The sectional curvature is the strongest and contains the most geometric information. Conditions such as strict positivity or negativity of the sectional curvature impose severe topological restrictions on the underlying manifold. The scalar curvature on the other hand, is the weakest of these curvatures. One piece of geometric information it does carry, concerns the volume growth of geodesic balls. In particular, the scalar curvature R at a point of a Riemannian n-manifold X, appears as a constant in an expansion

$$\frac{Vol(B_X(\epsilon))}{Vol(B_{\mathbb{R}^n}(\epsilon))} = 1 - \frac{R}{6(n+2)}\epsilon^2 + \cdots,$$

comparing the volume of a geodesic ball in X with the corresponding ball in Euclidean space, see [12]. Thus, positive scalar curvature implies that **small** geodesic balls have less volume than their Euclidean counterparts while for negative scalar curvature this inequality is reversed.

We will be interested in metrics of *positive* scalar curvature and in the problem of whether or not a given manifold admits such a metric. At this point, the reader may well ask why we focus on positivity. Why not consider metrics of negative, non-negative or zero scalar curvature? As a partial justification, we point out that there are no obstructions to the existence of metrics of negative scalar curvature in dimensions ≥ 3, see [29]. Furthermore, any closed manifold which admits a metric whose scalar curvature is non-negative and not identically zero, always admits a metric of positive scalar curvature. This follows from the Trichotomy theorem of Kazdan and Warner, see [25] and [26]. For a more thorough discussion of this matter, see section 2 of [36].

The existence problem for metrics of positive scalar curvature has been extensively studied. In the early 1960s, Lichnerowicz discovered that on a compact spin manifold, positive scalar curvature of the metric implies that the analytic index of the Dirac operator must be zero, see [28]. It then follows from the Atiyah-Singer Index Theorem that any compact spin manifold with non-vanishing $\hat{A}$-genus does not admit a metric of positive scalar curvature. In the 1970s, this fact was generalised by Hitchin in [18], who showed that the index of the Dirac operator for a compact spin manifold X, of dimension n, is represented by an element $\alpha(X)$ in the real K-theory group KO_n. As a geometric consequence, Hitchin exhibits exotic spheres (starting in dimension nine) which do not admit metrics of positive scalar curvature.

The other side of this problem concerns the construction of positive scalar curvature metrics when no obstructions exist. The principle tool for doing this is known as the Surgery Theorem.

This theorem was proved in the late 1970s by Gromov and Lawson [14] and, independently, by Schoen and Yau [38]. It provides an especially powerful device for building positive scalar curvature metrics. Before discussing this any further, we should say a few words about surgery.

A p-surgery (or codimension $q+1$-surgery) on a manifold X of dimension n is a process which involves removing an embedded product $S^p \times D^{q+1}$ and replacing it with $D^{p+1} \times S^q$, where $p+q+1=n$, see section II.3 for details. The result of this is a new n-dimensional manifold X' whose topology is usually very different from that of X. Importantly, surgery preserves the cobordism type of the original manifold. This means that if X' is obtained from X by surgery, there exists an $n+1$-dimensional manifold whose boundary is a disjoint union of X and X'. In Fig. I.1 we show a cobordism between a sphere S^2 and a torus T^2. The torus is obtained from the sphere by a 0-surgery.

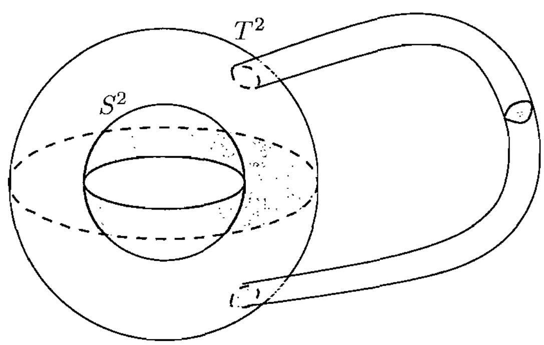

Figure I.1: A cobordism of the sphere S^2 and the torus T^2

Given a manifold X which admits a metric of positive scalar curvature, the Surgery Theorem gives a method for constructing further metrics of positive scalar curvature on every manifold which can be obtained from X by surgery in codimension ≥ 3. Under some restrictions, this includes every manifold which is cobordant to X. The Surgery Theorem therefore led to a dramatic increase in the number of examples of manifolds which were known to admit metrics of positive scalar curvature, effectively moving the problem from one of individual manifolds to one of cobordism classes. Since then, there has been a great deal of success in classifying which manifolds admit positive scalar curvature metrics, see [36] for a survey of this problem. Of particular interest to us is the case when X is a simply connected manifold of dimension ≥ 5. Here, the question of when X admits a positive scalar curvature metric is now completely answered, see [14], [36], [39].

In particular, when X is not a spin manifold, X always admits a such a metric and in the case when X is spin, X admits a metric of positive scalar curvature if and only if the above mentioned obstruction $\alpha(X)$ vanishes.

We conclude this section with some words about the analogous question for positive Ricci and sectional curvatures. Although some important progress has been made, the problem of constructing examples of metrics with positive Ricci and, in particular, positive sectional curvature is a very difficult one. There is no real analogue of the Surgery Theorem here as these curvatures do not exhibit the same flexibility as the scalar curvature. Positive Ricci and sectional curvatures do not behave well under surgery, as the following example shows.

Example I.1. The manifold $\mathbb{R}P^n$ admits a metric which is locally isometric to the round metric on S^n. When $n \geq 2$, it therefore has positive sectional, Ricci and scalar curvatures. By taking a connected sum of two copies of $\mathbb{R}P^n$, we obtain a new manifold $\mathbb{R}P^n \# \mathbb{R}P^n$. From the theorem of Van Kampen we know that, unlike $\mathbb{R}P^n$, this new manifold has infinite fundamental group. Assuming $n \geq 3$, the Surgery Theorem allows us to conclude that this manifold also admits a metric of positive scalar curvature. Indeed, we could take as many connected sums as we wished and still be confident that the resulting manifold admits a metric of positive scalar curvature. According to the theorem of Bonnet-Meyers however, positive sectional or Ricci curvature on a complete connected Riemannian manifold forces the fundamental group to be finite, see [34, Chapter 6, Theorem 25]. Hence, this manifold admits no metric of positive Ricci or sectional curvature.

Away from the classification problem, there are many other interesting questions in the area of positive scalar curvature where far less is known. In the next section we will discuss some of these.

I.2 Background

Henceforth, all manifolds are assumed to be compact and connected. Furthermore, the term positive scalar curvature will often be abbreviated as *psc*. Metrics of positive scalar curvature will usually be referred to as *psc-metrics* and manifolds which admit such metrics as *psc-manifolds*.

I.2.1 The space of psc-metrics

Let X be a smooth closed manifold of dimension n. We denote by $\mathcal{R}\mathrm{iem}(X)$, the space of Riemannian metrics on X, with its standard C^∞ topology. The set of all psc-metrics on X is denoted $\mathcal{R}\mathrm{iem}^+(X)$ and is an open subset of $\mathcal{R}\mathrm{iem}(X)$. In these terms, the above classification problem can be thought of as the problem of determining for which X, the space $\mathcal{R}\mathrm{iem}^+(X)$ is non-empty. In general very little is known about the topology of the space $\mathcal{R}\mathrm{iem}^+(X)$. This leads to the first problem we wish to consider.

Question I.2. *What is the topology of the space $\mathcal{R}\mathrm{iem}^+(X)$? In particular, is this space path connected and, if not, how many path components does it have?*

Some results have been obtained about this space when $X = S^n$. It is known that $\mathcal{R}\mathrm{iem}^+(S^2)$ is contractible (as is $\mathcal{R}\mathrm{iem}^+(\mathbb{R}P^2)$), see [36], and recent work by Botvinnik and Rosenberg indicates that this is also the case for S^3. When $n \geq 4$, the only known results are at the level of path connectedness. For example, Carr shows in [4] that $\mathcal{R}\mathrm{iem}^+(S^{4k-1})$ has an infinite number of path components when $k \geq 2$.

Suppose W is a smooth compact manifold and $\partial W \neq \emptyset$. The question of whether or not the space $\mathcal{R}\mathrm{iem}^+(W)$, of psc-metrics on W, is non-empty is not such an interesting question. It turns out that, without some condition on the boundary, W will not only always admit a metric of positive scalar curvature, but will in fact admit a metric of positive sectional curvature! This is a result of Gromov, see Theorem 4.5.1 of [13]. Thus, we will impose some boundary conditions on W. We denote by $\mathcal{R}\mathrm{iem}^+(W, \partial W)$, the subspace of $\mathcal{R}\mathrm{iem}^+(W)$ consisting of psc-metrics which have a product structure near ∂W. This means that if $g \in \mathcal{R}\mathrm{iem}^+(W, \partial W)$, $g = g|_{\partial W} + dt^2$ near ∂W. Here $g|_{\partial W}$ is the metric induced by the inclusion of ∂W into W.

I.2.2 Isotopy and concordance

When studying the space $\mathcal{R}\mathrm{iem}^+(X)$, one is immediately confronted with the notions of isotopy and concordance. Metrics which lie in the same path component of $\mathcal{R}\mathrm{iem}^+(X)$ are said to be *isotopic.* Two psc-metrics g_0 and g_1 on X are said to be *concordant* if there is a psc-metric $\bar{g}$ on the cylinder $X \times I$ ($I = [0,1]$), so that $\bar{g} = g_0 + dt^2$ near $X \times \{0\}$ and $\bar{g} = g_1 + dt^2$ near $X \times \{1\}$. It is well known that isotopic metrics are concordant, see Lemma II.2 below. It is also known that concordant metrics need not be isotopic when $\dim X = 4$, where the difference between isotopy and concordance is detected by the Seiberg-Witten invariant, see [37]. However, in the case when $\dim X \geq 5$, the question of whether or not concordance implies isotopy is an open problem and one we will attempt to shed some light on.

Before discussing this further, it is worth mentioning that the only known method for showing that two psc-metrics on X lie in distinct path components of $\mathcal{R}\mathrm{iem}^+(X)$, is to show that these metrics are not concordant. For example, Carr's proof in [4], that $\mathcal{R}\mathrm{iem}^+(S^{4k-1})$ has an infinite number of path components, involves using index obstruction methods to exhibit a countably infinite collection of distinct concordance classes on S^{4k-1}. This implies that the space $\mathcal{R}\mathrm{iem}^+(S^{4k-1})$ has at least as many path components. See also Example I.6 below for the case when $k = 2$. In [3], the authors show that if X is a connected spin manifold with $\dim X = 2k+1 \geq 5$ and if $\pi_1(X)$ is non-trivial and finite, then $\mathcal{R}\mathrm{iem}^+(X)$ has infinitely many path components provided $\mathcal{R}\mathrm{iem}^+(X)$ is non-empty. Again, this is done by exhibiting infinitely many distinct concordance classes. For a general smooth manifold X, understanding $\pi_0(\mathcal{R}\mathrm{iem}^+(X))$ is contingent on answering the following open questions.

Question I.3. *Are there more concordance classes undetected by the index theory?*

Question I.4. *When are concordant metrics isotopic?*

For more on the first of these problems, the reader is referred to [40] and [36]. We will focus our attention on the second problem.

A fundamental difficulty when approaching question I.4 is that an arbitrary concordance may be extraordinarily complicated. For example, let $g_s, s \in I$ denote an isotopy in the space $\mathcal{R}\mathrm{iem}^+(S^n)$. After an appropriate rescaling, see Lemma II.2, we may assume that the warped product metric $\bar{h} = g_t + dt^2$, on the cylinder $S^n \times I$, has positive scalar curvature and a product structure near the boundary, i.e. is a concordance of g_0 and g_1. Now let g be any psc-metric on the sphere S^{n+1} (this metric may be very complicated indeed). It is possible to construct a psc-metric $\bar{g}$ on $S^n \times I$ by taking a connected sum

$$\bar{g} = \bar{h} \# g,$$

see [14]. As this construction only alters the metric $\bar{h}$ on the interior of the cylinder, the resulting metric, $\bar{g}$, is still a concordance of g_0 and g_1, see Fig. I.2. Unlike the concordance $\bar{h}$ however, $\bar{g}$ could be arbitrarily complicated. In some sense, this makes $\bar{g}$ "unrecognisable" as an isotopy.

Consequently, we will not approach this problem at the level of arbitrary concordance. Instead, we will restrict our attention to concordances which are constructed by a particular application of the surgery technique of Gromov and Lawson. Such concordances will be called *Gromov-Lawson concordances.* Before discussing the relationship between surgery and concordance, it is worth recalling how the surgery technique alters a psc-metric.

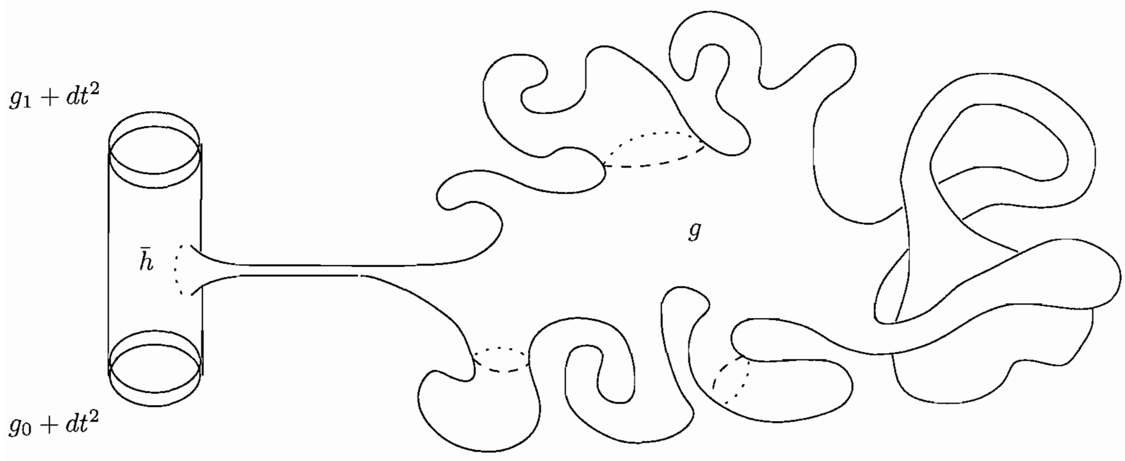

Figure I.2: The concordance $\bar{g}$ on $S^n \times I$, formed by taking a conncected sum of metrics $\bar{h}$ and g.

I.2.3 *Surgery and positive scalar curvature*

We begin by stating the Surgery Theorem of Gromov-Lawson and Schoen-Yau.

Surgery Theorem.([14], [38]) *Suppose X admits a psc-metric and X' is a manifold which is obtained from X by surgery in codimension ≥ 3. Then X' admits a psc-metric also.*

In their proof, Gromov and Lawson show that a psc-metric g on X can be replaced with a psc-metric g_{std} which is standard in a tubular neighbourhood of the embedded surgery sphere. More precisely, let ds_n^2 denote the standard round metric on the sphere S^n. We denote by $g_{tor}^n(\delta)$, the metric on the disk D^n which, near ∂D^n, is the Riemannian cylinder $\delta^2 ds_{n-1}^2 + dr^2$ and which near the centre of D^n is the round metric $\delta^2 ds_n^2$. The metric $g_{tor}^n(\delta)$ is known as a *torpedo metric*, see section II.2 for a detailed construction. For sufficiently small $\delta > 0$ and provided $n \geq 3$, the scalar curvature of this metric can be bounded below by an arbitrarily large positive constant. Now, let (X, g) be a smooth n-dimensional Riemannian manifold of positive scalar curvature and let S^p denote an embedded p-sphere in X with trivial normal bundle and with $p + q + 1 = n$ and $q \geq 2$.

The metric g can be replaced by a psc-metric g_{std} on X which, on a tubular neighbourhood of S^p, is the standard product $ds_p^2 + g_{tor}^{q+1}(\delta)$ for some appropriately small δ. In turn, surgery may be performed on this standard piece to obtain a psc-metric g' on X', which on the handle $D^{p+1} \times S^q$ is the standard product $g_{tor}^{p+1} + \delta^2 ds_q^2$, see Fig. I.3.

There is an important strengthenning of this technique whereby the metric g is extended over the trace of the surgery to obtain a psc-metric $\bar{g}$ which is a product metric near the boundary. This is sometimes referred to as the Improved Surgery Theorem, see [10]. Suppose $\{W; X_0, X_1\}$ is a smooth compact cobordism of closed n-manifolds X_0 and X_1, i.e. $\partial W = X_0 \sqcup X_1$, and $f : W \to I$ is a Morse function. All Morse functions are assumed to satisfy $f^{-1}(0) = X_0$, $f^{-1}(1) = X_1$ and have critical points only in the interior of W. The Morse function f gives rise to a decomposition of W into elementary cobordisms. Let us assume that each elementary cobordism is the trace of a codimension ≥ 3 surgery. This means that each critical point of f has index $\leq n - 2$. Roughly speaking, such Morse functions will be called "admissible". It is now possible to extend a psc-metric g_0 on X_0 to a psc-metric $\bar{g}$ on W which is a product near the boundary ∂W, see Theorem II.23 below. In particular, the restriction $g_1 = \bar{g}|_{X_1}$ is a psc-metric on X_1. Example I.6 below demonstrates that the metric g_1 may not be concordant (and therefore not isotopic) to g_0, an illustration of the power of the Surgery Theorem for generating new psc-metrics. This gives rise to the following question.

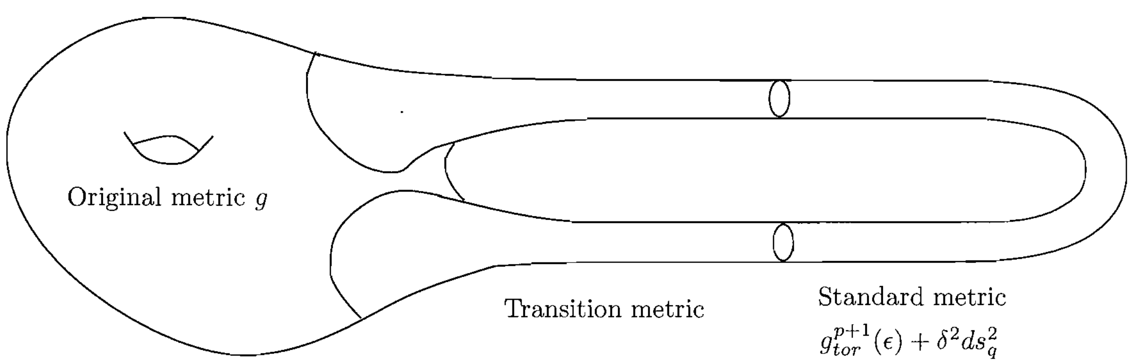

Figure I.3: The psc-metric g' obtained on X' by the Gromov-Lawson construction

Question I.5. *In the case when X_0 is diffeomorphic to X_1, when are the metrics g_0 and g_1 isotopic or concordant?*

Example I.6. Let $B = B^8$ be a Bott manifold, i.e. an 8-dimensional closed simply connected spin manifold with $\alpha(B) = 1$, see [24] for a geometric construction of such a manifold. Here α

is the obstruction discussed in section I.1 and so the fact that $\alpha(B) \neq 0$ means that B does not admit a psc-metric. Let $W = B \setminus (D_0 \sqcup D_1)$ denote the smooth manifold obtained by removing a disjoint pair of 8-dimensional disks D_0 and D_1 from B. The boundary of W is a pair of disjoint 7-dimensional smooth spheres, which we denote S_0^7 and S_1^7 respectively. It is possible, although we do not include the details here, to equip W with an admissible Morse function. This decomposes W into a union of elementary cobordisms, each the trace of a codimension ≥ 3 surgery. Thus, we can extend the standard round metric $g_0 = ds_7^2$ from the boundary component S_0^7 to a psc metric $\bar{g}$ on W, which is a product metric near both boundary components. In particular, the metric $\bar{g}$ restricts to a psc-metric g_1 on S_1^7. This metric however, is **not** concordant (and hence not isotopic) to g_0. This is because the existence of a concordance $\bar{h}$ of g_1 and $g_0 = ds_7^2$, would give rise to a psc-metric g_B on B (see Fig. I.4), defined by taking the union

$$(B, g_B) = (D_0, g_{tor}^8) \cup (W, \bar{g}) \cup (S^7 \times I, \bar{h}) \cup (D_1, g_{tor}^8),$$

something we know to be impossible.

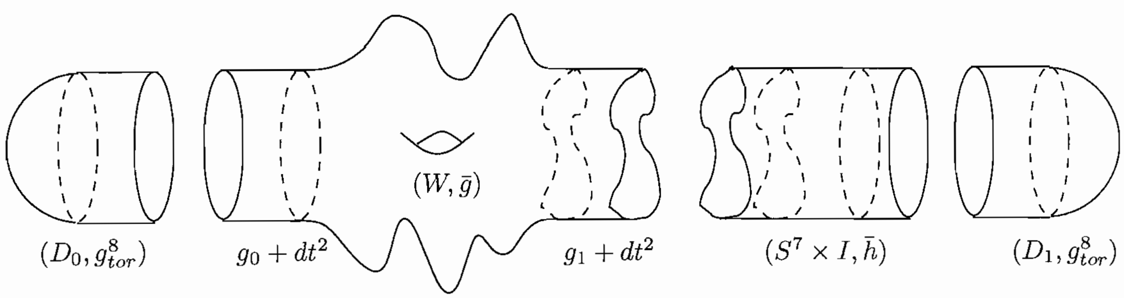

Figure I.4: The existence of a concordance $(S^7 \times I, \bar{h})$ between g_1 and $g_0 = ds_7^2$ would imply the existence of a psc-metric on B, which is impossible.

I.2.4 The structure of the thesis

This thesis is organised into two parts. Roughly speaking, Part One contains most of the geometric arguments and technical results about positive scalar curvature, while Part Two deals more with the topological applications of these geometric results. In particular, it is in Part Two that we explore the role of generalised Morse functions in this subject. We will now present the main results of Part One. Then, after some preliminary discussion, we will present the main results from Part Two.

I.3 Main Results of Part One

Part One begins with some important technical preliminaries, in particular, introducing a collection of metrics on the disk D^n and the sphere S^n which will be used throughout the thesis; see section II.2. These metrics are variations of the so-called torpedo metric discussed earlier. After proving some important results about spaces of such metrics, we proceed in II.3 to a careful analysis of the Surgery Theorem. From there we obtain the following results.

A Family Surgery Theorem

The proof of the Surgery Theorem involves replacing a psc-metric g on a manifold X with a psc-metric g_{std} on X which is standard near the embedded surgery sphere. After verifying that the metrics g and g_{std} are in fact isotopic (Theorem II.11), we show in Theorem II.19 that this technique can be applied continuously over a compact family of psc-metrics and with respect to a compact family of embedded surgery spheres.

Theorem II.19. *Let X be a smooth compact manifold of dimension n, and B and C a pair of compact spaces. Let $\mathcal{B} = \{g_b \in \mathcal{R}\mathrm{iem}^+(X) : b \in B\}$ be a continuous family of psc-metrics on X and $\mathcal{C} = \{i_c \in Emb(S^p, X) : c \in C\}$, a continuous family of embeddings each with trivial normal bundle, where with $p + q + 1 = n$ and $q \geq 2$. Finally, let g_p be any metric on S^p. Then, for some $\delta > 0$, there is a continuous map*

$$\begin{aligned} \mathcal{B} \times \mathcal{C} &\longrightarrow \mathcal{R}\mathrm{iem}^+(X) \\ (g_b, i_c) &\longmapsto g_{std}^{b,c} \end{aligned}$$

satisfying

(i) *Each metric $g_{std}^{b,c}$ has the form $g_p + g_{tor}^{q+1}(\delta)$ on a tubular neighbourhood of $i_c(S^p)$ and is the original metric g_b away from this neighbourhood.*

(ii) *For each $c \in C$, the restriction of this map to $\mathcal{B} \times \{i_c\}$ is homotopy equivalent to the inclusion $\mathcal{B} \hookrightarrow \mathcal{R}\mathrm{iem}^+(X)$.*

Applications of the Family Surgery Theorem

The above notion of generalising to compact families is necessary if one is to have any chance of understanding spaces of psc-metrics. Proving Theorem II.19 requires considerable preparation in the form of the rather long and technical Theorem II.11. Once established however, we can prove the following theorems without too much difficulty. The first of these is actually the main result in a paper by Chernysh; see [6].

Theorem II.21. *Let X be a smooth compact manifold of dimension n. Suppose X' is obtained from X by surgery on a sphere $S^p \hookrightarrow X$ with $p+q+1=n$ and $p, q \geq 2$. Then the spaces $\mathcal{R}\mathrm{iem}^+(X)$ and $\mathcal{R}\mathrm{iem}^+(X')$ are homotopy equivalent.*

It is now possible to show that for simply connected spin manifolds of dimension ≥ 5, the homotopy type of the space of psc-metrics is a spin cobordism invariant.

Theorem II.22. *Let X_0 and X_1 be a pair of compact simply-connected spin manifolds of dimension $n \geq 5$. Suppose also that X_0 is spin cobordant to X_1. Then the spaces $\mathcal{R}\mathrm{iem}^+(X_0)$ and $\mathcal{R}\mathrm{iem}^+(X_1)$ are homotopy equivalent.*

The Gromov-Lawson Cobordism Theorem

In Theorem II.23, we generalise the so-called Improved Surgery Theorem, as well as correct an error from the proof in [10]; see Remark II.3.4 in II.3.

Theorem II.23. *Let $\{W^{n+1}; X_0, X_1\}$ be a smooth compact cobordism. Suppose g_0 is a metric of positive scalar curvature on X_0 and $f : W \to I$ is an admissible Morse function. Then there is a psc-metric $\bar{g} = \bar{g}(g_0, f)$ on W which extends g_0 and has a product structure near the boundary.*

We call the metric $\bar{g} = \bar{g}(g_0, f)$, a *Gromov-Lawson cobordism (GL-cobordism) with respect to g_0 and f*. Essentially, the metric $\bar{g}$ restricts on a regular level set of f to the metric obtained by repeated application of the surgery technique with respect to each of the critical points below that level set. In the case when W is the cylinder $X \times I$, the metric $\bar{g}$ is a concordance of the metrics g_0 and $g_1 = \bar{g}|_{X \times \{1\}}$. It will be referred to as a *Gromov-Lawson concordance (GL-concordance)* *with respect to g_0 and f*; see Fig. I.5.

There are a number of obvious questions one may ask about the metric $\bar{g} = \bar{g}(g_0, f)$. In particular, the reader may wonder to what extent the metrics $\bar{g}$ and $g_1 = \bar{g}|_{X_1}$ depend on the choice of admissible Morse function. Different admissible Morse functions with different numbers of critical points will give rise to very different looking metrics. It is not hard to believe that isotopic admissible Morse functions (those connected by a path in the space of admissible Morse functions) should give rise to isotopic metrics. This is proven in Theorem II.25 below. The question of whether this holds for admissible Morse functions which are not isotopic (and containing possibly very different collections of critical points) is more difficult and one we will not address until Part Two.

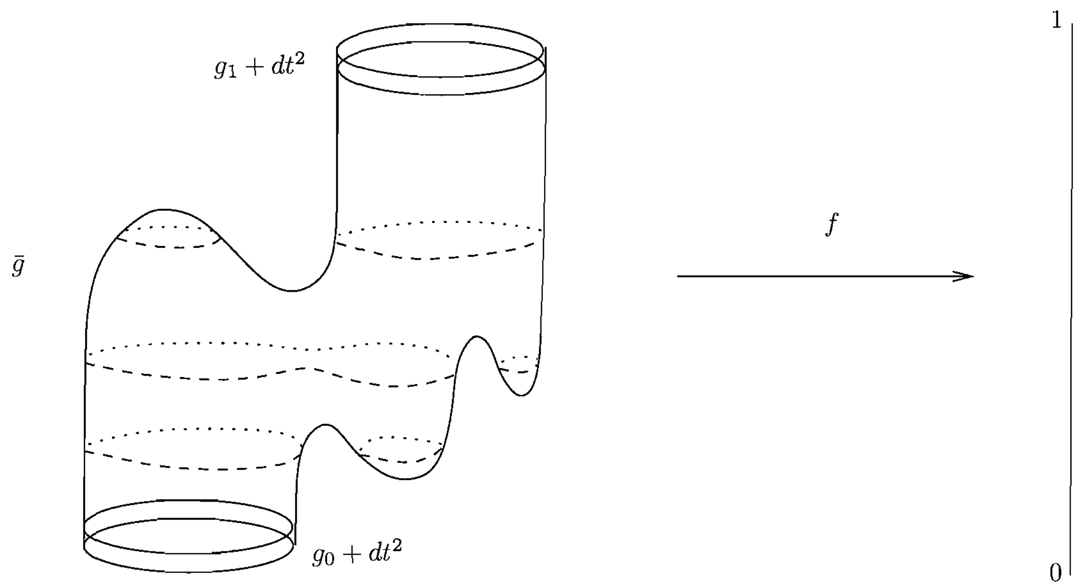

Figure I.5: Obtaining a Gromov-Lawson concordance on the cylinder $X \times I$ with respect to a Morse function f and a psc-metric g_0

Reversing a Gromov-Lawson cobordism

Any admissible Morse function f can be replaced by a Morse function denoted $1-f$, which has the gradient flow of f, but running in reverse. (Admissible Morse functions will come equipped with gradient-like vector fields.) This function has the same critical points as f, however, each critical point of index λ has been replaced with one of index $n+1-\lambda$. The following theorem can be obtained by "reversing" the construction from Theorem II.23.

Theorem II.24. *Let $\{W^{n+1}; X_0, X_1\}$ be a smooth compact cobordism, g_0 a psc-metric on X_0 and $f : W \to I$, an admissible Morse function. Suppose that $1-f$ is also an admissible Morse function. Let $g_1 = \bar{g}(g_0, f)|_{X_1}$ denote the restriction of the Gromov-Lawson cobordism $\bar{g}(g_0, f)$ to X_1. Let $\bar{g}(g_1, 1-f)$ be a Gromov-Lawson cobordism with respect to g_1 and $1-f$ and let $g_0' = \bar{g}(g_1, 1-f)|_{X_0}$ denote the restriction of this metric to X_0. Then g_0 and g_0' are canonically isotopic metrics in $\mathcal{R}\mathrm{iem}^+(X_0)$.*

A family version of the Gromov Lawson Cobordism Theorem

As shown in Theorem II.19 , the Gromov-Lawson construction can be applied continuously over a compact family of metrics as well as a compact family of embedded surgery spheres. In Theorem II.25, we show that the Gromov-Lawson cobordism construction of Theorem II.23 can also be applied continuously, over a contractible compact family of admissible Morse functions to obtain the following theorem.

Theorem II.25. *Let $\{W; X_0, X_1\}$ be a smooth compact cobordism, $\mathcal{B}$, a compact continuous family of psc-metrics on X_0 and $\mathcal{C}$, a compact continuous contractible family of admissible Morse functions on W. Then there is a continuous map*

$$\mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{R}\mathrm{iem}^+(W, \partial W)$$

$$(g_b, f_c) \longmapsto \bar{g}_{b,c} = \bar{g}(g_b, f_c)$$

so that for each pair (b, c), the metric $\bar{g}_{b,c}$ is a Gromov-Lawson cobordism.

Gromov-Lawson concordance implies isotopy

We now come to the main result of Part One. In section II.5 we construct an example of a GL-concordance on the cylinder $S^n \times I$. Here $g_0 = ds_n^2$, the standard round metric and f is an admissible Morse function with two critical points which have Morse indices $p+1$ and $p+2$ where $p+q+1 = n$ and $q \geq 3$. The critical point of index $p+1$ corresponds to a p-surgery on S^n resulting in a manifold diffeomorphic to $S^{p+1} \times S^q$. This is then followed by a $(p+1)$-surgery which restores the original manifold S^n. The restriction of the metric $\bar{g}(ds_n^2, f)$ to level sets of f below, between and above these critical points is denoted by g_0, g_0' and g_1 respectively; see Fig. I.6. The metric g_1 is also a psc-metric on S^n, but as Fig. I.6 suggests, looks radically different from the original metric g_0. Understanding why these metrics are in fact isotopic is crucial in proving our main result, stated below.

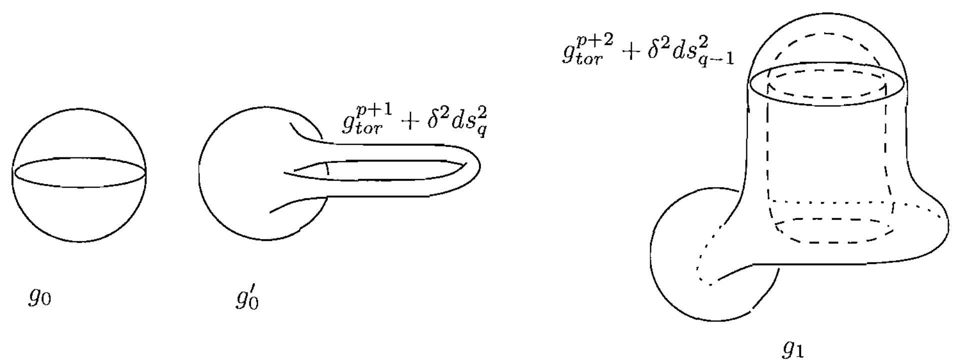

Figure I.6: Applying the Gromov-Lawson construction over a pair of cancelling surgeries of consecutive dimension

Theorem II.36. *Let X be a closed simply connected manifold of dimension $n \geq 5$ and let g_0 be a positive scalar curvature metric on X. Suppose $\bar{g} = \bar{g}(g_0, f)$ is a Gromov-Lawson concordance with respect to g_0 and an admissible Morse function $f : X \times I \to I$. Then the metrics g_0 and $g_1 = \bar{g}|_{X \times \{1\}}$ are isotopic.*

The proof of Theorem II.36 takes place in II.6 and II.7. In II.6 we prove the theorem in the case when f has exactly two "cancelling" critical points. This is the key geometric step and draws heavily from some important technical observations made in II.2. The general case then follows from Morse-Smale theory and the fact that the function f can be deformed to one whose critical points are arranged in cancelling pairs. Along with Theorem II.24, this result provides a partial answer to question I.5.

I.4 An Introduction to Part Two

As mentioned earlier, one motivation behind this work is to gain information about certain spaces of psc-metrics. In Part One, we develop a technique for building particular psc-metrics on a compact cobordism $\{W; X_0, X_1\}$. We call these metrics Gromov-Lawson cobordisms (*GL-*

cobordisms). In Part Two, we will attempt to better understand the space of GL-cobordisms, a subspace of $\mathcal{R}\text{iem}^+(W, \partial W)$. In a weak sense, Theorems II.23 and II.25 allow us to parametrise families of Gromov-Lawson cobordisms by admissible Morse functions. As it stands however, Theorem II.25 only works for compact contractible families of admissible Morse functions. This misses some very important structure. To see this we need to say some words about the space of Morse functions.

The space of admissible Morse functions on W is denoted $\mathcal{M}^{adm}(W)$ and can be thought of as a subspace of the space of Morse functions $W \to I$, denoted $\mathcal{M}(W)$. A good deal is understood about the topology of the space $\mathcal{M}(W)$, in particular; see [23]. It is clear that this space is not path connected, as functions in the same path component must have the same number of critical points of the same index. Thus, Theorem II.25 allows us to parametrise families of GL-cobordisms arising from a single path component of $\mathcal{M}^{adm}(W)$. This gives a rather misleading picture, as it is possible for appropriate pairs of Morse critical points to cancel, giving rise to a simpler handle decomposition of W. In Theorem II.36, we describe a corresponding "geometric cancellation" which simplifies a psc-metric associated to this Morse function. In order to obtain a more complete picture of the space of GL-cobordisms, we need to incorporate this cancellation property into our description.

Generalised Morse functions

There is a natural setting in which to consider the cancellation of Morse critical points. Recall that near a critical point w, a Morse function $f \in \mathcal{M}(W)$ is locally equivalent to the map

$$(x_1, \ldots, x_{n+1}) \longmapsto -x_1^2 \cdots - x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{n+1}^2.$$

A critical point w of a smooth function $f : W \to I$ is said to be of *birth-death* type if near w, f is equivalent to the map

$$(x_0, \ldots, x_n) \longmapsto x_0^3 - x_1^2 \cdots - x_{p+1}^2 + x_{p+2}^2 + \cdots + x_n^2.$$

A *generalised Morse function* $f : W \to I$ is a smooth function satisfying $f^{-1}(0) = X_0$, $f^{-1}(1) = X_1$ and whose singular set is contained in the interior of W and consists of only Morse and birth-death critical points. There is a natural embedding of $\mathcal{M}(W)$ into the space of generalised Morse functions $\mathcal{H}(W)$. This allows us to connect up distinct path components of $\mathcal{M}(W)$ since birth-death singularities allow for the cancellation of Morse critical points of consecutive index. Before going any further it is worth considering a couple of examples of this sort of cancellation.

Example I.7. The function $F(x,t) = x^3 + tx$ can be thought of as a smooth family of functions $x \longmapsto F(x,t)$ parametrised by t. When $t < 0$, the map $x \longmapsto F(x,t)$ is a Morse function with 2 critical points which cancel as a degenerate singularity of the function $x \longmapsto F(x,0)$. The function $x \longmapsto F(x,0)$ is an example of a generalised Morse function with a birth-death singularity at $x = 0$.

Example I.8. In Fig. I.7 we sketch using selected level sets, a path $f_t, t \in [-1,1]$, in the space $\mathcal{H}(S^n \times I)$ which connects a Morse function f_{-1} with two critical points of consecutive Morse index to a Morse function f_1 which has no critical points. We will assume that the critical points of f_{-1} lie on the level sets $f_{-1} = \frac{1}{4}$ and $f_{-1} = \frac{3}{4}$ and that f_0 has only a birth-death singularity on the level set $f_0 = \frac{1}{2}$.

Wrinkled maps

Our goal in Part Two will be to "extend" the notion of GL-cobordism to work for generalised Morse functions and so be able to parametrise families of GL-cobordisms over admissible Morse functions with varying numbers of critical points. A convenient setting in which to do this is described by Eliashberg and Mishachev in their work on "wrinklings" of smooth maps; see [8] and [9]. Let E and Q be smooth compact manifolds of dimension $n+1+k$ and k respectively. In section III.5, we specify a particular smooth fibre bundle $\pi : E \longrightarrow Q$, the fibre of which is the smooth cobordism W. Let $f : E \longrightarrow Q \times I$ be a smooth map so that $p_1 \circ f = \pi$, where p_1 is projection on the first factor. Roughly speaking, the map f is *wrinkled* if the singular set of f in E consists of a disjoint union of *folds* and *wrinkles*. We will not define the terms fold or wrinkle here except to say that under these conditions f restricts on fibres to a generalised Morse function $W \longrightarrow I$ of the type discussed earlier. Thus, a wrinkled map can be thought of as a family of generalised Morse functions. Note also that this family may be "twisted" in the event that π is a non-trivial bundle; see [11] for an example of this.

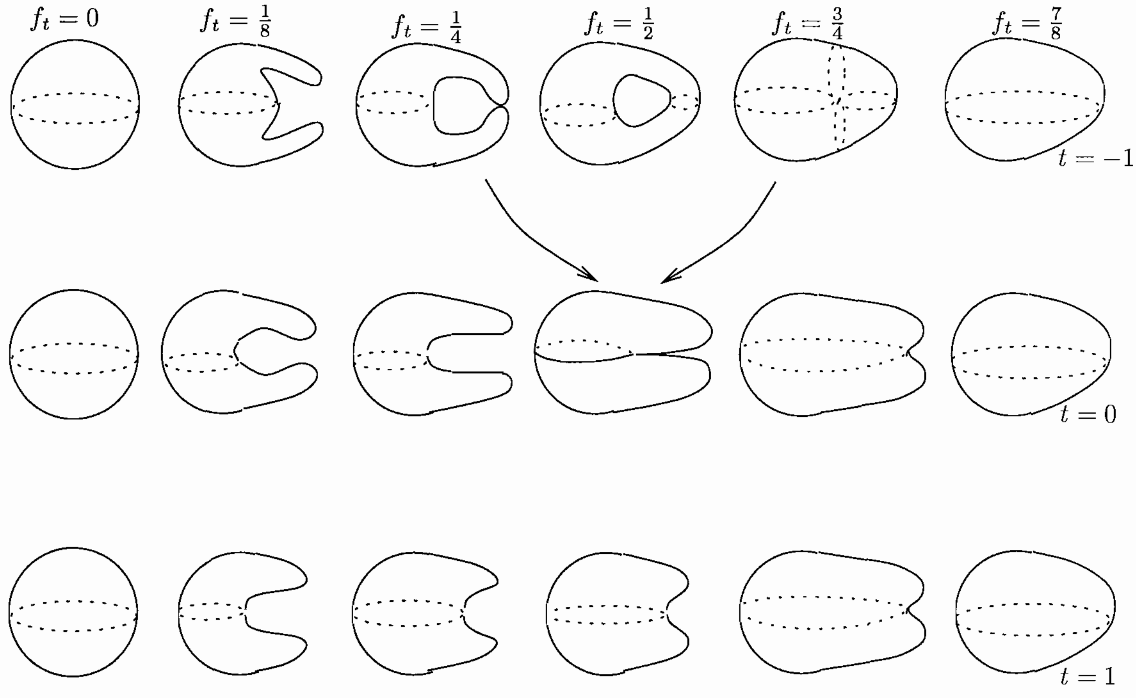

Figure I.7: Two Morse critical points cancelling at a birth-death singularity, from the point of view of selected level sets

I.5 Main Results of Part Two

With appropriate admissibility conditions on critical points of f, we can prove the following theorem. This is the main technical theorem in Part two.

Theorem III.6. *Let f be an admissible wrinkled map with respect to the submersion $\pi : E \to Q$. Let $g_0 : Q \to \mathcal{R}\mathrm{iem}^+(X_0)$ be a smooth map parameterising a compact family of psc-metrics on X_0. Then there is a metric G on the total space E which, for each $y \in Q$, restricts on the fibre $\pi^{-1}(y)$ to a regularised Gromov-Lawson cobordism $\bar{g}'(g_0(y), f|_{\pi^{-1}(y)})$. In the case when the bundle $\pi : E \to Q$ is trivial, there exists a smooth map*

$$Q \longrightarrow \mathcal{R}\mathrm{iem}^+(W, \partial W)$$
$$y \longmapsto \bar{g}'(y),$$

where each $\bar{g}'(y)$ is a regularised Gromov-Lawson cobordism.

As one might expect, the original construction of a Gromov-Lawson cobordism needs to be altered somewhat near cancelling critical points in order to prove such a theorem. This is the reason for the term *regularised* Gromov-Lawson cobordism, a slightly modified verion of the original construction. We will not go into details here except to say that if $\bar{g} = \bar{g}(g_0, f)$ is a GL-cobordism on W and $\bar{g}' = \bar{g}'(g_0, f)$ is its regularised analogue, then $\bar{g}|_{X_1} = \bar{g}'|_{X_1}$.

This last fact about regularised GL-cobordisms allows us to address a problem we discussed earlier. In what sense does the does the metric $g_1 = \bar{g}|_{X_1}$ depend on the choice of admissible Morse function f? Under the right conditions, it turns out that the isotopy type of g_1 is invariant of this choice.

Theorem III.9. *Let $\{W; X_0, X_1\}$ be a smooth compact cobordism where W, X_0 and X_1 are simply connected and W has dimension $n + 1 \geq 6$. Let $f_0, f_1 \in \mathcal{M}^{adm}(W)$ be a pair of admissible Morse functions. Suppose g_0 and g_1 are psc-metrics lying in the same path component of $\mathcal{R}\mathrm{iem}^+(X_0)$. If $\bar{g}_0 = \bar{g}(g_0, f_0)$ and $\bar{g}_1 = \bar{g}(g_1, f_1)$ are Gromov-Lawson cobordisms, then the psc-metrics $g_{0,1} = \bar{g}_0|_{X_1}$ and $g_{1,1} = \bar{g}_1|_{X_1}$ are isotopic metrics in $\mathcal{R}\mathrm{iem}^+(X_1)$.*

The proof of Theorem III.9 uses a number of deep results, in particular the 2-Index Theorem of Hatcher; see Theorem 1.1, Chapter VI, section 1 of [23]. The 2-Index theorem is necessary to show that f_0 and f_1 can be connected up by a path in the space of generalised Morse functions, each of which satisfies the admissibility condition on the indices of its critical points.

CHAPTER II

PART ONE: GROMOV-LAWSON CONCORDANCE IMPLIES ISOTOPY

II.1 Foreword to Part One

In Part One we deal with the construction of Gromov-Lawson cobordisms (GL-cobordisms) as well as prove that, in the case of closed simply connected manifolds of dimension ≥ 5, metrics which are Gromov-Lawson concordant are in fact isotopic. We will organise this as follows. In II.2, we introduce the notions of isotopy and concordance in the space of psc-metrics. We then construct a variety of different psc-metrics on the standard sphere and disk. Among them are metrics we will call, *torpedo*, *double torpedo* and *mixed torpedo* metrics. These metrics have some very nice properties with regard to the notion of isotopy and will play an important role throughout our work.

The construction of a GL-cobordism requires careful anaysis of the original surgery technique. This is done in II.3. In it, we prove some slightly stronger results, in particular Theorem II.11 and also the so-called Improved Surgery Theorem, Theorem II.10. In proving Theorem II.10, we fix the mistake contained in the original proof of this Theorem by Gajer in [10]; see Remark II.3.4. We also show, in Theorem II.19, that the surgery technique goes through for compact families of psc-metrics as well as compact families of embedded surgery spheres. As a consequence, we obtain some important results about how the homotopy type of the space of psc-metrics is affected by surgery on the underlying manifold; see Theorems II.21 and II.22.

In II.4, we finally prove the Gromov-Lawson cobordism Theorem, Theorem II.23, as well as a stronger theorem for compact families, Theorem II.25. This stronger theorem allows us to construct GL-cobordisms which are parametrised continuously by contractible families of admissible Morse functions. As discussed in the introduction, our goal in Part Two is to considerably strengthen this Theorem, to allow for admissible Morse functions with varying critical sets.

The main result in Part One is a partial answer to the question of whether or not concordant psc-metrics are isotopic. A Gromov-Lawson cobordism on the cylinder is a type of concordance which we call a Gromov-Lawson concordance (GL-concordance). Our main result, Theorem II.36, is that, in the case of closed simply connected manifolds of dimension ≥ 5, GL-concordant metrics are isotopic. The proof of this fact is long and technical and involves explicitly constructing an isotopy. In II.5, we introduce the notion of a GL-concordance and provide a simple but non-trivial example. This example illustrates a special case of GL-concordance where the underlying Morse function has just a pair of cancelling critical points. In II.6, we prove that GL-concordance always implies isotopy in this case; see Theorem II.34. Then, in II.7, we use Morse-Smale theory to show that, under the right hypotheses, the more general case reduces down to finitely applications of the special case, to prove the main result.

II.2 Definitions and Preliminary Results

II.2.1 Isotopy and concordance in the space of metrics of positive scalar curvature

Throughout this paper, X will denote a smooth closed compact manifold of dimension n. Later we will also require that X be simply connected and that $n \geq 5$. We will denote by $\mathcal{R}\mathrm{iem}(X)$, the space of all Riemannian metrics on X. The topology on this space is induced by the standard C^k-norm on Riemannian metrics and defined $|g|_k = \max_{i \leq k} \sup_X |\nabla^i g|$. Here ∇ is the Levi-Civita connection for some fixed reference metric and $|\nabla^i g|$ is the Euclidean tensor norm on $\nabla^i g$; see page 54 of [34] for a definition. Note that the topology on $\mathcal{R}\mathrm{iem}(X)$ does not depend on the choice of reference metric. For our purposes it is sufficient (and convenient) to assume that $k = 2$.

Contained inside $\mathcal{R}\mathrm{iem}(X)$, as an open subspace, is the space

$$\mathcal{R}\mathrm{iem}^+(X) = \{g \in \mathcal{R}\mathrm{iem}(X) : R_g > 0\}.$$

Here $R_g : X \to \mathbb{R}$ denotes the scalar curvature of the metric g, although context permitting we will sometimes denote the scalar curvature of a metric as simply R. The space $\mathcal{R}\mathrm{iem}^+(X)$ is the space of metrics on X whose scalar curvature function is strictly positive, i.e. the space of psc-metrics on X. As mentioned in the introduction, the problem of whether or not X admits any psc-metrics has been extensively studied and so unless otherwise stated, we will assume we are working with X so that $\mathcal{R}\text{iem}^+(X) \neq \emptyset$.

It is a straightforward exercise in linear algebra to show that $\mathcal{R}\text{iem}(X)$ is a convex space, i.e. for any pair $g_0, g_1 \in \mathcal{R}\text{iem}(X)$, the path $sg_0 + (1-s)g_1$, where $s \in I$, lies entirely in $\mathcal{R}\text{iem}(X)$. The topology of $\mathcal{R}\text{iem}^+(X)$ on the other hand is far less understood, even at the level of 0-connectedness. Before discussing this any further it is necessary to define the following equivalence relations on $\mathcal{R}\text{iem}^+(X)$.

Definition II.1. The metrics g_0 and g_1 are said to be *isotopic* if they lie in the same path component of $\mathcal{R}\text{iem}^+(X)$. A path $g_s, s \in I$ in $\mathcal{R}\text{iem}^+(X)$ connecting g_0 and g_1 is known as an *isotopy.*

Definition II.2. If there is a metric of positive scalar curvature $\bar{g}$ on the cylinder $X \times I$ so that for some $\delta > 0$, $\bar{g}|_{X\times[0,\delta]} = g_0 + ds^2$ and $\bar{g}|_{X\times[1-\delta,1]} = g_1 + ds^2$, then g_0 and g_1 are said to be *concordant.* The metric $\bar{g}$ is known as a *concordance.*

The following lemma is well known and proofs of various versions of it are found in [14], [10] and [36]. Given its importance to our work, we provide a detailed proof in appendix III.6.3.

Lemma II.1. *Let $g_r, r \in I$ be a smooth path in $\mathcal{R}\text{iem}^+(X)$. Then there exists a constant $0 < \Lambda \leq 1$ so that for every smooth function $f : \mathbb{R} \to [0, 1]$ with $|\dot{f}|, |\ddot{f}| \leq \Lambda$, the metric $g_{f(s)} + ds^2$ on $X \times \mathbb{R}$ has positive scalar curvature.*

Proof. See appendix III.6.3. □

Corollary II.2. *Metrics which are isotopic are also concordant.*

Proof. Let g_0 and g_1 be two psc-metrics, connected by the path g_r in $\mathcal{R}\text{iem}^+(X)$, where $r \in I$. Any continuous path in $\mathcal{R}\text{iem}^+(X)$ may be approximated by a smooth one and so we will assume that g_r is a smooth isotopy. Let f be a smooth increasing function which is of the form

$$f(s) = \begin{cases} 1 & \text{if } s \geq k_2 \\ 0 & \text{if } s \leq k_1 \end{cases}$$

where $k_1 < k_2$. The function f can be chosen with $|\dot{f}|$ and $|\ddot{f}|$ bounded by some arbitrarily small constant provided $k_2 - k_1$ is large enough. Now choose A_1, A_2 so that $A_1 < k_1 < k_2 < A_2$. By the lemma above, the metric $g_{f(s)} + ds^2$ on $X \times [A_1, A_2]$ has positive scalar curvature. This metric can easily be pulled back to obtain the desired concordance on $X \times I$. □

Whether or not the converse of this corollary holds, i.e. concordant metrics are isotopic, is a much more complicated question and one we discussed in the introduction. In particular, when $\dim X \geq 5$, the problem of whether or not concordance implies isotopy is completely open. Recall that a general concordance may be arbitrarily complicated. We will approach this problem restricting our attention to a particular type of concordance, which we construct using the surgery technique of Gromov and Lawson, and which we will call a Gromov-Lawson concordance. An important part of the surgery technique concerns modification of a psc-metric on or near an embedded sphere. For the remainder of this section we will consider a variety of psc-metrics both on the sphere and the disk. These metrics will play an important technical role in later sections.

II.2.2 Warped product metrics on the sphere

We denote by S^n, the standard n-dimensional sphere and assume that $n \geq 3$. We will study metrics on S^n which take the form of *warped* and *doubly warped* product metrics; see description below. All of the metrics we consider will have non-negative sectional and Ricci curvatures, positive scalar curvature and will be *isotopic* to the standard round metric on S^n. The latter fact will be important in the proof of the main theorem, Theorem II.36.

The standard round metric of radius 1, can be induced on S^n via the usual embedding into $\mathbb{R}^{n+1}$. We denote this metric ds_n^2. There are of course many different choices of coordinates with which to realise this metric. For example, the embedding

$$\begin{aligned} (0,\pi) \times S^{n-1} &\longrightarrow \mathbb{R} \times \mathbb{R}^n \\ (t, \theta) &\longmapsto (\cos t, \sin(t) \cdot \theta) \end{aligned}$$

gives rise to the metric $dt^2 + \sin^2(t) ds_{n-1}^2$ on $(0,\pi) \times S^{n-1}$. This extends uniquely to the round metric of radius 1 on S^n. Similarly, the round metric of radius ϵ has the form $dt^2 + \epsilon^2 \sin^2(\frac{t}{\epsilon}) ds_{n-1}^2$ on $(0, \epsilon\pi) \times S^{n-1}$. More generally, by replacing $\sin t$ with a suitable smooth function $f : (0, b) \to (0, \infty)$, we can construct other metrics on S^n. The following proposition specifies necessary and sufficent conditions on f which guarantee smoothness of the metric $dt^2 + f(t)^2 ds_{n-1}^2$ on S^n.

