THE CROSSED PRODUCT OF $C(X)$ BY A FREE MINIMAL ACTION OF $\mathbb{R}$

by

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In this dissertation, we will study the crossed product C*-algebras obtained from free and minimal $\mathbb{R}$ actions on compact metric spaces with finite covering dimension. We first define stable recursive subhomogeneous algebras (SRSHAs), which differ from recursive subhomogeneous algebras introduced by N. C. Phillips in that the irreducible representations of SRSHAs are infinite dimensional instead of finite dimensional. We show that simple inductive limits of SRSHAs with no dimension growth in which the connecting maps are injective and non-vanishing have topological stable rank one. We then construct C*-subalgebras of the crossed product that are analogous to the C*-subalgebras in the studies of free minimal $\mathbb{Z}$ actions on compact metric spaces with finite covering dimension. Finally, we prove that these C*-algebras are in fact simple inductive limits of SRSHAs in which the connecting maps are injective and non-vanishing. Thus these C*-subalgebras have topological stable rank one.

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For my parents and my wife.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

This dissertation is on crossed product C^*-algebras obtained from free and minimal $\mathbb{R}$ actions on unital abelian C^*-algebras, or transformation group C^*-algebras. A C^*-algebra can either be regarded as a subalgebra of the algebra $B(H)$ of the bounded operators on a Hilbert space H that are closed in norm and adjoint operation, or be defined abstractly using a set of axioms:

Definition I.0.1. *Let A be a Banach algebra with a *-operation $A \to A$, denoted $a \mapsto a^*$. We say A is a C^*-algebra if*

1. *the *-operation is conjugate linear;*
2. *for all $a, b \in A$, we have $(ab)^* = b^*a^*$;*
3. *for all $a \in A$, we have $(a^*)^* = a$;*
4. *for all $a \in A$, we have $\|a^*a\| = \|a\|^2$.*

The condition $\|a^*a\| = \|a\|^2$ for all $a \in A$ is called the C^*-norm condition, and a norm that satisfies this condition is called a C^*-norm.

When a topological group G acts on a C^*-algebra A by automorphisms, we can form the crossed product C^*-algebra. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a group homomorphism that is continuous when $\mathrm{Aut}(A)$ has the topology of pointwise convergence. For each $s \in G$, we use α_s to denote the image of s under α. Then on the linear space $C_c(G, A)$ of all continuous functions from G into A with compact support, we can define multiplication and a *-operation, which are often respectively called convolution and involution in this context, as follows: for all $f, g \in C_c(G, A)$, define convolution by

$$(f * g)(s) = \int_G f(t)\alpha_t(g(t^{-1}s))dt, \text{ for all } s \in G;$$

for all $f \in C_c(G, A)$, define involution by

$$f^*(s) = \Delta(s)^{-1}\alpha_s(f(s^{-1})^*), \text{ for all } s \in G,$$

where the measure on G is taken to be the left Haar measure, and where Δ is the modular function associated with the left Haar measure. With convolution and involution defined as above $C_c(G, A)$ becomes a *-algebra. On $C_c(G, A)$, we can put the norm defined by $\|f\|_1 = \int_G \|f(s)\|ds$, which is called the L^1-norm for obvious reasons. Denote the completion of $C_c(G, A)$ with respect to the L^1-norm by $L^1(G, A)$. Then $L^1(G, A)$ becomes a Banach-*-algebra. However $L^1(G, A)$ is not a C^*-algebra because the L^1-norm is not a C^*-norm.

In general, there are two different C^*-norms, the universal norm and the reduced norm, that we can put on $L^1(G, A)$ that will make $L^1(G, A)$ into a C^*-algebra after completion. The universal norm is defined to be

$$\|f\| = \sup\{\|\pi(f)\| : \pi \text{ is a representation of } L^1(G, A)\}.$$

A representation of a *-algebra B is a pair (π, H), where H is a Hilbert space, and where $\pi: B \to B(H)$ is a linear and multiplicative map that also preserves the *-operation. In order for the supremum to be well defined, we need to ensure that there is at least one representation of $L^1(G, A)$, and that $\{\|\pi(f)\| : \pi \text{ is a representation of } L^1(G, A)\}$ is a bounded set. The boundedness of $\{\|\pi(f)\| : \pi \text{ is a representation of } L^1(G, A)\}$ is automatic because any representation of any Banach-*-algebra is automatically norm reducing (Theorem 2.1.7 in [6]). To exhibit one representation of $L^1(G, A)$, we invoke the GNS construction. By the GNS construction, we know that any C^*-algebra has a representation (Section 3.4 in [6]), so the C^*-algebra A has a representation (π, H). The space $L^2(G, H)$ of all L^2 integrable measurable functions is a Hilbert space; or equivalently $L^2(G, H)$ is the Hilbert space tensor product $L^2(G) \otimes H$. Then we can define a representation $\lambda_\pi\colon C_c(G,A) \to B(L^2(G,H))$ by

$$\lambda_\pi(f)(\xi)(r) = \int_G [\pi(\alpha_r^{-1}(f(s)))][\xi(s^{-1}r)]ds$$

for all $f \in C_c(G,A)$, all $\xi \in L^2(G,H)$, and all $r \in G$. Routine calculations show that λ_π is L^1-norm decreasing, and hence extends to a representation of $L^1(G,A)$. See Chapter 2 in [17] for more details. The representation $(\lambda_\pi, L^2(G,H))$ obtained from the representation (π, H) is known as the left regular representation induced by (π, H). The reduced norm on $L^1(G,A)$ is defined to be

$$\|f\|_r = \sup\{\|\lambda_\pi(f)\|\colon \pi \text{ is a representation of } A\}.$$

The completion of $L^1(G,A)$ in the universal norm is called full crossed product, or just the crossed product, and will be denoted by $C^*(A,G,\alpha)$. The completion of $L^1(G,A)$ in the reduced norm is called the reduced crossed product and will be denoted $C^*_r(A,G,\alpha)$.

It is well known that when the group G is amenable, the universal norm and the reduced norm coincide (Theorem 7.13 in [17]). We will not go into details about amenability of groups, but it follows from Proposition A.16 in [17] that $\mathbb{R}$ is amenable. In this dissertation we only consider the group $\mathbb{R}$, so we will not distinguish the reduced crossed product from the full crossed product, nor the reduced norm from the universal norm. Further, we will only consider the crossed products of $C(X)$, the algebra of all continuous functions from X into $\mathbb{C}$, by $\mathbb{R}$, where X is a compact metric space with finite covering dimension, and where the action on $C(X)$ is induced by a free and minimal action of $\mathbb{R}$ on X. In this case, we will denote the crossed product by $C^*(X,\mathbb{R})$. We will use $s \cdot x$, or just simply sx to denote the action, for $s \in \mathbb{R}$ and $x \in X$. It is clear that we can identify the linear space $C_c(\mathbb{R}, C(X))$ with $C_c(\mathbb{R} \times X)$, the space of all continuous functions from the product space $\mathbb{R} \times X$ into the complex number $\mathbb{C}$ with compact support. Also, it is known that the Lebesgue measure on $\mathbb{R}$ is the left Haar measure for $\mathbb{R}$, and that the modular function Δ for the Lebesgue measure is the constant 1, i.e. $\mathbb{R}$ with the Lebesgue measure is unimodular. Then the convolution on $C_c(\mathbb{R} \times X)$ is given by the formula

$$(f * g)(r,x) = \int_{\mathbb{R}} f(t,x)g(r-t,(-t)x)dt; \tag{I.1}$$

and the involution on $C_c(\mathbb{R} \times X)$ is given by the formula

$$f^*(r,x) = \overline{f(-r,(-r)x)}. \tag{I.2}$$

It is known that the reduced normed on a crossed product $C^*(A, G, \alpha)$ can be obtained from just one of left regular representations induced by a faithful (injective) representations of A (Theorem 7.13 in [17]). That is, if (π, H) is a faithful representation of A, then $\|f\|_r = \|\lambda_\pi(f)\|$. Since the direct sum of all irreducible representations of $C(X)$ is faithful, and since the irreducible representations of $C(X)$ are the point evaluations, the universal norm (which is the same as the reduced norm) on $C_c(\mathbb{R} \times X)$ is given by

$$\|f\| = \sup_{x \in X} \|\lambda_x(f)\|, \tag{I.3}$$

where for each $x \in X$, the representation $\lambda_x \colon C_c(\mathbb{R} \times X) \to L^2(\mathbb{R})$ is the left regular representation induced by the evaluation map ev_x of $C(X)$ at x. We can quickly verify that λ_x is given by

$$\lambda_x(f)(\xi)(r) = \int_{\mathbb{R}} f(r-t, rx)\xi(t)dt, \tag{I.4}$$

for all $f \in C_c(\mathbb{R} \times X)$, all $\xi \in L^2(\mathbb{R})$, and all $r \in \mathbb{R}$.

If we consider the action of the group on a single orbit and forget about the topology, we quickly realize that the action is essentially the action of the group on itself by left translation. However, due to the minimality of the action, every orbit of the action is dense in the space X, and it becomes quite difficult to see how the orbits are tied together topologically. So we resort to the method of "orbit breaking" to simplify the dynamics, and obtain a structure theorem for certain distinguished C^*-subalgebras of $C^*(X, \mathbb{R})$.

The "orbit breaking" method was introduced by I. F. Putnam in the study of free and minimal actions of the group $\mathbb{Z}$ of integers on the Cantor set. Let $u \in C^*(X, \mathbb{Z})$ be the standard unitary, let $Y \subseteq C(X)$ be a closed subset, and let $C_0(X \setminus Y)$ be the space of all continuous functions from $X \setminus Y$ into $\mathbb{C}$ that vanish at infinity. In this case, finite dimensional C^*-subalgebras are constructed using partitions of the Cantor set X into clopen sets, and it is shown that the C^*-subalgebra A_Y of the crossed product generated by $C(X)$ and $uC_0(X \setminus Y)$ is an inductive limit of those finite dimensional subalgebras. See [10] and [11] for more details.

In [5], a similar idea is used on the crossed product of $C(X)$ by a free and minimal $\mathbb{Z}$ action, where X is an arbitrary compact metric space X with finite covering dimension, to obtain a structure theorem for the C^*-subalgebras A_Y generated by $C(X)$ and $uC_0(X \setminus Y)$. In this case, closed subsets Y of X with nonempty interior are used to break the orbits. Every orbit is broken into partial orbits that start and end in Y which do not go through Y in between. Upon collecting the partial orbits together, it is shown that the C^*-subalgebra A_Y is obtained by "gluing" finitely many homogeneous algebras together, i.e. is a recursive subhomogeneous algebra. Then shrinking a sequence of decreasing closed subsets with nonempty interior to the point y, it was shown that A_y, the C^*-subalgebra generated by $C(X)$ and $uC_0(X \setminus \{y\})$, is a simple inductive limit of recursive subhomogeneous algebras with no dimension growth.

Recursive subhomogeneous algebras were introduced by N. C. Phillips in [8]. This class of C^*-algebras is a useful technical tool for studying transformation group C^*-algebras. In [9], a stable rank reduction theorem is obtained, i.e. it is shown that a simple inductive limit of recursive subhomogeneous algebras with no dimension growth has topological stable rank one. (See [16] for the definition of topological stable rank.) In [3], H. Lin and N. C. Phillips show that the subalgebras A_y of the crossed product of $C(X)$ by a free and minimal action of $\mathbb{Z}$ have tracial rank zero given that certain hypothesis about traces hold. In the same paper, this result is used to show that the crossed product has tracial rank zero under the same hypothesis about traces.

In this dissertation, we similarly use the "orbit breaking" method to study the crossed products of $C(X)$ by free and minimal $\mathbb{R}$ actions. When the group that is acting is $\mathbb{R}$, the subalgebras A_Y are no longer obtained by "gluing" homogeneous algebras together; but rather, they are obtained by "gluing" algebras of the form $C(Z) \otimes \mathbb{K}$, where Z is a compact metric space with finite covering dimension, and $\mathbb{K}$ is the algebra of compact operators on the separable infinite dimensional Hilbert space. Thus we first define "stable recursive subhomogeneous algebras", analogous to recursive subhomogeneous algebras, to accommodate this change. We will also obtain a stable rank reduction theorem for simple inductive limits stable recursive subhomogeneous algebras with no dimension growth. Then we construct the analogs for actions of $\mathbb{R}$ of A_Y and A_y. Recall that, in the integer case, A_Y is defined to be the C^*-subalgebra generated by $C(X)$ and $uC_0(X \setminus Y)$, and A_y is defined to be the C^*-subalgebra generated by $C(X)$ and $uC_0(X \setminus \{y\})$. However, when the group is not discrete, the unitaries that implement the action and the algebra $C(X)$ are not contained in the crossed product. So we have to resort to other methods to define the analogous subalgebras. Finally we will show that the A_Y is a stable recursive homogeneous algebra, and that A_y is a simple inductive limit of the algebras A_Y with no dimension growth, and has have topological stable rank one.

CHAPTER II

STABLE RECURSIVE SUBHOMOGENEOUS ALGEBRAS

Recursive subhomogeneous algebras, abbreviated RSHA, are introduced by N. C. Phillips in [8]. Essentially, a RSHA is an iterated pull back of algebras of the form $C(X, M_n)$, where the spaces X are taken to be compact Hausdorff space, M_n is the algebra of $n \times n$-matrices, and $C(X, M_n)$ is the algebra of all continuous functions from X into M_n. It is well known that $C(X, A) = C(X) \otimes A$ for any C^*-algebra A. In some sense, a recursive subhomogeneous algebra is formed by "gluing" finitely many algebras of the form $C(X, M_n)$ together. In this chapter, we introduce an analogous "stable" version of RSHA, and establish a topological stable rank reduction result.

We will use $\mathbb{K}$ to denote the algebra of all compact operators on the separable infinite dimensional Hilbert space throughout the dissertation. If A is a C^*-algebra, we will take $C(\varnothing, A)$ to be the zero algebra.

II.1. Definitions

Definition II.1.1. *Let A, B be C^*-algebras, let X be a compact Hausdorff space, and let $\phi\colon A \to C(X, B)$ be a *-homomorphism. We say ϕ is* non-vanishing *if for all $x \in X$, there exists some $a \in A$ such that $\phi(a)(x) \neq 0$.*

Note that in the above definition, if $X = \varnothing$, then ϕ is vacuously non-vanishing.

Definition II.1.2. *Let H be a separable infinite dimensional Hilbert space and let $\mathbb{K}$ denote the set of all compact operators on H. Let n be a positive integer, let $X_1, \ldots, X_n$ be compact Hausdorff spaces, let $X_k^{(0)} \subseteq X_k$ be closed subspaces for $k = 2, \ldots, n$, and let $R_k\colon C(X_k, \mathbb{K}) \to C(X_k^{(0)}, \mathbb{K})$ be the restriction map for $k = 2, \ldots, n$. For each k with $2 \leq k \leq n$, let $\phi_k\colon A^{(k-1)} \to C(X_k^{(0)}, \mathbb{K})$ be a* *non-vanishing *-homomorphism, let* $A^{(1)} = C(X_1, \mathbb{K})$, *and inductively define*

$$A^{(k)} = \{(a, b) \in A^{(k-1)} \oplus C(X_k, \mathbb{K}) \colon \phi_k(a) = R_k(b)\}.$$

We call

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

a stable recursive sub-homogeneous system, *abbreviated SRSH system, and call the algebra* $A^{(n)}$ *the* stable recursive sub-homogeneous algebra, *abbreviated by SRSHA, corresponding to the system.*

Let A *be a* C^**-algebra. We say that* A *has a* stable recursive sub-homogeneous decomposition *if there exists a stable recursive sub-homogeneous system*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

such that $A \cong A^{(n)}$, *in which case we also say that* A *is a stable recursive sub-homogeneous algebra, and call the system a stable recursive sub-homogeneous decomposition of* A.

The integer n *is called the* length *of the system (or the decomposition). The spaces* $X_1, \ldots, X_n$ *are called the* bases spaces *of the system. The space* $X = \bigsqcup_{k=1}^{n} X_k$ *is called the* total space *of the system. The spaces* $X_2^{(0)}, \ldots, X_n^{(0)}$ *are called the* attaching spaces *of the system. The maps* $R_2, \ldots, R_k$ *are called the* restriction maps *of the system. The maps* $\phi_2, \phi_3, \ldots, \phi_n$ *are called the* attaching map *of the system. For each* $k \in \{1, \ldots, n\}$, *the algebra* $A^{(k)}$ *is called* k-th partial algebra *of the system.*

Note that a SRSH system of length 1 is simply $(X_1, C(X_1, \mathbb{K}))$. For a SRSHA A, the decomposition is by no means unique. We allow any or all of the attaching spaces to be the empty set. If $X_k^{(0)} = \varnothing$ for some k, then $A^{(k)}$ is simply $A^{(k-1)} \oplus C(X_k, \mathbb{K})$. If A has a stable SRSH decomposition

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right),$$

then A is a C^*-subalgebra of $\bigoplus_{k=1}^{n} C(X_k, \mathbb{K})$; also for each $k \in \{1, \ldots, n\}$, the k-th partial algebra is also a SRSHA with the decomposition being

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^{k}\right).$$

Let $a = (a_1, \ldots, a_n) \in A$ and let x be in the total space X. Then there exists unique k such that $x \in X_k$. We will use $a(x)$ to denote $a_k(x)$. So for each $x \in X$, the map $A \to \mathbb{K}$ sending $a \mapsto a(x)$ is a clearly *-homomorphism. If $1 \le k \le l \le n$, then it is easily verified that the map $p_{l,k}\colon A^{(l)} \to A^{(k)}$ defined by $p_{l,k}(a_1, \ldots, a_l) = (a_1, \ldots, a_k)$ is a surjective *-homomorphism. If $1 \le k \le l \le m \le n$, then $p_{m,k} = p_{l,k} \circ p_{m,l}$.

II.2. Ideals and Homomorphisms of SRSHAs

In this section we establish some results about the spectrum, primitive ideal space, and ideals of a SRSHA. We will use $\widehat{A}$ to denote the spectrum of A, i.e. the space of all irreducible representations of A, and if π is an irreducible representation of A, we will use $[\pi]$ to denote the corresponding element in $\widehat{A}$. We will use $\mathrm{Prim}(A)$ to denote the primitive ideal space of A. The next lemma is a standard result.

Lemma II.2.1. *Let X be a locally compact Hausdorff space and let $A = C_0(X, \mathbb{K})$. For each $x \in X$, let $\mathrm{ev}_x\colon A \to \mathbb{K}$ be defined by $\mathrm{ev}_x(f) = f(x)$. Then*

1. *the map $X \to \widehat{A}$ defined by $x \mapsto [\mathrm{ev}_x]$ is a well defined bijection;*
2. *the map $X \to \mathrm{Prim}(A)$ defined by $x \mapsto \{f \in A\colon f(x) = 0\}$ is a well-defined bijection.*

Lemma II.2.2. *Let n be a positive integer. Let*

$$(X_1, A^{(1)}, (X_k, X_k^{(0)}, \psi_k, R_k, A^{(k)})_{k=2}^n)$$

be a stable recursive sub-homogeneous system and let $A = A^{(n)}$. Let $X_1^{(0)} = \varnothing$. Then

1. *the map $M\colon \bigsqcup_{k=1}^n (X_k \setminus X_k^{(0)}) \to \mathrm{Prim}(A)$ defined by $M(x) = \{a \in A\colon a(x) = 0\}$ is a well defined bijection.*
2. *for each $x \in \bigsqcup_{k=1}^n (X_k \setminus X_k^{(0)})$, the evaluation map $\mathrm{ev}_x\colon A \to \mathbb{K}$, given by $a \mapsto a(x)$, is non-zero; also the map $S\colon \bigsqcup_{k=1}^n (X_k \setminus X_k^{(0)}) \to \widehat{A}$ defined by $S(x) = [\mathrm{ev}_x]$ is a well defined bijection.*

Proof: Induct on n. The case when $n = 1$ is given by Lemma II.2.1. Suppose that statement holds for some n, let

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \psi_k, R_k, A^{(k)}\right)_{k=2}^{n+1}\right)$$

be a SRSH system of length $n+1$ and let $A = A^{(n+1)}$.

Let $1 \le i \le n+1$ and let $x \in X_i \setminus X_i^{(0)}$. Define $\pi\colon A^{(n+1)} \to \mathbb{K}$ by $\pi(f_1, \dots, f_{n+1}) = f_i(x)$. Then π is a clearly a *-homomorphism. Let $a \in \mathbb{K}$. Choose $h \in C(X_i)$ such that $h(x) = 1$ and supp $h \subseteq X_i \setminus X_i^{(0)}$, and let $f \in C(X_i, \mathbb{K})$ be defined by $f(y) = h(y)a$. Then supp $f \subseteq X_i \setminus X_i^{(0)}$. Hence $R_i(f) = f|_{X_i^{(0)}} = 0 = \psi_i(0)$, and so $(0, \dots, 0, f) \in A^{(i)}$. Since the map $A^{(n+1)} \to A^{(i)}$ defined by $(g_1, \dots, g_{n+1}) \mapsto (g_1, \dots, g_i)$ is surjective, there exist $g_{i+1}, \dots, g_{n+1}$ such that $\xi = (0, \dots, 0, f, g_{i+1}, \dots, g_{n+1}) \in A^{(n+1)}$. Then $\pi(\xi) = f(x) = a$. Thus $\pi = \mathrm{ev}_x$ maps onto $\mathbb{K}$, and so π is non-zero and irreducible. This shows that the map S defined in part 2 of the statement of the lemma is well defined. Further, this also shows that

$$\{(g_1, \dots, g_{n+1}) \in A^{(n+1)} \colon g_i(x) = 0\} = \ker \pi \in \mathrm{Prim}(A^{(n+1)}),$$

and so M defined in part 1 of the statement of the lemma is well defined.

Now consider

$$I_{n+1} = \{(f_1, \dots, f_n, f_{n+1}) \in A^{(n+1)} \colon (f_1, \dots, f_n) = 0\}.$$

Then it is clear that I_{n+1} is a closed two sided ideal of A. Note that if $(f_1, \dots, f_{n+1}) \in I_{n+1}$, then $0 = \psi_{n+1}(f_1, \dots, f_n) = R_{n+1}(f_{n+1})$, and so f_{n+1} vanishes on $X_{n+1}^{(0)}$. Define

$$\phi\colon I_{n+1} \to C_0(X_{n+1} \setminus X_{n+1}^{(0)}, \mathbb{K})$$

by $\phi(f_1, \dots, f_{n+1}) = f_{n+1}|_{X_{n+1} \setminus X_{n+1}^{(0)}}$. This map is well defined because if $(f_1, \dots, f_{n+1}) \in I_{n+1}$, then f_{n+1} vanishes on $X_{n+1}^{(0)}$, so $f_{n+1} \in C_0(X_{n+1} \setminus X_{n+1}^{(0)}, \mathbb{K})$. Then it is clear that ϕ is a *-isomorphism.

Now let $\pi\colon A \to B(H)$ be a non-zero irreducible representation. First assume that $\pi|_{I_{n+1}}\colon I_{n+1} \to B(H)$ is not the zero representation. Then $\pi|_{I_{n_{n+1}}}$ is also irreducible. Thus $\pi \circ \phi^{-1}$ is an irreducible representation of $C_0(X_{n+1} \setminus X_{n+1}^{(0)}, \mathbb{K})$, and so by Lemma II.2.1 there exists $x \in X_{n+1} \setminus X_{n+1}^{(0)}$, such that $[\pi \circ \phi^{-1}] = [\mathrm{ev}_x]$. Then there exists a unitary u such that

$\pi \circ \phi^{-1} = \mathrm{Ad}(u) \circ \mathrm{ev}_x$, where $\mathrm{Ad}(u)\colon \mathbb{K} \to \mathbb{K}$ is defined by $\mathrm{Ad}(u)(a) = uau^*$. Define $\pi'\colon A \to B(H)$ by $\pi'(f_1, \ldots, f_{n+1}) = \mathrm{Ad}(u)(f_{n+1}(x))$. Then $\pi|_{I_{n+1}} = \pi'|_{I_{n+1}}$. Since $\pi|_{I_{n+1}} = \pi'|_{I_{n+1}}$ is irreducible, hence non-degenerate, we have $\pi = \pi'$. Then $S(x) = [\pi'] = [\pi]$.

Now suppose that $\pi|_{I_{n+1}} = 0$. Define $\psi\colon A^{(n+1)} \to A^{(n)}$ by $\psi(f_1, \ldots, f_{n+1}) = (f_1, \ldots, f_n)$. Consider the short exact sequence

$$0 \to I_{n+1} \to A^{(n+1)} \xrightarrow{\psi} A^{(n)} \to 0.$$

Since π restricts to zero on I_{n+1}, π factors through $A^{(n)}$. That is, there exists $\widetilde{\pi}\colon A^{(n)} \to B(H)$ such that $\widetilde{\pi} \circ \psi = \pi$. Then $\mathrm{Im}\ \pi = \mathrm{Im}\ \widetilde{\pi}$. Since π is irreducible, we see that $\widetilde{\pi}$ is also irreducible. Thus by the inductive hypothesis, we see that there exists some $1 \le i \le n$ and some $x \in X_i \setminus X_i^{(0)}$ such that $[\widetilde{\pi}] = [\mathrm{ev}_x]$. So there exists a unitary such that $\widetilde{\pi}(f) = \mathrm{Ad}(u)(f(x))$ for all $f \in A^{(n)}$. Then for all $f = (f_1, \ldots, f_n, f_{n+1}) \in A^{(n+1)}$, we have $\pi(f) = \widetilde{\pi}(\psi(f)) = \widetilde{\pi}(f_1, \ldots, f_n) = \mathrm{Ad}(u)(f_i(x))$. Thus $[\pi] = S(x)$, and hence S is surjective. If $J \in \mathrm{Prim}(A)$, then there exists some irreducible representation π of A such that $J = \ker \pi$. So there exists $x \in \bigsqcup_{k=1}^{n+1} \left(X_k \setminus X_k^{(0)} \right)$ such that $[\mathrm{ev}_x] = [\pi]$. It follows that

$$J = \ker \pi = \ker \mathrm{ev}_x = \{a \in A\colon a(x) = 0\} = M(x).$$

Thus M is also surjective.

Next we show that M and S are injective. Let $x, y \in \bigsqcup_{k=1}^{n+1} (X_k \setminus X_k^{(0)})$ and suppose that $x \ne y$. First assume that there exist $1 \le j < k \le n$ such that $x \in X_j \setminus X_j^{(0)}$ and $y \in X_k \setminus X_k^{(0)}$. Let $h \in C(X_k)$ satisfy $h(y) = 1$ and $\mathrm{supp}\ h \subseteq X_k \setminus X_k^{(0)}$, let $a \in \mathbb{K}$ be a non-zero element, let $f = ah$, and let $b = (0, \ldots, 0, f) \in A^{(k)}$. Let $f_{k+1}, \ldots, f_{n+1}$ be such that $g = (b, f_{k+1}, \ldots, f_{n+1}) \in A^{(n+1)}$. Then $g(x) = 0$, but $g(y) = a \ne 0$. Thus $g \in M(x)$, but $g \notin M(y)$, and so $M(x) \ne M(y)$. Since $M(x) = \ker \mathrm{ev}_x$ and $M(y) = \ker \mathrm{ev}_y$, we have $S(y) = [\mathrm{ev}_y] \ne [\mathrm{ev}_x] = S(x)$. Now suppose that $x, y \in X_k \setminus X_k^{(0)}$ for some $1 \le k \le n$. Since x, y are different, there exists an open $U \subseteq X_k \setminus X_k^{(0)}$ such that $y \in U$, but $x \notin U$. Choose $h \in C(X_k)$ such that $h(y) = 1$ and h vanishes outside of U. Let $a \in \mathbb{K}$ be non-zero. Let $f = ah$. Then f vanishes on $X_k^{(0)}$. So there exist

$$g_{k+1} \in C(X_{k+1}, \mathbb{K}), \ldots, g_{n+1} \in C(X_{n+1}, \mathbb{K})$$

such that $g = (0, \ldots, 0, f, g_{k+1}, \ldots, g_{n+1})$ belongs to A. Then $g(x) = f(x) = 0$ and $g(y) = f(y) = a$. It follows that $g \in M(x)$, but $g \notin M(y)$. So $M(y) \neq M(x)$, and consequently $S(x) \neq S(y)$. □

Corollary II.2.3. *Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \psi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

be a stable recursive sub-homogeneous system and let $A = A^{(n)}$. *Let* $X_1^{(0)} = \varnothing$. *Then for all* $x, y \in \bigsqcup_{k=1}^{n}(X_k \setminus X_k^{(0)})$ *with* $x \neq y$, *there exist some* $a, b \in A$ *such that* $a(x) = 0$, $a(y) \neq 0$, $b(x) \neq 0$, *and* $b(y) = 0$.

Proof: First suppose that $x \in X_j \setminus X_j^{(0)}$ and $y \in X_k \setminus X_k^{(0)}$, where $1 \leq j < k \leq n$. Then the element $a \in A$ needed is constructed in the last paragraph of the proof of II.2.2. Next we construct the element b. Let $h \in C(X_j)$ be such that $h(x) = 1$ and h vanishes on $X_j^{(0)}$, let $\xi \in \mathbb{K}$ be non-zero, and let $f = h\xi$. Then $(0, \ldots, 0, f) \in A^{(j)}$. Choose $b' \in A^{(k-1)}$ such that the first j entries of b' are $(0, \ldots, 0, f)$. Let $c = \phi_k(b')$. Let V be an open neighborhood of $X_k^{(0)}$ that does not contain y, and choose $h' \in C(X_k)$ such that $h'|_{X_k^{(0)}} = 1$ and h' vanishes outside of V. Let c' be any extension of c over X_k, and let $f' = h'c'$. Then $f'|_{X_k^{(0)}} = c = \phi_k(b')$. So $(b', f') \in A^{(k)}$. Choose $b \in A$ such that the first k entries of b are (b', f'). Then $b(x) = f(x) = \xi \neq 0$, and $b(y) = f'(y) = h'(y)c'(y) = 0$.

Now suppose that $x, y \in X_k \setminus X_k^{(0)}$. Let U_x and U_y be two disjoint open sets contained in $X_k \setminus X_k^{(0)}$ such that $x \in U_x$ and $y \in U_y$. Choose $h_x \in C(X_k)$ and $h_y \in C(X_k)$ such that $h_x(x) = 1$ and $h_y(y) = 1$, h_x vanishes outside of U_x, and h_y vanishes outside of U_y. Let $\xi \in \mathbb{K}$ be non-zero. Let $f_x = h_x\xi$, and $f_y = h_y\xi$. Then $a' = (0, \ldots, f_y) \in A^{(k)}$ and $b' = (0, \ldots, 0, f_x) \in A^{(k)}$. Let $a, b \in A$ be such that the first k entries of a and b are, respectively, a' and b'. Then

$$a(x) = a'(x) = f_y(x) = 0,$$

$$a(y) = a'(y) = f_y(y) = \xi \neq 0,$$

$$b(x) = b'(x) = f_x(x) = \xi \neq 0,$$

$$b(y) = b'(y) = f_x(y) = 0.$$

□

Corollary II.2.4. *Let n be a positive integer. Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \psi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

be a stable recursive sub-homogeneous system, and let $A = A^{(n)}$. Let $X_1^{(0)} = \varnothing$. Let $I \subseteq A$ be a closed two sided ideal of A. Then there exists a closed set $F \subseteq X = \bigsqcup_{k=1}^{n} X_k$ such that $I = \{a \in A \colon a|_F = 0\}$.

Proof: Let I be a closed two sided ideal of A. If $I = 0$, then take $F = X$. If $I = A$, then take $F = \varnothing$. Now assume that I is proper and non-zero. Recall that for any C^*-algebra B and for any closed two sided ideal I of B, the hull of I, denoted by $\mathrm{hull}(I)$, is the set of all primitive ideals of B that contain I; and for any subset $S \subseteq \mathrm{Prim}(B)$, the kernel of S, denoted by $\ker(S)$ is the intersection of all the members of S. We know that $I = \ker(\mathrm{hull}(I))$. Let M be as in Lemma II.2.2. Let $F = \overline{M^{-1}(\mathrm{hull}(I))}$. We will verify that $I = \{a \in A \colon a|_F = 0\}$. Let J denote $\{a \in A \colon a|_F = 0\}$.

Let $a \in I$, and let $x \in M^{-1}(\mathrm{hull}(I))$. Then $M(x) \in \mathrm{hull}(I)$, and so $a \in I \subseteq M(x)$. So $a(x) = 0$. This holds for all $x \in M^{-1}(\mathrm{hull}(I))$. Thus a vanishes on $M^{-1}(\mathrm{hull}(I))$. Since a is continuous, $a|_F = 0$. So $a \in J$, and so $I \subseteq J$. Now suppose that $a \in J$. Let $L \in \mathrm{hull}(I)$. Then there exists $x \in X$ such that $L = M(x)$, and so $x \in M^{-1}(\mathrm{hull}(I)) \subseteq F$. The condition $a \in J$ implies that $a(x) = 0$, which implies that $a \in M(x) = L$. This holds for all $L \in \mathrm{hull}(I)$, so $a \in \ker(\mathrm{hull}(I)) = I$. Thus $J \subseteq I$, and so $I = J$. □

The next theorem is a restatement of Theorem 1.4.4 in [1].

Theorem II.2.5. *Let H be an arbitrary Hilbert space, and let $A \subseteq K(H)$ be a non-zero C^*-subalgebra. Then there exists an index set I and a family $(p_i)_{i \in I}$ of mutually orthogonal projections in $B(H)$, indexed by I, such that*

1. *$p_i \in A'$ for all $i \in I$, where A' denotes the commutant of A;*
2. *$p_i A p_i = K(p_i H)$ for all $i \in I$ (we identify $K(p_i H)$ with $p_i K(H) p_i$ in an obvious way);*
3. *$\|a\| = \sup_{i \in I} \|p_i a p_i\|$ for all $a \in A$;*
4. *$\sum_{i \in I} p_i a p_i$ converges to a in norm for all $a \in A$;*
5. *for all $a \in A$ and for all $\epsilon > 0$, there exists a finite subset $F \subseteq I$ such that $\|p_i a p_i\| < \epsilon$ for all $i \notin F$.*

Proposition II.2.6. *Let H be a separable infinite dimension Hilbert space and let $\mathbb{K}$ denote the set of all compact operators on H. Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

*be a SRSH system whose underlying Hilbert space is H. Let $A = A^{(n)}$. Let $X_1^{(0)} = \varnothing$. Let $\phi\colon A \to K(H)$ be a non-zero *-homomorphism. Then there exists an index set I, a family $(p_i)_{i\in I}$ of mutually orthogonal projections in $B(H)$, a family $(w_i)_{i\in I}$ of isometries in $B(H)$, and a family $(x_i)_{i\in I}$ of elements in $\bigsqcup_{k=1}^{n}(X_k \setminus X_k^{(0)})$ (note that we do not assume that the x_i are mutually distinct) such that*

1. *$p_i \in \phi(A)'$ for all $i \in I$, where $\phi(A)'$ denotes the commutant of $\phi(A)$;*
2. *$w_i^* w_i = 1$ and $w_i w_i^* = p_i$ for all $i \in I$;*
3. *$\phi(a) = \sum_{i\in I} w_i a(x_i) w_i^*$ for all $a \in A$, where the convergence is in norm;*
4. *$\|\phi(a)\| = \sup_{i\in i} \|a(x_i)\|$ for all $a \in A$;*
5. *I is a finite set.*

Proof: It is clear that $\phi(A)$ is a non-zero C^*-subalgebra of $\mathbb{K}$. Apply Theorem II.2.5 to $\phi(A)$ to get the index set I and the family of mutually orthogonal projections $(p_i)_{i\in I}$. Then part 1 of the proposition holds holds. For each $i \in I$, define $\phi_i\colon A \to K(p_i H)$ by $\phi_i(a) = p_i \phi(a) p_i$. By part 1 of this proposition, ϕ_i is a well defined *-homomorphism. It is clear that

$$\phi_i(A) = p_i \phi(A) p_i \subseteq p_i K(H) p_i = K(p_i H).$$

Then part 2 of Theorem II.2.5 implies that $\phi_i(A) = K(p_i H)$. Thus $(\phi_i, p_i H)$ is an irreducible representation of A. So by Lemma II.2.2, there exists a unitary $w_i\colon H \to p_i H$ and some $x_i \in \bigsqcup_{k=1}^{n}(X_k \setminus X_k^{(0)})$ such that $\phi_i(a) = w_i a(x_i) w_i^*$ for all $a \in A$. Identifying w_i as an element of $B(H)$ in the obvious way (identify w_i with the composition inclusion $p_i H \to H$ followed by w_i), the element w_i is an isometry in $B(H)$. Then it is clear that part 2 of this proposition holds. By part 4 of Theorem II.2.5, we have

$$\phi(a) = \sum_{i\in I} p_i\phi(a)p_i = \sum_{i\in I}\phi_i(a) = \sum_{i\in I} w_i a(x_i) w_i^*$$

for all $a \in A$, where the convergence is in norm. So part 3 holds. By part 3 of Theorem II.2.5, we have

$$\|\phi(a)\| = \sup_{i\in I}\|p_i\phi(a)p_i\| = \sup_{i\in I}\|\phi_i(a)\| = \sup_{i\in I}\|w_i a(x_i)w_i^*\| = \sup_{i\in I}\|a(x_i)\|.$$

So 4 holds.

Finally we show that I is a finite set by contradiction. Suppose that I is an infinite set. Let S denote the set $\{x_i \in X : i \in I\}$, where $X = \bigsqcup_{k=1}^n X_k$. We claim that there are distinct $i_l \in I$ for $l \in \mathbb{N}$ such that $i_l \neq i_{l'}$ if $l \neq l'$, and that the sequence $(x_{i_l})_{l=1}^\infty$ converges to some $x_0 \in X$. To prove this claim, if S is finite, then there exists some $y \in S$ such that the set $\{i \in I \colon x_i = y\}$ is infinite. In this case take a sequence of mutually distinct indices $(i_l)_{l=1}^\infty$ in $\{i \in I \colon x_i = y\}$. Then clearly $x_{i_l} = y \to y$. If S is infinite, then, since X is compact, we can pick a countable mutually distinct subset elements $y_1, y_2, \ldots \in\subseteq S$ such that $y_n \to x_0$ for some $x \in X$. For each $l \geq 1$, choose $i_l \in I$ such that $x_{i_l} = y_l$. Then the indices $i_1, i_2, \ldots$ are necessarily mutually distinct, and $x_{i_l} = y_l \to x_0$. This proves the claim.

Now we show that for all $a \in A$, $\|a(x_{i_l})\| \to 0$. Let $a \in A$, and let $\epsilon > 0$. By part 5 of Theorem II.2.5, there exists a finite subset $F \subseteq I$ such that $i \notin F$ implies that

$$\|p_i\phi(a)p_i\| = \|\phi_i(a)\| = \|w_i a(x_i) w_i\| = \|a(x_i)\| < \epsilon.$$

Since F is finite, there exists $l_0 \geq 1$ such that if $l \geq l_0$ then $i_l \notin F$. Thus for all $l \geq l_0$, we have $\|a(x_{i_l})\| < \epsilon$. This shows that $\|a(x_{i_l})\| \to 0$ for all $a \in A$.

Since a is continuous for all $a \in A$, we have $a(x_0) = 0$ for all $a \in A$. Then the map $A \to \mathbb{K}$ defined by $a \mapsto a(x_0)$ is the zero map, hence $x_0 \in \bigsqcup_{k=1}^n X_k^{(0)}$, because by Lemma II.2.2, for all $y \in X \setminus \left(\bigsqcup_{k=1}^n X_k^{(0)}\right)$, the map $a \mapsto a(y)$ is an irreducible representation and hence cannot be the zero map. Suppose that $x_0 \in X_k^{(0)}$ for some $k \in \{1, \ldots, n\}$. Now, we assumed that the map $\phi_k \colon A^{(k-1)} \to C(X_k^{(0)}, \mathbb{K})$ is non-vanishing, so there exists some $b \in A^{(k-1)}$ such that $\phi_k(b)(x_0) \neq 0$. Then, since the map $A^{(n)} \to A^{(k-1)}$ defined by $(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_{k-1})$ is surjective, there exists some $a = (a_1, \ldots, a_n) \in A$ such that $(a_1, \ldots, a_{k-1}) = b$. Thus

$$a(x_0) = R_k(a_k)(x_0) = \phi_k(b)(x_0) \neq 0.$$

This contradicts the fact that $a(x_0) = 0$ for all $a \in A$. This means that I has to be finite. □

Definition II.2.7. *Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \psi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

*be a SRSH system, and let $A = A^{(n)}$. Let $\phi \colon A \to \mathbb{K}$ be a non-zero *-homomorphism. Then by Proposition II.2.6, there exists $x_1, \ldots, x_m \in \bigsqcup_{k=1}^{n}(X_k \setminus X_k^{(0)})$ and isometries $w_1, \ldots, w_m$ with orthogonal ranges such that $\phi(a) = \sum_{i=1}^{m} w_i a(x_i) w_i^*$ for all $a \in A$. We call the set $\{x_1, \ldots, x_n\}$ (not counting multiplicity) the* spectrum of ϕ, *and we will denote the spectrum of ϕ by* $\mathrm{sp}(\phi)$. *Let*

$$\left(Y_1, B^{(1)}, \left(Y_k, Y_k^{(k)}, \phi_k, Q_k, B^{(k)}\right)_{k=2}^{m}\right)$$

*be another SRSH system, let $B = B^{(m)}$, and let $\phi \colon A \to B$ be a *-homomorphism. We say that ϕ is* non-vanishing *if, for all $y \in \bigsqcup_{k=1}^{m} Y_k$, the map $A \to \mathbb{K}$ defined by* $\mathrm{ev}_y \circ \phi$ *is not the zero map. In this case, will call* $\mathrm{sp}(\mathrm{ev}_y \circ \phi)$ *the* spectrum of ϕ at y *and write* $\mathrm{sp}_y(\phi)$.

In the previous definition, it is not necessary to insist on ϕ being non-vanishing to define $\mathrm{sp}_y(\phi)$. If $\mathrm{ev}_y \circ \phi = 0$ for some y, then $\mathrm{sp}_y(\phi)$ would simply be the empty set. The condition that ϕ is non-vanishing guarantees that $\mathrm{sp}_y(\phi) \neq \varnothing$ for all $y \in \bigsqcup_{i=1}^{m} Y_i$.

The spectrum of a *-homomorphism between homogeneous algebras was used in [2] to show that simple inductive limits of homogeneous algebras with no dimension growth have topological stable rank one. One of the key steps is that if the inductive limit is simple, then the spectra of the connecting *-homomorphisms of the inductive system, in a sense, become more and more "dense" when we follow the connecting maps of the inductive limit further and further out. We will prove a similar result in our situation. We will first need a few preliminary results, and some results that will be used later in this dissertation.

Lemma II.2.8. *Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right),$$

$$\left(Y_1, B^{(1)}, \left(Y_k, Y_k^{(0)}, \psi_k, T_k, B^{(k)}\right)_{k=2}^{m}\right),$$

and

$$\left(Z_1, C^{(1)}, \left(Z_k, Z_k^{(0)}, \theta_k, S_k, C^{(k)}\right)_{k=2}^{l}\right)$$

be three SRSH systems, and let $A = A^{(n)}$, $B = B^{(m)}$, *and* $C = C^{(l)}$. *Let* $\phi\colon A \to B$ *and* $\psi\colon B \to C$ *be non-vanishing *-homomorphisms. Then* $\psi \circ \phi$ *is non-vanishing.*

Proof: Let $z \in \bigsqcup_{i=1}^{l} Z_k$. Since ψ is non-vanishing, the map $\mathrm{ev}_z \circ \psi$ is non-zero. So there exists $t \in \mathbb{N}$ with $t > 0$, and isometries $w_1, \ldots, w_t$, with orthogonal ranges such that $\psi(b)(z) = \sum_{i=1}^{t} w_i b(y_i) w_i^*$ for all $b \in B$, where $\{y_1, \ldots, y_t\} = \mathrm{sp}_z(\psi) \neq \varnothing$. Since ϕ is non-vanishing, there exists some $a \in A$ such that $\phi(a)(y_1) \neq 0$. Then $\|\psi(\phi(a))(z)\| \geq \|\phi(a)(y_1)\| > 0$, and hence $\psi \circ \phi$ is non-vanishing. □

Lemma II.2.9. *Let n be a positive integer. Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

be a SRSH system and let $A = A^{(n)}$. *Let* $X_1^{(0)} = \varnothing$ *and let* $X = \bigsqcup_{k=1}^{n} X_k$.

1. *Let* $U \subseteq X$ *be an open subset. Then* $I_U = \{a \in A\colon a|_{U^c} = 0\}$ *is a closed two sided ideal of* A. *Further, let* $U_k = U \cap X_k$ *for* $k \in \{1, \ldots, n\}$, *and let*

$$W_k = \left\{x \in X_k^{(0)}\colon \mathrm{sp}_x(\phi_k) \cap \left(\bigsqcup_{i=1}^{k-1} U_i\right) \neq \varnothing\right\}$$

for each $k = 2, \ldots, n$. *Suppose that*

$$U \neq \varnothing \text{ and } W_k = U_k \cap X_k^{(0)} \text{ for } k = 2, \ldots, n. \tag{II.1}$$

Then $I_U \neq 0$, *and*

$$U = \{x \in X\colon \text{there exists some } a \in I_U \text{ such that } a(x) \neq 0\}.$$

2. Let $I \subseteq A$ be a non-zero ideal. Then the set

$$U = \{x \in X \colon \textit{there exists some } a \in A \textit{ such that } a(x) \neq 0\}$$

is open in X and satisfies the condition II.1 in part 1. Also $I_U = I$.

Proof: For part 1, we induct on the length of the SRSH system. If $n = 1$, then result is trivial. Suppose that result holds for systems of length n, and let

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n+1}\right)$$

be a system of length $n+1$. Let U, $U_1, \dots, U_{n+1}$ and $W_1, \dots, W_{n+1}$ be as given in the statement of the lemma.

It is clear that I_U is a closed two sided ideal of A. Let $V = \bigsqcup_{k=1}^{n} U_k$. First suppose that $V \neq \varnothing$. Then by the induction hypothesis, $J_V = \{a \in A^{(n)} \colon a|_{V^c} = 0\}$ is a non-zero ideal. So let $b \in J_V$ be nonzero. Now, for all $x \in X_{n+1}^{(0)} \setminus W_{n+1}$, we have $\mathrm{sp}_x(\phi_{n+1}) \subseteq V^c$. Since b vanishes on V^c, the function $\phi_{n+1}(b)$ also vanishes outside of W_{n+1}. If $W_{n+1} = \varnothing$, then $\phi_{n+1}(b) = 0$. Thus $(b, 0) \in I_U$ and $(b, 0) \neq 0$. So assume that $W_{n+1} \neq \varnothing$. Since W_k is closed in U_{n+1}, we can extend $\phi_{n+1}(b)$ to some $f \in C_0(U_{n+1}, \mathbb{K})$. Since $U_{n+1} \subseteq X_{n+1}$ is open, we can define $f(x) = 0$ for all $x \notin U_{n+1}$, so that $f \in C(X_{n+1}, \mathbb{K})$. Then $R_{n+1}(f) = \phi_{n+1}(b)$, and so $(b, f) \in I_U$ and $(b, f) \neq 0$. Thus $I_U \neq 0$.

Now suppose that $V = \varnothing$. Then $W_{n+1} = \varnothing$, and so $U_{n+1} \subseteq X_{n+1} \setminus X_{n+1}^{(0)}$. Since $U_{n+1} \neq \varnothing$ (otherwise $U = \varnothing$), there exists $f \in C(X_{n+1}, \mathbb{K})$ such that f vanishes outside of U_{n+1} and $f \neq 0$. Then $(0, \dots, 0, f) \in I_U$ and $(0, \dots, 0, f) \neq 0$. So $I_U \neq 0$.

It is clear that

$$\{x \in X \colon \text{there exists some } a \in I_U \text{ such that } a(x) \neq 0\} \subseteq U.$$

Now let $x \in U$. Let k be the integer such that $x \in U_k$. First suppose that $1 \leq k \leq n$. Let $W = \bigsqcup_{i=1}^{n} U_i$. Then by the induction hypothesis, we have

$$W = \{x \in X \colon \text{there exists some } a \in I_W \subseteq A^{(n)} \text{ such that } a(x) \neq 0\}.$$

So there exists some $b \in I_W$ such that $b(x) \neq 0$. An argument similar to the one given in the second paragraph of this proof give some $f \in C(X_{n+1}, \mathbb{K})$ such that $a = (b, f) \in I_U$. Then $a(x) = b(x) \neq 0$. Therefore

$$x \in \{y \in X : \text{there exists some } a \in I_U \text{ such that } a(y) \neq 0\}.$$

Now suppose that $k = n + 1$. Assume that $x \in X_{n+1}^{(0)}$. Then $x \in W_{n+1}$, which means that there exists some $y \in \mathrm{sp}_x(\phi_{n+1}) \cap (\bigsqcup_{i=1}^{n} U_i)$. By what is shown in the previous paragraph, there exists some $a \in I_U$ such that $a(y) \neq 0$. Then

$$\|a(x)\| = \sup_{z \in \mathrm{sp}_x(\phi_{n+1})} \|a(z)\| \geq \|a(y)\| > 0,$$

so $a(x) \neq 0$, and so

$$x \in \{y \in X : \text{there exists some } a \in I_U \text{ such that } a(y) \neq 0\}.$$

Finally assume that $x \notin X_{n+1}^{(0)}$. Let $\xi \in \mathbb{K}$ be non-zero and choose $h \in C(X_{n+1})$ such that $h(x) = 1$ and h vanishes outside of $U_{n+1} \cap (X_{n+1} \setminus X_{n+1}^{(0)})$. Let $f = \xi h$. Then $a = (0, \ldots, 0, f) \in A$, and a vanishes outside of U. So $a \in I_U$, and $a(x) = f(x) = \xi \neq 0$. Therefore

$$x \in \{y \in X : \text{there exists some } a \in I_U \text{ such that } a(y) \neq 0\}.$$

Thus

$$U = \{x \in X : \text{there exists some } a \in I_U \text{ such that } a(x) \neq 0\}.$$

For part 2, we first note that $U = \bigcup_{a \in I}\{x \in X : a(x) \neq 0\}$ is open in X, and that U cannot be empty. Let $U_1, \ldots, U_{n+1}$ and $W_2, \ldots, W_n$ be as given in part 1. Let $k \in \{2, \ldots, n\}$. Let $x \in W_k$ and let $y \in \mathrm{sp}_x(\phi_k) \cap \left(\bigsqcup_{i=1}^{k-1} U_i\right)$. Let $a \in I$ satisfy $a(y) \neq 0$. Then

$$\|a(x)\| = \sup_{z \in \mathrm{sp}_x(\phi_k)} \|a(z)\| \geq \|a(y)\| > 0.$$

Thus $a(x) \neq 0$. So $x \in U_k$, and so $x \in U_k \cap X_k^{(0)}$.

Now suppose that $x \in U_k \cap X_k^{(0)}$. Then $a(x) \neq 0$ for some $a \in I$. Let $a = (b, g_1, \ldots, g_l)$, where $b \in A^{(k-1)}$. Then $\|a(x)\| = \sup_{z \in \mathrm{sp}_x(\phi_k)} \|b(z)\|$. Now, since b vanishes outside of $\bigsqcup_{i=1}^{k-1} U_i$, if

$\mathrm{sp}_x(\phi_k) \subseteq \left(\bigsqcup_{i=1}^{k-1} U_i\right)^c$, then $\|a(x)\| = 0$, and so $a(x) = 0$. Since $a(x) \neq 0$, we have

$$\mathrm{sp}_x(\phi_k) \cap \left(\bigsqcup_{i=1}^{k-1} U_i\right) \neq \varnothing.$$

So $x \in W_k$. Thus $W_k = U_k \cap X_k^{(0)}$.

It is clear that $I \subseteq I_U$. Now we know that there exists some closed subset $F \subseteq X$ such that $I = \{a \in A \colon a|_F = 0\}$. Since for all $x \in U$, there exists some $a \in I$ such that $a(x) \neq 0$, we have $F \subseteq U^c$. Then a belonging to I_U implies a vanishes on U^c, and so a vanishes on F. So $a \in I$. Thus $I_U \subseteq I$, and hence $I = I_U$. □

Lemma II.2.10. *Let*

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

be a SRSH system, and let $A = A^{(n)}$. Let $X = \bigsqcup_{k=1}^{n} X_k$. Then there exists some $a \in A$ such that $a(x) \neq 0$ for all $x \in X$.

Proof: Induct on the length of the system. The result clearly holds for $n = 1$. Suppose that result holds for systems of length n, let

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n+1}\right)$$

be a SRSH system, and let $A = A^{(n+1)}$.

