# COMPACT GROUP ACTIONS ON $C^*$-ALGEBRAS: CLASSIFICATION, NON-CLASSIFIABILITY, AND CROSSED PRODUCTS AND RIGIDITY RESULTS FOR $L^p$-OPERATOR ALGEBRAS.

by

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DISSERTATION ABSTRACT

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Title: Compact Group Actions on $C^*$-algebras: Classification, Non-classifiability, and Crossed Products and Rigidity Results for $L^p$-operator Algebras

This dissertation is concerned with representations of locally compact groups on different classes of Banach spaces.

The first part of this work considers representations of compact groups by automorphisms of $C^*$-algebras, also known as group actions on $C^*$-algebras. The actions we study enjoy a freeness-type of property, namely finite Rokhlin dimension. We investigate the structure of their crossed products, mainly in relation to their classifiability, and compare the notion of finite Rokhlin dimension with other existing notions of noncommutative freeness. In the case of Rokhlin dimension zero, also known as the Rokhlin property, we prove a number of classification theorems for these actions. Also, in this case, much more can be said about the structure of the crossed products. In the last chapter of this part, we explore the extent to which actions with Rokhlin dimension one can be classified. Our results show that even for $\mathbb{Z}_2$-actions on $\mathcal{O}_2$, their classification is not Borel, and hence it is intractable.

The second part of the present dissertation focuses on isometric representations of groups on $L^p$-spaces, for $p \in [1, \infty)$. For $p = 2$, these are the unitary representations on Hilbert spaces. We study the $L^p$-analogs of the full and reduced group $C^*$-algebras, particularly in connection to their rigidity. One of the main results of this work asserts that for $p \in [1, \infty) \setminus \{2\}$, the isometric isomorphism type of the reduced group $L^p$-operator algebra recovers the group. Our study of group algebras acting on $L^p$-spaces has also led us to answer a 20-year-old question of Le Merdy and Junge: for $p \neq 2$, the class of Banach algebras that can be represented on an $L^p$-space is not closed under quotients. We moreover study representations of groupoids, which are

a generalization of groups where multiplication is not always defined. The algebras associated to these objects provide new examples of $L^p$-operator algebras and recover some previously existing ones. Groupoid $L^p$-operator algebras are particularly tractable objects. For instance, while groupoid $L^p$-operator algebras can be classified by their $K_0$-group (an ordered, countable abelian group), we show that UHF-$L^p$-operator algebras not arising from groupoids cannot be classified by countable structures.

This dissertation includes unpublished coauthored material.

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Para mi mamá.

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# CHAPTER I

# INTRODUCTION

The study of group actions on $C^*$-algebras, as well as their associated crossed products, has been the object of very intensive research since the early beginnings of operator algebra theory. Both in the von Neumann algebra and in the $C^*$-algebra case, crossed products have provided a great many highly nontrivial examples via a construction that combines the dynamical properties of the action with the structural properties of the underlying algebra.

A crucial result in the context of measurable dynamics is the Rokhlin Lemma, which asserts that an aperiodic measure preserving action of $\mathbb{Z}$ can be "approximated", in a suitable sense, by finite cyclic shifts. Its reformulation in terms of outer automorphisms of (commutative) von Neumann algebras using partitions of unity consisting of orthogonal projections has led to a number of versions of the Rokhlin property, both in the von Neumann algebra and in the $C^*$-algebra case. Indeed, the Rokhlin property for discrete group actions on $C^*$-algebras first appeared in the late 1970's and early 1980's, in the work of Kishimoto [154], Fack and Maréchal [67], and Herman and Jones [113], on cyclic group actions on UHF-algebras, as well as in the work of Herman and Ocneanu [114] on integer actions on UHF-algebras. The term 'Rokhlin property' was used in later work of Kishimoto [157] and Izumi [131] for integer actions, and Izumi [132], [133] and Osaka-Phillips [191] in the finite group case. This notion was also extended to flows (actions of the reals) by Kishimoto in [158], as well as second countable compact groups by Hirshberg and Winter, in [122].

On the side of topological dynamics, the notion of freeness of a group action is central. Recall that an action of a group $G$ on a space $X$ is said to be *free* if no non-trivial group element of $G$ acts with fixed points. Unfortunately, the Rokhlin Lemma from ergodic theory no longer holds in the topological case. A useful substitute for it is Corollary 5.2 in [256], where Szabo shows that free homeomorphisms of finite dimensional compact metric spaces satisfy a higher-dimensional version of the Rokhlin Lemma. (When the homeomorphism is free and minimal, this result was first obtained by Hirshberg, Winter and Zacharias in [123].) The relevant notion here is that of 'Rokhlin dimension', introduced in [123] for finite group and integer actions on arbitrary $C^*$-algebras. Roughly speaking, the above mentioned result on topological dynamics asserts that free homeomorphisms of finite dimensional compact metric spaces can be approximated, in a suitable sense, by a finite number of cyclic shifts (and how many shifts are needed is essentially its Rokhlin dimension). It is believed that finite dimensionality of the underlying space is unnecessary, and that this condition should be replaced by the weaker assumption that the homeomorphism have "mean dimension zero".

Viewing $C^*$-algebras as noncommutative topological spaces, it is natural to look for generalizations of the concept of freeness to the case of group actions on $C^*$-algebras. It turns out that there is not a single version of noncommutative freeness. Indeed, the book [199] provides a detailed presentation and comparison of a number of them, mainly for compact Lie groups, including (locally) discrete $K$-theory, (total) $K$-freeness, (hereditary) saturation, and others. (Many of these are inspired in Atiyah-Segal's characterization of freeness from [5].) We refer the reader to [202] for a motivation of the study of free actions on $C^*$-algebras, as well of a survey that includes other more recent notions of freeness.

In Part I of this thesis, we study compact group actions with finite Rokhlin dimension, with the understanding that this could be the right notion of noncommutative freeness (particularly in its commuting towers version). We focus our study on their classification and the structure of the associated crossed products. Our stronger results assume the Rokhlin property, but we are nevertheless able to say a number of things in the more general setting of finite Rokhlin dimension.

### Classification of Actions

Classification is a major subject in all areas of mathematics, and has attracted the attention of many talented mathematicians. In the category of $C^*$-algebras, the program of classifying all simple amenable $C^*$-algebras was initiated by Elliott, first with the classification of AF-algebras, and later with the classification of certain simple $C^*$-algebras of real rank zero. His work was followed by many other classification results for nuclear $C^*$-algebras, both in the stably finite and the purely infinite case. (Of particular importance in this dissertation will be the classification theorem of Kirchberg and Phillips; see [150] and [200].)

The classification theory for von Neumann algebras precedes the classification program initiated by Elliott. In fact, the classification of amenable von Neumann algebras with separable predual, which is due to Connes, Haagerup, Krieger and Takesaki, was completed more than 30 years ago. Connes moreover classified automorphisms of the type $\mathrm{II}_1$ factor up to cocycle conjugacy in [30]. This can be regarded as the first classification result for actions on von Neumann algebras, which was complemented by his own work on the classification of pointwise outer actions of amenable groups on von Neumann algebras in [28].

Several people have since then tried to obtain similar classification results for actions on $C^*$-algebras. Early results in this direction include the work of Fack and Maréchal in [67] and [68] for cyclic groups actions on UHF-algebras, and the work of Handelman and Rossmann [109] for locally representable compact group actions on AF-algebras. Other results have been obtained by Elliott and Su in [61] for direct limit actions of $\mathbb{Z}_2$ on AF-algebras, and by Izumi in [132] and [133], where he proved a number of classification results for actions of finite groups on arbitrary unital separable $C^*$-algebras with the Rokhlin property, as well as for approximately representable actions. The classification result of Izumi for actions with the Rokhlin property has been extended recently by Nawata in [188] to cover actions on not necessarily unital separable $C^*$-algebras with what he called "almost stable rank one". It should be emphasized that the classification of group actions on $C^*$-algebras is a far less developed and generally more difficult subject than the classification of $C^*$-algebras, and it is even less developed than the classification of group actions on von Neumann algebras.

When trying to classify actions on $C^*$-algebras, one usually has to restrict oneself to a specific classifiable class of $C^*$-algebras, and also focus on a specific class of actions on them. The main feature that distinguishes the class of actions on which Izumi focused is the fact that they are not specified by the way they are constructed. Indeed, the previous known results only considered a rather limited class of finite (and sometimes compact) group actions, which in particular are direct limit actions. The class of actions considered by Izumi is the class of finite group actions with the Rokhlin property.

When the group is abelian, the dual action of a locally representable action has the Rokhlin property. This fact explains, at least heuristically, why locally representable actions can be classified in terms of their crossed products and dual actions. (In fact, Izumi obtains some of the classification theorems of the other cited authors as consequences of his results.) In this sense, it is fair to claim that the Rokhlin property is the main technical device in the majority of the classification results for finite group actions so far available. This should come as no surprise, in the light of Connes' classification of outer actions of amenable discrete groups on the hyperfinite $\mathrm{II}_1$-factor in [28], where the main technical tool is the fact that all such actions have the Rokhlin property.

It is natural to explore the classification of Rokhlin actions of compact groups on certain classes of classifiable $C^*$-algebras, complementing and generalizing Izumi's work for finite group actions with the Rokhlin property. This was probably the problem that attracted me the most in the past few years, and a number of chapters in this dissertation are devoted to it:

– In Chapter III, based on joint work with Luis Santiago, we develop the general theory of the equivariant Cuntz semigroup, which is an analog of equivariant $K$-theory using positive elements instead of projections. Although this result is not included here, Luis Santiago and I have successfully used the equivariant Cuntz semigroup to classify approximately representable actions of finite groups on direct limits of one-dimensional noncommutative CW-complexes (generalizing the results in [109], where the authors used equivariant $K$-theory for actions on AF-algebras).

– In Chapter VI, we study compact group actions with the Rokhlin property. Even though no classification results are proved here, we develop the theory and obtain important properties that are needed in Chapters XIX and X.

– In Chapter VIII, which is based on joint work with Luis Santiago, we generalize and extend Izumi's results from [132] and [133] to actions of finite groups on not necessarily unital $C^*$-algebras, classifying also equivariant homomorphisms.

– In Chapter XIX, we classify circle actions with the Rokhlin property on Kirchberg algebras. This is arguably the main chapter of this dissertation.

– In Chapter X, we classify actions of totally disconnected groups with the Rokhlin property, extending the results of Izumi for finite groups. We also prove the following automatic total disconnectedness result: if a compact group $G$ acts with the Rokhlin property on a unital $C^*$-algebra $A$ that has exactly one vanishing $K$-group, then $G$ must be totally disconnected.

– In Chapter XI, we make a connection between the Rokhlin property for a circle action and the Rokhlin property for its restrictions to finite cyclic groups, under the assumption that the underlying algebra tensorially absorbs a UHF-algebra of infinite type. Although this application will not appear here, we mention that this result enables one to use classifications results of Izumi in the finite group case (applied to the restrictions), to deduce some classification results for the circle action.

– In Chapter XII, which is based on joint work with Martino Lupini, we show that the problem of classifying finite group actions with finite Rokhlin dimension is intractable from the Borel complexity point of view.

**Structure of Crossed Products**

Crossed products have provided some of the most interesting examples of $C^*$-algebras, and studying their structure is a particularly active field of research within $C^*$-algebras. The focus is usually put on properties related to the classification program. This is relevant because it is useful to know what constructions preserve the different conditions and properties that appear as hypotheses in the main classification theorems.

Some properties are preserved under formation of crossed products in great generality. For example, crossed products of type I $C^*$-algebras by compact groups are type I, and crossed products of nuclear $C^*$-algebras by amenable groups are nuclear, regardless of the action. On the other hand, for preservation of other finer properties, one must assume some kind of freeness condition on the action. For example, reduced crossed products by pointwise outer actions of countable discrete groups preserve the class of purely infinite simple $C^*$-algebras, by Corollary 4.6 in [135] (here reproduced as part (2) of Theorem II.2.8). As a consequence, the class of Kirchberg algebras (separable, simple, nuclear and purely infinite $C^*$-algebras) is preserved by formation of crossed products by pointwise outer actions of countable discrete amenable groups.

In the finite group case, the Rokhlin property implies very strong structure preservation results for crossed products; see Theorem 2.3 in [202] for a list of properties that are preserved by Rokhlin actions, and see [122], [191] and [203] for the proofs of most of them. Some of these properties, specifically absorption of a strongly self-absorbing $C^*$-algebra and approximate divisibility, were shown in [122] to be preserved by Rokhlin actions of not just finite, but also compact groups.

On the side of finite Rokhlin dimension, one should expect far fewer preservation results in this context, due to the additional flexibility allowed. Moreover, there seems to be a significant difference between the commuting and noncommuting towers versions. As of positive results, the work of Hirshberg-Winter-Zacharias ([123]) shows that formation of crossed products by finite group actions with finite Rokhlin dimension (in any version) preserve finiteness of the nuclear dimension and decomposition rank. Furthermore, in the commuting towers version, the class of Jiang-Su stable $C^*$-algebras is also preserved. Little seems to be known regarding preservation of other properties. (This problem will be studied in [87].)

It should also be pointed out that the class of separable, nuclear, unital, simple $C^*$-algebras of tracial rank zero (in the sense of Lin; see [167]) is preserved under formation of crossed products by finite groups actions with the tracial Rokhlin property; see [203].

One of the projects I pursued during my studies was extending such results to the context of compact group actions. (In the case of finite Rokhlin dimension, this also required introducing a suitable definition.) The questions and problems addressed in each the works mentioned above are different, and consequently the approaches used by these authors are substantially distinct in some cases. Extending the results of [203], [191] and [194] to the case of arbitrary compact groups required new insights, since the main technical tool in all of these works (Theorem 3.2 in [191]) seems not to have a satisfactory analog in the compact group case. A similar problem arose when extending the results in [123], since the authors made essential use of the positive elements in the algebra coming from the different towers.

Chapters IV, V (finite Rokhlin dimension) and VII (Rokhlin property) are devoted to the study of these problems.

## Notation

Most of the notation will be established as it gets used. Here, we collect some standard symbols that will be assumed to be familiar throughout.

Let $A$ be a $C^*$-algebra. We denote by $\mathrm{Aut}(A)$ the automorphism group of $A$, and the identity map of $A$ is denoted $\mathrm{id}_A$. The suspension of $A$ is denoted by $SA$. We denote by $M(A)$ its multiplier algebra and by $\widetilde{A}$ its unitization (that is, the $C^*$-algebra obtained by adjoining a unit to $A$, even if $A$ is unital). If $A$ is unital, we denote by $\mathcal{U}(A)$ its unitary group. Homomorphisms of

$C^*$-algebras will always be assumed to preserve the adjoint operation (and hence they will always be contractive and have closed range).

An action of a topological group $G$ on a $C^*$-algebra $A$ will always mean a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$ which is continuous in the following sense: for $a \in A$, the map $G \to A$ given by $g \mapsto \alpha(g)(a)$ is continuous. As usual, we abbreviate $\alpha(g)$ to $\alpha_g$, and therefore write $\alpha_g(a)$ instead of the cumbersome $\alpha(g)(a)$. We denote by $A^G$ or $A^\alpha$ the subalgebra of $A$ consisting of those elements that are fixed by $\alpha_g$ for all $g \in G$.

For a positive integer $n \geq 2$, we denote by $\mathcal{O}_n$ the Cuntz algebra on $n$ generators. That is, the universal unital $C^*$-algebra generated by $n$ isometries whose range projections add up to the unit. The $C^*$-algebra $\mathcal{O}_\infty$ is the universal unital $C^*$-algebra generated by countably many isometries with orthogonal ranges.

For Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we write $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the Banach space of bounded, linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, we write $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ for the closed subspace of compact operators, and we write $\mathcal{U}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of unitaries from $\mathcal{H}_1$ to $\mathcal{H}_2$. When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, we write $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ for $\mathcal{B}(\mathcal{H}, \mathcal{H})$, $\mathcal{K}(\mathcal{H}, \mathcal{H})$ and $\mathcal{U}(\mathcal{H}, \mathcal{H})$. (In this case, $\mathcal{B}(\mathcal{H})$ is a $C^*$-algebra, $\mathcal{K}(\mathcal{H})$ is a closed, two-sided ideal in $\mathcal{B}(\mathcal{H})$, and $\mathcal{U}(\mathcal{H})$ is a group.) When $\mathcal{H}$ is separable and infinite dimensional, we usually write $\mathcal{K}$ for $\mathcal{K}(\mathcal{H})$.

We write $\mathbb{N}$ for $\{1, 2, \ldots\}$; we write $\overline{\mathbb{N}}$ for $\mathbb{N} \cup \{\infty\}$; we write $\mathbb{Z}_{\geq 0}$ for $\{0, 1, 2, \ldots\}$; and we write $\overline{\mathbb{Z}_{\geq 0}}$ for $\mathbb{Z}_{\geq 0} \cup \{\infty\}$.

For $n \in \mathbb{N}$, the finite cyclic group of order $n$ will be denoted by $\mathbb{Z}_n$. The circle will be denoted by $\mathbb{T}$ when it is regarded as a group, and by $S^1$ when it is regarded as a topological space.

# CHAPTER II

# BACKGROUND AND PRELIMINARIES

This chapter contains background material which is essentially well-known and established. It is nevertheless included here for the sake of completeness, and to establish the notation and terminology. In each section, the reader is referred to either the original source or a standard reference for further reading.

## Topological Groups: Representations, Dual Group, and Haar Measure

This dissertation is mostly concerned with group representations (or actions) on different kinds of objects ($C^*$-algebras, Banach algebras, Banach spaces). We devote the first preliminary subsection to recalling the basics of topological groups, particularly the dual group and the Haar measure.

The reader is referred to the first chapter of [268] for more about topological groups.

**Definition II.1.1.** A *topological group* (usually just 'group'), is a Hausdorff topological space whose underlying set has a group structure such that multiplication and inversion are continuous maps. A (topological) group is said to be locally compact, compact, discrete, connected, etc., if the underlying topological space has the corresponding property.

**Remark II.1.2.** Finite groups admit a unique topology making them into a topological groups, namely the discrete topology. It is also easy to see that a compact, discrete group must be finite.

For most of this dissertation, topological groups will be locally compact and second countable. (Many things work without any countability assumption, but second countable groups are enough for our purposes.) By a theorem of Birkoff-Kakutani (Theorem 1.22 in [184]), a topological group is metrizable if and only if it is first countable. In particular, all our groups will be metrizable.

It is well-known that a compact metrizable group admits a translation-invariant metric. We will implicitly choose such a metric on all our (compact) groups, which will be denoted by $d$.

**Definition II.1.3.** Let $G$ be a locally compact group. A *unitary representation* $u$ of $G$ on a Hilbert space $\mathcal{H}_u$ is a group homomorphism $u\colon G \to \mathcal{U}(\mathcal{H}_u)$ from $G$ to the unitary group of $\mathcal{H}_u$, such that for $\xi \in \mathcal{H}_u$, the function $G \to \mathcal{H}_u$ given by $g \mapsto u(g)\xi$, is continuous (in the norm topology of $\mathcal{H}_u$). We say that $u$ is *irreducible* if the only closed subspaces $E$ of $\mathcal{H}_u$ satisfying $u(g)E \subseteq E$ for all $g \in G$ are $E = \{0\}$ and $E = \mathcal{H}_u$.

Two unitary representations $u\colon G \to \mathcal{U}(\mathcal{H}_u)$ and $v\colon G \to \mathcal{U}(\mathcal{H}_v)$ are said to be *unitarily equivalent* if there exists a unitary operator $W\colon \mathcal{H}_u \to \mathcal{H}_v$ such that

$$u(g) = W^* v(g) W$$

for all $g \in G$.

We define the *dual of* $G$, which we denote by $\widehat{G}$, to be the set of unitary equivalence classes of irreducible unitary representations of $G$.

It is not completely trivial that $\widehat{G}$ is indeed a set, since the class of all unitary representations of $G$ is not a set even if $G$ is the trivial group. One possible argument is as follows. First, if a Hilbert space has an orthonormal basis whose cardinality is bigger than that of $G$, then there cannot be any irreducible representations of $G$ on it. (In particular, if $G$ is second countable, then the Hilbert space must be separable.) It follows that the dimension of the Hilbert spaces appearing in the definition of $\widehat{G}$ is bounded. Since all Hilbert spaces of the same dimension are isometrically isomorphic, we can choose one Hilbert space for each dimension not exceeding the cardinality of $G$. The resulting class of Hilbert spaces is then a set, and the class of all irreducible unitary representations of $G$ on elements of this set is also a set.

When $G$ is abelian, the dual of $G$ can be given a group structure as well as a (locally compact) topology, making $\widehat{G}$ into a group in its own right. A concrete picture of $\widehat{G}$ when $G$ is abelian is:

$$\widehat{G} = \{\gamma\colon G \to \mathbb{T}\colon \gamma \text{ is a continuous group homomorphism}\},$$

endowed with the topology of uniform convergence, and pointwise product. It is well-known that $G$ is compact (discrete) if and only if $\widehat{G}$ is discrete (compact), and that $G$ is connected (torsion-free) if and only $\widehat{G}$ is torsion-free (connected).

A celebrated theorem of Pontryagin asserts that there is a canonical topological group isomorphism $\varphi G \to \widehat{\widehat{G}}$ given by $\varphi_g(\gamma) = \gamma(g)$ for all $g \in G$ and all $\gamma \in \widehat{G}$.

*Haar measure*

Haar measures are a generalization of the Lebesgue measure on the Euclidean space to abstract locally compact groups. Their existence is in fact equivalent to local compactness of the group, so we will exclusively deal with this class. The Haar measure will be crucial in Chapter XIV.

Topological groups will always be endowed with their Borel $\sigma$-algebras.

**Definition II.1.4.** A measure $\mu$ on a topological group $G$ is said to be *left (respectively, right) translation invariant* if $\mu(gE) = \mu(E)$ (respectively, $\mu(Eg) = \mu(E)$) for every $g \in G$ and for every Borel set $E \subseteq G$.

A measure is said to be *translation invariant* if it is both left and right translation invariant.

Examples of Haar measures are the Lebesgue measure on $\mathbb{R}^n$, the arc length measure on $\mathbb{T}$, and the counting measure on any discrete group.

Positive left (right) invariant measures are called left (right) Haar measures, and they always exist on locally compact groups, thanks to the following fundamental result of Haar.

**Theorem II.1.5.** Every locally compact group admits a left (right) Haar measure, which is unique up to scalar factors.

In contrast to Haar's theorem, not every locally compact group admits a translation invariant measure. (One such example is the so-called $ax + b$ group.) The modular function, to be defined below, is a measurement of how far the left Haar measure is from being right invariant.

**Theorem II.1.6.** Let $G$ be a locally compact group, and fix a left Haar measure $\mu$. Then there exists a continuous group homomorphism $\Delta\colon G \to \mathbb{R}^+$ such that

$$\int_G f(gh)\ d\mu(g) = \Delta(h) \int_G f(hg)\ d\mu(g)$$

for all $f \in C_c(G)$.

This function moreover satisfies

$$\int_G f(g)\ d\mu(g) = \int_G \Delta(g^{-1})f(g^{-1})\ d\mu(g)$$

for all $f \in C_c(G)$.

Furthermore, the function $\Delta$ is independent of the choice of the left Haar measure.

**Corollary II.1.7.** Denote by $\mu$ and $\nu$ a left and a right Haar measure on $G$. Then $\mu$ and $\nu$ are mutually absolutely continuous, and its Radon-Nykodim derivative is given by $\Delta$. In other words, $\Delta$ satisfies

$$\int_G f(g)\ d\mu(g) = \int_G f(g)\Delta(g)\ d\nu(g)$$

for all $f \in C_c(G)$.

It is clear that $\Delta(g) = 1$ for all $g \in G$ if and only if $G$ has a left and right invariant Haar measure.

**Definition II.1.8.** The function $\Delta\colon G \to \mathbb{R}^+$ is called the *modular function* of $G$. A locally compact group is said to be *unimodular* if its modular function is identically one.

The following classes consist of unimodular groups:

1. Abelian groups (left and right translation agree);

2. Discrete groups (counting measure is left and right invariant);

3. Compact groups (the only compact subgroup of $\mathbb{R}^+$ is $\{1\}$, so it must be $\Delta(G) = \{1\}$);

4. Simple groups.

### Group Actions on $C^*$-algebras and Crossed Products

This section contains the basic definitions of group actions on $C^*$-algebras and their crossed products. The reader can find much more additional information in Chapter 2 of [268].

**Definition II.2.1.** Let $G$ be a locally compact group and let $A$ be a $C^*$-algebra. An *action* of $G$ on $A$ is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$ such that, for every $a \in A$, the map $G \to A$ given by $g \mapsto \alpha_g(a)$, is continuous. (This last condition is usually referred to as ‘strong continuity’.)

The continuity requirement in the definition above is designed in such a way that actions of a group $G$ on commutative $C^*$-algebras are in one-to-one correspondence with continuous actions of $G$ by homeomorphisms of the maximal ideal space.

Examples of actions on $C^*$-algebras are plentiful. The reader is referred to Section 3 in [212] for the construction of some interesting ones, and Section 10 there for the computation of some of their crossed products (which we define later).

For the rest of this section, we fix a locally compact group $G$, a $C^*$-algebra $A$, and an action $\alpha\colon G \to \mathrm{Aut}(A)$. The triple $(G, A, \alpha)$ is usually referred to as a *$C^*$-dynamical system.* Similarly, we may sometimes say that $(A, \alpha)$ is a $G$-algebra.

**Definition II.2.2.** We define the twisted convolution algebra $L^1(G, A, \alpha)$ as follows. Its underlying Banach space is just $L^1(G, A)$ (with the obvious norm), while its multiplication and involution are given by

$$(ab)(g) = \int_G a(h)\alpha_h(b(h^{-1}g))\ d\mu(g) \ \text{ and } \ a^*(g) = \Delta(g^{-1})\alpha_g(a(g^{-1}))^*$$

for all $a, b \in L^1(G, A, \alpha)$ and all $g \in G$.

The (full) crossed product of $(G, A, \alpha)$ will be defined as a the completion of $L^1(G, A, \alpha)$ with respect to contractive $*$-representations on Hilbert spaces. Such representations are in one-to-one correspondence with what is usually called a *covariant representation* of the dynamical system $(G, A, \alpha)$. We define these representations below.

**Definition II.2.3.** A *(nondegenerate) covariant representation* of $(G, A, \alpha)$ is a triple $(\mathcal{H}, u, \pi)$, consisting of a Hilbert space $\mathcal{H}$, a (nondegenerate) representation $\pi\colon A \to \mathcal{B}(\mathcal{H})$, and a unitary representation $u\colon G \to \mathcal{U}(\mathcal{H})$, that satisfy

$$\pi(\alpha_g(a)) = u_g\pi(a)u_g^*$$

for all $g \in G$ and all $a \in A$.

The proof of the following proposition is routine in the case of discrete groups, and requires some additional work in the locally compact case.

**Proposition II.2.4.** Nondegenerate covariant representations of $(G, A, \alpha)$ are in one-to-one correspondence with contractive, nondegenerate $*$-representations of $L^1(G, A, \alpha)$ on Hilbert spaces. For a nondegenerate covariant representation $(\mathcal{H}, u, \pi)$, the corresponding representation of $L^1(G, A, \alpha)$ is sometimes denoted

$$\pi \rtimes u \colon L^1(G, A, \alpha) \to \mathcal{B}(\mathcal{H}),$$

and it is given by

$$(\pi \rtimes u)(a)(\xi) = \int_G \pi(a(g)) u_g(\xi) \, dg$$

for all $a \in L^1(G, A, \alpha)$ and all $\xi \in \mathcal{H}$.

We will call the representation $\pi \rtimes u$ in the proposition above the *integrated form* of $(\mathcal{H}, u, \pi)$.

In principle, it is not clear whether covariant representations always exist, or whether there are 'many' of them. There is a class of them which is particularly easy to construct: the *regular* covariant representations.

**Example II.2.5.** Let $\mathcal{H}_0$ be a Hilbert space, and let $\pi_0 \colon A \to \mathcal{B}(\mathcal{H}_0)$ be a non-degenerate representation. Set $\mathcal{H} = L^2(G, \mathcal{H}_0)$. Let $u \colon G \to \mathcal{U}(\mathcal{H})$ be given by

$$u_g(\xi)(h) = \xi(g^{-1}h)$$

for all $g, h \in G$ and all $\xi \in \mathcal{H}$, and let $\pi \colon A \to \mathcal{B}(\mathcal{H})$ be given by

$$(\pi(a)\xi)(g) = \pi_0(\alpha_{g^{-1}}(a))(\xi(g))$$

for all $a \in A$, for all $\xi \in \mathcal{H}$, and for all $g \in G$. Then the triple $(\mathcal{H}, u, \pi)$ is a covariant representation, called the *(left) regular covariant representation* associated to the quintuple $(G, A, \alpha, \mathcal{H}_0, \pi_0)$.

The integrated form of a regular covariant representation is a nondegenerate contractive, homomorphism of $L^1(G, A, \alpha)$, and it will be called a *regular representation* of $L^1(G, A, \alpha)$.

**Remark II.2.6.** The term 'regular' in the example above is justified because the regular covariant representation associated to the quintuple $(G, \mathbb{C}, \mathrm{id}_{\mathbb{C}}, \mathbb{C}, \mathrm{id}_{\mathbb{C}})$ is the left regular representation of $G$.

We proceed to give the definition of full and reduced crossed products.

**Definition II.2.7.** The *full crossed product* of $(G, A, \alpha)$, denoted $A \rtimes_\alpha G$, is the completion of $L^1(G, A, \alpha)$ in the norm

$$\|a\|_{A\rtimes_\alpha G} = \sup\left\{\|\sigma(a)\| \colon \sigma\colon L^1(G,A,\alpha) \to \mathcal{B}(\mathcal{H}) \textit{ is a contractive } {*}\textit{-representation}\right\}.$$

The *reduced crossed product* of $(G, A, \alpha)$, denoted $A \rtimes_\alpha^\lambda G$, is the completion of $L^1(G, A, \alpha)$ in the norm

$$\|a\|_{A\rtimes_\alpha^\lambda G} = \sup\left\{\|\sigma(a)\| \colon \sigma\colon L^1(G,A,\alpha) \to \mathcal{B}(\mathcal{H}) \textit{ is a regular contractive } {*}\textit{-representation}\right\}.$$

By definition, it is clear that

$$\|a\|_{A\rtimes_\alpha^\lambda G} \leq \|a\|_{A\rtimes_\alpha G}$$

for all $a \in L^1(G, A, \alpha)$. It follows that the identity map on $L^1(G, A, \alpha)$ extends to a contractive homomorphism

$$\kappa\colon A \rtimes_\alpha G \to A \rtimes_\alpha^\lambda G,$$

whose range is dense, since it contains $L^1(G, A, \alpha)$. Basic properties of $C^*$-algebras imply that $\kappa$ is a quotient map.

It is well-known that $\kappa$ is an isomorphism whenever $G$ is amenable. This fact will be repeatedly used without particular reference throughout this work.

The structure of crossed products and fixed point algebras is one of the main concerns in the first part of this dissertation. Generally speaking, one is interested in finding suitable conditions on a group action, which ensure that relevant properties of the $C^*$-algebra that is being acted on, are inherited by the crossed product and the fixed point algebra. The results summarized in the following theorem are a first step in this direction, and will be used repeatedly throughout this work. The condition on the action is rather weak, which makes the result very useful in applications. On the other hand, for preservation of finer properties, stronger conditions on the actions must be imposed.

Recall that a simple $C^*$-algebra $A$ is said to be *purely infinite*, if for every $a, b \in A$ with $a \neq 0$, there exist $x, y \in A$ such that $xay = b$.

**Theorem II.2.8.** Let $A$ be a $C^*$-algebra, let $G$ be a countable discrete group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Assume that $\alpha_g$ is not inner for all $g \in G \setminus \{1\}$.

1. If $A$ is simple, then so is $A \rtimes_\alpha^\lambda G$. (Theorem 3.1 in [155].)

2. If $A$ is simple and purely infinite, then so is $A \rtimes_\alpha^\lambda G$. (Corollary 4.6 in [135].)

The conclusions of the above theorem fail in general for actions of arbitrary locally compact groups, or even compact groups. For example, the gauge action $\gamma$ of $\mathbb{T}$ on the Cuntz algebra $\mathcal{O}_\infty$, given by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta$ in $\mathbb{T}$ and all $j$ in $\mathbb{N}$, is pointwise outer by the Theorem in [181], and its crossed product $\mathcal{O}_\infty \rtimes_\gamma \mathbb{T}$ is a non-simple AF-algebra, so it is far from being (simple and) purely infinite.

The following result is useful when working with compact group actions. Since its proof is concise and very conceptual, we include it here for the sake of completeness.

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action of a compact group $G$ on a $C^*$-algebra $A$, and let $a \in A^G$. We denote by $c_a\colon G \to A$ the continuous function that is constantly equal to $a$. Then $c_a$ belongs to $L^1(G, A, \alpha)$, and that the assignment $a \mapsto c_a$ defines an injective homomorphism $c\colon A^G \to L^1(G, A, \alpha)$. (Recall that the product in $L^1(G, A, \alpha)$ is given by twisted convolution.)

**Theorem II.2.9.** (Theorem in [238].) Let $A$ be a $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then the image of $A^\alpha$ in $A \rtimes_\alpha G$ under the map $c$ is a corner.

*Proof.* Denote by 1 the unit of $M(A)$, and by $c_1$ the function on $G$ which is constantly equal to 1. Then $c_1$ belongs to $M(A \rtimes_\alpha G)$ (and it belongs to $A \rtimes_\alpha G$ if $A$ is unital). Since $c_a = c_1 c_a c_1$ for all $a \in A^\alpha$, we clearly have $c(A^\alpha) \subseteq c_1(A \rtimes_\alpha G)c_1$. Let us show the converse inclusion.

Given $f \in L^1(G, A, \alpha)$ and $t \in G$, we have

$$(f \cdot c_1)(t) = \int_G f(r)\alpha_s(c_1(r^{-1}t)\ dr = \int_G f(r)\ dr,$$

and thus

$$(c_1 \cdot f \cdot c_1)(t) = \int_G c_1(s)\alpha_s((f \cdot c(1))(s^{-1}t)\; ds = \int_G \int_G \alpha_s(f(r))\; drds.$$

In particular, $c_1 \cdot f \cdot c_1$ is a constant function whose constant value belongs to $A^\alpha$. It follows that $c_1(A \rtimes_\alpha G)c_1 \subseteq c(A^\alpha)$, and the proof is complete. $\square$

**Equivariant $K$-theory**

Equivariant $K$-theory for compact groups acting on topological spaces was introduced by Atiyah (the paper [249], by Segal, contains a basic treatment of the theory). One of the first applications of this theory was a striking characterization of freeness of a compact Lie group action ([5]; see also Theorem 1.1.1 in [199]). Equivariant $K$-theory was later defined and studied for actions of compact groups on noncommutative $C^*$-algebras. A fundamental result in this area is Julg's identification ([139]) of the equivariant $K$-theory of a given action with the ordinary $K$-theory of its associated crossed product. In a different direction, each of the different statements in Atiyah-Segal's characterization of freeness, interpreted in the context of $C^*$-algebras, can be taken as possible definitions of "noncommutative freeness". This is the approach taken by Phillips in [199]. Equivariant $K$-theory has also been used as an invariant for compact group actions ([109], [80]), and its definition has been extended to actions of more general objects, such as quantum groups ([64]).

We devote this subsection to recalling the definition and some basic facts about equivariant $K$-theory. A thorough development can be found in [199], whose notation we will follow. We denote the suspension of a $C^*$-algebra $A$ by $SA$.

**Definition II.3.1.** Let $G$ be a compact group and let $(G, A, \alpha)$ be a unital $G$-algebra. Let $\mathcal{P}_G(A)$ be the set of all $G$-invariant projections in all of the algebras $\mathcal{B}(V) \otimes A$, for all unitary finite dimensional representations $\lambda\colon G \to \mathcal{U}(V)$, the $G$-action on $\mathcal{B}(V) \otimes A$ being the diagonal action, that is, the one determined by $g \mapsto \mathrm{Ad}(\lambda(g)) \otimes \alpha_g$ for $g \in G$. There is no ambiguity about the tensor product norm on $\mathcal{B}(V) \otimes A$ since $V$ is finite dimensional.

Two $G$-invariant projections $p, q \in \mathcal{P}_G(A)$ are said to be *Murray-von Neumann equivalent* if there exists a $G$-invariant element $s \in \mathcal{B}(V, W) \otimes A$ such that $s^*s = p$ and $ss^* = q$. Given a unitary finite dimensional representation $\lambda\colon G \to \mathcal{U}(V)$ of $G$ and a $G$-invariant projection $p \in \mathcal{B}(V) \otimes A$, and to emphasize role played by the representation $\lambda$, we denote the element in

$\mathcal{P}_G(A)$ it determines by $(p, V, \lambda)$. We let $S_G(A)$ be the set of equivalence classes in $\mathcal{P}_G(A)$ with addition given by direct sum.

We define the *equivariant* $K_0$*-group* of $(G, A, \alpha)$, denoted $K_0^G(A)$, to be the Grothendieck group of $S_G(A)$.

Define the *equivariant* $K_1$*-group* of $(G, A, \alpha)$, denoted $K_1^G(A)$, to be $K_0^G(SA)$, where the action of $G$ on $SA$ is trivial in the suspension direction.

If confusion is likely to arise as to with respect to what action the equivariant $K$-theory of $A$ is being taken, we will write $K_0^\alpha(A)$ and $K_1^\alpha(A)$ instead of $K_0^G(A)$ and $K_1^G(A)$.

**Remark II.3.2.** The equivariant $K$-theory of $(G, A, \alpha)$ is a module over the representation ring $R(G)$ of $G$, which can be identified with $K_0^G(\mathbb{C})$, with the operation given by tensor product. This is, if $(p, V, \lambda) \in \mathcal{P}_G(A)$ and $(W, \mu)$ is a finite dimensional representation space of $G$, we define

$$(W, \mu) \cdot (p, V, \lambda) = (p \otimes 1_W, V \otimes W, \lambda \otimes \mu).$$

The induced operation $R(G) \times K_0^G(A) \to K_0^G(A)$ makes $K_0^G(A)$ into an $R(G)$-module. One defines the $R(G)$-module structure on $K_1^G(A)$ analogously.

The following result is Julg's Theorem (Theorem 2.6.1 in [199]).

**Theorem II.3.3.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then there is a natural ($\mathbb{Z}_2$-graded) isomorphism

$$J\colon K_j^G(A) \cong K_j(A \rtimes_\alpha G).$$

The isomorphism $J$ in Theorem II.3.3 (which we will usually suppress from the notation) induces a canonical $R(G)$-module structure on $K_*(A \rtimes_\alpha G)$ which is easiest to describe when $G$ is abelian. In this case, the dual group $\widehat{G}$ is discrete and there is a canonical identification $R(G) = \mathbb{Z}[\widehat{G}]$. It follows that the $R(G)$-module structure is determined by the action of the elements in $\widehat{G}$.

The following theorem specifies this module structure: it is given by the dual action of $\alpha$.

**Theorem II.3.4.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. For $\chi \in \widehat{G}$ and $x \in K_*(A \rtimes_\alpha G)$, we have

$$\chi \cdot J^{-1}(x) = J^{-1}(K_*(\widehat{\alpha}_\chi)(x)).$$

Let $A$ and $B$ be $C^*$-algebras, let $G$ be a locally compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be continuous actions of $G$ on $A$ and $B$ respectively. We say that $\alpha$ and $\beta$ are *conjugate* if there exists an isomorphism $\varphi\colon A \to B$ such that

$$\alpha_g = \varphi^{-1} \circ \beta_g \circ \varphi$$

for all $g$ in $G$. Isomorphisms of this form are called *equivariant*, and we usually use the notation $\varphi\colon (A, \alpha) \to (B, \beta)$ to mean that $\varphi$ satisfies the condition above.

A weaker form of equivalence for action is given by exterior equivalence, which we define below.

**Definition II.3.5.** Let $G$ be a locally compact group, let $A$ and $B$ be a $C^*$-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be continuous actions. We say that $\alpha$ and $\beta$ are *cocycle conjugate* if there exist an isomorphism $\theta\colon A \to B$ and a function $u\colon G \to \mathcal{U}(\mathcal{M}(B))$ such that:

1. $u_{gh} = u_g\theta(\alpha_g(\theta^{-1}(u_h)))$ for all $g, h \in G$,

2. For each $b \in B$, the map $G \to B$ given by $g \mapsto u_g b$ is continuous,

such that $\mathrm{Ad}(u_g) \circ \alpha_g = \beta_g$ for all $g \in G$.

The following result is folklore, and its proof is included here for the convenience of the reader.

**Proposition II.3.6.** Let $G$ be a locally compact abelian group, let $A$ and $B$ be a $C^*$-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be cocycle conjugate actions. Then there exists an isomorphism $\phi\colon A \rtimes_\alpha G \to A \rtimes_\beta G$ that intertwines the dual actions, that is, such that for every $\chi \in \widehat{G}$, we have $\widehat{\beta}_\chi \circ \phi = \phi \circ \widehat{\alpha}_\chi$.

*Proof.* Let $\theta\colon A \to B$ and let $u\colon G \to \mathcal{U}(M(B))$ be as in the definition of exterior equivalence above. Define

$$\phi_0\colon L^1(G, A, \alpha) \to L^1(G, B, \beta)$$

by $\phi_0(a)(g) = \theta(a(g))u_g^*$ for $a \in L^1(G,A,\alpha)$ and $g \in G$. One readily checks that $\phi_0$ is an isometric isomorphism of Banach $*$-algebras, and it therefore extends to an isomorphism $\phi\colon A \rtimes_\alpha G \to B \rtimes_\beta G$ of $C^*$-algebras.

In order to check that $\phi$ intertwines the dual actions, it is enough to check it on $C_c(G,A)$. (Note that $\phi_0(C_c(G,A)) \subseteq C_c(G,B)$.) Given a continuous function $a \in C_c(G,A)$, a group element $g \in G$ and a character $\chi \in \widehat{G}$, we have

$$\begin{aligned}
\phi\left(\widehat{\alpha}_\chi(a)\right)(g) &= \theta\left(\widehat{\alpha}_\chi(a)(g)\right)u_g^* \\
&= \chi(g)\theta(a(g))u_g^* \\
&= \chi(g)\phi(a)(g) \\
&= \widehat{\beta}_\chi(\phi(a))(g),
\end{aligned}$$

and the proof follows. □

**(Central) Sequence Algebras**

Let $A$ be a unital $C^*$-algebra. Let $\ell^\infty(\mathbb{N},A)$ denote the set of all bounded sequences $(a_n)_{n\in\mathbb{N}}$ in $A$, endowed with the supremum norm

$$\|(a_n)_{n\in\mathbb{N}}\| = \sup_{n\in\mathbb{N}} \|a_n\|$$

and pointwise operations. Then $\ell^\infty(\mathbb{N},A)$ is a unital $C^*$-algebra, the unit being the constant sequence $1_A$. Let

$$c_0(\mathbb{N},A) = \left\{(a_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N},A)\colon \lim_{n\to\infty}\|a_n\| = 0\right\}.$$

Then $c_0(\mathbb{N},A)$ is an ideal in $\ell^\infty(\mathbb{N},A)$, and we denote the quotient

$$\ell^\infty(\mathbb{N},A)/c_0(\mathbb{N},A)$$

by $A_\infty$. Write $\kappa_A\colon \ell^\infty(\mathbb{N},A) \to A_\infty$ for the quotient map. We identify $A$ with the unital subalgebra of $\ell^\infty(\mathbb{N},A)$ consisting of the constant sequences, and with a unital subalgebra of $A_\infty$ by taking its image under $\kappa_A$. We write $A_\infty \cap A'$ for the relative commutant of $A$ inside of

$A_\infty$. (We warn the reader that what we denote by $A_\infty$ is sometimes denoted by $A^\infty$, and what we denote by $A_\infty \cap A'$ is sometimes denoted by $A_\infty$; see, for example, [122].)

If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of $G$ on $A$, there are actions of $G$ on $\ell^\infty(\mathbb{N}, A)$ and on $A_\infty$, which we denote $\alpha^\infty$ and $\alpha_\infty$, respectively. Note that

$$(\alpha_\infty)_g(A_\infty \cap A') \subseteq A_\infty \cap A',$$

for all $g \in G$, so that $\alpha_\infty$ restricts to an action on $A_\infty \cap A'$, also denoted by $\alpha_\infty$.

When $G$ is not discrete, these actions are not necessarily continuous, as the next example shows.

**Example II.4.1.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T}))$ be the action induced by left translation. For $n \in \mathbb{N}$, let $u_n \in C(\mathbb{T})$ be the unitary given by $u_n(\zeta) = \zeta^n$ for all $\zeta \in \mathbb{T}$. Set $u = (u_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N}, C(\mathbb{T}))$. It is not difficult to check that the assignments

$$\zeta \mapsto (\alpha^\infty)_\zeta(u) \ \text{ and } \zeta \mapsto (\alpha_\infty)_\zeta(u),$$

are not continuous as a maps $\mathbb{T} \to \ell^\infty(\mathbb{N}, C(\mathbb{T}))$ and $\mathbb{T} \to C(\mathbb{T})_\infty = C(\mathbb{T})_\infty \cap C(\mathbb{T})'$, respectively. We leave the details to the reader.

To remedy this issue, we set

$$\ell^\infty_\alpha(\mathbb{N}, A) = \{a \in \ell^\infty(\mathbb{N}, A)\colon g \mapsto (\alpha_\infty)_g(a) \ \text{ is continuous}\},$$

and $A_{\infty,\alpha} = \kappa_A(\ell^\infty_\alpha(\mathbb{N}, A))$. By construction, $A_{\infty,\alpha}$ is invariant under $\alpha_\infty$, and the restriction of $\alpha_\infty$ to $A_{\infty,\alpha}$, which we also denote by $\alpha_\infty$, is continuous.

**Remark II.4.2.** We note that if $\varphi\colon A \to B$ is a unital homomorphism of unital $C^*$-algebras, then $\varphi$ induces unital homomorphisms $\ell^\infty(\mathbb{N}, \varphi)\colon \ell^\infty(\mathbb{N}, A) \to \ell^\infty(\mathbb{N}, B)$ and $\varphi_\infty\colon A_\infty \to B_\infty$. The assignments $A \mapsto \ell^\infty(\mathbb{N}, A)$ and $A \mapsto A_\infty$ are functorial for unital $C^*$-algebras and unital homomorphisms.

Functoriality of the assignment $A \mapsto A_\infty \cap A'$ is more subtle, since not every map between $C^*$-algebras induces a map between the corresponding central sequences. In Lemma II.4.3 and

Lemma II.4.4, we show two instances in which this is indeed the case. These two cases will be needed in the following section.

**Lemma II.4.3.** Let $\varphi\colon A \to B$ be a surjective unital homomorphism of unital $C^*$-algebras. Then the restriction of $\varphi_\infty$ to $A_\infty \cap A'$ induces a unital homomorphism $\varphi_\infty\colon A_\infty \cap A' \to B_\infty \cap B'$.

Moreover, if $G$ is a locally compact group and $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ are continuous actions of $G$ on $A$ and $B$ respectively, then $\varphi$ also induces a unital homomorphism $\varphi_\infty\colon A_{\infty,\alpha} \cap A' \to B_{\infty,\beta} \cap B'$, which is equivariant if $\varphi\colon A \to B$ is.

*Proof.* We only need to check that $\varphi_\infty(A_\infty \cap A') \subseteq B_\infty \cap B'$. Let $a = (a_n)_{n\in\mathbb{N}}$ be in $A_\infty \cap A'$ and let $b \in B$. We have to show that $\varphi_\infty(a)$ commutes with $\kappa_B(b)$. Choose $c \in A$ such that $\varphi(c) = b$. Since $\kappa_B \circ \varphi = \varphi_\infty \circ \kappa_A$, we have

$$[\varphi_\infty(a), b] = [\varphi_\infty(a), \kappa_B(\varphi(c))] = [a, \kappa_A(c)] = 0,$$

and the result follows.

The proof of the second claim is analogous. □

An important case in which a unital homomorphism between unital $C^*$-algebras induces a unital homomorphism between the central sequence algebras is that of the unital inclusion $A \hookrightarrow A \otimes B$ of a unital $C^*$-algebra as the first tensor factor. This homomorphism is not covered by the previous lemma, so we shall prove it separately.

**Lemma II.4.4.** Let $A$ and $B$ be unital $C^*$-algebras, let $A \otimes B$ be any $C^*$-algebra completion of the algebraic tensor product of $A$ and $B$, and let $\iota\colon A \to A \otimes B$ be given by $\iota(a) = a \otimes 1$ for all $a \in A$. Then $\iota_\infty$ restricts to a unital homomorphism

$$\iota_\infty\colon A_\infty \cap A' \to (A \otimes B)_\infty \cap (A \otimes B)'.$$

Moreover, if $G$ is a locally compact group and $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ are continuous actions of $G$ on $A$ and $B$ respectively, and if the tensor product action

$$g \mapsto (\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$$

extends to $A \otimes B$, then $\iota$ induces a unital equivariant homomorphism

$$\iota_\infty \colon A_{\infty,\alpha} \cap A' \to (A \otimes B)_{\infty,\alpha\otimes\beta} \cap (A \otimes B)'.$$

*Proof.* Let $a = (a_n)_{n\in\mathbb{N}}$ in $A_\infty \cap A'$ and let $x \in A \otimes B$. We may assume that $x$ is a simple tensor, say $x = c \otimes b$ for some $c \in A$ and some $b \in B$. Then

$$[\iota_\infty(a), x] = [(a_n \otimes 1)_{n\in\mathbb{N}}, \kappa_{A\otimes B}(c \otimes b)] = 0,$$

since $\lim\limits_{n\to\infty} \|[a_n, c]\| = 0$.

The proof of the second claim is straightforward. $\square$

The following proposition relates the crossed product functor with the sequence algebra functor.

**Proposition II.4.5.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. Then there is a canonical embedding

$$A_{\infty,\alpha} \rtimes_{\alpha_\infty} G \hookrightarrow (A \rtimes_\alpha G)_\infty.$$

*Proof.* Note that if $B$ is a $C^*$-algebra, then there is a unital map $M(B)_\infty \to M(B_\infty)$. The canonical maps $A \to M(A \rtimes_\alpha G)$ and $G \to M(A \rtimes_\alpha G)$ induce canonical maps

$$A_{\infty,\alpha} \to (M(A \rtimes_\alpha G))_\infty \to M((A \rtimes_\alpha G)_\infty)$$

and

$$G \to (M(A \rtimes_\alpha G))_\infty \to M(A \rtimes_\alpha G)_\infty$$

which satisfy the covariance condition for $\alpha_\infty$. It follows from the universal property of the crossed product $A_{\infty,\alpha} \rtimes_{\alpha_\infty} G$ that there is a map as in the statement. This map is injective because so is $A_{\infty,\alpha} \to (M(A \rtimes_\alpha G))_\infty$ and the group $G$ is amenable, being compact (see, for example, part (1) in Theorem 9.22 of [212]). $\square$

In the proposition above, the canonical embedding will in general not be surjective unless $G$ is the trivial group.

### Completely Positive Maps of Order Zero

We briefly recall some of the basics of completely positive order zero maps. See [271] for more details and further results.

Let $A$ be a $C^*$-algebra, and let $a, b$ be elements in $A$. We say that $a$ and $b$ are *orthogonal*, and write $a \perp b$, if $ab = ba = a^*b = ab^* = 0$. If $a, b \in A$ are selfadjoint, then they are orthogonal if and only if $ab = 0$.

**Definition II.5.1.** Let $A$ and $B$ be $C^*$-algebras, and let $\varphi\colon A \to B$ be a completely positive map. We say that $\varphi$ has *order zero* if for every $a$ and $b$ in $A$, we have $\varphi(a) \perp \varphi(b)$ whenever $a \perp b$.

**Remark II.5.2.** It is straightforward to check that $C^*$-algebra homomorphisms have order zero, and that the composition of two order zero maps again has order zero.

The following is the main result in [271].

**Theorem II.5.3.** (Theorem 2.3 and Corollary 3.1 in [271]) Let $A$ and $B$ be $C^*$-algebras. There is a bijection between completely positive contractive order zero maps $A \to B$ and $C^*$-algebra homomorphisms $C_0((0,1]) \otimes A \to B$. A completely positive contractive order zero map $\varphi\colon A \to B$ induces the homomorphism $\rho_\varphi\colon C_0((0,1]) \otimes A \to B$ determined by $\rho_\varphi(\mathrm{id}_{(0,1]} \otimes a) = \varphi(a)$ for all $a \in A$. Conversely, if $\rho\colon C_0((0,1]) \otimes A \to B$ is a homomorphism, then the induced completely positive contractive order zero map $\varphi_\rho\colon A \to B$ is the one given by $\varphi_\rho(a) = \rho(\mathrm{id}_{(0,1]} \otimes a)$ for all $a \in A$.

The following easy corollary will be used throughout without reference.

**Corollary II.5.4.** Let $A$ and $B$ be unital $C^*$-algebras, let $G$ be a locally compact group, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be continuous actions of $G$ on $A$ and $B$ respectively, and let $\varphi\colon A \to B$ be a completely positive order zero map. Denote by $\rho_\varphi\colon C_0((0,1]) \otimes A \to B$ the induced homomorphism given by Theorem II.5.3. Give $C_0((0,1])$ the trivial action of $G$, and give $C_0((0,1]) \otimes A$ the corresponding diagonal action. Then $\varphi$ is equivariant if and only if $\rho_\varphi$ is equivariant.

*Proof.* We denote by $\widetilde{\alpha}\colon G \to \mathrm{Aut}(C_0((0,1]) \otimes A)$ the diagonal action described in the statement. Assume that $\rho$ is equivariant. Given $g$ in $G$ and $a$ in $A$, we have

$$\begin{aligned}
\rho_\varphi(\widetilde{\alpha}_g(\mathrm{id}_{(0,1]} \otimes a)) &= \rho_\varphi(\mathrm{id}_{(0,1]} \otimes \alpha_g(a)) \\
&= \rho(\alpha_g(a)) \\
&= \beta_g(\rho(a)) \\
&= \beta_g(\rho_\varphi(\mathrm{id}_{(0,1]} \otimes a)).
\end{aligned}$$

Since $\mathrm{id}_{(0,1]}$ generates $C_0((0,1])$, we conclude that $\varphi_\rho$ is equivariant.

Conversely, if $\rho_\varphi$ is equivariant, it is clear that the map $A \to B$ given by $a \mapsto \rho_\varphi(\mathrm{id}_{(0,1]} \otimes a)$, which clearly agrees with $\rho$, is also equivariant. This finishes the proof. $\square$

It is a well-known fact that equivariant homomorphisms between dynamical systems induce homomorphisms between the crossed products, a fact that can be easily seen by considering the universal property of such objects. Using the structure of order zero maps, it follows that an analogous statement holds for completely contractive order zero maps that are equivariant.

**Proposition II.5.5.** Let $A$ and $B$ be $C^*$-algebras, let $G$ be a locally compact group, let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions, and let $\rho\colon A \to B$ be an equivariant completely positive contractive order zero map. Then $\rho$ induces a canonical completely positive contractive order zero map

$$\sigma\colon A \rtimes_\alpha G \to B \rtimes_\beta G.$$

*Proof.* Denote by $\varphi_\rho\colon C_0((0,1]) \otimes A \to B$ be the homomorphism determined by $\rho$ as in Theorem II.5.3. Denote by $\widetilde{\alpha}$ the diagonal action $\widetilde{\alpha} = \mathrm{id}_{C_0((0,1])} \otimes \alpha$ of $G$ on $C_0((0,1]) \otimes A$. Since $\varphi_\rho$ is equivariant with respect to this action by Corollary II.5.4, there is a canonical homomorphism

$$\psi\colon (C_0((0,1]) \otimes A) \rtimes_{\widetilde{\alpha}} G \to B \rtimes_\beta G.$$

Identify $(C_0((0,1]) \otimes A) \rtimes_{\widetilde{\alpha}} G$ with $C_0((0,1]) \otimes (A \rtimes_\alpha G)$ in the usual way. Then the order zero map $\sigma\colon A \rtimes_\alpha G \to B \rtimes_\beta G$, given by $\sigma(x) = \psi(\mathrm{id}_{(0,1]} \otimes x)$ for $x$ in $A \rtimes_\alpha G$, is the desired order zero map. $\square$

## The Cuntz Semigroup, the $\mathrm{Cu}^\sim$ Semigroup, and the Category Cu

The Cuntz semigroup $\mathrm{Cu}(A)$ of a $C^*$-algebra $A$, first considered by Cuntz in the 70's ([40]), has been intensively studied in the last decade since Toms successfully used it ([263]) to distinguish two non-isomorphic $C^*$-algebras with identical Elliott invariant (as well as identical real and stable ranks). Coward, Elliott and Ivanescu ([36]) suggested that the Cuntz semigroup could be used as an invariant for $C^*$-algebras (in many cases, finer than $K_0$), and this semigroup has since then been used to obtain positive classification results of not necessarily simple $C^*$-algebras. We refer the reader to [23] and [4] for thorough developments of the theory of the Cuntz semigroup. On the other hand, the $\mathrm{Cu}^\sim$ semigroup was introduced by Robert in [230], where he showed that the Cuntz semigroup is a complete invariant for (not necessarily simple) direct limits of 1-dimensional noncommutative CW-complexes with trivial $K_1$, a class that contains all AI-algebras. (When the algebra in question is unital, Robert showed that the $\mathrm{Cu}^\sim$ semigroup can be replaced by the Cuntz semigroup.)

In this section, we will recall the definitions of the Cuntz and $\mathrm{Cu}^\sim$ semigroups, as well as the category **Cu**, to which these semigroups naturally belong. The author is referred to the papers [36] and [23] for much more about the Cuntz semigroup and the category **Cu**, and to the paper [230] for more on the $\mathrm{Cu}^\sim$ semigroup.

### The category **Cu**

Let $S$ be an ordered semigroup and let $s, t \in S$. We say that $s$ is *compactly contained* in $t$, and denote this by $s \ll t$, if whenever $(t_n)_{n\in\mathbb{N}}$ is an increasing sequence in $S$ such that $t \leq \sup\limits_{n\in\mathbb{N}} t_n$, there exists $k \in \mathbb{N}$ such that $s \leq t_k$. A sequence $(s_n)_{n\in\mathbb{N}}$ is said to be *rapidly increasing* if $s_n \ll s_{n+1}$ for all $n \in \mathbb{N}$.

**Definition II.6.1.** An ordered abelian semigroup $S$ is an object in the category **Cu** if it has a zero element and it satisfies the following properties:

(O1) Every increasing sequence in $S$ has a supremum;

(O2) For every $s \in S$ there exists a rapidly increasing sequence $(s_n)_{n\in\mathbb{N}}$ in $S$ such that $s = \sup\limits_{n\in\mathbb{N}} s_n$;

(O3) If $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are increasing sequences in $S$, then

$$\sup_{n\in\mathbb{N}} s_n + \sup_{n\in\mathbb{N}} t_n = \sup_{n\in\mathbb{N}} (s_n + t_n);$$

(O4) If $s_1, s_2, t_1, t_2 \in S$ satisfy $s_1 \ll t_1$ and $s_2 \ll t_2$, then $s_1 + s_2 \ll t_1 + t_2$.

Let $S$ and $T$ be semigroups in the category **Cu**. An order preserving semigroup map $\varphi\colon S \to T$ is a morphism in the category **Cu** if it preserves the zero element and it satisfies the following properties:

(M1) If $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence in $S$, then

$$\varphi\left(\sup_{n\in\mathbb{N}} s_n\right) = \sup_{n\in\mathbb{N}} \varphi(s_n);$$

(M2) If $s, t \in S$ satisfy $s \ll t$, then $\varphi(s) \ll \varphi(t)$.

The following observation will be used repeatedly.

**Remark II.6.2.** Let $M$ be a partially ordered semigroup with identity element, and let $S$ be a semigroup in **Cu**. Suppose that there exists a semigroup morphism $\varphi\colon M \to S$ (or $\varphi\colon S \to M$) preserving the zero element and such that:

1. $\varphi$ preserves the order, that is, $x \leq y$ in $M$ implies $\varphi(x) \leq \varphi(y)$ in $S$;

2. $\varphi$ is an order embedding, that is, $\varphi(x) \leq \varphi(y)$ in $S$ implies $x \leq y$ in $M$ (this implies that $\varphi$ is injective); and

3. $\varphi$ is bijective.

Then $M$ belongs to **Cu**, and $\varphi$ is a Cu-isomorphism. In particular, $\varphi$ automatically preserves suprema of increasing sequences, and the compact containment relation.

The next result will be important in Chapter VIII:

**Theorem II.6.3.** (Theorem 2 in [36].) The category **Cu** is closed under sequential inductive limits.

The following description of inductive limits in the category **Cu** follows from the proof of this theorem.

**Proposition II.6.4.** Let $(S_n, \varphi_n)_{n\in\mathbb{N}}$, with $\varphi_n\colon S_n \to S_{n+1}$, be an inductive system in the category **Cu**. For $m, n \in \mathbb{N}$ with $m \geq n$, let $\varphi_{n,m}\colon S_n \to S_m$ denote the composition $\varphi_{n,m} = \varphi_{m-1} \circ \cdots \circ \varphi_n$. A pair $(S, (\varphi_{n,\infty})_{n\in\mathbb{N}})$, consisting of a semigroup $S$ and morphisms $\varphi_{n,\infty}\colon S_n \to S$ in the category **Cu** satisfying $\varphi_{n+1,\infty} \circ \varphi_n = \varphi_{n,\infty}$ for all $n \in \mathbb{N}$, is the inductive limit of the system $(S_n, \varphi_n)_{n\in\mathbb{N}}$ if and only if:

(1) For every $s \in S$ there exist elements $s_n \in S_n$ for $n \in \mathbb{N}$ such that $\varphi_n(s_n) \ll s_{n+1}$ for all $n \in \mathbb{N}$ and

$$s = \sup_{n\in\mathbb{N}} \varphi_{n,\infty}(s_n);$$

(2) Whenever $s, s', t \in S_n$ satisfy $\varphi_{n,\infty}(s) \leq \varphi_{n,\infty}(t)$ and $s' \ll s$, there exists $m \geq n$ such that $\varphi_{n,m}(s') \leq \varphi_{n,m}(t)$.

The Cuntz semigroup

Let $A$ be a $C^*$-algebra and let $a, b \in A$ be positive elements. We say that $a$ is *Cuntz subequivalent* to $b$, and denote this by $a \precsim b$, if there exists a sequence $(d_n)_{n\in\mathbb{N}}$ in $A$ such that $\lim\limits_{n\to\infty} \|d_n^* b d_n - a\| = 0$. We say that $a$ is *Cuntz equivalent* to $b$, and denote this by $a \sim b$, if $a \precsim b$ and $b \precsim a$. It is clear that $\precsim$ is a preorder relation on the set of positive elements of $A$, and thus $\sim$ is an equivalence relation. We denote by $[a]$ the Cuntz equivalence class of the element $a \in A_+$.

The following result, due to Rørdam, will be technically important in a number of our proofs. Given $\varepsilon > 0$, let $f_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be the continuous function given by

$$f_\varepsilon(t) = (t-\varepsilon)_+ = \begin{cases} t-\varepsilon, & \text{if } t \geq \varepsilon; \\ 0, & \text{otherwise.} \end{cases}$$

for $t \in \mathbb{T}$. For a positive (or selfadjoint) element $a$ in a $C^*$-algebra $A$, and for $\varepsilon > 0$, we denote by $(a-\varepsilon)_+$ the element $f_\varepsilon(a)$ obtained via continuous functional calculus.

**Proposition II.6.5.** (Rørdam's Lemma; Proposition 2.4 in [232].) Let $A$ be a $C^*$-algebra, and let $a, b \in A$ be positive elements. Then the following are equivalent:

1. $a \precsim b$;

2. There exist sequences $(r_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ in $A$ such that

$$\lim_{n\to\infty} \|r_n b s_n - a\| = 0.$$

3. For every $\varepsilon > 0$ there exist $\delta > 0$ and $d \in A$ such that

$$(a-\varepsilon)_+ = d(b-\delta)_+ d^*.$$

If, in addition, $A$ has stable rank one, then the above are also equivalent to

4. For every $\varepsilon > 0$ there exists a unitary $u \in \mathcal{U}(A)$ such that $u(a-\varepsilon)_+ u^*$ belongs to the hereditary subalgebra of $A$ generated by $b$.

The first conclusion of the following lemma was proved in [232, Proposition 2.2] (see also [152, Lemma 2.2]). The second statement was shown in [231, Lemma 1].

**Lemma II.6.6.** Let $A$ be a $C^*$-algebra and let $a$ and $b$ be positive elements in $A$ such that $\|a - b\| < \varepsilon$. Then $(a-\varepsilon)_+ \precsim b$. More generally, if $r$ is a non-negative real number, then $(a-r-\varepsilon)_+ \precsim (b-r)_+$.

The Cuntz semigroup of $A$, denoted by $\mathrm{Cu}(A)$, is defined as the set of Cuntz equivalence classes of positive elements of $A \otimes \mathcal{K}$. Addition in $\mathrm{Cu}(A)$ is given by

$$[a] + [b] = [a' + b'],$$

where $a', b' \in (A \otimes \mathcal{K})_+$ are orthogonal and satisfy $a' \sim a$ and $b' \sim b$. Furthermore, $\mathrm{Cu}(A)$ becomes an ordered semigroup when equipped with the order $[a] \leq [b]$ if $a \precsim b$. If $\phi\colon A \to B$ is a homomorphism, then $\phi$ induces an order preserving map $\mathrm{Cu}(\phi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$, given by $\mathrm{Cu}(\phi)([a]) = [(\phi \otimes \mathrm{id}_{\mathcal{K}})(a)]$ for every $a \in (A \otimes \mathcal{K})_+$.

**Remark II.6.7.** Let $A$ be a $C^*$-algebra, let $a \in A$ and let $\varepsilon > 0$. It can be checked that $[(a-\varepsilon)_+] \ll [a]$ and that $[a] = \sup\limits_{\varepsilon>0} [(a-\varepsilon)_+]$, thus showing that $\mathrm{Cu}(A)$ satisfies Axiom O2.

**Theorem II.6.8.** (Theorems 1 and 2 in [36].) Cu is a functor from the category of $C^*$-algebras to the category **Cu**, and it preserves inductive limits of sequences.

More recently, Antoine, Perera and Thiel have shown in Corollary 3.1.11 in [4], that the category **Cu** is in fact closed under arbitrary inductive limits, and that the functor Cu is continuous in full generality.

The $\mathrm{Cu}^{\sim}$-semigroup

Here we define the $\mathrm{Cu}^{\sim}$-semigroup of a $C^*$-algebra. This semigroup was introduced in [230] in order to classify certain inductive limits of 1-dimensional NCCW-complexes. [230]

**Definition II.6.9.** Let $A$ be $C^*$-algebra and let $\pi\colon \widetilde{A} \to \widetilde{A}/A \cong \mathbb{C}$ denote the quotient map. Then $\pi$ induces a semigroup homomorphism

$$\mathrm{Cu}(\pi)\colon \mathrm{Cu}(\widetilde{A}) \to \mathrm{Cu}(\mathbb{C}) \cong \overline{\mathbb{Z}_{\geq 0}}.$$

We define the semigroup $\mathrm{Cu}^{\sim}(A)$ by

$$\mathrm{Cu}^{\sim}(A) = \{([a], n) \in \mathrm{Cu}(\widetilde{A}) \times \mathbb{Z}_{\geq 0} \mid \mathrm{Cu}(\pi)([a]) = n\}/\sim,$$

where $\sim$ is the equivalence relation defined by $([a], n) \sim ([b], m)$ if there exists $k \in \mathbb{N}$ such that

$$[a] + m[1] + k[1] = [b] + n[1] + k[1],$$

The image of the element $([a], n)$ under the canonical quotient map is denoted by $[a] - n[1]$.

Addition in $\mathrm{Cu}^{\sim}(A)$ is induced by pointwise addition in $\mathrm{Cu}(\widetilde{A}) \times \mathbb{Z}_{\geq 0}$. The semigroup $\mathrm{Cu}^{\sim}(A)$ can be endowed with an order: we say that $[a]-n[1] \leq [b]-m[1]$ in $\mathrm{Cu}^{\sim}(A)$ if there exists $k$ in $\mathbb{Z}_{\geq 0}$ such that

$$[a] + (m+k)[1] \leq [b] + (n+k)[1]$$

in $\mathrm{Cu}(\widetilde{A})$.

The assignment $A \mapsto \mathrm{Cu}^{\sim}(A)$ can be turned into a functor as follows. Let $\phi\colon A \to B$ be a homomorphism and let $\widetilde{\phi}\colon \widetilde{A} \to \widetilde{B}$ denote the unital extension of $\phi$ to the unitizations of $A$ and $B$. Let us denote by $\mathrm{Cu}^{\sim}(\phi)\colon \mathrm{Cu}^{\sim}(A) \to \mathrm{Cu}^{\sim}(B)$ the map defined by

$$\mathrm{Cu}^{\sim}(\phi)([a] - n[1]) = \mathrm{Cu}(\widetilde{\phi})([a]) - n[1].$$

It is clear that $\mathrm{Cu}^{\sim}(\phi)$ is order preserving, and thus $\mathrm{Cu}^{\sim}$ becomes a functor from the category of $C^*$-algebras to the category of ordered semigroups.

It was shown in [230] that the $\mathrm{Cu}^{\sim}$-semigroup of a $C^*$-algebra with stable rank one belongs to the category **Cu**, that $\mathrm{Cu}^{\sim}$ is a functor from the category of $C^*$-algebras of stable rank one to the category **Cu**, and that it preserves inductive limits of sequences.

CHAPTER III

# THE EQUIVARIANT CUNTZ SEMIGROUP

This Chapter is based on joint work with Luis Santiago ([92]).

We introduce an equivariant version of the Cuntz semigroup, which takes an action of a compact group into account. The equivariant Cuntz semigroup is naturally a semimodule over the representation semiring of the given group. Moreover, this semimodule satisfies a number of additional structural regularity properties. We show that the equivariant Cuntz semigroup, as a functor, is continuous and stable. Moreover, cocycle conjugate actions have isomorphic associated equivariant Cuntz semigroups. One of our main results is an analog of Julg's theorem: the equivariant Cuntz semigroup is canonically isomorphic to the Cuntz semigroup of the crossed product. We compute the induced semimodule structure on the crossed product, which in the abelian case is given by the dual action. As an application of our results, we show that freeness of a compact Lie group action on a compact Hausdorff space can be characterized in terms of a canonically defined map into the equivariant Cuntz semigroup, extending results of Atiyah and Segal for equivariant $K$-theory.