Proposition II.3. (Chapter 1, section 3.4, [34]) *Let $f : (0,b) \to (0,\infty)$ be a smooth function with $f(0) = 0 = f(b)$. Then the metric $g = dt^2 + f(t)^2 ds_{n-1}^2$ is a smooth metric on the sphere S^n if and only if $f^{(even)}(0) = 0$, $\dot{f}(0) = 1$, $f^{(even)}(b) = 0$ and $\dot{f}(b) = -1$.*

Given the uniqueness of the extension, we will regard metrics of the form $dt^2 + f(t)^2 ds_{n-1}^2$ on $(0,b) \times S^{n-1}$ as simply metrics on S^n, provided f satisfies the conditions above. For a general smooth function $f : (0,b) \to (0,\infty)$, a metric of the form $dt^2 + f(t)^2 ds_{n-1}^2$ on $(0,b) \times S^{n-1}$ is known as a *warped product metric*. From page 69 of [34], we obtain the following formulae for the Ricci and scalar curvatures of such a metric. Let $\partial_t, e_1, \ldots, e_{n-1}$ be an orthonormal frame where ∂_t is tangent to the interval $(0,b)$ while each e_i is tangent to the sphere S^{n-1}. Then

$$\begin{aligned} Ric(\partial_t) &= -(n-1)\frac{\ddot{f}}{f}, \\ Ric(e_i) &= (n-2)\frac{1-\dot{f}^2}{f^2} - \frac{\ddot{f}}{f}\ \text{, when } i = 1, \ldots, n-1. \end{aligned}$$

Thus, the scalar curvature is

$$R = -2(n-1)\frac{\ddot{f}}{f} + (n-1)(n-2)\frac{1-\dot{f}^2}{f^2}. \tag{II.2.1}$$

Let $\mathcal{F}(0,b)$ denote the space of all smooth functions $f : (0,b) \to (0,\infty)$ which satisfy the following conditions.

$$\begin{aligned} f(0) &= 0, & f(b) &= 0, \\ \dot{f}(0) &= 1, & \dot{f}(b) &= -1, \\ f^{(even)}(0) &= 0, & f^{(even)}(b) &= 0, \\ \ddot{f} &\leq 0, & \dddot{f}(0) &< 0, \quad \dddot{f}(b) > 0, \\ \ddot{f}(t) &< 0, & &\text{when } t \text{ is near but not at } 0 \text{ and } b. \end{aligned} \tag{II.2.2}$$

Typical elements of $\mathcal{F}(0,b)$ are represented in Fig. II.1. For each function f in $\mathcal{F}(0,b)$, there is an associated smooth metric $g = dt^2 + f(t)^2 ds_{n-1}^2$ on S^n. We will denote the space of all such metrics by $\mathcal{W}(0,b)$. Note that $\mathcal{F}(0,b)$ is assumed to have the standard C^k function space topology with $k \geq 2$; see Chapter 2 of [17] for details.

Proposition II.4. *The space* $\mathcal{W}(0,b) = \{dt^2 + f(t)^2 ds_{n-1}^2 : f \in \mathcal{F}(0,b)\}$ *is a path connected subspace of* $\mathcal{R}\mathrm{iem}^+(S^n)$.

Proof. The first three conditions of (II.2.2) guarantee smoothness of such metrics on S^n, by Proposition II.3. We will now consider the scalar curvature when $0 < t < b$. Recall that $\ddot{f}(t) \leq 0$ and that near the endpoints this inequality is strict. This means that when $0 < t < b$, $|\dot{f}(t)| < 1$ and so while the first term in (II.2.1) is at worst non-negative, the second term is strictly positive. At the end points, several applications of l'Hospital's rule give that

$$\begin{array}{ll} \lim_{t\to 0^+} \frac{-\ddot{f}}{f} = -\dddot{f}(0), & \lim_{t\to 0^+} \frac{1-\dot{f}^2}{f^2} = -\dddot{f}(0) > 0, \\ \lim_{t\to b^-} \frac{-\ddot{f}}{f} = \dddot{f}(b), & \lim_{t\to b^-} \frac{1-\dot{f}^2}{f^2} = \dddot{f}(b) > 0. \end{array}$$

Thus, $\mathcal{W}(0,b) \subset \mathcal{R}\mathrm{iem}^+(S^n)$. Path connectedness now follows from the convexity of $\mathcal{F}(0,b)$ which in turn follows from an elementary calculation. □

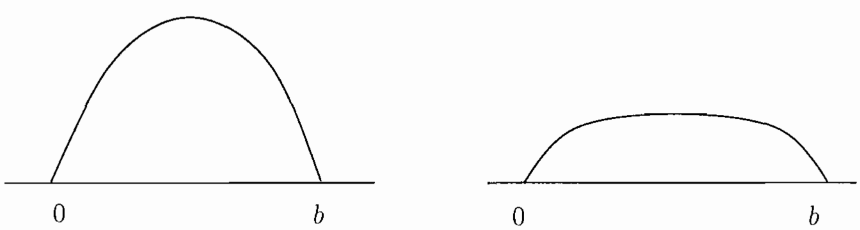

Figure II.1: Typical elements of $\mathcal{F}(0,b)$

It is convenient to allow b to vary. Thus, we will define $\mathcal{F} = \bigcup_{b\in(0,\infty)} \mathcal{F}(0,b)$ and $\mathcal{W} = \bigcup_{b\in(0,\infty)} \mathcal{W}(0,b)$. Each metric in $\mathcal{W}$ is defined on $(0,b)\times S^{n-1}$ for some $b>0$. In particular, the round metric of radius ϵ, $\epsilon^2 ds_n^2$, is an element of $\mathcal{W}(0,\epsilon\pi)$.

Proposition II.5. *The space* $\mathcal{W}$ *is a path connected subspace of* $\mathcal{R}\mathrm{iem}^+(S^n)$.

Proof. Let g be an element of $\mathcal{W}$. Then $g = dt^2 + f(t)^2 ds_{n-1}^2$ on $(0,b)\times S^{n-1}$ for some $f \in \mathcal{F}(0,b)$ and some $b > 0$. As $\mathcal{F}(0,b)$ is convex, there is a path connecting g to the metric $dt^2 + (\frac{b}{\pi})^2 \sin^2(\frac{\pi t}{b}) ds_{n-1}^2$, the round metric of radius $(\frac{b}{\pi})^2$ in $\mathcal{W}(0,b)$. As all round metrics on S^n are isotopic by an obvious rescaling, g can be isotopied to any metric in the space. □

II.2.3 Torpedo metrics on the disk

A *δ-torpedo metric* on a disk D^n, denoted $g^n_{tor}(\delta)$, is an $O(n)$ symmetric positive scalar curvature metric which is a product with the standard $n-1$-sphere of radius δ near the boundary of D^n and is the standard metric on the n-sphere of radius δ near the centre of the disk. It is not hard to see how such metrics can be constructed. Let f_δ be a smooth function on $(0,\infty)$ which satisfies the following conditions.

(i) $f_\delta(t) = \delta \sin\left(\frac{t}{\delta}\right)$ when t is near 0.

(ii) $f_\delta(t) = \delta$ when $t \geq \delta\frac{\pi}{2}$.

(iii) $\ddot{f}_\delta(t) \leq 0$.

From now on f_δ will be known as a *δ-torpedo function.*

Let r be the standard radial distance function on $\mathbb{R}^n$. It follows from Proposition II.3 that the metric $dr^2 + f_\delta(r)^2 ds^2_{n-1}$ on $(0,\infty)\times S^{n-1}$ extends smoothly as a metric on $\mathbb{R}^n$. The resulting metric is a torpedo metric of radius δ on $\mathbb{R}^n$. By restricting to $(0,b)\times S^{n-1}$ for some $b > \delta\frac{\pi}{2}$ we obtain a torpedo metric on a disk D^n; see Fig. II.2. From formula (II.2.1), it is clear that this metric has positive scalar curvature and moreover, the scalar curvature can be bounded below by an arbitrarily large constant by choosing δ sufficiently small.

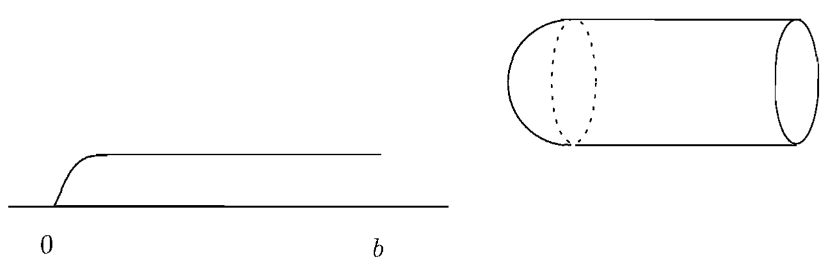

Figure II.2: A torpedo function and the resulting torpedo metric

We will refer to the cylindrical part of this metric as the *tube*, and the remaining piece as the *cap* of $g^n_{tor}(\delta)$. Notice that we can always isotopy $g^n_{tor}(\delta)$ to make the tube arbitrarily long. Strictly speaking then, $g^n_{tor}(\delta)$ denotes a collection of isotopic metrics, each with isometric cap of radius δ. It is convenient however, to think of $g^n_{tor}(\delta)$ as a fixed metric, the tube length of which may be adjusted if necessary.

The torpedo metric on a disk D^n can be used to construct a collection of psc-metrics on S^n which will be of use to us later on. The first of these is the double torpedo metric on S^n. By considering the torpedo metric as a metric on a hemisphere, we can obtain a metric on S^n by taking its double. More precisely let $\bar{f}_\delta(t)$ be the smooth function on $(0, b)$ which satisfies

(i) $\bar{f}_\delta(t) = f_\delta(t)$ on $[0, \frac{b}{2}]$

(ii) $\bar{f}_\delta(t) = f_\delta(b - t)$ on $[\frac{b}{2}, b]$,

where $\frac{b}{2} > \delta\frac{\pi}{2}$.

As $\bar{f} \in \mathcal{F}(0, b)$, the metric $dt^2 + \bar{f}_\delta(t)^2 ds_{n-1}^2$ on $(0, b) \times S^{n-1}$ gives rise to a smooth psc-metric on S^n. Such a metric will be called a *double torpedo metric of radius* δ and denoted $g_{Dtor}^n(\delta)$; see Fig. II.3. Then Proposition II.5 implies that $g_{Dtor}^n(\delta)$ is isotopic to ds_n^2.

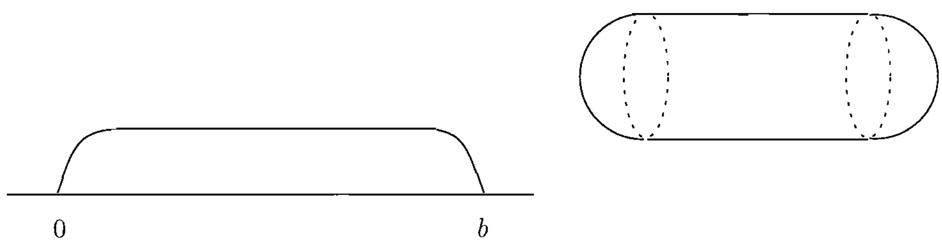

Figure II.3: A double torpedo function and the resulting double torpedo metric

II.2.4 Doubly warped products and mixed torpedo metrics

Henceforth p and q will denote a pair of non-negative integers satisfying $p + q + 1 = n$. The standard sphere S^n decomposes as a union of sphere-disk products as shown below.

$$\begin{aligned} S^n &= \partial D^{n+1} \\ &= \partial(D^{p+1} \times D^{q+1}), \\ &= (S^p \times D^{q+1}) \cup_{S^p \times S^q} (D^{p+1} \times S^q). \end{aligned}$$

We can utilise this decomposition to construct a new metric on S^n. Equip $S^p \times D^{q+1}$ with the product metric $\epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta)$. Then equip $D^{p+1} \times S^q$ with $g_{tor}^{p+1}(\epsilon) + \delta^2 ds_q^2$. These metrics glue together smoothly along the common boundary $S^p \times S^q$ to form a smooth metric on S^n.

Such metrics will be known as *mixed torpedo metrics* on S^n and denoted $g_{Mtor}^{p,q}$; see Fig. II.4. For the remainder of this section we will show how to realise these metrics in a more computationally useful form.

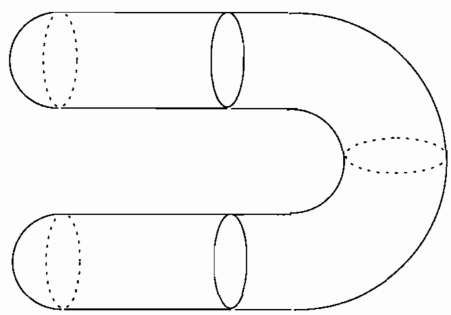

Figure II.4: S^n decomposed as $(S^p \times D^{q+1}) \cup_{S^p \times S^q} (D^{p+1} \times S^q)$ and equipped with a mixed torpedo metric $g_{Mtor}^{p,q}$

Recall that a metric of the form $dt^2 + f(t)^2 ds_{n-1}^2$ on $(0,b) \times S^{n-1}$, where $f:(0,b) \to (0,\infty)$ is a smooth function, is known as a *warped product metric*. We have observed that the standard round sphere metric: ds_n^2, can be represented as the warped product metric $dt^2 + \sin^2(t)ds_{n-1}^2$ on $(0,\pi) \times S^{n-1}$. The notion of a warped product metric on $(0,b) \times S^{n-1}$ generalises to something called a *doubly warped product metric* on $(0,b) \times S^p \times S^q$. Here the metric takes the form $dt^2 + u(t)^2 ds_p^2 + v(t)^2 ds_q^2$, where $u, v : (0,b) \to (0,\infty)$ are smooth functions.

From page 72 of [34], we obtain the following curvature formulae. Let $\partial_t, e_1, \ldots, e_p, e_1', \ldots, e_q'$ be an orthonormal frame where $e_1, \ldots, e_p$ are tangent to S^p and $e_1', \ldots, e_q'$ are tangent to S^q. Then

$$\begin{aligned} Ric(\partial_t) &= -(p)\frac{\ddot{u}}{u} - (q)\frac{\ddot{v}}{v}, \\ Ric(e_i) &= (p-1)\frac{1-\dot{u}^2}{u^2} - \frac{\ddot{u}}{u} - q\frac{\dot{u}\dot{v}}{uv} \quad, i = 1, \ldots, p, \\ Ric(e_i') &= (q-1)\frac{1-\dot{v}^2}{v^2} - \frac{\ddot{v}}{v} - p\frac{\dot{u}\dot{v}}{uv} \quad, i = 1, \ldots, q. \end{aligned}$$

Thus, the scalar curvature is

$$R = -2p\frac{\ddot{u}}{u} - 2q\frac{\ddot{v}}{v} + p(p-1)\frac{1-\dot{u}^2}{u^2} + q(q-1)\frac{1-\dot{v}^2}{v^2} - 2pq\frac{\dot{u}\dot{v}}{uv}. \tag{II.2.3}$$

We observe that the round metric ds_n^2 can be represented by a doubly warped product. Recalling that $p+q+1=n$, consider the map

$$\begin{aligned} (0, \frac{\pi}{2}) \times S^p \times S^q &\longrightarrow \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \\ (t, \phi, \theta) &\longmapsto (\cos(t)\cdot\phi, \sin(t)\cdot\theta) \end{aligned} \tag{II.2.4}$$

Here S^p and S^q denote the standard unit spheres in $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ respectively. The metric induced by this embedding is given by the formula

$$dt^2 + \cos^2(t)ds_p^2 + \sin^2(t)ds_q^2,$$

a doubly warped product representing the round metric on S^n. More generally the round metric of radius ϵ takes the form $dt^2 + \epsilon^2\cos^2(\frac{t}{\epsilon})ds_p^2 + \epsilon^2\sin^2(\frac{t}{\epsilon})ds_q^2$ on $(0, \epsilon\frac{\pi}{2}) \times S^p \times S^q$.

As before, by imposing appropriate conditions on the functions $u, v : (0, b) \to (0, \infty)$, the metric $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_q^2$ gives rise to a smooth metric on S^n. By combining propositions 1 and 2 on page 13 of [34], we obtain the following proposition which makes these conditions clear.

Proposition II.6. (Page 13, [34]) *Let $u, v : (0, b) \to (0, \infty)$ be smooth functions with $u(b) = 0$ and $v(0) = 0$. Then the metric $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_q^2$ on $(0, b) \times S^p \times S^q$ is a smooth metric on S^n if and only if the following conditions hold.*

$$u(0) > 0, \quad u^{(odd)}(0) = 0, \quad \dot{u}(b) = -1, \quad u^{(even)}(b) = 0. \tag{II.2.5}$$

$$v(b) > 0, \quad v^{(odd)}(b) = 0, \quad \dot{v}(0) = 1, \quad v^{(even)}(0) = 0. \tag{II.2.6}$$

Let $\mathcal{U}(0, b)$ denote the space of all functions $u : (0, b) \to (0, \infty)$ which satisfy (II.2.5) above and the condition that $\ddot{u} \leq 0$ with $\ddot{u}(t) < 0$ when t is near but not at b and $\dddot{u}(b) > 0$.

Similarly $\mathcal{V}(0, b)$ will denote the space of all functions $v : (0, b) \to (0, \infty)$ which satisfy (II.2.6) and for which $\ddot{v} \leq 0$ with $\ddot{v}(t) < 0$ when t is near but not at 0 and $\dddot{v}(0) < 0$.

Each pair u, v from the space $\mathcal{U}(0, b) \times \mathcal{V}(0, b)$ gives rise to a metric $dt^2 + u(t)^2ds_p^2 + v(t)^2ds_q^2$ on S^n. We denote the space of such metrics

$$\hat{\mathcal{W}}^{p,q}(0, b) = \{dt^2 + u(t)^2ds_p^2 + v(t)^2ds_q^2 : (u, v) \in \mathcal{U}(0, b) \times \mathcal{V}(0, b)\}.$$

We now obtain the following lemma.

Lemma II.7. *Let $n \geq 3$ and let p and q be any pair of non-negative integers satisfying $p+q+1 = n$. Then the space $\hat{\mathcal{W}}^{p,q}(0,b)$ is a path connected subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$.*

Proof. Let $g = dt^2 + u(t)^2 ds_p^2 + v(t)^2 ds_q^2$ be an element of $\hat{\mathcal{W}}^{p,q}(0,b)$. Smoothness of this metric on S^n follows from Proposition II.6. We will first show that g has positive scalar curvature when $0 < t < b$. Recall that u and v are both concave downward, that is $\ddot{u}, \ddot{v} < 0$. This means that the first two terms in (II.2.3) are at worst non-negative. Downward concavity and the fact that $\dot{u}(0) = 0$ and $\dot{u}(b) = -1$ imply that $-1 < \dot{u} \leq 0$. A similar argument gives that $0 \leq \dot{v} < 1$. This means that the fifth term in (II.2.3) is also non-negative and at least one of the third and fourth terms in (II.2.3) is strictly positive (the other may be 0 for dimensional reasons). When $t = 0$ and $t = b$, some elementary limit computations using l'Hospital's rule show that the scalar curvature is positive. Thus, $\hat{\mathcal{W}}(0,b)^{p,q} \subset \mathcal{R}\mathrm{iem}^+(S^n)$. Finally, path connectivity follows immediately from the convexity of the space $\mathcal{U}(0,b) \times \mathcal{V}(0,b)$. □

As before, it is convenient to allow b to vary. Thus, we define $\mathcal{U} \times \mathcal{V} = \bigcup_{b \in (0,\infty)} \mathcal{U}(0,b) \times \mathcal{V}(0,b)$ and $\hat{\mathcal{W}}^{p,q} = \bigcup_{b \in (0,\infty)} \hat{\mathcal{W}}^{p,q}(0,b)$. Finally we let $\hat{\mathcal{W}} = \bigcup_{p+q+1=n} \hat{\mathcal{W}}^{p,q}$ where $0 \leq p, q \leq n+1$.

Proposition II.8. *Let $n \geq 3$. The space $\hat{\mathcal{W}}$ is a path connected subspace of $\mathcal{R}\mathrm{iem}^+(S^n)$.*

Proof. The proof that $\hat{\mathcal{W}}^{p,q}$ is path connected is almost identical to that of Proposition II.5. The rest follows from the fact that each $\hat{\mathcal{W}}^{p,q}$ contains the round metric $ds_n^2 = dt^2 + \cos^2 t ds_p^2 + \sin^2 t ds_q^2$. □

At the beginning of this section we demonstrated that S^n could be decomposed into a union of $S^p \times D^{q+1}$ and $D^{p+1} \times S^q$. This can be seen explicitly by appropriate restriction of the embedding in (II.2.4). Thus, provided t is near 0, the metric $dt^2 + u(t)^2 ds_p^2 + v(t)^2 ds_q^2$, with $u, v \in \mathcal{U}(0,b) \times \mathcal{V}(0,b)$, is a metric on $S^p \times D^{q+1}$. When t is near b we obtain a metric on $D^{p+1} \times S^q$. We can now construct a mixed torpedo metric on S^n, as follows. Let f_ϵ and f_δ be the torpedo functions on $(0,b)$ defined in section II.2.3 with $b > \max\{\epsilon\pi, \delta\pi\}$. Then the metric

$$g_{Mtor}^{p,q} = dt^2 + f_\epsilon(b-t)^2 ds_p^2 + f_\delta(t)^2 ds_q^2 \tag{II.2.7}$$

is a mixed torpedo metric on S^n; see Fig. II.5.

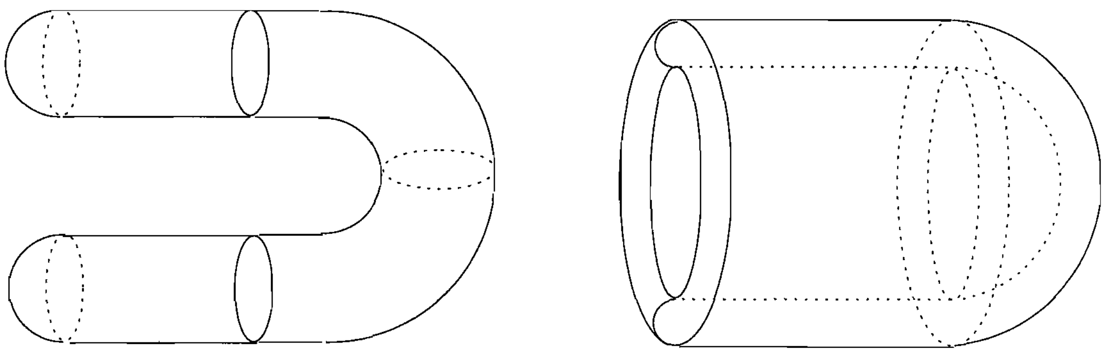

Figure II.5: The mixed torpedo metrics $g^{p,q}_{Mtor}$ and $g^{p+1,q-1}_{Mtor}$

Lemma II.9. *Let $n \geq 3$. For any non-negative integers p and q with $p+q+1=n$, the metric $g^{p,q}_{Mtor}$ is isotopic to ds_n^2.*

Proof. An elementary calculation shows that the functions $f_\epsilon(b-t)$ and $f_\delta(t)$ lie in $\mathcal{U}(0,b)$ and $\mathcal{V}(0,b)$ respectively. Thus, $g^{p,q}_{Mtor} \in \hat{\mathcal{W}}^{p,q}(0,b)$. As the standard round metric lies in $\hat{\mathcal{W}}^{p,q}(0,b)$, the proof follows from Proposition II.8. □

II.2.5 Inducing a mixed torpedo metric with an embedding

We close this section with a rather technical observation which will be of use later on. It is of course possible to realise mixed torpedo metrics on the sphere as the induced metrics of some embedding. Let $\mathbb{R}^{n+1}=\mathbb{R}^{p+1}\times\mathbb{R}^{q+1}$ where of course $p+q+1=n$. Let (ρ,ϕ) and (r,θ) denote standard spherical coordinates on $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ where ρ and r are the respective Euclidean distance functions and $\phi\in S^p$ and $\theta\in S^q$. Then equip $\mathbb{R}^{n+1}=\mathbb{R}^{p+1}\times\mathbb{R}^{q+1}$ with the metric $h=h^{p,q}$ defined

$$h^{p,q}=d\rho^2+f_\epsilon(\rho)^2ds_p^2+dr^2+f_\delta(r)^2ds_q^2, \tag{II.2.8}$$

shown in Fig. II.6, where $f_\epsilon, f_\delta:(0,\infty)\to(0,\infty)$ are the torpedo functions defined in section II.2.3.

We will now parametrise an embedded sphere S^n in $(\mathbb{R}^{n+1},h)$, the induced metric on which will be precisely the mixed torpedo metric described earlier. Let c_1 and c_2 be constants satisfying $c_1>\epsilon\frac{\pi}{2}$ and $c_2>\delta\frac{\pi}{2}$. Let $a=(a_1,a_2)$ denote a smooth unit speed curve in the first quadrant of $\mathbb{R}^2$ which begins at $(c_1,0)$ follows a vertical trajectory, bends by an angle of $\frac{\pi}{2}$ towards the vertical axis and continues as a horizontal line to end at $(0,c_2)$. We will assume that the bending takes

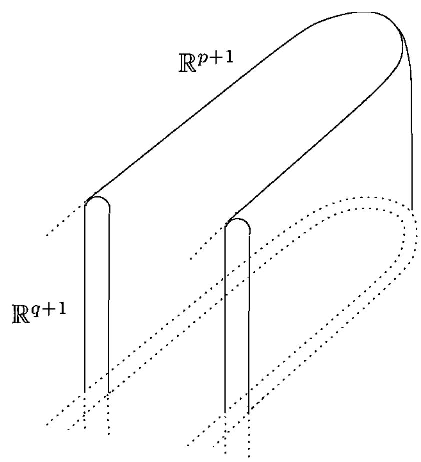

Figure II.6: The plane $\mathbb{R}^{n+1}$ equipped with the metric h

place above the horizontal line through $(0, \delta\frac{\pi}{2})$; see Fig. II.7. We also assume that $a_1 \in \mathcal{U}(0,b)$ and $a_2 \in \mathcal{V}(0,b)$ for sufficiently large $b > 0$.

We will now specify an embedding of the n-sphere into $(\mathbb{R}^{n+1}, h)$ which induces the mixed torpedo metric $g_{Mtor}^{p,q}$ described above. Let J be the embedding defined as follows

$$J : (0,b) \times S^p \times S^q \longrightarrow \mathbb{R}^{p+1} \times \mathbb{R}^{q+1},$$
$$(t, \theta, \phi) \longmapsto ((a_1(t), \phi), (a_2(t), \theta));$$

see Fig. II.8.

Provided that ϵ and δ are chosen sufficiently small, this embedding induces the mixed torpedo metric $g_{Mtor}^{p,q}$ on S^n. Indeed, we have

$$\begin{aligned} J^*h &= J^*(d\rho^2 + f_\epsilon(\rho)^2 ds_p^2 + dr^2 + f_\delta(r)^2 ds_q^2) \\ &= dt^2 + f_\epsilon(\alpha_1(t))^2 ds_p^2 + f_\delta(\alpha_2(t))^2 ds_q^2 \\ &= dt^2 + f_\epsilon(b-t)^2 ds_p^2 + f_\delta(t)^2 ds_q^2 \\ &= g_{Mtor}^{p,q}. \end{aligned}$$

The second equality follows from the fact that α is a unit speed curve and the third equality from the fact that $f_\epsilon(s)$ and $f_\delta(s)$ are both constant when $s > \max\{\epsilon\frac{\pi}{2}, \delta\frac{\pi}{2}\}$.

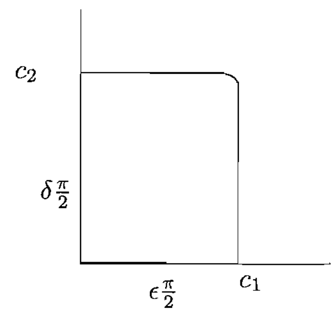

Figure II.7: The curve α

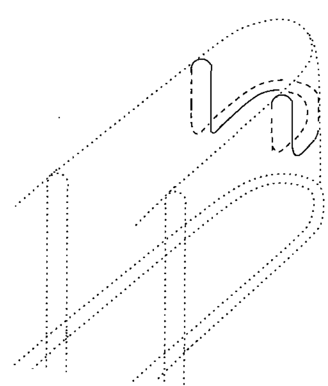

Figure II.8: The map J gives a parameterisation for S^n

II.3 Revisiting the Surgery Theorem

Over the the next two sections we will provide a proof of Theorem II.23. The proof involves the construction of a psc-metric on a compact cobordism $\{W^{n+1}; X_0, X_1\}$ which extends a psc-metric g_0 from X_0 and is a product near ∂W. A specific case of this is Theorem II.10 (stated below) which we prove in this section. It can be thought of as a building block for the more general case of the proof of Theorem II.23 which will be completed in II.4. Before stating Theorem II.10, it is worth briefly reviewing some basic notions about surgery and cobordism.

II.3.1 Surgery and cobordism

A *surgery* on a smooth manifold X of dimension n, is the construction of a new n-dimensional manfiold X' by removing an embedded sphere of dimension p from X and replacing it with a sphere of dimension q where $p + q + 1 = n$. More precisely, suppose $i : S^p \hookrightarrow X$ is an embedding. Suppose also that the normal bundle of this embedded sphere is trivial. Then we can extend i to an embedding $\bar{i} : S^p \times D^{q+1} \hookrightarrow X$. The map $\bar{i}$ is known as a *framed embedding* of S^p. By removing an open neighbourhood of S^p, we obtain a manifold $X \setminus \bar{i}(S^p \times \mathring{D}^{q+1})$ with boundary $S^p \times S^q$. Here $\mathring{D}^{q+1}$ denotes the interior of the disk D^{q+1}. As the handle $D^{p+1} \times S^q$ has the same boundary, we can use the map $\bar{i}|_{S^p \times S^q}$, to glue the manifolds $X \setminus \bar{i}(S^p \times \mathring{D}^{q+1})$ and $D^{p+1} \times S^q$ along their common boundary and obtain the manifold

$$X' = (X \setminus \bar{i}(S^p \times \mathring{D}^{q+1})) \cup_{\bar{i}} (D^{p+1} \times S^q).$$

The manifold X' can be taken as being smooth (although some minor smoothing of corners is necessary where the attachment took place). Topologically, X' is quite different from the manifold X. It is well known that the topology of X' depends on the embedding i and the choice of framing $\bar{i}$; see [35] for details. In the case when i embeds a sphere of dimension p we will describe a surgery on this sphere as either a p-*surgery* or a *surgery of codimenison* $q + 1$.

The *trace* of a p-surgery is a smooth $n+1$-dimensional manifold W with boundary $\partial W = X \sqcup X'$; see Fig. II.9. It is formed by attaching a solid handle $D^{p+1} \times D^{q+1}$ onto the cylinder $X \times I$, identifying the $S^p \times D^{q+1}$ part of the boundary of $D^{p+1} \times D^{q+1}$ with the embedded $S^p \times D^{q+1}$ in $X \times \{1\}$ via the framed embedding $\bar{i}$. The trace of a surgery is an example of a cobordism.

In general, a *cobordism* between n-dimensional manifolds X_0 and X_1 is an $n+1$-dimensional manifold $W^{n+1} = \{W^{n+1}; X_0, X_1\}$ with boundary $\partial W = X_0 \sqcup X_1$. Cobordisms which arise as the trace of a surgery are known as *elementary cobordisms.* By taking appropriate unions of elementary cobordisms it is clear that more general cobordisms can be constructed. An important consequence of Morse theory is that the converse is also true, that is any compact cobordism $\{W^{n+1}; X_0, X_1\}$ may be decomposed as a finite union of elementary cobordisms.

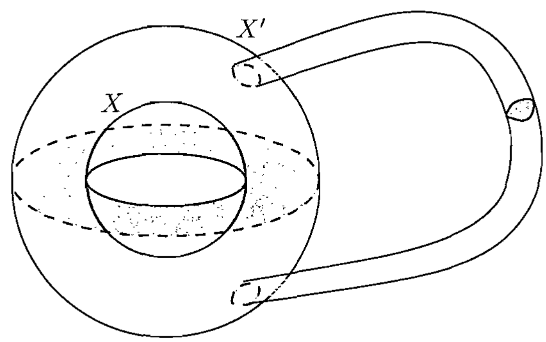

Figure II.9: The trace of a p-surgery on X

II.3.2 Surgery and positive scalar curvature

The Surgery Theorem of Gromov-Lawson and Schoen-Yau can now be stated as follows.

Surgery Theorem. ([14], [38]) *Let (X, g) be a Riemannian manifold of positive scalar curvature. Let X' be a manifold which has been obtained from X by a surgery of codimension at least 3. Then X' admits a metric g' which also has positive scalar curvature.*

Remark II.3.1. *We will concentrate on the technique used by Gromov and Lawson, however, the proof of the Surgery Theorem by Schoen and Yau in [38] is rather different and involves conformal methods. There is in fact another approach to the problem of classifying manifolds of positive scalar curvature which involves conformal geometry; see for example the work of Akutagawa and Botvinnik in [1].*

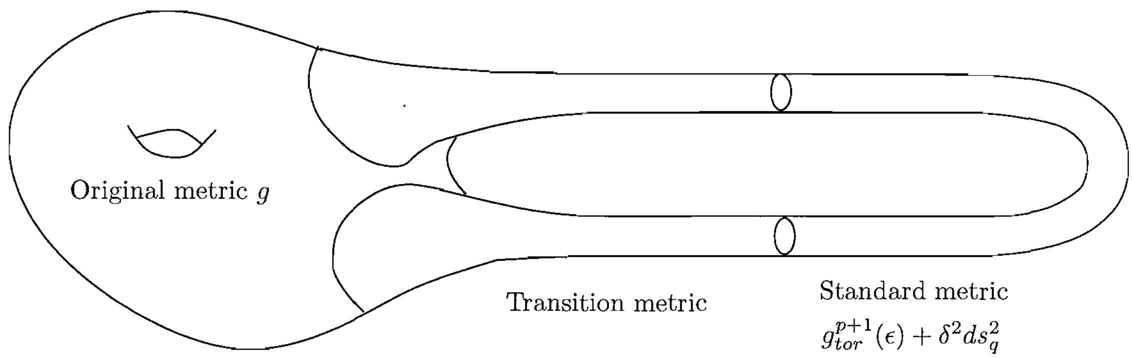

Figure II.10: The metric g', obtained by the Surgery Theorem

In their proof, Gromov and Lawson provide a technique for constructing the metric g'; see Fig. II.10. Their technique can be strengthened to yield the following theorem.

Theorem II.10. *Let (X, g) be a Riemannian manifold of positive scalar curvature. If W is the trace of a surgery on X in codimension at least 3, then we can extend the metric g to a metric $\bar{g}$ on W which has positive scalar curvature and is a product near the boundary.*

In fact, the restriction of the metric $\bar{g}$ to X', the boundary component of W which is the result of the surgery, is the metric g' of the Surgery Theorem. Theorem II.10 is sometimes referred to as the Improved Surgery Theorem and was originally proved by Gajer in [10]. We have two reasons for providing a proof of Theorem II.10. Firstly, there is an error in Gajer's original proof. Secondly, this construction will be used as a "building block" for generating concordances. In turn, it will allow us to describe a space of concordances; see section I.4 for a discussion of this.

The proof of Theorem II.10 will dominate much of the rest of this section. We will first prove a theorem which strengthens the Surgery Theorem in a slightly different way; see Fig. II.11. This is Theorem II.11 below, which will play a vital role throughout our work.

Theorem II.11. *Let (X, g) be an n-dimensional Riemannian manifold of positive scalar curvature and let g_p be any metric on the sphere S^p. Suppose $i : S^p \hookrightarrow X$ is an embedding of S^p, with trivial normal bundle. Suppose also that $p + q + 1 = n$ and that $q \geq 2$. Then, for some $\delta > 0$ there is an isotopy of g, to a psc-metric g_{std} on X, which has the form $g_p + g_{tor}^{q+1}(\delta)$ on a tubular neighbourhood of the embedded S^p and is the original metric g away from this neighbourhood.*

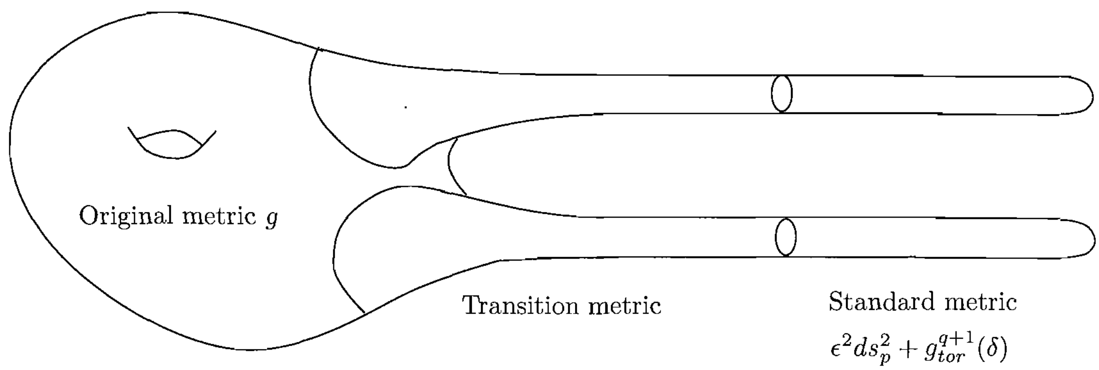

Figure II.11: The "surgery-ready" metric obtained by Theorem II.11

Corollary II.12. *There is a metric $\bar{g}$ on $X \times I$ satisfying*

(i) $\bar{g}$ has positive scalar curvature.

(ii) $\bar{g}$ restricts to g on $X \times \{0\}$, g_{std} on $X \times \{1\}$ and is product near the boundary.

$\bar{g}$ is therefore a concordance of g and g_{std}.

Proof. This follows immediately from Lemma II.2. □

Remark II.3.2. *The proof of Theorem II.11 is not made any simpler by choosing a particular metric for g_p. Indeed, the embedded sphere S^p can be replaced by any closed codimension≥ 3 submanifold with trivial normal bundle, and the result still holds with an essentially identical proof. That said, we are really only interested in the case of an embedded sphere and moreover, the case when g_p is the round metric $\epsilon^2 ds_p^2$.*

The proof of Theorem II.11 is long and technical. Contained in it is the proof of the original Surgery Theorem of Gromov and Lawson; see [14]. Their construction directly implies that the metric g can be replaced by the psc-metric g_{std} described in the statement of Theorem II.11, where in this case $g_p = \epsilon^2 ds_p^2$. Thus, Gromov and Lawson prepare the metric for surgery by making it standard near the surgery sphere. By performing the surgery entirely on the standard region, it is then possible to attach a handle $D^{p+1} \times S^q$ with a correponding standard metric, $g_{tor}^{p+1}(\epsilon) + \delta^2 ds_q^2$ onto $X \setminus \bar{i}(S^p \times \overset{\circ}{D}{}^{n-p})$, as in Fig. II.10. Rather than attaching a handle metric, Theorem II.11 states that the "surgery-ready" metric g_{std} on X; see Fig. II.11, is actually isotopic to the original metric g. Thus, the concordance $\bar{g}$ on $X \times I$, which is described in Corollary II.12, can be built. The proof of Theorem II.10 then proceeds by attaching a solid handle $D^{p+1} \times D^{q+1}$ to $X \times I$, with an appropriate standard metric. After smoothing, this will result in a metric of positive scalar curvature on the trace of the surgery on S^p. The only remaining task in the proof of Theorem II.10 is to show that this metric can be adjusted to also carry a product structure near the boundary.

II.3.3 Outline of the proof of Theorem II.11

Although the result is known, Theorem II.11 is based on a number of technical lemmas from a variety of sources, in particular [14], [36]. For the most part, it is a reworking of Gromov and Lawson's proof of the Surgery Theorem. To aid the reader we relegate many of the more technical proofs to the appendix. We begin with a brief summary.

Part 1: Using the exponential map we can specify a tubular neighbourhood $N \cong S^p \times D^{q+1}$, of the embedded sphere S^p. Henceforth, all of our work will take place in this neighbourhood. We construct a hypersurface M in $N \times \mathbb{R}$ where $N \times \mathbb{R}$ is equipped with the metric $g + dt^2$. Letting r denote the radial distance from $S^p \times \{0\}$ in N, this hypersurface is obtained by pushing out bundles of geodesic spheres of radius r in N along the t-axis with respect to some smooth curve γ of the type depicted in Fig. II.12. In Lemmas II.14 and II.15, we compute the scalar curvature of the metric g_γ which is induced on the hypersurface M.

Part 2: We recall the fact that γ can be chosen so that the metric g_γ has positive scalar curvature. This fact was originally proved in [14] although later, in [36], an error in the original proof was corrected. We will employ the method used by Rosenberg and Stolz in [36] to construct such a curve γ. We will then demonstrate that γ can be homotopied through appropriate curves back to the vertical axis, inducing an isotopy from the psc-metric g_γ back to the orginal psc-metric g. We will also comment on the error in the proof of the "Improved Surgery Theorem", Theorem 4 in [10]; see Remark II.3.4.

Part 3: We will now make a further deformation to the metric g_γ induced on M. Here we restrict our attention to the part of M arising from the torpedo part of γ. Lemma II.13 implies that M can be chosen so that the metric induced on the fibre disks can be made arbitrarily close to the standard torpedo metric of radius δ. It is therefore possible to isotopy the metric g, through psc-metrics, to one which, near S^p, is a Riemannian submersion with base metric $g|_{S^p}$ and fibre metric $g_{tor}^{q+1}(\delta)$. Using the formulae of O'Neill (Chapter 9 of [2]), we will show that the positivity of the curvature on the disk factor allows us to isotopy through psc-submersion metrics near S^p to obtain the desired metric $g_{std} = g_p + g_{tor}^{q+1}(\delta)$.

Proof. Let X^n be a manifold of dimension $n \geq 3$ and g a metric of positive scalar curvature on X.

II.3.4 Part 1 of the proof: Curvature formulae for the first deformation.

Let $i : S^p \hookrightarrow X$ be an embedding with trivial normal bundle, denoted by $\mathcal{N}$, and with $q \geq 2$ where $p + q + 1 = n$. By choosing an orthonormal frame for $\mathcal{N}$ over $i(S^p)$, we specify a bundle isomorphism $\tilde{i} : S^p \times \mathbb{R}^{q+1} \to \mathcal{N}$. Points in $S^p \times \mathbb{R}^{q+1}$ will be denoted (y, x). Let r denote the standard Euclidean distance function in $\mathbb{R}^{q+1}$ and let $D^{q+1}(\bar{r}) = \{x \in \mathbb{R}^{q+1} : r(x) \leq \bar{r}\}$ denote the standard Euclidean disk of radius $\bar{r}$ in $\mathbb{R}^{q+1}$. Provided $\bar{r}$ is sufficiently small, the composition $\exp \circ \tilde{i}|_{S^p \times D^{q+1}(\bar{r})}$, where exp denotes the exponential map with respect to the metric g, is an embedding. We will denote by $N = N(\bar{r})$, the image of this embedding and the coordinates (y, x) will be used to denote points on N. Note that curves of the form $\{y\} \times l$, where l is a ray in $D^q(\bar{r})$ emanating from 0, are geodesics in N.

Before proceeding any further we state a lemma concerning the metric induced on a geodesic sphere of a Riemannian manifold. Fix $z \in X$ and let D be a normal coordinate ball of radius $\bar{r}$ around z. Recall, this means first choosing an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_zX. This determines an isomorphism $E : (x_1, \ldots, x_n) \mapsto x_1e_1 + \cdots + x_ne_n$ from $\mathbb{R}^n$ to T_zX. The composition $E^{-1} \circ \exp^{-1}$ is a coordinate map provided we restrict it to an appropriate neighbourhood of z. Thus, we identify $D = \{x \in \mathbb{R}^n : |x| \leq \bar{r}\}$. The quantity $r(x) = |x|$ is the radial distance from the point z, and $S^{n-1}(\epsilon) = \{x \in \mathbb{R}^n : |x| = \epsilon\}$ will denote the geodesic sphere of radius ϵ around z.

Lemma II.13. (Lemma 1, [14])

(a) The principal curvatures of the hypersurfaces $S^{n-1}(\epsilon)$ in D are each of the form $\frac{-1}{\epsilon} + O(\epsilon)$ for ϵ small.

(b) Furthermore, let g_ϵ be the induced metric on $S^{n-1}(\epsilon)$ and let $g_{0,\epsilon}$ be the standard Euclidean metric of curvature $\frac{1}{\epsilon^2}$. Then as $\epsilon \to 0$, $\frac{1}{\epsilon^2} g_\epsilon \to \frac{1}{\epsilon^2} g_{0,\epsilon} = g_{0,1}$ in the C^2-topology.

Remark II.3.3. *We use the following notation. A function $f(r)$ is $O(r)$ as $r \to 0$ if $\frac{f(r)}{r} \to$ constant as $r \to 0$.*

Proof. See appendix III.6.3. □

This lemma was originally proved in [14]. In the appendix, we provide a complete proof, which includes details suppressed in the original; see Theorem ??. In order to deform the metric on N we will construct a hypersurface in $N \times \mathbb{R}$. Let r denote the radial distance from $S^p \times \{0\}$ on N and t the coordinate on $\mathbb{R}$. Let γ be a C^2 curve in the $t-r$ plane which satisfies the following conditions; see Fig. II.12.

1. For some $\bar{t} > 0$, γ lies entirely inside the rectangle $[0, \bar{r}] \times [0, \bar{t}]$, beginning at the point $(0, \bar{r})$ and ending at the point $(\bar{t}, 0)$. There are points $(0, r_1), (t_1', r_1'), (t_0, r_0)$ and (t_∞, r_∞) on the interior of γ with $0 < r_\infty < r_0 < \frac{r_1}{2} < r_1' < r_1 < \bar{r}$ and $0 < t_1' < t_0 < t_\infty < \bar{t}$. We will assume that $\bar{t} - t_\infty$ is much larger than r_∞.

2. When $r \in [r_0, \bar{r}]$, γ is the graph of a function f_0 with domain on the r-axis satisfying: $f_0(r) = 0$ when $r \in [r_1, \bar{r}]$, $f_0(r) = t_1' - \tan\theta_0(r - r_1')$ for some $\theta_0 \in (0, \frac{\pi}{2})$ when $r \in [r_0, r_1']$ and with $\dot{f}_0 \le 0$ and $\ddot{f}_0 \ge 0$.

3. When $r \in [0, r_\infty]$, γ is the graph of a function f_∞ defined over the interval $[t_\infty, \bar{t}]$ of the t-axis. The function f_∞ is given by the formula $f_\infty(t) = f_{r_\infty}(\bar{t} - t)$ where f_{r_∞} is an r_∞-torpedo function of the type described at the beginning of section II.2.3.

4. Inside the rectangle $[t_0, t_\infty] \times [r_\infty, r_0]$, γ is the graph of a C^2 function f with $f(t_0) = r_0$, $f(t_\infty) = r_\infty$, $\dot{f} \le 0$ and $\ddot{f} \ge 0$.

The curve γ specifies a hypersurface in $N \times \mathbb{R}$ in the following way. Equip $N \times \mathbb{R}$ with the product metric $g + dt^2$. Define $M = M_\gamma$ to be the hypersurface, shown in Fig. II.13 and defined

$$M_\gamma = \{(y, x, t) \in S^p \times D^{q+1}(\bar{r}) \times \mathbb{R} : (r(x), t) \in \gamma\}.$$

We will denote by g_γ, the metric induced on the hypersurface M. The fact that γ is a vertical line near the point $(0, \bar{r})$ means that $g_\gamma = g$, near ∂N. Thus, γ specifies a metric on X which is the orginal metric g outside of N and then transitions smoothly to the metric g_γ. Later we will show

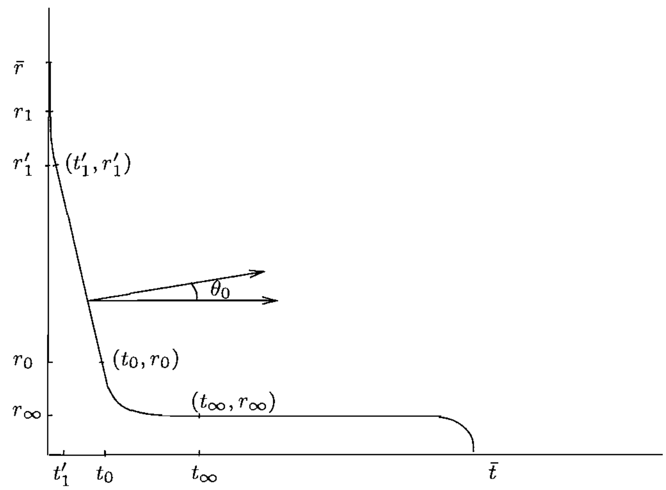

Figure II.12: The curve γ

that such a curve can be constructed so that g_γ has positive scalar curvature. In the meantime, we will derive an expression for the scalar curvature of g_γ, by computing principal curvatures for M with respect to the outward unit normal vector field and then utilising the Gauss curvature equation; see Lemmas II.14 and II.15. Details of these computations can be found in appendix III.6.3.

Lemma II.14. *The principal curvatures to M with respect to the outward unit normal vector field have the form*

$$\lambda_j = \begin{cases} k & \text{if } j = 1 \\ (-\frac{1}{r} + O(r))\sin\theta & \text{if } 2 \le j \le q+1 \\ O(1)\sin\theta & \text{if } q+2 \le j \le n. \end{cases} \tag{II.3.1}$$

Here k is the curvature of γ, θ is the angle between the outward normal vector η and the horizontal (or the outward normal to the curve γ and the t-axis) and the corresponding principal directions e_j are tangent to the curve γ when $j = 1$, the fibre sphere S^q when $2 \le j \le q+1$ and S^p when $q+2 \le j \le n$.

Proof. See appendix III.6.3. ∎

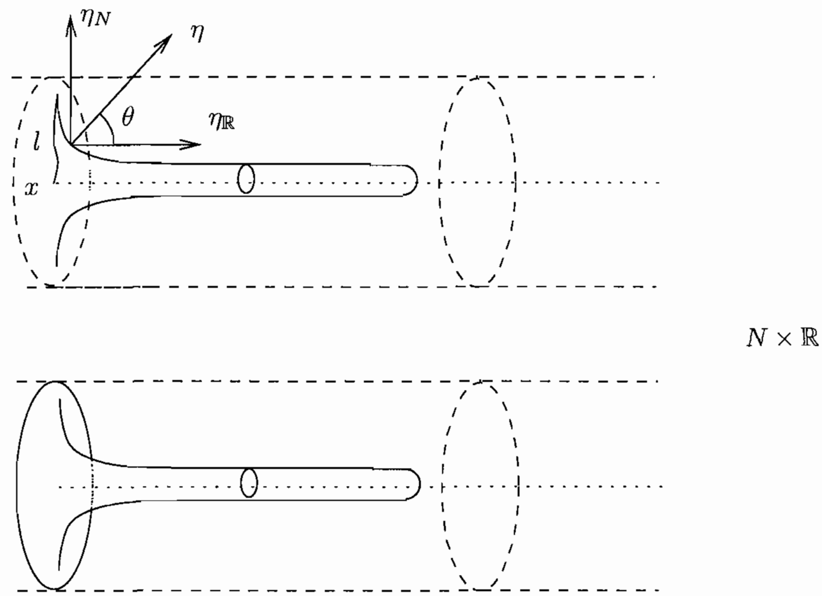

Figure II.13: The hypersurface M in $N \times \mathbb{R}$, the sphere S^p is represented schematically as a pair of points

Lemma II.15. *The scalar curvature of the metric induced on M is given by*

$$\begin{aligned} R^M = R^N + \sin^2\theta \cdot O(1) - 2k \cdot q\frac{\sin\theta}{r} \\ + 2q(q-1)\frac{\sin^2\theta}{r^2} + k \cdot qO(r)\sin\theta. \end{aligned} \tag{II.3.2}$$

Proof. See appendix III.6.3. □

II.3.5 Part 2 of the proof: A continuous bending argument

In this section we will prove the following lemma.

Lemma II.16. *The curve γ can be chosen so that the induced metric g_γ, on the hypersurface $M = M_\gamma$, has positive scalar curvature and is isotopic to the original metric g.*

Before proving this lemma, it is worth simplifying some of our formulae. From formula (II.3.2) we see that to keep $R^M > 0$ we must choose γ so that

$$k\left[2q\tfrac{\sin\theta}{r} + qO(r)\sin\theta\right] < R^N + \sin^2\theta \cdot O(1) + 2q(q-1)\tfrac{\sin^2\theta}{r^2}.$$

This inequality can be simplified to

$$k[\tfrac{\sin\theta}{r} + O(r)\sin\theta] < R_0 + \sin^2\theta \cdot O(1) + (q-1)\tfrac{\sin^2\theta}{r^2}$$

where

$$R_0 = \tfrac{1}{2q}[\inf_N(R^N)]$$

and $\inf_N(R^N)$ is the infimum of the function R^N on the neighbourhood N. Simplifying further, we obtain

$$k[1 + O(r)r] < R_0\tfrac{r}{\sin\theta} + r\sin\theta \cdot O(1) + (q-1)\tfrac{\sin\theta}{r}.$$

Replace $O(r)$ with $C'r$ for some constant $C' > 0$ and replace $O(1)$ with $-C$ where $C > 0$, assuming the worst case scenario that $O(1)$ is negative. Now we have

$$k[1 + C'r^2] < R_0\tfrac{r}{\sin\theta} + (q-1)\tfrac{\sin\theta}{r} - Cr\sin\theta. \tag{II.3.3}$$

The proof of Lemma II.16 is quite complicated and so it is worth giving an overview. We denote by γ^0, the curve which in the $t-r$-plane runs vertically down the r-axis, beginning at $(0,\bar{r})$ and finishing at $(0,0)$. Now consider the curve γ^{θ_0}, shown in Fig. II.14. This curve begins as γ^0, before smoothly bending upwards over some small angle $\theta_0 \in (0,\frac{\pi}{2})$ to proceed as a straight line segment before finally bending downwards to intersect the t-axis vertically. The corresponding hypersurface in $N \times \mathbb{R}$, constructed exactly as before, will be denoted by $M_{\gamma^{\theta_0}}$ and the induced metric by $g_{\gamma^{\theta_0}}$. The strict positivity of the scalar curvature of g means that provided we choose θ_0 to be sufficiently small, the scalar curvature of the metric $g_{\gamma^{\theta_0}}$ will be strictly positive. It will then be a relatively straightforward exercise to construct a homotopy of γ^{θ_0} back to γ^0 which induces an isotopy of the metrics $g_{\gamma^{\theta_0}}$ and g.

To obtain the curve γ, we must perform one final upward bending on γ^{θ_0}. This will take place on the straight line piece below the first upward bend. This time we will bend the curve right around by an angle of $\frac{\pi}{2}-\theta_0$ to proceed as a horizontal line segment, before bending downwards to intersect the t-axis vertically; see Fig. II.12. We must ensure throughout that inequality (II.3.3) is satisfied. In this regard, we point out that the downward bending, provided we maintain downward concavity, causes us no difficulty as here $k \leq 0$. The difficulty lies in performing an upward bending, where this inequality is reversed.

Having constructed γ, our final task will be to demonstrate that it is possible to homotopy γ back to γ^{θ_0} in such a way as to induce an isotopy between the metrics g_γ and $g_{\gamma^{\theta_0}}$. This, combined with the previously constructed isotopy of g_γ and g, will complete the proof.

Proof. **The initial bending:** For some $\theta_0 > 0$, γ^{θ_0} will denote the curve depicted in Fig. II.14, parametrised by the arc length parameter s. Beginning at $(0,\bar{r})$, the curve γ^{θ_0} runs downward along the vertical axis to the point $(0,r_1)$, for some fixed $0<r_1<\bar{r}$. It then bends upwards by an angle of θ_0, proceeding as a straight line segment with slope $m_0 = \frac{-1}{\tan\theta_0}$, before finally bending downwards and with downward concavity to intersect the t-axis vertically. The curvature of γ^{θ_0} at the point $\gamma^{\theta_0}(s)$ is denoted by $k(s)$ and $\theta = \theta(s)$ will denote the angle made by the normal vector to γ^{θ_0} and the t-axis, at the point $\gamma^{\theta_0}(s)$.