Now,

$$\left(X_1, A^{(1)}, \left(X_k, X_k^{(0)}, \phi_k, R_k, A^{(k)}\right)_{k=2}^{n}\right)$$

is a system of length n, so by inductive hypothesis, $A^{(n)}$ contains some a_0 such that $a_0(x) \neq 0$ for all $x \in \bigsqcup_{k=1}^{n} X_k$. Let $a = a_0^* a_0$. Then $a(x) \geq 0$ for all $x \in X$, and $a(x) \neq 0$ for all $x \in X$. Let $b = \phi_{n+1}(a)$. Because a vanishes nowhere, and because ϕ_{n+1} is non-vanishing, we have $b(x) \neq 0$ and $b(x) \geq 0$ for all $x \in X_{n+1}^{(0)}$. Extend b to some positive element $b' \in C(X_{n+1}, \mathbb{K})$. Let

$$U = \{x \in X_{n+1} \colon b'(x) \neq 0\}.$$

It is clear that U is an open neighborhood of $X_{n+1}^{(0)}$. Then $\{U, X_{n+1} \setminus X_{n+1}^{(0)}\}$ is an open cover for X_{n+1}. Let $\{h_1, h_2\}$ be a partition of unity subordinate to $\{U, X_{n+1} \setminus X_{n+1}^{(0)}\}$. (Without loss of generality, assume that supp $h_1 \subseteq U$, and supp $h_2 \subseteq X_{n+1} \setminus X_{n+1}^{(0)}$.) Let $\xi \in \mathbb{K}$ be a non-zero positive element. Let $f = h_1 b' + h_2 \xi$. Then if $x \in X_{n+1}^{(0)}$, we have

$$f(x) = h_1(x)b'(x) + h_2(x)\xi = b'(x) = b(x) = \phi_{n+1}(a)(x).$$

Thus $(a, f) \in A$. Now let $x \in X_{n+1}$. If $h_1(x) \neq 0$, then $x \in U$, and then $h_1(x)b'(x) \neq 0$. Since $f(x) \geq h_1(x)b'(x)$, we have $f(x) \neq 0$. If $h_1(x) = 0$, then $h_2(x) = 1$, and so $h_2(x)\xi = \xi \neq 0$. Since $f(x) \geq h_2(x)\xi$, we have $f(x) \neq 0$. Thus f vanishes nowhere. Then the element (a, f) vanishes nowhere on X. (That is (a, f) is not contained in any non-zero proper ideal of A.) □

The next proposition shows that in a simple inductive limit in which the connecting maps are injective and non-vanishing, the spectra of the connecting maps become more and more dense, in some sense. If A is a set and if B is a subset of A, we use B^c to denote the complement of B.

Proposition II.2.11. *Let (A_n, ψ_n) be an inductive system of SRSHAs and let A be the inductive limit. Let X_n be the total space for A_n. Suppose that ψ_n is injective for all n, that ψ_n is non-vanishing for all n, and that A is simple. Then for all $n \geq 1$, and for all open set $U \subseteq X_n$ such that $I_U = \{a \in A_n \colon a|_{U^c} = 0\}$ is a non-zero ideal, there exists $n_0 \geq n$ such that for all $k \geq n_0$ and for all $x \in X_k$, we have $\mathrm{sp}_x(\psi_{n,k}) \cap U \neq \varnothing$, where $\psi_{i,j} = \psi_{j-1} \circ \cdots \circ \psi_{i+1} \circ \psi_i$ for $i \leq j$.*

Proof: This will be a proof by contradiction. Suppose that there exists $m \geq 1$ and some open set $U \subseteq X_m$ with $I_U \neq 0$, such that for all $n \geq m$, there exists some $k_n \geq n$ and some $x \in X_{k_n}$ such that $\mathrm{sp}_x(\psi_{m,k_n}) \cap U = \varnothing$. Then U certainly cannot be the entire space X_n. Without loss of generality, we can assume that $k_n < k_{n+1} < k_{n+2} < \cdots$. Then, passing to a subsequence of the inductive system and truncating if necessary, we can assume that $m = 1$, and that $k_n = n$ for all $n \geq 1$. Thus we are assuming that there exists some open subset $U \subseteq X_1$ with $I_U \neq 0$ such that for all $n \geq 1$, there exists some $x \in X_n$ such that $\mathrm{sp}_x(\psi_{1,n}) \cap U = \varnothing$. It is clear that $U \neq X_1$.

For each $n \geq 1$, let $\psi^n \colon A_n \longrightarrow A$ be the natural injection that comes with the inductive limit. Also let

$$V = \{x \in X_1 \colon \text{ there exists some } b \in I_U \text{ such that } b(x) \neq 0\}.$$

It is clear that $V \subseteq U$. Then for all $n \geq 1$, there exists some $x \in X_n$ such that

$$\mathrm{sp}_x(\psi_{1,n}) \cap V \subseteq \mathrm{sp}_x(\psi_{1,n}) \cap U = \varnothing.$$

By Lemma II.2.9, we have $I_V = I_U \neq 0$. For each $n \geq 2$, let $F_n = \overline{\{x \in X_n \colon \mathrm{sp}_x(\psi_{1,n}) \cap V = \varnothing\}}$. Then $F_n \neq \varnothing$ for all $n \geq 2$. Let $I_n = \{a \in A_n \colon a|_{F_n} = 0\}$. Let $I_1 = I_V$. For each $n \geq 1$, let $J_n = \psi^n(I_n)$, and let $B_n = \psi^n(A_n)$. Then J_n is a closed two sided ideal of B_n. We first show that $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$. Fix $n \geq 1$, and let $a \in I_n$. Let $x_0 \in \{x \in X_{n+1} \colon \mathrm{sp}_x(\psi_{1,n+1}) \cap V = \varnothing\}$. Let $y \in \mathrm{sp}_{x_0}(\psi_n)$.

Suppose that $\mathrm{sp}_y(\psi_{1,n}) \cap V \neq \varnothing$. Let $z \in \mathrm{sp}_y(\psi_{1,n}) \cap V$, and let $b \in I_1 = I_V$ be such that $b(z) \neq 0$. Then

$$\|\psi_{1,n+1}(b)(x_0)\| = \|\psi_n(\psi_{1,n}(b))(x_0)\| \geq \|\psi_{1,n}(b)(y)\| \geq \|b(z)\| > 0.$$

But b vanishes outside of V, so if $x \in X_{n+1}$ satisfies $\mathrm{sp}_x(\psi_{1,n+1}) \cap V = \varnothing$, then

$$\|\psi_{1,n+1}(b)(x)\| = \sup_{z' \in \mathrm{sp}_x(\psi_{1,n+1})} \|b(z')\| = 0;$$

hence in particular $\psi_{1,n+1}(b)(x_0) = 0$. This contradicts the fact that $\|\psi_{1,n+1}(b)(x_0)\| > 0$. Thus $\mathrm{sp}_y(\psi_{1,n}) \cap V = \varnothing$.

Then $y \in F_n$, and so $a(y) = 0$. This holds for all $y \in \mathrm{sp}_{x_0}(\psi_n)$, so $\psi_n(a)(x_0) = 0$. This holds for all $x_0 \in X_{n+1}$ such that $\mathrm{sp}_{x_0}(\psi_{1,n+1}) \cap U = \varnothing$, so $\psi_n(a)|_{F_{n+1}} = 0$, and so $\psi_n(a) \in I_{n+1}$. Then $\psi^n(a) = \psi^{n+1}(\psi_n(a)) \in \psi^{n+1}(I_{n+1}) = J_{n+1}$. This holds for all $a \in I_n$, so $J_n = \psi^n(I_n) \subseteq J_{n+1}$. This holds for all $n \geq 1$, so we have $J_1 \subseteq J_2 \subseteq \cdots$.

Then $J = \overline{\bigcup_{n \geq 1} J_n}$ is an ideal of A. The ideal J cannot be 0, because ψ^1 is injective and $I_1 \neq 0$. Finally we show that $J \neq A$. Let $a \in A_1$ satisfy $a(x) \neq 0$ for all $x \in X_1$. Then compactness of X_1 gives that there exists $\epsilon > 0$ such that $\|a(x)\| \geq \epsilon$ for all $x \in X_1$. For all $n \geq 2$ and for all $x \in X_n$, we have $\|\psi_{1,n}(a)(x)\| = \sup_{y \in \mathrm{sp}_x(\psi_{1,n})} \|a(y)\| \geq \epsilon$. For all $n \geq 2$, and for all $b \in I_n$, we have

$$\|\psi_{1,n}(a) - b\| \geq \|\psi_{1,n}(a)|_{F_n} - b|_{F_n}\| = \|\psi_{1,n}(a)|_{F_n}\| \geq \epsilon.$$

Then for all $n \geq 1$ and for all $b \in I_n$, we have

$$\|\psi^1(a) - \psi^n(b)\| = \|\psi^n(\psi_{1,n}(a)) - \psi^n(b)\| = \|\psi_{1,n}(a) - b\| \geq \epsilon.$$

Thus $\psi^1(a) \notin J$. So $J \neq A$.

This shows that J is a non-zero proper ideal of A, which contradicts the simplicity of A. □

II.3. Topological Stable Rank of Simple Inductive Limits of SRSHAs

The first few lemmas of this section will be some trivial or nearly trivial results about functional calculus and semi-continuity of spectral projections at self-adjoint elements in $\mathbb{K}$, which the author of this dissertation has not encountered. These may or may be be written down explicitly in the literature. Then, through several lemmas, we adapt Lemma 3.3 in [9], which is the key lemma in showing that simple inductive limits of RSHAs with no dimension growth have topological stable rank one, to our situation. The last portion of the section will be dedicated to showing that if A is simple inductive limit of SRSHAs with no dimension growth such that all the connecting maps are injective and non-vanishing, then A has topological stable rank one.

Lemma II.3.1. *Let π be a polynomial with complex coefficients, let $M > 0$ be a positive real number, and let $\epsilon > 0$. Then there exists $\delta > 0$ such that if A is a unital C^*-algebra, and if $a, b \in A$ satisfy $\|a\| \leq M$, $\|b\| \leq M$, and $\|a - b\| < \delta$, then $\|\pi(a) - \pi(b)\| < \epsilon$.*

Proof: Let $n \in \mathbb{N}$ and let $\lambda_0, \lambda_1 \ldots, \lambda_n \in \mathbb{C}$ be such that $\pi(\xi) = \sum_{i=0}^{n} \lambda_i \xi^i$ for all $\xi \in \mathbb{C}$. If $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$, then $\pi(a) = 0$ for all $a \in A$, and the result follows trivially. So assume that not all of $\lambda_0, \ldots, \lambda_n$ are 0. Let

$$\delta = \frac{\epsilon}{\sum_{k=1}^{n}(k|\lambda_k|M^{k-1})} > 0.$$

Then $\delta > 0$. Let A be a unital C^*-algebra, and let $a, b \in A$ satisfy $\|a\| \leq M$, $\|b\| \leq M$, and

$\|a-b\|<\delta$. Then

$$\begin{aligned}
\|\pi(a)-\pi(b)\| &= \|\sum_{k=0}^{n}\lambda_k(a^k-b^k)\| = \|\sum_{k=1}^{n}\lambda_k(a^k-b^k)\| \\
&\leq \sum_{k=1}^{n}(|\lambda_k|\cdot\|a^k-b^k\|) \\
&= \sum_{k=1}^{n}[|\lambda_k|\cdot(\|a^k-a^{k-1}b+a^{k-1}b-a^{k-2}b^2 \\
&\qquad + a^{k-2}b^2-\cdots-ab^{k-1}+ab^{k-1}-b^k\|)] \\
&\leq \sum_{k=1}^{n}[|\lambda_k|\cdot(\|a^k-a^{k-1}b\|+\|a^{k-1}b-a^{k-2}b^2\|+\cdots+\|ab^{k-1}-b^k\|)] \\
&< \sum_{k=1}^{n}[|\lambda_k|\cdot kM^{k-1}\delta] \\
&= \delta\sum_{k=1}^{n}[|\lambda_k|\cdot kM^{k-1}] = \epsilon.
\end{aligned}$$

□

Corollary II.3.2. *Let $M>0$ be a real number, let $f\in C([-M,M])$, and let $\epsilon>0$. Then there exists a $\delta>0$ such that if A is a unital C^*-algebra, and if $a,b\in A$ are self-adjoint elements such that $\|a\|\leq M$, $\|b\|\leq M$, and $\|a-b\|<\delta$, then $\|f(a)-f(b)\|<\epsilon$.*

Proof: Since $[-M,M]$ is compact, there exists a polynomial π such that $\|\pi|_{[-M,M]}-f\|_\infty<\epsilon/3$. Apply Lemma II.3.1 to π, M, and $\epsilon/3$ to get $\delta>0$. Let A be a unital C^*-algebra, and let $a,b\in A$ be self-adjoint elements such that $\|a\|\leq M$, $\|b\|\leq M$, and $\|a-b\|<\delta$. Then

$$\begin{aligned}
\|f(a)-f(b)\| &\leq \|f(a)-\pi(a)\|+\|\pi(a)-\pi(b)\|+\|\pi(b)-f(b)\| \\
&\leq \|f|_{\mathrm{sp}(a)}-\pi|_{\mathrm{sp}(a)}\|_\infty+\epsilon/3+\|f|_{\mathrm{sp}(b)}-\pi|_{\mathrm{sp}(b)}\|_\infty \\
&< \epsilon/3+\epsilon/3+\epsilon/3 \\
&= \epsilon.
\end{aligned}$$

□

Corollary II.3.3. *Let $M>0$ be a real number, let $f\in C([0,M])$, and let $\epsilon>0$. Then there exists some $\delta>0$ such that if A is a unital C^*-algebra, and if $a,b\in A$ are positive elements such that*

$\|a\| \leq M$, $\|b\| \leq M$, *and* $\|a - b\| < \delta$, *then* $\|f(a) - f(b)\| < \epsilon$.

Proof: Extend f to f' over $[-M, M]$, then apply Corollary II.3.2 with f replaced by f'. □

Lemma II.3.4. *Let A be a C^*-algebra, let $\widetilde{A}$ denote the unitization of A, and let 1 be the adjoined identity. (Here, we add a new identity to A even if A is already unital.) Let $a \in A$ be self-adjoint and let $\widetilde{a} = a + 1$. Then*

1. $\mathrm{sp}(a) + 1 = \mathrm{sp}(\widetilde{a})$ *where both spectra are taken with respect to $\widetilde{A}$.*
2. *Let $h\colon \mathrm{sp}(\widetilde{a}) \to \mathrm{sp}(a)$ be defined by $h(\xi) = \xi - 1$ and let $h^*\colon C(\mathrm{sp}(a)) \to C(\mathrm{sp}(\widetilde{a}))$ be defined by $h^*(f) = f \circ h$. Let $F\colon C(\mathrm{sp}(a)) \to \widetilde{A}$ and let $\widetilde{F}\colon C(\mathrm{sp}(\widetilde{a})) \to \widetilde{A}$ be the functional calculus (with respect to $\widetilde{A}$) at a and $\widetilde{a}$ respectively. Then $F = \widetilde{F} \circ h^*$.*

Proof: Part 1 is trivial. To prove part 2, note that $\widetilde{a} = h^{-1}(a)$. Then if $f \in C(\mathrm{sp}(a))$, we have

$$\widetilde{F} \circ h^*(f) = h^*(f)(\widetilde{a}) = h^*(f)(h^{-1}(a)) = (f \circ h)(h^{-1}(a)) = (f \circ h \circ h^{-1})(a) = f(a) = F(f).$$

□

For all C^*-algebras A and all $a \in A$, we use $|a|$ to denote $(a^*a)^{1/2}$. We use $\chi_\alpha\colon \mathbb{R} \to \mathbb{R}$ to denote the characteristic function of $(-\infty, \alpha)$ for all $\alpha \in \mathbb{R}$. Also, for all C^*-algebras A and all self-adjoint $a \in A$, we use $p_\alpha(a)$ to denote $\chi_\alpha(a)$. Even though $p_\alpha(a)$ may not be in A for some combinations of a, A and α, it is still in the double commutant of A when A is faithfully represented on a Hilbert space. For our purposes, A will be either the algebras of compact operators on separable Hilbert spaces, or their unitization; and α will be less then the limit point of $\mathrm{sp}(a)$ (if any). In these cases $p_\alpha(a)$ will be a finite rank projection, and hence in A. Then the next corollary follows immediately from Lemma II.3.4.

Corollary II.3.5. *Let $a \in \mathbb{K}_{s.a.}$, let $1 > \alpha > 0$, and let $\widetilde{a} = a + 1$. Then $p_\alpha(\widetilde{a}) = p_{\alpha-1}(a)$.*

Lemma II.3.6. *Let A be a unital C^*-algebra and let $p_1, p_2 \in A$ be orthogonal projections such that $p_1 + p_2 = 1$. Let A_1 and A_2 be C^*-subalgebras of A such that p_i is the identity of A_i for $i = 1, 2$. Let $a_1 \in A_1$ and $a_2 \in A_2$.*

1. *Then $\mathrm{sp}_A(a_1 + a_2) = \mathrm{sp}_{A_1}(a_1) \cup \mathrm{sp}_{A_2}(a_2)$, where $\mathrm{sp}_B(b)$ denotes the spectrum of b with respect to B for all C^*-algebra B and any $b \in B$.*

2. *Suppose that a_1 and a_2 are self-adjoint. Let F_i be the functional calculus of a_i with respect to A_i, for $i = 1, 2$, and let F be the functional calculus of $a_1 + a_2$ with respect to A. Then for all $f \in C(\mathrm{sp}_A(a_1 + a_2))$, we have $F(f) = F_1(f) + F_2(f)$, that is, $f(a_1 + a_2) = f(a_1) + f(a_2)$.*

Proof: First assume that $A_i = p_iAp_i$ for $i = 1, 2$. Let $\lambda \in \mathbb{C}$. If $\lambda - (a_1 + a_2)$ is invertible in A, then there exists some $b \in A$ such that $b(\lambda - a_1 - a_2) = (\lambda - a_1 - a_2)b = 1 = p_1 + p_2$, and b commutes with p_1 and p_2. So p_1bp_1 and p_2bp_2 are the inverses of $\lambda p_1 - a_1$ and $\lambda p_2 - a_2$ in A_1 and A_2, respectively, and so $\lambda p_1 - a_1$ and $\lambda p_2 - a_2$ are both invertible. On the other hand, if both $\lambda p_1 - a_1$ and $\lambda p_2 - a_2$ are invertible, then there exists $b_i \in A_i$ such that $b_i = (\lambda p_i - a_i)^{-1}$ for $i = 1, 2$. Then $b_1 + b_2 = (\lambda - a_1 - a_2)^{-1}$. Thus $\lambda \notin \mathrm{sp}_A(a_1 + a_2)$ if and only if $\lambda \notin \mathrm{sp}_{A_1}(a_1) \cap \mathrm{sp}_{A_2}(a_2)$. So result follows. Now assume that A_i is an arbitrary C^*-algebra of A that contains p_i as its identity, for $i = 1, 2$. Then for $i = 1, 2$, A_i is a C^*-algebra of p_iAp_i that contains the identity of p_iAp_i, so $\mathrm{sp}_{p_iAp_i}(a_i) = \mathrm{sp}_{A_i}(a_i)$. Thus

$$\mathrm{sp}_A(a_1 + a_2) = \mathrm{sp}_{p_1Ap_1}(a_1) \cup \mathrm{sp}_{p_2Ap_2}(a_2) = \mathrm{sp}_{A_1}(a_1) \cup \mathrm{sp}_{A_2}(a_2),$$

and part 1 or the lemma is proven.

Since $a_1a_2 = a_2a_1 = 0$, it is easy to verify that if π is a polynomial on $\mathrm{sp}_A(a_1 + a_2)$, then $\pi(a_1) + \pi(a_2) = \pi(a_1 + a_2)$, where functional calculus on the left side of the equation is taken in the subalgebras A_i, $i = 1, 2$, and the functional calculus on the right side of the equation is taken in A. So the continuous map $C(\mathrm{sp}_A(a_1 + a_2)) \to A$ defined by $f \mapsto f(a_1) + f(a_2)$, where the respective functional calculus is taken in the subalgebra, agrees with the map $f \mapsto f(a_1 + a_2)$ on the set of all polynomials, which is dense in $C(\mathrm{sp}_A(a_1 + a_2))$. Hence the result follows. □

From II.3.6, a standard induction argument shows the following:

Corollary II.3.7. *Let A be a unital C^*-algebra, and let $p_1, \ldots, p_n \in A$ be orthogonal projections such that $p_1 + p_2 + \cdots + p_n = 1$. Let A_i be a C^*-subalgebra of A such that p_i is the identity of A_i for $i = 1, 2, \ldots, n$. Let $a_i \in A_i$, $i \in \{1, \ldots, n\}$.*

1. *Then $\mathrm{sp}_A\left(\sum_{i=1}^n fa_i\right) = \bigcup_{i=1}^n \mathrm{sp}_{A_i}(a_i)$.*

2. *Suppose that a_i is self-adjoint for $i \in \{1, \ldots, n\}$. Let F_i be the functional calculus of a_i with respect to A_i for $i \in \{1, \ldots, n\}$ and let F be the functional calculus of $\sum_{i=1}^n a_i$ with*

respect to A. Then for all $f \in C\left(\mathrm{sp}_A\left(\sum_{i=1}^n a_i\right)\right)$, *we have* $F(f) = \sum_{i=1}^n F_i(f)$, *that is,* $f\left(\sum_{i=1}^n a_i\right) = \sum_{i=1}^n f(a_i)$.

The next few results are about the semicontinuity of spectral projections.

Lemma II.3.8. *Let* $\epsilon > 0$, *let* $0 < \alpha_1 < \alpha_2 < 1$, *and let* $M \geq 1$ *be a real number . Then there exists some* $\delta > 0$ *such that if* $a, b \in \mathbb{K}_{s.a.}$, $\widetilde{a} = a+1$, $\widetilde{b} = b+1$, $\|\widetilde{a}\| \leq M$, $\|\widetilde{b}\| \leq M$, *and* $\|\widetilde{a}-\widetilde{b}\| < \delta$, *then*

$$\|p_{\alpha_1}(\widetilde{a})p_{\alpha_2}(\widetilde{b}) - p_{\alpha_1}(\widetilde{a})\| < \epsilon$$

and

$$\mathrm{rank}(p_{\alpha_1}(\widetilde{a})) \leq \mathrm{rank}(p_{\alpha_2}(\widetilde{b})).$$

Proof: We know that there exists a $\sigma_0 > 0$ such that if p, q are projections in $\mathbb{K}$ such that $\|pq - q\| < \sigma_0$, then $\mathrm{rank}(q) \leq \mathrm{rank}(p)$. Let $\sigma = \min\{\epsilon, \sigma_0\}$.

Define $f\colon [-M, M] \to [0,1]$ by

$$f(t) = \begin{cases} 1 & t \in [-M, \alpha_1] \\ \frac{\alpha_2 - t}{\alpha_2 - \alpha_1} & t \in [\alpha_1, \alpha_2] \\ 0 & t \in [\alpha_2, M]. \end{cases}$$

Then it is clear that $f \in C([-M, M])$. Apply Corollary II.3.2 to M, f, and $\sigma/2$, to get $\delta > 0$. Let $a, b \in \mathbb{K}_{s.a.}$, $\widetilde{a} = a + 1$, and $\widetilde{b} = b + 1$. Then $\widetilde{a}, \widetilde{b} \in \widetilde{\mathbb{K}}$, which is unital. Suppose that $\|\widetilde{a}\| \leq M$, $\|\widetilde{b}\| \leq M$, and that $\|\widetilde{a} - \widetilde{b}\| < \delta$. By the choice of δ, we have $\|f(\widetilde{a}) - f(\widetilde{b})\| < \sigma/2$. Now, $\chi_{\alpha_1} f = \chi_{\alpha_1}$ and $\chi_{\alpha_2} f = f$ on $[-M, M]$. Thus $p_{\alpha_1}(\widetilde{a})f(\widetilde{a}) = p_{\alpha_1}(\widetilde{a})$, and $p_{\alpha_2}(\widetilde{b})f(\widetilde{b}) = f(\widetilde{b})$. Then we have

$$\begin{aligned}\|p_{\alpha_1}(\widetilde{a}) - p_{\alpha_1}(\widetilde{a})p_{\alpha_2}(\widetilde{b})\| &= \|p_{\alpha_1}(\widetilde{a})f(\widetilde{a}) - p_{\alpha_1}(\widetilde{a})f(\widetilde{a})p_{\alpha_2}(\widetilde{b})\| \\ &\leq \|p_{\alpha_1}(\widetilde{a})f(\widetilde{a}) - p_{\alpha_1}(\widetilde{a})f(\widetilde{b})\| \\ &\qquad + \|p_{\alpha_1}(\widetilde{a})f(\widetilde{b}) - p_{\alpha_1}(\widetilde{a})f(\widetilde{a})p_{\alpha_2}(\widetilde{b})\| \\ &\leq \|f(\widetilde{a}) - f(\widetilde{b})\| + \|f(\widetilde{b}) - f(\widetilde{a})p_{\alpha_2}(\widetilde{b})\| \\ &= \|f(\widetilde{a}) - f(\widetilde{b})\| + \|f(\widetilde{b})p_{\alpha_2}(\widetilde{b}) - f(\widetilde{a})p_{\alpha_2}(\widetilde{b})\| \\ &\leq \|f(\widetilde{a}) - f(\widetilde{b})\| + \|f(\widetilde{b}) - f(\widetilde{a})\| \\ &< \sigma \leq \epsilon.\end{aligned}$$

Then by the choice of σ, we have $\mathrm{rank}(p_{\alpha_1}(\widetilde{a})) \leq \mathrm{rank}(p_{\alpha_2}(\widetilde{b}))$. □

Corollary II.3.9. *Let $\epsilon > 0$, let $0 \leq \alpha_1 < \alpha_2 < 1$, and let $M \geq 1$ be a real number. Then there exists $\delta > 0$ such that if X is compact Hausdorff space, and if $a, b \in C(X, \mathbb{K})_{s.a.}$, $\widetilde{a} = a + 1$, $\widetilde{b} = b + 1$, $\|\widetilde{a}\| \leq M$, $\|\widetilde{b}\| \leq M$, and $\|\widetilde{a} - \widetilde{b}\| < \delta$, then*

$$\|p_{\alpha_1}(\widetilde{a}(x))p_{\alpha_2}(\widetilde{b}(x)) - p_{\alpha_1}(\widetilde{a}(x))\| < \epsilon, \qquad \textit{for all } x \in X;$$

and

$$\mathrm{rank}(p_{\alpha_1}(\widetilde{a}(x))) \leq \mathrm{rank}(p_{\alpha_2}(\widetilde{b}(x))), \qquad \textit{for all } x \in X..$$

Proof: First of all, we identify $\widetilde{C(X, \mathbb{K})}$ as a subalgebra of $C(X, \widetilde{\mathbb{K}})$ by identifying $(a, \lambda) \in \widetilde{C(X, \mathbb{K})}$ with $a + \lambda 1_X$, where 1_X is the constant function on X at id_H. Then it is clear that $\widetilde{a}(x) = \widetilde{a(x)}$ for all $x \in X$.

Apply II.3.8 to $\epsilon, \alpha_1, \alpha_2$ and M to get a $\delta > 0$. The result follows. □

Corollary II.3.10. *Let X be a compact Hausdorff space, let $0 < \alpha < 1$, let $a \in C(X, \mathbb{K})_{s.a.}$, let $\widetilde{a} = a + 1$. Then there exists some $n \in \mathbb{N}$ such that $\mathrm{rank}(p_\alpha(\widetilde{a}(x))) \leq n$ for all $x \in X$.*

Proof: If $a = 0$, then nothing to prove. So assume $a \neq 0$.

Let $\alpha < \sigma < 1$. Apply Corollary II.3.8 to $\epsilon = 1$, $0 < \alpha < \sigma < 1$, and $M = \|\widetilde{a}\|$, to get $\delta > 0$. For each $x \in X$, let $U_x = \{y \in X \colon \|\widetilde{a}(x) - \widetilde{a}(y)\| < \delta\}$. Then there exists $x_1, \ldots, x_m \in X$ such that $\bigcup_{i=1}^m U_{x_i} = X$. Let $n = \max\{\mathrm{rank}(p_\sigma(\widetilde{a}(x_i))) \colon i = 1, \ldots, m\}$. Let $x \in X$. Then $x \in U_{x_k}$ for some k. So $\|\widetilde{a}(x) - \widetilde{a}(x_k)\| < \delta$. Also $\|\widetilde{a}(x)\| \leq \|\widetilde{a}\|$ and $\|\widetilde{a}(x_k)\| \leq \|\widetilde{a}\|$. So by the choice of δ, we have $\mathrm{rank}(p_\alpha(\widetilde{a}(x))) \leq \mathrm{rank}(p_\sigma(\widetilde{a}(x_k))) \leq n$. □

Lemma II.3.11. *Let $n \geq \mathbb{N}$, let $\alpha > 0$, let $M > 0$ be a real number, and let $a \in M_n$ be self-adjoint. Then $p_\alpha(a) = p_{\alpha/M}(a/M)$.*

Proof: Let $\mathrm{sp}(a) \cap (-\infty, \alpha) = \{r_1, \dots, r_k\}$. Then

$$\mathrm{sp}(a/M) \cap (-\infty, \alpha/M) = \{r_1/M, r_2/M, \dots, r_k/M\}.$$

Then $p_\alpha(a) = \sum_{i=1}^k p_i$, where p_i is the projection to the eigenspace of a corresponding to r_i, and $p_{\alpha/M}(a/M) = \sum_{i=1}^k q_i$, where q_i is the projection onto the eigenspace of a/M corresponding to r_i/M. But for all $i \in \{1, \dots, k\}$ and all $\xi \in \mathbb{C}^n$, $a(\xi) = r_i\xi$ if and only if $(a/M)(\xi) = (r_i/M)\xi$. So $p_i = q_i$ for all $i \in \{1, \dots, n\}$, and so the result follows. □

Lemma II.3.12. *Let $1 > \alpha > 0$, let $a \in \mathbb{K}_{s.a.}$, and let $\widetilde{a} = a + 1 \in \widetilde{\mathbb{K}}$. Then there exists a $\delta > 0$ such that if $b \in \widetilde{\mathbb{K}}_{s.a.}$, and if $\|b - \widetilde{a}\| < \delta$, then $\mathrm{rank}(p_\alpha(\widetilde{a})) \leq \mathrm{rank}(p_\alpha(b))$.*

Proof: Fix $1 > \alpha > 0$ and $a \in \mathbb{K}_{s.a.}$. Since $\alpha < 1$, $\mathrm{sp}(\widetilde{a}) \cap (-\infty, \alpha)$ is a finite set. So there exists $\delta_1 > 0$ such that $\mathrm{sp}(\widetilde{a}) \cap (\alpha - 3\delta_1, \alpha + 3\delta_1) \subseteq \{\alpha\}$. Let $F_1 = [-\|\widetilde{a}\| - \delta_1, \alpha - 2\delta_1]$, and $F_2 = [\alpha - \delta_1, \|\widetilde{a}\| + \delta_1]$. Then

$$\mathrm{sp}(\widetilde{a}) \subseteq (-\|\widetilde{a}\| - \delta_1, \alpha - 2\delta_1) \cup (\alpha - \delta_1, \|\widetilde{a}\| + \delta_1) \subseteq F_1 \cup F_2.$$

Let $K = F_1 \cup F_2$. Let $\phi = \chi_{F_1}$. Then $\phi \in C(K)$. Since $K \subseteq \mathbb{R}$ is compact, there exists a polynomial $\pi \in C(K)$ such that $\|\pi - \phi\|_\infty < 1/3$. The map $x \mapsto \pi(x)$ is continuous, so there exists $\delta_2 > 0$ such that if $\|x - \widetilde{a}\| < \delta_2$, then $\|\pi(x) - \pi(\widetilde{a})\| < 1/4$. Let $\delta = \min\{\delta_1/2, \delta_2\}$.

Let $b \in \widetilde{\mathbb{K}}_{s.a.}$ satisfy $\|b - \widetilde{a}\| < \delta$. Then $\mathrm{sp}(b) \subseteq \cup\{(r - \delta, r + \delta) \colon r \in \mathrm{sp}(\widetilde{a})\}$. If $r \in \mathrm{sp}(\widetilde{a})$, then $-\|\widetilde{a}\| \leq r \leq \alpha - 3\delta_1$ or $\alpha \leq r \leq \|\widetilde{a}\|$, and then

$$(r - \delta, r + \delta) \subseteq (-\|\widetilde{a}\| - \delta, \alpha - 3\delta_1 + \delta) \cup (\alpha - \delta, \|\widetilde{a}\| + \delta).$$

So

$$\begin{aligned}\mathrm{sp}(b) &\subseteq (-\|\widetilde{a}\| - \delta, \alpha - 3\delta_1 + \delta) \cup (\alpha - \delta, \|\widetilde{a}\| + \delta) \\ &\subseteq (-\|\widetilde{a}\| - \delta_1, \alpha - 2\delta_1) \cup (\alpha - \delta_1, \|\widetilde{a}\| + \delta_1) \subseteq K.\end{aligned}$$

Then

$$\|\phi(\widetilde{a}) - \phi(b)\| \leq \|\phi(\widetilde{a}) - \pi(\widetilde{a})\| + \|\pi(\widetilde{a}) - \pi(b)\| + \|\pi(b) - \phi(b)\| < 1.$$

Thus $\phi(\widetilde{a})$ and $\phi(b)$ are unitarily equivalent projections, and so $\mathrm{rank}(\phi(\widetilde{a})) = \mathrm{rank}(\phi(b))$. But $\phi(\widetilde{a}) = p_\alpha(\widetilde{a})$, so $\mathrm{rank}(p_\alpha(\widetilde{a})) = \mathrm{rank}(\phi(b))$. Also $\phi \leq \chi_{(-\infty,\alpha)}$, so $\phi(b) \leq p_\alpha(b)$, and so $\mathrm{rank}(p_\alpha(\widetilde{a})) = \mathrm{rank}(\phi(b)) \leq p_\alpha(b)$. ☐

The remaining portion of this section will be dedicated to obtaining a topological stable rank reduction theorem for SRSHAs. The idea is to obtain an approximate polar decomposition for elements a in a SRSHA such that the dimensions of the eigenspaces of $|a(x)|$ corresponding to small eigenvalues are large enough for every $x \in X$. This can be easily done in $\widetilde{C(X, \mathbb{K})}$, where X is just a one-point space and $\widetilde{C(X, \mathbb{K})}$ denotes the unitization of $C(X, \mathbb{K})$, which can always be taken to be the first base space of any SRSH system. We then have an approximate polar decomposition for the image of the first coordinate of a under the first attaching map. In order to obtain an approximate polar decomposition for a, we will need to be able to extend the image of the unitary used in the approximate polar decomposition for the first coordinate of the element a to a unitary in $\widetilde{C(X_2, \mathbb{K})}$, where X_2 is the second base space in the SRSH system. Thus we will need an extension result for such unitaries. This extension result for RSHAs is given by Lemma 3.3 in [9]. We will modify this lemma to suit our situation.

The following lemma is a slight modification of Lemma 3.3 in [9]. In fact, the original proof of Lemma 3.3 in [9] also proves the following lemma.

Lemma II.3.13. *Let $\epsilon, \alpha > 0$ and let $n \in \mathbb{N}$. Then there exists a $\delta > 0$ such that the following holds. Let X be a compact Hausdorff space with $\dim(X) = d < \infty$, and let $X^{(0)} \subseteq X$ be a closed subspace. Let $m \in \mathbb{N}$, and let $a \in C(X, M_m)$ satisfy $\|a\| \leq 1$. For each $x \in X$, let*

$$p(x) = \chi_{(-\infty,\alpha)}([a(x)^* a(x)]^{1/2}).$$

Suppose that $n \geq \mathrm{rank}(p(x)) \geq d/2$ for all $x \in X$. Let $u^{(0)} \in U_0(C(X, M_m))$ be a unitary such that

$$\|[u^{(0)}(x)[a(x)^* a(x)]^{1/2} - a(x)][1 - p(x)]\| < \delta$$

for every $x \in X^{(0)}$. Let $t \mapsto u_t^{(0)}$ be a homotopy from 1 to $u^{(0)}$ in $U(C(X^{(0)}, M_m))$. Then there *exists a unitary* $u \in U_0(C(X, M_m))$ *and a homotopy* $t \to u_t$ *in* $U(C(X, M_m))$ *from* 1 *to* u *such that* $u|_{X^{(0)}} = u^{(0)}$, $u_t|_{X^{(0)}} = u_t^{(0)}$ *for all* t, *and such that*

$$\|[u(x)[a(x)^*a(x)]^{1/2} - a(x)][1 - p(x)]\| < \epsilon$$

for all $x \in X$.

Now we remove the condition that the element $\|a\|$ has norm less or equal to 1 from Lemma II.3.13.

Corollary II.3.14. *Let* $\epsilon, \alpha > 0$, *let* $n \in \mathbb{N}$, *and let* $M \geq 1$ *be a real number. Then there exists a* $\delta > 0$ *such that the following holds. Let* X *be a compact Hausdorff space with* $\dim(X) = d < \infty$, *and let* $X^{(0)} \subseteq X$ *be a closed subspace. Let* $m \in \mathbb{N}$, *and let* $a \in C(X, M_m)$ *satisfy* $\|a\| \leq M$. *For each* $x \in X$, *let*

$$p(x) = p_\alpha(|a(x)|).$$

Suppose that $n \geq \operatorname{rank}(p(x)) \geq d/2$ *for all* $x \in X$. *Let* $u^{(0)} \in U_0(C(X^{(0)}, M_m))$ *be a unitary such that*

$$\|[u^{(0)}(x)|a(x)| - a(x)][1 - p(x)]\| < \delta$$

for every $x \in X^{(0)}$. *Let* $t \mapsto u_t^{(0)}$ *be a homotopy in* $U(C(X^{(0)}, M_m))$ *from 1 to* $u^{(0)}$. *Then there exists a unitary* $u \in U_0(C(X, M_m))$ *and a homotopy* $t \mapsto u_t$ *in* $U(C(X, M_m))$ *from 1 to* u *such that* $u|_{X^{(0)}} = u^{(0)}$, $u_t|_{X^{(0)}} = u_t^{(0)}$ *for all* t, *and that*

$$\|[u(x)|a(x)| - a(x)][1 - p(x)]\| < \epsilon$$

for all $x \in X$.

Proof: Apply Lemma II.3.13 to $\epsilon/M, \alpha/M$, n to get δ. Let X, $X^{(0)}$, m, a, p, $u^{(0)}$ be as given in the statement of this corollary. Let $t \mapsto u_t^{(0)}$ be a path from 1 to $u^{(0)}$.

Let $b = a/M$. Then $\|b\| \leq 1$. Let $q(x) = p_{\alpha/M}(|b(x)|)$. By Lemma II.3.11, we have $q(x) = p(x)$ for all $x \in X$. Then we have $n \geq \operatorname{rank}(q(x)) \geq d/2$ for all $x \in X$. Also,

$$\|[u^{(0)}(x)|b(x)| - b(x)][1 - q(x)]\| < \delta/M \leq \delta$$

for all $x \in X^{(0)}$. So by the choice of δ, there exists a unitary $u \in U_0(C(X, M_m))$, and a homotopy $t \mapsto u_t$ in $U(C(X, M_m))$ from 1 to u such that $u|_{X^{(0)}} = u^{(0)}$, $u_t|_{X^{(0)}}$ for all t, and that

$$\|[u(x)|b(x)| - b(x)][1 - q(x)]\| < \epsilon/M.$$

Then

$$\|[u(x)|a(x)| - a(x)][1 - p(x)]\| < M \cdot \frac{\epsilon}{M} = \epsilon.$$

□

The next lemma adapts the above to unitizations of $C(X) \otimes M_n$.

Lemma II.3.15. *Let* $1 > \alpha, \epsilon > 0$, *let* $n \in \mathbb{N}$, *and let* $M \in [1, \infty)$. *Then there exists* $\delta > 0$ *such that the following holds. Let* X *be a compact Hausdorff space such that* $\dim(X) = d < \infty$, *and let* Y *be a closed subspace. Let* $m \in \mathbb{N}$, *let* $a \in C(X, M_m)$, *and let* $\widetilde{a} = a + 1_X \in C(X, M_m)^\sim$, *where* 1_X *denotes the adjoined identity. Suppose that* $\|\widetilde{a}\| \leq M$. *For each* $x \in X$, *let* $\widetilde{p}(x) = p_\alpha(|\widetilde{a}(x)|)$. *Suppose that* $n \geq \operatorname{rank}(\widetilde{p}(x)) \geq d/2$. *Let* $u_0 \in U_0(C(Y, M_m)^\sim)$ *satisfy*

$$\|[u_0(x)|\widetilde{a}(x)| - \widetilde{a}(x)][1 - \widetilde{p}(x)]\| < \delta \quad \textit{for all } x \in Y. \tag{II.2}$$

Let $t \mapsto w_t$ *be a homotopy in* $U(C(Y, M_m)^\sim)$ *from 1 to* u_0. *Then there exists a unitary* u *contained in* $U_0(C(X, M_m)^\sim)$ *and a homotopy* $t \to v_t$ *in* $U(C(X, M_m)^\sim)$ *from 1 to* u *such that* $u|_Y = u_0$, $v_t|_Y = w_t$ *for all* t, *and that*

$$\|[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)][1 - \widetilde{p}(x)]\| < \delta \quad \textit{for all } x \in X, \tag{II.3}$$

Proof: Let $0 < \epsilon, \alpha < 1$, $n \in \mathbb{N}$, and $M \in [1, \infty)$ be given. Apply Corollary II.3.14 to ϵ, α, n, and M to obtain $\delta' > 0$, and let $\delta = \min\{\epsilon, \delta'/2\}$. Let X, Y, m, a, $\widetilde{p}$, and u_0 satisfy the conditions in the statement of the lemma. Let $t \mapsto w_t$ be a homotopy in $U(C(Y, M_m)^\sim)$ from 1 to u_0.

We set up some notations first. We use 1 to denote the adjoined identity of $\widetilde{M_m}$, and use e to denote the identity of M_m. Use 1_X and 1_Y to denote the adjoined identity of $C(X, M_m)^\sim$ and $C(Y, M_m)^\sim$, respectively. Use e_X and e_Y to denote the identities of $C(X, M_m)$ and $C(Y, M_m)$ respectively.

For each $x \in X$, or Y, use ev_x to denote the map $C(X, M_m) \to M_m$, or $C(Y, M_m) \to M_m$, defined by $\mathrm{ev}_x(a) = a(x)$. By identifying (a, λ) with $a + \lambda \cdot 1_X$, or $a + \lambda \cdot 1_Y$, we treat $C(X, M_m)^{\sim}$ and $C(Y, M_m)^{\sim}$ as subalgebras of $C(X, \widetilde{M_m})$ and $C(Y, \widetilde{M_m})$ respectively. For each $x \in X$, or Y, use $\widetilde{\mathrm{ev}_x}$ to denote the map $C(X, M_m)^{\sim} \to \widetilde{M_m}$ or $C(Y, M_m)^{\sim} \to \widetilde{M_m}$, defined by $\widetilde{\mathrm{ev}_x}(a) = a(x)$. Let τ denote the standard map from the unitization of any C^*-algebra to $\mathbb{C}$.

Define

$$\Phi_X \colon C(X, M_m)^{\sim} \to C(X, M_m) \oplus \mathbb{C} \quad \text{by } (a, \lambda) \mapsto (a + \lambda e_X, \lambda),$$

$$\Phi_Y \colon C(Y, M_m)^{\sim} \to C(Y, M_m) \oplus \mathbb{C} \quad \text{by } (a, \lambda) \mapsto (a + \lambda e_Y, \lambda),$$

and

$$\Phi \colon \widetilde{M_m} \to M_m \oplus \mathbb{C} \quad \text{by } (a, \lambda) \mapsto (a + \lambda e, \lambda).$$

Define $\widetilde{R} \colon C(X, M_m)^{\sim} \to C(Y, M_M)^{\sim}$ by $R(a + \lambda 1_X) = a|_Y + \lambda 1_Y$, and define $R \colon C(X, M_m) \to C(Y, M_m)$ by $R(a) = a|_Y$. Then for every $x \in X$ and every $y \in Y$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\widetilde{M_m} & \xleftarrow{\widetilde{\mathrm{ev}_x}} & C(X, M_m)^{\sim} & \xrightarrow{\widetilde{R}} & C(Y, M_m)^{\sim} & \xrightarrow{\widetilde{\mathrm{ev}_y}} & \widetilde{M_m} \\
\downarrow \Phi & & \downarrow \Phi_X & & \downarrow \Phi_Y & & \downarrow \Phi \\
M_m \oplus \mathbb{C} & \xleftarrow{\mathrm{ev}_x \oplus id} & C(X, M_m) \oplus \mathbb{C} & \xrightarrow{R \oplus \mathbb{C}} & C(Y, M_m) \oplus \mathbb{C} & \xrightarrow{\mathrm{ev}_y \oplus id} & M_m \oplus \mathbb{C}
\end{array}$$

Now, since for all $x \in X$, we have

$$\tau(\widetilde{p}(x)) = \tau(\chi_\alpha(|\widetilde{a}(x)|)) = \chi_\alpha(\tau(|\widetilde{a}(x)|)) = \chi_\alpha(|\tau(\widetilde{a}(x))|) = \chi_\alpha(1) = 0,$$

we see that for all $x \in X$, $\widetilde{p}(x) = (p(x), 0)$ for some projection $p(x) \in X$. Since $u_0 \in C(Y, M_m)^{\sim}$, there exists some $w_0 \in C(Y, M_m)$ and some unitary $\mu \in \mathbb{C}$ such that $u_0 = (w_0, \mu)$. Note that (II.2) implies that

$$|\mu - 1| = \left\| \tau \Big[[u_0(x)|\widetilde{a}(x)| - \widetilde{a}(x)][1 - \widetilde{p}(x)] \Big] \right\| < \delta \leq \epsilon, \quad \text{for all } x \in X. \tag{II.4}$$

Let $\widehat{v}_0 = w_0 + \mu e_Y$, so that $\Phi_Y(u_0) = (w_0 + \mu e_Y, \mu) = (\widehat{v}_0, \mu)$. Since Φ_Y is an isomorphism, we have $\widehat{v}_0 \in U_0(C(Y, M_m))$. Let $\widehat{a} = a + e_X$, so $(\widehat{a}, 1) = \Phi_X(\widetilde{a})$. Next we compute: for each $x \in Y$, we have

$$\begin{aligned}
&\Phi\left([u_0(x)|\widetilde{a}(x)| - \widetilde{a}(x)]\,[1 - \widetilde{p}(x)]\right) \\
&\quad = [\Phi(u_0(x))|\Phi(\widetilde{a}(x))| - \Phi(\widetilde{a}(x))]\,\Phi[1 - \widetilde{p}(x))\\
&\quad = \Big[(\widehat{v}_0(x), \mu)\cdot(|\widehat{a}(x)|, 1) - (\widehat{a}(x), 1)\Big](e - p(x), 1)\\
&\quad = \Big[(\widehat{v}_0(x)|\widehat{a}(x)|, \mu) - (\widehat{a}(x), 1)\Big](e - p(x), 1)\\
&\quad = \Big(\widehat{v}_0(x)|\widehat{a}(x)| - \widehat{a}(x), \mu - 1\Big)\cdot(e - p(x), 1)\\
&\quad = \Big(\Big[\widehat{v}_0(x)|\widehat{a}(x)| - \widehat{a}(x)\Big]\Big[e - p(x)\Big], \mu - 1\Big).
\end{aligned}$$

Thus, since Φ is isometric, we obtain the following from (II.2)

$$\left\| \Big[\widehat{v}_0(x)|\widehat{a}(x)| - \widehat{a}(x)\Big]\Big[e - p(x)\Big] \right\| < \delta < \delta', \quad \text{for all } x \in Y. \tag{II.5}$$

Now, let $\pi\colon M_m \oplus \mathbb{C} \to M_m$ be the standard map. Then we compute again: for every $x \in X$, we have

$$\begin{aligned}
p(x) &= \pi(p(x), 0) = \pi\circ\Phi(p(x), 0) = \pi\circ\Phi(\widetilde{p}(x))\\
&= \pi\circ\Phi(\chi_\alpha(|\widetilde{a}(x)|)) = \chi_\alpha(\pi\circ\Phi(|\widetilde{a}(x)|))\\
&= \chi_\alpha\big(|\pi\circ\Phi(\widetilde{a}(x))|\big) = \chi_\alpha\big(|\pi\circ\Phi(a(x), 1)|\big)\\
&= \chi_\alpha\big(|\pi(a(x) + e, 1)|\big) = \chi_\alpha\big(|\pi(\widehat{a}(x), 1)|\big)\\
&= \chi_\alpha(|\widehat{a}(x)|).
\end{aligned}$$

Also, we have $n \geq \operatorname{rank}(p(x)) = \operatorname{rank}(\widetilde{p}(x)) \geq d/2$ and $\|\widehat{a}\| \leq M$. Let $\widehat{w}_t = \pi(\Phi_Y(w_t))$ for each t. Then $t \mapsto \widehat{w}_t$ is a homotopy in $U(C(Y, M_m))$ from $\widehat{w}_t = \pi(\Phi_Y((0,1)) = \pi(e_Y, 1) = e_Y$, to $\widehat{w}_1 = \pi(\Phi_Y(u_0)) = \pi(\widehat{v}_0, \mu) = \widehat{v}_0$.

Thus by the choice of δ', there exist $\widehat{v} \in U_0(C(X, M_m))$ and a homotopy $t \mapsto \widehat{v}_t$ in $U(C(X, M_m))$ for e_X to $\widehat{v}$ such that $\widehat{v}|_Y = \widehat{v}_0$, $\widehat{v}_t|_Y = \widehat{w}_t$, and

$$\left\| \left[\widehat{v}(x)|\widehat{a}(x)| - \widehat{a}(x)\right]\left[e - p(x)\right] \right\| < \epsilon, \quad \text{for all } x \in X. \tag{II.6}$$

Let $u = (\widehat{v} - \mu e_X, \mu)$. Then $\Phi_X(u) = \Phi(\widehat{v} - \mu e_X, \mu) = (\widehat{v}, \mu)$. Since

$$(\widehat{v}, \mu) \in U_0(C(X, M_m) \oplus \mathbb{C}),$$

and since Φ_X is a *-isomorphism, we have $u \in U_0(C(X, M_m)^{\sim})$. Also for all $x \in Y$, we have

$$\begin{aligned} u(x) &= (\widehat{v}(x) - \mu e, \mu) = (\widehat{v}_0(x) - \mu e, \mu) \\ &= (w_0(x) + \mu e - \mu e, \mu) = (w_0(x), \mu) = u_0(x). \end{aligned}$$

Thus $u|_Y = u_0$.

Then for all $x \in X$, we have

$$\begin{aligned} &\Phi\left(\left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - \widetilde{p}(x)\right]\right) \\ &= \left[\Phi(u(x))|\Phi(\widetilde{a}(x))| - \Phi(\widetilde{a}(x))\right]\Phi(1 - \widetilde{p}(x)) \\ &= \left[(\widehat{v}(x), \mu)(|\widehat{a}(x)|, 1) - (\widehat{a}(x), 1)\right](e - p(x), 1) \\ &= \left[(\widehat{v}(x)|\widehat{a}(x)| - \widehat{a}(x), \mu - 1)\right](e - p(x), 1) \\ &= \left(\left[\widehat{v}(x)|\widehat{a}(x)| - \widehat{a}(x)\right]\left[e - p(x)\right], \mu - 1\right). \end{aligned}$$

Thus for all $x \in X$, we have, by (II.4), (II.6), and the fact that Φ is isometric,

$$\begin{aligned} &\left\|\left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - \widetilde{p}(x)\right]\right\| \\ &= \left\|\left(\left[\widehat{v}(x)|\widehat{a}(x)| - \widehat{a}(x)\right]\left[e - p(x)\right], \mu - 1\right)\right\| \\ &\qquad \text{(the norm above is now taken in } M_m \oplus \mathbb{C}) \\ &= \max\left\{\left\|\left[\widehat{v}(x)|\widehat{a}(x)| - \widehat{a}(x)\right]\left[e - p(x)\right]\right\|, |\mu - 1|\right\} \\ &< \epsilon. \end{aligned}$$

Let $v_t = \Phi_X^{-1}(\widehat{v}_t, \tau(w_t))$. Then $t \mapsto v_t$ is a homotopy in $U(C(X, M_m)^\sim)$. For each t and each $y \in Y$, we have $\widehat{v}_t(y) = \widehat{w}_t(y)$, so we have $(\widehat{v}_t(y), \tau(w_t)) = (\widehat{w}_t(y), \tau(w_t))$. So

$$R \oplus id(\widehat{v}_t, \tau(w_t)) = (\widehat{w}_t, \tau(w_t)) = \Phi_Y(w_t)$$

and

$$\Phi_Y(w_t) = R \oplus id(\Phi_X(v_t)) = \Phi_Y(\widetilde{R}(v_t)).$$

Thus $w_t = \widetilde{R}(v_t)$. So $w_t|_Y = v_t$. Also $v_0 = \Phi_X^{-1}(e_X, 1) = 1_X$ and $v_1 = \Phi_X^{-1}(\widehat{v}, \tau(u_0))) = \Phi_X^{-1}(\widehat{v}, \mu) = u$. This finishes the proof. □

The next lemma will "stabilize" the above lemma, and will be the one that we will need.

Lemma II.3.16. *Let $0 < \epsilon < 1$ and let $0 < \alpha_1 < \alpha_2 < 1$. Let X be a compact Hausdorff space with $\dim(X) = d < \infty$. Let $Y \subseteq X$ be a closed subset. Let $a \in C(X, \mathbb{K})$ and let $\widetilde{a} = a + 1 \in C(X, \mathbb{K})^\sim$. For all $x \in X$, let $p_1(x) = p_{\alpha_1}(|\widetilde{a}(x)|)$ and let $p_2(x) = p_{\alpha_2}(|\widetilde{a}(x)|)$. Suppose that for all $x \in X$, $\operatorname{rank}(p_1(x)) \geq d/2$. Then there exists $\delta > 0$ such that: if $u_0 \in U_0(C(Y, \mathbb{K})^\sim)$ is a unitary and $h_0\colon [0,1] \to U(C(Y, \mathbb{K})^\sim)$ is a homotopy such that $h_0(0) = 1$, $h_0(1) = u_0$, and*

$$\|[u_0(x)|\widetilde{a}(x)| - \widetilde{a}(x)][1 - p_1(x)]\| < \delta \quad \text{for all } x \in Y, \tag{II.7}$$

then there exists a unitary $u \in U_0(C(X, \mathbb{K})^\sim)$ and a homotopy $h\colon [0,1] \to U(C(X, \mathbb{K})^\sim)$ such that $h(0) = 1$, $h(1) = u$, that $h(t)|_Y = h_0(t)$ for all t, that $u|_Y = u_0$, and that

$$\|[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)][1 - p_2(x)]\| < \delta \quad \text{for all } x \in X.$$

Proof: Let ϵ, α_1, α_2, X, Y, a, p_1, and p_2 satisfy the hypothesis of the lemma, and let $M = 2\|\widetilde{a}\|$. Note that $M \geq \|\widetilde{a}\| \geq 1$.