## Introduction

In this chapter, which is based on [92], we study an equivariant version of the Cuntz semigroup for compact group actions on $C^*$-algebras. For an action $\alpha\colon G \to \mathrm{Aut}(A)$ of a compact group $G$ on a $C^*$-algebra $A$, we denote its equivariant Cuntz semigroup by $\mathrm{Cu}^G(A,\alpha)$. This is a partially ordered semigroup, and it has a natural semimodule structure over the representation semiring $\mathrm{Cu}(G)$ of $G$. We explore some basic properties of the functor $(A,\alpha) \mapsto \mathrm{Cu}^G(A,\alpha)$, such as continuity, stability, passage to full hereditary subalgebras, cocycle equivalence invariance, etc. One of the main results of this chapter (Theorem III.5.3) is an analog of Julg's theorem for the Cuntz semigroup: $\mathrm{Cu}^G(A,\alpha)$ is naturally isomorphic to $\mathrm{Cu}(A \rtimes_\alpha G)$. The induced $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$ is computed in Theorem III.5.13 (see Proposition III.5.15 for a simpler description when $G$ is abelian). Finally, we use the equivariant Cuntz semigroup to prove an analog of Atiyah-Segal's characterization of freeness; see Theorem III.6.6. Further applications of the equivariant Cuntz semigroup will appear in subsequent work.

We have organized this chapter as follows. In Section III.2, and after reviewing the definition of the Cuntz semigroup and the Cuntz category **Cu**, we introduce the equivariant Cuntz semigroup using positive invariant elements in suitable stabilizations of the algebra (Definition III.2.3). The main result of this section, Corollary III.2.8, asserts that the equivariant Cuntz semigroup is a functor from the category of $G$-$C^*$-algebras (that is, $C^*$-algebras with an action of $G$), to the category **Cu**.

In Section III.3, we introduce the representation semiring $\mathrm{Cu}(G)$ of $G$ (Definition III.3.1), which corresponds to the equivariant Cuntz semigroup of $G$ acting on $\mathbb{C}$ (Theorem III.3.3), and define a canonical $\mathrm{Cu}(G)$-action on $\mathrm{Cu}^G(A, \alpha)$; see Definition III.3.9. With this semimodule structure, the equivariant Cuntz semigroup becomes a functor from $G$-$C^*$-algebras to a distinguished category of $\mathrm{Cu}(G)$-semimodules (see Definition III.3.6 and Theorem III.3.10). We finish this section by showing that the functor $\mathrm{Cu}^G$ is stable (Proposition III.3.11) and continuous with respect to countable inductive limits (Proposition III.3.12).

Section III.4 is devoted to giving two pictures of the equivariant Cuntz semigroup using equivariant Hilbert modules. In one of these pictures, we identify $\mathrm{Cu}^G(A, \alpha)$ with the ordinary Cuntz semigroup of $\mathcal{K}(\mathcal{H}_A)^G$, where $\mathcal{H}_A$ is the universal equivariant Hilbert module for $(A, \alpha)$ introduced by Kasparov in [144] (see Definition III.4.3). In the second picture, we identify $\mathrm{Cu}^G(A, \alpha)$ with the $\mathrm{Cu}(G)$-semimodule obtained by taking a suitable equivalence relation (Definition III.4.2) on the class of equivariant Hilbert modules; see Corollary III.4.16.

Section III.5 contains some of our main results. In Theorem III.5.3, we construct a natural Cu-isomorphism $\mathrm{Cu}^G(A, \alpha) \cong \mathrm{Cu}(A \rtimes_\alpha G)$. (This is a Cuntz semigroup analog of Julg's theorem for $K$-theory; see [139].) The induced $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$ is computed in Theorem III.5.13, with an easier description available when $G$ is abelian; see Proposition III.5.15. As an application, we show that invariant full hereditary subalgebras have canonically isomorphic equivariant Cuntz semigroups (Proposition III.5.16).

Finally, in Section III.6, we apply the theory developed in the previous sections to prove a characterization of freeness of a compact Lie group action on a compact Hausdorff space, in terms of a certain canonical map into the equivariant Cuntz semigroup; see Theorem III.6.6. This characterization resembles (and depends on) Atiyah-Segal's characterization of freeness using equivariant $K$-theory ([5]).

Throughout this chapter, for a compact group $G$, we denote by $\mathrm{Cu}(G)$ the class of unitary equivalence classes of unitary representations of $G$ on separable Hilbert spaces. It is easy to see, fixing a separable Hilbert space and restricting to representations on it, that $\mathrm{Cu}(G)$ is in fact a set. This set has important additional structure that will not be discussed until it is needed in Section III.3. The set $\mathrm{Cu}(G)$ will play mostly a notational role in the first few sections.

We sometimes make a slight abuse of notation and do not distinguish between elements in $\widehat{G}$ (or $\mathrm{Cu}(G)$) and irreducible (separable) unitary representations of $G$. A unitary representation $\mu\colon G \to \mathcal{U}(\mathcal{H}_\mu)$ of $G$ on a Hilbert space $\mathcal{H}_\mu$ will usually be abbreviated to $(\mathcal{H}_\mu, \mu)$. We say that $(\mathcal{H}_\mu, \mu)$ is separable, or finite dimensional, if $\mathcal{H}_\mu$ is. The unitary equivalence class of $(\mathcal{H}_\mu, \mu)$ is denoted by $[\mu]$.

## The Equivariant Cuntz Semigroup

In this section, for a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$ of a compact group $G$ on a $C^*$-algebra $A$, we define its equivariant Cuntz semigroup $\mathrm{Cu}^G(A, \alpha)$ (Definition III.2.3), and explore some basic properties. The main result of this section, Corollary III.2.8, asserts that the equivariant Cuntz semigroup is a functor from the category of $G$-$C^*$-algebras to the Cuntz category **Cu**.

### *The equivariant Cuntz semigroup*

For the rest of this section, we fix a compact group $G$, a $C^*$-algebra $A$, and a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$.

Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$. We endow the Banach space $\mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu)$ with the $G$-action given by

$$g \cdot T = \nu_g \circ T \circ \mu_{g^{-1}},$$

for $g \in G$ and $T \in \mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu)$. It is clear that $\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu)$ is an invariant, closed subspace, which we will endow with the restricted $G$-action. With these actions, a $G$-invariant linear map is precisely a map $\mathcal{H}_\mu \to \mathcal{H}_\nu$ that is $\mu - \nu$ equivariant.

**Definition III.2.1.** Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ and $b \in (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$ be positive elements. We say that $a$ is *$G$-Cuntz subequivalent* to $b$, and denote this by $a \precsim_G b$, if there is a sequence $(d_n)_{n\in\mathbb{N}}$ in $(\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes A)^G$ such that $\lim\limits_{n\to\infty} \|d_n b d_n^* - a\| = 0$. We say that $a$ is *$G$-Cuntz equivalent* to $b$, and denote this by $a \sim_G b$, if $a \precsim_G b$ and $b \precsim_G a$. The $G$-Cuntz equivalence class of a positive element $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ will be denoted by $[a]_G$.

We claim that the relation $\precsim_G$ is transitive. To see this, let $(\mathcal{H}_\mu, \mu)$, $(\mathcal{H}_\nu, \nu)$ and $(\mathcal{H}_\lambda, \lambda)$ be separable unitary representation of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$, $b \in (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$, and $c \in \mathcal{K}(\mathcal{H}_\lambda \otimes A)^G$ satisfy $a \precsim_G b$ and $b \precsim_G c$. Fix $\varepsilon > 0$, and find $x \in (\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes A)^G$ such that $\|a - xbx^*\| < \frac{\varepsilon}{2}$. Also, since $b \precsim_G c$ there exists $y \in (\mathcal{K}(\mathcal{H}_\nu, \mathcal{H}_\lambda) \otimes A)^G$ such that

$$\|a - xycy^*x^*\| < \frac{\varepsilon}{2\|x\|}.$$

The element $z = xy$ belongs to $(\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\lambda) \otimes A)^G$, and satisfies $\|a - zcz^*\| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this implies that $a \precsim_G c$. In particular, it follows that $\sim_G$ is an equivalence relation.

The following lemma is a simple corollary of [152, Lemma 2.4] and the definition of $G$-Cuntz subequivalence.

**Proposition III.2.2.** Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ and $b \in (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$ be positive elements. The following are equivalent:

1. $a \precsim_G b$.
2. For every $\varepsilon > 0$, there exists $d \in (\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes A)^G$ such that
$$(a - \varepsilon)_+ = dbd^*.$$
3. For every $\varepsilon > 0$, there exist $\delta > 0$ and $d \in (\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes A)^G$, such that
$$(a - \varepsilon)_+ = d(b - \delta)_+ d^*.$$

**Definition III.2.3.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Define the *equivariant Cuntz semigroup* $\mathrm{Cu}^G(A,\alpha)$, of the dynamical system $(G,A,\alpha)$, to be the set of $G$-Cuntz equivalence classes of positive elements in all of the algebras of the form $(\mathcal{K}(\mathcal{H}_\mu)\otimes A)^G$, where $(\mathcal{H}_\mu,\mu)$ is a separable unitary representation of $G$.

We define addition on $\mathrm{Cu}^G(A,\alpha)$ as follows. Let $(\mathcal{H}_\mu,\mu)$ and $(\mathcal{H}_\nu,\nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu)\otimes A)^G$ and $b \in (\mathcal{K}(\mathcal{H}_\nu)\otimes A)^G$ be positive elements. Denote by $a\oplus b$ the positive element

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

in $(\mathcal{K}(\mathcal{H}_\mu\oplus\mathcal{H}_\nu)\otimes A)^G$, and set $[a]_G + [b]_G = [a\oplus b]_G$. (One must check that the definition is independent of the representatives, but this is routine.)

Finally, we endow $\mathrm{Cu}^G(A,\alpha)$ with the partial order given by $[a]_G \leq [b]_G$ if $a \precsim_G b$. (One has to again check that the order is well defined; we omit the proof.)

It is clear that if $\beta\colon G \to \mathrm{Aut}(B)$ is another continuous action of $G$ on a $C^*$-algebra $B$, and if $\psi\colon A \to B$ is an equivariant homomorphism, then $\psi$ induces an ordered semigroup homomorphism $\mathrm{Cu}^G(\psi)\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(B,\beta)$, given by

$$\mathrm{Cu}^G(\psi)([a]_G) = [(\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)}\otimes\psi)(a)]_G$$

for $a \in (\mathcal{K}(\mathcal{H}_\mu)\otimes A)^G$.

The rest of this section is devoted to proving that the equivariant Cuntz semigroup is a functor from the category of $G$-$C^*$-algebras to the category **Cu** (Definition II.6.1). This will be done by showing that $\mathrm{Cu}^G(A,\alpha)$ can be written as an inductive limit of semigroups in **Cu**; see Theorem III.2.6.

We point out that in Section III.3, we will show that the equivariant Cuntz semigroup has additional structure, and that $\mathrm{Cu}^G(A,\alpha)$ belongs to a certain category of semimodules; see Definition III.3.6 and Theorem III.3.10.

**Lemma III.2.4.** Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ be a positive element. Suppose that there exists $V \in (\mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes A)^G$ satisfying $V^*V = \mathrm{id}_{\mathcal{H}_\mu} \otimes 1_A$. Then $VaV^*$ is a positive element in $(\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$, and $a \sim_G VaV^*$.

Moreover, if $W \in (\mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes A)^G$ is another element satisfying $W^*W = \mathrm{id}_{\mathcal{H}_\mu} \otimes 1_A$, then $WaW^* \sim_G VaV^*$.

*Proof.* It is clear that $VaV^*$ is a $G$-invariant element in $\mathcal{K}(\mathcal{H}_\nu) \otimes A$. Likewise, for $n \in \mathbb{N}$, we have $a^{\frac{1}{n}} V^* \in (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$. Now,

$$\lim_{n \to \infty} \left\| \left(a^{1/n} V^*\right) (VaV^*) \left(a^{1/n} V^*\right)^* - a \right\| = \lim_{n \to \infty} \left\| a^{1/n} a a^{1/n} - a \right\| = 0,$$

so $a \precsim_G VaV^*$. Similarly, $Va^{\frac{1}{n}} \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$, and one shows that

$$\lim_{n \to \infty} \left\| \left(Va^{1/n}\right) a \left(Va^{1/n}\right)^* - VaV^* \right\| = 0.$$

We conclude that $a \sim_G VaV^*$, as desired.

The last part of the statement is immediate. $\square$

Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, such that $(\mathcal{H}_\mu, \mu)$ is unitarily equivalent to a subrepresentation of $(\mathcal{H}_\nu, \nu)$. Then there exists $W_{\mu,\nu} \in \mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu)^G$ satisfying $W_{\mu,\nu}^* W_{\mu,\nu} = \mathrm{id}_{\mathcal{H}_\mu}$. Set

$$V_{\mu,\nu}^A = W_{\mu,\nu} \otimes 1_A \in \mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu) \otimes M(A).$$

It is clear that $V_{\mu,\nu}^A$ is $G$-invariant and that $(V_{\mu,\nu}^A)^* V_{\mu,\nu}^A = \mathrm{id}_{\mathcal{H}_\mu} \otimes 1_A$.

Set

$$\iota_{\mu,\nu}^A = \mathrm{Ad}(V_{\mu,\nu}^A) \colon (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G \to (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G.$$

Then $\iota_{\mu,\nu}$ is a $*$-homomorphism, since $V_{\mu,\nu}$ is an isometry. Let

$$j_{\mu,\nu}^A = \mathrm{Cu}(\mathrm{Ad}(V_{\mu,\nu})) \colon \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G) \to \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G)$$

be the **Cu**-morphism given by $j_{\mu,\nu}^A = \mathrm{Cu}(\iota_{\mu,\nu}^A)$. Whenever $A$ and $\alpha$ are clear from the context, we will write $V_{\mu,\nu}$ for $V_{\mu,\nu}^A$, and similarly for $\iota_{\mu,\nu}$ and $j_{\mu,\nu}$.

**Lemma III.2.5.** Adopt the notation from the discussion above.

1. The map $j_{\mu,\nu}$ is independent of the choice of $V_{\mu,\nu}$.

2. If $(\mathcal{H}_\mu, \mu)$ is a separable unitary representation of $G$ which is unitarily equivalent to a subrepresentation of $(\mathcal{H}_\nu, \nu)$, then $j_{\mu,\nu} \circ j_{\lambda,\mu} = j_{\lambda,\nu}$.

*Proof.* The first part is an immediate consequence of Lemma III.2.4. The second one is straightforward. □

Let us consider the action on $\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathcal{H}_\mu \otimes A)^G$ induced by tensor product of the trivial action of $G$ on $\ell^2(\mathbb{N})$ and the given action of $G$ on $\mathcal{H}_\mu \otimes A$. Then $\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathcal{H}_\mu \otimes A)^G$ is naturally isomorphic to $\mathcal{K}(\ell^2(\mathbb{N})) \otimes ((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$. Thus, the Cuntz semigroup of $(\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ can be naturally identified with the set of (ordinary) Cuntz equivalences classes of positive elements in $(\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathcal{H}_\mu) \otimes A)^G$.

Fix $[\mu] \in \mathrm{Cu}(G)$. Then the inclusion

$$(\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathcal{H}_\mu) \otimes A)^G \hookrightarrow \bigsqcup_{[\nu] \in \mathrm{Cu}(G)} (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$$

induces a semigroup homomorphism

$$i_\mu \colon \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G) \to \mathrm{Cu}^G(A, \alpha), \tag{III.1}$$

which is given by $i_\mu([a]) = [a]_G$ for a positive element $a \in (\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathcal{H}_\mu) \otimes A)^G$. By Lemma III.2.4, the map $i_\mu$ satisfies

$$i_\mu \circ j_{\nu,\mu} = i_\nu$$

for all $\nu \in \mathrm{Cu}(G)$, whenever $\mu$ is equivalent to a subrepresentation of $\nu$. It is also clear that $i_\mu$ preserves the compact containment relation and that it is an order embedding.

Define a preorder $\leq$ on $\mathrm{Cu}(G)$ by setting $[\mu] \leq [\nu]$ if $\mu$ is equivalent to a subrepresentation of $\nu$. It is clear that $(\mathrm{Cu}(G), \leq)$ is a directed set. For use in the next theorem, we recall that by Corollary 3.1.11 in [4], the category **Cu** is closed under direct limits indexed over an arbitrary directed set.

**Theorem III.2.6.** The direct limit of the direct system

$$\left((\mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G))_{\mu \in \mathrm{Cu}(G)}, (j_{\mu,\nu})_{\mu,\nu \in \mathrm{Cu}(G)}\right),$$

in the category **Cu**, is naturally isomorphic to the pair

$$(\mathrm{Cu}^G(A, \alpha), (i_\mu)_{\mu \in \mathrm{Cu}(G)}).$$

In particular, $\mathrm{Cu}^G(A, \alpha)$ belongs to **Cu**.

*Proof.* We will show that $(\mathrm{Cu}^G(A, \alpha), (i_\mu)_{\mu \in \mathrm{Cu}(G)})$ satisfies the universal property of the direct limit in **Cu**. Let $(S, (\gamma_\mu)_{\mu \in \mathrm{Cu}(G)})$ be a pair consisting of a semigroup $S$ in the category **Cu** and **Cu**-morphisms

$$\gamma_\mu \colon \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G) \to S,$$

for $\mu \in \mathrm{Cu}(G)$, satisfying $\gamma_\nu \circ j_{\mu,\nu} = \gamma_\mu$ for all $\nu, \mu \in \mathrm{Cu}(G)$ with $\mu \leq \nu$. Define a map

$$\gamma \colon \mathrm{Cu}^G(A, \alpha) = \bigcup_{\mu \in \mathrm{Cu}(G)} i_\mu(\mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)) \to S$$

by

$$\gamma(i_\mu(s)) = \gamma_\mu(s)$$

for $s \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$.

The proof will be finished once we prove that $\gamma$ is a well-defined morphism in **Cu**. We divide the proof into a number of claims.

*Claim: $\gamma$ is a well defined order preserving map.* For this, it is enough to show the following. Given $\mu, \nu \in \mathrm{Cu}(G)$ and given $s \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$ and $t \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G)$, if $i_\mu(s) \leq i_\nu(t)$, then

$$\gamma(i_\mu(s)) \leq \gamma(i_\nu(t)).$$

Let $\mu, \nu, s$ and $t$ be as above. Then

$$i_{\mu\oplus\nu}(j_{\mu,\mu\oplus\nu}(s)) \leq i_{\mu\oplus\nu}(j_{\nu,\mu\oplus\nu}(t)).$$

Since $i_{\mu\oplus\nu}$ is an order embedding, we deduce that $j_{\mu,\mu\oplus\nu}(s) \leq j_{\nu,\mu\oplus\nu}(t)$. Hence,

$$\gamma(i_\mu(s)) = \gamma_\mu(s) = \gamma_{\mu\oplus\nu}(j_{\mu,\mu\oplus\nu}(s)) \leq \gamma_{\mu\oplus\nu}(j_{\nu,\mu\oplus\nu}(t)) = \gamma_\nu(t) = \gamma(i_\nu(t)).$$

The claim is proved.

*Claim: $\gamma$ is a semigroup homomorphism.* Given $\mu, \nu \in \mathrm{Cu}(G)$, given $s \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$, and given $t \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G)$, we have

$$\begin{aligned}
\gamma(i_\mu(s) + i_\nu(t)) &= \gamma_{\mu\oplus\nu}(j_{\mu,\mu\oplus\nu}(s) + j_{\nu,\mu\oplus\nu}(t)) \\
&= \gamma_{\mu\oplus\nu}(j_{\mu,\mu\oplus\nu}(s)) + \gamma_{\mu\oplus\nu}(j_{\nu,\mu\oplus\nu}(t)) \\
&= \gamma(i_\mu(s)) + \gamma(i_\nu(t)),
\end{aligned}$$

so the claim follows.

*Claim: $\gamma$ preserves suprema of increasing sequences (condition M1 in Definition II.6.1).* Let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathrm{Cu}^G(A,\alpha)$, and let $x \in \mathrm{Cu}^G(A,\alpha)$ be its supremum. For each $n \in \mathbb{N}$, choose $[\mu_n] \in \mathrm{Cu}(G)$ and an element $s_n \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_{\mu_n}) \otimes A)^G)$ such that $i_{\mu_n}(s_n) = x_n$. Likewise, choose $[\mu] \in \mathrm{Cu}(G)$ and an element $s \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$ such that $i_\mu(s) = x$.

Set $\nu = \bigoplus\limits_{n=1}^{\infty} \mu_n$. Then

$$i_{\mu_n}(s_n) = i_\nu(j_{\mu_n,\nu}(s_n)) \leq i_\nu(j_{\mu_{n+1},\nu}(s_{n+1})) = i_{\mu_n}(s_{n+1})$$

for all $n \in \mathbb{N}$. It follows that $j_{\mu_n,\nu}(s_n) \leq j_{\mu_{n+1},\nu}(s_{n+1})$ for all $n \in \mathbb{N}$, since $i_\nu$ is an order embedding. In other words, $(j_{\mu_n,\nu}(s_n))_{n\in\mathbb{N}}$ is an increasing sequence in $\mathrm{Cu}(\mathcal{K}(\mathcal{H}_\nu \otimes A)^G)$. Since suprema of increasing sequences exist in $\mathrm{Cu}(\mathcal{K}(\mathcal{H}_\nu \otimes A)^G)$ and $i_\nu$ and $\gamma_\nu$ are maps in **Cu** we get

$$\begin{aligned}
\gamma(x) =& \gamma(i_\mu(s)) = \gamma\left(\sup_{n\in\mathbb{N}} i_{\mu_n}(s_n)\right) = \gamma\left(\sup_{n\in\mathbb{N}} i_\nu(j_{\mu_n,\nu}(s_n))\right) \\
&= \gamma(i_\nu\left(\sup_{n\in\mathbb{N}} j_{\mu_n,\nu}(s_n)\right)) = \gamma_\nu(\sup_{n\in\mathbb{N}} j_{\mu_n,\nu}(s_n)) = \sup_{n\in\mathbb{N}} \gamma_\nu(j_{\mu_n,\nu}(s_n)) \\
&= \sup_{n\in\mathbb{N}} \gamma(i_{\mu_n}(s_n)) = \sup_{n\in\mathbb{N}} \gamma(x_n).
\end{aligned}$$

Hence $\gamma(x)$ is the supremum of $(\gamma(x_n))_{n\in\mathbb{N}}$, proving the claim.

*Claim: $\gamma$ preserves the compact containment relation (condition M2 in Definition II.6.1).* Given $\mu, \nu \in \mathrm{Cu}(G)$, and given $s \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$ and $t \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G)$, suppose that $i_\mu(s) \ll i_\nu(t)$. Then

$$i_{\mu\oplus\nu}(j_{\mu,\mu\oplus\nu}(s)) \ll i_\nu(j_{\nu,\mu\oplus\nu}(t)).$$

Since $i_{\mu\oplus\nu}$ is a morphism in the category **Cu** and it is an order embedding, we deduce that $j_{\mu,\mu\oplus\nu}(s) \ll j_{\nu,\mu\oplus\nu}(t)$. Hence,

$$\gamma(i_\mu(s)) = \gamma_{\mu\oplus\nu}(j_{\mu,\mu\oplus\nu}(s)) \ll \gamma_{\nu\oplus\nu}(j_{\nu,\mu\oplus\nu}(t)) = \gamma(i_\nu(t)).$$

We conclude that $\gamma$ is a morphism in **Cu**, so the proof is complete. □

We can now show that the semigroup homomorphism between the equivariant Cuntz semigroups induced by an equivariant $*$-homomorphism is a morphism in **Cu**.

**Proposition III.2.7.** Let $\beta\colon G \to \mathrm{Aut}(B)$ be a continuous action of $G$ on a $C^*$-algebra $B$, and let $\phi\colon A \to B$ be an equivariant homomorphism. Then the induced map $\mathrm{Cu}^G(\phi)\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(B,\beta)$ is a morphism in the category **Cu**.

*Proof.* For $[\mu] \in \mathrm{Cu}(G)$, set

$$\phi_\mu = \mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)} \otimes \phi\colon (\mathcal{K}(\mathcal{H}_\mu) \otimes A, \mathrm{Ad}(\mu) \otimes \alpha) \to (\mathcal{K}(\mathcal{H}_\mu) \otimes B, \mathrm{Ad}(\mu) \otimes \beta).$$

Then $\phi_\mu$ is equivariant. Its induced map

$$\mathrm{Cu}(\phi_\mu)\colon \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G) \to \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes B)^G),$$

between the Cuntz semigroups of the corresponding fixed point algebras, is a morphism in **Cu**.

For $[\mu] \leq [\nu]$, we have

$$j^B_{\mu,\nu} \circ \mathrm{Cu}(\phi_\mu) = \mathrm{Cu}(\phi_\nu) \circ j^A_{\mu,\nu}.$$

Consequently, the maps

$$i^B_\mu \circ \mathrm{Cu}(\phi_\mu)\colon \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G) \to \mathrm{Cu}^G(B,\beta)$$

satisfy

$$i^B_\mu \circ \mathrm{Cu}(\phi_\mu) = (i^B_\nu \circ \mathrm{Cu}(\phi_\nu)) \circ j^A_{\mu,\nu}$$

for $[\mu] \leq [\nu]$. The universal property of the direct limit provides a **Cu**-morphism

$$\kappa\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(B,\beta)$$

satisfying $\kappa \circ i^A_\mu = i^B_\mu \circ \mathrm{Cu}(\phi_\mu)$ for all $[\mu] \in \mathrm{Cu}(G)$. For $s \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$, we have

$$\kappa(i^A_\mu(s)) = i^B_\mu(\mathrm{Cu}(\phi_\mu)(s)) = i^B_\mu(\mathrm{Cu}(\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)} \otimes \phi)([a])) = \mathrm{Cu}^G(\phi)([a]).$$

We conclude that $\kappa = \mathrm{Cu}^G(\phi)$, and hence $\mathrm{Cu}^G(\phi)$ is a morphism in **Cu**. □

Since $\mathrm{Cu}^G$ obviously preserves composition of maps, we get the following.

**Corollary III.2.8.** The equivariant Cuntz semigroup $\mathrm{Cu}^G$ is a functor from the category of $G$-$C^*$-algebras to the category **Cu**.

## The Semiring $\mathrm{Cu}(G)$ and the Category $\mathbf{Cu}^G$

### *The semiring* $\mathrm{Cu}(G)$

Let $G$ be a compact group. Denote by $V(G)$ the semigroup of equivalence classes of finite dimensional representations of $G$, the operation being given by direct sum. Recall that the *representation ring* $R(G)$ of $G$ is the Grothendieck group of $V(G)$. The product structure on $R(G)$ is induced by the tensor product of representations. The construction of $R(G)$ resembles that of $K$-theory, while the object we define below is its Cuntz analog.

Recall that a *semiring* is a set $R$ with two binary operations $+$ and $\cdot$ on $R$, which satisfy all axioms of a unital ring except for the axiom demanding the existence of additive inverses.

**Definition III.3.1.** The *representation semiring* of $G$, denoted by $\mathrm{Cu}(G)$, is the set of all equivalence classes of unitary representations of $G$ on separable Hilbert spaces. Addition in $\mathrm{Cu}(G)$ is given by the direct sum of representations, while product in $\mathrm{Cu}(G)$ is given by the tensor product. We endow $\mathrm{Cu}(G)$ with the order: $[\mu] \leq [\nu]$ if $\mu$ is unitarily equivalent to a subrepresentation of $\nu$.

Since the tensor product of representations is associative, it is clear that $\mathrm{Cu}(G)$ is indeed a semiring.

**Lemma III.3.2.** Let $A$ be a $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$, and let $a \in \mathcal{K}(\mathcal{H}_\mu)^G$ be a positive element. Set $\mathcal{H} = \overline{a(\mathcal{H}_\mu)}$, and let $a'$ be the restriction of $a$ to $\mathcal{H}$. Then $a'$ is a $G$-invariant strictly positive element in $\mathcal{K}(\mathcal{H})$, and there exists a sequence $(d_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(\mathcal{H}_\mu, \mathcal{H})^G$ such that

$$\lim_{n\to\infty} \|d_n^* a' d_n - a\| = 0 \text{ and } \lim_{n\to\infty} \|d_n a d_n^* - a'\|.$$

*Proof.* Denote by $B_{\mathcal{H}_\mu}$ and $B_{\mathcal{H}}$ the unit balls $\mathcal{H}_\mu$ and $\mathcal{H}$, respectively. Since $a'(B_{\mathcal{H}}) \subseteq a(B_{\mathcal{H}_\mu})$, it is clear that $a'$ is compact.

Since $\lim\limits_{n\to\infty} a^{\frac{1}{n}}(\xi) = \xi$ for all $\xi \in \mathcal{H}$, we conclude that $a'$ is strictly positive. For $n \in \mathbb{N}$, let $d_n\colon \mathcal{H}_\mu \to \mathcal{H}$ be the operator defined by restricting the codomain of $a^{\frac{1}{n}}$ to $\mathcal{H}$. Then $d_n \in \mathcal{K}(\mathcal{H}_\mu, \mathcal{H})^G$, and $d_n^*\colon \mathcal{H} \to \mathcal{H}_\mu$ is given by $d_n^*(\xi) = a^{\frac{1}{n}}(\xi)$ for all $\xi \in \mathcal{H}$. It is now clear that

$$\lim_{n\to\infty} \|d_n^* a' d_n - a\| = 0 \text{ and } \lim_{n\to\infty} \|d_n a d_n^* - a'\|,$$

so the proof is complete. $\square$

Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$. Since $\mathcal{H}_\mu$ is separable, $\mathcal{K}(\mathcal{H}_\mu)$ has a strictly positive element $\widetilde{s}_\mu$. Moreover, by integrating $g \cdot \widetilde{s}_\mu$ over $G$, we get an invariant strictly positive element $s_\mu$ of $\mathcal{K}(\mathcal{H}_\mu)$.

**Theorem III.3.3.** Adopt the notation from the comments above. Then the map $s\colon \mathrm{Cu}(G) \to \mathrm{Cu}^G(\mathbb{C})$ given by $s([\mu]) = [s_\mu]$, for $[\mu] \in \mathrm{Cu}(G)$, is well defined. Moreover, it is an isomorphism of ordered semigroups.

*Proof.* We begin by showing that $s$ is well defined. Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, with $[\mu] \leq [\nu]$. Then there exists $V \in \mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu)^G$ such that $V^*V = \mathrm{id}_{\mathcal{H}_\mu}$. By Lemma III.2.4, we have $s_\mu \sim_G V s_\mu V^*$. Since $s_\nu$ is strictly positive, we also have $V s_\mu V^* \precsim_G s_\nu$. Thus, $s_\mu \precsim_G s_\nu$. It follows that $s$ is well defined and order preserving.

We now show that $s$ is an order embedding. Let $s_\mu \in \mathcal{K}(\mathcal{H}_\mu)^G$ and $s_\nu \in \mathcal{K}(\mathcal{H}_\nu)^G$ be strictly positive elements such that $s_\mu \precsim_G s_\nu$. By [27, Proposition 2.5], there is $x \in \mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu)$ such that

$s_\mu = x^*x$ and $xx^* \in \mathcal{K}(\mathcal{H}_\nu)^G$. Also, by simply inspecting the proof of that proposition, one sees that $x$ can be taken in $\mathcal{K}(\mathcal{H}_\mu, \mathcal{H}_\nu)^G$. Let $x = v(x^*x)^{\frac{1}{2}}$ be the polar decomposition of $x$. Then $v$ belongs to $\mathcal{B}(\mathcal{H}_\mu, \mathcal{H}_\nu)^G$, and $v^*v = \mathrm{id}_{\mathcal{H}_\mu}$. This implies that $[\mu] \leq [\nu]$. In particular, $s$ is injective.

To finish the proof, we show that $s$ is surjective. Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$, and let $a$ be a strictly positive element in $\mathcal{K}(\mathcal{H}_\mu)^G$. Set $\mathcal{H}_\nu = \overline{a(\mathcal{H}_\mu)}$, let $\nu$ be the compression of $\mu$ to $\mathcal{H}_\nu$, and let $a' \colon \mathcal{H}_\nu \to \mathcal{H}_\nu$ be the restriction of $a$. By Lemma III.3.2, $a'$ is a strictly positive element in $\mathcal{K}(\mathcal{H}_\nu)^G$, and $a' \sim_G a$. It follows that $s([\nu]) = [a]$, and the proof is complete. □

**Corollary III.3.4.** The semigroup $\mathrm{Cu}(G)$ is an object in **Cu**. In addition,

1. If $([\mu_n])_{n\in\mathbb{N}}$ is an increasing sequence in $\mathrm{Cu}(G)$, then $\sup\limits_{n\in\mathbb{N}}[\mu_n]$ exists. Moreover, $[\mu]$ is the supremum of $([\mu_n])_{n\in\mathbb{N}}$ if and only if $[s_\mu] = \sup\limits_{n\in\mathbb{N}}[s_{\mu_n}]$;
2. $[\mu] \ll [\nu]$ if and only if $[s_\mu] \ll [s_\nu]$.

Recall that when $G$ is compact, every unitary representation of $G$ is equivalent to a direct sum of finite dimensional representations.

**Corollary III.3.5.** Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$. Let $(\mathcal{H}_{\nu_k}, \nu_k)_{k\in\mathbb{N}}$ be a family of non-zero finite dimensional representations of $G$ such that

$$(\mathcal{H}_\mu, \mu) \cong \bigoplus_{k\in\mathbb{N}} (\mathcal{H}_{\nu_k}, \nu_k).$$

For $n \in \mathbb{N}$, set $\mu_n = \bigoplus\limits_{k=1}^{n} \nu_k$. Then $[\mu_n] \ll [\mu]$ for all $n \in \mathbb{N}$, and

$$[\mu] = \sup_{n\in\mathbb{N}} [\mu_n].$$

*Proof.* Let $s_\mu$ be a strictly positive element of $\mathcal{K}(\mathcal{H}_\mu)^G$. For each $n \in \mathbb{N}$, let $p_n$ be the unit of $\mathcal{B}(\mathcal{H}_{\mu_n})$. Then $[s_\mu] = \sup\limits_{n\in\mathbb{N}}[p_n]$ since $\mathcal{H}_\mu \cong \bigoplus\limits_{k\in\mathbb{N}} \mathcal{H}_{\nu_k}$. Also, $[p_n] \ll [s_\mu]$ for all $n \in \mathbb{N}$, because $p_n$ is a projection. The result then follows from Theorem III.3.3. □

*The* $\mathrm{Cu}(G)$*-semimodule structure on* $\mathrm{Cu}^G(A,\alpha)$

Throughout the rest of this section, we fix a compact group $G$, a $C^*$-algebra $A$, and a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$.

Recall that a (left) semimodule over a semiring $R$, or an $R$-semimodule, is a commutative monoid $S$ together with a function $\cdot\colon R \times S \to S$ satisfying all the axioms of a module over a ring, except for the axiom demanding the existence of additive inverses.

In this subsection, we show that $\mathrm{Cu}^G(A,\alpha)$ has a natural $\mathrm{Cu}(G)$-semimodule structure, which moreover satisfies a number of additional regularity properties. It follows that the equivariant Cuntz semigroups belong to a distinguished class of partially ordered semirings over $\mathrm{Cu}(G)$. We begin by defining this category, and then show that $\mathrm{Cu}^G(A,\alpha)$ belongs to it; see Theorem III.3.10.

**Definition III.3.6.** Denote by $\mathbf{Cu}^G$ the category defined as follows. The objects in $\mathbf{Cu}^G$ are partially ordered $\mathrm{Cu}(G)$-semimodules $(S,+,\cdot)$ such that:

(O1) $S$ is an object in $\mathbf{Cu}$;

(O2) if $x,y \in S$ and $s,t \in \mathrm{Cu}(G)$ satisfy $x \leq y$ and $r \leq s$, then $r \cdot x \leq s \cdot y$;

(O3) if $x,y \in S$ and $s,t \in \mathrm{Cu}(G)$ satisfy $x \ll y$ and $r \ll s$, then $r \cdot x \ll s \cdot y$;

(O4) if $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $S$, and $(r_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathrm{Cu}(G)$, then
$$\sup_{n\in\mathbb{N}}(r_n \cdot x_n) = \left(\sup_{n\in\mathbb{N}} r_n\right) \cdot \left(\sup_{n\in\mathbb{N}} x_n\right).$$

The morphisms in $\mathbf{Cu}^G$ between two $\mathrm{Cu}(G)$-semimodules $S$ and $T$ are all $\mathrm{Cu}(G)$-semimodule homomorphisms $\varphi\colon S \to T$ in the category $\mathbf{Cu}$.

**Lemma III.3.7.** Axiom O4 in Definition III.3.6 is equivalent to the following. If $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence in $S$, and $(r_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathrm{Cu}(G)$, then
$$\sup_{n\in\mathbb{N}}(r_n \cdot x) = \left(\sup_{n\in\mathbb{N}} r_n\right) \cdot x \text{ and } \sup_{n\in\mathbb{N}}(r \cdot x_n) = r \cdot \left(\sup_{n\in\mathbb{N}} x_n\right)$$
for all $r \in \mathrm{Cu}(G)$ and for all $x \in S$.

*Proof.* That Axiom (O4) implies the condition in the statement is immediate. Conversely, suppose that $S$ satisfies Axioms (O1), (O2) and (O3), and the condition in the statement. Let $(x_n)_{n\in\mathbb{N}}$ be an increasing sequence in $S$, and let $(r_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathrm{Cu}(G)$. For $m \in \mathbb{N}$, we have $r_m \cdot x_m \leq \sup\limits_{n\in\mathbb{N}} r_n \cdot \sup\limits_{n\in\mathbb{N}} x_n$ by Axiom (O2), so

$$\sup_{m\in\mathbb{N}} (r_m \cdot x_m) \leq \sup_{n\in\mathbb{N}} r_n \cdot \sup_{n\in\mathbb{N}} x_n.$$

For the opposite inequality, given $m \in \mathbb{N}$ we have

$$\sup_{n\in\mathbb{N}}(r_n \cdot x_n) \geq \sup_{n\in\mathbb{N}}(r_n \cdot x_m) = \left(\sup_{n\in\mathbb{N}} r_n\right) x_m.$$

By taking $\sup_{m\in\mathbb{N}}$, we conclude that Axiom (O4) is also satisfied. $\square$

We will need to know the following:

**Theorem III.3.8.** The category $\mathbf{Cu}^G$ is closed under countable direct limits.

*Proof.* Let $(S_n, \varphi_n)_{n\in\mathbb{N}}$ be a direct system in the category $\mathbf{Cu}^G$, with $\mathbf{Cu}^G$-morphisms $\varphi_n \colon S_n \to S_{n+1}$. For $m \geq n$, we write $\varphi_{m,n} \colon S_n \to S_{m+1}$ for the composition $\varphi_{m,n} = \varphi_m \circ \cdots \circ \varphi_n$. By Theorem 2 in [36], the limit of this direct system exists in the category **Cu**, and we denote it by $(S, (\psi_n)_{n\in\mathbb{N}})$, where $\psi_n \colon S_n \to S$ is a **Cu**-morphism satisfying $\psi_{n+1} \circ \varphi_n = \varphi_{n+1}$ for all $n \in \mathbb{N}$. We will use the description of the direct limit given in the proof of Theorem 2 in [36], in the form given in Proposition 2.2 in [91].

We define a $\mathrm{Cu}(G)$-semimodule structure on $S$ as follows. Let $x \in S$, and choose elements $x_n \in S_n$, for $n \in \mathbb{N}$, such that $\varphi_n(x_n) \ll x_{n+1}$ and $\sup\limits_{n\in\mathbb{N}} \psi_n(x_n) = x$. Given $r \in \mathrm{Cu}(G)$, set $r \cdot x = \sup\limits_{n\in\mathbb{N}} \psi_n(r \cdot x_n)$.

*Claim: the* $\mathrm{Cu}(G)$*-semimodule structure is well-defined and satisfies Axiom O2.* It is clearly enough to check Axiom O2. Let $x, y \in S$ with $x \leq y$, and let $r, s \in \mathrm{Cu}(G)$ with $r \leq s$. Choose elements $x_n, y_n \in S_n$, for $n \in \mathbb{N}$, satisfying $\varphi_n(x_n) \ll x_{n+1}$ and $\sup\limits_{n\in\mathbb{N}} \psi_n(x_n) = x$, as well as $\varphi_n(y_n) \ll y_{n+1}$ and $\sup\limits_{n\in\mathbb{N}} \psi_n(y_n) = y$.

Given $n \in \mathbb{N}$, we have $\psi_n(x_{n+1}) \ll x \leq y = \sup_{m\in\mathbb{N}} \psi_m(y_m)$, so there exists $m_0 \in \mathbb{N}$ such that $\psi_n(x_{n+1}) \leq \psi_m(y_m)$ for all $m \geq m_0$. Without loss of generality, we may assume $m_0 \geq n$. Since $\varphi_n(x_n) \ll x_{n+1}$, part (ii) of Proposition 2.2 in [91] implies that there exists $n_0 \in \mathbb{N}$ with

$n_0 \geq m$ such that $\varphi_{k,n}(x_n) \leq \varphi_{k,m}(y_m)$ for all $k \geq n_0$. It follows that

$$\varphi_{k,n}(r \cdot x_n) \leq \varphi_{k,m}(s \cdot y_m)$$

for all $k \geq n_0$. Taking the supremum over $k$, we deduce that $\psi_n(r \cdot x_n) \leq \psi_m(s \cdot y_m)$ for all $m \geq m_0$. Taking first the supremum over $m$, and then the supremum over $n$, we conclude that

$$\sup_{n\in\mathbb{N}} \psi_n(r \cdot x_n) \leq \sup_{m\in\mathbb{N}} \psi_m(s \cdot y_m).$$

The claim is proved.

*Claim: $S$ satisfies Axiom (O3).*

Let $x, y \in S$ with $x \ll y$, and let $r, s \in \mathrm{Cu}(G)$ with $r \ll s$. Then there exist $n \in \mathbb{N}$, and $x', y' \in S_n$ such that $x' \ll y'$ and $x \ll \psi_n(x') \ll \psi_n(y') \ll y$. Then $r \cdot x' \ll s \cdot y'$, and hence

$$r \cdot x \leq \psi_n(r \cdot x') \ll \psi_n(s \cdot y') \leq s \cdot y,$$

as desired.

*Claim: $S$ satisfies Axiom (O4).* It suffices to check the condition in the statement of Lemma III.3.7. Let $(r_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathrm{Cu}(G)$, and let $x \in S$. Choose elements $x_m \in S_m$, for $m \in \mathbb{N}$, such that $\varphi_m(x_m) \ll x_{m+1}$ and $\sup\limits_{m\in\mathbb{N}} \psi_m(x_m) = x$. Then

$$\sup_{n\in\mathbb{N}} (r_n \cdot x) = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \psi_m(r_n \cdot x_m) = \sup_{n\in\mathbb{N}} r_n \cdot \left( \sup_{m\in\mathbb{N}} \psi_m(x_m) \right) = \left( \sup_{n\in\mathbb{N}} r_n \right) \cdot x,$$

as desired. The other property in Lemma III.3.7 can be checked analogously, so we omit it. This concludes the proof. $\square$

We now define a $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}^G(A, \alpha)$.

**Definition III.3.9.** Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ be a positive element. In this definition, and to stress the role played by $\mu$, we write $[(\mathcal{H}_\mu, \mu, a)]_G$ for the $G$-Cuntz equivalence class of $a$. Use separability of $\mathcal{H}_\nu$ to choose a

$G$-invariant strictly positive element $s_\nu \in \mathcal{K}(\mathcal{H}_\nu)^G$. We set

$$[\nu] \cdot [(\mathcal{H}_\mu, \mu, a)]_G = [(\mathcal{H}_\nu \otimes \mathcal{H}_\mu, \nu \otimes \mu, s_\nu \otimes a)]_G .$$

The following is one of the main results in this section. For use in its proof, we recall that any tensor product of $C^*$-algebras respects Cuntz subequivalence.

**Theorem III.3.10.** The $\mathrm{Cu}(G)$-semimodule structure from Definition III.3.9 is well defined. Moreover, with this structure, the semigroup $\mathrm{Cu}^G(A, \alpha)$ becomes an object in $\mathbf{Cu}^G$, and the equivariant Cuntz semigroup is a functor from the category of $G$-$C^*$-algebras to the category $\mathbf{Cu}^G$.

*Proof.* We will prove that the $\mathrm{Cu}(G)$-semimodule structure is well defined together with condition O2 in Definition III.3.6. So let $[\mu], [\nu] \in \mathrm{Cu}(G)$ and $[a]_G, [b]_G \in \mathrm{Cu}^G(A, \alpha)$ satisfy $[\mu] \leq [\nu]$ and $[a]_G \leq [b]_G$. By Theorem III.3.3, we have $s_\mu \precsim_G s_\nu$. Since we also have $a \precsim_G b$, we get $s_\mu \otimes a \precsim_G s_\nu \otimes b$. Hence,

$$[\mu] \cdot [a]_G = [s_\mu \otimes a] \leq [s_\nu \otimes b] = [\nu] \cdot [b]_G,$$

as desired.

It is immediate that

$$[\mu] \cdot ([a]_G + [b]_G) = [\mu] \cdot [a]_G + [\mu] \cdot [b]_G$$

and

$$[\mu] \cdot ([\nu] \cdot [a]_G) = ([\mu] \cdot [\nu]) \cdot [a]_G,$$

for all $[\mu], [\nu] \in \mathrm{Cu}(G)$ and for all $[a]_G, [b]_G \in \mathrm{Cu}^G(A, \alpha)$. We conclude that $\mathrm{Cu}^G(A, \alpha)$ is a $\mathrm{Cu}(G)$-semimodule, and that it satisfies condition O2 in Definition III.3.6.

We now proceed to show that $\mathrm{Cu}^G(A, \alpha)$ is an object in $\mathbf{Cu}^G$. We already showed in Theorem III.2.6 that it is an object in $\mathbf{Cu}$, so condition O1 in Definition III.3.6 is satisfied.

We check condition O3. Suppose that $[\mu], [\nu] \in \mathrm{Cu}(G)$ and $[a]_G, [b]_G \in \mathrm{Cu}^G(A, \alpha)$ satisfy $[\mu] \ll [\nu]$ and $[a]_G \ll [b]_G$. By Theorem III.3.3, we get $[s_\mu] \ll [s_\nu]$ in $\mathrm{Cu}^G(\mathbb{C})$. Without loss of generality, we may assume that $s_\nu$ and $b$ are contractions. Recall that

$$[s_\nu] = \sup_{\varepsilon>0}[(s_\nu - \varepsilon)_+]_G \text{ and } [b]_G = \sup_{\varepsilon>0}[(b-\varepsilon)_+]_G.$$

Using the definition of the compact containment relation, find $\varepsilon > 0$ such that

$$[s_\mu] \leq [(s_\nu - \varepsilon)_+] \text{ and } [a]_G \leq [(b-\varepsilon)_+].$$

Use $\|s_\nu\| \leq 1$ and $\|b\| \leq 1$ at the third step to get

$$\begin{aligned}[\mu]\cdot[a]_G &= [s_\mu \otimes a] \\ &\leq [(s_\nu - \varepsilon)_+ \otimes (b-\varepsilon)_+] \\ &\leq [(s_\nu \otimes b - \varepsilon^2)_+] \\ &\ll [s_\nu \otimes b] = [\nu]\cdot[b]_G,\end{aligned}$$

so condition O3 is satisfied.

We now check condition O4. Let $([\mu_n])_{n\in\mathbb{N}}$ and $([a_n])_{n\in\mathbb{N}}$ be increasing sequences in $\mathrm{Cu}(G)$ and $\mathrm{Cu}^G(A,\alpha)$, respectively, and set $[\mu] = \sup\limits_{n\in\mathbb{N}}[\mu_n]$ and $[a] = \sup\limits_{n\in\mathbb{N}}[a_n]$. Without loss of generality, we may assume that $a$ is a contraction. For $n \in \mathbb{N}$, denote by $s_{\mu_n}$ an invariant strictly positive element in $\mathcal{K}(\mathcal{H}_{\mu_n})$, and denote by $s_\mu$ an invariant strictly positive element in $\mathcal{K}(\mathcal{H}_\mu)$. By part (1) of Corollary III.3.4, we have $[s_\mu] = \sup\limits_{n\in\mathbb{N}}[s_{\mu_n}]$ in $\mathrm{Cu}^G(\mathbb{C})$. As before, we may assume that $s_\mu$ is a contraction. Then the sequence $([s_{\mu_n}\otimes a_n])_{n\in\mathbb{N}}$ in $\mathrm{Cu}^G(A,\alpha)$ is increasing. Set

$$[c] = \sup_{n\in\mathbb{N}}[s_{\mu_n}\otimes a_n].$$

We claim that $[c] = [s_\mu \otimes a]$. It is clear that $[c] \leq [s_\mu \otimes a]$. To check the opposite inequality, let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that

$$[(s_\mu - \varepsilon)_+] \leq [s_{\mu_n}] \text{ and } [(a-\varepsilon)_+] \leq [a_n].$$

It follows that

$$[(s_\mu \otimes a - \varepsilon)_+] \leq [(s_\mu - \varepsilon)_+ \otimes (a-\varepsilon)_+] \leq [s_{\mu_n}\otimes a_n].$$

Hence, $[(s_\mu \otimes a - \varepsilon)_+] \leq [c]$. Since $[s_\mu \otimes a] = \sup\limits_{\varepsilon>0}[(s_\mu \otimes a - \varepsilon)_+]$, we deduce that $[s_\mu \otimes a] \leq [c]$, as desired. We have checked condition O4.

Since $\mathrm{Cu}^G(A, \alpha)$ is an object in **Cu** by Theorem III.2.6, we conclude that $\mathrm{Cu}^G(A, \alpha)$ is an object in $\mathbf{Cu}^G$.

It remains to argue that $\mathrm{Cu}^G$ is a functor into $\mathbf{Cu}^G$. Let $\beta \colon G \to \mathrm{Aut}(B)$ be a continuous action of $G$ on a $C^*$-algebra $B$, and let $\phi \colon A \to B$ be an equivariant homomorphism. By Corollary III.2.8, $\mathrm{Cu}^G(\phi)$ is a morphism in **Cu**, so we only need to check that it is a morphism of $\mathrm{Cu}(G)$-semimodules. This is immediate, so the proof is complete. $\square$

We mention here, without proof, that the isomorphism in Theorem III.2.6 becomes a $\mathbf{Cu}^G$-isomorphism when

$$\varinjlim \big( (\mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G))_{\mu \in \mathrm{Cu}(G)}, (j_{\mu,\nu})_{\mu,\nu \in \mathrm{Cu}(G)} \big)$$

is endowed with the following $\mathrm{Cu}(G)$-action. For separable representations $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ of $G$, and for $x \in \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G)$, we set $[\nu] \cdot x = j_{\nu \otimes \mu, \mu}(x)$.

*Toolkit for computing examples*

It would be desirable to compute now some examples of equivariant Cuntz semigroups together with their $\mathrm{Cu}(G)$-semimodule structure. However, equivariant Cuntz semigroups are in general hard to compute, and a major tool to do this will be the Cuntz semigroup analog of Julg's Theorem, together with the computation of the $\mathrm{Cu}(G)$-semimodule structure of $\mathrm{Cu}(A \rtimes_\alpha G)$; see Theorem III.5.3 and Proposition III.5.15.

We mention two trivial cases: when $G = \{1\}$, then $\mathrm{Cu}^G(A, \alpha) = \mathrm{Cu}(A)$, and when $A = \mathbb{C}$, then $\mathrm{Cu}^G(\mathbb{C}) = \mathrm{Cu}(G)$ (with the obvious $\mathrm{Cu}(G)$-action). The next step would be computing $\mathrm{Cu}^G(A, \mathrm{id}_A)$, where $G$ acts trivially on $A$. The answer is

$$\mathrm{Cu}^G(A, \mathrm{id}_A) \cong \{f \colon \widehat{G} \to \mathrm{Cu}(A) \colon f \text{ has countable support}\},$$

where multiplication by elements in $\mathrm{Cu}(G)$ is applied to the input (decomposing a separable representation as sums of irreducibles). We could carry out this computation here, but we choose to delay it until the end of Section III.5, since it will then be an easy consequence of Theorem III.5.13; see Proposition III.5.14.

Here, we give some tools for computing further examples in Proposition III.3.11 and Proposition III.3.12. (These propositions will be needed in Section III.5.)

**Proposition III.3.11.** Let $q$ be any rank one projection on $\ell^2(\mathbb{N})$, and denote by

$$\iota_q\colon A \to A \otimes \mathcal{K}(\ell^2(\mathbb{N}))$$

the inclusion obtained by identifying $A$ with $q(A \otimes \mathcal{K}(\ell^2(\mathbb{N})))q$. Give $\ell^2(\mathbb{N})$ the trivial $G$-representation. Then $\iota_q$ induces a natural $\mathbf{Cu}^G$-isomorphism

$$\mathrm{Cu}^G(\iota_q)\colon \mathrm{Cu}^G(A, \alpha) \to \mathrm{Cu}^G(A \otimes \mathcal{K}(\ell^2(\mathbb{N})), \alpha \otimes \mathrm{id}_{\mathcal{K}(\ell^2(\mathbb{N}))}).$$

*Proof.* We abbreviate $\mathcal{K}(\ell^2(\mathbb{N}))$ to $\mathcal{K}$. By Theorem III.3.10, $\mathrm{Cu}^G(\iota_q)$ is a morphism in $\mathbf{Cu}^G$. It thus suffices to check that it is an isomorphism in $\mathbf{Cu}$.

Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$. Denote by

$$\kappa^q_\mu\colon \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G) \to \mathrm{Cu}((\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G \otimes \mathcal{K})$$

the $\mathbf{Cu}$-morphism induced by the inclusion as the corner associated to $q$. Then $\kappa^q_\mu$ is an isomorphism (see Appendix 6 in [36]). With the notation from Theorem III.3.10, it is clear that

$$j^{A\otimes\mathcal{K}}_{\mu,\nu} \circ \kappa^q_\nu = \kappa^q_\mu \circ j^A_{\mu,\nu}.$$

By the universal property of the direct limit in $\mathbf{Cu}$, applied to the object $\mathrm{Cu}^G(A \otimes \mathcal{K}, \alpha \otimes \mathrm{id}_{\mathcal{K}})$ and the maps $i^{A\otimes\mathcal{K}}_\mu \circ \kappa^q_\mu$, for $\mu \in \mathrm{Cu}(G)$, it follows that there is a $\mathbf{Cu}$-morphism

$$\kappa^q\colon \mathrm{Cu}^G(A, \alpha) \to \mathrm{Cu}^G(A \otimes \mathcal{K}, \alpha \otimes \mathrm{id}_{\mathcal{K}})$$

satisfying $\kappa^q \circ i^A_\mu = i^{A\otimes\mathcal{K}}_\mu \circ \kappa^q_\mu$. Since $\kappa^q$ is induced by $\iota_q$, we must have $\kappa^q = \mathrm{Cu}^G(\iota_q)$. Finally, since $\kappa^q_\mu$ is an isomorphism for all $\mu$, the same holds for $\kappa^q$, so the proof is complete. ☐

**Proposition III.3.12.** Let $(A_n, \iota_n)_{n\in\mathbb{N}}$ be a direct system of $C^*$-algebras with connecting maps $\iota_n\colon A_n \to A_{n+1}$. For $n \in \mathbb{N}$, let $\alpha^{(n)}\colon G \to \mathrm{Aut}(A_n)$ be a continuous action, and suppose that $\alpha^{(n+1)} \circ \iota_n = \iota_n \circ \alpha^{(n)}$ for all $n \in \mathbb{N}$. Set $A = \varinjlim(A_n, \iota_n)$, and $\alpha = \varinjlim \alpha^{(n)}$. Then there is a natural

$\mathbf{Cu}^G$-isomorphism

$$\varinjlim \mathrm{Cu}^G(A_n, \alpha^{(n)}) \cong \mathrm{Cu}^G(A, \alpha).$$

*Proof.* There is an inductive system

$$\left(\mathrm{Cu}^G(A_n, \alpha^{(n)}), \mathrm{Cu}^G(\iota_n)\right)_{n\in\mathbb{N}}$$

in $\mathbf{Cu}^G$. By Theorem III.3.8, its inductive limit exists in $\mathbf{Cu}^G$, and we will denote it by $\varinjlim \mathrm{Cu}^G(A_n, \alpha^{(n)})$.

For $n \in \mathbb{N}$, denote by $\iota_{\infty,n} \colon A_n \to A$ the equivariant map into the direct limit. Then $\mathrm{Cu}^G(\iota_{\infty,n})$ is a morphism in $\mathbf{Cu}^G$, and the universal property of inductive limits, there is a $\mathbf{Cu}^G$-morphism $\varphi\colon \varinjlim \mathrm{Cu}^G(A_n, \alpha^{(n)}) \to \mathrm{Cu}^G(A, \alpha)$.

We claim that $\varphi$ is an isomorphism. For this, it is enough to check that it is an isomorphism in $\mathbf{Cu}$. The rest of the proof is analogous to that of Proposition III.3.11, using Theorem 2 in [36]. We omit the details. □

We point out that in the previous proposition, we may allow direct limits over arbitrary directed sets, using Corollary 3.1.11 in [4] instead of Theorem 2 in [36].

### A Hilbert Module Picture of $\mathrm{Cu}^G(A, \alpha)$

In analogy with the non-equivariant case, the equivariant Cuntz semigroup can be constructed in terms of equivariant Hilbert modules. The goal of this section is to present this construction and identify it with $\mathrm{Cu}^G(A, \alpha)$ in a canonical way.

The description of $\mathrm{Cu}^G(A, \alpha)$ provided in this section will be needed in Section III.5, where we will prove that $\mathrm{Cu}^G(A, \alpha)$ can be naturally identified with $\mathrm{Cu}(A \rtimes_\alpha G)$ (Theorem III.5.3).

#### *Equivariant Hilbert $C^*$-modules*

Throughout this section, we fix a $C^*$-algebra $A$, a compact group $G$, and an action $\alpha\colon G \to \mathrm{Aut}(A)$. All modules will be right modules and a Hilbert $A$-module will mean a Hilbert C*-module over $A$. The reader is referred to [162] for the basics of Hilbert C*-modules.

Given Hilbert $A$-modules $E$ and $F$, we let $\mathcal{L}(E, F)$ and $\mathcal{K}(E, F)$ denote the spaces of adjointable operators and compact operators from $E$ to $F$, respectively. We write $\mathcal{U}(E, F)$ for the set of unitaries between $E$ and $F$. When $E = F$, we write $\mathcal{L}(E)$, $\mathcal{K}(E)$, and $\mathcal{U}(E)$ for $\mathcal{L}(E,E)$, $\mathcal{K}(E,E)$, and $\mathcal{U}(E,E)$, respectively.

**Definition III.4.1.** A *Hilbert* $(G, A, \alpha)$*-module* is a pair $(E, \rho)$ consisting of

1. a Hilbert $A$-module $E$, and

2. a strongly continuous group homomorphism $\rho\colon G \to \mathcal{U}(E)$, satisfying

   (a) $\rho_g(x \cdot a) = \rho_g(x) \cdot \alpha_g(a)$ for all $g \in G$, all $x \in E$ and all $a \in A$, and

   (b) $\langle \rho_g(x), \rho_g(y)\rangle_E = \alpha_g\left(\langle x, y\rangle_E\right)$ for all $g \in G$ and all $x, y \in E$.

(The continuity condition for $\rho$ means that for $x \in E$, the map $G \to E$ given by $g \mapsto \rho_g(x)$ is continuous.)

A pair $(F, \eta)$ consisting of a Hilbert submodule $F$ of $E$ satisfying $\rho_g(F) \subseteq F$ for all $g \in G$, and an action $\eta\colon G \to \mathrm{Aut}(F)$ with $\eta_g = \rho_g|_F$ for all $g \in G$, will be called a *Hilbert* $(G, A, \alpha)$*-submodule* of $(E, \rho)$.

We say that $E$ is *countably generated* if there exists a countable subset $\{\xi_n\}_{n\in\mathbb{N}} \subseteq E$ such that

$$\left\{\sum_{n=1}^{k} \xi_n a_n \colon a_n \in A, k \in \mathbb{N}\right\}$$

is dense in $E$.

We will sometimes call Hilbert $(G, A, \alpha)$-modules $G$-Hilbert $(A, \alpha)$-modules, or just $G$-Hilbert $A$-modules if the action $\alpha$ is understood.

Given $G$-Hilbert A-modules $(E, \rho)$ and $(F, \eta)$, we let $\mathcal{L}(E,F)^G$ and $\mathcal{K}(E,F)^G$ denote the subsets of $\mathcal{L}(E,F)$ and $\mathcal{K}(E,F)$, respectively, consisting of the equivariant operators. That is,

$$\mathcal{L}(E,F)^G = \{T \in \mathcal{L}(E,F)\colon T \circ \rho_g = \eta_g \circ T \text{ for all } g \in G\},$$
$$\mathcal{K}(E,F)^G = \{T \in \mathcal{K}(E,F)\colon T \circ \rho_g = \eta_g \circ T \text{ for all } g \in G\}.$$

(Note that $\mathcal{L}(E,F)^G$ is the set of fixed points of $\mathcal{L}(E,F)$, where for an adjointable operator $T\colon E \to F$ and $g \in G$, we set $g \cdot T = \eta_g \circ T \circ \rho_{g^{-1}}$.)

As before, $\mathcal{L}(E)^G$ and $\mathcal{K}(E)^G$ denote $\mathcal{L}(E,E)^G$ and $\mathcal{K}(E,E)^G$, respectively.

**Definition III.4.2.** Let $(E,\rho)$ and $(F,\eta)$ be $G$-Hilbert A-modules. We say that $(E,\rho)$ is *isomorphic* to $(F,\eta)$, in symbols $(E,\rho) \cong (F,\eta)$, if there exists a unitary in $\mathcal{L}(E,F)^G$. We say that $(E,\rho)$ is *subequivalent* to $(F,\eta)$, in symbols $(E,\rho) \preceq (F,\eta)$, if $(E,\rho)$ is isomorphic to a direct summand of $(F,\eta)$. (That is, if there exists $V \in \mathcal{L}(E,F)^G$ such that $V^*V = \mathrm{id}_E$.)

Let $I$ be a set and let $(E_j,\rho_j)_{j\in I}$ be a family of Hilbert $A$-modules. Then the Hilbert direct sum $\left(\bigoplus\limits_{j\in I} E_j, \bigoplus\limits_{j\in I}\rho_j\right)$ is the completion of the corresponding algebraic direct sum with respect to the norm defined by the scalar product

$$\left\langle \bigoplus_{j\in I}\xi_j, \bigoplus_{j\in I}\zeta_j \right\rangle = \sum_{j\in I}\langle \xi_j, \zeta_j\rangle.$$

Let $\mathcal{H}$ be a Hilbert space. By convention, the scalar product on $\mathcal{H}$ is linear in the second argument and conjugate linear in the first one. Let $\mathcal{H}\otimes A$ denote the exterior tensor product of $\mathcal{H}$ and $A$, where $A$ is considered as a right $A$-module over itself ([162, Chapter 4]). That is, $\mathcal{H}\otimes A$ is the completion of the algebraic tensor product $\mathcal{H}\otimes_{\mathrm{alg}} A$ in the norm given by the $A$-valued product

$$\langle \xi_1\otimes a_1, \xi_2\otimes a_2\rangle = \langle \xi_1,\xi_2\rangle a_1^* a_2$$

for $\xi_1,\xi_2\in\mathcal{H}$ and $a_1,a_2\in A$.

**Definition III.4.3.** For each element $[\pi]\in\widehat{G}$, choose a representative $\pi\colon G\to\mathcal{U}(\mathcal{H}_\pi)$. Denote by $\mathcal{H}_{\mathbb{C}}$ the Hilbert space direct sum

$$\mathcal{H}_{\mathbb{C}} = \bigoplus_{[\pi]\in\widehat{G}}\bigoplus_{n=1}^{\infty}\mathcal{H}_\pi,$$

and let $\pi_{\mathbb{C}}\colon G\to\mathcal{U}(\mathcal{H}_{\mathbb{C}})$ be the unitary representation given by

$$\pi_{\mathbb{C}} = \bigoplus_{[\pi]\in\widehat{G}}\bigoplus_{n=1}^{\infty}\pi.$$

The unitary representation $(\mathcal{H}_{\mathbb{C}},\pi_{\mathbb{C}})$ is easily seen not to depend on the choices of representatives $\pi\colon G\to\mathcal{U}(\mathcal{H}_\pi)$ up to unitary equivalence.

We define the *universal $G$-Hilbert $(A,\alpha)$-module* $(\mathcal{H}_A,\pi_A)$ to be $\mathcal{H}_A = \mathcal{H}_{\mathbb{C}}\otimes A$ and $\pi_A = \pi_{\mathbb{C}}\otimes\alpha$.

**Remark III.4.4.** It is easy to check, using the Peter-Weyl theorem, that $(\mathcal{H}_{\mathbb{C}}, \pi_{\mathbb{C}})$ is (unitarily equivalent) to the representation $(L^2(G) \otimes \ell^2, \lambda \otimes \mathrm{id}_{\ell^2})$.

An equivalent presentation of $\mathcal{H}_{\mathbb{C}}$ (and therefore of $\mathcal{H}_A$), which will be more convenient for our purposes, is

$$\mathcal{H}_{\mathbb{C}} = \bigoplus_{[\mu] \in \mathrm{Cu}(G)} \mathcal{H}_\mu,$$

with $\pi_{\mathbb{C}} = \bigoplus\limits_{[\mu] \in \mathrm{Cu}(G)} \mu$. (This presentation will be used when defining a $\mathrm{Cu}(G)$-module structure on $\mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G)$.)

**Remark III.4.5.** It is a classical result of Kasparov that when $G$ is second countable, then every countably generated $G$-Hilbert $A$-module is isomorphic to a direct summand of $(\mathcal{H}_A, \pi_A)$; see [144, Theorem 2].

**Lemma III.4.6.** Let $(E, (\pi_A)|_E)$ be a countably generated $G$-Hilbert $A$-submodule of $(\mathcal{H}_A, \pi_A)$. Then there exists a separable subrepresentation $(\mathcal{H}_\mu, \mu)$ of $(\mathcal{H}_{\mathbb{C}}, \pi_{\mathbb{C}})$ such that $E \subseteq \mathcal{H}_\mu \otimes A$ and $(\pi_A)|_E = (\mu \otimes \alpha)|_E$.

*Proof.* Since $\mathcal{H}_A = \bigoplus\limits_{[\nu] \in \mathrm{Cu}(G)} (\mathcal{H}_\nu \otimes A)$ for any $\xi \in \mathcal{H}_A$, there exists a countable set $X_\xi \subseteq \mathrm{Cu}(G)$ such that $\xi$ belongs to $\bigoplus\limits_{[\nu] \in X_\xi} (\mathcal{H}_\nu \otimes A)$.

Now let $\{\xi_n\}_{n \in \mathbb{N}}$ be a countable generating subset of $E$. Then $X = \bigcup\limits_{n \in \mathbb{N}} X_{\xi_n}$ is a countable subset of $\mathrm{Cu}(G)$. Set

$$(\mathcal{H}_\mu, \mu) = \bigoplus_{[\nu] \in X} (\mathcal{H}_\nu, \nu).$$

Then $\mathcal{H}_\mu$ is separable. It is immediate that $E \subseteq \mathcal{H}_\mu \otimes A$ and $(\pi_A)|_E = (\mu \otimes \alpha)|_E$, so the proof is complete. □

Let $(E, \rho)$ be a $G$-Hilbert $A$-module. Then the action $G$ on $E$ induces an action of $G$ on the $C^*$-algebra $\mathcal{K}(E)$ by conjugation. The fixed point algebra of this action will be denoted by $\mathcal{K}(E)^G$. When $(E, \rho)$ is the $G$-Hilbert $A$-module $(\mathcal{H}_\mu \otimes A, \mu \otimes \alpha)$, for some separable unitary representation $(\mathcal{H}_\mu, \mu)$ of $G$, then the induced action on $\mathcal{K}(\mathcal{H}_\mu \otimes A)$ will be denoted by $\mathrm{Ad}(\mu \otimes \alpha)$.

Let $E$ be a Hilbert $A$-module, and let $\xi, \zeta \in E$. We denote by $\Theta_{\xi,\zeta} \colon E \to E$ the $A$-rank one operator given by $\Theta_{\xi,\zeta}(\eta) = \xi \cdot \langle \zeta, \eta \rangle$ for $\eta \in E$.

**Lemma III.4.7.** Let $a \in \mathcal{K}(\mathcal{H}_A)^G$, and set $E = \overline{\mathrm{span}\{a(\mathcal{H}_A) \cup a^*(\mathcal{H}_A)\}}$, endowed with the restricted $G$-representation $\rho$. Then $(E, \rho)$ is a countably generated $G$-Hilbert $A$-module and $a|_E \in \mathcal{K}(E)^G$.

*Proof.* It is clear that $E$ is invariant under $(\pi_A)_g$ for all $g \in G$; thus $(E, \rho)$ is a $G$-Hilbert $A$-module. Let $\varepsilon > 0$. Since $a \in \mathcal{K}(\mathcal{H}_A)^G$, there exist $k \in \mathbb{N}$, and $\xi_1, \ldots, \xi_k, \zeta_1, \ldots, \zeta_k \in \mathcal{H}_A$ satisfying

$$\left\| a - \sum_{j=1}^{k} \Theta_{\xi_j, \zeta_j} \right\| < \frac{\varepsilon}{8}.$$

Use $a \in \overline{(aa^*)\mathcal{K}(\mathcal{H}_A)}$ and $a \in \overline{\mathcal{K}(\mathcal{H}_A)(a^*a)}$ to choose $n \in \mathbb{N}$ with

$$\|aa^*\|^{\frac{1}{n}} < 2 \text{ and } \|a^*a\|^{\frac{1}{n}} < 2$$

such that, in addition,

$$\left\| a - (aa^*)^1/na(a^*a)^1/n \right\| < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned}
\left\| a - \sum_{j=1}^{k} \Theta_{(aa^*)^1/n(\xi_j),(a^*a)^1/n(\zeta_j)} \right\| &= \left\| a - \sum_{j=1}^{k} (aa^*)^1/n\Theta_{\xi_j,\zeta_j}(a^*a)^1/n \right\| \\
&\leq \left\| a - (aa^*)^1/na(a^*a)^1/n \right\| \\
&\quad + \|aa^*\|^1/n \left\| a - \sum_{j=1}^{k} \Theta_{\xi_j,\zeta_j} \right\| \|a^*a\|^1/n \\
&< \varepsilon.
\end{aligned}$$

For $j = 1, \ldots, k$, the map $\Theta_{(aa^*)^{\frac{1}{n}}(\xi_j),(a^*a)^{\frac{1}{n}}(\zeta_j)}$ leaves $E$ invariant, so its restriction to $E$ is a rank one operator in $\mathcal{K}(E)$. It follows that $a|_E$ is the limit of a sequence finite rank operators on $E$, and hence it is compact. In particular, $E$ is countably generated, because the range of each finite rank operator is finitely generated. Finally, it is clear that $a$ is invariant, so $a \in \mathcal{K}(E)^G$. $\square$

The following observation will be used throughout without particular reference. Recall that a positive element $x$ in a $C^*$-algebra $A$ is said to be *strictly positive* if $\tau(x) > 0$ for every continuous linear map $\tau\colon A \to \mathbb{C}$.

**Remark III.4.8.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Denote by $\mu$ the normalized Haar measure on $G$. If $x \in A$ is a strictly positive element, then $y = \int\limits_G \alpha_g(x)\, d\mu(g)$ is strictly positive in $A$. Indeed, let $\tau\colon A \to \mathbb{C}$ be a linear map. For $g \in G$, the map $\tau \circ \alpha_g\colon A \to \mathbb{C}$ is also linear, and so $\tau(\alpha_g(x)) > 0$. Since $g \mapsto \tau(\alpha_g(x))$ is continuous, we deduce that

$$\tau(y) = \tau\left(\int\limits_G \alpha_g(x)\, d\mu(g)\right) = \int\limits_G \tau(\alpha_g(x))\, d\mu(g) > 0,$$

so $y$ is strictly positive, as desired.