The bending itself will be obtained by choosing a very small bump function for k, with support on an interval of length $\frac{r_1}{2}$; see Fig. II.15. This will ensure that the entire upward bending takes place over some interval $[r_1', r_1]$ which is contained entirely in $[\frac{r_1}{2}, r_1]$. The downward bending

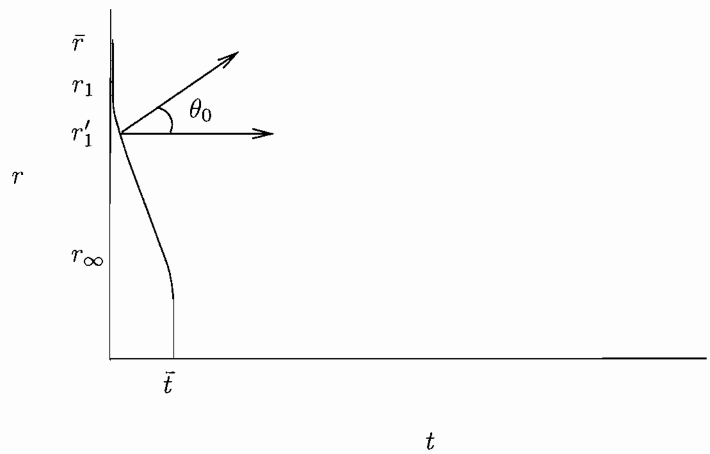

Figure II.14: The curve γ^{θ_0} resulting from the initial bend

will then begin at $r = r_\infty$, for some $r_\infty \in (0, \frac{r_1}{2})$.

We will first show that the parameters $\theta_0 \in (0, \frac{\pi}{2})$ and $r_1 \in (0, \bar{r})$ can be chosen so that inequality (II.3.3) holds for all $\theta \in [0, \theta_0]$ and all $r \in (0, r_1]$. Begin by choosing some $\theta_0 \in (0, \arcsin\sqrt{\frac{R_0}{C}})$. This guarantees that the right hand side of (II.3.3) remains positive for all $\theta \in [0, \theta_0]$. For now, the variable θ is assumed to lie in $[0, \theta_0]$. Provided θ is close to zero, the term $R_0 \frac{r}{\sin\theta}$ is positively large and dominates. When $\theta = 0$ the right hand side of (II.3.3) is positively infinite. Once θ becomes greater than zero, the term $(q-1)\frac{\sin\theta}{r}$ can be made positively large by choosing r small, and so can be made to dominate. Recall here that $q \geq 2$ by the assumption that the original surgery sphere had codimension at least three. It is therefore possible to choose $r_1 > 0$ so that inequality (II.3.3) holds for all $\theta \in [0, \theta_0]$ and for all $r \in (0, r_1]$. Note also that without the assumption that the scalar curvature of the original metric g is strictly positive, this argument fails.

We will now bend γ^0 to γ^{θ_0}, smoothly increasing θ from 0 to θ_0. We do this by specifying a bump function k which describes the curvature along γ^{θ_0}; see Fig. II.15. This gives

$$\Delta\theta = \int k ds \approx \frac{1}{2} r_1 \cdot k_{max}.$$

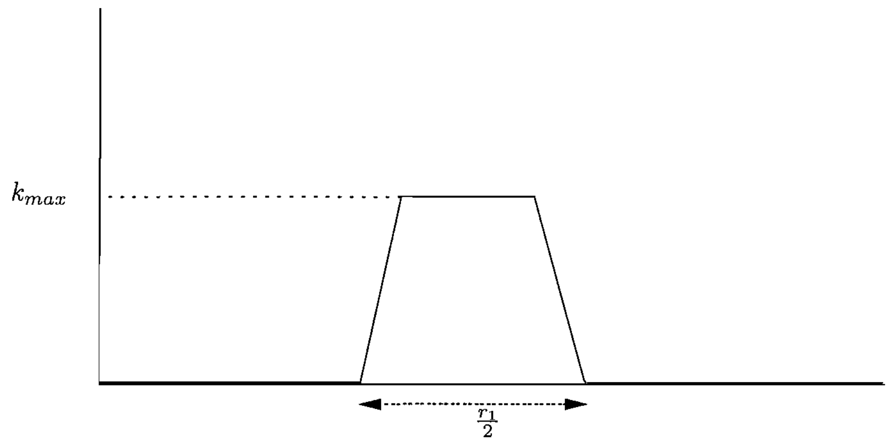

Figure II.15: The bump function k

This approximation can be made as close to equality as we wish. If necessary re-choose θ_0 so that $\theta_0 < \frac{1}{2}r_1 \cdot k_{max}$. Note that r_1 has been chosen to make inequality (II.3.3) hold for all $\theta \in [0, \theta_0]$ and so rechoosing a smaller θ_0 does not affect the choice of r_1. We need to show that $k_{max} > 0$ can be found so that

$$k_{max}[1 + C'{r_1}^2] < R_0\frac{r_1}{\sin\theta} + (q-1)\frac{\sin\theta}{r_1} - Cr_1\sin\theta,$$

for all $\theta \in [0, \theta_0]$. From the earlier argument, r_1 and θ_0 have been chosen so that the right-hand side of this inequality is positive for all $\theta \in [0, \theta_0]$. So some such $k_{max} > 0$ exists. This completes the initial upward bending.

The curve γ^{θ_0} then proceeds as a straight line before bending downwards, with downward concavity, to vertically intersect the t-axis. This downward concavity ensures that $k \leq 0$ and so inequality (II.3.3) is easily satisfied, completing the construction of γ^{θ_0}.

The initial isotopy:

Next we will show that γ^{θ_0} can be homotopied back to γ^0 in such a way as to induce an isotopy between the metrics $g_{\gamma^{\theta_0}}$ and g. Treating γ^{θ_0} as the graph of a smooth function f_0 over the interval $[0, \bar{r}]$, we can compute the curvature k, this time in terms of r, as

$$k = \frac{\ddot{f_0}}{(1 + \dot{f_0}^2)^{\frac{3}{2}}}.$$

By replacing f_0 with λf_0 where $\lambda \in [0,1]$, we obtain a homotopy from γ^{θ_0} back to γ^0. To ensure that the induced metric has positive scalar curvature at each stage in this homotopy, it is enough to show that on the interval $[\frac{r_1}{2}, r_1]$, $k^\lambda \leq k$ for all $\lambda \in [0,1]$, where k^λ is the curvature of λf_0. Note that away from this interval, downward concavity means that $k^\lambda \leq 0$ for all λ and so inequality (II.3.3) is easily satisfied.

We wish to show that for all $\lambda \in [0,1]$,

$$\frac{\lambda \ddot{f_0}}{(1+\lambda^2 \dot{f_0}^2)^{\frac{3}{2}}} \leq \frac{\ddot{f_0}}{(1+\dot{f_0}^2)^{\frac{3}{2}}}.$$

A slight rearrangement of this inequality gives

$$\frac{\ddot{f_0}}{((\lambda^{\frac{-2}{3}})(1+\lambda^2 \dot{f_0}^2))^{\frac{3}{2}}} \leq \frac{\ddot{f_0}}{(1+\dot{f_0}^2)^{\frac{3}{2}}},$$

and hence, it is enough to show that

$$\frac{1+\lambda^2 \dot{f_0}^2}{1+\dot{f_0}^2} \geq \lambda^{\frac{2}{3}} \qquad \text{for all } \lambda \in [0,1].$$

Replacing $\lambda^{\frac{2}{3}}$ with μ and $\dot{f_0}^2$ with b we obtain the following inequality.

$$\mu^3 b - \mu b - \mu + 1 \geq 0. \tag{II.3.4}$$

The left hand side of this inequality is zero when $\mu = 1$ or when $\mu = \frac{-b \pm \sqrt{b^2+4b}}{2b}$. A simple computation then shows that, provided θ_0 has been chosen sufficiently small, the left hand side of (II.3.4) is non-zero when μ (and thus λ) is in $[0,1]$, and so the inequality holds.

The final bending:

We will now construct the curve γ so that the induced metric g_γ has positive scalar curvature. From the description of γ given in Part 1, we see that it is useful to regard γ as consisting of three pieces. When $r > r_0$, γ is just the curve γ^{θ_0} constructed above and when $r \in [0, r_\infty]$, γ is the graph of the concave downward function f_∞. In both of these cases, inequality (II.3.3) is easily satisfied. The third piece is where the difficulty lies. In the rectangle $[t_0, t_\infty] \times [r_\infty, r_0]$, we must specify a curve which connects the previous two pieces to form a C^2 curve, and satisfies inequality

(II.3.3). This will be done by constructing an appropriate C^2 function $f : [t_0, t_\infty] \longrightarrow [r_\infty, r_0]$. Before discussing the construction of f we observe that inequality (II.3.3) can be simplified even further.

Choose $r_0 \in (0, \frac{r_1}{2})$ so that $0 < r_0 < min\{\frac{1}{\sqrt{4C}}, \frac{1}{\sqrt{2C'}}\}$. Now, when $r \in (0, r_0]$ and $\theta \geq \theta_0$, we have

$$\begin{aligned}(q-1)\tfrac{\sin\theta}{r} - Cr\sin\theta &\geq \sin\theta[\tfrac{q-1}{r} - Cr] \\ &\geq \tfrac{\sin\theta}{r}[1 - Cr^2].\end{aligned}$$

When $r < \frac{1}{\sqrt{4C}}$, $r^2 < \frac{1}{4C}$. So $Cr^2 < \frac{1}{4}$ and $1 - Cr^2 > \frac{3}{4}$. Thus,

$$(q-1)\tfrac{\sin\theta}{r} - Cr\sin\theta \geq \tfrac{3}{4}\tfrac{\sin\theta}{r}.$$

Also $r < \sqrt{\frac{1}{2C'}}$. So $r^2 < \frac{1}{2C'}$ giving that $2C'r^2 < 1$. Thus, $1 + C'r^2 < \frac{3}{2}$. Hence from inequality (II.3.3) we get

$$k < \frac{2}{3}.\frac{3}{4}\frac{\sin\theta}{r} = \frac{\sin\theta}{2r}.$$

So, if we begin the second bend when $r \in [0, r_0]$, it suffices to maintain

$$k < \frac{\sin\theta}{2r}. \tag{II.3.5}$$

It should be pointed out that inequality (II.3.5) **only holds when $\theta > \theta_0 > 0$ and does not hold for only $\theta > 0$, no matter how small r is chosen.** The following argument demonstrates this. Assuming θ is close to zero and using the fact that $k(s) = \frac{d\theta}{ds}$, we can assume (II.3.5) is

$$\frac{d\theta}{ds} < \frac{\theta}{2r}.$$

But this is

$$\frac{d\log(\theta)}{ds} < \frac{1}{2r},$$

the left hand side of which is unbounded as θ approaches 0. It is for this reason that the initial bend and hence, the **strict positivity** of the scalar curvature of g, is so important.

Remark II.3.4. *From the above one can see that the inequality on page 190 of [10] breaks down when θ is near 0. In this case the bending argument aims at maintaining non-negative mean* *curvature. Since apriori the mean curvature is not strictly positive, an analogous initial bend to move θ away from 0 is not possible.*

We will now restrict our attention entirely to the rectangle $[t_0, t_\infty] \times [r_\infty, r_0]$. Here we regard γ as the graph of a function f. Thus, we obtain

$$\sin\theta = \frac{1}{\sqrt{1+\dot{f}^2}}$$

and

$$k = \frac{\ddot{f}}{(1+\dot{f}^2)^{\frac{3}{2}}}.$$

Hence, (II.3.5) gives rise to the following differential inequality

$$\frac{\ddot{f}}{(1+\dot{f}^2)^{\frac{3}{2}}} < \frac{1}{\sqrt{1+\dot{f}^2}}\frac{1}{2f}.$$

This simplifies to

$$\ddot{f} < \frac{1+\dot{f}^2}{2f}. \tag{II.3.6}$$

Of course to ensure that γ is a C^2 curve we must insist that as well as satisfying (II.3.6), f must also satisfy conditions (II.3.7), (II.3.8) and (II.3.9) below.

$$f(t_0) = r_0, \qquad f(t_\infty) > 0, \tag{II.3.7}$$

$$\dot{f}(t_0) = m_0, \qquad \dot{f}(t_\infty) = 0, \tag{II.3.8}$$

$$\ddot{f}(t_0) = 0, \qquad \ddot{f}(t_\infty) = 0, \tag{II.3.9}$$

where $m_0 = \frac{-1}{\tan\theta_0}$. The fact that such a function can be constructed is the subject of the following lemma. Having constructed such a function, r_∞ will then be set equal to $f(t_\infty)$ and the construction of γ will be complete.

Lemma II.17. *For some $t_\infty > t_0$, there is a C^2 function $f : [t_0, t_\infty] \longrightarrow [0, r_0]$ which satisfies inequality (II.3.6) as well as conditions (II.3.7), (II.3.8) and (II.3.9).*

Proof. The following formula describes a family of functions, all of which satisfy inequality (II.3.6).

$$f(t) = c + \tfrac{C_1}{4}(t - C_2)^2, \qquad \text{where } C_1, C_2 > 0 \text{ and } c \in (0, \tfrac{1}{C_1}).$$

Such a function f has first and second derivatives

$$\dot{f} = \tfrac{C_1}{2}(t - C_2) \text{ and } \ddot{f} = \tfrac{C_1}{2}.$$

We will shortly see that C_1 and C_2 can be chosen so that on the interval $[t_0, C_2]$, $f(t_0) = r_0$, $\dot{f}(t_0) = m_0$, $\dot{f}(C_2) = 0$ and $f(C_2) = c > 0$. The choice of C_1 needs to be very large which makes $\ddot{f}$ a large positive constant. Thus, some adjustment is required near the end points if such a function is to satisfy the requirements of the lemma. We will achieve this by restricting the function to some proper subinterval $[t_0', t_\infty'] \subset [t_0, C_2]$ and pasting in appropriate transition functions on the intervals $[t_0, t_0']$ and $[t_\infty', t_\infty]$ (where t_∞ is close to C_2).

More precisely, let $t_0' - t_0 = \delta_0$, $t_\infty - t_\infty' = \delta_\infty$ and $C_2 - t_\infty' = \frac{\delta_\infty}{2}$. We will now show that for appropriate choices of C_1, C_2, δ_0 and δ_∞, the following function satisfies the conditions of the lemma. To aid the reader, we include the graph of the second derivative of this function; see Fig. II.16.

$$f(t) = \begin{cases} r_0 + m_0(t - t_0) + \frac{C_1}{12\delta_0}(t - t_0)^3, & \text{if } t \in [t_0, t_0'] \\ c + \frac{C_1}{4}(t - C_2)^2, & \text{if } t \in [t_0', t_\infty'] \\ c - \frac{C_1}{48}{\delta_\infty}^2 - \frac{C_1}{12\delta_\infty}(t - t_\infty)^3, & \text{if } t \in [t_\infty', t_\infty]. \end{cases} \qquad \text{(II.3.10)}$$

A simple check shows that $f(t_0) = r_0$, $\dot{f}(t_0) = m_0$ and $\ddot{f}(t_0) = 0$. Now we must show that C_1 can be chosen so that this function is C^2 at t_0'. We begin by solving, for t_0', the equation

$$c + \tfrac{C_1}{4}(t_0' - C_2)^2 = r_0 + m_0(t_0' - t_0) + \tfrac{C_1}{12\delta_0}(t_0' - t_0)^3.$$

This results in the following formula for t_0',

$$t_0' = C_2 - \sqrt{\tfrac{4}{C_1}(r_0 + m_0\delta_0 + \tfrac{C_1}{12}\delta_0^2 - c)}.$$

Equating the first derivatives of the first two components of (III.4.3) at t_0' and replacing t_0' with

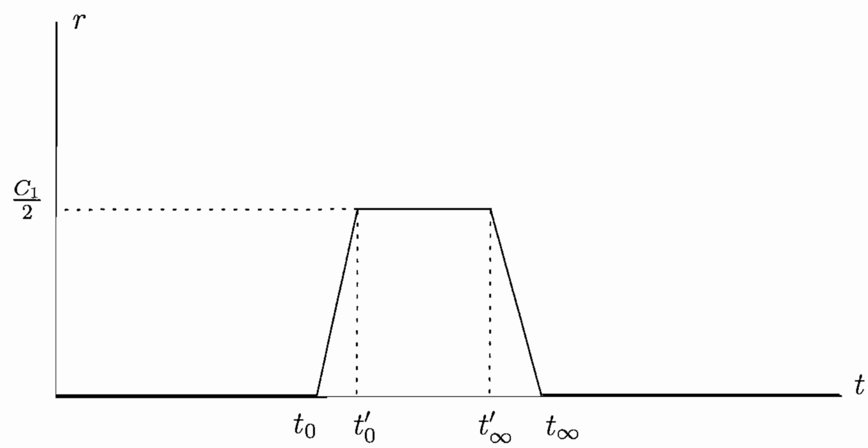

Figure II.16: The second derivative of f

the expression above, results in the following equation.

$$C_1(r_0 - c) - m_0^2 = -\tfrac{C_1}{2}\delta_0 m_0 - \tfrac{C_1^2}{48}\delta_0^2. \tag{II.3.11}$$

The second derivatives of the first two components of (III.4.3) agree at t_0' and so provided C_1 and δ_0 are chosen to satisfy (II.3.11), f is C^2 at t_0'. It remains to show that δ_0 can be chosen so that f satisfies inequality (II.3.6) on $[t_0, t_0']$. The parameter C_1 varies continuously with respect to δ_0. Denoting by $\bar{C_1}$, the solution to the equation $\bar{C_1}(r_0 - c) - m_0^2 = 0$, it follows from equation (II.3.11), that for small δ_0, C_1 is given by a formula $C_1(\delta_0) = \bar{C_1} + \epsilon(\delta_0)$ for some continuous parameter ϵ with $\epsilon(0) = 0$ and $\epsilon(\delta_0) > 0$ when $\delta_0 > 0$. When $\delta_0 = 0$, we obtain the strict inequality

$$\bar{C_1} < \frac{1 + m_0^2}{r_0}.$$

Thus, there exists some sufficiently small δ_0, so that for all $s \in [0, 1]$,

$$C_1 = \bar{C_1} + \epsilon(\delta_0) < \frac{1 + (m_0 + \frac{C_1}{4}\delta_0 s)^2}{r_0},$$

while at the same time,

$$-r_0 < m_0\delta_0 s + \tfrac{C_1}{12}\delta_0^2 s^3 \leq 0.$$

Hence,

$$C_1 < \frac{1 + (m_0 + \frac{C_1}{4}\delta_0 s)^2}{r_0 + m_0\delta_0 s + \frac{C_1}{12}\delta_0^2 s^3} \tag{II.3.12}$$

holds for all $s \in [0, 1]$. Replacing s with $\frac{t-t_0}{\delta_0}$ in (II.3.12) yields inequality (II.3.6) for $t \in [t_0, t_0']$ and so f satisfies (II.3.6) at least on $[t_0, t_\infty']$.

Given r_0, the only choice we have made so far in the construction of f, is the choice of δ_0. This choice determines uniquely the choices of C_1 and C_2. Strictly speaking we need to choose some c in $(0, \frac{1}{C_1})$ but we can always regard this as given by the choice of C_1, by setting $c = \frac{1}{2C_1}$ say. There is one final choice to be made and that is the choice of δ_∞. Some elementary calculations show that f is C^2 at t_∞'. The choice of δ_∞ is completely independent of any of the choices we have made so far and so can be made arbitrarily small. Thus, an almost identical argument to the one made when choosing δ_0 shows that for a sufficiently small choice of δ_∞, inequality (II.3.6) is satisfied when $t \in [t_\infty', t_\infty]$. Also, the independence of δ_0 and C_1 means that $f(t_\infty) = c - \frac{C_1}{12}\delta_\infty^2$ can be kept strictly positive by ensuring δ_∞ is sufficiently small. The remaining conditions of the lemma are then trivial to verify. □

The final isotopy:

The final task in the proof of Lemma II.16, is the construction of a homotopy between γ and γ^{θ_0} which induces an isotopy between the metrics g_γ and $g_{\gamma^{\theta_0}}$. We will begin by making a very slight adjustment to γ. Recall that the function f has as its second derivative: a bump function with support on the interval $[t_0, t_\infty]$; see Fig. II.16. By altering this bump function on the region $[t_\infty', t_\infty]$, we make adjustments to f. In particular, we will replace f with the C^2 function which agrees with f on $[t_0, t_\infty']$ but whose second derivative is the bump function shown in Fig. II.17, with support on $[t_0, t_\infty'']$, where $t_\infty'' \in [C_2, t_\infty]$. We will denote this new function f^∞.

When $t_\infty'' = t_\infty$, no change has been made and $f^\infty = f$. When $t_\infty'' < t_\infty$, the derivative of f^∞ on the interval $[t_\infty'', t_\infty]$ is a negative constant, causing the formerly horizontal straight line piece of γ to tilt downwards with negative slope. Thus, by continuously decreasing t_∞'' from t_∞ by some sufficiently small amount, we can homotopy γ to a curve of the type shown in Fig. II.18, where the second straight line piece now has small negative slope, before bending downwards to intersect the t-axis vertically at $\bar{t}$. Note that the rectangle $[t_0, t_\infty] \times [r_\infty, r_0]$ is now replaced by the rectangle $[t_0, t_\infty''] \times [r_\infty'', r_0]$, where $f^\infty(t_\infty'') = r_\infty''$. It is easy to see how, on $[t_\infty'', \bar{t}]$, γ can be

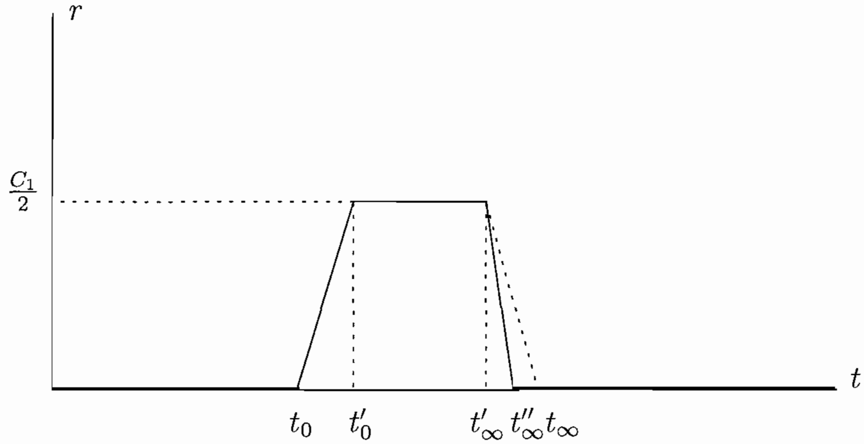

Figure II.17: The second derivative of the function f^{∞}

homotopied through curves each of which is the graph of a C^2 function with non-positive second derivative, thus satisfying inequality (II.3.6). We do need to verify however, that on $[t'_{\infty}, t''_{\infty}]$, this inequality is valid. Recall, this means showing that

$$\ddot{f}^{\infty} < \frac{1 + {\dot{f}^{\infty}}^2}{2f^{\infty}}.$$

When $t''_{\infty} = t_{\infty}$, $f^{\infty} = f$ and so this inequality is already strict on the interval $[t'_{\infty}, t''_{\infty}]$. Now suppose t''_{∞} is slightly less than t_{∞}. Then, on $[t'_{\infty}, t''_{\infty}]$, $\ddot{f}^{\infty} \leq \ddot{f}$, while the 2-jets of f and f^{∞} agree at t'_{∞}. This means that $\dot{f}^{\infty} \leq \dot{f}$ and $f^{\infty} \leq f$ on $[t'_{\infty}, t''_{\infty}]$. But $\dot{f} < 0$ on this interval and so $(\dot{f}^{\infty})^2 \geq \dot{f}^2$. Also, provided t''_{∞} is sufficiently close to t_{∞}, we can keep $f^{\infty} > 0$ and sufficiently large on this interval so that the curve γ can continue as the graph of a decreasing non-negative concave downward function all the way to the point $\bar{t}$. Thus, the inequality in (II.3.6) actually grows as t''_{∞} decreases.

It remains to show that this slightly altered γ can be homotopied back to γ^{θ_0} in such a way as to induce an isotopy of metrics. To ease the burden of notation we will refer to the function f^{∞} as simply f and the rectangle $[t_0, t''_{\infty}] \times [r''_{\infty}, r_0]$ as simply $[t_0, t_{\infty}] \times [r_{\infty}, r_0]$. It is important to remember that f differs from the function constructed in Lemma II.17 in that $m_0 \leq \dot{f} < 0$ on $[t_0, t_{\infty}]$. We wish to continuously deform the graph of f to obtain the straight line of slope m_0 intersecting the point (t_0, r_0). We will denote this straight line segment by l, given by the formula $l(r) = r_0 + m_0(t - t_0)$. We will now construct a homotopy by considering the functions which

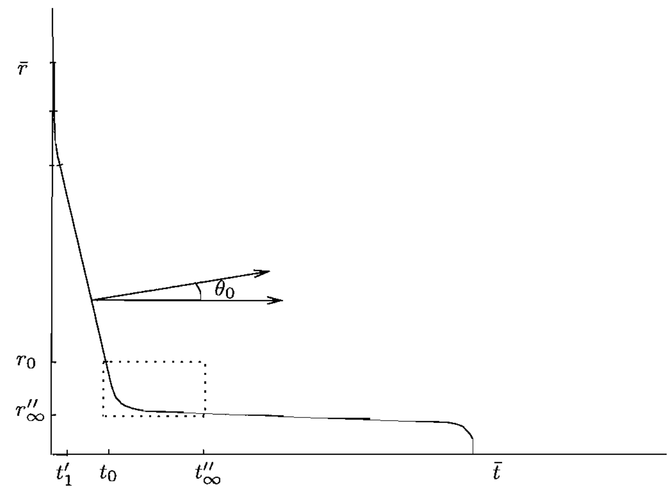

Figure II.18: The effect of such an alteration on the curve γ

are inverse to f and l; see Fig. II.19. Consider the linear homotopy $h_s^{-1} = (1-s)f^{-1} + sl^{-1}$, where $s \in [0,1]$. Let h_s denote the corresponding homotopy from f to l, where for each s, h_s is inverse to h_s^{-1}. Note that the domain of h_s is $[t_0, (1-s)t_\infty + sl^{-1}(r_\infty)]$. For each $r \in [r_\infty, r_0]$, $\dot{h_s^{-1}}(r) \leq \dot{f^{-1}}(r)$. This means that for any $s \in [0,1]$ and any $r \in [r_\infty, r_0]$, $\dot{h_s}(t_s) \geq \dot{f}(t)$, where $h_s(t_s) = f(t) = r$. As the second derivative of h_s is bounded by $\ddot{f}$, this means that inequality (II.3.6) is satisfied throughout the homotopy.

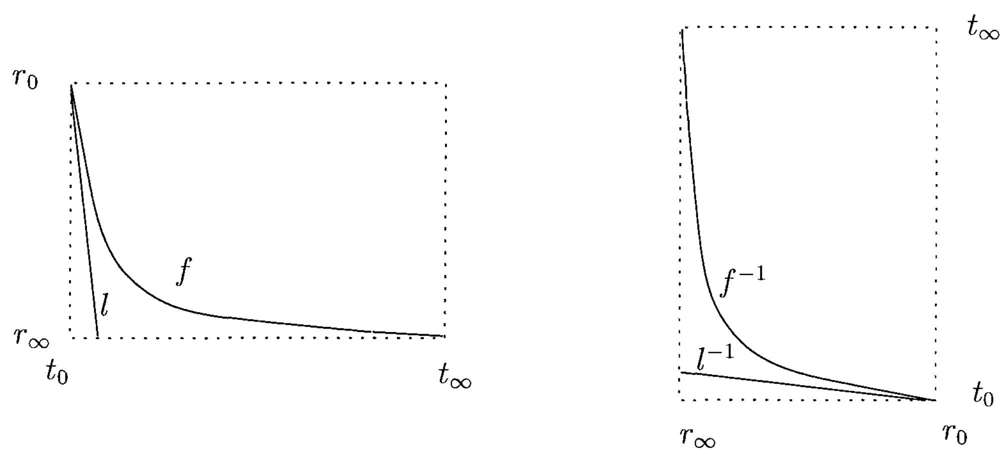

Figure II.19: The graphs of the functions f and l and their inverses

This homotopy extends easily to the part of γ on the region where $t \geq (1-s)t_\infty + sl^{-1}(r_\infty)$, which can easily be homotopied through curves, each the graph of a concave downward decreasing non-negative function. The result is a homtopy between γ and γ^{θ_0}, through curves which satisfy inequality (II.3.3) at every stage. This, combined with the initial isotopy, induces an isotopy through metrics of positive scalar curvature between g and g_γ, completing the proof of Lemma II.16. □

II.3.6 Part 3 of the proof: Isotopying to a standard product

Having constructed the psc-metric g_γ and having demonstrated that g_γ is isotopic to the original metric g, one final task remains. We must show that the metric g_γ can be isotopied to a psc-metric which, near the embedded sphere S^p, is the product $g_p + g_{tor}^{q+1}(\delta)$. Composing this with the isotopy from Part 2 yields the desired isotopy from g to g_{std} and proves Theorem II.11.

We denote by $\pi : \mathcal{N} \to S^p$, the normal bundle to the embedded S^p in X. The Levi-Civita connection on X, with respect to the metric g_γ, gives rise to a normal connection on the total space of this normal bundle . This gives rise to a horizontal distribution $\mathcal{H}$ on the total space of $\mathcal{N}$. Equip the fibres of the bundle $\mathcal{N}$ with the metric $g_{tor}^{q+1}(\delta)$. Equip S^p with the metric $g_\gamma|_{S^p}$, the induced metric on S^p. The projection $\pi : (\mathcal{N}, \tilde{g}) \to (S^p, \breve{g})$ is now a Riemannian submersion with base metric $\breve{g} = g_\gamma|_{S^p}$ and fibre metric $\hat{g} = g_{tor}^{q+1}(\delta)$. The metric $\tilde{g}$ denotes the unique submersion metric arising from $\breve{g}, \hat{g}$ and $\mathcal{H}$. See Chapter 9 of [2] for details about Riemannian submersions.

Our focus will mostly be on the restriction of this Riemannian submersion to the disk bundle, $\pi : D\mathcal{N}(\epsilon) \to S^p$. We will retain $\tilde{g}$, $\hat{g}$ and $\breve{g}$ to denote the relevant restrictions. Before saying anything more specific about this disk bundle, it is worth introducing some useful notation. For some $t_L \in (t_\infty, \bar{t} - r_\infty)$, we define the following submanifolds of M (see Fig. II.20),

$$M(t_L, \bar{t}) = \{(y, x, t) \in S^p \times D^{q+1}(\bar{r}) \times \mathbb{R} : (r(x), t) \in \gamma \text{ and } t \geq t_L\}.$$

and

$$M(t_\infty, t_L) = \{(y, x, t) \in S^p \times D^{q+1}(\bar{r}) \times \mathbb{R} : (r(x), t) \in \gamma \text{ and } t_\infty \leq t \leq t_L\}.$$

Note that $M(t_L, \bar{t})$ is, for appropriately small ϵ, the disk bundle $D\mathcal{N}(\epsilon)$ and $M(t_\infty, t_L)$ is a cylindrical region (diffeomorphic to $S^p \times S^q \times [t_\infty, t_L]$) which connects this disk bundle with the rest of M. We will make our primary adjustments on the disk bundle $D\mathcal{N}(\epsilon)$, where we will construct an isotopy from the metric g_γ to a metric which is a product. The cylindrical piece will be then

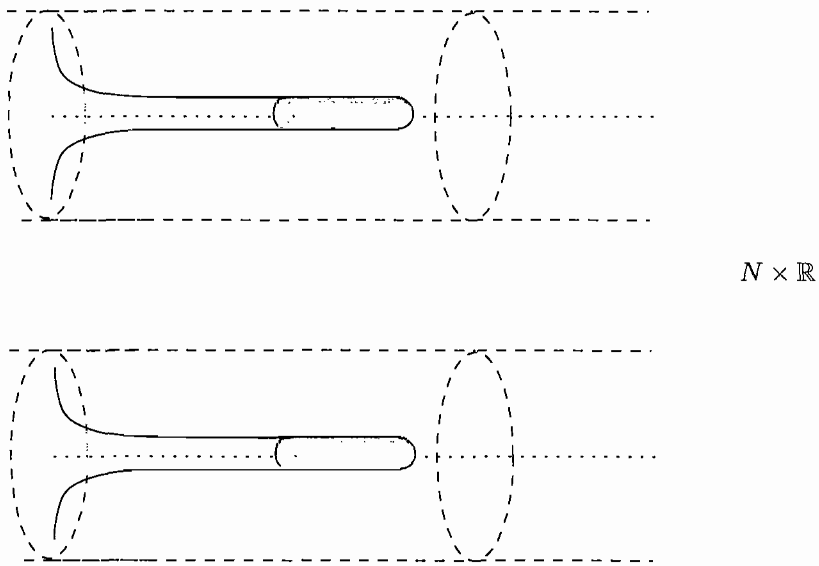

Figure II.20: The shaded piece denotes the region $M(t_L, \bar{t})$

be used as a transition region.

On $D\mathcal{N}(\epsilon)$ we can use the exponential map to compare the metrics g_γ and $\tilde{g}$. Replacing the term r_∞ with δ, we observe the following convergence.

Lemma II.18. *There is C^2 convergence of the metrics g_γ and $\tilde{g}$ as $\delta \to 0$.*

Proof. Treating g_γ as a submersion metric (or at least the metric obtained by pulling back g_γ via the exponential map), it suffices to show convergence of the fibre metrics. In the case of g_γ, the metric on each fibre $D^{q+1}(\epsilon)_y$, where $y \in S^p$, is of the form

$$\hat{g_\gamma} = dt^2 + g|_{S^q(f_\delta(\bar{t}-t))_y}.$$

Here $t \in [t_L, (\bar{t})]$ and recall that $S^q(f_\delta(\bar{t}-t))_y$ is the geodesic fibre sphere of radius $f_\delta(\bar{t}-t)$ at the point $y \in S^p$. In the same coordinates, the fibre metric for $\tilde{g}$ is

$$\hat{g} = dt^2 + f_\delta(\bar{t}-t)^2 ds_q^2.$$

We know from Lemma II.13 that as $r \to 0$,

$$g|_{S^q(r)_y} \longrightarrow r^2 ds_q^2$$

in the C^2 topology. Now, $0 < f_\delta(\bar{t}-t) \leq \delta$ and so as $\delta \to 0$, we get that

$$g|_{S^q(f_\delta(\bar{t}-t))_y} \longrightarrow f_\delta(\bar{t}-t)^2 ds_q^2.$$

□

Hence, we can isotopy g_γ, through submersion metrics, to one which pulls back on $S^p \times D^{q+1}(\epsilon)$ to the submersion metric $\tilde{g}$. In fact, we can do this with arbitrarily small curvature effects and so maintain positive scalar curvature. Furthermore, the fact that there is C^2 convergence of $g_\gamma|_{S^q(f(\bar{t}-t))_x}$ to $f(\bar{t}-t)^2 ds_q^2$ means that we can ensure a smooth transition along the cylinder $M(t_\infty, t_L)$, although this may necessitate making the cylindrical piece very long.

Now, by the formulae of O'Neill, we get that the scalar curvature of $\tilde{g}$ is

$$\tilde{R} = \check{R} \circ \pi + \hat{R} - |A|^2 - |T|^2 - |\bar{n}|^2 - 2\check{\delta}(\bar{n}).$$

where $\tilde{R}, \check{R}$ and $\hat{R}$ are the scalar curvatures of $\tilde{g}, \check{g}$ and $\hat{g}$ respectively. For full definitions and formulae for $A, T, \bar{n}$ and $\check{\delta}$; see chapter 9 of [2]. Briefly, the terms T and A are the first and second tensorial invariants of the submersion. Here T is the obstruction to the bundle having totally geodesic fibres and so by construction $T = 0$, while A is the obstruction to the integrability of the distribution. The $\bar{n}$ term is also 0 as $\bar{n}$ is the mean curvature vector and vanishes when T vanishes. We are left with

$$\tilde{R} = \check{R} \circ \pi + \hat{R} - |A|^2. \tag{II.3.13}$$

We wish to deform $\tilde{g}$ through Riemannian submersions to one which has base metric g_p, preserving positive scalar curvature throughout the deformation. We can make $\hat{R}$ arbitrarily positively large by choosing small δ. As the deformation takes place over a compact interval, curvature arising from the base metric is bounded. We must ensure, however, as we shrink the fibre metric, that $|A|^2$ is not affected in a significant way.

Letting $\tau = \bar{t} - t$, the metric on the fibre is

$$\begin{aligned} g_{tor}^{q+1}(\delta) &= d\tau^2 + f_\delta(\tau)^2 ds_q^2 \\ &= \delta^2 d(\tfrac{\tau}{\delta})^2 + \delta^2 f_1(\tfrac{\tau}{\delta})^2 ds_q^2 \\ &= \delta^2 g_{tor}^{q+1}(1). \end{aligned}$$

The canonical variation formula, Chapter 9 of [2], now gives that

$$\tilde{R} = R_{\delta^2} = \frac{1}{\delta^2}\hat{R} + \check{R} \circ \pi - \delta^2 |A|^2$$

Thus, far from the $|A|^2$ term becoming more significant as we shrink δ, it actually diminishes.

Having isotopied $\tilde{g}$ through positive scalar curvature Riemannian submersions to obtain a submersion metric with base metric $\check{g} = g_p$ and fibre metric $\hat{g} = g_{tor}^{q+1}(\delta)$, we finish by isotopying through Riemannian submersions to the product metric $g_p + g_{tor}^{q+1}(\delta)$. This involves a linear homotopy of the distribution to one which is flat i.e. where A vanishes. As $|A|^2$ is bounded throughout, we can again shrink δ if necessary to ensure positivity of the scalar curvature.

At this point we have constructed an isotopy between $\tilde{g} = g_{std}|_{D\mathcal{N}(\epsilon)}$ and $g_p + g_{tor}^{q+1}(\delta)$. In the original Gromov-Lawson construction, this isotopy is performed only on the sphere bundle $S\mathcal{N}(\epsilon)$ and so the resulting metric is the product $g_p + \delta^2 ds_q^2$ (in this case $g_p = ds_p^2$). In fact, the restriction of the above isotopy to the boundary of the disk bundle $D\mathcal{N}(\epsilon)$ is precisely this Gromov-Lawson isotopy. Thus, as the metric on $D\mathcal{N}(\epsilon)$ is being isotopied from g_γ to $g_p + g_{tor}^{q+1}(\delta)$, we can continuously transition along the cylinder $M(t_\infty, t_L)$ from this metric to the orginal metric g_γ. Again, this may require a considerable lengthenning of the cylindrical piece. This completes the proof of Theorem II.11. □

II.3.7 The Family Surgery Theorem

Before proceeding with the proof of Theorem II.10 we make an important observation. Theorem II.11 can be extended to work for a compact family of positive scalar curvature metrics on X as well as a compact family of embedded surgery spheres. A compact family of psc-metrics on X will be specified with a continuous map from some compact indexing space B into the space $\mathcal{R}\text{iem}^+(X)$. In the case of a compact family of embedded surgery spheres, we need to introduce some notation. The set of smooth maps $C^\infty(W, Y)$ between the compact manifolds W and Y can be equipped with a standard C^∞ topology; see Chapter 2 of [17]. Note that as W is compact there is no difference between the so called "weak" and "strong" topologies on this space. Contained in $C^\infty(W, Y)$, as an open subspace, is the space $Emb(W, Y)$ of smooth embeddings. We can now specify a compact family of embedded surgery spheres on a compact manifold X, with a continuous map from some compact indexing space C into $Emb(S^p, X)$.

Theorem II.19. *Let X be a smooth compact manifold of dimension n, and B and C a pair of compact spaces. Let $\mathcal{B} = \{g_b \in \mathcal{R}\text{iem}^+(X) : b \in B\}$ be a continuous family of psc-metrics on X and $\mathcal{C} = \{i_c \in Emb(S^p, X) : c \in C\}$, a continuous family of embeddings each with trivial normal bundle, where $p+q+1 = n$ and $q \geq 2$. Finally, let g_p be any metric on S^p. Then, for some $\delta > 0$, there is a continuous map*

$$\mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{R}\text{iem}^+(X)$$

$$(g_b, i_c) \longmapsto g_{std}^{b,c}$$

satisfying

(i) Each metric $g_{std}^{b,c}$ has the form $g_p + g_{tor}^{q+1}(\delta)$ on a tubular neighbourhood of $i_c(S^p)$ and is the original metric g_b away from this neighbourhood.

(ii) For each $c \in C$, the restriction of this map to $\mathcal{B} \times \{i_c\}$ is homotopy equivalent to the inclusion $\mathcal{B} \hookrightarrow \mathcal{R}\text{iem}^+(X)$.

Proof. For each pair b, c, the exponential map $\exp_b$ of the metric g_b can be used to specify a tubular neighbourhood $N_{b,c}(\bar{r})$ of the embedded sphere $i_c(S^p)$, exactly as in Part 1 of Theorem II.11. Compactness gives that the infimum of injectivity radii over all metrics g_b on X is some positive constant and so a single choice $\bar{r} > 0$ can be found, giving rise to a continuous family of such tubular neighbourhoods $\{N_{b,c} = N_{b,c}(\bar{r}) : b, c \in B \times C\}$. Each metric g_b may be adjusted in $N_{b,c}$ by specifying a hypersurface $M_\gamma^{b,c} \subset N_{b,c} \times \mathbb{R}$ constructed with respect to a curve γ, exactly as described in the proof of Theorem II.11. Equipping each $N_{b,c} \times \mathbb{R}$ with the metric $g_b|_{Nb,c} + dt^2$ induces a continuous family of metrics $g_\gamma^{b,c}$ on the respective hypersurfaces $M_\gamma^{b,c}$.

We will first show that a single curve γ can be chosen so that the resulting metrics $g_\gamma^{b,c}$ have positive scalar curvature for all b and c. The homotopy of γ to the vertical line segment in Part 2 of the proof Theorem II.11 can be applied exactly as before, inducing an isotopy between $g_\gamma^{b,c}$ and g_b which varies continuously with respect to b and c. Finally, Part 3 of Theorem II.11 can be generalised to give rise to an isotopy between $g_\gamma^{b,c}$ and $g_{std}^{b,c}$, which again varies continuously with respect to b and c. Recall from the proof of Theorem II.11 that for any curve γ, the scalar curvature on the hypersurface $M = M_\gamma$ is given by:

$$R^M = R^N + \sin^2\theta.O(1) - 2k\cdot q\tfrac{\sin\theta}{r}$$
$$+2q(q-1)\tfrac{\sin^2\theta}{r^2} + k\cdot qO(r)\sin\theta.$$

The $O(1)$ term comes from the principal curvatures on the embedded surgery sphere S^p and the Ricci curvature on N, both of which are bounded. Over a compact family of psc-metrics $g_b, b \in B$ and and a compact family of embeddings $i_c, c \in C$, these curvatures remain bounded and so the $O(1)$ term is unchanged. Here, the tubular neighbourhood N is replaced with the continuous family of tubular neighbourhoods $N_{b,c}$ described above. Recall that we can specify all of these neighbourhoods with a single choice of radial distance $\bar{r}$. The $O(r)$ term comes from the principal curvatures on the geodesic spheres $S^{q-1}(r)$, which were computed in Lemma II.13. This computation works exactly the same for a compact family of metrics and so this $O(r)$ term is unchanged. The expression now becomes

$$R^{M_{b,c}} = R^{N_{b,c}} + \sin^2\theta.O(1) - 2k\cdot q\tfrac{\sin\theta}{r}$$
$$+2q(q-1)\tfrac{\sin^2\theta}{r^2} + k\cdot qO(r)\sin\theta.$$

Inequality (II.3.3) can be obtained as before as

$$k[1 + C'r^2] < R_0\tfrac{r}{\sin\theta} + (q-1)\tfrac{\sin\theta}{r} - Cr\sin\theta,$$

where in this case $R_0 = \frac{1}{2q}[\inf(R^{N_{b,c}})]$, taken over all pairs b, c. The important thing is that R_0 is still positive. The construction of a curve γ which satisfies this inequality then proceeds exactly as in Part 2 of Theorem II.11. The resulting curve γ specifies a family of hypersurfaces $M_\gamma^{b,c} \subset N_{b,c} \times \mathbb{R}$. For each (b,c), the induced metric on $M_\gamma^{b,c}$ has positive scalar curvature. The curve γ can then be homotopied back to the vertical line, exactly as in Part 2 of Theorem II.11, inducing a continuous deformation of the family $\{g_\gamma^{b,c}\}$ to the family $\{g_b\}$.

Part 3 of Theorem II.11, can be applied almost exactly as before. The bundle $\mathcal{N}$ and distribution $\mathcal{H}$ are now replaced with continuous families $\mathcal{N}_{b,c}$ and $\mathcal{H}_{b,c}$, giving rise to a continuous family of Riemannian submersions $\pi_{b,c} : (\mathcal{N}_{b,c}, \tilde{g}_{b,c}) \to (i_c(S^p), \check{g}_{b,c})$ where the base metric $\check{g}_{b,c} = g_b|_{i_c(S^p)}$, the fibre metric is $\hat{g} = g_{tor}^{q+1}(\delta)$ as before and $\tilde{g}_{b,c}$ is the respective submersion metric.

By compactness, a single choice of ϵ gives rise to a family of disk bundles $D\mathcal{N}_{b,c}(\epsilon)$ all specifying appropriate submanifolds $M_{\gamma}^{b,c}[t_{\infty}, t_L]$ and $M_{\gamma}^{b,c}[t_L, \bar{t}]$ of $M_{\gamma}^{b,c}$ (see Part 3 of Theorem II.11 for details). Lemma II.18 easily generalises to show that as $\delta \to 0$ there is uniform C^2 convergence $g_{\gamma}^{b,c} \to \tilde{g}_{b,c}$. Thus, there is a continuously varying family of isotopies over b and c, through psc-submersion metrics, deforming each $g_{\gamma}^{b,c}$ into $\tilde{g}_{b,c}$.

Formula II.3.13 now generalises to give the following formula for the scalar curvature of $\tilde{g}_{b,c}$, varying continuously over b and c.

$$\tilde{R_{b,c}} = \check{R_{b,c}} \circ \pi_{b,c} + \hat{R} - |A_{b,c}|^2. \tag{II.3.14}$$

Here $\tilde{R_{b,c}}, \check{R_{b,c}}$ and $\hat{R}$ denote the scalar curvatures of $\tilde{g}_{b,c}, \check{g}_{b,c}$ and $\hat{g}$ respectively. The term $A_{b,c}$ satisfies all of the properties of A in formula (II.3.13), namely $|A_{b,c}|$ is bounded and in fact diminishes uniformly as δ decreases. Thus, there is a sufficiently small $\delta > 0$, so that the family $\{\tilde{g}_{b,c}\}$ can be isotopied through families of psc-submersion metrics to the desired family $\{g_{std}^{b,c}\}$, as in the proof of Theorem II.11. □

Remark II.3.5. *Note that Theorem II.19 claims only the existence of such a map. To write down a well-defined function of this type means incorporating the various parameter choices made in the construction of Theorem II.11. For our current purposes, in this paper, that is not necessary.*

II.3.8 Applications of the Family Surgery Theorem

There are a number of important applications to Theorem II.19. The first is a rather obvious corollary which will be of use to us later on.

Corollary II.20. *Let g and h be isotopic psc-metrics on X. Let g' and h' be respectively, the psc-metrics obtained by application of the Surgery Theorem on a codimension ≥ 3 surgery. Then g' and h' are isotopic.*

Proof. This is just Theorem II.19 where $B = I$ and C is a point. □

A more interesting application is Theorem II.21 below. This theorem is actually the main result in a paper by Chernysh; see [6].

Theorem II.21. *Let X be a smooth compact manifold of dimension n. Suppose X' is obtained from X by surgery on a sphere $S^p \hookrightarrow X$ with $p+q+1=n$ and $p,q \geq 2$. Then the spaces $\mathcal{R}\mathrm{iem}^+(X)$ and $\mathcal{R}\mathrm{iem}^+(X')$ are homotopy equivalent.*

Proof. We will first prove weak homotopy equivalence. Let $\bar{i} : S^p \times D^{q+1} \hookrightarrow X$ be a framed embedding of the sphere S^p. We will assume that $p, q \geq 2$, where as always $p+q+1=n$. Denote by X', the manifold obtained by surgery on S^p with respect to this embedding. Recall that X' is defined as

$$X' = (X \setminus \bar{i}(S^p \times \overset{\circ}{D}{}^{q+1})) \cup_{\bar{i}} D^{p+1} \times S^q.$$

This surgery can be canonically reversed by performing a surgery on the embedded S^q of the attached handle. As $p, q \geq 3$, both surgeries are in codimension ≥ 3.

Let $Y = X \setminus (\bar{i}(S^p \times D^{q+1}))$ and let $\mathcal{R}\mathrm{iem}^+_{std}(Y)$ denote the space of all psc-metrics on Y which, near the boundary ∂Y, have the form $dt^2 + \epsilon^2 ds_p^2 + \delta^2 ds_q^2$ for some $\epsilon, \delta > 0$. Note that ϵ and δ are allowed to vary. Let $g_{\epsilon,\delta}$ denote such a metric in $\mathcal{R}\mathrm{iem}^+_{std}(X \setminus (\bar{i}(S^p \times D^{q+1})))$. This metric can be canonically extended to a metric on X by attaching $(S^p \times D^{q+1}, \epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta))$ with respect to the isometry $\bar{i}|_{S^p \times S^q}$. This gives rise to a map $j : \mathcal{R}\mathrm{iem}^+_{std}(Y) \to \mathcal{R}\mathrm{iem}^+(X)$. Similarly $g_{\epsilon,\delta}$ can be canonically extended to a metric on X' by an analogous attachment of $(D^{p+1} \times S^q, g_{tor}^{p+1}(\epsilon) + \delta^2 ds_q^2)$. We will denote by j' the corresponding map $\mathcal{R}\mathrm{iem}^+_{std}(Y) \to \mathcal{R}\mathrm{iem}^+(X')$.

$$\begin{array}{ccc} & \mathcal{R}\mathrm{iem}^+_{std}(Y) & \\ {\scriptstyle j}\swarrow & & \searrow{\scriptstyle j'} \\ \mathcal{R}\mathrm{iem}^+(X) & & \mathcal{R}\mathrm{iem}^+(X') \end{array}$$

It will now be enough to show that the groups $\pi_k(j)$ and $\pi_k(j')$, in the homotopy long exact sequences of j and j', are trivial for all k. Recall that an element α of $\pi_k(j)$, is an equivalence class of pairs of maps (ϕ, ψ) which form the commutative diagram shown in Fig. II.21.

Let (ϕ, ψ) be a representative pair of some element $\alpha \in \pi_k(j)$. Then $\psi : D^k \to \mathcal{R}\mathrm{iem}^+(X)$ parametrises a compact family of psc-metrics on X. By Theorem II.19, this family can be continuously deformed into a family of psc-metrics which are standard near the embedded sphere S^p. It is of course important that the metrics parametrised by the boundary ∂D^k remain in $\mathcal{R}\mathrm{iem}^+_{std}(Y)$.

$$\begin{array}{ccc} S^{k-1} & \hookrightarrow & D^k \\ \downarrow \phi & & \downarrow \psi \\ \mathcal{R}\mathrm{iem}^+_{std}(Y) & \xrightarrow{j} & \mathcal{R}\mathrm{iem}^+(X) \end{array}$$

Figure II.21: An element α of $\pi_k(j)$

This is almost immediate. Although the initial "bending" part of the Gromov-Lawson construction does alter the standard torpdedo metric on the fibres somewhat, this alteration is very minor. The problem is easily solved by extending $\mathcal{R}\mathrm{iem}^+_{std}(Y)$ to include these altered metrics. The resulting space is homotopy equivalent to $\mathcal{R}\mathrm{iem}^+_{std}(Y)$ via a deformation retract obtained by reversing the bending construction on fibres near the embedded surgery sphere.

Thus, ψ is homotopic to a map $\psi_{std} : D^k \to \mathcal{R}\mathrm{iem}^+(X)$, with $\psi_{std}(D^k)$ contained in the image of j, and so $\alpha = 0$. Hence, $\pi_k(j) = 0$. An analogous argument can be performed on any $\alpha' \in \pi_k(j')$. In this case, Theorem II.19 is used to standardise a compact family of metrics near the embedded S^q. Again, α' is shown to be 0, completing the proof for weak homotopy equivalence.

It follows from the work of Palais in [33] that the spaces $\mathcal{R}\mathrm{iem}^+_{std}(Y)$, $\mathcal{R}\mathrm{iem}^+(X)$ and $\mathcal{R}\mathrm{iem}^+(X_0)$ are all dominated by CW-complexes. Thus, by the theorem of Whitehead, we obtain the desired homotopy equivalence. □

Interestingly, when X is a simply connected spin manifold of dimension ≥ 5, the homotopy type of the space $\mathcal{R}\mathrm{iem}^+(X)$ is an invariant of spin cobordism. This fact is proved by the following theorem.

Theorem II.22. *Let X_0 and X_1 be a pair of compact simply-connected spin manifolds of dimension $n \geq 5$. Suppose also that X_0 is spin cobordant to X_1. Then the spaces $\mathcal{R}\mathrm{iem}^+(X_0)$ and $\mathcal{R}\mathrm{iem}^+(X_1)$ are homotopy equivalent.*

Proof. Let W be a spin cobordism of X_0 and X_1. Then, by Morse-Smale theory, W can be decomposed into a union of elementary cobordisms; see [30]. Each elementary cobordism corresponds to surgery on a sphere S^p. To apply Theorem II.21, we must ensure that $p, q \geq 2$, where $p+q+1 = n$. Each elementary cobordism in the decomposition of W gives rise to an element in $H_*(W, X_0)$, or, viewed from the other direction, an element of $H_*(W, X_1)$. To satisfy the relevant conditions on p

and q, we must show that W can be altered by surgery to make $H_1(W, X_0) = H_2(W, X_0) = 0$ and $H_{n-1}(W, X_0) = H_n(W, X_0) = 0$.

Consider the long exact sequence in homology, of the pair (W, X_0).

$$\cdots \to H_2(W) \to H_2(W, X_0) \to H_1(X_0) \to H_1(W) \to H_1(W, X_0) \to 0$$

By the theorem of Hurewicz, $H_1(X_0) = 0$ and so it is clear that to kill generators in $H_1(W, X_0)$ and $H_2(W, X_0)$, it is enough to kill generators in $H_1(W)$ and $H_2(W)$. Let $\alpha \in \pi_1(W)$. It follows from a theorem of Whitney, Theorem 2 of [41], that α can be represented by an embedding $S^1 \to W$. As W is orientable, i.e. the first Stiefel Whitney class $w_1(W) = 0$, this embedding can be extended to a framed embedding $S^1 \times D^n \to W$ and the generator killed by surgery. This can be repeated to kill all generators in $\pi_1(W)$ and, hence, all generators in $H_1(W)$.