First of all, it is clear that there exists some $c \in C(X, \mathbb{K})_{s.a.}$ such that $|\widetilde{a}| = c + 1$. Denote $c + 1$ by $\widetilde{c}$. Note that $\|\widetilde{c}\| = \|\widetilde{a}\|$, since $(\widetilde{c})^2 = (\widetilde{a})^*(\widetilde{a})$. Let $\alpha' = \frac{\alpha_1 + \alpha_2}{2}$, and for each $x \in X$, let $p'(x) = p_{\alpha'}(|\widetilde{a}(x)|)$. Note that for all $x \in X$, we have $p_2(x) \geq p'(x) \geq p_1(x) \geq d/2$, and so we have

$$\operatorname{rank}(p_2(x)) \geq \operatorname{rank}(p'(x)) \geq \operatorname{rank}(p_1(x)) \geq d/2.$$

By Lemma II.3.10, there exists $n \in \mathbb{N}$ such that $\mathrm{rank}(p_2(x)) = \mathrm{rank}(p_{\alpha_2}(\widetilde{c})) \leq n$ for all $x \in X$. Apply Lemma II.3.15 to $\epsilon/(16M) > 0$, $1 > \alpha' > 0$, n, and M, to get $\delta_1 > 0$. Without loss of generality, assume that $\delta_1 < \epsilon/(16M)$. Apply Corollary II.3.9 to $\delta_1/(4M)$ in place of ϵ, α_1, α' in place of α_2, and M, to get $\sigma_1 > 0$. Apply Corollary II.3.9 again to $\delta_1/(4M)$ in place of ϵ, α' in place of α_1, α_2, and M to get $\sigma_2 > 0$. Let

$$\delta = \min\{\epsilon/(16M), \delta_1/(16M), \sigma_1/(16M), \sigma_2/(16M), \alpha_2/(16M)\}.$$

Now let $u_0 \in U_0(C(Y,\mathbb{K})^{\sim})$ be a unitary such that (II.7) holds, and let $h_0 \colon [0,1] \to U(C(Y,\mathbb{K})^{\sim})$ be a homotopy from 1 to u_0.

For each $k \in \mathbb{N}$, embed M_k into M_{k+1} in the standard, and embed M_k into $\mathbb{K}$ in the standard way. Then we have $\mathbb{K} = \overline{\bigcup_{k\geq 1} M_k}$ and $\widetilde{\mathbb{K}} = \overline{\bigcup_{k\geq 1} \widetilde{M_k}}$, where the adjoined identity of each $\widetilde{M_k}$ is the same as the adjoined identity of $\widetilde{\mathbb{K}}$. We will use 1 to denote the adjoined identity of $\widetilde{\mathbb{K}}$ and $\widetilde{M_k}$, for $k \geq 1$. The above embeddings give the embedding of $C(X, M_k)$ into $C(X, M_{k+1})$ and into then $C(X,\mathbb{K})$. Then $C(X,\mathbb{K}) = \overline{\bigcup_{k\geq 1} C(X, M_k)}$ and $C(X,\mathbb{K})^{\sim} = \overline{\bigcup_{k\geq 1} C(X, M_k)^{\sim}}$. Again, we assume that the adjoined identity of $C(X,\mathbb{K})^{\sim}$ is the same as the adjoined identity of $C(X, M_k)^{\sim}$ for every $k \geq 1$. We will use 1_X to denote the adjoined identity of $C(X,\mathbb{K})^{\sim}$ and $C(X, M_k)^{\sim}$ for all $k \geq 1$. Similarly, we use 1_Y to denote the adjoined identity of $C(Y,\mathbb{K})^{\sim}$ and $C(Y, M_k)^{\sim}$ for all $k \geq 1$.

Then, we can find some $m \in \mathbb{N}$, some $b \in C(X, M_m)$, and some homotopy

$$f_0 \colon [0,1] \to U(C(Y, M_m)^{\sim})$$

such that

$$\|a - b\| < \delta/(8M), \quad \|\widetilde{a} - \widetilde{b}\| < \delta/(8M), \quad \left\| |\widetilde{b}| - \widetilde{c} \right\| < \delta/(8M) \tag{II.8}$$

$$\|\widetilde{b}\| \leq M \tag{II.9}$$

$$f_0(0) = 1 \text{ and } \|f_0 - h_0\| < \delta/(8M), \tag{II.10}$$

where $\widetilde{b} = b + 1$. Let $v_0 = f_0(1)$. Then $\|v_0 - u_0\| < \delta/(8M)$. Let $b' \in C(X, M_m)_{s.a.}$ be such that

$|\widetilde{b}| = b' + 1$. Then $\|b' + 1\| = \|\widetilde{b}\| \leq M$. Then (II.8) implies that

$$\|b' - c\| < \delta/(8M). \tag{II.11}$$

For each $x \in X$, let $q'(x) = p_{\alpha'}(|\widetilde{b}(x)|)$ and let $q_2(x) = p_{\alpha_2}(|\widetilde{b}(x)|)$. By the choice of σ_1, which is greater than $\delta/(8M)$, we have (the space X, and elements a and b in Corollary II.3.9 are taken to be X, c and b', respectively)

$$\|p_1(x)q'(x) - p_1(x)\| < \delta_1/(4M) \text{ and } \operatorname{rank}(p_1(x)) \leq \operatorname{rank}(q'(x)), \tag{II.12}$$

for all $x \in X$. By the choice of σ_2, we have (the space X, and the elements a and b in Corollary II.3.9 are taken to be X, b' and c, respectively)

$$\|q'(x)p_2(x) - q'(x)\| \leq \delta_1/(4M) \text{ and } \operatorname{rank}(q'(x)) \leq \operatorname{rank}(p_2(x)), \tag{II.13}$$

for all $x \in X$. Then

$$n \geq \operatorname{rank}(p_2(x)) \geq \operatorname{rank}(q'(x)) \geq \operatorname{rank}(p_1(x)) \geq d/2. \tag{II.14}$$

Now, by (II.8), for all $x \in Y$, we have

$$\begin{aligned}
&\left\| \left[v_0(x)|\widetilde{b}(x)| - \widetilde{b}(x)\right] - \left[u_0(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right] \right\| \\
&\leq \left\| v_0(x)|\widetilde{b}(x)| - u_0(x)|\widetilde{a}(x)| \right\| + \|\widetilde{b}(x) - \widetilde{a}(x)\| \\
&\leq \left\| v_0(x)|\widetilde{b}(x)| - v_0(x)|\widetilde{a}(x)| \right\| + \left\| v_0(x)|\widetilde{a}(x)| - u_0(x)|\widetilde{a}(x)| \right\| + \delta/(8M) \\
&\leq \left\| |\widetilde{b}(x)| - |\widetilde{a}(x)| \right\| + M \|v_0(x) - u_0(x)\| + \delta/(8M) \\
&< 2\delta/(8M) + \delta/8 \leq 3\delta/8.
\end{aligned}$$

Also, by (II.12), for all $x \in X$, we have

$$\begin{aligned}\|(1-p_1(x))(1-q'(x))-(1-q'(x))\| &= \|1-q'(x)-p_1(x)+p_1q'(x)-1+q'(x)\| \\ &= \|p_1(x)q'(x)-p_1(x)\| \\ &< \delta_1/(4M).\end{aligned}$$

Then combining the above two calculations and (II.7), we have

$$\begin{aligned}&\left\|\left[v_0(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]\left[1-q'(x)\right]\right\| \\ &\leq \left\|\left[v_0(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]\left[1-p_1(x)\right]\left[1-q'(x)\right]\right\| \\ &\qquad + \left\|\left[v_0(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]\left\{[1-q'(x)]-\left[1-p_1(x)\right]\left[1-q'(x)\right]\right\}\right\| \\ &\leq \left\|\left[v_0(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]\left[1-p_1(x)\right]\right\| + 2M\left\|[1-q'(x)]-\left[1-p_1(x)\right]\left[1-q'(x)\right]\right\| \\ &< \left\|\left\{\left[v_0(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]-\left[u_0(x)|\widetilde{a}(x)|-\widetilde{a}(x)\right]\right\}\left[1-p_1(x)\right]\right\| \\ &\qquad + \left\|\left[u_0(x)|\widetilde{a}(x)|-\widetilde{a}(x)\right]\left[1-p_1(x)\right]\right\| + \delta_1/2 \\ &\leq \left\|\left[v_0(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]-\left[u_0(x)|\widetilde{a}(x)|-\widetilde{a}(x)\right]\right\| + \delta + \delta_1/2 \\ &< 3\delta/8 + \delta + \delta_1/2 \\ &\leq 3\delta_1/(16\cdot 8M) + \delta_1/(16M) + \delta_1/2 \\ &< \delta_1/16 + \delta_1/16 + \delta_1/2 < \delta_1,\end{aligned}$$

for all $x \in Y$. Then by the choice of δ_1 (with X, Y, m, a $\widetilde{p}$, w_t, and u_0 in Lemma II.3.15 taken to be, respectively, X, Y, m, b, q', f_0 and v_0), there exists a unitary $v \in U_0(C(X, M_m)^\sim) \subseteq U_0(C(X,\mathbb{K})^\sim)$ and a homotopy $f\colon [0,1] \to U(C(X, M_m)^\sim) \subseteq U(C(X,\mathbb{K})^\sim)$, such that $f(0) = 1$, $f(1) = v$, $f(t)|_Y = f_0(t)$ for all t, and $v|_Y = v_0$, and that

$$\left\|\left[v(x)|\widetilde{b}(x)|-\widetilde{b}(x)\right]\left[1-q'(x)\right]\right\| < \epsilon/(16M), \quad \text{for all } x \in X. \tag{II.15}$$

Since, by (II.10), $\|f_0 - h_0\| < \delta/(8M)$, and since $f(t)|_Y = f_0(t)$ for all $t \in [0,1]$, there exists $h\colon [0,1] \to U(C(X,\mathbb{K})^\sim)$ such that $h(0) = 1$, $h(t)|_Y = h_0(t)$ for all t, and $\|h - f\| < \delta/(4M)$.

Let $u = h(1)$. Then $\|u - v\| < \delta/(4M)$, and $u|_Y = h_0(1) = u_0$. By (II.8), we have

$$\begin{aligned}
&\left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right] - \left[v(x)|\widetilde{b}(x)| - \widetilde{b}(x)\right] \right\| \\
&\quad \le \left\| u(x)|\widetilde{a}(x)| - v(x)|\widetilde{b}(x)| \right\| + \left\| \widetilde{a}(x) - \widetilde{b}(x) \right\| \\
&\quad \le \left\| u(x)|\widetilde{a}(x)| - u(x)|\widetilde{b}(x)| \right\| + \left\| u(x)|\widetilde{b}(x)| - v(x)|\widetilde{b}(x)| \right\| + \delta/(8M) \\
&\quad \le \left\| |\widetilde{a}(x)| - |\widetilde{b}(x)| \right\| + M \|u(x) - v(x)\| + \delta/(8M) \\
&\quad < 2\delta/(8M) + \delta/4 \le \delta/2,
\end{aligned}$$

for all $x \in X$. Also by (II.13), we have

$$\|[1 - q'(x)][1 - p_2(x)] - [1 - p_2(x)]\| < \delta_1/(4M)$$

for all $x \in X$. Thus by the two estimates above and (II.15), for all $x \in X$, we have

$$\begin{aligned}
&\left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - p_2(x)\right] \right\| \\
&\quad \le \left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - q'(x)\right]\left[1 - p_2(x)\right] \right\| \\
&\qquad + \left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left\{\left[1 - p_2(x)\right] - \left[1 - q'(x)\right]\left[1 - p_2(x)\right]\right\} \right\| \\
&\quad \le \left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - q'(x)\right] \right\| + 2M\delta_1/(4M) \\
&\quad \le \left\| \left\{\left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right] - \left[v(x)|\widetilde{b}(x)| - \widetilde{b}(x)\right]\right\}\left[1 - q'(x)\right] \right\| \\
&\qquad + \left\| \left[v(x)|\widetilde{b}(x)| - \widetilde{b}(x)\right]\left[1 - q'(x)\right] \right\| + 2M\delta_1/(4M) \\
&\quad < \left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right] - \left[v(x)|\widetilde{b}(x)| - \widetilde{b}(x)\right] \right\| \\
&\qquad + \epsilon/(16M) + 2M\delta_1/(4M) \\
&\quad < \delta/2 + \epsilon/(16M) + 2M\delta_1/(4M) \\
&\quad \le \delta + \epsilon/16 + \delta_1/2 \\
&\quad \le \epsilon/16 + \epsilon/16 + \epsilon/16 < \epsilon.
\end{aligned}$$

This finishes the proof. $\square$

Let A, B, and C be C^*-algebras. Let $\phi\colon A \to C$ and $R\colon B \to C$ be *-homomorphisms. Let $D = \{(a,b) \in A \oplus B\colon \phi(a) = R(b)\}$. If we unitize A, B, C, ϕ and R, and let

$$E = \{((a,\lambda),(b,\mu)) \in \widetilde{A} \oplus \widetilde{B}\colon \widetilde{\phi}(a) = \widetilde{R}(b)\},$$

then $((a,\lambda),(b,\mu)) \in E$ if and only if $(a,b) \in D$ and $\lambda = \mu$. So the map $E \to \widetilde{D}$ defined by $((a,\lambda),(b,\lambda)) \mapsto ((a,b),\lambda)$ is a *-isomorphism. Thus, given a SRSH system

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^n\right)$$

and $A = A^{(n)}$, we can inductively unitize all the algebras and maps to obtain the unitized system

$$\left(X_1, \widetilde{A^{(1)}}, \left(X_i, X_i^{(0)}, \widetilde{\phi}_i, \widetilde{R}_i, \widetilde{A^{(i)}}\right)_{i=2}^n\right).$$

Then $(a_i,\lambda_i)_{i=1}^n \in \widetilde{A}$ if and only if $(a_i)_{i=1}^n \in A$ and $\lambda_1 = \cdots = \lambda_n$; and each element $((a_i)_{i=1}^n, \lambda) \in \widetilde{A}$ can be uniquely written as $(a_i,\lambda)_{i=1}^n$. Also, if $a \in \widetilde{A}$ and $x \in X_k$ for some k, then $a = (a_i,\lambda)_{i=1}^n$ for some $(a_1,\ldots,a_n) \in A$, and we will use $a(x)$ to denote $(a_k,\lambda)(x) = (a_k(x),\lambda)$.

Lemma II.3.17. *Let*

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^n\right)$$

*be a SRSH system and let $A = A^{(n)}$. Let Y be a compact Hausdorff space and let $\phi\colon A \to C(Y,\mathbb{K})$ be a *-homomorphism (not necessarily non-vanishing). Let $\widetilde{\phi}$ denote the unitization of ϕ. Let $\epsilon > 0$, let $1 > \alpha > 0$, let $a \in A$, and let $\widetilde{a} = a + 1 \in \widetilde{A}$. Let $u \in U_0(\widetilde{A})$ be a unitary such that for all $x \in \bigsqcup_{i=1}^n (X_i \setminus X_i^{(0)})$,*

$$\left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - p_\alpha(|\widetilde{a}(x)|)\right] \right\| < \epsilon. \tag{II.16}$$

Then $\widetilde{\phi}(u) \in U_0(\widetilde{C(Y,\mathbb{K})})$ and all $y \in Y$, we have

$$\left\| \left[\widetilde{\phi}(u)(y)|\widetilde{\phi}(\widetilde{a})(y)| - \widetilde{\phi}(\widetilde{a})(y)\right]\left[1 - p_\alpha(|\widetilde{\phi}(\widetilde{a})(y)|)\right] \right\| < \epsilon. \tag{II.17}$$

Proof: Let H denote the separable infinite dimensional Hilbert space and let 1 denote the identity of $B(H)$. We identify the $\widetilde{\mathbb{K}}$ with $\mathbb{K} \oplus (\mathbb{C}\cdot 1)$ using the map $(a,\lambda) \mapsto a + \lambda\cdot 1$. For any compact Hausdorff space Z, let 1_Z denote the identity of $C(Z, B(H))$. We identify the algebra

$C(Z,\mathbb{K}) \oplus (\mathbb{C} \cdot 1_Z)$ as a subalgebra of $C(Z, B(H))$ using the map $(a, \lambda \cdot 1_Z) \mapsto a + \lambda \cdot 1_Z$. Then we identify $\widetilde{C(Z,\mathbb{K})}$ with $C(Z,\mathbb{K}) \oplus (\mathbb{C} \cdot 1_Z) \subseteq C(Z, B(H))$ using the map $(f, \lambda) \mapsto f + \lambda \cdot 1_Z$.

Let

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^{n}\right)$$

be a SRSH system and let $A = A^{(n)}$. Let Y be a compact Hausdorff space and let $\phi\colon A \to C(Y,\mathbb{K})$ be a *-homomorphism (not necessarily non-vanishing). Let $\widetilde{\phi}$ denote the unitization of ϕ. Let $\epsilon > 0$, let $1 > \alpha > 0$, let $a \in A$, and let $\widetilde{a} = a + 1 \in \widetilde{A}$. Let $u \in U_0(\widetilde{A})$ be a unitary that satisfies (II.16) for all $x \in \bigsqcup_{i=1}^{n}(X_i \setminus X_i^{(0)})$. With the above identifications, we can treat $\widetilde{A}$ as a subalgebra of $C(X, B(H))$ using the maps $(b, \lambda) \mapsto b + \lambda 1_X$, where X is the total space of A, and then the identity of $\widetilde{A}$ is 1_X. So every element in $\widetilde{A}$ can be uniquely written as $((a_1, \lambda 1_{X_1}), \ldots, (a_n, \lambda 1_{X_n}))$, where $\lambda \in \mathbb{C}$ and $(a_1, \ldots, a_n) \in A$. Then for all $b + \lambda 1_X \in \widetilde{A}$, we have $\widetilde{\phi}(b + \lambda 1_X) = \phi(b) + \lambda 1_Y$.

It is clear that $\widetilde{\phi}(u) \in U_0(C(Y,\mathbb{K})^{\sim})$. Fix $y \in Y$. If the map $A \to \mathbb{K}$ defined by $b \mapsto \phi(b)(y)$ is the zero map, then for all $b \in A$, we have $\widetilde{\phi}(\widetilde{b})(y) = 1 = |\widetilde{\phi}(\widetilde{a})(y)|$, and so $p_\alpha(|\widetilde{\phi}(\widetilde{a})(y)|) = p_\alpha(1) = 0$. Since $u = (v, \mu) \in U_0(\widetilde{A})$ satisfies (II.16), we have $|\mu - 1| < \epsilon$, and then the left side of (II.17) reduces to $\|[\mu \cdot 1 - 1][1 - 0]\| = |\mu - 1| < \epsilon$. So we can assume that the map $A \to \mathbb{K}$ given by $b \mapsto \phi(b)(y)$ is not the zero map.

Let $(p_i)_{i=1}^{m}$ be the family of mutually orthogonal projections in $B(H)$, let $(w_i)_{i=1}^{m}$ be the family of isometries in $B(H)$ and let $(x_i)_{i=1}^{m}$ be the family of elements of $\bigsqcup_{k=1}^{n}(X_k \setminus X_k^{(0)})$ that satisfy the conclusion of Proposition II.2.6. Let $p_{m+1} = 1 - \sum_{i=1}^{m} p_i$. Then $(p_i)_{i=1}^{m+1}$ is still a mutually orthogonal family of projections. For all $b + \lambda 1_X \in \widetilde{A}$, we have

$$\begin{aligned}
\widetilde{\phi}(b + \lambda 1_X)(y) = \phi(b)(y) + \lambda 1 &= \sum_{i=1}^{m} w_i b(x_i) w_i^* + \lambda \sum_{i=1}^{m} p_i + \lambda p_{m+1} \\
&= \sum_{i=1}^{m} w_i b(x_i) w_i^* + \lambda \sum_{i=1}^{m} w_i w_i^* + \lambda p_{m+1} \\
&= \sum_{i=1}^{m} w_i (b(x_i) + \lambda \cdot 1) w_i^* + \lambda p_{m+1} \\
&= \sum_{i=1}^{m} w_i (b + \lambda 1_X)(x_i) w_i^* + \lambda p_{m+1}.
\end{aligned}$$

Let $v \in A$ and $\mu \in \mathbb{C}$ satisfy $v + \mu 1_X = u$. Then

$$\widetilde{\phi}(u)(y) = \widetilde{\phi}(v + \mu 1_X) = \sum_{i=1}^{m} w_i u(x_i) w_i^* + \mu p_{m+1}. \tag{II.18}$$

Also, we have

$$\widetilde{\phi}(\widetilde{a})(y) = \widetilde{\phi}(a + 1_X) = \sum_{i=1}^{m} w_i \tilde{a}(x_i) w_i^* + p_{m+1} \tag{II.19}$$

and

$$|\widetilde{\phi}(\widetilde{a})(y)| = \widetilde{\phi}(|\widetilde{a}|)(y) = \sum_{i=1}^{m} w_i |\widetilde{a}|(x_i) w_i^* + p_{m+1} = \sum_{i=1}^{m} w_i |\widetilde{a}(x_i)| w_i^* + p_{m+1}. \tag{II.20}$$

Then (II.18) and (II.20) give

$$\widetilde{\phi}(u)(y)|\widetilde{\phi}(\widetilde{a})(y)| = \sum_{i=1}^{m} w_i u(x_i) |\widetilde{a}(x_i)| w_i^* + \mu p_{m+1}. \tag{II.21}$$

Also, by Corollary II.3.7, we have

$$p_\alpha(|\widetilde{\phi}(\widetilde{a})(y)|) = p_\alpha\left(\sum_{i=1}^{m} w_i |\widetilde{a}(x_i)| w_i^* + p_{m+1}\right) = \sum_{i=1}^{m} p_\alpha(w_i |\widetilde{a}(x_i)| w_i^*) + p_\alpha(p_{m+1}),$$

where the functional calculus in the last expression is taken in $p_i B(H) p_i$ for $i \in \{1, \ldots, m+1\}$. Now, for each $i \in \{1, \ldots, m\}$, the map $B(H) \to p_i B(H) p_i$ defined by $T \mapsto w_i T w_i^*$ is a unital *-isomorphism, so we have $p_\alpha(w_i |\widetilde{a}(x_i)| w_i^*) = w_i p_\alpha(|\widetilde{a}(x_i)|) w_i^*$, where the last functional calculus is now taken in $B(H)$. So we have

$$p_\alpha(|\widetilde{\phi}(\widetilde{a})(y)|) = \sum_{i=1}^{m} w_i p_\alpha(|\widetilde{a}(x_i)|) w_i^*, \tag{II.22}$$

(functional calculus on both sides is taken in $B(H)$, i.e. the identity used in the functional calculus is id_H on both sides).

Note that (II.16) implies that $|\mu - 1| < \epsilon$. Then from (II.16), (II.19), (II.21), and (II.22), we have

$$\begin{aligned}
&\left\| \left[\widetilde{\phi}(u)(y)|\widetilde{\phi}(\widetilde{a})(y)| - \widetilde{\phi}(\widetilde{a})(y)\right]\left[1 - p_\alpha(|\widetilde{\phi}(\widetilde{a})(y)|)\right] \right\| \\
&\quad = \left\| \left[(\mu - 1)p_{m+1} + \sum_{i=1}^{m} w_i \left[u(x_i)|\widetilde{a}(x_i)| - \widetilde{a}(x_i)\right] w_i^*\right] \right. \\
&\qquad\qquad \left. \cdot \left[p_{m+1} + \sum_{i=1}^{m} w_i \left[1 - p_\alpha(|\widetilde{a}(x_i)|)\right] w_i^*\right] \right\| \\
&\quad = \left\| (\mu - 1)p_{m+1} + \sum_{i=1}^{m} w_i \left[u(x_i)|\widetilde{a}(x_i)| - \widetilde{a}(x_i)\right]\left[1 - p_\alpha(|\widetilde{a}(x_i)|)\right] w_i^* \right\| \\
&\quad = \max\left(\{|\mu - 1|\} \cup \left\{ \left\| \left[u(x_i)|\widetilde{a}(x_i)| - \widetilde{a}(x_i)\right]\left[1 - p_\alpha(|\widetilde{a}(x_i)|)\right] \right\| : 1 \le i \le m \right\}\right) \\
&\quad < \epsilon.
\end{aligned}$$

This estimate holds for all $y \in Y$, so result follows. □

Lemma II.3.18. *Let*

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^{n}\right)$$

be a SRSH system, let $A = A^{(n)}$ *and let* X *be the total space. Suppose that* $\dim(X) = d < \infty$. *Let* $1 > \epsilon > 0$ *and let* $1 > \alpha > 0$. *Let* $a \in A$, *and let* $\widetilde{a} = a + 1 \in \widetilde{A}$. *Suppose that for all* $x \in X$, *we have* $\operatorname{rank}(p_{\alpha/2}(|\widetilde{a}(x)|)) \ge d/2$. *Then there exists* $u \in U_0(\widetilde{A})$ *such that for all* $x \in X$, *we have*

$$\left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right]\left[1 - p_\alpha(|\widetilde{a}(x)|)\right] \right\| < \epsilon. \tag{II.23}$$

Proof: First of all, if we let $x_0 \in X_1$, let $X_1^{(0)} = X_0 = \{x_0\}$, let $R_1 \colon C(X_1, \mathbb{K}) \to C(X_1^{(0)}, \mathbb{K})$ be the restriction map, let $\phi_1 \colon C(X_0, \mathbb{K}) \to C(X_1^{(0)}, \mathbb{K})$ be the identity map, and let $A^{(0)} = C(X_0, \mathbb{K})$, then

$$\left(X_0, A^{(0)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{k=1}^{n}\right)$$

is again a SRSH system that gives the same SRSHA as the original system. This change does not affect any of the hypotheses or the conclusion of the lemma. Thus without loss of generality, assume that X_1 is just one point set, and so $A^{(1)} \cong \mathbb{K}$.

Now suppose

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^{n}\right),$$

where X_1 is a one-point set, $1 > \epsilon > 0$, $1 > \alpha > 0$, and $a \in A$ satisfy the hypothesis of the lemma. Write $a = (a_1, \ldots, a_n)$ with $a_k \in C(X_k, \mathbb{K})$ for $k \in \{1, \ldots n\}$.

Choose $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $0 < \alpha/2 = \alpha_1 < \cdots < \alpha_n = \alpha$. Now we inductively pick $\delta_1, \ldots, \delta_n > 0$. Let $\delta_n = \epsilon/2$. Suppose that $\delta_k > 0$ is picked. Note that $\dim(X_k) \leq \dim(X) = d$, and that for each $x \in X_k$, we have

$$\operatorname{rank}(p_{\alpha_{k-1}}(|\widetilde{a}_k(x)|)) = \operatorname{rank}(p_{\alpha_{k-1}}(|\widetilde{a}(x)|)) \geq \operatorname{rank}(p_{\alpha/2}(|\widetilde{a}(x)|)) \geq d/2.$$

So we can apply Lemma II.3.16, with $\epsilon, \alpha_1, \alpha_2$, X, Y, and a in Lemma II.3.16 respectively taken to be $\min\{\delta_k/2, \epsilon/(2^k)\}$, α_{k-1}, α_k, X_k, $X_k^{(0)}$, and a_k, to obtain δ'_{k-1}. Set $\delta_{k-1} = \min\{\delta_k/2, \delta'_{k-1}\}$. Next we inductively choose $u_k \in C(X_k, \mathbb{K})^\sim$ for $k \in \{1, \ldots, n\}$, and homotopies $h_k \colon [0,1] \to U(C(X_k, \mathbb{K})^\sim$ for $k \in \{1, \ldots, n\}$, such that

$$h_k(0) = 1, h_k(1) = u_k, \quad \text{for } k \in \{1, \ldots, n\}, \tag{II.24}$$

$$(h_1(t), \ldots h_k(t)) \in U(\widetilde{A^{(k)}}), \quad \text{for } t \in [0,1] \tag{II.25}$$

$$(u_1, \ldots, u_k) \in U_0(\widetilde{A^{(k)}}), \quad \text{for } k \in \{1, \ldots, n\}, \tag{II.26}$$

$$\left\| \left[u_k(x)|\widetilde{a}_k(x)| - \widetilde{a}_k(x)\right](1 - p_{\alpha_k}(|\widetilde{a}_k(x)|))\right\| < \delta_k, \quad \text{for all } x \in X_k. \tag{II.27}$$

For each $\xi = (\xi_1, \ldots, \xi_n) \in \widetilde{A}$, we will use $\xi^{(k)}$ to denote the first k entries of ξ. Note that $(\xi_1, \ldots, \xi_k) \in \widetilde{A^{(k)}}$. Since X_1 is just a one-point space, it is clear that there exists $u_1 \in U_0(\widetilde{A^{(1)}})$ and a homotopy $h_1 \colon [0,1] \to U(\widetilde{A^{(1)}})$ such that $h_1(0) = 1$ and $h_1(1) = u_1$, and that (II.24), (II.26), and (II.27) hold for $k = 1$. Suppose that u_k and h_k are chosen to satisfy (II.24), (II.25), (II.26), and (II.27).

Let $v = \widetilde{\phi}_{k+1}(u^{(k)})$, where $u^{(k)} = (u_1, \ldots, u_k) \in \widetilde{A^{(k)}}$, and define

$$f_0 \colon [0,1] \to U(C(X_{k+1}^{(0)}, \mathbb{K})^\sim)$$

by $f_0(t) = \widetilde{\phi}_{k+1}(h_1(t), \ldots, h_k(t))$. Then $v \in U_0(C(X_{k+1}^{(0)}, \mathbb{K})^\sim)$ and f_0 is a homotopy in $U(C(X_{k+1}^{(0)}, \mathbb{K})^\sim)$ from 1 to v. Also, applying Lemma (II.3.17) to $A^{(k)}$ in place of A, $X_{k+1}^{(0)}$ in place of Y, ϕ_{k+1} in place of ϕ, $a^{(k)}$ in place of a, δ_k in place of ϵ, α_k in place of α, and $u^{(k)} = (u_1, \ldots, u_k)$ in place of u, we have

$$\left\| \left[v(x)|\widetilde{\phi}(\widetilde{a}^{(k)})(x))| - \widetilde{\phi}(\widetilde{a}^{(k)})(x)\right]\left[1 - p_{\alpha_k}(|\widetilde{\phi}(\widetilde{a}^{(k)}(x)|)\right]\right\| < \delta_k,$$

for all $x \in X_{k+1}^{(0)}$. Since $\widetilde{\phi}_{k+1}(\widetilde{a}^{(k)}) = \widetilde{R}(\widetilde{a}_{k+1})$, we have

$$\left\| \left[v(x)|\widetilde{a}_{k+1}(x)| - \widetilde{a}_{k+1}(x)\right]\left[1 - p_{\alpha_k}(|\widetilde{a}_{k+1}(x)|)\right]\right\| < \delta_k,$$

for all $x \in X_{k+1}^{(0)}$. Then by the choice of δ_k, there exists $u_{k+1} \in U_0(C(X_{k+1}, \mathbb{K})^\sim)$ and a homotopy h_{k+1} in $U(C(X_{k+1}, \mathbb{K})^\sim)$ such that $h_{k+1}(0) = 1$, such that $h_{k+1}(1) = u_{k+1}$, such that $h_{k+1}(t)|_{X_{k+1}^{(0)}} = f_0(t)$ for all $t \in [0, 1]$, such that $u_{k+1}|_{X_{k+1}^{(0)}} = v$, and such that

$$\left\| \left[u_{k+1}(x)|\widetilde{a}_{k+1}(x)| - \widetilde{a}_{k+1}(x)\right]\left[1 - p_{\alpha_{k+1}}(|\widetilde{a}_{k+1}(x)|)\right]\right\| < \delta_{k+1},$$

for all $x \in X_{k+1}$. It is clear that $(u_1, \ldots, u_k, u_{k+1})$ is a unitary $A^{(k+1)}$, and that for each $t \in [0, 1]$, we have

$$(h_1(t), \ldots, h_k(t), h_{k+1}(t)) \in U(C(X_{k+1}, \mathbb{K})^\sim).$$

Then $t \mapsto (h_1(t), \ldots, h_{k+1}(t))$ is a homotopy in $U(C(X_{k+1}, \mathbb{K})^\sim)$ from 1 to $(u_1, \ldots, u_k)$. So $(u_1, \ldots, u_k) \in U_0(\widetilde{A^{(k+1)}})$. This completes the inductive step.

Now take $u = (u_1, \ldots, u_n)$. Since for all $k \in \{1, \ldots, n\}$ and for all $x \in X_k$, we have $1 - p_{\alpha_k}(|\widetilde{a}(x)|) \geq 1 - p_\alpha(|\widetilde{a}(x)|)$, and since $\delta_1 < \delta_2 < \cdots < \delta_k < \epsilon$, (II.27) implies (II.23). This finishes the proof. ☐

As a consequence of the above lemma, the next proposition will give an approximate polar decomposition for elements a in a SRSHA such that the dimension of the the eigenspaces of the small eigenvalues of $|a(x)|$ is large enough.

Proposition II.3.19. *Let*

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)}\right)_{i=2}^{n}\right)$$

be a SRSH system, let $A = A^{(n)}$, and let X be the total space. Suppose that $\dim(X) = d < \infty$. *Let*

$1 > \epsilon > 0$ and let $1 > \alpha > 0$. Let $a \in A$, and let $\widetilde{a} = a + 1 \in \widetilde{A}$. Suppose that for all $x \in X$, we have $\operatorname{rank}(p_{\alpha/2}(|\widetilde{a}(x)|)) \geq d/2$. Then there exists $u \in U_0(\widetilde{A})$ such that $\| u|\widetilde{a}| - \widetilde{a} \| < \epsilon + 2\alpha$.

Proof: Let u be the unitary obtained using Lemma II.3.18. Then for all $x \in X$ and all $\xi \in H$, where H is the underlying Hilbert space, we have

$$\begin{aligned} &\| [u(x)|\widetilde{a}(x)| - \widetilde{a}(x)](\xi) \| \\ &\quad \leq \left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right] \left(1 - p_\alpha(|\widetilde{a}(x)|)(\xi)\right) \right\| \\ &\qquad\qquad + \left\| \left[u(x)|\widetilde{a}(x)| - \widetilde{a}(x)\right] p_\alpha(|\widetilde{a}(x)|)(\xi) \right\| \\ &\quad < \epsilon \|\xi\| + \| (|\widetilde{a}(x)|) p_\alpha(|\widetilde{a}(x)|)(\xi) \| + \| \widetilde{a}(x) p_\alpha(|\widetilde{a}(x)|)(\xi) \| \\ &\quad \leq \epsilon \|\xi\| + 2\alpha \|\xi\|. \end{aligned}$$

Thus $\| [u(x)|\widetilde{a}(x)| - \widetilde{a}(x)] \| \leq \epsilon + 2\alpha$ for all $x \in X$. So $\| u|\widetilde{a}| - \widetilde{a} \| \leq \epsilon + 2\alpha$. □

Corollary II.3.20. *Let*

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)} \right)_{i=2}^{n} \right)$$

be a SRSH system, let $A = A^{(n)}$, and let X be the total space. Suppose that $\dim(X) = d < \infty$. Let $1 > \epsilon > 0$. Let $a \in A$ and let $\widetilde{a} = a + 1 \in \widetilde{A}$. Suppose that for all $x \in X$, we have $\operatorname{rank}(p_{\epsilon/8}(|\widetilde{a}(x)|)) \geq d/2$. Then there exists $b \in \widetilde{A}$ such that b is invertible and $\| \widetilde{a} - b \| < \epsilon$.

Proof: Apply Proposition II.3.19 to A, $\epsilon/4$ in place of ϵ, $\epsilon/4$ in place of α, and $a \in A$, to obtain a unitary $u \in U_0(\widetilde{A})$ such that $\| u|\widetilde{a}| - \widetilde{a} \| < \epsilon/4 + \epsilon/2 = 3\epsilon/4$. Let $b = u(|\widetilde{a}| + \epsilon/4)$. Then b is invertible and

$$\| b - \widetilde{a} \| \leq \left\| b - u|\widetilde{a}| \right\| + \| u|\widetilde{a}| - \widetilde{a} \| < \epsilon/4 + 3\epsilon/4 = \epsilon.$$

□

Lemma II.3.21. *Let*

$$\left(X_1, A^{(1)}, \left(X_i, X_i^{(0)}, \phi_i, R_i, A^{(i)} \right)_{i=2}^{n} \right)$$

be a SRSH system, let $A = A^{(n)}$, and let X be the total space. Let $a \in A$ and let $\widetilde{a} = a + 1 \in \widetilde{A}$. Let $1 > \alpha > 0$. Then the set $U = \{ x \in X \colon \operatorname{rank}(p_\alpha(|\widetilde{a}(x)|) \geq 1 \}$ is open. Further, if $U \neq \varnothing$, then $I_U = \{ a \in A \colon a|_{U^c} = 0 \}$ is a non-zero ideal of A.

Proof: If $U = \{x \in X : \operatorname{rank}(p_\alpha(|\widetilde{a}(x)|)) \geq 1\}$ is empty, then we are done. So assume that $U \neq \varnothing$. To show that U is open, it is enough to show that every $x \in U$ is an interior point, i.e. there exists some open $V \subseteq U$ such that $x \in V$. Fix $x_0 \in U$.

Apply Lemma II.3.12 to α and $|\widetilde{a}(x_0)|$ to obtain $\delta > 0$. The map $x \mapsto |\widetilde{a}(x)|$ is continuous, and the set $V = \left\{x \in X : \left\| |\widetilde{a}(x)| - |\widetilde{a}(x_0)| \right\| < \delta\right\}$ is open and contains x_0. If $x \in V$, then the choice of δ implies that $1 \leq \operatorname{rank}(p_\alpha(|\widetilde{a}(x_0)|)) \leq \operatorname{rank}(p_\alpha(|\widetilde{a}(x)|))$. Therefore $V \subseteq U$, and hence U is open.

To show that $I_U \neq 0$, we verify the condition in part 1 of Lemma II.2.9. For each $k \in \{1, \ldots, n\}$, let $U_k = X_k \cap U$, and for each $k = 2, \ldots, n$, let

$$W_k = \left\{x \in X_k^{(0)} : \operatorname{sp}_x(\phi_k) \cap \left(\bigsqcup_{i=1}^{k-1} U_i\right) \neq \varnothing\right\}.$$

Let $2 \leq k \leq n$ and let $x \in W_k$. Then $\operatorname{sp}_x(\phi_k) \cap U \neq \varnothing$, so let $y_0 \in \operatorname{sp}_x(\phi_k) \cap U$. Let $w_1, \ldots, w_l$ be the family of isometries with orthogonal ranges such that $\phi_k(f) = \sum_{i=1}^{l} w_i f(y_i) w_i^*$ for all $f \in A^{(k-1)}$, where $y_i \in \operatorname{sp}_x(\phi_k)$ for $i \in \{1, \ldots, l\}$. Let i_0 be an integer such that $1 \leq i_0 \leq l$ and $y_{i_0} = y_0$. Let $c \in A_{s.a.}$ be such that $|\widetilde{a}| = c + 1$. Then

$$\begin{aligned} p_\alpha(|\widetilde{a}(x)|) &= p_\alpha(c(x) + 1) = p_{\alpha-1}(c(x)) \\ &= \sum_{i=1}^{l} w_i p_{\alpha-1}(c(y_i)) w_i^* \geq w_{i_0} p_{\alpha-1}(c(y_0)) w_{i_0}^* \\ &= w_{i_0} p_\alpha(c(y_0) + 1) w_{i_0}^* = w_{i_0} p_\alpha(|\widetilde{a}(y_0)|) w_{i_0}^*. \end{aligned}$$

So, since $y_0 \in U$, we have $\operatorname{rank}(p_\alpha(|\widetilde{a}(x)|)) \geq \operatorname{rank}(p_\alpha(|\widetilde{a}(y_0)|)) \geq 1$. Hence $x \in U_k$, and so $x \in U_k \cap X_k^{(0)}$. Therefore $W_k \subseteq U_k \cap X_k^{(0)}$.

Now let $x \in U_k \cap X_k^{(0)}$. Let $w_1, \ldots, w_l$ be the family of isometries with orthogonal ranges such that $\phi_k(f) = \sum_{i=1}^{l} w_i f(y_i) w_i^*$ for all $f \in A^{(k-1)}$, where $y_i \in \operatorname{sp}_x(\phi_k)$ for all $i \in \{1, \ldots, l\}$. Then

$$\operatorname{rank}(p_\alpha(|\widetilde{a}(x)|)) = \operatorname{rank}\left(\sum_{i=1}^{l} w_i p_\alpha(|\widetilde{a}(y_i)|) w_i^*\right) = \sum_{i=1}^{l} \operatorname{rank}(p_\alpha(|\widetilde{a}(y_i)|)).$$

Since $x \in U$, for some $i \in \{1, \ldots, l\}$, we have $\operatorname{rank}(p_\alpha(|\widetilde{a}(y_i)|)) \geq 1$. Thus $y_i \in \bigsqcup_{j=1}^{k-1} U_j$. So $\operatorname{sp}_x(\phi_k) \cap \left(\bigsqcup_{j=1}^{k-1} U_j\right) \neq \varnothing$, and so $x \in W_k$. Hence $U_k \cap X_k^{(0)} \subseteq W_k$.

Thus by Lemma II.2.9, $I_U \neq 0$. ☐

Lemma II.3.22. *Let (A_n, ψ_n) be an inductive system of SRSHAs and let A be the inductive limit. Let X_n be the total space for A_n. Suppose that ψ_n is injective for all n, that ψ_n is non-vanishing for all n, and suppose that A is simple. Let $1 > \alpha > 0$. Then for all $n \geq 1$ and all $a \in A_n$ such that $\widetilde{a} = a + 1$ is not invertible in $\widetilde{A}_n$, there exists some $m \geq n$ such that for all $k \geq m$ and all $x \in X_k$, we have $\operatorname{rank}(p_\alpha(|\widetilde{\psi}_{n,k}(\widetilde{a})(x)|)) \geq 1$, where $\widetilde{\psi}_{n,k}$ is the unitization of the map $\psi_{n,k}$.*

Proof: Let $U = \{x \in X_n \colon \operatorname{rank}(p_\alpha(|\widetilde{a}(x)|)) \geq 1\}$. We first show that $U \neq \varnothing$. Since $\widetilde{a}$ is not invertible, there exists some x_0 in the total space of A_n such that $\widetilde{a}(x_0)$ is not invertible. Then by the Fredholm Alternative, the operator $\widetilde{a}(x_0)$ is not injective, which implies that $|\widetilde{a}(x_0)|$ is not injective. Then $p_\alpha(|\widetilde{a}(x_0)|) \neq 0$, which implies that $x_0 \in U$. This shows that $U \neq \varnothing$.

By Lemma II.3.21, $I_U = \{a \in A_n \colon a|_{U^c} = 0\}$ is a non-zero ideal. Then by Proposition II.2.11, there exists $m \geq N$ such that for all $k \geq m$, and for all $x \in X_k$, we have $\operatorname{sp}_x(\psi_{n,k}) \cap U \neq \varnothing$. Let $k \geq m$, let $x \in X_k$, and let $w_1, \ldots, w_l$ be the family of isometries with orthogonal ranges such that $\psi_{n,k}(f)(x) = \sum_{i=1}^{l} w_i f(y_i) w_i^*$ for all $f \in A_n$, where $\{y_i \colon i = 1, \ldots, l\} = \operatorname{sp}_x(\psi_{n,k})$. Let $y_0 \in \operatorname{sp}_x(\psi_{n,k}) \cap U$ and choose $1 \leq i_0 \leq l$ such that $y_{i_0} = y_0$. Let $c \in (A_n)_{s.a.}$ be such that $|\widetilde{a}| = \widetilde{c}$. Then $|\widetilde{\psi}_{n,k}(\widetilde{a})| = \widetilde{\psi}_{n,k}(|\widetilde{a}|) = \widetilde{\psi}_{n,k}(\widetilde{c}) = \psi_{n,k}(c) + 1$. Thus

$$\begin{aligned}
\operatorname{rank}(p_\alpha(|\widetilde{\psi}_{n,k}(\widetilde{a})(x)|)) &= \operatorname{rank}(p_\alpha(|\widetilde{\psi}_{n,k}(\widetilde{a})|(x))) \\
&= \operatorname{rank}(p_\alpha(\psi_{n,k}(c)(x) + 1)) \\
&= \operatorname{rank}(p_{\alpha-1}(\psi_{n,k}(c)(x))) \\
&= \sum_{i=1}^{l} \operatorname{rank}(p_{\alpha-1}(c(y_i))) \geq \operatorname{rank}(p_{\alpha-1}(c(y_{i_0})) \\
&= \operatorname{rank}(p_{\alpha-1}(c(y_0)) \\
&= \operatorname{rank}(p_\alpha(c(y_0) + 1)) \\
&= \operatorname{rank}(p_\alpha(\widetilde{c}(y_0))) = \operatorname{rank}(p_\alpha(|\widetilde{a}(y_0)|)) \geq 1.
\end{aligned}$$

The last inequality above holds because $y_0 \in U$. □

Theorem II.3.23. *Let (A_n, ψ_n) be an inductive system of SRSHAs and let A be the inductive limit. Let X_n be the total space for A_n. Suppose that ψ_n is injective and non-vanishing for all n, and suppose that A is simple. Also assume that there exists $d \in \mathbb{N}$ such that $\dim(X_n) \leq d$ for all $n \geq 1$. Then A has topological stable rank one.*

Proof: We first show that an element of the form $b+1 \in \widetilde{A}$, where $b \in A$, can be approxmiated arbitrarily closely by some invertible element in $\widetilde{A}$.

Let $b \in A$, let $1 > \epsilon > 0$, and let $\widetilde{b} = b+1$. Let $n \geq 1$, and let $a \in A_n$ satisfy $\|\widetilde{\psi^n}(\widetilde{a}) - \widetilde{b}\| < \epsilon/2$, where $\psi^n : A_n \to A$ is the standard map that comes with the inductive limit. If $\widetilde{a}$ is invertible in A_n, then $\widetilde{\psi^n}(\widetilde{a})$ is invertible in $\widetilde{A}$, and we are done. So assume that $\widetilde{a}$ is not invertible in A_n. Then by Lemma II.3.22, using $\epsilon/16$ as α, find some $m_1 \geq n$ such that for all $k \geq m_1$, $\operatorname{rank}(p_{\epsilon/16}(|\widetilde{\psi}_{n,k}(\widetilde{a})(x)|)) \geq 1$ for all $x \in X_k$.

For each $n \geq 1$, let $X_{n,1}, \ldots, X_{n,l(n)}$ be the base spaces of A_n, let $X^{(0)}_{n,2}, \ldots, X^{(0)}_{n,l(n)}$ be the attaching spaces, and let $X^{(0)}_{n,1} = \varnothing$. If for all $k \geq m_1$, the set $\bigsqcup_{i=1}^{l(k)}(X_{k,i} \setminus X^{(0)}_{k,i})$ is a finite set, then for all $k \geq m_1$ the algebra A_k is simply a finite direct sum of copies of $\mathbb{K}$. This means that A_k has topological stable rank one for all $k \geq m_1$, which implies that A has topological stable rank one, and we are done. So we can assume that there exists some $m_2 \geq m_1$ such that $\bigsqcup_{i=1}^{l(m_2)}(X_{m_2,i} \setminus X^{(0)}_{m_2,i})$ is infinite. Let $1 \leq l \leq l(m_2)$ be the largest integer such that $X_{m_2,l} \setminus X^{(0)}_{m_2,l}$ is infinite. Then A_{m_2} is isomorphic to $A^{(l)}_{m_2} \oplus \left(\bigoplus_{i=1}^{l'} \mathbb{K}\right)$ for some $l' \in \mathbb{N} \cup \{0\}$, via some isomorphism

$$h \colon A_{m_2} \to A^{(l)}_{m_2} \oplus \left(\bigoplus_{i=1}^{l'} \mathbb{K}\right)$$

such that the composition $A_{m_2} \xrightarrow{h} A^{(l)}_{m_2} \oplus \left(\bigoplus_{i=1}^{l'} \mathbb{K}\right) \to A^{(l)}_{m_2}$ (the map on the right is the standard projection) is the restriction map $A_{m_1} \to A^{(l)}_{m_1}$. Let d_1 be an integer greater that $d/2$ and let $x_1, \ldots, x_{d_1} \in X_{m_2,l} \setminus X^{(0)}_{m_2,l}$. For each $i \in \{1, \ldots, d_1\}$, let $V_i \subseteq X_{m_2,l} \setminus X^{(0)}_{m_2,l}$ be an open neighborhood of x_i such that $\{V_i \colon i = 1, \ldots, d_1\}$ is disjoint. For each $i \in \{1, \ldots, d_1\}$, let

$$J_i = \{a \in A^{(l)}_{m_2} \colon a|_{V_i^c} = 0\}.$$

Then each J_i is a non-zero closed two sided ideal of $A^{(l)}_{m_2} \oplus \left(\bigoplus_{i=1}^{l'} \mathbb{K}\right)$. For each $i \in \{1, \ldots, d_1\}$, let $I_i = h^{-1}(J_i)$. Since $\{J_i \colon i = 1, \ldots, d_1\}$ is orthogonal, so is $\{I_i \colon i = 1, \ldots, d_1\}$. For each $i \in \{1, \ldots, d_1\}$, let

$$W_i = \{x \in X_{m_1} \colon \text{ there exists some } a \in I_i \text{ such that } a(x) \neq 0\}.$$

Then for each $i = 1, \ldots, d_1$, we have $V_i \subseteq W_i$ and $W_i \cap \left(\bigsqcup_{j=1}^{l(m_2)}(X_{m_2,j} \setminus X^{(0)}_{m_2,j})\right) = V_i$.

Now, for each $i \in \{1, \ldots, d_1\}$, apply Proposition II.2.11, to obtain some $n_i \geq m_2$ such that for all $k \geq n_i$, and for all $x \in X_k$, $\mathrm{sp}_x(\psi_{m_1,k}) \cap W_i \neq \varnothing$. Let $n_0 = \max\{n_1, \ldots, n_{d_1}\}$. Let $k \geq n_0$ and let $x \in X_k$. Then $\mathrm{sp}_x(\psi_{m_2,k}) \cap W_i \neq \varnothing$ for each $i \in \{1, \ldots, d_1\}$. So for each $i \in \{1, \ldots, d_1\}$, we can choose $y_i \in \mathrm{sp}_x(\psi_{m_2,k}) \cap W_i$. Since for each $i \in \{1, \ldots, d_1\}$,

$$y_i \in W_i \cap \left(\bigsqcup_{i=1}^{l(m_2)} (X_{m_2,i} \setminus X_{m_2,i}^{(0)}) \right) = V_i,$$

and since $V_1, \ldots, V_{d_2}$ are pairwise disjoint, we see that $y_1, \ldots, y_{d_1}$ are distinct. Let $w_1, \ldots, w_t$ be isometries with mutually orthogonal ranges such that for all $f \in A_{m_2}$ we have $\psi_{m_2,k}(f)(x) = \sum_{i=1}^t w_i f(z_i) w_i^*$, where $\{z_i \colon i = 1, \ldots, t\} = \mathrm{sp}_x(\psi_{m_2,k})$. Since $m_2 \geq m_1$, we have $\mathrm{rank}(p_{\epsilon/16}(|\widetilde{\psi}_{n,m_2}(\widetilde{a})(y_i)|)) \geq 1$ for each $i \in \{1, \ldots, d_1\}$. Let $c \in (A_{m_2})_{s.a.}$ satisfy $|\widetilde{\psi}_{n,m_2}(\widetilde{a})| = \widetilde{c}$. Then

$$\begin{aligned}
\mathrm{rank}(p_{\epsilon/16}(|\widetilde{\psi}_{n,k}(\widetilde{a})(x)|) &= \mathrm{rank}(p_{\epsilon/16}(|\widetilde{\psi}_{m_2,k}(\widetilde{\psi}_{n,m_2}(\widetilde{a}))(x)|)) \\
&= \mathrm{rank}(p_{\epsilon/16}(\widetilde{\psi}_{m_2,k}(|\widetilde{\psi}_{n,m_2}(\widetilde{a})|)(x))) \\
&= \mathrm{rank}(p_{\epsilon/16}(\widetilde{\psi}_{m_2,k}(\widetilde{c})(x))) \\
&= \mathrm{rank}(p_{\epsilon/16}(\psi_{m_2,k}(c)(x) + 1)) \\
&= \mathrm{rank}(p_{(\epsilon/16)-1}(\psi_{m_2,k}(c)(x))) \\
&= \mathrm{rank}\left(p_{(\epsilon/16)-1}\left(\sum_{i=1}^t w_i c(z_i) w_i^*\right)\right) \\
&= \sum_{i=1}^t \mathrm{rank}(p_{(\epsilon/16)-1}(c(z_i))) \\
&\geq \sum_{i=1}^{d_2} \mathrm{rank}(p_{(\epsilon/16)-1}(c(y_i))) \\
&= \sum_{i=1}^{d_2} \mathrm{rank}(p_{\epsilon/16}(\widetilde{c}(y_i))) \\
&= \sum_{i=1}^{d_2} \mathrm{rank}(p_{\epsilon/16}(|\widetilde{\psi}_{n,m_2}(\widetilde{a})(y_i)|)) \\
&= d_1 \geq d/2 \geq \dim(X_k)/2.
\end{aligned}$$

Then by Corollary II.3.20, there exists some invertible element $c \in \widetilde{A}_k$ such that $\|\widetilde{\psi}_{n,k}(\widetilde{a}) - c\| < \epsilon/2$.