**Lemma III.4.9.** Let $(E, \rho)$ be a countably generated $G$-Hilbert $A$-module, and let $(F, \mu|_F)$ be a countably generated $G$-Hilbert submodule of $E$. Then there exists $a \in \mathcal{K}(E)^G$ such that $F = \overline{a(F)} = \overline{a(E)}$.

*Proof.* Use [162, Proposition 6.7] to choose a strictly positive element $c \in \mathcal{K}(F)$. Then the element $c' = \int_G (g \cdot c)\, dg$ is strictly positive and $G$-invariant Since $c(F) = c'(F)$ and $c(E) = c'(E)$, we may assume that $c$ is invariant.

Using strict positivity of $c$, choose a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathcal{K}(F)$ such that

$$\lim_{n\to\infty} \|cb_n - c^{\frac{1}{n}}\| = 0.$$

By [162, Equation 1.5], we have $\lim\limits_{n\to\infty} cb_n\xi = \xi$ for all $\xi \in F$. It follows that $\overline{c(F)} = F$. Using [144, Theorem 2], one shows that $c$ can be extended to an element $b \in \mathcal{K}(E)$ satisfying $\overline{b(E)} = \overline{c(F)} = F$. Now, the desired element is obtained by integrating $g \cdot b$ over $G$ (using normalized Haar measure). $\square$

*The Hilbert module picture of* $\mathrm{Cu}^G(A, \alpha)$

We now define the relevant equivalence and subequivalence relations of $G$-Hilbert modules that will give rise to a different description of the equivariant Cuntz semigroup.

**Definition III.4.10.** Let $(E, \rho)$ be a $G$-Hilbert A-module, and let $(F, \eta)$ be a $G$-Hilbert $A$-submodule. We say that $(F, \eta)$ is *$G$-compactly contained* in $(E, \rho)$, and denote this by $(F, \eta) \Subset (E, \rho)$, if there exists a contraction $T \in \mathcal{K}(E)$ whose restriction to $F$ is $\mathrm{id}_F$.

We claim that the operator $T$ in the definition above can be taken in the fixed point algebra $\mathcal{K}(E)^G$. To see this, first note that if $\xi \in F$, then

$$(g \cdot T)(\xi) = \rho_g(T(\rho_{g^{-1}}(\xi))) = \xi.$$

With $dg$ denoting the normalized Haar measure on $G$, it follows that $T' = \int_G (g \cdot T)\, dg$ is invariant and its restriction to $F$ is the identity.

**Definition III.4.11.** Let $(E, \rho)$ and $(F, \eta)$ be $G$-Hilbert A-modules. We say that $(E, \rho)$ is *$G$-Cuntz subequivalent* to $(F, \eta)$, and denote this by $(E, \rho) \precsim_G (F, \eta)$, if every compactly contained $G$-Hilbert submodule of $(E, \rho)$ is unitarily equivalent to a $G$-Hilbert submodule of $(F, \eta)$.

We say that $(E, \rho)$ is *$G$-Cuntz equivalent* to $(F, \eta)$, and denote this by $(E, \rho) \sim_G (F, \eta)$, if $(E, \rho) \precsim_G (F, \eta)$ and $(F, \mu) \precsim_G (E, \nu)$. The $G$-Cuntz equivalence class of the $G$-Hilbert $A$-module $(E, \rho)$ is denoted by $[(E, \rho)]$.

We denote by $\mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$ the set of Cuntz equivalence classes of $G$-Hilbert $A$-modules.

It is easy to check that the direct sum of $G$-Hilbert $A$-modules induces a well defined operation on $\mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$. Endow $\mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$ with the partial order given by $[(E, \rho)] \leq [(F, \eta)]$ if $(E, \rho) \precsim_G (F, \eta)$. With this structure, it is clear that $\mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$ is a partially ordered abelian semigroup.

The proof of the following lemma is easy, and it is left to the reader.

**Lemma III.4.12.** The notion of $G$-compact containment for equivariant Hilbert modules from Definition III.4.10 induces the compact containment relation on $\mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$.

We now define a $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$. For $[(E, \rho)] \in \mathrm{Cu}^G_{\mathcal{H}}(A, \alpha)$ and $[\mu] \in \mathrm{Cu}(G)$, we set

$$[\mu] \cdot [(E, \rho)] = [(\mathcal{H}_\mu \otimes E, \mu \otimes \rho)].$$

Similarly, $\mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G)$ has a natural $\mathrm{Cu}(G)$-semimodule structure (see Definition III.4.3 for the definition of $\mathcal{H}_A$). Let $a \in \mathcal{K}(\mathcal{H}_A)^G$ be a positive element. For a separable unitary representation $(\mathcal{H}_\mu, \mu)$ of $G$, let $s_\mu \in \mathcal{K}(\mathcal{H}_\mu)^G$ be a strictly positive element. Identify $\mathcal{H}_\mu \otimes \mathcal{H}_A$ with a submodule of $\mathcal{H}_A$ using the product in $\mathrm{Cu}(G)$, and set

$$[\mu] \cdot [a] = [s_\mu \otimes a].$$

Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$. Then $(\mathcal{H}_\mu, \mu)$ is unitarily equivalent to a subrepresentation of $(\mathcal{H}_{\mathbb{C}}, \pi_{\mathbb{C}})$, and hence there exists an operator

$$V_\mu = V_{\mu,\pi_A} \in \mathcal{L}(\mathcal{H}_\mu \otimes A, \mathcal{H}_A)^G$$

satisfying $V_\mu^* V_\mu = \mathrm{id}_{\mathcal{H}_\mu}$.

Define a map $\chi\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G)$ as follows. Given a separable unitary representation $(\mathcal{H}_\mu, \mu)$ of $G$, and given a positive element $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$, set

$$\chi([a]_G) = [V_\mu a V_\mu^*].$$

**Proposition III.4.13.** The map $\chi\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G)$, described above, is well defined. Moreover, it is an isomorphism in $\mathbf{Cu}^G$.

*Proof.* We divide the proof into a number of claims.

*Claim: $\chi$ is well defined, and it preserves the order.* Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ and $b \in (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$ satisfy $a \precsim_G b$. Then there exists a sequence $(d_n)_{n\in\mathbb{N}}$ in $(\mathcal{K}(\mathcal{H}_\nu, \mathcal{H}_\mu) \otimes A)^G$, such that $\lim\limits_{n\to\infty} \|d_n b d_n^* - a\| = 0$. It follows that $V_\mu d_n V_\nu^*$ belongs to $\mathcal{K}(\mathcal{H}_A)^G$, and

$$(V_\mu d_n V_\nu^*)(V_\nu b V_\nu^*)(V_\mu d_n V_\nu^*)^* = V_\mu d_n b d_n V_\mu^* \to V_\mu a V_\mu^*,$$

in the norm of $(\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$, as $n \to \infty$. This shows that $V_\mu a V_\mu^* \precsim V_\nu b V_\nu^*$, and the claim is proved.

*Claim: $\chi$ is an order embedding.* Let $(\mathcal{H}_\mu, \mu)$ and $(\mathcal{H}_\nu, \nu)$ be separable unitary representations of $G$, and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ and $b \in (\mathcal{K}(\mathcal{H}_\nu) \otimes A)^G$ satisfy $\chi([a]_G) \leq \chi([b]_G)$. Set $a' = V_\mu a V_\mu^*$ and $b' = V_\nu b V_\nu^*$. Then there exists a sequence $(d_n')_{n\in\mathbb{N}}$ in $\mathcal{K}(\mathcal{H}_A)^G$ such that $\lim\limits_{n\to\infty} \|d_n' b' (d_n')^* - a'\| = 0$.

For each $n \in \mathbb{N}$, set $E_n = \overline{\mathrm{span}}\,(d_n'(\mathcal{H}_A) \cup (d_n')^*(\mathcal{H}_A))$. Then $E_n$ is a countably generated $G$-Hilbert $A$-module by Lemma III.4.7. Use Lemma III.4.6 to choose a separable subrepresentation $(\mathcal{H}_n, (\pi_{\mathbb{C}})_{\mathcal{H}_n})$ of $(\mathcal{H}_{\mathbb{C}}, \pi_{\mathbb{C}})$, satisfying $E \subseteq \mathcal{H}_n \otimes A \subseteq \mathcal{H}_A$ as $G$-Hilbert $A$-modules. Let $W_{\mu,\pi_A} \in \mathcal{B}(\mathcal{H}_\mu)$ and $W_{\nu,\pi_A} \in \mathcal{B}(\mathcal{H}_\nu)$ be the partial isometries implementing the isomorphisms of $\mathcal{H}_\mu$ and

$\mathcal{H}_\nu$ with Hilbert subspaces $\mathcal{H}'_\mu$ and $\mathcal{H}'_\nu$ of $\mathcal{H}_\mathbb{C}$, respectively. Set

$$\mathcal{H} = \overline{\operatorname{span}}\left(\mathcal{H}'_\mu \cup \mathcal{H}'_\nu \cup \bigcup_{n\in\mathbb{N}} \mathcal{H}_n\right) \subseteq \mathcal{H}_\mathbb{C}.$$

Then $\mathcal{H}$ is separable and the operators $a'$, $b'$, $d'_n$, and $(d'_n)^*$, for $n \in \mathbb{N}$, map $\mathcal{H} \otimes A$ to itself. Moreover, the restrictions $a''$, $b''$, $d''_n$ of $a'$, $b'$, $d'_n$, for $n \in \mathbb{N}$, to $\mathcal{H} \otimes A$, belong to $\mathcal{K}(\mathcal{H} \otimes A)^G$. Moreover, we have

$$\lim_{n\to\infty} \|d''_n b''(d''_n)^* - a''\| = 0,$$

and thus $a'' \precsim b''$ in $\mathcal{K}(\mathcal{H}\otimes A)^G$. Consequently, $a'' \precsim_G b''$. Lemma III.2.4 implies that $a \sim_G a''$ and $b \sim_G b''$. We conclude that $a \precsim_G b$, as desired.

*Claim: $\chi$ is surjective.* Let $a \in \mathcal{K}(\mathcal{H}_A)^G$. By Lemma III.4.9, there exists a subrepresentation $(\mathcal{H}_\mu, \mu)$ of $(\mathcal{H}_\mathbb{C}, \pi_\mathbb{C})$ such that $\overline{a(\mathcal{H}_A)} \subseteq \mathcal{H}_\mu \otimes A$ as $G$-Hilbert $A$-modules. Let $a'$ be the restriction of $a$ to $\mathcal{H}_\mu \otimes A$. It is then clear that $\chi([a']_G) = [a]$, so the claim is proved.

It follows that $\chi$ is a **Cu**-isomorphism.

*Claim: $\chi$ is a* $\mathrm{Cu}(G)$*-semimodule morphism (and hence a* $\mathbf{Cu}^G$*-isomorphism).* It is enough to check that $\chi$ preserves the $\mathrm{Cu}(G)$-action. This is immediate from the definitions. □

We record here the following useful corollary.

**Corollary III.4.14.** Let $[\mu] \in \mathrm{Cu}(G)$ and let $a \in (\mathcal{K}(\mathcal{H}_\mu) \otimes A)^G$ be a positive element. Then $[(a-\varepsilon)_+]_G \ll [a]_G$ for all $\varepsilon > 0$, and

$$[a]_G = \sup_{\varepsilon>0} [(a-\varepsilon)_+]_G.$$

*Proof.* Choose an operator $V_\mu \in \mathcal{L}(\mathcal{H}_\mu \otimes A, \mathcal{H}_A)^G$ satisfying $V_\mu^* V_\mu = \mathrm{id}_{\mathcal{H}_\mu \otimes A}$. Note that $\mathrm{Ad}(V_\mu)$ is a $*$-homomorphism, and hence it commutes with functional calculus. Use Proposition III.4.13 at the first and last step to get

$$\begin{aligned}
[a]_G &= \chi^{-1}([V_\mu a V_\mu^*]) \\
&= \sup_{\varepsilon>0} \chi^{-1}([(V_\mu a V_\mu^* - \varepsilon)_+]) \\
&= \sup_{\varepsilon>0} \chi^{-1}([V_\mu (a-\varepsilon)_+ V_\mu^*]) \\
&= \sup_{\varepsilon>0} [(a-\varepsilon)_+]_G.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
[(a-\varepsilon)_+]_G &= \chi^{-1}([V_\mu (a-\varepsilon)_+ V_\mu^*]) \\
&= \chi^{-1}([(V_\mu a V_\mu^* - \varepsilon)_+]) \\
&\ll \chi^{-1}([V_\mu a V_\mu^*]) \\
&= [a]_G,
\end{aligned}$$

as desired. □

For $a \in \mathcal{K}(\mathcal{H}_A)^G$, we denote by $\mathcal{H}_{A,a}$ the $G$-Hilbert $A$-module $\overline{a(\mathcal{H}_A)}$, and we let $\pi_{A,a}$ be the compression of $\pi_A$ to $\overline{a(\mathcal{H}_A)}$.

**Theorem III.4.15.** Suppose that $G$ is second countable. Then the map

$$\tau \colon \mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G) \to \mathrm{Cu}_{\mathcal{H}}^G(A,\alpha),$$

defined by $\tau([a]) = [(\mathcal{H}_{A,a}, \pi_{A,a})]$ for a positive element $a \in \mathcal{K} \otimes (\mathcal{K}(\mathcal{H}_A)^G)$, is a well defined natural isomorphism in $\mathbf{Cu}^G$.

*Proof.* We divide the proof into a number of claims.

*Claim: $\tau$ is well defined and it preserves the partial order.* To show this, it suffices to prove that if $a, b \in \mathcal{K}(\mathcal{H}_A)^G$ are positive elements with $a \precsim_G b$, then

$$(\mathcal{H}_{A,a}, \pi_{A,a}) \precsim_G (\mathcal{H}_{A,b}, \pi_{A,b}).$$

(See Definition III.4.2.)

Let $(E, (\pi_A)|_E)$ be a countably generated $G$-Hilbert $A$-module which is compactly contained in $(\mathcal{H}_{A,a}, \pi_{A,a})$. By Lemma III.4.9, there exists

$$c \in \mathcal{K}\left(\mathcal{H}_{A,a}\right)^G \cong \left(\overline{a(\mathcal{K}(\mathcal{H}_{\mathbb{C}}) \otimes A)a}\right)^G$$

such that $c|_E$ is strictly positive and $\overline{c(\mathcal{H}_{A,a})} = E$. Then $\mathcal{K}(E) \cong \overline{c(\mathcal{K}(\mathcal{H}_{\mathbb{C}}) \otimes A)c}$. Use the definition of compact containment of $G$-Hilbert $A$-modules (Definition III.4.10; see also Lemma III.4.12) to choose $d \in \left(\overline{a(\mathcal{K}(\mathcal{H}_{\mathbb{C}}) \otimes A)a}\right)^G$ with $\|d\| = 1$ and $dc = c$. In particular, we have $(d + c - 1)_+ = c$. Apply Proposition III.2.2 with $\varepsilon = 1$ to the elements $d + c \precsim_G b$ to find $f \in \mathcal{K}\left(\mathcal{H}_{A,b}, \mathcal{H}_{A,a}\right)^G$ such that $(d + c - 1)_+ = fbf^*$. Set $x = b^{\frac{1}{2}} f^*$, which is an element in $\mathcal{K}\left(\mathcal{H}_{A,a}, \mathcal{H}_{A,b}\right)^G$. Then

$$x^* x = c \text{ and } xx^* \in \overline{b(\mathcal{K}(\mathcal{H}_{\mathbb{C}}) \otimes A)b}^G.$$

Set $F = \overline{x(E)}$, and let $y\colon E \to F$ be the operator obtained from $x$ by restricting its domain to $E$ and its codomain to $F$. Since $x$ is invariant, we have $y \in \mathcal{L}(E, F)^G$. It is clear that $y$ has dense range. Moreover, $y^* = (x^*)|_F \in \mathcal{L}(F, E)^G$, and hence

$$\overline{y^*(F)} = \overline{y^*(\overline{x(E)})} = \overline{y^*(x(E))} = \overline{x^* x(E)} = \overline{c(E)} = E.$$

It follows that both $y$ and $y^*$ have dense range. By [162, Proposition 3.8], it follows that $E$ and $F$ are unitarily equivalent. Moreover, it can be seen from the proof of that proposition that the unitary can be chosen in $\mathcal{L}_G(E, F)$. This shows that $(E, (\pi_A)|_E)$ is $G$-equivalent to a submodule of $(\mathcal{H}_{A,b}, \pi_{A,b})$, as desired. This proves the claim.

*Claim: $\tau$ is an order embedding.* Let $a, b \in \mathcal{K}(\mathcal{H}_A)^G$ satisfy

$$(\mathcal{H}_{A,a}, \pi_{A,a}) \precsim_G (\mathcal{H}_{A,b}, \pi_{A,b}).$$

Let $\varepsilon > 0$ and let $f \in \mathrm{C}_0(0, \|a\|]$ be a function that is linear on $[0, \varepsilon]$ and constant equal to 1 on $[\varepsilon, \|a\|]$. Then $f(a)$ belongs to $\left(\overline{a\mathcal{K}(\mathcal{H}_A)a}\right)^G$ and satisfies $f(a)(a - \varepsilon)_+ = (a - \varepsilon)_+$. It follows that $\left(\mathcal{H}_{A,(a-\varepsilon)_+}, \pi_{A,(a-\varepsilon)_+}\right)$ is compactly contained in $(\mathcal{H}_{A,a}, \pi_{A,a})$, so there exists an equivariant unitary

$$U\colon \left(\mathcal{H}_{A,(a-\varepsilon)_+}, \pi_{A,(a-\varepsilon)_+}\right) \to (\mathcal{H}_{A,b}, \pi_{A,b}).$$

Set $x = (a-\varepsilon)_+ U^*$, which is an element in $\mathcal{K}(\mathcal{H}_{A,(a-\varepsilon)_+}, \overline{b(\mathcal{H}_A)})$. Then

$$(a-\varepsilon)_+ = xx^* \text{ and } x^*x = U(a-\varepsilon)_+U^* \in \mathcal{K}(\overline{b\mathcal{H}_A})^G.$$

It follows that $\lim\limits_{n\to\infty} \left\| b^{\frac{1}{n}} x^*x b^{\frac{1}{n}} - x^*x \right\| = 0$. Therefore $x^*x \precsim_G b$. Since we also have $(a-\varepsilon)_+ = xx^* \sim_G x^*x$, it follows that $(a-\varepsilon)_+ \precsim_G b$. Since $\varepsilon > 0$ is arbitrary, we conclude that

$$[a] = \sup_{\varepsilon>0} [(a-\varepsilon)_+] \leq [b],$$

and the claim is proved.

*Claim: $\tau$ is surjective.* Let $(E, (\pi_A)|_E)$ be a countably generated $G$-Hilbert $A$-module. Since $G$ is assumed to be second countable, $(E, (\pi_A)|_E)$ is isomorphic to a $G$-Hilbert submodule of $(\mathcal{H}_A, \pi_A)$, by [144, Theorem 2]. Use Lemma III.4.9 to find $a \in \mathcal{K}(\mathcal{H}_A)^G$ such that $(E, (\pi_A)|_E) \cong (\mathcal{H}_{A,a}, \pi_{A,a})$. It follows that

$$\tau([a]) = [(\mathcal{H}_{A,a}, \pi_{A,a})] = [(E, (\pi_A)|_E)],$$

and the claim follows.

We deduce that $\tau$ is an isomorphism in **Cu**.

*Claim: $\tau$ is a* $\mathrm{Cu}(G)$*-semimodule morphism (and hence an isomorphism in* $\mathbf{Cu}^G$*).* We only check that $\tau$ preserves the $\mathrm{Cu}(G)$-action. Let $(\mathcal{H}_\mu, \mu)$ be a separable unitary representation of $G$, and let $s_\mu$ be a strictly positive element in $\mathcal{K}(\mathcal{H}_\mu)^G$. For $a \in \mathcal{K}(\mathcal{H}_A)^G$, we have

$$\tau([\mu]\cdot[a]) = \tau([s_\mu \otimes a]) = [\overline{(s_\mu \otimes a)(\mathcal{H}_\mu \otimes \mathcal{H}_A)}] = [\mathcal{H}_\mu \otimes \mathcal{H}_{A,a}] = [\mu]\cdot[\mathcal{H}_{A,a}].$$

This concludes the proof of the claim and of the theorem. □

The following is the main result of this section.

**Corollary III.4.16.** Suppose that $G$ is second countable. Then there is a natural $\mathbf{Cu}^G$-isomorphism

$$\delta\colon \mathrm{Cu}^G(A,\alpha) \cong \mathrm{Cu}^G_{\mathcal{H}}(A,\alpha).$$

*Proof.* By Theorem III.4.15 and Proposition III.4.13, the map $\delta = \tau \circ \chi$ is the desired $\mathbf{Cu}^G$-isomorphism. $\square$

### Julg's Theorem and the $\mathrm{Cu}(G)$-semimodule Structure on $\mathrm{Cu}(A \rtimes_\alpha G)$

The goal of this section is to prove that if $\alpha\colon G \to \mathrm{Aut}(A)$ is a continuous action of a compact group $G$ on a $C^*$-algebra $A$, then there is a natural $\mathbf{Cu}$-isomorphism between its equivariant Cuntz semigroup $\mathrm{Cu}^G(A, \alpha)$ and $\mathrm{Cu}(A \rtimes_\alpha G)$; see Theorem III.5.3. This isomorphism allows us to endow $\mathrm{Cu}(A \rtimes_\alpha G)$ with a canonical $\mathrm{Cu}(G)$-semimodule structure, and we compute it explicitly in Theorem III.5.13.

When $G$ is abelian, this semimodule structure is particularly easy to describe: it is given by the dual action of $\alpha$; see Proposition III.5.15. We will prove these results using the equivariant Hilbert module picture of $\mathrm{Cu}^G(A, \alpha)$ studied in the previous section.

#### *Julg's Theorem*

For the rest of this section, we fix a compact group $G$, a $C^*$-algebra $A$, and a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$. The goal of this section is to prove the Cuntz analog of Julg's theorem; see Theorem III.5.3. Most of the work has already been done in the previous section, and the only missing ingredients are Remark III.5.1, which is essentially the Peter-Weyl theorem, and Proposition III.5.2, which is noncommutative duality.

Let $L^2(G)$ denote the Hilbert space of square integrable functions on $G$ with respect to its normalized Haar measure, and let $\lambda\colon G \to \mathcal{U}(L^2(G))$ denote the left regular representation.

**Remark III.5.1.** By the Peter-Weyl Theorem ([73, Theorem 5.12]), the $G$-Hilbert module $(\mathcal{H}_{\mathbb{C}}, \pi_{\mathbb{C}})$ is unitarily equivalent (see Definition III.4.2) to

$$(\ell^2(\mathbb{N}) \otimes L^2(G), \mathrm{id}_{\ell^2(\mathbb{N})} \otimes \lambda).$$

Therefore there is an equivalence

$$(\mathcal{K}(\mathcal{H}_A), \pi_A) \sim_G (\mathcal{K}(\ell^2(\mathbb{N}) \otimes L^2(G) \otimes A), \mathrm{Ad}(\mathrm{id}_{\ell^2(\mathbb{N})} \otimes \lambda \otimes \alpha)).$$

It follows that there is a natural $*$-isomorphism

$$\theta\colon \mathcal{K}(\mathcal{H}_A)^G \to \mathcal{K}(\ell^2(\mathbb{N}) \otimes L^2(G) \otimes A)^G.$$

The following result is standard, and it is a consequence of Landstad's duality. See, for example, Theorem 2.7 in [141], and specifically Example 2.9 in [141]. (The result can also be derived from Katayama's duality; see Theorem 8 in [146].)

**Proposition III.5.2.** Let $G$ be a locally compact group, let $B$ be a $C^*$-algebra, and let $\delta\colon B \to M(B \otimes C^*(G))$ be a normal coaction. Denote by $B \rtimes_\delta G$ the corresponding cocrossed product, and by $\widehat{\delta}\colon G \to \mathrm{Aut}(B \rtimes_\delta G)$ the dual action. Then there is a canonical $*$-isomorphism

$$\psi\colon (B \rtimes_\delta G)^{\widehat{\delta}} \to B,$$

which is moreover $\delta^{j_G} - \delta$ equivariant (see Definition 2.8 in [141]).

The following result is an analog of Julg's Theorem ([139]; see also [199, Theorem 2.6.1]) for the equivariant Cuntz semigroup.

**Theorem III.5.3.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then there is a natural **Cu**-isomorphism

$$\sigma\colon \mathrm{Cu}^G(A, \alpha) \to \mathrm{Cu}(A \rtimes_\alpha G).$$

*Proof.* Endow $\mathcal{K}(L^2(G))$ with the action of conjugation by the left regular representation of $G$, endow $\mathcal{K}(\ell^2(\mathbb{N}))$ with the trivial $G$-action, and endow $\mathcal{K}(L^2(G)) \otimes A$ and $\mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(L^2(G)) \otimes A$ with the corresponding tensor product actions. Then there is a natural identification

$$\mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(L^2(G)) \otimes A)^G = \mathcal{K}(\ell^2(\mathbb{N})) \otimes (\mathcal{K}(L^2(G)) \otimes A)^G).$$

Since Cu is a stable functor, there exists a natural **Cu**-isomorphism

$$\kappa\colon \mathrm{Cu}(\mathcal{K}(\ell^2(\mathbb{N}) \otimes L^2(G) \otimes A)^G) \to \mathrm{Cu}(\mathcal{K}(L^2(G) \otimes A)^G).$$

By Remark III.5.1, there exists a natural $*$-isomorphism

$$\theta\colon \mathcal{K}(\mathcal{H}_A)^G \to \mathcal{K}(\ell^2(\mathbb{N}) \otimes L^2(G) \otimes A)^G.$$

Denote by $\psi\colon (\mathcal{K}(L^2(G)) \otimes A)^G \to A \rtimes_\alpha G$ the natural $*$-isomorphism obtained from Proposition III.5.2 for $B = A \rtimes_\alpha G$ and $\delta = \widehat{\alpha}$. (Recall that coactions of compact groups are automatically normal; see, for example, the end of the proof of Lemma 4.8 in [85].) With $\chi\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G)$ denoting the natural **Cu**-isomorphism given by Proposition III.4.13, define $\sigma$ to be the following composition:

$$\begin{tikzcd}
\mathrm{Cu}^G(A,\alpha) \arrow[r, "\chi"] \arrow[dr, dashed, "\sigma"'] & \mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G) \arrow[r, "\mathrm{Cu}(\theta)"] & \mathrm{Cu}((\mathcal{K}(\ell^2(\mathbb{N})) \otimes \mathcal{K}(L^2(G)) \otimes A)^G) \arrow[d, "\kappa"] \\
& \mathrm{Cu}(A \rtimes_\alpha G) & \mathrm{Cu}((\mathcal{K}(L^2(G)) \otimes A)^G). \arrow[l, "\mathrm{Cu}(\psi)"]
\end{tikzcd}$$

It is clear that $\sigma$ is a natural isomorphism in the category **Cu**. $\square$

*Semimodule structure on the crossed product*

Theorem III.5.3 provides an isomorphism $\mathrm{Cu}^G(A,\alpha) \cong \mathrm{Cu}(A \rtimes_\alpha G)$ as **Cu**-semigroups. We can give $\mathrm{Cu}(A \rtimes_\alpha G)$ the unique $\mathrm{Cu}(G)$-semimodule structure that makes this isomorphism into a $\mathbf{Cu}^G$-isomorphism. To make this result useful, we must describe this semimodule structure internally. This takes some work, and we will need a series of intermediate results. This subsection is based, to some extent, on [199].

The main technical difficulties are the absence of short exact sequences in the context of semigroups, and the fact that in the construction of the equivariant Cuntz semigroup, representations of the group $G$ on infinite dimensional Hilbert spaces are allowed. Compactness of $G$ is crucial in overcoming the latter.

We need a standard definition. For a $C^*$-algebra $A$, we denote by $M(A)$ its multiplier algebra.

**Definition III.5.4.** Let $\alpha, \beta\colon G \to \mathrm{Aut}(A)$ be continuous actions of a locally compact group $G$ on a $C^*$-algebra $A$. We say that $\alpha$ and $\beta$ are *cocycle equivalent*, if there exists a function $\omega\colon G \to \mathcal{U}(M(A))$ satisfying:

1. $\omega_{gh} = \omega_g \alpha_g(\omega_h)$ for all $g, h \in G$;

2. $\alpha_g = \mathrm{Ad}(\omega_g) \circ \beta_g$ for all $g \in G$;

3. For $a \in A$, the map $G \to A$ given by $g \mapsto \omega_g a$ is continuous.

**Remark III.5.5.** It is well known that cocycle equivalent actions have isomorphic associated crossed products. Nevertheless, it does not follow from this that cocycle equivalent actions have isomorphic equivariant Cuntz semigroups, because we do not know how to compute the $\mathrm{Cu}(G)$-semimodule structure of the crossed products. (That this is indeed the case is a consequence of Theorem III.5.13.) In order to prove said result, however, we do need to know that some specific cocycle conjugate actions yield isomorphic equivariant Cuntz semigroups; see Proposition III.5.6.

For the rest of the subsection, we fix a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$ of a compact group $G$ on a $C^*$-algebra $A$.

**Proposition III.5.6.** Let $\beta$ be an action of $G$ on $A$ which is cocycle equivalent to $\alpha$. Suppose that $A$ has an increasing countable approximate identity consisting of projections which are invariant for both $\alpha$ and $\beta$. Then there is a natural $\mathbf{Cu}^G$-isomorphism $\mathrm{Cu}^G(A, \alpha) \cong \mathrm{Cu}^G(A, \beta)$.

*Proof.* Suppose first that $A$ is unital. Choose a cocycle $\omega\colon G \to U(A)$ such that $\alpha_g = \mathrm{Ad}(\omega_g) \circ \beta_g$ for all $g \in G$. Let $(E, \rho)$ be a countably generated $G$-Hilbert $(A, \alpha)$-module. Define a representation $\rho^\omega\colon G \to \mathcal{U}(E)$ by $\rho^\omega_g(x) = \rho_g(x)\omega_g$ for all $g \in G$ and all $x$ in $E$.

We claim that $(E, \rho^\omega)$ is a countably generated $G$-Hilbert $(A, \beta)$-module. Since $E$ was chosen to be countably generated to begin with, we shall only check that $\rho^\omega$ is compatible with $\beta$ and with the Hilbert module structure. Given $a \in A$, given $x \in E$ and given $g \in G$, we have

$$\rho^\omega_g(xa) = \rho_g(xa)\omega_g = \rho_g(x)\alpha_g(a)\omega_g = \rho_g(x)\omega_g\beta_g(a) = \rho^\omega_g(x)\beta_g(a),$$

as desired. Moreover, for $g \in G$ and $x, y \in E$, we have

$$\begin{aligned}
\langle \rho^\omega_g(x), \rho^\omega_g(y)\rangle_E &= \langle \rho_g(x)\omega_g, \rho_g(y)\omega_g\rangle_E \\
&= \omega_g^* \langle \rho_g(x), \rho_g(y)\rangle_E \omega_g \\
&= (\mathrm{Ad}(\omega_g^*) \circ \alpha_g)(\langle x, y\rangle_E) \\
&= \beta_g(\langle x, y\rangle_E),
\end{aligned}$$

thus proving the claim.

The assignment $(E,\rho) \mapsto (E,\rho^{\omega})$ is clearly surjective, and hence every $G$-Hilbert $(A,\beta)$-module has the form $(E,\rho^{\omega})$ for some $G$-Hilbert $(A,\alpha)$-module $(E,\rho)$.

We claim that the assignment $(E,\rho) \mapsto (E,\rho^{\omega})$ preserves $G$-Cuntz subequivalence. Let $(E,\rho)$ and $(E',\rho')$ be countably generated $G$-Hilbert $(A,\alpha)$-modules, and suppose that $(E,\rho) \precsim_G (E',\rho')$. If $(F,\eta^{\omega})$ is a $G$-Hilbert $(A,\beta)$-module that is compactly contained in $(E,\rho^{\omega})$, then it is straightforward to check that $(F,\eta)$ is compactly contained in $(E,\rho)$. If $(F',\eta')$ is a $G$-Hilbert $(A,\alpha)$-module compactly contained in $(E',\rho')$ such that $(F',\eta') \sim_G (F,\eta)$, then one readily checks that $(F',(\eta')^{\omega}) \sim_G (F,\eta^{\omega})$. This shows that $(E,\rho^{\omega}) \precsim_G (E',(\rho')^{\omega})$, and proves the claim.

Denote by $\varphi\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(A,\beta)$ the map given by $[(E,\rho)] \mapsto [(E,\rho^{\omega})]$, where $\rho^{\omega}$ is given by $\rho^{\omega}_g(x) = \rho_g(x)\omega_g$ for all $g \in G$ and all $x$ in $E$. We claim that $\varphi$ is a $\mathbf{Cu}^G$-morphism.

We already showed that $\varphi$ preserves the order. It is also easy to show that is preserves compact containment and suprema of increasing sequences. The only non-trivial part is showing that it is a morphism of $\mathrm{Cu}(G)$-semimodules. Given a separable unitary representation $(\mathcal{H}_\mu,\mu)$ of $G$, we must show that the diagram

$$\begin{tikzcd}
\mathrm{Cu}^G(A,\alpha) \arrow[r, "\varphi"] \arrow[d, "{[\mu]\cdot}"'] & \mathrm{Cu}^G(A,\beta) \arrow[d, "{[\mu]\cdot}"] \\
\mathrm{Cu}^G(A,\alpha) \arrow[r, "\varphi"'] & \mathrm{Cu}^G(A,\beta)
\end{tikzcd}$$

commutes. Given a countably generated $G$-Hilbert $(A,\alpha)$-module $(E,\rho)$, we have

$$[(\mathcal{H}_\mu,\mu)] \cdot \varphi([(E,\rho)]) = [(E \otimes \mathcal{H}_\mu, \rho^{\omega} \otimes \mu)]\,.$$

On the other hand, the element $\varphi\left([(\mathcal{H}_\mu,\mu)] \cdot [(E,\rho)]\right)$ is represented by the $G$-Hilbert $(A,\beta)$-module $(E \otimes \mathcal{H}_\mu, (\rho \otimes \mu)^{\omega})$, so it is enough to check that both actions on $\mathcal{H}_\mu \otimes E$ agree.

Let $g \in G$, let $x \in E$ and let $\xi \in \mathcal{H}_\mu$. Then

$$\begin{aligned}
((\rho \otimes \mu)^\omega)_g (x \otimes \xi) &= (\rho_g(x) \otimes \mu_g(\xi))\omega_g \\
&= (\rho_g(x)\omega_g) \otimes \mu_g(\xi) \\
&= \rho_g^\omega(x) \otimes \mu_g(\xi) \\
&= (\rho^\omega \otimes \mu)_g(x \otimes \xi),
\end{aligned}$$

thus showing that $\varphi$ is a $\mathbf{Cu}^G$-morphism. Since $\varphi$ is clearly bijective, it follows that it is an isomorphism. Naturality is also clear. This proves the unital case.

For the general case, let $(e_n)_{n\in\mathbb{N}}$ be an increasing approximate identity in $A$ consisting of projections that are invariant for both $\alpha$ and $\beta$. For $n \in \mathbb{N}$, let $\alpha^{(n)} \colon G \to \mathrm{Aut}(e_nAe_n)$ be the action given by $\alpha_g^{(n)}(a) = \alpha_g(a)$ for all $g \in G$ and $a \in e_nAe_n$, and similarly for $\beta^{(n)} \colon G \to \mathrm{Aut}(e_nAe_n)$. We claim that $\alpha^{(n)}$ and $\beta^{(n)}$ are exterior equivalent for all $n \in \mathbb{N}$.

Choose a cocycle $\omega \colon G \to \mathcal{U}(M(A))$ as in Definition III.5.4. For $g \in G$ and $n \in \mathbb{N}$, one checks that

$$\omega_g e_n \omega_g^* = (\mathrm{Ad}(\omega_g) \circ \alpha_g)(e_n) = \beta_g(e_n) = e_n.$$

Define $\omega^{(n)} \colon G \to \mathcal{U}(e_nAe_n)$ by $\omega_g^{(n)} = e_n u_g e_n$ for $g \in G$. One readily checks that $\mathrm{Ad}(\omega_g^{(n)}) \circ \beta_g^{(n)} = \alpha_g^{(n)}$ for all $n \in \mathbb{N}$ and all $g \in G$. The cocycle condition is also easy to verify, so the claim is proved.

Note that there are natural equivariant $*$-isomorphisms

$$(A, \alpha) = \varinjlim(e_nAe_n, \alpha^{(n)}) \text{ and } (A, \beta) = \varinjlim(e_nAe_n, \beta^{(n)}).$$

By Proposition III.3.12, there are natural $\mathrm{Cu}^G$-isomorphisms

$$\mathrm{Cu}^G(A, \alpha) \cong \varinjlim \mathrm{Cu}^G(e_nAe_n, \alpha^{(n)}) \text{ and } \mathrm{Cu}^G(A, \beta) \cong \varinjlim \mathrm{Cu}^G(e_nAe_n, \beta^{(n)}).$$

For $n \in \mathbb{N}$, denote by $\varphi^{(n)} \colon \mathrm{Cu}^G(e_nAe_n, \alpha^{(n)}) \to \mathrm{Cu}^G(e_nAe_n, \beta^{(n)})$ the natural $\mathbf{Cu}^G$-isomorphism provided by the unital case of this proposition. Naturality implies that the following diagram in

$\mathbf{Cu}^G$ is commutative:

$$\begin{tikzcd}
\mathrm{Cu}^G(e_1Ae_1,\alpha^{(1)}) \arrow[r] \arrow[d, "\varphi^{(1)}"'] & \mathrm{Cu}^G(e_2Ae_2,\alpha^{(2)}) \arrow[r] \arrow[d, "\varphi^{(2)}"'] & \cdots \arrow[r] & \mathrm{Cu}^G(A,\alpha) \arrow[d, dashed, "\varphi"] \\
\mathrm{Cu}^G(e_1Ae_1,\beta^{(1)}) \arrow[r] & \mathrm{Cu}^G(e_2Ae_2,\beta^{(2)}) \arrow[r] & \cdots \arrow[r] & \mathrm{Cu}^G(A,\beta).
\end{tikzcd}$$

The universal property of the inductive limit in $\mathbf{Cu}^G$ shows that there exists a natural $\mathbf{Cu}^G$-morphism $\varphi\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(A,\beta)$. This map is easily seen to be an isomorphism in $\mathbf{Cu}^G$, so the proof is complete. $\square$

**Corollary III.5.7.** Suppose that $A$ is unital and let $(\mathcal{H}_\mu,\mu)$ be a separable unitary representation of $G$. Let $q$ be any $\mu$-invariant rank one projection on $\mathcal{H}_\mu$, and denote by $\iota_q\colon A \to A\otimes\mathcal{K}(\mathcal{H}_\mu)$ the inclusion obtained by identifying $A$ with $q(A\otimes\mathcal{K}(\mathcal{H}_\mu))q$. Then $\iota_q$ induces a natural $\mathbf{Cu}^G$-isomorphism

$$\mathrm{Cu}^G(\iota_q)\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}^G(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{Ad}(\mu)).$$

*Proof.* Since $\mu$ can be decomposed as a direct sum of finite dimensional representations by the Peter-Weyl Theorem, it follows that $A\otimes\mathcal{K}(\mathcal{H}_\mu)$ has a countable approximate identity consisting of projections that are invariant under both $\alpha\otimes\mathrm{Ad}(\mu)$ and $\alpha\otimes\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)}$. Using Proposition III.5.6 for $(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{Ad}(\lambda))$ and $(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)})$, it follows that it is enough to assume that the representation of $G$ on $\mathcal{H}$ is trivial. The result now follows from Proposition III.3.11. $\square$

**Lemma III.5.8.** Suppose that $A$ is unital and let $(\mathcal{H}_\mu,\mu)$ be a separable unitary representation of $G$. Let $p\in\mathcal{K}(\mathcal{H}_\mu)$ be a $\mu$-invariant projection and let $q$ in $\mathcal{K}(\mathcal{H}_\mu)$ be any $\mu$-invariant rank one projection. Define equivariant homomorphisms

$$\varphi\colon (A,\alpha)\to(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{Ad}(\mu)) \text{ and } \psi\colon (A,\alpha)\to(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)})$$

by $\varphi(a)=a\otimes p$ and $\psi(a)=a\otimes q$ for all $a\in A$. Then $\mathrm{Cu}^G(\psi)$ is an isomorphism via which $\mathrm{Cu}^G(\varphi)$ is identified with multiplication by the class $[(p\mathcal{H}_\mu,\mu|_{p\mathcal{H}_\mu})]\in\mathrm{Cu}(G)$. More explicitly, if we let

$$\epsilon\colon \mathrm{Cu}^G(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{Ad}(\mu))\to\mathrm{Cu}^G(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)})$$

be the exterior equivalence isomorphism given by Proposition III.5.6, then the following $\mathbf{Cu}^G$-diagram commutes:

$$\begin{tikzcd}
\mathrm{Cu}^G(A,\alpha) \arrow[r, "{[\mu|_{p\mathcal{H}}]\cdot}"] \arrow[d, "\mathrm{Cu}^G(\varphi)"'] & \mathrm{Cu}^G(A,\alpha) \arrow[d, "\mathrm{Cu}^G(\psi)"] \\
\mathrm{Cu}^G(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{Ad}(\mu)) \arrow[r, "\epsilon"'] & \mathrm{Cu}^G(A\otimes\mathcal{K}(\mathcal{H}_\mu),\alpha\otimes\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)}).
\end{tikzcd}$$

*Proof.* That $\psi$ is a natural isomorphism in $\mathbf{Cu}^G$ follows from Proposition III.3.11. We only need to check that

$$\mathrm{Cu}^G(\psi)^{-1}\circ\epsilon\circ\mathrm{Cu}^G(\varphi)\colon \mathrm{Cu}^G(A,\alpha)\to\mathrm{Cu}^G(A,\alpha)$$

is multiplication by $[\mu|_{p\mathcal{H}_\mu}]$.

Denote by $(\mathcal{H}_A,\pi_A)$ the canonical countably generated $G$-Hilbert $(A,\alpha)$-module from Definition III.4.3. Let $(E,\rho)$ be a countably generated $G$-Hilbert $(A,\alpha)$-module. Use Kasparov's absorption theorem (Theorem 2 in [144]) to choose a $\pi_A$-invariant projection $r$ in $\mathcal{L}(\mathcal{H}_A)^G$ such that $(E,\rho)\cong(r\mathcal{H}_A,\pi_A|_{r\mathcal{H}_A})$. Then

$$\begin{aligned}\mathrm{Cu}^G(\varphi)([(E,\rho)]) &= [((r\otimes p)\left(\mathcal{K}(\mathcal{H}_A)\otimes\mathcal{K}(\mathcal{H}_\mu)\right),\lambda\otimes\mu)]\\ &= [E\otimes p\mathcal{K}(\mathcal{H}_\mu),\rho\otimes\mu_{p\mathcal{H}_\mu}].\end{aligned}$$

Denote by $\widetilde{\mu}$ the unitary representation of $G$ on $p\mathcal{K}(\mathcal{H}_\mu)$ given by $\widetilde{\mu}_g(x)=\mu_g x\mu_g^*\mu_g=\mu_g x$ for all $g\in G$ and all $x\in p\mathcal{K}(\mathcal{H}_\mu)$. The computation above then shows that $(\epsilon\circ\mathrm{Cu}^G(\varphi))([E,\rho])$ can be identified with the class of the $G$-Hilbert $(A\otimes\mathcal{K}(\mathcal{H}_\mu))$-module $(E\otimes p\mathcal{K}(\mathcal{H}_\mu),\rho\otimes\widetilde{\mu})$. We must compare the class of $(E\otimes p\mathcal{K}(\mathcal{H}_\mu),\rho\otimes\widetilde{\mu})$ with the class of $\mathrm{Cu}^G(\psi)([E\otimes p\mathcal{K}(H),\rho\otimes\mathrm{Ad}(\mu)])$, and show that they agree.

One checks that $\mathrm{Cu}^G(\psi)\left([(p\mathcal{H}_\mu,\mu|_{p\mathcal{H}_\mu})]\cdot[(E,\rho)]\right)$ is represented by

$$\left(E\otimes p\mathcal{H}_\mu\otimes q\mathcal{K}(\mathcal{H}_\mu),\rho\otimes\mathrm{Ad}(\mu)\otimes\mathrm{id}_{\mathcal{K}(\mathcal{H}_\mu)}\right).$$

Evaluating the diagram in the statement at $(E, \rho)$, we get

$$
\begin{tikzcd}
(E,\rho) \arrow[r, mapsto] \arrow[dd, mapsto] & (E \otimes p\mathcal{H}_\mu, \rho \otimes \mu) \arrow[d, mapsto] \\
& (E \otimes p\mathcal{H}_\mu \otimes q\mathcal{K}(\mathcal{H}_\mu), \rho \otimes \mathrm{Ad}(\mu) \otimes \mathrm{id}_{\mathcal{K}}) \arrow[d, dashed, "\cong"', "?"] \\
(E \otimes p\mathcal{K}(\mathcal{H}_\mu), \rho \otimes \mathrm{Ad}(\mu)) \arrow[r, mapsto] & (E \otimes p\mathcal{K}(\mathcal{H}_\mu), \rho \otimes \widetilde{\mu})\,.
\end{tikzcd}
$$

It is therefore enough to check that

$$\left(p\mathcal{H}_\mu \otimes q\mathcal{K}(\mathcal{H}_\mu), \mu|_{p\mathcal{H}_\mu} \otimes \mathrm{id}_{q\mathcal{K}(\mathcal{H}_\mu)}\right) \cong (p\mathcal{K}(\mathcal{H}_\mu), \widetilde{\mu})$$

as $G$-Hilbert $\mathcal{K}(\mathcal{H}_\mu)$-modules. Fix a unit vector $\xi^{(0)} \in \mathcal{H}_\mu$ in the range of $q$ and define

$$\sigma \colon p\mathcal{H}_\mu \otimes q\mathcal{K}(\mathcal{H}_\mu) \to p\mathcal{K}(\mathcal{H}_\mu)$$

by $\sigma(\xi \otimes b)(\eta) = \langle b^*(\xi^{(0)}, \eta)\rangle\xi$ for all $\xi \in p\mathcal{H}_\mu$, for all $b \in q\mathcal{K}(\mathcal{H}_\mu)$ and for all $\eta \in \mathcal{H}_\mu$, and extended linearly and continuously. Note that

$$\left(p \circ (\sigma(\xi \otimes b))\right)(\eta) = \langle b^*(\xi^{(0)}), \eta\rangle p(\xi) = \langle b^*(\xi^{(0)}), \eta\rangle\xi,$$

so the range of $\sigma$ is really contained in $p\mathcal{K}(\mathcal{H}_\mu)$.

We claim that $\sigma$ is injective. Assume that $\sigma(\xi \otimes b) = 0$ and $\xi \neq 0$. It follows that $\langle \eta, b^*(\xi^{(0)})\rangle = 0$ for all $\eta \in \mathcal{H}_\mu$ and hence $b^*(\xi^{(0)}) = 0$. Thus $b^*$ vanishes on $\mathrm{span}\{\xi^{(0)}\} = q\mathcal{H}_\mu$, and in particular $b = 0$.

We claim that $\sigma$ is surjective. Given $a$ in $\mathcal{K}(\mathcal{H}_\mu)$, the map $pa \colon \mathcal{H}_\mu \to p\mathcal{H}_\mu$ is a linear map with finite rank. It follows from the Riesz Representation Theorem that there exist $m \in \mathbb{N}$, rank one projections $r_1, \ldots, r_m$, unit vectors $\xi_j^{(0)}$ in the range of $q_j$ for $j = 1, \ldots, m$, vectors $\xi_1, \ldots, \xi_m$ and linear maps $c_j \in q_j\mathcal{K}(\mathcal{H}_\mu)$ such that

$$pa(\eta) = \sum_{j=1}^{m} \langle \eta, c_j^*(\xi_j^{(0)})\rangle\xi_j$$

for all $\eta \in \mathcal{H}_\mu$. Since any two rank one projections are unitarily equivalent, it follows that there are linear maps $b_1, \ldots, b_m \in q\mathcal{K}(\mathcal{H}_\mu)$ such that

$$pa(\eta) = \sum_{j=1}^{m} \langle b_j^*(\xi^{(0)}), \eta \rangle \xi_j$$

and thus $pa = \sigma\left(\sum_{j=1}^{m} \xi_j \otimes b_j\right)$, showing that $\sigma$ is surjective.

To show that $\sigma$ is a $G$-Hilbert $\mathcal{K}(\mathcal{H}_\mu)$-homomorphism, let $\xi \in p\mathcal{H}_\mu$, let $b \in q\mathcal{K}(\mathcal{H}_\mu)$, let $c \in \mathcal{K}(\mathcal{H}_\mu)$ and let $\eta \in \mathcal{H}_\mu$. Then

$$\begin{aligned} \sigma(\xi \otimes b)(c\eta) &= \langle cb^*(\xi^{(0)}), \eta \rangle \xi \\ &= \langle (bc)^*(\xi^{(0)}), \eta \rangle \xi \\ &= \sigma(\xi \otimes bc)(\eta). \end{aligned}$$

Finally, for $g \in G$, for $b \in q\mathcal{K}(\mathcal{H}_\mu)$, for $c \in \mathcal{K}(\mathcal{H}_\mu)$, for $\xi \in p\mathcal{H}_\mu$ and for $\eta \in \mathcal{H}_\mu$, one has

$$\begin{aligned} \sigma\left((\mu|_{p\mathcal{H}_\mu} \otimes \mathrm{id}_{q\mathcal{K}(\mathcal{H}_\mu)})_g(\xi \otimes b)\right) &= \sigma\left(\mu_g \xi \otimes b\right)(\eta) \\ &= \langle , b^*(\xi^{(0)}), \eta \rangle \mu_g \xi \\ &= \mu_g \langle b^*(\xi^{(0)}), \eta \rangle \xi \\ &= \widetilde{\mu}_g(\sigma(\xi \otimes b))(\eta), \end{aligned}$$

which shows that $\sigma$ is equivariant. This finishes the proof. $\square$

The above result leads to a method for computing the $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$. This description makes essential use of the exterior equivalence isomorphism $\epsilon$, and similarly to what happens with equivariant $K$-theory, it is inconvenient when trying to use it. To remedy this, we give an alternative description of the $\mathrm{Cu}(G)$-action, which, even though it is not as transparent as the one in Lemma III.5.8, has the advantage that all **Cu**-maps involved are induced by $*$-homomorphisms.

**Definition III.5.9.** We define a $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$ as follows. Let $(\mathcal{H}_\mu, \mu)$ be a *finite dimensional* unitary representation of $G$, and denote by $\mu^+ \colon G \to \mathcal{U}(\mathcal{H}_\mu \oplus \mathbb{C})$ its direct sum with the trivial representation on $\mathbb{C}$. Let $p_{\mathcal{H}_\mu}, p_{\mathbb{C}} \in \mathcal{K}(\mathcal{H}_\mu \oplus \mathbb{C})$ be the projections onto $\mathcal{H}_\mu$ and $\mathbb{C}$, respectively. Define equivariant $*$-homomorphism $\varphi_{\mathcal{H}_\mu}, \varphi_{\mathbb{C}} \colon A \to A \otimes \mathcal{K}(\mathcal{H}_\mu \oplus \mathbb{C})$ by

$$\varphi_{\mathcal{H}_\mu}(a) = a \otimes p_{\mathcal{H}_\mu} \text{ and } \varphi_{\mathbb{C}}(a) = a \otimes p_{\mathbb{C}}$$

for $a \in A$. Denote by $\overline{\varphi}_{\mathcal{H}_\mu}$ and $\overline{\varphi}_{\mathbb{C}}$ the corresponding maps on the crossed products by $G$. The $\mathrm{Cu}(\varphi_{\mathbb{C}})$ is invertible by Lemma III.5.8, since it corresponds to multiplication by the class of the trivial representation. By considering these maps at the level of the Cuntz semigroups, we have

$$\mathrm{Cu}(A \rtimes_\alpha G) \xrightarrow{\mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_\mu})} \mathrm{Cu}\left((A \otimes \mathcal{K}(\mathcal{H}_\mu \oplus \mathbb{C})) \rtimes_{\alpha \otimes \mathrm{Ad}(\mu^+)} G\right) \underset{\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})}{\overset{\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})^{-1}}{\rightleftarrows}} \mathrm{Cu}(A \rtimes_\alpha G).$$

For $s \in \mathrm{Cu}(A \rtimes_\alpha G)$, we set

$$[\mu] \cdot s = \left(\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})^{-1} \circ \mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_\mu})\right)(s).$$

For an arbitrary separable unitary representation $(\mathcal{H}_\nu, \nu)$, use compactness of $G$ to choose finite dimensional unitary representations $(\mathcal{H}_{\mu_n}, \mu_n)$ of $G$ such that $\nu \cong \bigoplus\limits_{n=1}^{\infty} \mu_n$. For $m \in \mathbb{N}$, set $\nu_m = \bigoplus\limits_{n=1}^{m} \mu_n$. For $s \in \mathrm{Cu}(A \rtimes_\alpha G)$, we set

$$[\nu] \cdot s = \sup_{m \in \mathbb{N}} \left([\nu_m] \cdot s\right).$$

We must first check that in the above definition, $\sup\limits_{m \in \mathbb{N}} ([\nu_m] \cdot s)$ is independent of the decomposition $\nu \cong \bigoplus\limits_{n=1}^{\infty} \mu_n$.

**Lemma III.5.10.** Let $(\mathcal{H}_\nu, \nu)$ be a separable unitary representation of $G$, and find finite dimensional unitary representations $(\mathcal{H}_{\mu_n}, \mu_n)$ of $G$ as in Definition III.5.9. For $m \in \mathbb{N}$, set $\nu_m = \bigoplus\limits_{n=1}^{m} \mu_n$. Let $s \in \mathrm{Cu}(A \rtimes_\alpha G)$.

1. The sequence $([\nu_m] \cdot s)_{n \in \mathbb{N}}$ is increasing in $\mathrm{Cu}(A \rtimes_\alpha G)$.

2. The element $[\nu] \cdot s = \sup\limits_{m \in \mathbb{N}} ([\nu_m] \cdot s)$ is independent of the decomposition $\nu \cong \bigoplus\limits_{n \in \mathbb{N}} \mu_n$.

*Proof.* Both parts are immediate, using that Definition III.5.9 gives a $\mathrm{Cu}(G)$-semimodule structure when restricted to finite dimensional unitary representations. The proof is left to the reader. ∎

**Lemma III.5.11.** The $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$ described above is compatible with taking suprema in $\mathrm{Cu}(G)$.

*Proof.* Let $(\mathcal{H}_{\mu_n}, \mu_n)_{n\in\mathbb{N}}$ be a sequence of separable unitary representations of $G$ such that $([\mu_n])_{n\in\mathbb{N}}$ is increasing in $\mathrm{Cu}(G)$. Set $[\mu] = \sup\limits_{n\in\mathbb{N}}[\mu_n]$. Without loss of generality, we can assume that $\mu_n$ is a subrepresentation of $\mu_{n+1}$ for all $n \in \mathbb{N}$. In particular, we may assume that $\mathcal{H}_\mu = \overline{\bigcup\limits_{n\in\mathbb{N}} \mathcal{H}_{\mu_n}}$ with $\mathcal{H}_{\mu_n} \subseteq \mathcal{H}_{\mu_{n+1}}$ for all $n \in \mathbb{N}$. It follows that $\mu^+ = \sup\limits_{n\in\mathbb{N}}(\mu_n^+)$ as representations of $G$ on $\mathcal{H}_\mu \oplus \mathbb{C}$. Thus, for $a \in A$, we have

$$\varphi_{\mathcal{H}_\mu}(a) = a \otimes p_{\mathcal{H}_\mu} = \sup_{n\in\mathbb{N}}(a \otimes p_{\mathcal{H}_{\mu_n}}) = \sup_{n\in\mathbb{N}}\left(\varphi_{\mathcal{H}_{\mu_n}}(a)\right).$$

Finally, since $\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})$ is an isomorphism in **Cu**, we conclude that

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\left([\mu_n]\cdot s\right) &= \sup_{n\in\mathbb{N}}\left(\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})^{-1} \circ \mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_{\mu_n}})(s)\right) \\
&= \mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})^{-1}\left(\sup_{n\in\mathbb{N}}\left(\mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_{\mu_n}})(s)\right)\right) \\
&= \mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})^{-1} \circ \mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_\mu})(s) \\
&= [\mu]\cdot s,
\end{aligned}$$

for all $s \in \mathrm{Cu}(A \rtimes_\alpha G)$, as desired. □

**Lemma III.5.12.** Let $S_1$ and $S_2$ be semigroups in **Cu**, and let $J_1$ and $J_2$ be order ideals in $S_1$ and $S_2$ respectively. For $j = 1, 2$, denote by $\iota_j \colon J_j \to S_j$ the canonical inclusion, and by $\pi_j \colon S_j \to S_j/J_j$ the quotient map. Assume that there is a commutative diagram in **Cu**

$$\begin{tikzcd}
J_1 \arrow[r, "\iota_1"] \arrow[d, dashed, "\theta"'] & S_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"] & S_1/J_1 \arrow[d, "\psi"] \\
J_2 \arrow[r, "\iota_2"'] & S_2 \arrow[r, "\pi_2"'] & S_2/J_2,
\end{tikzcd}$$

such that $\varphi$ and $\psi$ are isomorphisms. Then there exists a unique morphism $\theta \colon J_1 \to J_2$ in **Cu** making the resulting diagram commute. Moreover, $\theta$ is an isomorphism.

*Proof.* Let $x \in J_1$. Since $(\pi_2 \circ \varphi \circ \iota_1)(x) = (\psi \circ \pi_1 \circ \iota_1)(x) = 0$, it follows that $(\varphi \circ \iota_1)(x)$ belongs to $\iota_2(J_2)$, that is, there exists $y \in J_2$ such that $(\varphi \circ \iota_1)(x) = \iota_2(y)$. Define $\theta(x) = y$. The map $\theta$ is well-defined because $\iota_2(y) = \iota_2(y')$ implies $y = y'$. The left square commutes by construction. We claim that $\theta$ is an isomorphism in **Cu**.

Let us start by showing injectivity. Let $x, x' \in J_1$ satisfy $\theta(x) = \theta(x')$. Since $\iota_2 \circ \theta = \varphi \circ \iota_1$, we conclude that $(\varphi \circ \iota_1)(x) = (\varphi \circ \iota_1)(x')$. Now, $\varphi$ and $\iota_1$ are injective, and thus $x = x'$. For surjectivity, let $y \in J_2$. Then

$$(\pi_1 \circ \varphi^{-1} \circ \iota_2)(y) = (\psi^{-1} \circ \pi_2 \circ \iota_2)(y) = 0,$$

so $(\varphi^{-1} \circ \iota_2)(y) \in \iota_1(J_1)$. If $x \in J_1$ satisfies $\iota_1(x) = (\varphi^{-1} \circ \iota_2)(y)$, then $\theta(x) = y$. Hence $\theta$ is surjective.

That $\theta$ is a semigroup homomorphism is clear. We claim that it is an order isomorphism. Given $x$ and $x'$ in $J_1$ with $x \leq x'$, it follows that $(\varphi \circ \iota_1)(x) \leq (\varphi \circ \iota_1)(x')$. Set $y = \theta(x)$ and $y' = \theta(x')$. Since $\iota_2$ is an order embedding and $\iota_2(y) \leq \iota_2(y')$, this implies that $y \leq y'$ as desired.

Conversely, given $x$ and $x'$, suppose that $\theta(x) \leq \theta(x')$. Then

$$(\varphi^{-1} \circ \iota_2 \circ \theta)(x) = \iota_1(x) \leq (\varphi^{-1} \circ \iota_2 \circ \theta)(x') = \iota_1(x'),$$

which implies that $x \leq x'$ since $\iota_1$ is an order embedding. This finishes the proof. $\square$

We have now arrived at the main result of this section.

**Theorem III.5.13.** Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then there is a natural $\mathbf{Cu}^G$-isomorphism

$$\mathrm{Cu}^G(A, \alpha) \cong \mathrm{Cu}(A \rtimes_\alpha G),$$

where the $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$ is given by Definition III.5.9.

*Proof.* Assume first that $A$ is unital. Let $(\mathcal{H}_\mu, \mu)$ be a finite dimensional unitary representation of $G$, and let $\varphi_{\mathcal{H}_\mu}, \varphi_{\mathbb{C}}, \overline{\varphi}_{\mathcal{H}_\mu}$ and $\overline{\varphi}_{\mathbb{C}}$ be as in Definition III.5.9. By naturality of the isomorphism in

Theorem III.5.3, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Cu}^G(A,\alpha) \arrow[r, "\mathrm{Cu}^G(\varphi_{\mathcal{H}_\mu})"] \arrow[d] & \mathrm{Cu}^G\left(A\otimes\mathcal{K}(\mathcal{H}_\mu\oplus\mathbb{C}),\alpha\otimes\mathrm{Ad}(\mu^+)\right) \arrow[d] & \mathrm{Cu}^G(A,\alpha) \arrow[l, "\mathrm{Cu}^G(\varphi_{\mathbb{C}})"'] \arrow[d] \\
\mathrm{Cu}(A\rtimes_\alpha G) \arrow[r, "\mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_\mu})"'] & \mathrm{Cu}\left((A\otimes\mathcal{K}(\mathcal{H}_\mu\oplus\mathbb{C}))\rtimes_{\alpha\otimes\mathrm{Ad}(\mu^+)} G\right) & \mathrm{Cu}(A\rtimes_\alpha G), \arrow[l, "\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})"]
\end{tikzcd}$$

where all vertical arrows are given by Theorem III.5.3. By Lemma III.5.8, $\mathrm{Cu}^G(\varphi_{\mathbb{C}})$ corresponds to multiplication by the class of the trivial representation in the $\mathrm{Cu}(G)$-semimodule $\mathrm{Cu}^G\left(A\otimes\mathcal{K}(\mathcal{H}_\mu\oplus\mathbb{C}),\alpha\otimes\mathrm{Ad}(\mu^+)\right)$. It follows that $\mathrm{Cu}^G(\varphi_{\mathbb{C}})$ is invertible. Thus $\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})$ is also invertible, since the vertical arrows are invertible. By definition, $\mathrm{Cu}(\overline{\varphi}_{\mathbb{C}})^{-1}\circ\mathrm{Cu}(\overline{\varphi}_{\mathcal{H}_\mu})$ is multiplication by $[\mu]$ on $\mathrm{Cu}(A\rtimes_\alpha G)$. Commutativity of the diagram implies that the left vertical arrow $\mathrm{Cu}^G(A,\alpha)\to\mathrm{Cu}(A\rtimes_\alpha G)$ commutes with multiplication by $[\mu]$.

Assume now that $(\mathcal{H}_\nu,\nu)$ is a separable unitary representation of $G$. Since $G$ is compact, it follows that $\mathcal{K}(\mathcal{H}_\nu)$ has an increasing approximate identity $(e_n)_{n\in\mathbb{N}}$ consisting of $G$-invariant projections. For $n\in\mathbb{N}$, denote by $\mu_n\colon G\to\mathcal{U}(e_n\mathcal{H}_\nu)$ the restriction of $\nu$. It follows that

$$[\nu]=\sup_{n\in\mathbb{N}}[\mu_n]$$

in $\mathrm{Cu}(G)$. Since the $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A\rtimes_\alpha G)$ described above is compatible with taking suprema in $\mathrm{Cu}(G)$ by Lemma III.5.11, it follows that the left-most vertical arrow $\mathrm{Cu}^G(A,\alpha)\to\mathrm{Cu}(A\rtimes_\alpha G)$ commutes with multiplication by $[\nu]$, since it commutes with multiplication by $[\mu_n]$ for all $n\in\mathbb{N}$ by the above paragraph. This shows that this map is a $\mathrm{Cu}(G)$-semimodule homomorphism.

Now suppose that $A$ is non-unital, and denote by $\widetilde{A}$ its unitization. Define an extension $\widetilde{\alpha}\colon G\to\mathrm{Aut}(\widetilde{A})$ of $\alpha$ to $\widetilde{A}$ by setting $\widetilde{\alpha}_g(a+\lambda 1)=\alpha_g(a)+\lambda 1$ for all $a\in A$ and all $\lambda\in\mathbb{C}$. The short exact sequence of $G$-$C^*$-algebras

$$0\to A\to\widetilde{A}\to\mathbb{C}\to 0$$

induces the short exact sequence of crossed products

$$0\to A\rtimes_\alpha G\to\widetilde{A}\rtimes_{\widetilde{\alpha}} G\to C^*(G)\to 0.$$

(Recall that $\mathbb{C} \rtimes_{\mathrm{id}} G \cong C^*(G)$ for a locally compact group $G$.) It follows that $\mathrm{Cu}(A \rtimes_\alpha G)$ is an order ideal in $\mathrm{Cu}(\widetilde{A} \rtimes_{\widetilde{\alpha}} G)$, and the quotient is isomorphic to $\mathrm{Cu}(C^*(G))$. The following diagram in **Cu** is commutative:

$$\begin{tikzcd} \mathrm{Cu}(A \rtimes_\alpha G) \arrow[r] \arrow[d, "\overline{\varphi}^A_{\mathbb{C}}"'] & \mathrm{Cu}(\widetilde{A} \rtimes_{\widetilde{\alpha}} G) \arrow[r] \arrow[d, "\overline{\varphi}^{\widetilde{A}}_{\mathbb{C}}"'] & \mathrm{Cu}(C^*(G)) \arrow[d, "\overline{\varphi}^{\mathbb{C}}_{\mathbb{C}}"'] \\ \mathrm{Cu}^G(A, \alpha) \arrow[r] & \mathrm{Cu}^G(\widetilde{A}, \widetilde{\alpha}) \arrow[r] & \mathrm{Cu}^G(\mathbb{C}, \mathrm{id}_{\mathbb{C}}). \end{tikzcd}$$

(Note that $\overline{\varphi}^A_{\mathbb{C}}$ exists even if $A$ does not have a unit.) The vertical maps $\overline{\varphi}^{\widetilde{A}}_{\mathbb{C}}$ and $\overline{\varphi}^{\mathbb{C}}_{\mathbb{C}}$ are isomorphisms by the unital case. By Lemma III.5.12, it follows that $\overline{\varphi}^A_{\mathbb{C}}$ is also invertible. Denote by $\theta\colon \mathrm{Cu}^G(A, \alpha) \to \mathrm{Cu}(A \rtimes_\alpha G)$ the (unique) isomorphism given by Lemma III.5.12 making the following diagram commute

$$\begin{tikzcd} \mathrm{Cu}^G(A, \alpha) \arrow[r] \arrow[d, "\theta"'] & \mathrm{Cu}^G(\widetilde{A}, \widetilde{\alpha}) \arrow[r] \arrow[d] & \mathrm{Cu}^G(\mathbb{C}, \mathrm{id}_{\mathbb{C}}) \arrow[d] \\ \mathrm{Cu}(A \rtimes_\alpha G) \arrow[r] & \mathrm{Cu}(\widetilde{A} \rtimes_{\widetilde{\alpha}} G) \arrow[r] & \mathrm{Cu}(C^*(G)), \end{tikzcd}$$

where the middle and right vertical arrows are the isomorphisms given by Theorem III.5.3. Then $\theta$ commutes with multiplication by $[\mu]$, because the other vertical arrows commute with multiplication by $[\mu]$ by the unital case, and the diagram is commutative. This finishes the proof. $\square$

We illustrate these methods by computing an easy example announced in Subsection III.3.2. Let $G$ be a compact group, and regard $\widehat{G}$ as a set (with no topology). If $A$ is a $C^*$-algebra, then we write $\mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$ for the $\mathrm{Cu}(G)$-semimodule

$$\mathrm{Cu}(G) \otimes \mathrm{Cu}(A) = \{f\colon \widehat{G} \to \mathrm{Cu}(A)\colon f \text{ has countable support}\},$$

again with pointwise addition and partial order. The $\mathrm{Cu}(G)$-action on $\mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$ can be described as follows. Given $[\mu] \in \mathrm{Cu}(G)$ and $[\pi] \in \widehat{G}$, let $m_\pi(\mu) \in \overline{\mathbb{Z}_{\geq 0}}$ be the multiplicity of $\pi$ in $\mu$. Then

$$[\mu] = \sum_{[\pi] \in \widehat{G}} m_\pi(\mu) \cdot [\pi].$$

For $f \in \mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$, we set

$$([\mu] \cdot f)([\pi]) = m_\pi(\mu) f([\pi])$$

for $\pi \in \widehat{G}$.

The tensor product notation is justified because of the following. One can check that

$$\mathrm{Cu}(G) \cong \{f \colon \widehat{G} \to \overline{\mathbb{N}} \colon f \text{ has countable support}\},$$

with pointwise operations and partial order. Moreover, it is easy to check that $\mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$ really is the tensor product in the category **Cu** of the semiring $\mathrm{Cu}(G)$ and the semigroup $\mathrm{Cu}(A)$, in the sense of Theorem 6.3.3 in [4].

**Proposition III.5.14.** Suppose that $G$ acts trivially on $A$. Then $\mathrm{Cu}^G(A, \mathrm{id}_A) \cong \mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$.

*Proof.* Since $G$ acts trivially on $A$, we have $A \rtimes_\alpha G \cong A \otimes C^*(G)$ canonically. For $[\pi] \in \widehat{G}$, denote by $d_\pi$ the dimension of $\pi$. Then $C^*(G) \cong \bigoplus\limits_{[\pi] \in \widehat{G}} M_{d_\pi}$, so

$$A \rtimes_\alpha G \cong \bigoplus_{[\pi] \in \widehat{G}} M_{d_\pi}(A).$$

For $[\tau] \in \widehat{G}$, let

$$\rho_\tau \colon \bigoplus_{[\pi] \in \widehat{G}} M_{d_\pi}(A) \to M_{d_\tau}(A)$$

be the corresponding surjective $*$-homomorphism.

We define a map $\psi \colon \mathcal{K} \otimes (A \rtimes_\alpha G) \to \mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$ as follows. Let $a$ be a positive element in

$$\mathcal{K} \otimes (A \rtimes_\alpha G) \cong \bigoplus_{[\pi] \in \widehat{G}} \mathcal{K} \otimes M_{d_\pi}(A).$$

Suppose first that there exists a finite subset $X$ of $\widehat{G}$ such that $a$ belongs to $\bigoplus\limits_{[\pi] \in X} (\mathcal{K} \otimes M_{d_\pi}(A))$. Define $\psi(a)([\pi]) = [\rho_\pi(a)]$ for $[\pi] \in \widehat{G}$. Then $\psi(a)$ has finite support, as a function $\widehat{G} \to \mathrm{Cu}(A)$, so it belongs to $\mathrm{Cu}(G) \otimes \mathrm{Cu}(A)$.