As $\pi_1(W) = H_1(W) = 0$, the Hurewicz theorem now tells us that $\pi_2(W) \cong H_2(W)$. We now consider a generator $\beta \in \pi_2(W)$. Again, Whitney's theorem tells us that β can be represented by an embedding $S^2 \to W$. The fact that W is spin, i.e. $w_2(W) = 0$, means that this embedding can be extended to a framed embedding $S^2 \times D^{n-1}$ and the generator killed by surgery. This can be done for all such generators to give $H_1(W, X_0) = H_2(W, X_0) = 0$. It now follows from duality and the Universal Coefficient Theorem, that $H_{n-1}(W, X_0) = H_n(W, X_0) = 0$ also, completing the proof. □

II.3.9 The proof of Theorem II.10(The Improved Surgery Theorem)

Proof. Recall that g denotes a positive scalar curvature metric on the closed manifold X^n, $i : S^p \hookrightarrow X$ denotes an embedding of the sphere S^p with trivial normal bundle and that $p+q+1 = n$ with $q \geq 2$. Let W denote the trace of a surgery on X with respect to this embedded sphere. We wish to extend g over W to obtain a psc-metric which is product near the boundary.

Corollary II.12 implies the existence of a psc-metric $\bar{g}$ on the cylinder $X \times I$ so that near $X \times \{0\}$, $\bar{g} = g + ds^2$ and near $X \times \{1\}$, $\bar{g} = g_{std} + ds^2$ where g_{std} is the metric obtained in Theorem II.11. Thus, by choosing $g_p = \epsilon^2 ds_p^2$, near S^p the metric g_{std} has the form $\epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta)$ for some sufficiently small $\delta > 0$. Using the exponential map for g_{std} we can specify a tubular neighbourhood of S^p , $N = S^p \times D^{q+1}(\bar{r})$, so that the restriction of g_{std} on N is precisely the metric $\epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta)$. As before, N is equipped with the coordinates (y, x) where $y \in S^p$, $x \in D^{q+1}(\bar{r})$

and $D^{q+1}(\bar{r})$ is the Euclidean disk of radius $\bar{r}$. The quantity r will denote the Euclidean radial distance on $D^{q+1}(\bar{r})$. Moreover, we may assume that δ is arbitrarily small and that the tube part of $g_{tor}^{q+1}(\delta)$ is arbitrarily long, thus the quantity $\bar{r}-\delta$ can be made as large as we like.

We will now attach a handle $D^{p+1}\times D^{q+1}$ to the cylinder $X\times I$. Recall that in section II.2, we equipped the plane $\mathbb{R}^{n+1}=\mathbb{R}^{p+1}\times\mathbb{R}^{q+1}$ with a metric $h=g_{tor}^{p+1}(\epsilon)+g_{tor}^{q+1}(\delta)$. By equipping $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ with standard spherical coordinates (ρ,ϕ) and (r,θ), we realised the metric h as

$$h=d\rho^2+f_\epsilon(\rho)^2ds_p^2+dr^2+f_\delta(r)^2ds_q^2,$$

where $f_\epsilon, f_\delta:(0,\infty)\to(0,\infty)$ are the torpedo curves defined in section II.2. The restriction of h to the disk product $D^{p+1}(\bar{\rho})\times D^{q+1}(\bar{r})$ is the desired handle metric, where $\bar{\rho}$ is as large as we like. We can then glue the boundary component $\partial(D^{p+1}(\bar{\rho}))\times D^{q+1}(\bar{r})$ to N with the isometry

$$\begin{aligned} S^p\times D^{q+1}(\bar{r}) &\longrightarrow N\\ (y,x)&\longmapsto (i(y),L_y(x)),\end{aligned}$$

where $L_y\in O(q+1)$ for all $y\in S^p$. Different choices of map $y\mapsto L_y\in O(q+1)$ give rise to different framings of the embedded surgery sphere S^p in X. The resulting manifold (which is not yet smooth) is represented in Fig. II.22. Recall that $\bar{\rho}$ and $\bar{r}$ are radial distances with respect to the Euclidean metric on $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ respectively. By choosing ϵ and δ sufficiently small and the corresponding tubes long enough, we can ensure that $\frac{\pi}{2}\epsilon<\bar{\rho}$ and $\frac{\pi}{2}\delta<\bar{r}$.

Two tasks remain. Firstly, we need to make this attaching smooth near the corners. This will be done in the obvious way by specifying a smooth hypersurface inside $D^{p+1}(\bar{\rho})\times D^{q+1}(\bar{r})$ which meets N smoothly near its boundary, as shown by the dashed curve in Fig. II.22. This is similar to the hypersurface M constructed in the original Gromov-Lawson construction. Again we must ensure that the metric induced on this hypersurface has positive scalar curvature. This is considerably easier than in the case of M, given the ambient metric we are now working with. We will in fact show that the metric induced on this hypersurface is precisely the metric obtained by the Gromov-Lawson construction. The second task is to show that this metric can be adjusted to have a product structure near the boundary.

The spherical coordinates (ρ,ϕ,r,θ) on the handle $D^{p+1}(\bar{\rho})\times D^{q+1}(\bar{r})$ can be extended to overlap with $X\times I$ on $N(\bar{r})\times[1-\epsilon_1,1]$, where ϵ_1 is chosen so that $\bar{g}|_{X\times[1-\epsilon_1,1]}=g_{std}+dt^2$. We

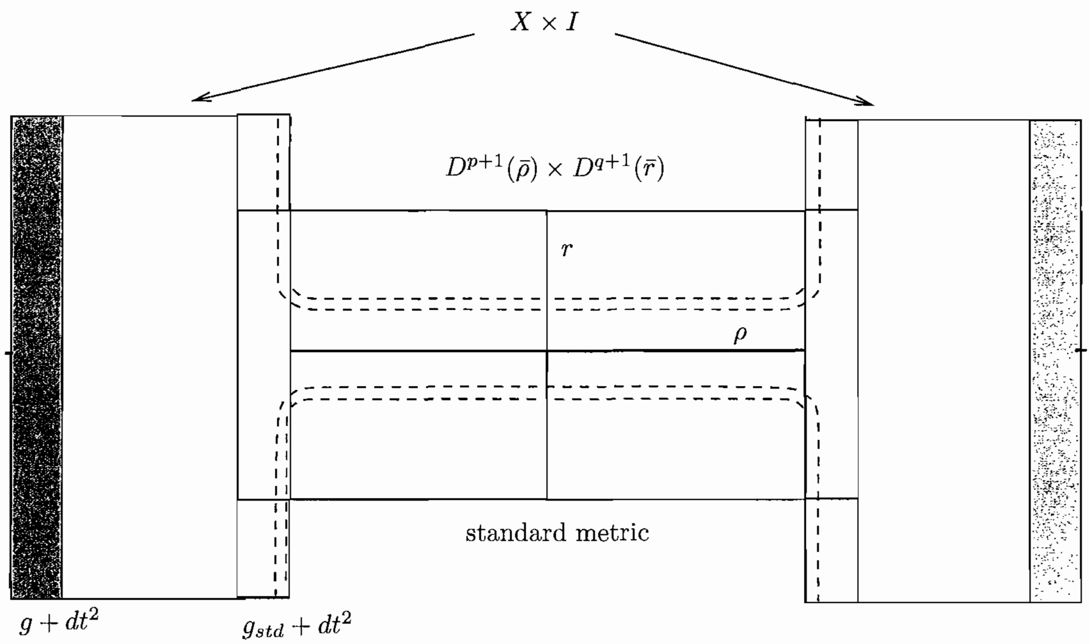

Figure II.22: The metric $(X \times I, \bar{g}) \cup (D^{p+1}(\bar{\rho}) \times D^{q+1}(\bar{r}), h)$ and the smooth handle represented by the dashed curve

denote this region $D^{p+1}(\bar{\rho}) \times D^{q+1}(\bar{r})$. Let E be the embedding

$$\begin{aligned} E : [0, \epsilon_1] \times [0, \infty) &\longrightarrow \mathbb{R} \times \mathbb{R} \\ (s, t) &\longmapsto (a_1(s, t), a_2(s, t)) \end{aligned} \tag{II.3.15}$$

shown in Fig. II.23. The map E will satisfy the following conditions.

(1) For each $s_0 \in [0, \epsilon_1]$, $E(s_0, t)$ is the curve $(t, c_2(s_0))$ when $t \in [0, \frac{\epsilon\pi}{2}]$, and ends as the unit speed vertical line $(c_1(s_0), t)$. Here c_1 and c_2 are functions on $[0, \epsilon_1]$ defined as follows. For each s, $c_1(s) = \bar{\rho} + s$ and $c_2(s) = c_2(0) - s$, where $c_1(0) - \epsilon_1 > \frac{\pi}{2}\epsilon$ and $c_2(s) > \frac{\pi}{2}\delta$.

(2) For each $t_0 \in [0, \infty)$, the path $E(s, t_0)$ runs orthogonally to the levels $E(s_0, t)$ for each $s_0 \in [0, \epsilon_1]$. That is, for each (s_0, t_0), $\frac{\partial E}{\partial s}(s_0, t_0) \cdot \frac{\partial E}{\partial t}(s_0, t_0) = 0$.

Provided ϵ_1 is chosen sufficiently small, the map

$$\begin{aligned} Z : [0, \epsilon_1] \times (0, \infty) \times S^p \times S^q &\to D^{p+1}(\bar{\rho} + \epsilon_1) \times D^{q+1}(\bar{r}) \\ (s, t, \phi, \theta) &\mapsto (a_1(s, t), \phi, a_2(s, t), \theta) \end{aligned}$$

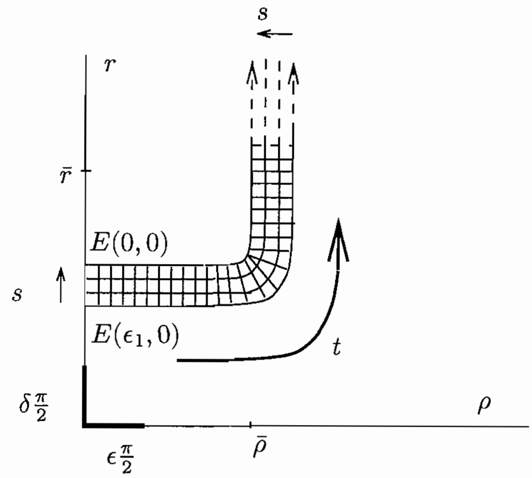

Figure II.23: The embedding E

parametrises a region in $D^{p+1}(\bar{\rho}+\epsilon_1)\times D^{q+1}(\bar{r})$. Consider the hypersurface parametrised by the restriction of Z to $\{0\}\times(0,\infty)\times S^p\times S^q$. The metric induced on the region bounded by this hypersurface extends $\bar{g}$ as a psc-metric over the trace of the surgery. Now we need to show that this metric can be deformed to one which is a product near the boundary while maintaining positive scalar curvature.

We begin by computing the metric near the boundary with respect to the parameterisation Z. Letting

$$Y_s=\frac{\partial a_1}{\partial s}^2+\frac{\partial a_2}{\partial s}^2 \quad \text{and} \quad Y_t=\frac{\partial a_1}{\partial t}^2+\frac{\partial a_2}{\partial t}^2,$$

$$\begin{aligned}Z^*(d\rho^2+f_\epsilon(\rho)^2ds_p^2+dr^2+f_\delta(r)^2ds_q^2) &= da_1{}^2+f_\epsilon(a_1)^2ds_p^2+da_2{}^2+f_\delta(a_2)^2ds_q^2\\ &= Y_s(s,t)ds^2+Y_t(s,t)dt^2+f_\epsilon(a_1)^2ds_p^2+f_\delta(a_2)^2ds_q^2.\end{aligned}$$

On the straight pieces of our neighbourhood, it is clear that $Y_s=1$ and $Y_t=1$. Thus, on the straight region running parallel to the horizontal axis, the metric is

$$\begin{aligned}ds^2+dt^2+f_\epsilon(a_1)^2ds_p^2+f_\delta(a_2)^2ds_q^2 &= ds^2+dt^2+f_\epsilon(t)^2ds_p^2+f_\delta(c_2(s))^2ds_q^2\\ &= ds^2+dt^2+f_\epsilon(t)^2ds_p^2+\delta^2ds_q^2, \quad \text{since } c_2>\tfrac{\pi}{2}\delta.\end{aligned}$$

On the straight region running parallel to the vertical axis, the metric is

$$\begin{aligned} ds^2 + dt^2 + f_\epsilon(a_1)^2 ds_p^2 + f_\delta(a_2)^2 ds_q^2 &= ds^2 + dt^2 + f_\epsilon(c_1(s))^2 ds_p^2 + f_\delta(t)^2 ds_q^2 \\ &= ds^2 + dt^2 + \epsilon^2 ds_p^2 + \delta^2 ds_q^2, \\ &= ds^2 + dt^2 + f_\epsilon(t)^2 ds_p^2 + \delta^2 ds_q^2. \end{aligned}$$

The second equality holds because $c_1 > \frac{\pi}{2}\epsilon$ and $t > \frac{\pi}{2}\delta$. The last equality follows from the fact that $t > c_1 > \frac{\pi}{2}\epsilon$ and $t > \frac{\pi}{2}\delta$. As we do not have unit speed curves in s and t, the best we can say about the remaining "bending" region is that the metric is of the form

$$Y_s ds^2 + Y_t dt^2 + \epsilon^2 ds_p^2 + \delta^2 ds_q^2.$$

The graphs of Y_s and Y_t are surfaces, shown schematically in Fig. II.24. Outside of a compact region, $Y_s = 1$ and $Y_t = 1$. We can replace Y_s and Y_t with smooth functions Y_s' and Y_t', so that on $[\epsilon_2, \epsilon_1] \times (0, \infty)$, $Y_s = Y_s'$ and $Y_t = Y_t'$ and so that on $[0, \epsilon_3] \times (0, \infty)$, $Y_s' = Y_t' = 1$ for some $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$. Moreover, this can be done so that $Y_s - Y_s'$ and $Y_t - Y_t'$ have support in a compact region.

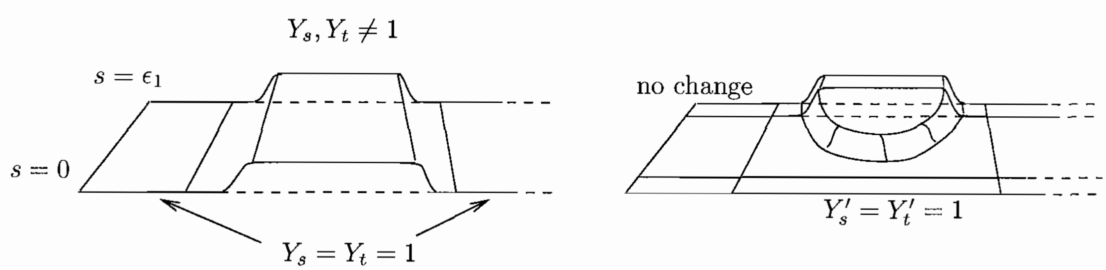

Figure II.24: Adjusting Y_s and Y_t

Any curvature resulting from these changes is bounded and completely independent of the metric on the sphere factors. Thus, we can always choose δ sufficiently small to guarantee the positive scalar curvature of the resulting metric

$$Y_s' ds^2 + Y_t' dt^2 + f_\epsilon(t)^2 ds_p^2 + \delta^2 ds_q^2$$

which, when $s \in [0, \epsilon_3]$, is the metric

$$ds^2 + dt^2 + f_\epsilon(t)^2 ds_p^2 + \delta^2 ds_n^2.$$

This is of course the desired product metric $ds^2 + g_{tor}^{p+1}(\epsilon) + \delta^2 ds_q^2$, completing the proof of Theorem II.10. □

II.4 Constructing Gromov-Lawson Cobordisms

In section II.3 we showed that a psc-metric g on X can be extended over the trace of a codimension ≥ 3 surgery to a psc-metric with product structure near the boundary. Our goal in section II.4 is to generalise this result in the form of Theorem II.23. Here $\{W^{n+1}; X_0, X_1\}$ denotes a smooth compact cobordism and g_0 is a psc-metric on X_0. If W can be decomposed into a union of elementary cobordisms, each the trace of a codimension ≥ 3 surgery, then we should be able to extend g_0 to a psc-metric on W, which is product near the boundary, by repeated application of Theorem II.10. Two questions now arise. Assuming W admits such a decomposition, how do we realise it? Secondly, how many such decompositions can we realise? In order to answer these questions, it is worth briefly reviewing some basic Morse Theory. For a more thorough account of this, see [30] and [16].

II.4.1 Morse Theory and admissible Morse functions

Let $\mathcal{F} = \mathcal{F}(W)$ denote the space of smooth functions $f : W \to I$ on the cobordism $\{W; X_0, X_1\}$ with the property that $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$, and having no critical points near ∂W. The space $\mathcal{F}$ is a subspace of the space of smooth functions on W with its standard C^∞ topology; see Chapter 2 of [17] for the full definition. A function $f \in \mathcal{F}$ is a Morse function if, whenever w is a critical point of f, $det(D^2 f(w)) \neq 0$. Here $D^2 f(w)$ denotes the Hessian of f at w. The Morse index of the critical point w is the number of negative eigenvalues of $D^2 f(w)$. The well known Morse Lemma, Lemma 2.2 of [30], then says that there is a coordinate chart $\{x = (x_1, x_2, \ldots, x_{n+1})\}$ near w, with w identified with $(0, \ldots, 0)$, so that in these coordinates,

$$f(x) = c - x_1^2 - \cdots - x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{n+1}^2, \tag{II.4.1}$$

where $c = f(w)$. Here $p+1$ is the Morse index of w and this coordinate chart is known as a *Morse coordinate chart.*

Inside of this coordinate chart it is clear that level sets below the critical level are diffeomorphic to $S^p \times D^{q+1}$ and that level sets above the critical level are diffeomorphic to $D^{p+1} \times S^q$ where $p+q+1 = n$; see Fig. II.25. In the case where f has exactly one critical point w of index $p+1$, the cobordism W is diffeomorphic to the trace of a p-surgery on X_0. If W admits a Morse function f with no critical points then by theorem 3.4 of [30], W is diffeomorphic to the cylinder $X_0 \times I$ (and consequently X_0 is diffeomorphic to X_1).

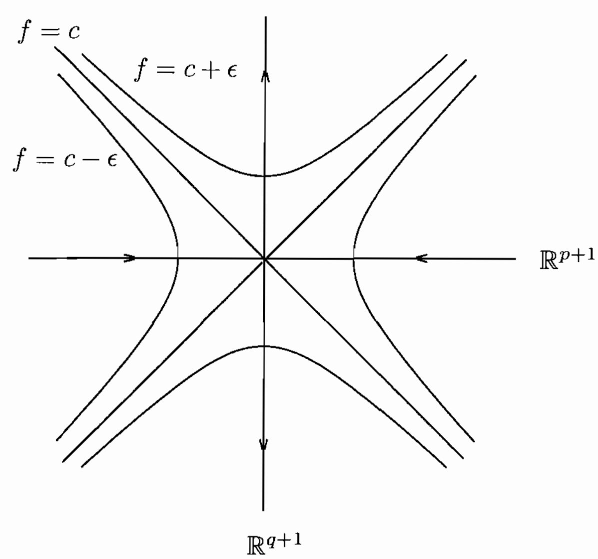

Figure II.25: Morse coordinates around a critical point

The critical points of a Morse function are isolated and as W is compact, f will have only finitely many. Denote the critical points of f as $w_0, w_1, \dots, w_k$ where each w_i has Morse index $p_i + 1$. We will assume that $0 < f(w_0) = c_0 \leq f(w_1) = c_1 \leq \cdots f(w_k) = c_k < 1$.

Definition II.3. The Morse function f is *well-indexed* if critical points on the same level set have the same index and for all i, $p_i \leq p_{i+1}$.

In the case when the above inequalities are all strict, f decomposes W into a union of elementary cobordisms $C_0 \cup C_1 \cup \cdots \cup C_k$. Here each $C_i = f^{-1}([c_{i-1}+\tau, c_i+\tau])$ when $0 < i < k$, and

$C_0 = f^{-1}([0, c_0 + \tau])$ and $C_k = f^{-1}([c_{k-1} + \tau, 1])$, for some appropriately small $\tau > 0$. Each C_i is the trace of a p_i-surgery. When these inequalities are not strict, in other words f has level sets with more than one critical point, then W is decomposed into a union of cobordisms $C_0' \cup C_1' \cup \cdots \cup C_l'$ where $l < k$. A cobordism C_i' which is not elementary, is the trace of several distinct surgeries. It is of course possible, with a minor adjustment of f, to make the above inequalities strict.

By equipping W with a Riemannian metric m, we can define $\text{grad}_m f$ the gradient vector field for f. This metric is called a *background metric* for f and has no relation to any of the other metrics mentioned here. In particular, no assumptions are made about its curvature. More generally, we define *gradient-like* vector fields on W with respect to f and m, as follows.

Definition II.4. A *gradient-like* vector field with respect to f and m is a vector field V on W satisfying the following properties.

(1) $df_x(V_x) > 0$ for all x in W which are not critical points of f.

(2) Each critical point w of f, lies in a neighbouhood U so that for all $x \in U$, $V_x = \text{grad}_m f(x)$.

We point out that the space of background metrics for a particular Morse function $f : W \to I$ is a convex space. So too, is the space of gradient-like vector fields associated with any particular pair (f, m); see Chapter 2, section 2 of [16]. We can now define an admissible Morse function on W.

Definition II.5. An *admissible Morse function* f on a compact cobordism $\{W; X_0, X_1\}$ is a triple $f = (f, m, V)$ where $f : W \to I$ is a Morse function, m is a background metric for f, V is a gradient like vector field with respect to f and m, and finally, any critical point of f has Morse index less than or equal to $n - 2$

Remark II.4.1. *We emphasise the fact that an admissible Morse function is actually a triple consisting of a Morse function, a Riemannian metric and a gradient-like vector field. However, to ease the burden of notation, an admissible Morse function (f, m, V) will be denoted as simply f.*

Associated to each critical point w of index $p+1$, is a pair of trajectory spheres $S_-^p(w)$ and $S_+^q(w)$, respectively converging towards and emerging from w; see Fig. II.26. As usual $p + q + 1 = n$. Let us assume for simplicity that f has exactly one critical point w and that w has Morse index $p + 1$. Then associated to w is an embedded sphere $S^p = S_-^p(w)$ in X_0 which follows

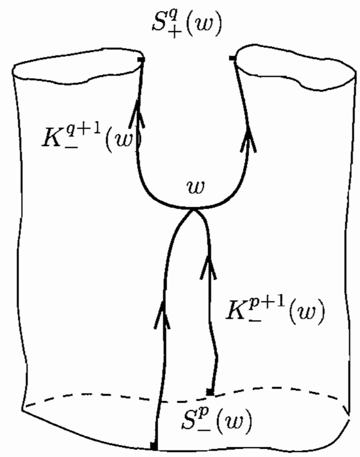

Figure II.26: Trajectory spheres for a critical point w on an elementary cobordism

a trajectory towards w. The trajectory itself consists of the union of segments of integral curves of the gradient vector field beginning at the embedded $S^p \subset X_0$ and ending at w. It is topologically a $(p+1)$-dimensional disk D^{p+1}. We denote it $K_-^{p+1}(w)$. Similarly, there is an embedded sphere $S^q = S_+^q \subset X_1$ which bounds a trajectory $K_+^q(w)$ (homeomorphic to a disk D^q) emerging from w. Both spheres are embedded with trivial normal bundle and the elementary cobordism W is in fact diffeomorphic to the trace of a surgery on X_0 with respect to S^p.

We are now in a position to prove Theorem II.23. This is the construction, given a positive scalar curvature metric g_0 on X_0 and an admissible Morse function f on W, of a psc-metric $\bar{g} = \bar{g}(g_0, f)$ on W which extends g_0 and is a product near the boundary. As pointed out in the introduction, the metric $\bar{g}$ is known as a *Gromov-Lawson cobordism with respect to* g_0 *and* f. The resulting metric induced on X_1, $g_1 = \bar{g}|_{X_1}$, is said to be *Gromov-Lawson cobordant* to g_0.

Theorem II.23. *Let* $\{W^{n+1}; X_0, X_1\}$ *be a smooth compact cobordism. Suppose* g_0 *is a metric of positive scalar curvature on* X_0 *and* $f : W \to I$ *is an admissible Morse function. Then there is a psc-metric* $\bar{g} = \bar{g}(g_0, f)$ *on* W *which extends* g_0 *and has a product structure near the boundary.*

Proof. Let f be an admissible Morse function on W. Let m be the background metric on W, as described above. Around each critical point w_i of f we choose mutually disjoint Morse coordinate balls $B(w_i) = B_m(w_i, \bar{\epsilon})$ where $\bar{\epsilon} > 0$ is some sufficiently small constant. In each case we will assume that the background metric m agrees with the metric obtained by pulling back the standard Euclidean metric via the Morse coordinate diffeomorphism. This is reasonable since the metric m can always be adjusted via a linear homotopy to obtain such a metric. For the moment, we may assume that f has only one critical point w of Morse index $p+1$ where as usual $p+q+1=n$ and $q\geq 2$. Let $c=f(w)\in(0,1)$. Associated to w are the trajectory spheres $S^p=S^p_-(w)$ and $S^q_+(w)$, defined earlier in this section. Let $N=S^p\times D^{q+1}(\bar{r})\subset X_0$ denote the tubular neighbourhood defined in the proof of Theorem II.11, constructed using the exponential map for the metric g_0. The *normalised* gradient-like flow of f (obtained by replacing V with $\frac{V}{m(V,V)}$ away from critical points and smoothing with an appropriate bump function) gives rise to a diffeomorphism from $f^{-1}([0,\epsilon_0])$ to $f^{-1}([0,c-\tau])$ where $0<\epsilon_0<c-\tau<c$. In particular, normalisation means that it maps $f^{-1}([0,\epsilon_0])$ diffeomorphically onto $f^{-1}([c-\tau-\epsilon_0,c-\tau])$. For sufficiently small $\bar{r}$, ϵ_0 and τ, the level set $f^{-1}(c-\tau)$ may be chosen to intersect with $B(w)$ so as to contain the image of $N\times[0,\epsilon_0]$ under this diffeomorphism; see Fig. II.27.

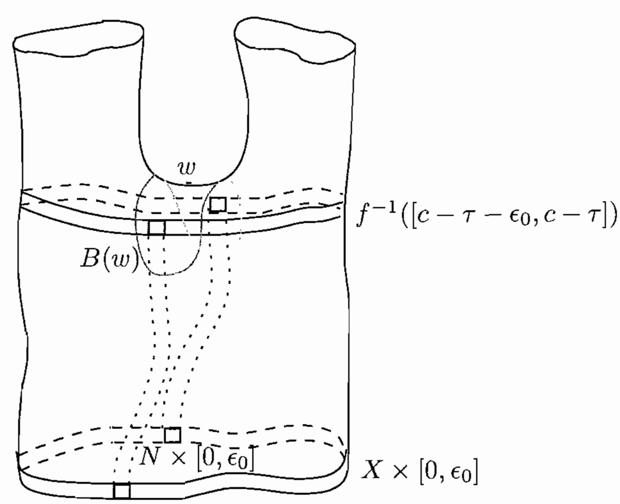

Figure II.27: The action of the gradient-like flow on $N\times[0,\epsilon_0]$

We may use the normalised gradient-like flow to construct a diffeomorphism between $X_0\times[0,c-\tau]$ and $f^{-1}([0,c-\tau])$ which for each $s\in[0,c-\tau]$, maps $X\times\{s\}$ diffeomorphically onto $f^{-1}(s)$. Corollary II.12 then allows us to extend the metric g_0 from X_0 as a psc-metric over $f^{-1}([0,c-\tau])$ which is product near the boundary. Moreover, this extension can be constructed so that the resulting metric $\bar{g_0}$, is the product g_0+dt^2 outside of $B(w)$ and on $f^{-1}([c-\tau-\epsilon_0,c-\tau])$ is the metric $(g_0)_{std}+dt^2$ where $(g_0)_{std}$ is the metric constructed in Theorem II.11 with respect to N. Recall that on X_0, the metric $(g_0)_{std}$ is the original metric g_0 outside of N but that near S^p, $(g_0)_{std} = \epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta)$. Choose $r_0 \in (0, \bar{r})$, so that on the neighbourhood $N(r_0) = S^p \times D^{q+1}(r_0) \subset N$, $(g_0)_{std} = \epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta)$. Observe that the trajectories beginning at $X_0 \setminus N(r_0)$ do not pass any critical points. Thus, it is possible to extend $\bar{g}_0$ as $(g_0)_{std} + dt^2$ along this trajectory up to the level set $f^{-1}(c+\tau)$. To extend this metric over the rest of $f^{-1}([0, c+\tau])$, we use a diffeomorphism of the type described in Fig. II.28 to adjust coordinates near w. Thus, away from the origin, the level sets and flow lines of f are the vertical and horizontal lines of the standard Cartesian plane. Also, the extension along the trajectory of $X_0 \setminus N(r_0)$ is assumed to take place on this region; see Fig. II.29. Over the rest of $f^{-1}(c+\tau)$, the metric $\bar{g}_0$ can be extended extended as the metric constructed in Theorem II.10.

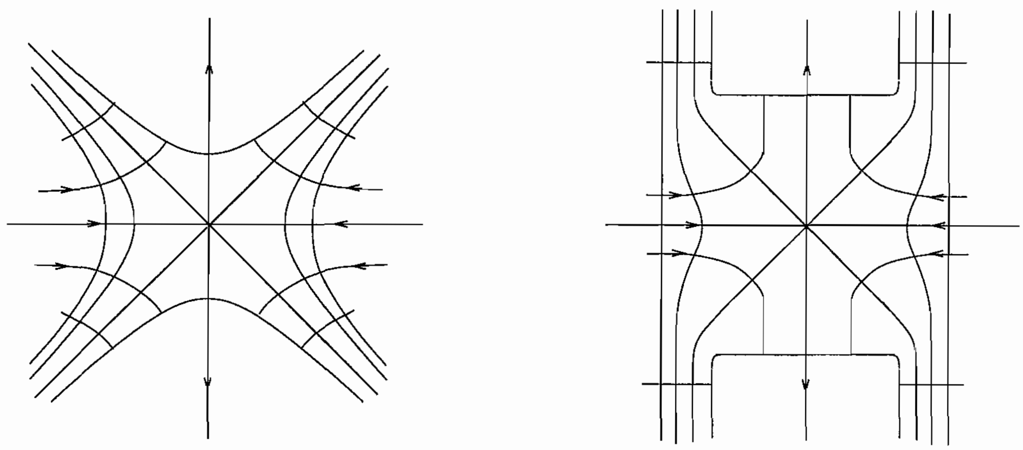

Figure II.28: A diffeomorphism on the handle.

At this stage we have constructed a psc-metric on $f^{-1}(c+\tau)$, which extends the original metric g_0 on X_0 and is product near the boundary. As $f^{-1}([c+\tau, 1])$ is diffeomorphic to the cylinder $X_1 \times [c+\tau, 1]$, this metric can then be extended as a product metric over the rest of W. This construction is easily generalised to the case where f has more than one critical point on the level set $f^{-1}(c)$. In the case where f has more than one critical level, and thus decomposes W into cobordisms $C_0' \cup C_1' \cup \cdots \cup C_l'$ as described above, repeated application of this construction over each cobordism results in the desired metric $\bar{g}(g_0, f)$. □

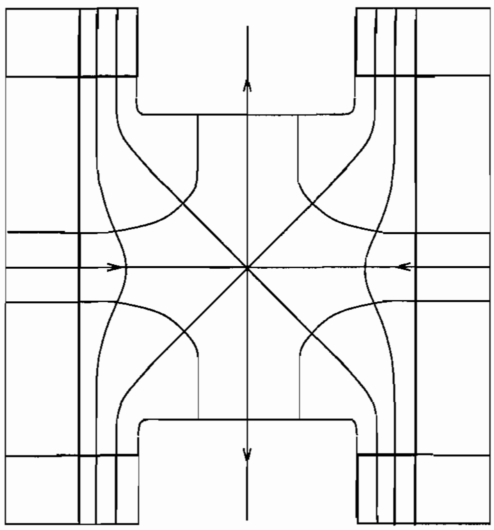

Figure II.29: Extending $\bar{g}_0$ along the trajectory of $X_0 \setminus N(r_0)$ to the level set $f^{-1}(c+\tau)$.

II.4.2 A reverse Gromov-Lawson cobordism

Given a Morse triple $f = (f, m, V)$ on a smooth compact cobordism $\{W; X_0, X_1\}$, with $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$, we denote by $1 - f$, the Morse triple $(1 - f, m, -V)$ which has the gradient-like flow of f, but running in the opposite direction. In particular, $(1-f)^{-1}(0) = X_1$ and $(1-f)^{-1}(1) = X_0$ and so it is easier to think of this as simply "turning the cobordism W upside down". Although $1 - f$ has the same critical points as f, there is a change in the indices. Each critical point of f with index $p+1$ is a critical point of $1-f$ with index $q+1$, where $p+q+1 = n$ and $\dim W = n + 1$. Just as f describes a sequence of surgeries which turns X_0 into X_1, $1 - f$ describes a sequence of complementary surgeries which reverses this process and turns X_1 back into X_0.

Given an admissible Morse function f on a cobordism $\{W; X_0, X_1\}$, Theorem II.23 allows us to construct, from a psc-metric g_0 on X_0, a new psc-metric g_1 on X_1. Suppose now that $1 - f$ is also an admissible Morse function. The following theorem describes what happens if we reapply the construction of Theorem II.23 on the metric g_1 with respect to the function $1 - f$.

Theorem II.24. *Let $\{W^{n+1}; X_0, X_1\}$ be a smooth compact cobordism, g_0 a psc-metric on X_0 and $f : W \rightarrow I$, an admissible Morse function. Suppose that $1-f$ is also an admissible Morse function. Let $g_1 = \bar{g}(g_0, f)|_{X_1}$ denote the restriction of the Gromov-Lawson cobordism $\bar{g}(g_0, f)$ to X_1. Let $\bar{g}(g_1, 1-f)$ be a Gromov-Lawson cobordism with respect to g_1 and $1-f$ and let $g_0' = \bar{g}(g_1, 1-f)|_{X_0}$ denote the restriction of this metric to X_0. Then g_0 and g_0' are canonically isotopic metrics in $\mathcal{R}\mathrm{iem}^+(X_0)$.*

Proof. It is enough to consider the case where f has a single critical point w of index $p+1$. The metric g_1 is the restriction of the metric $\bar{g}(g_0, f)$, constructed in Theorem II.23, to X_1. In constructing the metric g_0' we apply the Gromov-Lawson construction to this metric with respect to surgery on an embedded sphere S^q. The admissible Morse function $1-f$ determines a neighbourhood $S^q \times D^{p+1}$ on which this surgery takes place. Recall that, by construction, the metric g_1 is already the standard metric $\delta^2 ds_q^2 + g_{tor}^{p+1}(\epsilon)$ near this embedded sphere. Thus, g_0' is precisely the metric obtained by applying the Gromov-Lawson construction on this standard piece. Removing a tubular neighbourhood of S^q in this standard region results in a metric on $X_1 \setminus S^q \times D^{p+1}$, which is the standard product $\delta^2 ds_q^2 + \epsilon^2 ds_p^2$. The construction is completed by attaching the product $D^{q+1} \times S^p$ with the standard metric $g_{tor}^{q+1}(\delta) + \epsilon^2 ds_p^2$. In Fig. II.30 we represent this, using a dashed curve, as a hypersurface of the standard region. The resulting metric is isotopic to the metric $(g_0)_{std}$, the metric obtained from g_0 in Theorem II.11, by a continuous rescaling of the tube length of the torpedo factor. In turn $(g_0)_{std}$ can then be isotopied back to g_0 by Theorem II.11.

□

II.4.3 Continuous families of Morse functions

The construction of Theorem II.23 easily generalises to the case of a compact contractible family of admissible Morse functions. Before doing this we should briefly discuss the space $\mathcal{M} = \mathcal{M}(W)$ of Morse functions in $\mathcal{F} = \mathcal{F}(W)$. It is well known that $\mathcal{M}$ is an open dense subspace of $\mathcal{F}$; see theorem 2.7 of [30]. Let $\tilde{\mathcal{F}}$ denote the space of triples (f, m, V) so that $f \in \mathcal{F}$, m is a backgound metric for f and V is a gradient-like vector field with respect to f and m. The space $\tilde{\mathcal{F}}$ is then homotopy equivalent to the space $\mathcal{F}$. In fact, by equipping W with a fixed background metric $\bar{m}$, the inclusion map

$$f \longmapsto (f, \bar{m}, \mathrm{grad}_{\bar{m}} f) \qquad \text{(II.4.2)}$$

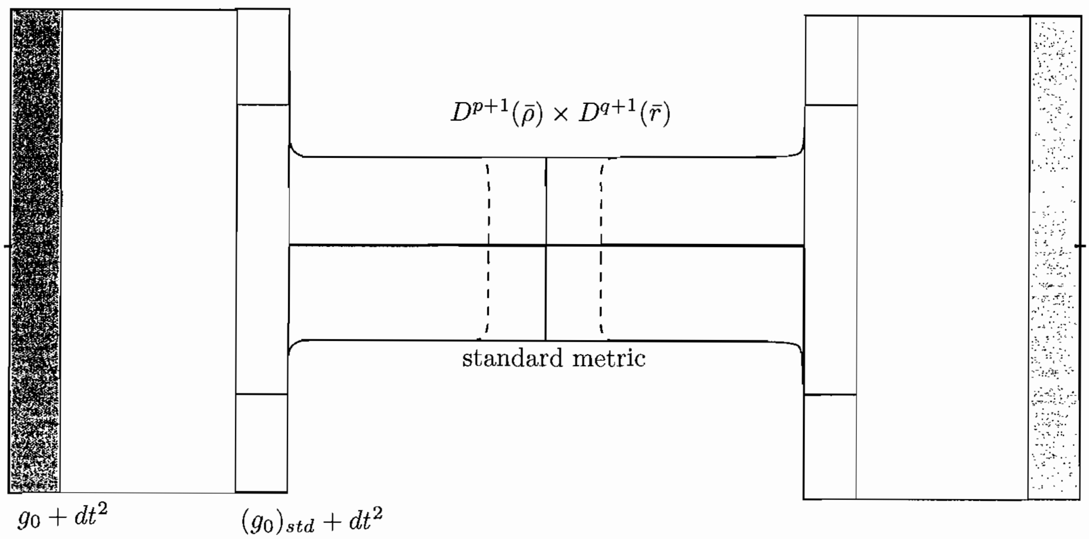

Figure II.30: The metric induced by $\bar{g}(g_1, -f)$ on a level set below the critical level

forms part of a deformation retract of $\tilde{\mathcal{F}}$ down to $\mathcal{F}$; see Chapter 2, section 2 of [16] for details.

Denote by $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(W)$, the subspace of $\tilde{\mathcal{F}}$, of triples (f, m, V) where f is a Morse function. Elements of $\tilde{\mathcal{M}}$ will be known as *Morse triples*. The above deformation retract then restricts to a deformation retract of $\tilde{\mathcal{M}}$ to $\mathcal{M}$. The subspace of $\tilde{\mathcal{M}}$ consisting of admissible Morse functions will be denoted $\tilde{\mathcal{M}}^{adm} = \tilde{\mathcal{M}}^{adm}(W)$. To economise in notation we will shorten (f, m, V) to simply f. Let $f_0, f_1 \in \tilde{\mathcal{M}}$.

Definition II.6. The Morse triples f_0 and f_1 are *isotopic* if they lie in the same path component of $\tilde{\mathcal{M}}$. A path $f_t, t \in I$ connecting f_0 and f_1 is called an *isotopy* of Morse triples.

Remark II.4.2. *This dual use of the word isotopy is unfortunate, however, it should be clear from context which meaning we wish to employ.*

In order for two Morse triples f_0 and f_1 to lie in the same path component of $\tilde{\mathcal{M}}$, it is necessary that both have the same number of index p critical points for each $p \in \{0, 1, \ldots n+1\}$. Thus, if f_0 and f_1 are both admissible Morse functions, an isotopy of Morse triples connecting f_0 to f_1 is contained entirely in $\tilde{\mathcal{M}}^{adm}$. We now prove Theorem II.25.

Theorem II.25. *Let $\{W, X_0, X_1\}$ be a smooth compact cobordism and let B a compact space. Let $\mathcal{B} = \{g_b \in \mathcal{R}\text{iem}^+(X_0) : b \in B\}$ be a continuous family of psc-metrics on X_0 and let $\mathcal{C} = \{f_c \in \tilde{\mathcal{M}}^{adm}(W) : c \in D^k\}$ be a smooth family of admissible Morse functions on W, parametrised by a k-dimensional disk D^k. Then there is a continuous map*

$$\mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{R}\text{iem}^+(W)$$

$$(g_b, f_c) \longmapsto \bar{g}_{b,c} = \bar{g}(g_b, f_c)$$

so that for each pair (b, c), the metric $\bar{g}_{b,c}$ is a Gromov-Lawson cobordism.

Proof. For each $c \in D^k$, f_c will have the same number of critical points of the same index. There is therefore a smooth rearrangement of the critical points $w_1(c), \ldots, w_l(c)$ as c varies over D^k. In turn this means a smooth rearrangement of embedded surgery spheres. The proof then follows almost directly from Lemma II.19. There is however, a compatability issue to address. In order to carry out the construction of Theorem II.23 on f_c with respect to any psc-metric $g_b \in \mathcal{B}$, we must specify disjoint Morse coordinate neighbourhoods $U(w_i(c))$ around each critical point of f_c. As c varies over D^k we must be sure that we can vary these coordinate neighbourhoods. The fact that the parameterising space is a disk means that this is certainly possible and follows from Theorem 1.4 in the appendix of [23]. For each critical point $w_i(c) \in f_c$, this theorem guarantees the existence of a smoothly varying family of embeddings $\psi_c : \mathbb{R}^{n+1} \to W$ so that $\psi_c(0) = w_i(c)$ and the composition $f_c \circ \psi_c(x) = f(w_i(c)) - \sum_{j=1}^{p+1} x_j^2 + \sum_{j=p+2}^{n+1} x_j^2$, where $w_i(c)$ has index $p+1$. □

Corollary II.26. *Let $f_t, t \in I$ be an isotopy in the space admissible Morse functions, $\tilde{\mathcal{M}}^{adm}(W)$. Then there is a continuous family of psc-metrics $\bar{g}_t$ on W so that for each t, $\bar{g}_t = \bar{g}(g_0, f_t)$ is a Gromov-Lawson cobordism of the type constructed in Theorem II.23. In particular, $\bar{g}_t|_{X_1}, t \in I$ is an isotopy of psc-metrics on X_1.*

Definition II.7. A Morse triple (f, m, V) is *well indexed* if the Morse function f is well indexed.

Theorem II.27. *[30] Let $f \in \tilde{\mathcal{M}}$. Then there is a well-indexed Morse triple $\bar{f}$ which lies in the same path component of $\tilde{\mathcal{M}}$.*

This is basically theorem 4.8 of [30], which proves this fact for Morse functions. We only add that it holds for Morse triples also.

Proof. It is sufficient to consider the case where f has exactly two critical points w and w' with $0 < f(w) = c < \frac{1}{2} < f(w') = c' < 1$. The proof of the more general case is exactly the same. Now suppose that w has index $p+1$, w' has index $p'+1$ and $p \geq p'$. Denote by K_w, the union of trajectories $K_-^{p+1}(w)$ and $K_+^{q+1}(w)$ associated with w. As always, $p+q+1 = n$. Similarly $K_{w'}$ will denote the union of trajectories $K_-^{p'+1}(w')$ and $K_+^{q'+1}(w')$ associated with w' where $p'+q'+1 = n$.

We begin with the simpler case when K_w and $K_{w'}$ do not intersect; see Fig. II.31. For any $0 < a' < a < 1$, Theorem 4.1 of [30] provides a construction for a well-indexed function $\bar{f}$ with critical points w and w' but with $f(w') = a'$ and $f(w) = a$. The construction can be applied continuously and so replacing $0 < a' < a < 1$ with a pair of continuous functions $0 < a'_t < a_t < 1$, with $a'_0 = c$, $a_0 = c'$, $a'_1 = a'$, $a_1 = a$ and $t \in I$ results in the desired isotopy.

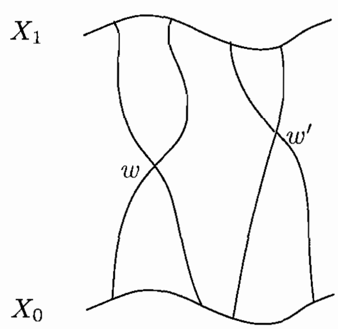

Figure II.31: Non-intersecting trajectories K_w and $K_{w'}$

In general, the trajectory spheres of two distinct Morse critical points may well intersect; see Fig. II.32. However, provided certain dimension restrictions are satisfied, it is possible to continuously move one trajectory sphere out of the way of the other trajectory sphere. This is theorem 4.4 of [30]. We will not reprove it here, except to say that the main technical tool required in the proof is lemma 4.6 of [30], which we state below.

Lemma II.28. *[30] Suppose M and N are two submanifolds of dimension m and n in a manifold V of dimension v. If M has a product neighbourhood in V, and $m+n < v$, then there exists a diffeomorphism h of V onto itself which is smoothly isotopic to the identity, such that $h(M)$ is disjoint from N.*

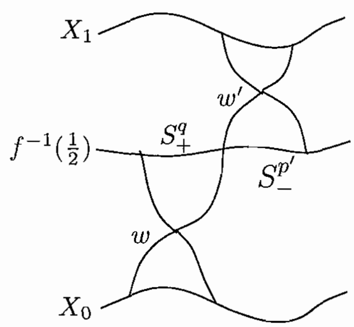

Figure II.32: Intersecting trajectories

The following observation then makes theorem 4.4 of [30] possible. Let S_+^q and $S_-^{p'}$ denote the respective intersections of $f^{-1}(\frac{1}{2})$ with $K_+^{q+1}(w)$ and $K_-^{p'+1}(w')$. Adding up dimensions, we see that

$$\begin{aligned} q + p' &= n - p - 1 + p' \\ &\leq n - p' + 1 + p' \\ &\leq n - 1. \end{aligned}$$

We can now isotopy f to have disjoint K_w and $K_{w'}$, before proceeding as before. □

Corollary II.29. *Any Gromov-Lawson cobordism $\bar{g}(g_0, f)$ can be isotopied to a Gromov-Lawson cobordism $\bar{g}(g_0, \bar{f})$ which is obtained from a well-indexed admissible Morse function $\bar{f}$.*

Proof. This follows immediately from Theorem II.27 above and Corollary II.26. □

II.5 Constructing Gromov-Lawson Concordances

Replacing X_0 with X and the metric g_0 with g, we now turn our attention to the case when the cobordism $\{W; X_0, X_1\}$ is the cylinder $X \times I$. By equipping $X \times I$ with an admissible Morse function f, we can use Theorem II.23 to extend the psc-metric g over $X \times I$ as a Gromov-Lawson cobordism $\bar{g} = \bar{g}(g, f)$. The resulting metric is known as a *Gromov-Lawson concordance* or more specifically, a *Gromov-Lawson concordance of g with respect to f* and the metrics $g_0 = \bar{g}|_{X\times\{0\}}$ and $g_1 = \bar{g}|_{X\times\{1\}}$ are said to be *Gromov-Lawson concordant.*

II.5.1 Applying the Gromov-Lawson technique over a pair of cancelling surgeries

In this section, we will construct a Gromov-Lawson concordance on the cylinder $S^n \times I$. It is possible to decompose this cylinder into the union of two elementary cobordisms, one the trace of a p-surgery, the other the trace of a $(p+1)$-surgery. The second surgery therefore undoes the topological effects of the first surgery. Later in the section, we will show how such a decomposition of the cylinder can be realised by a Morse function with two "cancelling" critical points. Assuming that $n - p \geq 4$, the standard round metric ds_n^2 can be extended over the union of these cobordisms by the technique of Theorem II.23, resulting in a Gromov-Lawson concordance. To understand this concordance we need to analyse the geometric effects of applying the Gromov-Lawson construction over the two cancelling surgeries.

Example II.30. Let S^n represent the standard smooth n-sphere equipped with the round metric $g = ds_n^2$. We will perform two surgeries, the first a p-surgery on S^n and the second, a $p+1$-surgery on the resulting manifold. The second surgery will have the effect of undoing the topological change made by the first surgery and restoring the original topology of S^n. Later we will see that the union of the resulting traces will in fact form a cylinder $S^n \times I$.

In section II.2 we saw that S^n can be decomposed as a union of sphere-disk products. Assuming that $p + q + 1 = n$ we obtain,

$$
\begin{aligned}
S^n &= \partial D^{n+1}, \\
&= \partial(D^{p+1} \times D^{q+1}), \\
&= S^p \times D^{q+1} \cup_{S^p \times S^q} D^{p+1} \times S^q.
\end{aligned}
$$

Here we are are assuming that $q \geq 3$. Let $S^p \times \mathring{D}^{q+1} \hookrightarrow S^n$ be the embedding obtained by the inclusion

$$S^p \times \mathring{D}^{q+1} \hookrightarrow S^p \times D^{q+1} \cup_{S^p \times S^q} D^{p+1} \times S^q.$$

We will now perform a surgery on this embedded p-sphere. This involves first removing the embedded $S^p \times \mathring{D}^{q+1}$ to obtain $S^n \setminus (S^p \times \mathring{D}^{q+1}) = D_-^{p+1} \times S^q$, and then attaching $(D_+^{p+1} \times S^q)$

along the common boundary $S^p \times S^q$. The attaching map here is given by restriction of the orginal embedding to the boundary. The resulting manifold is of course the sphere product $S^{p+1} \times S^q$ where the disks D_-^{p+1} and D_+^{p+1} are hemispheres of the S^{p+1} factor.

By performing a surgery on an embedded p-sphere in S^n we have obtained a manifold which is diffeomorphic to $S^{p+1} \times S^q$. By applying the Gromov-Lawson construction to the metric g we obtain a positive scalar curvature metric g' on $S^{p+1} \times S^q$; see Fig. II.33. This metric is the original round metric on an $S^n \setminus (S^p \times D^{q+1})$ piece and is $g_{tor}^{p+1}(\epsilon) \times \delta^2 ds_q^2$ on a $D^{p+1} \times S^q$ piece for some small $\delta > 0$. There is also a piece diffeomorphic to $S^p \times S^q \times I$ where the metric smoothly transitions between the two forms, the construction of which took up much section II.3.

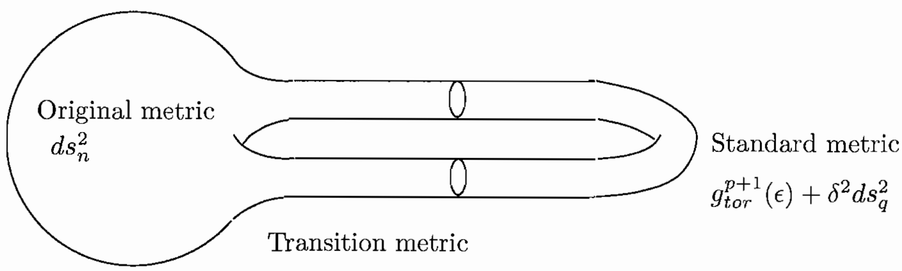

Figure II.33: The geometric effect of the first surgery

We will now perform a second surgery, this time on the manifold $(S^{p+1} \times S^q, g')$. We wish to obtain a manifold which is diffeomorphic to the orginal S^n, that is, we wish to undo the p-surgery we have just performed. Consider the following decomposition of $S^{p+1} \times S^q$.

$$\begin{aligned} S^{p+1} \times S^q &= S^{p+1} \times (D_-^q \cup D_+^q) \\ &= (S^{p+1} \times D_-^q) \cup (S^{p+1} \times D_+^q). \end{aligned}$$

Again, the inclusion map gives us an embedding of $S^{p+1} \times D_-^q$. Removing $S^{p+1} \times \mathring{D}_-^q$ and attaching $D^{p+2} \times S^{q-1}$ along the boundary gives

$$\begin{aligned} (D^{p+2} \times S^{q-1}) \cup (S^{p+1} \times D_+^q) &= \partial(D^{p+2} \times D^q) \\ &= \partial(D^{n+1}) \\ &= S^n. \end{aligned}$$

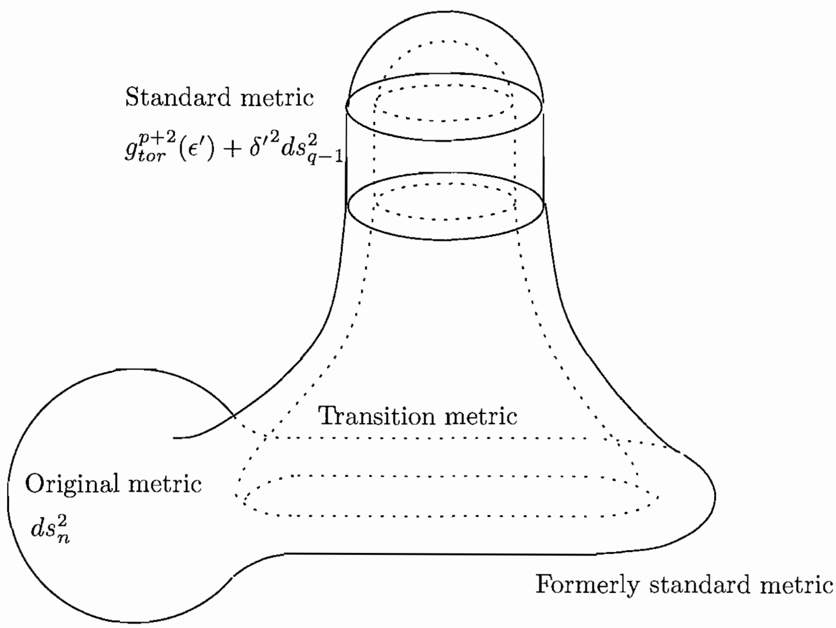

Figure II.34: The geometric effect of the second surgery: a different metric on S^n

Applying the Gromov-Lawson construction to the metric g' with respect to this second surgery produces a metric which looks very different to the orginal round metric on S^n; see Fig. II.34. Roughly speaking, g'' can be thought of as consisting of four pieces: the original piece where $g'' = g$ and which is diffeomorphic to a disk D^n, the new standard piece where $g'' = g_{tor}^{p+2}(\epsilon) + \delta'^2 ds_{q-1}^2$ and which is diffeomorphic to $D^{p+2} \times S^{q-1}$, the old standard piece where $g'' = g_{tor}^{p+1}(\epsilon') + \delta^2 ds_q^2$, this time only on a region diffeomorphic to $D^{p+1} \times D^q$ and finally, a transition metric which connects these pieces. Later on we will need to describe this metric in more detail.