So $\widetilde{\psi}^k(c)$ is invertible in $\widetilde{A}$, and

$$\begin{aligned}\|\widetilde{\psi}^k(c) - \widetilde{b}\| &\leq \|\widetilde{\psi}^k(c) - \widetilde{\psi}^k(\widetilde{\psi}_{n,k}(\widetilde{a}))\| + \|\widetilde{\psi}^k(\widetilde{\psi}_{n,k}(\widetilde{a})) - b\| \\ &= \|c - \widetilde{\psi}_{n,k}(\widetilde{a})\| + \|\widetilde{\psi}^n(\widetilde{a}) - b\| \\ &< \epsilon/2 + \epsilon/2.\end{aligned}$$

Thus we have shown that for all $b \in A$ and all $\epsilon > 0$, there exists some invertible element $c \in \widetilde{A}$ such that $\|\widetilde{b} - c\| < \epsilon$. Next will show that for all $b \in A$ and all $\epsilon > 0$, there exists some $c \in A$ such that $c + 1$ is invertible and $\|\widetilde{c} - \widetilde{b}\| < \epsilon$.

Let $b \in A$ and let $1 > \epsilon > 0$. By what we just proved above, $\widetilde{b} \in \overline{\text{inv}(\widetilde{A})}$, where $\text{inv}(\widetilde{A})$ denote the set of all invertible elements of $\widetilde{A}$. So there exists a sequence $(a_n, \lambda_n) \in \text{inv}(\widetilde{A})$ such that $\|(a_n, \lambda_n) - (b, 1)\| \to 0$. Then $\lambda_n \to 1$. So $(\lambda_n^{-1} a_n, 1) = \lambda_n^{-1}(a_n, \lambda_n) \to \widetilde{b}$. Thus we can pick some n such that $\|(\lambda_n^{-1} a_n, 1) - \widetilde{b}\| < \epsilon$. Setting $c = \lambda_n^{-1} a_n$, we see that $\widetilde{c} = \lambda_n^{-1}(a_n, \lambda_n)$ is invertible and $\|\widetilde{c} - \widetilde{b}\| < \epsilon$. Then by Proposition 4.2 of [16], the algebra A has topological stable rank one. □

Many arguments in this chapter may be simplified greatly if every SRSHA is the tensor product of a RSHA with $\mathbb{K}$; however we were not able to determine whether every SRSHA is the tensor product of a RSHA with $\mathbb{K}$. In the approach we used when trying to resolve this question, we found that in order to show that a SRSHA is the tensor product of a RSHA with $\mathbb{K}$, we needed to extend projection valued functions over a closed subspace of a compact metric space to the entire space. This cannot be done in general, and so we feel that it is not true that every SRSHA is the tensor product of a RSHA with $\mathbb{K}$.

Also, SRSHAs are likely to be $\mathbb{K}$-stable. If A is a SRSHA, then A is contained in $B = \bigoplus_{i=1}^n C(X_i, \mathbb{K})$ as a C^*-subalgebra, which implies that $A \otimes \mathbb{K}$ is a C^*-subalgebra of $B \otimes \mathbb{K}$. The obvious *-isomorphism from $B \otimes \mathbb{K}$ to B restricted to $A \otimes \mathbb{K}$ may very well be a *-isomorphism from $A \otimes \mathbb{K}$ to A.

CHAPTER III

STABLE RECURSIVE SUBHOMOGENEOUS C^*-SUBALGEBRAS OF $C^*(X,\mathbb{R})$

In general, when X is a compact metric space, and G is a topological group acting on X freely and minimally, the structure and properties of the crossed product $C^*(X, G)$ are often very difficult to study, even if G is as familiar as $\mathbb{Z}$ or $\mathbb{R}$. So we would like to look at certain distinguished C^*-subalgebras of the crossed product instead. Often, properties and the structure of those C^*-subalgebras can be used to study the entire crossed product.

In [10], X was taken to by the Cantor set, G was taken to be $\mathbb{Z}$, and the action was assumed to be free and minimal. For $Y \subseteq X$ closed, define A_Y to be the C^*-subalgebra of the crossed product $C^*(X, \mathbb{Z})$ generated by $C(X)$ and $uC_0(X \setminus Y)$. When Y is also open, it was shown that A_Y is an AF-algebra. For $y \in X$, let A_y denote $A_{\{y\}}$. If $(Y_n)_{n\geq 1}$ is a decreasing sequence of clopen sets such that $\bigcap_{n\geq 1} Y_n = \{y\}$, then it is easy to see that A_y is the closure of the increasing union $\bigcup_{n\geq 1} A_{Y_n}$. Hence, A_y is an AF-algebra as well.

When $\mathbb{Z}$ acts freely and minimally on a arbitrary compact metric space X with finite covering dimension, it is shown in [5] that the C^*-subalgebra A_Y generated by $C(X)$ and $uC_0(X\setminus Y)$ is a RSHA. This fact is used in [3] to show that, under certain hypothesis, the crossed product has tracial rank zero.

When we consider free minimal actions of $\mathbb{R}$ on compact metric spaces with finite covering dimension, we would like to look at C^*-subalgebras of the crossed product that are analogous to the ones mentioned above. However, we immediately run into a difficulty: the algebra $C(X)$ and the unitaries that implement the action are not contained in the crossed product; they are contained in the multiplier algebra of the crossed product instead. So we cannot define the C^*-algebras A_Y and A_y as the C^*-algebras generated by certain sets of elements of the crossed product. We need to take a more explicit approach. In retrospect, we realize that the subalgebra A_Y in the integer case, in some sense, is the "algebra of partial orbits": orbits are broken at a chosen subset Y, then partial orbits are grouped together according to their lengths to make C^*-subalgebras of the crossed product. This is the approach we take in this chapter to construct the C^*-subalgebras analogous to A_Y and A_y in the integer case.

In the rest of this dissertation, we fix a compact metric space X, and fix a free minimal action of $\mathbb{R}$ on X. The construction that we will describe in this chapter requires that the action admits "pseudo-transversals," which we define below.

Definition III.0.1. *Let X be a compact metric space and let $\mathbb{R}$ act on X freely and minimally. A nonempty closed subset Z of X is called a* pseudo-transversal *if*

1. *For all $x \in X$, the set $(\mathbb{R} \cdot x) \cap Z$ is dense in Z.*
2. *There exists $\sigma > 0$ such that for all $x \in Z$, we have $([-\sigma, \sigma] \cdot x) \cap Z = \{x\}$.*

The existence of pseudo-transversals is essentially guaranteed by Lemma 3.1 in [12]. Only the density condition is not explicitly stated in the statement of that lemma. We include the proof of the existence of pseudo-transversals here, applying Lemma 3.1 in [12].

Lemma III.0.2. *Let X be a compact metric space. Let $\mathbb{R}$ act freely and minimally on X. Then the action admits a pseudo-transversal.*

Proof: By Lemma 3.1 in [12], there exist a real number $\epsilon > 0$, an element $x_0 \in X$, and a closed subset $S \subseteq X$ containing x_0 such that the map $\Gamma\colon (-\epsilon, \epsilon) \times S \to X$ defined by $\Gamma(r, x) = rx$ is a homeomorphism onto a neighborhood of x_0.

We first claim that any subset $T \subseteq S$ satisfies condition 2 in Definition III.0.1. Take $\sigma = \epsilon/2$. Let $x \in T$. Suppose that $y \in ([-\sigma, \sigma] \cdot x) \cap T$. Then $y = rx$ for some $r \in [-\sigma, \sigma] \subseteq (-\epsilon, \epsilon)$. So $(r, x) \in (-\epsilon, \epsilon) \times S$. Therefore $y = \Gamma(r, x) = \Gamma(0, y)$. It follows from the injectivity of Γ that $x = y$. This proves the claim.

Next we claim that if $x, y \in S$ and $r \in \mathbb{R}$ satisfy $y = rx$, then either $r = 0$ or $|r| \geq 2\epsilon$. Let $x, y \in S$ and $r \in \mathbb{R}$ satisfy $y = rx$. Also assume that $|r| < 2\epsilon$. Then $-r/2, r/2 \in (-\epsilon, \epsilon)$. Since $y = rx$, we have

$$\Gamma\left(\left(\frac{r}{2}\right), x\right) = \left(\frac{r}{2}\right) \cdot x = \left(-\frac{r}{2}\right) \cdot y = \Gamma\left(-\left(\frac{r}{2}\right), y\right).$$

By the injectivity of Γ, we have $r = 0$. This proves the claim.

Let d be the metric on X. For each $r > 0$ and each $x \in X$, let $B(x, r)$ denote the open ball $\{y \in X : d(x, y) < r\}$. Now, since $(-\epsilon, \epsilon) \cdot S$ is a neighborhood of x_0, there exists some $\delta > 0$ such that $B(x_0, \delta) \subseteq (-\epsilon, \epsilon) \cdot S$. Let $Z = \overline{B(x_0, \delta/2) \cap S}$. Note that since S is closed in X, the set Z is contained in S. With $\sigma = \epsilon/2$, condition 2 in Definition III.0.1 holds by the first claim above.

We now show that Z satisfies condition 1 in Definition III.0.1. Fix some $x \in X$ and some $z \in B(x_0, \delta/2) \cap S$. Note that $z \in B(x_0, \delta) \cap S$. Choose a sequence $\{r_n\}$ of strictly positive real numbers such that $B(z, r_n) \subseteq B(x_0, \delta)$ for all n and such that $\lim_{n \to \infty} r_n = 0$. Since the action is minimal, the set $(\mathbb{R} \cdot x) \cap B(z, r_n)$ is nonempty for all $n \geq 1$. So for each $n \geq 1$, we can choose some $z_n \in B(z, r_n) \cap (\mathbb{R} \cdot x)$. Then z_n is in the image of the map Γ for each $n \geq 1$. Thus, for each $n \geq 1$, there exists $(s_n, y_n) \in (-\epsilon, \epsilon) \times S$ such that $\Gamma(s_n, y_n) = z_n$. It is clear that $z_n \to z$. That is, we have $\Gamma(s_n, y_n) \to \Gamma(0, z)$. Then, since Γ is a homeomorphism, we have $y_n \to z$. Because $s_n y_n = z_n \in \mathbb{R} \cdot x$ for all $n \geq 1$, we have $y_n \in \mathbb{R} \cdot x$ for all $n \geq 1$. Now, because $y_n \to z$ and $z \in B(x_0, \delta/2)$, we can assume, passing to a subsequence if necessary, that $y_n \in B(x_0, \delta/2)$ for all $n \geq 1$. Then we have $y_n \in Z \cap (\mathbb{R} \cdot x)$ for all $n \geq 1$. We have now shown that for all $z \in B(x_0, \delta/2) \cap S$ there is a sequence in $Z \cap \mathbb{R} \cdot x$ that converges to z. Then it is clear that $Z \cap \mathbb{R} \cdot x$ is dense in Z. This finishes the proof of the lemma. □

For the rest of the chapter, fix a pseudo-transversal Z, and use σ to denote the real number in the second condition of the definition above. Before we start describing the construction, we look as some examples of $\mathbb{R}$ actions.

The most trivial example is $\mathbb{R}$ acting trivially on an arbitrary metric space X. That is, for every $r \in \mathbb{R}$ and every $x \in X$, we have $rx = x$. In this case, the action is not free and is minimal only when X contains only one element. The corresponding crossed product $C^*(X, \mathbb{R})$ is well known (for instance, Example 2.53 in [17]) to be isomorphic to $C(X) \otimes C^*(\mathbb{R})$, where $C^*(\mathbb{R})$ is the group C^*-algebra of $\mathbb{R}$, which we will not describe here. (See Section 3.1 in [17] for the definition of the group C^*-algebra.) It is also well known (for instance, Proposition 3.1 in [17]) that $C^*(\mathbb{R})$ is isomorphic to $C_0(\mathbb{R})$. So $C^*(X, \mathbb{R})$ is isomorphic to $C(X) \otimes C_0(\mathbb{R}) = C_0(X \times \mathbb{R})$.

When $\mathbb{R}$ acts on itself by translation, the action is free and minimal. The corresponding crossed product $C^*(\mathbb{R}, \mathbb{R})$ is isomorphic to the algebra of all compact operators on $L^2(\mathbb{R})$. In fact, more generally, when a locally compact group G acts on itself by left translation, the crossed product is isomorphic to the algebra of compact operators on $L^2(G)$. This fact is essentially proven in [14], and is the motivation behind the map defined by Equation III.9 in this chapter .

Another class of examples is the class of flows under ceiling functions. Take a locally compact space X. Let $h\colon X \to X$ be a homeomorphism. Then h induces a $\mathbb{Z}$ action on X. Let Y be the quotient space $([0,1] \times X)/{\sim}$, where the equivalence relation $\sim$ is given by $(1, x) \sim (0, h(x))$. Now let points in Y flow upward at unit speed. When a point reaches the ceiling (i.e. the set $\{1\} \times X$), it jumps to the floor (i.e. the set $\{0\} \times X$) and keeps moving up at unit speed. This gives a flow under the ceiling function that is constantly one. When the $\mathbb{Z}$ action on X is free and minimal, the $\mathbb{R}$ action on Y is also free and minimal. If X is compact, then so is Y. It was shown in [15] that the crossed product $C^*(Y, \mathbb{R})$ is stably isomorphic to $C^*(X, \mathbb{Z})$. So this class of examples is also essentially trivial. A similar construction can be used to allow the ceiling function to be an arbitrary strictly positive continuous function from X to $\mathbb{R}$. In this case, the corresponding crossed product $C^*(Y, \mathbb{R})$ is still stably isomorphic to $C^*(X, \mathbb{Z})$. See [15] for more details.

The examples we have described so far are all more or less trivial. Less trivial examples would be free minimal actions on compact metric spaces that are not flows under ceiling functions. It was shown in an unpublished work by N. C. Phillips that such actions indeed exist.

III.1. Entering Times and Return Times

Definition III.1.1. *Let X be a compact metric space, let $\mathbb{R}$ act on X freely and minimally, and let $Z \subseteq X$ be a pseudo-transversal. Let Z^c denote the complement of Z with respect to X. Define the* forward entering time $\beta\colon Z^c \to \mathbb{R}$ *by*

$$\beta(x) = \inf\{r > 0 \colon rx \in Z\};$$

define the backward entering time $\alpha\colon Z^c \to \mathbb{R}$ *by*

$$\alpha(x) = \sup\{r < 0 \colon rx \in Z\};$$

and define the return time $R\colon Z \to \mathbb{R}$ *by*

$$R(x) = \inf\{r > 0 \colon rx \in Z\}.$$

Note that the entering times are well defined because Z meets every orbit of the action. Now we fix some notation for the rest of the chapter

Notation III.1.2. For the rest of the chapter we use α and β to denote, respectively, the forward and backward entering times associated with the pseudo-transversal, and use R to denote the return time for the transversal. We first establish some elementary properties of α, β and R.

Lemma III.1.3. *For all $x \in Z^c$, we have $(\alpha(x), \beta(x)) \cdot x \subseteq Z^c$. For all $z \in Z$, we have $(0, R(z)) \cdot z \subseteq Z^c$. (We use the notation $(\alpha(x), \beta(x))$ and $(0, R(z))$ to denote open intervals of the real line, the notation $(\alpha(x), \beta(x)) \cdot x$ to denote the set $\{rx\colon r \in (\alpha(x), \beta(x))\}$, and the notation $(0, R(z)) \cdot z$ to denote the set $\{rz\colon r \in (0, R(z))\}$.)*

Proof: Let $x \in Z^c$ and let $r \in (\alpha(x), \beta(x))$. Suppose that $rx \in Z$. If $r > 0$, then

$$\beta(x) = \inf\{s > 0\colon sx \in Z\} \leq r < \beta(x),$$

a contradiction. So $r \leq 0$. If $r < 0$, then

$$\alpha(x) = \sup\{r < 0\colon rx \in Z\} \geq r > \alpha(x),$$

contradiction. So $r = 0$. But then $x = rx \in Z$, contradicting the assumption. Thus $(\alpha(x), \beta(x)) \cdot x \subseteq Z^c$.

Let $z \in Z$ and let $r \in (0, R(z))$. Suppose that $rz \in Z$. Then $R(z) \leq r < R(z)$, a contradiction. So $rz \in Z^c$. Thus $(0, R(z)) \cdot z \subseteq Z^c$. □

Lemma III.1.4. *For all $x \in Z^c$, we have $\alpha(x) < 0$ and $\beta(x) > 0$. Also, for all $z \in Z$, we have $R(z) \geq \sigma$.*

Proof: Let $x \in Z^c$. There exists $\epsilon > 0$ such that $(-\epsilon, \epsilon)x \subseteq Z^c$. Then by definition, $\alpha(x) \leq -\epsilon < 0$ and $\beta(x) \geq \epsilon > 0$.

Let $z \in Z$. It is clear that we have $(0, \sigma)z \subseteq Z^c$. Then by definition, $R(z) \geq \sigma$. □

Lemma III.1.5. *For all $x \in Z^c$, we have $\alpha(x) \cdot x \in Z$ and $\beta(x) \cdot x \in Z$. Also, for all $z \in Z$, we have $R(z) \cdot z \in Z$.*

Proof: We know that $\beta > 0$ and $\alpha < 0$, by Lemma III.1.4.

Let $x \in Z^c$. Suppose that $\alpha(x) \cdot x \notin Z$. The map $r \mapsto r \cdot (\alpha(x) \cdot x)$ is a continuous map from $\mathbb{R}$ to X, so the inverse image of Z^c under the map, which contains 0 since we assumed

$\alpha(x) \cdot x \in Z^c$, is open in $\mathbb{R}$. Thus there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cdot (\alpha(x)x) \subseteq Z^c$. Then for all $r \in (-\epsilon + \alpha(x), \alpha(x)]$, we have $rx \notin Z^c$. Now Lemma III.1.3 implies that $(\alpha(x), 0) \cdot x \subseteq Z^c$. So for all $r \in (-\epsilon+\alpha(x), 0)$, we have $rx \notin Z$. Then $\alpha(x) - \epsilon$ is an upper bound to the set $\{r < 0 \colon rx \in Z\}$, contradicting the fact that $\alpha(x) = \sup\{r < 0 \colon rx \in Z\}$. Thus $\alpha(x)x \in Z$.

Very similar arguments show that $\beta(x) \cdot x \in Z$ for all $x \in Z^c$ and $R(z) \cdot z \in Z$ for all $z \in Z$. $\square$

Lemma III.1.6. *The map α is upper semi-continuous, and the maps β and R are lower semi-continuous.*

Proof: Let $r \in \mathbb{R}$. We will show that $\alpha^{-1}([r, \infty))$ is closed in Z^c. If $r \geq 0$, then by Lemma III.1.4, we know that $\alpha^{-1}([r, \infty)) = \varnothing$, and then we are done. So assume that $r < 0$. Suppose that $\{x_n\}_{n \geq 1}$ is a sequence in Z^c such that $\alpha(x_n) \geq r$ for all $n \geq 1$, and suppose that there is $x \in Z^c$ such that $x_n \to x$. Since the sequence $\{\alpha(x_n)\}$ is bounded, it has a convergent subsequence $\{\alpha(x_{k_n})\}$. Say $\alpha(x_{k_n}) \to s$ with $s \in [r, 0]$. Then $\alpha(x_{k_n})x_{k_n} \to sx$. By Lemma III.1.5, we have $\alpha(x_{k_n})x_{k_n} \in Z$ for all $n \geq 1$. So $sx \in Z$, since Z is closed. Also, $s \neq 0$, since $x \notin Z$. Then by the definition of α, we have $\alpha(x) \geq s \geq r$. Thus $x \in \alpha^{-1}([r, \infty))$, and so α is upper semi-continuous.

Let $r \in (0, \infty)$ and let $\{x_n\}$ be a sequence in $\beta^{-1}((-\infty, r])$ such that $x_n \to x$ for some $x \in Z^c$. Then $\{\beta(x_n)\}$ has a subsequence $\{\beta(x_{k_n})\}$ such that $\beta(x_{k_n}) \to s$ for some $s \in [0, r]$. For each $n \geq 1$, we have $\beta(x_{k_n})x_{k_n} \in Z$, so $sx \in Z$. Also, $x \in Z^c$ implies that $s \neq 0$. So $\beta(x) \leq s \leq r$. This shows that β is lower semi-continuous.

In the previous paragraph, if we replace all occurrences of β by R and suppose that $x \in Z$ instead of Z^c, then we get the argument that shows that R is lower semi-continuous. $\square$

Lemma III.1.7. *For all $x \in Z^c$ and for all $r \in (\alpha(x), \beta(x))$, we have $\alpha(rx) = \alpha(x) - r$ and $\beta(rx) = \beta(x) - r$.*

Proof: Let $x \in Z^c$ and let $r \in (\alpha(x), \beta(x))$. We know that $\beta(x) - r > 0$ and $(\beta(x) - r)(rx) = \beta(x)x \in Z$. Therefore by the definition of β, we have $\beta(x) - r \geq \beta(rx)$. Also, $\alpha(x) - r < 0$ and $(\alpha(x) - r)(rx) \in Z$ imply that $\alpha(rx) \geq \alpha(x) - r$. Then it follows from Lemma III.1.3 that

$$(\alpha(x) - r, \beta(x) - r) \cdot (rx) = (\alpha(x), \beta(x)) \cdot x \subseteq Z^c.$$

Then $\beta(rx) \geq \beta(x) - r$ and $\alpha(rx) \leq \alpha(x) - r$. $\square$

Lemma III.1.8. *Let* $\widehat{Z} = [-\sigma, \sigma] \cdot Z$, *let* $\widehat{Z}_- = [-\sigma, 0] \cdot Z$, *and let* $\widehat{Z}^+ = [0, \sigma] \cdot Z$. *Then* $\widehat{Z}$, $\widehat{Z}_+$ *and* $\widehat{Z}_-$ *are all closed and have nonempty interior.*

Proof: It is clear that $\widehat{Z}$, $\widehat{Z}_+$ and $\widehat{Z}_-$ are all closed, because they are all continuous images of compact sets. Suppose that $\widehat{Z}_-$ has empty interior. Then for every $n \in \mathbb{Z}$, the set $(\sigma n) \cdot \widehat{Z}_-$ has empty interior also, since the map $x \mapsto (\sigma n)x$ is a homeomorphism. Now

$$X \supseteq \bigcup_{n\in\mathbb{Z}} ((\sigma n) \cdot \widehat{Z}_-) \supseteq \left(\bigcup_{n\in\mathbb{Z}} [\sigma n - \sigma, \sigma n] \right) \cdot Z = \mathbb{R} \cdot Z = X.$$

So $X = \bigcup_{n\in\mathbb{Z}} ((\sigma n) \cdot \widehat{Z}_-)$. Since each $(\sigma n)\widehat{Z}_-$ is closed and has empty interior, $(\sigma n)\widehat{Z}_-$ is nowhere dense for each $n \in \mathbb{Z}$. Then we see that X is a countable union of nowhere dense set. But X is a compact metric space, hence complete. This contradicts the Baire Category Theorem. Thus $\widehat{Z}_-$ has nonempty interior. Similarly, $\widehat{Z}_+$ and $\widehat{Z}$ have nonempty interior also. □

Lemma III.1.9. *The functions* α, β *and* R *are all bounded functions.*

Proof: Let U be the interior of $\widehat{Z}_-$. Then U is open in X. By Lemma III.1.8, $U \neq \varnothing$. Since the action is minimal, for each $x \in X$, there exists some $r \in [0, \infty)$ such that $rx \in U$. That is, for all $x \in X$, there exists $r \in [0, \infty)$ such that $x \in (-r)U$. So $\{(-r)U : r \in [0, \infty)\}$ is an open cover for X. Since X is compact, there exist $r_1, \ldots, r_n \in \mathbb{R}$ such that $X = \bigcup_{i=1}^{n} (-r_i)U$. Let $r = \max\{r_1, \ldots, r_n\}$. Then

$$X = [-r, 0]U \subseteq [-r, 0] \cdot ([-\sigma, 0]Z) \subseteq [-r - \sigma, 0]Z.$$

Thus, if $x \in Z^c$, we have $x = (-t)z$ for some $t \in (0, r + \sigma]$ and some $z \in Z$. Then $\beta(x) \le t \le r + \sigma$. Thus β is bounded above by $\sigma + r$. It is clear that β is bounded below by 0. If $z \in Z$, then $(\sigma/2)z \in Z^c$ and $(\sigma/2)z = (-s)z'$ for some $z' \in Z$ and some $s \in (0, r + \sigma]$. Then $(s + \sigma/2)z = z'$. We have $s + \sigma/2 > 0$, so then $R(z) \le s + \sigma/2 \le r + \sigma + \sigma/2$. So R is bounded.

An argument similar to the one that shows β is bounded shows that α is bounded. □

Notation III.1.10. For the rest of the chapter, let M denote some positive real number such that $M \ge |\beta(x)|$ for all $x \in Z^c$, $M \ge |\alpha(x)|$ for all $x \in Z^c$, and $M \ge |R(z)|$ for all $z \in Z$. Also for the rest of the chapter, define

$$G_Z = \{(r, x) \in \mathbb{R} \times X : x \in Z^c, -r \in (\alpha(x), \beta(x))\}. \tag{III.1}$$

Lemma III.1.11. *The set G_Z is an open subset of $\mathbb{R} \times X$ with compact closure. Further, if $(r, x), (s, y) \in G_Z$ satisfy $x = (-s)y$, then $(r + s, y) \in G_Z$; also $(r, x) \in G_Z$ if and only if $(-r, (-r)x) \in G_Z$.*

Proof: Let $(r, x) \in G_Z$. Then $x \in Z^c$ and $-r \in (\alpha(x), \beta(x))$. Let

$$\epsilon = (1/2) \min\{\beta(x) + r, -r - \alpha(x)\}.$$

It is clear that $\epsilon > 0$. Let

$$U = \beta^{-1}((-r + \epsilon, \infty)) \cap \alpha^{-1}((-\infty, -r - \epsilon)).$$

Note that U contains x. Also, since β is lower semi-continuous, and since α is upper semi-continuous, we see that U is open. Let $(t, y) \in (r - \epsilon, r + \epsilon) \times U$. Then $\alpha(y) < -r - \epsilon < -t < -r + \epsilon < \beta(y)$. So $(t, y) \in G_Z$. Thus $(r - \epsilon, r + \epsilon) \times U \subseteq G_Z$. Then we have $(r, x) \in (r - \epsilon, r + \epsilon) \times U \subseteq G_Z$. So (r, x) is an interior point of G_Z. This holds for all $(r, x) \in G_Z$, so G_Z is open. To see that G_Z has compact closure, note that $G_Z \subseteq [-M, M] \times X$, which is compact.

Let $(r, x), (s, y) \in G_Z$ satisfy $x = (-s)y$. Then

$$(\alpha(x), \beta(x)) = (\alpha((-s)y), \beta((-s)y)) = (\alpha(y), \beta(y)) + s.$$

So $-r \in (\alpha(x), \beta(x))$ implies that $-s - r \in (\alpha(y), \beta(y))$, whence $(r + s, y) \in G_Z$.

If $(r, x) \in G_Z$, then $(\alpha((-r)x), \beta((-r)x)) = (\alpha(x), \beta(x)) + r$. Since $0 \in (\alpha(x), \beta(x))$, we have $r \in (\alpha(x), \beta(x)) + r = (\alpha((-r)x), \beta((-r)x))$. So $(-r, (-r)x) \in G_Z$. Applying the previous argument to $(-r, (-r)x)$, we see that if $(-r, (-r)x) \in G_Z$, then $(r, x) \in G_Z$. □

It follows from Lemma III.1.11 that $C_0(G_Z)$ is a linear subspace of $C_c(\mathbb{R} \times X)$. Recall that the linear space $C_c(\mathbb{R} \times X)$ is endowed with a multiplication and a *-operation, as defined by the formulas in I.1 and I.2 respectively. Thus $C_c(\mathbb{R} \times X)$ is a *-algebra.

Lemma III.1.12. *$C_0(G_Z)$ is a *-subalgebra of $C_c(\mathbb{R} \times X)$.*

Proof: We only need to show that $C_0(G_Z)$ is closed under involution and convolution.

Let $f, g \in C_0(G_Z)$. We only need to show that $(f * g)|_{(G_Z)^c} = 0$ and $(f^*)|_{(G_Z)^c} = 0$. Let $(r, x) \in (G_Z)^c$. Then $(-r, (-r)x) \notin G_Z$ by Lemma III.1.11, so $f^*(r, x) = \overline{f(-r, (-r)x)} = 0$. Now, suppose that $(f * g)(r, x) \neq 0$. Since f and g are continuous, for some $t \in \mathbb{R}$, we have $f(t, x) \neq 0$ and $g(r - t, (-t)x) \neq 0$. Then $(t, x) \in G_Z$ and $(r - t, (-t)x) \in G_Z$. So $(r, x) \in G_Z$ by Lemma III.1.11 again, a contradiction. Hence $(f * g)(r, x) = 0$. Therefore $f * g, f^* \in C_0(G_Z)$. □

For the rest of the chapter define

$$A_Z = \overline{C_0(G_Z)}, \tag{III.2}$$

where the closure is taken in the crossed product $C^*(X, \mathbb{R})$.

By Lemma III.1.12, it is clear that A_Z is a C^*-subalgebra of the crossed product. This subalgebra A_Z will be the subalgebra that is analogous to the subalgebras A_Y in [3]. In fact, the subset G_Z of $\mathbb{R} \times X$ is a subgroupoid of the transformation groupoid $\mathbb{R} \times X$. See [13] for definitions of groupoids and groupoid C^*-algebras. We find it more convenient to work directly with the construction we have given then to formulate the construction in terms of groupoids. In particular, we will not use any machinery from the theory of groupoids.

III.2. Continuous Extensions of the Entering Times

We wish to obtain a stable recursive subhomogeneous decomposition for A_Z. We first find finitely many subsets of G_Z that are closed in G_Z whose union covers G_Z. We will show that each of those subsets is locally compact with compact closure, and that spaces of continuous functions on those subsets that vanish at infinity are in fact pre-C^*-algebras whose closures have the form $C(F, \mathbb{K})$, where F is a compact metric space. Finally, we show that G_Z is obtained by gluing these C^*-algebras together.

To obtain the subset of G_Z mentioned above, we first need to cut Z^c into finitely many pieces so that α and β are continuous on each piece, and can be extended continuously to the closure of the each piece. The continuity of the entering times is required if we want to identify the components of the of a stable recursive subhomogeneous decomposition of A_Z as "continuous" functions from a compact metric space into $\mathbb{K}$.

Lemma III.2.1. *For every $D \in (0,\infty)$ and for every $z \in Z$, there exists a compact neighborhood K of z (K contains a set U that is open in X and $z \in U$) such that $\{(0,D]\cdot(K\cap Z)\}\cap(K\cap Z) = \varnothing$.*

Proof: This will be a proof by contradiction. Suppose that the statement is not true. Then there exists $D \in (0,\infty)$ and some $z \in Z$ such that for every compact neighborhood of K of z we have $\{(0,D]\cdot(K\cap Z)\}\cap(K\cap Z) \neq \varnothing$. For each $n \in \mathbb{N}$, let $K_n = \{x \in X : d(x,z) \leq 1/n\}$. Then K_n is a compact neighborhood of z for every $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$, there exists $r_n \in (0,D]$ and $z_n \in K_n \cap Z$ such that $r_n z_n \in K_n \cap Z$. Then $z_n \to z$ and $r_n z_n \to z$. Since $\{r_n\}$ is a bounded sequence, it has a subsequence $\{r_{k_n}\}$ such that $r_{k_n} \to r$ for some $r \in [0,D]$. Then $\lim_{n\to\infty} z_{k_n} = \lim_{n\to\infty}(-r_{k_n})(r_{k_n}z_{k_n}) = (-r)z$. But $\lim_{n\to\infty} z_{k_n} = z$, so $(-r)z = z$. Since the action is free, we have $r = 0$. Therefore there exists $m \in \mathbb{N}$ such that $0 < r_{k_m} < \sigma$. Now $z_{k_m} \in Z$, $r_{k_m}z_{k_m} \in Z$, and $z_{k_m} \neq r_{k_m}z_{k_m}$, so $([-\sigma,\sigma]\cdot z_{k_m})\cap Z$ contains two distinct elements, namely z_{k_m} and $r_{k_m}z_{k_m}$, which contradicts the definition of Z. ☐

Lemma III.2.2. *There exist $n_V \in \mathbb{N}$ and $Z_1, Z_2, \ldots, Z_{n_V} \subseteq Z$ such that*

1. *for every $i \in \{1,\ldots,n_V\}$, the set Z_i is compact;*
2. *for every $i \in \{1,\ldots,n_V\}$, every $x \in Z_i$, and every $r \in (0,8M]$, we have $rx \notin Z_i$;*
3. *$\bigcup_{i=1}^{n_V} Z_i = Z$;*
4. *for every $i \in \{1,\ldots,n_V\}$, the map $[0,8M]\times Z_i \to [0,8M]\cdot Z_i$ defined by $(r,z)\mapsto rz$ is a homeomorphism.*

Proof: For each $z \in Z$, let K_z be the compact neighborhood obtained from Lemma III.2.1 where the real number D in Lemma III.2.1 is taken to be $8M$. Use K_z° to denote the interior of K_z for each $z \in Z$. Now, the collection $\{K_z^\circ : z \in Z\}$ is an open cover for Z, which is compact, so there exists n_V such that $\bigcup_{i=1}^{n_V} K_{z_i} \supset Z$. For each $i \in \{1,\ldots,n_V\}$, let $Z_i = K_{z_i}\cap Z$. Then part 1 and part 3 of the lemma hold. Also, by the choice of the sets K_{z_i}, we have

$$\varnothing = \left[(0,8M]\cdot(K_{z_i}\cap Z)\right]\cap(K_{z_i}\cap Z) = ((0,8M]\cdot Z_i)\cap Z_i$$

for $i \in \{1,\ldots,n_V\}$. So part 2 holds.

The map $[0,8M]\times Z_i \to [0,8M]\cdot Z_i$, defined by $(r,z)\mapsto rz$, is certainly continuous and surjective. Now suppose that $(r,x)\in[0,8M]\times Z_i$ and $(s,y)\in[0,8M]\times Z_i$, and that $rx = sy$.

Without loss of generality, assume that $r \leq s$. Then $s-r \in [0, 8M]$ and $x = (s-r)y$. But $x, y \in Z_i$ and $s - r \in [0, 8M]$, so, by part 2, we have $s - r = 0$. Therefore $(r, x) = (s, y)$. Thus the map is injective. Since both $[0, 8M] \times Z_i$ and $[0, 8M] \cdot Z_i$ are compact and Hausdorff, the map is a homeomorphism. Hence part 4 holds. $\square$

Notation III.2.3. Now we use the return time function R to partition Z. For each $i \in \mathbb{N}$, let

$$T^i = R^{-1}\left(\left(\frac{(i-1)\sigma}{16}, \frac{i\sigma}{16}\right]\right).$$

Note that, because R is bounded above by M and below by σ, we have $T^i = \varnothing$ for all but finitely many i, and that $\{T^i : i \in \mathbb{N}\}$ partitions Z. Let $\Lambda = \{n \in \mathbb{N} : n \geq 1, T^n \neq \varnothing\}$. Then for some $n_R \in \mathbb{N}$, we have $\Lambda = \{k_1, k_2, \ldots, k_{n_R}\}$. Re-indexing if necessary, we can assume that $k_1 < k_2 < \cdots < k_{n_R}$. For each $i \in \{1, \ldots, n_R\}$, let $Z^i = T^{k_i}$. Then it is clear that $\{Z^1, \ldots, Z^{n_R}\}$ partitions Z. For each $i \in \{1, \ldots, n_V\}$ and each $j \in \{1, \ldots, n_R\}$, let $Y_{i,j} = Z_i \cap Z^j$. It is clear that $\{Y_{i,j} : 1 \leq i \leq n_V, 1 \leq j \leq n_R\}$ covers Z.

Lemma III.2.4. *Let n_V be as in Lemma III.2.2 and let n_R, Z^j and $Y_{i,j}$ be as given above. For each $i \in \{1, \ldots, n_V\}$ and each $j \in \{1, \ldots, n_R\}$, we have:*

1. *if $\{z_n\}$ is a Cauchy sequence in Z^j, then $\{R(z_n)\}$ is Cauchy;*
2. *$R|_{Z^j}$ is continuous;*
3. *the map $[0, 8M] \times \overline{Y_{i,j}} \to [0, 8M] \cdot \overline{Y_{i,j}}$ is a homeomorphism.*

Proof: Fix $i \in \{1, \ldots, n_V\}$ and $j \in \{1, \ldots, n_R\}$. We show part 1 first. Suppose that $\{x_n\}$ is a Cauchy sequence in Z^j. Then $x_n \to x$ for some $x \in Z$. Since $\{R(x_n)\}$ is a bounded sequence, it has a convergent subsequence $\{R(x_{k_n})\}$. Let $r = \lim_{n\to\infty} R(x_{k_n})$. Then $R(x_{k_n})x_{k_n} \to rx$. Since $R(x_{k_n})x_{k_n} \in Z$ for all $n \geq 1$, we have $rx \in Z$. Suppose that $\{R(x_n)\}$ does not converge to r. Then there exists $\epsilon > 0$ and a subsequence $\{R(x_{j_n})\}$ of $\{R(x_n)\}$ such that $|R(x_{j_n}) - r| \geq \epsilon$ for every $n \geq 1$. We know that $\{R(x_{j_n})\}$ is bounded, so it has a convergent subsequence $\{R(x_{j_{l_n}})\}$. Say $R(x_{j_{l_n}}) \to s$. Then $sx \in Z$. Also, $|R(x_{j_{l_n}}) - r| \geq \epsilon$ for all $n \geq 1$ implies that $r \neq s$, and so $rx \neq sx$.

Then by the second condition in the definition of a pseudo-transversal, we have $|r - s| > \sigma$. But

$$|r - s| \leq |r - R(x_{k_n})| + |R(x_{k_n}) - R(x_{j_{l_n}})| + |R(x_{j_{l_n}}) - s|$$
$$\leq |r - R(x_{k_n})| + \sigma/16 + |R(x_{j_{l_n}}) - s|,$$

which converges to $\sigma/16$, a contradiction. Thus $R(x_n) \to r$. So $\{R(x_n)\}$ is Cauchy, and part 1 is proven.

Now suppose that $\{x_n\}$ is a sequence in Z^j such that $x_n \to x$ for some $x \in Z^j$. Then by part 1, $R(x_n) \to r$ for some $r \in \mathbb{R}$. We will show that $r = R(x)$. Suppose that $r \neq R(x)$. Then $rx \neq R(x)x$. Also we have $R(x)x \in Z$, $rx \in Z$, and $(R(x) - r)(rx) = R(x)x$. So we have $|R(x) - r| \geq \sigma$. But

$$|R(x) - r| \leq |R(x) - R(x_n)| + |R(x_n) - r| \leq \sigma/16 + |R(x_n) - r|,$$

and the last expression converges to $\sigma/16$, a contradiction. So $R(x) = r$, and so part 2 holds.

For the last part, we note that the map in part 3 is the restriction of the map in part 4 of Lemma III.2.2, and that the map in part 3 is surjective. □

Now we fix some more notation for this chapter.

Notation III.2.5. Recall the definition of the integers n_V and n_R from Lemma III.2.2 and Notation III.2.3, respectively. Enumerate the collection of sets $\{Y_{i,j} \colon 1 \leq i \leq n_V, 1 \leq j \leq n_R\}$ by Y_k in the following order: $Y_1 = Y_{1,1}, Y_2 = Y_{2,1}, \ldots, Y_{n_V} = Y_{n_V,1}, Y_{n_V+1} = Y_{1,2}, Y_{n_V+2} = Y_{2,2}, \ldots, Y_{2n_V} = Y_{n_V,2}, \ldots, Y_{(n_R-1)n_V+1} = Y_{1,n_R}, \ldots, Y_{n_R n_V} = Y_{n_V,n_R}$. Throw away the empty members of $\{Y_k \colon 1 \leq k \leq n_R n_V\}$, and let N be the number of nonempty sets in the collection, then relabel the nonempty members of $\{Y_k \colon 1 \leq k \leq n_R n_V\}$ without changing the relative order. That is, if we let $\iota \colon \{1, \ldots, N\} \to \{1, \ldots, n_V n_R\}$ be a strictly increasing function such that $\{Y_{\iota(k)} \colon 1 \leq k \leq N\}$ is the collection of all nonempty members of $\{Y_k \colon 1 \leq k \leq n_V n_R\}$, then we relabel $Y_{\iota(k)}$ as Y_k. It is clear that $\{Y_k \colon 1 \leq k \leq N\}$ covers Z. For the rest of the chapter, let Y_k denote the sets just mentioned, and for each $i \in \{1, \ldots, N\}$, let $C_i = \{(\frac{R(z)}{2})z \colon z \in Y_i\}$, let $W_i = \{rz \colon z \in Y_i, r \in (0, R(z))\}$ and let $X_i = \overline{C_i}$.

Lemma III.2.6. *Let α and β are the maps in III.1.2. For each $i \in \{1, \ldots, N\}$, let $\alpha_i^\circ = \alpha|_{W_i}$, and let $\beta_i^\circ = \beta|_{W_i}$. Then*

1. *For each $i \in \{1, \ldots, N\}$, the map $R|_{Y_i}$ is continuous.*

2. *For each $i \in \{1, \ldots, N\}$, we have $C_i \subseteq W_i \subseteq Z^c$.*

3. *We have $\bigcup_{i=1}^N W_i = Z^c$.*

4. *For each $i \in \{1, \ldots, N\}$, for each $z \in Y_i$ and each $r \in (0, R(z))$, we have $\alpha_i^\circ(rz) = -r$, and $\beta_i^\circ(rz) = R(z) - r$.*

5. *For each $i \in \{1, \ldots, N\}$, the map α_i° and β_i° are continuous.*

6. *For each $i \in \{1, \ldots, N\}$, each $x \in W_i$, and each $r \in (\alpha_i^\circ(x), \beta_i^\circ(x))$, we have $rx \in W_i$.*

Proof: For each $i \in \{1, \ldots, N\}$, the set Y_i is contained in some Z^j, and $R|_{Z^j}$ is continuous, so $R|_{Y_i}$ is continuous. It is clear that for each $i \in \{1, \ldots, N\}$ we have $C_i \subseteq W_i$; and $W_i \subseteq Z^c$ follows from Lemma III.1.3.

Let $x \in Z^c$. Then $\alpha(x)x \in Y_i$ for some $i \in \{1, \ldots, N\}$. Since $(\alpha(x), \beta(x))x \subseteq Z^c$, we have $(0, \beta(x) - \alpha(x))(\alpha(x)x) \subseteq Z^c$. Also, $\beta(x) - \alpha(x) > 0$ and $(\beta(x) - \alpha(x)) \cdot (\alpha(x)x) \in Z$, so $R(\alpha(x)x) = \beta(x) - \alpha(x)$. So $-\alpha(x) \in (0, R(\alpha(x)x))$, and then $x = (-\alpha(x)) \cdot (\alpha(x)x) \in W_i$. Thus $Z^c = \bigcup_{i=1}^N W_i$. (Here we used the fact that $\alpha < 0 < \beta$.)

Now fix $i \in \{1, \ldots, N\}$. Let $z \in Y_i$ and let $r \in (0, R(z))$. Then by Lemma III.1.3, we have $(-r, R(z) - r) \cdot (rz) = (0, R(z))z \subseteq Z^c$. Also, we have $(-r)(rz), (R(z) - r)(rz) \in Z$, and $-r < 0 < R(z) - r$. So by the definition of α and β, we have $\alpha_i^\circ(rz) = -r$ and $\beta_i^\circ(rz) = R(z) - r$.

Now let $\{x_n\}$ be a sequence in W_i such that $x_n \to x$ for some $x \in W_i$. Then for each $n \geq 1$, there exist $z_n \in Y_i$ and $r_n \in (0, R(z_n))$ such that $x_n = r_n z_n$; and there exist $z \in Y_i$ and $r \in (0, R(z))$ such that $x = rz$. By Lemma III.2.4, we have $z_n \to z$ and $r_n \to r$. Now, by part 4, for each $n \geq 1$, we have $\alpha(x_n) = -r_n$ and $\beta(x_n) = R(z_n) - r_n$; and also $\alpha(x) = -r$ and $\beta(x) = R(z) - r$. Then we have $\alpha(x_n) = -r_n \to -r = \alpha(x)$; and since $R|_{Y_i}$ is continuous, we have $\beta(x_n) = R(x_n) - r_n \to R(x) - r = \beta(x)$. Thus α_i° and β_i° are continuous.

Now let $x \in W_i$, and let $r \in (\alpha_i^\circ(x), \beta_i^\circ(x))$. Then $x = sz$ for some $z \in Y_i$ and some $s \in (0, R(z))$. So $\alpha_i^\circ(x) = -s$ and $\beta_i^\circ(x) = R(z) - s$. Then $r \in (\alpha_i^\circ(x), \beta_i^\circ(x))$ implies that $r \in (-s, R(z) - s)$, and so $r + s \in (0, R(z))$. Therefore $rx = (r + s)z \in W_i$. □

The next lemma is used to extend the entering times. It is a well known result in analysis, so we will omit its proof here.

Lemma III.2.7. *Let X be any metric space, and let $Y \subseteq X$ be an arbitrary subset. Let $f\colon Y \to \mathbb{R}$ be a continuous function. Suppose that for every Cauchy sequence $\{y_n\}$ in Y, the sequence $\{f(y_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Then there exists $g\colon \overline{Y} \to \mathbb{R}$ such that $g|_Y = f$, and g is continuous. Moreover, $g(y) = \lim_{n\to\infty} f(y_n)$, where $\{y_n\}$ is any sequence in Y that converges to y.*

Now we extend the maps α_i° and β_i° in Lemma III.2.6 continuously to the closures of W_i.

Lemma III.2.8. *For each $i \in \{1, \ldots, N\}$, the maps α_i° and β_i° from Lemma III.2.6 can be extended to continuous functions on $\overline{W_i}$.*

Proof: Fix $i \in \{1, \ldots, N\}$. By Lemma III.2.7 and III.2.6, we only need to show that α_i° and β_i° preserve Cauchy sequences.

Let $\{x_n\}$ be a Cauchy sequence in W_i. Note that $W_i \subseteq [0, 8M] \cdot \overline{Y_i}$, which is compact, so $x_n \to x$ for some $x \in [0, 8M] \cdot \overline{Y_i}$. For each $n \geq 1$, we have $x_n = r_n z_n$ for some $z_n \in Y_i$ and some $r_n \in (0, R(z_n))$; and $x = rz$ for some $z \in \overline{Y_i}$ and some $r \in [0, 8M]$. Then by Lemma III.2.4, $r_n \to r$ and $z_n \to z$. Now, by Lemma III.2.6, we have $\alpha_i^\circ(x_n) = \alpha_i^\circ(r_n z_n) = -r_n$, and $\beta_i^\circ(x_n) = R(z_n) - r_n$. Then it follows that $\{\alpha_i^\circ(x_n)\}$ is Cauchy. By Lemma III.2.4, the sequence $\{R(z_n)\}$ is Cauchy, so then $\{\beta_i^\circ(x_n)\} = \{R(z_n) - r_n\}$ is also Cauchy. The lemma now follows from Lemma III.2.7. $\square$

Notation III.2.9. For the rest of the chapter, let α_i and β_i denote the extensions of α_i° and β_i°, respectively, obtained from Lemma III.2.8. We will let $V_i = \{rc\colon c \in X_i, r \in (\alpha_i(c), \beta_i(c))\}$ for each $i \in \{1, \ldots, N\}$. Note that $V_i \subseteq \overline{W_i}$, but in general, we do not expect V_i to equal to $\overline{W_i}$ or W_i.

III.3. Properties of α_i, β_i, W_i and V_i

Lemma III.3.1. *Let $i \in \{1, \ldots, N\}$, let $x \in \overline{W_i}$, and let $r \in [\alpha_i(x), \beta_i(x)]$. Then:*

1. $\alpha_i(x)x \in \overline{Y_i}$ *and* $\beta_i(x)x \in Z$.

2. $-M \leq \alpha_i(x) \leq 0 \leq \beta_i(x) \leq M$.

3. $\beta_i(x) - \alpha_i(x) \geq \sigma$.

4. $rx \in \overline{W_i}$.

5. *$\alpha_i(rx) = \alpha_i(x) - r$ and $\beta_i(rx) = \beta_i(x) - r$.*

6. *If $\alpha_i(x) = 0$ then $x \in \overline{Y_i}$; if $\beta_i(x) = 0$, then $x \in Z$.*

7. *If $x \in Z^c$, then $\alpha_i(x) \leq \alpha(x) < 0 < \beta(x) \leq \beta_i(x)$, where α and β are the maps in III.1.2.*

Proof: Let $\{x_n\}$ be a sequence in W_i that converges to x. For each $n \geq 1$, we have $x_n = r_n z_n$ for some $z_n \in Y_i$ and some $r_n \in (0, R(z_n))$. Then $\alpha_i(x_n) = -r_n$ and $\beta_i(x_n) = R(x_n) - r_n$ for each $n \geq 1$ by Lemma III.2.6(4)

Since $\alpha_i(x_n)x_n = z_n \to \alpha_i(x)x$, and $z_n \in Y_i$ for each $n \geq 1$, we have $\alpha_i(x)x \in \overline{Y_i}$. Also, $\beta_i(x_n)x_n \in Z$ for all $n \geq 1$, and $\beta_i(x_n)x_n \to \beta_i(x)x$, so $\beta_i(x)x \in Z$. So part 1 holds.

Note that $-M \leq \alpha(y) < 0 < \beta(y) \leq M$ for all $y \in Z^c$ and $0 < R(z) \leq M$ for all $z \in Z$. So $-M \leq \alpha(x_n) = \alpha_i(x_n) < 0 < \beta(x_n) = \beta_i(x_n) \leq M$, for every n. Then part 2 follows from continuity of α_i and β_i.

For each $n \geq 1$, $\beta_i(x_n) - \alpha_i(x_n) = R(z_n) \geq \sigma$. Part 3 now follows from continuity of α_i and β_i.

We first claim that $(\alpha_i(x), \beta_i(x))x \subseteq \overline{W_i}$. Let $s \in (\alpha_i(x), \beta_i(x))$. Since $\alpha_i(x_n) \to \alpha_i(x)$ and $\beta_i(x_n) \to \beta_i(x)$, we can assume that, taking a subsequence if necessary, $s \in (\alpha_i(x_n), \beta_i(x_n))$ for all $n \geq 1$. Then $sx_n \in W_i$ for all $n \geq 1$. Since $sx_n \to sx$, we have $sx \in \overline{W_i}$. This proves the claim. Now, for each $n \geq 1$, let $s_n = \beta_i(x) - \frac{\beta_i(x) - \alpha_i(x)}{n+1}$. Then $s_n \in (\alpha_i(x), \beta_i(x))$ for all $n \geq 1$, so $s_n x \in \overline{W_i}$ for all $n \geq 1$. Since $s_n x \to \beta_i(x)x$, we have $\beta_i(x)x \in \overline{W_i}$. Similarly, taking $s_n = \alpha_i(x) + \frac{\beta_i(x) - \alpha_i(x)}{n+1}$, we have $\alpha_i(x)x \in \overline{W_i}$. So part 4 holds.

Next we prove part 5. First assume that $r \in (\alpha_i(x), \beta_i(x))$. Recall from the beginning of the proof that $\{x_n\}$ is a sequence in W_i that converges to x. Without loss of generality, assume that $r \in (\alpha_i(x_n), \beta_i(x_n))$ for all $n \geq 1$. Then $rx_n \in W_i$ for all $n \geq 1$, and then

$$\alpha_i(rx) = \lim_{n \to \infty} \alpha_i(rx_n) = \lim_{n \to \infty} \alpha(rx_n) = \lim_{n \to \infty} \alpha(x_n) - r = \alpha_i(x) - r.$$

Similarly, $\beta_i(rx) = \beta_i(x) - r$. Now, let $s_n = \beta_i(x) - \frac{\beta_i(x) - \alpha_i(x)}{n+1}$. Then $s_n \in (\alpha_i(x), \beta_i(x))$, so $\beta_i(s_n x) = \beta_i(x) - s_n$, and $\alpha_i(s_n x) = \alpha_i(x) - s_n$, by what we just proved. Thus

$$\alpha_i(\beta_i(x)x) = \lim_{n \to \infty} \alpha_i(s_n x) = \lim_{n \to \infty} \alpha_i(x) - s_n = \alpha_i(x) - \beta_i(x),$$

and

$$\beta_i(\beta_i(x)) = \lim_{n\to\infty} \beta_i(s_n x) = \lim_{n\to\infty} \beta_i(x) - s_n = \beta_i(x) - \beta_i(x).$$

By taking $s_n = \alpha_i(x) + \frac{\beta_i(x)-\alpha_i(x)}{n+1}$, we have $\alpha_i(\alpha_i(x)x) = \alpha_i(x) - \alpha_i(x)$, and

$$\beta_i(\alpha_i(x)) = \beta_i(x) - \alpha_i(x).$$

So part 5 holds.