In the general case, for $0 < \varepsilon < \|a\|$, there exists a finite subset $X_\varepsilon$ of $\widehat{G}$ such that the element $(a-\varepsilon)_+$ belongs to $\bigoplus_{[\pi]\in X_\varepsilon} \mathcal{K}\otimes M_{d_\pi}(A)$. Set $\psi(a) = \sup_{n\in\mathbb{N}} \psi\left(\left(a - \frac{\|a\|}{n}\right)_+\right)$. Then $\psi(a)$ is a supremum of an increasing sequence of functions $\widehat{G} \to \mathrm{Cu}(A)$ with finite support, so $\psi(a)$ has countable support.

*Claim: $\psi$ preserves Cuntz subequivalence.* Let $a$ and $b$ be positive elements in $\bigoplus_{[\pi]\in\widehat{G}} (\mathcal{K}\otimes M_{d_\pi}(A))$ satisfying $a \precsim b$.

Without loss of generality, we may assume that $\|a\| = \|b\| = 1$. Given $n\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that $\left(a-\frac{1}{m}\right)_+ \precsim \left(b-\frac{1}{n}\right)_+$. Hence, $\rho_\pi\left(\left(a-\frac{1}{m}\right)_+\right) \precsim \rho_\pi\left(\left(b-\frac{1}{n}\right)_+\right)$ for all $[\pi]\in\widehat{G}$. It follows that

$$\psi\left(\left(a-\frac{1}{m}\right)_+\right) \leq \psi\left(\left(b-\frac{1}{n}\right)_+\right),$$

and thus $\psi(a)\leq\psi(b)$, and the claim is proved.

It follows that there is an order preserving semigroup morphism

$$\varphi\colon \mathrm{Cu}(A\rtimes_\alpha G) \to \mathrm{Cu}(G)\otimes \mathrm{Cu}(A)$$

given by $\varphi([a]) = \psi(a)$ for all positive elements $a\in\mathcal{K}\otimes(A\rtimes_\alpha G)$. It is clear that the restriction of $\varphi$ to the image in $\mathrm{Cu}(A\rtimes_\alpha G)$ of the positive elements in $\bigoplus_{[\pi]\in\widehat{G}} (\mathcal{K}\otimes M_{d_\pi}(A))$ with finitely many nonzero coordinates preserves the compact containment relation and is an order embedding.

*Claim: $\varphi$ is an order embedding.* Let $a$ and $b$ be positive elements in

$$\bigoplus_{[\pi]\in\widehat{G}} (\mathcal{K}\otimes M_{d_\pi}(A)),$$

and assume that $\varphi([a]) \leq \varphi([b])$ in $\mathrm{Cu}(G)\otimes\mathrm{Cu}(A)$. If there exists a finite subset $X\subseteq\widehat{G}$ such that $a$ and $b$ belong to $\bigoplus_{[\pi]\in X} (\mathcal{K}\otimes M_{d_\pi}(A))$, then it is clear that we must have $[a]\leq[b]$. For the general case, we can assume without loss of generality that $\|a\| = \|b\| = 1$. The sequence $\left(\varphi\left(\left[\left(a-\frac{1}{n}\right)_+\right]\right)\right)_{n\in\mathbb{N}}$ is rapidly increasing by the comments before this claim. In particular, for fixed $n\in\mathbb{N}$, we have $\varphi\left(\left[\left(a-\frac{1}{n}\right)_+\right]\right) \ll \varphi([a])$. Since $\varphi([b]) = \sup_{n\in\mathbb{N}}\varphi\left(\left[\left(b-\frac{1}{n}\right)_+\right]\right)$ by definition of $\varphi$, there exists $n_0\in\mathbb{N}$ such that

$$\varphi\left(\left[\left(a-\frac{1}{n}\right)_+\right]\right) \ll \varphi\left(\left[\left(b-\frac{1}{m}\right)_+\right]\right)$$

for all $m \geq n_0$. It follows that

$$\left[\left(a - \frac{1}{n}\right)_+\right] \leq \left[\left(b - \frac{1}{m}\right)_+\right],$$

because $\left(a - \frac{1}{n}\right)_+$ and $\left(b - \frac{1}{m}\right)_+$ have only finitely many nonzero coordinates. By taking the supremum over $m$ first, and then over $n$, we deduce that $[a] \leq [b]$, as desired.

*Claim: $\varphi$ is surjective.* Let $f\colon \widehat{G} \to \mathrm{Cu}(A)$ be a function with countable support. Let $(\pi_n)_{n\in\mathbb{N}}$ be an enumeration of the support of $f$. For $n \in \mathbb{N}$, let $a_n \in \mathcal{K} \otimes A$ be a positive element with $\|a\| = \frac{1}{n}$ satisfying $[a_n] = f(\pi_n)$ in $\mathrm{Cu}(A)$. Let

$$a \in \mathcal{K} \otimes (A \rtimes_\alpha G) \cong \bigoplus_{[\pi]\in\widehat{G}} (\mathcal{K} \otimes M_{d_\pi}(A))$$

be the positive element determined by $\rho_{\pi_n}(a) = a_n$ for $n \in \mathbb{N}$, and $\rho_\pi(a) = 0$ for $\pi \notin \mathrm{supp}(f)$. It is then clear that $\varphi([a]) = f$.

It follows that $\varphi$ is a **Cu**-isomorphism, and the proof will be complete once we prove the following claim.

*Claim: $\varphi$ is a $\mathrm{Cu}(G)$-morphism.* This is immediate, because $A \rtimes_\alpha G \cong A \otimes C^*(G)$, and the module structure on the crossed product (Definition III.5.9) is easily seen to be trivial on $A$, and the usual multiplication on $C^*(G)$. We leave the details to the reader. $\square$

Similarly to what happens in equivariant $K$-theory, the $\mathrm{Cu}(G)$-semimodule structure on $\mathrm{Cu}(A \rtimes_\alpha G)$ has a more concrete expression when $G$ is abelian.

We saw that $\mathrm{Cu}(G)$ consists of the suprema of all finite linear combinations of elements of $\widehat{G}$ with coefficients in $\mathbb{Z}_{\geq 0}$, with coordinate-wise addition and multiplication. In particular, it follows that a $\mathrm{Cu}(G)$-semimodule structure on a partially ordered abelian semigroup that is compatible with suprema is necessarily completely determined by multiplication by the elements of $\widehat{G}$.

We denote by $\widehat{\alpha}\colon \widehat{G} \to \mathrm{Aut}(A \rtimes_\alpha G)$ the dual action of $\alpha$. In the following proposition, we use the identification

$$\mathcal{H}_A = \left(\bigoplus_{n\in\mathbb{N}} \bigoplus_{\pi\in\widehat{G}} \mathcal{H}_\pi\right) \otimes A.$$

**Proposition III.5.15.** Let $G$ be a compact abelian group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then for $\tau \in \widehat{G}$ and $s \in \mathrm{Cu}(A \rtimes_\alpha G)$, we have $\tau \cdot s = \mathrm{Cu}(\widehat{\alpha}_\tau)(s)$. More precisely, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cu}^G(A,\alpha) \arrow[r, "\tau\cdot"] \arrow[d, "\sigma"'] & \mathrm{Cu}^G(A,\alpha) \arrow[d, "\sigma"] \\
\mathrm{Cu}(A \rtimes_\alpha G) \arrow[r, "\mathrm{Cu}(\widehat{\alpha}_\tau)"'] & \mathrm{Cu}(A \rtimes_\alpha G),
\end{tikzcd}$$

where $\sigma\colon \mathrm{Cu}^G(A,\alpha) \to \mathrm{Cu}(A \rtimes_\alpha)$ is the natural **Cu**-isomorphism given by Theorem III.5.3.

*Proof.* Fix $\tau \in \widehat{G}$. By the construction of the Cu-isomorphism $\mathrm{Cu}^G(A,\alpha) \cong \mathrm{Cu}(A \rtimes_\alpha G)$ in Theorem III.5.3, and adopting the notation in its proof, it is enough to show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Cu}^G(\mathcal{K}(\mathcal{H}_A)^G) \arrow[r, "\tau\cdot"] \arrow[d, "\mathrm{Cu}(\theta)"'] & \mathrm{Cu}^G(\mathcal{K}(\mathcal{H}_A)^G) \arrow[d, "\mathrm{Cu}(\theta)"] \\
\mathrm{Cu}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes (\mathcal{K}(L^2(G)) \otimes A)^G) \arrow[d, "\kappa"'] & \mathrm{Cu}(\mathcal{K}(\ell^2(\mathbb{N})) \otimes (\mathcal{K}(L^2(G)) \otimes A)^G) \arrow[d, "\kappa"] \\
\mathrm{Cu}((\mathcal{K}(L^2(G)) \otimes A)^G) \arrow[d, "\mathrm{Cu}(\psi)"'] & \mathrm{Cu}((\mathcal{K}(L^2(G)) \otimes A)^G) \arrow[d, "\mathrm{Cu}(\psi)"] \\
\mathrm{Cu}(A \rtimes_\alpha G) \arrow[r, "\mathrm{Cu}(\widehat{\alpha}_\tau)"'] & \mathrm{Cu}(A \rtimes_\alpha G).
\end{tikzcd}$$

Fix a $G$-invariant positive element $a \in \mathcal{K}(\mathcal{H}_A)^G$, and write it as an infinite matrix $a = \left(a^{n,m}_{\pi,\gamma}\right)_{n,m\in\mathbb{N},\pi,\gamma\in\widehat{G}}$. It is easy to see that

$$\tau \cdot [a] = \left[\left(a^{n,m}_{\pi\tau,\gamma\tau}\right)_{n,m\in\mathbb{N},\pi,\gamma\in\widehat{G}}\right].$$

With these identifications, $\mathrm{Cu}(\theta)(\tau \cdot [a])$ is represented with the same matrix coefficients.

Recall that $\kappa$ is induced by the inclusion

$$\iota\colon (\mathcal{K}(L^2(G)) \otimes A)^G \to \mathcal{K}(\ell^2(\mathbb{N})) \otimes (\mathcal{K}(L^2(G)) \otimes A)^G$$

as the upper left corner. It follows that

$$(\kappa \circ \mathrm{Cu}(\theta))(\tau \cdot [a]) = \left[ \left( a^{(1,1)}_{\pi\tau,\gamma\tau} \right)_{\pi,\gamma \in \widehat{G}} \right].$$

Finally, since the isomorphism $\psi \colon (\mathcal{K}(L^2(G)) \otimes A)^G \to A \rtimes_\alpha G$ provided by Proposition III.5.2 for $(B,\delta) = (A \rtimes_\alpha G, \widehat{\alpha})$ is equivariant, we conclude that

$$(\mathrm{Cu}(\psi) \circ \kappa \circ \mathrm{Cu}(\theta))(\tau \cdot [a]) = [\widehat{\alpha}_\tau((\mathrm{Cu}(\psi) \circ \kappa \circ \mathrm{Cu}(\theta))(a))].$$

This concludes the proof. □

We close this section with an application to invariant hereditary subalgebras. The result is a Cuntz analog of Proposition 2.9.1 in [199].

**Proposition III.5.16.** Suppose that $A$ is separable and $G$ is second countable. Let $B \subseteq A$ be an $\alpha$-invariant hereditary subalgebra of $A$, and denote by $\beta \colon G \to \mathrm{Aut}(B)$ the compression of $\alpha$. If $B$ is full, then the canonical inclusion induces a natural $\mathbf{Cu}^G$-isomorphism $\mathrm{Cu}^G(B,\beta) \to \mathrm{Cu}^G(A,\alpha)$.

*Proof.* Under the canonical identification given by Theorem III.5.3, the map in the statement becomes the map $\mathrm{Cu}(B \rtimes_\beta G) \to \mathrm{Cu}(A \rtimes_\alpha G)$ induced by the inclusion. Now, Proposition 2.9.1 in [199] shows that $B \rtimes_\beta G$ is a full hereditary subalgebra of $A \rtimes_\alpha G$. Separability of the objects implies, by Brown's stability theorem, that they are stably isomorphic. It follows that the canonical map $\mathrm{Cu}(B \rtimes_\beta G) \to \mathrm{Cu}(A \rtimes_\alpha G)$, which belongs to $\mathbf{Cu}^G$ by Theorem III.3.10, is an isomorphism. □

### A Characterization of Freeness Using the Equivariant Cuntz Semigroup

In this section, we give an application of the equivariant Cuntz semigroup in the context of free actions of locally compact spaces, which resembles Atiyah-Segal's characterization of freeness using equivariant $K$-theory; see [5]. Indeed, in Theorem III.6.6, we characterize freeness of a compact Lie group action on a commutative $C^*$-algebra in terms of a certain canonical map to the equivariant Cuntz semigroup. We define this map, for arbitrary $C^*$-algebras, below.

**Definition III.6.1.** Let $\alpha \colon G \to \mathrm{Aut}(A)$ be a continuous action of a compact group $G$ on a $C^*$-algebra $A$. We define a natural $\mathbf{Cu}$-map $\phi \colon \mathrm{Cu}(A^G) \to \mathrm{Cu}^G(A,\alpha)$ as follows. Given a positive element $a \in \mathcal{K}(\ell^2(\mathbb{N})) \otimes A^G$, regard it as an element in $(\mathcal{K}(\ell^2(\mathbb{N})) \otimes A)^G$ by giving $\ell^2(\mathbb{N})$ the trivial $G$-representation, and set $\phi([a]) = [a]_G$.

**Remark III.6.2.** Here is an alternative description of $\phi$. Let $\iota\colon A^G \to A$ be the canonical inclusion. Since $\iota$ is equivariant, it induces a $\mathbf{Cu}^G$-morphism

$$\mathrm{Cu}(\iota)\colon \mathrm{Cu}^G(A^G) \to \mathrm{Cu}^G(A, \alpha)$$

between the equivariant Cuntz semigroups. Now, by Proposition III.5.14, there is a natural $\mathbf{Cu}^G$-isomorphism $\mathrm{Cu}^G(A^G) \cong \mathrm{Cu}(G) \otimes \mathrm{Cu}(A^G)$. Then $\phi$ is the restriction of $\mathrm{Cu}(\iota)$ to the second tensor factor.

We need a proposition first, which is interesting in its own right. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action of a compact group $G$ on a $C^*$-algebra $A$, and let $a \in A^G$. We denote by $c_a\colon G \to A$ the continuous function with constant value equal to $a$. Note that $c_a$ belongs to $L^1(G, A, \alpha)$, and the assignment $a \mapsto c_a$ defines a $*$-homomorphism $c\colon A^G \to L^1(G, A, \alpha)$. (Recall that the product in $L^1(G, A, \alpha)$ is given by twisted convolution.)

**Proposition III.6.3.** Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action of a compact group $G$ on a $C^*$-algebra $A$. Denote by $\sigma\colon \mathrm{Cu}^G(A, \alpha) \to \mathrm{Cu}(A \rtimes_\alpha G)$ the canonical $\mathbf{Cu}$-isomorphism constructed in Theorem III.5.3. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Cu}(A^G) \arrow[r, "\phi"] \arrow[d, "\mathrm{Cu}(c)"'] & \mathrm{Cu}^G(A, \alpha) \arrow[dl, "\sigma"] \\
\mathrm{Cu}(A \rtimes_\alpha G). &
\end{tikzcd}$$

*Proof.* Abbreviate $\mathcal{K}(\ell^2(\mathbb{N}))$ (with the trivial $G$-action) to $\mathcal{K}$. By the construction of the map $\sigma$, we need to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Cu}(A^G) \arrow[r, "\phi"] \arrow[d, "\mathrm{Cu}(c)"'] & \mathrm{Cu}^G(A, \alpha) \arrow[r, "\chi"] & \mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G) \arrow[d, "\mathrm{Cu}(\theta)"] \\
\mathrm{Cu}(A \rtimes_\alpha G) & \mathrm{Cu}((\mathcal{K}(L^2(G)) \otimes A)^G) \arrow[l, "\mathrm{Cu}(\psi)"] & \mathrm{Cu}((\mathcal{K} \otimes \mathcal{K}(L^2(G)) \otimes A)^G). \arrow[l, "\kappa"]
\end{tikzcd}$$

For an irreducible representation $(\mathcal{H}_\pi, \pi)$ of $G$, set $d_\pi = \dim(\mathcal{H}_\pi)$. Write

$$\mathcal{H}_{\mathbb{C}} = \ell^2(\mathbb{N}) \otimes \left( \bigoplus_{[\pi] \in \widehat{G}} \left( \bigoplus_{j=1}^{d_\pi} \mathcal{H}_\pi \right) \right).$$

Let $1_G \colon G \to \mathcal{U}(\mathbb{C})$ denote the trivial representation, and let

$$W \colon \ell^2(\mathbb{N}) \cong \ell^2(\mathbb{N}) \otimes \mathcal{H}_{1_G} \hookrightarrow \mathcal{H}_{\mathbb{C}}$$

be the isometry corresponding to the canonical inclusion ($\mathcal{H}_{1_G}$ is just $\mathbb{C}$). Write $V \colon \ell^2(\mathbb{N}) \otimes A \to \mathcal{H}_A$ for $W \otimes \mathrm{id}_A$.

Let a positive element $a \in \mathcal{K} \otimes A^G$ be given. Then $(\chi \circ \phi)([a])$ corresponds to the class of $[VaV^*]$ in $\mathrm{Cu}(\mathcal{K}(\mathcal{H}_A)^G)$. Denote by $e \colon L^2(G) \to L^2(G)$ the projection onto the constant functions. With the presentation of $\mathcal{H}_{\mathbb{C}}$ used above, it is clear that $\mathrm{Cu}(\theta)$ maps $[VaV^*]$ to the class of

$$a \otimes e \in (\mathcal{K} \otimes A \otimes \mathcal{K}(L^2(G)))^G \cong (\mathcal{K} \otimes \mathcal{K}(L^2(G)) \otimes A)^G.$$

Since $\kappa$ is induced by the embedding $(\mathcal{K}(L^2(G)) \otimes A)^G \to \mathcal{K} \otimes (\mathcal{K}(L^2(G)) \otimes A)^G$ as the upper left corner, and since $\psi$ is equivariant (see Proposition III.5.2), it is now not difficult to check that $\sigma(\phi([a]))$ agrees with $[c_a]$ in $\mathrm{Cu}(A \rtimes_\alpha G)$. $\square$

We recall a version of the Atiyah-Segal completion theorem that is convenient for our purposes. If a compact group $G$ acts on a compact Hausdorff space $X$, then there is a canonical map $K^*(X/G) \to K^*_G(X)$ obtained by regarding a vector bundle on $X/G$ as a $G$-vector bundle on $X$ (using the trivial action).

For a compact group $G$, we denote by $I_G$ the augmentation ideal in $R(G)$. That is, $I_G$ is the kernel of the dimension map $R(G) \to \mathbb{Z}$.

**Theorem III.6.4.** (Atiyah-Segal). Let $X$ be a compact Hausdorff space and let a compact Lie group $G$ act on $X$. The the following statements are equivalent:

1. The action of $G$ on $X$ is free.

2. The natural map $K^*(X/G) \to K^*_G(X)$ is an isomorphism.

3. The natural map $K^0(X/G) \to K^0_G(X)$ is an isomorphism.

*Proof.* That (1) implies (2) is proved in Proposition 2.1 in [249]. That (2) implies (3) is obvious. Let us show that (3) implies (1), so assume that the natural map $K^0(X/G) \to K^0_G(X)$ is an isomorphism.

An inspection of the proof of the implication (4) $\Rightarrow$ (1) in Proposition 4.3 of [5] shows that, in our context, there exists $n \in \mathbb{N}$ such that $I_G^n \cdot K_0^G(X) = 0$. Now, the $R(G)$-module $K_G^*(X) = K_G^0(X) \oplus K_G^1(X)$ is in fact an $R(G)$-algebra, where multiplication is given by tensor product (with diagonal $G$-actions). In this algebra, the class of the trivial $G$-bundle over $X$ is the unit, so it belongs to $K_G^0(X)$. In particular, $I_G^n$ annihilates the unit of $K_G^*(X)$, and hence it annihilates all of $K_G^*(X)$, that is, $I_G^n \cdot K_*^G(X) = 0$. In other words, $K_*^G(X)$ is discrete in the $I_G$-adic topology. The implication (1) $\Rightarrow$ (4) in Proposition 4.3 of [5] now shows that the $G$-action is free. $\square$

We mention here that the implication (1) $\Rightarrow$ (2) holds even if $G$ is not a Lie group, and even if $X$ is merely locally compact (this is essentially due to Rieffel; see the proof of Theorem III.6.6 below for a similar argument). However, the equivalence between (2) and (3) may fail if $X$ is not compact: the trivial action on $\mathbb{R}$ is a counterexample. This can happen even for free actions.

Recall that a unital $C^*$-algebra $A$ is said to be *finite* if $u \in A$ and $u^*u = 1$ imply $uu^* = 1$. A nonunital $C^*$-algebra is finite if its unitization is. Finally, a $C^*$-algebra $A$ is *stably finite* if $M_n(A)$ is finite for all $n \in \mathbb{N}$. Commutative $C^*$-algebras and AF-algebras are stably finite, and it is obvious that a subalgebra of a stably finite $C^*$-algebra is stably finite. It is an open problem whether the tensor product of two stably finite $C^*$-algebras is stably finite.

**Remark III.6.5.** By Theorem 3.5 in [22], if $A$ is a stably finite $C^*$-algebra, then the set of compact elements in $\mathrm{Cu}(A)$ can be naturally identified with the Murray-von Neumann semigroup $V(A)$ of $A$.

In the next theorem, for an abelian semigroup $V$, we denote by $\mathcal{G}(V)$ its Grothendieck group.

**Theorem III.6.6.** Let $X$ be a locally compact, metric space and let a compact group $G$ act on $X$. Consider the following statements:

1. The action of $G$ on $X$ is free.

2. The canonical map $\phi\colon \mathrm{Cu}(C_0(X/G)) \to \mathrm{Cu}^G(C_0(X))$ is a **Cu**-isomorphism.

Then (1) implies (2). If $G$ is a Lie group and $X$ is compact, then the converse is also true.

*Proof.* Assume that the action of $G$ on $X$ is free, and denote by $\alpha\colon G \to \mathrm{Aut}(C_0(X))$ the induced action. By Proposition III.6.3, under the identification $\mathrm{Cu}^G(C_0(X),\alpha) \cong \mathrm{Cu}(C_0(X) \rtimes_\alpha G)$ provided by Theorem III.5.3, the canonical map $\phi$ becomes the map at the level of the (ordinary) Cuntz semigroup induced by $c\colon C_0(X)^G = C_0(X/G) \to C_0(X) \rtimes_\alpha G$. Denote by $e \in \mathcal{K}(L^2(G))$ the projection onto the constant functions. By the Theorem in [238], we have $c(C_0(X)^G) = e(C_0(X) \rtimes_\alpha G)e$. Since Cu is a stable functor, it is enough to show that $e$ is a full projection in $C_0(X) \rtimes_\alpha G$.

For $a, b \in C_0(X)$, denote by $f_{a,b}\colon G \to C_0(X)$ the function given by

$$f_{a,b}(g)(x) = a(x)b(g^{-1} \cdot x)$$

for $g \in G$ and $x \in X$. It is an easy consequence of the Stone-Weierstrass theorem that the set

$$\{f_{a,b}\colon a, b \in C_0(X)\}$$

has dense linear span in $C_0(X) \rtimes G$. In the language of [199], this amounts to the well-known fact that free actions of compact groups are saturated (Definition 7.1.4 in [199]).

Denote by $I$ the ideal in $C_0(X) \rtimes G$ generated by $e$. Let $(a_\lambda)_{\lambda\in\Lambda}$ be an approximate identity for $C_0(X)$. Upon averaging over $G$, we may assume that $a_\lambda$ belongs to $C_0(X)^G$ for all $\lambda \in \Lambda$. Let $a, b \in C_0(X)$. Then $f_{a,a_\lambda b} = c_{a_\lambda} f_{a,b}$ for $\lambda \in \Lambda$, and hence

$$f_{a,b} = \lim_{\lambda\in\Lambda} f_{a,a_\lambda b} = \lim_{\lambda\in\Lambda} \left(c_{a_\lambda} f_{a,b}\right),$$

so $f_{a,b}$ belongs to $I$. We conclude that $C_0(X) \rtimes G = I$, as desired.

Assume now that $G$ is a compact Lie group, that $X$ is compact, and that the canonical map $\mathrm{Cu}(C(X/G)) \to \mathrm{Cu}^G(C(X))$ is an isomorphism in **Cu**. We claim that the canonical map $K^0(X/G) \to K^0_G(X)$ is an isomorphism.

Under the natural identification given by Theorem III.5.3, the canonical inclusion $c\colon C(X/G) \to C(X) \rtimes G$ induces an isomorphism $\mathrm{Cu}(c)$ at the level of the Cuntz semigroup (see also Proposition III.6.3). The algebra $C(X/G)$ is clearly stably finite. On the other hand, $C(X) \rtimes_\alpha G$ is also stably finite because it is a subalgebra of the stable finite $C^*$-algebra $C(X) \otimes \mathcal{K}(L^2(G))$. By Remark III.6.5, the restriction of the isomorphism $\mathrm{Cu}(c)$ to the compact elements of $C(X/G)$ yields an isomorphism

$$\psi\colon V(C(X/G)) \to V(C(X) \rtimes G)$$

between the respective Murray-von Neumann semigroups of projections. By taking the Grothendieck construction, one gets an isomorphism

$$\varphi\colon \mathcal{G}(V(C(X/G))) \to \mathcal{G}(V(C(X) \rtimes G))$$

between the respective Grothendieck groups. We want to conclude from this that $\varphi$ induces an isomorphism between the $K_0$-groups of these $C^*$-algebras.

Since $C(X/G)$ is unital, we have $\mathcal{G}(V(C(X/G))) = K_0(C(X/G))$. However, $C(X) \rtimes G$ is not unital unless $G$ is finite, and it is even not clear whether it (or its stabilization) has an approximate identity consisting of projections. (This would also imply that its $K_0$-group is obtained as the Grothendieck group of its Murray-von Neummann semigroup.)

Instead, we appeal to Julg's theorem for equivariant $K$-theory. Indeed, the proof given in Theorem 2.6.1 in [199] shows that if $A$ is a unital $C^*$-algebra, $G$ is a compact group, and $\alpha\colon G \to \mathrm{Aut}(A)$ is a continuous action, then there is a canonical isomorphism of semigroups

$$V^G(A) \cong V(A \rtimes G).$$

Since $K_0^G(A)$ is the Grothendieck group of $V^G(A)$, it follows that the same is true for $K_0(A \rtimes_\alpha G)$. In our context, this shows that $\varphi$ induces an isomorphism

$$\theta\colon K^0(X/G) \to K_0^G(C(X)) \cong K_G^0(X).$$

Now, the implication (3) $\Rightarrow$ (1) in Theorem III.6.4 shows that the action of $G$ on $X$ is free. $\square$

# CHAPTER IV

# ROKHLIN DIMENSION FOR COMPACT GROUP ACTIONS

We study the notion of Rokhlin dimension (with and without commuting towers) for compact group actions on $C^*$-algebras. This notion generalizes the one introduced by Hirshberg, Winter and Zacharias for finite groups, and contains the Rokhlin property as the zero dimensional case. We show, by means of an example, that commuting towers cannot always be arranged, even in the absence of $K$-theoretic obstructions. For a compact Lie group action on a compact Hausdorff space, freeness is equivalent to finite Rokhlin dimension of the induced action. We compare the notion of finite Rokhlin dimension to other existing definitions of noncommutative freeness for compact group actions. We obtain further $K$-theoretic obstructions to having an action of a non-finite compact Lie group with finite Rokhlin dimension with commuting towers, and use them to confirm a conjecture of Phillips. Furthermore, we obtain a Rokhlin-dimensional inequality that allows us to show that every pointwise outer action of a finite group on a Kirchberg algebra has Rokhlin dimension at most one.

## Introduction

Hirshberg, Winter and Zacharias introduced in [123] the notion of finite Rokhlin dimension for finite group actions (as well as automorphisms), as a generalization of the Rokhlin property. This more general notion has the Rokhlin property as its zero dimensional case, and moreover has the advantage of not requiring the existence of projections in the underlying algebra. Finite Rokhlin dimension is in particular much more common than the Rokhlin property.

The paper [120] consists of a further study of finite Rokhlin dimension, where the authors extend the notion to the non-unital setting, and also derive some $K$-theoretical obstructions in the commuting tower version. These obstructions are used to show that no non-trivial finite group acts with finite Rokhlin dimension with commuting towers on either the Jiang-Su algebra $\mathcal{Z}$, or the Cuntz algebra $\mathcal{O}_\infty$. There, Hirshberg and Phillips introduce the notion of $X$-Rokhlin property for an action of a compact Lie group $G$ and a compact free $G$-space $X$. They show that when $G$ is finite and $X$ is finite dimensional, the $X$-Rokhlin property is equivalent to having finite Rokhlin dimension with commuting towers in the sense of Hirshberg-Winter-Zacharias.

This chapter, based on [83] develops the concept of Rokhlin dimension for compact group actions on unital $C^*$-algebras. This notion generalizing the case of finite group actions of [123], the Rokhlin property in the compact group case as in [122], and, most of the times, including the $X$-Rokhlin property from [120], which is shown to be equivalent to finite Rokhlin dimension with commuting towers in our sense, at least for Lie groups. The starting point of this project was the simple observation that if $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action of the circle with the Rokhlin property on a unital $C^*$-algebra $A$, and if $n$ is a positive integer, then the restriction of $\alpha$ to the finite subgroup $\mathbb{Z}_n \subseteq \mathbb{T}$ has Rokhlin dimension at most one. Theorem IV.2.9 can be regarded as a significant generalization of this fact.

This chapter is organized as follows. In Section IV.2, we introduce and systematically study the notion of Rokhlin dimension for compact group actions on unital $C^*$-algebras. In particular, we show that finite Rokhlin dimension is preserved under a number of constructions, namely tensor products, direct limits, passage to quotients by invariant ideals, and restriction to closed subgroups of finite codimension. In Section IV.3, we compare the notion of having finite Rokhlin dimension (mostly in the commuting tower case) with other existing forms of freeness of group actions on $C^*$-algebras. We show that in the commutative case, finite Rokhlin dimension is equivalent to freeness of the action on the maximal ideal space; see Theorem IV.3.2. Moreover, for a compact Lie group action, the formulation with commuting towers is equivalent to the $X$-Rokhlin property introduced in [120]; see Theorem IV.3.5. We apply this to deduce that actions with finite Rokhlin dimension with commuting towers have discrete $K$-theory and are totally $K$-free. Theorem IV.3.22 establishes $K$-theoretic obstructions that are complementary to the ones established in [120]. We use this result to confirm a Conjecture of Phillips from [199]; see Remark IV.3.23. Our results in fact show that Phillips' conjecture holds for a class of $C^*$-algebras which is much larger than the class of AF-algebras, without assuming that the action is specified by the way it is constructed.

We show in Example IV.3.10 that commuting towers cannot always be arranged, even at the cost of considering additional towers, and even in the absence of $K$-theoretic obstructions. Indeed, this example (originally constructed by Izumi in [132]) has Rokhlin dimension 1 with noncommuting towers, and infinite Rokhlin dimension with commuting towers. In Theorem IV.3.19, we obtain a Rokhlin-dimensional inequality relating the $\mathcal{O}_\infty$ and

$\mathcal{O}_2$ stabilizations of a given action. As a consequence of this, we deduce in Theorem IV.3.20 that every pointwise outer action of a finite group on a Kirchberg algebra has Rokhlin dimension at most one, generalizing Theorem 2.3 in [7], which is the case $G = \mathbb{Z}_2$.

Theorem IV.3.27 collects and summarizes the known implications between the notions considered in Section IV.3, and it also references counterexamples that show that no other implications hold in full generality. Finally, in Section IV.4, we give some indication of possible directions for future work, and raise some natural questions related to our findings.

Some applications of the results in this chapter will be presented in Chapter V, and some more will appear in [87].

## Rokhlin Dimension For Compact Group Actions

We begin by recalling the definition of finite Rokhlin dimension for finite groups.

**Definition IV.2.1.** (See Definition 1.1 in [123].) Let $G$ be a finite group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of $G$ on $A$. Given a non-negative integer $d$, we say that $\alpha$ has *Rokhlin dimension* $d$, and denote this by $\dim_{\mathrm{Rok}}(\alpha) = d$, if $d$ is the least integer with the following property: for every $\varepsilon > 0$ and for every finite subset $F$ of $A$, there exist positive contractions $f_g^{(\ell)}$ for $g \in G$ and $\ell = 0, \ldots, d$, satisfying the following conditions for every $\ell = 0, \ldots, d$, for every $g, h \in G$, and for every $a \in F$:

1. $\left\| \alpha_h\left(f_g^{(\ell)}\right) - f_{hg}^{(\ell)} \right\| < \varepsilon$;
2. $\left\| f_g^{(\ell)} f_h^{(\ell)} \right\| < \varepsilon$ whenever $g \neq h$;
3. $\left\| \sum\limits_{g\in G} \sum\limits_{\ell=0,\ldots,d} f_g^{(\ell)} - 1 \right\| < \varepsilon$;
4. $\left\| \left[ f_g^{(\ell)}, a \right] \right\| < \varepsilon$.

If one can always choose the positive contractions $f_g^{(\ell)}$ above to moreover satisfy

$$\left\| \left[ f_g^{(\ell)}, f_h^{(k)} \right] \right\| < \varepsilon$$

for every $g, h \in G$ and every $k, \ell = 0, \ldots, d$, then we say that $\alpha$ has *Rohlin dimension $d$ with commuting towers*, and denote this by $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) = d$.

Given a compact group $G$, we denote by $\mathtt{Lt}\colon G \to \mathrm{Homeo}(G)$ the action of left translation. With a slight abuse of notation, we will also denote by $\mathtt{Lt}$ the induced action of $G$ on $C(G)$.

Definition IV.2.1 can be generalized to the case of second countable compact groups as follows.

**Definition IV.2.2.** Let $G$ be a second countable, Hausdorff compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. We say that $\alpha$ has *Rokhlin dimension* $d$, if $d$ is the least integer such that there exist equivariant completely positive contractive order zero maps

$$\varphi_0, \ldots, \varphi_d\colon (C(G), \mathtt{Lt}) \to (A_{\infty,\alpha} \cap A', \alpha_\infty)$$

such that $\varphi_0(1) + \ldots + \varphi_d(1) = 1$.

We denote the Rokhlin dimension of $\alpha$ by $\dim_{\mathrm{Rok}}(\alpha)$. If no integer $d$ as above exists, we say that $\alpha$ has *infinite Rokhlin dimension*, and denote it by $\dim_{\mathrm{Rok}}(\alpha) = \infty$. If one can always choose the maps $\varphi_0, \ldots, \varphi_d$ to have commuting ranges, then we say that $\alpha$ has *Rokhlin dimension $d$ with commuting towers*, and write $\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) = d$.

**Remark IV.2.3.** It is an easy exercise to check that if $G$ is a finite group, then Definition IV.2.2 agrees with Definition 1.1 in [123].

It is clear that if $A$ is commutative, then the notions of Rokhlin dimension with and without commuting towers agree. Nevertheless, Example IV.3.10 below shows that commuting towers cannot always be arranged, even for $\mathbb{Z}_2$-actions on $\mathcal{O}_2$ with Rokhlin dimension 1. In fact, it seems that there really is a big difference between these two notions, although we do not know how much they differ in general.

**Remark IV.2.4.** It follows from Theorem 2.3 in [271] that a unital completely positive contractive order zero map is necessarily a homomorphism. In particular, Rokhlin dimension zero is equivalent to the Rokhlin property as in Definition 2.3 of [122]. (We point out that the requirement that $\varphi$ be injective in Definition 2.3 of [122] is unnecessary: its kernel is a translation-invariant ideal of $C(G)$, so it must be either $\{0\}$ or $C(G)$. Since $\varphi$ is assumed to be unital, it must be $\ker(\varphi) = \{0\}$.)

The following result is probably well-known to the experts. Since we have not been able to find a reference, we present its proof here.

**Lemma IV.2.5.** Let $A$ and $B$ be $C^*$-algebras, and let $\varphi\colon A \to B$ be a completely positive contractive order zero map. Then

$$\ker(\varphi) = \{a \in A \colon \varphi(a) = 0\}$$

is a closed two-sided ideal in $A$.

*Proof.* That $\ker(\varphi)$ is closed follows easily by continuity of $\varphi$. Let us now show that it is a two-sided ideal.

Let $\pi\colon C_0((0,1]) \otimes A \to B$ be the homomorphism determined by $\pi(\mathrm{id}_{(0,1]} \otimes a) = \varphi(a)$ for all $a \in A$ (see Theorem II.5.3 above). Then $\ker(\pi)$ is an ideal of $C_0((0,1]) \otimes A$. Let $a \in \ker(\varphi)$ and let $x \in A$, and assume that $a$ is positive. Then $ax$ belongs to $\ker(\varphi)$ if and only if $\mathrm{id}_{(0,1]} \otimes ax$ belongs to $\ker(\pi)$. Denote by $t^{1/2}$ the map $(0,1] \to (0,1]$ given by $x \mapsto \sqrt{x}$. By functional calculus, $t^{1/2} \otimes a^{1/2}$ belongs to $\ker(\pi)$. It follows that

$$\mathrm{id}_{(0,1]} \otimes ax = \left(t^{1/2} \otimes a^{1/2}\right)\left(t^{1/2} \otimes a^{1/2}x\right) \in \ker(\pi)$$

since $\ker(\pi)$ is an ideal in $C_0((0,1]) \otimes A$, and hence $\varphi(ax) = \pi(\mathrm{id}_{(0,1]} \otimes ax) = 0$. A similar argument shows that $\varphi(xa) = 0$ as well, proving that $\ker(\varphi)$ is a two-sided ideal in $A$. ☐

**Corollary IV.2.6.** Adopt the notation of Definition IV.2.2 above. Then the order zero maps $\varphi_0, \ldots, \varphi_d$ are either zero or injective.

*Proof.* For $j = 0, \ldots, d$, the kernel $I_j$ of $\varphi_j$ is a translation invariant ideal in $C(G)$, since $\varphi_j$ is equivariant. The result now follows from the fact that the only translation invariant ideals of $C(G)$ are $\{0\}$ and $C(G)$. ☐

In particular, if $\dim_{\mathrm{Rok}}(\alpha) = d < \infty$, then the maps $\varphi_0, \ldots, \varphi_d$ from Definition IV.2.2 are injective.

We start by presenting some permanence properties for actions of compact groups with finite Rokhlin dimension. Not surprisingly, finite Rokhlin dimension is far more flexible than the Rokhlin property, and it is preserved by several constructions. Most notably, finite Rokhlin dimension for finite dimensional compact groups (in particular, for Lie groups) is inherited by the restriction to any closed subgroup, except that the actual dimension may increase.

We begin with a technical lemma which characterizes finite Rokhlin dimension in terms of elements in the $C^*$-algebra itself, rather than its central sequence algebra.

**Lemma IV.2.7.** Let $G$ be a compact group, let $A$ be a separable unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of $G$ on $A$. Let $d$ be a non-zero integer.

1. We have $\dim_{\mathrm{Rok}}(\alpha) \leq d$ if and only if for every $\varepsilon > 0$, for every finite subset $S$ of $C(G)$, and for every compact subset $F$ of $A$, there exist completely positive contractive maps $\psi_0, \ldots, \psi_d\colon C(G) \to A$ satisfying the following conditions:

   (a) $\|\psi_j(f)a - a\psi_j(f)\| < \varepsilon$ for all $j = 0, \ldots, d$, for all $f$ in $S$, and all $a$ in $F$.

   (b) $\|\psi_j(\mathtt{Lt}_g(f)) - \alpha_g(\psi_j(f))\| < \varepsilon$ for all $j = 0, \ldots, d$, for all $g$ in $G$, and for all $f$ in $S$.

   (c) $\|\psi_j(f_1)\psi_j(f_2)\| < \varepsilon$ whenever $f_1$ and $f_2$ in $S$ are orthogonal.

   (d) $\left\|\sum\limits_{j=0}^{d} \psi_j(1_{C(G)}) - 1_A\right\| < \varepsilon$.

2. We have $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$ if and only if for every $\varepsilon > 0$, for every finite subset $S$ of $C(G)$, and for every compact subset $F$ of $A$, there exist completely positive contractive maps $\psi_0, \ldots, \psi_d\colon C(G) \to A$ satisfying the conditions listed above in addition to

$$\|\psi_j(f_1)\psi_k(f_2) - \psi_k(f_2)\psi_j(f_1)\| < \varepsilon$$

   for all $j, k = 0, \ldots, d$ and all $f_1$ and $f_2$ in $S$.

*Proof.* We prove (1) first. Assume that for every $\varepsilon > 0$, for every finite subset $S$ of $C(G)$, and every finite subset $F$ of $A$, there exist completely positive contractive maps $\varphi_0, \ldots, \varphi_d\colon C(G) \to A$ satisfying the conditions of the statement. Choose increasing sequences $(F_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ of finite subsets of $A$ and $C(G)$, whose union is dense in $A$ and in $C(G)$, respectively. Let $\psi_0^{(n)}, \ldots, \psi_d^{(n)}\colon C(G) \to A$ be as in the statement for the choices $F_n$ and $\frac{1}{n}$. For $j = 0, \ldots, d$, denote by $\varphi_j\colon C(G) \to A_\infty$ the linear map given by

$$\varphi_j\left(\kappa_A((a_n)_{n\in\mathbb{N}})\right) = \kappa_A\left(\left(\psi_j^{(n)}(a_n)\right)_{n\in\mathbb{N}}\right)$$

for all $(a_n)_{n\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N}, A)$. Then $\varphi_j$ is easily seen to be completely positive contractive and order zero. It is also straightforward to check that its image is contained in $A_{\infty,\alpha} \cap A'$, and that it is equivariant. Finally, it is immediate that $\sum_{j=0}^{d} \varphi_j(1) = 1$.

Conversely, suppose that $\alpha$ has Rokhlin dimension at most $d$. Choose completely positive contractive order zero maps $\varphi_0, \ldots, \varphi_d \colon C(G) \to A_{\infty,\alpha} \cap A'$ as in the definition if finite Rokhlin dimension. Fix $j$ in $\{0, \ldots, d\}$. By Choi-Effros, there exist completely positive contractive maps $\psi_j = (\psi_j^{(n)})_{n\in\mathbb{N}} \colon C(G) \to \ell^\infty(\mathbb{N}, A)$ for $j = 0, \ldots, d$ such that for all $f, f_1, f_2$ in $C(G)$ with $f_1$ orthogonal to $f_2$, for all $a$ in $A$, and for all $g$ in $G$, we have

$$\begin{aligned}
\left\| \psi_j^{(n)}(f)a - a\psi_j^{(n)}(f) \right\| &\to 0 \\
\left\| \psi_j^{(n)}(\mathtt{Lt}_g(f)) - \alpha_g(\psi_j^{(n)}(f)) \right\| &\to 0 \\
\left\| \psi_j^{(n)}(f_1)\psi_j^{(n)}(f_2) \right\| &\to 0 \\
\left\| \sum_{j=0}^{d} \psi_j^{(n)}(1_{C(G)}) - 1_A \right\| &\to 0
\end{aligned}$$

Given $\varepsilon > 0$, given a finite subset $S$ of $C(G)$ and given a finite subset $F$ of $A$, choose a positive integer $n$ such that the quantities above are all less than $\varepsilon$ on the elements of $S$ and $F$, respectively, and set $\psi_j = \psi_j^{(n)}$ for $j = 0, \ldots, d$. This finishes the proof of (1).

The proof of (2) is analogous. In particular, in the "only if" implication, one has to use that the completely positive contractive maps $\psi_j = (\psi_j^{(n)})_{n\in\mathbb{N}} \colon C(G) \to \ell^\infty(\mathbb{N}, A)$ for $j = 0, \ldots, d$ obtained from Choi-Effros, moreover satisfy

$$\lim_{n\to\infty} \left\| \psi_j^{(n)}(f_1)\psi_k^{(n)}(f_2) - \psi_k^{(n)}(f_2)\psi_j^{(n)}(f_1) \right\| = 0$$

for all $f_1$ and $f_2$ in $C(G)$, and all $j, k = 0, \ldots, d$. We omit the details. $\square$

Regarding finite Rokhlin dimension as a noncommutative analog of freeness of group actions on topological spaces, we give the following interpretation of Theorem IV.2.8 below. Part (1) is the analog of the fact that a diagonal action on a product space is free if only of the factors is free; part (2) is the analog of the fact that the restriction of a free action to an invariant closed subset is also free; and part (3) is the analog of the fact that an inverse limit of free actions is again free.

**Theorem IV.2.8.** Let $A$ be a unital $C^*$-algebra let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action of $G$ on $A$.

1. Let $B$ be a unital $C^*$-algebra, and let $\beta\colon G \to \mathrm{Aut}(B)$ be a continuous action of $G$ on $B$. Let $A \otimes B$ be any $C^*$-algebra completion of the algebraic tensor product of $A$ and $B$ for which the tensor product action $g \mapsto (\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$ is defined. Then

$$\dim_{\mathrm{Rok}}(\alpha \otimes \beta) \leq \min\left\{\dim_{\mathrm{Rok}}(\alpha), \dim_{\mathrm{Rok}}(\beta)\right\}$$

   and

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha \otimes \beta) \leq \min\left\{\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha), \dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta)\right\}.$$

2. Let $I$ be an $\alpha$-invariant ideal in $A$, and denote by $\overline{\alpha}\colon G \to \mathrm{Aut}(A/I)$ the induced action on the quotient. Then

$$\dim_{\mathrm{Rok}}(\overline{\alpha}) \leq \dim_{\mathrm{Rok}}(\alpha)$$

   and

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\overline{\alpha}) \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha).$$

Furthermore,

3. Let $(A_n, \iota_n)_{n\in\mathbb{N}}$ be a direct system of unital $C^*$-algebras with unital connecting maps, and for each $n \in \mathbb{N}$, let $\alpha^{(n)}\colon G \to \mathrm{Aut}(A_n)$ be a continuous action such that $\iota_n \circ \alpha_g^{(n)} = \alpha_g^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $g \in G$. Suppose that $A = \varinjlim A_n$ and $\alpha = \varinjlim \alpha^{(n)}$. Then

$$\dim_{\mathrm{Rok}}(\alpha) \leq \liminf_{n\to\infty} \dim_{\mathrm{Rok}}(\alpha^{(n)})$$

   and

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq \liminf_{n\to\infty} \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha^{(n)}).$$

*Proof.* We only prove the results for the noncommuting tower version; the proofs for the commuting tower version are analogous and are left to the reader.

Part (1). The statement is immediate if both $\alpha$ and $\beta$ have infinite Rokhlin dimension, so assume that $\dim_{\mathrm{Rok}}(\alpha) = d < \infty$. Then there are equivariant completely positive contractive order zero maps

$$\varphi_0, \ldots, \varphi_d \colon C(G) \to A_{\infty,\alpha} \cap A',$$

such that $\varphi_0(1) + \ldots + \varphi_d(1) = 1$. Denote by $\iota \colon A \to A \otimes B$ the canonical embedding as the first tensor factor. By Lemma II.4.4, this inclusion induces a unital homomorphism $\iota_\infty \colon A_\infty \cap A' \to (A \otimes B)_{\infty,\alpha\otimes\beta} \cap (A \otimes B)'$, and $\iota_\infty$ is moreover equivariant with respect to $\alpha_\infty$ and $(\alpha \otimes \beta)_\infty$. For $j = 0, \ldots d$, set

$$\psi_j = \iota_\infty \circ \varphi_j \colon C(G) \to (A \otimes B)_{\infty,\alpha\otimes\beta} \cap (A \otimes B)'.$$

Then $\psi_0, \ldots, \psi_d$ are equivariant completely positive contractive order zero maps, and $\psi_0(1) + \ldots + \psi_d(1) = \varphi_0(1) + \ldots + \varphi_d(1) = 1$. Hence $\dim_{\mathrm{Rok}}(\alpha \otimes \beta) \leq d$ and the result follows.

Part (2). The statement is immediate if $\alpha$ has infinite Rokhlin dimension, so suppose that there exist a positive integer $d$ in $\mathbb{N}$ and equivariant completely positive contractive order zero maps

$$\varphi_0, \ldots, \varphi_d \colon C(G) \to A_{\infty,\alpha} \cap A'$$

such that $\varphi_0(1) + \ldots + \varphi_d(1) = 1$. Denote by $\pi \colon A \to A/I$ the quotient map. Lemma II.4.3 implies that $\pi$ induces a unital homomorphism

$$\pi_\infty \colon A_\infty \cap A' \to (A/I)_{\infty,\overline{\alpha}} \cap (A/I)'.$$

Moreover, this homomorphism is easily seen to be equivariant. For $j = 0, \ldots, d$, set $\psi_j = \pi_\infty \circ \varphi_j \colon C(G) \to (A/I)_{\infty,\overline{\alpha}} \cap (A/I)'$. Then $\psi_j$ is an equivariant completely positive contractive order zero map for all $j = 0, \ldots, d$, and $\psi_0(1) + \ldots + \psi_d(1) = \varphi_0(1) + \ldots + \varphi_d(1) = 1$. It follows that $\dim_{\mathrm{Rok}}(\overline{\alpha}) \leq d$, as desired.

Part (3). The statement is immediate if $\liminf\limits_{n\to\infty} \dim_{\mathrm{Rok}}(\alpha^{(n)}) = \infty$. We shall therefore assume that there exists $d \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, there is $n \geq m$ in $\mathbb{N}$ with $\dim_{\mathrm{Rok}}(\alpha^{(n)}) \leq d$. By passing to a subsequence, we may also assume that $\dim_{\mathrm{Rok}}(\alpha^{(n)}) \leq d$ for all $n \in \mathbb{N}$.

We use Lemma IV.2.7. Let $\varepsilon > 0$, let $S$ be a finite subset of $C(G)$, and let $F$ be a finite subset of $A$. With $L = \mathrm{card}(F)$, write $F = \{a_1, \ldots, a_L\}$, and find a positive integer $n$ in $\mathbb{N}$ and elements $b_1, \ldots, b_L$ in $A_n$ such that $\|a_j - b_j\| < \frac{\varepsilon}{2}$ for all $j = 1, \ldots, L$. Choose completely positive contractive maps $\psi_0, \ldots, \psi_d \colon C(G) \to A_n$ satisfying conditions (a) through (d) in part (1) of Lemma IV.2.7 for $\frac{\varepsilon}{2}$ and the finite set $F' = \{b_1, \ldots, b_L\} \subseteq A_n$. If $\iota_{n,\infty} \colon A_n \to A$ denotes the canonical map, then it is easy to check that the completely positive contractive maps

$$\iota_{n,\infty} \circ \psi_0, \ldots, \iota_{n,\infty} \circ \psi_d \colon C(G) \to A$$

satisfy conditions (a) through (d) in part (1) of Lemma IV.2.7 for $\varepsilon$ and the finite set $F$. This shows the result in the case of non commuting towers.

The proof in the commuting towers case is analogous, using also the extra condition in part (2) of Lemma IV.2.7. We omit the details. $\square$

In relation to part (1) of Theorem IV.2.8, we briefly describe what can go wrong when defining the tensor product of two actions (or even just the tensor product of two automorphisms). We are thankful to Chris Phillips for pointing this issue to us, and for providing the following example.

Let $A$ and $B$ be $C^*$-algebras, let $\varphi$ and $\psi$ be automorphisms of $A$ and $B$ respectively, and let $\|\cdot\|_q$ be a $C^*$-norm on the algebraic tensor product $A \otimes_{alg} B$. There is in general no reason why the tensor product automorphism $\varphi \otimes \psi$ of $A \otimes_{alg} B$ should extend to the completion $\overline{A \otimes_{alg} B}^{\|\cdot\|_q} = A \otimes_q B$. For example, choose $A_0$ and $B$ such that $A_0 \otimes_{max} B$ and $A_0 \otimes_{min} B$ are not isomorphic. Set $A = A_0 \oplus A_0$ and define the flip automorphism $\varphi$ on $A$, sendining $(a, b)$ to $(b, a)$ for $(a, b)$ in $A$. Let $\|\cdot\|_q$ be the $C^*$-norm on $A \otimes_{alg} B$ such that

$$A \otimes_q B \cong (A_0 \otimes_{max} B) \oplus (A_0 \otimes_{min} B).$$

It is straightforward to check that the automorphism $\varphi \otimes \mathrm{id}_B$ of the algebraic tensor product, does not extend to its completion with respect to $\|\cdot\|_q$. Note that the automorphism $\varphi$, when regarded as an action of $\mathbb{Z}_2$ on $A$, has the Rokhlin property.

On the other hand, automorphisms, and more generally, actions of locally compact groups, always extend to the maximal and minimal tensor products. In particular, if one of the factors is nuclear, then no issues like the one exhibited above can possibly arise.

We now turn to restrictions of actions in relation to Rokhlin dimension. The following result is the analog of the fact that the restriction of a free action to a (closed) subgroup is again free.

**Theorem IV.2.9.** Let $A$ be a unital $C^*$-algebra, let $G$ be a finite dimensional compact group, let $H$ be a closed subgroup of $G$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then

$$\dim_{\mathrm{Rok}}(\alpha|_H) \leq (\dim(G) - \dim(H) + 1)(\dim_{\mathrm{Rok}}(\alpha) + 1) - 1$$

and

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha|_H) \leq (\dim(G) - \dim(H) + 1)(\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) + 1) - 1$$

*Proof.* Without loss of generality, we may assume that $\dim_{\mathrm{Rok}}(\alpha)$ is finite.

Being a closed subspace of a finite dimensional subspace, $H$ is finite dimensional. Let $d = \dim(G/H) = \dim(G) - \dim(H)$. We will produce $d+1$ completely positive contractive $H$-equivariant order zero maps $\varphi_0, \ldots, \varphi_d \colon C(H) \to C(G)$ with $\varphi_0(1) + \cdots + \varphi_d(1) = 1$. Once we have done this, and since these maps will obviously have commuting ranges, both claims will follow by composing each of the maps $\varphi_0, \ldots, \varphi_d$ with the $\dim_{\mathrm{Rok}}(\alpha) + 1$ maps as in the definition of finite Rokhlin dimension for $\alpha$. The result will then be $(d+1)(\dim_{\mathrm{Rok}}(\alpha) + 1)$ maps which will satisfy the definition of finite Rokhlin dimension for $\alpha|_H$.

Denote by $\pi\colon G \to G/H$ the canonical surjection. By part (1) of Theorem 2 in [143], the map $\pi\colon G \to G/H$ is a principal $H$-bundle. In particular, there exist local cross-sections from the orbit space $G/H$ to $G$. For every $x \in G$, let $V_{\pi(x)}$ be a neighborhood of $\pi(x)$ in $G/H$ where $\pi$ is trivial. Using compactness of $G/H$, let $\mathcal{U}$ be a finite subcover of $G/H$. Use Proposition 1.5 in [153] to refine $\mathcal{U}$ to a $d$-decomposable covering $\mathcal{V}$. In other words, $\mathcal{V}$ can be written as the disjoint union of $d+1$ families $\mathcal{V}_0 \cup \cdots \cup \mathcal{V}_d$ of open sets, in such a way that for every $k = 0, \ldots, d$, the elements of $\mathcal{V}_k$ are pairwise disjoint.

For $k = 0, \ldots, d$, let $V_k$ denote the union of all the open sets in $\mathcal{V}_k$, and note that there is a cross-section defined on $V_k$. Let $\{f_0, \ldots, f_d\}$ be a partition of unity of $G/H$ subordinate to the cover $\{V_0, \ldots, V_d\}$. Upon replacing $V_k$ with the open set $f_k^{-1}((0,1]) \subseteq V_k$, we may assume that $V_k = f_k^{-1}((0,1])$ for all $k = 0, \ldots, d$. For each $k = 0, \ldots, d$, set $U_k = \pi^{-1}(V_k) \subseteq G$, and observe that there is an equivariant homeomorphism $U_k \cong V_k \times H$, where the $H$-action on $V_k \times H$ is diagonal with the trivial action on $V_k$ and translation on $H$. Define a continuous function $\phi_k \colon V_k \times H \cong U_k \to (0,1] \times H$ by

$$\phi_k(x,h) = (f_k(x), h)$$

for all $(x,h)$ in $V_k \times H \cong U_k$. Then $\phi_k$ is continuous because the cross-section is continuous. Moreover, $\phi_k$ is clearly equivariant.

Identify $C_0((0,1]) \otimes C(H)$ with $C_0((0,1] \times H)$, and for $k = 0, \ldots, d$ define

$$\psi_k \colon C_0((0,1]) \otimes C(H) \to C(G)$$

by

$$\psi_k(f)(x) = \begin{cases} (f \circ \phi_k)(x), & \text{if } x \in U_k; \\ 0, & \text{else.} \end{cases}$$

Then $\psi_k$ is a homomorphism, and it is equivariant since $\phi_k$ is. The map $\varphi_k \colon C(H) \to C(G)$ given by $\varphi_k(f) = \psi_k(\mathrm{id}_{(0,1]} \otimes f)$ for $f \in C(H)$ is an equivariant completely positive contractive order zero map. Finally, using that $(f_k)_{k=0}^d$ is a partition of unity of $G/H$ at the last step, we have

$$\sum_{k=0}^{d} \varphi_k(1) = \sum_{k=0}^{d} \psi_k(\mathrm{id}_{(0,1]} \otimes 1) = \sum_{k=0}^{d} f_k = 1.$$

It follows that the maps $\varphi_0, \ldots, \varphi_d$ have the desired properties, and the proof is finished. $\square$

In some cases, restricting to a subgroup does not increase the Rokhlin dimension. In the following proposition, the group is not assumed to be finite dimensional.

**Proposition IV.2.10.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a continuous action. Let $H$ be a closed subgroup of $G$, and assume that at least one of the following holds:

1. the coset space $G/H$ is zero dimensional (this is the case whenever $H$ has finite index in $G$).

2. $G = \prod\limits_{i \in I} G_i$ or $G = \bigoplus\limits_{i \in I} G_i$, and $H = G_j$ for some $j \in I$.

3. $H$ is the connected component of $G$ containing its unit.

Then

$$\dim_{\mathrm{Rok}}(\alpha|_H) \leq \dim_{\mathrm{Rok}}(\alpha) \quad \text{and} \quad \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha|_H) \leq \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha).$$

*Proof.* In all these cases, we will produce a unital $H$-equivariant homomorphism $C(H) \to C(G)$, where the $H$ action on both $C(H)$ and $C(G)$ is given by left translation. This is easily seen to be equivalent to the existence of a continuous map $\phi\colon G \to H$ such that $\phi(hg) = h\phi(g)$ for all $h \in H$ and all $g \in G$.

Assuming the existence of such a homomorphism $C(H) \to C(G)$, the result will follow by composing it with the completely positive contractive order zero maps associated with $\alpha$, similarly to what was done in parts (1) and (2) of Theorem IV.2.8.

(1). Assume that $G/H$ is zero dimensional. By Theorem 8 in [185], there exists a continuous section $\lambda\colon G/H \to G$. Denote by $\pi\colon G \to G/H$ the quotient map, and define $\phi\colon G \to H$ by

$$\phi(g) = g(\lambda(\pi(g))^{-1}$$

for all $g \in G$. We check that the range of $\phi$, which a priori is contained in $G$, really lands in $H$:

$$\pi(\phi(g)) = \pi(g)\pi\left(\lambda(\pi(g))^{-1}\right) = \pi(g)\pi(g)^{-1} = 1$$

for all $g \in G$, so $\phi(G) \subseteq H$. Continuity of $\phi$ follows from continuity of $\lambda$ and from continuity of the group operations on $G$. Finally, if $h \in H$ and $g \in G$, then

$$\phi(hg) = hg\lambda(\pi(hg))^{-1} = hg\lambda(\pi(g))^{-1} = h\phi(g),$$

as desired.

(2). Both cases follow from the fact that there is a group homomorphism $G \to G_j$ determined by $(g_i)_{i\in I} \mapsto g_j$.

(3). This follows from (1) and the fact that $G/G_0$ is totally disconnected. ∎

Rokhlin dimension can indeed increase when passing to a subgroup, even if the original action has the Rokhlin property.

**Example IV.2.11.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T}))$ be given by $\alpha_\zeta(f)(\omega) = f(\zeta^{-1}\omega)$ for $\zeta, \omega \in \mathbb{T}$ and $f \in C(\mathbb{T})$. Then $\alpha$ has Rokhlin dimension zero. Given $n \in \mathbb{N}$ with $n > 1$, identify $\mathbb{Z}_n$ with the subgroup of $\mathbb{T}$ consisting of the $n$-th roots of unity. Then

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha|_{\mathbb{Z}_n}) = \dim_{\mathrm{Rok}}(\alpha|_{\mathbb{Z}_n}) = 1.$$

Indeed, $\dim_{\mathrm{Rok}}(\alpha|_{\mathbb{Z}_n}) \leq 1$ by Theorem IV.2.9. If $\dim_{\mathrm{Rok}}(\alpha|_{\mathbb{Z}_n}) = 0$, then $\alpha|_{\mathbb{Z}_n}$ would have the Rokhlin property, which in particular would imply the existence of a non-trivial projection in $C(S^1)$, which is a contradiction.

Even for circle actions with the Rokhlin property, there are less obvious $K$-theoretic obstructions for the restriction of a circle action with the Rokhlin property to have the Rokhlin property, besides merely the lack of projections. The reader is referred to Chapter XI for examples and results regarding restrictions of circle actions with the Rokhlin property to finite cyclic groups.

### Comparison With Other Notions of Noncommutative Freeness

In this section, we compare the notion of finite Rokhlin dimension (with and without commuting towers) with some of the other forms of freeness of group actions on $C^*$-algebras that have been studied. The properties we discuss here include freeness of actions on compact Hausdorff spaces in the commutative case, the Rokhlin property, discrete $K$-theory, local discrete $K$-theory, total $K$-freenesss, and pointwise outerness.

We begin by comparing finite Rokhlin dimension on unital commutative $C^*$-algebras with freeness of the induced action on the maximal ideal space. Notice that in the case of commutative $C^*$-algebras, the distinction between commuting and non-commuting towers is irrelevant.

**Lemma IV.3.1.** Let $G$ be a compact group acting on a compact Hausdorff space $X$. Denote by $\alpha\colon G \to \mathrm{Aut}(C(X))$ the induced action of $G$ on $X$ and let $n$ be a non-negative integer. Then $\alpha$ has Rokhlin dimension at most $n$ if and only if there are an open cover $\{U_0, \ldots, U_n\}$ of $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ consisting of $G$-invariant open sets, and continuous, proper, equivariant functions $\phi_j\colon U_j \to G \times (0,1]$, where the action of $G$ on $G \times (0,1]$ is translation on $G$ and trivial on $(0,1]$.

*Proof.* Note that

$$C(X)_{\infty,\alpha} \cap C(X)' = C(X)_{\infty,\alpha} = C((\beta\mathbb{N} \setminus \mathbb{N}) \times X),$$

and that the induced action on $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ is trivial on $\beta\mathbb{N} \setminus \mathbb{N}$ and the $G$-action on $X$. The existence of a completely positive contractive order zero map $\varphi\colon C(G) \to C((\beta\mathbb{N} \setminus \mathbb{N}) \times X)$ is easily seen to be equivalent to the existence of an open set $U$ in $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ and a continuous function $\phi\colon U \to G \times (0, 1]$. With this in mind, it is easy to see that $\varphi$ is equivariant if and only if $U$ is $G$-invariant and $\phi$ is equivariant. The rest of the proof is straightforward, and is omitted. $\square$

**Theorem IV.3.2.** Let $G$ be a compact Lie group and let $X$ be a compact Hausdorff space. Let $G$ act on $X$ and denote by $\alpha\colon G \to \mathrm{Aut}(C(X))$ the induced action of $G$ on $C(X)$.

1. If $\alpha$ has finite Rokhlin dimension, then the action of $G$ on $X$ is free.
2. If the action of $G$ on $X$ is free, then $\alpha$ has finite Rokhlin dimension. In fact, there are a non-negative integer $d$ and equivariant completely positive contractive order zero maps
$$\varphi_0, \ldots, \varphi_d\colon C(G) \to C(X)$$
such that $\sum\limits_{j=0}^{d} \varphi_j(1) = 1$. Moreover, if $\dim(X) < \infty$, we have
$$\dim_{\mathrm{Rok}}(\alpha) \leq \dim(X) - \dim(G).$$

We point out that the conclusion in part (2) above really is stronger than $\alpha$ having finite Rokhlin dimension, since one can choose the maps to land in $C(X)$ rather than in its (central) sequence algebra $C(X)_{\infty,\alpha} = C(X)_{\infty,\alpha} \cap C(X)'$.

*Proof.* Part (1). Assume that there exist $g \in G$ and $x \in X$ with $g \cdot x = x$. Choose an open cover $U_0, \ldots, U_n$ of $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ consisting of $G$-invariant open sets, and continuous equivariant functions $\phi_j\colon U_j \to G \times (0, 1]$ as in Lemma IV.3.1. Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and choose $j \in \{0, \ldots, n\}$ such that $(\omega, x) \in U_j$. Write $\phi_j\colon U_j \to G \times (0, 1]$ as $\phi = \left(\phi^{(1)}, \phi^{(2)}\right)$. Note that $\phi^{(1)}\colon U_j \to G$ is equivariant, where the action of $G$ on itself is given by left translation (and in particular, it is free). We have

$$\left(\phi_j^{(1)}(\omega, x), \phi_j^{(2)}(\omega, x)\right) = \phi_j(\omega, x) = \phi_j(\omega, g \cdot x) = \left(g\phi_j^{(1)}(\omega, x), \phi_j^{(2)}(\omega, x)\right),$$

which implies that $\phi_j^{(1)}(\omega, x) = g\phi_j^{(1)}(\omega, x)$ and hence $g = 1$. The action of $G$ on $X$ is therefore free.

Part (2). The proof is almost identical to that of Theorem IV.2.9, using Theorem 1.1 in [202] in place of part (1) of Theorem 2 in [143]. (Since $G$ is a Lie group, we do not need $X$ to be finite dimensional for the quotient map $X \to X/G$ to be a principal $G$-bundle.) When $X$ is not necessarily finite dimensional, we simply take $d$ to be the carinality of some open cover $\mathcal{U}$ consisting of open subsets of $X/G$ over which the fiber bundle $X \to X/G$ is trivial. When $\dim(X) < \infty$, we have $\dim(X/G) = \dim(X) - \dim(G)$, so we can again use Proposition 1.5 in [153] to refine $\mathcal{U}$ to a $(\dim(X) - \dim(G))$-decomposable open cover of $X/G$, and proceed as in the proof of Theorem IV.2.9. We omit the details. $\square$

**Remark IV.3.3.** It follows from the dimension estimate in part (2) of the above theorem that whenever a compact Lie group $G$ acts freely on a compact Hausdorff space $X$ of the same dimension as $G$, then the induced action of $G$ on $C(X)$ has the Rokhlin property. In fact, in this case it follows that $X$ is equivariantly homeomorphic to $G \times (X/G)$, where the $G$-action on $G$ is left translation and the action on $X/G$ is trivial. Indeed, $\dim(X/G) = \dim(X) - \dim(G)$, so $X/G$ is zero dimensional. If $\pi\colon X \to X/G$ denotes the canonical quotient map, then by Theorem 8 in [185], there exists a continuous map $\lambda\colon X/G \to X$ such that $\pi \circ \lambda = \mathrm{id}_{X/G}$. One easily checks that the map $X \to G \times (X/G)$ given by $x \mapsto (\lambda(\pi(x)), \pi(x))$ for $x \in X$, is a homeomorphism. It is also readily verified that it is equivariant, thus proving the claim.

Theorem IV.3.5 below leads to a useful criterion to determine when a given action of a compact group has finite Rokhlin dimension with commuting towers, although it is less useful if one is interested in the actual value of the Rokhlin dimension. For most applications, however, having the exact value is not as important as knowing that it is finite. In particular, it will follow from said theorem that for a compact Lie group, the $X$-Rokhlin property for a finite dimensional compact Hausdorff space $X$ (as defined in Definition 1.5 in [120]), is equivalent to finite Rokhlin dimension with commuting towers in our sense.

We first need a lemma about universal $C^*$-algebras generated by the images of completely positive contractive order zero maps. We present the non-commuting tower version, as well as its commutative counterpart, for use in a later result. In the case of a finite group action with finite

Rokhlin dimension with commuting towers, the result below was first obtained by Ilan Hirshberg, and its proof is contained in the proof of Lemma 1.6 in [120].

**Lemma IV.3.4.** Let $G$ be a compact group and let $d$ be a non-negative integer.

1. There exist a unital $C^*$-algebra $C$ and an action $\gamma\colon G \to \mathrm{Aut}(C)$ of $G$ on $C$ with the following universal property. Let $B$ be a unital $C^*$-algebra, let $\beta\colon G \to \mathrm{Aut}(B)$ be an action of $G$ on $B$, and let $\varphi_0, \ldots, \varphi_d\colon A \to B$ be equivariant completely positive contractive order zero maps such that $\varphi_0(1) + \cdots + \varphi_d(1) = 1$. Then there exists a unital equivariant homomorphism $\varphi\colon C \to B$.

2. There exists a compact metrizable free $G$-space $X$ with the following universal property. Let $B$ be a unital $C^*$-algebra, let $\beta\colon G \to \mathrm{Aut}(B)$ be an action of $G$ on $B$, and let $\varphi_0, \ldots, \varphi_d\colon A \to B$ be equivariant completely positive contractive order zero maps with commuting ranges such that $\varphi_0(1) + \cdots + \varphi_d(1) = 1$. Then there exists a unital equivariant homomorphism $\varphi\colon C(X) \to B$.
   Moreover, the space $X$ in part (2) satisfies
$$\dim(X) \leq (d+1)(\dim(G)+1) - 1.$$

*Proof.* Part (1). Set
$$D = *_{j=0}^{d} C_0((0,1] \times G),$$
and let $\delta\colon G \to \mathrm{Aut}(D)$ be the action obtained by letting $G$ act on each of the free factors diagonally, with trivial action on $(0,1]$ and translation on $G$. Denote by $I$ the ideal in $D$ generated by
$$\left\{ \left( \sum_{j=0}^{d} \mathrm{id}_{(0,1]} * 1_{C(G)} \right) c - c \colon c \in D \right\}.$$
Then $I$ is $\delta$-invariant, and hence there is an induced action $\gamma$ of $G$ on the unital quotient $C = D/I$. It is clear that the $C^*$-algebra $C$ and the action $\gamma$ are as desired.

Part (2). Set
$$D = \bigotimes_{j=0}^{d} C_0((0,1] \times G),$$

and let $\delta\colon G \to \mathrm{Aut}(D)$ be the action obtained by letting $G$ act on each of the tensor factors diagonally. Then $D$ is a commutative $C^*$-algebra and the action on its maximal ideal space induced by $\delta$ is free. Denote by $I$ the ideal in $D$ generated by

$$\left\{ \left( \sum_{j=0}^{d} \mathrm{id}_{(0,1]} \otimes 1_{C(G)} \right) c - c \colon c \in D \right\}.$$

Then $I$ is $\delta$-invariant, and hence there is an induced action $\gamma$ of $G$ on the unital quotient $C = D/I$. Set $X = \mathrm{Max}(C)$, which is a compact metrizable space. The action on $X$ induced by $\gamma$ is free, being the restriction of a free action to an invariant closed subset. It is clear that $X$ is the desired free $G$-space.