II.5.2 Cancelling Morse critical points: The weak and strong cancellation theorems

We will now show that the cylinder $S^n \times I$ can be decomposed into a union of two elementary cobordisms which are the traces of the surgeries described above. This decomposition is obtained from a Morse function $f : S^n \times I \to I$ which satisfies certain properties. The following theorem, known as the *weak cancellation theorem* is proved in Chapter 5 of [30]. It is also discussed, in much greater generality, in Chapter 5, Section 1 of [16].

Theorem II.31. *[30] Let $\{W^{n+1}; X_0, X_1\}$ be a smooth compact cobordism and $f : W \to I$ be a Morse triple on W. Letting $p + q + 1 = n$, suppose that f satisfies the following conditions.*

(a) The function f has exactly 2 critical points w and z and $0 < f(w) < c < f(z) < 1$.

(b) The points w and z have Morse index $p + 1$ and $p + 2$ respectively.

(c) On $f^{-1}(c)$, the trajectory spheres $S^p_+(w)$ and $S^q_-(z)$, respectively emerging from the critical point w and converging toward the critical point z, intersect transversely as a single point.

Then the critical points w and z cancel and W is diffeomorphic to $X_0 \times I$.

The proof of II.31 in [30] is attributed to Morse. The fact that $S^p_+(w)$ and $S^q_-(z)$ intersect transversely as a point means that there is a single *trajectory arc* connecting w and z. It is possible to alter the vector field V on an arbitrarily small neighbourhood of this arc to obtain a nowhere zero gradient-like vector field V' which agrees with V outside of this neighbourhood. This in turn gives rise to a Morse function f' with gradient-like vector field V', which agrees with f outside this neighbourhood and has **no** critical points; see Fig. II.35. The desired decomposition of $S^n \times I$

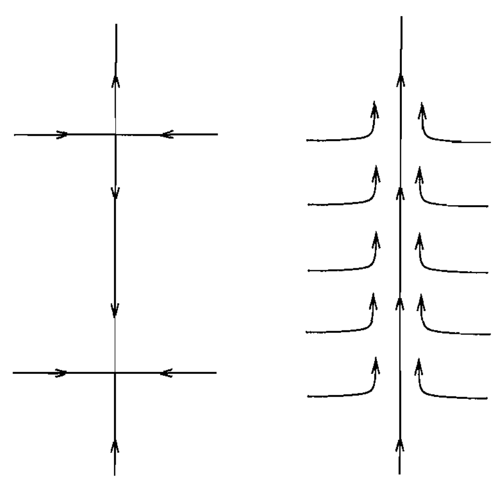

Figure II.35: Altering the gradient-like vector field along the trajectory arc

can now be realised by a Morse function $f : S^n \times I \to I$ which satisfies (a), (b) and (c) above as well as the condition that $n - p \geq 4$. Application of Theorem II.23 with respect to an admissible Morse function f which satisfies (a), (b) and (c) will result in a Gromov-Lawson concordance on

$S^n \times I$ between $g = ds_n^2$ and the metric g'' described above. Equivalently, one can think of this as obtained by two applications of Theorem II.10, one for each of the elementary cobordisms specified by f.

II.5.3 A strengthening of Theorem II.31

There is a strengthening of theorem II.31 in the case where W, X_0 and X_1 are simply connected and of dimension ≥ 5. Before stating it, we should recall what is meant by the intersection number of two manifolds. Let M and M' be two smooth submanifolds of dimensions r and s in a smooth manifold N of dimension $r+s$ and suppose that M and M' intersect transversely as the set of points $\{n_1, n_2, \ldots, n_l\}$ in N. Suppose also that M is oriented and that the normal bundle $\mathcal{N}(M')$ of M' in N is oriented. At n_i, choose a positively oriented r-frame $v_1, \ldots, v_r$ of linearly independent vectors which span the tangent space $T_{n_i}M$. Since the intersection at n_i is transverse, this frame is a basis for the normal fibre $\mathcal{N}_{n_i}(M')$.

Definition II.8. The *intersection number of M and M' at n_i* is defined to be $+1$ or -1 according as the vectors $v_1, \ldots, v_r$ represent a positively or negatively oriented basis for the fibre $\mathcal{N}_{n_i}(M')$. The *intersection number $M' \cdot M$ of M and M'* is the sum of intersection numbers over all n_i.

Remark II.5.1. *In the expression $M'.M$, we adopt the convention that the manifold with oriented normal bundle is written first.*

We now state the *strong cancellation theorem.* This is theorem 6.4 of [30].

Theorem II.32. *[30] Let $\{W; X_0, X_1\}$ be a smooth compact cobordism where W, X_0 and X_1 are simply connected manifolds and W has dimension $n+1 \geq 6$. Let $f: W \to I$ be a Morse triple on W. Letting $p+q+1=n$, suppose that f satisfies the following conditions.*

(a′) The function f has exactly 2 critical points w and z and $0 < f(w) < c < f(z) < 1$.

(b′) The points w and z have Morse index $p+1$ and $p+2$ respectively and $1 \leq p \leq n-4$.

(c′) On $f^{-1}(c)$, the trajectory spheres $S_+^p(w)$ and $S_-^q(z)$ have intersection number $S_+^p(w).S_-^q(z) = 1$ or -1.

Then the critical points w and z cancel and W is diffeomorphic to $X_0 \times I$. In fact, f can be altered near $f^{-1}(c)$ so that the trajectory spheres intersect transversely at a single point and the conclusions of theorem II.31 then apply.

Simple connectivity plays an important role in the proof. It of course guarantees the orientability conditions we need but more importantly it is used to simplify the intersection of the trajectory spheres. Roughly speaking, if n_1 and n_2 are two intersection points with opposite intersection, there are arcs connecting these points in each of the trajectory spheres, whose union forms a loop contractible in $f^{-1}(c)$ which misses all other intersection points. An isotopy can be constructed (which involves contracting this loop) to move the trajectory spheres to a position where the intersection set contains no new elements but now excludes n_1 and n_2.

Remark II.5.2. *The hypothesis that critical points of f have index at least 2 is necessary, as the presence of index* 1 *critical points would spoil the assumption of simple connectivity.*

II.5.4 Standardising the embedding of the second surgery sphere

In Example II.30, the second surgery sphere S^{p+1} was regarded as the union of two hemispheres D_-^{p+1} and D_+^{p+1}, the latter hemisphere coming from the handle attachment. It was assumed in the construction of the metric g'', that the disk D_+^{p+1} was embedded so that the metric induced by g' was precisely the $g_{tor}^{p+1}(\epsilon)$ factor of the handle metric. Now let f be an admissible Morse function on $X \times I$ which satisfies conditions (a), (b) and (c) above. This specifies two trajectory spheres S_-^p and S_-^{p+1} corresponding to the critical points w and z respectively. On the level set $f^{-1}(c)$, the spheres S_+^q and S_-^{p+1} intersect transversely at a single point α. Suppose we extend a psc-metric g on X over $f^{-1}([0,c])$ in the manner of Theorem II.23, denoting by g' the induced metric on $f^{-1}(c)$. In general, the metric induced by g' on S_-^{p+1} near α will not be $g_{tor}^{p+1}(\epsilon)$. We will now show that such a metric can be obtained with a minor adjustment of the Morse function f.

Let $\mathbb{R}^{n+1} = \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ denote the Morse coordinate neighbourhood near w. Here $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ denote the respective inward and outward trajectories at w. Let $\mathbb{R}$ denote the 1-dimensional subspace of $\mathbb{R}^{q+1}$ spanned by the vector based at zero and ending at α. Finally, let D^{p+1} denote the intersection of $f^{-1}(c)$ with the plane $\mathbb{R} \times \mathbb{R}^{p+1}$; see Fig. II.36. The metric induced by g' on D^{p+1} is precisely the $g_{tor}^{p+1}(\epsilon)$ factor of the handle metric.

Lemma II.33. *It is possible to isotopy the trajectory sphere $S_-^{p+1}(z)$, so that on $f^{-1}(c)$ it agrees, near α, with the disk D^{p+1}.*

Proof. Choose a coordinate chart $\mathbb{R}^n$ in $f^{-1}(c)$ around α, where α is identified with 0 and $\mathbb{R}^n$

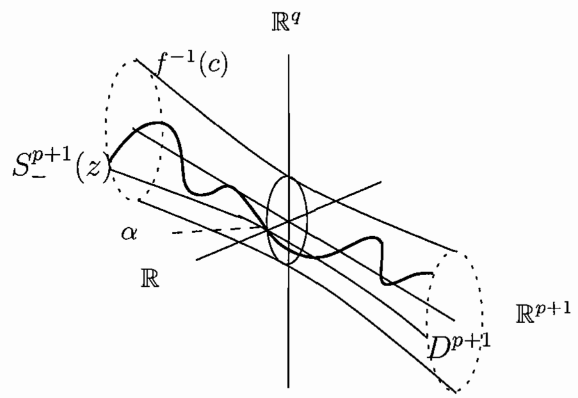

Figure II.36: Isotopying S_-^{p+1} near α to coincide with D^{p+1}.

decomposes as $\mathbb{R}^{p+1} \times \mathbb{R}^q$. The intersection of $S_-^{p+1}(z)$ with this chart is a $(p+1)$-dimensional disk in $\mathbb{R}^n$ which intersects with $\mathbb{R}^q$ transversely at the orgin. Thus, near the orgin, $S_-^{p+1}(z)$ is the graph of a function over $\mathbb{R}^{p+1}$ and so we can isotopy it to an embedding which is the plane $\mathbb{R}^{p+1}$ on some neighbourhood of 0, and the original $S_-^{p+1}(z)$ away from this neighbourhood. □

Thus, the Morse function f can be isotopied to a Morse function where the standard part of the metric g' induces the $g_{tor}^{p+1}(\epsilon)$ factor of the handle metric on $S_-^{p+1}(\alpha)$; see Fig. II.37.

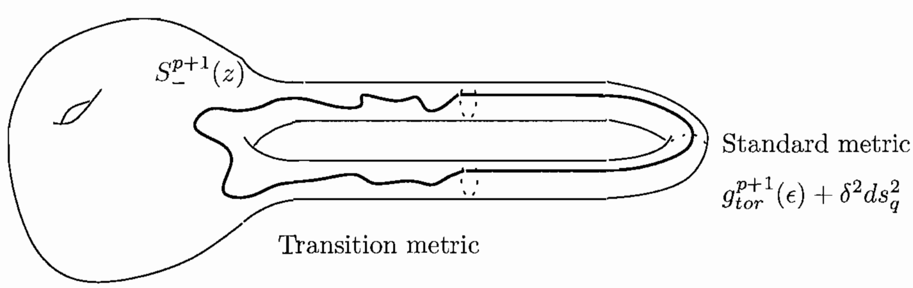

Figure II.37: The embedded sphere $S_-^{p+1}(z)$ after adjustment

II.6 Gromov-Lawson Concordance Implies Isotopy for Cancelling Surgeries

We continue to employ the notation of the previous section in stating the following theorem.

Theorem II.34. *Let $f : X \times I \to I$ be an admissible Morse function which satisfies conditions (a),(b) and (c) of Theorem II.31 above. Let g be a metric of positive scalar curvature on X and $\bar{g} = \bar{g}(g, f)$, a Gromov-Lawson concordance with respect to f and g on $X \times I$. Then the metric $g'' = \bar{g}|_{X \times \{1\}}$ on X is isotopic to the original metric g.*

We will postpone the proof of Theorem II.34 for now. Later we will show that this theorem contains the main geometric argument needed to prove that any metrics which are Gromov-Lawson concordant are actually isotopic. Before doing this, we need to introduce some more terminology.

II.6.1 Connected sums of psc-metrics

Suppose (X, g_X) and (Y, g_Y) are Riemannian manifolds of positive scalar curvature with $\dim X = \dim Y \geq 3$. A *psc-metric connected sum* of g_X and g_Y is a positive scalar curvature metric on the connected sum $X \# Y$, obtained using the Gromov-Lawson technique for connected sums on g_X and g_Y. Recall that on X, the Gromov-Lawson technique involves modifying the metric on a disk $D = D^n$ around some point $w \in X$, by pushing out geodesic spheres around w to form a tube. It is possible to construct this tube so that the metric on it has positive scalar curvature and so that it ends as a Riemannian cylinder $S^{n-1} \times I$. Furthermore the metric induced on the S^{n-1} factor can be chosen to be arbitrarily close to the standard round metric and so we can isotopy this metric to the round one. By Lemma II.1, we obtain a metric on $X \setminus D^n$ which has positive scalar curvature and which near the boundary is the standard product $\delta^2 ds_{n-1}^2 + dt^2$ for some (possibly very small) δ. Repeating this procedure on Y allows us to form a psc-metric connected sum of (X, g_X) and (Y, g_Y) which we denote

$$(X, g_X) \# (Y, g_Y).$$

II.6.2 An analysis of the metric g'' obtained from the second surgery

Recall that f specifies a pair of cancelling surgeries. The first surgery is on an embedded sphere S^p and we denote the resulting surgery manifold by X'. Applying the Gromov-Lawson construction results in a metric g' on X', which is the orginal metric g away from S^p and transitions on a region diffeomorphic to $S^p \times S^q \times I$ to a metric which is the standard product $g_{tor}^{p+1}(\epsilon) \times \delta^2 ds_q^2$ on the handle $D^{p+1} \times S^q$. The second surgery sphere, embedded in X', is denoted S^{p+1}. In section II.5.4, we showed that it is reasonable to assume that on the standard region, the restriction of g' to the sphere S^{p+1} is precisely the $g_{tor}^{p+1}(\epsilon)$ factor of the standard metric $g_{tor}^{p+1}(\epsilon) + \delta^2 ds_q^2$; see Fig. II.37. Applying the Gromov-Lawson construction to g' with respect to this second surgery, results in a metric g'' on X (see Fig. II.38) which is concordant to g.

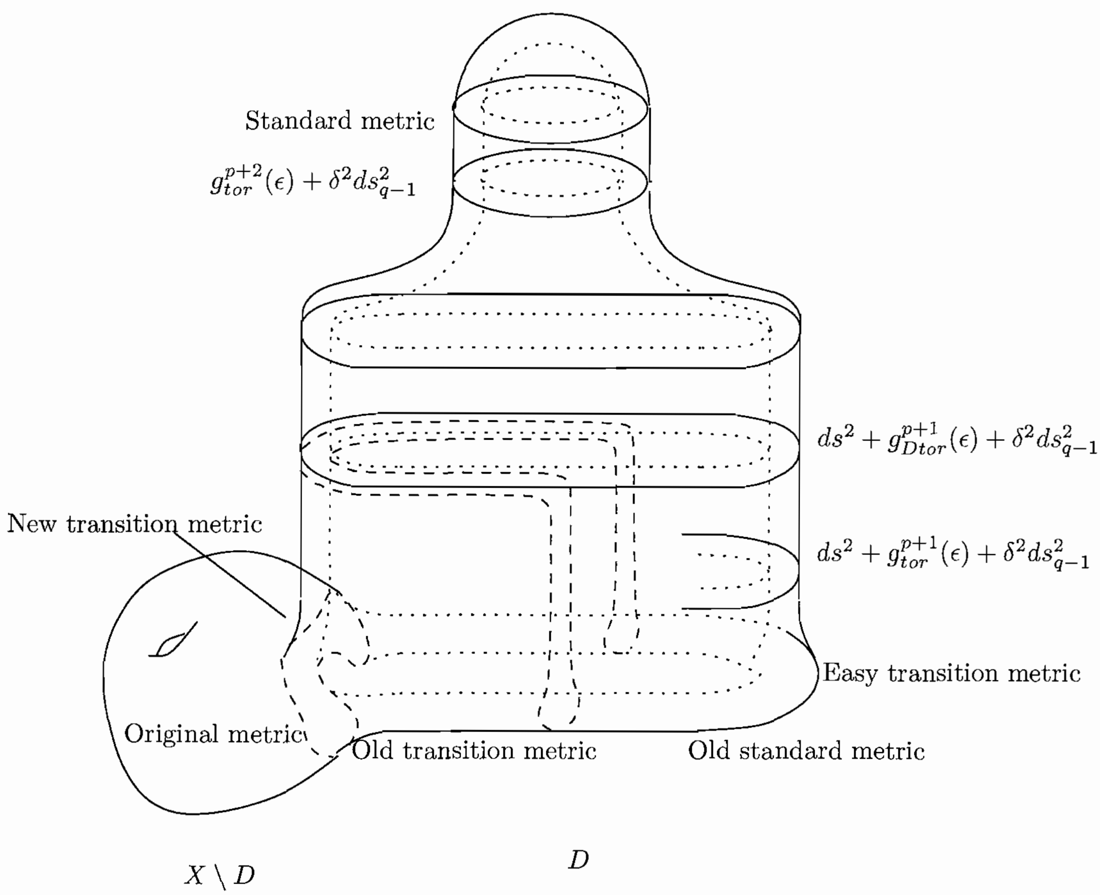

Figure II.38: The metric g''

In Fig. II.38, we describe a metric which is obtained from g'' by a only a very minor adjustment. We will discuss the actual adjustment a little later but, as it can be obtained through an isotopy, to ease the burden of notation we will still denote the metric by g''. This metric agrees with g outside of an n-dimensional disk D; see Fig. II.38. The restriction of g'' to this disk can be thought of as consisting of several regions. Near the boundary of the disk, and represented schematically by two dashed curves, is a cylindrical region which is diffeomorphic to $S^{n-1} \times I$. This cylindrical region will be known as the *connecting cylinder*; see Fig. II.39. We will identify the sphere which bounds $X \setminus D$ with $S^{n-1} \times \{1\}$. This sphere is contained in a region where $g'' = g$ and so we know very little about the induced metric on this sphere.

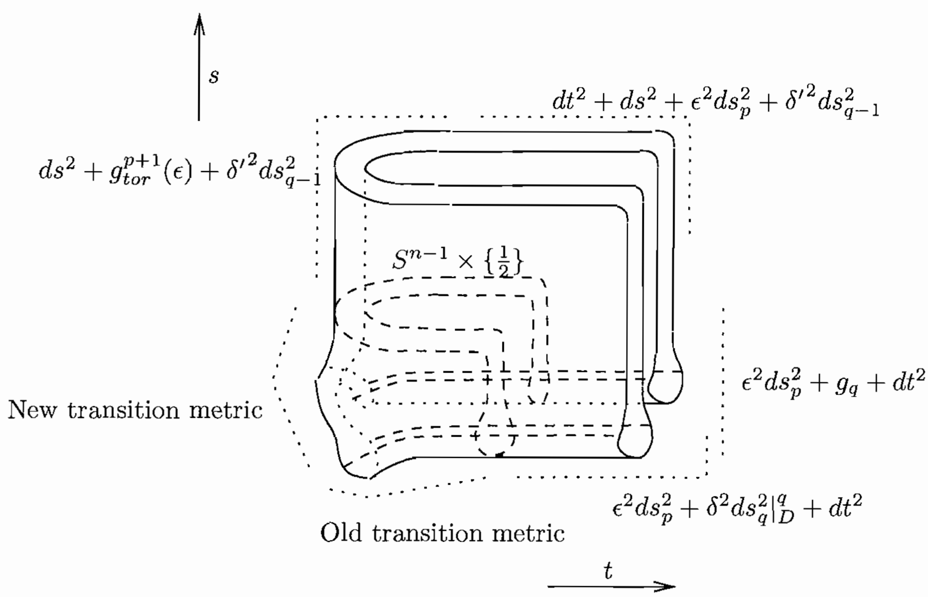

Figure II.39: The connecting cylinder $S^{n-1} \times I$

The region $S^{n-1} \times [\frac{1}{2}, 1]$ is where most of the transitioning happens from the old metric g to the standard form. This transition metric consists in part of the *old transition metric* from the first surgery and the *new transition metric* from the second surgery. The old transition metric is on a region which is diffeomorphic to $S^p \times D^q \times [\frac{1}{2}, 1]$ (schematically this is the region below the horizontal dashed lines near the bottom of Fig. II.39) while the new transition metric is on a region which is diffeomorphic to $D^{p+1} \times S^{q-1} \times [\frac{1}{2}, 1]$. On the second cylindrical piece $S^{n-1} \times [0, \frac{1}{2}]$, the metric g'' is much closer to being standard.

Turning our attention away from the connecting cylinder for a moment, it is clear that the metric g'' agrees with the standard part of the metric g' on a region which is diffeomorphic to $D^{p+1} \times D^q$; see Fig. II.38. Here g'' is the metric $g_{tor}^{p+1}(\epsilon) + \delta^2 ds_q^2|_{D^q}$ and we call this piece the *old standard metric*. The old standard metric transitions through an *easy transition metric* on a region diffeomorphic to $I \times D^{p+1} \times S^q$ to take the form $ds^2 + g_{tor}^{p+1}(\epsilon') + \delta'^2 ds_{q-1}^2$. This particular transition is known as the *easy transition metric* as it is far simpler than the previous transitions.

Returning now to the second cylindrical piece of the connecting cylinder, we see that there is a neighbourhood of $S^{n-1} \times [0, \frac{1}{2}]$, containing both the old standard and easy transition regions where the metric g'' takes the form of a product $\epsilon^2 ds_p^2 + dt^2 + g_q$, where the metric g_q is a metric on the disk D^q; see Fig. II.40. Shortly, we will write represent g_q more explicitly.

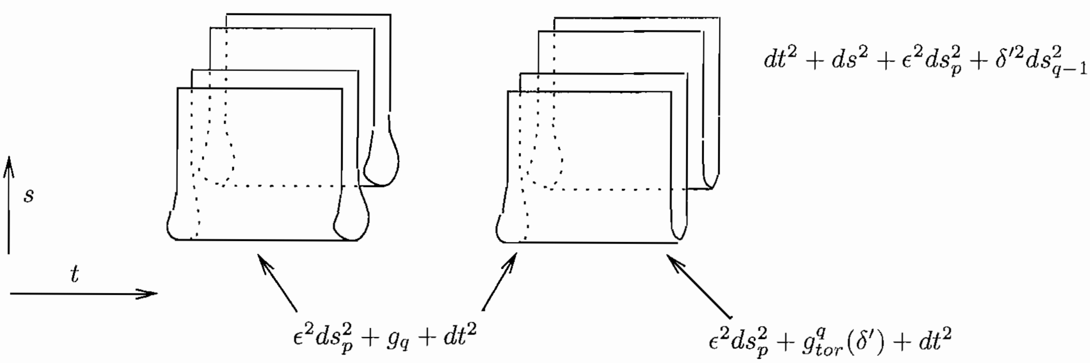

Figure II.40: A neighbourhood of $S^{n-1} \times [0, \frac{1}{2}]$ on which g'' has a product structure (left) and the metric resulting from an isotopy on this neighbourhood (right)

Returning once more to the metric g'' on D, we observe that outside of $S^{n-1} \times I$ and away from the old standard and easy transition regions, the metric is almost completely standard. The only difference between this metric and the metric g'' constructed in Example II.30, is the fact that the metric on the second surgery sphere S^{p+1} is first isotopied to the double torpedo metric $g_{Dtor}^{p+1}(\epsilon)$, before finally transitioning to the round metric $\epsilon^2 ds_{p+1}^2$. This gives a concordance between the metric $g_{Dtor}^{p+1} + \delta'^2 ds_{q-1}^2$ and $\epsilon^2 ds_{p+1}^2 + \delta'^2 ds_{q-1}^2$ which is capped off on the remaining $D^{p+2} \times S^{q-1}$ by the *new standard metric* $g_{tor}^{p+2}(\epsilon) + \delta'^2 ds_{q-1}^2$. This completes our initial analysis of g''.

II.6.3 The proof of Theorem II.34

We now proceed with the proof of Theorem II.34.

Proof. We will perform a sequence of adjustments on each of the metrics g and g''. Beginning with the metric g'', we will construct g_1'' and g_2'' each of which is isotopic to the previous one. Similarly, we will make sequence adjustments to the metric g, resulting in isotopic metrics g_1, g_2 and g_3. The metrics g_3 and g_2'' will then be demonstrably isotopic.

The initial adjustment of g''.

We will begin by making some minor adjustments to the metric g'' to obtain the metric g_1''. Recall that on the part of the connecting cylinder identified with $S^{n-1} \times [0, \frac{1}{2}]$, the metric g'' is somewhat standard. We observed that on a particular region of $S^{n-1} \times [0, \frac{1}{2}]$, g'' takes the form $\epsilon^2 ds_p^2 + g_q + dt^2$. Here g_q can be written more explicitly as

$$g_q = dr^2 + F(r)^2 ds_{q-1}^2,$$

where r is the radial distance cordinate and F is a function of the type shown in Fig. II.41. A

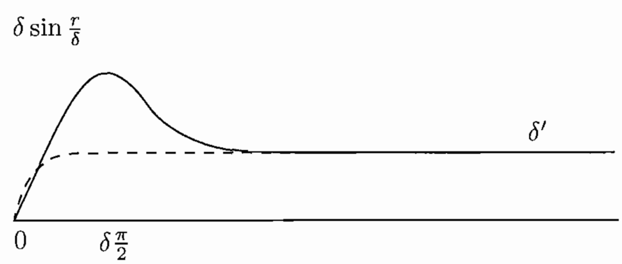

Figure II.41: The function F, with $f_{\delta'}$ shown as the dashed curve

linear homotopy of F to the torpedo function $f_{\delta'}$ induces an isotopy from the metric g_q to the metric $g_{tor}^q(\delta')$. With an appropriate rescaling, it is possible to isotopy the metric $\epsilon^2 ds_p^2 + g_q + dt^2$ to one which is unchanged near $S^{n-1} \times \{\frac{1}{2}\}$ but near $S^{n-1} \times \{0\}$, is the standard product $\epsilon^2 ds_p^2 + g_{tor}^q(\delta') + dt^2$. This isotopy then easily extends to an isotopy of g'' resulting in a metric which, on the old standard and easy transition regions, is now $g_{tor}^{p+1}(\epsilon) + g_{tor}^q(\delta')$ away from $S^{n-1} \times \{\frac{1}{2}\}$; see Fig. II.42.

The embedding $\bar{J}$.

For sufficiently small λ, the cylindrical portion $S^{n-1} \times [0, \lambda]$ of the connecting cylinder $S^{n-1} \times I$ is contained entirely in a region where $g'' = g_{tor}^{p+1}(\epsilon) + g_{tor}^q(\delta')$; see Fig. II.43. Recall that in section II.2.5, we equipped the plane $\mathbb{R}^n = \mathbb{R}^{p+1} \times \mathbb{R}^q$ with this metric, then denoted by $h = g_{tor}^{p+1}(\epsilon) + g_{tor}^q(\delta')$. In standard spherical coordinates $(\rho, \phi), (r, \theta)$ for $\mathbb{R}^{p+1}$ and $\mathbb{R}^q$ respectively,

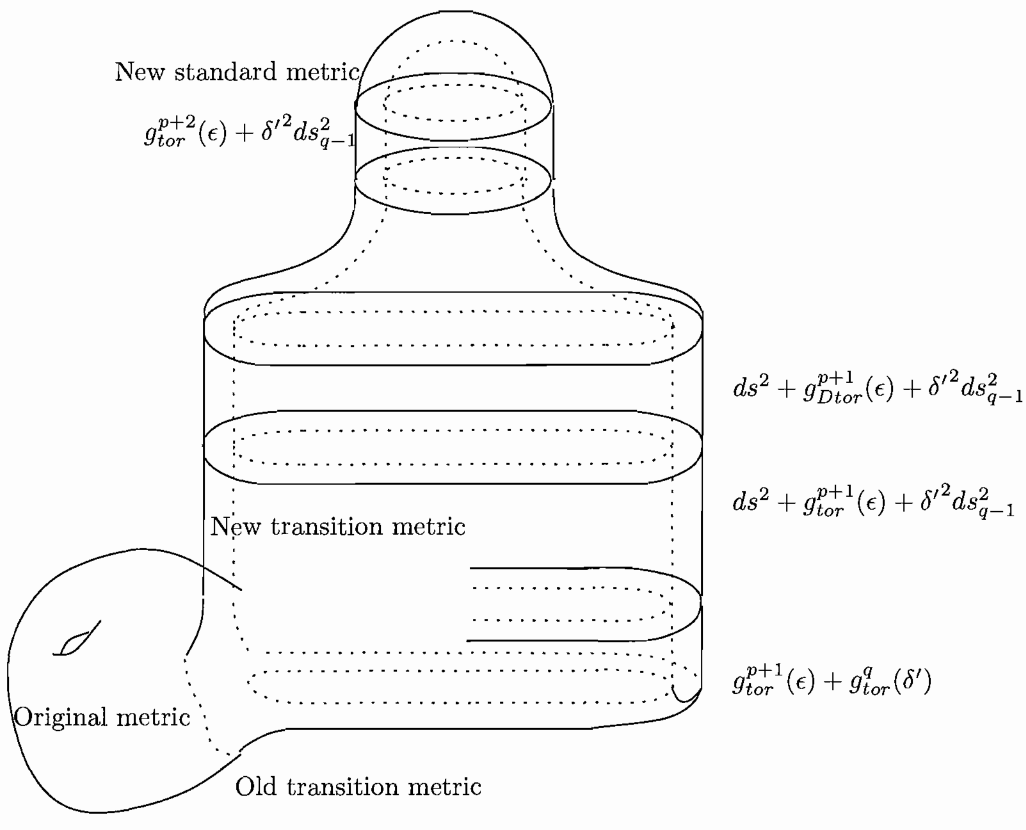

Figure II.42: The metric g_1'' resulting from the initial adjustment we can represent this metric with the explicit formula

$$h = d\rho^2 + f_\epsilon(\rho)^2 ds_p^2 + dr^2 + f_{\delta'}(r)^2 ds_q^2, \tag{II.6.1}$$

where $f_\epsilon, f_{\delta'}$ are the standard ϵ and δ-torpedo functions defined on $(0, \infty)$. The restriction of g'' to the region $S^{n-1} \times [0, \lambda]$ is now isometric to an annular region of $(\mathbb{R}^n, h)$ shown in Fig. II.44. For a more geometrically accurate schematic of $(\mathbb{R}^n, h)$; see Fig. II.6 in section II.2.

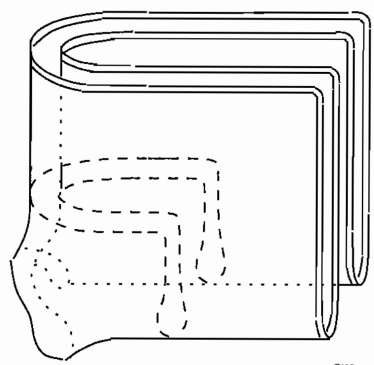

Figure II.43: The collar neighbourhood $S^{n-1} \times [0, \lambda]$

There is an isometric embedding $\bar{J}$ of the cylindrical portion $S^{n-1} \times [0, \lambda]$ of the connecting cylinder $S^{n-1} \times I$ into $(\mathbb{R}^n, h)$. Let $\bar{a}$ denote an embedding

$$\bar{a} : [0, \lambda] \times [0, b] \to \mathbb{R} \times \mathbb{R}$$
$$(t_1, t_2) \mapsto (a_1(t_1, t_2), a_2(t_1, t_2))$$

which satisfies the following properties.

(1) For each $t_1 \in [0, \lambda]$, the restriction of $\bar{a}$ to $\{t_1\} \times [0, b]$ is a smooth curve in the first quadrant of $\mathbb{R}^2$ which begins at $(c_1 + t_1, 0)$, follows a vertical trajectory, bends by an angle of $\frac{\pi}{2}$ towards the vertical axis in the form of a circular arc and continues as a horizontal line to end at $(0, c_1 + t_1)$. We will assume that $c_1 > \max\{\frac{\pi}{2}\epsilon, \frac{\pi}{2}\delta'\}$ and that the bending takes place above the horizontal line through line $(0, \delta\frac{\pi}{2})$ as in Fig. II.7.

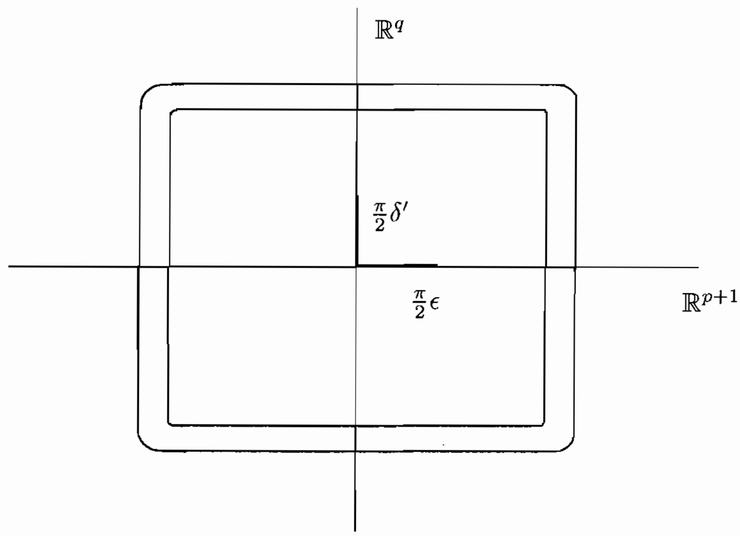

Figure II.44: The image of $\bar{J}$

(2) At each point (t_1, t_2), the products $\frac{\partial a_1}{\partial t_1}.\frac{\partial a_1}{\partial t_2}$ and $\frac{\partial a_1}{\partial t_1}.\frac{\partial a_1}{\partial t_2}$ are both zero.

For some such $\bar{a}$, there is a map $\bar{J}$ defined

$$\bar{J} : [0, \lambda] \times [0, b] \times S^p \times S^{q-1} \longrightarrow \mathbb{R}^{p+1} \times \mathbb{R}^q$$
$$(t_1, t_2, \phi, \theta) \longmapsto (a_1(t_1, t_2), \phi, a_2(t_1, t_2), \theta)$$

which isometrically embeds the cylindrical piece $(S^{n-1} \times [0, \lambda], g''|_{S^{n-1} \times [0,\lambda]})$ into $(\mathbb{R}^n, h)$; see Fig. II.44. Furthermore, assumption (2) above means that the metric $g''|_{S^{n-1} \times [0,\lambda]}$ can be foliated as $dt_1^2 + g''_{t_1}$ where g''_{t_1} is the metric induced on the restriction of $\bar{J}$ to $\{t_1\} \times [0, \lambda] \times S^p \times S^q$. For each $t_1 \in [0, \lambda]$, the metric g''_{t_1} is a mixed torpedo metric $g^{p,q-1}_{Mtor}$. These metrics are of course not isometric, but differ only in that the tube lengths of the various torpedo parts vary.

Isotopying the metric on $S^{n-1} \times [0, \lambda]$ to the "connected sum" metric g''_2.

Given two copies of the plane $\mathbb{R}^n$, each equipped with the metric h, we can apply the Gromov-Lawson technique to construct a connected sum $(\mathbb{R}^n, h)\#(\mathbb{R}^n, h)$. This technique determines a psc-metric by removing a disk around each origin and gluing the resulting manifolds together with an appropriate psc-metric on the cylinder $S^{n-1} \times I$. In this section, we will isotopy the metric $g''|_{S^{n-1}\times[0,\lambda]}$ to obtain precisely this cylinder metric; see Fig. II.45. Importantly, this isotopy will fix the metric near the ends of the cylinder and so will extend easily to an isotopy of g_1'' on all of X to result in the metric g_2''.

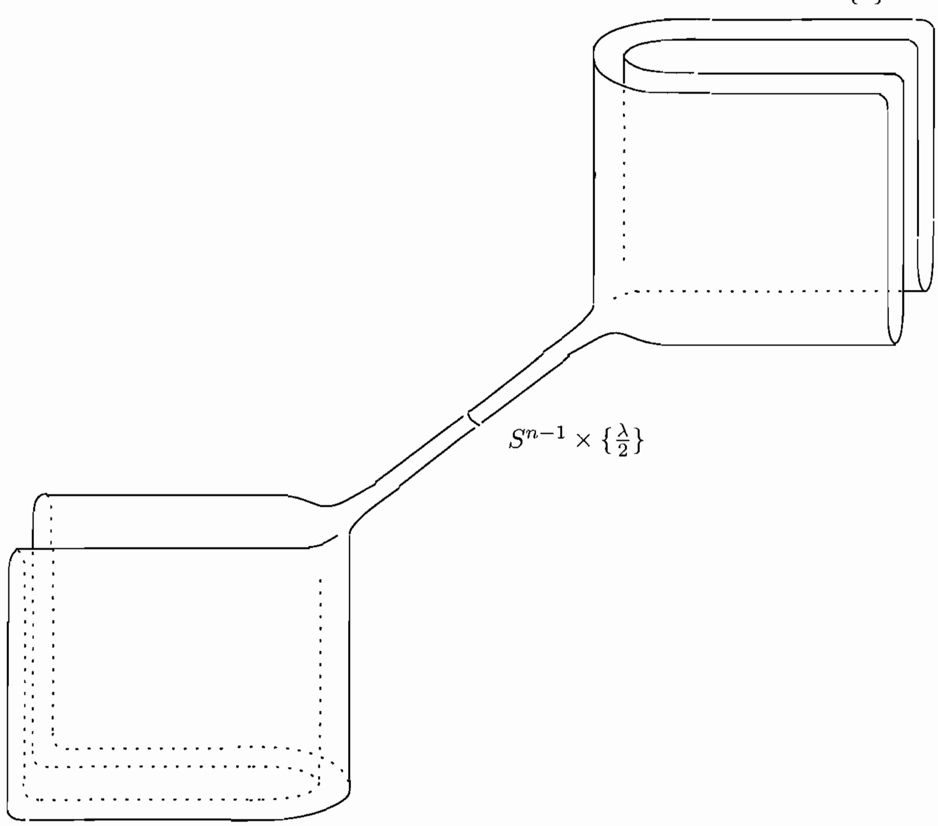

Figure II.45: The metric obtained by isotopying $g''|_{S^{n-1}\times[0,\lambda]}$ to the cylinder metric of Gromov-Lawson "connected sum" construction

Let $\bar{a}^{t_1}$ denote the curve which is the image of the map $\bar{a}$ restricted to $\{t_1\} \times [0, b]$ and $\bar{J}^{t_1}$ will denote the embedded sphere in $\mathbb{R}^n$ which is the image of the map $\bar{J}$ on $\{t_1\} \times [0, b] \times S^p \times S^{q-1}$. We define the map K^τ as

$$\begin{aligned} K^\tau : [0, \tfrac{\pi}{2}\tau] \times S^p \times S^{q-1} &\longrightarrow \mathbb{R}^{p+1} \times \mathbb{R}^q \\ (t, \phi, \theta) &\longmapsto (\tau \cos(\tfrac{t}{\tau}), \phi, \tau \sin(\tfrac{t}{\tau}), \theta) \end{aligned}$$

For each $\tau > 0$, the image of K^τ in $(\mathbb{R}^n, h)$ is a geodesic sphere of radius τ. Now consider the region, shown in Fig. II.47, bounded by the embedded spheres $\bar{J}^{\frac{\lambda}{2}}$ and K^{τ} where τ is assumed to be very small. Let c^{τ} denote the circular arc given by $c^{\tau}(t) = (\tau\cos(\frac{t}{\tau}), \tau\sin(\frac{t}{\tau}))$, for $t \in [0, \frac{\pi}{2}\tau]$. It is easy to construct a smooth homotopy between $\bar{a}^{\frac{\lambda}{2}}$ and c^{τ} through curves $(x_{\nu}, y_{\nu}), \nu \in I$ where $c^{\tau} = (x_0, y_0)$ and $\bar{a}^{\frac{\lambda}{2}} = (x_1, y_1)$. For example, this can be done by smoothly shrinking the straight edge pieces of $\bar{a}^{\frac{\lambda}{2}}$ to obtain a piece which is within arbitrarily small smoothing adjustments of being a circular arc, the radius of which can then be smoothly shrunk as required; see Fig. II.46.

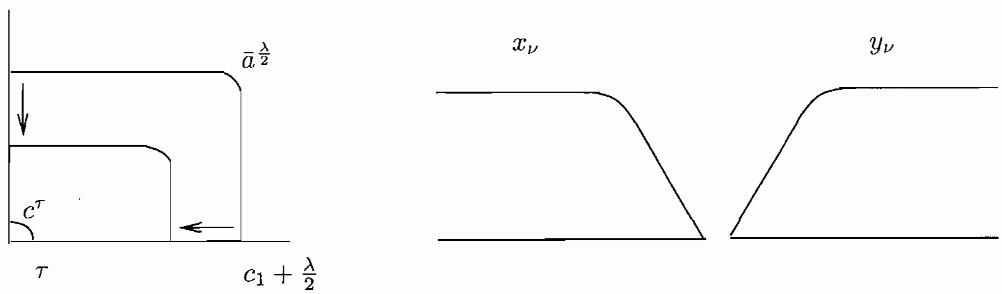

Figure II.46: Homotopying the curve $\bar{a}^{\frac{\lambda}{2}}$ to c^{τ}

By smoothly varying the length of domain intervals of x_{ν} and y_{ν} with respect to ν, we can ensure that the curve (x_{ν}, y_{ν}) is unit speed for all ν. The above homotopy gives rise to a foliation of the region contained between $\bar{J}^{\frac{\lambda}{2}}$ and K^{τ}; see Fig. II.47, and a corresponding foliation of the metric h on this region. Letting $l \in [0, 1]$ denote the coordinate running orthogonal to the curves given by the above homotopy, we can write the metric $h = dl^2 + h_l$. Moreover, the metric h_l can be computed explicitly as

$$h_l = dt^2 + f_{\epsilon}(x(t))^2 ds_p^2 + f_{\delta'}(y(t))^2 ds_q^2$$

where $x = x_{\nu}$ and $y = y_{\nu}$ for some ν. An elementary calculation shows that $-1 \leq \dot{x_{\nu}} \leq 0$, $0 \leq \dot{y_{\nu}} \leq 1$, $\ddot{x_{\nu}} \leq 0$ and $\ddot{y_{\nu}} \leq 0$. A further elementary calculation now shows that the functions $f_{\epsilon}(x(t))$ and $f_{\delta'}(y(t))$ belong to the spaces $\mathcal{U}$ and $\mathcal{V}$ defined in section II.2. Thus, by Lemma II.7, the metric h_l has positive scalar curvature and so the decomposition of h into $dl^2 + h_l$ induces an isotopy between the metric $h_1 = g''_{\frac{\lambda}{2}}$ and the metric h_0 induced by h on the geodesic sphere of radius τ.

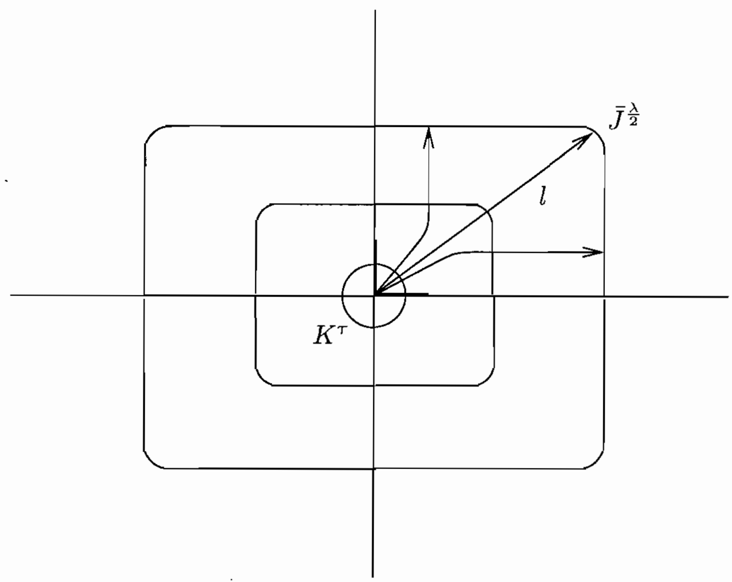

Figure II.47: The region bounded by $\bar{J}^{\frac{\lambda}{2}}$ and K^{τ}

Recall that the restriction of the metric g'' to $S^{n-1} \times [0, \frac{\lambda}{2}]$, isometrically embeds into $(\mathbb{R}^n, h)$ as the region between the curves $\bar{J}^0$ and $\bar{J}^{\frac{\lambda}{2}}$. Using the foliation $h = dl^2 + h_l$, this metric can now be continuously extended as the metric h over the rest of the region between $\bar{J}^0$ and K^{τ}; see Fig. II.48. As the curve K^{τ} is a geodesic sphere with respect to h, this metric can then be continuously extended as the metric obtained by the Gromov-Lawson construction, to finish as a round cylinder metric. The metric $g''|_{S^{n-1}\times[0,\frac{\lambda}{2}]}$ has now been isotopied to one half of the metric depicted in Fig. II.45 without making any adjustment near $S^{n-1} \times \{0\}$.

An analogous construction can be performed on $g''|_{S^{n-1}\times[\frac{\lambda}{2},\lambda]}$, this time making no alteration to the metric near $S^{n-1} \times \{\lambda\}$. Both constructions can be combined to form the desired isotopy by making a minor modifcation to ensure that at each stage, the metric near $S^{n-1} \times \{\frac{\lambda}{2}\}$ is a psc-Riemannian cylinder. Such a modification is possible because of the fact that the above foliation decomposes h into an isotopy of psc-metrics.

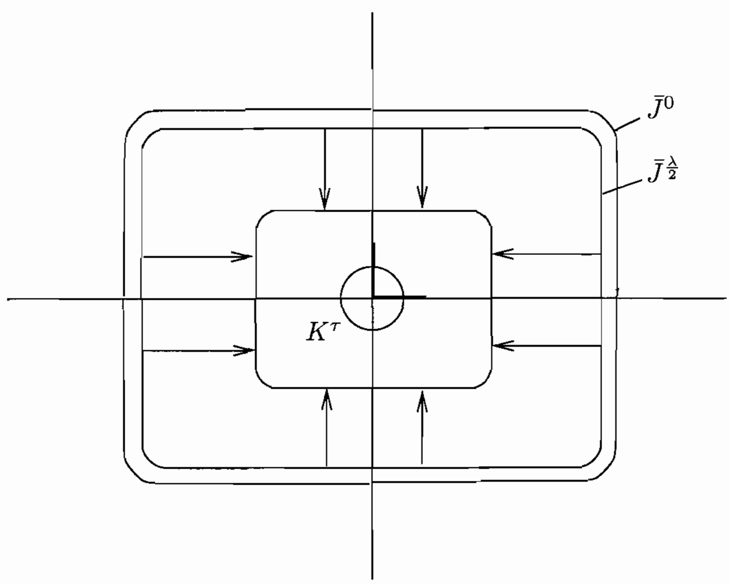

Figure II.48: Isotopying the metric $g''|_{S^{n-1}\times[0,\frac{\lambda}{2}]}$ to the metric h on the region bounded by $\bar{J}^0$ and K^τ

Isotopying the metric g.

In this step we will perform three successive adjustments on the metric g, resulting in successive positive scalar curvature metrics g_1, g_2 and g_3. Each adjustment will result in a metric which is isotopic to the previous one and thus to g.

In adjusting the metric g, we wish to mimic, as closely as possible, the Gromov-Lawson technique applied in the construction of g''. The main difficulty is that we are prevented from making any topological change to the manifold X. Thus, the first adjustment is one we have seen before. The metric g_1 is precisely the metric g_{std} constructed in Theorem II.11, this being the closest we can get to the original Gromov-Lawson construction without changing the topology of X; see Fig. II.49. The metric g_1 is the original metric g outside of a tubular neighbourhood of the embedded S^p. It then transitions to a standard form so that near S^p it is $\epsilon^2 ds_p^2 + g_{tor}^{q+1}(\delta)$ for some suitably small $\delta > 0$. We will refer to this region as the *standard region* throughout this proof; see Fig. II.49. From Theorem II.11, we know that g_1 is isotopic to the original g. We make two important observations.

(i) All of the data regarding the effects of the Gromov-Lawson construction on (X, g), is contained in the metric g_1.

(ii) The embedded disk D_-^{p+1} agrees entirely with the non-standard part of the embedded sphere S^{p+1}.

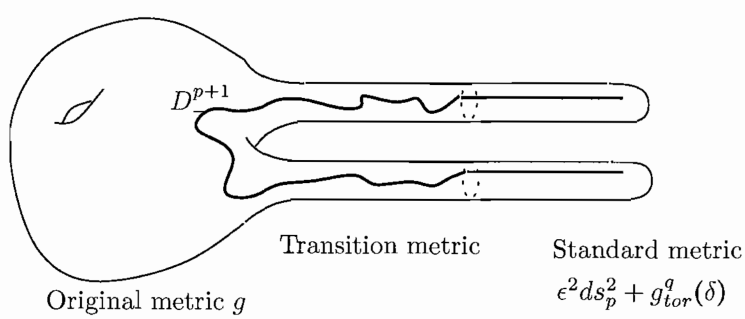

Figure II.49: The metric g_1 on X, made standard near the embedded S^p

The aim of the next adjustment is to mimic as closely as possible the metric effects of the second surgery. The boundary of D_-^{p+1} lies at the end of the standard region of (X, g_1). Application of Theorem II.11 allows us adjust the metric near D_-^{p+1} exactly as in the construction of g''. Near the boundary of D_-^{p+1}, the induced metric is standard and so we can transition (possibly very slowly) back to the metric g_1; see Fig. II.50. The connecting cylinder $S^{n-1} \times I$ can be specified exactly as before and it is immediately obvious that the metric g_2 agrees with g'' on this region. The metric g_3 is now obtained by making precisely the adjustments made to the metric g'' in the region of $S^{n-1} \times [0, \frac{1}{2}]$.

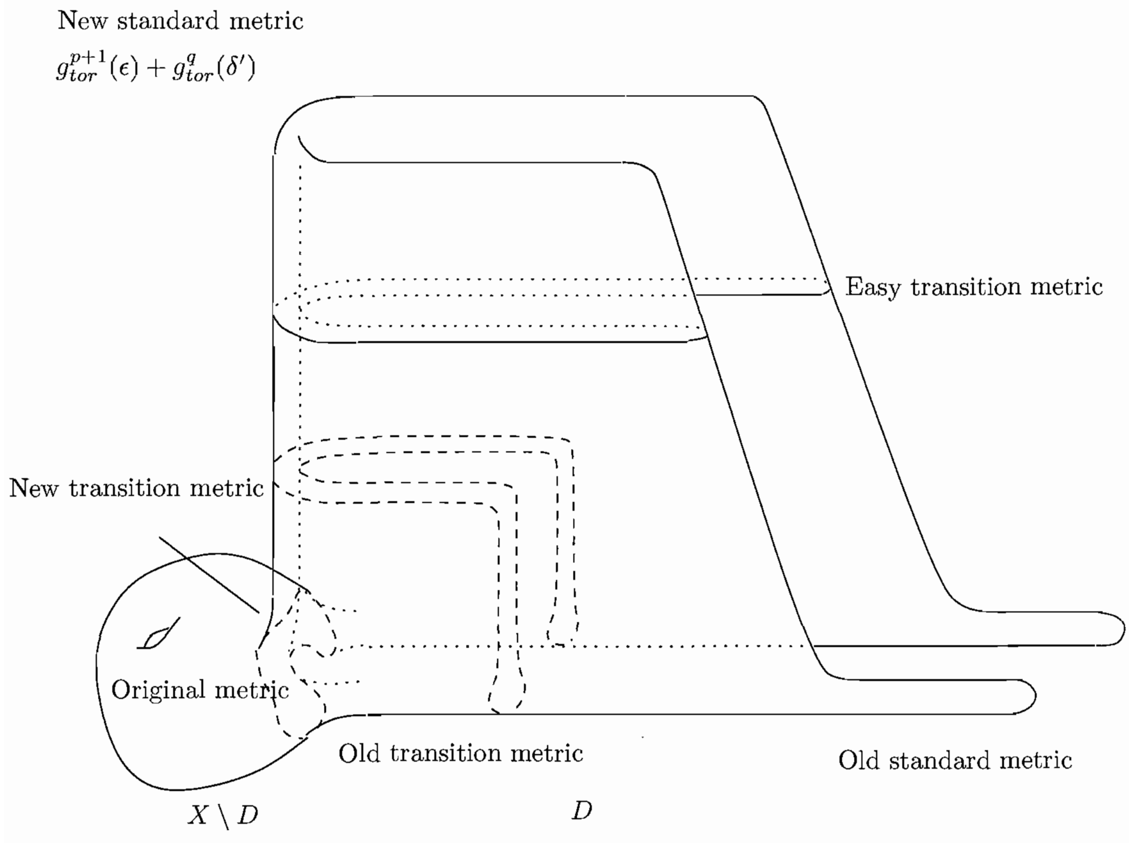

Figure II.50: Adjusting the metric g_1 on a neighbourhood of the embedded disk D_-^{p+1}: Notice how no change is made near the boundary of this disk.

Comparing the metrics g_2'' and g_3.

At this stage we have constructed two metrics g_2'' and g_3 on X which agree on $(X \setminus D) \cup (S^{n-1} \times [\frac{\lambda}{2}, \lambda])$. Near $S^{n-1} \times \{\frac{\lambda}{2}\}$, both metrics have the form of a standard round cylinder. The remaining region of X is an n-dimensional disk which we denote D'. Here the metrics g_2'' and g_3 are quite different. Henceforth g_2'' and g_3 will denote the restriction of these metrics to the disk D'. As g_2'' and g_3 agree near the boundary of D', to complete the proof it is enough to show that there is an isotopy from g_2'' to g_3 which fixes the metric near the boundary.

Both g_2'' and g_3 are obtained from metrics on the sphere S^n by removing a point and pushing out a tube in the manner of the Gromov-Lawson connected sum construction. In both cases, the point itself is the origin of a region which is isometrically identified with a neighbourhood of the origin in $(\mathbb{R}^n, h)$. We will denote by $\bar{g}_2''$ and $\bar{g}_3$, the respective sphere metrics which give rise to g_2'' and g_3 in this way; see Fig. II.51 and Fig. II.52.