By part 1, we see that $\alpha_i(x) = 0$ implies that $x = \alpha_i(x)x \in \overline{Y_i}$; and that $\beta_i(x) = 0$ implies that $x = \beta_i(x)x \in Z$. So part 6 holds.

By part 1, we know that $\alpha_i(x)x \in Z$, and $\beta_i(x)x \in Z$. Since $x \in Z^c$, by part 6, we have $\alpha_i(x) \neq 0$, and $\beta_i(x) \neq 0$. Then part 2 implies that $\alpha_i(x) < 0$, and $\beta_i(x) > 0$. Part 7 follows from the definition of α and β. □

Lemma III.3.2. *Let $i \in \{1, \ldots, N\}$. Then:*

1. *$X_i \subseteq \overline{W_i}$, $\overline{Y_i} \subseteq \overline{W_i}$, and $\overline{W_i} \subseteq [0, M] \cdot \overline{Y_i} \subseteq [0, 8M] \cdot \overline{Y_i}$.*
2. *If $z \in Y_i$, then $\beta_i(z) = R(z)$. If $z \in \overline{Y_i}$, then $\alpha_i(z) = 0$.*
3. *$X_i = \left\{ \left(\frac{\alpha_i(x)+\beta_i(x)}{2} \right) \cdot x \colon x \in \overline{W_i} \right\}$; and*

   $$\overline{W_i} = \{rc\colon c \in X_i, r \in [\alpha_i(c), \beta_i(c)]\} = \{rz\colon z \in \overline{Y_i}, r \in [0, \beta_i(z)]\}.$$

4. *The map $X_i \to \overline{Y_i}$ defined by $c \mapsto \alpha_i(c)c$ is a homeomorphism.*
5. *Suppose that $c, c' \in X_i$, that $c \neq c'$, and that $rc = c'$. Then $|r| \geq 6M$.*
6. *The map*

   $$\{(r, c) \in \mathbb{R} \times X \colon c \in X_i, r \in [\alpha_i(c), \beta_i(c)]\} \to \overline{W_i},$$

   defined by $(r, c) \mapsto rc$, is a homeomorphism.
7. *For all $c \in X_i$, we have $\alpha_i(c) = -\beta_i(c)$.*

Proof: We already know that $C_i \subseteq W_i$, so $X_i = \overline{C_i} \subseteq \overline{W_i}$. If $z \in Y_i$, then $(R(z)/2n)z \in W_i$; and then $z \in \overline{W_i}$, since $(R(z)/2n)z \to z$. So $\overline{Y_i} \subseteq \overline{W_i}$. It is clear that $W_i \subseteq [0, M] \cdot \overline{Y_i}$, so $\overline{W_i} \subseteq [0, M] \cdot \overline{Y_i} \subseteq [0, 8M] \cdot \overline{Y_i}$. So part 1 holds.

Now we show part 2. Let $z \in Y_i$. For each $n \geq 1$, let $s_n = \frac{R(z)}{2n}$. Then $s_n \in (0, R(z))$ for all $n \geq 1$, and $s_n z \to z$; so by continuity, we have $\beta_i(s_n z) \to \beta_i(z)$, and that $\alpha_i(s_n z) \to \alpha_i(z)$. By Lemma III.2.6, we have $\beta_i(s_n z) = R(z) - s_n$ and $\alpha_i(s_n z) = -s_n$. So

$$\beta_i(z) = \lim_{n\to\infty} \beta_i(s_n z) = \lim_{n\to\infty} R(z) - s_n = R(z);$$

and

$$\alpha_i(z) = \lim_{n\to\infty} \alpha_i(s_n z) = \lim_{n\to\infty} -s_n = 0.$$

Then it is clear that $\alpha_i|_{\overline{Y_i}} = 0$. So part 2 holds.

Now we show part 3. Let

$$A = \left\{ \left(\frac{\alpha_i(x) + \beta_i(x)}{2} \right) \cdot x \colon x \in \overline{W_i} \right\},$$

let

$$B = \{rc \colon c \in X_i, r \in [\alpha_i(c), \beta_i(c)]\},$$

and let

$$C = \{rz \colon z \in \overline{Y_i}, r \in [0, \beta_i(z)]\}.$$

Let $c \in X_i$. If $c \in C_i$, then $c = (R(z)/2)z$ for some $z \in Y_i$, and $c \in W_i$. Thus $\alpha_i(c) = -R(z)/2$, and $\beta_i(c) = R(z)/2$. Then $c = \left(\frac{\alpha_i(c)+\beta_i(c)}{2}\right) \cdot c \in A$. Thus $C_i \subseteq A$. Now, $\overline{W_i}$ is compact, and A is the image of the continuous map $x \mapsto \left(\frac{\alpha_i(x)+\beta_i(x)}{2}\right) \cdot x$, so A is compact, and hence closed. Then $X_i = \overline{C_i} \subseteq A$. Thus $X_i = A$.

Let

$$B' = \{(r, c) \in \mathbb{R} \times X \colon c \in X_i, r \in [\alpha_i(c), \beta_i(c)]\},$$

and let

$$C' = \{(r, z) \colon z \in \overline{Y_i}, r \in [0, \beta_i(z)]\}.$$

Note that $B' \subseteq [-M, M] \times X$, and $C' \subseteq [0, M] \times X$. We first show that B' and C' are closed. Suppose that $(r_n, c_n) \in B'$, and $(r_n, c_n) \to (r, c)$. Then $c \in X_i$, since X_i is closed. Now, $\alpha_i(x_n) \leq r_n \leq \beta_i(x_n)$ for each $n \geq 1$; also $\alpha_i(x_n) \to \alpha_i(x)$, $\beta_i(x_n) \to \beta_i(x)$, and $r_n \to r$. So $\alpha_i(x) \leq r \leq \beta_i(x)$, and so $(r, c) \in B'$. Thus B' is closed. Similarly, C' is closed. Then both B' and C' are compact, since both are contained in compact sets. So B and C are also compact, because they are the images of B' and C' under a continuous map, and so B and C are closed.

From part 2 and the definition of W_i, it is clear that $W_i \subseteq C$, and so $\overline{W_i} \subseteq C$. Now let $z \in \overline{Y_i}$, and let $r \in [0, \beta_i(z)]$. Let $c = (\beta_i(z)/2)z$. Now, there exists a sequence $\{z_n\}$ in Y_i such that $z_n \to z$. Then $(R(z_n)/2)z_n \in C_i$ for each $n \geq 1$. But by part 2, $(R(z_n)/2)z_n = (\beta_i(z_n)/2)z_n \to c$, so $c \in X_i$. Now

$$[\alpha_i(c), \beta_i(c)] = [\alpha_i(z) - (\beta_i(z)/2), \beta_i(z) - (\beta_i(z)/2)] = [-(\beta_i(z)/2), (\beta_i(z)/2)].$$

The $r \in [0, \beta_i(z)]$ implies that $r - \beta_i(z) \in [\alpha_i(c), \beta_i(c)]$. Then $rz = (r - \beta_i(z))c \in B$. Thus $C \subseteq B$. By part 1 and part 4 of Lemma III.3.1, we have $B \subseteq \overline{W_i}$. Thus $\overline{W_i} = B = C$. So part 3 holds.

Now we show part 4. By Lemma III.3.1, we see that the map does map to $\overline{Y_i}$. Continuity is clear. If $z \in \overline{Y_i}$, then $(\beta_i(z)/2)z \in X_i$, and $z = \alpha_i\left(\left(\frac{\beta_i(z)}{2}\right)z\right) \cdot \left(\left(\frac{\beta_i(z)}{2}\right)z\right)$. So the map is surjective. Now suppose that $\alpha_i(c)c = \alpha_i(c')c'$ with $c, c' \in X_i$. Then $\alpha_i(c)c, \alpha_i(c)c' \in \overline{Y_i} \subseteq Z_k$ for some k. Since $(\alpha_i(c') - \alpha_i(c))(\alpha_i(c)c) = \alpha_i(c')c'$, by Lemma III.2.2, we see that $|\alpha_i(c) - \alpha_i(c')| = 0$ or $|\alpha_i(c) - \alpha_i(c')| \geq 8M$. But $|\alpha_i(c) - \alpha_i(c')| \leq 2M$, so $\alpha_i(c) = \alpha_i(c')$. Then $c = c'$ by freeness of the action. So the map is bijective and continuous, and since both X_i and $\overline{Y_i}$ are compact and Hausdorff, it is a homeomorphism. Part 4 is proven.

Now we show part 5. Since $rc = c'$, we have

$$\alpha_i(c')(c') = \alpha_i(c')(rc) = r(\alpha_i(c')c) = (r + \alpha_i(c') - \alpha_i(c))(\alpha_i(c)c).$$

Both $\alpha_i(c')c'$ and $\alpha_i(c)c$ are in $\overline{Y_i} \subseteq Z_k$ for some k, so by Lemma III.2.2, we have $|r+\alpha_i(c')-\alpha_i(c)| = 0$ or $|r+\alpha_i(c')-\alpha_i(c)| \geq 8M$. If $|r+\alpha_i(c')-\alpha_i(c)| \geq 8M$, then we done, since $|\alpha_i(c)-\alpha_i(c')| \leq 2M$. So suppose that $r = \alpha_i(c) - \alpha_i(c')$. Then $c' = rc = (\alpha_i(c) - \alpha_i(c'))c$. So $\alpha_i(c')c' = \alpha_i(c)c$. Then part 4 implies that $c = c'$, contradicting the hypothesis $c \neq c'$. So part 5 is proven.

In part 6, the map is well defined and surjective by part 3, and continuity is clear. Suppose that $rc = r'c'$. Then $(r - r')c = c'$. By part 5, either $c' = c'$, or $|r| \geq 6M$. But $|r - r'| \leq 2M$, so $c = c'$. So the map is injective. We have already shown in the proof of part 3 that the domain is compact. Thus the map is a homeomorphism.

Part 7 follows directly from part 3. □

Notation III.3.3. For the rest of the chapter, let $\pi_i \colon \overline{W_i} \to X_i$ denote the map

$$x \mapsto \left(\frac{\alpha_i(x) + \beta_i(x)}{2}\right) \cdot x.$$

Lemma III.3.4. *Let $i \in \{1, \ldots, N\}$. Then:*

1. *We have*

$$\begin{aligned} V_i &= \overline{W_i} \setminus \left(\{\alpha_i(c)c \colon c \in X_i\} \cup \{\beta_i(c)c \colon c \in X_i\}\right) \\ &= \overline{W_i} \setminus \left(\{\alpha_i(x)x \colon x \in \overline{W_i}\} \cup \{\beta_i(x)x \colon x \in \overline{W_i}\}\right) \\ &= \overline{W_i} \setminus \left(\{\beta_i(z)z \colon z \in \overline{Y_i}\} \cup \overline{Y_i}\right) \\ &= \overline{W_i} \setminus \left\{x \in \overline{W_i} \colon \alpha_i(x) = 0 \text{ or } \beta_i(x) = 0\right\} \\ &= \{rz \colon z \in \overline{Y_i}, r \in (0, \beta_i(z))\}. \end{aligned}$$

2. $X_i \subseteq V_i$.

3. *For all $x \in V_i$, we have $-M \leq \alpha_i(x) < 0 < \beta_i(x) < M$.*

4. *For all $x \in V_i$, and for all $r \in (\alpha_i(x), \beta_i(x))$, we have $rx \in V_i$.*

5. *The map*

$$\{(r, c) \in \mathbb{R} \times X \colon c \in X_i, r \in (\alpha_i(c), \beta_i(c))\} \to V_i$$

   defined by $(r, c) \mapsto rc$ is a homeomorphism.

6. $Z^c \subseteq \bigcup_{j=1}^{N} V_i$.

7. *$V_i \cap Z^c$ is closed in Z^c.*

Proof: We first show part 1. Let

$$\begin{aligned} A &= \overline{W_i} \setminus \big(\{\alpha_i(c)c\colon c \in X_i\} \cup \{\beta_i(c)c\colon c \in X_i\}\big); \\ B &= \overline{W_i} \setminus \big(\{\beta_i(z)z\colon z \in \overline{Y_i}\} \cup \overline{Y_i}\big); \\ C &= \overline{W_i} \setminus \{x \in \overline{W_i}\colon \alpha_i(x) = 0 \text{ or } \beta_i(x) = 0\}; \\ D &= \{rz\colon z \in \overline{Y_i}, r \in (0, \beta_i(z))\} \\ E &= \overline{W_i} \setminus \big(\{\alpha_i(x)x\colon x \in \overline{W_i}\} \cup \{\beta_i(x)x\colon x \in \overline{W_i}\}\big). \end{aligned}$$

Let $A_1 = \{\alpha_i(c)c\colon c \in X_i\}$, let $A_2 = \{\beta_i(c)c\colon c \in X_i\}$, let $B_1 = \{\beta_i(z)z\colon z \in \overline{Y_i}\}$, let $B_2 = \overline{Y_i}$, let $C_1 = \{x \in \overline{W_i}\colon \alpha_i(x) = 0\}$, let $C_2 = \{x \in \overline{W_i}\colon \beta_i(x) = 0\}$, let $E_1 = \{\alpha_i(x)x\colon x \in \overline{W_i}\}$, and let $E_2 = \{\beta_i(x)x\colon x \in \overline{W_i}\}$. It is clear that $C_1 \subseteq B_2 \subseteq A_1 \subseteq E_1 \subseteq C_1$. Now, if $x \in C_2$, then we have

$$x = \beta_i(x)x = (\beta_i(x) - \alpha_i(x)) \cdot (\alpha_i(x)x) = \beta_i(\alpha_i(x)x) \cdot (\alpha_i(x)x) \in B_1.$$

So $C_2 \subseteq B_1$. Let $z \in \overline{Y_i}$. Let $r = (\alpha_i(z) + \beta_i(z))/2$. Then $rz \in X_i$. So

$$\beta_i(rz)(rz) = (\beta_i(z) - r)(rz) = \beta_i(z)z,$$

which implies that $\beta_i(z)z \in A_2$. Thus $B_1 \subseteq A_2$. Then it is clear that $C_2 \subseteq B_1 \subseteq A_2 \subseteq E_2 \subseteq C_2$; and so it follows that $A = B = C = E$.

Let $x \in V_i$, then $x = rc$ for some $c \in X_i$, and some $r \in (\alpha_i(c), \beta_i(c))$. Thus

$$r - \alpha_i(c) \in (0, \beta_i(c) - \alpha_i(c)) = (0, \beta_i(\alpha_i(c)c).$$

Then $rc = (r - \alpha_i(c)) \cdot (\alpha_i(c)c) \in D$. Thus $V_i \subseteq D$. Let $x \in D$. Then $x = rz$ for some $z \in \overline{Y_i}$, and some $r \in (0, \beta_i(z))$. So

$$r - \beta_i(z)/2 \in (-\beta_i(z)/2, \beta_i(z)/2) = (\alpha_i((\beta_i(z)/2)z), \beta_i((\beta_i(z)/2)z)).$$

Also, $(\beta_i(z)/2)z \in X_i$, so $x = (r - \beta_i(z)/2) \cdot ((\beta_i(z)/2)z) \in V_i$. Thus $V_i = D$.

If $x \in V_i$, then $x = rc$ for some $c \in X_i$ and some $r \in (\alpha_i(c), \beta_i(c))$. We thus have $\alpha_i(x) = \alpha_i(c) - r \neq 0$, and $\beta_i(x) = \beta_i(c) - r \neq 0$. Thus $x \notin C_1 \cup C_2$, so $x \in C$. So $V_i \subseteq C$. Now let $x \in C$. Then $\alpha_i(x) \neq 0$, and $\beta_i(x) \neq 0$. Let $r = (\alpha_i(x) + \beta_i(x))/2$. Then $c = rx \in X_i$. Also $(\alpha_i(c), \beta_i(c)) = (\alpha_i(x), \beta_i(x)) - r$. Since $\alpha_i(x) \neq 0$, and $\beta_i(x) \neq 0$, so $0 \in (\alpha_i(x), \beta_i(x))$, and so $-r \in (\alpha_i(x), \beta_i(x)) - r = (\alpha_i(c), \beta_i(c))$. Then $x = (-r)(rx) \in V_i$. Thus $V_i = C$. Part is 1 proven.

For part 2, let $c \in X_i$. By part 3 of Lemma III.3.1, we have $\beta_i(c) - \alpha_i(c) \geq \sigma$. Since $\alpha_i(c) = -\beta_i(c)$, we have $\alpha_i(c) \neq 0$, and $\beta_i(c) \neq 0$. So $c \in C = V_i$.

Part 3 follows immediately from part 1 and part 2 of Lemma III.3.1. Part 4 follows from part 1 and part 5 of Lemma III.3.1. Part 5 follows from part 1 and part 6 of Lemma III.3.2.

For part 6, let $x \in Z^c$. Then $x \in W_j$ for some $j \in \{1, \ldots, N\}$. So $\alpha_i(x) = \alpha(x) < 0 < \beta(x) = \beta_i(x)$. Therefore $x \in V_j$.

For part 7, let $\{x_n\}$ be a sequence in $V_i \cap Z^c$ that converges to x for some $x \in Z^c$. Since $V_i \subseteq \overline{W_i}$, we see that $x \in \overline{W_i}$. Since $x \in Z^c$, $\alpha_i(x) \neq 0$ and $\beta_i(x) \neq 0$. So $x \in V_i$. $\square$

Lemma III.3.5. *Let $i, j \in \{1, \ldots, N\}$. Then $\pi_i(V_i \cap V_j)$ is closed in X_i.*

Proof: We only need to show that $\pi_i(V_i \cap V_j)$ is closed in X_i; the other statements follows from symmetry. If $V_i \cap V_j = \varnothing$, then we are done. So assume that $V_i \cap V_j \neq \varnothing$.

Let $\{w_n\}$ be a sequence in $\pi_i(V_i \cap V_j)$ that converges to some $w \in X$. Since X_i is compact, $w \in X_i$. Choose $x_n \in V_i \cap V_j$ such that $\pi_i(x_n) = w_n$. But $V_i \cap V_j \subseteq \overline{W_i} \cap \overline{W_j}$, which is compact, so x_n has a subsequence, say $\{y_n\}$, that converges to some $y \in \overline{W_i} \cap \overline{W_j}$. We claim that

$$(\alpha_i(y), \beta_i(y)) \cap (\alpha_j(y), \beta_j(y)) \neq \varnothing.$$

Suppose that $(\alpha_i(y), \beta_i(y)) \cap (\alpha_j(y), \beta_j(y)) = \varnothing$. But $0 \in [\alpha_i(y), \beta_i(y)] \cap [\alpha_j(y), \beta_j(y)]$, so either $\beta_i(y) = \alpha_j(y) = 0$ or $\alpha_i(y) = \beta_j(y) = 0$. First assume that $\beta_i(y) = \alpha_j(y)$. Then we have $\beta_i(y_n) - \alpha_j(y_n) \to \beta_i(y) - \alpha_j(y) = 0$. Now, $y_n \in V_i \cap V_j$, so $\beta_i(y_n) > 0$ and $\alpha_j(y_n) < 0$ for all $n \geq 1$. Then $\beta_i(y_n) - \alpha_j(y_n) > 0$ for all $n \geq 1$. For each $n \geq 1$, let $z_n = \alpha_j(y_n)y_n$. Then $R(z_n) \leq \beta_i(y_n) - \alpha_j(y_n) \to 0$, which contradicts the fact that $R \geq \sigma$. Similarly, we get a contradiction if we assume $\beta_j(y) = \alpha_i(y)$. Therefore $(\alpha_i(y), \beta_i(y)) \cap (\alpha_j(y), \beta_j(y)) \neq \varnothing$.

Let $r \in (\alpha_i(y), \beta_i(y)) \cap (\alpha_j(y), \beta_j(y))$. Then $ry \in V_i \cap V_j$. Now, $y_n \to y$, so $\alpha_i(y_n) \to \alpha_i(y)$, and $\beta_i(y_n) \to \beta_i(y)$. Passing to a subsequence if necessary, we can assume that $r \in (\alpha_i(y_n), \beta_i(y_n))$ for all $n \geq 1$. Then $ry_n \in V_i$ for all $n \geq 1$, and so $\pi_i(ry_n) \to \pi_i(ry)$. But $\pi_i(ry_n) = \pi_i(y_n) \to w$, so $w = \pi_i(ry) \in \pi_i(V_i \cap V_j)$. We have shown that $\pi_i(V_i \cap V_j)$ is closed in X_i. $\square$

Notation III.3.6. We fix the following notation for the rest of the chapter. For each $x \in X$, let $T^x = \{r \in \mathbb{R}\colon rx \in Z\}$. Then T^x is an infinite discrete set, hence countable. So index T^x as

$$\cdots < a^x_{-n} < a^x_{-n+1} < \cdots < a^x_{-1} < a^x_0 < a^x_1 < \cdots < a^x_{n-1} < a^x_n < \cdots .$$

Also note that for each $n \in \mathbb{Z}$, we have $a^x_{n+1} - a^x_n \geq \sigma$. For $i \in \{1, \ldots, N\}$ and for each $x \in V_i$, let $V_i^x = \{rx\colon r \in (\alpha_i(x), \beta_i(x))\} = (\alpha_i(x), \beta_i(x)) \cdot x$.

The following lemma shows that the sets V_i are ordered in the correct order.

Lemma III.3.7. *Let $k \in \{2, \ldots, N\}$, and let $x \in V_k$. Suppose that $T^x \cap [\alpha_k(x), \beta_k(x)]$ contains 3 or more elements. Then $Z^c \cap V_k^x = \bigcup_{i=1}^{k-1}(V_k^x \cap V_i) \cap Z^c$.*

Proof: Let $T = T^x \cap [\alpha_k(x), \beta_k(x)]$. Then for some $m, l \in \mathbb{Z}$ with $m < l$, there exist $a^x_m, a^x_{m+1}, \ldots, a^x_l \in [\alpha_k(x), \beta_k(x)]$ such that $\alpha_k(x) = a^x_m < a^x_{m+1} < \cdots < a^x_l = \beta_k(x)$ and $T = \{a^x_m, a^x_{m+1}, \ldots, a^x_l\}$. For each $n \in \{m, m+1, \ldots, l-1\}$, let $z_n = a^x_n x$.

Then for each $n \in \{m, m+1, \ldots, l-1\}$, we have

$$R(z_n) = a^x_{n+1} - a^x_n \leq (\beta_k(x) - \alpha_k(x)) - \sigma.$$

We claim that for each $n \in \{m, m+1, \ldots, l-1\}$, there exists $k_n < k$ such that $z_n \in Y_{k_n}$. So fix $n \in \{m, m+1, \ldots, l-1\}$.

Now $Y_k = Y_{i,j}$ for some $1 \leq i \leq n_V$ and some $1 \leq j \leq n_R$. Also, $Y_{i,j} = Z_i \cap Z^j \subseteq Z_j = T^{t_j}$ for some $1 \leq t_j \leq n_R$. See Lemma III.2.2, Notation III.2.3 and Notation III.2.5 for the definitions of Z_i, Z^j, $Y_{i,j}$, T^{t_j} and n_R. If $y \in W_k$, then $\alpha_k(y)y \in Y_k \subseteq T^{t_j}$, and

$$\beta_k(y) - \alpha_k(y) = \beta_k(\alpha_k(y)y) = R(\alpha(y)y) \in \left(\frac{(t_j-1)\sigma}{16}, \frac{t_j\sigma}{16}\right] .$$

Then $\{\beta_k(y) - \alpha_k(y)\colon y \in V_k\} \subseteq \left[\frac{(t_j-1)\sigma}{16}, \frac{t_j\sigma}{16}\right]$. In particular, $\beta_k(x) - \alpha_k(x) \in \left[\frac{(t_j-1)\sigma}{16}, \frac{t_j\sigma}{16}\right]$. Thus

$$R(z_n) \leq (\beta_k(x) - \alpha_k(x)) - \sigma \leq \frac{t_j\sigma}{16} - \sigma = \frac{(t_j-16)\sigma}{16}.$$

Then there exists some h with $1 \leq h < t_j$ such that $R(z_n) \in \left(\frac{(h-1)\sigma}{16}, \frac{h\sigma}{16}\right]$, which implies that $z_n \in T^h$. In particular, T^h is not empty, hence it is relabeled as Z^d for some $d < j$ (see Notation

III.2.3). So $z_n \in Y_{t,h}$ for some $1 \leq t \leq n_V$. From the definition of Y_s for $s \in \{1, \dots, N\}$ (see Notation III.2.5), it is clear that $Y_{t,h} = Y_{k_n}$ for some $k_n < k$. This proves the claim.

Now, if $y \in V_k^x \cap Z^c$, then there exists some $r \in (a_n^x, a_{n+1}^x)$ such that $y = rx$. Then $r - a_n^x \in (0, a_{n+1}^x - a_n^x) = (0, R(z_n))$. So we have

$$y = rx = (r - a_n^x)(a_n^x x) = (r - a_n^x) z_n \in V_{k_n} \subseteq \bigcup_{i=1}^{k-1} V_i.$$

□

Lemma III.3.8. *Let $k \in \{2, \dots, N\}$ and let $x \in V_k \cap \left(\bigcup_{i=1}^{k-1} V_i \right)$. Then*

$$Z^c \cap V_k^x = \bigcup_{i=1}^{k-1} (V_k^x \cap V_i) \cap Z^c.$$

Proof: Note that $T = T^x \cap [\alpha_k(x), \beta_k(x)]$ contains 2 or more elements. First suppose that T contains only 2 elements. Since $x \in V_k$, we see that $0 \in (\alpha_k(x), \beta_k(x))$. Also $(\alpha_k(x), \beta_k(x))x \subseteq Z^c$ by assumption, so we see that $x = 0 \cdot x \in (\alpha_k(x), \beta_k(x)) \cdot x \subseteq Z^c$. Then we have $(\alpha_k(x), \beta_k(x)) \subseteq (\alpha(x), \beta(x)) \subseteq (\alpha_i(x), \beta_i(x))$ for every $i \in \{1, \dots, N\}$ such that $x \in V_i$. Since $x \in V_i$ for some $1 \leq i < k$, we have $V_k^x = (\alpha_k(x), \beta_k(x))x \subseteq (\alpha_i(x), \beta_i(x))x = V_i^x$. Then we are done.

If T contains 3 or more elements, then we are done by Lemma III.3.7. □

III.4. Properties of G_i, $F^{(k)}$ and $G^{(k)}$

Now we define the subspaces G_i of $\mathbb{R} \times X$ which will be used to define the components of the stable recursive subhomogeneous decomposition of A_Z.

Notation III.4.1. For each $i \in \{1, \dots, N\}$, let

$$G_i = \{(r, x) \in \mathbb{R} \times X : x \in V_i, -r \in (\alpha_i(x), \beta_i(x))\}. \tag{III.3}$$

For each $k \in \{1, \dots, N-1\}$, let

$$F^{(k)} = \pi_{k+1} \left(V_{k+1} \cap \bigcup_{i=1}^{k} V_i \right). \tag{III.4}$$

Note that by Lemma III.3.5 the set $F^{(k)}$ is closed in X_{k+1}. For each $i \in \{1, \dots, N\}$ and each

$F \subseteq X_i$, let

$$G_{i,F} = \{(r, sc) \colon c \in F, s \in (\alpha_i(c), \beta_i(c)), s - r \in (\alpha_i(c), \beta_i(c))\}. \tag{III.5}$$

Note that $G_i = G_{i,X_i}$. (Lemma III.4.4 part 1.) For each $k \in \{1, \ldots, N-1\}$, let

$$G^{(k)} = G_{k+1,F^{(k)}}. \tag{III.6}$$

The subsets G_i of $\mathbb{R} \times X$ defined above are in fact subgroupoids of the transformation groupoid $\mathbb{R} \times X$. For each i, the subgroupoid G_i is contained in $(\mathbb{R} \times X)_{V_i}^{V_i}$, where $(\mathbb{R} \times X)_{V_i}^{V_i}$ is the set of all elements of $\mathbb{R} \times X$ whose sources and ranges are both contained in V_i. Due to minimality of the action, the subgroupoid $(\mathbb{R} \times X)_{V_i}^{V_i}$ is too large. The subgroupoid G_i, in some sense, is the largest continuous piece in $(\mathbb{R} \times X)_{V_i}^{V_i}$. See [13] for more details about groupoids.

Recall that $G_Z = \{(r, x) \colon x \in Z^c, -r \in (\alpha(x), \beta(x))\}$.

Lemma III.4.2. $G_1 \subseteq G_Z$.

Proof: First of all, we know that Y_1 is closed in X. By Lemma III.2.6, for all $z \in \overline{Y_1}$, we have $R(z) = \beta_1(z)$, and so by Lemma III.3.4, we have

$$V_1 = \{rz \colon z \in \overline{Y_1}, r \in (0, \beta_i(z))\} = \{rz \colon z \in Y_i, r \in (0, R(z))\} = W_1 \subseteq Z^c.$$

Then if $(r, x) \in G_1$, we have $x \in V_1 = W_1 \subseteq Z^c$ and $-r \in (\alpha_1(x), \beta_1(x)) = (\alpha(x), \beta(x))$, since $\alpha_i|_{W_i} = \alpha|_{W_i}$, so $(r, x) \in G_Z$, and thus $G_1 \subseteq G_Z$. ☐

Lemma III.4.3. $G_Z \subseteq \bigcup_{i=1}^{N} G_i$.

Proof: Let $(r, x) \in G_Z$. Then $x \in Z^c$, and $-r \in (\alpha(x), \beta(x))$. So $x \in V_i$ for some $1 \leq i \leq N$. Then $x \in Z^c \cap V_i$ implies that $\alpha_i(x) \leq \alpha(x) < -r < \beta(x) \leq \beta_i(x)$. So $(r, x) \in G_i$. ☐

Part 2 and part 3 of next lemma essentially show that $G_{i,F}$ is a subgroupoid of $\mathbb{R} \times X$ for $i \in \{1, \ldots, N\}$ and $F \subseteq X_i$.

Lemma III.4.4. *Let $i \in \{1, \ldots, N\}$. Then the following hold:*

1. $G_i = \{(r, tc) \colon c \in X_i, t, t - r \in (\alpha_i(c), \beta_i(c))\}$.

2. *Let $F \subseteq X_i$. Let $c_1, c_2 \in F$. Let r_1, t_1, r_2 and t_2 be real numbers that satisfy*

$$t_1, t_1 - r_1 \in (\alpha_i(c_1), \beta_i(c_1));$$

$$t_2, t_2 - r_2 \in (\alpha_i(c_2), \beta_i(c_2));$$

and

$$t_1 c_1 = (t_2 - r_2) c_2.$$

Then $(r_1 + r_2, t_2 c_2) \in G_{i,F}$, and $(-r_1, (t_1 - r_1)c_1) \in G_{i,F}$. Let $s\colon \mathbb{R} \times X \to X$ be defined by $(r, x) \mapsto x$. Then $G_{i,F} = G_i \cap s^{-1}(\pi_i^{-1}(F))$ and $G_{i,F}$ has compact closure.

3. *If $F \subseteq X_i$, $c \in X_i$, $t \in (\alpha_i(c), \beta_i(c))$, and $-r \in (\alpha_i(c), \beta_i(c)) - t$, then $(r, tc) \in G_{i,F}$ if and only if $c \in F$.*

4. *If $F, F' \subseteq X_i$, then $G_{i,F \cup F'} = G_{i,F} \cup G_{i,F'}$ and $G_{i,F \cap F'} = G_{i,F} \cap G_{i,F'}$.*

Proof: Part 1 is clear.

Now we show part 2. Since $(r_1, t_1 c_1), (r_2, t_2 c_2) \in G_{i,F}$, we see that $c_1, c_2 \in F$, that $t_1, t_1 - r_1 \in (\alpha_i(c_1), \beta_i(c_1))$, and that $t_2, t_2 - r_2 \in (\alpha_i(c_2), \beta_i(c_2))$. Now $t_1 - r_1 \in (\alpha_i(c_1), \beta_i(c_1))$ implies that

$$\begin{aligned} -r_1 &\in (\alpha_i(c_1), \beta_i(c_1)) - t_1 = (\alpha_i(t_1 c_1), \beta_i(t_1 c_1)) \\ &= (\alpha_i((t_2 - r_2)c_2), \beta_i((t_2 - r_2)c_2)) = (\alpha_i(c_2), \beta_i(c_2)) - (t_2 - r_2). \end{aligned}$$

So $t_2 - (r_1 + r_2) \in (\alpha_i(c_2), \beta_i(c_2))$ and $(r_1 + r_2, t_2 c_2) \in G_{i,F}$. Also,

$$t_1 - r_1 \in (\alpha_i(c_1), \beta_i(c_1)) \text{ and } (t_1 - r_1) - (-r_1) = t_1 \in (\alpha_i(c_1), \beta_i(c_1))$$

imply that $(-r_1, (t_1 - r_1)c_1) \in G_{i,F}$.

To see that $G_{i,F}$ is pre-compact, note that $G_{i,F} \subseteq [-M, M] \times X$.

Let $(r, tc) \in G_{i,F}$. Then $c \in F$ and $t, t - r \in (\alpha_i(c), \beta_i(c))$. Also,

$$\pi_i(s(r, tc)) = \pi_i(tc) = \pi_i(c) = c \in F.$$

So $G_{i,F} \subseteq G_i \cap s^{-1}(\pi_i^{-1}(F))$. Let $(r,x) \in G_i \cap s^{-1}(\pi_i^{-1}(F))$. Then $x \in V_i$ and $-r \in (\alpha_i(x), \beta_i(x))$. Therefore $x = tc$ for some $c \in X_i$ and some $t \in (\alpha_i(c), \beta_i(c))$. Thus $\pi_i(s(r,x)) = \pi_i(tc) = \pi_i(c) = c \in F$. Since

$$-r \in (\alpha_i(x), \beta_i(x)) = (\alpha_i(tc), \beta_i(tc)) = (\alpha_i(c), \beta_i(c)) - t,$$

we see that $t - r \in (\alpha_i(c), \beta_i(c)) - s$, and so $(r,x) = (r,tc) \in G_{i,F}$.

For part 3, $(r,tc) \in G_{i,F}$ implies that there exists $c' \in F$ and $t', t'-r' \in (\alpha_i(c), \beta_i(c))$ such that $(r,tc) = (r', t'c')$. Then $c = \pi_i(tc) = \pi_i(t'c') = c' \in F$. Thus $(r,tc) \in G_{i,F}$ implies that $c \in F$. The other direction is trivial.

Let $F, F' \subseteq X_i$. Then

$$\begin{aligned} G_{i,F\cup F'} &= G_i \cap s^{-1}(\pi_i^{-1}(F \cup F')) \\ &= G_i \cap [s^{-1}(\pi_i^{-1}(F)) \cup s^{-1}(\pi_i^{-1}(F'))] = G_{i,F} \cup G_{i,F'}. \end{aligned}$$

Also, since $(r,tc) \in G_{i,F\cap F'}$ if and only if $c \in F \cap F'$, if and only if $(r,tc) \in G_{i,F} \cap G_{i,F'}$, part 4 follows. □

Corollary III.4.5. *For each $i \in \{1, \ldots, N\}$ and each $F \subseteq X_i$, if F is closed (open) in X_i, then $G_{i,F}$ is closed (open) in G_i.*

Lemma III.4.6. *Let $i \in \{1, \ldots, N\}$. Then $G_i \cap G_Z$ is closed in G_Z.*

Proof: Let $\{(r_n, x_n)\}$ be a sequence in $G_i \cap G_Z$ that converges to some $(r,x) \in G_Z$. Then $x_n \in V_i \cap Z^c$ for all $n \geq 1$, and $x \in Z^c$. By part 7 of Lamma III.3.4, we have $x \in V_i$. Since $x \in Z^c$, and since $(r,x) \in G_Z$, we see that $-r \in (\alpha(x), \beta(x)) \subseteq (\alpha_i(x), \beta_i(x))$. Thus $(r,x) \in G_i$, and so $G_i \cap G_Z$ is closed in G_Z. □

Lemma III.4.7. *Let $k \in \{1, \ldots, N-1\}$. Then for all $i \in \{1, \ldots, k\}$, we have $G_i \cap G^{(k)} = G_i \cap G_{k+1}$; and $G_Z \cap G_i \cap G^{(k)}$ is closed in $G^{(k)} \cap G_Z$, in $G_i \cap G_Z$, and in $G_{k+1} \cap G_Z$.*

Proof: Fix $k \in \{1, \ldots, N-1\}$, and fix $i \in \{1, \ldots, k\}$. We first show that $G_i \cap G^{(k)} = G_i \cap G_{k+1}$. The inclusion $G_i \cap G^{(k)} \subseteq G_i \cap G_{k+1}$ is clear. Let $(r,x) \in G_i \cap G_{k+1}$. Then by the definition of sets G_i (Notation III.4.1), we have $x \in V_i \cap V_{k+1}$. So $\pi_{k+1}(s(r,x)) = \pi_{k+1}(x)$, which is contained in $\pi_{k+1}(V_i \cap V_{k+1}) \subseteq F^{(k)}$. Thus $(r,x) \in G^{(k)}$.

Now we claim that if A is any topological space, and $B, C, D \subseteq A$ are arbitrary subspaces such that B is closed in C, then $B \cap D$ is closed in $C \cap D$. To prove this, since B is closed in C, there exists F closed in A such that $F \cap C = B$. Then $B \cap D = F \cap C \cap D$ is closed in $C \cap D$. This proves the claim.

Now we know that $G_i \cap G_Z$ is closed in G_Z, and $G_{k+1} \cap G_Z$ is closed in G_Z. So $G_i \cap G_{k+1} \cap G_Z$ is closed in G_Z. Then by the claim above,

$$G_i \cap G_{k+1} \cap G_Z = (G_i \cap G_{k+1} \cap G_Z) \cap G_i$$

is closed in $G_i \cap G_Z$. Similarly $G_i \cap G_{k+1} \cap G_Z$ is closed in $G_{k+1} \cap G_Z$.

Then by the first statement of the lemma, $G_Z \cap G_i \cap G^{(k)}$ is closed in $G_i \cap G_Z$, and in $G_{k+1} \cap G_Z$. But then $G_Z \cap G_i \cap G^{(k)} = (G_Z \cap G_i \cap G^{(k)}) \cap G^{(k)}$ is closed in $G^{(k)} \cap G_{k+1} \cap G_Z = G^{(k)} \cap G_Z$. □

Lemma III.4.8. *Let $k \in \{1, \ldots, N-1\}$. Then*

$$G^{(k)} \cap G_Z = \bigcup_{i=1}^{k} (G_i \cap G^{(k)} \cap G_Z) = \bigcup_{i=1}^{k} (G_i \cap G_{k+1} \cap G_Z).$$

Proof: The last equality of the lemma follows from Lemma III.4.7. Also it is clear that

$$\bigcup_{i=1}^{k} (G_i \cap G^{(k)} \cap G_Z) \subseteq G^{(k)} \cap G_Z.$$

We will show that $G^{(k)} \cap G_Z \subseteq \bigcup_{i=1}^{k} (G_i \cap G^{(k)} \cap G_Z)$.

Let $(r, x) \in G^{(k)} \cap G_Z$. Then $x \in V_{k+1} \cap Z^c$ and $-r \in (\alpha(x), \beta(x))$. Now consider

$$V_{k+1}^x = \{rx \colon r \in (\alpha_{k+1}(x), \beta_{k+1}(x))\}.$$

We first check that $V_{k+1}^x = V_{k+1} \cap \pi_{k+1}^{-1}(\pi_{k+1}(x))$. It is clear that $V_{k+1}^x \subseteq V_{k+1} \cap \pi_{k+1}^{-1}(\pi_{k+1}(x))$. Let $y \in V_{k+1} \cap \pi_{k+1}^{-1}(\pi_{k+1}(x))$, let $r_x = \frac{\alpha_{k+1}(x)+\beta_{k+1}(x)}{2}$, let $r_y = \frac{\alpha_{k+1}(y)+\beta_{k+1}(y)}{2}$, let $c_x = r_x x$, and let $c_y = r_y y$. Then $c_x = \pi_{k+1}(x)$ and $c_y = \pi_{k+1}(y)$. By assumption, $c_x = c_y$. Part 3 of Lemma III.3.1 implies that $r_x \in (\alpha_{k+1}(x), \beta_{k+1}(x))$ and $r_y \in (\alpha_{k+1}(y), \beta_{k+1}(y))$, so we have $-r_x \in (\alpha_{k+1}(c_x), \beta_{k+1}(c_x))$ and $-r_y \in (\alpha_{k+1}(c_y), \beta_{k+1}(c_y))$. Note that $x, y \in V_{k+1}$ implies that

$\alpha_{k+1}(x) < 0 < \beta_{k+1}(x)$ and that $\alpha_{k+1}(y) < 0 < \beta_{k+1}(y)$. Now, if $r_x - r_y \geq \beta_{k+1}(x)$, we have $(\alpha_{k+1}(x) + \beta_{k+1}(x)) - (\alpha_{k+1}(y) + \beta_{k+1}(y)) \geq 2\beta_{k+1}(x)$, and so

$$\begin{aligned}\alpha_{k+1}(y) + \beta_{k+1}(y) &\leq \alpha_{k+1}(x) - \beta_{k+1}(x) = \alpha_{k+1}(c_x) - \beta_{k+1}(c_x)\\ &= \alpha_{k+1}(c_y) - \beta_{k+1}(c_y) = \alpha_{k+1}(y) - \beta_{k+1}(y).\end{aligned}$$

Then $\beta_{k+1}(y) \leq 0$, contradiction. Similarly, $r_x - r_y \leq \alpha_{k+1}(x)$ implies that $\alpha_{k+1}(y) \geq 0$, also a contradiction. So $r_x - r_y \in (\alpha_{k+1}(x), \beta_{k+1}(x))$. Thus $y = (r_x - r_y)(x) \in V_{k+1}^x$, and $V_{k+1}^x = V_{k+1} \cap \pi_{k+1}^{-1}(\pi_{k+1}(x))$. Also, note that if $y \in V_{k+1}^x$, then $y = sx$ for some $s \in (\alpha_{k+1}(x), \beta_{k+1}(x))$, and then

$$\begin{aligned}V_{k+1}^y &= (\alpha_{k+1}(y), \beta_{k+1}(y))y = (\alpha_{k+1}(sx), \beta_{k+1}(sx))(sx)\\ &= (\alpha_{k+1}(x) - s, \beta_{k+1}(x) - s)(sx) = V_{k+1}^x.\end{aligned}$$

Now, $(r, x) \in G^{(k)}$ implies that $\pi_{k+1}(x) \in \pi_{k+1}\left(V_{k+1} \cap \left(\bigcup_{i=1}^k V_i\right)\right)$. Thus there exists $y \in V_{k+1} \cap \left(\bigcup_{i=1}^k V_i\right)$ such that $\pi_{k+1}(x) = \pi_{k+1}(y)$. Then $y \in V_{k+1}^x$, so $V_{k+1}^y = V_{k+1}^x$. But by Lemma III.3.8, we know that $Z^c \cap V_{k+1}^y = Z^c \cap \left(\bigcup_{i=1}^k V_{k+1}^y \cap V_i\right)$. So we have $Z^c \cap V_{k+1}^x = Z^c \cap \left(\bigcup_{i=1}^k V_{k+1}^x \cap V_i\right)$. Since $x \in V_{k+1}^x \cap Z^c$, there exists $i \in \{1, \ldots, k\}$ such that $x \in V_{k+1}^x \cap V_i \subseteq V_{k+1} \cap V_i$. Then $x \in V_i \cap Z^c$, and then $(\alpha(x), \beta(x)) \subseteq (\alpha_i(x), \beta_i(x))$, and so $-r \in (\alpha_i(x), \beta_i(x))$. Thus $(r, x) \in G_i$. Hence

$$(r, x) \in G^{(k)} \cap G_Z \cap G_i \subseteq \bigcup_{i=1}^k (G_i \cap G^{(k)} \cap G_Z).$$

□

Lemma III.4.9. *Let $k \in \{1, \ldots, N-1\}$. Then $G_{k+1} \setminus G^{(k)} \subseteq G_Z$.*

Proof: Let $(r, x) \in G_{k+1} \setminus G^{(k)}$. First of all, if $V_{k+1}^x \cap \left(\bigcup_{i=1}^k V_i\right) \neq \varnothing$, there exists

$$y \in V_{k+1}^x \cap \left(\bigcup_{i=1}^k V_i\right) \subseteq V_{k+1} \cap \left(\bigcup_{i=1}^k V_i\right).$$

Then $\pi_{k+1}(x) = \pi_{k+1}(y) \in F^{(k)}$. Hence $(r, x) \in G^{(k)}$. This contradicts our assumption that (r, x) is not contained in $G^{(k)}$. Therefore $V_{k+1}^x \cap \left(\bigcup_{i=1}^k V_i\right) = \varnothing$.

Now, if $Z^c \cap V^x_{k+1} = Z^c \cap \left(\bigcup_{i=1}^k (V^x_{k+1} \cap V_i)\right)$, then

$$Z^c \cap V^x_{k+1} = Z^c \cap V^x_{k+1} \cap \left(\bigcup_{i=1}^k V_i\right) = \varnothing,$$

a contradiction. So $V^x_{k+1} \subseteq Z$. That is, $(\alpha_{k+1}(x), \beta_{k+1}(x))x \subseteq Z$. Let

$$\omega = \min\{\sigma, \beta_{k+1}(x)/2, -\alpha_{k+1}(x)/2\}.$$

Since $x \in V_{k+1}$, we have $\alpha_{k+1}(x) < 0 < \beta_{k+1}(x)$. So $\omega > 0$. Then $[-\omega, \omega]x \subseteq Z$. But

$$([-\omega, \omega]x) \cap Z \subseteq ([-\sigma, \sigma]x) \cap Z = \{x\}.$$

So, because the action is free, $\omega = 0$, which is a contradiction. Therefore

$$Z^c \cap V^x_{k+1} \neq Z^c \cap \left(\bigcup_{i=1}^k V^x_{k+1} \cap V_i\right).$$

By Lemma III.3.7, the set $T^x \cap [\alpha_{k+1}(x), \beta_{k+1}(x)]$ contains only 2 elements, namely $\alpha_{k+1}(x)$ and $\beta_{k+1}(x)$. Then for all $s \in (\alpha_{k+1}(x), \beta_{k+1}(x))$, we have $sx \in Z^c$. So $x \in Z^c$ (because $\alpha_{k+1}(x) < 0 < \beta_{k+1}(x)$), $\alpha(x) = \alpha_{k+1}(x)$, and $\beta(x) = \beta_{k+1}(x)$. Since $(r, x) \in G_{k+1}$, we have $-r \in (\alpha_{k+1}(x), \beta_{k+1}(x)) = (\alpha(x), \beta(x))$, and so $(r, x) \in G_Z$. ☐

Lemma III.4.10. *Let $i \in \{1, \ldots, N\}$, and let $F \subseteq X_i$ be closed. Then:*

1. *we have $\overline{G_{i,F}} = \{(r, x) \in \mathbb{R} \times \overline{W_i} \colon \pi_i(x) \in F, -r \in [\alpha_i(x), \beta_i(x)]\}$.*

2. *we have*

$$\begin{aligned}\overline{G_{i,F}} \setminus G_{i,F} = &\{(r, x) \in \overline{G_{i,F}} \colon \alpha_i(x) = 0\} \\ &\cup \{(r, x) \in \overline{G_{i,F}} \colon \beta_i(x) = 0\} \\ &\cup \{(r, x) \in \overline{G_{i,F}} \colon -r = \alpha_i(x)\} \\ &\cup \{(r, x) \in \overline{G_{i,F}} \colon -r = \beta_i(x)\}.\end{aligned}$$

3. *the set $\overline{G_{i,F}} \setminus G_{i,F}$ is closed in $\mathbb{R} \times X$, and $G_{i,F}$ is open in $\overline{G_{i,F}}$.*

Proof: Let

$$A = \{(r,x) \in \mathbb{R} \times \overline{W_i} \colon \pi_i(x) \in F, -r \in [\alpha_i(x), \beta_i(x)]\}.$$

We first show that A is closed. Well, if $(r_n, x_n) \in A$, and $(r_n, x_n) \to (r,x)$ for some $(r,x) \in \mathbb{R} \times X$, then $x \in \overline{W_i}$ and $\pi_i(x) \in F$, because F and $\overline{W_i}$ are closed in X, and because $\alpha_i(x_n) \to \alpha_i(x)$, $\beta_i(x_n) \to \beta_i(x)$, and $-r_n \to -r$. Since $-r_n \in [\alpha_i(x_n), \beta_i(x_n)]$ for all $n \geq 1$, we have $-r \in [\alpha_i(x), \beta_i(x)]$. Hence $(r,x) \in A$, and so A is closed.

Now let $(r,x) \in A$. Let $s = (\alpha_i(x) + \beta_i(x))/2$, and let $c = sx = \pi_i(x) \in F \subseteq V_i$. Since $-r \in [\alpha_i(x), \beta_i(x)]$, there exists a sequence $\{r_n\}$ in $(-\beta_i(x), -\alpha_i(x))$ such that $r_n \to r$. Now since $\alpha_i(x) < \beta_i(x)$, we see that $\alpha_i(x) < s < \beta_i(x)$. Since $\alpha_i(x) \leq 0$, we see that $\alpha_i(x) \leq \alpha_i(x)/(2n)$ for all $n \geq 1$; since $\beta_i(x) \geq 0$, we have $\beta_i(x)/(2n) \leq \beta_i(x)$ for all $n \geq 1$. Then

$$\alpha_i(x) \leq \alpha_i(x)/(2n) < s/(2n) < \beta_i(x)/(2n) \leq \beta_i(x)$$

for all $n \geq 1$. Thus $s/(2n) \in (\alpha_i(x), \beta_i(x))$ for all $n \geq 1$. Then $(\frac{s}{2n})x \in \overline{W_i}$, $\alpha_i((\frac{s}{2n})x) \neq 0$, and $\beta_i((\frac{s}{2n})x) \neq 0$ for all $n \geq 1$. Thus $(\frac{s}{2n})x \in V_i$ for all $n \geq 1$. Since $-r_n \in (\alpha_i(x), \beta_i(x))$ for all $n \geq 1$, we have

$$-r_n - s/(2n) \in (\alpha_i(x), \beta_i(x)) - s/(2n) = \left(\alpha_i\left(\left(\frac{s}{2n}\right)x\right), \beta_i\left(\left(\frac{s}{2n}\right)x\right)\right)$$

for all $n \geq 1$, so $(r_n + s/(2n), (s/2n)x) \in G_i$ for all $n \geq 1$. Since

$$\pi_i(s((r_n + s/(2n), (s/2n)x)) = \pi_i(x) \in F,$$

we have $(r_n + s/(2n), (s/2n)x) \in G_{i,F}$ for all $n \geq 1$. Since $(r_n + s/(2n), (\frac{s}{2n})x) \to (r,x)$, we see that $(r,x) \in \overline{G_{i,F}}$. Thus part 1 holds.

Let $A_1 = \{(r,x) \in \overline{G_{i,F}} \colon \alpha_i(x) = 0\}$, let $A_2 = \{(r,x) \in \overline{G_{i,F}} \colon \beta_i(x) = 0\}$, let $A_3 = \{(r,x) \in \overline{G_{i,F}} \colon -r = \alpha_i(x)\}$, let $A_4 = \{(r,x) \in \overline{G_{i,F}} \colon -r = \beta_i(x)\}$, and let $A = A_1 \cup \cdots \cup A_4$. To show part 2, we only need to show that $G_{i,F} \cap A = \varnothing$ and $G_{i,F} \cup A = \overline{G_{i,F}}$. We first show that $G_{i,F} \cap A_j = \varnothing$ for all $j \in \{1, \ldots, 4\}$.

Note that

$$\begin{aligned} G_{i,F} &= \{(r, sc) \in \mathbb{R} \times X \colon c \in F, s, s - r \in (\alpha_i(c), \beta_i(c))\} \\ &= \{(r, x) \in G_i \colon \pi_i(x) \in F\}. \end{aligned}$$

If $(r, x) \in G_{i,F}$, then $x \in V_i$, and so $\alpha_i(x) \neq 0$ and $\beta_i(x) \neq 0$. Then $(r, x) \notin A_1$ and $(r, x) \notin A_2$. Thus $A_1 \cap G_{i,F} = \varnothing$, and $A_2 \cap G_{i,F} = \varnothing$. Also, $(r, x) \in G_{i,F}$ implies that $-r \neq \alpha_i(x)$ and $-r \neq \beta_i(x)$. Then $(r, x) \notin A_3$ and $(r, x) \notin A_4$. Thus $A_3 \cap G_{i,F} = \varnothing$, and $A_4 \cap G_{i,F} = \varnothing$. Then $G_{i,F} \cap A = \varnothing$.

Now let $(r, x) \in \overline{G_{i,F}}$. Then $x \in \overline{W_i}$, $\pi_i(x) \in F$, and $-r \in [\alpha_i(x), \beta_i(x)]$. Suppose that $(r, x) \notin A$. Then $\alpha_i(x) \neq 0$, $\beta_i(x) \neq 0$, $-r \neq \alpha_i(x)$, and $-r \neq \beta_i(x)$. So $x \in V_i$, $-r \in (\alpha_i(x), \beta_i(x))$, and $(r, x) \in G_i$. Since $\pi_i(x) \in F$, we see that $(r, x) \in G_{i,F}$. Thus $\overline{G_{i,F}} = A \cup G_{i,F}$, and part 2 holds.