The dimension estimate for $X$ follows from the fact that it is a closed subset of $\bigotimes_{j=0}^{d} (0,1] \times G$. ☐

**Theorem IV.3.5.** Let $G$ be a compact Lie group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then $\alpha$ has finite Rokhlin dimension with commuting towers if and only if there exist a finite dimensional compact free $G$-space $X$ and an equivariant unital embedding

$$\varphi\colon C(X) \to A_{\infty,\alpha} \cap A'.$$

Moreover, we have the following relations between the dimension of $X$ and the Rokhlin dimension of $\alpha$:

$$\dim(X) \leq (\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) + 1)(\dim(G) + 1) - 1$$
$$\dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha) \leq \dim(X) - \dim(G)$$

*Proof.* We begin by showing the "only if" implication. Let $d = \dim^{\mathrm{c}}_{\mathrm{Rok}}(\alpha)$, and denote by $Y$ the compact metrizable free $G$-space obtained as in the conclusion of part (2) in Lemma IV.3.4. By universality of $Y$, there is a unital equivariant homomorphism

$$C(Y) \to A_{\infty,\alpha} \cap A'.$$

The kernel of this homomorphism is a $G$-invariant ideal of $C(Y)$ which has the form $C_0(U)$ for some $G$-invariant open subset $U$ of $Y$. Set $X = Y \setminus U$ and denote by $\varphi\colon C(X) \to A_{\infty,\alpha} \cap A'$ the induced homomorphism. Then the $G$-action on $X$ is free and $\varphi$ is unital, equivariant and injective. Finally, we have

$$\dim(X) \leq \dim(Y) \leq (d+1)(\dim(G)+1) - 1,$$

and since $G$ is a compact Lie group, it follows that $X$ is finite dimensional.

We now show the "if" implication. Set $d = \dim(X) - \dim(G) + 1$ and choose completely positive contractive order zero maps

$$\varphi_0, \ldots, \varphi_d\colon C(G) \to C(X)$$

as in the conclusion of part (2) of Theorem IV.3.2. It is immediate to show that the completely positive contractive order zero maps

$$\varphi \circ \varphi_0, \ldots, \varphi \circ \varphi_d\colon C(G) \to A_{\infty,\alpha} \cap A'$$

satisfy the conditions in the definition of finite Rokhlin dimension with commuting towers for $\alpha$. This finishes the proof. □

In particular, it follows from Theorem IV.3.5 that for a compact Lie group, the $X$-Rokhlin property for a compact Hausdorff space $X$ (as defined in Definition 1.5 of [120]), is equivalent to finite Rokhlin dimension with commuting towers.

**Corollary IV.3.6.** Let $G$ be a compact Lie group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$. Then $\alpha$ has discrete $K$-theory, that is, there is $n$ in $\mathbb{N}$ such that $I_G^n \cdot K_*^G(A) = 0$.

*Proof.* Corollary 4.3 in [120] asserts that compact Lie group actions with the $X$-Rokhlin property have discrete $K$-theory. The result follows from the fact that finite Rokhlin dimension with commuting towers implies the $X$-Rokhlin property by Theorem IV.3.5. □

As an application, we show that the gauge action of $\mathbb{T}$ on $\mathcal{O}_2$ does not have finite Rokhlin dimension with commuting towers:

**Example IV.3.7.** Denote by $s_1$ and $s_2$ the canonical generators of the Cuntz algebra $\mathcal{O}_2$. Let $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2)$ be the gauge action, that is, the action determined by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta \in \mathbb{T}$ and for $j = 1, 2$. We claim that $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\gamma) = \infty$. For this, it is enough to show that no power of the augmentation ideal $I_{\mathbb{T}}$ annihilates $K_*^{\mathbb{T}}(\mathcal{O}_2)$. Recall that the crossed product of $\mathcal{O}_2$ by the gauge action is isomorphic to $M_{2^\infty} \otimes \mathcal{K}$. For $n \in \mathbb{N}$, and under the canonical identifications given by Julg's Theorem (here reproduced as Theorem II.3.3), we have

$$I_{\mathbb{T}}^n \cdot K_*^{\mathbb{T}}(\mathcal{O}_2) \cong \mathrm{Im}\left(\left(\mathrm{id}_{K_0(M_{2^\infty}\otimes\mathcal{K})} - K_*(\widehat{\gamma})\right)^n\right).$$

It is a well known fact that $\widehat{\gamma}$ is the unilateral shift on $M_{2^\infty} \otimes \mathcal{K}$, whose induced action on $K_0$ is multiplication by 2. Thus $\mathrm{id}_{K_0(M_{2^\infty}\otimes\mathcal{K})} - K_0(\widehat{\gamma})$ is multiplication by -1, which is an isomorphism of $\mathbb{Z}\left[\frac{1}{2}\right] \cong K_*^{\mathbb{T}}(\mathcal{O}_2)$. In particular, any of its powers is also an isomorphism, and thus

$$I_{\mathbb{T}}^n \cdot K_0^{\mathbb{T}}(\mathcal{O}_2) = K_0^{\mathbb{T}}(\mathcal{O}_2) \neq \{0\}$$

for all $n$ in $\mathbb{N}$. This proves the claim.

We do not know whether the Rokhlin dimension (with noncommuting towers) of the gauge action on $\mathcal{O}_2$ is finite, but we strongly suspect it is not.

**Remark IV.3.8.** Although we will not prove it here, it may be interesting to point out that if $\gamma\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2)$ is the gauge action from the example above, then $\gamma \otimes \mathrm{id}_{\mathcal{O}_2}\colon \mathbb{T} \to \mathrm{Aut}(\mathcal{O}_2 \otimes \mathcal{O}_2)$ has the Rokhlin property. Thus, the tensor product of two actions with infinite Rokhlin dimension can have the Rokhlin property.

It is proved in Theorem VI.3.9 that if a compact Lie group $G$ acts on a unital $C^*$-algebra $A$ with the Rokhlin property, then already $I_G$ annihilates $K_*^G(A)$. It may therefore be tempting to conjecture that in the context of Corollary IV.3.6 above, one has

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) + 1 = \min\left\{n\colon I_G^n \cdot K_*^G(A) = 0\right\},$$

or at least that the right-hand side determines $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha)$. This is unfortunately not in general the case, even for finite group actions on Kirchberg algebras that satisfy the UCT, as the following example shows.

**Example IV.3.9.** Let $B$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be the $C^*$-algebra and the circle action from Example XI.3.7. As mentioned there, $\beta$ has the Rokhlin property and for all $m$ in $\mathbb{N}$, its restriction $\beta|_m$ to $\mathbb{Z}_m \subseteq \mathbb{T}$ does not have the Rokhlin property. Fix $m$ in $\mathbb{N}$. It follows from Theorem IV.2.9 that $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta|_m) = 1$. Moreover, by Lemma XI.2.7 and part (1) of Proposition XI.2.11, the dual action $\widehat{\beta|_n}\colon \mathbb{Z}_m \to \mathrm{Aut}(B \rtimes \mathbb{Z}_m)$ is approximately inner. In particular $1 - K_*(\widehat{\beta|_{n1}}) = 0$ and thus $I_{\mathbb{Z}_m} \cdot K_*^{\mathbb{Z}_m}(B) = 0$. If $\min\left\{n\colon I_{\mathbb{Z}_m}^n \cdot K_*^{\mathbb{Z}_m}(B) = 0\right\}$ determined the Rokhlin dimension of $\beta|_m$, we should have $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta|_m) = 0$, and this would be a contradiction.

The phenomenon exhibited above can also be encountered among free actions on spaces, as the action of $\mathbb{Z}_2$ on $S^1$ by rotation shows. Finally, we mention that it is nevertheless conceivable that one has

$$\min\left\{n\colon I_G^n \cdot K_*^G(A) = 0\right\} \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) + 1,$$

but we have not explored this direction any further.

Having discrete $K$-theory is special to actions with finite Rokhlin dimension *with commuting towers*, as the following example, which was constructed by Izumi in a different context, shows.

Let $p$ be a projection in $\mathcal{O}_\infty$ whose class in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ is 0. We recall that the *standard Cuntz algebra* $\mathcal{O}_\infty^{st}$ is defined to be the corner $p\mathcal{O}_\infty p$. It follows from Kirchberg-Phillips classification of Kirchberg algebras in the UCT class (see [150] and [200]), that $\mathcal{O}_\infty^{st}$ is the unique unital Kirchberg algebra satisfying the UCT with $K$-theory given by

$$(K_0(\mathcal{O}_\infty^{st}), \left[1_{\mathcal{O}_\infty^{st}}\right], K_1(\mathcal{O}_\infty^{st})) \cong (\mathbb{Z}, 0, \{0\}).$$

Since the unit of $\mathcal{O}_\infty^{st}$ is trivial in $K_0$, there is a unital homomorphism $\mathcal{O}_2 \to \mathcal{O}_\infty^{st}$. Hence there is an approximately central embedding of $\mathcal{O}_2$ into $\bigotimes\limits_{n\in\mathbb{N}} \mathcal{O}_\infty^{st}$, so it follows from Theorem 3.8 in [151] that $\bigotimes\limits_{n\in\mathbb{N}} \mathcal{O}_\infty^{st} \cong \mathcal{O}_2$.

**Example IV.3.10.** (See the example on page 262 of [132].) Let $p$ be a projection in $\mathcal{O}_\infty$ whose class in $K_0(\mathcal{O}_\infty)$ is 0, and set $u = 2p - 1$. Then $u$ is a unitary of $\mathcal{O}_\infty$ which leaves the corner

$p\mathcal{O}_\infty p \cong \mathcal{O}_\infty^{st}$ invariant. Since $\bigotimes\limits_{n\in\mathbb{N}} \mathcal{O}_\infty^{st}$ is isomorphic to $\mathcal{O}_2$, if we let $\alpha$ be the infinite tensor product automorphism $\alpha = \bigotimes\limits_{n\in\mathbb{N}} \mathrm{Ad}(u)$ of $\mathcal{O}_2$, then $\alpha$ determines a $\mathbb{Z}_2$ action on $\mathcal{O}_2$.

It is shown in Remark 2.5 in [7] that $\alpha$ has Rokhlin dimension 1 with non-commuting towers. We claim that $\dim_{\mathrm{Rok}}^c(\alpha) = \infty$, that is, that $\alpha$ has infinite Rokhlin dimension with commuting towers. We show that $\alpha$ does not have discrete $K$-theory, and the result will then follow from Corollary IV.3.6.

It is shown in [132] that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to a direct limit of $B_n = \mathcal{O}_\infty^{st} \oplus \mathcal{O}_\infty^{st}$ with connecting maps that on $K_0$ are stationary and given by the matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Moreover, the dual action $\widehat{\alpha}\colon \widehat{\mathbb{Z}_2} \cong \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)$ is the direct limit of the actions $\gamma_n\colon \mathbb{Z}_2 \to \mathrm{Aut}(B_n)$ given by $\gamma_n(a,b) = (b,a)$ for all $(a,b) \in B_n = \mathcal{O}_\infty^{st} \oplus \mathcal{O}_\infty^{st}$. It follows that $\widehat{\alpha}$ is multiplication by $-1$ on $K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)$ (it is given by exchanging the columns in the above matrix). It is shown in Lemma 4.7 in [132] that $K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}\left[\frac{1}{2}\right]$. Given $n$ in $\mathbb{N}$, we have

$$I_{\mathbb{Z}_2}^n \cdot K_0^{\mathbb{Z}_2}(\mathcal{O}_2) \cong \mathrm{Im}\left(\mathrm{id}_{K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)} - K_0(\widehat{\alpha})\right)^n.$$

Now, $\mathrm{id}_{K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)} - K_0(\widehat{\alpha})$ is multiplication by 2 on $\mathbb{Z}\left[\frac{1}{2}\right]$, so

$$\left(\mathrm{id}_{K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)} - K_0(\widehat{\alpha})\right)^n$$

is an isomorphism for all $n$ in $\mathbb{N}$, and in particular $\alpha$ does not have discrete $K$-theory. This proves the claim.

It follows from the example above that the notions of Rokhlin dimension with and without commuting towers do not in general agree. Even more, having *finite* Rokhlin dimension without commuting towers is really weaker than having *finite* Rokhlin dimension with commuting towers. Such phenomenon can happen even if the $K$-theoretic obstructions found in [120] vanish. This answers a question that was implicitly left open in [123], at least in the finite (and compact) group case. We do not know whether there are similar examples for automorphisms.

Finite Rokhlin dimension with commuting towers is not in general equivalent to having discrete $K$-theory, since the trivial action on $\mathcal{O}_2$ clearly does not have finite Rokhlin dimension but has discrete $K$-theory for trivial reasons. The following example, which was originally constructed by Phillips with a different purpose, shows that absence of $K$-theory is not the only thing that can go wrong.

**Example IV.3.11.** We recall the construction in Example 9.3.9 in [199] of an AF-action of $\mathbb{Z}_4$ on the CAR algebra $M_{2^\infty}$ whose restriction to $\mathbb{Z}_2$ is not $K$-free, and show that it has other interesting properties.

For $n$ in $\mathbb{N}$, let $A_n = M_{2^n}(\mathbb{C} \oplus \mathbb{C})$ and set $u_n = \bigotimes\limits_{k=1}^{n} \mathrm{diag}(1, -1)$, which is a unitary in $M_{2^n}$ (not in $A_n$). Define connecting maps $\iota_n \colon A_n \to A_{n+1}$ by

$$\iota_n(a,b) = \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & u_n a u_n^* \end{pmatrix} \right)$$

for $(a,b) \in A_n$. Define an automorphism $\alpha^{(n)}$ of $A_n$ by $\alpha^{(n)}(a,b) = (u_n b u_n^*, a)$ for $(a,b) \in A_n$. Since $u_n$ has order two, it is easy to see that $\alpha^{(n)}$ has order four, so it defines an action of $\mathbb{Z}_4$ on $A_n$. It is also readily checked that there is a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $\mathbb{Z}_4$ on $A = \varinjlim A_n$. Finally, the direct limit algebra $A$ is easily seen to be isomorphic to the CAR algebra $M_{2^\infty}$ by classification.

As proved in [199], with $p = (1,0) \in \mathbb{C} \oplus \mathbb{C} \subseteq A$, it is easy to show that $\alpha^2$ is the action of conjugation by the unitary $2p-1$, so $\alpha|_{\mathbb{Z}_2}$ is in fact inner. In particular, $\alpha$ is not pointwise outer so it does not have finite Rokhlin dimension, with or without commuting towers, by Theorem IV.3.16 below.

The crossed product $A \rtimes_\alpha \mathbb{Z}_4$ is the direct limit of the inductive system

$$A_1 \otimes C^*(\mathbb{Z}_4) \to A_2 \otimes C^*(\mathbb{Z}_4) \to \cdots \to A \rtimes_\alpha \mathbb{Z}_4.$$

The computation of the connecting maps is routine, and yields an isomorphism $A \rtimes_\alpha \mathbb{Z}_4 \cong M_{2^\infty}$, which is best seen using Bratteli diagrams. (Alternatively, one can compute the equivariant $K$-theory of $A$, as is done in [199].) To show that $\alpha$ has discrete $K$-theory, it suffices to observe that the dual action acts via approximately inner automorphisms, since every automorphism of a UHF-algebra is approximately inner. In particular,

$$I_{\mathbb{Z}_4} \cdot K_*^{\mathbb{Z}_4}(A) \cong \mathrm{Im}\,(1 - K_*(\widehat{\alpha}_1)) = 0,$$

as desired.

We turn to the comparison with locally discrete $K$-theory and total $K$-freeness.

**Definition IV.3.12.** (See Definitions 4.1.1, 4.2.1 and 4.2.4 of [199].) Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action.

1. We say that $\alpha$ has *locally discrete $K$-theory* if for every prime ideal $P$ of $R(G)$ not containing the augmentation ideal $I_G$, the localization $K_*^G(A)_P$ is zero.

2. We say that $\alpha$ is *$K$-free* if for every invariant ideal $I$ of $A$, the induced action $\alpha|_I\colon G \to \mathrm{Aut}(I)$ has locally discrete $K$-theory.

3. We say that $\alpha$ is *totally $K$-free* if for every closed subgroup $H$ of $G$, the restriction $\alpha|_H$ is $K$-free.

**Corollary IV.3.13.** Let $G$ be a compact Lie group, let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with finite Rokhlin dimension with commuting towers. Then $\alpha$ has locally discrete $K$-theory.

*Proof.* The action $\alpha$ has discrete $K$-theory by Corollary IV.3.6. It then follows from the equivalence between (1) and (2) in Proposition 4.1.3 of [199] that $\alpha$ has locally discrete $K$-theory. □

**Corollary IV.3.14.** Let $G$ be a compact Lie group, let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with finite Rokhlin dimension with commuting towers. Then $\alpha$ is totally $K$-free.

*Proof.* Let $H$ be a closed subgroup of $G$ and let $I$ be an $H$-invariant ideal of $A$. Since the restriction of $\alpha$ to $H$ has finite Rokhlin dimension with commuting towers by Theorem IV.2.9, we may assume that $H = G$, so that $I$ is $G$-invariant. We have to show that the induced action of $G$ on $I$ has locally discrete $K$-theory.

Since the induced action of $G$ on $A/I$ has finite Rokhlin dimension with commuting towers by part (2) of Theorem IV.2.8, it follows from the corollary above that it has locally discrete $K$-theory. In particular, the extension

$$0 \to I \to A \to A/I \to 0$$

is $G$-equivariant, and the actions on $A$ and $A/I$ have locally discrete $K$-theory. The result now follows from Lemma 1.4 of [199]. $\square$

Even total $K$-freeness is not equivalent to finite Rokhlin dimension.

**Example IV.3.15.** Let $\alpha$ be the trivial action of $\mathbb{Z}_2$ on $\mathcal{O}_2$. Then $\alpha$ is readily seen to be totally $K$-free, but it clearly does not have finite Rokhlin dimension, with or without commuting towers, by Theorem IV.3.16 below.

Recall that an action of a locally compact group $G$ on a $C^*$-algebra $A$ is said to be *pointwise outer* (and sometimes just *outer*), if for every $g \in G \setminus \{1\}$, the automorphism $\alpha_g$ of $A$ is not inner.

**Theorem IV.3.16.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact Lie group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. If $\dim_{\mathrm{Rok}}(\alpha) < \infty$, then $\alpha$ is pointwise outer.

We point out that we do not assume that $\alpha$ has finite Rokhlin dimension with commuting towers, unlike in most other results in this section.

*Proof.* Let $g \in G \setminus \{1\}$ and assume that $\alpha_{g_0}$ is inner, say $\alpha_{g_0} = \mathrm{Ad}(u)$ for some $u \in \mathcal{U}(A)$. Let $C$, let $\gamma\colon G \to \mathrm{Aut}(C)$ and let

$$\varphi\colon C \to A_{\infty,\alpha} \cap A'$$

be the unital $C^*$-algebra, the action of $G$ on $C$, and the unital equivariant homomorphism obtained as in the conclusion of part (1) of Lemma IV.3.4.

We claim that there exists a positive element $a$ in $C$ with the following properties:

- The elements $a$ and $\gamma_g(a)$ are orthogonal.

- $\|\varphi(a)\| = \|\varphi(\gamma_g(a))\| = \|\varphi(a) - \varphi(\gamma_g(a))\| = 1$.

Set $d = \dim_{\mathrm{Rok}}(\alpha)$. Recall from the proof of Lemma IV.3.4 that $C$ is the quotient of the $C^*$-algebra

$$D = *_{j=0}^{d} C_0((0,1] \times G)$$

by the ideal $I$ generated by

$$\left\{ \left( \sum_{j=0}^{d} \mathrm{id}_{(0,1]} * 1_{C(G)} \right) c - c \colon c \in D \right\}.$$

Denote by $\pi\colon D \to C$ the quotient map. Choose a positive function $f$ in $C(G)$ such that the supports of $\mathtt{Lt}_g(f)$ and $f$ are disjoint. Set $b = \mathrm{id}_{(0,1]} \otimes f \in C_0((0,1]) \otimes C(G)$, and regard it as an element in $D$ via the embedding of $C_0((0,1]) \otimes C(G)$ as the first free factor. We claim that $\pi(b) \neq 0$. Indeed, if $\pi(b) = 0$, then $\pi(\delta_h(b)) = 0$ for all $h$ in $G$. Since the action of translation of $G$ on itself is transitive, we conclude that the first free factor of $D$ is contained in the kernel of $\pi$. Now, this contradicts the fact that $d = \dim_{\mathrm{Rok}}(\alpha)$, since it shows that the definition of finite Rokhlin dimension for $\alpha$ is satisfied with $d-1$ order zero maps. This shows that $\pi(b) \neq 0$.

Upon renormalizing $b$, we may assume that $a = \pi(b)$ is positive and has norm 1. It is clear that $a$ and $\gamma_g(a) = \pi(\delta_g(b))$ are orthogonal, and that $\gamma_g(a)$ is positive and has norm 1. Finally, it follows from orthogonality of $a$ and $\gamma_g(a)$ that $\|\varphi(a) - \varphi(\gamma_g(a))\| = 1$. This proves the claim.

Let $\varepsilon = \frac{1}{3}$. Using Choi-Effros lifting theorem, find a completely positive contractive map $\psi\colon C \to A$ satisfying the following conditions:

1. $\|[\psi(a), u]\| < \varepsilon$;
2. $\|\psi(\gamma_g(a)) - \alpha_g(\psi(a))\| < \varepsilon$;
3. $|\|\psi(a) - \psi(\gamma_g(a))\| - 1| < \varepsilon$.

We have

$$\begin{aligned}\frac{2}{3} &= 1 - \varepsilon < \|\psi(a) - \psi(\gamma_g(a))\| \leq \varepsilon + \|\psi(a) - \alpha_g(\psi(a))\| \\ &= \varepsilon + \|\psi(a) - u\psi(a)u^*\| \leq 2\varepsilon = \frac{2}{3},\end{aligned}$$

which is a contradiction. This contradiction implies that $\alpha_g$ is not inner, thus showing that $\alpha$ is pointwise outer. ☐

In the case of commuting towers, the converse to the theorem above fails quite drastically, and there are many examples of compact group actions that are pointwise outer and have infinite Rokhlin dimension with commuting towers. See Example IV.3.7, where it is shown that the gauge action on $\mathcal{O}_2$ has infinite Rokhlin dimension with commuting towers, and see Example IV.3.10 for an example where the acting group is $\mathbb{Z}_2$. The second one has finite Rokhlin dimension with non-commuting towers (in fact, Rokhlin dimension 1). We do not know whether the Rokhlin dimension of the gauge action on $\mathcal{O}_2$ is finite. It is known, however, that all of its restrictions to finite subgroups of $\mathbb{T}$ have Rokhlin dimension with non commuting towers equal to 1.

On the other hand, we do not know exactly how badly the converse to the theorem above fails in the case of non-commuting towers, although we know it does not hold in full generality.

**Example IV.3.17.** The action $\alpha$ of $\mathbb{Z}_2$ on $S^1$ given by conjugation has two fixed points, so it is not free, and hence $\dim_{\mathrm{Rok}}(\alpha) = \infty$. On the other hand, $\alpha$ is certainly pointwise outer since it is not trivial.

We now proceed to obtain a dimensional inequality that will allow us to show that for finite group actions on Kirchberg algebras, pointwise outerness is equivalent to having Rokhlin dimension at most one; see Theorem IV.3.20.

We need the following lemma, whose proof is implicit in [7].

**Lemma IV.3.18.** Let $\varepsilon > 0$. Then there exist injective homomorphisms $\iota_0, \iota_1 \colon \mathcal{O}_2 \to \mathcal{O}_\infty$, and positive contractions $h_0, h_1 \in \mathcal{O}_2$, satisfying $\|\iota_0(h_0) + \iota_1(h_1) - 1_{\mathcal{O}_\infty}\| < \varepsilon$.

*Proof.* See the first part of the proof of Theorem 3.3 in [7]. ☐

It is not clear whether we can choose $\iota_0$ and $\iota_1$ to have approximately commuting ranges. If one were able to do this, in the following theorem one would be able to prove a similar inequality with $\dim_{\mathrm{Rok}}^{\mathrm{c}}$.

The following is our main dynamical dimensional inequality. The technique boils down to a "doubling color" argument, which is standard among the experts.

**Theorem IV.3.19.** Let $A$ be a separable unital $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then

$$\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}) \leq 2\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{O}_2}) + 1.$$

*Proof.* Since $(\mathcal{O}_2 \otimes \mathcal{O}_\infty, \mathrm{id}_{\mathcal{O}_2} \otimes \mathrm{id}_{\mathcal{O}_\infty})$ is equivariantly isomorphic to $(\mathcal{O}_2, \mathrm{id}_{\mathcal{O}_2})$, and $(\mathcal{O}_\infty \otimes \mathcal{O}_\infty, \mathrm{id}_{\mathcal{O}_\infty} \otimes \mathrm{id}_{\mathcal{O}_\infty})$ is equivariantly isomorphic to $(\mathcal{O}_\infty, \mathrm{id}_{\mathcal{O}_\infty})$ by Kirchberg's absorption theorems ([151]), we may assume that $(A \otimes \mathcal{O}_\infty, \alpha \otimes \mathrm{id}_{\mathcal{O}_\infty})$ is equivariantly isomorphic to $(A, \alpha)$.

*Claim:* there is a sequence $\theta_n\colon A \otimes \mathcal{O}_\infty \to A$ of equivariant homomorphisms satisfying

$$\lim_{n\to\infty} \|\theta_n(a \otimes 1_{\mathcal{O}_\infty}) - a\| = 0$$

for all $a \in A$. (This is an equivariant version of Remark 2.7 in [265], whose proof we adapt.)

Denote by $\varphi_n\colon \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to \mathcal{O}_\infty$ the sequence of homomorphisms constructed in part (iii) of Proposition 1.9 in [265]. Define homomorphisms $\widetilde{\theta}_n\colon A \otimes \mathcal{O}_\infty \otimes \mathcal{O}_\infty \to A \otimes \mathcal{O}_\infty$ by $\widetilde{\theta}_n = \mathrm{id}_A \otimes \varphi_n$. Then $\widetilde{\theta}_n$ is clearly equivariant with respect to the actions $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty} \otimes \mathrm{id}_{\mathcal{O}_\infty}$ and $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}$, and satisfies

$$\lim_{n\to\infty} \|\widetilde{\theta}_n(a \otimes x \otimes 1_{\mathcal{O}_\infty}) - a \otimes x\| = 0$$

for all $a \in A$ and for all $x \in \mathcal{O}_\infty$. The desired homomorphisms $\theta_n$ are obtained by appropriately composing $\widetilde{\theta}_n$ with an equivariant isomorphism $A \otimes \mathcal{O}_\infty \cong A$.

Since the statement is trivial when $\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{O}_2}) = \infty$, we may assume that $d = \dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{O}_2}) < \infty$. We use Lemma IV.2.7. Let $\varepsilon > 0$, let $F \subseteq A$ and $S \subseteq C(G)$ be finite subsets. Without loss of generality, the sets $F$ and $S$ contain only contractions. Use Lemma IV.3.18 to choose homomorphisms $\iota_0, \iota_1\colon \mathcal{O}_2 \to \mathcal{O}_\infty$, and positive contractions $h_0, h_1 \in \mathcal{O}_2$ satisfying $\|\iota_0(h_0) + \iota_1(h_1) - 1_{\mathcal{O}_\infty}\| < \frac{\varepsilon}{4}$.

Choose completely positive contractive maps

$$\psi_0, \ldots, \psi_d\colon C(G) \to A \otimes \mathcal{O}_2$$

satisfying conditions (a) through (d) in Lemma IV.2.7 for the action $\alpha \otimes \mathrm{id}_{\mathcal{O}_2}$, tolerance $\frac{\varepsilon}{4}$, finite set $S \subseteq C(G)$, and finite set

$$\left\{a \otimes h_k^{\frac{1}{2}} : a \in F, k = 0,1\right\} \cup \left\{1_A \otimes h_k^{\frac{1}{2}} : k = 0,1\right\}.$$

For $j = 0, \ldots, d$ and $k = 0,1$, define a linear map $\widetilde{\varphi}_j^{(k)} \colon C(G) \to A \otimes \mathcal{O}_\infty$ by

$$\widetilde{\varphi}_j^{(k)}(f) = \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)(\mathrm{id}_A \otimes \iota_k)(\psi_j(f))\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right),$$

for $f \in C(G)$. It is easy to check that $\widetilde{\varphi}_j^{(k)}$ is completely positive and contractive. Using the claim above, choose an equivariant homomorphism $\theta \colon A \otimes \mathcal{O}_\infty \to A$ such that

$$\|\theta(x \otimes 1_{\mathcal{O}_\infty}) - x\| < \frac{\varepsilon}{4}$$

for all $x \in F$. For $j = 0, \ldots, d$ and $k = 0,1$, define a completely positive contractive map $\varphi_j^{(k)} \colon C(G) \to A$ by $\varphi_j^{(k)} = \theta \circ \widetilde{\varphi}_j^{(k)}$. We claim that the completely positive contractive maps $\varphi_j^{(k)}$, for $j = 0, \ldots, d$ and $k = 0,1$, satisfy conditions (a) through (d) in Lemma IV.2.7 for $\varepsilon$, $F$ and $S$.

Let $j \in \{0, \ldots, d\}$, let $k \in \{0,1\}$, let $a \in F$ and let $f \in S$. Then

$$\begin{aligned}
\widetilde{\varphi}_j^{(k)}(f)(a \otimes 1_{\mathcal{O}_\infty}) &= \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)(\mathrm{id}_A \otimes \iota_k)(\psi_j(f))\left(a \otimes \iota_k(h_k)^{\frac{1}{2}}\right) \\
&= \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)(\mathrm{id}_A \otimes \iota_k)\left(\psi_j(f)\left(a \otimes h_k^{\frac{1}{2}}\right)\right) \\
&\approx_{\frac{\varepsilon}{4}} \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)(\mathrm{id}_A \otimes \iota_k)\left(\left(a \otimes h_k^{\frac{1}{2}}\right)\psi_j(f)\right) \\
&= (a \otimes 1_{\mathcal{O}_\infty})(1_A \otimes \iota_k(h_k))(\mathrm{id}_A \otimes \iota_k)(\psi_j(f)) \\
&\approx_{\frac{\varepsilon}{4}} (a \otimes 1_{\mathcal{O}_\infty})\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)(\mathrm{id}_A \otimes \iota_k)(\psi_j(f))\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right) \\
&= (a \otimes 1_{\mathcal{O}_\infty})\widetilde{\varphi}_j^{(k)}(f).
\end{aligned}$$

It follows that

$$\left\|\varphi_j^{(k)}(f)a - a\varphi_j^{(k)}(f)\right\| \leq 2\frac{\varepsilon}{4} + \left\|\widetilde{\varphi}_j^{(k)}(f)(a \otimes 1_{\mathcal{O}_\infty}) - (a \otimes 1_{\mathcal{O}_\infty})\widetilde{\varphi}_j^{(k)}(f)\right\| < \varepsilon,$$

so condition (a) is satisfied.

We proceed to check condition (b), so let $j \in \{0, \ldots, d\}$, let $k \in \{0, 1\}$, let $g \in G$ and let $f \in S$. Then

$$\begin{aligned}
\widetilde{\varphi}_j^{(k)}\left(\mathtt{Lt}_g(f)\right) &\approx_{\frac{\varepsilon}{4}} \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\left(\mathrm{id}_A \otimes \iota_k\right)\left(\psi_j(\mathtt{Lt}_g(f))\right)\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right) \\
&\approx_{\frac{\varepsilon}{4}} \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\left(\mathrm{id}_A \otimes \iota_k\right)\left((\alpha_g \otimes \mathrm{id}_{\mathcal{O}_2})(\psi_j(f))\right)\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right) \\
&= \left(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_\infty}\right)\left(\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\left(\mathrm{id}_A \otimes \iota_k\right)(\psi_j(f))\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\right) \\
&= \left(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_\infty}\right)\left(\widetilde{\varphi}_j^{(k)}(f)\right).
\end{aligned}$$

Since $\theta\colon A \otimes \mathcal{O}_\infty \to A$ is equivariant, we conclude that

$$\left\|\varphi_j^{(k)}\left(\mathtt{Lt}_g(f)\right) - \alpha_g\left(\varphi_j^{(k)}(f)\right)\right\| \leq \left\|\widetilde{\varphi}_j^{(k)}\left(\mathtt{Lt}_g(f)\right) - \left(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_\infty}\right)\left(\widetilde{\varphi}_j^{(k)}(f)\right)\right\| < \frac{\varepsilon}{2},$$

as desired.

To check condition (c), let $j \in \{0, \ldots, d\}$, let $k \in \{0, 1\}$, and let $f_1, f_2 \in S$ be orthogonal. Then

$$\begin{aligned}
&\widetilde{\varphi}_j^{(k)}(f_1)\widetilde{\varphi}_j^{(k)}(f_2) \\
&\approx_{2\frac{\varepsilon}{4}} \left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\left(\mathrm{id}_A \otimes \iota_k\right)(\psi_j(f_1))\left(\mathrm{id}_A \otimes \iota_k\right)(\psi_j(f_2))\left(1_A \otimes \iota_k(h_k)^{\frac{3}{2}}\right) \approx_{\frac{\varepsilon}{4}} 0.
\end{aligned}$$

Thus,

$$\left\|\varphi_j^{(k)}(f_1)\varphi_j^{(k)}(f_2)\right\| \leq \left\|\widetilde{\varphi}_j^{(k)}(f_1)\widetilde{\varphi}_j^{(k)}(f_2)\right\| < \varepsilon,$$

as desired.

Finally,

$$\begin{aligned}
\sum_{k=0}^{1}\sum_{j=0}^{d}\widetilde{\varphi}_j^{(k)}(1_{C(G)}) &= \sum_{k=0}^{1}\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\left(\mathrm{id}_A \otimes \iota_k\right)\left(\sum_{j=0}^{d}\psi_j(1_{C(G)})\right)\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right) \\
&\approx_{\frac{\varepsilon}{2}} \sum_{k=0}^{1}\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right)\left(\mathrm{id}_A \otimes \iota_k\right)(1)\left(1_A \otimes \iota_k(h_k)^{\frac{1}{2}}\right) \\
&= \sum_{k=0}^{1}(1_A \otimes h_k) \approx_{\frac{\varepsilon}{4}} 1_A \otimes 1_{\mathcal{O}_\infty},
\end{aligned}$$

so the proof is complete.

□

The following result generalizes one of the main results in [7], namely Theorem 2.3, which is the case $G = \mathbb{Z}_2$. Our argument is different: instead of proving that a faithful quasifree action of a finite group on $\mathcal{O}_\infty$ has Rokhlin dimension at most one, and invoking results of Goldstein-Izumi, we prove apply Theorem IV.3.19, together with a result of Izumi from [132] concerning the Rokhlin property.

Notice that we do not assume that $A$ satisfies the UCT.

**Theorem IV.3.20.** Let $G$ be a finite group, let $A$ be a unital Kirchberg algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. Then $\alpha$ is pointwise outer if and only if

$$\dim_{\mathrm{Rok}}(\alpha) \leq 1.$$

*Proof.* The "if" implication follows from Theorem IV.3.16. Assume now that $\alpha$ is pointwise outer. By Corollary 2.11 in [132], there is an equivariant isomorphism $(A\otimes\mathcal{O}_\infty, \alpha\otimes\mathrm{id}_{\mathcal{O}_\infty})$. It thus follows from Theorem IV.3.19 that $\dim_{\mathrm{Rok}}(\alpha) \leq 2\dim_{\mathrm{Rok}}(\alpha\otimes\mathrm{id}_{\mathcal{O}_2})+1$. Since $\alpha$ is outer, Corollary 4.3 in [132] implies that $\alpha\otimes\mathrm{id}_{\mathcal{O}_2}$ has the Rokhlin property, so $\dim_{\mathrm{Rok}}(\alpha\otimes\mathrm{id}_{\mathcal{O}_2}) = 0$. We deduce that $\dim_{\mathrm{Rok}}(\alpha) \leq 1$, as desired. □

We point out that the statement of Theorem IV.3.20 fails for compact group actions. First, pointwise outer actions of compact groups on Kirchberg algebras do not in general absorb $\mathrm{id}_{\mathcal{O}_\infty}$. Even worse, a pointwise outer action of $\mathbb{T}$ on $\mathcal{O}_2$, which moreover absorbs $\mathrm{id}_{\mathcal{O}_2}$, need not even have finite Rokhlin dimension.

For the sake of comparison, we present here the following result. The proof, which also relies on a "doubling color" argument, was first observed by Barlak and Szabo in the context of finite groups (and the proof appeared in the original preprint version of [8]). A variant of their argument is included here for the sake of completeness, and we thank Barlak and Szabo for their permission to do so.

We denote by $\mathcal{Z}$ the Jiang-Su algebra ([136]).

**Proposition IV.3.21.** Let $A$ be a separable unital $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Let $\mathcal{U}$ be a UHF-algebra of infinite type. Then

$$\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}}) \leq 2\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{U}}) + 1.$$

*Proof.* Again, we can assume that $(A \otimes \mathcal{Z}, \alpha \otimes \mathrm{id}_{\mathcal{Z}})$ is equivariantly isomorphic to $(A, \alpha)$. Since the case $\dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}})\infty$ is trivial, we may assume that $d = \dim_{\mathrm{Rok}}(\alpha \otimes \mathrm{id}_{\mathcal{Z}}) < \infty$. Let $\varepsilon > 0$, let $F \subseteq A$ and $S \subseteq C(G)$ be finite subsets consisting of contractions. Use Lemma 6.2 of [247] with $e = 0$ to choose completely positive contractive order zero maps $\rho_0, \rho_1\colon \mathcal{U} \to \mathcal{Z}$ satisfying

$$\|\rho_0(1) + \rho_1(1) - 1\| < \frac{\varepsilon}{4}.$$

Choose completely positive contractive maps

$$\psi_0, \ldots, \psi_d\colon C(G) \to A \otimes \mathcal{U}$$

satisfying conditions (a) through (d) in Lemma IV.2.7 for the action $\alpha \otimes \mathrm{id}_{\mathcal{U}}$, tolerance $\frac{\varepsilon}{4}$, finite set $S \subseteq C(G)$, and finite set

$$\{a \otimes \rho_k(1)\colon a \in F, k = 0, 1\} \cup \{1_A \otimes \rho_k(1)\colon k = 0, 1\}.$$

For $j = 0, \ldots, d$ and $k = 0, 1$, define a completely positive contractive map $\widetilde{\varphi}_j^{(k)}\colon C(G) \to A \otimes \mathcal{Z}$ by

$$\widetilde{\varphi}_j^{(k)}(f) = (\mathrm{id}_A \otimes \rho_k)(\psi_j(f)),$$

for $f \in C(G)$. Using an argument similar to the one used in Theorem IV.3.19, choose an equivariant homomorphism $\theta\colon A \otimes \mathcal{Z} \to A$ such that

$$\|\theta(x \otimes 1_{\mathcal{Z}}) - x\| < \frac{\varepsilon}{4}$$

for all $x \in F$. For $j = 0, \ldots, d$ and $k = 0, 1$, define a completely positive contractive map $\varphi_j^{(k)}\colon C(G) \to A$ by $\varphi_j^{(k)} = \theta \circ \widetilde{\varphi}_j^{(k)}$. It is then easy to check that these maps satisfy conditions (a) through (d) in Lemma IV.2.7 for $\varepsilon$, $F$ and $S$. □

It should be pointed out that the conclusion of Lemma 6.2 in [247] for $e = 0$ can also be deduced from the computations performed in Section 5 of [183].

*A rigidity result*

Using the fact that compact group actions with finite Rokhlin dimension with commuting towers are totally $K$-free by Corollary IV.3.14, we are able to obtain a certain $K$-theoretical obstruction for a unital $C^*$-algebra to admit such an action. As a consequence of this $K$-theoretical obstruction, we confirm a conjecture of Phillips.

**Theorem IV.3.22.** Let $G$ be a compact Lie group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Assume that one and only one of either $K_0(A)$ or $K_1(A)$ vanishes. If $\alpha$ is totally $K$-free, then $G$ is finite.

*Proof.* We will show the result assuming that $K_1(A) = 0$; the corresponding proof for the case $K_0(A) = 0$ is analogous.

By restricting to the connected component of the identity in $G$, and recalling that $K$-freeness passes to subgroups, we can assume that $G$ is connected. Assume that $G$ is not the trivial group. By further restricting to any copy of the circle inside a maximal torus, we may assume that $G = \mathbb{T}$. Having discrete $K$-theory, there exists $n \in \mathbb{N}$ such that $I_{\mathbb{T}}^n \cdot K_*^{\mathbb{T}}(A) = 0$. Equivalently,

$$\ker((\mathrm{id}_{K_*(A\rtimes_\alpha \mathbb{T})} - \widehat{\alpha}_*)^n) = K_*(A \rtimes_\alpha \mathbb{T}).$$

Using that $K_1(A) = 0$, it follows from the Pimsner-Voiculescu exact sequence associated to $\alpha$ (see Subsection 10.6 in [13]),

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r, "1-K_0(\widehat{\alpha})"] & K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{T}), \arrow[l, "1-K_1(\widehat{\alpha})"]
\end{tikzcd}$$

that the map $\mathrm{id}_{K_0(A\rtimes_\alpha \mathbb{T})} - \widehat{\alpha}_0$ is injective. This implies that $K_0(A \rtimes_\alpha \mathbb{T}) = 0$ and the remaining potentially non-zero terms in the Pimsner-Voiculescu exact sequence yield the short exact sequence

$$0 \to K_0(A) \to K_1(A \rtimes_\alpha \mathbb{T}) \to K_1(A \rtimes_\alpha \mathbb{T}) \to 0,$$

where the last map is $\mathrm{id}_{K_1(A\rtimes_\alpha \mathbb{T})} - \widehat{\alpha}_1$. Being surjective, every power of it is surjective, and hence the identity

$$\ker((\mathrm{id}_{K_1(A\rtimes_\alpha \mathbb{T})} - \widehat{\alpha}_1)^n) = K_1(A \rtimes_\alpha \mathbb{T})$$

forces $K_1(A \rtimes_\alpha \mathbb{T}) = 0$. In this case, it must be $K_0(A) = 0$ as well, which contradicts the fact that $K_0(A)$ is not zero. □

Recall that an AF-action is an action on an AF-algebra obtained as a direct limit of actions on finite dimensional $C^*$-algebras. It was shown in [12] that not every action on an AF-algebra is an AF-action, even when the group is $\mathbb{Z}_2$.

**Remark IV.3.23.** Conjecture 9.4.9 in [199] says that there does not exist a totally $K$-free AF-action of a non-trivial connected compact Lie group on an AF-algebra. Theorem IV.3.22 above confirms this conjecture of Phillips, for a much larger class of $C^*$-algebras, and without assuming that the action is specified by the way it is constructed.

**Corollary IV.3.24.** No non-finite compact Lie group admits an action with finite Rokhlin dimension with commuting towers on a $C^*$-algebra with exactly one vanishing $K$-group. In particular, there are no such actions on AF-algebras, AI-algebras, the Cuntz algebras $\mathcal{O}_n$ for $n \in \{3, \ldots, \infty\}$, or the Jiang-Su algebra $\mathcal{Z}$.

We make some comments about what happens for finite groups. Many AF-algebras (although not all of them) as well as all Cuntz algebras $\mathcal{O}_n$ with $n < \infty$, admit finite group actions with finite Rokhlin dimension. In fact, they even admit actions of some finite groups with the Rokhlin property (although there are severe restrictions on the cardinality of the group in each case). On the other hand, Theorem 4.7 in [120] asserts that $\mathcal{O}_\infty$ and $\mathcal{Z}$ do not admit *any* finite group action with finite Rokhlin dimension.

We specialize to AF-algebras now, since a little more can be said in this case. Recall that an action $\alpha$ of a locally compact group $G$ on a unital $C^*$-algebra is said to be *inner* if there exists a continuous group homomorphism $u\colon G \to \mathcal{U}(A)$ such that $\alpha_g = \mathrm{Ad}(u(g))$ for all $g \in G$.

**Definition IV.3.25.** An AF-action $\alpha$ of a locally compact group $G$ on a unital $C^*$-algebra $A$ is said to be *locally representable* if there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of unital finite dimensional subalgebras of $A$ such that

– $\bigcup_{n\in\mathbb{N}} A_n$ is dense in $A$

– $\alpha_g(A_n) \subseteq A_n$ for all $g \in G$ and all $n$ in $\mathbb{N}$.

– $\alpha|_{A_n}$ is inner for all $n$ in $\mathbb{N}$.

Product type actions on UHF-algebras are examples of locally representable actions. Such actions have been classified in terms of their equivariant $K$-theory by Handelman and Rossmann in [109].

Using Theorem IV.3.22 and a result from [199], we are able to describe all locally representable actions $\alpha$ of a compact Lie group $G$ on an AF-algebra with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$: the group $G$ must be finite, and all such actions are conjugate to a specific model action $\mu^G$, so in particular they all have the Rokhlin property. The model action $\mu^G \colon G \to \mathrm{Aut}(M_{|G|^\infty})$ is the infinite tensor product of copies of the left regular representation. (We identify $M_{|G|}$ with $\mathcal{K}(\ell^2(G))$ in the usual way.) It is well known that $\mu^G$ (and any tensor product of it with any other action) has the Rokhlin property; see [132].

**Corollary IV.3.26.** Let $G$ be a compact Lie group, let $A$ be a unital AF-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a locally representable AF-action. Then the following are equivalent:

1. $\alpha$ has the Rokhlin property;
2. $\alpha$ has finite Rokhlin dimension with commuting towers;
3. $\alpha$ is totally $K$-free;
4. $\alpha$ has discrete $K$-theory.

Moreover, if any of the above holds, then $G$ must be finite and there is an equivariant isomorphism

$$(A, \alpha) \cong (A \otimes M_{|G|^\infty}, \mathrm{id}_A \otimes \mu^G).$$

In particular, $\alpha$ absorbs $\mu^G$ tensorially.

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are true in general. The equivalence between (3) and (4) follows from Theorem 9.2.4 of [199]. In particular, any of the conditions (1) through (4) implies that $\alpha$ is totally $K$-free, so $G$ must be finite by Theorem IV.3.22. Now, the fact that the second condition implies the fifth in Theorem 9.2.4 of [199], which in turn follows from the classification results in [109], shows that (3) implies the existence of an equivariant isomorphism

$$(A, \alpha) \cong (A \otimes M_{|G|^\infty}, \mathrm{id}_A \otimes \mu^G).$$

We conclude that $\alpha$ has the Rokhlin property, so (3) implies (1).

The final claim follows from the fact that $\mu^G$ absorbs itself tensorially. $\square$

We close this section by summarizing the known implications between the notions we have studied in this section.

**Theorem IV.3.27.** Let $G$ be a compact Lie group, let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Consider the following conditions for the action $\alpha$:

($a$) Rokhlin property.

($b$) Finite Rokhlin dimension with commuting towers, $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$.

($c$) $X$-Rokhlin property for some free $G$-space $X$.

($d$) Finite Rokhlin dimension, $\dim_{\mathrm{Rok}}(\alpha) < \infty$.

($e$) Pointwise outerness.

($f$) Discrete $K$-theory.

($g$) Total $K$-freeness.

We have the following implications, where a theorem or corollary is referenced when it proves the implication in question:

$$
\begin{tikzcd}
 & (c) \arrow[d, Leftrightarrow, "\text{Theorem IV.3.5}"'] & (d) \arrow[r, Rightarrow, "\text{Theorem IV.3.16}"] & (e) \\
(a) \arrow[r, Rightarrow, "\text{Remark IV.2.4}"] & (b) \arrow[ur, Rightarrow] \arrow[rr, Rightarrow, "\text{Corollary IV.3.6}"] \arrow[dr, Rightarrow, "\text{Corollary IV.3.14}"'] & & (f) \\
 & & (g). &
\end{tikzcd}
$$

None the above arrows can be reversed in full generality, and presumably there are no other implications between the stated conditions. In the diagram below, a dotted arrow means that the implication does not hold in general, and in each case a counterexample is referenced:

$$
\begin{tikzcd}
 & & (d) \arrow[dl, dashed, "\text{Example IV.3.10}"'] \arrow[dd, dashed, "\text{Example IV.3.10}"] & (e) \arrow[l, dashed, "\text{Example IV.3.17}"'] \\
(a) & (b) \arrow[l, dashed, "\text{Example IV.2.11}"'] & & (f) \arrow[ll, dashed, "\text{Example IV.3.11}"'] \\
 & & (g). \arrow[ul, dashed, "\text{Example IV.3.15}"] \arrow[ur, dashed, "\text{Example 4.1.7 in [199]}"'] &
\end{tikzcd}
$$

Finally, some of the arrows in the first diagram can be reversed in special situations:

1. If $A$ is commutative, then conditions $(b), (c), (d), (f)$ and $(g)$ are all equivalent to each other, and equivalent to freeness of the action on the maximal ideal space, by Theorem IV.3.5 and Atiyah-Segal completion theorem (see, for example, Theorem 1.1.1 in [199]). Condition $(e)$ is not equivalent to the others by Example IV.3.17, and neither is condition $(a)$ by Example IV.2.11.

2. If $A$ is an AF-algebra and $\alpha$ is a locally representable AF-action (see Definition IV.3.25), then conditions $(a),(b),(c),(f)$ and $(g)$ are equivalent by Corollary IV.3.26.

3. If $A$ is a Kirchberg algebra and $G=\mathbb{Z}_2$ (and possibly also if $G$ is any finite group), then $(e)$ and $(f)$ are equivalent by Theorem 2.3 in [7].

**Outlook and Open Problems**

In this last section, we give some indication of possible directions for future work, and raise some natural questions related to our findings.

Although some of our results, particularly in Section 4, assume that the acting group is a Lie group, this is probably not needed everywhere. Our first suggested problem is then:

**Problem IV.4.1.** Extend some of the results in this chapter to actions of not necessarily finite dimensional compact groups.

We point out that the assumption that $G$ be a Lie group in Corollary 4.3 in [120] is necessary, since it relies on Atiyah-Segal completion Theorem (see [5], and see Theorem 1.1.1 in [199]), for which one needs the representation ring $R(G)$ to be finitely generated. We suspect that Corollary IV.3.6 is not true in general for arbitrary compact groups (it probably already fails for actions on compact spaces), but it may be the case that all compact group actions with finite Rokhlin dimension with commuting towers have *locally* discrete $K$-theory.

Somewhat related, we ask:

**Question IV.4.2.** Is finite Rokhlin dimension with commuting towers equivalent to the $X$-Rokhlin property for *arbitrary* compact groups?

Maybe one should start with the commutative case:

**Question IV.4.3.** Is finite Rokhlin dimension for actions on commutative $C^*$-algebras, equivalent to freeness of the induced action on the maximal ideal space for *arbitrary* compact groups?

For compact Lie groups, the answer is yes in both cases; see Theorem IV.3.5 and Theorem IV.3.2. It was crucial in the proofs of said theorems that free actions of compact Lie groups have local cross-sections. We suspect that the answer to Question IV.4.2 and Question IV.4.3 is 'no', and it is possible that the action of $G=\prod\limits_{n\in\mathbb{N}}\mathbb{Z}_n$ on $X=\prod\limits_{n\in\mathbb{N}}S^1$ by coordinate-wise rotation is a counterexample (to both questions). Such action has the $X$-Rokhlin property and is free, but the quotient map $X \to X/G$ is known not to have local cross-sections, since $X$ is not locally homeomorphic to $X/G \times G$. We have not checked, however, whether this action has finite Rokhlin dimension.

The results in [7] show that for $\mathbb{Z}_2$-actions on Kirchberg algebras, pointwise outerness implies Rokhlin dimension at most 1 with noncommuting towers. It is conceivable that a similar result holds for a larger class of finite groups, and presumably all of them.

**Question IV.4.4.** Is pointwise outerness equivalent to finite Rokhlin dimension (with noncommuting towers) for finite group actions on Kirchberg algebras?

If the question above has an affirmative answer, as the results in [7] suggest, one may try to prove the corresponding result for simple unital separable nuclear $C^*$-algebras with tracial rank zero, where presumably some additional assumptions will be needed.

Alternatively,

**Problem IV.4.5.** Find obstructions (not necessarily $K$-theoretical) for having an action of a finite (or compact) group with finite Rokhlin dimension with noncommuting towers.

The results in Section XI.2 suggest the following:

**Conjecture IV.4.6.** Let $G$ be a compact Lie group and let $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_2)$ be an action with finite Rokhlin dimension with commuting towers. Then $\alpha$ has the Rokhlin property.

Example IV.3.10 shows that the corresponding statement for noncommuting towers is in general false.

Based on Corollary IV.3.6 and the comments and examples after it, we ask:

**Question IV.4.7.** If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a compact Lie group on a unital $C^*$-algebra $A$ and $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) < \infty$, does one have

$$\min\left\{n\colon I_G^n \cdot K_*^G(A) = 0\right\} \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) + 1,$$

or any other relationship between these quantities?

Example IV.3.9 shows that one cannot in general expect equality to hold.

Finally, the following problem is likely to be challenging:

**Problem IV.4.8.** Can actions with finite Rokhlin dimension with commuting towers on unital Kirchberg algebras satisfying the UCT be classified, in a way similar to what was done in [133] for finite group actions with the Rokhlin property, or in Chapter IX for circle actions with the Rokhlin property?

Some of these questions will be addressed in [87].

CHAPTER V

# REGULARITY PROPERTIES AND ROKHLIN DIMENSION

We show that formation of crossed products and passage to fixed point algebras by compact group actions with finite Rokhlin dimension preserve the following regularity properties: finite decomposition rank, finite nuclear dimension, and tensorial absorption of the Jiang-Su algebra, the latter in the formulation with commuting towers.

## Introduction

The Elliott conjecture predicts that simple, separable, nuclear $C^*$-algebras may be classified by their so-called Elliott invariant, which is essentially $K$-theoretical in nature. Despite the great success that the classification program enjoyed in its beginnings (see Section 4 of [62] for a detailed account), the first counterexamples appeared in the mid to late 1990's, due to Rørdam ([236]) and Toms ([263]). These examples suggest two alternatives: either the invariant should be enlarged (to include, for example, the Cuntz semigroup), or the class of $C^*$-algebras should be restricted, assuming further regularity properties (stronger than nuclearity). Significant effort has been put into both directions, and the present chapter is a contribution to the second one of these (in particular, to the verification of certain regularity properties for specific crossed product $C^*$-algebras).

The regularity properties that have been studied are of very different nature: topological, analytical and algebraic. These are: finite nuclear dimension (or finite decomposition rank, in the stably finite case); tensorial absorption of the Jiang-Su algebra; and strict comparison of positive elements. These notions are surveyed in [62], and we briefly recall what we will use in Section V.2.

Despite their seemingly different flavors, Toms and Winter conjectured these notions to be equivalent for all unital, nuclear, separable, non-elementary, simple $C^*$-algebras. Some implications hold in full generality, as was shown by Rørdam ([237]) and Winter ([269] and [270])), and several partial results are available for the remaining implications ([182], [152], [246], and [264]). More recently, Sato, White and Winter showed that the Toms-Winter conjecture is true if one moreover assumes that the $C^*$-algebra in question has at most one trace (Corollary C in

[247]). It should also be pointed out that all three regularity properties are satisfied by every $C^*$-algebra in any of the classes considered by the existing classification theorems.

In view of their importance in the classification program, it is useful to know what constructions preserve these regularity properties. In this chapter, which is based on [86], we show that formation of crossed products and passage to fixed point algebras preserve finiteness of nuclear dimension (Theorem V.3.4), finiteness of decomposition rank (Theorem V.3.3), and tensorial absorption of the Jiang-Su algebra (Theorem V.4.4), provided that the action has finite Rokhlin dimension in the sense of Definition IV.2.2 (for Jiang-Su absorption, one needs to assume the formulation with commuting towers). Our work generalizes results for finite groups of Hirshberg, Winter and Zacharias from [123], where they also studied similar questions for crossed products by automorphisms.

The last two sections contain, respectively, examples where our methods can be applied to obtain information that does not follow from known methods, and a couple of suggestions for future work.

## Preliminaries on Regularity Properties

This section contains some background notions on regularity properties for $C^*$-algebras. We limit ourselves to collecting the definitions and results that will be needed in the rest of the chapter, but we refer the reader to [62] for a more detailed account of these and other regularity properties that are relevant to the classification programm.

### *Covering dimension for $C^*$-algebras*

In this subsection, we describe two related noncommutative versions of covering dimension for $C^*$-algebras: decomposition rank (due to Kirchberg and Winter; see [153]), and nuclear dimension (due to Winter and Zacharias; see [**?** ]). Both notions enjoy good permanence properties with respect to quotients, inductive limits, hereditary subalgebras, tensor products, and, as we shall see, crossed products by compact group actions with finite Rokhlin dimension.

Here are the definitions:

**Definition V.2.1.** Let $A$ be a $C^*$-algebra, and let $n \in \mathbb{Z}_{\geq 0}$. We say that $A$ has *decomposition rank at most* $n$, written $\mathrm{dr}(A) \leq n$, if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist finite dimensional $C^*$-algebras $B_0, \ldots, B_n$, and completely positive order zero maps $\psi\colon A \to B = \bigoplus_{j=0}^{n} B_j$ and $\varphi_j\colon B_j \to A$, satisfying the following:

1. $\varphi = \sum_{j=0}^{n} \varphi_j\colon B \to A$ is contractive,
2. $\|(\varphi \circ \psi)(a) - a\| < \varepsilon$ for all $a \in F$.

In other words, the following diagram commutes up to $\varepsilon$ on $F$:

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, "\psi"'] & & A \\
& B_0 \oplus \cdots \oplus B_n \arrow[ur, "\varphi = \sum_{j=0}^{n} \varphi_j"'] &
\end{tikzcd}.$$

We say that $A$ has *nuclear dimension at most* $n$, written $\dim_{\mathrm{nuc}}(A) \leq n$, if in the definition of decomposition rank, we drop condition (1) (that is, we do not insist that the map $\varphi$ be contractive).

Clearly one has $\dim_{\mathrm{nuc}}(A) \leq \mathrm{dr}(A)$ for every $C^*$-algebra $A$. For a locally compact metric space $X$, we have $\mathrm{dr}(C_0(X)) = \dim_{\mathrm{nuc}}(C_0(X)) = \dim(X)$, so decomposition rank and nuclear dimension can be regarded as noncommutative analogs of covering dimension. (In fact, the definitions themselves are modeled after the notion of covering dimension.)

The distinction between decomposition rank and nuclear dimension may seem like a minor one that could in general be arranged upon rescaling. However, the two notions are quite different: while Kirchberg algebras have finite nuclear dimension, only strongly quasidiagonal $C^*$-algebras (in particular, stably finite) can have finite decomposition rank.

*Absorption of the Jiang-Su algebra.*

The Jiang-Su algebra $\mathcal{Z}$ is a simple, separable, unital, nuclear, infinite dimensional $C^*$-algebra with the same $K$-theory as the complex numbers. It was introduced by Jiang and Su in [136], and was subsequently studied by a number of authors. One of the main reasons why this algebra is relevant in the context of the classification program is the following theorem of Gong, Jiang and Su. Recall that an ordered group $G$ is said to be *weakly unperforated* if whenever $x$ is an element in $G$ for which $nx \in G_+$ for some $n \in \mathbb{N}$, then $x \in G_+$.

**Theorem V.2.2.** ([104].) Let $A$ be a simple, unital $C^*$-algebra, such that $K_0(A)$ is weakly unperforated. Then the Elliott invariants of $A$ and $A \otimes \mathcal{Z}$ are canonically isomorphic.

The definition of the Jiang-Su algebra is a bit technical. Since we will only need to use and prove absorption of it, we give a convenient characterization in Lemma V.4.1.

We close this section by recalling that the Toms-Winter regularity conjecture asserts that for infinite dimensional, nuclear, simple, separable, (stably finite) unital $C^*$-algebras, finiteness of the nuclear dimension (decomposition rank) is equivalent to absorption of the Jiang-Su algebra. That $\mathcal{Z}$-stability follows from finiteness of the nuclear dimension (and hence, from finiteness of the decomposition rank), has been known for some time, while the converse implication seems to be more elusive. For $C^*$-algebras with at most one tracial state, this has been confirmed in recent work of Sato, White and Winter ([247]).

### Preservation of Finite Nuclear Dimension and Decomposition Rank

In this section, we explore the structure of the crossed product and fixed point algebra of an action of a compact group with finite Rokhlin dimension in relation to their nuclear dimension and decomposition rank. Specifically, we show that finite nuclear dimension and finite decomposition rank are inherited by the crossed product and fixed point algebra by any such action.

We introduce some notation that will be used in this subsection.

Let $A$ be a $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Give $C(G)\otimes A$ the diagonal action $\mathtt{Lt}\otimes\alpha$ of $G$. Then the canonical inclusion $A \to C(G)\otimes A$ is equivariant. Identify $C(G) \otimes A$ with $C(G, A)$ in the usual way, and let

$$\theta\colon (C(G,A), \mathtt{Lt} \otimes \alpha) \to (C(G,A), \mathtt{Lt} \otimes \mathrm{id}_A)$$

be given by $\theta(\xi)(g) = \alpha_{g^{-1}}(\xi(g))$ for all $\xi$ in $C(G, A)$ and all $g$ in $G$. Then $\theta$ is clearly an isomorphism, and it is moreover equivariant, since

$$\begin{aligned}
\theta\left((\mathtt{Lt}_g \otimes \alpha_g)(\xi)\right)(h) &= \alpha_{h^{-1}}\left((\mathtt{Lt}_g \otimes \alpha_g)(\xi)(h)\right) \\
&= \alpha_{h^{-1}}\left(\alpha_g(\xi(g^{-1}h))\right) \\
&= \theta(\xi)(g^{-1}h) \\
&= (\mathtt{Lt}_g \otimes \mathrm{id}_A)(\xi)(h)
\end{aligned}$$

for all $g$ and $h$ in $G$, and all $\xi$ in $C(G, A)$. It follows that there are isomorphisms

$$(C(G) \otimes A) \rtimes_{\mathtt{Lt}\otimes\alpha} G \cong (C(G) \rtimes_{\mathtt{Lt}} G) \otimes A \cong \mathcal{K}(L^2(G)) \otimes A.$$

We conclude that the canonical inclusion $A \to C(G) \otimes A$ induces an injective homomorphism

$$\iota\colon A \rtimes_\alpha G \to \mathcal{K}(L^2(G)) \otimes A,$$

which we will refer to as the *canonical* embedding of $A \rtimes_\alpha G$ into $A \otimes \mathcal{K}(L^2(G))$. This terminology is justified by the following observation.

**Remark V.3.1.** Adopt the notation from the discussion above, and denote by $\lambda\colon G \to \mathcal{U}(L^2(G))$ the left regular representation, and identify $A \rtimes_\alpha G$ with its image under $\iota$. Then

$$A \rtimes_\alpha G = \left(A \otimes \mathcal{K}(L^2(G))\right)^{\alpha\otimes\mathrm{Ad}(\lambda)}.$$

The following lemma will be the main technical device we will use to prove Theorem V.3.3. When the group $G$ is finite (as is considered in [123]), the desired "almost" order zero maps are constructed using the elements of appropriately chosen towers in the definition of finite Rokhlin dimension. In the case of an arbitrary compact group, some work is needed to get such maps.

Lemma V.3.2 can be thought of as an analog of Osaka-Phillips' approximation of crossed products of actions with the Rokhlin property by matrices over corners of the underlying algebra, which was used in [191]. This approximation technique is implicit in the paper [122], and further applications of it will appear in [87]. We would like to thank Luis Santiago for suggesting such an approach.

**Lemma V.3.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group, let $d$ be a nonnegative integer, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with $\dim_{\mathrm{Rok}}(\alpha) \leq d$. Denote by $\iota\colon A \rtimes_\alpha G \to A \otimes \mathcal{K}(L^2(G))$ the canonical embedding.

Given compact sets $F \subseteq A \rtimes_\alpha G$ and $S \subseteq A \otimes \mathcal{K}(L^2(G))$, and given $\varepsilon > 0$, there are completely positive maps

$$\rho_0, \ldots, \rho_d \colon A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G$$

such that

1. $\|\rho_j(a)\rho_j(b)\| < \varepsilon$ whenever $a$ and $b$ are positive elements in $S$ with $ab = 0$;
2. $\left\| \sum\limits_{j=0}^{d} (\rho_j \circ \iota)(x) - x \right\| < \varepsilon$ for all $x$ in $F$;
3. The map
$$\sum_{j=0}^{d} \rho_j \colon \bigoplus_{j=0}^{d} A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G$$
is completely positive and contractive.

In other words, the maps $\iota$ and $\rho_0, \ldots, \rho_d$ induce a diagram

$$\begin{tikzcd} A \rtimes_\alpha G \arrow[rr, "\mathrm{id}_{A \rtimes_\alpha G}"] \arrow[dr, "\iota"'] & & A \rtimes_\alpha G \\ & A \otimes \mathcal{K}(L^2(G)) \arrow[ur, "\sum_{j=0}^{d} \rho_j"'] & \end{tikzcd}$$

that approximately commutes on $F$ up to $\varepsilon$, and such that the completely positive contractive maps $\rho_j$ are "almost" order zero on $S$.

*Proof.* Let $\varphi_0, \ldots, \varphi_d \colon C(G) \to A_{\infty,\alpha} \cap A'$ be the equivariant completely positive contractive order zero maps as in the definition of Rokhlin dimension at most $d$ for $\alpha$. Upon tensoring with $\mathrm{id}_A$, we obtain equivariant completely positive contractive order zero maps

$$\psi_0, \ldots, \psi_d \colon A \otimes C(G) \to A_{\infty,\alpha},$$

which satisfy $\sum_{j=0}^{d} \psi_j(a \otimes 1) = a$ for all $a$ in $A$. (The action on $A \otimes C(G)$ is the diagonal, using translation on $C(G)$.) With $e \in \mathcal{K}(L^2(G))$ denoting the projection onto the constant functions on $G$, use Proposition II.5.5 and Proposition II.4.5 to obtain completely positive contractive order zero maps

$$\sigma_0, \ldots, \sigma_d \colon A \otimes \mathcal{K}(L^2(G)) \to (A \rtimes_\alpha G)_\infty$$

which satisfy $\sum_{j=0}^{d} \sigma_j(x \otimes e) = x$ for all $x$ in $A \rtimes_\alpha G$, and such that $\sum_{j=0}^{d} \sigma_j$ is contractive.

For $j = 0, \ldots, d$, use nuclearity of $A$, together with Choi-Effros lifting theorem, to lift $\sigma_j$ to a completely positive contractive map

$$\rho_j \colon A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G,$$

which satisfies conditions (1) and (2) of the statement with $\frac{\varepsilon}{2}$ in place of $\varepsilon$, and such that

$$\left\| \sum_{j=0}^{d} \rho_j \right\| < 1 + \frac{\varepsilon}{2}.$$

Dividing each of the maps $\rho_j$ by the above norm introduces an additional error of $\frac{\varepsilon}{2}$, and the resulting rescaled maps are the desired order zero maps. □

With the aid of Lemma V.3.2, the proof of the following theorem can be proved using ideas similar to the ones used to prove Theorem 1.3 in [123].

**Theorem V.3.3.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be a continuous action with finite Rokhlin dimension. Then

$$\mathrm{dr}(A^\alpha) \leq \mathrm{dr}(A \rtimes_\alpha G) \leq (\dim_{\mathrm{Rok}}(\alpha) + 1)(\mathrm{dr}(A) + 1) - 1.$$

*Proof.* The first inequality is a consequence of the fact that $A^\alpha$ is isomorphic to a corner of $A \rtimes_\alpha G$ by compactness of $G$ (see the Theorem in [238]), together with Proposition 3.8 in [153].

In order to show the second inequality, it is enough to do it when $\mathrm{dr}(A) < \infty$ and $\dim_{\mathrm{Rok}}(\alpha) < \infty$. Set $N = \mathrm{dr}(A)$ and set $d = \dim_{\mathrm{Rok}}(\alpha)$.

Let $\varepsilon > 0$ and let $F$ be a compact subset of $A \rtimes_\alpha G$. Choose finite dimensional $C^*$-algebras $\mathcal{F}_0, \ldots, \mathcal{F}_N$, a completely positive contractive map

$$\psi \colon A \to \mathcal{F} = \mathcal{F}_0 \oplus \cdots \oplus \mathcal{F}_N,$$

and completely positive contractive order zero maps $\phi_\ell \colon \mathcal{F}_\ell \to A$ for $\ell = 0, \ldots, N$, such that $\phi = \phi_0 + \ldots + \phi_N \colon \mathcal{F} \to A$ is completely positive and contractive, and satisfies

$$\|(\phi \circ \psi)(a) - a\| < \frac{\varepsilon}{2}$$

for all $a$ in $F$. Denote by $\iota \colon A \rtimes_\alpha G \to A \otimes \mathcal{K}(L^2(G))$ the canonical inclusion. We will construct completely positive approximations for $A \rtimes_\alpha G$ of the form

$$\begin{tikzcd}
A \rtimes_\alpha G \arrow[rrrr, "\mathrm{id}_{A \rtimes_\alpha G}"] \arrow[dr, "\iota"'] & & & & A \rtimes_\alpha G, \\
& A \otimes \mathcal{K}(L^2(G)) \arrow[dr, "\bigoplus_{j=0}^{d} \psi"'] & & \bigoplus_{j=0}^{d} A \otimes \mathcal{K}(L^2(G)) \arrow[ur, "\rho"'] & \\
& & \bigoplus_{j=0}^{d} \mathcal{F} \arrow[ur, "\bigoplus_{j=0}^{d} \phi"'] & &
\end{tikzcd}$$

where the map $\rho \colon \bigoplus_{j=0}^{d} A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G$ will be constructed later using that $\dim_{\mathrm{Rok}}(\alpha) \leq d$, in such a way that $\rho \circ \phi \colon \mathcal{F} \to A \rtimes_\alpha G$ is the sum of "almost" order zero maps. We will then use projectivity of the cone over finite dimensional $C^*$-algebras to replace the map $\rho \circ \phi$ with maps that are decomposable into completely positive contractive order zero maps.

Set

$$\varepsilon_1 = \frac{\varepsilon}{8(d+1)(N+1)}.$$

Using stability of order zero maps from finite dimensional $C^*$-algebras, choose $\delta > 0$ such that whenever $\sigma \colon \mathcal{F} \to A \rtimes_\alpha G$ is a completely positive contractive map satisfying

$$\|\sigma(x)\sigma(y)\| < \delta$$

for all positive orthogonal contractions $x$ and $y$ in $\mathcal{F}$, there exists a completely positive contractive order zero map $\sigma'\colon \mathcal{F} \to A \rtimes_\alpha G$ with $\|\sigma' - \sigma\| < \varepsilon_1$.

Let $B_{\mathcal{F}}$ denote the unit ball of $\mathcal{F}$, and set

$$S = \bigcup_{g \in G} \bigcup_{\ell=0}^{N} \left(\alpha_g \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}\right)(\phi_\ell(B_{\mathcal{F}})),$$

which is a compact subset of $A \otimes \mathcal{K}(L^2(G))$.

Set $\varepsilon_2 = \min\left\{\delta, \frac{\varepsilon}{4}\right\}$. Use Lemma V.3.2 to find completely positive contractive maps

$$\rho_0, \ldots, \rho_d \colon A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G$$

such that

1. $\|\rho_j(a)\rho_j(b)\| < \varepsilon_2$ whenever $a$ and $b$ are positive elements in $S$ with $ab = 0$;
2. $\left\| \sum_{j=0}^{d} (\rho_j \circ \iota)(x) - x \right\| < \varepsilon_2$ for all $x$ in $F$;
3. The map
$$\sum_{j=0}^{d} \rho_j \colon \bigoplus_{j=0}^{d} A \otimes \mathcal{K}(L^2(G)) \to A \rtimes_\alpha G$$
is completely positive and contractive.