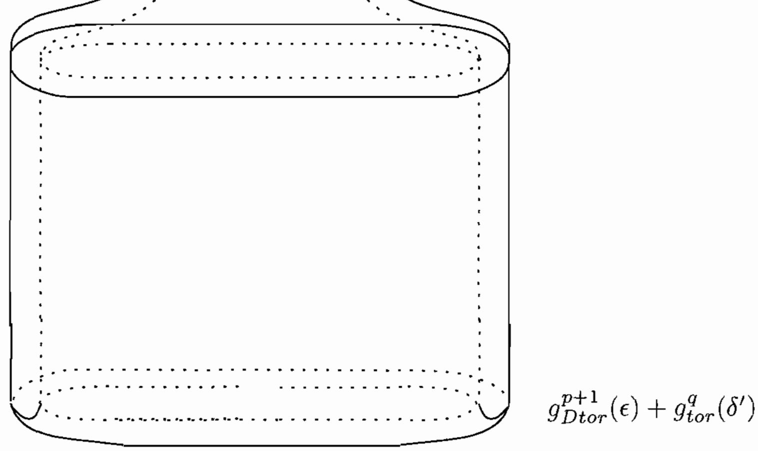

Figure II.51: The metric $\bar{g}_2''$

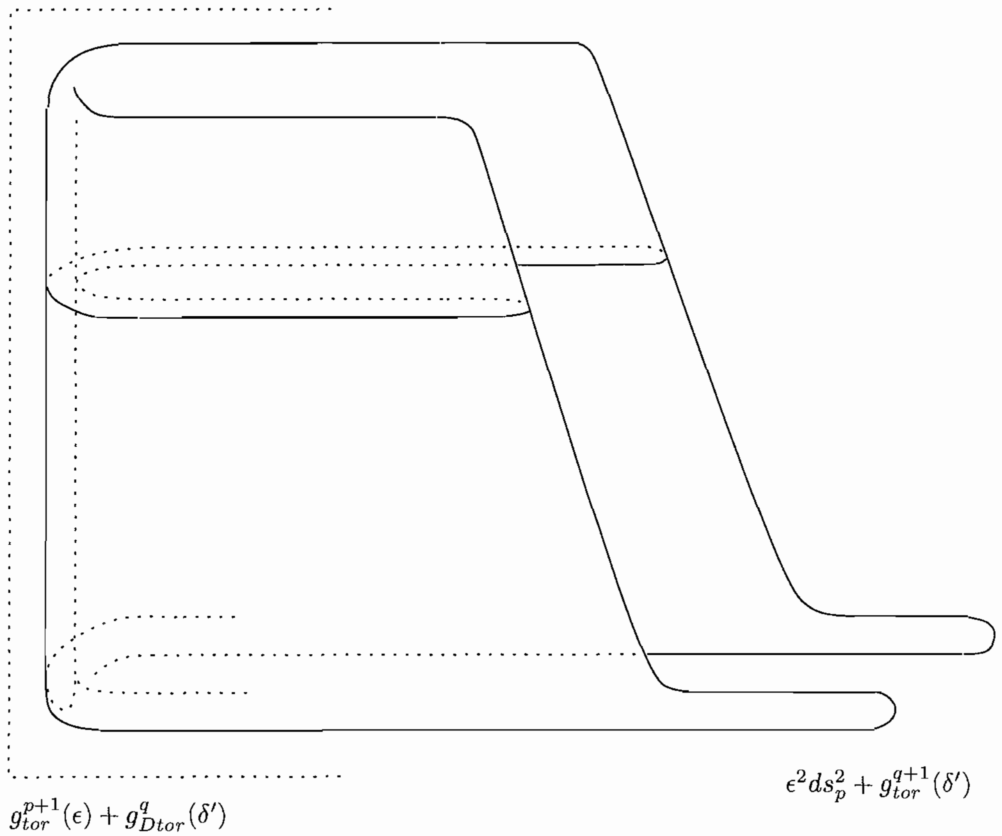

Figure II.52: The metric $\bar{g}_3$

The metrics $\bar{g}_2''$ and $\bar{g}_3$ isotopy easily to the respective mixed torpedo metrics $g_{tor}^{p+1,q-1}$ and $g_{tor}^{p,q}$ on S^n (Fig. II.53), and are thus isotopic to each other by the results of section II.2, in particular Lemma II.9. The proof of Theorem II.34 then follows from Theorem II.19, where we showed that the Gromov-Lawson construction goes through for a compact family of psc-metrics.

□

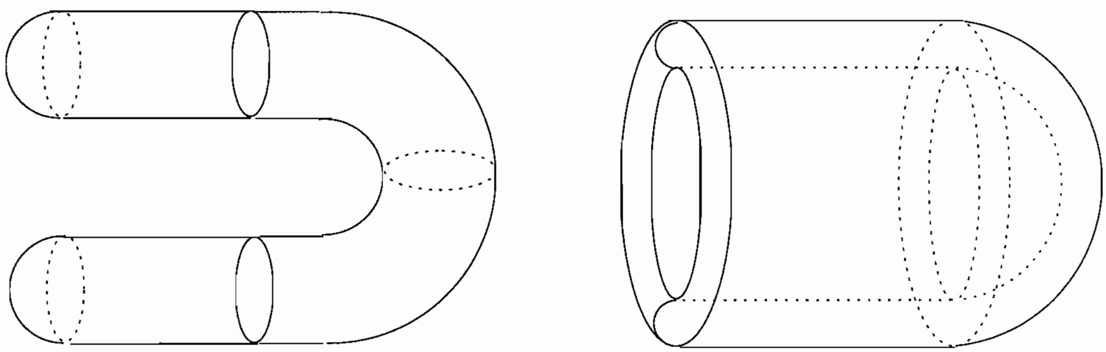

Figure II.53: The mixed torpedo metrics $g_{Mtor}^{p,q}$ and $g_{Mtor}^{p+1,q-1}$

II.7 Gromov-Lawson Concordance Implies Isotopy in the General Case

Theorem II.34 is the main tool needed in the proof of Theorem II.36. The rest of the proof follows from Morse-Smale theory and all of the results needed to complete it are to be found in [30]. Before we proceed with the proof of Theorem II.36, it is worth discussing some of these results.

II.7.1 A weaker version of Theorem II.36

Throughout, $\{W^{n+1}, X_0, X_1\}$ is a smooth compact cobordism where X_0 and X_1 are closed manifolds of dimension n. Later on we will also need to assume that X_0, X_1 and W are simply connected and that $n \geq 5$, although that is not necessary yet. Let f denote a Morse triple on W, as defined in section II.4. Recall this means that $f: W \to I$ is a Morse function which comes with extra data, a Riemannian metric m on W and a gradient-like vector field V with respect to f and m. Now by Theorem II.27, f can be isotopied to a Morse triple which is well-indexed. We will retain the name f for this well-indexed Morse triple. As discussed in section II.4, f decomposes W into a union of cobordisms $C_0 \cup C_1 \cup \cdots \cup C_{n+1}$ where each C_k contains at most one critical level (contained in its interior) and all critical points of f on this level have index k. For each $0 \leq k \leq n+1$, we denote by W_k, the union $C_0 \cup C_1 \cup \cdots \cup C_k$. By setting $W_{-1} = X_0$, we obtain the following sequence of inclusions

$$X_0 = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{n+1} = W,$$

describing this decomposition.

Suppose that f has l critical points of index k. Then for some a, b, c with $a < c < b$, the cobordism $C_k = f^{-1}[a, b]$, where c is the only critical value between a and b. The level set $f^{-1}(c)$ has l critical points $w_1, \ldots, w_l$, each of index k. Associated to these critical points are trajectory disks $K_-^k(w_1), \ldots K_-^k(w_l)$ where each $K_-^k(w_i)$ has its boundary sphere $S_-^{k-1}(w_i)$ embedded in $f^{-1}(a)$. These trajectory disks determine a basis, by theorem 3.15 of [30], for the relative integral homology group $H_k(W_k, W_{k-1})$ which is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (l summands).

We can now construct a chain complex $\mathcal{C}_* = \{\mathcal{C}_k, \partial\}$, where $\mathcal{C}_k = H_k(W_k, W_{k-1})$ and $\partial : \mathcal{C}_k \to \mathcal{C}_{k-1}$ is the boundary homomorphism of the long exact sequence of the triple $W_{k-2} \subset W_{k-1} \subset W_k$. The fact that $\partial^2 = 0$ is proved in theorem 7.4 of [30]. Furthermore, this theorem gives that $H_k(\mathcal{C}_*) = H_k(W, X_0)$.

Theorem II.35. *Let X be a closed simply connected manifold with dimension $n \geq 5$ and let g_0 be a positive scalar curvature metric on X. Let f be an admissible Morse function on $X \times I$ with no critical points of index 0 or 1. Let $\bar{g} = \bar{g}(g_0, f)$ be a Gromov-Lawson concordance on $X \times I$. Then the metrics g_0 and $g_1 = \bar{g}|_{X \times \{1\}}$ are isotopic.*

Proof. By Corollary II.29, we may assume that f is well indexed. Using the notation above, f gives rise to a decomposition $X \times I = C_2 \cup C_3 \cup \cdots \cup C_{n-2}$ which in turn gives rise to a chain complex

$$\mathcal{C}_{n-2} \to \mathcal{C}_{n-3} \to \cdots \to \mathcal{C}_{k+1} \to \mathcal{C}_k \to \cdots \to \mathcal{C}_2$$

where each $\mathcal{C}_k$ is a free abelian group. (Recall that all critical points of an admissible Morse function have index which is less than or equal to $n-2$.) Since $H_*(X \times I, X) = 0$, it follows that the above sequence is exact. Thus, for each $\mathcal{C}_{k+1}$ we may choose elements $z_1^{k+1}, \ldots, z_{l_{k+1}}^{k+1} \in \mathcal{C}_{k+1}$ and $b_1^{k+1}, \ldots, b_{l_k}^{k+1} \in \mathcal{C}_{k+1}$ so that $\partial(b_i^{k+1}) = z_i^k$ for $i = 1, \ldots, l_k$. Then $z_1^{k+1}, \ldots, z_{l_{k+1}}^{k+1}, b_1^{k+1}, \ldots, b_{l_k}^{k+1}$ is a basis for $\mathcal{C}_{k+1}$.

We will now restrict our attention to the cobordism $C_k \cup C_{k+1}$. Let $w_1^{k+1}, w_2^{k+1}, \ldots, w_{l_{k+1}+l_k}^{k+1}$ denote the critical points of f inside of C_{k+1} and $w_1^k, w_2^k, \ldots, w_{l_k+l_{k-1}}^k$ denote the critical points of f inside of C_k. As $2 \leq k < k+1 \leq n-2$, it follows from theorem 7.6 of [30], that f can be perturbed so that the trajectory disks $K_-(w_1^{k+1}), \ldots K_-(w_{l_{k+1}+l_k}^{k+1})$ and $K_-(w_1^k), \ldots K_-(w_{l_k+l_{k-1}}^k)$ represent the chosen bases for $\mathcal{C}_{k+1}$ and $\mathcal{C}_k$ respectively.

Denote by $w_1^k, w_2^k, \ldots, w_{l_k}^k$, those critical points on C_k which correspond to the elements $z_1^k, z_2^k \ldots, z_{l_k}^k$ of $\mathcal{C}_k$, i.e. the kernel of $\partial : \mathcal{C}_k \to \mathcal{C}_{k-1}$. Denote by $w_1^{k+1}, w_2^{k+2}, \ldots, w_{l_k}^{k+1}$, those critical points in C_{k+1} which correspond to the elements $b_1^{k+1}, \ldots, b_{l_k}^{k+1} \in \mathcal{C}_{k+1}$. A slight perturbation of f replaces $C_k \cup C_{k+1}$ with the decomposition $C_k' \cup C_k'' \cup C_{k+1}'' \cup C_{k+1}'$; see Fig. II.54. Here $C_k' \cup C_k''$ is diffeomorphic to C_k, however, the critical points $w_1^k, w_2^k, \ldots, w_{l_k}^k$ have been moved to a level set above their orginal level, resulting in a pair of cobordisms each with one critical level. Similarly, we can move the critical points $w_1^{k+1}, w_2^{k+2}, \ldots, w_{l_k}^{k+1}$ down to a level set below their original level to replace C_{k+1} with $C_{k+1}'' \cup C_{k+1}'$.

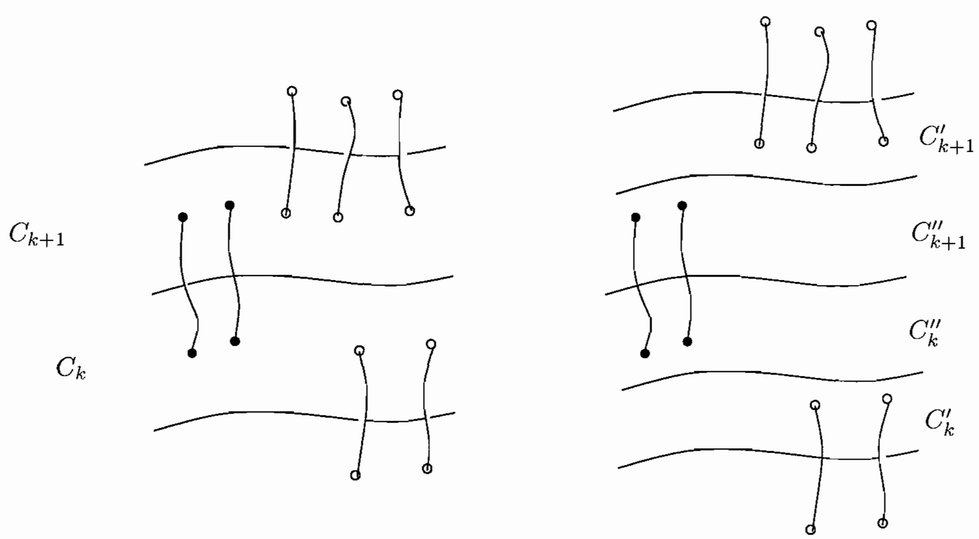

Figure II.54: Replacing $C_k \cup C_{k+1}$ with $C_k' \cup C_k'' \cup C_{k+1}'' \cup C_{k+1}'$

We now consider the the cobordism $C_k'' \cup C_{k+1}''$. For some $a < c_k < c < c_{k+1} < b$, $C_k'' \cup C_{k+1}'' = f^{-1}[a, b]$, where $f^{-1}(c_k)$ contains all of the critical points of index k and $f^{-1}(c_{k+1})$ contains all of the critical points of index $k+1$. Each critical point w_i^k of index k is associated with a critical point w_i^{k+1} of index $k+1$. Using Van Kampen's theorem, we can show that $f^{-1}([a, b]), f^{-1}(a)$ and $f^{-1}(b)$ are all simply connected; see remark 1 on page 70 of [30].

Since $\partial(b_i^{k+1}) = z_i^k$, each pair of trajectory spheres has intersection 1 or -1. The strong cancellation theorem, Theorem II.32, now gives that f can be perturbed so that each pair of trajectory spheres intersects transversely on $f^{-1}(c)$ at a single point and that $f^{-1}([a,b])$ is diffeomorphic to $f^{-1}(a) \times [a,b]$.

Consider the restriction of the metric $\bar{g} = \bar{g}(g_0, f)$ to $f^{-1}([a,b])$. Let g_a and g_b denote the induced metrics on $f^{-1}(a)$ and $f^{-1}(b)$ respectively. The trajectories connecting the critical points of the first critical level with trajectory spheres in $f^{-1}(a)$ are mutually disjoint, as are those connecting critical points on the second critical level with the trajectory spheres on $f^{-1}(b)$. In turn, pairs of cancelling critical points can be connected by mutually disjoint arcs where each arc is the union of intersection points of the corresponding trajectory spheres. The metric g_b is therefore obtained from g_a by finitely many independent applications of the construction in Theorem II.34 and so g_a and g_b are isotopic. By repeating this argument as often as necessary we show that g_0 is isotopic to g_1. □

II.7.2 The proof of the main theorem of Part One

We can now complete the proof of Theorem II.36. To do this, we must extend Theorem II.35 to deal with the case of index 0 and index 1 critical points.

Theorem II.36. *Let X be a closed simply connected manifold with $dim X = n \geq 5$ and let g_0 be a positive scalar curvature metric on X. Suppose $\bar{g} = \bar{g}(g_0, f)$ is a Gromov-Lawson concordance with respect to g_0 and an admissible Morse function $f : X \times I \to I$. Then the metrics g_0 and $g_1 = \bar{g}|_{X \times \{1\}}$ are isotopic.*

Proof. We will assume that f is a well-indexed admissible Morse function on W. Using the notation of the previous theorem, f decomposes W into a union of cobordisms $C_0 \cup C_1 \cup \cdots \cup C_{n-2}$. In the case where f has index 0 critical points, f can be perturbed so that for some $\epsilon > 0$, $f^{-1}([0, \epsilon])$ contains all index 0 critical points along with an equal number of index 1 critical points. These critical points are arranged so that all index 0 critical points are on the level $f^{-1}(c_0)$ and all index 1 critical points are on the level set $f^{-1}(c_1)$, where $0 < c_0 < c < c_1 < \epsilon$. In theorem 8.1 of [30], it is proved that these critical points can be arranged into pairs of index 0 and index 1 critical points where each pair is connected by mutually disjoint arcs and each pair satisfies the conditions of theorem II.31. Thus, Theorem II.34 gives that the metric g_0 is isotopic to the metric

$g_\epsilon = \bar{g}(g_0, f)|_{f^{-1}(\epsilon)}$.

If f has no other critical points of index 1, then Theorem II.35 gives that g_ϵ is isotopic to g_1, completing the proof. We thus turn our attention to the case where f has excess index 1 critical points which do not cancel with critical points of index 0. Each of these critical points is asscociated with a critical point of index 2 and the intersection number of the corresponding trajectory spheres is 1 or -1. Unfortunately, theorem II.32 does not apply here as the presence of index 1 critical points means the upper boundary component of W_1 is not simply connected. In turn, this prevents us from applying Theorem II.34.

There is however, another way to deal with these excess index 1 critical points which we will now summarise. It is possible to add in auxiliary pairs of index 2 and index 3 critical points. This can be done so that the newly added pairs have trajectory spheres which intersect transversely at a point and so satisfy the conditions of theorem II.31. Furthermore, for each excess index 1 critical point, such a pair of auxiliary critical points may be added so that the newly added index 2 critical point has an incoming trajectory sphere which intersects transversely at a single point with the outgoing trajectory sphere of the index 1 critical point. This allows us to use theorem II.31 and hence Theorem II.34 with respect to these index 1, index 2 pairs. The old index 2 critical points now all have index 3 critical points with which to cancel and so we can apply Theorem II.35 to complete the proof. In effect, the excess index 1 critical points are replaced by an equal number of index 3 critical points. The details of this construction are to be found in the proof of theorem 8.1 of [30] and so we will provide only a rough outline. The key result which makes this possible is a theorem by Whitney, which we state below.

Theorem II.37. *[41] If two smooth embeddings of a smooth manifold M of dimension m into a smooth manifold N of dimension n are homotopic, then they are smoothly isotopic provided $n \geq 2m + 2$.*

Choose $\delta > 0$ so that the metric $\bar{g}$ is a product metric on $X \times [1 - \delta, 1]$. Thus, f has no critical points here either. On any open neighbourhood U contained inside $f^{-1}([1 - \delta, 1])$, it is possible to replace the function f with a new function f_1, so that outside U, $f_1 = f$, but inside U, f has a pair of critical points, y and z with respective indices 2 and 3 and so that on the cylinder $f^{-1}([1 - \delta, 1])$, f_1 satisfies the conditions of Theorem II.31. For a detailed proof of this fact; see lemma 8.2 of [30].

Remark II.7.1. *The Morse functions f and f_1 are certainly not isotopic, as they have different numbers of critical points. However, this is not a problem as the following comment makes clear.*

Recall that the metric $\bar{g}|_{X\times\{1-\delta\}} = g_1$ and that our goal is to show that g_1 is isotopic to g_0. By Theorem II.34, the metric $\bar{g}|_{X\times\{1-\delta\}}$ is isotopic to $\bar{g}(g_0, f_1)|_{X\times\{1\}}$ and so it is enough to show that g_0 is isotopic to $\bar{g}(g_0, f_1)|_{X\times\{1\}}$ for some such f_1.

For simplicity, we will assume that f has no index 0 critical points. We will assume that all of the critical points of index 1 are on the level $f = c_1$. Choose points $a < c_1 < b$ so that $f^{-1}([a, b])$ contains no other critical levels except $f^{-1}(c_1)$. Let w be an index 1 critical point of f. Emerging from w is an outward trajectory whose intersection with the level set $f^{-1}(b)$ is an $n-1$-dimensional sphere $S_{+}^{n-1}(z)$. The following lemma is lemma 8.3 of [30].

Lemma II.38. *There exists an embedded 1-sphere $S = S^1$ in $f^{-1}(b)$ which intersects transversely with $S_{+}^{n-1}(z)$ at a single point and meets no other outward trajectory sphere.*

Replace f with the function f_1 above. By Theorem II.27, the function f_1 can be isotopied through admissible Morse functions to a well-indexed one $\bar{f}_1$. Consequently, the metric $\bar{g}(g_0, f_1)$ can be isotopied to a Gromov-Lawson concordance $\bar{g}(g_0, \bar{f}_1)$. The critical points y and z have now been moved so that y is on the same level as all of the other index 2 critical points. There is a trajectory sphere $S_{-}^{1}(y)$, which is converging to y, embedded in $f^{-1}(b)$. Theorem II.37 implies that $\bar{f}_1$ can be isotopied so as to move $S_{-}^{1}(y)$ onto the embedded sphere S of Lemma II.38. The resulting well-indexed admissible Morse function has the property that the outward trajectory spheres of index 1 critical points intersect the inward trajectory spheres of their corresponding index 2 critical points transversely at a point.

We can make an arbitrarily small adjustment to $\bar{f}_1$ so that the index 2 critical points which correspond to the kernel of the map $\partial : \mathcal{C}_3 \to \mathcal{C}_2$, are on a level set just above the level containing the remaining index 2 critical points. Let $f^{-1}(c)$ denote a level set between these critical levels. Then $f^{-1}([0, c])$ is diffeomorphic to $X \times [0, c]$ and, by Theorem II.34, the metric g_0 is isotopic to the metric $\bar{g}(g_0, \bar{f}_1)|_{f^{-1}(c)}$. Furthermore, the cobordism $f^{-1}([c, 1])$ is diffeomorphic to $X \times [c, 1]$ and the restriction of $\bar{f}_1$ satisfies all of the conditions of Theorem II.35. This means that $\bar{g}(g_0, \bar{f}_1)|_{f^{-1}(c)}$ is isotopic to $\bar{g}(g_0, \bar{f}_1)|_{f^{-1}(1)}$, completing the proof. □

CHAPTER III

PART TWO: FAMILIES OF GROMOV-LAWSON COBORDISMS

III.1 Foreword to Part Two

Our main goal in Part Two is to develop tools for parameterising families of Gromov-Lawson cobordisms by admissible Morse functions. This was done to an extent in Theorem II.25 of Part One. This theorem allows for the parametrisation of a family of GL-cobordisms by a compact contractible family of admissible Morse functions. Unfortunately, all admissible Morse functions in this family must have the same number of critical points of the same index. As it is possible for certain pairs of Morse critical points to cancel in the form of birth-death singularities, this theorem gives us a rather limited picture.

In order to connect up admissible Morse functions which have different critical sets, we must allow for this cancellation. This means working in the space of admissible *generalised* Morse functions. A generalised Morse function has Morse and birth-death singularities; see below for a definition. By utilising the "geometric cancellation" described in the proof of Theorem II.36, we will describe a *regularised* Gromov-Lawson cobordism; see Theorem III.2 and Corollary III.3 below. This is a type of GL-cobordism which has been adapted to vary continuously over a cancellation of Morse critical points.

A convenient setting for describing families of admissible generalised Morse functions arises from the work of Eliashberg and Mishachev on wrinklings of smooth maps in [8] and [9]. Roughly speaking, a wrinkled map gives rise to a particular smooth bundle of admissible generalised Morse functions. In our main result, Theorem III.6, we perform a construction on the total space of this bundle, which restricts on each fibre to a regularised Gromov-Lawson cobordism.

The final result of Part Two provides a partial answer to a question we posed in the introduction. Namely, how does the choice of admissible Morse function affect the isotopy type of a Gromov-Lawson cobordism? In Theorem III.9, we show that when the cobordism and its boundary components are simply connected and of dimension ≥ 5, the isotopy type of the metric is unaffected by the choice of admissible Morse function.

III.2 A Review of Part One

We begin by very briefly reviewing some notions from Part One which will be of use to us in Part Two. In particular, we review what we mean by an admissible Morse function, before reexamining the structure of a Gromov-Lawson cobordism. This will be especially useful when it comes to proving Theorem III.2.

III.2.1 Admissible Morse functions

In this section we review what we mean by an admissible Morse function. We begin with an important piece of terminology. Let M and N be smooth manifolds of dimensions m and n respectively. Let $f : M \to N$ be a smooth map and let $w \in M$. We say that f is *locally equivalent near* w to a smooth map $f' : \mathbb{R}^m \to \mathbb{R}^n$, if there exist neighbourhoods $U \subset M$, $V \subset N$ with $w \in U$ and $f(w) \in V$, along with diffeomorphisms $\psi_1 : \mathbb{R}^m \to U$, $\psi_2 : \mathbb{R}^n \to V$ with $\psi_1(0) = w$ and $\psi_2(0) = f(w)$, for which the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & V \\ {\scriptstyle \psi_1}\Big\uparrow & & \Big\uparrow{\scriptstyle \psi_2} \\ \mathbb{R}^m & \xrightarrow[\quad f' \quad]{} & \mathbb{R}^n \end{array}$$

Let $(W^{n+1}; X_0, X_1)$ be a smooth compact cobordism. Recall that we let $\mathcal{F} = \mathcal{F}(W)$ denote the space of smooth functions $f : W \to I$ satisfying $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$, and having no critical points near ∂W. The space $\mathcal{F}$ is a subspace of the space of smooth functions on W with its standard C^∞ topology; see Chapter 2 of [17] for the full definition. A critical point $w \in W$ of a smooth function $f : W \to I$ is a Morse critical point if, near w, the map f is locally equivalent to the map

$$\begin{aligned}\mathbb{R}^{n+1} &\longrightarrow \mathbb{R}\\ x &\longmapsto -\sum_{i=1}^{p+1} {x_i}^2 + \sum_{i=p+2}^{n+1} {x_i}^2.\end{aligned}$$

The integer $p+1$ is called the *Morse index* of w and is an invariant of the critical point. A function $f \in \mathcal{F}$ is a *Morse function* if every critical point of f is a Morse critical point. By equipping W with a Riemannian metric m, we can define $\text{grad}_m f$, the gradient vector field for f with respect to m. More generally, we define *gradient-like* vector fields on W with respect to f and m, as follows.

Definition III.1. A *gradient-like* vector field with respect to f and m is a vector field V on W satisfying the following properties.

(1) $df_x(V_x) > 0$ when x is not a critical point of f.

(2) Each critical point w of f lies in a neighbourhood U so that for all $x \in U$, $V_x = \text{grad}_m f(x)$.

Definition III.2. An *admissible Morse function* f on a compact cobordism $\{W; X_0, X_1\}$ is a triple $f = (f, m, V)$ where $f : W \to I$ is a Morse function, m is a background metric for f, V is a gradient like vector field with respect to f and m, and finally, any critical point of f has Morse index less than or equal to $n - 2$.

We emphasise the fact that an admissible Morse function is actually a triple consisting of a Morse function, a Riemannian metric and a gradient-like vector field. However, to ease the burden of notation, an admissible Morse function (f, m, V) will be denoted simply by f.

We conclude with some comments on the space of Morse functions $\mathcal{M} = \mathcal{M}(W) \subset \mathcal{F}$. Recall that this an open dense subspace of $\mathcal{F}$; see theorem 2.7 of [30]. We let $\tilde{\mathcal{F}}$ denote the space of triples (f, m, V) so that $f \in \mathcal{F}$, m is a backgound metric for f and V is a gradient-like vector field with respect to f and m. Recall that the space $\tilde{\mathcal{F}}$ is homotopy equivalent to the space $\mathcal{F}$. In fact, by equipping W with a fixed background metric $\bar{m}$, the inclusion map

$$f \longmapsto (f, \bar{m}, \text{grad}_{\bar{m}} f) \qquad \text{(III.2.1)}$$

forms part of a deformation retract of $\tilde{\mathcal{F}}$ down to $\mathcal{F}$; see Chapter 2, section 2 of [16] for details.

We denote by $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(W)$, the subspace of $\tilde{F}$, consisting of triples (f, m, V) where f is a Morse function. Elements of $\tilde{\mathcal{M}}$ are known as *Morse triples*. The subspace of $\tilde{\mathcal{M}}$ consisting of admissible Morse functions is denoted $\tilde{\mathcal{M}}^{adm} = \tilde{\mathcal{M}}^{adm}(W)$. Successive restrictions of the deformation retract above give rise to respective deformation retracts of $\tilde{\mathcal{M}}$ onto $\mathcal{M}$ and $\tilde{\mathcal{M}}^{adm}$ onto $\mathcal{M}^{adm}$. Here $\mathcal{M}^{adm}$ is the space of Morse functions with all critical points having index $\leq n-2$.

III.2.2 A brief review of the Gromov-Lawson cobordism Theorem

Let $(W; X_0, X_1)$ be as before and let g_0 be a psc-metric on X_0. In Part One we discuss the problem of extending the metric g_0 to a psc-metric $\bar{g}$ on W which has a product structure near ∂W. In particular we have proved the following theorem.

Theorem II.23. *Let $\{W^{n+1}; X_0, X_1\}$ be a smooth compact cobordism. Suppose g_0 is a metric of positive scalar curvature on X_0 and $f : W \to I$ is an admissible Morse function. Then there is a psc-metric $\bar{g} = \bar{g}(g_0, f)$ on W which extends g_0 and has a product structure near the boundary.*

We call the metric $\bar{g}$ a *Gromov-Lawson cobordism* with respect to g_0 and f. It is worth briefly reviewing the structure of this metric.

We begin with a few topological observations. For simplicity, let us assume for now that f has only a single critical point w of index $p+1$. Intersecting transversely at w are a pair of trajectory disks K_{-}^{p+1} and K_{+}^{q+1}; see Fig. III.1. The lower disk K_{-}^{p+1} is a $p+1$-dimensional disk which is bounded by an embedded p-sphere $S_{-}^{p} \subset X_0$. It consists of the union of segments of integral curves of the gradient-like vector field beginning at the bounding sphere and ending at w. Similarly, K_{+}^{q+1} is a $q+1$-dimensional disk bounded by an embedded $q+1$-sphere $S_{+}^{q} \subset X_1$. The bounding spheres S_{-}^{p} and S_{+}^{q} are known as trajectory spheres.

Let N denote a small tubular neighbourhood of S_{-}^{p}, defined with respect to the metric $m|_{X_0}$. Consider the region $X_0 \setminus N$. For each point $x \in X_0 \setminus N$, there is a unique maximal integral curve of the vector field V, $\psi_x : [0,1] \to W$ satisfying $f \circ \psi_x(t) = t$; see section 3 of [30] for details.

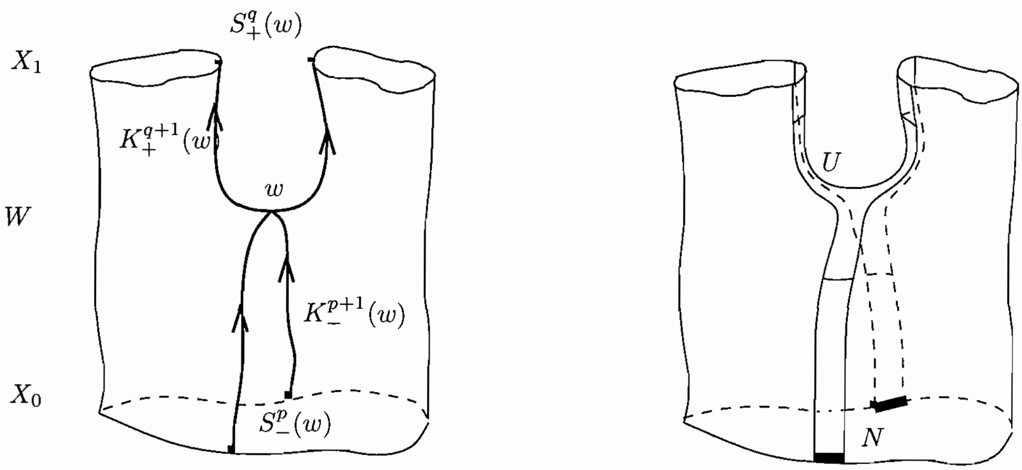

Figure III.1: Trajectory disks of the critical point w contained inside a disk U

This gives rise to an embedding

$$\psi : (X_0 \setminus N) \times I \longrightarrow W$$
$$(x,t) \longmapsto (\psi_x(t)).$$

We denote by U, the complement of this embedding in W, and observe that U is a neighbourhood of $K^{p+1}_- \cup K^{q+1}_+$; see Fig. III.1. Indeed, a continuous shrinking of the radius of N down to 0 induces a deformation retract of U onto $K^{p+1}_- \cup K^{q+1}_+$.

We now define the metric $\bar{g}$ on the region $W \setminus U$ to be simply $g_0|_{X \setminus N} + dt^2$ where the t coordinate comes from the embedding ψ above. Of course, the real challenge lies in extending this metric over the region U. Notice that the boundary of U decomposes as

$$\partial U = (S^p \times D^{q+1}) \cup (S^p \times S^q \times I) \cup (D^{p+1} \times S^q).$$

The $S^p \times D^{q+1}$ part of this decomposition is of course the tubular neighbourhood N while the $D^{p+1} \times S^q$ piece is a tubular neighbourhood of the outward trajectory sphere $S^q_+ \subset X_1$. Without loss of generality, assume that $f(w) = \frac{1}{2}$. Let c_0 and c_1 be constants satisfying $0 < c_0 < \frac{1}{2} < c_1 < 1$. The level sets $f = c_0$ and $f = c_1$ divide U into three regions: $U_0 = f^{-1}([0, c_1]) \cap U$,

$U_w = f^{-1}([c_0, c_1]) \cap U$ and $U_1 = f^{-1}([c_1, 1]) \cap U$.

The region U_0 can be diffeomorphically identified with $N \times [0, c_1]$ in exactly the way we identified $W \setminus U$ with $X_0 \setminus N \times I$. Thus, on U_0, we define $\bar{g}$ as simply the product $g_0|_N + dt^2$. Indeed we can extend this metric $g_0|_N + dt^2$ near the $S^p \times S^q \times I$ part of the boundary also where, again, t is the trajectory coordinate. Inside the region U_w, which is identified with the disk product $D^{p+1} \times D^{q+1}$, the metric smoothly transitions to a standard product $g_{tor}^{p+1}(\epsilon) + g_{tor}^{q+1}(\delta)$ for some appropriately chosen $\epsilon, \delta > 0$. This is done so that the induced metric on the level set $f^{-1}(c_1)$, denoted g_1, is precisely the metric obtained by application of the Gromov-Lawson construction on g_0. Furthermore, near $f^{-1}(c_1)$, $\bar{g} = g_1 + dt^2$. Finally, on U_1, which is identified with $D^{p+1} \times S^q \times [c_1, 1]$ in the usual manner, the metric $\bar{g}$ is simply the product $g_1 + dt^2$. See Fig. III.2 for an illustration.

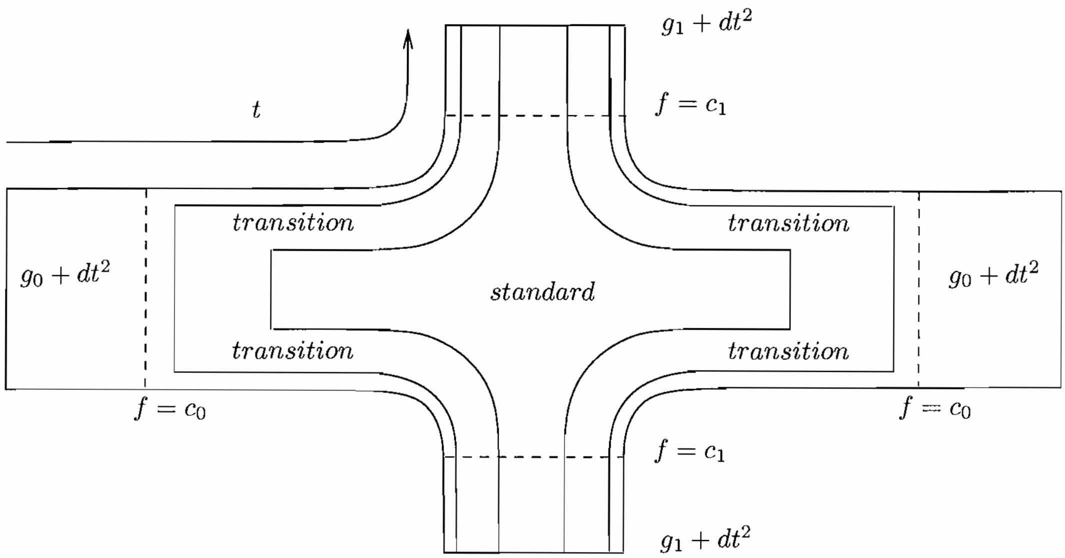

Figure III.2: The metric $\bar{g}$ on the disk U

We should point out that this construction can be carried out for a tubular neighbourhood N of arbitrarily small radius and for c_0 and c_1 chosen arbitrarily close to $\frac{1}{2}$. Thus, the region U_w, on which the metric $\bar{g}$ is not simply a product and is undergoing some kind of transition, can be made arbitrarily small with respect to the background metric m. As critical points of a Morse function are isolated, it follows that this construction generalises easily to Morse functions with more than one critical point.

III.2.3 Continuous families of Gromov-Lawson cobordisms

A careful analysis of the Gromov-Lawson construction shows that it can be applied continuously over a compact family of metrics as well as a compact family of embedded surgery spheres; see Theorem II.19 in section II.3. It then follows that the construction of Theorem II.23 can be applied continuously over certain compact families of admissible Morse functions to obtain Theorem II.25. Before stating it we introduce some notation. Let $\mathcal{B} = \{g_b \in \mathcal{R}\text{iem}^+(X_0) : b \in B\}$ be a compact continuous family of psc-metrics on X_0, parametrised by a compact space B. Let $\mathcal{C} = \{f_c \in \tilde{\mathcal{M}}^{adm}(W) : c \in D^k\}$ be a smooth compact family of admissible Morse functions on W, parametrised by the disk D^k.

Theorem II.25. *There is a continuous map*

$$\mathcal{B} \times \mathcal{C} \longrightarrow \mathcal{R}\text{iem}^+(W)$$

$$(g_b, f_c) \longmapsto \bar{g}_{b,c} = \bar{g}(g_b, f_c)$$

so that for each pair (b, c), the metric $\bar{g}_{b,c}$ is a Gromov-Lawson cobordism.

III.2.4 A brief review of Gromov-Lawson concordance

We now consider the case when W is the cylinder $X \times I$ for some closed smooth manifold X. If g_0 is a psc-metric on X and $f : W \to I$ is an admissible Morse function, then the metric $\bar{g} = \bar{g}(g_0, f)$ obtained by application of Theorem II.23, is a concordance. We call this metric a *Gromov-Lawson concordance* with respect to g_0 and f. The main result of Part One can now be stated as follows.

Theorem II.36. *Let X be a closed simply connected manifold of dimension $n \geq 5$. Let g_0 be a positive scalar curvature metric on X. Suppose $\bar{g} = \bar{g}(g_0, f)$ is a Gromov-Lawson concordance with respect to g_0 and an admissible Morse function $f : X \times I \to I$. Then the metrics g_0 and $g_1 = \bar{g}|_{X \times \{1\}}$ are isotopic.*

The key geometric fact used in the proof of Theorem II.36 is Theorem II.34 below.

Theorem II.34. *Let $f : W \to I$ be an admissible Morse function which satisfies conditions (a),(b) and (c) below.*

(a) *The function f has exactly 2 critical points w and z and $0 < f(w) < f(z) < 1$.*

(b) *The critical points w and z have Morse index $p+1$ and $p+2$ respectively.*

(c) *For each $t \in (f(w), f(z))$, the trajectory spheres $S^q_{t,+}(w)$ and $S^{p+1}_{t,-}(z)$ on the level set $f^{-1}(t)$, respectively emerging from the critical point w and converging toward the critical point z, intersect transversely as a single point.*

Let g be a metric of positive scalar curvature on X and let $\bar{g} = \bar{g}(g, f)$ be a Gromov-Lawson cobordism with respect to f and g on W. Then $\bar{g}$ is a concordance and the metric $g'' = \bar{g}|_{X\times\{1\}}$ on X is isotopic to the original metric g.

The fact that $\bar{g}$ is a concordance follows immediately from Theorem 5.4 of [30] as conditions (a), (b) and (c) force W to be diffeomorphic to the cylinder $X_0 \times I$. The rest of the proof of Theorem II.34 is long and technical and involves explicitly constructing an isotopy between the metrics g and g''. Roughly speaking, simple connectivity and the fact that $n \geq 5$ mean that the proof of Theorem II.36 can be reduced down to finitely many applications of the case considered in Theorem II.34.

III.3 Folds, Cusps and Wrinkles

In this section we review some basic singularity theory. For the most part this section summarises a discussion by Eliashberg and Mishachev in [8]. We will employ much of the same notation.

III.3.1 Birth-death singularities

Let M be a smooth manifold of dimension n and $f : M \to \mathbb{R}$ a smooth function. The singular set of f is the set $\Sigma f = \{w \in M : df_w = 0\}$ and a point $w \in \Sigma f$ is said to be a non-degenerate singularity if $\det d^2 f_w \neq 0$ and a degenerate singularity otherwise. Non-degenerate singularities of are of course just the Morse singularities discussed earlier. This is proved in a lemma of Morse; see Lemma 2.2 of [31]. Degenerate singularities on the other hand may be much more complicated. We will restrict our attention mostly to one type of degenerate singularity, the so-called birth-death singularity. A critical point $w \in \Sigma f$ is said to be *birth-death* of index $s + \frac{1}{2}$ if, near w, f is locally equivalent to the map

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^{n-1} &\longrightarrow \mathbb{R} \\ (z, x) &\longmapsto z^3 - \sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-1} {x_i}^2. \end{aligned}$$

The assignment of a non-integer index to w conveys the fact that at a birth-death critical point, regular Morse critical points of index s and $s+1$ may cancel.

Definition III.3. The smooth function $f : M \to \mathbb{R}$ is said to be a *generalised Morse function* if all of its degenerate singularities are of birth-death type.

Later, we will insist that M is a smooth cobordism $\{W, X_0, X_1\}$ of the type discussed earlier and that $f : W \to I$ with $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$, but for now the more general definition will suffice.

III.3.2 Fold singularities

Let M and Q be smooth manifolds of dimension n and k respectively. Let $f : M \to Q$ be a smooth map. The singular set Σf is the set $\{w \in M : \operatorname{rank} df_w < k\}$.

Definition III.4. A point $w \in \Sigma f$ is called a *fold type singularity* of index s if, near w, the map f is locally equivalent to

$$\begin{aligned} \mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1} &\longrightarrow \mathbb{R}^{k-1} \times \mathbb{R} \\ (y, x) &\longmapsto \left(y, -\sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k+1} {x_i}^2 \right). \end{aligned}$$

Definition III.5. A *fold* of f is a connected component of Σf which contains only fold-type singularities.

In the case when $Q = \mathbb{R}$, a fold singularity is just a Morse singularity of index s and is thus non-degenerate, i.e. $\det d^2 f_w \neq 0$. When $k \geq 2$, this is a degenerate singularity with $\dim(\ker d^2 f_w) = k - 1$. In this case, it is often useful to regard f locally as a constant $k-1$-

parameter family of Morse functions

$$\mathbb{R}^{n-k+1} \longrightarrow \mathbb{R}$$
$$x \longmapsto -\sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k+1} {x_i}^2,$$

over $\mathbb{R}^{k-1}$.

III.3.3 Cusp singularities

In defining a cusp singularity we will assume that $k > 1$. See Fig. III.3 for the case when $k = 2$.

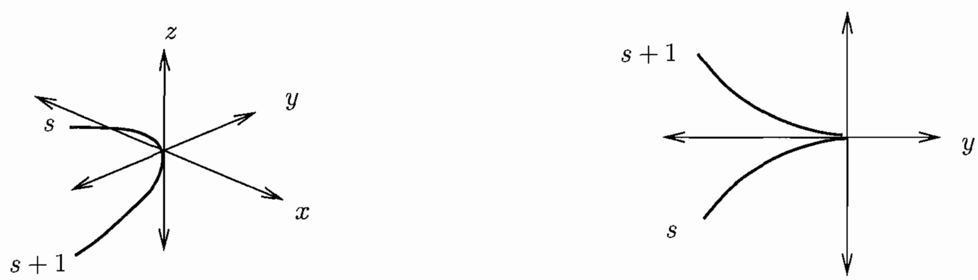

Figure III.3: A cusp singularity and its image where $k = 2$.

Definition III.6. A point $w \in \Sigma f$ is called a *cusp type singularity* of index $s + \frac{1}{2}$ if near w, the map f is locally equivalent to

$$\mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}^{k-1} \times \mathbb{R}$$
$$(y, z, x) \longmapsto \left(y, z^3 + 3y_1 z - \sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k} {x_i}^2 \right).$$

As before, it is often useful to regard f as a $k-1$-parameter family of functions, although unlike the fold case this family is not constant. In the above coordinates, the singular set of f is

$$\Sigma f = \{(y, z, x) : z^2 + y_1 = 0, x = 0\}.$$

Thus, when $y_1 > 0$, the function f is locally a $k-1$-parameter family of Morse functions with no critical points, parametrised by $y \in (0, \infty) \times \mathbb{R}^{k-2}$. At $y_1 = 0$, the function f is a $k-2$-parameter family of generalised Morse functions each with exactly one birth-death critical point occurring at $(z = 0, x = 0)$. When $y_1 < 0$, f is a $k-1$-parameter family of Morse functions each with exactly two critical points, parametrised by $y \in (-\infty, 0) \times \mathbb{R}^{k-2}$. Each Morse function in this family has a critical point of index s at $(z = \sqrt{-y_1}, x = 0)$ and a critical point of index $s+1$ at $(z = -\sqrt{-y_1}, x = 0)$. Thus, as $y_1 \to 0^-$, these pairs of Morse critical points converge and cancel as a $k-2$-parameter family of birth-death singularities. The case when $k = 2$ is illustrated in Figures III.3 and III.4.

This is the standard unfolding of a birth-death singularity and is best thought of as a 1-parameter family of functions

$$\begin{aligned} q_y : \mathbb{R} \times \mathbb{R}^{n-2} &\longrightarrow \mathbb{R} \\ (z, x) &\longmapsto z^3 + 3yz - \sum_{i=1}^{s} x_i{}^2 + \sum_{i=s+1}^{n-2} x_i{}^2, \end{aligned}$$

parametrised by $y \in \mathbb{R}$. In these coordinates, the singular set Σf is the curve $z^2 + y = 0$ on the plane $x = 0$, shown in Fig. III.3. The topological effects of the unfolding are illustrated in Fig. III.4 by selected level sets $q_y = q_y(\sqrt{c}, 0) - \epsilon$, $q_y = 0$ and $q_y = q_y(-\sqrt{c}, 0) + \epsilon$ for $y = -c, 0$ and c, where c and ϵ are both positive constants. The critical points of index s and $s+1$ occur at $z = \sqrt{c}$ and $z = -\sqrt{c}$ respectively for the function q_{-c}. The birth-death singularity occurs on the level set $q_0 = 0$ shown in the centre of this figure while the function q_c has no critical points.

III.3.4 Wrinkles and wrinkled maps

Let ω denote the map

$$\begin{aligned} \omega : \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k} &\longrightarrow \mathbb{R}^{k-1} \times \mathbb{R} \\ (y, z, x) &\longmapsto \left(y, z^3 + 3(|y|^2 - 1)z - \sum_{i=1}^{s} x_i{}^2 + \sum_{i=s+1}^{n-k} x_i{}^2 \right). \end{aligned}$$

Here $|y|$ is the standard Euclidean norm on $\mathbb{R}^{k-1}$. The singular set $\Sigma\omega$, shown with its image in Fig. III.5, is the standard $k-1$-dimensional sphere

$$\{x = 0, z^2 + |y|^2 = 1\} \subset \mathbb{R}^{n-k} \times \mathbb{R} \times \mathbb{R}^{k-1}.$$

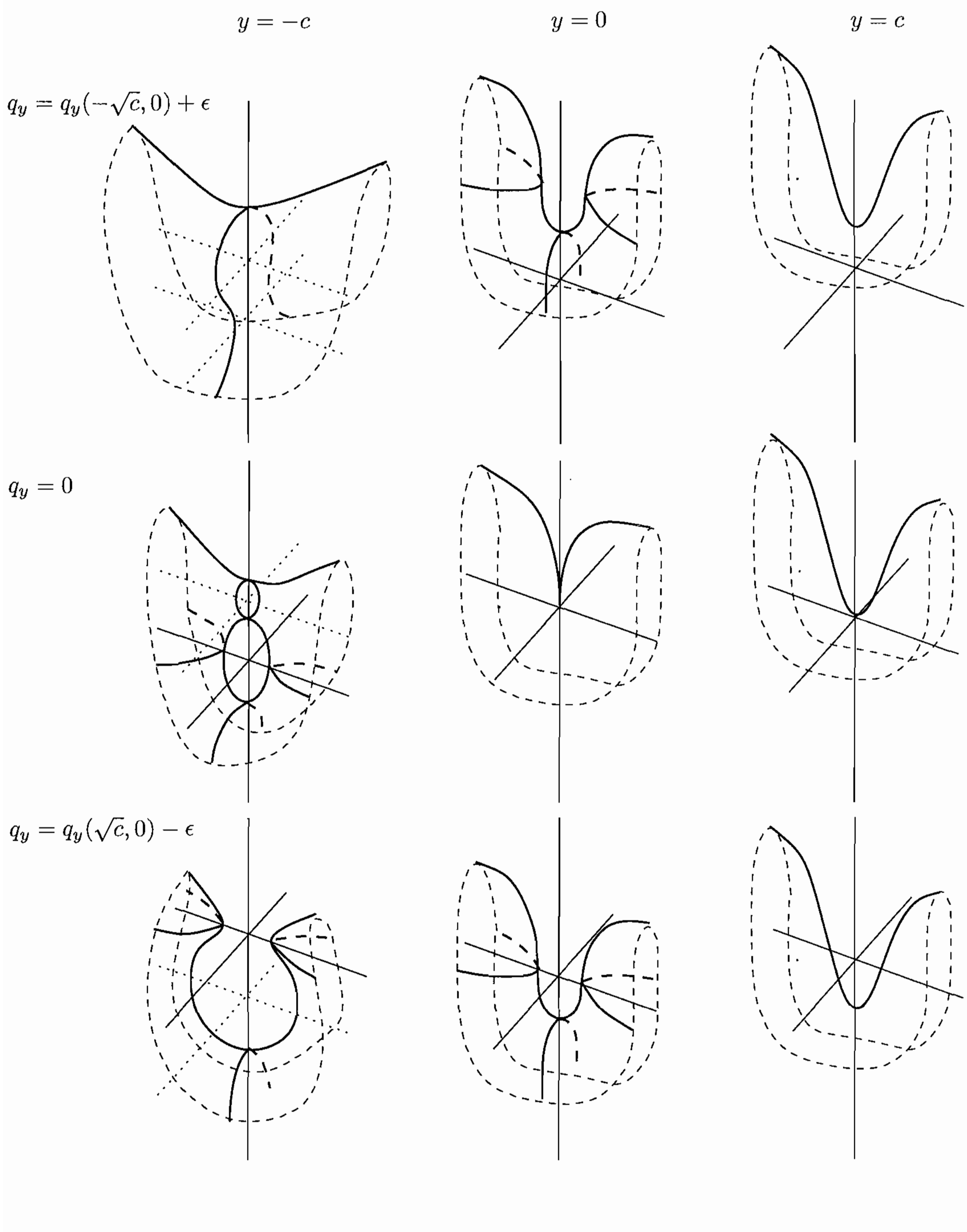

Figure III.4: Selected level sets showing the unfolding of a birth-death singularity

The equator of this sphere is the $k-2$-dimensional sphere

$$\{x = 0, z = 0, |y|^2 = 1\} \subset \mathbb{R}^{n-k} \times \mathbb{R} \times \mathbb{R}^{k-1}$$

consisting of cusp points of index $s + \frac{1}{2}$. The upper hemisphere $\Sigma\omega \cap \{z > 0\}$ consists of folds of index s and the lower hemisphere consists of folds of index $s + 1$. Alternatively, the map ω can be regarded as a smooth $k-1$-parameter family of smooth functions

$$\omega_y : \mathbb{R} \times \mathbb{R}^{n-k} \longrightarrow \mathbb{R}$$
$$(z, x) \longmapsto z^3 + 3(|y|^2 - 1)z - \sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k} {x_i}^2.$$

When $|y| < 1$, ω_y has a pair of non-degenerate critical points of index s and $s + 1$. When $|y| = 1$, the function ω_y has a single birth-death singularity of index $s + \frac{1}{2}$ and when $|y| > 1$, ω_y has no critical points. Let D denote the disk $\{x = 0, z^2 + |y|^2 \leq 1\} \subset \mathbb{R}^{n-k} \times \mathbb{R} \times \mathbb{R}^{k-1}$ bounded by $\Sigma\omega$.

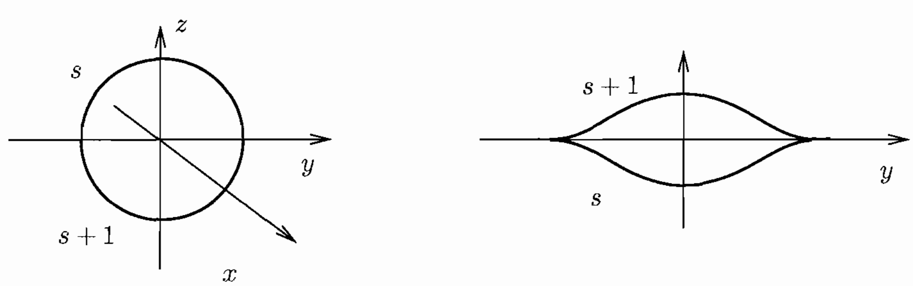

Figure III.5: The singular set $\Sigma\omega$ and its image in the case when $k = 2$.

Let U be an open neighbourhood of M.

Definition III.7. A map $f : U \to Q$ is called a *wrinkle* of index $s + \frac{1}{2}$ if f is equivalent to the restriction of the map ω on some open neighbourhood V, so that $D \subset V$.

When it is not confusing the term wrinkle will also be used to denote the singular set of f. More generally, a map $f : M \to Q$ is called a *wrinkled map* if there exists disjoint open neighbourhoods $U_1 \ldots, U_l \subset M$ so that $f|_{M \setminus U}(U = \bigcup_{i=1}^{l} U_i)$ is a submersion and for each $i = 1, \ldots, l$, the restriction $f|_{U_i}$ is a wrinkle.

III.3.5 Regularising a wrinkled map

In this section we describe a procedure for replacing a wrinkled map f with a submersion f'. We describe it here in the form of Lemma III.1. The submersion f' constructed in this lemma is known as the *regularisation* of the wrinkled map f.