Now let $\{(r_n, x_n)\}$ be a sequence in A_1 that converges to some $(r, x) \in \mathbb{R} \times X$. Since $\overline{G_{i,F}}$ is closed, we see that $(r, x) \in \overline{G_{i,F}}$. Then by continuity of α_i, we have $\alpha_i(x) = 0$. So $(r, x) \in A_1$, and so A_1 is closed in $\mathbb{R} \times X$. Similarly, A_2 is closed. Now let $\{(r_n, x_n)\}$ be a sequence in A_3 that converges to some $(r, x) \in \mathbb{R} \times X$. Then $(r, x) \in \overline{G_{i,F}}$. Since $r_n = \alpha_i(x_n)$ for all $n \geq 1$, since $\alpha_i(x_n) \to \alpha_i(x)$, and since $r_n \to r$, we have $\alpha_i(x) = r$. Thus $(r, x) \in A_3$. So A_3 is closed in $\mathbb{R} \times X$. Similarly A_4 is closed in $\mathbb{R} \times X$; and so A is closed in $\mathbb{R} \times X$. Then $G_{i,F} = \overline{G_{i,F}} \cap A^c$ is open in $\overline{G_{i,F}}$. □

Corollary III.4.11. *Let $i \in \{1, \ldots, N\}$. Then*

1. *we have $\overline{G_i} = \{(r, x) \in \mathbb{R} \times \overline{W_i} \colon -r \in [\alpha_i(x), \beta_i(x)]\}$,*

2. *we have*

$$\begin{aligned} \overline{G_i} \setminus G_i = &\{(r, x) \in \overline{G_i} \colon \alpha_i(x) = 0\} \\ &\cup \{(r, x) \in \overline{G_i} \colon \beta_i(x) = 0\} \\ &\cup \{(r, x) \in \overline{G_i} \colon -r = \alpha_i(x)\} \\ &\cup \{(r, x) \in \overline{G_i} \colon -r = \beta_i(x)\}, \end{aligned}$$

3. *the set $\overline{G_i} \setminus G_{i,F}$ is closed in $\mathbb{R} \times X$, and G_i is open in $\overline{G_i}$.*

III.5. The C^*-Algebra of G_i

In this section we will define *-algebra structures and C^*-norms on $C_0(G_i)$ and $C_0(G^{(k)})$. Let $f, g \in C(\overline{G_{i,F}})$, and let $(r, x) \in \overline{G_{i,F}}$. For each $t \in [-\beta_i(x), -\alpha_i(x)]$, (t, x) and $(r - t, (-t)x)$ are elements of $\overline{G_{i,F}}$ (by Lemma III.4.4), so we can define $h\colon [-\beta_i(x), -\alpha_i(x)] \to \mathbb{C}$ by $h(t) = f(t, x)g(r - t, (-t)x)$. Then h is certainly continuous, and hence in $L^1([-\beta_i(x), -\alpha_i(x)])$, and so $\int_{-\beta_i(x)}^{-\alpha_i(x)} f(t, x)g(r-t, (-t)x)dt$ exists. Also, $(-r, (-r)x)$ is also an element of $\overline{G_{i,F}}$, so $\overline{f(-r, (-r)x)}$ exists. Then we can define convolution on $\overline{G_{i,F}}$ by

$$(f * g)(r, x) = \int_{-\beta_i(x)}^{-\alpha_i(x)} f(t, x)g(r - t, (-t)x)\, dt, \tag{III.7}$$

and involution by

$$f^*(r, x) = \overline{f(-r, (-r)x)}. \tag{III.8}$$

We verify through the next three lemmas that the above formulas make $C_0(G_{i,F})$ into a *-algebra. In fact, if we take the groupoid structure of $G_{i,F}$ into consideration, the above formulas are the ones used in the construction of groupoid C^*-algebras in [13].

Lemma III.5.1. *Let $i \in \{1, \ldots, N\}$, let $F \neq \varnothing$ be a closed subspace of X_i, and let $f, g \in C(\overline{G_{i,F}})$. Then $f * g$ and f^* are continuous. That is $f * g, f^* \in C(\overline{G_{i,F}})$.*

Proof: It is clear that f^* is continuous.

Let $\{(r_n, x_n)\}$ be a sequence in $\overline{G_{i,F}}$ that converges to some $(r, x) \in \overline{G_{i,F}}$. Let $\epsilon > 0$. For each $n \geq 1$, let $h_n\colon \mathbb{R} \to \mathbb{C}$ be defined by $h_n(t) = f(t, x_n)g(r_n - t, (-t)x_n)$ if $t \in [-\beta_i(x_n), -\alpha_i(x_n)]$, and $h_n(t) = 0$ otherwise. Then h_n is measurable for each $n \geq 1$. Define $h\colon \mathbb{R} \to \mathbb{C}$ by $h(t) = f(t, x)g(r - t, (-r)x)$ for $t \in [-\beta_i(x), -\alpha_i(x)]$, and $h_n(t) = 0$ otherwise. Then h is measurable. Let $\delta = \min\left\{\frac{\epsilon}{8\|f\|_\infty \|g\|_\infty}, \frac{\beta_i(x) - \alpha_i(x)}{4}\right\}$. Then $\delta > 0$. Since $\alpha_i(x_n) \to \alpha_i(x)$, and $\beta_i(x_n) \to \beta_i(x)$, there exists $M \geq 1$ such that $n \geq M$ implies that $|\alpha_i(x_n) - \alpha_i(x)| < \delta$, and $|\beta_i(x) - \beta_i(x_n)| < \delta$. Now, if $t \in [-\beta_i(x) + \delta, -\alpha_i(x) - \delta]$, then $t \in [-\beta_i(x_n), -\alpha_i(x_n)]$ for all $n \geq M'$, and $t \in [-\beta_i(x), -\alpha_i(x)]$. Therefore

$$h_n(t) = f(t, x_n)g(r_n - t, (-t)x_n) \to f(t, x)g(r - t, (-r)x) = h(t).$$

Since $|h_n(t)| \leq \|f\|_\infty \|g\|_\infty$ for all $n \geq 1$ and all $t \in [-\beta_i(x)+\delta, -\alpha_i(x)-\delta]$, and since

$$\|f\|_\infty \|g\|_\infty \in L^1([-\beta_i(x)+\delta, -\alpha_i(x)-\delta]),$$

by the Lebesgue Dominated Convergence Theorem, we have

$$\int_{-\beta_i(x)+\delta}^{\alpha_i(x)-\delta} |h_n(t)-h(t)|\,dt \to 0.$$

So there exists $M' \geq 1$ such that $n \geq M'$ implies that

$$\int_{-\beta_i(x)+\delta}^{\alpha_i(x)-\delta} |h_n(t)-h(t)|\,dt < \epsilon/2.$$

Let $M'' = M' + M$. Then if $n \geq M''$, we have

$$\begin{aligned}
&\left| \int_{-\beta_i(x_n)}^{-\alpha_i(x_n)} h_n(t)\,dt - \int_{-\beta_i(x)}^{-\alpha_i(x)} h(t)\,dt \right| \\
&\qquad \leq 2\delta\|h_n\|_\infty + 2\delta\|h\|_\infty + \left| \int_{-\beta_i(x_n)+\delta}^{-\alpha_i(x_n)-\delta} (h_n(t)-h(t))\,dt \right| \\
&\qquad < 4\delta\|f\|_\infty\|g\|_\infty + \epsilon/2 \leq \epsilon/2 + \epsilon/2 \\
&\qquad = \epsilon.
\end{aligned}$$

So $|(f*g)(r_n,x_n) - (f*g)(r,x)| < \epsilon$ for all $n \geq M''$. Thus $(f*g)(r_n,x_n) \to (f*g)(r,x)$. Therefore $f*g$ is continuous. □

Lemma III.5.2. *Let $i \in \{1,\ldots,N\}$, and let $F \neq \varnothing$ be a closed subset of X_i. Let $f,g \in C_0(G_{i,F})$. Then $f*g \in C_0(G_{i,F})$ and $f^* \in C_0(G_{i,F})$.*

Proof: By Lemma III.4.10, we have

$$\begin{aligned}
\overline{G_{i,F}} \setminus G_{i,F} = &\{(r,x) \in \overline{G_{i,F}} \colon \alpha_i(x) = 0\} \\
&\cup \{(r,x) \in \overline{G_{i,F}} \colon \beta_i(x) = 0\} \\
&\cup \{(r,x) \in \overline{G_{i,F}} \colon -r = \alpha_i(x)\} \\
&\cup \{(r,x) \in \overline{G_{i,F}} \colon -r = \beta_i(x)\}.
\end{aligned}$$

Next we define four different subsets of $\overline{G_{i,F}}$, which can be thought of as the faces of $\overline{G_{i,F}}$. Define

$$A_1 = \{(r,x) \in \overline{G_{i,F}} \colon \alpha_i(x) = 0\},$$

$$A_2 = \{(r,x) \in \overline{G_{i,F}} \colon \beta_i(x) = 0\},$$

$$A_3 = \{(r,x) \in \overline{G_{i,F}} \colon -r = \alpha_i(x)\},$$

and

$$A_4 = \{(r,x) \in \overline{G_{i,F}} \colon -r = \beta_i(x)\}.$$

To show that $f * g, f^* \in C_0(G_{i,F})$, we just need to show that $(f * g)|_{A_j} = 0$ and $f^*|_{A_j} = 0$ for $j \in \{1, \ldots, 4\}$.

Let $(r,x) \in A_1 \cup A_2$. Either $\alpha_i(x) = 0$ or $\beta_i(x) = 0$. Then for all $t \in [-\beta_i(x), -\alpha_i(x)]$, we have $(t,x) \in A_1 \cup A_2$. So $f(t,x) = 0$ for all $t \in [-\beta_i(x), -\alpha_i(x)]$, and so

$$(f * g)(r,x) = \int_{-\beta_i(x)}^{-\alpha_i(x)} f(t,x) g(r-t, (-t)x)\, dt = 0.$$

Thus $(f * g)|_{A_1 \cup A_2} = 0$.

Let $(r,x) \in A_3 \cup A_4$. Then either $(r,x) = (-\alpha_i(x), x)$ or $(r,x) = (-\beta_i(x), x)$. So for all $t \in [-\beta_i(x), -\alpha_i(x)]$, we have

$$r - t = -\alpha_i(x) - t = -(\alpha_i(x) + t) = -(\alpha_i((-t)x)),$$

or

$$r - t = -\beta_i(x) - t = -(\beta_i(x) + t) = -(\beta_i((-t)x)),$$

and so $(r-t, (-t)x) \in A_3 \cup A_4$; and then $g(r-t, (-t)x) = 0$. Therefore we have

$$(f * g)(r,x) = \int_{-\beta_i(x)}^{-\alpha_i(x)} f(t,x) g(r-t, (-t)x)\, dt = 0.$$

Thus $(f * g)|_{A_3 \cup A_4} = 0$, and so $f * g \in C_0(G_{i,F})$.

Next we consider f^*. Now $(r,x) \in A_1 \cup A_2$ implies $\alpha_i(x) = 0$ or $\beta_i(x) = 0$, which implies that $r = \alpha_i((-r)x)$ or $r = \beta_i((-r)x)$, which in turn implies that $(-r,(-r)x) \in A_3 \cup A_4$. Also, $(r,x) \in A_3 \cup A_4$ implies that $-r = \alpha_i(x)$ or $-r = \beta_i(x)$, which implies that $\alpha_i((-r)x) = 0$ or $\beta_i((-r)x) = 0$, which means that $(r,x) \in A_1 \cup A_2$. Thus if $(r,x) \in \overline{G_{i,F}}$, then so is $(-r,(-r)x)$, and so $f^*(r,x) = \overline{f(-r,(-r)x))} = 0$. Therefore $f^* \in C_0(G_{i,F})$. □

Lemma III.5.3. *The set $C(\overline{G_{i,F}})$ is a *-algebra, and $C_0(G_{i,F})$ is a *-subalgebra of $C(\overline{G_{i,F}})$.*

Proof: It is clear that $C(\overline{G_{i,F}})$ is a linear space. Lemma III.5.1 shows that convolution and involution are well-defined.

Let $f, g, h \in C(\overline{G_{i,F}})$, let $(r,x) \in \overline{G_{i,F}}$, and let $\lambda \in \mathbb{C}$. To simplify the notation, let $a = \alpha_i(x)$ and $b = \beta_i(x)$. It is clear that $\lambda(f * g) = (\lambda f) * g = f * (\lambda g)$. Now, applying the Fubini Theorem to interchange integrals, we check that convolution is associative:

$$\begin{aligned}
[(f*g)*h](r,x) &= \int_{-b}^{-a} (f*g)(t,x)h(r-t,(-t)x)\,dt \\
&= \int_{-b}^{-a} \left(\int_{-b}^{-a} f(s,x)g(t-s,(-s)x)\,ds \right) h(r-t,(-t)x)\,dt \\
&= \int_{-b}^{-a} \int_{-b}^{-a} f(s,x)g(t-s,(-s)x)h(r-t,(-t)x)\,dt\,ds \\
&= \int_{-b}^{-a} \int_{-b-s}^{-a-s} f(s,x)g(t,(-s)x)h(r-(t+s),(-(t+s))x)\,dt\,ds \\
&= \int_{-b}^{-a} f(s,x) \left(\int_{-\beta_i((-s)x)}^{-\alpha_i((-s)x)} g(t,(-s)x)h((r-s)-t),(-t)((-s)x))\,dt \right) ds \\
&= \int_{-b}^{-a} f(s,x)(g*h)(r-s,(-s)x)\,ds \\
&= [f*(g*h)](r,x).
\end{aligned}$$

Thus convolution is associative. It is clear that convolution is distributive. Now we check that involution is anti-commutative:

$$\begin{aligned}
(f*g)^*(r,x) &= \overline{(f*g)(-r,(-r)x)} \\
&= \overline{\int_{-\beta_i((-r)x)}^{-\alpha_i((-r)x)} f(t,(-r)x)g(-r-t,(-r-t)x)\,dt} \\
&= \overline{\int_{-(\beta_i(x)+r)}^{-(\alpha_i(x)+r)} f(t,(-r)x)g(-r-t,(-r-t)x)\,dt} \\
&= \int_{-b}^{-a} \overline{f(s-r,(-r)x)}\,\overline{g(-s,(-s)x)}\,ds \\
&= \int_{-b}^{-a} f^*(r-s,(-s)x)g^*(s,x)\,ds \\
&= (g^* * f^*)(r,x).
\end{aligned}$$

So involution is anti-commutative. It is clear that involution is conjugate linear. It is also clear that $(f^*)^* = f$ for all $f \in C(\overline{G_{i,F}})$. Thus $C(\overline{G_{i,F}})$ is a *-algebra. By Lemma III.5.2, $C_0(G_{i,F})$ is a *-subalgebra of $C(\overline{G_{i,F}})$. □

Next, we will define a family of *-representations of $G_{i,F}$ for each $i = 1, \ldots, N$, and each $F \subseteq X_i$. For each $i \in \{1, \ldots, N\}$ and for each $x \in X_i$, let $\chi_i^x \colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of the interval $(\alpha_i(x), \beta_i(x)) \subseteq \mathbb{R}$, and define a projection in $p_i^x \in B(L^2(\mathbb{R}))$ by $p_i^x(\xi) = \chi_i^x \xi$. For each $i \in \{1, \ldots, N\}$, each nonempty closed subset $F \subseteq X_i$, and each $x \in F$, define

$$\lambda_{i,F}^x \colon C_0(G_{i,F}) \to B(L^2(\mathbb{R}))$$

by, for $f \in C_0(G_{i,F})$, $\xi \in L^2(\mathbb{R})$, and $r \in \mathbb{R}$,

$$\lambda_{i,F}^x(f)(\xi)(r) = \int_{\alpha_i(x)}^{\beta_i(x)} \chi_i^x(r)\chi_i^x(t)\xi(t)f(r-t,rx)\,dt. \tag{III.9}$$

Notation III.5.4. For the rest of the chapter, let λ_i^x denote λ_{i,X_i}^x for each $i \in \{1, \ldots, N\}$, and let $\lambda^{(k),x}$ denote $\lambda_{k+1,F^{(k)}}^x$ for each $k = 1, \ldots, N-1$.

Lemma III.5.5. *For each $i \in \{1, \ldots, N\}$, each nonempty closed subset $F \subseteq X_i$ and each $x \in F$, the map $\lambda_{i,F}^x$ is a *-homomorphism. Further, if $f \in C_0(G_{i,F})$, and if $\{x_n\}$ is a sequence in F that converges to some $x \in F$, then $\lambda_{i,F}^{x_n}(f) \to \lambda_{i,F}^x(f)$. Moreover, if $f \in C_0(G_{i,F})$ and $x \in F$, then*

$\lambda^x_{i,F}(f) = 0$ *if and only if* $f|_{H_x} = 0$, *where*

$$H_x = \{(r-t, rx) \colon r, t \in (\alpha_i(x), \beta_i(x))\} = \{(t, rx) \colon r \in (\alpha_i(x), \beta_i(x)), r - t \in (\alpha_i(x), \beta_i(x)\}.$$

Proof: Fix $i \in \{1, \ldots, N\}$ and $F \subseteq X$ closed for the entire proof.

Let $x \in F$. Linearity of $\lambda^x_{i,F}$ is clear. Now let $f, g \in C_0(G_{i,F})$. Then for all $\xi \in L^2(\mathbb{R})$ and all $r \in (\alpha_i(x), \beta_i(x))$, we have, applying the Fubini Theorem,

$$\begin{aligned}
\lambda^x_{i,F}(f * g)(\xi)(r) &= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t)(f * g)(r-t, rx)\,dt \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) \left(\int_{-\beta_i(rx)}^{-\alpha_i(rx)} f(s, rx) g(r-t-s, (-s+r)x)\,ds \right) dt \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) \left(\int_{\alpha_i(rx)}^{\beta_i(rx)} f(-s, rx) g(r-t+s, (s+r)x)\,ds \right) dt \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) \left(\int_{\alpha_i(x)}^{\beta_i(x)} f(r-s, rx) g(s-t, sx)\,ds \right) dt \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) f(r-s, rx) g(s-t, sx)\,dt\,ds \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} f(r-s, rx)\chi^x_i(r) \left(\int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(t)\xi(t) g(s-t, sx)\,dt \right) ds.
\end{aligned}$$

Now we show that for all $s \in \mathbb{R}$, we have $\chi^x_i(s) f(r-s, rx) = f(r-s, rx)$. If $f(r-s, rx) = 0$, then we are done, so assume that $f(r-s, rx) \neq 0$. Then $(r-s, rx) \in G_{i,F}$. So $s - r \in (\alpha_i(rx), \beta_i(rx)) = (\alpha_i(x), \beta_i(x)) - r$, and thus $s \in (\alpha_i(x), \beta_i(x))$. Then $\chi^x_i(s) = 1$. So $\chi^x_i(s) f(r-s, rx) = f(r-s, rx)$ for all $s \in \mathbb{R}$. Then

$$\begin{aligned}
\lambda^x_{i,F}(f * g)(\xi)(r) &= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(s)^2 f(r-s, rx)\chi^x_i(r) \left(\int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(t)\xi(t) g(s-t, sx)\,dt \right) ds \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(s) f(r-s, rx)\chi^x_i(r) \left(\int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(s)\chi^x_i(t)\xi(t) g(s-t, sx) dt \right) ds \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(s) f(r-s, rx)\chi^x_i(r)\lambda^x_{i,F}(g)(\xi)(s)\,ds \\
&= \lambda^x_{i,F}(f)[\lambda^x_{i,F}(g)(\xi)](r).
\end{aligned}$$

If $r \notin (\alpha_i(x), \beta_i(x))$, then $\lambda^x_{i,F}(f * g)(\xi)(r) = 0 = \lambda^x_{i,F}(f)[\lambda^x_{i,F}(g)(\xi)](r)$. Thus

$$\lambda^x_{i,F}(f * g)(\xi) = \lambda^x_{i,F}(f)[\lambda^x_{i,F}(g)(\xi)]$$

for all $\xi \in L^2(\mathbb{R})$. So $\lambda^x_{i,F}(f * g) = \lambda^x_{i,F}(f)\lambda^x_{i,F}(g)$. Therefore $\lambda_{i,F}$ is multiplicative.

For all $f \in C_0(G_{i,F})$ and all $\xi, \eta \in L^2(\mathbb{R})$, we have, applying the Fubini Theorem,

$$\begin{aligned}
\langle \lambda^x_{i,F}(f^*)(\xi), \eta\rangle &= \int_{\mathbb{R}} \lambda^x_{i,F}(f^*)(\xi)(r)\overline{\eta(r)}\, dr \\
&= \int_{\mathbb{R}} \left(\int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) f^*(r-t, rx) dt \right) \overline{\eta(r)}\, dr \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) f^*(r-t, rx)\overline{\eta(r)}\, dr\, dt \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t)\xi(t) \overline{f(t-r, tx)\eta(r)}\, dr\, dt \\
&= \int_{\alpha_i(x)}^{\beta_i(x)} \xi(t)\chi^x_i(t) \int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\overline{\chi^x_i(t) f(t-r, tx)\eta(r)}\, dr\, dt \\
&= \int_{\mathbb{R}} \xi(t) \left(\int_{\alpha_i(x)}^{\beta_i(x)} \chi^x_i(r)\chi^x_i(t) \overline{f(t-r, tx)\eta(r)}\, dr \right) dt \\
&= \int_{\mathbb{R}} \xi(t)\overline{\lambda^x_{i,F}(f)(\eta)(t)}\, dt \\
&= \langle \xi, \lambda_{i,F}(f)(\eta)\rangle.
\end{aligned}$$

So $\lambda^x_{i,F}(f^*) = \lambda^x_{i,F}(f)^*$. Thus $\lambda^x_{i,F}$ is a *-homomorphism.

Let $f \in C_0(G_{i,F})$, and let $\{x_n\}$ be a sequence in F that converges to $x \in F$. We now show that $\|\lambda^{x_n}_{i,F}(f) - \lambda^x_{i,F}(f)\| \to 0$. For each $n \geq 1$, let $\chi_n \colon \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function of $(\alpha_i(x_n), \beta_i(x_n)) \times (\alpha_i(x_n), \beta_i(x_n)) \subseteq \mathbb{R}^2$, and let $\chi \colon \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function of $(\alpha_i(x), \beta_i(x)) \times (\alpha_i(x), \beta_i(x)) \subseteq \mathbb{R}^2$. Because β_i is continuous on F and because $x_n \to x$, we see that the sequence $\{\beta_i(x_n)\}$ is bounded. Let $D = \sup_{n\geq 1} \beta_i(x_n)$ and let $\chi_D \colon \mathbb{R}^2 \to \mathbb{R}$ be the characteristic function of the square $(-D, D) \times (-D, D)$. Since $\alpha_i(y) = -\beta_i(y)$ for all $y \in X_i$, we see that $\chi_n \leq \chi_D$ for all $n \geq 1$ and $\chi \leq \chi_D$. For each $n \geq 1$, define $h_n \colon \mathbb{R}^2 \to \mathbb{C}$ by $h_n(r, t) = f(r - t, rx_n)$. Also define $h \colon \mathbb{R}^2 \to \mathbb{C}$ by $h(r, t) = f(r - t, rx)$.

It is clear that for all $n \geq 1$, either $\chi_n \geq \chi$ or $\chi \geq \chi_n$. Then either

$$\int_{\mathbb{R}^2} |\chi_n - \chi| = \int_{\mathbb{R}^2} \chi_n - \int_{\mathbb{R}^2} \chi = (2\beta_i(x_n))^2 - (2\beta_i(x))^2,$$

or

$$\int_{\mathbb{R}^2} |\chi_n - \chi| = \int_{\mathbb{R}^2} \chi - \int_{\mathbb{R}^2} \chi_n = (2\beta_i(x))^2 - (2\beta_i(x_n))^2.$$

But in either case $\int_{\mathbb{R}^2} |\chi_n - \chi| \to 0$, and so

$$\|\chi_n - \chi\|_2 = \left(\int_{\mathbb{R}^2} |\chi_n - \chi|^2\right)^{1/2} = \left(\int_{\mathbb{R}^2} |\chi_n - \chi|\right)^{1/2} \to 0.$$

Therefore $\|\chi_n h - \chi h\|_2 \leq \|h\|_\infty \cdot \|\chi_n - \chi\|_2 \to 0$. Also, for every $n \geq 1$, we have $|\chi_D h_n - \chi_D h|^2 = \chi_D \cdot |h_n - h|^2 \leq 4\chi_D \|f\|_\infty^2$. Since $4\chi_D\|f\|_\infty^2 \in L^1(\mathbb{R}^2)$ and since h_n converges to h point-wise, it follows from the Lebesgue's Dominated Convergence Theorem that $\|\chi_D h_n - \chi_D h\|_2 \to 0$. Then

$$\|\chi_n h_n - \chi_n h\|_2 = \|\chi_n \chi_D h_n - \chi_n \chi_D h\|_2 \leq \|\chi_D h_n - \chi_D h\|_2 \to 0.$$

Thus we have

$$\|\chi_n h_n - \chi h\|_2 \leq \|\chi_n h_n - \chi_n h\|_2 + \|\chi_n h - \chi h\|_2 \to 0.$$

Note that $\chi_n(r,t) = \chi_i^{x_n}(r)\chi_i^{x_n}(t)$ and $\chi(r,t) = \chi_i^x(r)\chi_i^x(t)$. So for each $\xi \in L^2(\mathbb{R})$, we have

$$\begin{aligned}
&\|(\lambda_{i,F}^{x_n}(f) - \lambda_{i,F}^x(f))(\xi)\|^2 \\
&\quad= \int_{\mathbb{R}} \left|(\lambda_{i,F}^{x_n}(f) - \lambda_{i,F}^x(f))(\xi)(r)\right|^2 dr \\
&\quad= \int_{\mathbb{R}} \left|\lambda_{i,F}^{x_n}(f)(\xi)(r) - \lambda_{i,F}^x(f)(\xi)(r)\right|^2 dr \\
&\quad= \int_{\mathbb{R}} \left| \int_{\alpha_i(x_n)}^{\beta_i(x_n)} \chi_i^{x_n}(r)\chi_i^{x_n}(t)\xi(t)f(r-t, rx_n)dt \right. \\
&\qquad\qquad \left. - \int_{\alpha_i(x)}^{\beta_i(x)} \chi_i^x(r)\chi_i^x(t)\xi(t)f(r-t, rx)\, dt \right|^2 dr \\
&\quad= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \chi_n(r,t)\xi(t)h_n(r,t)dt - \int_{\mathbb{R}} \chi(r,t)\xi(t)h(r,t)\, dt \right|^2 dr \\
&\quad= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} [\chi_n(r,t)\xi(t)h_n(r,t) - \chi(r,t)\xi(t)h(r,t)]\, dt \right|^2 dr \\
&\quad= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \xi(t)[\chi_n(r,t)h_n(r,t) - \chi(r,t)h(r,t)]\, dt \right|^2 dr \\
&\quad\le \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\xi(t)| \cdot |\chi_n(r,t)h_n(r,t) - \chi(r,t)h(r,t)|\, dt \right]^2 dr \\
&\quad\le \int_{\mathbb{R}} \left[\left(\int_{\mathbb{R}} |\xi(t)|^2 dt \right)^{1/2} \cdot \left(\int_{\mathbb{R}} |\chi_n(r,t)h_n(r,t) - \chi(r,t)h(r,t)|^2 dt \right)^{1/2} \right]^2 dr \\
&\quad\le \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\xi(t)|^2 dt \right] \cdot \left[\int_{\mathbb{R}} |\chi_n(r,t)h_n(r,t) - \chi(r,t)h(r,t)|^2 dt \right] dr \\
&\quad\le \|\xi\|^2 \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} |\chi_n(r,t)h_n(r,t) - \chi(r,t)h(r,t)|^2\, dt\, dr \\
&\quad= \|\xi\|^2 \cdot \|\chi_n h_n - \chi h\|_2^2.
\end{aligned}$$

Thus, $\|\lambda_{i,F}^{x_n}(f) - \lambda_{i,F}^x(f)\| \le \|\chi_n h_n - \chi h\|_2 \to 0$.

Next we show that for all $x \in F$, if $\xi \in L^2(\mathbb{R})$ is continuous on $(\alpha_i(x), \beta_i(x))$ and bounded, then $\lambda_{i,F}^x(f)(\xi)$ is continuous on $(\alpha_i(x), \beta_i(x))$. Let $x \in F$, and let $\xi \in L^2(\mathbb{R})$ be continuous on $(\alpha_i(x), \beta_i(x))$ and bounded. Suppose that $r_n \to r$ in $(\alpha_i(x), \beta_i(x))$. Then $h_n(t) = \chi_i^x(t)\chi_i^x(r_n)\xi(t)f(r_n - t, r_n x)$ converges to

$$h(t) = \chi_i^x(t)\chi_i^x(r)\xi(t)f(r-t, rx)$$

pointwise on $(\alpha_i(x), \beta_i(x))$. Therefore, since $|h_n| \le \chi_i^x \|\xi\|_\infty \|f\|_\infty \in L^1((\alpha_i(x), \beta_i(x))$, by the

Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned}\lim_{n\to\infty} \lambda^x_{i,F}(f)(\xi)(r_n) &= \lim_{n\to\infty} \int_{\mathbb{R}} \chi^x_i(t)\chi^x_i(r_n)\xi(t)f(r_n - t, r_n x)\, dt \\ &= \int_{\mathbb{R}} \chi^x_i(t)\chi^x_i(r)\xi(t)f(r-t, rx)\, dt = \lambda^x_{i,F}(f)(\xi)(r).\end{aligned}$$

Thus $\lambda^x_{i,F}(f)(\xi)$ is continuous on $(\alpha_i(x), \beta_i(x))$.

Now, let $f \in C_0(G_{i,F})$, and let $x \in F$. Suppose that $\lambda^x_{i,F}(f) = 0$. Let $r \in (\alpha_i(x), \beta_i(x))$. Define $\xi\colon \mathbb{R} \to \mathbb{C}$ by $\xi(t) = f(r-t, rx)$ for $t \in (\alpha_i(x), \beta_i(x))$, and zero otherwise. Then ξ is continuous on $(\alpha_i(x), \beta_i(x))$, and ξ is bounded. Therefore $\lambda^x_{i,F}(f)(\xi)$ is continuous. Since $\lambda^x_{i,F}(f)(\xi) = 0$, we have

$$0 = \lambda^x_{i,F}(f)(\xi)(r) = \int_{\mathbb{R}} \chi^x_i(t)\chi^x_i(r)|f(r-t, rx)|^2 dt = \int_{\alpha_i(x)}^{\beta_i(x)} |f(r-t, rx)|^2 dt.$$

But $t \mapsto |f(r-t, rx)|^2$ is continuous on $(\alpha_i(x), \beta_i(x))$, so $f(r-t, rx) = 0$ for all $t \in (\alpha_i(x), \beta_i(x))$. This holds for all $r \in (\alpha_i(x), \beta_i(x))$, so $f(r-t, rx) = 0$ for all $r, t \in (\alpha_i(x), \beta_i(x))$. That is $f|_{H_x} = 0$. It is clear that if $f|_{H_x} = 0$, then $\lambda^x_{i,F}(f) = 0$. □

The following proposition is an immediate consequence of Lemma III.5.5.

Proposition III.5.6. *For each $i \in \{1, \dots, N\}$ and each nonempty closed subset $F \subseteq X$, define $\phi_{i,F}\colon C_0(G_{i,F}) \to C(F, K(L^2(\mathbb{R})))$ by*

$$\phi_{i,F}(f)(x) = \lambda^x_{i,F}(f).$$

*If $F = \varnothing$, put $\phi_{i,F} = 0$. Then $\phi_{i,F}$ is a *-homomorphism such that $\|\phi_{i,F}(f)\| = \sup_{x\in F} \|\lambda^x_{i,F}(f)\|$ for all $f \in C_0(G_{i,f})$.*

III.6. Stable Recursive Subhomogeneous Decomposition of A_Z

Notation III.6.1. We fix the following notations for the rest of the chapter. Now for each $i \in \{1, \dots, N\}$, and each closed $F \subseteq X_i$ define a C^*- norm $\|\cdot\|_{i,F}$ on $C_0(G_{i,F})$ by $\|f\|_{i,F} = \sup_{x\in F} \|\lambda^x_{i,F}(f)\|$. Note that Lemma III.5.5 ensures that $\|\cdot\|_{i,F}$ is a C^*-norm. Let $\|\cdot\|_i = \|\cdot\|_{i,X_i}$, for each $i \in \{1, \dots, N\}$; and let $\|\cdot\|^{(k)} = \|\cdot\|_{k+1,F^{(k)}}$ for each $k \in \{1, \dots, N-1\}$. (If $F^{(k)} = \varnothing$, let $\|\cdot\|^{(k)}$ be the obvious norm on $C_0(G^{(k)})$.) For each $i \in \{1, \dots, N\}$ and each closed $F \subseteq X_i$, let

$A_{i,F}$ be the completion of $C_0(G_{i,F})$ with respect to $\|\cdot\|_{i,F}$. For each $i \in \{1,\ldots,N\}$ let A_i denote A_{i,X_i}, and for each $k \in \{1,\ldots,N-1\}$ let $A^{(k)}$ denote $A_{k+1,F^{(k)}}$. For each $i \in \{1,\ldots,N\}$ and each nonempty closed subset $F \subseteq X_i$, let $\phi_{i,F}$ denote the map in Proposition III.5.6. It is then clear that $\phi_{i,F}$ is isometric and extends to an injective *-homomorphism from $A_{i,F}$ into $C(F,\mathbb{K}(L^2(\mathbb{R})))$, and we will also use $\phi_{i,F}$ to denote the extension. Let ϕ_i denote ϕ_{i,X_i} for $i \in \{1,\ldots,N\}$, and let $\phi^{(k)}$ denote $\phi_{k+1,F^{(k)}}$. For each $i \in \{1,\ldots,N\}$ and each nonempty closed subset $F \subseteq X$, let

$$K_{i,F} = \{f \in C(F,K(L^2(\mathbb{R}))) \colon p_i^x f(x) p_i^x = f(x) \text{ for all } x \in F\}.$$

If $F = \varnothing$, then let $K_{i,F} = 0$. Let K_i denote K_{i,X_i} and let $K^{(k)}$ denote $K_{k+1,F^{(k)}}$.

The C^*-algebras A_i will be the components of a SRSH decomposition of A_Z. We proceed to obtain a SRSH decomposition of A_Z as follows: We first identify A_i with $C(X_i,\mathbb{K})$ for each $i \in \{1,\ldots,N\}$. Note that Proposition III.5.6 already shows that $C_0(G_i)$ is isometrically *-isomorphic to a *-subalgebra of $C(X_i,\mathbb{K})$. Thus we only need to identify the range of the map, and show that the norm closure of the range is isomorphic to $C(X_i,\mathbb{K})$. Then we glue the *-algebras $C_0(G_i)$ to obtain $C_0(G_Z)$. After the gluing, we extend the gluing to the A_i to obtain a decomposition of A_Z. Finally, we use the identifications between the algebras A_i and the algebras $C(X_i,\mathbb{K})$ to obtain a SRSH decomposition of A_Z.

The next lemma is a standard result in operator algebra.

Lemma III.6.2. *Let H be a Hilbert space, let $\{a_n\}$ be a sequence in $B(H)$ that converges to some $a \in B(H)$ in strong operator topology, and let $\{b_n\}$ be a sequence in $K(H)$ that converges to some $b \in K(H)$ in the norm topology. Then $a_n b_n a_n^* \to aba^*$ in the norm topology.*

Lemma III.6.3. *For each $i \in \{1,\ldots N\}$, and for each nonempty closed subset $F \subseteq X$, let $K_{i,F}$ be as in III.6.1. Then we have:*

1. *$K_{i,F}$ is a C^*-subalgebra of $C(F,K(L^2(\mathbb{R})))$.*
2. *$\phi_{i,F}(C_0(G_{i,F})) \subseteq K_{i,F}$.*
3. *$\overline{\phi_{i,F}(C_0(G_{i,F}))} = K_{i,F}$.*
4. *For each $i \in \{1,\ldots,N\}$, and for each $x \in X_i$, define $u_{i,x} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $u_{i,x}(\xi)(r) = \frac{\xi(r/\beta_i(x))}{(\beta_i(x))^{1/2}}$. Then for each $i \in \{1,\ldots,N\}$, and each $x \in X_i$, $u_{i,x}$ is a unitary, with $u_{i,x}^*$ given*

by $u_{i,x}^*(\xi)(r) = (\beta_i(x))^{1/2}\xi(\beta_i(x)r)$. *Further, for each* $i = 1, \ldots, N$, *if* $\{x_n\}$ *is a sequence in* X_i *that converges to some* $x \in X_i$, *then* $\{u_{i,x_n}\}$ *and* $\{u_{i,x_n}^*\}$ *converge to* $u_{i,x}$ *and* $u_{i,x}^*$, *respectively, in the strong operator topology.*

5. *Let* $I = (-1, 1)$, *let* $p_I \in B(L^2(\mathbb{R}))$ *be the projection given by* $p_I(\xi) = \chi_I \xi$, *and let*

$$\Omega \colon p_I K(L^2(\mathbb{R}))p_I \to K(L^2(I))$$

*be the canonical *-isomorphism. For each* $i \in \{1, \ldots, N\}$, *and each closed subset* $F \subseteq X_i$, *define* $\Phi_{i,F} \colon K_{i,F} \to C(F, K(L^2(I)))$ *by* $\Phi_{i,F}(f)(x) = \Omega(u_{i,x}^* f(x) u_{i,x})$. *Then* $\Phi_{i,F}$ *is a well defined *-isomorphism for all* $i \in \{1, \ldots, N\}$, *and all closed* $F \subseteq X_i$. *(If* $F = \varnothing$, *take* $C(F, K(L^2(I))) = 0$, *and* $\Phi_{i,F} = 0$.*)*

Proof: Part 1 and part 2 are clear.

Now we show that for each $x \in F$, the set $S_x = \{\phi_{i,F}(f)(x) \colon f \in C_0(G_{i,F})\}$ is dense in $T_x = \{a \in K(L^2(\mathbb{R})) \colon p_i^x a p_i^x = a\}$. Let $I_i^x = (\alpha_i(x), \beta_i(x))$. Note that $T_x = p_i^x K(L^2(\mathbb{R}))p_i^x = K(L^2(I_i^x))$ is C^*-subalgebra of $K(L^2(\mathbb{R}))$. Let $\xi, \eta \in C_c((\alpha_i(x), \beta_i(x))$. Let $E = \{(r, tx) \in \mathbb{R} \times X \colon t, r-t \in I_i^x\}$. Then $E \subseteq G_{i,F}$. It follows from Lemma III.3.2 that the map $h \colon I_i^x \times I_i^x \to E$ defined by $h(r, t) = (t - r, tx)$ is a homeomorphism. (The inverse is given by $(r, tx) \mapsto (t - r, t)$.) Let $f'' \colon I \times I \to \mathbb{C}$ be defined by $f''(r, t) = \xi(t)\overline{\eta(r)}$. Then $f'' \in C_0(I \times I)$. Let $f' \colon E \to \mathbb{C}$ be defined by $f' = f'' \circ h^{-1}$. Then $f' \in C_0(E)$, and $f'(r, tx) = f''(t - r, tx) = \xi(t)\overline{\eta(t - r)}$. Now E is closed in $G_{i,F}$, so there exists $f \in C_0(G_{i,F})$ such that $f|_E = f'$. Then for all $r \in \mathbb{R}$ and all $\zeta \in L^2(I_i^x)$ we have

$$\begin{aligned}
\phi_{i,F}(f)(x)(\zeta)(r) &= \lambda_{i,F}^x(f)(\zeta)(r) \\
&= \int_{\mathbb{R}} \chi_i^x(r)\chi_i^x(t)\zeta(t)f(r - t, rx)\,dt \\
&= \int_{\mathbb{R}} \chi_i^x(r)\chi_i^x(t)\zeta(t)\xi(r)\overline{\eta(t)}\,dt \\
&= \int_{\mathbb{R}} \zeta(t)\xi(r)\overline{\eta(t)}\,dt \\
&= \langle \zeta, \eta \rangle \xi(r).
\end{aligned}$$

For any Hilbert space H and any $\xi', \eta' \in H$, we use the notation $\xi' \otimes \eta'$ to denote the rank one operator defined by $\zeta \mapsto \langle \zeta, \eta' \rangle \xi'$. Then $\phi_{i,F}(f)(x) = \xi \otimes \eta$, and $\xi \otimes \eta \in S_x$. Since $C_c(I_i^x)$ is dense in $L^2(I_i^x) = p_i^x(L^2(\mathbb{R}))$, we see that $\xi \otimes \eta \in \overline{S_x}$ for all $\xi, \eta \in p_i^x(L^2(\mathbb{R}))$. Since

$$\{p_i^x(\xi) \otimes p_i^x(\eta) \colon \xi, \eta \in L^2(\mathbb{R})\}$$

spans a dense subset of T_x, we see that S_x is dense in T_x.

Now we show that for all $f \in K_{i,F}$, for all $x \in F$, and for all $\epsilon > 0$, there exists an open subset $U \subseteq F$ containing x and $g \in C_0(G_{i,F})$ such that for all $y \in U$, we have $\|\phi_{i,F}(g)(y) - f(y)\| < \epsilon$. Let $f \in K_{i,F}$, $x \in F$ and $\epsilon > 0$ be given. Then, by the paragraph above, there exists $g \in C_0(G_{i,F})$ such that $\|\phi_{i,F}(g)(x) - f(x)\| < \epsilon/2$. Now the map $y \mapsto \|\phi_{i,F}(g)(y) - f(y)\|$ is continuous, so $U = \{y \in F \colon \|\phi_{i,F}(g)(y) - f(y)\| < \epsilon\}$ is an open set containing x. It is clear that for all $y \in U$, we have $\|\phi_{i,F}(g)(y) - f(y)\| < \epsilon$.

Now we show that if $f \in C_0(G_{i,F})$ and $h \in C(F)$, then $h\phi_{i,F}(f) \in \operatorname{Im} \phi_{i,F}$. Define $\widetilde{h} \colon \overline{G_{i,F}} \to \mathbb{C}$ by $\widetilde{h}(r,x) = h(\pi_i(x))$. Then $\widetilde{h} \in C(\overline{G_{i,F}})$, and $\widetilde{h}f \in C_0(G_{i,F})$. So for all $x \in F$, all $\xi \in L^2(\mathbb{R})$, and all $r \in \mathbb{R}$, we have

$$\begin{aligned}
\phi_{i,F}(\widetilde{h}f)(x)(\xi)(r) &= \lambda_{i,F}^x(\widetilde{h}f)(\xi)(r) \\
&= \int_{\mathbb{R}} \chi_i^x(r)\chi_i^x(t)\xi(t)\widetilde{h}(r-t, rx)f(r-t, rx)\, dt \\
&= \int_{\mathbb{R}} \chi_i^x(r)\chi_i^x(t)\xi(t)h(x)f(r-t, rx)\, dt \\
&= h(x)\int_{\mathbb{R}} \chi_i^x(r)\chi_i^x(t)\xi(t)f(r-t, rx)\, dt \\
&= (h(x)\lambda_{i,F}^x(f))(\xi)(r) \\
&= (h(x)\phi_{i,F}(f)(x))(\xi)(r).
\end{aligned}$$

Thus $h\phi_{i,F}(f) = \phi_{i,F}(\widetilde{h}f) \in \operatorname{Im} \phi_{i,F}$.

Now we finish the proof of part 3. Let $g \in K_{i,F}$, and let $\epsilon > 0$. For each $x \in F$, let $V_x \subseteq F$ be an open subset containing x, and let $f_x \in C_0(G_{i,F})$ be such that for all $y \in V_x$ we have $\|\phi_{i,F}(f_x)(y) - g(y)\| < \epsilon$. The existence of V_x and f_x are shown above. Then $\{V_x \colon x \in F\}$ is an open cover of F, which is compact; so there exist $y_1, \ldots, y_m$ such that $F = \bigcup_{j=1}^m V_{y_j}$. Let $\{h_j \colon 1 \le j \le m\}$ be a partition of unity subordinate to $\{V_j \colon 1 \le j \le m\}$. By what is shown above, we have $h_j\phi_{i,F}(f_{y_j}) \in \operatorname{Im} \phi_{i,F}$ for each $j \in \{1, \ldots, m\}$. Then $f = \sum_{j=1}^m h_j\phi_{i,F}(f_{y_j}) \in \operatorname{Im} \phi_{i,F}$. Now let $x \in F$, and let $1 \le j \le m$. If $x \notin V_{y_j}$, then $h_j(x) = 0$ and $h_j(x)\|\phi_{i,F}(f_{y_j})(x) - g(x)\| = 0$;

and if $x \in V_{y_j}$, then $h_j(x)\|\phi_{i,F}(f_{y_j})(x) - g(x)\| \leq \epsilon h_j(x)$ Thus, for all $x \in F$, we have

$$\begin{aligned}\|f(x) - g(x)\| &= \left\| \sum_{j=1}^{m} h_j(x)\phi_{i,F}(f_{y_j})(x) - g(x) \right\| \\ &= \left\| \sum_{j=1}^{m} h_j(x)\phi_{i,F}(f_{y_j})(x) - \sum_{j=1}^{m} h_j(x)g(x) \right\| \\ &\leq \sum_{j=1}^{m} h_j(x)\|\phi_{i,F}(f_{y_j})(x) - g(x)\| \\ &< \sum_{j=1}^{m} h_j(x)\epsilon = \epsilon.\end{aligned}$$

Part 3 proven.

Now we show part 4. It is clear that for each $i \in \{1, \dots, N\}$ and each $x \in X_i$, $u_{i,x}$ is a unitary, and that $u_{i,x}^*$ is given by the formula in the statement. Fix $i \in \{1, \dots, N\}$. Now we show that if $x_n \to x$ in X_i, then $u_{i,x_n} \to u_{i,x}$ in strong operator topology, and $u_{i,x_n}^* \to u_{i,x}$ in strong operator topology.

Let $x_n \to x$ in X_i, and let $\xi \in C_c(\mathbb{R})$. Since $\beta_i(x_n) \to \beta_i(x)$, we have

$$\left| \frac{\xi(r/\beta_i(x_n))}{\beta_i(x_n)^{1/2}} - \frac{\xi(r/\beta_i(x))}{\beta_i(x)^{1/2}} \right|^2 \to 0$$

for every $r \in \mathbb{R}$. Suppose that supp $\xi \subseteq [-b, b]$. Since β_i is continuous and strictly positive on the compact set X_i, it is bounded above by some real number M and below by some real number $L > 0$. Then

$$\left| \frac{\xi(r/\beta_i(x_n))}{\beta_i(x_n)^{1/2}} - \frac{\xi(r/\beta_i(x))}{\beta_i(x)^{1/2}} \right|^2 \leq \frac{4 \cdot \chi_{[-Mb,Mb]}(r) \cdot \|\xi\|_\infty^2}{L},$$

for all $r \in \mathbb{R}$. Since $\left(4 \cdot \chi_{[-Mb,Mb]} \cdot \|\xi\|_\infty^2\right) L^{-1} \in L^1(\mathbb{R})$, by the Lebesgue Dominated Convergence Theorem, we have

$$\int_{\mathbb{R}} \left| \frac{\xi(r/\beta_i(x_n))}{\beta_i(x_n)^{1/2}} - \frac{\xi(r/\beta_i(x))}{\beta_i(x)^{1/2}} \right|^2 dr \to 0.$$

That is, $\|u_{i,x_n}(\xi) - u_{i,x}(\xi)\| \to 0$. Thus $\|u_{i,x_n}(\xi) - u_{i,x}(\xi)\| \to 0$ for all $\xi \in C_c(\mathbb{R})$.

Now let $\xi \in L^2(\mathbb{R})$, and let $\epsilon > 0$. Choose $\eta \in C_c(\mathbb{R})$ such that $\|\eta - \xi\| < \epsilon/3$. Let $N \geq 1$ be an integer such that $n \geq N$ implies that $\|u_{i,x_n}(\eta) - u_{i,x}(\eta)\| < \epsilon/3$. Then for all $n \geq N$, we have

$$\begin{aligned} &\|u_{i,x_n}(\xi) - u_{i,x}(\xi)\| \\ &\qquad \leq \|u_{i,x_n}(\xi) - u_{i,x_n}(\eta)\| + \|u_{i,x_n}(\eta) - u_{i,x}(\eta)\| + \|u_{i,x}(\eta) - u_{i,x}(\xi)\| \\ &\qquad < \|\xi - \eta\| + \epsilon/3 + \|\xi - \eta\| = \epsilon. \end{aligned}$$

Thus $u_{i,x_n} \to u_{i,x}$ in the strong operator topology. Since the strong and *-strong operator topologies agree on the set of all unitaries in $B(L^2(\mathbb{R}))$, we have $u_{i,x_n}^* \to u_{i,x}^*$ in the strong operator topology as well. This proves Part 4.

Now we show part 5. Fix $i \in \{1, \ldots, N\}$ and fix a nonempty closed subset $F \subseteq X_i$. Note that for all $x \in F$, we have $u_x^* p_i^x u_x = p_I$. Now define $\psi\colon C(F, K(L^2(\mathbb{R}))) \to C(F, K(L^2(\mathbb{R})))$ by $\psi(f)(x) = u_x^* f(x) u_x$. Continuity of $\psi(f)$ follows from the previous three paragraphs and Lemma III.6.2. It is clear that ψ is a *-isomorphism. We claim that $\psi(K_{i,F}) = C(F, p_I K(L^2(\mathbb{R})) p_I)$. Let $f \in K_{i,F}$. Then

$$\psi(f)(x) = u_x^* f(x) u_x = u_x^* p_i^x f(x) p_i^x u_x = p u_x^* f(x) u_x p \in p_I K(L^2(\mathbb{R})) p_I$$

for all $x \in F$. Thus $\psi(K_i, F) \subseteq C(F, p_I K(L^2(\mathbb{R})) p_I)$. Now let $f \in C(F, p_I K(L^2(\mathbb{R})) p_I)$. Then for all $x \in F$, we have $f(x) = p f(x) p = u_x^* p_i^x u_x f(x) u_x^* p_i^x u_x$. Define $g\colon F \to K(L^2(\mathbb{R}))$ by $g(x) = p_i^x u_x f(x) u_x^* p_i^x$. Then $g \in K_{i,F}$ (continuity follows from the fact that if $x_n \to x$ in F, then $p_i^{x_n} \to p_i^x$ in the strong operator topology), and $\psi(g) = f$. Thus $\psi(K_{i,F}) = C(F, p_I K(L^2(\mathbb{R})) p_I)$.

Since for all $f \in K_{i,F}$ and all $x \in F$, we have $\Phi_{i,F}(f)(x) = \Omega([\psi(f)](x))$, it is clear that $\Phi_{i,F}$ is a well defined *-homomorphism. It is also clear that $\Phi_{i,F}$ is invertible. □

Notation III.6.4. For the rest of the chapter, let $\Phi_{i,F}$ be the *-isomorphism from Lemma III.6.3. Use Φ_i to denote Φ_{i,X_i} for each $i \in \{1, \ldots, N\}$, and use $\Phi^{(k)}$ to denote $\Phi_{k+1,F^{(k)}}$ for all k with $1 \leq k \leq N-1$.

Lemma III.6.5. *For each $k \in \{1, \ldots, N-1\}$, if $G^{(k)} \neq \varnothing$, define $R_k\colon C_0(G_{k+1}) \to C_0(G^{(k)})$ by $R_k(f) = f|_{G^{(k)}}$; if $G^{(k)} = \varnothing$, let $R_k\colon C_0(G_{k+1}) \to C_0(G^{(k)})$ be the zero map. Then for each $k \in \{1, \ldots, N-1\}$, the map R_k is a norm decreasing surjective *-homomorphism.*

Proof: Fix $k \in \{1, \ldots, N-1\}$. Since $G^{(k)}$ is closed in G_{k+1}, the map R_k is a well defined surjective linear map.

Let $f, g \in C_0(G_{k+1})$. Note that if $(r, x) \in G^{(k)}$, then $(t, x), (r-t, (-t)x), (-r, (-r)x) \in G^{(k)}$ for all $t \in (-\beta_i(x), -\alpha_i(x))$. Then for all $(r, x) \in G^{(k)}$, we have

$$\begin{aligned} R_k(f * g)(r, x) = (f * g)(r, x) &= \int_{-\beta_i(x)}^{-\alpha_i(x)} f(t, x) g(r - t, (-t)x) \, dt \\ &= \int_{-\beta_i(x)}^{-\alpha_i(x)} R_k(f)(t, x) R_k(g)(r - t, (-t)x) \, dt \\ &= (R_k(f) * R_k(g))(r, x); \end{aligned}$$

and

$$R_k(f^*)(r, x) = f^*(r, x) = \overline{f(-r, (-r)x)} = \overline{R_k(f)(-r, (-r)x)} = R_k(f)^*(r, x).$$

Thus R_k is a *-homomorphism.

Let $f \in C_0(G_{k+1})$. Then for each $x \in F^{(k)}$, we have $\lambda^{(k),x}(R_k(f)) = \lambda^x_{k+1}(f)$. Thus

$$\begin{aligned} \|R_k(x)\|^{(k)} &= \sup_{x \in F^{(k)}} \|\lambda^{(k),x}(R_k(f))\| \\ &= \sup_{x \in F^{(k)}} \|\lambda_{k+1}(f)\| \\ &\leq \sup_{x \in X_i} \|\lambda_{k+1}(f)\| = \|f\|_{k+1}. \end{aligned}$$

So R_k is norm-decreasing. □

Lemma III.6.6. *Let $k \in \{1, \ldots, N-1\}$. For each $\epsilon > 0$, and for each $f \in C_0(G^{(k)})$ with $\|f\|^{(k)} < \epsilon$, there exists $g \in C_0(G_{k+1})$ such that $\|g\|_{k+1} \leq \epsilon$ and $R_k(g) = f$, where R_k is the map defined in Lemma III.6.6.*

Proof: Fix $k \in \{1, \ldots, N-1\}$. First note that for all $f \in C_0(G_{k+1})$ we have $\phi^{(k)}(R_k(f)) = \phi_{k+1}(f)|_{F^{(k)}}$.