Fix indices $\ell$ in $\{0, \ldots, N\}$ and $j$ in $\{0, \ldots, d\}$, and fix positive orthogonal elements $x$ and $y$ in $S$. Set $a = \phi_\ell(x)$ and $b = \phi_\ell(y)$. Since $\phi_\ell$ is order zero, we have $ab = 0$. Then

$$\|(\rho_j \circ \phi_\ell)(x)(\rho_j \circ \phi_\ell)(y)\| = \|\rho_j(a)\rho_j(b)\| < \varepsilon_2.$$

By the choice of $\delta$, there are completely positive contractive order zero maps $\sigma_{j,\ell} \colon \mathcal{F} \to A \rtimes_\alpha G$ satisfying

$$\|\sigma_{j,\ell} - \rho_j \circ \phi_\ell\| < \varepsilon_1$$

for $j = 0, \ldots, d$ and $\ell = 0, \ldots, N$. For $j = 0, \ldots, d$, define a linear map $\sigma_j \colon \mathcal{F} \to A \rtimes_\alpha G$ by

$$\sigma_j = \sum_{\ell=0}^{N} \sigma_{j,\ell},$$

and let $\sigma\colon \bigoplus_{j=0}^{d} \mathcal{F} \to A \rtimes_\alpha G$ be given by $\sigma = \sum_{j=0}^{d} \sigma_j$. Then $\sigma$ is completely positive, and moreover

$$\|\sigma\| < 1 + (d+1)(N+1)\varepsilon_1.$$

Set $\tau = \frac{\sigma}{\|\sigma\|}$, which is completely positive contractive and order zero, and satisfies

$$\left\| \tau - \rho \circ \left( \sum_{j=0}^{d} \phi \right) \right\| \leq \|\tau - \sigma\| + \left\| \sigma - \rho \circ \left( \sum_{j=0}^{d} \phi \right) \right\| < 2(d+1)(N+1)\varepsilon_1.$$

Finally, we claim that

$$\begin{tikzcd} A \rtimes_\alpha G \arrow[rr, "\mathrm{id}"] \arrow[dr, "\bigoplus_{j=0}^{d} \psi\circ\iota"'] & & A \rtimes_\alpha G \\ & \bigoplus_{\ell=0}^{d} \mathcal{F} \arrow[ur, "\tau"'] & \end{tikzcd}$$

approximately commutes on the set $F$ within $\varepsilon$, and that $\tau$ can be decomposed into $(d+1)(N+1)-1$ order zero summands. The only thing that remains to be checked is that $\|(\tau\circ\psi)(a) - a\| < \varepsilon$ for all $a$ in $F$. Given $a$ in $F$, we estimate as follows:

$$\begin{aligned} \left( \tau \circ \left( \bigoplus_{j=0}^{d} \psi \circ \iota \right) \right)(a) &\approx_{2(d+1)(N+1)\varepsilon_1} \left( \rho \circ \left( \sum_{j=0}^{d} \phi \right) \circ \left( \bigoplus_{j=0}^{d} \psi \circ \iota \right) \right)(a) \\ &\approx_{\frac{\varepsilon}{2}} (\rho \circ \iota)(a) \\ &\approx_{\varepsilon_2} a. \end{aligned}$$

Hence

$$\begin{aligned} \left\| \left( \tau \circ \left( \bigoplus_{j=0}^{d} \psi \circ \iota \right) \right)(a) - a \right\| &< 2(d+1)(N+1)\varepsilon_1 + \frac{\varepsilon}{2} + \varepsilon_2 \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon, \end{aligned}$$

and the claim is proved. This finishes the proof. $\square$

The corresponding statement for nuclear dimension is true. Its proof is analogous to that of Theorem V.3.3 and is therefore omitted. The difference is that one does not need to take care of the norms of the components of the approximations.

**Theorem V.3.4.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action with finite Rokhlin dimension. Then

$$\dim_{\mathrm{nuc}}(A^\alpha) \leq \dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq (\dim_{\mathrm{Rok}}(\alpha)+1)(\dim_{\mathrm{nuc}}(A)+1)-1.$$

**Remark V.3.5.** It should be pointed out that the inequalities $\mathrm{dr}(A^\alpha) \leq \mathrm{dr}(A \rtimes_\alpha G)$ and $\dim_{\mathrm{nuc}}(A^\alpha) \leq \dim_{\mathrm{nuc}}(A \rtimes_\alpha G)$ are likely to be equalities whenever $\alpha$ has finite Rokhlin dimension. Indeed, it is probably the case, although we have not checked, that finite Rokhlin dimension implies saturation (see Definition 5.2 in [202]), from which it would follow that $A^\alpha$ and $A \rtimes_\alpha G$ are Morita equivalent, and hence they have the same nuclear dimension and decomposition rank. We remark that saturation is automatic whenever the crossed product is simple, so the equalities $\mathrm{dr}(A^\alpha) = \mathrm{dr}(A \rtimes_\alpha G)$ and $\dim_{\mathrm{nuc}}(A^\alpha) = \dim_{\mathrm{nuc}}(A \rtimes_\alpha G)$ hold in many cases of interest. In particular, this is the case whenever $G$ is finite and $A$ is simple by Theorem IV.3.16.

An example in which one can apply these results to deduce finite nuclear dimension and decomposition rank of the crossed product is that of free actions of compact Lie groups on compact metric spaces with finite covering dimension. Indeed, such actions have finite Rokhlin dimension by Theorem IV.3.5, and since $A = C(X)$ has finite nuclear dimension and decomposition rank, we deduce that $A \rtimes G$ does as well. (Note that since the action is free, Situation 2 in [228] implies that $A^G$ and $A \rtimes G$ are Morita equivalent.)

Nevertheless, there is a much simpler proof of this fact, which even yields a better estimate of the nuclear dimension and decomposition rank. Indeed, if $X$ is a compact free $G$-space, then the fixed point algebra of $C(X)$ is $C(X/G)$. Moreover, the orbit space $X/G$ has covering dimension at most $\dim(X) - \dim(G)$, and hence $C(X/G)$ has finite nuclear dimension and decomposition rank (and equal to each other). We conclude that

$$\dim_{\mathrm{nuc}}(C(X) \rtimes G) = \mathrm{dr}(C(X) \rtimes G) \leq \dim(X) - \dim(G).$$

More interesting applications of our results will be presented in Section V.5.

## Preservation of $\mathcal{Z}$-absorption

We now turn to preservation of $\mathcal{Z}$-absorption, under the (stronger) assumption that the action have finite Rokhlin dimension with commuting towers. In this section, compact groups will be second countable. By a theorem of Birkoff-Kakutani (Theorem 1.22 in [184]), a topological group is metrizable if and only if it is first countable. In particular, all our groups will be metrizable. It is well-known that a compact metrizable group admits a left translation-invariant metric. We will implicitly choose such a metric on all our groups, which will be denoted by $d$.

We will need a technical lemma characterizing $\mathcal{Z}$-absoprtion in a form that is useful in our context.

**Lemma V.4.1.** Let $A$ be a unital separable $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Let $d$ be a non-negative integer, and suppose that for any $r$ in $\mathbb{N}$, for any compact subset $F \subseteq A$, and for any $\varepsilon > 0$, there exist completely positive contractive maps

$$\theta_0, \ldots, \theta_d \colon M_r \to A_{\infty,\alpha} \quad \text{and} \quad \eta_0, \ldots, \eta_d \colon M_{r+1} \to A_{\infty,\alpha}$$

such that the following properties hold for all $x, x'$ in $M_r$ and for all $y, y'$ in $M_{r+1}$ with $\|x\|, \|x'\|, \|y\|, \|y'\| \leq 1$, for all $g$ in $G$, for all $a$ in $F$ and for all $j, k = 0, \ldots, d$:

$$\|[\theta_j(x), \eta_k(y)]\| < \varepsilon; \tag{V.1}$$

$$\text{if} \;\; x, x', y, y' \geq 0 \;\; \text{and} \;\; x \perp x' \;,\; y \perp y', \;\text{then} \tag{V.2}$$

$$\|\theta_j(x)\theta_j(x')\| < \varepsilon \quad \text{and} \quad \|\eta_k(y)\eta_k(y')\| < \varepsilon;$$

$$\|(\alpha_\infty)_g(\theta_k(x)) - \theta_k(x)\| < \varepsilon \quad \text{and} \quad \|(\alpha_\infty)_g(\eta_k(y)) - \eta_k(y)\| < \varepsilon; \tag{V.3}$$

$$\|a\theta_k(x) - \theta_k(x)a\| < \varepsilon \quad \text{and} \quad \|a\eta_k(y) - \eta_k(y)a\| < \varepsilon; \tag{V.4}$$

$$\left\| \sum_{k=0}^{d} \theta_k(1) + \eta_k(1) - 1 \right\| < \varepsilon. \tag{V.5}$$

Then $A \rtimes_\alpha G$ is $\mathcal{Z}$-stable.

*Proof.* Using stability of completely positive order zero maps from matrix algebras, we may assume that the maps $\theta_0, \ldots, \theta_d$ and $\eta_0, \ldots, \eta_d$ can always be chosen to satisfy condition (2) exactly.

Let $r$ in $\mathbb{N}$. We claim that there are order zero maps maps

$$\theta_0, \ldots, \theta_d \colon M_r \to (A_\infty \cap A')^{\alpha_\infty} \quad \text{and} \quad \eta_0, \ldots, \eta_d \colon M_{r+1} \to (A_\infty \cap A')^{\alpha_\infty}$$

with $\sum\limits_{k=0}^{d} \theta_k(1) + \eta_k(1) = 1$. Once we prove the claim, the rest of the proof goes exactly as in Lemma 5.7 in [123]. (There, the authors assumed the group $G$ to be discrete, but since the order zero maps we will produce land in the fixed point algebra of $A_\infty \cap A'$, and in particular, in $A_{\infty,\alpha} \cap A'$, the fact that $G$ is not discrete in this lemma is not an issue.)

Choose an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of compact subsets of $A$ such that $A_0 = \bigcup\limits_{n\in\mathbb{N}} F_n$ is dense in $A$. Without loss of generality, we may assume that $A_0$ is closed under multiplication, addition and involution. For each $n$ in $\mathbb{N}$, let

$$\theta_0^{(n)}, \ldots, \theta_d^{(n)} \colon M_r \to A_{\infty,\alpha} \quad \text{and} \quad \eta_0^{(n)}, \ldots, \eta_d^{(n)} \colon M_{r+1} \to A_{\infty,\alpha}$$

be completely positive contractive order zero maps satisfying conditions (1) and (3)-(5) in the statement for $F_n$ and $\varepsilon = \frac{1}{n}$. For each $n$ in $\mathbb{N}$ and each $j = 0, \ldots, d$, let

$$\widetilde{\theta}_j^{(n)} \colon M_r \to \ell^\infty_\alpha(\mathbb{N}, A) \quad \text{and} \quad \widetilde{\eta}_j^{(n)} \colon M_{r+1} \to \ell^\infty_\alpha(\mathbb{N}, A)$$

be completely positive contractive lifts of $\theta_j^{(n)}$ and $\eta_j^{(n)}$ respectively. As in the proof of Lemma 2.4 in [122], we can find a strictly increasing sequence $n_k$ of natural numbers such that the following hold for all $k$ in $\mathbb{N}$, for all $j = 0, \ldots, d$ and for all $g$ in $G$:

- $\left\| \alpha_g \left( (\widetilde{\theta}_j^{(k)}(n_k))(x) \right) - \left( \widetilde{\theta}_j^{(k)}(n_k) \right)(x) \right\| < \frac{1}{k}$ for all $x$ in $M_r$ with $\|x\| \leq 1$.
- $\left\| \alpha_g \left( (\widetilde{\eta}_j^{(k)}(n_k))(y) \right) - \left( \widetilde{\eta}_j^{(k)}(n_k) \right)(y) \right\| < \frac{1}{k}$ for all $y$ in $M_{r+1}$ with $\|y\| \leq 1$.

$$-\left\|\sum_{j=0}^{d}\left(\widetilde{\theta}_j^{(k)}(n_k)\right)(1)+\left(\widetilde{\eta}_j^{(k)}(n_k)\right)(1)-1\right\|<\frac{1}{k}$$

With $\kappa_A\colon \ell^\infty_\alpha(\mathbb{N},A)\to A_{\infty,\alpha}$ denoting the quotient map, it follows that for $j=0,\ldots,d$, the maps

$$\theta_j=\kappa_A\circ\left(\widetilde{\theta}_j^{(1)}(n_1),\widetilde{\theta}_j^{(2)}(n_2),\ldots\right)\colon M_r\to(A_\infty\cap A')^{\alpha_\infty}$$

and

$$\eta_j=\kappa_A\circ\left(\widetilde{\eta}_j^{(1)}(n_1),\widetilde{\eta}_j^{(2)}(n_2),\ldots\right)\colon M_{r+1}\to(A_\infty\cap A')^{\alpha_\infty}$$

are completely positive contractive order zero, and satisfy $\sum\limits_{j=0}^{d}\theta_j(1)+\eta_j(1)=1$. This proves the claim, and finishes the proof of the lemma. $\square$

We now need to introduce a certain averaging technique that will allow us to take averages over the group in such a way that $*$-algebraic relations are approximately preserved.

Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G\to\mathrm{Aut}(A)$ be a continuous action. Identify $C(G)\otimes A$ with $C(G,A)$, and denote by $\gamma\colon G\to\mathrm{Aut}(C(G,A))$ the diagonal action, this is, $\gamma_g(a)(h)=\alpha_g(a(g^{-1}h))$ for all $g,h\in G$ and all $a\in C(G,A)$. Define an *averaging process* $\phi\colon C(G,A)\to C(G,A)$ by

$$\phi(a)(g)=\alpha_g(a(1))$$

for all $a$ in $C(G,A)$ and all $g$ in $G$.

For use in the proof of the following lemma, we recall the following standard fact about self-adjoint elements: if $A$ is a unital $C^*$-algebra and $a,b\in A$ with $b^*=b$, then $-\|b\|a^*a\leq a^*ba\leq\|b\|a^*a$.

**Lemma V.4.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group and let $\alpha\colon G\to\mathrm{Aut}(A)$ be a continuous action. Denote by $\phi\colon C(G,A)\to C(G,A)$ the averaging process defined above, and by $\gamma\colon G\to\mathrm{Aut}(C(G,A))$ the diagonal action. Given $\varepsilon>0$ and given a compact set $F\subseteq C(G,A)$, there exist a positive number $\delta>0$, a finite subset $K\subseteq G$ and continuous functions $f_k$ in $C(G)$ for $k$ in $K$ such that

1. If $g$ and $h$ in $G$ satisfy $d(g,h)<\delta$, then $\|\gamma_g(a)-\gamma_h(a)\|<\varepsilon$ for all $a$ in $\bigcup\limits_{g\in G}\gamma_g(F)$

2. We have $0 \leq f_k \leq 1$ for all $k$ in $K$.

3. The family $(f_k)_{k\in K}$ is a partition of unity for $G$.

4. For $k_1$ and $k_2$ in $K$, whenever $f_{k_1}f_{k_2} \neq 0$, then $d(k_1, k_2) < \delta$.

5. For every $g \in G$ and every $a$ in $\bigcup\limits_{g\in G} \gamma_g(F)$, we have

$$\left\| \phi(a)(g) - \sum_{k\in K} f_k(g)\alpha_k(a(1)) \right\| < \varepsilon.$$

*Proof.* We observe that the averaging process $\phi\colon C(G, A) \to C(G, A)$ is a homomorphism, since it is the composition of the homomorphism $C(G, A) \to A$ obtained as the evaluation at the unit of $G$, with the homomorphism $\rho\colon A \to C(G, A)$ given by

$$\rho(a)(g) = \alpha_g(a)$$

for all $a \in A$ and all $g \in G$.

We claim that $\gamma_g(\phi(a)) = \phi(a)$ for all $g$ in $G$ and all $a$ in $C(G, A)$. Indeed, for $h$ in $G$, we have

$$\begin{aligned}\gamma_g(\phi(a))(h) &= \alpha_g(\phi(a)(g^{-1}h)) = \alpha_g(\alpha_{g^{-1}h}(a(1))) \\ &= \alpha_h(a(1)) = \phi(a)(h),\end{aligned}$$

which proves the claim.

Set $F' = \bigcup\limits_{g\in G} \gamma_g(F)$, which is a compact subset of $C(G, A)$. Since every element in a $C^*$-algebra is the linear combination of two self-adjoint elements, we may assume without loss of generality that every element of $F'$ is self-adjoint. Set

$$F'' = \{a(g)\colon a \in F', g \in G\},$$

which is a compact subset of $A$. Using continuity of $\alpha$, choose $\delta > 0$ such that whenever $g$ and $h$ in $G$ satisfy $d(g, h) < \delta$, then $\|\alpha_g(a) - \alpha_h(a)\| < \varepsilon$ for all $a$ in $F''$. Given $g$ in $G$, denote by $U_g$ the open ball centered at $g$ with radius $\frac{\delta}{2}$. Let $K \subseteq G$ be a finite subset such $\bigcup\limits_{k\in K} U_k = G$, and let

$(f_k)_{k\in K}$ be a partition of unity subordinate to $\{U_k\}_{k\in K}$. Given $g$ in $G$ and $a$ in $F'$, we have

$$\begin{aligned}\phi(a)(g) - \sum_{k\in K} f_k(g)\alpha_k(a(1)) &= \sum_{k\in K} f_k(g)^{1/2}(\alpha_k(a(1)) - \alpha_g(a(1)))f_k(g)^{1/2} \\ &\leq \sum_{j=1}^{n} \|\alpha_k(a(1)) - \alpha_g(a(1))\| f_k(g).\end{aligned}$$

Now, for $k \in K$, if $f_k(g) \neq 0$, then $d(g,k) < \delta$, and hence $\|\alpha_k(a(1)) - \alpha_g(a(1))\| < \varepsilon$. In particular, we conclude that

$$-\varepsilon < \phi(a)(g) - \sum_{k\in K} f_k(g)\alpha_k(a(1)) < \varepsilon.$$

This shows that condition (5) in the statement is satisfied, and finishes the proof. $\square$

Let $A$ be a $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. We denote by $E\colon A \to A^\alpha$ the standard conditional expectation. If $\mu$ denotes the normalized Haar measure on $G$, then $E$ is given by

$$E(a) = \int_G \alpha_g(a)\, d\mu(g)$$

for all $a$ in $A$.

**Proposition V.4.3.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group, let $d$ be a non-negative integer, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with $\dim_{\mathrm{Rok}}(\alpha) \leq d$. For every $\varepsilon > 0$ and every be a compact subset $F$ of $A$, there exist $\delta > 0$, a finite subset $K \subseteq G$, continuous functions $f_k$ in $C(G)$ for $k$ in $K$, and completely positive contractive linear maps $\psi_0, \ldots, \psi_d\colon C(G) \to A$ such that

1. If $g$ and $g'$ in $G$ satisfy $d(g,g') < \delta$, then $\|\alpha_g(a) - \alpha_{g'}(a)\| < \varepsilon$ for all $a$ in $F$.
2. We have $0 \leq f_k \leq 1$ for all $k$ in $K$.
3. Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} \neq 0$, then $d(k,k') < \delta$.
4. For every $g \in G$, for every $j = 0, \ldots, d$, and for every $a \in F$, we have

$$\left\| \alpha_g\left(\sum_{k\in K} \psi_j(f_k)^{1/2}\alpha_k(a)\psi_j(f_k)^{1/2}\right) - \sum_{k\in K} \psi_j(f_k)^{1/2}\alpha_k(a)\psi_j(f_k)^{1/2} \right\| < \varepsilon.$$

5. For every $a \in F$, for all $k \in K$, and for every $j = 0, \ldots, d$, we have

$$\|a\psi_j(f_k) - \psi_j(f_k)a\| < \frac{\varepsilon}{|K|} \quad \text{and} \quad \left\|a\psi_j(f_k)^{1/2} - \psi_j(f_k)^{1/2}a\right\| < \frac{\varepsilon}{|K|}.$$

6. Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} = 0$, then for all $j = 0, \ldots, d$ we have

$$\left\|\psi_j(f_k)^{1/2}\psi_j(f_{k'})^{1/2}\right\| < \frac{\varepsilon}{|K|}.$$

7. The family $(f_k)_{k \in K}$ is a partition of unity for $G$, and moreover,

$$\left\|\sum_{j=0}^{d}\sum_{k \in K}\psi_j(f_k) - 1\right\| < \frac{\varepsilon}{|K|}.$$

Moreover, if $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$, then the choices above can be made so that in addition to conditions (1) through (7) above, we have:

8. For all $j, \ell = 0, \ldots, d$ and for all $k, k'$ in $K$,

$$\|[\psi_j(f_k), \psi_\ell(f_{k'})]\| < \varepsilon \quad \text{and} \quad \left\|\left[\psi_j(f_k)^{1/2}, \psi_\ell(f_{k'})^{1/2}\right]\right\| < \frac{\varepsilon}{|K|}.$$

*Proof.* Without loss of generality, we may assume that $F$ is $\alpha$-invariant. Using Lemma V.4.2, choose a positive number $\delta > 0$, a finite subset $K \subseteq G$, and continuous functions $f_k$ in $C(G)$ for $k$ in $K$, such that conditions (1) through (5) in Lemma V.4.2 are satisfied for $F$ and $\frac{\varepsilon}{2}$. Set $S = \left\{f_k, f_k^{1/2} \colon k \in K\right\} \subseteq C(G)$, and for every $m$ in $\mathbb{N}$, choose completely positive contractive maps

$$\psi_0^{(m)}, \ldots, \psi_d^{(m)} \colon C(G) \to A$$

as in the conclusion of part (1) of Lemma IV.2.7 for the choices of finite set $S \subseteq C(G)$, compact subset $F \subseteq A$, and tolerance $\frac{1}{m}$. Identify $C(G, A)$ with $C(G) \otimes A$, and for $j = 0, \ldots, d$, define a completely positive contractive order zero map $\phi_j \colon C(G, A) \to A_{\infty,\alpha}$ by

$$\phi_j(f \otimes a) = \psi_j(f)a$$

for $f \in C(G)$ and $a \in A$. (Note that the range of $\psi$ commutes with the copy of $A$ in $A_{\infty,\alpha}$.) It is clear that $\phi^{(m)}$ is equivariant, where we take $C(G,A)$ to have the diagonal action $\gamma$ of $G$. For $m \in \mathbb{N}$, choose a completely positive contractive map $\phi_j^{(m)} \colon C(G,A) \to A$ satisfying

$$\kappa_A\left(\left(\phi_j^{(m)}(\xi)\right)_{m\in\mathbb{N}}\right) = \phi_j(\xi)$$

for all $\xi \in C(G,A)$. Then $\limsup\limits_{m\to\infty} \left\|\phi_j^{(m)}(f\otimes a) - \psi_j^{(m)}(f)a\right\| = 0$ for all $f \in C(G)$ and for all $a \in A$, and

$$\limsup_{m\to\infty} \left\|\alpha_g\left(\phi_j^{(m)}(\xi)\right) - \phi_j^{(m)}\left(\gamma_g(\xi)\right)\right\| = 0$$

for all $g \in G$ and all $\xi \in C(G,A)$.

Given $a$ in $F$, given $j = 0,\ldots,d$, and given $g$ in $G$, we have the following, where use condition (5) in the conclusion of Lemma V.4.2 at the last step:

$$\begin{aligned}
\limsup_{m\to\infty} \Bigg\| \alpha_g\left(\sum_{k\in K} \psi_j^{(m)}(f_k)^{1/2}\alpha_k(a)\psi_j^{(m)}(f_k)^{1/2}\right) & \\
- \sum_{k\in K} \psi_j^{(m)}(f_k)^{1/2}\alpha_k(a)\psi_j^{(m)}(f_k)^{1/2}\Bigg\| & \\
= \limsup_{m\to\infty} \left\|\alpha_g\left(\phi_j^{(m)}\left(\sum_{k\in K} f_k\otimes\alpha_k(a)\right)\right) - \phi_j^{(m)}\left(\sum_{k\in K} f_k\otimes\alpha_k(a)\right)\right\| & \\
= \limsup_{m\to\infty} \left\|\phi_j^{(m)}\left(\gamma_g\left(\sum_{k\in K} f_k\otimes\alpha_k(a)\right) - \sum_{k\in K} f_k\otimes\alpha_k(a)\right)\right\| & \\
\leq \limsup_{m\to\infty} \left\|\gamma_g\left(\sum_{k\in K} f_k\otimes\alpha_k(a)\right) - \sum_{k\in K} f_k\otimes\alpha_k(a)\right\| \leq \frac{\varepsilon}{2} &
\end{aligned}$$

The result in the case that $\dim_{\mathrm{Rok}}(\alpha) \leq d$ follows by choosing $m > \frac{|K|}{\varepsilon}$ large enough so that

$$\left\|\alpha_g\left(\sum_{k\in K} \psi_j^{(m)}(f_k)^{1/2}\alpha_k(a)\psi_j^{(m)}(f_k)^{1/2}\right) - \sum_{k\in K} \psi_j^{(m)}(f_k)^{1/2}\alpha_k(a)\psi_j^{(m)}(f_k)^{1/2}\right\| < \varepsilon.$$

If one moreover has $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq d$, one uses part (2) of Lemma IV.2.7, and the same argument shows that the choices can be made so that condition (8) in this proposition is also satisfied. We omit the details. $\square$

We are now ready to prove that absorption of the Jiang-Su algebra $\mathcal{Z}$ passes to crossed products and fixed point algebras by compact group actions with finite Rokhlin dimension with commuting towers. This generalizes Theorem 5.9 in [123], and it partially generalizes part (1) of Corollary 3.2 in [122].

We do not know whether commuting towers are really necessary in the theorem below. In view of the Toms-Winter conjecture and Theorem V.3.4, this condition should not be necessary if both $A$ and $A \rtimes_\alpha G$ are simple and $A$ is nuclear.

It should also be mentioned here that the commuting towers assumption imposes restrictions on the equivariant $K$-theory. See the comments after Definition IV.2.2.

**Theorem V.4.4.** Let $A$ be a separable unital $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action with finite Rokhlin dimension with commuting towers. Suppose that $A$ is $\mathcal{Z}$-absorbing. Then the crossed product $A \rtimes_\alpha G$ and the fixed point algebra $A^\alpha$ are also $\mathcal{Z}$-absorbing.

*Proof.* We show first that the crossed product $A \rtimes_\alpha G$ is $\mathcal{Z}$-absorbing. Our proof combines the methods of Theorem 5.9 in [123] and, to a lesser extent, Theorem 3.3 in [122]. We will produce maps as in the statement of Lemma V.4.1.

Let $d = \dim_{\mathrm{Rok}}^c(\alpha)$. Fix a positive integer $r$ in $\mathbb{N}$ and a compact subset $F \subseteq A$, which, without loss of generality, we assume to be $\alpha$-invariant. We may also assume that $F$ contains only self-adjoint elements of norm at most 1. Choose order zero maps $\theta\colon M_r \to \mathcal{Z}$ and $\eta\colon M_{r+1} \to \mathcal{Z}$ with commuting ranges satisfying $\theta(1) + \eta(1) = 1$. Define unital homomorphisms

$$\iota_0, \ldots, \iota_d \colon \mathcal{Z} \to A \hookrightarrow A_{\infty,\alpha}$$

as follows. Start with any unital homomorphism $\iota_0\colon \mathcal{Z} \to A$ satisfying

$$\|\iota_0(z)a - a\iota_0(z)\| < \varepsilon$$

for all $z$ in $\mathcal{Z}$ and all $a$ in $F$, which exists because $A$ is $\mathcal{Z}$-absorbing. Once we have constructed $\iota_j$ for $j = 0, \ldots, k-1$, we choose $\iota_k\colon \mathcal{Z} \to A$ such that

$$\|\iota_k(z)b - b\iota_k(z)\| < \varepsilon$$

for all $z$ in $\mathcal{Z}$ and for all $b$ in the compact $\alpha$-invariant set

$$F \cup \bigcup_{j=0}^{k-1} \bigcup_{g\in G} \alpha_g \left( \{ (\iota_j \circ \theta)(x), (\iota_j \circ \eta)(y) \colon x \in M_r, y \in M_{r+1}, \|x\|, \|y\| \leq 1 \} \right).$$

Set

$$F' = F \cup \bigcup_{j=0}^{d} \bigcup_{g\in G} \alpha_g \left( \{ (\iota_j \circ \theta)(x), (\iota_j \circ \eta)(y) \colon x \in M_r, y \in M_{r+1}, \|x\|, \|y\| \leq 1 \} \right),$$

which we regard as a subset of $A_{\infty,\alpha}$ via the inclusion $A \hookrightarrow A_{\infty,\alpha}$. Choose a finite subset $K \subseteq G$, continuous functions $f_k$ in $C(G)$ for $k$ in $K$, and unital completely positive contractive maps $\varphi_0, \ldots, \varphi_d \colon C(G) \to A \hookrightarrow A_{\infty,\alpha} \cap A'$ as in the conclusion of Proposition V.4.3 for the choices of compact set $F' \subseteq A_{\infty,\alpha}$ and tolerance $\varepsilon$. Then the ranges of $\varphi_0, \ldots, \varphi_d$ commute with the ranges of the homomorphisms $\iota_0, \ldots, \iota_d$.

For $j = 0, \ldots, d$, define

$$\theta_j \colon M_r \to A_\infty \cap A' \quad \text{and} \quad \eta_j \colon M_{r+1} \to A_\infty \cap A'$$

by

$$\theta_j(x) = \sum_{k\in K} \varphi_j(f_k)\alpha_k((\iota_j \circ \theta)(x)) \quad \text{and} \quad \eta_j(y) = \sum_{k\in K} \varphi_j(f_k)\alpha_k((\iota_j \circ \eta)(y))$$

for all $x$ in $M_r$ and all $y$ in $M_{r+1}$. We claim that these maps satisfy the conditions in the statement of Lemma V.4.1.

Condition (1). Let $j, \ell \in \{0, \ldots, d\}$, let $x$ in $M_r$ and let $y \in M_{r+1}$ satisfy $\|x\|, \|y\| \leq 1$. Without loss of generality, we may assume that $x^* = x$ and $y^* = -y$. Then $[\theta_j(x), \eta_\ell(y)]$ is self-adjoint and

$$\begin{aligned}
[\theta_j(x), \eta_\ell(y)] &= \sum_{k,k' \in K} [\varphi_j(f_k)\alpha_k((\iota_j \circ \theta)(x)), \varphi_\ell(f_{k'})\alpha_{k'}((\iota_\ell \circ \eta)(y))] \\
&= \sum_{k,k' \in K} \varphi_j(f_k)\varphi_\ell(f_{k'})\alpha_k\left([(\iota_j \circ \theta)(x), \alpha_{k^{-1}k'}((\iota_\ell \circ \eta)(y))]\right) \\
&\leq \sum_{k,k' \in K} \varphi_j(f_k)\varphi_\ell(f_{k'}) \left\|[(\iota_j \circ \theta)(x), \alpha_{k^{-1}k'}((\iota_\ell \circ \eta)(y))]\right\| \\
&< \left(\sum_{k,k' \in K} \varphi_j(f_k)\varphi_k(f_{k'})\right) \varepsilon = \varepsilon.
\end{aligned}$$

Likewise, $[\theta_j(x), \eta_\ell(y)] > -\varepsilon$, so $\| [\theta_j(x), \eta_\ell(y)] \| < \varepsilon$, as desired.

Condition (2). Given $j = 0, \ldots, d$ and given positive orthogonal elements $x, x' \in M_r$ with $\|x\|, \|x'\| \leq 1$, set $a = (\iota_j \circ \theta)(x)$ and $b = (\iota_j \circ \theta)(x')$. Then $ab = 0$ because $\theta$ is an order zero map and $\iota_j$ is a homomorphism. Using that $f_k f_{k'} \neq 0$ implies $d(k, k') < \delta$ for $k, k' \in K$ at the fourth step, we have

$$\begin{aligned}
\|\theta_j(x)\theta_j(x')\| &= \left\| \sum_{k,k' \in K} \varphi_j(f_k)\alpha_h(a)\varphi_j(f_{k'})\alpha_{k'}(b) \right\| \\
&= \left\| \sum_{f_k f_{k'} \neq 0, h \neq h'} \varphi_j(f_k)\varphi_j(f_{k'})\alpha_k(a)\alpha_{k'}(b) \right\| \\
&< |K|\frac{\varepsilon}{|K|} + \left\| \sum_{f_k f_{k'} \neq 0, k \neq k'} \varphi_j(f_k)\varphi_j(f_{k'})\alpha_k(ab) \right\| = \varepsilon,
\end{aligned}$$

as desired. A similar computation shows that $\|\eta_j(y)\eta_j(y')\| < \varepsilon$ whenever $y$ and $y'$ are as in Condition (2) of Lemma V.4.1.

Condition (3). Given $x$ in $M_r$ with $\|x\| \leq 1$, given $g$ in $G$, and given $j = 0, \ldots, d$, the same computation carried out in the verification of condition (4) in Proposition V.4.3 shows that $\|(\alpha_\infty)_g(\theta_j(a)) - \theta_j(a)\| < \varepsilon$, as desired. A similar computation shows that $\|(\alpha_\infty)_g(\eta_j(y)) - \eta_j(y)\| < \varepsilon$ whenever $y$ is as in Condition (3) of Lemma V.4.1.

Condition (4). Let $a$ in $F$, let $j = 0, \ldots, d$ and let $x$ in $M_r$ satisfy $\|x\| \leq 1$ and $x^* = -x$. Then $[a, \theta_j(x)]$ is self-adjoint because $a^* = a$, and moreover we have

$$\begin{aligned}
[a, \theta_j(x)] &= \sum_{k \in K} \varphi_j(f_k)[a, \alpha_k((\iota_j \circ \theta)(x))] \\
&\leq \sum_{k \in K} \varphi_j(f_k) \, \|[a, \alpha_k((\iota_j \circ \theta)(x))]\| \\
&< \varepsilon \left( \sum_{k \in K} \varphi_j(f_k) \right) = \varepsilon.
\end{aligned}$$

Likewise, $[a, \theta_j(x)] > -\varepsilon$, so $\| \, [a, \theta_j(x)] \, \| < \varepsilon$, as desired. A similar computation shows that $\|[a, \eta_j(y)]\| < \varepsilon$ whenever $y$ is as in Condition (4) of Lemma V.4.1.

Condition (5). We use that the family $(f_k)_{k \in K}$ is a partition of unity of $G$ in the third step to conclude that

$$\begin{aligned}
\sum_{j=0}^{d} \theta_j(1) + \eta_j(1) &= \sum_{j=0}^{d} \sum_{k \in K} \varphi_j(f_k) \alpha_k(\iota_j(\theta(1) + \eta(1))) \\
&= \sum_{j=0}^{d} \sum_{k \in K} \varphi_j(f_k) \\
&= \sum_{j=0}^{d} \varphi_j(1) \\
&= 1.
\end{aligned}$$

The result for $A \rtimes_\alpha G$ now follows from Lemma V.4.1.

Since the fixed point algebra $A^\alpha$ is a corner in $A \rtimes_\alpha G$ by the Theorem in [238], it follows from Corollary 3.1 in [265] that $A^\alpha$ is also $\mathcal{Z}$-absorbing. ☐

**Examples**

In this section, we proceed to give a few applications of the results in this chapter that yield new information in some cases of interest.

In preparation, we present an intermediate result.

**Proposition V.5.1.** Let $A$ be a unital $C^*$-algebra, let $G$ be a compact group, let $\alpha \colon G \to \mathrm{Aut}(A)$ be a continuous action, and let $p \in A$ be a $G$-invariant projection. Set $B = pAp$, which is a

$G$-invariant corner in $A$, and denote by $\beta\colon G \to \mathrm{Aut}(B)$ the compressed action. Then

$$\dim_{\mathrm{Rok}}(\beta) \leq \dim_{\mathrm{Rok}}(\alpha) \quad \text{and} \quad \dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta) \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha).$$

*Proof.* We prove the proposition for $\dim_{\mathrm{Rok}}$; the proof for $\dim_{\mathrm{Rok}}^{\mathrm{c}}$ is analogous and is left to the reader.

We check the conditions in part (1) of Lemma IV.2.7. Let $\varepsilon > 0$, let $S \subseteq C(G)$ be a finite subset, let $F \subseteq B$ be a compact subset, and set $d = \dim_{\mathrm{Rok}}(\alpha)$. Regard $F$ as a compact subset of $A$ and choose completely positive contractive maps

$$\varphi_0, \ldots, \varphi_d\colon C(G) \to A$$

satisfying conditions (a) through (d) in part (1) of Lemma IV.2.7 for $\alpha$ with $S \cup \{p\}$ in place of $S$, and $\varepsilon/3$ in place of $\varepsilon$. Define completely positive contractive maps

$$\psi_0, \ldots, \psi_d\colon C(G) \to B = pAp$$

by $\psi_j(f) = p\varphi_j(f)p$ for $f \in C(G)$ and $j = 0, \ldots, d$. One readily checks that these maps satisfy the desired conditions for $\beta$, $S$, $F$ and $\varepsilon$. We omit the details. $\square$

Recall that a $C^*$-algebra $A$ is said to be *homogeneous* if there are positive integers $m, k_1, \ldots, k_m$, compact Hausdorff spaces $X_1, \ldots, X_{n_m}$, and projections $p_j \in M_{k_j}(C(X_j))$ such that

$$A \cong p_1 M_{k_1}(C(X_1))p_1 \oplus \cdots \oplus p_m M_{k_m}(C(X_m))p_m.$$

A $C^*$-algebra is said to be an *approximately homogeneous algebra*, or *AH-algebra* for short, if it is isomorphic to a direct limit of homogeneous algebras.

An AH-algebra $A$ is said to have *no dimension growth* if there exists an AH-decomposition of it for which the compact Hausdorff spaces appearing in the building blocks have uniformly bounded (covering) dimension. (AH-algebras of no dimension growth may have AH-decompositions that do not witness the fact that it has no dimension growth.)

We will consider a special kind of compact group actions on AH-algebras of no dimension growth:

**Definition V.5.2.** Let $A$ be a unital AH-algebra and let

$$\left(A_n = p_1^{(n)} M_{k_1^{(n)}}\left(C(X_1^{(n)})\right) p_1^{(n)} \oplus \cdots \oplus p_{m^{(n)}}^{(n)} M_{k_{m^{(n)}}^{(n)}}\left(C(X_{m^{(n)}}^{(n)})\right) p_{m^{(n)}}^{(n)}, \phi_n\right)_{n\in\mathbb{N}}$$

be an AH-decomposition of $A$, with unital connecting maps $\phi_n \colon A_n \to A_{n+1}$. Given a compact group $G$, we construct a direct limit action of $G$ on $A$ as follows. For $n \in \mathbb{N}$, consider actions

$$\gamma_{k_j}^{(n)} \colon G \to \mathrm{Aut}\left(M_{k_j^{(n)}}\right) \quad \text{and} \quad \delta_{k_j}^{(n)} \colon G \to \mathrm{Homeo}\left(X_j^{(n)}\right)$$

for $j = 1, \ldots, m^{(n)}$. Give $M_{k_j^{(n)}}\left(C\left(X_j^{(n)}\right)\right)$ the diagonal $G$-action. Let $\alpha^{(n)}$ be the $G$-action on $A_n$ obtained as the direct sum of such actions. Assume that

$$\alpha^{(n+1)} \circ \phi_n = \phi_n \circ \alpha^{(n)}$$

for all $n \in \mathbb{N}$, and set $\alpha = \varinjlim \alpha^{(n)} \colon G \to \mathrm{Aut}(A)$.

Actions on AH-algebras constructed in this way will be called *AH-actions*. An AH-action $\alpha \colon G \to \mathrm{Aut}(A)$ is said to have *no dimension growth*, if $A$ has no dimension growth and there is an AH-decomposition of $(A, \alpha)$ which witnesses the fact that $A$ has no dimension growth. Finally, $\alpha$ is said to be a *free* AH-action with no dimension growth, if $A$ has no dimension growth and there is an AH-decomposition of $(A, \alpha)$ which witnesses the fact that $A$ has no dimension growth, and where the resulting $G$-actions on the compact Hausdorff spaces are free.

We now give some applications of our main results. In our examples, particularly the first two, it is usually easier to obtain information about the fixed point algebras. One has to nevertheless prove some form of saturation of the actions involved, to deduce that the crossed product and the fixed point algebra are Morita equivalent, in order to obtain useful information about the crossed product. Although one may be able to prove this in concrete examples, our argument is applicable in situations where one cannot even say much about the algebra of fixed points, such as Example V.5.5 and Example V.5.6.

Throughout, we fix a compact Lie group $G$.

**Example V.5.3.** Let $X$ be a compact Hausdorff space and let $k \in \mathbb{N}$. Let $G$ act freely on $X$ via $\delta\colon G \to \mathrm{Aut}(C(X))$, and let $\gamma\colon G \to \mathrm{Aut}(M_k)$ be an arbitrary action. Let $\beta\colon G \to \mathrm{Aut}(M_k(C(X))$ denote the diagonal action. Let $p \in M_k(C(X))$ be an invariant projection, and denote by $\alpha\colon G \to \mathrm{Aut}(pM_k(C(X))p)$ the compression of $\beta$. Using Proposition V.5.1 at the first step, part (1) of Theorem IV.2.8 at the second step, and part (2) of Theorem IV.3.5 at the third step, we get

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\beta) \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\delta) < \infty.$$

(Even if $X$ is infinite dimensional.)

When $\dim(X) < \infty$, it follows from Theorem V.3.3 and Theorem V.3.4 that

$$\mathrm{dr}\left(pM_k(C(X))p \rtimes_\alpha G\right) \leq (\dim(X)+1)(\dim(X)-\dim(G)+1)-1$$

and

$$\dim_{\mathrm{nuc}}\left(pM_k(C(X))p \rtimes_\alpha G\right) \leq (\dim(X)+1)(\dim(X)-\dim(G)+1)-1.$$

**Example V.5.4.** Let $A$ be a unital AH-algebra with no dimension growth, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a free AH-action with no dimension growth. Denote by $X_j^{(n)}$, with $1 \leq j \leq m^{(n)}$ and $n \in \mathbb{N}$, the compact Hausdorff spaces arising in some AH-decomposition of $\alpha$ witnessing the fact that it is free with no dimension growth. Set

$$M = \sup_{n\in\mathbb{N}} \sup_{1\leq j\leq m^{(n)}} \dim\left(X_j^{(n)}\right) < \infty.$$

Using Example V.5.3 and part (3) of Theorem IV.2.8, we deduce that

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq M - \dim(G).$$

We conclude from Theorem V.3.3 and Theorem V.3.4 that

$$\mathrm{dr}\left(A \rtimes_\alpha G\right) \leq (M+1)(M-\dim(G)+1)-1, \quad \text{and}$$
$$\dim_{\mathrm{nuc}}\left(A \rtimes_\alpha G\right) \leq (M+1)(M-\dim(G)+1)-1.$$

In addition, if $A$ is $\mathcal{Z}$-stable (for example, if it is simple; see Theorem 6.1 in [45]), then $A \rtimes_\alpha G$ is also $\mathcal{Z}$-stable, by Theorem V.4.4.

We remark that in the following example, finite dimensionality of $X$ is not needed to deduce $\mathcal{Z}$-stability of the crossed product $C(X,B) \rtimes_\alpha G$ as long as $B$ is $\mathcal{Z}$-stable. In particular, and even if $C(X,B) \rtimes_\alpha G$ can be shown to be a locally trivial bundle over $X/G$ with fiber $B \otimes \mathcal{K}(L^2(G))$, absorption of the Jiang-Su algebra cannot be concluded from the results in [121] since the base space is allowed to be infinite dimensional.

**Example V.5.5.** Let $X$ be a compact Hausdorff space, and let $B$ be a separable, unital $C^*$-algebra. (We do not assume $B$ to be simple or even nuclear.) Let $\delta\colon G \to \mathrm{Aut}(C(X))$ be induced by a free action of $G$ on $X$, and let $\beta\colon G \to \mathrm{Aut}(B)$ be an arbitrary action. If $\alpha\colon G \to \mathrm{Aut}(C(X,B))$ denotes the diagonal action, then

$$\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq \dim_{\mathrm{Rok}}^{\mathrm{c}}(\delta) < \infty$$

by Theorem IV.3.5.

In particular, if $\dim(X) < \infty$ and $B$ has finite decomposition rank or nuclear dimension, then so does $C(X,B) \rtimes_\alpha G$. On the other hand, if $B$ is $\mathcal{Z}$-stable, and regardless of whether $X$ is finite dimensional, then $C(X,B) \rtimes_\alpha G$ is $\mathcal{Z}$-stable.

In the following example, one could replace the free AH-action action of no dimension growth with any other action with finite Rokhlin dimension (with commuting towers for the conclusion involving $\mathcal{Z}$-stability).

**Example V.5.6.** Let $A$ be a unital AH-algebra with no dimension growth, let $\alpha\colon G \to \mathrm{Aut}(A)$ be a free AH-action with no dimension growth, and let $B$ be a separable, unital $C^*$-algebra. (We do not assume $B$ to be simple or even nuclear.) Let $\beta\colon G \to \mathrm{Aut}(B)$ be an arbitrary action, and denote by $\gamma\colon G \to \mathrm{Aut}(A \otimes B)$ be the diagonal $G$-action. By part (1) of Theorem IV.2.8 and Example V.5.4, $\gamma$ has finite Rokhlin dimension with commuting towers.

It follows that $(A \otimes B) \rtimes_\gamma G$ has finite nuclear dimension or decomposition rank whenever $B$ does, independently of $\beta$. Additionally, $(A \otimes B) \rtimes_\gamma G$ is $\mathcal{Z}$-stable whenever $A$ is $\mathcal{Z}$-stable (for example, when $A$ is simple), and independently of $B$ and $\beta$.

## Open Problems

In this last section, we give some indication of possible directions for future work, and raise some natural questions related to our findings. Some of these questions will be addressed in [87].

Theorem V.4.4 and Corollary 3.4 in [122] suggest that the following conjecture may be true.

**Conjecture V.6.1.** Let $G$ be a second countable compact group, let $A$ be a separable unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of $G$ on $A$ with finite Rokhlin dimension with commuting towers. Let $\mathcal{D}$ be a strongly self-absorbing $C^*$-algebra and suppose that $A$ is $\mathcal{D}$-absorbing. Then $A \rtimes_\alpha G$ is $\mathcal{D}$-absorbing.

We point out that the corresponding result for noncommuting towers is not in general true. Indeed, Example IV.3.10 shows that $\mathcal{O}_2$-absorption is not preserved. It moreover shows that absorption of UHF-algebras other than $M_{2^\infty}$ is also not preserved. For $M_{2^\infty}$-absorption, one may adapt the construction of Izumi to produce a $\mathbb{Z}_3$-action $\beta$ on $\mathcal{O}_2$ with $\dim_{\mathrm{Rok}}(\beta) = 1$, such that $K_*(\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_3)$ is not uniquely 2-divisible. (See, for example, Section XII.6.)

One should also explore preservation of other structural properties besides $\mathcal{D}$-absorption.

**Problem V.6.2.** Can one generalize some of the parts of Theorem 2.6 in [202] to compact (or finite) group actions with finite Rokhlin dimension with commuting towers?

As pointed out before, one should not expect much if only noncommuting towers are assumed (except for (1) and (8) – without UCT, which are true for arbitrary pointwise outer actions of discrete amenable groups). Also, it is probably easy to construct counterexamples to several parts of Theorem 2.6 in [202] even in the case of finite Rokhlin dimension with commuting towers.

We provide one such counterexample here.

**Proposition V.6.3.** There exist a unital $C^*$-algebra $A$ and an action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ with $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) = 1$, such that the map $K_*(A^\alpha) \to K_*(A)$ induced by the canonical inclusion $A^\alpha \to A$, is not injective. In particular, Theorem 3.13 of [132] (where simplicity of $A$ is not needed – see Theorem VI.3.3) does not hold for finite group actions with finite Rokhlin dimension with commuting towers.

*Proof.* Denote by $\mathbb{RP}^2$ the real projective plane, and set $X = \mathbb{T}\times\mathbb{RP}^2$. Define an action $\alpha$ of $\mathbb{Z}_2$ on $X$ via $(\zeta, r) \mapsto (-\zeta, r)$ for all $(\zeta, r) \in X$. Then $\alpha$ is the restriction of the product action $\mathtt{Lt}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{RP}^2}$

of $\mathbb{T}$ on $X$. Since this product action has the Rokhlin property, it follows from Theorem 3.10 in [83] that $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) \leq 1$. Since $X$ has no non-trivial projections, it must be $\dim_{\mathrm{Rok}}^{\mathrm{c}}(\alpha) = 1$.

One has $X/\mathbb{Z}_2 \cong (\mathbb{T}/\mathbb{Z}_2) \times \mathbb{RP}^2$, and the canonical quotient map $\pi\colon X \to X/\mathbb{Z}_2$ is given by $\pi(\zeta, r) = (\zeta^2, r)$ for $(\zeta, r) \in X$. The induced map

$$K^1(\pi)\colon K^0(\mathbb{RP}^2) \oplus K^1(\mathbb{RP}^2) \to K^0(\mathbb{RP}^2) \oplus K^1(\mathbb{RP}^2)$$

is easily seen to be given by $K^1(\pi)(a, b) = (2a, b)$ for $(a, b) \in K^0(\mathbb{RP}^2) \oplus K^1(\mathbb{RP}^2)$. Since $K^0(\mathbb{RP}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, we conclude that $K^1(\pi)$ is not injective, and hence neither is $K_1(\iota)$. $\square$

# CHAPTER VI

# THE ROKHLIN PROPERTY FOR COMPACT GROUP ACTIONS

We study compact group actions with the Rokhlin property. This is a strong form of noncommutative freeness, since already the augmentation ideal in the group ring annihilates the equivariant $K$-theory of such actions. Additionally, our characterization of commutative dynamical systems with the Rokhlin property illustrates how rigid this notion is. Despite its restrictive nature, we show that the Rokhlin property is in some cases generic, and provide a family of examples on simple AH-algebras with no dimension growth.

We study the canonical inclusion of the fixed point algebra at the level of $K$-theory: the induced map is an order embedding, and it has a splitting whenever it is restricted to finitely generated subgroups. We establish similar properties for the induced map on the Cuntz semigroup. Dual actions of actions with the Rokhlin property can be completely characterized, extending results of Izumi in the finite group case. Finally, we establish some connections between the Rokhlin property and (equivariant) semiprojectivity. As an application, it is shown that every Rokhlin action of a compact Lie group of dimension at most one, is a dual action.

A number of our results, particularly those in the last two sections, are new even in the much better understood case of finite groups.

## Introduction

The purpose of this chapter, which is based on [85], is to provide a systematic study of compact group actions with the Rokhlin property, with a spirit similar to the one in Chapter IV, where we explored compact group actions with finite Rokhlin dimension. To a certain extent, what we do here is also motivated by the book [199], in that the Rokhlin property is a strong form of noncommutative freeness. The results contained in this chapter, we hope, provide the necessary technical tools to attack problems in which the Rokhlin property of a compact group action can be proved to have a relevant role. For example, the Rokhlin property for a particular action of $\mathbb{Z}_3$ was used in [214] to compute, among other things, the Elliott invariant and the Cuntz semigroup of a certain simple separable exact $C^*$-algebra not anti-isomorphic to itself. Similarly, the tracial

Rokhlin property played a crucial role in Phillips' proof [201] that all simple, higher dimensional noncommutative tori are $A\mathbb{T}$-algebras.

The approach used in this work yields new information even for actions of finite groups. Indeed, a number of our results, particularly those in Sections 6.3 and 6.4, had not been noticed even in the well-studied case of finite groups.

This chapter is structured as follows. In Section VI.2, we recall the definition of the Rokhlin property for a compact group action, and establish some permanence properties (Theorem VI.2.3 and Proposition VI.2.4), which are consequences of more general results in Chapter IV. These results are complemented by Theorem VI.2.6, which shows that for actions on the Cuntz algebra $\mathcal{O}_2$, the Rokhlin property passes to (finite) subgroups. We also construct examples of compact group actions with the Rokhlin property on several simple $C^*$-algebras. One of the main results of this section, Corollary VI.2.21, asserts that for compact group actions on a unital, separable $\mathcal{O}_2$-absorbing $C^*$-algebra, the Rokhlin property is generic. A similar results holds when one replaces $\mathcal{O}_2$ with any other strongly self-absorbing $C^*$-algebra $\mathcal{D}$, provided that there exists at least one action of $G$ on $\mathcal{D}$ with the Rokhlin property (which is not always the case for $\mathcal{D} \not\cong \mathcal{O}_2$).

In Section VI.3, we study the $K$-theory and Cuntz semigroup of fixed point algebras and crossed products, as well as the equivariant $K$-theory. In Theorem VI.3.3, we show that $K_0(A \rtimes_\alpha G)$ can be naturally identified with a subgroup of $K_0(A)$, and that this subgroup is a direct summand if $K_0(A)$ is finitely generated. These results, as well as the main technical device used to prove them (Theorem VI.3.1), seem not to have been noticed even in the context of finite group actions (with the exception of part (1) in Theorem VI.3.3, which was proved in the simple case by Izumi; see Theorem 3.13 in [132]). Actions with the Rokhlin property are shown in Theorem VI.3.9 to have a strong form of discrete $K$-theory (which again suggests that the Rokhlin property is a rather strong form of freeness). In Theorem VI.3.10, we show that the Cuntz semigroup of $A \rtimes_\alpha G$ is naturally order isomorphic to a subsemigroup of the Cuntz semigroup of $A$. Simplicity of the crossed product and fixed point algebra is established in Proposition VI.3.14.

In Section VI.4, we explore duality and other applications involving equivariant semiprojectivity. In Theorem VI.4.2, we characterize the dual or predual actions of actions with the Rokhlin property, as those actions that are approximately representable (Definition VI.4.1).

In Corollary VI.4.7, we use equivariant semiprojectivity to show that if $G$ is a compact, abelian Lie group with $\dim(G) \leq 1$, then every action of $G$ with the Rokhlin property is a dual action. To our surprise, this result is new even in the case of finite groups. The techniques in this section are used in Theorem VI.4.9 to characterize those topological dynamical systems with the Rokhlin property.

Groups will be assumed to be second countable, and in particular metrizable. It is a standard fact that compact metrizable groups admit a (left) translation invariant metric.

### Permanence Properties, Examples and Genericity

If $G$ is a locally compact group, we denote by $\mathtt{Lt}\colon G \to \mathrm{Aut}(C_0(G))$ the action induced by left translation of $G$ on itself. In some situations (particularly in Theorem VI.4.9), we make a slight abuse of notation and also denote by $\mathtt{Lt}$ the action of $G$ on itself by left translation.

The following is essentially Definition 3.2 of [122].

**Definition VI.2.1.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. We say that $\alpha$ has the *Rokhlin property* if there is an equivariant unital homomorphism

$$\varphi\colon (C(G), \mathtt{Lt}) \to (A_{\infty,\alpha} \cap A', \alpha_\infty).$$

**Remark VI.2.2.** Definition VI.2.1 is formally weaker than Definition 3.2 in [122], since we do not require the map $\varphi$ to be injective. However, this condition is automatic: the kernel of $\varphi$ is a translation invariant ideal in $C(G)$, so it must be either $\{0\}$ or all of $C(G)$. It follows that the two notions are in fact equivalent.

Since unital completely positive maps of order zero are necessarily homomorphisms, it is easy to see that the Rokhlin property for a compact group action agrees with Rokhlin dimension zero in the sense of Definition IV.2.2. In particular, the following is a consequence of Theorem IV.2.8 and Proposition V.5.1.

**Theorem VI.2.3.** Let $A$ be a unital $C^*$-algebra let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action of $G$ on $A$.

1. Let $B$ be a unital $C^*$-algebra, and let $\beta\colon G \to \mathrm{Aut}(B)$ be a continuous action of $G$ on $B$. Let $A \otimes B$ be any $C^*$-algebra completion of the algebraic tensor product of $A$ and $B$ for which the tensor product action $g \mapsto (\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$ is defined. If $\alpha$ has the Rokhlin property, then so does $\alpha \otimes \beta$.

2. Let $I$ be an $\alpha$-invariant ideal in $A$, and denote by $\overline{\alpha}\colon G \to \mathrm{Aut}(A/I)$ the induced action on $A/I$. If $\alpha$ has the Rokhlin property, then so does $\overline{\alpha}$.

3. Suppose that $\alpha$ has the Rokhlin property and let $p$ be an $\alpha$-invariant projection in $A$. Set $B = pAp$ and denote by $\beta\colon G \to \mathrm{Aut}(B)$ the compressed action of $G$. Then $\beta$ has the Rokhlin property.

Furthermore,

4. Let $(A_n, \iota_n)_{n\in\mathbb{N}}$ be a direct system of unital $C^*$-algebras with unital connecting maps, and for each $n \in \mathbb{N}$, let $\alpha^{(n)}\colon G \to \mathrm{Aut}(A_n)$ be a continuous action such that $\iota_n \circ \alpha_g^{(n)} = \alpha_g^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $g \in G$. Suppose that $A = \varinjlim A_n$ and that $\alpha = \varinjlim \alpha^{(n)}$. If $\alpha^{(n)}$ has the Rokhlin property for infinitely many values of $n$, then $\alpha$ has the Rokhlin property as well.

It is not in general the case that the Rokhlin property for compact group actions is preserved by restricting to a closed subgroup. The reader is referred to Section IV.2 for a discussion about the interaction between Rokhlin dimension and restriction to closed subgroups.

The Rokhlin property is nevertheless preserved by passing to a subgroup in some special cases.

**Proposition VI.2.4.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $H$ be a closed subgroup of $G$, and assume that at least one of the following holds:

1. The coset space $G/H$ is zero dimensional (for example, if $[G : H] < \infty$).

2. We have $G = \prod\limits_{i\in I} G_i$ or $G = \bigoplus\limits_{i\in I} G_i$, and $H = G_j$ for some $j \in I$.

3. The subgroup $H$ is the connected component of $G$ containing its unit.

Then $\alpha|_H\colon H \to \mathrm{Aut}(A)$ has the Rokhlin property.

*Proof.* This follows immediately from Proposition IV.2.10. $\square$

Note that in the proposition above, we did not make any assumptions on the $C^*$-algebra in question, but rather on the subgroup. In the following theorem, we assume that the $C^*$-algebra is the Cuntz algebra $\mathcal{O}_2$, but the subgroups we consider are arbitrary finite subgroups. We need an easy lemma first. Its proof is standard, and we include it here for the sake of completeness.

**Lemma VI.2.5.** Let $A$ be a $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Then there is a canonical identification

$$(A^\alpha)_\infty \cap A' = (A_{\infty,\alpha} \cap A')^{\alpha_\infty}.$$

*Proof.* It is clear that $(A^\alpha)_\infty \cap A' \subseteq (A_{\infty,\alpha} \cap A')^{\alpha_\infty}$. Conversely, let $x \in (A_{\infty,\alpha} \cap A')^{\alpha_\infty}$ be given. Choose a bounded sequence $(a_n)_{n\in\mathbb{N}} \in \ell^\infty_\alpha(\mathbb{N}, A)$ such that $\kappa_A((a_n)_{n\in\mathbb{N}}) = x$. Then

$$\lim_{n\to\infty} \max_{g\in G} \|\alpha_g(a_n) - a_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|a_n a - a a_n\| = 0$$

for all $a \in A$.

Denote by $\mu$ the normalized Haar measure on $G$, and for $n \in \mathbb{N}$, set

$$b_n = \int_G \alpha_g(a_n)\ d\mu(g).$$

Then $b_n \in A^\alpha$ and $\lim\limits_{n\to\infty} \|a_n - b_n\| = 0$. In particular, $(b_n)_{n\in\mathbb{N}}$ is in $\ell^\infty_\alpha(\mathbb{N}, A)^{\alpha^\infty}$, and it is clear that $\kappa((b_n)_{n\in\mathbb{N}}) = x$. The result follows. ∎

**Theorem VI.2.6.** Let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(\mathcal{O}_2)$ be an action with the Rokhlin property. Let $H$ be a finite subgroup of $G$. Then $\alpha|_H\colon H \to \mathrm{Aut}(\mathcal{O}_2)$ has the Rokhlin property.

*Proof.* Set $\beta = \alpha|_H\colon H \to \mathrm{Aut}(\mathcal{O}_2)$. Since $\alpha$ is pointwise outer by Theorem IV.3.16, so is $\beta$. The crossed product $\mathcal{O}_2 \rtimes_\beta H$ is therefore (separable, unital, nuclear) purely infinite and simple by Corollary 4.6 in [135] (here reproduced as part (2) of Theorem II.2.8), and it absorbs $\mathcal{O}_2$ by Theorem XI.2.16. Thus, we must have $\mathcal{O}_2 \rtimes_\beta H \cong \mathcal{O}_2$.

We wish to apply Theorem 4.2 in [132] (specifically the direction (3) implies (1)), and for this it will suffice to prove that there exists a unital embedding

$$\mathcal{O}_2 \to (\mathcal{O}_2{}^{\beta})_\infty \cap \mathcal{O}_2{}'.$$

By Lemma VI.2.5, we have $(\mathcal{O}_2{}^{\beta})_\infty \cap \mathcal{O}_2{}' = (\mathcal{O}_{2\infty} \cap \mathcal{O}_2{}')^{\beta}$, and hence the existence of such an embedding is guaranteed by Lemma XI.2.15. It follows that $\beta$ has the Rokhlin property, as desired. □

*Examples, non-existence, and genericity*

Compact group actions with the Rokhlin property are rare (and they seem to be even less common if the group is connected). In a forthcoming paper ([84]), we will show that there are many $C^*$-algebras of interest that do not admit any non-trivial compact group action with the Rokhlin property (such as the Cuntz algebra $\mathcal{O}_\infty$ and the Jiang-Su algebra $\mathcal{Z}$; see [120] for a stronger statement valid for compact Lie groups), while there are many $C^*$-algebras that only admit actions with the Rokhlin property of *totally disconnected* compact groups, such as the Cuntz algebras $\mathcal{O}_n$ for $n \geq 3$, UHF-algebras, AF-algebras, AI-algebras, etc. See Chapters X and XI for some non-existence results of circle actions with the Rokhlin property.

In this section, we shall construct a family of examples of compact group actions with the Rokhlin property on certain simple AH-algebras of no dimension growth, and on certain Kirchberg algebras, including $\mathcal{O}_2$. We also show that compact group actions on $\mathcal{O}_2$-absorbing $C^*$-algebras are generic, in a suitable sense; see Theorem VI.2.20.

**Example VI.2.7.** Given a second countable compact group $G$, the action $\texttt{Lt}\colon G \to \mathrm{Aut}(C(G))$ has the Rokhlin property, essentially by definition.

From the example above, we can construct more interesting ones using Theorem VI.2.3.

**Example VI.2.8.** Let $G$ be a second countable compact group. For $n \in \mathbb{N}$, set $A_n = C(G) \otimes M_{2^n}$. Set $\alpha^{(n)} = \texttt{Lt} \otimes \mathrm{id}_{M_{2^n}}\colon G \to \mathrm{Aut}(A_n)$. Then $\alpha^{(n)}$ has the Rokhlin property by part (1) of Theorem VI.2.3 and Example VI.2.7. Fix a countable subset $X = \{x_1, x_2, x_3, \ldots\}$ of $G$ such that

$\{x_m, x_{m+1}, \ldots\}$ is dense in $G$ for all $m \in \mathbb{N}$. Given $n \in \mathbb{N}$, define a map $\iota_n \colon A_n \to A_{n+1}$ by

$$\iota_n(f) = \begin{pmatrix} f & 0 \\ 0 & \mathtt{Lt}_{x_n}(f) \end{pmatrix}$$

for every $f$ in $A_n$. Then $\iota_n$ is unital and injective. The direct limit $A = \varinjlim(A_n, \iota_n)$ is clearly a unital AH-algebra of no dimension growth, and it is simple by Proposition 2.1 in [44].

It is easy to check that

$$\iota_n \circ \alpha_g^{(n)} = \alpha_g^{(n+1)} \circ \iota_n$$

for all $n \in \mathbb{N}$ and all $g \in G$, and hence $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $G$ on $A$. Then $\alpha$ has the Rokhlin property by part (3) of Theorem VI.2.3.

The (graded) $K$-theory of $A$ is easily seen to be $K_*(A) \cong K_*(C(G)) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. Additionally, $A^\alpha$ is isomorphic to the CAR algebra, and the inclusion $A^\alpha \to A$, at the level of $K_0$, induces the canonical embedding $\mathbb{Z}\left[\frac{1}{2}\right] \to K_0(C(G)) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ as the second tensor factor.

In the example above, the $2^\infty$ UHF pattern can be replaced by any other UHF or (simple) AF pattern, and the resulting $C^*$-algebra is also a (simple) AH-algebra with no dimension growth. If the group is totally disconnected, the direct limit algebra will be an AF-algebra. For non-trivial groups, these AF-algebras will nevertheless not be UHF-algebras, even if a UHF pattern is followed.

**Example VI.2.9.** Given a second countable compact group $G$, let $A$ and $\alpha$ be as in Example VI.2.8. Then

$$\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty} \colon G \to \mathrm{Aut}(A \otimes \mathcal{O}_\infty)$$

has the Rokhlin property by part (1) of Theorem VI.2.3, and $A \otimes \mathcal{O}_\infty$ is a Kirchberg algebra. One can obtain actions of $G$ on other Kirchberg algebras by following a different UHF or AF pattern in Example VI.2.8.

Using the absorption properties of $\mathcal{O}_2$, we can construct an action of the circle on $\mathcal{O}_2$ with the Rokhlin property.

**Example VI.2.10.** Let $G$ be a second countable compact group, and let $A$ and $\alpha$ be as in Example VI.2.8. Use Theorem 3.8 in [151] to choose an isomorphism $\varphi \colon A \otimes \mathcal{O}_2 \to \mathcal{O}_2$, and define an action $\beta\colon G \to \mathrm{Aut}(\mathcal{O}_2)$ by $\beta_g = \varphi \circ (\alpha_g \otimes \mathrm{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$ for $g \in G$. Then $\beta$ has the Rokhlin property by part (1) of Theorem VI.2.3.

More generally, the action constructed in Example VI.2.8 can be used to construct an action of $G$ on any $\mathcal{O}_2$-absorbing $C^*$-algebra.

In contrast, it follows from the following proposition that only finite groups act with the Rokhlin property on finite dimensional $C^*$-algebras (and, in this case, the action must be a permutation of the simple summands). The result is not surprising, but our proof allows us to show that the result is true even under the much weaker assumption that the action be pointwise outer. In particular, by Theorem IV.3.16, this applies to compact group actions with finite Rokhlin dimension (not necessarily with commuting towers).

**Proposition VI.2.11.** Let $G$ be a compact group, let $A$ be a finite dimensional $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. If $\alpha_g$ is outer for all $g \in G\setminus\{1\}$, then $G$ must be finite.

*Proof.* Choose positive integers $m, n_1, \ldots, n_m$ such that $A \cong \bigoplus\limits_{j=1}^{m} M_{n_j}$. Denote by $G_0$ the connected component of the identity of $G$. By Proposition XI.3.6, the restriction $\alpha|_{G_0}$ acts trivially on $K$-theory. This is easily seen to be equivalent to $\alpha_g(M_{n_j}) = M_{n_j}$ for all $g \in G_0$ and all $j = 1, \ldots, m$. Given $g \in G_0$, and since every automorphism of a matrix algebra is inner, it follows that $\alpha_g|_{M_{n_j}}$ is inner for all $j = 1, \ldots, m$. Hence $\alpha_g$ is inner, and we conclude that $G_0 = \{1\}$. In other words, $G$ is totally disconnected.

If $G$ is infinite, then there is a strictly decreasing sequence

$$G \supseteq H_1 \subseteq H_2 \supseteq \cdots \supseteq \{1\}$$

of infinite closed subgroups of $G$. Therefore there is an increasing sequence of inclusions

$$A^G \subseteq A^{H_1} \subseteq \cdots \subseteq A^{\{1\}} = A.$$

Since $A$ is finite dimensional, this sequence must stabilize, and hence there exists an infinite subgroup $H$ of $G$ such that $A^H = A$. We conclude that $\alpha_g = \mathrm{id}_A$ for all $g \in H$. This is a contradiction, and hence $G$ is finite. $\square$

The following technical definition will be needed in the next subsection.

**Definition VI.2.12.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. We say that $\alpha$ has the *strong Rokhlin property* if there exists a unital equivariant homomorphism

$$\varphi\colon (C(G), \mathtt{Lt}) \to (A_{\infty,\alpha} \cap A', \alpha_\infty)$$

whose image lands in the canonical copy of $A$ inside of $A_{\infty,\alpha}$.

**Remark VI.2.13.** It is easy to check that an action $\alpha\colon G \to \mathrm{Aut}(A)$ has the strong Rokhlin property if and only if for every finite subset $F \subseteq A$, for every finite subset $S \subseteq C(G)$, and for every $\varepsilon > 0$, there exists a unital equivariant homomorphism

$$\varphi\colon C(G) \to A$$

such that

$$\|\varphi(f)a - a\varphi(f)\| < \varepsilon$$

for all $f \in S$ and for all $a \in F$.

It follows from Proposition 5.26 in [205], that any action of a *finite* group with the Rokhlin property has the strong Rokhlin property. By Proposition 3.3 in [80], the same is true for circle actions. More generally, it will follow from Theorem VI.4.6 that the Rokhlin property is equivalent to the strong Rokhlin property for (abelian) compact Lie groups with $\dim(G) \leq 1$. We do not have an example of an action with the Rokhlin property that does not have the strong Rokhlin property, although we suspect it exists. On the other hand, for our purposes, all we need to know is that every group admits an action on $\mathcal{O}_2$ with the strong Rokhlin property.

**Lemma VI.2.14.** Let $G$ be a second countable compact group. Then there exists a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$ with the strong Rokhlin property.

*Proof.* By Theorem 3.8 in [151], it is enough to construct an action of $G$ on a simple, unital, separable, nuclear $C^*$-algebra, with the strong Rokhlin property. It is a straightforward exercise to check that the action constructed in Example VI.2.8 satisfies the desired condition. We omit the details. ☐

*The Rokhlin property is generic.*

In this subsection, we specialize to $\mathcal{O}_2$-absorbing $C^*$-algebras. We show that if $A$ is a separable, unital $C^*$-algebra absorbing $\mathcal{O}_2$ and if $G$ is a second countable compact group, then $G$-actions on $A$ with the Rokhlin property are generic, in a suitable sense; see Corollary VI.2.21. Specifically for totally disconnected groups, one can obtain a similar conclusion for $C^*$-algebras that absorb a UHF-algebra of a certain infinite type naturally associated to the group; see [84].

Throughout, $A$ will be a separable, unital $C^*$-algebra, and $G$ will be a second countable compact group.

**Definition VI.2.15.** Given an enumeration $X = \{a_1, a_2, \ldots\}$ of a countable dense subset of the unit ball of $A$, define metrics on $\mathrm{Aut}(A)$ by

$$\rho_X^{(0)}(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{\|\alpha(a_k) - \beta(a_k)\|}{2^k}$$

and

$$\rho_X(\alpha, \beta) = \rho_X^{(0)}(\alpha, \beta) + \rho_X^{(0)}(\alpha^{-1}, \beta^{-1})$$

for $\alpha, \beta \in \mathrm{Aut}(A)$.