Lemma III.1. *Let $f : M \to Q$ be a wrinkled map. Let $U_1 \dots, U_l \subset M$ be a collection of open neighbourhoods so that each $f|_{U_i}$ is a wrinkle and $\Sigma f \subset U = \bigcup_{i=1}^{l} U_i$. Then there is a smooth submersion $f' : M \to Q$ which agrees with f on $M \setminus U$.*

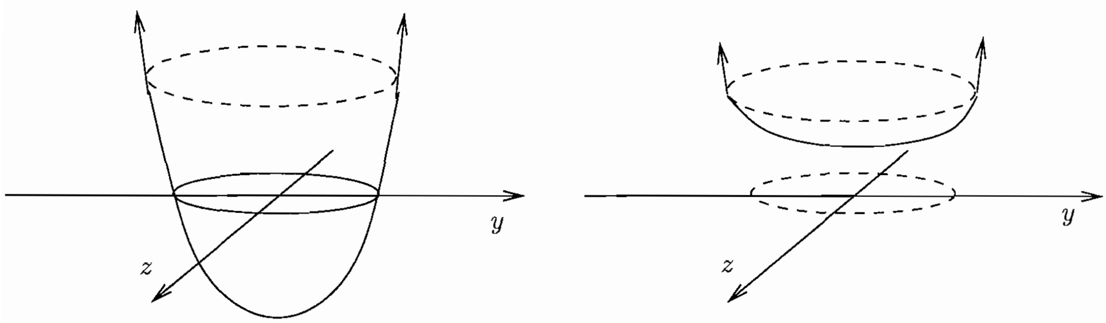

Figure III.6: The graphs of the term $3(z^2 + |y|^2 - 1)$ and its replacement $T(y, z, x)$ when $x = 0$

Proof. As wrinkles of f are isolated, it is enough to consider the case when f is the function ω defined above. Consider the differential, $d\omega : T(\mathbb{R}^n) \to T(\mathbb{R}^q)$. This map is degenerate when the element $3(z^2 + |y|^2 - 1)$ of the Jacobian matrix is 0. The differential $d\omega$ can be regularised by replacing this term with one which agrees with $3(z^2 + |y|^2 - 1)$ outside of a neighbourhood of D, but which is never zero; see Fig. III.6. Let $\alpha : \mathbb{R}^{n-k} \to [0, 1]$ and $\beta : \mathbb{R} \times \mathbb{R}^{k-1} \to [0, \infty)$

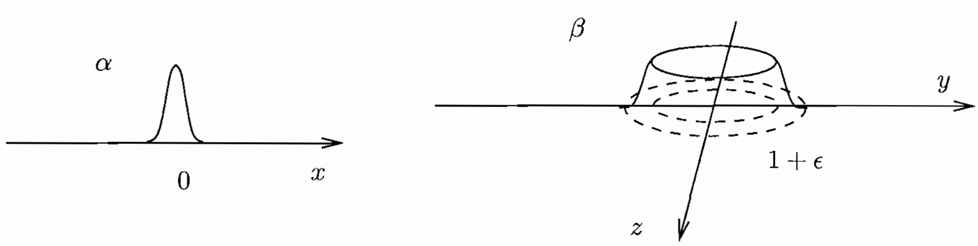

Figure III.7: The bump functions α and β

be bump functions of the type shown in Fig. III.7. In particular, $\alpha(0) = 1$, $\alpha(x) = \beta(z, y) = 0$

when $|x| > \epsilon, |(z,y)| > 1+\epsilon$, for some small constant $\epsilon > 0$, and $\beta(z,y) > |3(z^2+|y|^2-1)|$ when $z^2+y^2 \leq 1$. Define the function $\tau : \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k} \to \mathbb{R}$ by the formula

$$\tau(y,z,x) = 3(z^2+|y|^2-1) + \alpha(x)\beta(z,y).$$

The function $\tau(y,z,x) > 0$ for all (y,z,x) and agrees with $3(z^2+|y|^2-1)$ outside of $B_{\mathbb{R}^{n-k}}(0,\epsilon) \times B_{\mathbb{R}^k}(0,1+\epsilon)$. Replacing the term $3(z^2+|y|^2-1)$ with $\tau(y,z,x)$ in the Jacobian matrix, results in the desired "regularised" differential $\mathcal{R}(d\omega)$.

We can now define a new map ω' so that $\omega' = \omega$ outside $B_{\mathbb{R}^{n-k}}(0,\epsilon) \times B_{\mathbb{R}^k}(0,1+\epsilon)$ and $\Sigma\omega' = \emptyset$. This map is defined

$$\begin{aligned} \omega' : \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R}^{n-k} &\longrightarrow \mathbb{R}^{k-1} \times \mathbb{R} \\ (y,z,x) &\longmapsto \left(y, T(y,z,x) - \sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k} {x_i}^2\right), \end{aligned}$$

where the term $T(y,z,x)$ is given by the formula

$$T(y,z,x) = \int_0^z (3(t^2+|y|^2-1) + \alpha(x)\beta(t,y))dt.$$

This completes the proof. □

III.4 Regularising a Gromov-Lawson Cobordism

In this section we discuss a notion of regularisation for admissible Morse functions as well as a geometric analogue which applies to Gromov-Lawson cobordisms. In particular, we will prove a slightly stronger version of Theorem II.34. We begin by discussing some of the topological implications of conditions (a), (b) and (c) of that theorem.

III.4.1 Regularisation of admissible Morse functions

Let $f : W \to I$ be an admissible Morse function satisfying conditions (a), (b) and (c) of Theorem II.34. Recall these conditions are as follows.

(a) The function f has exactly 2 critical points w and z and $0 < f(w) < f(z) < 1$.

(b) The points w and z have Morse index $p+1$ and $p+2$ respectively.

(c) For each $t \in (f(w), f(z))$, the trajectory spheres $S^q_{t,+}(w)$ and $S^{p+1}_{t,-}(z)$ on the level set $f^{-1}(t)$, respectively emerging from the critical point w and converging toward the critical point z, intersect transversely as a single point.

Let $K^{p+1}_-(w) \subset f^{-1}([0, f(w)])$ denote the inward trajectory disks of w. This disk is bounded by a trajectory sphere which we denote $S^p_- \subset X_0$. Let $t \in (f(w), f(z))$. Emerging from w is an outward trajectory disk $K^{q+1}_{t,+}(w) \subset f^{-1}([f(w), t])$ which is bounded by an outward trajectory sphere $S^q_{t,+} \subset f^{-1}(t)$. Similarly, associated to z is an inward trajectory disk $K^{p+2}_{t,-}(w) \subset f^{-1}([t, f(z)])$ bounded by an inward trajectory sphere $S^{p+1}_{t,-} \subset f^{-1}(t)$ and an outward trajectory disk $K^q_+(z) \subset f^{-1}([f(z), 1])$ bounded by an outward trajectory sphere $S^{q-1}_+ \subset X_1$. We define a smooth *trajectory arc* $\gamma : [f(w), f(z)] \to W$ by the formula

$$\gamma(t) = \begin{cases} w, & \text{when } t = f(w) \\ S^q_{t,+} \cap S^{p+1}_{t,-}, & \text{when } t \in (f(w), f(z)) \\ z, & \text{when } t = f(z). \end{cases}$$

Condition (c) means that for each $t \in (f(w), f(z))$, the intersection $S^q_{t,+} \cap S^{p+1}_{t,-}$ is a single point and so this formula makes sense.

The embedded sphere S^p_- in $X \times \{0\}$ bounds a particular embedded disk which we denote D^{p+1}_-. This disk is determined as follows. Let $t \in (f(w), f(z))$. Each point in $S^{p+1}_{t,-} \setminus \gamma(t) \subset f^{-1}(t)$ is the end point of an integral curve of V beginning in X_0. Thus, applying in reverse the trajectory flow generated by V, to $S^{p+1}_{t,-} \setminus \gamma(t)$, specifies a diffeomorphism

$$S^{p+1}_{t,-} \setminus \gamma(t) \longrightarrow D^{p+1}_- \subset X_0.$$

The boundary of this disk is of course the inward trajectory sphere S^p_- which collapses to a point at w.

Let N_w and N_z denote respective tubular neighbourhoods in X_0 of the sphere S^p_- and the disk D^{p+1}_- with respect to the background metric m. We will assume that $N_w \subset N_z$. Note that N_z is topologically a disk and the radii of these neighbourhoods can be chosen to be arbitrarily small. Each point $x \in X_0 \setminus N_z$ is the starting point of a maximal integral curve $\psi_x : [0, 1] \to W$ of V, which ends in X_1. As before, this gives rise to an embedding $\psi : (X_0 \setminus N_z) \times I \to W$. We denote by U, the complement in W of the image of this embedding. The region U contains both critical points w and z, the trajectory disks $K_-^{p+1}(w)$ and $K_+^q(z)$ as well as the trajectory arc γ; see Fig. III.8. It is immediately clear that U is diffeomorphic to $N_z \times I$, however, the gradient-like vector field V has zeroes in U and so we cannot use its trajectory to construct an explicit diffeomorphism here in the way we can outside of U. It is possible *regularise* the admissible Morse function f,

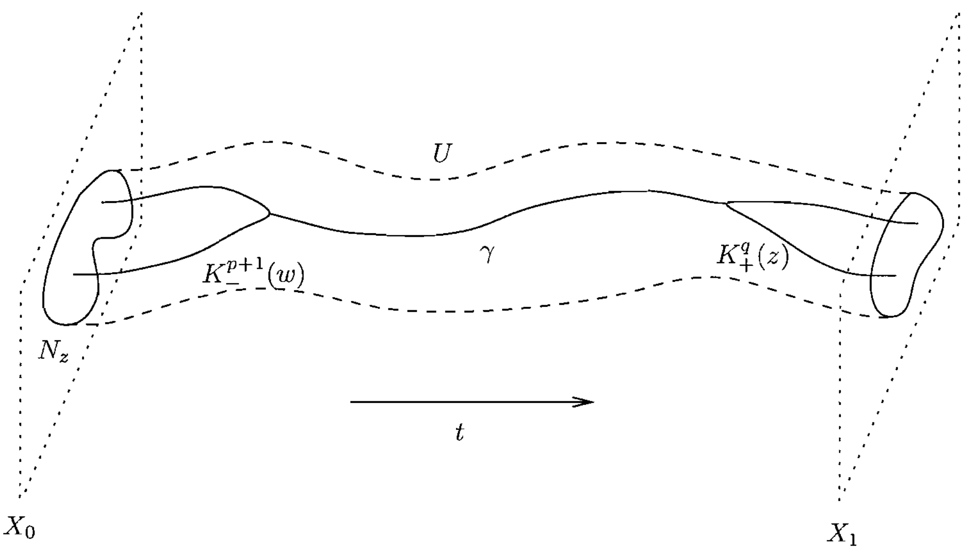

Figure III.8: The neighbourhood U, diffeomorphic to the cylinder $N_z \times I$

replacing it with an admissible Morse function f' which agrees with f on $W \setminus U$ and near X_0 and X_1, but which has no critical points. This is Theorem 5.4 of [30]. The key point, which requires much work to show, is that there is a coordinate neighbourhood $U' \subset U$, containing the trajectory arc γ, on which $f|_{U'}$ takes the form

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (z, x) &\longmapsto z^3 + 3yz - \sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-2} {x_i}^2, \end{aligned}$$

for some constant $y < 0$. This function can then be regularised as in the previous section. The effect of this regularisation on the gradient-like vector field V, replacing it with a non-vanishing vector field V' which agrees with V on $W \setminus U$ and near X_0 and X_1, is shown schematically in Fig. III.9. The map ψ can now be extended to a diffeomorphism $\psi : X \times I \to W$, satisfying

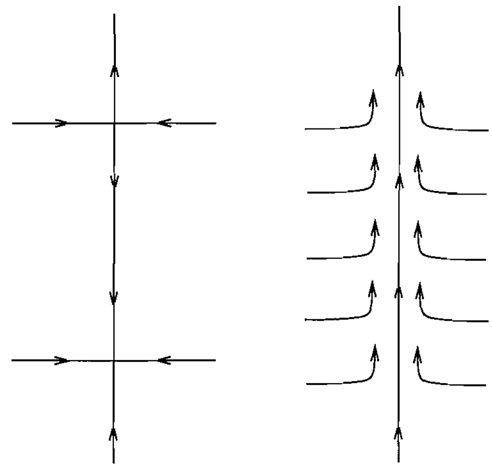

Figure III.9: The gradient-like vector fields V and V'

$f' \circ \psi(x,t) = t$ and providing a foliation of U with leaves which are diffeomorphic to N_z. We now turn our attention to an important geometric analogue of regularisation.

III.4.2 A geometric analogue of regularisation

We retain the notation of the previous section. As before, $f : W \to I$ is an admissible Morse function which satisfies conditions (a), (b) and (c) of Theorem II.34. Furthermore ψ, U, f', V' and ψ' are as defined above.

Let $g \in \mathcal{R}\text{iem}^+(X_0)$ and let $\bar{g} = \bar{g}(g,f)$ be a Gromov-Lawson cobordism on W. This metric is constructed so that on $W \setminus U$, $\bar{g}|_{W \setminus U} = g_0|_{X_0 \setminus N_z} + dt^2$, the t coordinate coming from the identification $\psi : (X_0 \setminus N_z) \times I \to W \setminus U$. Also, near X_0 and X_1, the metric has respectively the product structure $g + dt^2$ and $g'' + dt^2$, where g'' is obtained by two applications of the Gromov-Lawson construction to g. Inside U, and away from ∂W, the metric $\bar{g}$ has a more complicated structure. Later on we will wish to describe certain families of these metrics. It will then be useful that our metrics have a more regular structure in this region. This is the goal of Theorem III.2. Here we replace $\bar{g}$ with a "regularised" metric $\bar{g}'$. This metric agrees with $\bar{g}$ on $W \setminus U$ and also near ∂W, but takes the form of a particular warped product metric.

Theorem III.2. *Let $g \in \mathcal{R}\mathrm{iem}^+(X_0)$ and let $f : W \to I$ be a smooth map satisfying conditions (a), (b) and (c) of Theorem II.34. Let $\bar{g} = \bar{g}(g, f)$ be a Gromov-Lawson cobordism with respect to g and f. Finally, let c_0 and c_1 be constants satisfying $0 < c_0 < f(w) < f(z) < c_1 < 1$. There exists a diffeomorphism $\psi : X_0 \times I \to W$ and a psc-metric $\bar{g}'$ on W satisfying the following conditions.*

(i) *On $(W \setminus U) \cup (f^{-1}([0, c_0])) \cup (f^{-1}([c_1, 1]))$, the composition $f \circ \psi$ satisfies $f \circ \psi(x, t) = t$.*

(ii) *On $(W \setminus U) \cup (f^{-1}([0, c_0])) \cup (f^{-1}([c_1, 1]))$, the metric $\bar{g}'$ satisfies $\bar{g}' = \bar{g}$.*

(iii) *There exists a smooth family of psc-metrics h_t, for $t \in I$ in $\mathcal{R}\mathrm{iem}^+(X_0)$ and a smooth function $\alpha : [0, 1] \to [1, \infty)$, so that $\psi^*(\bar{g}') = h_t + \alpha(t)^2 dt^2$.*

Proof. The diffeomorphism ψ is precisely the one described in the previous section. As explained above, the first critical point of f, w, determines a p-dimensional embedded surgery sphere S^p_- in $X \times \{0\}$ bounding a disk D^{p+1}_- which is determined by the second critical point z. More precisely, as we follow the trajectory with respect to V, of D^{p+1}_-, the effect of passing the first critical point w is to collapse the boundary of this disk. This in turn gives rise to the inward trajectory sphere $S^{p+1}_{t,+} \subset f^{-1}(t)$, with $t \in (f(w), f(z))$, of the second critical point z. Recall that N_w and N_z denote respective tubular neighbourhoods of S^p_- and D^{p+1}_- in X_0, with $N_w \subset N_z$.

The restriction of the metric $\bar{g} = \bar{g}(g, f)$ to the level sets $f = 0$, $f = c$ and $f = 1$, where $f(w) < c < f(z)$, is shown schematically in Fig. III.10. The induced metrics are denoted g, g' and g'' respectively. The constant c can always be chosen so that the metric g' is the metric obtained by a single application of the Gromov-Lawson construction with respect to the sphere S^p. Thus, outside of the neighbourhood N_w the metric g' is precisely the original metric $g|_{X_0 \setminus N_w}$. Finally, the metric g'' is obtained by application of the Gromov-Lawson construction to the metric g' with respect to the trajectory sphere S^{p+1}_-. The restriction $g''|_{X \setminus N_z} = g|_{X \setminus N_z}$.

In Theorem II.34, we construct a smooth isotopy $g_s, s \in [c_0, c_1]$, in the space $\mathcal{R}\mathrm{iem}^+(X)$, which connects the metrics g and g''. That is, $g_{c_0} = g$ and $g_{c_1} = g''$. Moreover, this isotopy fixes the metric g on $X \setminus N_z$, i.e. $g_s|_{(X \setminus N_z)} = g|_{(X \setminus N_z)}$, for all $s \in I$. By Lemma II.1, there exists a smooth bump function $\nu : [0, b] \to [0, 1]$ of the type shown in Fig. III.11 so that the metric $g_{\nu(l)} + dl^2$ is a psc-metric on $X \times [0, b]$. In particular, $\nu = 0$ on $[0, k_1]$ and $\nu = 1$ on $[k_2, b]$. The metric $g_{\nu(l)} + dl^2$ pulls back to a psc-metric $h_t + \dot{\mu}(t)^2 dt^2$ on $X \times [c_0, c_1]$, where $\mu : [c_0, c_1] \to [0, b]$ is the smooth map shown in Fig. III.11 and $h_t = g_{\nu(\mu(t))}$. The function μ can be chosen so that

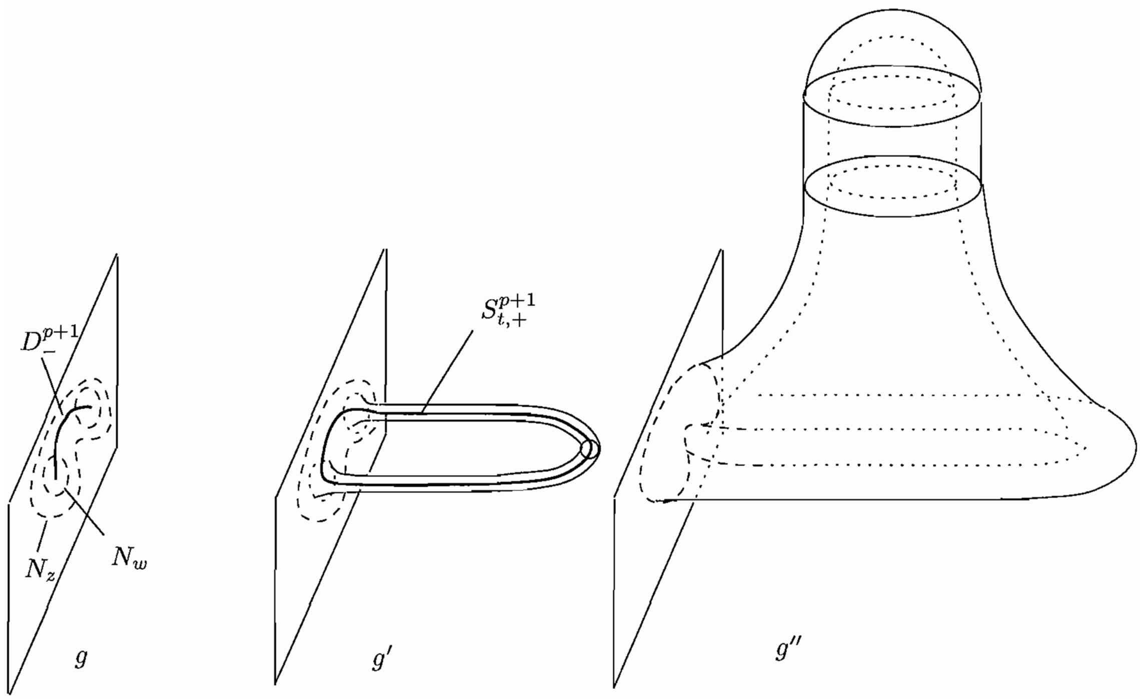

Figure III.10: The metrics g, g' and g'' induced by restriction of $\bar{g}$ to level sets $f = 0$, $f = c$ and $f = 1$

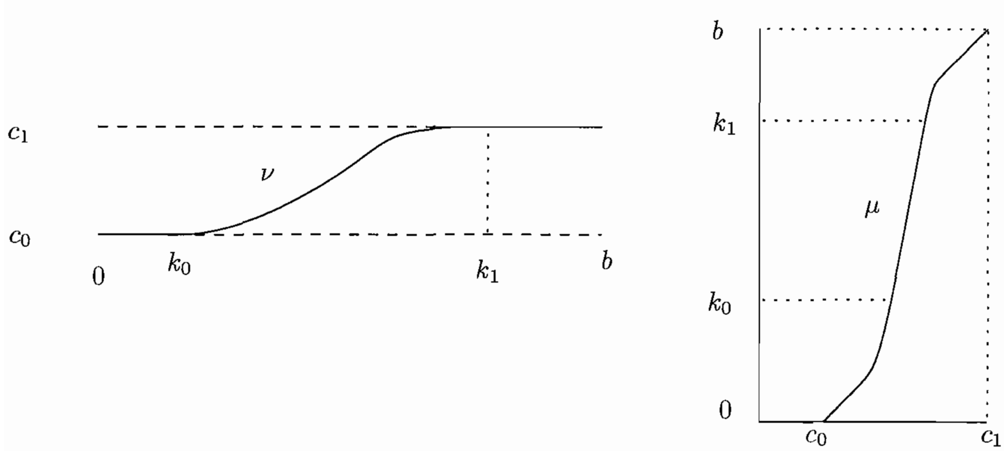

Figure III.11: The functions ν and μ

$\dot{\mu} = 1$ when u is near c_0 and c_1, and so this metric is a product near the boundary. Thus, it extends smoothly over $X \times I$, giving rise to the the metric $\bar{h}$, defined

$$\bar{h} = \begin{cases} g + dt^2, & \text{on } X \times [0, c_0] \\ h_t + \dot{\mu}(t)^2 dt^2, & \text{on } X \times [c_0, c_1] \\ g'' + dt^2, & \text{on } X \times [c_1, 1]. \end{cases}$$

The metric $\bar{h}$ can now be pulled back onto W via the diffeomorphism ψ^{-1}, to obtain the desired metric $\bar{g}'$. □

It is not difficult to generalise this notion of regularisation to a Gromov-Lawson cobordism arising from an admissible Morse function with many critical points. Two critical points w and z of f are said to be in *cancelling position* if they satisfy conditions (a), (b) and (c) of Theorem II.34. We descibe as a *cancelling pair*, any two critical points of f which can be moved into cancelling position by a smooth isotopy of f in the space of admissible Morse functions. Now suppose f is an admissible Morse function so that every cancelling pair of critical points is in cancelling position. Denote these cancelling pairs $\{(w_i, z_i)\}_{i=1}^{l}$ and denote by γ_i, the trajectory arc connecting w_i to z_i. Let $\psi_i : D^n \times [a_i, b_i] \to W$ denote a family of embeddings which satisfy the following.

(i) The images of the maps ψ_i are disjoint.

(ii) Each trajectory arc γ_i is contained inside the image of ψ_i.

(iii) The constants a_i and b_i are chosen so that $0 < a_i < f(w_i) < f(z_i) < b_i < 1$ and so that $f^{-1}([a_i, f(w_i)))$ and $f^{-1}(f(z_i), b_i]$ contain no critical points.

(iv) Near $(D^n \times \{a_i\}) \cup (D^n \times \{b_i\}) \cup (\partial D^n \times I)$, the composition $f \circ \psi_i$ is projection onto $[a_i, b_i]$.

Corollary III.3. *Let $\bar{g}(g_0, f)$ be a Gromov-Lawson cobordism with respect to an admissible Morse function f and a psc-metric g_0 on X_0. Suppose also that all cancelling pairs of critical points of f are in cancelling position. Then there is a psc-metric $\bar{g}' = \bar{g}'(g_0, f)$ on W and a collection of embeddings $\psi_i : D^n \times [a_i, b_i] \to W$ satisfying conditions (i), (ii), (iii) and (iv) above so that:*

(1) *The metrics $\bar{g}$ and $\bar{g}'$ agree on $W \setminus \bigcup_i \psi_i(D^n \times [a_i, b_i])$*

(2) *The metric $\psi_i^*(\bar{g}'|_{\psi_i(D^n \times [a_i, b_i])})$ is a warped product $h_t^i + \alpha_i(t)^2 dt^2$ where each $h_t^i \in \mathcal{R}\text{iem}^+(D^n)$ and $\alpha_i : [a_i, b_i] \to [1, \infty)$ is the constant function 1 near a_i and b_i.*

Proof. By Theorem 5.4 of [30], the embeddings ψ_i may be chosen to allow for a regularisation of the function f. In other words, the admissible Morse function f can be replaced by an admissible Morse function f' which satisfies the following conditions.

(i) On $W \setminus \bigcup_i \psi_i(D^n \times [a_i, b_i])$, $f' = f$.

(ii) For each i, the composition $f' \circ \psi_i$ is projection onto $[a_i, b_i] \subset I$.

The proof then follows by application of Theorem III.2 inside each neighbourhood $\psi_i(D^n \times [a_i, b_i])$.

□

The metric $\bar{g}'$ constructed in this corollary will be called a *regularised Gromov-Lawson cobordism* with respect to g_0 and f.

III.4.3 Arc-length dependent regularisation

We will now describe a slight variation of the construction from Corollary III.3 which will be of use when we come to prove our main theorem. Let $f : W \to I$ be an admissible Morse function satisfying conditions (a), (b) and (c) of Theorem II.34. Let L denote the length of the trajectory arc γ connecting the critical points w and z, with respect to the reference metric m. Now consider the metric $\bar{g}'$ obtained by Theorem III.2 with respect to f and a psc-metric $g_0 \in \mathcal{R}\mathrm{iem}^+(X_0)$. On $f^{-1}([c_0, c_1]) \cap U$, this metric takes the form

$$\bar{g}' = g_{\nu(\mu(t))} + \dot{\mu}(t)^2 dt^2,$$

where $t \in [c_0, c_1]$ is the coordinate coming from the regularised trajectory flow and ν and μ are the functions defined in the proof of Theorem III.2 and shown in Fig. III.11. Let $\xi : [0, \infty) \to [0, 1]$ be a standard cut-off function so that for some interval $[\epsilon_0, \epsilon_1]$, $\xi(s) = 0$ when $s \leq \epsilon_0$ and $\xi(s) = 1$ when $s \geq \epsilon_1$. Let $\bar{h}(L)$ denote the metric defined

$$\bar{h}(L) = \begin{cases} g_0 + dt^2, & \text{on } X \times [0, c_0] \\ g_{\xi(L)\nu(\mu(t))} + \dot{\mu}(t)^2 dt^2, & \text{on } X \times [c_0, c_1] \\ g_{\xi(L)c_1} + dt^2, & \text{on } X \times [c_1, 1]. \end{cases}$$

This metric can be pulled back to a psc-metric on W by the regularised trajectory diffeomorphism $\psi' : X_0 \times I \to W$ as in Theorem III.2. We denote the resulting metric $\bar{g}'(L)$.

Lemma III.4. *The metric $\bar{g}'(L)$ has positive scalar curvature.*

Proof. By construction, the metric $g_{\nu(\mu(t))} + \dot{\mu}(t)^2 dt^2$ has positive scalar curvature, although it is worth recalling that the positivity of the scalar curvature depends upon the fact that the function $\nu : [0, b] \to [c_0, c_1]$ has been chosen so that $|\dot{\nu}|$ and $|\ddot{\nu}|$ are small. It is therefore enough to ensure that for each L, $|\frac{d}{dt}(\xi(L)\nu)| < |\frac{d}{dt}\nu|$ and $|\frac{d^2}{dt^2}(\xi(L)\nu)| < |\frac{d^2}{dt^2}\nu|$. This follows from the fact that $0 < c_0 < \xi(L) < c_1 < 1$. □

Thus, when L is very small, the metric $\bar{g}'(L)$ is just the standard product $g_0 + dt^2$. As the length L varies from ϵ_0 to ϵ_1, we get a smooth transition through positive scalar curvature metrics back to the metric $\bar{g}'$. It is important to realise however, that the replacement of the metric $\bar{g}'$ with the metric $\bar{g}'(L)$ changes the metric on $f^{-1}([c_1, 1])$. This is unlike the construction of $\bar{g}'$ from $\bar{g}$, where the metric was only altered locally; see Fig. III.12.

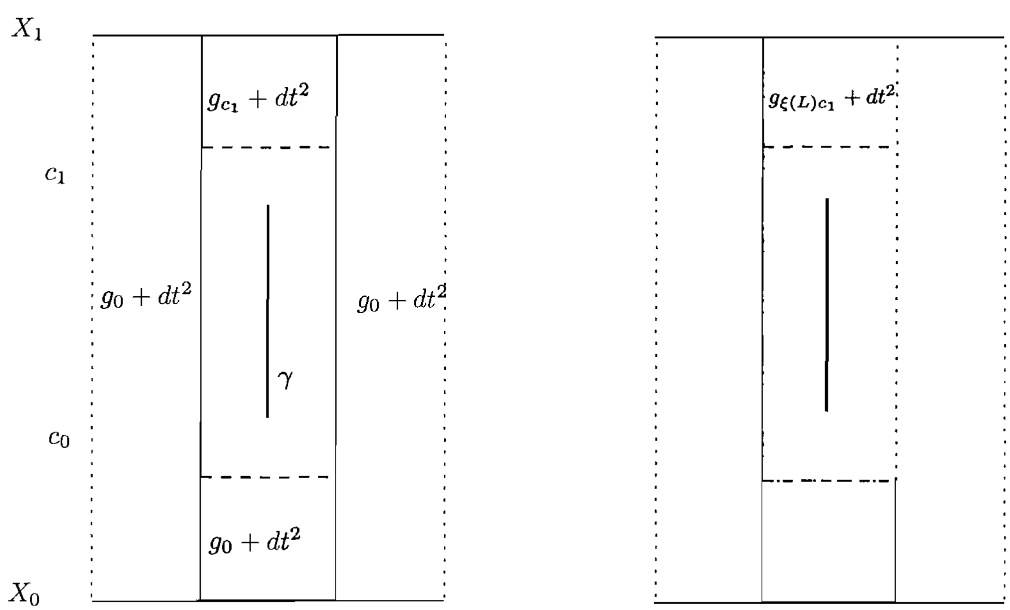

Figure III.12: The metrics $\bar{g}'$ and $\bar{g}'(L)$ with the shaded region representing U and the darkly shaded region denoting where these metrics differ

III.5 Families of Regularised Gromov-Lawson Cobordisms

In this section we will prove our main technical theorem.

III.5.1 Admissible wrinkled maps

Let W^{n+1} be as before and let E^{n+k+1} and Q^k be a pair of smooth compact manifolds of dimension $n+1+k$ and k respectively. The manifolds E and Q form part of a smooth fibre bundle with fibre W, arising from a submersion $\pi : E \to Q$. We will assume also that the boundary of E, ∂E, contains a pair of disjoint smooth submanifolds E_0 and E_1. The restriction of π to these submanifolds is denoted π_0 and π_1 respectively. These maps are still submersions onto Q and give rise to a pair of smooth subbundles of π with respective fibres $X_0, X_1 \subset W$. These form the commutative diagram represented in Fig. III.13.

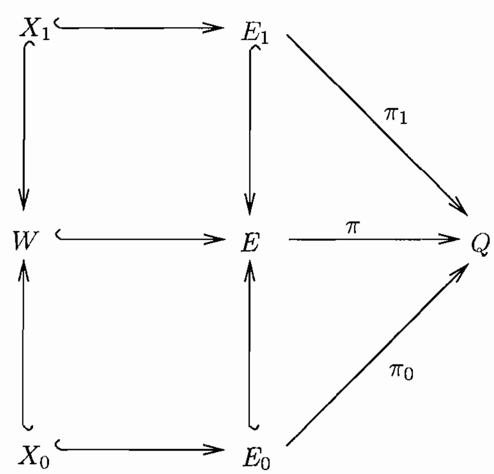

Figure III.13: The smooth fibre bundle π and subbundles π_i where $i = 0, 1$.

The union of tangent bundles to $T(\pi^{-1}(y))$ over $y \in Q$ forms a smooth subbundle of TE, the tangent bundle to E. This subbundle is denoted $Vert$.

Definition III.8. A smooth map $f : E \to Q \times I$ is said to be *moderate* if it satisfies the following conditions.

(i) The diagram shown in Fig. III.14 commutes, where p_1 is projection on the first factor.

(ii) The pre-images $f^{-1}(Q \times \{0\})$ and $f^{-1}(Q \times \{1\})$ are the submanifolds E_0 and E_1 respectively.

(iii) The singular set Σf is contained entirely in $E \setminus (E_0 \sqcup E_1)$ and is a union of folds and wrinkles.

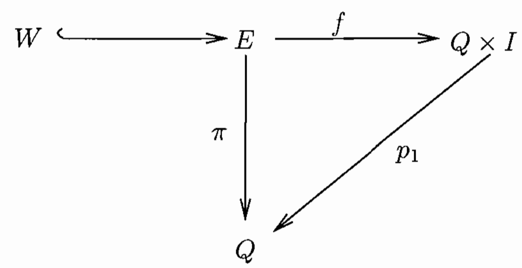

Figure III.14: The moderate map f

(iv) For each $y \in Q$, the restriction $f|_{\pi^{-1}(y)}$ is a generalised Morse function whose critical points have index $\leq n-2$.

Let V be a vector field on E. We say that V is *gradient-like* with respect to f and m if for each $y \in Q$, the restriction $V|_{\pi^{-1}(y)}$ is gradient-like with respect to $f|_{\pi^{-1}(y)}$ and $m|_{\pi^{-1}(y)}$.

Definition III.9. An *admissible wrinkled map* is a triple $f = (f, m, V)$ where f is a moderate map with respect to the submersion $\pi : E \to Q$, m is a Riemannian metric on E and V is a gradient-like vector field on E with respect to f and m.

Example III.5. Let $\tau : TS^4 \to S^4$ denote the tangent bundle to the sphere S^4. Equipping S^4 with a Riemannian metric allows us to define an annular bundle $E = D_1(TS^4) \setminus D_0(TS^4) \to S^4$, where $D_0(TS^4) \subset D_1(TS^4)$ are disk bundles. The total space E can now be thought of as a product of sphere bundles $S_0(TS^4) \times I$ and we may define a function f on this space as $f(x,t) = (\tau(x), t)$. In this case, $\Sigma f = \emptyset$. On any local trivialiation, $\tau^{-1}(D^4) \cong D^4 \times S^4 \times I$ where $D^4 \subset S^4$, it is easy to replace the function f with one which contains a wrinkle inside $\tau^{-1}(D^4)$ and which agrees with f outside of this neighbourhood.

For a more interesting example, where f has only fold singuarities, see section 5.a of [11]. A minor modification to the example there results in a non-trivial $S^n \times I$ bundle E over a sphere $Q = S^k$ with f restricting on each fibre as a Morse function with a pair of cancelling critical points.

III.5.2 The main theorem of Part Two

We are now in a position to state our main theorem. This will allow us to describe a family of regularised Gromov-Lawson cobordisms arising from admissible Morse functions with varying numbers of critical points.

Theorem III.6. *Let f be an admissible wrinkled map with respect to the submersion $\pi : E \to Q$. Let $g_0 : Q \to \mathcal{R}\mathrm{iem}^+(X_0)$ be a smooth map parameterising a compact family of psc-metrics on X_0. Then there is a metric G on the total space E which, for each $y \in Q$, restricts on the fibre $\pi^{-1}(y)$ to a regularised Gromov-Lawson cobordism $\bar{g}'(g_0(y), f|_{\pi^{-1}(y)})$. In the case when the bundle $\pi : E \to Q$ is trivial, there exists a smooth map*

$$Q \longrightarrow \mathcal{R}\mathrm{iem}^+(W, \partial W)$$
$$y \longmapsto \bar{g}'(y),$$

where each $\bar{g}'(y)$ is a regularised Gromov-Lawson cobordism.

The metric G will be constructed in a method which is quite similar to that employed in the proof of Theorem II.23. We begin by equipping the boundary component E_0 with a particular Riemannian metric G_0. Using the trajectory flow of the gradient-like vector field V, we extend G_0 as a product metric away from critical points of f. Near critical points of f, some modification must be made. Roughly speaking however, the entire construction goes through in such a way that the restriction to any fibre, is the construction of Corollary III.3.

Before beginning the proof, we need to make some observations about Σf. The singular set Σf forms a smooth k-dimensional submanifold of E, with possibly many path components; see Fig. III.15. These path components are either folds or wrinkles of f. The condition that $f|_{\pi^{-1}(y)}$ is a generalised Morse function, for all $y \in Q$, puts some further restrictions on the types of singularities that can occur. Near any fold singularity, f is equivalent to the map

$$\begin{aligned} \mathbb{R}^k \times \mathbb{R}^{n+1} &\longrightarrow \mathbb{R}^k \times \mathbb{R} \\ (y, x) &\longmapsto \left(y, -\sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k+1} {x_i}^2\right), \end{aligned} \tag{III.5.1}$$

for some $s \in \{0, 1, \ldots, n-2\}$. The index s will be consistent throughout any particular fold of f and so such a fold may be regarded as an *s-fold.* Each wrinkle is contained in a neighbourhood in which f is equivalent to the map

$$\begin{aligned} \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^k \times \mathbb{R} \\ (y, z, x) &\longmapsto \left(y, z, z^3 + y_1 z - \sum_{i=1}^{s} {x_i}^2 + \sum_{i=s+1}^{n-k+1} {x_i}^2 \right). \end{aligned}$$

where in this case $s \in \{0, 1, \ldots, n-3\}$. In both cases, regions parametrised by the $\mathbb{R}^k$ factor are mapped diffeomorphically onto their images in Q, by π.

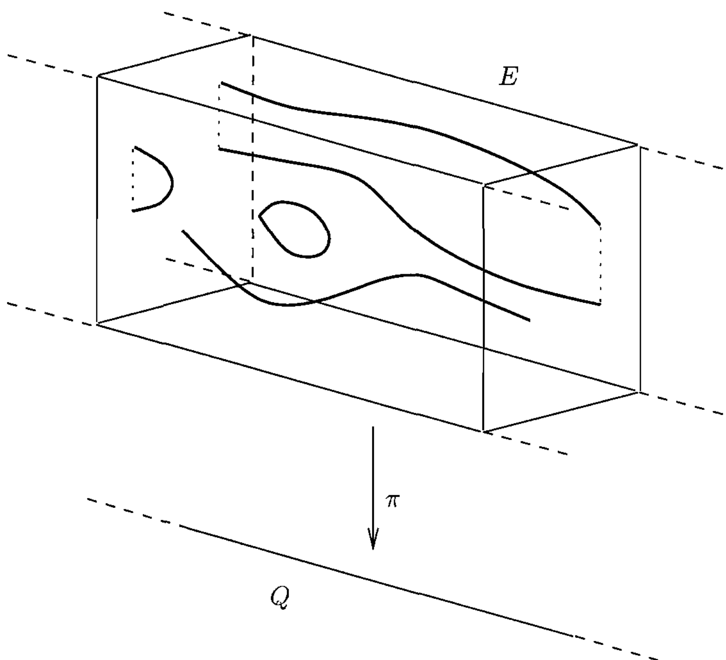

Figure III.15: The singular set Σf

The background Riemannian metric m on E gives a reduction of the structure group on $Vert$ to $SO(n+1)$. There is a further reduction of this structure group on folds of f. Suppose $F \subset \Sigma f$ is a fold of f. In other words, near any point in F, f is locally equivalent to the map (III.5.1). The fold F is thus a smooth k-dimensional submanifold of E, and each point $w \in F$ is an index s Morse singularity of the function $f|_{\pi^{-1}(\pi(w))}$. In keeping with our earlier notation, we will assume that $s = p+1$ and that $p+q+1 = n$. Associated to each tangent space $Vert_w = T_w \pi^{-1}(\pi(w))$ of $w \in F$ is an orthogonal splitting (with respect to m) of the tangent space into positive and negative eigenspaces of the Hessian $d^2 f_w$. We denote these spaces $Vert_w^+$ and $Vert_w^-$. They have respective dimensions $p+1$ and $q+1$ and give the restriction of $Vert$ to the fold F the structure of an $SO(p+1) \times SO(q+1)$-bundle.

III.5.3 *The proof of the main Theorem*

We now proceed with the proof of Theorem III.6.

Proof. Let π_0 denote the restriction of π to $E_0 = f^{-1}(Q \times \{0\})$. Recall this is a subbundle of E with fibre X_0. Let H be an integrable horizontal distribution for the submersion $\pi : E \rightarrow Q$. The restriction of H to E_0 is an integrable horizontal distribution for the submersion π_0, which we denote H_0. We begin by giving the bundle π_0, the structure of a Riemannian submersion $\pi_0 : (E_0, G_0) \rightarrow (Q, m_Q)$. Here G_0 is the unique submersion metric with respect to m_Q, the distribution H_0 and the smooth family of fibre metrics specified by the map $g_0 : Q \rightarrow \mathcal{R}\text{iem}^+(X_0)$; see chaper 9 of [2] for details. Since H_0 is an integrable distribution, we get that G_0 is (locally at least) isometric to $m_Q + g_0(y)$; see section 9.26 of [2].

Now consider integral curves of the gradient-like vector field V starting at E_0. As Σf is contained entirely in the interior of E, all of these integral curves run for some time and so we may specify a diffeomorphism

$$\begin{aligned} \phi_0 : E_0 \times [0, \delta_0] &\longrightarrow f^{-1}(Q \times [0, \delta_0]) \\ (w, t) &\longmapsto (h_w(t)), \end{aligned}$$

for some $\delta_0 \in (0,1)$, where h_w is the integral curve beginning at w. In particular, $f \circ \phi_0$ is projection onto $[0, \delta_0]$. Each fibre metric $g_0(y)$ on $\pi_0^{-1}(y)$ can now be extended fibrewise as a product metric $g_0(y) + dt^2$ along $\pi^{-1}(y) \cap f^{-1}(Q \times [0, \delta_0])$, in the manner of the proof of Theorem II.23. The restriction of H to $f^{-1}(Q \times [0, \delta_0])$ allows us to glue these fibre metrics together and so extend G_0 as a submersion metric over $f^{-1}(Q \times [0, \delta_0])$. We may continue extending G_0 over E in this way until we encounter elements of Σf and can no longer extend some of our integral curves. At this stage we must adapt our construction. There are two cases to consider here, either we run into a fold of f or we encounter a wrinkle.

III.5.4 *Case 1: Extending the metric past a fold of f*

Suppose that for some $c \in (0, 1)$, the level set $f^{-1}(Q \times \{c\})$ contains a fold F. It could contain more than one fold or even a cusp, but as folds and wrinkles are disjoint, it is enough to consider the case when this level set contains a single fold. Let δ_c be chosen sufficiently small so that $f^{-1}(Q \times ([c-\delta_c, c) \cup (c, c+\delta_c]))$ contains no critical points. We will assume inductively that we have extended the metric G_0 to a metric $G_{c-\delta_c}$ on $f^{-1}([0, c-\delta_c])$ so that for each $y \in Q$, the metric induced by $G_{c-\delta_c}$ on $\pi^{-1}(y) \cap f^{-1}(Q \times [0, c-\delta_c])$ is a psc-metric and is a product near $\pi^{-1}(y) \cap f^{-1}(Q \times \{c-\delta_c\})$. Our goal is to construct a metric $G_{c+\delta_c}$ on $f^{-1}(Q \times [0, c+\delta_c])$, so that on each fibre $\pi^{-1}(y) \cap f^{-1}(Q \times [0, c+\delta_c])$, the induced metric has positive scalar curvature and is a product near the boundary.

Fibrewise, this is precisely the situation dealt with in Theorem II.23. Thus, on any fibre we can choose a Morse coordinate neighbourhood of the critical point w and perform a parametrised version of the Gromov-Lawson construcion on this neighbourhood to extend the metric past the critical point, exactly as we did in Theorem II.23. This works perfectly well for a single Morse critical point. For a family of Morse critical points however, we must ensure compatibility of our construction over the entire family.

It is important to point out that the construction of Theorem II.23 depends specifically on an orthogonal decomposition of the plane $\mathbb{R}^{n+1}$ into $\mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ and a diffeomorphism of $\mathbb{R}^{n+1}$ onto a neighbourhood of the critical point so that $\mathbb{R}^{p+1}$ and $\mathbb{R}^{q+1}$ parametrise the respective inward and outward trajectory disks near w. The construction itself is $SO(p+1) \times SO(q+1)$ symmetric with respect to this decomposition. Thus, to perform this construction fibrewise over all critical points in the fold F we must establish a canonical way of assigning a smoothly varying diffeomorphism of the type just described for each $w \in F$. This will be done with the aid of the exponential map (with respct to m) near F.

Denote by $Vert(F)$, the restriction of the vertical bundle $Vert$ to the fold F. For some $\epsilon_c > 0$, let $D(Vert(F)) \subset Vert$ denote the disk bundle of radius ϵ_c with respect to the background metric m. Provided ϵ_c is small enough, the exponential map exp_m embeds $DVert(F)$ into E. We denote by N the tubular neighbourhood of F that is the image of this embedding. Let $w \in F$ and let $N_w = \exp_m(D_w(Vert)) \subset \pi^{-1}(\pi(w))$. We will now make some adjustments to the metric m and the function f inside this tubular neighbourhood. These adjustments should be thought of as standardising m and f near the fold.

The decomposition of $Vert_w$ into negative and positive eigenspaces of the Hessian: $Vert_w^-$

and $Vert_w^+$, as w varies over F can be thought of as a smooth map

$$Q \longrightarrow \frac{SO(n+1)}{SO(p+1) \times SO(q+1)}.$$

For a definition of this map, see [23]. In turn this gives a smooth family of isomorphisms $T_wW \to \mathbb{R}^{n+1} \cong \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$, which identify each $Vert_w^-$ with $\mathbb{R}^{p+1}$ and each $Vert_w^+$ with $\mathbb{R}^{q+1}$, as w varies over F. Pulling back the Euclidean metric on $\mathbb{R}^{n+1}$ results in a smooth family of Euclidean metrics on the fibres $Vert_w$, for which the subspaces $Vert_w^-$ and $Vert_w^+$ are orthogonal. The distribution H (along with the base metric $m|_F$) allows us to glue these fibre metrics together to construct a submersion metric on the total space $Vert(F)$. Using the exponential map with respect to the original metric m on the disk bundle $DVert(F)$, we can pull this metric back to the tubular neighbourhood N. By way of a partition of unity, this metric can then be extended over the rest of E as the original metric m (with analogous constructions taking place near other folds of F). Abusing notation, we will retain the name m for this standardised background metric.

Let $w \in F$. Contained in N_w are a pair of trajectory disks D_w^{p+1} and D_w^{q+1} arising from the vector field $V|_{\pi^{-1}(\pi(w))}$ and intersecting orthogonally at w. We may assume that δ_c is sufficiently small that $f^{-1}(c-\delta_c)$ is contained entirely in the interior of N. Thus, on each neighbourhood N_w, the psc-metric induced by $G_{c-\delta_c}$ is a product metric defined on a region (diffeomorphic to $S^{p+1} \times D^{q+1}$) below the critical level exactly as in Theorem II.23; see Fig. III.16.

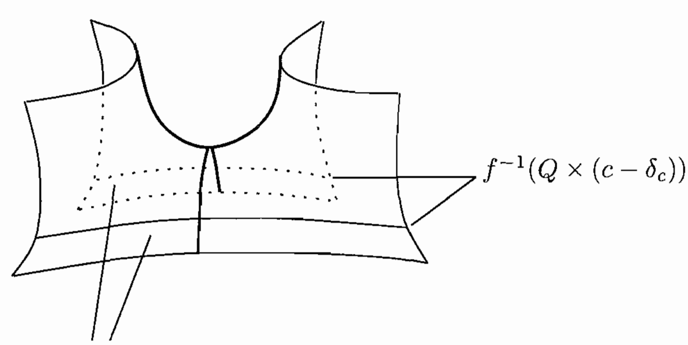

Figure III.16: The neighbourhood $U_w \subset \pi^{-1}(\pi(w))$ containing the Morse singularity w of $f|_{\pi^{-1}(\pi(w))}$

Using the exponential map, we pull back the metric $G_{c-\delta_c}$ on $N \cap f^{-1}[0, c-c_\delta]$ to the bundle $D(Vert(F_{p+1}))$. We will now work entirely inside $D(Vert(F_{p+1}))$. The inverse exponential map embeds D_w^{p+1} and D_w^{q+1} into $D_w(Vert(F_{p+1}))$. Abusing notation, we will retain the names D_w^{p+1} and D_w^{q+1} for the image disks. Contained inside the vertical tangent disk $D_w(Vert(F))$ are a pair of eigen-disks of the Hessian $d^2 f_w$, $DVert_w^- = Vert_w^- \cap D(Vert_w(F_{p+1}))$ and $DVert_w^+ = Vert_w^+ \cap D(Vert_w(F_{p+1}))$. These are restrictions of the negative and positive eigenspaces of $d^2 f_w$. We will now compare $DVert_w^+$ and $DVert_w^-$ with D_w^{p+1} and D_w^{q+1} near w; see Fig. III.17 and Fig. III.18.

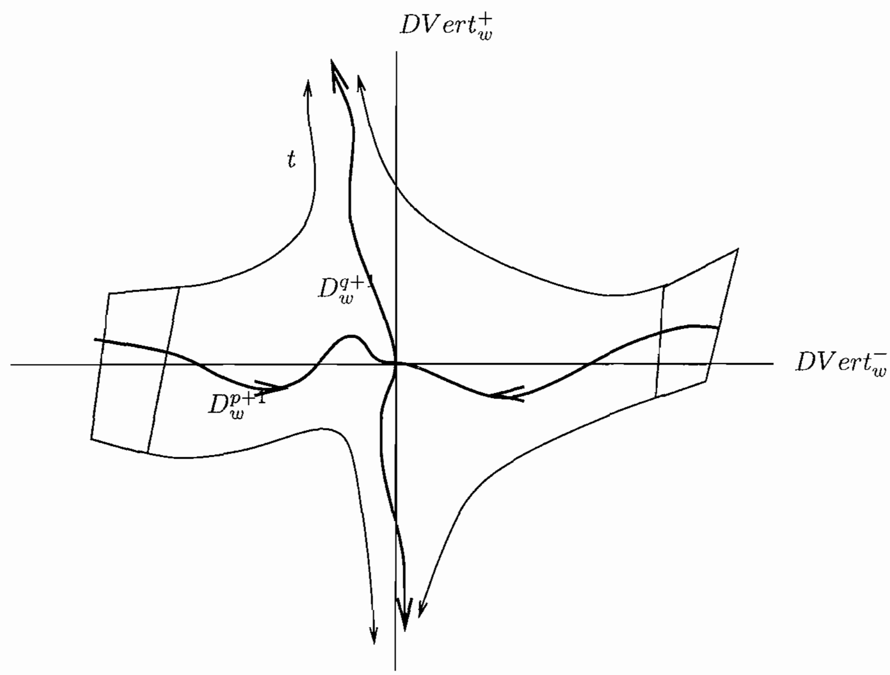

Figure III.17: The images of the trajectory disks D_w^{p+1} and D_w^{q+1} in $D_w Vert(F)$ after application of the inverse exponential map

For each $w \in F$, the trajectory disks D_w^{p+1} and D_w^{q+1} intersect orthogonally at the origin. Furthermore, (D_w^{p+1}) intersects tangentially with $DVert_w^-$, as does (D_w^{p+1}) with $DVert_w^+$. Thus, provided ϵ_c is chosen sufficiently small, inside $D_w Vert(F)$ and for all $w \in F$, D_w^{p+1} and D_w^{p+1} are the graphs of smooth functions on $Vert_w^-$ and $Vert_w^+$ repectively. The function f can now be easily perturbed near F so that inside the disk bundle $DVert(F)$, the eigen-disks $DVert_w^-$ and $DVert_w^+$ agree with the respective trajectory disks D_w^{p+1} and D_w^{q+1}, for all $w \in F$; see Fig. III.19.

This gives to each $w \in F$, the desired association of disk neighbourhoods $D_w Vert(F)$, each with an orthogonal splitting $D_w Vert(F) = DVert_w^- \times DVert_w^+$, varying smoothly over w.

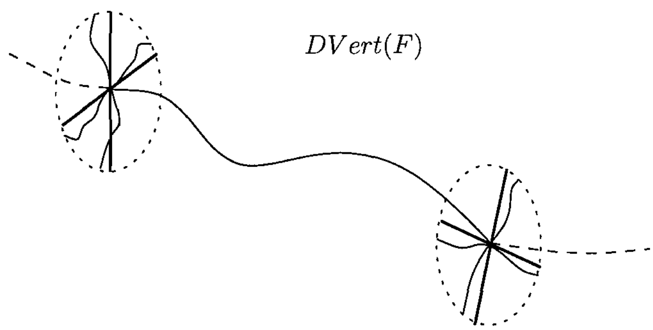

Figure III.18: Comparing the trajectory disks with the eigen-disks (heavy) in $D_wVert(F)$ as w varies over F

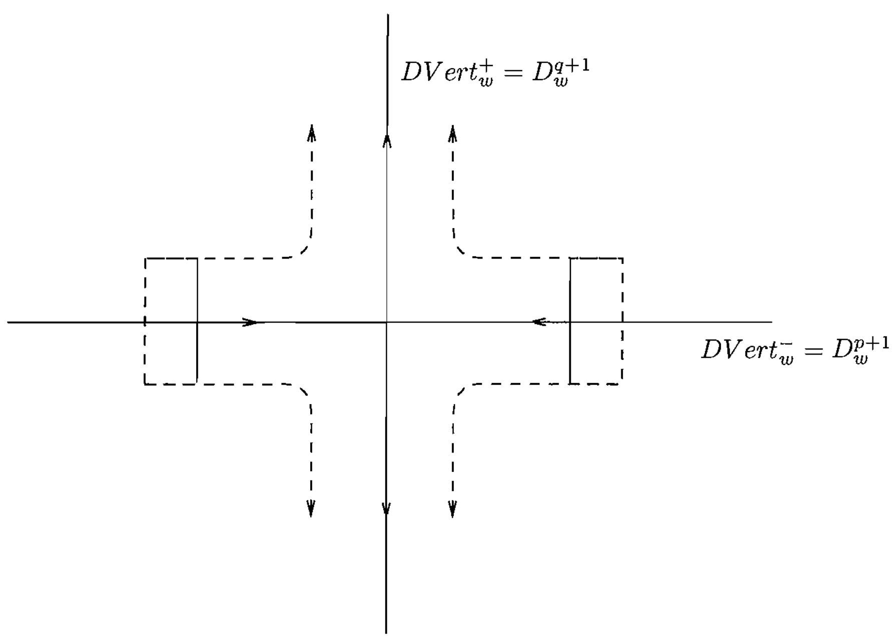

Figure III.19: The shaded region denotes the region of the fibre $D_wVert(F)$ on which the induced metric is defined.

Extending the metric fibrewise in the manner of Theorem II.23 and pulling back via the exponential map, gives a smooth family of fibre metrics, which, with respect to the distribution H and the base metric m_Q, give the desired submersion metric on $f^{-1}(Q \times [0, c+\delta_c])$.