Let $\epsilon > 0$, and let $f \in C_0(G^{(k)})$. Extend f to $f' \in C_0(G_{k+1})$. Let

$$U = \{x \in X_i \colon \|\phi_{k+1}(f')(x)\| < \epsilon\}.$$

Then U is an open set in X_i. If $x \in F^{(k)}$, then

$$\|\phi_{k+1}(f')(x)\| = \|\phi^{(k)}(R_k(f'))(x)\| = \|\phi^{(k)}(f)(x)\| \leq \|\phi^{(k)}(f)\| = \|f\|^{(k)} < \epsilon.$$

Thus $F^{(k)} \subseteq U$.

Let $h \in C_c(X_i)$ satisfy $0 \leq h \leq 1$, supp $h \subseteq U$, and $h|_{F^{(k)}} = 1$. Define $h' \in C(\overline{G_{k+1}})$ by $h'(r, y) = h(\pi_{k+1}(y))$. Then $g = h'f' \in C_0(G_{k+1})$. Note that $\phi_{k+1}(g) = h\phi_{k+1}(f')$. Now, if $x \in X_i \setminus U$, then $\phi_{k+1}(g)(x) = h(x)\phi_{k+1}(f')(x) = 0$; if $x \in U$, then

$$\|\phi_{k+1}(g)(x)\| = \|h(x)\phi_{k+1}(f')(x)\| = \|\phi_{k+1}(f')(x)\| < \epsilon.$$

Thus $\|g\|_{k+1} = \|\phi_{k+1}(g)\| \leq \epsilon$. Also,

$$R_k(g)(r, x) = h'(r, x)f'(r, x) = h(\pi_{k+1}(x))f(r, x) = f(r, x).$$

So $R_k(g) = f$. □

Lemma III.6.7. *For each $i \in \{1, \ldots, N\}$, define $Q_i\colon C_0(G_Z) \to C_0(G_i)$ by $Q_i(f) = f|_{G_i \cap G_Z}$. Then Q_i is a norm decreasing *-homomorphism for each $i \in \{1, \ldots, N\}$.*

Proof: We first show that Q_i is a *-homomorphism. Let $i \in \{1, \ldots, N\}$.

By Lemma III.4.6, the set $G_i \cap G_Z$ is closed in G_Z. Thus we see that $Q_i(f) \in C_0(G_i \cap G_Z)$ for all $f \in C_0(G_Z)$. Since $G_i \cap G_Z$ is open in G_i, we see that $Q_i(f) \in C_0(G_i)$. So Q_i is well defined. Linearity of Q_i is clear.

Let $f, g \in C_0(G_Z)$. Note that if $(r, x) \in G_Z \cap G_i$, then $(\alpha(x), \beta(x)) \subseteq (\alpha_i(x), \beta_i(x))$, and so for all $t \in (-\beta(x), -\alpha(x))$, we have $(t, x) \in G_Z \cap G_i$, $(r - t, (-t)x) \in G_i \cap G_Z$, and $(-r, (-r)x) \in G_i \cap G_Z$. Thus for all $(r, x) \in G_Z \cap G_i$ and all $t \in (-\beta(x), -\alpha(x))$, we have $Q_i(f)(t, x) = f(t, x)$ and $Q_i(g)(r - t, (-t)x) = g(r - t, (-t)x)$. Then for every (r, x) contained in

$G_Z \cap G_i$, we have

$$\begin{aligned} Q_i(f*g)(r,x) &= (f*g)(r,x) = \int_{\mathbb{R}} f(t,x)g(r-t,(-t)x)\,dt \\ &= \int_{-\beta(x)}^{-\alpha(x)} f(t,x)g(r-t,(-t)x)\,dt \\ &= \int_{-\beta(x)}^{-\alpha(x)} Q_i(f)(t,x)Q_i(g)(r-t,(-t)x)\,dt \\ &= \int_{-\beta_i(x)}^{-\alpha_i(x)} Q_i(f)(t,x)Q_i(g)(r-t,(-t)x)\,dt \\ &= (Q_i(f)*Q_i(g))(r,x). \end{aligned}$$

Also, for all $(r,x) \in G_i \cap G_Z$, we have

$$Q_i(f^*)(r,x) = f^*(r,x) = \overline{f(-r,(-r)x)} = \overline{Q_i(f)(-r,(-r)x)} = Q_i(f)^*(r,x).$$

Now we consider what happens if $(r,z) \in G_i \setminus (G_Z \cap G_i)$. Suppose that

$$(Q_i(f)*Q_i(g))(r,x) \neq 0$$

for some $(r,x) \in G_i$. Then for some $t \in (-\beta_i(x), -\alpha_i(x))$, we have $(t,x) \in G_i \cap G_Z$ and (r-t, (-t)x) $\in G_i \cap G_Z$. Thus, by the first statement in part 2 of Lemma III.4.4, we have $(r,x) = (r-t,(-t)x)(t,x) \in G_i \cap G_Z$. So if $(r,x) \in G_i \setminus G_Z$, then $(Q_i(f)*Q_i(g))(r,x) = 0$; and clearly $Q_i(f*g)(r,x) = 0$ for all $(r,x) \in G_i \setminus G_Z$ as well. Thus for all $(r,x) \in G_i$, we have $Q_i(f*g)(r,x) = (Q_i(f)*Q_i(g))(r,x)$. Also, if $(r,x) \notin G_i \cap G_Z$, then $(-r,(-r)x) = (r,x)^{-1} \notin G_Z \cap G_i$. So $(r,x) \notin G_i \cap G_Z$ implies that $Q_i(f^*)(r,x) = 0 = Q_i(f)^*(r,x)$. Thus Q_i is a *-homomorphism.

Now we prove that Q_i is norm-decreasing. Let $x \in X_i$, let $r \in \mathbb{R}$, and let $t \in \mathbb{R}$. If $r \notin (\alpha_i(x), \beta_i(x))$ or $t \notin (\alpha_i(x), \beta_i(x))$, then

$$\chi_i^x(t)\chi_i^x(r)f(r-t,rx) = 0 = \chi_i^x(t)\chi_i^x(r)Q_i(f)(r-t,rx).$$

If $r,t \in (\alpha_i(x), \beta_i(x))$, then $(r-t,rt) \in G_i$, and then $Q_i(f)(r-t,rx) = f(r-t,rx)$. Thus for each

$x \in X_i$, each $f \in C_0(G_Z)$, each $\xi \in L^2(\mathbb{R})$, and each $r \in \mathbb{R}$, we have

$$\begin{aligned}
\lambda_i^x(Q_i(f))(\xi)(r) &= \int_{-\beta_i(x)}^{-\alpha_i(x)} \chi_i^x(r)\chi_i^x(t)\xi(t)Q_i(f)(r-t, rx)\,dt \\
&= \int_{-\beta_i(x)}^{-\alpha_i(x)} \chi_i^x(r)\chi_i^x(t)\xi(t)f(r-t, rx)\,dt \\
&= \chi_i^x(r)\int_{-\beta_i(x)}^{-\alpha_i(x)} \chi_i^x(t)\xi(t)f(r-t, rx)\,dt \\
&= \chi_i^x(r)\int_{-\beta_i(x)}^{-\alpha_i(x)} p_i^x(\xi)(t)f(r-t, rx)\,dt \\
&= \chi_i^x(r)(\lambda_x(f)(p_i^x(\xi)))(r) \\
&= (p_i^x\lambda_x(f)p_i^x)(\xi)(r).
\end{aligned}$$

Then for each $x \in X_i$, we have $\|\lambda_i^x(Q_i(f))\| = \|p_i^x\lambda_x(f)p_i^x\| \leq \|\lambda_x(f)\|$. Thus $\|Q_i(f)\|_i \leq \|f\|_r$. So Q_i is norm-decreasing. $\square$

Lemma III.6.8. *Let H be a Hilbert space. For each $n \in \mathbb{Z}$, let $p_n \in B(H)$ be a projection. Suppose that $p_m p_n = 0$ for all $m \neq n$, and that $\sum_{n\in\mathbb{Z}} p_n$ converges to 1 in the strong operator topology. Let $a \in B(H)$ satisfy $p_n a p_n = a p_n$ for all $n \in \mathbb{Z}$. Then $\|a\| = \sup_{n\in\mathbb{Z}} \|p_n a p_n\|$.*

Proof: We first show that $\sum_{n\in Z} p_n a p_n$ converges to a in the strong operator topology. Let $\xi \in H$. Then $\lim_{k\to\infty}\sum_{n=-k}^{k} p_n(\xi) = \xi$, so $\lim_{k\to\infty} a(\sum_{n=-k}^{k} p_n(\xi)) = a(\xi)$. Thus

$$\begin{aligned}
\lim_{k\to\infty}\sum_{n=-k}^{k} p_n a p_n(\xi) &= \lim_{k\to\infty}\sum_{n=-k}^{k} a p_n(\xi) \\
&= \lim_{k\to\infty} a\left(\sum_{n=-k}^{k} p_n(\xi)\right) = a(\xi).
\end{aligned}$$

So $\sum_{n\in\mathbb{Z}} p_n a p_n$ converges to a in the strong operator topology.

Now, let $\xi \in H$. For each $k \geq 1$, let $\xi_k = \sum_{n=-k}^{k} p_n(\xi)$. Then by assumption, $\xi_k \to \xi$. For each $k \geq 1$, we have

$$\begin{aligned}
\langle \xi_k, \xi_k \rangle = \left\langle \sum_{n=-k}^{k} p_n(\xi), \sum_{m=-k}^{k} p_m(\xi) \right\rangle &= \sum_{m,n=-k}^{k} \langle p_m(\xi), p_n(\xi) \rangle \\
= \sum_{n=-k}^{n} \langle p_n(\xi), p_n(\xi) \rangle &= \sum_{n=-k}^{k} \|p_n(\xi)\|^2.
\end{aligned}$$

Since $\langle \xi_k, \xi_k \rangle \to \|\xi\|^2$, we see that $\|\xi\|^2 = \sum_{n\in\mathbb{Z}} \|p_n(\xi)\|^2$. Thus for all $\xi \in H$, we have $\|\xi\|^2 = \sum_{n\in\mathbb{Z}} \|p_n(\xi)\|^2$.

For each $k \geq 1$, let $a_k = \sum_{n=-k}^{k} p_n a p_n$. Then we have shown that $a_k \to a$ in the strong operator topology. Let $R = \sup_{n\in\mathbb{Z}} \|p_n a p_n\|$. For each $n \in \mathbb{Z}$, we have $\|p_n a p_n\| \leq \|a\|$, so $R \leq \|a\|$. Now for each $k \geq 1$ and each $\xi \in H$, we have

$$\begin{aligned}
\|a_k(\xi)\|^2 &= \sum_{n\in\mathbb{Z}} \|p_n(a_k(\xi))\|^2 \\
&= \sum_{n\in\mathbb{Z}} \left\| p_n \left(\sum_{m=-k}^{k} p_m a p_m(\xi) \right) \right\|^2 \\
&= \sum_{n\in\mathbb{Z}} \left\| \sum_{m=-k}^{k} p_n p_m a p_m(\xi) \right\|^2 \\
&= \sum_{n=-k}^{k} \|p_n a p_n(\xi)\|^2 \\
&\leq \sum_{n=-k}^{k} \|p_n a p_n\|^2 \|p_n(\xi)\|^2 \\
&\leq R^2 \sum_{n=-k}^{n} \|p_n(\xi)\|^2 \\
&\leq R^2 \|\xi\|^2.
\end{aligned}$$

Thus for each $k \geq 1$, $\|a_k\| \leq R$. Let $B = \{b \in B(H) \colon \|b\| \leq R\}$. Now, $a_k \in B$ for all k, and $a_k \to a$ in the strong operator topology. Since B is closed in the strong operator topology, we have $a \in B$, and so $\|a\| \leq R$. □

Notation III.6.9. Recall from III.3.6 that for each $x \in X$, the set $T^x = \{r \in \mathbb{R} \colon rx \in Z\}$ is indexed by $\mathbb{Z}$ in the increasing order:

$$T^x = \{\cdots < a_{-n}^x < a_{-n+1}^x < \cdots a_{-1}^x < a_0 < a_1^x < \cdots a_n^x < \cdots\}.$$

For each $x \in X$ and each $n \in \mathbb{Z}$, define a projection $q_n^x \in B(L^2(\mathbb{R}))$ by $q_n^x(\xi) = \chi_{(a_n^x, a_{n+1}^x)}\xi$.

Proposition III.6.10. *1. Let $r, t \in \mathbb{R}$, and let $x \in X$. Suppose that $(r - t, rx) \in G_Z$. Then for all $n \in \mathbb{Z}$, we have $r \in (a_n^x, a_{n+1}^x)$ if and only if $t \in (a_n^x, a_{n+1}^x)$, where a_n^x is as defined in III.6.9.*

2. *For all $x \in X$, and for all $n \neq m$, we have $q_m^x q_n^x = 0$; and $\sum_{n\in\mathbb{Z}} q_n^x$ converges to 1 in strong operator topology, where q_n^x is as defined in III.6.9.*

3. *For all $f \in C_0(G_Z)$, all $x \in X$, and all $n \in \mathbb{Z}$, we have $q_n^x \lambda_x(f) q_n^x = \lambda_x(f) q_n^x$.*

4. *For all $f \in C_0(G_Z)$ and all $x \in X$, we have $\|\lambda_x(f)\| = \sup_{n\in\mathbb{Z}} \|q_n^x \lambda_x(f) q_n^x\|$, where λ_x is as defined by Equation (I.4).*

Proof: Part 1: Suppose that $r \in (a_n^x, a_{n+1}^x)$. Then $\beta(rx) = a_{n+1}^x - r$, and $\alpha(rx) = a_n^x - r$. Since $(r-t, rx) \in G_Z$, we see that $t - r \in (\alpha(rx), \beta(rx)) = (a_n^x - r, a_{n+1}^x - r)$. Thus $t \in (a_n^x, a_{n+1}^x)$. Thus $r \in (a_n^x, a_{n+1}^x)$ implies that $t \in (a_n^x, a_{n+1}^x)$. Now suppose that $r \notin (a_n^x, a_{n+1}^x)$. Then $r \in (a_m^x, a_{m+1}^x)$ for some $m \neq n$, whence $t \in (a_m^x, a_{m+1}^x)$, and so $t \notin (a_n^x, a_{n+1}^x)$.

Part 2: It is clear that $q_n^x q_m^x = 0$ if $m \neq n$. For each $k \geq 1$, let $q_k = \sum_{n=-k}^{k} q_n^x$. Then q_k is an increasing sequence of projections, hence converges in the strong operator topology to some projection q (Theorem 4.1.2 in [6]). It is clear that $q_k q = q_k$ for all $k \geq 1$. Suppose that $q(\xi) = 0$ for some ξ. Then $q_k(\xi) = q_k q(\xi) = 0$ for all $k \geq 1$. So $\chi_{(a_{-k}^x, a_k^x)}\xi = 0$ for all $k \geq 1$. That is $\int_{a_{-k}^x}^{a_k^x} |\xi|^2 = 0$ for all k. So $\xi = 0$. Thus $q = 1$.

Part 3: Fix $f \in C_0(G_Z)$, $x \in X$, and $n \in \mathbb{Z}$. Let $\chi_n \colon \mathbb{R} \to \mathbb{R}$ denote the characteristic function of (a_n^x, a_{n+1}^x). Now, if $r \in (a_n^x, a_{n+1}^x)$, then

$$\chi_n(r)\chi_n(t) f(r-t, rx) = \chi_n(t) f(r-t, rx)$$

for all $t \in \mathbb{R}$. If $r \notin (a_n^x, a_{n+1}^x)$, then

$$\chi_n(r)\chi_n(t) f(r-t, rx) = 0.$$

If $t \in (a_n^x, a_{n+1}^x)$, then by part 1, we have $(r-t, rx) \notin G_Z$, and so $f(r-t, rx) = 0$; then $\chi_n(r)\chi_n(t) f(r-t, rx) = 0 = \chi_n(t) f(r-t, rx)$. If $t \notin (a_n^x, a_{n+1}^x)$, then

$$\chi_n(r)\chi_n(t) f(r-t, rx) = 0 = \chi_n(t) f(r-t, rx)$$

also. Thus for all $r, t \in \mathbb{R}$, we have $\chi_n(r)\chi_n(t) f(r-t, rx) = \chi_n(t) f(r-t, rx)$. Then for all $r \in \mathbb{R}$

we have

$$\begin{aligned}
\lambda_x(f)q_n^x(\xi)(r) &= \int_{\mathbb{R}} q_n^x(\xi)(t)f(r-t,rx)\,dt \\
&= \int_{\mathbb{R}} \chi_n(t)\xi(t)f(r-t,rx)\,dt \\
&= \int_{\mathbb{R}} \chi_n(r)\chi_n(t)\xi(t)f(r-t,rx)\,dt \\
&= \chi_n(r)\int_{\mathbb{R}} \chi_n(t)\xi(t)f(r-t,rx)\,dt \\
&= \chi_n(r)\int_{\mathbb{R}} q_n^x(\xi)(t)f(r-t,rx)\,dt \\
&= \chi_n(r)\lambda_x(f)q_n^x(\xi)(r) \\
&= q_n^x\lambda(f)q_n^x(\xi)(r).
\end{aligned}$$

So $q_n^x\lambda_x(f)q_n^x = \lambda_x(f)q_n^x$.

Part 4: This follows from part 2 and 3, and Lemma III.6.8. □

Proposition III.6.11. *Let Q_i be the map defined in Lemma III.6.7. Define*

$$Q\colon C_0(G_Z) \to \bigoplus_{i=1}^{N} C_0(G_i)$$

*by $Q(f) = (Q_1(f), Q_2(f), \ldots, Q_N(f))$. Then Q is an isometric *-homomorphism.*

Proof: Since each Q_i is a *-homomorphism, so is Q.

Recall that $\|\cdot\|_r$ denotes that reduced norm on $C_c(\mathbb{R}\times X)$, which contains $C_0(G_Z)$ as a *-subalgebra. We now show that $\|Q(f)\| \geq \|f\|_r$. Let $f \in C_0(G_Z)$, let $x \in X$, and let $n \in \mathbb{Z}$. Let $r_0 \in (a_{n+1}^x, a_n^x)$. Then $r_0x \in V_i$ for some $i \in \{1,\ldots,N\}$. Let $c = \pi_i(r_0x) \in X_i$, let $s_0 = (\alpha_i(r_0x) + \beta_i(r_0x))/2$, and let $s = r_0 + s_0$. Then $c = (s_0 + r_0)x = sx$. Let $\chi_n\colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of (a_n^x, a_{n+1}^x). Define $\chi(t) = \chi_n(t+s)$. We first show that $\chi\chi_i^c = \chi$. Let $t \in \mathbb{R}$. First suppose that $\chi(t) \neq 0$. Then $t + s \in (a_n^x, a_{n+1}^x)$, and

$$t + s_0 \in (a_n^x - r_0, a_{n+1}^x - r_0) = (\alpha(r_0x), \beta(r_0x)) \subseteq (\alpha_i(r_0x), \beta_i(r_0x)).$$

So $t \in (\alpha_i(r_0x) - s_0, \beta(r_0x) - s_0) = (\alpha_i(c), \beta_i(c))$. Thus $\chi_i^c(t) = 1$. So $\chi_i^c(t)\chi(t) = \chi(t)$. If $\chi(t) = 0$, then $\chi(t)\chi_i^c(t) = 0 = \chi(t)$. Thus $\chi\chi_i^c = \chi$.

Let $p \in B(L^2(\mathbb{R}))$ be the projection defined by $p(\xi) = \chi\xi$. Define $v\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $v(\xi)(r) = \xi(r+s)$. It is easily checked that v is a unitary with v^* defined by $v^*(\xi)(r) = \xi(r-s)$. Then for all $\xi \in L^2(\mathbb{R})$ and all $r \in \mathbb{R}$, we have

$$\begin{aligned}
[vq_n^x\lambda_x(f)q_n^x v^*](\xi)(r) &= [q_n^x\lambda_x(f)q_n^x v^*](\xi)(r+s) \\
&= \chi_n(r+s)\lambda_x(f)q_n^x v^*(\xi)(r+s) \\
&= \chi(r)\int_{\mathbb{R}} q_n^x(v^*(\xi))(t)f(r+s-t,(r+s)x)\,dt \\
&= \chi(r)\int_{\mathbb{R}} \chi_n(t)\xi(t-s)f(r+s-t,rc)\,dt \\
&= \chi(r)\int_{\mathbb{R}} \chi_n(t+s)\xi(t)f(r-t,rc)\,dt \\
&= \chi(r)\int_{\mathbb{R}} \chi(t)\xi(t)f(r-t,rc)\,dt \\
&= \chi(r)\int_{\mathbb{R}} \chi_i^c(r)\chi(t)\chi_i^c(t)\xi(t)f(r-t,rc)\,dt \\
&= \chi(r)\int_{\mathbb{R}} \chi_i^c(r)\chi(t)\chi_i^c(t)\xi(t)Q_i(f)(r-t,rc)\,dt \\
&= \chi(r)\int_{\mathbb{R}} \chi_i^c(r)\chi_i^c(t)p(\xi)(t)Q_i(f)(r-t,rc)\,dt \\
&= \chi(r)\lambda_i^c(Q_i(f))(p(\xi))(r) \\
&= (p\lambda_i^c(Q_i(f))p)(\xi))(r).
\end{aligned}$$

Thus $vq_n^x\lambda_x(f)q_n^x v^* = p\lambda_i^c(Q_i(f))p$, and hence

$$\begin{aligned}
\|q_n^x\lambda_x(f)q_n^x\| &= \|vq_n^x\lambda_x(f)q_n^x v^*\| = \|p\lambda_i^c(Q_i(f))p\| \\
&\leq \|\lambda_i^c(Q_i(f))\| \leq \|Q_i(f)\|_i \leq \|Q(f)\|.
\end{aligned}$$

This holds for all $n \in \mathbb{Z}$, so $\|\lambda_x(f)\| = \sup_{n\in\mathbb{Z}} \|q_n^x\lambda_x(f)q_n^x\| \leq \|Q(f)\|$. This holds for all $x \in X$, so $\|f\|_r = \sup_{x\in X}\|\lambda_x(f)\| \leq \|Q(f)\|$.

For $\|Q(f)\| \leq \|f\|_r$, we have shown in Lemma III.6.7 that $\|Q_i(f)\|_i \leq \|f\|_r$ for all $i \in \{1,\ldots,N\}$. So $\|Q(f)\| = \sup\{\|Q_i(f)\|_i \colon i = 1,\ldots,N\} \leq \|f\|_r$. Thus Q is isometric. □

At this point, we are almost ready to glue the *-algebras $C_0(G_i)$ together to form $C_0(G_Z)$. Before we do that, let us recall some of the notation that we have used in this chapter so far, and let us fix further notation for the rest of this chapter.

Notation III.6.12. For each $i \in \{1, \ldots, N\}$, the set $C_0(G_i)$ (G_i is defined in Notation III.4.1) is a *-algebra; A_i is the completion of $C_0(G_i)$ with respect to $\|\cdot\|_i$ ($\|\cdot\|_i$ is defined in Notation III.6.1);

$$K_i = \{f \in C(X_i, K(L^2(\mathbb{R}))) \colon p_i^x f(x) p_i^x = f(x) \text{ for all } x \in X_i\};$$

$\phi_i \colon C_0(G_i) \to K_i$ is an isometric *-homomorphism with dense range (ϕ_i is defined in Notation III.6.1); $A_i \cong K_i$ via the extension of ϕ_i; and $\Phi_i \colon K_i \to C(X_i, K(L^2(I)))$ is a *-isomorphism, where I is the interval $(-1, 1)$ (Φ_i is defined in Notation III.6.4).

For each $k \in \{1, \ldots, N-1\}$ the space $C_0(G^{(k)})$ is a *-algebra ($G^{(k)}$ is defined in III.4.1); $A^{(k)}$ is the completion of $C_0(G^{(k)}) = C_0(G_{k+1, F^{(k)}})$ with respect to $\|\cdot\|^{(k)}$ ($\|\cdot\|^{(k)}$ is defined in Notation III.6.1);

$$K^{(k)} = K_{k+1, F^{(k)}} = \{f \in C(F^{(k)}, K(L^2(\mathbb{R}))) \colon p_i^x f(x) p_i^x = f(x) \text{ for all } x \in F^{(k)}\};$$

$\phi^{(k)} \colon C_0(G^{(k)}) \to K^{(k)}$ is an isometric *-homomorphism with dense range ($\phi^{(k)}$ is defined in Notation III.6.1); $A^{(k)} \cong K^{(k)}$ via the extension of $\phi^{(k)}$; $\Phi^{(k)} \colon K^{(k)} \to C(F^{(k)}, K(L^2(I)))$ is a *-isomorphism ($\Phi^{(k)}$ is defined in Notation III.6.4); the restriction map $R_k \colon C_0(G_{k+1}) \to C_0(G^{(k)})$ is a norm-decreasing surjective *-homomorphism such that an element with small norm lifts to some element with small norm.

Let Q_i be the map defined in III.6.7, and let Q be the map defined in III.6.11. Then $Q_i \colon C_0(G_Z) \to C_0(G_i)$ is a norm-decreasing *-homomorphism, and $Q \colon C_0(G_Z) \to \bigoplus_{i=1}^N C_0(G_i)$ is an isometric *-homomorphism.

The next statement is used in the decompostion of $C_0(G_Z)$. The proof is easy and is omitted.

Lemma III.6.13. *Let X be any locally compact Hausdorff space, and let $F_1, \ldots, F_n$ be closed subsets of X such that $\bigcup_{i=1}^n F_i = X$. Let $f \colon X \to \mathbb{C}$ an arbitrary function. Also suppose that $f|_{F_i} \in C_0(F_i)$ for each $i \in \{1, \ldots, n\}$. Then $f \in C_0(X)$.*

Proposition III.6.14. *Let $E_1 = C_0(G_1)$. For each $k = 2, \ldots, N$, there exists a *-subalgebra $E_k \subseteq C_0(G_1) \oplus \cdots \oplus C_0(G_k)$ and a *-homomorphism $\psi_{k-1} \colon E_{k-1} \to C_0(G^{(k-1)})$ such that*

1. *ψ_{k-1} is norm decreasing.*

2. $E_k = E_{k-1} \oplus_{C_0(G^{(k-1)})} C_0(G_k) = \{(e,f) \in E_{k-1} \oplus C_0(G_k) : \psi_{k-1}(e) = R_{k-1}(f)\}$.

3. *If $(f_1, \ldots, f_k) \in E_k$, then for all $i \in \{1, \ldots, k\}$, we have $f_i \in C_0(G_i \cap G_Z)$. (We treat $C_0(G_i \cap G_Z)$ as a subspace of $C_0(G_i)$.)*

4. *If $(f_1, \ldots, f_k) \in E_k$, then for all $i, j \in \{1, \ldots, k\}$, we have $f_i|_{G_i \cap G_j \cap G_Z} = f_j|_{G_i \cap G_j \cap G_Z}$.*

5. *If $(f_1, \ldots, f_k) \in E_k$, then for all $j \in \{1, \ldots, k-1\}$, we have $(f_1, \ldots, f_j) \in E_j$.*

Proof: This is a proof by induction. We first simplify the base case of the induction by making the first algebra of the gluing process trivial. Fix some $x_0 \in X_1$. Let $F^{(0)} = \{x_0\}$ and let $G_0 = G^{(0)} = G_{1,F^{(0)}}$. It is clear that $G_0 = G^{(0)}$ is a closed subset of G_1. Then by Lemma III.5.3, we see that $C_0(G_0) = G_0(G^{(0)})$ is a *-algebra with the involution and convolution given by Equations III.7 and III.8. Let $R_0 : C_0(G_1) \to C_0(G^{(0)})$ be the restriction map. Then an argument identical to the one given in Lemma III.6.5 shows that R_0 is a norm decreasing surjective *-homomorphism.

Now, instead of proving the statement of this lemma, we prove the following instead, which is the same as the the statement of the lemma except that the index k ranges from 1 through n instead of 2 through n. The statement of this lemma follows immediately.

Let $E_0 = C_0(G_0)$. For each $k \in \{1, \ldots, N\}$, there exists a *-subalgebra

$$E_k \subseteq C_0(G_1) \oplus \cdots \oplus C_0(G_k)$$

and a *-homomorphism $\psi_{k-1} : E_{k-1} \to C_0(G^{(k-1)})$ such that

1. ψ_{k-1} is norm decreasing.

2. $E_k = E_{k-1} \oplus_{C_0(G^{(k-1)})} C_0(G_k) = \{(e,f) \in E_{k-1} \oplus C_0(G_k) : \psi_{k-1}(e) = R_{k-1}(f)\}$.

3. If $(f_0, \ldots, f_k) \in E_k$, then for all $i \in \{0, \ldots, k\}$, we have $f_i \in C_0(G_i \cap G_Z)$. (We treat $C_0(G_i \cap G_Z)$ as a subspace of $C_0(G_i)$.)

4. If $(f_0, \ldots, f_k) \in E_k$, then for all $i, j \in \{0, \ldots, k\}$, we have $f_i|_{G_i \cap G_j \cap G_Z} = f_j|_{G_i \cap G_j \cap G_Z}$.

5. If $(f_0, \ldots, f_k) \in E_k$, then for all $j \in \{0, \ldots, k-1\}$, we have $(f_0, \ldots, f_j) \in E_j$.

Induct on k. For the base case when $k = 1$, let $\psi_0 \colon E_0 \to C_0(G^{(0)})$ be the identity map and let $E_1 = \{(f, g) \in E_0 \oplus C_0(G_1) \colon \psi_0(f) = R_0(g)\}$. Then conditions 1 through 5 hold trivially. This proves the base case.

Inductive step: Suppose that for $1 < k < N$, there exist E_k and ψ_{k-1} that satisfy conditions 1 through 5 in the statement.

If $F^{(k)} = \varnothing$, then let $\psi_k = 0$, and let $E_{k+1} = E_k \oplus C_0(G_{k+1})$. Then condition 1, 2, 4, and 5 are clear; and condition 3 follows from Lemma III.4.9.

Now assume that $F^{(k)} \neq \varnothing$. Then $G^{(k)} \neq \varnothing$.

Define $\psi_k \colon E_k \to C_0(G^{(k)})$ by $\psi_k(f_0, \ldots, f_k)(w) = f_i(w)$ if $w \in G_i$ for some $i = 0, \ldots, k$, and 0 otherwise. We first show that for all $(f_0, \ldots, f_k) \in E_k$, $\psi_k(f_0, \ldots, f_k)$ is a well defined function. We only need to show that the definition does not depend on the choice of i. Let $(f_1, \ldots, f_k) \in E_k$, and suppose that $w \in G_i \cap G_j$. If $w \notin G_Z$, then $f_i(w) = 0 = f_j(w)$ by condition 3 in the inductive hypothesis. So suppose that $w \in G_Z$. Then $w \in G_i \cap G_j \cap G_Z$, and then $f_i(w) = f_j(w)$ by condition 4 in the inductive hypothesis. Thus $\psi_k(f_0, \ldots, f_k)$ is well defined.

Note that if $(r, x) \in G^{(k)} \setminus G_Z$, then for all $i = 0, \ldots, k$, we have $(r, x) \notin G_i \cap G_Z$; and then $\psi_k(f_0, \ldots, f_k)(r, x) = 0$ by condition 3 in the inductive hypothesis and by the definition of $\psi_k(f_0, \ldots, f_k)$.

Next we show that if $(f_0, \ldots, f_k) \in E_k$, then $\psi_k(f_0, \ldots, f_k) \in C_0(G_Z \cap G^{(k)})$. Now we know, by Lemma III.4.8, that

$$G^{(k)} \cap G_Z = \bigcup_{i=1}^{k} G_i \cap G^{(k)} \cap G_Z = \bigcup_{i=0}^{k} G_i \cap G^{(k)} \cap G_Z,$$

and by Lemma III.4.7, that $G_i \cap G^{(k)} \cap G_Z$ is closed in $G^{(k)} \cap G_Z$. From the definition of $\psi_k(f_0, \ldots, f_k)$, we see that

$$\psi_k(f_0, \ldots, f_k)|_{G_i \cap G_Z \cap G^{(k)}} = f_i|_{G_i \cap G_Z \cap G^{(k)}}.$$

Now $G_i \cap G_Z \cap G^{(k)}$ is closed in $G_i \cap G_Z$, by Lemma III.4.7. By condition 3 in the inductive hypothesis, each f_i is in $C_0(G_i \cap G_Z)$. So $f_i|_{G_i \cap G_Z \cap G^{(k)}} \in C_0(G_i \cap G_Z \cap G^{(k)})$. By Lemma III.6.13, we have $\psi_k(f_0, \ldots, f_k) \in C_0(G_Z \cap G^{(k)}) \subseteq C_0(G^{(k)})$. Thus ψ_k is a well defined map.

Next we show that ψ_k is a *-homomorphism. Linearity is clear. Also, ψ_k preserves the involution because $(r,x) \in G_i$ if and only if $(-r,(-r)x) \in G_i$ (by the first statement in part 2 of Lemma III.4.4). Let $(f_0,\dots,f_k),(g_0,\dots,g_k) \in E_k$. Let $h_f = \psi_k(f_0,\dots,f_k)$, let $h_g = \psi_k(g_0,\dots,g_k)$, and let $h = \psi_k(f_0 * g_0,\dots,f_k * g_k)$. We only need to show that $h = h_f * h_g$. Note that $h_g, h_f, h \in C_0(G_Z \cap G^{(k)})$. Let $(r,x) \in G^{(k)}$. If

$$(h_f * h_g)(r,x) = \int_{-\beta_{k+1}(x)}^{-\alpha_{k+1}(x)} h_f(t,x)h_g(r-t,(-t)x)\,dt \neq 0,$$

then for some $t \in (-\beta_{k+1}(x), -\alpha_{k+1}(x))$, we have $(t,x),(r-t,(-t)x) \in G_Z$. Then by the first statement in part 2 of Lemma III.4.4, we have $(r,x) \in G_Z$. Thus if $(r,x) \notin G_Z$, then $h(r,x) = 0 = (h_f * h_g)(r,x)$. Now suppose that $(r,x) \in G_Z$. Then by Lemma III.4.8, we have $(r,x) \in G_i \cap G^{(k)} \cap G_Z$ for some $i \in \{1,\dots,k\}$. So $h(r,x) = (f_i * g_i)(r,x)$. Also, we have

$$(h_f * h_g)(r,x) = \int_{-\beta_{k+1}(x)}^{-\alpha_{k+1}(x)} h_f(t,x)h_g(r-t,(-t)x)\,dt.$$

If $t \notin (-\beta(x), -\alpha(x))$, then $(t,x) \notin G_Z$, and then $h_f(t,x) = 0$. So we have

$$(h_f * h_g)(r,x) = \int_{-\beta(x)}^{-\alpha(x)} h_f(t,x)h_g(r-t,(-t)x)\,dt.$$

Now, $(r,x) \in G_i \cap G_Z \cap G^{(k)}$, so $x \in V_i \cap V_{k+1} \cap Z^c$. Then for all $t \in (-\alpha(x), -\beta(x))$, we have $t \in (-\beta_i(x), -\alpha_i(x))$, and $t \in (-\beta_{k+1}(x), -\alpha_{k+1}(x))$, since $\alpha_j(y) \leq \alpha(y) < 0 < \beta(y) \leq \beta_j(y)$ for all $j \in \{1,\dots,N\}$ and all $y \in Z^c \cap V_j$. Thus for all $t \in (-\beta(x), -\alpha(x))$, we have $(t,x) \in G_Z \cap G_i \cap G^{(k)}$. Then by Lemma III.4.4, $(r-t,(-t)x) \in G_Z \cap G_i \cap G^{(k)}$ for all $t \in (-\beta(x), -\alpha(x))$. Thus we have

$$(h_f * h_g)(r,x) = \int_{-\beta(x)}^{-\alpha(x)} f_i(t,x)g_i(r-t,(-t)x)\,dt.$$

Now, by condition 3 in the inductive hypothesis, f_i vanishes outside of $G_i \cap G_Z$. Then we have

$$(h_f * h_g)(r,x) = \int_{-\beta_i(x)}^{-\alpha_i(x)} f_i(t,x)g_i(r-t,(-t)x)\,dt = (f_i * g_i)(r,x) = h(r,x).$$

Therefore ψ_k preserves convolution, and so ψ_k is a *-homomorphism.

Next we show that ψ_k is norm decreasing. Let $(f_0, \dots, f_k)$ be an element of E_k, and let $h = \psi_k(f_0, \dots, f_k)$. Let $x \in F^{(k)}$. Note that there exist $m < n$ and $a_m^x, a_{m+1}^x, \dots, a_n^x \in \mathbb{R}$ such that

$$T^x \cap [\alpha_{k+1}(x), \beta_{k+1}(x)] = \{a_m^x, a_{m+1}^x, \dots, a_n^x\}$$

and $\alpha_{k+1}(x) = a_m^x < a_{m+1}^x < \cdots < a_n^x = \beta_{k+1}(x)$. For each $l = m, \dots, n-1$, let $\chi_l \colon \mathbb{R} \to \mathbb{R}$ be the characteristic function of (a_l^x, a_{l+1}^x), and let q_l be the projection in $B(L^2(\mathbb{R}))$ defined by $q_l(\xi) = \chi_l \xi$. It is clear that $q_l q_{l'} = 0$ if $l \neq l'$, and that $\sum_{l=m}^{n-1} q_l = p_{k+1}^x$. (Recall that p_{k+1}^x is the projection in $B(L^2(\mathbb{R}))$ defined by $p_{k+1}^x(\xi) = \chi_{k+1}^x \xi$.) Then it is clear that $\lambda^{(k),x}(h) = p_{k+1}^x \lambda^{(k),x}(h) p_{k+1}^p$. We claim that

$$\|\lambda^{(k),x}(h)\| = \sup\{\|q_l \lambda^{(k),x}(h) q_l\| \colon l = m, \dots, n-1\}.$$

Let $l \in \{m, \dots, n-1\}$. Let $r, t \in \mathbb{R}$. If $(r-t, rx) \notin G_Z$, then $h(r-t, rx) = 0$, and so $\chi_l(r)h(r-t, rx) = 0 = \chi_l(t)h(r-t, rx)$. Suppose that $(r-t, rx) \in G_Z$. By Proposition III.6.10 part 1, we have $r \in (a_l^x, a_{l+1}^x)$ if and only if $t \in (a_l^x, a_{l+1}^x)$. Therefore $\chi_l(t) = 1$ if and only if $\chi_l(r) = 1$, and $\chi_l(t)h(r-t, rx) = \chi_l(r)h(r-t, rx)$. Thus $\chi_l(r)h(r-t, rx) = \chi_l(t)h(r-t, rx)$ for all $r, t \in \mathbb{R}$. Then for all $\xi \in L^2(\mathbb{R})$ and all $r \in \mathbb{R}$, we have

$$\begin{aligned}
\lambda^{(k),x}(h) q_l(\xi)(r) &= \int_{\alpha_{k+1}(x)}^{\beta_{k+1}(x)} \chi_{k+1}^x(r) \chi_{k+1}^x(t) \chi_l(t) \xi(t) h(r-t, rx)\, dt \\
&= \int_{\alpha_{k+1}(x)}^{\beta_{k+1}(x)} \chi_{k+1}^x(r) \chi_{k+1}^x(t) \chi_l(r) \xi(t) h(r-t, rx)\, dt \\
&= \chi_l(r) \int_{\alpha_{k+1}(x)}^{\beta_{k+1}(x)} \chi_{k+1}^x(r) \chi_{k+1}^x(t) \xi(t) h(r-t, rx)\, dt \\
&= \chi_l(r) \lambda^{(k),x}(h)(\xi)(r).
\end{aligned}$$

Thus $\lambda^{(k),x}(h) q_l = q_l \lambda^{(k),x}(h)$ for all $l \in \{m, \dots, n-1\}$. Then it is clear that

$$\|\lambda^{(k),x}(h)\| = \|p_{k+1}^x \lambda^{(k),x}(h) p_{k+1}^x\| = \sup\{\|q_l \lambda^{(k),x}(h) q_l\| \colon l = m, \dots, n-1\}.$$

Now we show that for each $l \in \{m, \dots, n-1\}$, we have $\|q_l \lambda^{(k),x}(h) q_l\| \leq \|(f_0, f_1, \dots, f_k)\|$. Let $l \in \{m, \dots, n-1\}$. Since $x \in F^{(k)}$, there exists $x_0 \in V_{k+1} \cap \left(\bigcup_{i=1}^k V_i\right)$ such that $\pi_{k+1}(x_0) = x$. Let $r_0 \in (a_l^x, a_{l+1}^x)$. Then $r_0 x \in Z^c \cap V_{k+1}^x = Z^c \cap V_{k+1}^{x_0}$. Thus by Lemma III.3.8, there exists some i with $1 \leq i \leq k$ such that $r_0 x \in Z^c \cap V_i$. Let $s_0 = (\alpha_i(r_0 x) + \beta_i(r_0 x))/2$, let $c = (s_0 + r_0)x$, and let $s = r_0 + s_0$. Then c belongs to X_i. We claim that for every real number r, we have $\chi_l(r+s)\chi_i^c(r) = \chi_l(r+s)$.

Let $r \in \mathbb{R}$. If $\chi_l(r+s) = 0$, then we are done. Suppose that $\chi_l(r+s) \neq 0$. Then $r+s \in (a_l^x, a_{l+1}^x)$, and then

$$r + s_0 \in (a_l^x - r_0, a_{l+1}^x - r_0) = (\alpha(r_0x), \beta(r_0x)) \subseteq (\alpha_i(r_0x), \beta_i(r_0x)).$$

Because $s_0 \in (\alpha_i(r_0x), \beta_i(r_0x))$, we have

$$r \in (\alpha_i(r_0x) - s_0, \beta_i(r_0x) - s_0) = (\alpha_i(c), \beta_i(c)).$$

So $\chi_i^c(r) = 1$, and so $\chi_l(r+s)\chi_i^c(r) = \chi_l(r+s)$.

Define $u\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $u(\xi)(r) = \xi(r+s)$. Then u is a unitary with u^* given by $u^*(\xi)(r) = \xi(r-s)$. For all $\xi \in L^2(\mathbb{R})$, and for all $r \in \mathbb{R}$, we have

$$\begin{aligned}
&[uq_l\lambda^{(k),x}(h)q_lu^*](\xi)(r)\\
&\quad= [q_l\lambda^{(k),x}(h)q_lu^*](\xi)(r+s)\\
&\quad= \chi_l(r+s)[\lambda^{(k),x}(h)q_lu^*(\xi)](r+s)\\
&\quad= \chi_l(r+s)\int_{\mathbb{R}} \chi_{k+1}^x(r+s)\chi_{k+1}^x(t)q_l(u^*(\xi))(t)h(r+s-t,(r+s)x)\,dt\\
&\quad= \chi_l(r+s)\int_{\mathbb{R}} \chi_{k+1}^x(r+s)\chi_{k+1}^x(t)\chi_l(t)\xi(t-s)h(r+s-t,(r+s)x)\,dt\\
&\quad= \chi_l(r+s)\int_{\mathbb{R}} \chi_{k+1}^x(r+s)\chi_{k+1}^x(t+s)\chi_l(t+s)\xi(t)h(r-t,rc)\,dt\\
&\quad= \chi_l(r+s)\int_{\mathbb{R}} \chi_l(t+s)\xi(t)h(r-t,rc)\,dt\\
&\quad= \chi_l(r+s)\int_{\mathbb{R}} \chi_i^c(r)\chi_i^c(t)\chi_l(t+s)\xi(t)h(r-t,rc)\,dt.
\end{aligned}$$

Now for all $r, t \in (\alpha_i(c), \beta_i(c))$, we have $(r-t, rc) \in G_i$, so $h(r-t, rc) = f_i(r-t, rc)$ for all $r, t \in (\alpha_i(c), \beta_i(c))$. Then, letting p be the projection in $B(L^2(\mathbb{R}))$ given by $p(\xi)(r') = \chi_l(r'+s)\xi(r')$, we have

$$[uq_l\lambda^{(k),x}(h)q_lu^*](\xi)(r) = \chi_l(r+s)\int_{\mathbb{R}} \chi_i^c(r)\chi_i^c(t)p(\xi)(t)f_i(r-t,rc)\,dt$$
$$= \chi_l(r+s)\lambda_i^c(f_i)(p(\xi))(r)$$
$$= [p\lambda_i^c(f_i)p](\xi)(r).$$

Thus $uq_l\lambda^{(k),x}(h)q_lu^* = p\lambda_i^c(f_i)p$. Then

$$\|q_l\lambda^{(k),x}(h)q_l\| = \|uq_l\lambda^{(k),x}(h)q_lu^*\| = \|p\lambda_i^c(f_i)p\|$$
$$\leq \|\lambda_i^c(f_i)\| \leq \|f_i\|_i \leq \|(f_0,\dots f_k)\|.$$

Thus $\|\lambda^{(k),x}(h)\| \leq \|(f_0,\dots,f_k)\|$ for all $x \in F^{(k)}$, and so

$$\|\psi_k(f_0,\dots,f_k)\|^{(k)} = \|h\|^{(k)} \leq \|(f_0,\dots,f_k)\|.$$

So ψ_k is norm-decreasing.

Now, let

$$E_{k+1} = E_k \oplus_{C_0(G^{(k)})} C_0(G_{k+1}) = \{(e,f) \in E_k \oplus C_0(G_{k+1})\colon \psi_k(e) = R_k(f)\}.$$

Condition 5 is clear.

Now let $(f_0,\dots,f_{k+1}) \in E_{k+1}$. By condition 5 and inductive hypothesis (condition 3), $f_i \in C_0(G_i \cap G_Z)$ for all $i = 0,\dots,k$. To show that $f_{k+1} \in C_0(G_{k+1} \cap G_Z)$, we only need to show that f_{k+1} vanishes outside G_Z, since $f_{k+1} \in C_0(G_{k+1})$ and $G_Z \cap G_{k+1}$ is open in G_{k+1}. Let $w \in G_{k+1} \setminus G_Z$. Then by Lemma III.4.9, $w \in G^{(k)}$, and

$$f_{k+1}(w) = R_k(f_{k+1})(w) = \psi_k(f_0,\dots,f_k)(w).$$

If $w \notin G_i$ for all $i = 0,\dots,k$, then $\psi_k(f_1,\dots,f_k)(w) = 0$ by the definition of ψ_k. Suppose that $w \in G_i$ for some $i \in \{0,\dots,k\}$. Then $\psi_k(f_0,\dots,f_k)(w) = f_i(w)$. But $f_i \in C_0(G_Z \cap G_i)$, so $f_i(w) = 0$. Thus f_{k+1} vanishes outside of G_Z, and so $f_{k+1} \in C_0(G_Z \cap G_{k+1})$. So condition 3 holds.

Now we show that condition 4 holds. Let $(f_0, \ldots, f_k, f_{k+1})$ be an element of E_{k+1}, and let $i, j \in \{0, \ldots, k+1\}$. Without loss of generality, assume that $i < j$. If $j < k+1$, then by condition 4 in the inductive hypothesis and condition 5, $f_i|_{G_Z \cap G_i \cap G_j} = f_j|_{G_Z \cap G_i \cap G_j}$. So assume that $j = k+1$. Let $w \in G_Z \cap G_i \cap G_{k+1}$. By Lemma III.4.7, if $i \geq 1$, then we have $G_Z \cap G_i \cap G_{k+1} = G_Z \cap G_i \cap G^{(k)}$. Also,

$$G_Z \cap G_0 \cap G_{k+1} = G_Z \cap G_0 \cap G_1 \cap G_{k+1} = G_Z \cap G_0 \cap G_1 \cap G^{(k)} = G_Z \cap G_0 \cap G^{(k)}.$$

Then

$$f_{k+1}(w) = R_k(f_{k+1})(w) = \psi_k(f_1, \ldots, f_k)(w) = f_i(w).$$

So $f_i|_{G_Z \cap G_i \cap G_{k+1}} = f_{k+1}|_{G_Z \cap G_i \cap G_{k+1}}$. This proves condition 4 and finishes the proof. □

Lemma III.6.15. *For each $k \in \{1, \ldots, N\}$, let Q_k be the map defined in Lemma III.6.7 and let E_k be the algebra defined in Proposition III.6.14. For each $k \in \{1, \ldots, N\}$, define a map $\rho_k \colon C_0(G_Z) \to \bigoplus_{i=1}^{k} C_0(G_i)$ by $\rho_k(f) = (Q_1(f), \ldots, Q_k(f))$. (Note that ρ_N is the same as the map Q defined in Proposition III.6.11.) Then for each $k = 1, \ldots, N$, we have $\operatorname{Im} \rho_k \subseteq E_k$. Further, ρ_N is an isometric *-isomorphism from $C_0(G_Z)$ onto E_N.*

Proof: To show that $\operatorname{Im} \rho_k \subseteq E_k$, induct on k. This is clear when $k = 1$, since $\rho_1 = Q_1$ and $E_1 = C_0(G_1) = C_0(G_1 \cap G_Z)$.

Let k satisfy $1 < k < N$, and suppose that $\operatorname{Im} \rho_k \subseteq E_k$. Let $f \in C_0(G_Z)$. Then $\rho_k(f) \in E_k$. Let ψ_k be the map defined in Proposition III.6.14. Let $w \in G^{(k)}$. If $w \notin G_Z$. Then $\psi_k(\rho_k(f))(w) = 0 = R_k(Q_{k+1}(f))(w)$. Suppose that $w \in G_Z$, then $w \in G_Z \cap G^{(k)}$. By Lemma III.4.8, there exists some i with $1 \leq i \leq k$ such that $w \in G_i \cap G_Z \cap G^{(k)}$. Then

$$\psi_k(\rho_k(f))(w) = \psi_k(Q_1(f), \ldots, Q_k(f))(w) = Q_i(f)(w) = (f|_{G_i \cap G_Z})(w) = f(w),$$

and

$$R_k(Q_{k+1}(f))(w) = Q_{k+1}(f)(w) = (f|_{G_{k+1} \cap G_Z})(w) = f(w).$$

Thus $\psi_k(\rho_k(f)) = R_k(Q_{k+1}(f))$, and so $\rho_{k+1}(f) = (\rho_k(f), Q_{k+1}(f)) \in E_{k+1}$. Thus $\operatorname{Im} \rho_{k+1} \subseteq E_{k+1}$.

Next we show that ρ_N is an isometric *-isomorphism. First of all, $\rho_N = Q$ is an isometric *-homomorphism. So we just need to show that the range of ρ_N is E_N.

Let $(f_1, \ldots, f_N) \in E_N$. Define $f\colon G_Z \to \mathbb{C}$ by $f(w) = f_i(w)$ if $w \in G_i \cap G_Z$. We first show that f is well-defined. Well, we know that $G_Z = \bigcup_{i=1}^N G_Z \cap G_i$ by Lemma III.4.3, so $f(w)$ exists. Suppose that $w \in G_i \cap G_j \cap G_Z$. By Proposition III.6.14, we have

$$f_i(w) = (f_i|_{G_i \cap G_j \cap G_Z})(w) = (f_j|_{G_i \cap G_j \cap G_Z})(w) = f_j(w).$$

Thus f is a well defined function. It is clear that $f|_{G_i \cap G_Z} = f_i|_{G_Z \cap G_i} \in C_0(G_i \cap G_Z)$.

Now $G_i \cap G_Z$ is closed in G_Z for all $i \in \{1, \ldots, N\}$ by Lemma III.4.6. Applying Lemma III.6.13 to G_Z, $G_1 \cap G_Z, \ldots, G_N \cap G_Z$, and f, we see that $f \in C_0(G_Z)$.

Finally, we check that $\rho_N(f) = (f_1, \ldots, f_N)$. Let $1 \leq i \leq N$, and let $w \in G_i$. If $w \notin G_Z$, then $f_i(w) = 0 = Q_i(f)(w)$; if $w \in G_Z$, then $f_i(w) = f(w) = Q_i(f)(w)$. Thus $f_i = Q_i(f)$ for all $i = 1, \ldots, N$, and so

$$\rho_N(f) = (Q_1(f), \ldots, Q_N(f)) = (f_1, \ldots, f_N).$$

Hence ρ_N is surjective.

This finishes the proof. □

The previous two lemmas give a recursive decomposition of $C_0(G_Z)$ with components $C_0(G_i)$. Next we use the fact that A_Z and A_i are closures of, respectively, $C_0(G_n)$ and $C_0(G_i)$ in $C^*(X, \mathbb{R})$ to extend the decomposition to A_Z with components A_i. We need a technical lemma first.

Lemma III.6.16. *Let B, D, and F be C^*-algebras. Let A, C and E be dense *-subalgebras of B, D, and F, respectively. Let $\phi_A\colon A \to E$ and $\phi_C\colon C \to E$ be norm-decreasing *-homomorphisms. Let $G = A \oplus_E C = \{(a, c) \in A \oplus C\colon \phi_A(a) = \phi_C(c)\}$. Let $\phi_B\colon B \to F$ and $\phi_D\colon D \to F$ be continuous extensions of ϕ_A and ϕ_C, respectively. Let $H = B \oplus_F D = \{(b, d) \in B \oplus D\colon \phi_B(b) = \phi_D(d)\}$. Suppose that ϕ_C is surjective, and that for every $\epsilon > 0$ and every $e \in E$ with $\|e\| < \epsilon$, there exists $c \in C$ such that $\phi_C(c) = e$ and $\|c\| \leq \epsilon$. Then G is a *-subalgebra of H, and $\overline{G} = H$.*

Proof: It is clear that G is a *-subalgebra of H. Let $(b, d) \in H$, and let $\epsilon > 0$. Since A is dense in B and C is dense in D, there exist $a \in A$ and $c \in C$ such that $\|a - b\| < \epsilon/4$ and $\|c - d\| < \epsilon/4$.