Denote by $\mathrm{Act}_G(A)$ the set of all continuous actions of $G$ on $A$. Define a metric on $\mathrm{Act}_G(A)$ by

$$\rho_{G,S}(\alpha, \beta) = \max_{g \in G} \rho_X(\alpha_g, \beta_g),$$

for $\alpha, \beta \in \mathrm{Act}_G(A)$.

**Lemma VI.2.16.** For any enumeration $X$ as above, the function $\rho_{G,X}$ is a complete metric on $\mathrm{Act}_G(A)$.

*Proof.* Let $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Act}_G(A)$, so that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that, for every $n, m \geq n_0$, we have $\rho_{G,X}\left(\alpha^{(n)}, \alpha^{(m)}\right) < \varepsilon$. We want to show that there is $\alpha \in \mathrm{Act}_G(A)$ such that $\lim\limits_{n\to\infty} \rho_{G,X}\left(\alpha, \alpha^{(n)}\right) = 0$.

Given $g \in G$, we have $\rho_X\left(\alpha_g^{(n)}, \alpha_g^{(m)}\right) \leq \rho_{G,X}\left(\alpha^{(n)}, \alpha^{(m)}\right)$, and hence $\left(\alpha_g^{(n)}\right)_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{Aut}(A)$. By Lemma 3.2 in [206], the pointwise norm limit of the sequence $\left(\alpha_g^{(n)}\right)_{n\in\mathbb{N}}$ exists, and we denote it by $\alpha_g$. It also follows from Lemma 3.2 in [206] that $\alpha_g$ is an automorphism of $A$, with inverse $\alpha_{g^{-1}}$. Moreover, the map $\alpha\colon G \to \mathrm{Aut}(A)$ given by $g \mapsto \alpha_g$, is a group homomorphism, since it is the pointwise norm limit of group homomorphisms. It remains to check that it is continuous, and this follows using an $\frac{\varepsilon}{3}$ argument from the equation $\lim\limits_{n\to\infty} \left\|\alpha_g^{(n)}(a_k) - \alpha_g(a_k)\right\| = 0$ for all $k$ in $\mathbb{N}$, and the fact that $\alpha^{(n)}\colon G \to \mathrm{Aut}(A)$ is continuous for all $n \in \mathbb{N}$. We omit the details. □

**Notation VI.2.17.** Given a finite subset $F \subseteq A$, given a finite subset $S \subseteq C(G)$, and given $\varepsilon > 0$, we let $W_G(F, S, \varepsilon)$ denote the set of all actions $\alpha \in \mathrm{Act}_G(A)$ such that there exists a unital completely positive linear map $\varphi\colon C(G) \to A$ satisfying

1. $\|\varphi(f)a - a\varphi(f)\| < \varepsilon$ for all $a \in F$ and for all $f \in S$;
2. $\|\varphi(f_1f_2) - \varphi(f_1)(\varphi(f_2)\| < \varepsilon$ for all $f_1, f_2 \in S$; and
3. $\|\alpha_g(\varphi(f)) - \varphi(\mathtt{Lt}_g(f))\| < \varepsilon$ for all $g \in G$ and for all $f \in S$.

It is easy to check that an action $\alpha \in \mathrm{Act}_G(A)$ has the Rokhlin property if and only if it belongs to $W_G(F, S, \varepsilon)$ for all finite subsets $F \subseteq A$ and $S \subseteq C(G)$, and for all positive numbers $\varepsilon > 0$.

If $Z$ is a set, we denote by $\mathcal{P}_f(Z)$ the set of all finite subsets of $Z$. Note that $|\mathcal{P}_f(Z)| = |Z|$ if $Z$ is infinite.

**Lemma VI.2.18.** Let $X$ be a countable dense subset of the unit ball of $A$, and let $Y$ be a countable dense subset of the unit ball of $C(G)$. Then $\alpha \in \mathrm{Act}_G(A)$ has the Rokhlin property if and only if it belongs to the countable intersection

$$\bigcap_{F\in\mathcal{P}_f(X)} \bigcap_{S\in\mathcal{P}_f(Y)} \bigcap_{n=1}^{\infty} W_G\left(F, S, \frac{1}{n}\right).$$

*Proof.* One just needs to approximate any finite subset of $A$ by scalar multiples of elements in a finite subset of $X$, and likewise for finite subsets of $C(G)$. (We are implicitly using that both $A$ and $C(G)$ are separable.) We omit the details. □

For use in the following proposition, we denote by $d\colon G\times G \to \mathbb{R}$ a (left) invariant metric on $G$.

**Proposition VI.2.19.** Let $A$ and $\mathcal{D}$ be unital, separable $C^*$-algebras, such that there is an action $\gamma\colon G \to \mathrm{Aut}(\mathcal{D})$ with the strong Rokhlin property. Suppose that there exists an isomorphism $\theta\colon A \otimes \mathcal{D} \to A$ such that $a \mapsto \theta(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to $\mathrm{id}_A$. Then for every finite subset $F \subseteq A$ and every $\varepsilon > 0$, the set $W_G(F, S, \varepsilon)$ is open and dense in $\mathrm{Act}_G(A)$.

*Proof.* We first check that $W_G(F, S, \varepsilon)$ is open. Choose an enumeration $X = \{a_1, a_2, \ldots\}$ of a countable dense subset of the unit ball of $A$, and an enumeration $Y = \{f_1, f_2, \ldots\}$ of a dense subset of the unit ball of $C(G)$. Let $\alpha$ in $W_G(F, S, \varepsilon)$, and choose a unital completely positive linear map $\varphi\colon C(G) \to A$ satisfying

1. $\|\varphi(f)a - a\varphi(f)\| < \varepsilon$ for all $a \in F$ and for all $f \in S$;
2. $\|\varphi(fh) - \varphi(f)(\varphi(h)\| < \varepsilon$ for all $f, h \in S$; and
3. $\|\alpha_g(\varphi(f)) - \varphi(\mathtt{Lt}_g(f))\| < \varepsilon$ for all $g \in G$ and for all $f \in S$.

Set

$$\varepsilon_0 = \max_{g\in G} \max_{f\in S} \|\alpha_g(\varphi(f)) - \varphi(\mathtt{Lt}_g(f))\|,$$

so that $\varepsilon_1 = \varepsilon - \varepsilon_0$ is positive. For $f \in S$, let $k_f \in \mathbb{N}$ satisfy $\|a_{k_f} - \varphi(f)\| < \frac{\varepsilon_1}{3}$. Set $M = \max\limits_{f\in S} k_f$.

We claim that if $\alpha' \in \mathrm{Act}_G(A)$ satisfies $\rho_{G,X}(\alpha', \alpha) < \frac{\varepsilon_1}{2^M 3}$, then $\alpha'$ belongs to $W_G(F, S, \varepsilon)$.

Let $\alpha' \in \mathrm{Act}_G(A)$ satisfy $\rho_{G,X}(\alpha', \alpha) < \frac{\varepsilon_1}{2^M 3}$, and let $f \in S$. Then

$$\begin{aligned}
\|\alpha'_g(\varphi(f)) - \varphi(\mathtt{Lt}_g(f))\| &\leq \|\alpha'_g(\varphi(f)) - \alpha_g(\varphi(f))\| + \|\alpha_g(\varphi(f)) - \varphi(\mathtt{Lt}_g(f))\| \\
&\leq \frac{2\varepsilon_1}{3} + \|\alpha'_g(a_{k_f}) - \alpha_g(a_{k_f})\| + \varepsilon_0 \\
&\leq \frac{2\varepsilon_1}{3} + 2^M \rho_{G,X}(\alpha, \alpha') + \varepsilon_0 \\
&= \varepsilon_1 + \varepsilon_0 = \varepsilon.
\end{aligned}$$

The claim is proved. It follows that $W_G(F, S, \varepsilon)$ is open.

We will now show that $W_G(F, S, \varepsilon)$ is dense in $\mathrm{Act}_G(A)$. Let $\alpha$ be an arbitrary action in $\mathrm{Act}_G(A)$, let $E \subseteq A$ be a finite subset, and let $\delta > 0$. We want to find $\beta \in \mathrm{Act}_G(A)$ such that $\beta$ belongs to $W_G(F, S, \varepsilon)$ and $\|\alpha_g(a) - \beta_g(a)\| < \delta$ for all $g \in G$ and all $a \in E$.

Fix $0 < \delta' < \min\{\delta, \varepsilon\}$. Since $G$ is compact and second countable, it admits a left-invariant metric, which we denote by $d\colon G \times G \to \mathbb{R}$. Since $\alpha$ is continuous, there is $\delta_0 > 0$ such that whenever $g, g' \in G$ satisfy $d(g, g') < \delta_0$, then

$$\|\alpha_g(a) - \alpha_{g'}(a)\| < \frac{\delta'}{4}$$

for all $a \in E$. Choose $m \in \mathbb{N}$ and $g_1, \ldots, g_m \in G$, such that for every $g \in G$, there is $j \in \mathbb{N}$, with $1 \leq j \leq m$, satisfying $d(g, g_j) < \delta_0$. Choose $w \in \mathcal{U}(A)$ with

$$\|w\theta(1_A \otimes a)w^* - a\| < \frac{\delta'}{2}$$

for all $a \in E \cup \bigcup\limits_{j=1}^{m} \alpha_{g_j}(E)$. Set $\rho = \mathrm{Ad}(w) \circ \theta$ and define an action $\beta \in \mathrm{Act}_G(A)$ by

$$\beta_g = \rho \circ (\gamma_g \otimes \alpha_g) \circ \rho^{-1}$$

for $g \in G$.

We claim that $\beta$ belongs to $W_G(F, S, \varepsilon)$. Choose $r \in \mathbb{N}$, $d_1, \ldots, d_r \in \mathcal{D}$, and $x_1, \ldots, x_r \in A$, such that $w' = \sum\limits_{\ell=1}^{r} x_\ell \otimes d_\ell$ satisfies $\|w - w'\| < \frac{\delta}{3}$. Use the strong Rokhlin property of $\gamma$ to find a unital equivariant homomorphism $\varphi \colon C(G) \to \mathcal{D}$ such that

$$\|\varphi(f)d_\ell - d_\ell\varphi(f)\| < \frac{\varepsilon}{4}$$

for all $f \in S$ and for all $\ell = 1, \ldots, r$. Then

$$\|(1_A \otimes \varphi(f))w' - w'(1_A \otimes \varphi(f))\| < \frac{\delta}{3}$$

for all $f \in S$, and hence $\|(1_A \otimes \varphi(f))w - w(1_A \otimes \varphi(f))\| < \delta$. Define a unital homomorphism $\psi \colon C(G) \to A$ by

$$\psi(f) = \theta(1_A \otimes \varphi(f))$$

for $f \in C(G)$. Given $g \in G$ and $f \in S$, we have

$$\begin{aligned}
&\|\beta_g(\psi(f)) - \psi(\mathtt{Lt}_g(f))\| \\
&= \left\|w\theta\left((\alpha_g \otimes \gamma_g)(\theta^{-1}(w^*\theta(1_A \otimes \varphi(f))w))\right)w^* - \theta(1_A \otimes \varphi(\mathtt{Lt}_g(f)))\right\| \\
&\leq \left\|w\theta\left((\alpha_g \otimes \gamma_g)(\theta^{-1}(w^*\theta(1_A \otimes \varphi(f))w))\right)w^* - w\theta\left((\alpha_g \otimes \gamma_g)(1_A \otimes \varphi(f))\right)w^*\right\| \\
&\quad + \|w\theta\left((\alpha_g \otimes \gamma_g)(1_A \otimes \varphi(f))\right)w^* - \theta(1_A \otimes \varphi(\mathtt{Lt}_g(f)))\| \\
&< \frac{\delta'}{2} + \|w\theta\left(g1_A \otimes \varphi(f)\right)w^* - g\theta(1_A \otimes \varphi(f))\| \\
&< \frac{\delta'}{2} + \frac{\delta'}{2} = \delta' < \varepsilon,
\end{aligned}$$

and thus $\|\beta_g(\psi(f)) - \psi(\mathtt{Lt}_g(f))\| < \varepsilon$ for all $g \in G$ and for all $f \in S$. On the other hand, given $a \in F$ and $f \in S$, we use the identity

$$(a \otimes 1_{\mathcal{D}})(1_A \otimes \varphi(f)) = (1_A \otimes \varphi(f))(a \otimes 1_{\mathcal{D}})$$

at the third step, to obtain

$$\begin{aligned}
\|\psi(f)a - a\psi(f)\| &= \|\theta(1_A \otimes \varphi(f))a - a\theta(1_A \otimes \varphi(f))\| \\
&\leq \|\theta(1_A \otimes \varphi(f))a - \theta(1_A \otimes \varphi(f))w\theta(a \otimes 1_{\mathcal{D}})w^*\| \\
&\quad + \|\theta(1_A \otimes \varphi(f))w\theta(a \otimes 1_{\mathcal{D}})w^* - w\theta(a \otimes 1_{\mathcal{D}})w^*\theta(1_A \otimes \varphi(f))\| \\
&\quad + \|w\theta(a \otimes 1_{\mathcal{D}})w^*\theta(1_A \otimes \varphi(f)) - a\theta(1_A \otimes \varphi(f))\| \\
&< \frac{\delta'}{2} + 0 + \frac{\delta'}{2} = \delta' < \varepsilon.
\end{aligned}$$

This proves the claim.

It remains to prove that $\|\beta_g(a) - \alpha_g(a)\| < \delta$ for all $a$ in $E$ and all $g$ in $G$. For fixed $g \in G$, choose $j \in \{1, \ldots, m\}$ such that $d(g, g_j) < \delta_0$. Then, for $a \in E$, we have

$$\begin{aligned}
\|\beta_g(a) - \alpha_g(a)\| &= \left\|w\theta\left((\alpha_g \otimes \gamma_g)(\theta^{-1}(w^*aw))\right) - \alpha_g(a)\right\| \\
&\leq \left\|w\theta\left((\alpha_g \otimes \gamma_g)(\theta^{-1}(w^*aw))\right) - w\theta\left((\alpha_g \otimes \gamma_g)(a \otimes 1_{\mathcal{D}})\right)\right\| \\
&\quad + \left\|w\theta\left((\alpha_g \otimes \gamma_g)(a \otimes 1_{\mathcal{D}})\right) - \alpha_g(a)\right\| \\
&< \frac{\delta'}{2} + \|w\theta(\alpha_g(a) \otimes 1_{\mathcal{D}})w^* - \alpha_g(a)\| \\
&\leq \frac{\delta'}{2} + \|w\theta(\alpha_g(a) \otimes 1_{\mathcal{D}})w^* - w\theta(\alpha_{g_j}(a) \otimes 1_{\mathcal{D}})w^*\| \\
&\quad + \|w\theta(\alpha_{g_j}(a) \otimes 1_{\mathcal{D}})w^* - \alpha_{g_j}(a)\| + \|\alpha_{g_j}(a) - \alpha_g(a)\| \\
&< \frac{\delta'}{2} + \frac{\delta'}{4} + \frac{\delta'}{4} = \delta' < \delta.
\end{aligned}$$

This finishes the proof. □

**Theorem VI.2.20.** Let $A$ and $\mathcal{D}$ be unital, separable $C^*$-algebras, such that there is an action $\gamma\colon G \to \mathrm{Aut}(\mathcal{D})$ with the strong Rokhlin property. Suppose that there exists an isomorphism $\theta\colon A \otimes \mathcal{D} \to A$ such that $a \mapsto \theta(a \otimes 1_{\mathcal{D}})$ is approximately unitarily equivalent to $\mathrm{id}_A$. Then the set of actions of $G$ on $A$ with the Rokhlin property is a dense $G_\delta$-set in $\mathrm{Act}_G(A)$.

*Proof.* By Lemma VI.2.18, the set of all circle actions on $A$ that have the Rokhlin property is precisely the countable intersection

$$\bigcap_{F \in \mathcal{P}_f(X)} \bigcap_{S \in \mathcal{P}_f(Y)} \bigcap_{n \in \mathbb{N}} W_G\left(F, S, \frac{1}{n}\right).$$

By Proposition VI.2.19, each $W_G\left(F, S, \frac{1}{n}\right)$ is open and dense in $\mathrm{Act}_G(A)$, which is a complete metric space by Lemma VI.2.16. The result then follows from the Baire Category Theorem. □

Recall that a unital, separable $C^*$-algebra $\mathcal{D}$ is said to be *strongly self-absorbing*, if it is infinite dimensional and the map $\mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$, is approximately unitarily equivalent to an isomorphism. The only known examples are the Jiang-Su algebra $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF-algebras of infinite type, and tensor products of $\mathcal{O}_\infty$ by such UHF-algebras. (The reader is referred to [265] for more on strongly self-absorbing $C^*$-algebras.)

Out of the known examples of strongly self-absorbing $C^*$-algebras, $\mathcal{O}_\infty$ and $\mathcal{Z}$ do not admit any actions of any compact group with the Rokhlin property (see part (1) of Theorem 4.6 in [120] for a more general statement in the case of a Lie group; and see [84] for a proof specifically for the Rokhlin property, valid for all compact groups). On the other hand, only totally disconnected groups can act with the Rokhlin property on UHF-algebras, or their tensor products with $\mathcal{O}_\infty$; see [84]. Hence, only on $\mathcal{O}_2$ can we construct actions of an arbitrary compact group with the Rokhlin property, and, as it turns out, the Rokhlin property is generic in this case.

**Corollary VI.2.21.** Let $A$ be a separable unital $C^*$-algebra such that $A \otimes \mathcal{O}_2 \cong A$. Then the set of all circle actions on $A$ with the Rokhlin property is a dense $G_\delta$-set in $\mathrm{Act}_G(A)$.

*Proof.* By Lemma VI.2.14, there is an action $\gamma\colon G \to \mathrm{Aut}(\mathcal{O}_2)$ with the strong Rokhlin property. Since $A$ absorbs $\mathcal{O}_2$ tensorially, the hypotheses of Theorem VI.2.20 are met by Theorem 7.2.2 in [235], and the result follows. ☐

## $K$-theory and Cuntz Semigroups of Crossed Products

We begin this section by proving the main technical theorem that will be used in the proofs of essentially every other result in this section. Roughly speaking, Theorem VI.3.1 will allow us to take averages over the group $G$, in such a way that elements of the fixed point algebra are left fixed, and also such that $*$-polynomial relations in the algebra are approximately preserved.

**Theorem VI.3.1.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Given a compact subset $F \subseteq A$ and $\varepsilon > 0$, there exists a unital, continuous linear map $\psi\colon A \to A^\alpha$ satisfying

1. $\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon$ for all $a, b \in F$;
2. $\|\psi(a^*) - \psi(a)^*\| < \varepsilon$ for all $a \in F$;
3. $\|\psi(a)\| \leq 2\|a\|$ for all $a \in F$; and
4. $\psi(a) = a$ for all $a \in A^\alpha$.

If $A$ is separable, it follows that there exists an approximate homomorphism $(\psi_n)_{n\in\mathbb{N}}$ consisting of bounded, unital linear maps $\psi_n\colon A \to A^\alpha$ satisfying $\psi_n(a) = a$ for all $a \in A^\alpha$.

*Proof.* Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in F$. For the compact set $F$ and the tolerance $\varepsilon_0 = \frac{\varepsilon}{6}$, use Proposition V.4.3 to find a positive number $\delta > 0$, a finite subset $K \subseteq G$, a partition of unity $(f_k)_{k \in K}$ in $C(G)$, and a unital completely positive map $\varphi\colon C(G) \to A$, such that the following conditions hold:

(a) If $g$ and $g'$ in $G$ satisfy $d(g, g') < \delta$, then $\|\alpha_g(a) - \alpha_{g'}(a)\| < \varepsilon_0$ for all $a \in F$.

(b) Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} \neq 0$, then $d(k, k') < \delta$.

(c) For every $g \in G$ and for every $a \in F \cup F^*$, we have

$$\left\| \alpha_g \left( \sum_{k \in K} \varphi(f_k)\alpha_k(a) \right) - \sum_{k \in K} \varphi(f_k)\alpha_k(a) \right\| < \varepsilon_0.$$

(d) For every $a \in F \cup F^*$ and for every $k \in K$, we have

$$\|a\varphi(f_k) - \varphi(f_k)a\| < \frac{\varepsilon_0}{|K|} \quad \text{and} \quad \left\| a\varphi(f_k)^{\frac{1}{2}} - \varphi(f_k)^{\frac{1}{2}}a \right\| < \frac{\varepsilon_0}{|K|}.$$

(e) Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} = 0$, then

$$\|\varphi(f_k)\varphi(f_{k'})\| < \frac{\varepsilon_0}{|K|}.$$

Define a unital linear map $\psi\colon A \to A^\alpha$ by

$$\psi(a) = E\left( \sum_{k \in K} \varphi(f_k)\alpha_k(a) \right)$$

for all $a \in A$. We claim that $\psi$ has the desired properties. It is immediate that $\psi(a) = a$ for all $a \in A^\alpha$, using the properties of the conditional expectation $E$, so condition (4) is guaranteed.

We proceed to check condition (1) in the statement. Given $a, b \in F$, we use condition (c) at the second and fifth step, conditions (a), (b), (d) and (e) at the third step, and the fact that $\varphi$ is unital and $(f_k)_{k\in K}$ is a partition of unity of $C(G)$ at the fourth step, to get

$$\begin{aligned}
\psi(a)\psi(b) &= E\left(\sum_{k\in K}\varphi(f_k)\alpha_k(a)\right)E\left(\sum_{k'\in K}\varphi(f_{k'})\alpha_{k'}(b)\right)\\
&\approx_{2\varepsilon_0} \sum_{k\in K}\sum_{k'\in K}\varphi(f_k)\alpha_k(a)\varphi(f_{k'})\alpha_{k'}(b)\\
&\approx_{3\varepsilon_0} \sum_{k\in K}\sum_{k'\in K}\varphi(f_k)\alpha_k(ab)\varphi(f_{k'})\\
&= \sum_{k\in K}\varphi(f_k)\alpha_k(ab)\\
&\approx_{\varepsilon_0} E\left(\sum_{k\in K}\varphi(f_k)\alpha_k(ab)\right)\\
&= \psi(ab).
\end{aligned}$$

Hence $\|\psi(ab)-\psi(a)\psi(b)\| < 6\varepsilon_0 = \varepsilon$, and condition (1) is proved.

To prove condition (2), let $a \in F$. In the following computation, we use condition (d) at the second step, and the fact that $\varphi$ and $E$ are positive at the third step:

$$\begin{aligned}
\psi(a^*) &= E\left(\sum_{k\in K}\varphi(f_k)\alpha_k(a)^*\right)\\
&\approx_{\varepsilon_0} E\left(\sum_{k\in K}\alpha_k(a)^*\varphi(f_k)\right)\\
&= E\left(\sum_{k\in K}(\varphi(f_k)\alpha_k(a))^*\right)^* = \psi(a)^*.
\end{aligned}$$

Thus, condition (2) is verified.

To check condition (3), let $a \in F$. Then

$$\begin{aligned}
\psi(a) &= E\left(\sum_{k\in K}\varphi(f_k)\alpha_k(a)\right)\\
&\approx_{\varepsilon_0} E\left(\sum_{k\in K}\varphi(f_k)^{\frac{1}{2}}\alpha_k(a)\varphi(f_k)^{\frac{1}{2}}\right).
\end{aligned}$$

Since the assignment $a \mapsto E\left(\sum_{k\in K} \varphi(f_k)^{\frac{1}{2}}\alpha_k(a)\varphi(f_k)^{\frac{1}{2}}\right)$ defines a unital (completely) positive map $\widetilde{\psi}\colon A \to A^\alpha$, we conclude that

$$\|\psi(a)\| \leq \|\widetilde{\psi}(a)\| + \varepsilon_0 \leq \|a\| + \varepsilon_0 \leq 2\|a\|,$$

as desired. This finishes the proof. $\square$

*K-theory and fixed point algebras.*

Our first applications of Theorem VI.3.1 are, first, to the maps induced by the canonical inclusion $A^\alpha \hookrightarrow A$ at the level of $K$-theory (Theorem VI.3.3), and, second, to the equivariant $K$-theory $K_*^G(A)$ (Theorem VI.3.9).

We start off with an intermediate result. Recall that for two projections $p$ and $q$ in a $C^*$-algebra $B$, we say that $p$ is *Murray-von Neumann subequivalent (in B)* to $q$, written $p \precsim_{\mathrm{M-vN}} q$, if there exists a third projection $q_0 \in B$ such that $p \sim_{\mathrm{M-vN}} q_0$ and $q_0 \leq q$. Murray-von Neumann subequivalence is easily seen to be transitive. (We warn the reader that $p \precsim_{\mathrm{M-vN}} q$ and $q \precsim_{\mathrm{M-vN}} p$ do not in general imply $p \precsim_{\mathrm{M-vN}} q$.)

**Proposition VI.3.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $p, q \in A^\alpha$ be two projections.

1. Suppose that $p \precsim_{\mathrm{M-vN}} q$ in $A$. Then $p \precsim_{\mathrm{M-vN}} q$ in $A^\alpha$.

2. Suppose that $p \sim_{\mathrm{M-vN}} q$ in $A$. Then $p \sim_{\mathrm{M-vN}} q$ in $A^\alpha$.

*Proof.* (1). Find a projection $q_0 \in A$ such that $p \sim_{\mathrm{M-vN}} q_0$ and $q_0 \leq q$. Find a partial isometry $s \in A$ such that $s^*s = p$ and $ss^* = q_0$. For $\varepsilon = \frac{1}{21}$, find $\delta_0 > 0$ such that whenever $x \in A^\alpha$ satisfies $\|x^*x - x\| < \delta_0$, then there exists a projection $r \in A^\alpha$ such that $\|r - x\| < \varepsilon$. Set $\delta = \min\left\{\frac{\delta_0}{3}, \varepsilon\right\}$.

Find a unital, continuous linear map $\psi\colon A \to A^\alpha$ as in the conclusion of Theorem VI.3.1 for $\delta$ and $F = \{p, q, q_0, s, s^*\}$. Then

$$\begin{aligned}\|\psi(q_0)^*\psi(q_0) - \psi(q_0)\| &\leq \|\psi(q_0)^*\psi(q_0) - \psi(q_0^*)\psi(q_0)\| + \|\psi(q_0)\psi(q_0) - \psi(q_0^2)\| \\ &< 2\delta + \delta < \delta_0.\end{aligned}$$

By the choice of $\delta_0$, there exists a projection $\widetilde{q}_0$ in $A^\alpha$ such that $\|\psi(q_0) - \widetilde{q}_0\| < \varepsilon$. We claim that $\widetilde{q}_0 \precsim_{\mathrm{M-vN}} q$ in $A^\alpha$

We use the identity $q_0 = q_0 q$ at the second step to show that

$$\|\widetilde{q}_0 - \widetilde{q}_0 q\| \leq 2\|\widetilde{q}_0 - \psi(q_0)\| + \|\psi(q_0) - \psi(q_0)q\| < 2\varepsilon + \delta \leq 1.$$

It follows from Lemma 2.5.2 in [166] that $\widetilde{q}_0 \precsim_{\mathrm{M-vN}} q$ in $A^\alpha$.

Now we claim that $p \sim_{\mathrm{M-vN}} \widetilde{q}_0$ in $A^\alpha$. We use the identity $ps^* q_0 sp = p$ at the last step to show

$$\begin{aligned}\|(\widetilde{q}_0\psi(s)p)^*(\widetilde{q}_0\psi(s)p) - p\| &\leq \|p\psi(s)^*\widetilde{q}_0^2\psi(s)p - p\psi(s^*)\widetilde{q}_0^2\psi(s)p\| + 4\|\widetilde{q}_0 - \psi(q_0)\| \\ &\quad + \|\psi(p)\psi(s^*)\psi(q_0)\psi(s)\psi(p) - \psi(s^*s)\| \\ &< 2\delta + 4\varepsilon + 15\delta + \|\psi(ps^*q_0sp) - \psi(s^*s)\| < 21\varepsilon = 1.\end{aligned}$$

Likewise, $\|(\widetilde{q}_0\psi(s)p)(\widetilde{q}_0\psi(s)p)^* - q\| < 1$. By Lemma 2.5.3 in [166] applied to $\widetilde{q}_0\psi(s)p$, there exists a partial isometry $t$ in $A^\alpha$ such that $t^*t = p$ and $tt^* = \widetilde{q}_0$.

We conclude that

$$p \sim_{\mathrm{M-vN}} \widetilde{q}_0 \precsim_{\mathrm{M-vN}} q$$

in $A^\alpha$, so the proof is complete.

(2). The proof of this part is analogous, but simpler. We therefore omit the details. ☐

Part (1) of the following theorem generalizes Theorem 3.13 in [132] in two ways: we do not assume our algebras to be simple, and we consider compact groups. The remaining parts are new even when $G$ is finite.

**Theorem VI.3.3.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the following assertions hold:

1. The canonical inclusion $\iota\colon A^\alpha \to A$ induces an injective map

$$K_*(\iota)\colon K_*(A^\alpha) \to K_*(A).$$

2. The map $K_0(\iota)\colon K_0(A^\alpha) \to K_0(A)$ is an order embedding; that is, whenever $x, y \in K_0(A^\alpha)$ satisfy $K_0(\iota)(x) \leq K_0(\iota)(y)$ in $K_0(A)$, then $x \leq y$ in $K_0(A^\alpha)$.

3. Let $j \in \{0, 1\}$, and let $H$ be a finitely generated subgroup of $K_j(A)$. Then there exists a group homomorphism $\pi\colon H \to K_0(A^\alpha)$ such that

$$\pi \circ \iota|_{K_0(\iota)^{-1}(H)} = \mathrm{id}_{K_0(\iota)^{-1}(H)}.$$

*Proof.* (1). The result for $K_1$ follows from the result for $K_0$, by tensoring with any unital $C^*$-algebra $B$ satisfying the UCT with $K$-theory $(0, \mathbb{Z})$, and using the Künneth formula. We will therefore only prove the theorem for $K_0$.

Let $x \in K_0(A^\alpha)$ satisfy $K_0(\iota)(x) = 0$. Choose $n \in \mathbb{N}$ and projections $p, q \in M_n(A^\alpha)$ such that $x = [p] - [q]$. Then $[p] = [q]$ in $K_0(A)$. Without loss of generality, we may assume that $p$ and $q$ are Murray-von Neumann equivalent in $M_n(A)$. Since the action $\alpha \otimes \mathrm{id}_{M_n}$ of $G$ on $M_n(A)$ has the Rokhlin property by part (1) of Theorem VI.2.3, it follows from part (2) of Proposition VI.3.2 that $p$ and $q$ are Murray-von Neumann equivalent in $M_n(A^\alpha)$. Hence $x = 0$ and $K_0(\iota)$ is injective.

(2). Let $x, y \in K_0(A^\alpha)$ and suppose that $K_0(\iota)(x) \leq K_0(\iota)(y)$. Choose $n \in \mathbb{N}$ and projections $p, q, e$, and $f$ in $M_n(A^\alpha)$, such that $x = [p]-[q]$ and $y = [e]-[f]$. Then $[p]+[f] \leq [e]+[q]$ in $K_0(A)$. Without loss of generality, we can assume that $p \oplus f \precsim_{\mathrm{M-vN}} e \oplus q$ in $M_{2n}(A)$. Since the action induced by $\alpha$ on $M_{2n}(A)$ has the Rokhlin property by part (1) of Theorem VI.2.3, it follows from part (2) of Proposition VI.3.2 that $p \oplus f \precsim_{\mathrm{M-vN}} e \oplus q$ in $M_{2n}(A^\alpha)$. Hence $[p] + [f] \leq [e] + [q]$ in $K_0(A^\alpha)$ and thus $x \leq y$ in $K_0(A^\alpha)$, as desired.

(3). We prove the statement only for $K_0$, which without loss of generality we assume is not the trivial group. So let $H$ be a finitely generated subgroup of $K_0(A)$, and choose an integer $m \geq 0$, and $k_0, \ldots, k_m \in \mathbb{N}$ such that $H$ is isomorphic to $\mathbb{Z}^{k_0} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_m}$ as abelian groups. For $1 \leq i \leq k_0$ and $1 \leq j \leq m$, choose projections

$$p_{0,i}, q_{0,i}, p_j, q_j \in M_\infty(A),$$

such that $[p_{0,i}] - [q_{0,i}]$ is a (free) generator of the $i$-th copy of $\mathbb{Z}$; and $[p_j] - [q_j]$ is a generator (of order $k_j$) of $\mathbb{Z}_{k_j}$. Without loss of generality, we may assume that there are unitaries $u_j \in M_\infty(A)$, for $j = 1, \ldots, m$, such that

$$u_j \mathrm{diag}(p_j, \ldots, p_j) u_j^* = \mathrm{diag}(q_j, \ldots, q_j).$$

(In the equation above, there are $k_j$ repetitions of $p_j$ and $q_j$.)

For $i = 1, \ldots, k_0$, let $\mathbb{Z}^{(i)}$ denote the $i$-th copy of $\mathbb{Z}$ in $H$. Then the intersection $\mathbb{Z}^{(i)} \cap K_0(\iota)(K_0(A^\alpha))$ is a subgroup of $\mathbb{Z}^{(i)}$, so there exists $n_{i,0} \geq 0$ such that

$$\mathbb{Z}^{(i)} \cap K_0(\iota)(K_0(A^\alpha)) = n_i \mathbb{Z}^{(i)}.$$

Find projections $r_{i,0}, s_{i,0} \in M_\infty(A^\alpha)$ such that

$$[r_{i,0}] - [s_{i,0}] = n_i([p_{0,i}] - [q_{0,i}]) \in K_0(A).$$

Without loss of generality, we may assume that there exists a unitary $v_{i,0} \in M_\infty(A)$ such that

$$v_{i,0} \mathrm{diag}(r_{i,0}, q_{i,0}, \ldots, q_{i,0}) v_{i,0}^* = \mathrm{diag}(s_{i,0}, p_{i,0}, \ldots, p_{i,0}).$$

(In the equation above, there are $n_i$ repetitions of $p_{i,0}$ and $q_{i,0}$.)

Likewise, for $j = 1, \ldots, m$, there exists $\ell_j$ dividing $k_j$, such that

$$\mathbb{Z}_{k_j} \cap K_0(\iota)(K_0(A^\alpha)) = \ell_j \mathbb{Z}_{k_j}.$$

Find projections $r_j, s_j \in M_\infty(A^\alpha)$ such that

$$[r_j] - [s_j] = \ell_j([p_j] - [q_j]) \in K_0(A).$$

Without loss of generality, we may assume that there exists a unitary $v_j \in M_\infty(A)$ such that

$$v_j \mathrm{diag}(r_j, q_j, \ldots, q_j) v_j^* = \mathrm{diag}(s_j, p_j, \ldots, p_j).$$

(In the equation above, there are $\ell_j$ repetitions of $p_j$ and $q_j$.)

*Claim 1:* the set

$$\{[r_{i,0}]-[s_{i,0}]\colon i=1,\ldots,k,n_i\neq 0\}\cup\{[r_j]-[s_j]\colon j=1,\ldots,m,\ell_j\neq 0\}\subseteq K_0(A^\alpha)$$

generates $K_0(A^\alpha)\cap K_0(\iota)^{-1}(H)$. By construction, the above set, when regarded as a subset of $K_0(A)$ under $K_0(\iota)$, generates $K_0(\iota)(K_0(A^\alpha))\cap H$. Moreover, since $K_0(\iota)$ is injective by part (1) of this theorem, the claim follows.

An analogous argument shows that $[r_{i,0}]-[s_{i,0}]$, for $i=1,\ldots,k_0$, has infinite order, and that $[r_j]-[s_j]$, for $j=1,\ldots,m$, has order $\ell_j$.

Find $n$ large enough such that the set

$$F=\{p_{0,i},q_{0,i},r_{0,i},s_{0,i},v_{0,i}\colon i=1,\ldots,k_0\}\cup\{p_j,q_j,u_j,r_j,s_j,v_j\colon j=1,\ldots,m\}$$

is contained in $M_n(A)$. Since the amplification of $\alpha$ to $M_n(A)$ has the Rokhlin property by part (1) of Theorem VI.2.3, we can replace $M_n(A)$ with $A$.

Set $\varepsilon=\frac{1}{10}$. Find $\delta_1>0$ such that whenever $x\in A$ satisfies $\|x^*x-x\|<\delta_1$, then there exists a projection $p\in A$ with $\|p-x\|<\varepsilon$. Find $\delta_2>0$ such that whenever $y\in A$ satisfies $\|y^*y-1\|<\delta_2$ and $\|yy^*-1\|<\delta_2$, then there exists a unitary $u\in A$ with $\|u-y\|<\varepsilon$. Set $\delta=\min\left\{\frac{\varepsilon}{3},\frac{\delta_1}{3},\frac{\delta_2}{3}\right\}$. Use Theorem VI.3.1 to find a unital, linear map $\psi\colon A\to A^\alpha$ satisfying

(a) $\|\psi(a)\psi(b)-\psi(ab)\|<\delta$ for all $a,b\in F\cup F^2$;

(b) $\|\psi(a)^*-\psi(a^*)\|<\delta$ for all $a\in F\cup F^2$;

(c) $\|\psi(a)\|\leq 2\|a\|$ for all $a\in F\cup F^2$; and

(d) $\psi(a)=a$ for all $a\in A^\alpha$.

Let $x\in\{p_{i,0},q_{i,0},p_j,q_j\colon i=1,\ldots,k_0,j=1,\ldots,m\}$. Then

$$\begin{aligned}\|\psi(x)^*\psi(x)-\psi(x)\|&\leq\|\psi(x)^*\psi(x)-\psi(x)\psi(x)\|+\left\|\psi(x)\psi(x)-\psi(x^2)\right\|\\&\leq 2\delta+\delta<\delta_1.\end{aligned}$$

Using the choice of $\delta_1$, find (and fix) a projection $\widetilde{x}\in A^\alpha$ such that $\|\widetilde{x}-\psi(x)\|<\varepsilon$.

For $y \in \{u_j, v_j \colon j = 1, \ldots, m\}$, we have

$$\begin{aligned}\|\psi(y)^*\psi(y) - 1\| &\leq \|\psi(y)^*\psi(y) - \psi(y^*)\psi(y)\| + \|\psi(y^*)\psi(y) - 1\| \\ &\leq 2\delta + \delta < \delta_2.\end{aligned}$$

Likewise, $\|\psi(y)\psi(y)^* - 1\|$. Using the choice of $\delta_2$, find (and fix) a unitary $\widetilde{y} \in A^\alpha$ such that $\|\widetilde{y} - \psi(y)\| < \varepsilon$.

*Claim 2:* Let $j = 1, \ldots, m$. Then $k_j\left([\widetilde{p}_j] - [\widetilde{q}_j]\right) = 0$ in $K_0(A^\alpha)$. In the estimates below, there are $k_j$ repetitions on each diagonal matrix:

$$\begin{aligned}&\left\|\widetilde{u}_j \mathrm{diag}(\widetilde{p}_j, \ldots, \widetilde{p}_j)\widetilde{u}_j^* - \mathrm{diag}(\widetilde{q}_j, \ldots, \widetilde{q}_j)\right\| \\ &\leq \left\|\widetilde{u}_j \mathrm{diag}(\widetilde{p}_j, \ldots, \widetilde{p}_j)\widetilde{u}_j^* - \psi(u_j)\mathrm{diag}(\psi(p_j), \ldots, \psi(p_j))\psi(u_j)^*\right\| \\ &\quad + \|\psi(u_j)\mathrm{diag}(\psi(p_j), \ldots, \psi(p_j))\psi(u_j)^* - \mathrm{diag}(\psi(q_j), \ldots, \psi(q_j))\| \\ &\quad + \|\mathrm{diag}(\psi(q_j), \ldots, \psi(q_j)) - \mathrm{diag}(\widetilde{q}_j, \ldots, \widetilde{q}_j)\| \\ &\leq 7\varepsilon + 3\delta + \varepsilon \leq 9\varepsilon < 1.\end{aligned}$$

It follows from Lemma 2.5.3 in [166] that $\widetilde{u}_j \mathrm{diag}(\widetilde{p}_j, \ldots, \widetilde{p}_j)\widetilde{u}_j^*$ is Murray-von Neumann equivalent to $\mathrm{diag}(\widetilde{q}_j, \ldots, \widetilde{q}_j)$ in $A^\alpha$, and the claim is proved.

Define a group homomorphism $\pi \colon H \to K_0(A^\alpha)$ by

$$\pi([p_{i,0}] - [q_{i,0}]) = [\widetilde{p}_{i,0}] - [\widetilde{q}_{i,0}],$$

for $i = 1, \ldots, k_0$, and

$$\pi([p_j] - [q_j]) = [\widetilde{p}_j] - [\widetilde{q}_j],$$

for $j = 1, \ldots, m$. (This is a well-defined group homomorphism by the previous claim.)

*Claim 3:* We have $\pi \circ K_0(\iota)|_{K_0(\iota)^{-1}(H)} = \mathrm{id}_{K_0(\iota)^{-1}(H)}$. For this, it is enough to check that

$$\pi\left(K_0(\iota)([r_{i,0}] - [s_{i,0}])\right) = [r_{i,0}] - [s_{i,0}]$$

for all $i = 1, \ldots, k_0$, and

$$\pi\left(K_0(\iota)([r_j] - [s_j])\right) = [r_j] - [s_j]$$

for all $j = 1, \ldots, m$.

Fix $i \in \{1, \ldots, k_0\}$. Then $K_0(\iota)([r_{i,0}] - [s_{i,0}]) = n_i([p_{i,0}] - [q_{i,0}])$ in $K_0(A)$, so we shall prove that

$$n_i([\widetilde{p}_{i,0}] - [\widetilde{q}_{i,0}]) = [r_{i,0}] - [s_{i,0}]$$

in $K_0(A^\alpha)$. The following estimate can be shown in a way similar to what was done in the proof of Claim 2 (there are $n_i + 1$ entries on each diagonal matrix):

$$\left\| \widetilde{v}_j \mathrm{diag}(s_{i,0}, \widetilde{p}_{i,0}, \ldots, \widetilde{p}_{i,0}) \widetilde{v}_j^* - \mathrm{diag}(\widetilde{r}_{i,0}, q_{i,0}, \ldots, \widetilde{q}_{i,0}) \right\| < 1.$$

Again, it follows from Lemma 2.5.3 in [166] that $\widetilde{v}_j \mathrm{diag}(s_{i,0}, \widetilde{p}_{i,0}, \ldots, \widetilde{p}_{i,0}) \widetilde{v}_j^*$ is Murray-von Neumann equivalent to $\mathrm{diag}(\widetilde{r}_{i,0}, q_{i,0}, \ldots, \widetilde{q}_{i,0})$ in $A^\alpha$, and the claim is proved. The argument for $[r_j] - [s_j]$, for $j = 1, \ldots, m$, is analogous, and we omit it. $\square$

It follows that certain features of the $K$-groups of $A$ are inherited by the $K$-groups of $A^\alpha$ and $A \rtimes_\alpha G$:

**Corollary VI.3.4.** Let $A$ be a unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $j \in \{0, 1\}$ and suppose that $K_j(A)$ is either:

1. Free;
2. Torsion;
3. Torsion-free;
4. Finitely generated;
5. Zero.

Then the same holds for $K_j(A^\alpha)$ and $K_j(A \rtimes_\alpha G)$.

*Proof.* This follows immediately from part (1) of Theorem VI.3.3. $\square$

**Corollary VI.3.5.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Suppose that $K_j(A)$ is finitely generated for some $j \in \{0, 1\}$. Then $K_j(A^\alpha)$ is isomorphic to a direct summand in $K_j(A)$.

*Proof.* This is consequence of part (3) of Theorem VI.3.3, together with the fact that a short exact sequence $0 \to G_1 \to G_2 \to G_3 \to 0$ of abelian groups with a section $G_2 \to G_1$ must split. $\square$

The splitting constructed in part (3) of Theorem VI.3.3 is not natural with respect to equivariant homomorphisms between $G$-algebras with the Rokhlin property. On the other hand, the splitting *can* be shown to be natural with respect to certain maps. We state it below, but omit its proof because it is straightforward (although a bit messy).

**Proposition VI.3.6.** Let $B$ be unital $C^*$-algebra, let $A$ be a unital subalgebra of $B$ (with the same unit), and denote by $\mu\colon A \to B$ the canonical inclusion. Let $G$ be a second countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions with the Rokhlin property. Let $\tau\colon G \to G$ be a surjective group homomorphism satisfying

$$\beta_g(\mu(a)) = \mu(\alpha_{\tau(g)}(a))$$

for all $g \in G$ and all $a \in A$. (Note, in particular, that $\mu$ restricts to the inclusion $A^\alpha \subseteq B^\beta$.) Write $\iota_A\colon A^\alpha \to A$ and $\iota_B\colon B^\beta \to B$ for the canonical inclusions.

Suppose that there are unital, equivariant homomorphisms

$$\varphi_A\colon C(G) \to A_{\infty,\alpha} \cap B' \subseteq A_{\infty,\alpha} \cap A' \quad \text{and} \quad \varphi_B\colon C(G) \to B_{\infty,\beta} \cap B'$$

making the following diagram commute:

$$\begin{tikzcd} C(G) \arrow[r, "\tau^*"] \arrow[d, "\varphi_A"'] & C(G) \arrow[d, "\varphi_A"] \\ A_{\infty,\alpha} \cap B' \arrow[r, hook] & B_{\infty,\beta} \cap B'. \end{tikzcd}$$

Let $j \in \{0,1\}$ and let $H$ be a finitely generated subgroup of $K_j(A)$. Set $\widetilde{H} = K_j(\mu)(H)$, which is a finitely generated subgroup of $K_j(B)$. Then there exist group homomorphisms $\pi_A\colon H \to K_0(A^\alpha)$ and $\pi_B\colon \widetilde{H} \to K_0(B^\beta)$, satisfying

$$\pi_A \circ K_j(\iota_A)|_{K_j(\iota_A)^{-1}(H)} = \mathrm{id}_{K_j(\iota_A)^{-1}(H)}$$

and

$$\pi_B \circ K_j(\iota_B)|_{K_j(\iota_B)^{-1}(\widetilde{H})} = \mathrm{id}_{K_j(\iota_B)^{-1}(\widetilde{H})},$$

which moreover make the following diagram commute:

$$\begin{tikzcd}
K_j(A^\alpha) \arrow[d, "K_j(\mu)"'] \arrow[rr, bend left, "K_j(\iota_A)"] & H \arrow[l, "\pi_A"'] \arrow[r, hook] \arrow[d, "K_j(\mu)"] & K_j(A) \arrow[d, "K_j(\mu)"] \\
K_j(B^\beta) \arrow[rr, bend right, "K_j(\iota_B)"'] & \widetilde{H} \arrow[l, "\pi_B"] \arrow[r, hook] & K_j(B).
\end{tikzcd}$$

Though we do not have an immediate application for the following observation, we record it here for use in the future. The first cohomology group $H^1_\alpha(G,\mathcal{U}(A))$ of a compact group action $\alpha\colon G \to \mathrm{Aut}(A)$ is defined as the quotient of the set of $\alpha$-cocycles on $A$, by the set of $\alpha$-coboundaries. (See, for example, Subsection 2.1 in [132] for the precise definitions.)

**Corollary VI.3.7.** Let $A$ be a separable, simple, purely infinite unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then its first cohomology group $H^1_\alpha(G,\mathcal{U}(A))$ is trivial.

*Proof.* We proceed to check the hypotheses of Proposition 2.5 in [132].

By Theorem IV.3.16, the action $\alpha_g$ is outer for all $g \in G \setminus \{1\}$, so in particular $\alpha$ is faithful. By Proposition VII.4.10, the crossed product $A \rtimes_\alpha G$ is purely infinite, and it is simple by Proposition VI.3.14. Finally, $K_0(\iota)$ is injective by part (1) of Theorem VI.3.3, so the result follows from Proposition 2.5 in [132]. ☐

We recall here a result of Izumi (Theorem 3.13 in [132]). Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on a unital, simple $C^*$-algebra $A$ with the Rokhlin property, and denote by $\iota\colon A^\alpha \to A$ the canonical inclusion. Then

$$K_*(\iota)(K_*(A^\alpha)) = \{x \in K_*(A)\colon K_*(\alpha_g)(x) = x \text{ for all } g \in G\}.$$

The analogous statement for compact group actions fails quite drastically:

**Example VI.3.8.** Let $A$ be a unital Kirchberg algebra satisfying the UCT, with $K$-theory given by $K_0(A) \cong K_1(A) \cong \mathbb{Z}_6$, with the class of the unit in $K_0(A)$ corresponding to $3 \in \mathbb{Z}_6$. By Theorem IX.9.3, there exists a circle action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ with the Rokhlin property, such that $K_0(A^\alpha) \cong \mathbb{Z}_2$, with the class of the unit of $A^\alpha$ corresponding to $1 \in \mathbb{Z}_2$, and $K_1(A^\alpha) \cong \mathbb{Z}_3$.

By Proposition XI.3.6, if $\zeta \in \mathbb{T}$ and $j \in \{0,1\}$, then $K_j(\alpha_\zeta) = \mathrm{id}_{K_j(A)}$. In particular,

$$K_j(\iota)(K_j(A^\alpha)) \cong \mathbb{Z}_{j+2} \not\cong \mathbb{Z}_6 \cong \{x \in K_j(A)\colon K_j(\alpha_\zeta)(x) = x \text{ for all } \zeta \in \mathbb{T}\}.$$

What goes wrong in the example above (whose notation we keep), is that if $p$ is a projection in $A$ such that $\alpha_\zeta(p)$ is unitarily equivalent to $p$ for all $\zeta \in \mathbb{T}$, then the unitaries $u_\zeta$, for $\zeta \in \mathbb{T}$, which implement the unitary equivalence, cannot in general be chosen to depend continuously on $\zeta$.

Next, we show that the Rokhlin property implies discrete $K$-theory (Definition 4.1.2 in [199]) in a very strong way. (The reader who is not familiar with equivariant $K$-theory is referred to Section 2.3.)

In the following theorem, $I(G)$ denotes the augmentation ideal in $R(G)$, that is, the kernel of the ring homomorphism $\dim\colon R(G) \to \mathbb{Z}$ given by

$$\dim([(V,v)] - [(W,w)]) = \dim(V) - \dim(W)$$

for $[(V,v)] - [(W,w)] \in R(G)$.

**Theorem VI.3.9.** Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $I(G) \cdot K_*^G(A) = 0$.

*Proof.* We show the result for $K_0^G$; the result for $K_1^G$ is analogous. (It also follows by replacing $A$ with $A \otimes B$, where $B$ is any unital $C^*$-algebra satisfying the UCT with $K_0(B) = \{0\}$ and $K_1(B) = \mathbb{Z}$, and endowing it with the $G$-action $\alpha \otimes \mathrm{id}_B$.)

We show first that $I(G) \cdot V_G(A) = 0$. Let $v\colon G \to \mathcal{U}(V)$ be a finite dimensional representation, and let $p$ be a $G$-invariant projection in $\mathcal{B}(V) \otimes A$. Since the action $g \mapsto \mathrm{Ad}(w) \otimes \alpha_g$ of $G$ on $\mathcal{B}(V) \otimes A$ has the Rokhlin property by part (1) of Theorem VI.2.3, we may assume that $V = \mathbb{C}$, so that $p$ is a $G$-invariant projection in $A$.

Let $x = [(W_1, w_1)] - [(W_2, w_2)] \in I(G)$ be given. Since $W_1 \cong W_2$ as vector spaces, it is clear that

$$x = ([(W_1, w_1)] - [(W_1, \mathrm{id}_{W_1})]) - ([(W_2, w_2)] - [(W_2, \mathrm{id}_{W_2})]) .$$

In particular, it is enough to show that if $w\colon G \to \mathcal{U}(W)$ is a finite dimensional representation, then $([(W_1, w_1)] - [(W_1, \mathrm{id}_{W_1})])\, [p] = 0$ in $K_0^G(A)$. We identify $M_2(\mathcal{B}(W)) \otimes A$ with $M_2(\mathcal{B}(W, A))$ in the usual way. We will show that the elements

$$\begin{pmatrix} \mathrm{id}_W & 0 \\ 0 & 0 \end{pmatrix} \otimes p = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{B}(W, A))$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{id}_W \end{pmatrix} \otimes p = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \in M_2(\mathcal{B}(W, A))$$

are Murray-von Neumann equivalent in the fixed point algebra of $M_2(\mathcal{B}(W, A))$. The action $\beta\colon G \to \mathrm{Aut}(M_2(\mathcal{B}(W)) \otimes A)$ is given by

$$\beta_g = \mathrm{Ad} \begin{pmatrix} w_g & 0 \\ 0 & 1 \end{pmatrix} \otimes \alpha_g,$$

which again has the Rokhlin property.

Let $0 < \varepsilon < \frac{1}{3}$, and find $\delta_0 > 0$ such that whenever $B$ is a $C^*$-algebra and $s \in B$ satisfies $\|s^*s - 1\| < \delta_0$ and $\|ss^* - 1\| < \delta$, then there exists a unitary $u$ in $B$ such that $\|u - s\| < \varepsilon$. Set $\delta = \min\{\delta_0, \frac{1}{5}\}$.

Set $r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1_A$, and observe that

$$\beta_g(r) = \begin{pmatrix} 0 & w_g^* \\ w_g & 0 \end{pmatrix} \otimes 1_A$$

for all $g \in G$. Set $F = \left\{ r, r^*, \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \right\}$. Use Theorem VI.3.1 to find a continuous, unital linear map

$$\psi \colon M_2(\mathcal{B}(W)) \otimes A \to (M_2(\mathcal{B}(W)) \otimes A)^\beta$$

which is the identity on $(M_2(\mathcal{B}(W)) \otimes A)^\beta$, and moreover satisfies

$$\|\psi(ab) - \psi(a)\psi(b)\| < \frac{\delta}{3} \ \text{ and } \ \|\psi(a^*) - \psi(a)^*\| < \frac{\delta}{2}$$

for all $a, b \in F$.

We have

$$\begin{aligned} \|\psi(r)^*\psi(r) - 1\| &\leq \|\psi(r)^*\psi(r) - \psi(r^*)\psi(r)\| + \|\psi(r^*)\psi(r) - \psi(r^*r)\| \\ &\leq 2\|\psi(r)^* - \psi(r^*)\| + \frac{\delta}{3} < \delta \leq \delta_0. \end{aligned}$$

Likewise, $\|\psi(r)\psi(r)^* - 1\| < \delta_0$. By the choice of $\delta_0$, there exists a unitary $u \in M_2(\mathcal{B}(W, A))^\beta$ such that $\|u - \psi(r)\| < \varepsilon$.

Recall that $\psi\left(\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and similarly for $\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$. Using this at the third step, we compute

$$\begin{aligned} \left\| u \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* - \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \right\| &\leq 2\|u - \psi(r)\| + \left\| \psi(r) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* - \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \right\| \\ &\leq 2\varepsilon + \left\| \psi(r) \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} u^* - \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \right\| \\ &\leq 2\varepsilon + 5\frac{\delta}{3} < 1. \end{aligned}$$

By Lemma 2.5.3 in [166], the projections $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$ are Murray-von Neumann equivalent in $M_2(\mathcal{B}(W, A))^\beta$.

Since $K_0^G(A)$ is generated, as a group, by (the image of) $V_G(A)$, the result follows. $\square$

Suppose that $G$ is abelian, and set $\Gamma = \widehat{G}$. It follows from Theorem VI.3.9 and the canonical $R(G)$-module identification $K_*^G(A) \cong K_*(A \rtimes_\alpha G)$ given by Julg's Theorem, that $K_*(\widehat{\alpha}_\gamma) = \mathrm{id}_{K_*(A\rtimes_\alpha G)}$ for all $\gamma \in \Gamma$.

*Cuntz semigroup and fixed point algebras.*

In this section, we study the map induced by the inclusion $A^\alpha \to A$ at the level of the Cuntz semigroup (Theorem VI.3.10). In Corollary VI.3.12, we relate the $K$-theory and Cuntz semigroup of the crossed product to those of the original algebra. (The reader who is not familiar with equivariant $K$-theory is referred to Section 2.6.)

**Theorem VI.3.10.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the canonical inclusion $\iota\colon A^\alpha \to A$ induces an order embedding $\mathrm{Cu}(\iota)\colon \mathrm{Cu}(A^\alpha) \to \mathrm{Cu}(A)$.

*Proof.* Let $x$ and $y$ be positive elements in $A^\alpha \otimes \mathcal{K}$ such that $x \precsim y$ in $A \otimes \mathcal{K}$. By Rørdam's Lemma (here reproduced as Proposition II.6.5), given $\varepsilon > 0$ there exist $k$ in $\mathbb{N}$, a positive number $\delta > 0$ and $s$ in $A \otimes M_k$ such that $(x-\varepsilon)_+ = s^*s$ and $ss^*$ belongs to the hereditary subalgebra of $A \otimes M_k$ generated by $(y-\delta)_+$. Note that the action $\alpha \otimes \mathrm{id}_{M_k}$ of $G$ on $A \otimes M_k$ has the Rokhlin property by part (1) of Theorem VI.2.3, and that $M_k(A)^{\alpha\otimes \mathrm{id}_{M_k}}$ can be canonically identified with $M_k(A^\alpha)$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $A \to A^\alpha$ as in the conclusion of Theorem VI.3.1. For $n \in \mathbb{N}$, we denote by $\psi_n^{(k)}\colon M_k(A) \to M_k(A^\alpha)$ the tensor product of $\psi_n$ with $\mathrm{id}_{M_k}$. Since $s^*s = (x-\varepsilon)_+$, we have

$$\lim_{n\to\infty} \left\| \psi_n^{(k)}(s)^* \psi_n^{(k)}(s) - (x-\varepsilon)_+ \right\| = 0.$$

We can therefore find a sequence $(t_m)_{m\in\mathbb{N}}$ in $M_k(A^\alpha)$ such that

$$\|t_m^* t_m - (x-\varepsilon)_+\| < \frac{1}{m} \quad \text{and} \quad \|t_m t_m^* - ss^*\| < \frac{1}{m}$$

for all $m \in \mathbb{N}$. We deduce that

$$\left[\left(x - \varepsilon - \frac{1}{m}\right)_+\right] \le [t_m^* t_m] = [t_m t_m^*]$$

in $\mathrm{Cu}(A^\alpha)$. Taking limits as $m \to \infty$, and using Rørdam's Lemma (here reproduced as Proposition II.6.5) again, we conclude that

$$[(x-\varepsilon)_+] \leq [ss^*] \leq [y]$$

in $\mathrm{Cu}(A^\alpha)$. Since $\varepsilon > 0$ is arbitrary, this implies that $[x] \leq [y]$ in $\mathrm{Cu}(A^\alpha)$, as desired. This finishes the proof. □

**Remark VI.3.11.** In the context of the theorem above, one can show that if $\mathrm{Cu}(A)$ is finitely generated as a Cuntz semigroup, then $\mathrm{Cu}(\iota)(\mathrm{Cu}(A^\alpha))$ is a direct summand of $\mathrm{Cu}(A)$ (although the splitting is not natural). However, very few $C^*$-algebras have finitely generated Cuntz semigroups (this, in particular, implies that $\mathrm{Cu}(A)$ is countable), and hence we do not prove this assertion here.

**Corollary VI.3.12.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

1. There is a canonical identification of $K_*(A \rtimes_\alpha G)$ with an order subgroup of $K_*(A)$;
2. There is a canonical identification of $\mathrm{Cu}(A \rtimes_\alpha G)$ with a sub-semigroup of $\mathrm{Cu}(A)$.

*Proof.* Recall that two Morita equivalent separable $C^*$-algebras have canonically isomorphic $K$-groups and Cuntz semigroup. The result then follows from Proposition VII.2.2 together with Theorem VI.3.3 or Theorem VI.3.10. □

Recall that an ordered semigroup $S$ is said to be *almost unperforated* if for every $x, y \in S$, if there exists $n \in \mathbb{N}$ such that $(n+1)x \leq ny$, then $x \leq y$. Since almost unperforation passes to sub-semigroups (with the induced order), the following is immediate.

**Corollary VI.3.13.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $\mathrm{Cu}(A)$ is almost unperforated, then so are $\mathrm{Cu}(A^\alpha)$ and $\mathrm{Cu}(A \rtimes_\alpha G)$.

It is a standard fact (see Theorem 3.1 in [156]) that formation of (reduced) crossed products by pointwise outer actions of discrete groups preserves simplicity. However, the corresponding statement for not necessarily discrete groups is false, even in the compact case.

For example, consider the gauge action $\gamma$ of $\mathbb{T}$ on $\mathcal{O}_\infty$, which is given by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta$ in $\mathbb{T}$ and all $j \in \mathbb{N}$. Then $\gamma$ is pointwise outer by the Theorem in [181], but the crossed product is well-known to be non-simple.

Intuitively speaking, sufficiently outer actions of, say, compact groups, ought to preserve simplicty. (And this should be a test for what "sufficiently outer" means for a compact group action.) Since the Rokhlin property is a rather strong form of outerness, the next result should come as no surprise.

**Proposition VI.3.14.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ is simple, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

*Proof.* We show the statement only for $A^\alpha$, since the result for $A \rtimes_\alpha G$ follows from the fact that it is Morita equivalent to $A^\alpha$; see Proposition VII.2.2.

Let $a \in A^\alpha$ with $a \neq 0$. We will show that the ideal generated by $a$ in $A^\alpha$, which by definition is $\overline{A^\alpha a A^\alpha}$, equals $A^\alpha$. Since $A$ is simple, we have $\overline{AaA} = A$. Find a positive integer $m \in \mathbb{N}$, and elements $x_1, \ldots, x_m, y_1, \ldots, y_m \in A$ such that

$$\left\| \sum_{j=1}^{m} x_j a y_j - 1 \right\| < \frac{1}{2}.$$

Set

$$F = \{x_j, y_j \colon j = 1, \ldots, m\} \cup \{a\} \cup \left\{ \sum_{j=1}^{m} x_j a y_j - 1 \right\},$$

and let $\varepsilon = \frac{1}{6m}$. Find a unital linear map $\psi\colon A \to A^\alpha$ as in the conclusion of Theorem VI.3.1. Then

$$\begin{aligned}
\left\| \sum_{j=1}^{m} \psi(x_j) a \psi(y_j) - 1 \right\| &= \left\| \sum_{j=1}^{m} \psi(x_j)\psi(a)\psi(y_j) - \sum_{j=1}^{m} \psi(x_j a y_j) \right\| \\
&\quad + \left\| \psi\left( \sum_{j=1}^{m} x_j a y_j - 1 \right) \right\| \\
&\leq \sum_{j=1}^{m} \|\psi(x_j)\psi(a)\psi(y_j) - \psi(x_j a y_j)\| + \frac{1}{2} \\
&< 3\varepsilon m + \frac{1}{2} = 1.
\end{aligned}$$

It follows that $b = \sum_{j=1}^{m} \psi(x_j)a\psi(y_j)$ is invertible. Since $b$ clearly belongs to $A^\alpha aA^\alpha$, the result follows. ☐

In particular, we deduce that strict comparison of positive elements is preserved under formation of crossed products and passage to fixed point algebras by the actions here considered.

**Corollary VI.3.15.** Let $A$ be a simple, unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ has strict comparison of positive elements, then so do $A^\alpha$ and $A \rtimes_\alpha G$.

*Proof.* By Lemma 6.1 in [262], strict comparison of positive elements for a simple $C^*$-algebra is equivalent to almost unperforation of its Cuntz semigroup. Since $A^\alpha$ and $A \rtimes_\alpha G$ are simple by Proposition VI.3.14, the result follows from Corollary VI.3.13. ☐

### Duality and (Equivariant) Semiprojectivity

We now turn to dual actions. Recall that the Pontryagin dual of a compact abelian group is a discrete (abelian) group. For a $C^*$-algebra $B$, we denote by $M(B)$ its multiplier algebra.

**Definition VI.4.1.** Let $\Gamma$ be a discrete abelian group, let $B$ be a $C^*$-algebra, and let $\beta\colon \Gamma \to \mathrm{Aut}(B)$ be an action. We say that $\beta$ is *approximately representable*, if there exists a unitary representation $u\colon \Gamma \to \mathcal{U}\left((M(B)^\beta)_\infty\right)$ such that

$$\beta_\gamma(b) = u_\gamma^* b u_\gamma$$

for every $\gamma \in \Gamma$ and every $b \in B$.

The following theorem is a generalization of Lemma 3.8 in [132] to compact groups.

**Theorem VI.4.2.** Let $A$ and $B$ be separable unital $C^*$-algebras, let $G$ be a second countable compact abelian group, and set $\Gamma = \widehat{G}$.

1. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a strongly continuous action. Then $\alpha$ has the Rokhlin property if and only if its dual action $\widehat{\alpha}\colon \Gamma \to \mathrm{Aut}(A \rtimes_\alpha G)$ is approximately representable.

2. Let $\beta\colon \Gamma \to \mathrm{Aut}(B)$ be an action. Then $\beta$ is approximately representable if and only if its dual action $\widehat{\beta}\colon G \to \mathrm{Aut}(B \rtimes_\beta \Gamma)$ has the Rokhlin property.

Note that the crossed product $B \rtimes_\beta \Gamma$ is unital because $B$ is and $\Gamma$ is discrete.

*Proof.* (1). Throughout the proof of this part, for $g \in G$ we denote by $v_g$ the canonical unitary in $M(A \rtimes_\alpha G)$ implementing the automorphism $\alpha_g$ in $A \rtimes_\alpha G$.

Suppose that $\alpha$ has the Rokhlin property, and let

$$\varphi\colon C(G) \to A_{\infty,\alpha} \cap A'$$

be a unital, equivariant homomorphism. Since $C(G)$ can be canonically identified with $C^*(\Gamma)$, the map $\varphi$ induces a unitary representation $u$ of $\Gamma$ on $A_{\infty,\alpha} \cap A'$, and the fact that $\varphi$ is equivariant implies that

$$(\alpha_\infty)_g(u(\gamma)) = \gamma(g)u(\gamma)$$

for all $g \in G$ and for all $\gamma \in \Gamma$. Given $g \in G$ and $\gamma \in \Gamma$, we have

$$v_g u(\gamma) v_g^* = \gamma(g)u(\gamma).$$

We deduce that

$$u(\gamma)^* v_g u(\gamma) = \gamma(g) v_g = \widehat{\alpha}_\gamma(v_g).$$

On the other hand, given $a \in A$, and regarded as an element in $M(A \rtimes_G)$, it is clear that

$$u_\gamma^* a u_\gamma = a = \widehat{\alpha}_\gamma(a).$$

Note that $M(A \rtimes_\alpha G)^{\widehat{\alpha}}$ coincides with the canonical copy of $A$ in $M(A \rtimes_\alpha G)$. It follows that the unitary representation $u^*\colon \Gamma \to \mathcal{U}(M(A \rtimes_\alpha G)^{\widehat{\alpha}}_\infty)$ implements $\widehat{\alpha}$, and thus $\widehat{\alpha}$ is approximately representable.

Conversely, assume that $\widehat{\alpha}$ is approximately representable, and let $u\colon \Gamma \to \mathcal{U}(M(A \rtimes_\alpha G)^{\widehat{\alpha}}_\infty)$ be a unitary representation as in Definition VI.4.1. For $\gamma \in \Gamma$, the unitary $u_\gamma$ satisfies

$$u_\gamma a u_\gamma^* = a$$

in $A_{\infty,\alpha}$, so $u_\gamma \in A_{\infty,\alpha} \cap A'$. Thus, the unitary representation $u$ induces a unital homomorphism $\varphi\colon C^*(\Gamma) \cong C(G) \to A_{\infty,\alpha} \cap A'$.

On the other hand, since

$$u_\gamma v_g u_\gamma = \widehat{\alpha}_\gamma(v_g) = \gamma(g) v_g$$

for all $\gamma \in \Gamma$ and all $g \in G$, we deduce that $\varphi$ is equivariant, and hence $\alpha$ has the Rokhlin property.

(2). Throughout the proof of this part, for $\gamma \in \Gamma$ we denote by $w_\gamma$ the canonical unitary in $B \rtimes_\beta \Gamma$ implementing the automorphism $\beta_\gamma$ in $B \rtimes_\beta \Gamma$.

Suppose that $\beta$ is approximately representable, and let

$$u\colon \Gamma \to \mathcal{U}((B^\beta)_\infty)$$

be a unitary representation implementing $\beta$ in $B_\infty$. Given $\gamma, \sigma \in \Gamma$, we have

$$w_\sigma u_\gamma w_\sigma^* = u_\gamma$$

in $(B \rtimes_\beta \Gamma)_{\infty,\widehat{\beta}}$, so $u_\gamma$ asymptotically commutes with the canonical unitaries in the crossed product. Given $b \in B$ and $\gamma \in \Gamma$, we have

$$(w_\gamma u_\gamma^*) b (w_\gamma u_\gamma^*)^* = w_\gamma \beta_{\gamma^{-1}}(b) w_\gamma^* = \beta_\gamma(\beta_{\gamma^{-1}}(b)) = b.$$

For $\gamma \in \Gamma$, set

$$v_\gamma = w_\gamma u_\gamma^* \in (B \rtimes_\beta \Gamma)_{\infty,\widehat{\beta}}.$$

Then $v_\gamma$ asymptotically commutes with the copy of $B$ in the crossed product. In addition, since $\Gamma$ is abelian and $u_\gamma$ asymptotically commutes with $w_\sigma$ for $\sigma \in \Gamma$, the same is true for $v_\gamma$. It follows that $v_g$ belongs to $(B \rtimes_\beta \Gamma)_{\infty,\widehat{\beta}} \cap (B \rtimes_\beta \Gamma)'$, so it defines a unital homomorphism

$$\varphi\colon C^*(\Gamma) \cong C(G) \to (B \rtimes_\beta \Gamma)_{\infty,\widehat{\beta}} \cap (B \rtimes_\beta \Gamma)',$$

which is easily seen to be equivariant.

The converse is proved in a similar way, using that $(B \rtimes_\beta \Gamma)^{\widehat{\beta}} = B$. We omit the details. $\square$

In particular, it follows from Theorem VI.4.2 that dual actions of actions with the Rokhlin property are approximately inner.

We now turn to equivariant semiprojectivity in connection with duality for actions with the Rokhlin property. The following is essentially Definition 1.1 in [205]; see also [213]

**Definition VI.4.3.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Let $\mathcal{B}$ be a class of $C^*$-algebras. We say that the triple $(G, A, \alpha)$ is *equivariantly semiprojective with respect to* $\mathcal{B}$, if the following holds: given an action $\beta\colon G \to \mathrm{Aut}(B)$ of $G$ on a $C^*$-algebra $B$ in $\mathcal{B}$, given an increasing sequence $J_1 \subseteq J_2 \subseteq \cdots \subseteq B$ of $G$-invariant ideals such that $B/J_n$ is in $\mathcal{B}$ for all $n \in \mathbb{N}$, and, with $J = \overline{\bigcup\limits_{n\in\mathbb{N}} J_n}$, an equivariant homomorphism $\phi\colon A \to B/J$, there exist $n \in \mathbb{N}$ and an equivariant homomorphism $\psi\colon A \to B/J_n$ such that, with $\pi_n\colon B/J_n \to B/J$ denoting the canonical quotient map, we have $\phi = \pi_n \circ \psi$. In other words, the following lifting problem can be solved:

$$\begin{tikzcd} & B \arrow[d] \\ & B/J_n \arrow[d, "\pi_n"] \\ A \arrow[ur, dashed, "\psi"] \arrow[r, "\phi"'] & B/J. \end{tikzcd}$$

In the diagram above, full arrows are given, and $n \in \mathbb{N}$ and the dotted arrow are supposed to exist and make the lower triangle commute.