III.5.5 *Case 2: Extending the metric past a wrinkle of f*

We will assume that $c_0 \in (0,1)$ is so that all wrinkles of f lie outside of $f^{-1}(Q \times [0, c_0])$ and so that $f^{-1}(Q \times \{c_0\})$ contains no critical points. From Case 1, we can construct a metric G_{c_0} on $f^{-1}(Q \times [0, c_0])$ so that the metric induced on fibres has positive scalar curvature and product structure near the boundary. Suppose P is a wrinkle of f which is contained in the interior of $f^{-1}(Q \times [c_0, c_1])$ for some $c_1 \in (c_0, 1)$. We wish to extend G_{c_0} to a metric G_{c_1} on $f^{-1}(Q \times [c_0, c_1])$ so that the once again, the induced metric on fibres has positive scalar curvature and is a product near the boundary. Away from the wrinkle, we can extend this metric as a standard product in the usual way. We will focus our attention therefore, on extending this metric near P.

Recall that a wrinkle is a path component $P \subset \Sigma f$ which satisfies the following property. There is a pair of embeddings $\psi_1 : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n \to E$ and $\psi_2 : \mathbb{R}^k \times \mathbb{R} \to Q \times I$ so that P is contained in the image of ψ_1 and so that the composition $\psi_2^{-1} \circ f \circ \psi_1$ is the map ω defined

$$\begin{aligned} \omega : \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n &\longrightarrow \mathbb{R}^k \times \mathbb{R} \\ (y, z, x) &\longmapsto \left(y, z^3 + 3(|y|^2 - 1) - \sum_{i=1}^{p+1} x_i^2 + \sum_{p+2}^{n+1} x_i^2 \right). \end{aligned}$$

In these coordinates, the wrinkle P is the k-dimensional sphere given by $\{z^2 + |y|^2 = 1, x = 0\}$ and the function f is locally a k-parameter family of generalised Morse functions.

We will now regularise the wrinkle P in the manner of Theorem III.1. Let $B_{\mathbb{R}^{k+1}}(0,1)$ denote the closed ball in $\mathbb{R}^k \times \mathbb{R} \times \{0\}$ which is bounded by this sphere. Theorem III.1 now gives that the map ω can be replaced by a map ω' which for some $\epsilon > 0$ agrees with ω outside of $B_{\mathbb{R}^k}(0, 1+\epsilon) \times [-1-\epsilon, 1+\epsilon] \times B_{\mathbb{R}^n}(0, \epsilon)$ and which has no critical points. Let D^k denote the closed ball $B_{\mathbb{R}^k}(0,2)$ in the plane $\mathbb{R}^k$. It follows from the regularisation of ω that there is an embedding $\phi : D^k \times [c_0, c_1] \times D^n \to \mathbb{R}^k \times \mathbb{R}^{n+1}$ so that the composition $p_2 \circ f \circ \psi_1 \circ \phi$ is projection onto $[c_0, c_1]$; see Fig. III.20. Here p_2 is projection onto the second factor.

Now, for each $y \in D^k$, with $|y| < 1$, the function f restricts to a Morse function with two

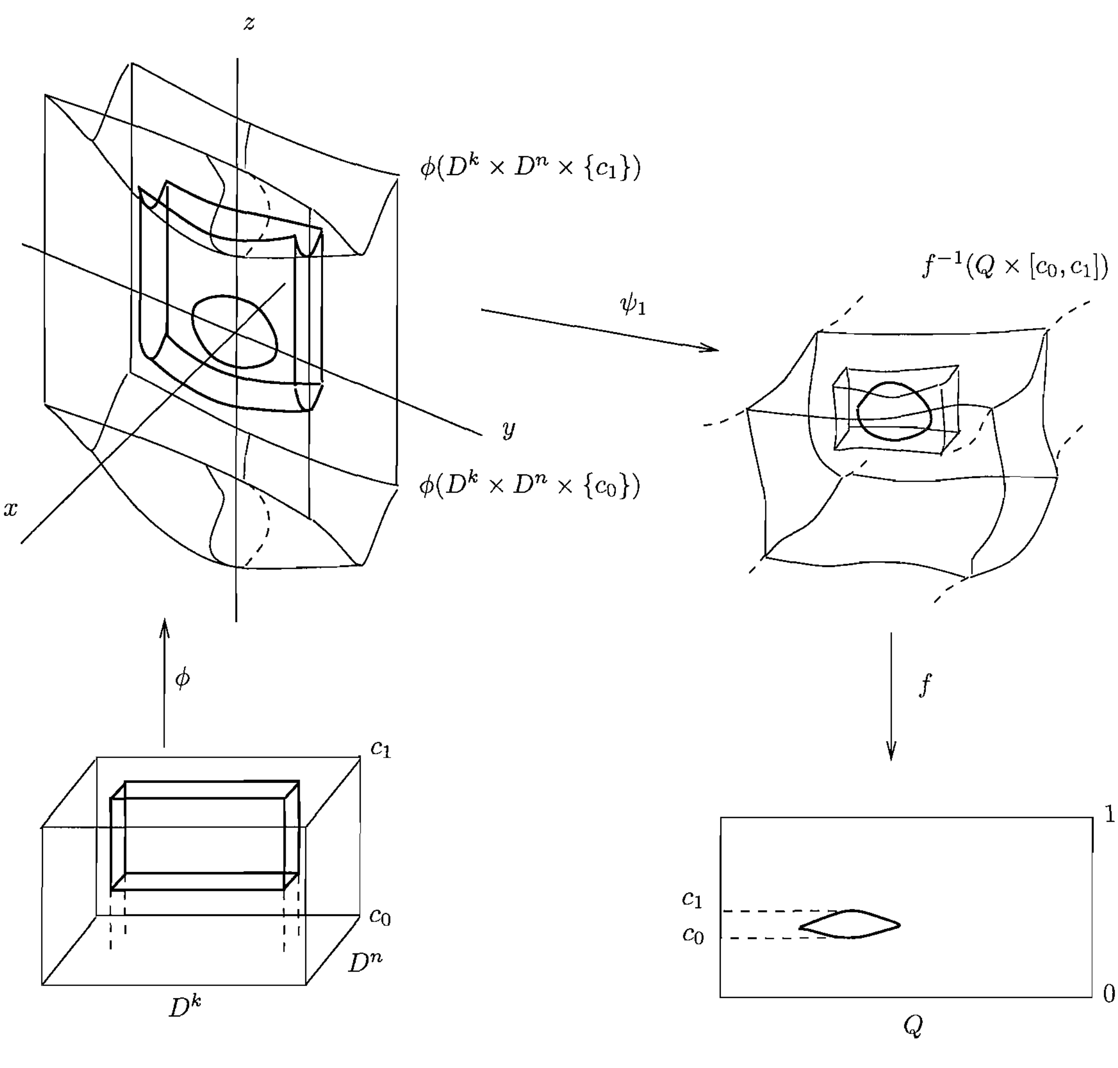

Figure III.20: The composition $\psi_1 \circ \phi$ parametrises a region containing the wrinkle P

critical points in cancelling position and connected by a trajectory arc γ_y given by the formula

$$\gamma_y : \left[-\sqrt{1-|y|^2}, \sqrt{1-|y|^2}\right] \longrightarrow E$$
$$t \longmapsto \psi_1(y,t,0).$$

Let ξ denote the cut-off function described in section III.4.3 with respect to constants ϵ_0 and ϵ_1 and let $L(\gamma_y)$ denote the length, with respect to m, of the trajectory arc γ_y. We may assume that $0 < \epsilon_0 < \epsilon_1 < \frac{\bar{L}}{2}$ where $\bar{L} = \sup_{D^k}(L(\gamma_y))$.

Equip $D^k \times D^n \times \{c_0\}$ with the metric obtained by pulling back $G_{c_0}|_{f^{-1}(Q\times\{c_0\})}$ via $\psi_1 \circ \phi$. On fibres $\{y\} \times D^n \times \{c_0\}$, the induced metric is a psc-metric denoted h_y. On each vertical slice $\{y\} \times D^n \times [c_0, c_1]$, we extend the metric h_y as the metric $\bar{h_y}'(L)$, defined in section III.4.3. When $L < \epsilon_0$, this metric is just a standard product and so this construction transitions smoothly fibrewise over all of D^k. Near ∂D^k, the metric on the fibres pulls back to precisely the one obtained by standard product extension of the metric G_{c_0} away from the wrinkle. As $|y|$ decreases, the fibre metric smoothly transitions into the regularised Gromov-Lawson cobordism obtained in Theorem III.2. This is indicated by the smaller darker rectangle in the bottom left drawing of Fig. III.20. Outside of this rectangle the metric extends fibreswise as the product $h_y + dt^2$. Inside of this rectangle, the fibre metric is smoothly altered as $|y| \to \epsilon_0$. Pulling back this smooth family of metrics via $\psi_1 \circ \phi$ results, via H and m_Q in the desired submersion metric G_{c_1}. This completes the proof of Theorem III.6. □

III.6 Gromov-Lawson Cobordism and Isotopy

In this section we will consider the psc-metrics obtained on X_1 via application of the technique of Theorem II.23 with respect to a psc-metric $g_0 \in \mathcal{R}\text{iem}^+(W)$ and an admissible Morse function f. Recall that Theorem II.23 allows us to construct a psc-metric $\bar{g} = \bar{g}(g_0, f)$ on W which has a product structure near the boundary. In particular, the metric $g_1 = \bar{g}|_{X_1}$ is a psc-metric on X_1. It is worth considering to what extent the metric g_1 depends on the admissible Morse function f. In other words, if f_a and f_b are two distinct admissible Morse functions (with possibly different numbers of critical points), what can we say about the metrics $g_1(a) = \bar{g}(g_0, f_a)|_{X_1}$ and $g_1(b) = \bar{g}(g_0, f_b)$? In particular are these metrics concordant or even isotopic? We have already answered this question in the case when W is a cylinder $X \times I$ and X is simply connected with

$n \geq 5$; see Theorem II.36. In that case $g_1(a)$ and $g_1(b)$ are always isotopic. In this section we will show that this result holds for more general W.

In section III.2.1 we reviewed the notion of an admissible Morse function and described the space of such functions on W. Recall that $\mathcal{F} = \mathcal{F}(W)$ denotes the space of smooth functions $W \to I$, so that if $f \in \mathcal{F}$, Σf is contained in the interior of W and $f^{-1}(0) = X_0$ and $f^{-1}(1) = X_1$. Furthermore, $\tilde{F}$ denoted the space of triples (f, m, V) where $f \in \mathcal{F}$, $m \in \mathcal{R}\text{iem}(W)$ and V is a gradient-like vector field on W with respect to f and m. Contained in $\mathcal{F}$ as as open dense subspace is the space $\mathcal{M}$ of Morse functions. The subspace of $\mathcal{M}$ consisting of Morse functions all of whose critical points have index $\leq n-2$ is denoted $\mathcal{M}^{adm}$. The space of admissible Morse functions, denoted $\tilde{\mathcal{M}}^{adm} = \tilde{\mathcal{M}}^{adm}(W)$ is the space of triples $\{(f, m, V) \in \tilde{F} : f \in \mathcal{M}^{adm}\}$. Finally, recall that $\tilde{\mathcal{M}}^{adm}$ is homotopy equivalent to $\mathcal{M}^{adm}$.

III.6.1 The space of generalised Morse functions

Throughout this section, W, X_0 and X_1 are simply connected and of dimension ≥ 5. The space $\mathcal{M}$ is of course not path connected as functions lying in the same path component of M must have the same number of critical points of the same index. There is however, a natural setting in which to consider the cancellation of Morse critical points. Let $\mathcal{H} = \mathcal{H}(W)$ denote the subspace of $\mathcal{F}$ which consists of all generalised Morse functions. Recall that the singular set of a generalised Morse function consists of both Morse and birth-death singularities and so $\mathcal{M} \subset \mathcal{H}$. It follows from Theorem 4.6.3 of [19], that any two Morse functions in $\mathcal{M}$ can be connected by a path in $\mathcal{H}$. Furthermore, all but finitely many points on this path are Morse functions. Note that a great deal of work has been done in understanding the homotopy type of the space $\mathcal{H}$; see for example [7], [21] and [23].

We will be particularly interested in generalised Morse functions whose critical points satisfy certain index requirements. Let $\mathcal{H}_{i,j}$ denote the subspace of $\mathcal{H}$ consisting of all generalised Morse functions with only critical points of index between i and j inclusively. Of special interest to us is the space $\mathcal{H}^{adm} = \mathcal{H}_{0,n-2}$. Furthermore, let $\tilde{\mathcal{H}}_{i,j} = \{(f, m, V) \in \tilde{F} : f \in \mathcal{H}_{i,j}\}$. As before, $\tilde{H}_{i,j}$ is homotopy equivalent to the space $\mathcal{H}_{i,j}$. The space of *admissible generalised Morse functions*, denoted $\tilde{\mathcal{H}}^{adm}$, is the space $\tilde{\mathcal{H}}_{0,n-2}$.

III.6.2 Hatcher's 2-Index Theorem

It will be important for us to be able to connect up an arbitrary pair of admissible Morse functions with a path through admissible generalised Morse functions. To do this we will need the following corollary of Hatcher's 2-Index Theorem; see Theorem 1.1, Chapter VI, Section 1 of [23].

Theorem III.7. (Corollary 1.4, Chapter VI, [23]) *Under the following conditions the inclusion map $\mathcal{H}_{i,j-1} \longrightarrow \mathcal{H}_{i,j}$ is k-connected.*

(a) (W, X_1) *is* $(n-j+1)$*-connected.*

(b) $j \geq i+2$.

(c) $n-j+1 \leq n-k-1-\min(j-1, k-1)$.

(d) $n-j+1 \leq n-k-3$.

We can now prove the following lemma.

Lemma III.8. *Let $\{W; X_0, X_1\}$ be a smooth compact cobordism where W, X_0 and X_1 are simply connected and W has dimension $n+1 \geq 6$. Let $f_0, f_1 \in \mathcal{M}^{adm} = \mathcal{M}^{adm}(W)$. Then there is a path $s \mapsto f_s$, for $s \in I$, in $\mathcal{H}^{adm} = \mathcal{H}^{adm}(W)$ which connects f_0 and f_1 and which lies in $\mathcal{M}^{adm}$ for all but finitely many points $s_0, \ldots, s_l \in (0,1)$.*

Proof. The existence of such a path in $\mathcal{H}$ which connects f_0 and f_1 is given to us by Theorem 4.6.3 of [19]. We need to show that such a path can be deformed to one which lies entirely inside $\mathcal{H}^{adm}$. A careful analysis of the statement of Theorem III.7 gives that the inclusions

$$\mathcal{H}^{adm} = \mathcal{H}_{0,n-2} \longrightarrow \mathcal{H}_{0,n-1} \longrightarrow \mathcal{H}_{0,n} \longrightarrow \mathcal{H}_{0,n+1} = \mathcal{H}$$

are 0, 1 and 2-connected respectively. Note that condition (a) of Theorem III.7 is satisfied by the existence of f_0 and f_1 on W. This gives that any path in $\mathcal{H}$, connecting f_0 and f_1, can be deformed into one which lies entirely in $\mathcal{H}^{adm}$. □

III.6.3 An application of Hatcher's 2-Index Theorem

We will now prove a theorem concerning the problem of how the choice of admissible Morse function affects the isotopy type of a Gromov-Lawson cobordism.

Theorem III.9. *Let $\{W; X_0, X_1\}$ be a smooth compact cobordism where W, X_0 and X_1 are simply connected and W has dimension $n \geq 5$. Let $f_0, f_1 \in \tilde{M}^{adm}(W)$. Suppose g_0 and g_1 are psc-metrics lying in the same path component of $\mathcal{R}\mathrm{iem}^+(X_0)$. If $\bar{g}_0 = \bar{g}(g_0, f_0)$ and $\bar{g}_1 = \bar{g}(g_1, f_1)$ are Gromov-Lawson cobordisms, then the psc-metrics $g_{0,1} = \bar{g}_0|_{X_1}$ and $g_{1,1} = \bar{g}_1|_{X_1}$ are isotopic metrics in $\mathcal{R}\mathrm{iem}^+(X_1)$.*

Proof. Let g_s, $s \in I$ denote a path in $\mathcal{R}\mathrm{iem}^+(X_0)$ connecting the metrics g_0 and g_1. Recall that $\tilde{\mathcal{H}}^{adm}$ is homotopy equivalent to $\mathcal{H}^{adm}$. Thus, by lemma III.8, there is a smooth path f_s in $\tilde{\mathcal{H}}^{adm}$, with $s \in I$, connecting f_0 and f_1. It will be useful to regard the family f_s as a smooth map f, defined by

$$f : W \times I \to I \times I$$
$$(w, s) \to (s, f_s(w)).$$

Recall that the path f_s lies in $\tilde{\mathcal{M}}^{adm}$ for all but finitely many points $s_0, \ldots s_l \in (0,1)$. Choose $\epsilon > 0$ sufficiently small so that for all $i \in \{0, 1 \ldots, l\}$, the intervals $(s_i - \epsilon, s_i + \epsilon)$ are disjoint subintervals of $(0,1)$. On $[0, s_0 - \epsilon]$, f_s is a family of admissible Morse functions. Thus, by Theorem II.25, we can extend the Gromov-Lawson cobordism $\bar{g}_0 = \bar{g}(g_0, f_0)$ as a compact family of Gromov-Lawson cobordisms $\bar{g}_s = \bar{g}(g_s, f_s)$. Similarly, we can do this for all $s \in I \setminus \bigcup_i (s_i - \epsilon, s_i + \epsilon)$.

This gives a disjoint collection of paths $g_{s,1} = \bar{g}_s|_{X_1}$ in $\mathcal{R}\mathrm{iem}^+(X_1)$. It remains to show that these paths can be connected along the intervals $(s_i - \epsilon, s_i + \epsilon)$. Without loss of generality we may assume that $l = 0$ and that $g_{s,1}$ is defined for all s except on $(s_0 - \epsilon, s_0 + \epsilon)$. Furthermore we may assume that f_{s_0} has only one birth-death singularity at the point $(w, s_0) \in W \times I$. This is possible since singularities of a generalised Morse function are isolated. Provided ϵ is chosen sufficiently small, Lemma 3.5 of [23] gives that there is a coordinate map

$$\psi : (-\epsilon, \epsilon) \times \mathbb{R} \times \mathbb{R}^n \longrightarrow W \times I$$
$$(s, z, x) \longmapsto (\psi_s(z, x), s),$$

so that the composition $f_s \circ \psi_s$ is given by the rule

$$f_s \circ \psi_s(z, x) = z^3 \pm sz - \sum_{0}^{p+1} x_i^2 + \sum_{p+2}^{n} x_i^2.$$

Thus, the point (w, s_0) is a cusp singularity of f. The $\pm sz$ term in the equation above is determined by the cancellation direction. Without loss of generality we will take it as $+sz$. Thus, the admissible Morse function $f_{s_0-\epsilon}$ contains a pair of cancelling critical points, in cancelling position and connected by a trajectory arc $\gamma_{s_0-\epsilon}$. These critical points cancel at (w, s_0). Replace the Gromov-Lawson cobordism $\bar{g}_{s_0-\epsilon}$ with the regularised arc-length dependent Gromov-Lawson cobordism $\bar{g}'_{s_0-\epsilon}(L(s_0-\epsilon))$ constructed in section III.4.3. Here $L_{s_0-\epsilon}$ is the length of the trajectory arc $\gamma_{s_0-\epsilon}$. We will assume that the cut-off function ξ, associated with this metric, has been chosen with constants ϵ_0 and ϵ_1 satisfying $0 < \epsilon_0 < \epsilon_1 < L(s_0 - \epsilon)$. This will ensure that the metrics $\bar{g}'_{s_0-\epsilon}(L)$ and $\bar{g}_{s_0-\epsilon}$ agree near $X_1 \times \{s_0\}$. Similarly, replace $\bar{g}_{s_0+\epsilon}$ with $\bar{g}'_{s_0+\epsilon}(L(s_0 + \epsilon))$. Using the technique of Theorem III.6, we may now extend the regularised arc-length dependent Gromov-Lawson cobordism $\bar{g}'_{s_0}(L)$ to obtain a smooth family of regularised Gromov-Lawson cobordisms $\bar{g}'_s(L(s))$ over $W \times [s_0 - \epsilon, s_0 + \epsilon]$. The restriction of this family to $X_1 \times [s_0 - \epsilon, s_0 + \epsilon]$ provides an isotopy connecting $g_{s_0-\epsilon,1}$ to $g_{s_0+\epsilon,1}$. This completes the proof. □

APPENDIX A

ISOTOPY IMPLIES CONCORDANCE

We will prove the following lemma from section II.2, an easy corollary of which is that isotopic psc-metrics on a smooth compact manifold X are necessarily concordant.

Lemma II.1. *Let $g_r, r \in I$ be a smooth path in $\mathcal{R}\mathrm{iem}^+(X)$. Then there exists a constant $0 < \Lambda \leq 1$ so that for every smooth function $f : \mathbb{R} \to [0,1]$ with $|\dot{f}|, |\ddot{f}| \leq \Lambda$, the metric $G = g_{f(t)} + dt^2$ on $X \times \mathbb{R}$ has positive scalar curvature.*

Proof. Choose a point $(x_0, t_0) \in X \times \mathbb{R}$. Denote by $(x_0^1, \ldots, x_0^n, x_0^{n+1} = t)$, coordinates around (x_0, t_0), where $x_0^1, \ldots, x_0^n$ are normal coordinates on X with respect to the metric $g_{f(t_0)}$. The respective coordinate vector fields will be denoted $\partial_1, \ldots, \partial_n, \partial_{n+1} = \partial_t$. Let $\bar{\nabla}$ denote the Levi-Civita connection of the metric G on $X \times \mathbb{R}$ and let ∇ denote the Levi-Civita connection of the metric $g_{f(t_0)}$ on $X \times \{t_0\}$. All of our calculations will take place at the point (x_0, t_0).

We need to compute the scalar curvature of G in terms of the scalar curvature of the metric $g_{f(t_0)}$, and the first and second derivatives of the function f. We begin by computing the Christoffel symbols $\bar{\Gamma}_{i,j}^k$ of the connection $\bar{\nabla}$. Recall that these are given in terms of the metric G by the formula

$$\bar{\Gamma}_{i,j}^k = \frac{1}{2} G^{kl}(\partial_j G_{il} + \partial_i G_{jl} - \partial_l G_{ij}).$$

When $i, j, k \leq n$, it is clear that $\bar{\Gamma}_{i,j}^k = \Gamma_{i,j}^k$. We now turn our attention to the remaining cases. Suppose $i, j \leq n$ and $k = n+1$. Then

$$\begin{aligned}\bar{\Gamma}_{i,j}^{n+1} &= \frac{1}{2} G^{n+1,n+1}(0 + 0 - \partial_t G_{ij}(x_0, t_0)) \\ &= \frac{-1}{2} \partial_r g_{r(ij)}(x_0, f(t_0)).\dot{f}(t_0).\end{aligned}$$

When $i \leq n$ and $j, k = n+1$,

$$\bar{\Gamma}^{n+1}_{i,n+1} = \frac{1}{2} G^{n+1,n+1}(0+0-0)$$
$$= 0.$$

In the case when $i, k \leq n$ and $j = n+1$, we obtain

$$\bar{\Gamma}^{k}_{i,n+1} = \frac{1}{2} G^{kl}(\partial_t G_{il}(x_0, t_0))$$
$$= \frac{1}{2} G^{kl} \partial_r g_{r(il)}(x_0, f(t_0)).\dot{f}(t_0).$$

Finally, when $k \leq n$ and $i, j = n+1$

$$\bar{\Gamma}^{k}_{n+1,n+1} = \frac{1}{2} G^{kl}(0+0-0)$$
$$= 0.$$

Thus, $\bar{\Gamma}^{n+1}_{ij}$ and $\bar{\Gamma}^{k}_{i,n+1}$ are both $O(|\dot{f}|)$, while $\bar{\Gamma}^{n+1}_{i,n+1} = 0 = \bar{\Gamma}^{k}_{n+1,n+1}$. Let $\bar{K}_{ij}$ and K_{ij} denote the respective sectional curvatures for the metrics G and $g_{f(t_0)}$. Viewing $X \times \{t_0\}$ as a hypersurface of $X \times \mathbb{R}$, the Gauss curvature equation gives us the following formula for $\bar{K}_{ij}$, when $i, j \leq n$.

$$\bar{K}_{ij} = K_{ij} - G(\Pi(\partial_i, \partial_i), \Pi(\partial_j, \partial_j)) + G(\Pi(\partial_i, \partial_j), \Pi(\partial_i, \partial_j)),$$

where Π denotes the second fundamental form on $X \times \{t_0\}$. In this case,

$$\Pi(\partial_i, \partial_j) = G(\bar{\nabla}_{\partial_i} \partial_j, \partial_{n+1})\partial_{n+1}$$
$$= \bar{\Gamma}^{n+1}_{ij} \partial_{n+1}.$$

Hence,

$$\bar{K}_{ij} = K_{ij} + O(|\dot{f}|^2).$$

In the case when $i \leq n$ and $j = n+1$, we use the following formula, derived in Proposition A.1 below, for the sectional curvature.

$$\bar{K}_{i,n+1} = \partial_i \bar{\Gamma}^{i}_{n+1,n+1} - \partial_{n+1} \bar{\Gamma}^{i}_{i,n+1} + \sum_{k=1}^{n+1} \left(\bar{\Gamma}^{k}_{n+1,n+1} \bar{\Gamma}^{i}_{ik} - \bar{\Gamma}^{k}_{i,n+1} \bar{\Gamma}^{i}_{n+1,k} \right)$$

As the expression

$$-\partial_{n+1}\bar{\Gamma}^i_{i,n+1} = -\partial_t \frac{1}{2} G^{il} \partial_r g_{r(il)}(x_0, f(t_0)) \cdot \dot{f}(t_0),$$

we obtain

$$\bar{K}_{i,n+1} = O(|\dot{f}|) + O(|\ddot{f}|).$$

Finally, let $\bar{R}$ denote the the scalar curvature of the metric G, while R denotes the scalar curvature of $g_{f(t_0)}$. It now follows that, at the point (x_0, t_0),

$$\bar{R} = R + O(|\dot{f}|) + O(|\dot{f}|^2) + O(|\ddot{f}|).$$

This completes the proof. □

Proposition A.1. *Let (M, g) be a Riemannian n-manifold. Let $(x_1, \ldots, x_n)$ denote a normal coordinate neighbourhood about a point p. The sectional curvature K_{ij} of the metric g at p is given by the formula*

$$K_{ij}(p) = \partial_i \Gamma^i_{jj} - \partial_j \Gamma^i_{ij} + \sum_{k=1}^{n} \left(\Gamma^k_{jj} \Gamma^i_{ik} - \Gamma^k_{ij} \Gamma^i_{jk} \right).$$

Proof. In these coordinates,

$$K_{ij} = \frac{g(\mathcal{R}(\partial_i, \partial_j)\partial_j, \partial_i)}{g_{ii} g_{jj} - g_{ij}^2}$$

where $\mathcal{R}$ is the Riemannian curvature tensor for the metric g. At p this simplifies to

$$\begin{aligned}
K_{ij}(p) &= g(\mathcal{R}(\partial_i, \partial_j)\partial_j, \partial_i) \\
&= g(\nabla_{\partial_i} \nabla_{\partial_j} \partial_j - \nabla_{\partial_j} \nabla_{\partial_i} \partial_j, \partial_i) \\
&= g(\nabla_{\partial_i}(\Gamma^k_{jj} \partial_k) - \nabla_{\partial_j}(\Gamma^k_{ij} \partial_k), \partial_i) \\
&= g(\Gamma^k_{jj} \Gamma^l_{ik} \partial_l + \partial_i(\Gamma^k_{jj} \partial_k) - \Gamma^k_{ij} \Gamma^l_{jk} \partial_l - \partial_j(\Gamma^k_{ij} \partial_k), \partial_i) \\
&= \partial_i \Gamma^i_{jj} - \partial_j \Gamma^i_{ij} + \Gamma^k_{jj} \Gamma^i_{ik} - \Gamma^k_{ij} \Gamma^i_{jk}.
\end{aligned}$$

□

APPENDIX B

CURVATURE CALCULATIONS FOR THE SURGERY THEOREM

Below we provide detailed proofs of Lemmas used in the proof of Theorem II.11 from section II.3. In particular, Lemma II.13 is exactly Lemma 1 from [14]. The proof below is due to Gromov and Lawson although we include details which are suppressed in the original. Lemmas II.14 and II.15 are curvature calculations. The resulting formulae arise in Gromov and Lawson's original proof of the Surgery Theorem; see [14] or [36].

Let (X, g) be a Riemannian manifold. Fix a point $z \in X$ and let D be a normal coordinate ball of radius $\bar{r}$ around z. Recall, this means first choosing an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_zX. This determines an isomorphism $E : (x_1, \ldots, x_n) \mapsto x_1e_1 + \ldots + x_ne_n$ from $\mathbb{R}^n$ to T_zX. The composition $E^{-1} \circ \exp^{-1}$ is a coordinate map provided we restrict it to an appropriate neighbourhood of z. Thus, we identify $D = \{x \in \mathbb{R}^n : |x| \leq \bar{r}\}$. The quantity $r(x) = |x|$ is the radial distance from the point z, and $S^{n-1}(\epsilon) = \{x \in \mathbb{R}^n : |x| = \epsilon\}$ will denote the geodesic sphere of radius ϵ around z.

Lemma II.13. (Lemma 1, [14])

(a) The principal curvatures of the hypersurfaces $S^{n-1}(\epsilon)$ in D are each of the form $\frac{-1}{\epsilon} + O(\epsilon)$ for ϵ small.

(b) Furthermore, let g_ϵ be the induced metric on $S^{n-1}(\epsilon)$ and let $g_{0,\epsilon}$ be the standard Euclidean metric of curvature $\frac{1}{\epsilon^2}$. Then as $\epsilon \to 0$, $\frac{1}{\epsilon^2}g_\epsilon \to \frac{1}{\epsilon^2}g_{0,\epsilon} = g_{0,1}$ in the C^2-topology.

Below we use the following notation. A function $f(r)$ is $O(r)$ as $r \to 0$ if $\frac{f(r)}{r} \to$ constant as $r \to 0$.

Proof. We begin with the proof of (a). On D, in coordinates $x_1, \ldots, x_n$, the metric g has the form

$$g_{ij}(x) = \delta_{ij} + \sum a_{ij}^{kl} x_k x_l + O(|x|^3) = \delta_{ij} + O(|x|^3). \tag{B.1}$$

This follows from the Taylor series expansion of $g_{ij}(x)$ around 0 and the fact that in a normal coordinate neighbourhood of $p = 0$, $g_{ij}(0) = \delta_{ij}$ and $\Gamma_{ij}^k(0) = 0$.

Next we will show that the Christoffel symbols of the corresponding Levi-Civita connection have the form

$$\Gamma_{ij}^k = \textstyle\sum_l \gamma_{ijk,l} + O(|x|^2) = O(|x|).$$

Recall that the Christoffel symbols are given by the formula

$$\Gamma_{ij}^k = \tfrac{1}{2} \textstyle\sum_l g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}).$$

Differentiating (B.1), gives

$$\tfrac{\partial}{\partial x_i} g_{jl} = 0 + \textstyle\sum a_{jl}^{st}(x_s \delta_{ti} + x_t \delta_{si}) + \ldots$$

Hence,

$$g_{il,j} + g_{jl,i} - g_{ij,l} = O(|x|).$$

We must now deal with the g^{kl} terms. Let $(g_{kl}) = I + Y$ where the I is the identity matrix and Y is the matrix $(a_{kl}^{ij} x_i x_j + O(|x|^3))$. Recall the following elementary fact.

$$(1 + a)^{-1} = 1 - a + a^2 - a^3 + \ldots$$

Thus we can write

$$(g^{kl}) = I - Y + Y^2 - Y^3 + \ldots$$

Each component of this matrix has the form

$$g^{kl} = \delta_{kl} + O(|x|^2).$$

Finally we obtain

$$\Gamma^k_{ij} = \tfrac{1}{2}\sum_l (\delta_{ij} + O(|x|^2))(O(|x|)) = O(|x|).$$

We will now compute the scalar second fundamental form on tangent vectors to the geodesic sphere $S^{n-1}(\epsilon)$. Consider the smooth curve α on $S^{n-1}(\epsilon)$ given by

$$\alpha(s) = (\epsilon \cos \tfrac{s}{\epsilon}, \epsilon \sin \tfrac{s}{\epsilon}, 0, \ldots, 0).$$

Velocity vectors of this curve are tangent vectors to $S^{n-1}(\epsilon)$ and have the form

$$\dot{\alpha}(s) = (-\sin \tfrac{s}{\epsilon}, \cos \tfrac{s}{\epsilon}, 0, \ldots, 0).$$

Letting ξ denote the exterior unit normal vector field to $S^{n-1}(\epsilon)$, we have

$$\xi(\alpha(0)) = \tfrac{\alpha(0)}{|\alpha(0)|} = (1, 0, \ldots, 0) = e_1.$$

and

$$\dot{\alpha}(0) = (0, 1, 0, \ldots, 0) = e_2.$$

We will now proceed to compute the scalar second fundamental form at $\alpha(0)$. We denote by

$$A : T_{\alpha(0)}S^{n-1}(\epsilon) \times T_{\alpha(0)}S^{n-1}(\epsilon) \to \mathbb{R},$$

the scalar second fundamental form and by

$$S : T_{\alpha(0)}S^{n-1}(\epsilon) \to T_{\alpha(0)}S^{n-1}(\epsilon),$$

the shape operator, for the hypersurface $S^{n-1}(\epsilon) \subset D$. Recall that,

$$S(X_{\alpha(0)}) = -\nabla_X \xi,$$

$$A(X_{\alpha(0)}, Y_{\alpha(0)}) = g(S(X_{\alpha(0)}), Y)$$

where X, Y are tangent vector fields on $S^{n-1}(\epsilon)$, and that A only depends on X and Y at p. We now compute

$$
\begin{aligned}
A(\dot\alpha(0),\dot\alpha(0)) &= g(S(\dot\alpha(0)),\dot\alpha) \\
&= g(-\nabla_{\dot\alpha}\xi,\dot\alpha)_{\alpha(0)} \\
&= -\dot\alpha[g(\xi,\dot\alpha)](\alpha(0)) + g(\nabla_{\dot\alpha}\dot\alpha,\xi)_{\alpha(0)} \\
&= 0 + g(\nabla_{\dot\alpha}\dot\alpha, e_1) \\
&= \ddot\alpha^{(1)}(0) + \textstyle\sum_{i,j}\alpha^1_{ij}\dot\alpha^{(i)}\dot\alpha^{(j)}(0).
\end{aligned}
$$

The components of the velocity vector are

$$
\dot\alpha^{(1)} = -\sin\tfrac{s}{\epsilon}, \qquad \dot\alpha^{(2)} = \cos\tfrac{s}{\epsilon}, \qquad \dot\alpha^{(j)} = 0, \;\; j \geq 3,
$$

while

$$
\ddot\alpha^{(1)} = -\tfrac{1}{\epsilon}\cos\tfrac{s}{\epsilon}.
$$

Thus,

$$
(\ddot\alpha^{(1)} + \textstyle\sum_{i,j}\alpha^1_{ij}\dot\alpha^{(i)}\dot\alpha^{(j)})(s) = -\tfrac{1}{\epsilon}\cos\tfrac{s}{\epsilon} + \alpha^1_{11}\sin^2\tfrac{s}{\epsilon} - 2\alpha^1_{12}\sin\tfrac{s}{\epsilon}\cos\tfrac{s}{\epsilon} + \alpha^1_{22}\cos^2\tfrac{s}{\epsilon}.
$$

Hence,

$$
(\ddot\alpha^{(1)} + \textstyle\sum_{i,j}\alpha^1_{ij}\dot\alpha^{(i)}\dot\alpha^{(j)})(0) = -\tfrac{1}{\epsilon} + \alpha^1_{22}(\epsilon,0,\ldots,0) = -\tfrac{1}{\epsilon} + O(\epsilon).
$$

We now have that $A(\dot\alpha(0),\dot\alpha(0)) = -\frac{1}{\epsilon} + O(\epsilon)$. Finally we need to normalise the vector $\dot\alpha(0)$. We can write

$$
A(\dot\alpha(0),\dot\alpha(0)) = |\dot\alpha(0)|^2 A(v,v)
$$

where v is the unit length vector $\frac{\dot\alpha(0)}{|\dot\alpha(0)|}$.

$$
\begin{aligned}
|\dot\alpha(0)|^2 &= g(\dot\alpha(0),\dot\alpha(0)) \\
&= g(e_2,e_2)_{(\epsilon,0,\ldots,0)} \\
&= g_{22}(\epsilon,0,\ldots,0) \\
&= \delta_{22} + (\textstyle\sum_{k,l} a^{kl}_{22}x_k x_l + O(|x|^3))(\epsilon,0,\ldots,0) \\
&= 1 + a^{11}_{22}\epsilon^2 + O(|x|^3) \\
&= 1 + O(\epsilon^2).
\end{aligned}
$$

We now have that

$$A(\dot{\alpha}(0), \dot{\alpha}(0)) = (1 + O(\epsilon^2))A(v, v).$$

That is

$$-\tfrac{1}{\epsilon} + O(\epsilon) = (1 + O(\epsilon^2))A(v, v).$$

This means that

$$A(v, v) = -\tfrac{1}{\epsilon} + O(\epsilon) + O(\epsilon^2) = -\tfrac{1}{\epsilon} + O(\epsilon).$$

By an orthogonal change of coordinates (another choice of orthonormal basis $\{e_1, \ldots, e_n\}$), this computation is valid for any unit vector. In particular, it holds if v is a principal direction. Hence, the principal curvatures have the desired form. This proves part (a).

The second part of the lemma is more straightforward. We can compare the induced metrics g_ϵ on $S^{n-1}(\epsilon)$ for decreasing values of ϵ by pulling back onto $S^{n-1}(1)$ via the map

$$f_\epsilon : S^{n-1}(1) \longrightarrow S^{n-1}(\epsilon)$$

$$x \longmapsto \epsilon x$$

Then at a point x where $|x| = 1$, we have

$$\begin{aligned}\tfrac{1}{\epsilon^2} f_\epsilon^*(g_\epsilon)(x) &= \textstyle\sum_{i,j} g_{ij}(\epsilon x) dx_i dx_j \\ &= \textstyle\sum_{i,j}(\delta_{ij} + \epsilon^2 \sum_{i,j} a_{ij}^{kl} x_k x_l) dx_i dx_j + \epsilon^3(\text{higher order terms}).\end{aligned}$$

In the C^2-topology (that is, in the zeroth, first and second order terms of the Taylor series expansion), $\frac{1}{\epsilon^2} f_\epsilon^*(g_\epsilon)$ converges to the standard Euclidean metric in some neighbourhood of $S^{n-1}(1)$ as $\epsilon \to 0$. As f_ϵ is a diffeomorphism, the metric $\frac{1}{\epsilon^2}(g_\epsilon)$ is isometric to $\frac{1}{\epsilon^2} f_\epsilon^*(g_\epsilon)$ and converges (in C^2) to the standard metric in some neighbourhood of $S^{n-1}(\epsilon)$. This proves part (b) and completes the proof of Lemma II.13. □

Recall, in the proof of Theorem II.11, we deform a psc-metric g on a smooth manifold X inside a tubular neighbourhood $N = S^p \times D^{q+1}$ of an embedded sphere S^p. Here $q \geq 2$. We do this by specifying a hypersurface M inside $N \times \mathbb{R}$, shown in Fig. B.2 and inducing a metric from the ambient metric $g + dt^2$. The hypersurface M is defined as

$$M_\gamma = \{(y, x, t) \in S^p \times D^{q+1}(\bar{r}) \times \mathbb{R} : (r(x), t) \in \gamma\}.$$

where γ is the curve shown in Fig. B.1 and r denotes radial distance from S^p on N. The induced metric is denoted g_γ. The fact that γ is a vertical line near the point $(0, \bar{r})$ means that $g_\gamma = g$, near ∂N. Thus γ specifies a metric on X which is the orginal metric g outside of N and then transitions smoothly to the metric g_γ. For a more detailed description; see section II.3. In the following lemmas we compute the scalar curvature of g_γ.

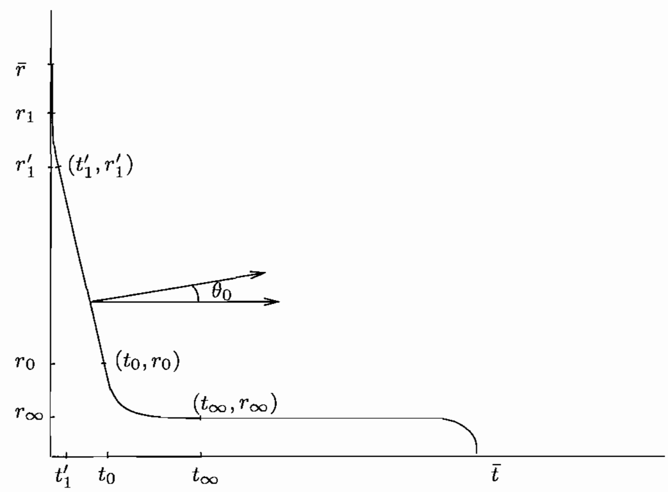

Figure B.1. The curve γ

Lemma II.14. *The principal curvatures to M with respect to the outward unit normal vector field have the form*

$$\lambda_j = \begin{cases} k & \text{if } j = 1 \\ (-\frac{1}{r} + O(r)) \sin\theta & \text{if } 2 \leq j \leq q+1 \\ O(1)\sin\theta & \text{if } q+2 \leq j \leq n. \end{cases}$$

Here k is the curvature of γ, θ is the angle between the outward normal vector η and the horizontal (or the outward normal to the curve γ and the t-axis) and the corresponding principal directions e_j are tangent to the curve γ when $j = 1$, the fibre sphere S^q when $2 \leq j \leq q+1$ and S^p when $q + 2 \leq j \leq n$.

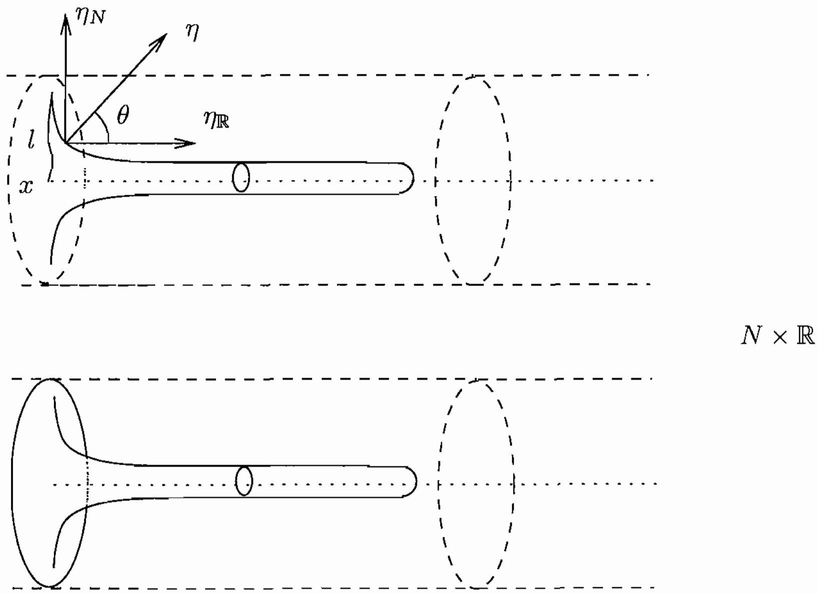

Figure B.2. The hypersurface M in $N \times \mathbb{R}$

Proof. Let $w = (y, x, t) \in S^p \times D^{q+1} \times \mathbb{R}$ be a point on M. Let l be the geodesic ray emanating from $y \times \{0\}$ in N through the point (y, x). The surface $l \times \mathbb{R}$ in $N \times R$ can be thought of as an embedding of $[0, \bar{r}) \times \mathbb{R}$, given by the map $(r, t) \mapsto (l_r, t)$ where l_r is the point on l of length r from $y \times \{0\}$. We will denote by γ_l, the curve $M \cap l \times \mathbb{R}$. This can be parametrised by composing the parameterisation of γ with the above embedding. We will denote by $\dot{\gamma}_{l_w}$, the velocity vector of this curve at w. Finally we denote by η, the outward pointing unit normal vector field to M.

We now make a couple of observations.

(a) The surface $l \times \mathbb{R}$ is a totally geodesic surface in $N \times \mathbb{R}$. This can be seen from the fact that any geodesic in $l \times \mathbb{R}$ projects onto geodesics in l and $\mathbb{R}$. But $D \times \mathbb{R}$ is a Riemannian product and so such a curve is therefore a geodesic in $D \times \mathbb{R}$.

(b) The vector η is tangential to $l \times \mathbb{R}$. This can be seen by decomposing η into orthogonal components

$$\eta = \eta_N + \eta_{\mathbb{R}}.$$

Here η_N is tangent to N and $\eta_{\mathbb{R}}$ is tangent to $\mathbb{R}$. Now η_N is orthogonal to the geodesic sphere $S^q(r)_y$, centered at $y \times \{0\}$ with radius $r = |x|$. By Gauss's Lemma, we know that l runs orthogonally through $S^q(r)_y$ and so η_N is tangent to l. Hence, η is tangent to $l \times \mathbb{R}$.

We will now show that γ_l is a principal curve in M. Let S^M denote the shape operator for M in $N \times \mathbb{R}$ and S^{γ_l}, the shape operator for γ_l in $l \times \mathbb{R}$. Both shape operators are defined with respect to η.

$$\begin{aligned} S^M(\dot{\gamma}_l) &= -\nabla^{N\times\mathbb{R}}_{\dot{\gamma}_l}\eta \\ &= (-\nabla^{N\times\mathbb{R}}_{\dot{\gamma}_l}\eta)^T + (-\nabla^{N\times\mathbb{R}}_{\dot{\gamma}_l}\eta)^\perp \\ &= -\nabla^{l\times\mathbb{R}}_{\dot{\gamma}_l}\eta + 0 \\ &= S^{\gamma_l}(\dot{\gamma}_l). \end{aligned}$$

The third equality is a direct consequence of the fact that $l \times \mathbb{R}$ is a totally geodesic surface in $N \times \mathbb{R}$. Now as $T\gamma_l$, the tangent bundle of the curve γ_l, is a one-dimensional bundle, $\dot{\gamma}_{l_w}$ must be an eigenvector of S^M. Hence, γ_l is a principal curve. The corresponding principal curvature is of course the curvature of γ, which we denote by k.

At w, we denote the principal direction $\dot{\gamma}_{l_w}$ by e_1. The other principal directions we denote by $e_2, \ldots, e_n$, where $e_2, \ldots, e_{q+1}$ are tangent to the $S^q(r)$ factor and $e_{q+2}, \ldots, e_n$ are tangent to S^p. Recall that the set $\{e_1, \ldots, e_n\}$ forms an orthonormal basis for T_wM. The corresponding principal curvatures will be denoted $\lambda_1 = k, \lambda_2, \ldots, \lambda_n$. Our next task is to compute these principal curvatures.

Let A denote the second fundamental form for M in $N \times \mathbb{R}$ with respect to the outward normal vector η. Let A^N denote the second fundamental form for $S^p \times S^q(r)$ in N, again with respect to η, and λ_j^N the corresponding principal curvatures. When $2 \le j \le n$,

$$\begin{aligned} \lambda_j &= A(e_j, e_j) \\ &= -g(\nabla^{N\times\mathbb{R}}_{e_j}\eta, e_j) \\ &= -g(\nabla^{N\times\mathbb{R}}_{e_j}(\cos\theta\partial_t + \sin\theta\partial_r), e_j) \\ &= -g(\nabla^{N\times\mathbb{R}}_{e_j}\cos\theta\partial_t, e_j) - g(\nabla^{N\times\mathbb{R}}_{e_j}\sin\theta\partial_r, e_j). \end{aligned}$$

where ∂_t and ∂_r are the coordinate vector fields for the t and r coordinates respectively. Now,

$$\begin{aligned} \nabla^{N\times\mathbb{R}}_{e_j}\cos\theta\partial_t &= \cos\theta\nabla^{N\times\mathbb{R}}_{e_j}\partial_t + \partial_j(\cos\theta)\cdot\partial_t \\ &= \cos\theta\cdot 0 + 0\cdot\partial_t \\ &= 0. \end{aligned}$$

However,

$$\nabla^{N\times\mathbb{R}}_{e_j}\sin\theta\partial_r = \sin\theta\nabla^{N\times\mathbb{R}}_{e_j}\partial_r + \partial_j(\sin\theta)\cdot\partial_r$$
$$= \sin\theta\nabla^{N\times\mathbb{R}}_{e_j}\partial_r + 0\cdot\partial_r$$
$$= \sin\theta\nabla^{N\times\mathbb{R}}_{e_j}\partial_r.$$

Hence,

$$\lambda_j = -\sin\theta . g(\nabla^{N\times\mathbb{R}}_{e_j}\partial_r, e_j)$$
$$= \sin\theta\cdot A^N(e_j, e_j)$$
$$= \sin\theta\cdot\lambda_j^N.$$

We know from Lemma II.13 that when $2 \le j \le q+1$, $\lambda_j^N = -\frac{1}{r} + O(r)$. When $q+2 \le j \le n$, $\lambda_j^N = O(1)$ as here the curvature is bounded. Hence, the principal curvatures to M are

$$\lambda_j = \begin{cases} k & \text{if } j = 1 \\ (-\frac{1}{r} + O(r))\sin\theta & \text{if } 2 \le j \le q+1 \\ O(1)\sin\theta & \text{if } q+2 \le j \le n. \end{cases}$$

□

Lemma II.15. *The scalar curvature of the metric induced on M is given by*

$$R^M = R^N + \sin^2\theta\cdot O(1) - 2k\cdot q\frac{\sin\theta}{r}$$
$$+ 2q(q-1)\frac{\sin^2\theta}{r^2} + k\cdot qO(r)\sin\theta.$$

Proof. The Gauss Curvature Equation gives that

$$\tfrac{1}{2}R^M = \textstyle\sum_{i<j}(K_{ij}^{N\times\mathbb{R}} + \lambda_i\lambda_j)$$

where $K^{N\times\mathbb{R}}$ denotes sectional curvature on $N\times\mathbb{R}$. Before we continue we should examine $K^{N\times\mathbb{R}}$. When $2 \le i, j \le n$,

$$K_{ij}^{N\times\mathbb{R}} = K_{ij}^N.$$

When $2 \leq j \leq n$,

$$\begin{aligned}
K_{ij}^{N\times\mathbb{R}} &= Rm^{N\times\mathbb{R}}(e_1, e_j, e_j, e_1) \\
&= Rm^{N\times\mathbb{R}}(-\cos\theta\partial_r + \sin\theta\partial_t, e_j, e_j, -\cos\theta\partial_r + \sin\theta\partial_t) \\
&= \cos^2\theta \cdot Rm^{N\times\mathbb{R}}(\partial_r, e_j, e_j, \partial_r) + \sin^2\theta \cdot Rm^{N\times\mathbb{R}}(\partial_t, e_j, e_j, \partial_t) \\
&= \cos^2\theta \cdot Rm^{N\times\mathbb{R}}(\partial_r, e_j, e_j, \partial_r) + \sin^2\theta \cdot 0 \\
&= \cos^2\theta \cdot Rm^{N}(\partial_r, e_j, e_j, \partial_r) \\
&= \cos^2\theta \cdot K_{\partial_r j}^{N}.
\end{aligned}$$

Now,

$$\begin{aligned}
\tfrac{1}{2}R^M &= \textstyle\sum_{i<j}(K_{ij}^{N\times\mathbb{R}} + \lambda_i\lambda_j) \\
&= \textstyle\sum_{j\leq 2} K_{1j}^{N\times\mathbb{R}} + \sum_{1\neq i<j} K_{ij}^{N\times\mathbb{R}} + \sum_{i<j}\lambda_i\lambda_j.
\end{aligned}$$

We know from earlier that

$$\textstyle\sum_{j\leq 2} K_{1j}^{N\times\mathbb{R}} = (1-\sin^2\theta)\sum_{j\geq 2} K_{\partial_r j}^{N}.$$

Hence,

$$\textstyle\sum_{j\leq 2} K_{1j}^{N\times\mathbb{R}} + \sum_{1\neq i<j} K_{ij}^{N\times\mathbb{R}} = \frac{1}{2}R^N - \sin^2\theta \cdot Ric^N(\partial_r, \partial_r).$$

Next we deal with $\sum_{i<j}\lambda_i\lambda_j$.

$$\textstyle\sum_{i<j}\lambda_i\lambda_j = k\sum_{j\geq 2}\lambda_j + \sum_{2\leq i<j\leq q+1}\lambda_i\lambda_j + \sum_{2\leq i\leq q+1, q+2\leq j\leq n}\lambda_i\lambda_j + \sum_{q+2\leq i<j\leq n}\lambda_i\lambda_j$$

$$\begin{aligned}
&= k\cdot q(-\tfrac{1}{r} + O(r))\sin\theta + kO(1)\sin\theta \\
&\quad + q(q-1)(-\tfrac{1}{r} + O(r))^2\sin^2\theta \\
&\quad + q(-\tfrac{1}{r} + O(r))O(1)\sin^2\theta \\
&\quad + O(1)\sin^2\theta.
\end{aligned}$$

Thus,

$$\begin{aligned}
\tfrac{1}{2}R^M = &\tfrac{1}{2}R^N - \sin^2\theta \cdot Ric^N(\partial_r, \partial_r) \\
&-q\tfrac{k\sin\theta}{r} + k \cdot qO(r)\sin\theta + k \cdot O(1)\sin\theta \\
&+q(q-1)(\tfrac{1}{r^2} + O(1))\sin^2\theta \\
&-\tfrac{q}{r}\sin^2\theta \cdot O(1) + q\sin^2\theta \cdot O(r) \\
&+\sin^2\theta \cdot O(1)
\end{aligned}$$

$$\begin{aligned}
= &\tfrac{1}{2}R^N - \sin^2\theta \cdot [Ric^N(\partial_r, \partial_r) + q(q-1)O(1) + O(1) + qO(r)] \\
&-k \cdot q\tfrac{\sin\theta}{r} + q(q-1)\tfrac{\sin^2\theta}{r^2} + k \cdot qO(r)\sin\theta \\
&+k \cdot O(1)\sin\theta - \tfrac{q}{r}\sin^2\theta \cdot O(1)
\end{aligned}$$

$$\begin{aligned}
= &\tfrac{1}{2}R^N + \sin^2\theta \cdot O(1) - [k \cdot q\tfrac{\sin\theta}{r} - k \cdot O(1)\sin\theta] \\
&+[q(q-1)\tfrac{\sin^2\theta}{r^2} - \tfrac{q}{r}\sin^2\theta \cdot O(1)] \\
&+k \cdot qO(r)\sin\theta.
\end{aligned}$$

When r is small, this reduces to

$$\begin{aligned}
R^M = &R^N + \sin^2\theta \cdot O(1) - 2k \cdot q\tfrac{\sin\theta}{r} \\
&+2q(q-1)\tfrac{\sin^2\theta}{r^2} + k \cdot qO(r)\sin\theta.
\end{aligned}$$

□

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