Let $e = \phi_A(a) - \phi_C(c)$. Then

$$\|e\| \leq \|\phi_A(a) - \phi_B(b)\| + \|\phi_D(d) - \phi_C(c)\| < \epsilon/2.$$

By assumption, there exists $f \in C$ such that $\|f\| \leq \epsilon/2$ and $\phi_C(f) = e$. Then

$$\phi_C(f + c) = \phi_C(f) + \phi_C(c) = e + \phi_C(c) = \phi_A(a).$$

Thus $(a, f + c) \in G$, and

$$\|f + c - d\| \leq \|c - d\| + \|f\| < \epsilon/4 + \epsilon/2 < \epsilon.$$

So $\|(a, c + f) - (b, d)\| < \epsilon$, and hence G is dense in H. □

Lemma III.6.17. *For each $k \in \{1, \ldots N\}$, let $R_k \colon C_0(G_{k+1}) \to C_0(G^{(k)})$ be the restriction map defined in Lemma III.6.5. Let $D_1 = A_1$, and let $\widetilde{R}_k \colon A_{k+1} \to A^{(k)}$ be the continuous extension of R_k. Then $\widetilde{R}_k$ is surjective. Moreover for each $k \in \{2, \ldots, N\}$, there exists a *-subalgebra $D_k \subseteq \bigoplus_{i=1}^k A_i$ and a *-homomorphism $\widetilde{\psi}_{k-1} \colon D_{k-1} \to A^{(k-1)}$ such that*

1. $D_k = D_{k-1} \oplus_{A^{(k-1)}} A_k = \{(a, b) \colon D_{k-1} \oplus A_k \colon \widetilde{\psi}_{k-1}(a) = \widetilde{R}_{k-1}(b)\}$.
2. *E_k is a dense *-subalgebra of D_k.*
3. *$\widetilde{\psi}_{k-1}|_{E_{k-1}} = \psi_{k-1}$, where the map ψ_k is the one defined in Proposition III.6.14 for each $k \in \{1, \ldots, N - 1\}$.*

Proof: It is clear from Lemma III.6.5 that $\widetilde{R}_k$ is surjective for all k.

We prove other statements by induction on k. The base case is when $k = 2$. Let $\widetilde{\psi}_1$ be the continuous extension of ψ_1, and let $D_2 = \{(a, b) \in D_1 \oplus A_2 \colon \widetilde{\psi}_1(a) = \widetilde{R}_1(b)\}$. It is clear that E_2 is a *-subalgebra of D_2. Condition 1 is clear, condition 2 follows from Lemma III.6.16 and Lemma III.6.6, and condition 3 follows immediately from condition 2.

Suppose that result holds from some k. By the inductive hypothesis, E_k is dense in D_k, so we can extend $\psi_k \colon E_k \to C_0(G^{(k)})$ continuously to $\widetilde{\psi}_k \colon D_k \to A^{(k)}$. Let

$$D_{k+1} = \{(a, b) \in D_k \oplus A_{k+1} \colon \widetilde{\psi}_k(a) = \widetilde{R}_k(b)\}.$$

It is clear that E_{k+1} is a *-subalgebra of D_{k+1}. Condition 1 is clear, and condition 2 follows from Lemma III.6.16 and Lemma III.6.6. Condition 3 is also clear. □

Corollary III.6.18. *$A_Z \cong D_N$ as C^*-algebras, where D_N is the C^*-algebra obtained in Lemma III.6.17.*

Proof: The map $\rho_N \colon C_0(G_Z) \to E_N$ is an isometric *-isomorphism, $C_0(G_Z)$ is dense in A_Z, and E_N is dense in D_N. So ρ_N extends to a *-isomorphism from A_Z to D_N. □

Lemma III.6.17 and Corollary III.6.18 give a recursive decomposition of A_Z. Now we use the fact that each of the components A_i in the decomposition is isomorphic to the corresponding $C(X_i, \mathbb{K})$ to obtain a stable recursive subhomogeneous decomposition of A_Z.

Theorem III.6.19. *Let $K = K(L^2((-1,1)))$. For each $k \in \{1, \ldots, N-1\}$, let*

$$\gamma_k \colon C(X_{k+1}, K) \to C(F^{(k)}, K)$$

*be the restriction map. For $k \in \{1, \ldots, N\}$, let Φ_k be the map defined in Notation III.6.12. Let $B_1 = C(X_1, K)$, and let $\theta_1 \colon D_1 \to B_1$ be given by $\theta_1 = \Phi_1 \circ \phi_1$. For each $k = 2, \ldots, N$, there exists a *-subalgebra of $B_k \subseteq \bigoplus_{i=1}^{k} C(X_i, K)$, a *-homomorphism $\Psi_{k-1} \colon B_{k-1} \to C(F^{(k-1)}, K)$, and a *-homomorphism $\theta_k \colon D_k \to B_k$ such that*

1. *$B_k = B_{k-1} \oplus_{C(F^{(k-1)},K)} C(X_k, K) = \{(a,b) \in B_{k-1} \oplus C(X_k, K) \colon \Psi_{k-1}(a) = \gamma_{k-1}(b)\}$.*
2. *θ_k is a *-isomorphism.*

Proof: First of all, some routine computation shows that for all $k \in \{1, \ldots, N-1\}$, and all $f \in C_0(G_{k+1})$, we have $\gamma_k\left(\Phi_{k+1}(\phi_{k+1}(f))\right) = \Phi^{(k)}\left(\phi^{(k)}(R_k(f))\right)$, where Φ_k, ϕ_k, R_k, $\Phi^{(k)}$, and $\phi^{(k)}$ are as defined in Notation III.6.12. Since $C_0(G_{k+1})$ is dense in A_{k+1}, for each $k \in \{1, \ldots, N-1\}$, we have the following commutative diagram:

$$\begin{array}{ccccc} A_{k+1} & \xrightarrow{\phi_{k+1}} & K_{k+1} & \xrightarrow{\Phi_{k+1}} & C(X_{k+1}, K) \\ \downarrow \widetilde{R}_k & & & & \downarrow \gamma_k \\ A^{(k)} & \xrightarrow{\phi^{(k)}} & K^{(k)} & \xrightarrow{\Phi^{(k)}} & C(F^{(k)}, K). \end{array}$$

Let ψ_k and $\widetilde{\psi}_k$ be the maps obtained from Proposition III.6.14 and Lemma III.6.17, respectively.

Now we proceed to induct on k. When $k=2$, let $\Psi_1\colon B_1 = C(X_1,K) \to C(F^{(k-1)},K)$ be defined by $\Psi_1 = \left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \widetilde{\psi}_1 \circ (\Phi_1 \circ \phi_1)^{-1}$; let

$$B_2 = B_1 \oplus_{C(F^{(1)},K)} C(X_2,K) = \{(a,b) \in B_1 \oplus C(X_2,K) \colon \Psi_1(a) = \gamma_k(b)\};$$

and let $\theta_2\colon D_2 \to B_2$ be defined by $\theta_2 = (\Phi_1 \circ \phi_1) \oplus (\Phi_2 \circ \phi_2)$.

We first show that θ_2 does map into B_2. Let $(a,b) \in D_2$ Then $\widetilde{\psi}_1(a) = \widetilde{R}_1(b)$. Then

$$\Psi_1(\Phi_1 \circ \phi_1(a)) = \left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \tilde{\psi}_1(a) = \left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \tilde{R}_1(b) = \gamma_1(\Phi_2 \circ \phi_2(b)).$$

Thus $\theta_2(a,b) = (\Phi_1 \circ \phi_1(a), \Phi_2 \circ \phi_2(b)) \in B_2$. So θ_k maps into B_2.

Next we show that θ_2 is surjective. Let $(c,d) \in B_2$, and let

$$(a,b) = \left((\Phi_1 \circ \phi_1)^{-1}(c), (\Phi_2 \circ \phi_2)^{-1}(d)\right).$$

Now, $(c,d) \in B_2$ implies that $\Psi_1(c) = \gamma_1(d)$, that is $\Psi_1((\Phi_1 \circ \phi_1)(a)) = \gamma_1((\Phi_2 \circ \phi_2)(b))$. But $\Psi_1((\Phi_1 \circ \phi_1(a)) = \left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \tilde{\psi}_1(a)$, and $\gamma_1((\Phi_2 \circ \phi_2)(b)) = \left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \tilde{R}_1(b)$. So

$$\left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \tilde{\psi}_1(a) = \left(\Phi^{(1)} \circ \phi^{(1)}\right) \circ \tilde{R}_1(b).$$

Thus $\widetilde{\psi}_1(a) = \widetilde{R}_1(b)$, since $\Phi^{(1)} \circ \phi^{(1)}$ is injective. Therefore $(a,b) \in D_2$. It is clear that $\theta_2(a,b) = (c,d)$. Hence θ_2 is surjective.

It is clear that θ_2 is an injective *-homomorphism. So θ_2 is a *-isomorphism.

Now suppose that result holds for some k with $2 < k < N$. Let $\Psi_k\colon B_k \to C(F^{(k)},K)$ be given by $\Psi_k = (\Phi^{(k)} \circ \phi^{(k)}) \circ \widetilde{\psi}_k \circ \theta_k^{-1}$, let

$$B_{k+1} = B_k \oplus_{C(F^{(k)},K)} C(X_{k+1},K) = \{(a,b) \in B_k \oplus C(X_{k+1},K) \colon \Psi_k(a) = \gamma_k(b)\},$$

and let $\theta_{k+1}\colon D_{k+1} \to B_{k+1}$ be given by $\theta_{k+1} = \theta_k \oplus (\Phi_{k+1} \circ \phi_{k+1})$.

We first show that θ_{k+1} maps D_{k+1} to B_{k+1}. Let $(a,b) \in D_{k+1}$. Then

$$\Psi_k(\theta_k(a)) = \left(\Phi^{(k)} \circ \phi^{(k)}\right) \circ \tilde{\psi}_k(a)$$
$$= \Phi^{(k)} \circ \phi^{(k)} \left(\widetilde{R}_k(b)\right) = \gamma_k((\Phi_{k+1} \circ \phi_{k+1})(b)).$$

Thus $\theta_{k+1}(a,b) = (\theta_k(a), (\Phi_{k+1} \circ \phi_{k+1})(b)) \in B_{k+1}$.

Next we show that θ_{k+1} is surjective. Let $(c,d) \in B_{k+1}$, and let $(a,b) = (\theta_k^{-1}(c), (\Phi_{k+1} \circ \phi_{k+1})^{-1}(d))$. Since

$$\Psi_k(c) = \Psi_k(\theta_k(a)) = \left(\Phi^{(k)} \circ \phi^{(k)}\right) \widetilde{\psi}_k(a) = \gamma_k(d)$$
$$= \gamma_k((\Phi_{k+1} \circ \phi_{k+1})(b)) = \left(\Phi^{(k)} \circ \phi^{(k)}\right) \circ \tilde{R}_k(b),$$

we see that $\widetilde{\psi}_k(a) = \widetilde{R}_k(b)$. Thus $(a,b) \in D_{k+1}$, and it is clear that $\theta_{k+1}(a,b) = (c,d)$. Therefore θ_{k+1} is surjective. Since θ_{k+1} is clearly an injective *-homomorphism, we see that θ_{k+1} is a *-isomorphism. □

Corollary III.6.20. *Let θ_N and ρ_N be the *-isomorphisms obtained in Corollary III.6.18 and Lemma III.6.15, respectively. Then $\theta_N \circ \rho_N$ is a *-isomorphism between A_Z and B_N.*

At this moment, we essentially have a SRSH decomposition of A_Z. We only need to verify that the attaching maps are non-vanishing:

Lemma III.6.21. *Let θ_N be as in Lemma III.6.19, let ρ_N be as in Corollary III.6.18, and let Φ_k, ϕ_k, Q_k be as in Notation III.6.12. Let $f \in C_0(G_Z)$.*

1. *We have $\theta_N \circ \rho_N(f) = (\Phi_1 \circ \phi_1 \circ Q_1(f), \Phi_2 \circ \phi_2 \circ Q_2(f), \ldots, \Phi_N \circ \phi_N \circ Q_N(f))$.*
2. *Let $1 \leq k \leq N$, let $x \in X_k$, and let*

$$T_x = \{(r, sx) \colon s \in (\alpha_k(x), \beta_k(x)), s - r \in (\alpha_k(x), \beta_k(x))\}.$$

   Then $T_x = G_{k,\{x\}}$ is a closed subset of G_k, $T_x \cap G_Z \neq \varnothing$, and $\Phi_k \circ \phi_k \circ Q_k(f)(x) = 0$ if and only if $\phi_k \circ Q_k(f)(x) = 0$, which happens if and only if $f|_{G_Z \cap T_x} = 0$.

3. *For each $k = 2, \ldots, N$, and for each $x \in F^{(k-1)}$, there exists some $a \in B_{k-1}$ such that $\Psi_{k-1}(a)(x) \neq 0$, where Ψ_{k-1} is the map defined in Lemma III.6.19.*

Proof: From the construction of the maps θ_k in the proof of Lemma III.6.19, we see that

$$\theta_N(f_1,\dots,f_N) = (\Phi_1 \circ \phi_1(f_1),\dots,\Phi_N \circ \phi_N(f_N)),$$

for all $(f_1,\dots,f_N) \in D_N$. From the definition of the maps ρ_k in Lemma III.6.15, we see that $\rho_N(f) = (Q_1(f),\dots,Q_N(f))$ for all $f \in C_0(G_Z)$. So part 1 is clear.

It is clear that $T_x = G_{k,\{x\}}$ is a closed subset of G_k, and $T_x \cap G_Z$ is nonempty. From the definition of the the maps Φ_i, it is clear that $\Phi_k \circ \phi_k \circ Q_k(f)(x) = 0$ if and only if $\phi_k \circ Q_k(f)(x) = 0$. By Lemma III.5.5, we have $\phi_k((Q_k(f))(x) = \lambda_k^x(Q_k(f)) = 0$ if and only if $Q_k(f)|_{T_x} = 0$. So $\Phi_k \circ \phi_k \circ Q_k(f)(x) = 0$ if and only if $Q_k(f)|_{T_x} = 0$, if and only if $Q_k(f)|_{T_x \cap G_Z} = 0$ ($Q_k(f)$ vanishes outside of G_Z), if and only if $(f|_{G_Z \cap G_k})|_{T_x \cap G_Z} = 0$, if and only if $f|_{T_x \cap G_Z} = 0$.

For part 3, we use the notation in Lemma III.6.19. Note that $\Psi_{k-1} = \Phi^{(k-1)} \circ \phi^{(k-1)} \circ \tilde{\psi}_{k-1} \circ \theta_{k-1}^{-1}$. It is clear that there exists some $f \in C_c(G_Z)$ such that $f|_{T_x \cap G_Z} \neq 0$. Let $a = \theta_{k-1} \circ \rho_{k-1}(f)$. Then $a \in B_{k-1}$. By part 2 we have

$$\begin{aligned}
\Psi_{k-1}(a)(x) &= \Phi^{(k-1)} \circ \phi^{(k-1)} \circ \tilde{\psi}_{k-1}(\rho_{k-1}(f))(x) \\
&= \Phi^{(k-1)} \circ \phi^{(k-1)} \circ \tilde{\psi}_{k-1}(Q_1(f),\dots,Q_{k-1}(f))(x) \\
&= \gamma_{k-1}((\Phi_k \circ \phi_k(Q_k(f)))(x) \\
&= (\Phi_k \circ \phi_k(Q_k(f))(x) \\
&\neq 0.
\end{aligned}$$

□

Corollary III.6.22. *A_Z is a SRSHA.*

Proof: By Lemma III.6.19, and part 3 of Lemma III.6.21, we see that

$$\left(X_1, B_1, \left(X_k, F^{(k-1)}, \Psi_{k-1}, \gamma_{k-1}, B_k\right)_{k=2}^{N}\right)$$

is a SRSH system, so $A_Z \cong B_N$ is a SRSHA. □

The following lemma is known as the gluing lemma. It a standard result in point-set topology, so we will omit its proof.

Lemma III.6.23. *Let X be a topological space. Let Y and Z be two subsets of X. Let $f\colon Y \to \mathbb{C}$ and $g\colon Z \to \mathbb{C}$ be continuous functions such that $f|_{Y\cap Z} = g|_{Y\cap Z}$. If either both Y and Z are closed in X or both Y and Z are both open in X, then the function $h : X \to \mathbb{C}$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x \in Y \\ g(x) & \text{if } x \in Z \end{cases}$$

is continuous.

The next lemma will be used in the next chapter.

Lemma III.6.24. *Let*

$$\left(X_1, B_1, \left(X_k, F^{(k-1)}, \Psi_{k-1}, \gamma_{k-1}, B_k\right)_{k=2}^{N}\right)$$

be the SRSH decomposition for A_Z as in III.6.22. For each $k \in \{1, \dots k\}$, let $H_k = G_Z \cap \left(\bigcup_{i=1}^{k} G_i\right)$. For each k with $1 \le k \le N$, if $I \subseteq B_k$ is a non-zero ideal, then $I \cap \widetilde{\theta}_k(C_c(H_k)) \neq 0$.

Proof: Define $\tau_k\colon C_0(H_k) \to E_k$ by $\tau_k(f) = (f|_{G_i \cap G_Z})_{i=1,\dots,k}$. By Lemma III.4.6, for each k with $1 \le k \le N$, the set H_k is a closed subset of G_Z. Hence each $f \in C_0(H_k)$ extends to some $f' \in C_0(G_Z)$. Thus $\tau(f) = \rho_k(f')$, where ρ_k is the map in the proof of Lemma III.6.15. Thus we see that τ_k indeed sends elements of $C_0(H_k)$ into E_k. It is clear that τ_k is injective. Also, since $G_Z \cap G_i$ is closed in H_k for every i with $1 \le i \le k+N$, surjectivity of τ_k follows easily from Lemma III.6.23. Linearity of τ_k is clear as well.

For each k with $1 \le k \le n$, define $\widetilde{\theta}_k\colon C_0(H_k) \to B_k$ by $\widetilde{\theta}_k = \theta_k \circ \tau_k$, where B_k and θ_k are as in Lemma III.6.19. We will also use τ_k and $\widetilde{\theta}_k$ to denote their restrictions to $C_c(H_k)$.

Now we proceed by induction. If $k = 1$, then there exists a closed subset $F \subsetneq X_1$ such that $I = \{f \in B_1 \colon f|_F = 0\}$. Then $G_{1,F}$ is a closed subset of $G_1 = G_1 \cap G_Z$ by Corollary III.4.5. If $G_{1,F} = G_1$, then it is clear that $F = X_1$, which implies that $I = 0$. Thus $F \neq X_1$, and so $G_{1,F} \neq G_1 = H_1$. Then there exists $f \in C_c(G_1) = C_c(H_1)$ such that $f|_{G_{i,F}} = 0$ and $f \neq 0$. So $\widetilde{\theta}_1(f) \in I \cap \widetilde{\theta}_1(C_0(H_1))$ and $\widetilde{\theta}_1(f) \neq 0$. Thus the lemma holds for $k = 1$.

Now suppose that the lemma holds for some k with $1 < k < N$. Let $I \subseteq B_{k+1}$ be a non-zero ideal. We can assume that $I \neq B_{k+1}$. Then we know that for each i with $1 \le i \le k+1$, there exists a closed subset $F_i \subseteq X_i$ such that

$$I = \{(f_1, \dots, f_{k+1}) \in B_{k+1} \colon f_i|_{F_i} = 0 \text{ for } i = 1, \dots, k+1\}.$$

First assume that $X_{k+1} \setminus F^{(k)}$ is not contained in F_{k+1}. (Recall that $F^{(k)}$ is the k-th attaching space.) Now, by Lemma III.4.6, we know that $G_i \cap G_Z$ is closed in G_Z for every i with $1 \leq i \leq k+1$. So $\bigcup_{i=1}^{k}(G_i \cap G_Z)$ is closed in G_Z. Thus $\bigcup_{i=1}^{k}(G_i \cap G_Z)$ is closed in H_{k+1}, because H_{k+1} is also contained in G_Z. Similarly, $G_Z \cap G_{k+1}$ is closed in H_{k+1} as well. Also, by Corollary III.4.5, we know that $G_{k+1,F_{k+1}}$ is closed in G_{k+1}. Thus $G_Z \cap G_{k+1,F_{k+1}}$ is closed in $G_{k+1} \cap G_Z$, which implies that $G_Z \cap G_{k+1,F_{k+1}}$ is closed in H_{k+1}. Therefore $G_Z \cap \left[G_{k+1,F_{k+1}} \cup \left(\bigcup_{i=1}^{k} G_i\right)\right]$ is closed in H_{k+1}.

If $G_Z \cap \left[G_{k+1,F_{k+1}} \cup \left(\bigcup_{i=1}^{k} G_i\right)\right] = H_{k+1}$, then we have, by Lemma III.4.8 and Lemma III.4.4,

$$\begin{aligned}
G_Z \cap G_{k+1} &= G_{k+1} \cap G_Z \cap H_{k+1} \\
&= G_{k+1} \cap G_Z \cap \left[G_{k+1,F_{k+1}} \cup \left(\bigcup_{i=1}^{k} G_i\right)\right] \\
&= [G_{k+1} \cap G_Z \cap G_{k+1,F_{k+1}}] \cup \left[G_{k+1} \cap G_Z \cap \left(\bigcup_{i=1}^{k} G_i\right)\right] \\
&= (G_Z \cap G_{k+1,F_{k+1}}) \cup (G^{(k)} \cap G_Z) \\
&= G_Z \cap (G_{k+1,F_{k+1}} \cup G^{(k)}) \\
&= G_Z \cap G_{k+1,F_{k+1} \cup F^{(k)}}.
\end{aligned}$$

Then $X_{k+1} = F^{(k)} \cup F_{k+1}$, which contradicts our assumption that $X_{k+1} \setminus F^{(k)} \not\subseteq F_{k+1}$. Thus $G_Z \cap \left[G_{k+1,F_{k+1}} \cup \left(\bigcup_{i=1}^{k} G_i\right)\right] \neq H_{k+1}$. Then there exists a nonzero element $f \in C_c(H_{k+1})$ such that $f|_{G_Z \cap [G_{k+1,F_{k+1}} \cup (\bigcup_{i=1}^{k} G_i)]} = 0$. Then $\widetilde{\theta}_{k+1}(f) \neq 0$ and $\widetilde{\theta}_{k+1}(f)$ vanishes on F_i for all i with $1 \leq i \leq k+1$. Thus $I \cap \widetilde{\theta}_{k+1}(C_0(H_{k+1})) \neq 0$.

Now assume that $X_{k+1} \setminus F^{(k)} \subseteq F_{k+1}$. Let $F = \overline{X_{k+1} \setminus F^{(k)}}$ and let

$$J = \{(f_1, \dots, f_{k+1}) \in B_{k+1} \colon f_{k+1}|_F = 0\}.$$

Let $P \colon B_{k+1} \to B_k$ be defined by $P(f_1, \dots, f_{k+1}) = (f_1, \dots, f_k)$. Then P is surjective (this follows because the map $\gamma_k \colon C(X_{k+1}, K) \to C(F^{(k)}, K)$ is surjective) and $P|_J$ is injective (this follows from the construction of B_{k+1} and the definition of J). Also J is an ideal of B_{k+1}, $F \subseteq F_{k+1}$, and $I \subseteq J$. Note that

$$\ker P = \{(0, \ldots, 0, f_{k+1}) \colon f_{k+1}|_{F^{(k)}} = 0\}.$$

If $I \subseteq \ker P$, then for every $a = (f_1, \ldots, f_{k+1}) \in I$, we have $f_{k+1}|_{F^{(k)}} = 0$ and $f_i = 0$ for every i with $1 \leq i \leq k$. But f_{k+1} also vanishes on F_{k+1}, which contains $X_{k+1} \setminus F^{(k)}$ as a subset. So $f_{k+1} = 0$. Consequently, we have $a = 0$. This contradicts the assumption that $I \neq 0$. Thus I is not contained in $\ker P$, which implies that $P(I)$ is a non-zero ideal of B_k. Therefore we have $P(I) \cap \widetilde{\theta}_k(C_c(H_k)) \neq 0$ by the inductive hypothesis. So pick $g \in C_c(H_k)$ such that $g \neq 0$ and $\widetilde{\theta}_k(g) \in P(I)$. Now we prove some claims.

Claim 1: Let $R\colon C_0(H_{k+1}) \to C_0(H_k)$ be defined by $R(f) = f|_{H_k}$. Then R is a linear surjection. Also the following diagram commutes:

$$\begin{array}{ccc} C_0(H_{k+1}) & \xrightarrow{\widetilde{\theta}_{k+1}} & B_{k+1} \\ \downarrow R & & \downarrow P \\ C_0(H_k) & \xrightarrow{\widetilde{\theta}_k} & B_k. \end{array}$$

It is clear that R is a linear surjection. If $f \in C_0(H_{k+1})$, then

$$\begin{aligned} P(\widetilde{\theta}_{k+1}(f)) &= P(\theta_{k+1}(\tau_{k+1}(f))) \\ &= P(\theta_{k+1}(f|_{G_1 \cap G_Z}, \ldots, f|_{G_{k+1} \cap G_Z})) \\ &= P(\theta_k(f|_{G_1 \cap G_Z}, \ldots, f|_{G_k}), \Phi_{k+1} \circ \phi_{k+1}(f|_{G_{k+1} \cap G_Z})) \\ &= \theta_k(f|_{G_1 \cap G_Z}, \ldots, f|_{G_k \cap G_Z}) \\ &= \theta_k(R(f)|_{G_1 \cap G_Z}, \ldots, R(f)|_{G_k \cap G_Z}) \\ &= \theta_k(\tau_k(R(f))) \\ &= \widetilde{\theta}_k(R(f)). \end{aligned}$$

So Claim 1 is proven.

Claim 2: We have $P^{-1}(P(I)) \subseteq \{(f_1, \ldots, f_k, f_{k+1}) \in B_{k+1} \colon f_{k+1}|_{F \cap F^{(k)}} = 0\}$.

Suppose that $f = (f_1, \ldots, f_{k+1}) \in P^{-1}(P(I))$. Then there exists $a = (g_1, \ldots, g_{k+1}) \in I$ such that $P(f) = P(a)$. So $(f_1, \ldots, f_k) = (g_1, \ldots g_k)$. Then by the construction of B_{k+1}, we have

$f_{k+1}|_{F^{(k)}} = g_{k+1}|_{F^{(k)}}$. Therefore $f_{k+1}|_{F\cap F^{(k)}} = g_{k+1}|_{F\cap F^{(k)}} = 0$, since $a \in I$ and $F \cap F^{(k)} \subseteq F_{k+1}$. Claim 2 is proven.

Claim 3: (Recall that the element g is chosen, right before Claim 1 above, to satisfy $g \neq 0$ and $\widetilde{\theta}_k(g) \in P(I)$.) We have $g|_{G_{k+1,F}\cap H_k} = 0$ or $G_{k+1,F} \cap H_k = \varnothing$.

Suppose that $G_{k+1,F} \cap H_k \neq \varnothing$ and $g|_{G_{k+1,F}\cap H_k} \neq 0$. Using Claim 1, choose $h \in C_0(H_{k+1})$ such that $R(h) = g$. Note that

$$\begin{aligned} G_{k+1,F} \cap H_k &= G_{k+1,F} \cap G_Z \cap G_{k+1} \cap H_k \\ &= (G_{k+1,F} \cap G_Z) \cap (G_Z \cap G_{k+1,F^{(k)}}) \\ &= G_Z \cap G_{k+1,F\cap F^{(k)}}. \end{aligned}$$

So $h|_{G_{k+1,F\cap F^{(k)}}} \neq 0$. Then by Lemma III.5.5, $\widetilde{\theta}_{k+1}(h)|_{F\cap F^{(k)}} \neq 0$. By Claim 2, $P(\widetilde{\theta}_{k+1}(h)) \notin P(I)$. But by Claim 1, $P(\widetilde{\theta}_{k+1}(h)) = \widetilde{\theta}_k(R(h)) = \widetilde{\theta}_k(g) \in P(I)$. This is a contradiction, so Claim 3 is proven.

Now,

$$\begin{aligned} G_{k+1} \cap [(G_{k+1,F} \cap G_Z) \cup H_k] &= G_{k+1} \cap G_Z \cap \left[G_{k+1,F} \cup \left(\bigcup_{i=1}^{k} G_i \right) \right] \\ &= (G_Z \cap G_{k+1,F}) \cup \left[(G_Z \cap G_{k+1}) \cap \left(\bigcup_{i=1}^{k} G_i \right) \right] \\ &= (G_Z \cap G_{k+1,F}) \cup (G_Z \cap G_{k+1,F^{(k)}}) \\ &= G_Z \cap G_{k+1,F^{(k)}\cup F} \\ &= G_Z \cap G_{k+1}. \end{aligned}$$

So we have

$$\begin{aligned}
H_{k+1} &= (G_Z \cap G_{k+1}) \cup H_k \\
&= \{G_{k+1} \cap [(G_{k+1,F} \cap G_Z) \cup H_k]\} \cup H_k \\
&= \{G_{k+1} \cup H_k\} \cap \{[(G_{k+1,F} \cap G_Z) \cup H_k] \cup H_k\} \\
&= \{G_{k+1} \cup H_k\} \cap \{(G_{k+1,F} \cap G_Z) \cup H_k\} \\
&= [G_{k+1} \cap (G_{k+1,F} \cap G_Z)] \cup H_k \\
&= (G_{k+1,F} \cap G_Z) \cup H_k.
\end{aligned}$$

Both $G_{k+1,F} \cap G_Z$ and H_k are closed in H_{k+1}. Also, by Claim 3, regardless of whether or not $G_{k+1,F} \cap H_k = (G_{k+1,F} \cap G_Z) \cap H_k$ is empty, the function g agrees with the zero function on

$$G_{k+1,F} \cap H_k = (G_{k+1,F} \cap G_Z) \cap H_k.$$

Thus by Lemma III.6.23, g can be extended to some $g' \in C_0(H_{k+1})$ such that $g'|_{G_{k+1,F} \cap G_Z} = 0$. Then by Lemma III.5.5, $\widetilde{\theta}_{k+1}(g')$ vanishes on F. So $\widetilde{\theta}_{k+1}(g') \in J$. It is clear that $\widetilde{\theta}_{k+1}(g') \neq 0$. Also, since g' vanishes outside of H_k, the support of g' is the same as g, so $g' \in C_c(H_{k+1})$.

Finally we check that $\widetilde{\theta}_{k+1}(g') \in I$. By Claim 1, $P(\widetilde{\theta}_{k+1}(g')) = \widetilde{\theta}_k(g) \in P(I)$. So there exists some $g'' \in I$ such that $P(\widetilde{\theta}_{k+1}(g')) = P(g'')$. But $P|_J$ is injective, and both $\widetilde{\theta}_{k+1}(g')$ and g'' are in J, so $\widetilde{\theta}_{k+1}(g') = g'' \in I$. This completes the proof. □

Corollary III.6.25. *If $I \subseteq A_Z$ is a non-zero ideal, then $I \cap C_c(G_Z) \neq 0$.*

Proof: Let $I \subseteq A_Z$ be a non-zero ideal. Note that $(\theta_N \circ \rho_N)|_{C_0(G_Z)} = \widetilde{\theta}_N$. Since $\theta_N \circ \rho_N(I)$ is a non-zero ideal of B_N, we see that

$$\begin{aligned}
0 &\neq \theta_N \circ \rho_N(I) \cap \widetilde{\theta}_N(C_c(G_Z)) \\
&= \theta_N \circ \rho_N(I) \cap \theta_N \circ \rho_N(C_c(G_Z)) \\
&= \theta_N \circ \rho_N(I \cap C_c(G_Z)).
\end{aligned}$$

So $I \cap C_c(G_Z) \neq 0$. □

CHAPTER IV

INDUCTIVE LIMITS OF SRSHAS AS C^*-SUBALGEBRAS OF $C^*(X, \mathbb{R})$

In this chapter, we show that when X is a compact metric space and when $\mathbb{R}$ acts on X freely and minimally, the crossed product $C^*(X, \mathbb{R})$ contains C^*-subalgebras that are isomorphic to simple inductive limits of SRSHAs. These subalgebras are the analogs of the algebras $A_y = C^*(C(X), uC_0(X \setminus \{y\}))$, the C^*-subalgebra generated by $C(X)$ and $uC_0(X \setminus \{y\})$, in the crossed product obtained from a free minimal action of $\mathbb{Z}$ on a compact metric space X.

IV.1. Definition of the Subalgebra A_y

To define the subalgebras A_y, we will first need a different description of the set G_Z defined in Notation III.1.10.

Lemma IV.1.1. *Let Z be a pseudo-transversal of a free minimal action of $\mathbb{R}$ on a compact metric space. Let G_Z be the set defined in Notation III.1.10. For each $r \in [0, \infty)$, let $D_r = [0, r] \cdot Z$, and for each $r \in (-\infty, 0]$, let $D_r = [r, 0] \cdot Z$, where we take $[0, 0]$ to be the degenerate closed interval $\{0\}$. Then $G_Z = \left(\bigcup_{s \in \mathbb{R}} (\{s\} \times D_s)\right)^c$.*

Proof: Let $H = \left(\bigcup_{s \in \mathbb{R}} (\{s\} \times D_s)\right)^c$. Let $(r, x) \in G_Z$. Then $x \in Z^c$, and $-r \in (\alpha(x), \beta(x))$, where α and β are the backward and forward entering times for Z, respectively. First assume that $r \geq 0$. If $(r, x) \notin H$, then $(r, x) \in \bigcup_{s \in \mathbb{R}} (\{s\} \times D_s)$, and then $x \in D_r = [0, r] \cdot Z$, so there exists $t \in [0, r]$ and $z \in Z$ such that $x = tz$. Then $(-t)x = z \in Z$. Since $x \in Z^c$, we see that $t \neq 0$, and so $-t < 0$. Then $\alpha(x) \geq -t$ by the definition of the backward entering time. But $-r > \alpha(x) \geq -t$, so $r < t$, contradicting the fact that $t \in [0, r]$. Thus $(r, x) \in H$. With a very similar argument, we see that $(r, x) \in H$ when $r \leq 0$. So $G_Z \subseteq H$.

Now suppose that $(r, x) \in H$. Then $x \notin D_r$. First assume that $r \geq 0$. Since $x \notin D_r = [0, r] \cdot Z$, for all $s \in [-r, 0]$, we have $sx \notin Z$. In particular $x \notin Z$ and $(-r)x \notin Z$. Also, $\alpha(x) \leq -r$.

But $\alpha(x) \neq -r$, for otherwise, $(-r)x = \alpha(x)x \in Z$. Thus $\alpha(x) < -r \leq 0 < \beta(x)$. So $(r, x) \in G_Z$. With a very similar argument, we see that $(r, x) \in G_Z$ if $r \leq 0$. So $H \subseteq G_Z$. □

Notation IV.1.2. Let Z be a compact pseudo-transversal of a free minimal action of $\mathbb{R}$ on a compact metric space X. For each $y \in X$, let $D_r^y = [0, r] \cdot y$ if $r \geq 0$, let $D_r^y = [r, 0] \cdot y$ if $r \leq 0$, where $[0, 0] = \{0\}$, and let $G_y = \left(\bigcup_{r \in \mathbb{R}}(\{r\} \times D_r^y)\right)^c$. For each $y \in Z$ and each $r > 0$, let $B(y, r) = \{x \in X : d(x, y) < r\}$, let $\widetilde{Z}_r^y = Z \cap B(y, r)$, and let $Z_r^y = \overline{\widetilde{Z}_r^y}$.

Lemma IV.1.3. *Using the notation in Notation IV.1.2, for all $y \in Z$, all $r > 0$, and all $x \in X$, we have*

1. $(\mathbb{R} \cdot x) \cap \widetilde{Z}_r^y \neq \varnothing$.
2. $\widetilde{Z}_r^y \subseteq \overline{\widetilde{Z}_r^y \cap (\mathbb{R} \cdot x)}$.
3. $\overline{Z_r^y \cap (\mathbb{R} \cdot x)} = Z_r^y$.
4. *Z_r^y is a pseudo-transversal, and $Z_r^y \subseteq Z$.*

Proof: Fix $y \in Z$ $r > 0$ and $x \in X$. Let $S = (\mathbb{R} \cdot x) \cap Z$.

Since Z is a pseudo-transversal, we have $\overline{S} = Z$. This implies that $S \cap B(y, r) \cap Z \neq \varnothing$, which implies that $(\mathbb{R} \cdot x) \cap \widetilde{Z}_r^y \neq \varnothing$. This proves part 1.

Let $z \in \widetilde{Z}_r^y$. Then there exists $\epsilon > 0$ such that $B(z, \epsilon) \subseteq B(y, r)$. By part 1, for all $n \geq 1$, we have $(\mathbb{R} \cdot x) \cap \widetilde{Z}_{\epsilon/2^n}^z \neq \varnothing$. So for each $n \geq 1$, choose $x_n \in (\mathbb{R} \cdot x) \cap \widetilde{Z}_{\epsilon/2^n}^z$. Now, for each $n \geq 1$, we have $B(z, \epsilon/2^n) \subseteq B(z, \epsilon) \subseteq B(y, r)$, so $x_n \in \widetilde{Z}_r^y \cap (\mathbb{R} \cdot x)$ for all $n \geq 1$. Since $d(x_n, z) < \epsilon/2^n$ for each $n \geq 1$, we see that $x_n \to z$. So part 2 holds. Then $Z_r^y = \overline{\widetilde{Z}_r^y} \subseteq \overline{(\mathbb{R} \cdot x) \cap \widetilde{Z}_r^y} \subseteq \overline{(\mathbb{R} \cdot x) \cap Z_r^y}$. Since $(\mathbb{R} \cdot x) \cap Z_r^y \subseteq Z_r^y$, and since Z_r^y is clearly compact, we see that $\overline{(\mathbb{R} \cdot x) \cap Z_r^y} \subseteq Z_r^y$. So part 3 holds. Part 4 follows immediately from part 3. This finishes the proof. □

IV.2. Simplicity and Topological Stable Rank of A_y

Notation IV.2.1. For the rest of the chapter, we fix a pseudo-transversal Z, a point $y \in Z$, and a strictly decreasing sequence $\{r_n\}$ of positive real numbers that converges to 0. For each $n \geq 1$, let $Z_n = Z_{r_n}^y$, where $Z_{r_n}^y$ is as in Notation IV.1.2, let G_{Z_n} be the set defined in Notation III.1.10, let $A_n = \overline{C_0(G_{Z_n})}$, and let $A_y = \overline{C_c(G_y)}$. Note that $Z_1 \supseteq Z_2 \supseteq \cdots$, and that $\bigcap_{n=1}^{\infty} Z_n = \{y\}$.

Lemma IV.2.2. *We have*

1. $G_{Z_1} \subseteq G_{Z_2} \subseteq \cdots$ *and* $\bigcup_{n\geq 1} G_{Z_n} = G_y$.

2. $C_0(G_{Z_1}) \subseteq C_0(G_{Z_2}) \subseteq \cdots$ *and* $C_c(G_y) = \bigcup_{n\geq 1} C_c(G_{Z_n})$.

3. $A_1 \subseteq A_2 \subseteq \cdots$ *and* $A_y = \overline{\bigcup_{n\geq 1} A_n}$.

Proof: For each $r \in \mathbb{R}$, let D_r^y be as in Notation IV.1.2; and for each $n \geq 1$, and each $r \in \mathbb{R}$, let D_r^n be the set D_r in Lemma IV.1.1 for the pseudo-transversal Z_n. Then by Lemma IV.1.1, we have $G_{Z_n} = \left(\bigcup_{r\in\mathbb{R}}(\{r\} \times D_r^n)\right)^c$. We first claim that for all $r \in \mathbb{R}$, we have $D_r^y = \bigcap_{n=1}^{\infty} D_r^n$.

It is clear that for all $r \in \mathbb{R}$, we have $D_r^y \subseteq \bigcap_{n=1}^{\infty} D_r^n$. So we just need to prove the other inclusion. Let $r \in \mathbb{R}$. We will only prove the inclusion for the case when $r > 0$, because the case when $r < 0$ is similar, and the case when $r = 0$ is trivial. Let $x \in \bigcap_{n\geq 1} D_r^n$. Then for each $n \geq 1$, there exist $s_n \in [0, r]$ and $z_n \in Z_n$ such that $x = s_n z_n$. It is clear that $z_n \to y$. Since $\{s_n\}$ is a bounded sequence, we can assume, passing to a subsequence if necessary, that $s_n \to s$ for some $s \in [0, r]$. Then $x = s_n z_n \to sy \in D_r^y$. Thus $\bigcap_{n\geq 1} D_r^n \subseteq D_r^y$. So the claim is proven.

Thus $(s, x) \in (G_y)^c$ if and only if $(s, x) \in \bigcup_{r\in\mathbb{R}}(\{r\} \times D_r^y)$, if and only if $x \in D_s^y$, if and only if $x \in \bigcap_{n\geq 1} D_s^n$, if and only if $(s, x) \in \bigcap_{n\geq 1}\{s\} \times D_s^n$, if and only if (s, x) belongs to $\bigcap_{n\geq 1}\left(\bigcup_{r\in\mathbb{R}}(\{r\} \times D_r^n)\right)$, if and only if $(s, x) \in \bigcap_{n\geq 1}(G_{Z_n}^c) = \left(\bigcup_{n\geq 1} G_{Z_n}\right)^c$. So $\bigcup_{n\geq 1} G_{Z_n} = G_y$. Since $D_r^1 \supseteq D_r^2 \supseteq \cdots$ for all $r \in \mathbb{R}$, it follows immediately that $G_{Z_1} \subseteq G_{Z_2} \subseteq \cdots$. Part 1 is proven.

The first statement of part 2 and the first statement of part 3 follow immediately from the first statement of part 1. Now let $f \in C_c(G_y)$, and let K be the support of f. Then $K \subseteq G_y = \bigcup_{n\geq 1} G_{Z_n}$. Since G_{Z_n} is open, and since K is compact, there exists $N \geq 1$ such that $K \subseteq \bigcup_{n=1}^{N} G_{Z_n} = G_{Z_N}$. So $f \in C_c(G_{Z_n}) \subseteq \bigcup_{n\geq 1} C_c(G_{Z_n})$. It is clear that $\bigcup_{n\geq 1} C_c(G_{Z_n}) \subseteq C_c(G_y)$. So part 2 is proven.

It follows immediately from part 1 and 2 and the first statement of part 3 that $A_y \subseteq \overline{\bigcup_{n\geq 1} A_n}$. For the other inclusion, note that for each $n \geq 1$, $C_c(G_{Z_n}) \subseteq C_c(\mathbb{R} \times X)$ is dense in $C_0(G_{Z_n}) \subseteq C_c(\mathbb{R} \times X)$ when $C_c(\mathbb{R} \times X)$ has the inductive limit topology, and so $C_c(G_{Z_n})$ is dense in $C_0(G_{Z_n})$ in the norm topology. Then for all $n \geq 1$, we have $A_n = \overline{C_c(G_{Z_n})} \subseteq \overline{C_c(G_y)} = A_y$. The desired inclusion follows. □

Lemma IV.2.3. *If $I \subseteq A_y$ is a non-zero ideal, then $I \cap C_c(G_y) \neq 0$.*

Proof: Since $I = \overline{\bigcup_{n\geq 1}(A_n \cap I)}$, we know that for some $n \geq 1$, $I \cap A_n \neq 0$. Then $I \cap A_n$ is a non-zero ideal in A_n, so by Corollary III.6.25, we have $I \cap A_n \cap C_c(G_{Z_n}) \neq 0$. But $I \cap A_n \cap C_c(G_{Z_n}) \subseteq I \cap C_c(G_y)$, so $C_c(G_y) \cap I \neq 0$. □

Lemma IV.2.4. *Let U be an open set in $\mathbb{R} \times X$. For each $n \geq 1$, let R_n denote the return time for Z_n, and for each $n \geq 1$ and each $z \in Z_n$, let*

$$T_z^n = \{(r, sz) \colon s \in (0, R_n(z)), s - r \in (0, R_n(z))\}.$$

Then there exists $N \geq 1$ such that for all $n \geq N$ and all $z \in Z_n$, we have $T_z^n \cap U \neq \varnothing$.

Proof: We first show that for each $\Gamma \in (0, \infty)$, there exists $m \geq 1$ such that $R_m(z) \geq \Gamma$ for all $z \in Z_m$. By Lemma III.2.1, there exists a compact neighborhood K of y that satisfies $[(0, \Gamma] \cdot (K \cap Z)] \cap (K \cap Z) = \varnothing$. Let $\delta > 0$ satisfy $B(y, \delta) \subseteq K$, and let $m \geq 1$ satisfy $r_m < \delta$. Then

$$Z_m = \overline{B(y, r_m) \cap Z} \subseteq \overline{B(y, \delta) \cap Z} \subseteq \overline{B(y, \delta)} \cap Z \subseteq K \cap Z.$$

So for all $z \in Z_m$, we have

$$[(0, \Gamma] \cdot z] \cap Z_m \subseteq [(0, \Gamma] \cdot (K \cap Z)] \cap (K \cap Z) = \varnothing,$$

and so $R_m(z) \geq \Gamma$.

Now let $I \subseteq \mathbb{R}$ be a nonempty bounded open interval, and let $V \subseteq X$ be an open set such that $I \times V \subseteq U$. Let $r_0 > 0$ be such that $I \subseteq (-r_0, r_0)$, and let $s_0 > r_0$ be such that $s_0 \cdot y \in V$. (The existence of s_0 is guaranteed by the minimality of the action.) Pick N such that $s_0 \cdot B(y, r_N) \subseteq V$ and $R_N(z) \geq s_0 + r_0$ for all $z \in Z_N$. Note that $R_1 \leq R_2 \leq \cdots$. Let $n > N$. Then $s_0 \cdot Z_n \subseteq V$. Now let $z \in Z_n$. Then $s_0 \cdot z \in V$. Let $t \in I$. Then $-r_0 < -t < r_0$, so

$$0 < s_0 - r_0 < s_0 - t < s_0 + r_0 \leq R_N(z) \leq R_n(z).$$

Also $R_n(z) \geq r_0 + s_0 > s_0 > 0$, so $(t, s_0 z) \in T_z^n$. It is clear that $(t, s_0 z) \in I \times V \subseteq U$. Thus $T_z^n \cap U \neq \varnothing$. □

Proposition IV.2.5. *Let A_y be the C^*-algebra defined in Notation IV.2.1. Then A_y is simple.*

Proof: Recall that for each $n \geq 1$, the set Z_n denotes the pseudo-transversal that gives rise to A_n. Let $I \subseteq A_y$ be a non-zero ideal. By Lemma IV.2.3, we have $I \cap C_c(G_y) \neq 0$. So let $0 \neq f \in C_c(G_y) \cap I$. Let $U = \{x \in \mathbb{R} \times X : f(x) \neq 0\}$. Then U is open. Use Part 2 of Lemma IV.2.2 and Lemma IV.2.4 to get N such that for all $n \geq N$, the function f belongs to $C_c(G_{Z_n})$, and for all $n \geq N$ and for all $z \in Z_n$, we have $T_z^n \cap U \neq \varnothing$, where $T_z^n = \{(r, sz) : s, s-r \in (0, R_n(z))\}$. Now fix $n \geq N$.

Let $X_1, X_2, \ldots, X_m$ be the compact subsets of X associated with the pseudo-transversal Z_n as defined in Notation III.2.5. Let $\alpha_1, \ldots, \alpha_m$ be the extensions of the backward entering times associated with $X_1, \ldots, X_m$, as obtained in Lemma III.2.8. Let $\beta_1, \ldots, \beta_m$ be the extensions of the forward entering times associated with $X_1, \ldots, X_m$, as obtained in Lemma III.2.8. Then $X_1, \ldots, X_m$ are the base spaces of the stable recursive decomposition of A_n with components $C(X_i, K)$, for $i = 1, \ldots, m$, as in Corollary III.6.22. For each $i \in \{1, \ldots, m\}$ and each $x \in X_i$, let $H_i^x = \{(r, sx) : s, s-r \in (\alpha_i(x), \beta_i(x))\}$. We claim that $H_i^x \cap G_Z \cap U \neq \varnothing$ for each $i \in \{1, \ldots, m\}$ and each $x \in X_i$.

Let $i \in \{1, \ldots, m\}$, and let $x \in X_i$. Let $z = \alpha_i(x)x \in Z_n$. Then $R_n(z) \leq \beta_i(x) - \alpha_i(x)$. Let $(r, sz) \in T_z^n$. Then $(r, sz) = (r, (s + \alpha_i(x))x)$. Since $0 < s < R_n(z)$, we see that $\alpha_i(x) < s + \alpha_i(x) < R_n(z) + \alpha_i(x) \leq \beta_i(x)$, so $s + \alpha_i(x) \in (\alpha_i(x), \beta_i(x))$. Since $0 < s - r < R_n(z)$, we have

$$\alpha_i(x) < \alpha_i(x) + s - r < R_n(z) + \alpha_i(x) \leq \beta_i(x).$$

So $(r, sz) = (r, (\alpha_i(x) + s)x) \in H_i^x$. Thus $T_z^n \subseteq H_i^x$. Then, since $T_z^n \subseteq G_Z$, we see that $T_z^n \subseteq H_i^x \cap G_Z$. Thus $\varnothing \neq U \cap T_z^n \subseteq U \cap H_i^x \cap G_Z$. This proves the claim.

To finish the proof, let $(f_1, \ldots, f_m)$ be the image of f in the recursive decomposition B of A_n. Let $i \in \{1, \ldots, m\}$ and let $x \in X_i$. We just showed that $H_i^x \cap G_Z \cap U \neq \varnothing$. So $f|_{H_i^x \cap G_Z} \neq 0$. Then by Lemma III.6.21, we have $f_i(x) \neq 0$. This holds for all $i \in \{1, \ldots, m\}$ and all $x \in X_i$. So $(f_1, \ldots, f_m)$ is not contained in any primitive ideal of B, so $(f_1, \ldots, f_m)$ is not contained any proper closed ideal B, so neither can f be contained in any proper closed ideal of A_n. Therefore $I \cap A_n = A_n$. This holds for all $n \geq N$. So $I = \overline{\bigcup_{n=1}(I \cap A_n)} = \overline{\bigcup_{n \geq N} A_N} = A_y$. Thus A_y is simple. $\square$

The next lemma shows that the connecting maps in the direct system (A_n, ι_n), where A_n is as in Notation IV.2.1 and ι_n is the inclusion map, are non-vanishing.

Lemma IV.2.6. *Let A_n and A_y be as in IV.2.1. Let $\iota_n\colon A_n \to A_{n+1}$ be the inclusion. For each $n \geq 1$, let $X_1^n, \ldots, X_{l_n}^n$ be the spaces associated with the pseudo-transversal Z_{n+1} as defined in Notation III.2.5. Then for each $n \geq 1$, for each $k \in \{1, \ldots, l_n\}$, and for each $x \in X_k$, there exists some $f \in C_c(G_{Z_n})$ such that $\iota_n(f)|_{T_x} \neq 0$, where*

$$T_x = \{(r, sx)\colon s \in ((\alpha_{n+1}(x), \beta_{n+1}(x)), s - r \in (\alpha_{n+1}(x), \beta_{n+1}(x))\},$$

and where α_{n+1} and β_{n+1} are the entering times (not the extensions) associated with the pseudo-transversal Z_{n+1}.

Proof: We know that $G_{Z_n} \subseteq G_{Z_{n+1}}$. We show that $T_x \cap G_{Z_n}$ is nonempty. Because Z_n and Z_{n+1} are pseudo-transversals, there exists some $s \in (\alpha_{n+1}(x), \beta_{n+1}(x))$ such that $sx \notin Z_n$. Take $r > 0$ small enough so that $-r \in (\alpha_{n+1}(sx), \beta_{n+1}(sx))$, and that $(-2r, 2r) \cdot (sx) \subseteq Z_n^c$. Then $(r, sx) \in G_{Z_n} \cap T_x$. Thus $T_x \cap G_{Z_n} \neq \varnothing$.

Then it is clear that there exists some $f \in C_c(G_{Z_n})$ such that $f|_{T_x} \neq 0$. □

Theorem IV.2.7. *The algebra A_y is isomorphic to a simple inductive limit of SRSHAs such that all connecting the maps of the inductive system are injective and non-vanishing. Let X_n be the total space of the n-th SRSHA in the inductive system. Then $\dim(X_n) \leq d$ for some $d \in \mathbb{N}$. Moreover, A_y has topological stable rank one.*

Proof: For each $n \geq 1$, let $\iota_n\colon A_n \to A_{n+1}$ be the inclusion map. Let B_n be the SRSHA associated with the SRSH decomposition obtained in previous chapter, and let $h_n\colon A_n \to B_n$ be the isomorphism in Corollary III.6.20. Define $\zeta_n\colon B_n \to B_{n+1}$ by $\zeta_n = h_{n+1} \circ \iota_n \circ h_n^{-1}$.

It is clear that the total space of B_n has dimension less or equal to the dimension of X, which is finite. It is also clear that ζ_n is injective. Lemmas III.6.21 and IV.2.6 show that ζ_n is non-vanishing.

So the first statement of the theorem holds. It follows from Theorem II.3.23 and Porposition IV.2.5 that A_y has topological stable rank one. □

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