We will mostly use the above definition in the case where $\mathcal{B}$ is the class of all $C^*$-algebras (in which case we speak about equivariantly semiprojective actions). However, in the proof of Theorem VI.4.9, it will be convenient to take $\mathcal{B}$ to be the class of all commutative $C^*$-algebras.

**Remark VI.4.4.** Let $G$ be a compact group, let $A$ and $B$ be $C^*$-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be continuous actions. If $\varphi\colon A \to B$ is a surjective, equivariant homomorphism, then $\varphi(A^\alpha) = B^\beta$. Indeed, it is clear that $\varphi(A^\alpha) \subseteq B^\beta$. For the reverse inclusion, given $b \in B^\beta$, let $x \in A$ satisfy $\phi(x) = b$. Then $a = \int\limits_G \alpha_g(x)\, dg$ (using normalized Haar measure on $G$) is fixed by $\alpha$ and satisfies $\phi(a) = b$.

**Lemma VI.4.5.** Let $G$ be a locally compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a contiuous action. Let $H$ be compact group, and denote by $\gamma\colon G \times H \to \mathrm{Aut}(A)$ the action given by $\gamma_{(g,h)}(a) = \alpha_g(a)$ for all $(g,h) \in G \times H$ and for all $a \in A$. If $(G, A, \alpha)$ is equivariantly semiprojective, then so is $(G \times H, A, \gamma)$.

*Proof.* Let $(G \times H, B, \beta)$ be a $G \times H$-algebra, let $(J_n)_{n\in\mathbb{N}}$ be an increasing sequence of $\beta$-invariant ideals in $B$, and set $J = \overline{\bigcup_{n\in\mathbb{N}} J_n}$. For $n \in \mathbb{N}$, denote by $\pi_n\colon B/J_n \to B/J$ the canonical quotient map. Let $\varphi\colon A \to B/J$ be an equivariant homomorphism. For ease of notation, we identify $H$ with $\{1_G\} \times H$. Since $H$ acts trivially on $A$, we must have $\varphi(A) \subseteq (B/J)^H$. By Remark VI.4.4, and since $H$ is compact, we have

$$\pi_n\left((B/J_n)^H\right) = (B/J)^H.$$

By averaging over $H$, similarly to what was done in the comments before this lemma, it is easy to show that $\overline{\bigcup_{n\in\mathbb{N}} J_n^H} = J^H$. Also, for $n \in \mathbb{N}$, it is clear that $J_n^H$ is an ideal in $B^H$, and that there is a canonical identification

$$(B/J_n)^H \cong B^H/J_n^H,$$

under which $\pi_n\colon B/J_n \to B/J$ restricts to the quotient map

$$\pi_n^H\colon B^H/J_n^H \to B^H/J^H.$$

Denote by $\iota_n\colon B^H/J_n^H \to B/J$ the canonical inclusion as the $H$-fixed point algebra, and likewise for $\iota\colon B^H/J^H \to B/J$. We thus have an associated diagram

$$\begin{tikzcd}
 & B \arrow[d] & B^H \arrow[l] \arrow[d] \\
 & B/J_n \arrow[d, "\pi_n"'] & B^H/J_n^H \arrow[l, "\iota_n"'] \arrow[d, "\pi_n^H"] \\
A \arrow[r, "\varphi"] \arrow[uurr, dashed, bend left, "\psi_0"] \arrow[rr, bend right, "\varphi_0"'] & B/J & B^H/J^H. \arrow[l, "\iota"']
\end{tikzcd}$$

Regard $A$ and $B$ as $G$-algebras. Since the range of $\varphi$ really is contained in $(B/J)^H \cong B^H/J^H$, there is a $G$-equivariant homomorphism $\varphi_0\colon A \to B^H/J^H$ such that $\varphi = \iota \circ \varphi_0$. Use equivariant semiprojectivity of $(G, A, \alpha)$ to find $n \in \mathbb{N}$ and a $G$-equivariant homomorphism $\psi_0\colon A \to B^H/J_n^H$ such that $\varphi_0 = \pi_n^H \circ \psi_0$.

Set $\psi = \iota_n \circ \psi_0\colon A \to B/J_n$. Then $\psi$ is $G \times H$-equivariant, and satisfies $\varphi = \pi_n \circ \psi$. We conclude that $(G \times H, A, \gamma)$ is equivariantly semiprojective. $\square$

Next, we characterize those compact groups $G$ for which $(G, C(G), \mathtt{Lt})$ is equivariantly semiprojective. The application we have in mind is to pre-dual actions (Corollary VI.4.7), so we only deal with abelian groups, although this restriction is almost certainly unnecessary.

Some parts of the proof of the following result could be simplified by using Theorem 2.6 in [205]. However, since this preprint remains unpublished and the proof of the referred theorem is rather long, we give here elementary (and short) proofs in the cases where it is needed.

**Theorem VI.4.6.** Let $G$ be a second countable compact abelian group. Then the dynamical system $(G, C(G), \mathtt{Lt})$ is equivariantly semiprojective if and only if $G$ is a Lie group with $\dim(G) \leq 1$.

*Proof.* Suppose that $(G, C(G), \mathtt{Lt})$ is equivariantly semiprojective. Then $C(G)$ is semiprojectivey by Corollary 3.12 in [213]. Now, by Theorem 1.2 in [253], $G$ must be an ANR with $\dim(G) \leq 1$. Now, ANR's are locally contractible by Theorem 2 in [76]. Since the action of $G$ on itself by translation is faithful and transitive, it follows from Corollary 3.7 in [128] that $G$ is a Lie group.

Conversely, let $G$ be an abelian compact Lie group with $\dim(G) \leq 1$. By Theorem 5.2 (a) in [250], $G$ is isomorphic to a product of copies of finite cyclic groups and, in the one-dimensional case, also with the circle $\mathbb{T}$.

*Claim: it is enough to prove the result for $G = \mathbb{Z}_m$ and $G = \mathbb{T}$.* Suppose we have proved it for all finite cyclic groups and for $\mathbb{T}$. Let $m_1, m_2 \in \mathbb{N}$, and let us show that the result holds for $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$. There is an equivariant isomorphism between $(G, C(G), \mathtt{Lt})$ and

$$\left(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}, C(\mathbb{Z}_{m_1}), \mathtt{Lt} \otimes \mathrm{id}_{C(\mathbb{Z}_{m_2})}\right) \otimes \left(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}, C(\mathbb{Z}_{m_2}), \mathrm{id}_{C(\mathbb{Z}_{m_1})} \otimes \mathtt{Lt}\right).$$

Note that $\mathbb{Z}_{m_j}$ acts trivially on $C(\mathbb{Z}_{m_{3-j}})$ for $j = 1, 2$. Thus, each of the tensor factors is equivariantly semiprojective by Lemma VI.4.5. Now, since $C(\mathbb{Z}_{m_1})$ is finite dimensional, the result follows from Theorem 2.8 in [205]. (Note that, when applying Theorem 2.8 in [205], we do not really need to use Theorem 2.6, just Lemma 1.4 in [205], since we already know that $C(\mathbb{Z}_{m_1})$ is equivariantly semiprojective.) An easy argument by induction shows that the result holds whenever $G$ is a finite abelian group.

Now let $F$ be a finite abelian group, and let $G = F \times \mathbb{T}$. A similar argument as in the case $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ shows that $C(G)$ is equivariantly isomorphic to the tensor product of two equivariantly semiprojective $G$-algebras (again, using Lemma VI.4.5), one of which is finite dimensional. The result then follows from Theorem 2.8 in [205] (where, once again, we do not need to use Theorem 2.6 in [205], since $C(F)$ is already known to be equivariantly semiprojective). The claim is proveed.

Let $(G, B, \beta)$ be a $G$-algebra, let $(J_n)_{n\in\mathbb{N}}$ be an increasing sequence of $\beta$-invariant ideals in $B$, and set $J = \overline{\bigcup\limits_{n\in\mathbb{N}} J_n}$. For $n \in \mathbb{N}$, let $\pi_n \colon B/J_n \to B/J$ denote the canonical quotient map, and write $\beta^{(n)} \colon G \to \mathrm{Aut}(B/J_n)$ for the induced action on $B/J_n$. Let $\varphi \colon C(G) \to B/J$ be an equivariant homomorphism. By Lemma 1.5 in [213], we may assume that $B$ and $\varphi$ are unital.

The proofs of the two cases $G = \mathbb{Z}_m$ and $G = \mathbb{T}$, depend crucially on the fact that (non-equivariant) semiprojectivity of $C(\mathbb{Z}_m)$ and $C(\mathbb{T})$ can be proved using functional calculus, which commutes with automorphisms (and hence actions).

*Case 1: $G = \mathbb{Z}_m$ for some $m \in \mathbb{N}$.* For $j \in \mathbb{Z}_m$, let $p_j \in C(\mathbb{Z}_m)$ denote the $j$-th vector of the standard basis. We will use the following descriptions of $C(\mathbb{Z}_m)$: it is the universal unital $C^*$-algebra generated by $m$ projections adding up to 1; and it is the universal unital $C^*$-algebra generated by a unitary of order $m$. (The unitary is $z = \sum\limits_{j\in\mathbb{Z}_m} e^{\frac{2\pi i j}{m}} p_j$.) With this in mind, an equivariant unital homomorphism $\phi$ from $C(\mathbb{Z}_m)$ is determined by a unitary $w = \phi(z)$ of order $m$, such that the automorphism corresponding to $j \in \mathbb{Z}_m$ sends $w$ to $e^{-\frac{2\pi i j}{m}} w$.

Use semiprojectivity of the $C^*$-algebra $C(\mathbb{Z}_m)$ (in the unital category) to find $n \in \mathbb{N}$ and a unital homomorphism $\psi_0 \colon C(\mathbb{Z}_m) \to B/J_n$ such that $\pi_n \circ \psi_0 = \varphi$:

$$\begin{tikzcd}
 & B \arrow[d] \\
 & B/J_n \arrow[d, "\pi_n"] \\
C(\mathbb{Z}_m) \arrow[ur, "\psi_0"] \arrow[r, "\varphi"'] & B/J.
\end{tikzcd}$$

Fix $\varepsilon > 0$ such that

$$2^m m \left( \frac{4\varepsilon + 6\varepsilon^2}{1 + 3\varepsilon} \right) < 1.$$

(In particular, $\varepsilon < 1$.) For $j \in \mathbb{Z}_m$, set $q_j = \varphi(p_j)$. By increasing $n$, we may assume that

$$\max_{j,k \in \mathbb{Z}_m} \left\| \beta_j^{(n)}(q_k) - q_{j+k} \right\| < \frac{\varepsilon}{m}.$$

Set $u = \sum_{j \in \mathbb{Z}_m} e^{\frac{2\pi i j}{m}} q_j$, which is a unitary in $B/J_n$. For $j \in \mathbb{Z}_m$, we have

$$\left\| \beta_j^{(n)}(u) - e^{\frac{-2\pi i j}{m}} u \right\| \leq \sum_{k \in \mathbb{Z}_m} \left\| \beta_j^{(n)}(q_k) - q_{j+k} \right\| < \varepsilon.$$

Set

$$x = \frac{1}{m} \sum_{j \in \mathbb{Z}_m} e^{\frac{2\pi i j}{m}} \beta_j^{(n)}(u).$$

Then

$$\|x - u\| \leq \frac{1}{m} \sum_{j \in \mathbb{Z}_m} \left\| e^{\frac{2\pi i j}{m}} \beta_j^{(n)}(u) - u \right\| < \varepsilon.$$

Since $u$ is a unitary, $x$ is invertible and $\|x\| \leq (1+\varepsilon)$. Hence $v = x(x^*x)^{-\frac{1}{2}}$ is a unitary in $B/J_n$. Since $\|x^*x - 1\| < 3\varepsilon$, we have

$$\begin{aligned}
\|v - x\| &= \left\| x(x^*x)^{-\frac{1}{2}} - x \right\| \\
&\leq (1+\varepsilon) \left\| (x^*x)^{-\frac{1}{2}} - 1 \right\| \\
&\leq (1+\varepsilon) \left\| (x^*x)^{-1} - 1 \right\| \\
&= (1+\varepsilon) \left\| (x^*x)^{-1}(x^*x - 1) \right\| \\
&\leq (1+\varepsilon) \frac{1}{1+3\varepsilon} 3\varepsilon.
\end{aligned}$$

Moreover, given $j \in \mathbb{Z}_m$, it is clear that $\beta_j^{(n)}(v) = v$, since $\beta_j^{(n)}(x) = x$. On the other hand,

$$\begin{aligned}
\pi_n(x) &= \frac{1}{m} \sum_{j \in \mathbb{Z}_m} e^{\frac{2\pi i j}{m}} \pi_n(\beta_j^{(n)}(u)) \\
&= \frac{1}{m} \sum_{j \in \mathbb{Z}_m} e^{\frac{2\pi i j}{m}} e^{-\frac{2\pi i j}{m}} \pi_n(u) \\
&= \pi_n(u) = \varphi(z),
\end{aligned}$$

and thus

$$\pi_n(v) = \pi_n(x) = \pi_n(u) = \varphi(z).$$

We use the identity $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1})$ at the second step, to estimate as follows:

$$\begin{aligned}
\|v^m - 1\| &\leq \|v^m - x^m\| + \|x^m - u^m\| + \|u^m - 1\| \\
&\leq 2^m m \|v - x\| + 2^m m \|x - u\| \\
&< 2^m m \left( \frac{3\varepsilon + 3\varepsilon^2}{1+3\varepsilon} + \varepsilon \right) < 1.
\end{aligned}$$

Find a continuous function $f$ on the spectrum of $v$ such that $f(v)$ is a unitary of order $m$. Since continuous functional calculus commutes with homomorphisms, we have

$$\pi_n(f(v)) = f(\pi_n(v)) = f(z) = z,$$

and

$$\beta_j^{(n)}(f(v)) = f(\beta_j^{(n)}(v)) = e^{-\frac{2\pi i j}{m}} f(v)$$

for all $j \in \mathbb{Z}_m$. Hence $f(v)$ determines a unital homomorphism $\psi\colon C(\mathbb{Z}_m) \to B/J_n$ by $\psi(z) = f(v)$, which is moreover equivariant and satisfies $\pi_n \circ \psi = \varphi$. Hence $(\mathbb{Z}_m, C(\mathbb{Z}_m), \mathtt{Lt})$ is equivariantly semiprojective, as desired.

*Case 2: $G = \mathbb{T}$.* The argument is similar to the case $G = \mathbb{Z}_m$, but somewhat simpler. Denote by $z$ the canonical unitary generating $C(\mathbb{T})$. Use semiprojectivity of the $C^*$-algebra $C(\mathbb{T})$ (in the unital category) to find $n \in \mathbb{N}$ and a unitary $u \in B/J_n$ such that $\pi_n(u) = \varphi(z)$. (This is equivalent to having a unital homomorphism $\psi_0\colon C(\mathbb{T}) \to B/J_n$ satisfying $\pi_n \circ \psi_0 = \varphi$.) By increasing $n$, we may assume that

$$\max_{\zeta \in \mathbb{T}} \left\| \beta_\zeta^{(n)}(u) - \zeta^{-1} u \right\| < 1.$$

Let $\mu$ denote the normalized Haar measure on $\mathbb{T}$, and set $x = \int_{\mathbb{T}} \zeta \beta_\zeta^{(n)}(u)\, d\mu(\zeta)$. Then $\|x - u\| < 1$ and thus $x$ is invertible. Set $v = x(x^*x)^{-\frac{1}{2}}$, which is a unitary in $B/J_n$ satisfying $\beta_\zeta^{(n)}(v) = \zeta v$ for all $\zeta \in \mathbb{T}$. Finally, since $\pi_n(u) = \varphi(z)$, we have

$$\pi_n(x) = \int_{\mathbb{T}} \zeta \pi_n(\beta_\zeta^{(n)}(u))\, d\mu(\zeta) = \int_{\mathbb{T}} \zeta\zeta^{-1} \varphi(z)\, d\mu(\zeta) = \varphi(z),$$

and thus $\pi_n(v) = \pi_n(x) = \varphi(z)$. It follows that the unitary $v$ determines a unital homomorphism $\psi\colon C(\mathbb{T}) \to B/J_n$ satisfying $\pi_n \circ \psi = \varphi$, which is moreover equivariant. This finishes the proof. □

In particular, it follows that if $G$ is a compact Lie group with $\dim(G) \leq 1$, then any action of $G$ with the Rokhlin property, automatically has the strong Rokhlin property (Definition VI.2.12).

We apply Theorem VI.4.6 to deduce that certain Rokhlin actions are always dual actions. Our result is new even in the well-studied case of finite groups.

**Corollary VI.4.7.** Let $A$ be a unital $C^*$-algebra, let $G$ be an abelian compact Lie group with $\dim(G) \leq 1$, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there exist an action $\breve{\alpha}$ of $G$ on $A^\alpha$, and an isomorphism

$$\varphi\colon A^\alpha \rtimes_{\breve{\alpha}} G \to A$$

such that

$$\varphi \circ \widehat{\breve{\alpha}}_g = \alpha_g \circ \varphi$$

for all $g \in G$. In other words, $\alpha$ is a dual action.

*Proof.* We claim that there is a unital, equivariant homomorphism $C(G) \to A$. Once we show this, the result will be an immediate consequence of Theorem 4 in [163]. (One should check that the algebra provided by Landstad's theorem is really $A^\alpha$, but this is a straighforward verification.)

Let $\varphi\colon C(G) \to A_{\infty,\alpha} \cap A'$ be an equivariant unital homomorphism as in Definition VI.2.1. Note that $c_0(\mathbb{N}, A) \cong \bigoplus\limits_{m\in\mathbb{N}} A$. Now, $(G, C(G), \mathtt{Lt})$ is equivariantly semiprojective by Theorem VI.4.6, so there exist $n \in \mathbb{N}$ and a unital equivariant homomorphism $\psi\colon C(G) \to \ell^\infty_\alpha(\mathbb{N}, A)/\bigoplus\limits_{m=1}^{n} A$ making the following diagram commute:

$$\begin{tikzcd}
 & \ell^\infty_\alpha(\mathbb{N}, A) \arrow[d] \\
 & \ell^\infty_\alpha(\mathbb{N}, A)/\bigoplus_{m=1}^n A \arrow[d] \\
C(G) \arrow[ur, "\psi"] \arrow[r, "\varphi"'] & A_{\infty,\alpha}.
\end{tikzcd}$$

Since $\ell^\infty_\alpha(\mathbb{N}, A)/\bigoplus\limits_{m=1}^{n} A$ is again isomorphic to $\ell^\infty_\alpha(\mathbb{N}, A)$, it follows that there is a unital equivariant homomorphism $C(G) \to A$, as desired. $\square$

Even though we did not define approximate representability for coactions of compact groups, a reasonable definition should include the predual coactions obtained in Corollary VI.4.7 whenever $G$ is not abelian.

We need an easy lemma first. (We are thankful to Tron Omland for providing its proof.) We refer the reader to Appendix A in [53] for the definitions of coactions and their crossed products, as well as for some of their basic features.

**Lemma VI.4.8.** Let $G$ be a compact group, let $A$ be a commutative $C^*$-algebra, and let

$$\delta\colon C(Y) \to M(A \otimes C^*(G))$$

be a coaction of $G$ on $A$. If $A \rtimes_\delta G$ is commutative, then $\delta$ is trivial, that is, $\delta(a) = a \otimes 1$ for all $a \in A$. In this case, there is a canonical isomorphism

$$A \rtimes_\delta G \cong A \otimes C(G).$$

*Proof.* Denote by $j_A\colon A \to M(A \rtimes_\delta G)$ and $j_G\colon C(G) \to M(A \rtimes_\delta G)$ the universal maps (see, for example, Definition A.39 in [53]), which satisfy the covariance condition

$$((j_A \otimes \mathrm{id}_G) \circ \delta)(a) = [(j_G \otimes \mathrm{id}_G)(w_G)]\,(j_A(a) \otimes 1)\,[(j_G \otimes \mathrm{id}_G)(w_G)^*]$$

for all $a \in A$; see Definition A.32 in [53]. Since $j_A(a) \otimes 1$ is in the center of $A \otimes M(C^*(G))$, which is dense in $M(A \otimes C^*(G))$, the above identity becomes

$$((j_A \otimes \mathrm{id}_G) \circ \delta)(a) = j_A(a) \otimes 1$$

for all $a \in A$. This is equivalent to $(j_A \otimes \mathrm{id}_G)(\delta(a) - a \otimes 1) = 0$ for all $a \in A$.

We claim that $\delta$ is normal (that is, that $j_A$ is injective; see Definition 2.1 in [221]). Once we prove the claim, it will follow that $\delta(a) = a \otimes 1$ for all $a \in A$, so $\delta$ is the trivial coaction.

We prove the claim. Since $G$ is amenable, $\delta$ is both a full and reduced coaction. Now, $\delta$ admits a faithful covariant representation by Proposition 3.2 in [221], so it is normal by Lemma 2.2 in [221].

The last part of the statement follows immediately from the definition of the cocrossed product. ☐

As an application of the techniques used in this section, we are able to describe all topological dynamical systems with the Rokhlin property. We are thankful to Hannes Thiel for providing the reference [186].

**Theorem VI.4.9.** Let $X$ be a compact Hausdorff space, and let $G$ be a compact Lie group acting on $X$. Denote by $\alpha\colon G \to \mathrm{Aut}(C(X))$ the associated action. Then $\alpha$ has the Rokhlin property if and only if there is a homeomorphism

$$\sigma\colon X/G \times G \to X$$

such that

$$g \cdot \sigma(Gx, h) = \sigma(Gx, gh)$$

for all $g, h \in G$ and for all $x \in X$.

*Proof.* The "if" implication follows from part (1) of Theorem VI.2.3, since the assumptions imply that there is an equivariant isomorphism

$$(G, C(X), \alpha) \cong (G, C(X/G) \otimes C(G), \mathrm{id}_{C(X/G)} \otimes \mathtt{Lt}).$$

Let us show the "only if" implication. By Theorem 8.8 in [186], $G$ is an equivariant ANR (see the definition right before Proposition 4.1 in [186]) when equipped with the $G$-action $\mathtt{Lt}$. Denote by $\mathcal{B}$ the class of all commutative $C^*$-algebras. It is an easy exercise to check that $(G, C(G), \mathtt{Lt})$ is equivariantly semiprojective with respect to $\mathcal{B}$.

Let $\varphi\colon C(G) \to C(X)_{\infty,\alpha} \cap C(X)' = C(X)_{\infty,\alpha}$ be an equivariant unital homomorphism as in Definition VI.2.1. Since

$$C(X)_{\infty,\alpha} = \ell^\infty_\alpha(\mathbb{N}, C(X)) / \bigoplus_{m\in\mathbb{N}} C(X),$$

by equivariant semiprojectivity there exist $n \in \mathbb{N}$ and a unital equivariant homomorphism

$$\psi\colon C(G) \to \ell^\infty_\alpha(\mathbb{N}, C(X)) / \bigoplus_{m=1}^{n} C(X)$$

that lifts $\varphi$ via the canonical quotient map $\ell^\infty_\alpha(\mathbb{N}, C(X)) / \bigoplus\limits_{m=1}^{n} C(X) \to C(X)_{\infty,\alpha}$. In particular, there is a unital equivariant homomorphism $C(G) \to C(X)$. Dually, there is an equivariant, continuous map $\rho\colon X \to G$, which is necessarily surjective.

We show two ways of finishing the proof.

*(First argument.)* Denote by $\pi\colon X \to X/G$ the canonical quotient map onto the orbit space. (This map is a fiber bundle, but we do not need this here.) Define an equivariant continuous map $\kappa\colon X \to X/G \times G$ by

$$\kappa(x) = (\pi(x), \rho(x))$$

for all $x \in X$. We claim that $\kappa$ is a homeomorphism.

To check surjectivity, let $(y, g) \in X/G \times G$ be given. Choose $x \in X$ such that $\pi(x) = y$, and find $h \in G$ such that $\rho(h \cdot x) = g$. (Such element $h$ exists because the action of $G$ on itself is transitive.) It is then clear that $\kappa(h \cdot x) = (y, g)$, so $\kappa$ is surjective.

We now check injectivity. Let $x_1, x_2 \in X$ satisfy $\kappa(x_1) = \kappa(x_2)$. Since $\pi(x_1) = \pi(x_2)$, it follows that there exists $g \in G$ such that $g \cdot x_1 = x_2$. Now, since $\rho(x_1) = \rho(x_2)$, we must have $g = 1_G$ and hence $x_1 = x_2$.

It follows that $\kappa$ is a continuous bijection. Since $X$ and $X/G\times G$ are compact metric spaces, it follows that $\kappa^{-1}$ is continuous, and the claim is proved. This finishes the proof.

*(Second argument).* Since there is a unital equivariant homomorphism $C(G) \to C(X)$, it follows from Theorem 3 in [163], that there are a coaction $\beta$ of $G$ on $C(X/G)$, and an isomorphism

$$C(X/G) \rtimes_\beta G \cong C(X)$$

that intertwines the dual $G$-action of $\beta$ and $\alpha$. (The verification of the hypotheses of Theorem 3 in [163] takes slightly more work than for Theorem 4 in [163], which was needed in Corollary VI.4.7, but it is nevertheless not difficult. With the notation of Theorem 4 in [163], observe that $\delta$ is will be nondegenerate because $A$ is unital.) Since the crossed product $C(X/G) \rtimes_\beta G$ is commutative, Lemma VI.4.8 implies that the coaction $\beta$ must be trivial. In this case, there is a canonical identification $C(X/G) \rtimes_\beta G \cong C(X/G) \otimes C(G)$, which is moreover equivariant, with $\widehat{\beta}$ on the left-hand side, and the action $\mathrm{id}_{C(X/G)} \otimes \mathtt{Lt}$ on the right-hand side. The result follows. $\square$

**Remark VI.4.10.** We note that in the proof of Theorem VI.4.9, we did not use the full strength of equivariant semiprojectivity (which can only happen for Lie groups); in fact, all we used is that $(G, C(G), \mathtt{Lt})$ is equivariantly *weakly* semiprojective (in the commutative category). There are many more compact groups that act equivariantly weakly semiprojectively on themselves. For example, totally disconnected groups of the form $\prod\limits_{n\in\mathbb{N}} \mathbb{Z}_{m_n}$ can be shown to have this property, and the conclusion of Theorem VI.4.9 applies to these, with exactly the same proof.

It follows that if $G$ acts on $C(X)$ with the Rokhlin property, then the induced action of $G$ on $X$ is free. The converse is not in general true: consider, for example the circle on the Möbius cylinder $M$ which is given by rotating each copy of the circle, and acting trivially on the non-orientable direction. This action is free, and the orbit space is homeomorphic to $\mathbb{T}$. However, $M$ is not homeomorphic to $\mathbb{T}\times\mathbb{T}$, and thus this action does not have the Rokhlin property.

On the other hand, we have the following partial converse:

**Proposition VI.4.11.** Let a Lie group $G$ act freely on a compact Hausdorff space $X$. If $\dim(G) = \dim(X)$, then the induced action of $G$ on $C(X)$ has the Rokhlin property.

*Proof.* Since $G$ is a Lie group, we have $\dim(X/G) = \dim(X) - \dim(G) = 0$. Denote by $\pi\colon X \to X/G$ the canonical quotient map. By Theorem 8 in [185], there exists a continuous cross-section $s\colon X/G \to X$. Given $x \in X$, there exists $g_x \in G$ such that $g_x \cdot x = (s\circ\pi)(x)$. Moreover, since the action is free, $g_x$ is uniquely determined by $x$ and $s$, and it is easy to verify that the assignment $x \mapsto g_x$ is continuous, using continuity of $s$ and of the group operations.

One readily checks that the map $\kappa\colon X \to X/G \times G$, given by $\kappa(x) = (Gx, g_x)$ for all $x \in X$, is an equivariant homeomorphism. We conclude that $G \to \mathrm{Aut}(C(X))$ has the Rokhlin property. $\square$

CHAPTER VII

# CROSSED PRODUCTS BY COMPACT GROUP ACTIONS WITH THE ROKHLIN PROPERTY

We present a systematic study of the structure of crossed products and fixed point algebras by compact group actions with the Rokhlin property. Our main technical result is the existence of an approximate homomorphism from the algebra to its subalgebra of fixed points, which is a left inverse for the canonical inclusion. Upon combining this with known results regarding local approximations, we show that a number of classes characterized by inductive limit decompositions with weakly semiprojective building blocks, are closed under formation of crossed products by such actions. Similarly, in the presence of the Rokhlin property, if the algebra has any of the following properties, then so do the crossed product and the fixed point algebra: being a Kirchberg algebra, being simple and having tracial rank zero or one, having real rank zero, having stable rank one, absorbing a strongly self-absorbing $C^*$-algebra, and being weakly semiprojective. The ideal structure of crossed products and fixed point algebras by Rokhlin actions is also studied.

Our methods unify, under a single conceptual approach, the work of a number of authors, who used rather different techniques. Our methods yield new results even in the case of finite group actions with the Rokhlin property.

## Introduction

Izumi [132], Hirshberg and Winter [122], Phillips [203], Osaka and Phillips [191], and Pasnicu and Phillips [194], explored the structure of crossed products by finite group actions with the Rokhlin property on unital $C^*$-algebras, while Santiago [243] addressed similar questions in the non-unital case. The questions and problems addressed in each of these works are different, and consequently the approaches used by the above mentioned authors are substantially distinct in some cases.

In [122], Hirshberg and Winter also introduced the Rokhlin property for a compact group action on a unital $C^*$-algebra, and their definition coincides with Izumi's in the case of finite groups. They showed that approximate divisibility and $\mathcal{D}$-stability, for a strongly self-absorbing $C^*$-algebra $\mathcal{D}$, are preserved under formation of crossed product by compact group actions with the Rokhlin property. Extending the results of [203], [191], and [194] to the case of arbitrary compact groups requires new insights, since the main technical tool in all of these works (Theorem 3.2 in [191]) seems not to have a satisfactory analog in the compact group case.

In this chapter, which is based on [82], we generalize and extend the results on finite group actions with the Rokhlin property of the above mentioned papers, to the case of compact group actions. Our contribution is two-fold. First, most of the results we prove here were known only in some special cases (mostly for finite or circle group actions; see [80] and [81] for the circle case), and some of them had not been noticed even in the context of finite groups. Second, our methods represent a uniform treatment of the study of crossed products by actions with the Rokhlin property, where the attention is shifted from the crossed product itself, to the algebra of fixed points.

Our results can be summarized as follows (the list is not exhaustive). We point out that (14) below was first obtained, with different techniques, by Hirshberg and Winter as part (1) of Corollary 3.4 in [122]. Also, (10) and (15) were already obtained in Chapter VII.

**Theorem.** The following classes of separable $C^*$-algebras are closed under formation of crossed products and passage to fixed point algebras by actions of second countable compact groups with the Rokhlin property:

1. Simple $C^*$-algebras (Corollary VII.2.9). More generally, the ideal structure can be completely determined (Theorem VII.2.7);
2. $C^*$-algebras that are direct limits of certain weakly semiprojective $C^*$-algebras (Theorem VII.3.10). This includes UHF-algebras (or matroid algebras), AF-algebras, AI-algebras, A$\mathbb{T}$-algebras, countable inductive limits of one-dimensional NCCW-complexes, and several other classes (Corollary VII.3.11);
3. Kirchberg algebras (Corollary VII.4.11);
4. Simple $C^*$-algebras with tracial rank at most one (Theorem VII.4.5);
5. Simple, separable, nuclear $C^*$-algebras satisfying the Universal Coefficient Theorem (Theorem VII.3.13);
6. $C^*$-algebras with nuclear dimension at most $n$, for $n \in \mathbb{N}$ (Theorem VII.2.3);

7. $C^*$-algebras with decomposition rank at most $n$, for $n \in \mathbb{N}$ (Theorem VII.2.3);

8. $C^*$-algebras with real rank zero (Proposition VII.4.12);

9. $C^*$-algebras with stable rank one (Proposition VII.4.12);

10. $C^*$-algebras with strict comparison of positive elements (Corollary VI.3.15);

11. $C^*$-algebras whose order on projections is determined by traces (Proposition VII.4.14);

12. (Not necessarily simple) purely infinite $C^*$-algebras (Proposition VII.4.10);

13. Stably finite $C^*$-algebras (Corollary VII.2.5);

14. $\mathcal{D}$-absorbing $C^*$-algebras, for a strongly self-absorbing $C^*$-algebra $\mathcal{D}$ (Theorem VII.4.3);

15. $C^*$-algebras whose $K$-groups are either: trivial, free, torsion-free, torsion, or finitely generated (Corollary VI.3.4);

16. Weakly semiprojective $C^*$-algebras (Proposition VII.4.17).

Our work yields new results even in the case of finite groups. For example, in (15) above, we do not require the algebra $A$ to be simple, unlike in Theorem 3.13 of [132]. In addition, the classes of $C^*$-algebras considered in Theorem VII.3.10 may consist of simple $C^*$-algebras, unlike in Theorem 3.5 in [191] (we also do not impose any conditions regarding corners of our algebras). Additionally, in Proposition VII.4.17, we show that the inclusion $A^\alpha \to A$ is sequence algebra extendible (Definition VII.4.15) whenever $\alpha$ has the Rokhlin property, and hence weak semiprojectivity passes from $A$ to $A^\alpha$. Our conclusion seem not to be obtainable with the methods developed in [191] and related works, since it is not in general true that a corner of a weakly semiprojective $C^*$-algebra is weakly semiprojective.

Given that our results all follow as easy consequences of our main technical observation, Theorem VII.2.6, we believe that the methods in this chapter unify the work of a number of authors, who used rather different methods, under a single systematic and conceptual approach.

**First Results on Crossed Products and the Averaging Process**

It is well known (see the Theorem in [238]) that if a compact group $G$ acts on a $C^*$-algebra $A$, then $A^G$ is a corner in $A \rtimes G$. Using this, one can many times obtain information about the fixed point algebra through the crossed product. However, since this corner is not in general full, Rosenberg's theorem is not always useful if one is interested in transferring structure from $A^G$ to $A \rtimes G$. Saturation for compact group actions is the basic notion that allows one to do this, up to Morita equivalence. The definition, which is essentially due to Rieffel, is as in Definition 7.1.4 in [199]. What we reproduce below is the equivalent formulation given in Lemma 7.1.9 in [199]. We point out that saturation is equivalent to the corner $A^G \subseteq A \rtimes G$ being full.

**Definition VII.2.1.** (Definition 7.1.4 in [199].) Let $G$ be a compact group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that $\alpha$ is *saturated*, if the set

$$\{f_{a,b}\colon G \to A; f_{a,b}(g) = a\alpha_g(b) \text{ for all } g \in G, \text{ with } a, b \in A\} \subseteq L^1(G, A, \alpha)$$

spans a dense subspace of $A \rtimes_\alpha G$.

It is an easy exercise to check that if a compact group $G$ acts freely on a compact Hausdorff space $X$, then the induced action on $C(X)$ is saturated. For this, it suffices to prove that the set

$$\left\{ f_{a,b} \in C(G \times X)\colon \begin{array}{c} f_{a,b}(g,x) = a(x)b(g \cdot x) \text{ for all} \\ (g,x) \in G \times X, \text{ with } a, b \in C(X) \end{array} \right\}$$

spans a dense subset of $C(G \times X)$. This linear span is closed under multiplication and contains the constant functions regardless of whether the action of $G$ is free or not, and it is easy to see that it separates the points of $G \times X$ if and only if it is free. The claim then follows from the Stone-Weierstrass theorem. See Theorem 7.2.6 in [199] for a more general result involving $C(X)$-algebras.

The following result will be used repeatedly throughout.

**Proposition VII.2.2.** Let $G$ be a second countable compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $\alpha$ is saturated.

In particular, the fixed point algebra and the crossed product by a compact group action with the Rokhlin property are Morita equivalent, and thus stably isomorphic whenever the original algebra is separable.

*Proof.* We begin by proving the statement for finite $G$, because we believe the reader will gain better intuition from this particular case. Indeed, finiteness of $G$ allows one to construct the approximations explicitly.

Suppose that $G$ is finite. Fix $g \in G$, and denote by $u_g$ the canonical unitary in the crossed product $A \rtimes_\alpha G$ implementing $\alpha_g$. We claim that it is enough to show that $u_g$ is in the closed linear span of the functions $f_{a,b}$ from Definition VII.2.1. Indeed, if this is the case, and if $x \in A$, then $xu_g$ also belongs to the closed linear span, and elements of this form span $A \rtimes_\alpha G$.

For $n \in \mathbb{N}$, find projections $e_g^{(n)} \in A$, for $g \in G$, such that

1. $\left\|\alpha_g(e_h^{(n)}) - e_{gh}^{(n)}\right\| < \frac{1}{n}$ for all $g, h \in G$; and
2. $\sum\limits_{g\in G} e_g^{(n)} = 1$.

For $a, b \in A$, the function $f_{a,b}$ corresponds to the sum $a\left(\sum\limits_{h\in G}\alpha_h(b)u_h\right)$. Thus, for $n \in \mathbb{N}$ and $k \in G$, we have

$$f_{e_{gk}^{(n)}, e_k^{(n)}} = e_{gk}^{(n)}\left(\sum_{h\in G}\alpha_h(e_k^{(n)})u_h\right).$$

We use pairwise orthogonality of the projections $e_g^{(n)}$, for $g \in G$, at the third step, to get

$$\begin{aligned}
\left\|f_{e_{gk}^{(n)}, e_k^{(n)}} - e_{gk}^{(n)}u_g\right\| &= \left\|e_{gk}^{(n)}\left(\sum_{h\in G} e_{gk}^{(n)}\alpha_h(e_k^{(n)})u_h\right) - e_{gk}^{(n)}u_g\right\| \\
&\leq \left\|e_{gk}^{(n)}\alpha_g(e_k^{(n)})u_h - e_{gk}^{(n)}u_h\right\| + \sum_{h\in G, h\neq g}\left\|e_{gk}^{(n)}\alpha_h(e_k^{(n)})u_h\right\| \\
&< \left\|\alpha_g(e_k^{(n)}) - e_{gk}^{(n)}\right\| + \sum_{h\in G, h\neq g}\left\|\alpha_h(e_k^{(n)}) - e_{hk}^{(n)}\right\| \\
&< \frac{1}{n} + (|G|-1)\frac{1}{n} = \frac{|G|}{n}.
\end{aligned}$$

It follows from condition (2) above that

$$\limsup_{n\to\infty}\left\|\sum_{k\in G} f_{e_{gk}^{(n)}, e_k^{(n)}} - u_g\right\| \leq \limsup_{n\to\infty}\frac{|G|^2}{n} = 0.$$

Hence $u_g$ belongs to the closed linear span of the $f_{a,b}$, and $\alpha$ is saturated.

For $G$ compact and second countable, we are not able to describe so explicitly the approximating functions $f_{a,b}$. (In fact, their existence is a consequence of the Stone-Weierstrass theorem.) Our proof consists in showing that one can build approximating functions in $A \rtimes_\alpha G$ using approximating functions in $C(G) \rtimes_{\mathtt{Lt}} G$.

So suppose that $G$ is compact. Since $\|\cdot\|_{L^1(G,A,\alpha)}$ dominates $\|\cdot\|_{A\rtimes_\alpha G}$, it is enough to show that the span of the functions $f_{a,b}$, with $a, b \in A$, is dense in $L^1(G, A, \alpha)$. Denote by $\chi_E$ the characteristic function of a Borel set $E \subseteq G$. It is a standard fact that the linear span of

$$\{x\chi_E \colon x \in A, E \subseteq G \text{ Borel}\}$$

is dense in $L^1(G, A, \alpha)$. Since $f_{x,1}f_{a,b} = f_{xa,b}$ for $x, a, b \in A$, it is enough to show that for a Borel set $E \subseteq G$, the function $\chi_E$ belongs to the linear span of the functions $f_{a,b}$.

Fix $\varepsilon > 0$. Since $\mathtt{Lt}\colon G \to \mathrm{Aut}(C(G))$ is saturated (see the comments before this proposition), there exist $m \in \mathbb{N}$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in C(G)$ such that

$$\left\| \sum_{j=1}^{m} f_{a_j,b_j} - \chi_E \right\|_{C(G)\rtimes_{\mathtt{Lt}}G} < \varepsilon.$$

Let $\varphi\colon C(G) \to A_{\infty,\alpha} \cap A' \subseteq A_{\infty,\alpha}$ be a unital equivariant homomorphism as in the definition of the Rokhlin property for $\alpha$. Then $\varphi$ induces a homomorphism

$$\psi\colon C(G) \rtimes_{\mathtt{Lt}} G \to A_{\infty,\alpha} \rtimes_{\alpha_\infty} G,$$

which, under the canonical embedding

$$A_{\infty,\alpha} \rtimes_{\alpha_\infty} G \hookrightarrow (A \rtimes_\alpha G)_\infty$$

provided by Proposition II.4.5, we will regard as a homomorphism

$$\psi\colon C(G) \rtimes_{\mathtt{Lt}} G \to (A \rtimes_\alpha G)_\infty.$$

It is clear that $\psi(f_{a_j,b_j}) = f_{\varphi(a_j),\varphi(b_j)}$ for all $j = 1, \ldots, m$, and that $\psi(\chi_E) = \chi_E$. Hence

$$\begin{aligned}\left\| \sum_{j=1}^{m} f_{\varphi(a_j),\varphi(b_j)} - \chi_E \right\|_{(A \rtimes_\alpha G)_\infty} &= \left\| \psi \left( \sum_{j=1}^{m} f_{a_j,b_j} - \chi_E \right) \right\|_{(A \rtimes_\alpha G)_\infty} \\ &\leq \left\| \sum_{j=1}^{m} f_{a_j,b_j} - \chi_E \right\|_{C(G) \rtimes_{\mathrm{Lt}} G} < \varepsilon.\end{aligned}$$

To finish the proof, for $j = 1, \ldots, m$, choose bounded sequences $(\varphi(a_j)_n)_{n\in\mathbb{N}}$ and $(\varphi(b_j)_n)_{n\in\mathbb{N}}$ in $A$, which represent $\varphi(a)$ and $\varphi(b)$, respectively. Then

$$\kappa_{A \rtimes_\alpha G} \left( \left( f_{\varphi(a_j)_n,\varphi(b_j)_n} \right)_{n\in\mathbb{N}} \right) = f_{\varphi(a_j),\varphi(b_j)}.$$

It follows that for $n$ large enough, we have

$$\left\| \sum_{j=1}^{m} f_{\varphi(a_j)_n,\varphi(b_j)_n} - \chi_E \right\|_{A \rtimes_\alpha G} < \varepsilon,$$

showing that $\alpha$ is saturated.

The last part of the statement follows from Rieffel's original definition of saturation (Definition 7.1.4 in [199]; see also Proposition 7.1.3 in [199]). □

Since unital completely positive maps of order zero are necessarily homomorphisms, it is easy to see that the Rokhlin property for a compact group action agrees with Rokhlin dimension zero in the sense of Definition IV.2.2. In particular, combining Proposition VII.2.2 with results from Chapter V, we obtain estimates of the nuclear dimension and decomposition rank of crossed products by Rokhlin actions.

**Theorem VII.2.3.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then

$$\begin{aligned}\dim_{\mathrm{nuc}}(A^\alpha) = \dim_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq \dim_{\mathrm{nuc}}(A), \quad \text{and} \\ \mathrm{dr}(A^\alpha) = \mathrm{dr}(A \rtimes_\alpha G) \leq \mathrm{dr}(A).\end{aligned}$$

*Proof.* That $\dim_{\mathrm{nuc}}(A^\alpha) = \dim_{\mathrm{nuc}}(A \rtimes_\alpha G)$ and $\mathrm{dr}(A^\alpha) = \mathrm{dr}(A \rtimes_\alpha G)$ follows from Proposition VII.2.2. The rest is an immediate consequence of Theorem V.3.4 and Theorem V.3.3, since $\dim_{\mathrm{Rok}}(\alpha) = 0$. □

**Corollary VII.2.4.** Let $A$ be a unital AF-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ and $A \rtimes_\alpha G$ are AF-algebras.

*Proof.* Since a separable $C^*$-algebra has decomposition rank zero if and only if it is an AF-algebra (Example 4.1 in [153]), the result follows from Theorem VII.2.3. □

Here is another consequence of the fact that $A^\alpha$ and $A \rtimes_\alpha G$ are Morita equivalent whenever $\alpha$ has the Rokhlin property.

**Corollary VII.2.5.** Let $A$ be a unital, separable, stably finite $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ and $A \rtimes_\alpha G$ are stably finite.

*Proof.* Unital subalgebras of stably finite $C^*$-algebras are stably finite in full generality, so $A^\alpha$ is stably finite even if $\alpha$ does not have the Rokhlin property. Stable finiteness of the crossed product then follows from Proposition VII.2.2. □

The following result will be crucial in obtaining further structure properties for crossed products by actions with the Rokhlin property.

**Theorem VII.2.6.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Given a compact subset $F_1 \subseteq A$, a compact subset $F_2 \subseteq A^\alpha$ and $\varepsilon > 0$, there exists a unital completely positive map $\psi\colon A \to A^\alpha$ such that

1. For all $a, b \in F_1$, we have
$$\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon;$$
2. For all $a \in F_2$, we have $\|\psi(a) - a\| < \varepsilon$.

In other words, there exists an approximate homomorphism $(\psi_n)_{n\in\mathbb{N}}$ consisting of unital, completely positive linear maps $\psi_n\colon A \to A^\alpha$ for $n \in \mathbb{N}$, such that $\lim\limits_{n\to\infty} \|\psi_n(a) - a\| = 0$ for all $a \in A^\alpha$.

*Proof.* Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in F_1 \cup F_2$. For the compact set $F = F_1^2 \cup F_2$, and the tolerance $\varepsilon_0 = \frac{\varepsilon}{6}$, use Proposition V.4.3 to find a positive number $\delta > 0$, a finite subset $K \subseteq G$, a partition of unity $(f_k)_{k\in K}$ of $C(G)$, and a unital completely positive map $\varphi\colon C(G) \to A$, such that

(a) If $g$ and $g'$ in $G$ satisfy $d(g,g') < \delta$, then $\|\alpha_g(a) - \alpha_{g'}(a)\| < \varepsilon_0$ for all $a \in F$.

(b) Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} \neq 0$, then $d(k,k') < \delta$.

(c) For every $g \in G$ and for every $a \in F$, we have

$$\left\| \alpha_g\left(\sum_{k\in K} \varphi(f_k)^{1/2}\alpha_k(a)\varphi(f_k)^{1/2}\right) - \sum_{k\in K} \varphi(f_k)^{1/2}\alpha_k(a)\varphi(f_k)^{1/2} \right\| < \varepsilon_0.$$

(d) For every $a \in F$ and for every $k \in K$, we have

$$\|a\varphi(f_k) - \varphi(f_k)a\| < \frac{\varepsilon_0}{|K|} \quad \text{and} \quad \left\|a\varphi(f_k)^{1/2} - \varphi(f_k)^{1/2}a\right\| < \frac{\varepsilon_0}{|K|}.$$

(e) Whenever $k$ and $k'$ in $K$ satisfy $f_k f_{k'} = 0$, then

$$\left\|\varphi(f_k)^{1/2}\varphi(f_{k'})^{1/2}\right\| < \frac{\varepsilon_0}{|K|}.$$

Define a linear map $\psi\colon A \to A^\alpha$ by

$$\psi(a) = E\left(\sum_{k\in K} \varphi(f_k)^{\frac{1}{2}}\alpha_k(a)\varphi(f_k)^{\frac{1}{2}}\right)$$

for all $a \in A$. We claim that $\psi$ has the desired properties.

It is clear that $\psi$ is unital and completely positive. It follows from condition (c) above that

$$\left\|\psi(a) - \sum_{k\in K} \varphi(f_k)^{1/2}\alpha_k(a)\varphi(f_k)^{1/2}\right\| < \varepsilon$$

for all $a \in F$. We proceed to check that conditions (1) and (2) in the statement are satisfied.

Given $a, b \in F_1$, we use condition (c) at the second and fifth step (in the form of the observation above), conditions (a), (b), (d) and (e) at the third step, and the fact that $\varphi$ is unital and $(f_k)_{k\in K}$ is a partition of unity of $C(G)$ at the fourth step, to get

$$\begin{aligned}
\psi(a)\psi(b) &= E\left(\sum_{k\in K}\varphi(f_k)^{\frac{1}{2}}\alpha_k(a)\varphi(f_k)^{\frac{1}{2}}\right)E\left(\sum_{k'\in K}\varphi(f_{k'})^{\frac{1}{2}}\alpha_{k'}(b)\varphi(f_{k'})^{\frac{1}{2}}\right)\\
&\approx_{2\varepsilon_0}\sum_{k\in K}\sum_{k'\in K}\varphi(f_k)^{\frac{1}{2}}\alpha_k(a)\varphi(f_k)^{\frac{1}{2}}\varphi(f_{k'})^{\frac{1}{2}}\alpha_{k'}(b)\varphi(f_{k'})^{\frac{1}{2}}\\
&\approx_{3\varepsilon_0}\sum_{k\in K}\sum_{k'\in K}\varphi(f_k)^{\frac{1}{2}}\alpha_k(ab)\varphi(f_k)^{\frac{1}{2}}\varphi(f_{k'})\\
&=\sum_{k\in K}\varphi(f_k)^{\frac{1}{2}}\alpha_k(ab)\varphi(f_k)^{\frac{1}{2}}\\
&\approx_{\varepsilon_0} E\left(\sum_{k\in K}\varphi(f_k)^{\frac{1}{2}}\alpha_k(ab)\varphi(f_k)^{\frac{1}{2}}\right)\\
&=\psi(ab).
\end{aligned}$$

Hence $\|\psi(ab)-\psi(a)\psi(b)\|<6\varepsilon_0=\varepsilon$, and condition (1) is proved. For the second condition, let $a\in F_2\subseteq A^\alpha$. Then

$$\begin{aligned}
\psi(a) &= E\left(\sum_{k\in K}\varphi(f_k)^{\frac{1}{2}}a\varphi(f_k)^{\frac{1}{2}}\right)\\
&\approx_{\varepsilon_0} E\left(\sum_{k\in K}\varphi(f_k)a\right)\\
&= E\left(\sum_{k\in K}\varphi(f_k)\right)a = a.
\end{aligned}$$

Thus $\|\psi(a)-a\|<\varepsilon$ for all $a\in F_2$, and the proof is complete. $\square$

Our first application of Theorem VII.2.6 is to the ideal structure of crossed products and fixed point algebras. In the presence of the Rokhlin property, we can describe all ideals: they are naturally induced by invariant ideals in the original algebra.

**Theorem VII.2.7.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G\to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

1. If $I$ is an ideal in $A^\alpha$, then there exists an $\alpha$-invariant ideal $J$ in $A$ such that $I=J\cap A^\alpha$.
2. If $I$ is an ideal in $A\rtimes_\alpha G$, then there exists an $\alpha$-invariant ideal $J$ in $A$ such that $I=J\rtimes_\alpha G$.

*Proof.* (1). Let $I$ be an ideal in $A^\alpha$. Then $J = \overline{AIA}$ is an $\alpha$-invariant ideal in $A$. We claim that $J \cap A^\alpha = I$. It is clear that $I \subseteq J \cap A^\alpha$. For the reverse inclusion, let $x \in J \cap A^\alpha$, that is, an $\alpha$-invariant element in $\overline{AIA}$. For $n \in \mathbb{N}$, choose $m_n \in \mathbb{N}$, elements $a_1^{(n)}, \ldots, a_{m_n}^{(n)}, b_1^{(n)}, \ldots, b_{m_n}^{(n)}$ in $A$, and elements $x_1^{(n)}, \ldots, x_{m_n}^{(n)}$ in $I$, such that

$$\left\| x - \sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \right\| < \frac{1}{n}.$$

Set $M_n = \max_{j=1,\ldots,m_n} \{\|a_j^{(n)}\|, \|b_j^{(n)}\|, 1\}$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n \colon A \to A^\alpha$ as in the conclusion of Theorem VII.2.6 for the choices $\varepsilon_n = \frac{1}{n m_n M_n^2}$ and

$$F_1^{(n)} = \{a_j^{(n)}, x_j^{(n)}, b_j^{(n)} \colon j = 1, \ldots, m_n\} \cup \{x\}$$

and $F_2^{(n)} = \{x_j^{(n)} \colon j = 1, \ldots, m_n\} \cup \{x\}$. Then

$$\left\| \psi_n \left( \sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \right) - x \right\| < \frac{1}{n} + \frac{1}{n m_n M_n^2} \leq \frac{2}{n}$$

and

$$\begin{aligned}
&\left\| \psi_n \left( \sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \right) - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) x_j^{(n)} \psi_n(b_j^{(n)}) \right\| \\
&\quad \leq \frac{1}{n} + \left\| \psi_n \left( \sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \right) - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) \psi_n(x_j^{(n)}) \psi_n(b_j^{(n)}) \right\| \\
&\quad \leq \frac{1}{n} + \frac{1}{n M_n} + \left\| \psi_n \left( \sum_{j=1}^{m_n} a_j^{(n)} x_j^{(n)} b_j^{(n)} \right) - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)} x_j^{(n)}) \psi_n(b_j^{(n)}) \right\| \\
&\quad \leq \frac{1}{n} + \frac{2}{n M_n} \leq \frac{3}{n}.
\end{aligned}$$

We conclude that

$$\left\| x - \sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) x_j^{(n)} \psi_n(b_j^{(n)}) \right\| < \frac{5}{n}.$$

Since $\sum_{j=1}^{m_n} \psi_n(a_j^{(n)}) x_j^{(n)} \psi_n(b_j^{(n)})$ belongs to $I$, it follows that $x$ is a limit of elements in $I$, and hence it belongs to $I$ itself.

(2). This follows from (1) together with the fact that $\alpha$ is saturated (see Proposition VII.2.2). We omit the details. ☐

In the following corollary, hereditary saturation is as in Definition 7.2.2 in [199], while the strong Connes spectrum for an action of a non-abelian compact group (which is a subset of the set $\widehat{G}$ of irreducible representations of the group) is as in Definition 1.2 of [105]. (For abelian groups, the notion of strong Connes spectrum was introduced earlier by Kishimoto in [155].)

**Corollary VII.2.8.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $\alpha$ has full strong Connes spectrum: $\widetilde{\Gamma}(\alpha) = \widehat{G}$, and it is hereditarily saturated.

*Proof.* That $\widetilde{\Gamma}(\alpha) = \widehat{G}$ follows from Theorem 3.3 in [105]. Hereditary saturation of actions with full strong Connes spectrum is established in the comments after Lemma 3.1 in [105]. ☐

**Corollary VII.2.9.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ is simple, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

## Generalized Local Approximations

We now turn to the study of preservation of certain structural properties that have been studied in the context of Elliott's classification program. In order to provide a conceptual approach, it will be necessary to introduce some convenient terminology.

**Definition VII.3.1.** Let $\mathcal{C}$ be a class of separable $C^*$-algebras and let $A$ be a $C^*$-algebra.

1. We say that $A$ is an *(unital) approximate $\mathcal{C}$-algebra*, if $A$ is isomorphic to a direct limit of $C^*$-algebras in $\mathcal{C}$ (with unital connecting maps).

2. We say that $A$ is a *(unital) local $\mathcal{C}$-algebra*, if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist a $C^*$-algebra $B$ in $\mathcal{C}$ and a not necessarily injective (unital) homomorphism $\varphi\colon B \to A$ such that $\mathrm{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in F$.

3. We say that $A$ is a *generalized (unital) local $\mathcal{C}$-algebra*, if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist a $C^*$-algebra $B$ in $\mathcal{C}$ and sequence $(\varphi_n)_{n\in\mathbb{N}}$ of (unital) completely

positive contractive maps $\varphi_n \colon B \to A$ that $\mathrm{dist}(a, \varphi_n(B)) < \varepsilon$ for all $a \in F$ and for all $n$ sufficiently large.

**Remark VII.3.2.** The term 'local $\mathcal{C}$-algebra' is sometimes used to mean that the local approximations are realized by *injective* homomorphisms. For example, in [259] Thiel says that a $C^*$-algebra $A$ is '$\mathcal{C}$-like', if for every finite subset $F \subseteq A$ and for every $\varepsilon > 0$, there exist a $C^*$-algebra $B$ in $\mathcal{C}$ and an *injective* homomorphism $\varphi \colon B \to A$ such that $\mathrm{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in F$. Finally, we point out that what we call here 'approximate $\mathcal{C}$' is called 'A$\mathcal{C}$' in [259].

The Rokhlin property is related to the above definition in the following way. Note that the approximating maps for $A^\alpha$ that we obtain in the proof are not necessarily injective, even if we assume that the approximating maps for $A$ are.

**Proposition VII.3.3.** Let $\mathcal{C}$ be a class of $C^*$-algebras, let $A$ be a $C^*$-algebra, let $G$ be a second countable group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ is a (unital) local $\mathcal{C}$-algebra, then $A^\alpha$ is a generalized (unital) local $\mathcal{C}$-algebra.

*Proof.* Let $F \subseteq A^\alpha$ be a finite subset, and let $\varepsilon > 0$. Find a $C^*$-algebra $B$ in $\mathcal{C}$ and a (unital) homomorphism $\varphi \colon B \to A$ such that $\mathrm{dist}(a, \varphi(B)) < \frac{\varepsilon}{2}$ for all $a \in F$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n \colon A \to A^\alpha$ as in the conclusion of Theorem VII.2.6. Then $(\psi_n \circ \varphi)_{n\in\mathbb{N}}$ is a sequence of (unital) completely positive contractive maps $B \to A^\alpha$ as in the definition of generalized local $\mathcal{C}$-algebra. $\square$

Let $\mathcal{C}$ be a class of $C^*$-algebras. It is clear that any (unital) approximate $\mathcal{C}$-algebra is a (unital) local $\mathcal{C}$-algebra, and that any (unital) local $\mathcal{C}$-algebra is a generalized (unital) local $\mathcal{C}$-algebra.

While the converses to these implications are known to fail in general, the notions in Definition VII.3.1 agree under fairly mild conditions on $\mathcal{C}$; see Proposition VII.3.9.

**Definition VII.3.4.** Let $\mathcal{C}$ be a class of $C^*$-algebras. We say that $\mathcal{C}$ has *(unital) approximate quotients* if whenever $A \in \mathcal{C}$ and $I$ is an ideal in $A$, the quotient $A/I$ is a (unital) approximate $\mathcal{C}$-algebra, in the sense of Definition VII.3.1.

The term 'approximate quotients' has been used in [191] with a considerably stronger meaning. Our weaker assumptions still yield an analog of Proposition 1.7 in [191]; see Proposition VII.3.9.

We need to recall a definition due to Loring. The original definition appears in [173], while in Theorem 3.1 in [55] it is proved that weak semiprojectivity is equivalent to a condition that is more resemblant of semiprojectivity. For the purposes of this chapter, the original definition is better suited.

**Definition VII.3.5.** A $C^*$-algebra $A$ is said to be *weakly semiprojective (in the unital category)* if given a $C^*$-algebra $B$ and given a (unital) homomorphism $\psi\colon A \to B_\infty$, there exists a (unital) homomorphism $\varphi\colon A \to \ell^\infty(\mathbb{N}, B)$ such that $\kappa_B \circ \varphi = \psi$. In other words, the following lifting problem can always be solved:

$$\begin{tikzcd} & \ell^\infty(\mathbb{N}, B) \arrow[d, "\kappa_B"] \\ A \arrow[ur, dashed, "\varphi"] \arrow[r, "\psi"'] & B_\infty. \end{tikzcd}$$

The proof of the following observation is left to the reader. It states explicitly the formulation of weak semiprojectivity that will be used in our work, specifically in Proposition VII.3.9.

**Remark VII.3.6.** Using the definition of the sequence algebra $B_\infty$, it is easy to show that if $A$ is a weakly semiprojective $C^*$-algebra, and $(\psi_n)_{n\in\mathbb{N}}$ is an asymptotically $*$-multiplicative sequence of linear maps $\psi_n\colon A \to B$ from $A$ to another $C^*$-algebra $B$, then there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of homomorphisms $\varphi_n\colon A \to B$ such that

$$\lim_{n\to\infty} \|\varphi_n(a) - \psi_n(a)\| = 0$$

for all $a \in A$. If each $\psi_n$ is unital and $A$ is weakly semiprojective in the unital category, then $\varphi_n$ can also be chosen to be unital.

We proceed to give some examples of classes of $C^*$-algebras that will be used in Theorem VII.3.10. We need a definition first, which appears as Definition 11.2 in [56].

**Definition VII.3.7.** A $C^*$-algebra $A$ is said to be a *one-dimensional noncommutative cellular complex*, or one-dimensional NCCW-complex for short, if there exist finite dimensional $C^*$-algebras $E$ and $F$, and unital homomorphisms $\varphi, \psi\colon E \to F$, such that $A$ is isomorphic to the pull back $C^*$-algebra

$$\{(a,b) \in E \oplus C([0,1],F) \colon b(0) = \varphi(a) \ \text{ and } \ b(1) = \psi(a)\}.$$

It was shown in Theorem 6.2.2 of [56] that one-dimensional NCCW-complexes are semiprojective (in the unital category).

**Examples VII.3.8.** The following are examples of classes of weakly semiprojective $C^*$-algebras (in the unital category) which have approximate quotients.

1. The class $\mathcal{C}$ of matrix algebras. The (unital) approximate $\mathcal{C}$-algebras are precisely the matroid algebras (UHF-algebras).

2. The class $\mathcal{C}$ of finite dimensional $C^*$-algebras. The (unital) approximate $\mathcal{C}$-algebras are precisely the (unital) AF-algebras.

3. The class $\mathcal{C}$ of interval algebras, that is, algebras of the form $C([0,1]) \otimes F$, where $F$ is a finite dimensional $C^*$-algebra. The (unital) approximate $\mathcal{C}$-algebras are precisely the (unital) AI-algebras.

4. The class $\mathcal{C}$ of circle algebras, that is, algebras of the form $C(\mathbb{T}) \otimes F$, where $F$ is a finite dimensional $C^*$-algebra. The (unital) approximate $\mathcal{C}$-algebras are precisely the (unital) A$\mathbb{T}$-algebras.

5. The class $\mathcal{C}$ of one-dimensional NCCW-complexes. We point out that certain approximate $\mathcal{C}$-algebras have been classified, in terms of a variant of their Cuntz semigroup, by Robert in [230].

The following result is well-known for several particular classes.

**Proposition VII.3.9.** Let $\mathcal{C}$ be a class of separable $C^*$-algebras which has (unital) approximate quotients (see Definition VII.3.4). Assume that the $C^*$-algebras in $\mathcal{C}$ are weakly semiprojective (in the unital category). For a separable (unital) $C^*$-algebra $A$, the following are equivalent:

1. $A$ is an (unital) approximate $\mathcal{C}$-algebra;

2. $A$ is a (unital) local $\mathcal{C}$-algebra;

3. $A$ is a generalized (unital) local $\mathcal{C}$-algebra.

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are true in full generality. Weak semiprojectivity of the algebras in $\mathcal{C}$ implies that any generalized local approximation by $C^*$-algebras in $\mathcal{C}$ can be perturbed to a genuine local approximation by $C^*$-algebras in $\mathcal{C}$ (see Remark VII.3.6), showing (3) $\Rightarrow$ (2).

For the implication (2) $\Rightarrow$ (1), note that since $\mathcal{C}$ has approximate quotients, every a local $\mathcal{C}$-algebra is A$\mathcal{C}$-like, in the sense of Definition 3.2 in [259] (see also Paragraph 3.6 there). It then follows from Theorem 3.9 in [259] that $A$ is an approximate $\mathcal{C}$-algebra.

For the unital case, one uses Remark VII.3.6 to show that (3) $\Rightarrow$ (2) when units are considered. Moreover, for (2) $\Rightarrow$ (1), one checks that in the proof of Theorem 3.9 in [259], if one assumes that the building blocks are weakly semiprojective in the unital category, then the conclusion is that a unital A$\mathcal{C}$-like algebra is a unital A$\mathcal{C}$-algebra. With the notation and terminology of the proof of Theorem 3.9 in [259], suppose that $A$ is a unital $A\mathcal{C}$-like algebra, and suppose that $\varphi\colon C \to A$ is a unital homomorphism, with $C \in \mathcal{C}$. Since $C$ is assumed to be weakly semiprojective in the unital category, the morphism $\alpha\colon C \to B$ can be chosen to be unital. For the same reason, one can arrange that the morphism $\widetilde{\alpha}\colon C \to C_{k_1}$ be unital (possible by changing the choice of $k_1$). Now, since the connecting maps $\gamma_k$ are also assumed to be unital, it is easily seen that the one-sided approximate intertwining constructed has unital connecting maps. Finally, when applying Proposition 3.5 in [259], if the algebras $A_i$, with $i \in I$, are weakly semiprojective in the unital category, then the morphisms $\psi_k\colon A_{i(k)} \to A_{i(k+1)}$ can be chosen to be unital as well. We leave the details to the reader. $\square$

The following is the main application of our approximations results.

**Theorem VII.3.10.** Let $\mathcal{C}$ be a class of separable weakly semiprojective $C^*$-algebras (in the unital category), and assume that $\mathcal{C}$ has (unital) approximate quotients. Let $A$ be a (unital) local $\mathcal{C}$-algebra, let $G$ be a second countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ is a (unital) approximate $\mathcal{C}$-algebra.

*Proof.* This is an immediate consequence of Proposition VII.3.3 and Proposition VII.3.9. $\square$

An alternative proof of part (2) of the corollary below is given in Corollary VII.2.4.

**Corollary VII.3.11.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

1. If $A$ is a UHF-algebra, then $A^\alpha$ is a UHF-algebra and $A \rtimes_\alpha G$ is a matroid algebra. If $G$ is finite, then $A \rtimes_\alpha G$ is also a UHF-algebra.
2. If $A$ is an AF-algebra, then so are $A^\alpha$ and $A \rtimes_\alpha G$.
3. If $A$ is an AI-algebra, then so are $A^\alpha$ and $A \rtimes_\alpha G$.
4. If $A$ is an A$\mathbb{T}$-algebra, then so are $A^\alpha$ and $A \rtimes_\alpha G$.
5. If $A$ is a direct limit of one-dimensional NCCW-complexes, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

*Proof.* Since the classes in Examples VII.3.8 have approximate quotients and contain only weakly semiprojective $C^*$-algebras, the claims follow from Theorem VII.3.10. □

Theorem VII.3.10 allows for far more flexibility than Theorem 3.5 in [191], since we do not assume our classes of $C^*$-algebras to be closed under direct sums or by taking corners, nor do we assume that our algebras are semiprojective. In particular, the class $\mathcal{C}$ of weakly semiprojective purely infinite, simple algebras satisfies the assumptions of Theorem VII.3.10, but appears not to fit into the framework of flexible classes discussed in [191].

Recall that a $C^*$-algebra is said to be a *Kirchberg algebra* if it is purely infinite, simple, separable and nuclear.

The following lemma is probably standard, but we include its proof here for the sake of completeness.

**Lemma VII.3.12.** Let $A$ be a Kirchberg algebra satisfying the Universal Coefficient Theorem. Then $A$ is isomorphic to a direct limit of weakly semiprojective Kirchberg algebras satisfying the Universal Coefficient Theorem.

*Proof.* Since every non-unital Kirchberg algebra is the stabilization of a unital Kirchberg algebra, by Proposition 3.11 in [42], it is enough to prove the statement when $A$ is non-unital. For $j = 0, 1$, set $G_j = K_j(A)$. Write $G_j$ as a direct limit $G_j \cong \varinjlim (G_j^{(n)}, \gamma_j^{(n)})$ of finitely generated abelian groups $G_j^{(n)}$, with connecting maps

$$\gamma_j^{(n)}\colon G_j^{(n)} \to G_j^{(n+1)}.$$

For $j = 0, 1$, sse Theorem 4.2.5 in [200] to find, for $n \in \mathbb{N}$, Kirchberg algebras $A_n$ satisfying the Universal Coefficient Theorem with $K_j(A_n) \cong G_j^{(n)}$, and homomorphisms

$$\varphi_n \colon A_n \to A_{n+1}$$

such that $K_j(\varphi_n)$ is identified with $\gamma_j^{(n)}$ under the isomorphism $K_j(A_n) \cong G_j^{(n)}$.

The direct limit $\varinjlim(A_n, \varphi_n)$ is isomorphic to $A$ by Theorem 4.2.4 in [200]. On the other hand, each of the algebras $A_n$ is weakly semiprojective by Theorem 2.2 in [254] (see also Corollary 7.7 in [170]), so the proof is complete. □

**Theorem VII.3.13.** Let $A$ be a separable, nuclear, unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ satisfies the Universal Coefficient Theorem, then so do $A^\alpha$ and $A \rtimes_\alpha G$.

*Proof.* We claim that it is enough to prove the statement when $A$ is a Kirchberg algebra. Indeed, a $C^*$-algebra $B$ satisfies the Universal Coefficient Theorem if and only if $B \otimes \mathcal{O}_\infty$ does, since $\mathcal{O}_\infty$ is $KK$-equivalent to $\mathbb{C}$. On the other hand, $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}$ has the Rokhlin property, and

$$(A \otimes \mathcal{O}_\infty)^{\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}} = A^\alpha \otimes \mathcal{O}_\infty.$$

Suppose then that $A$ is a Kirchberg algebra. Denote by $\mathcal{C}$ the class of all unital weakly semiprojective Kirchberg algebras satisfying the Universal Coefficient Theorem. Note that $\mathcal{C}$ has approximate quotients. By Lemma VII.3.12, $A$ is a unital approximate $\mathcal{C}$-algebra. By Theorem VII.3.10, $A^\alpha$ is also a unital approximate $\mathcal{C}$-algebra. Since the Universal Coefficient Theorem passes to direct limits, we conclude that $A^\alpha$ satisfies it. Since $A \rtimes_\alpha G$ is Morita equivalent to $A^\alpha$, the same holds for the crossed product. □

### Further Structure Results

We now turn to preservation of classes of $C^*$-algebras that are not necessarily defined in terms of an approximation by weakly semiprojective $C^*$-algebras. The classes we study can all be dealt with using Theorem VII.2.6.

The following is Definition 1.3 in [265].

**Definition VII.4.1.** A unital, separable $C^*$-algebra $\mathcal{D}$ is said to be *strongly self-absorbing*, if it is infinite dimensional and the map $\mathcal{D} \to \mathcal{D} \otimes_{\min} \mathcal{D}$, given by $d \mapsto d \otimes 1$ for $d \in \mathcal{D}$, is approximately unitarily equivalent to an isomorphism.

It is a consequence of a result of Effros and Rosenberg that strongly self-absorbing $C^*$-algebras are nuclear, so that the choice of the tensor product in the definition above is irrelevant. The only known examples of strongly self-absorbing $C^*$-algebras are the Jiang-Su algebra $\mathcal{Z}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF-algebras of infinite type, and tensor products of $\mathcal{O}_\infty$ by such UHF-algebras. It has been conjectured that these are the only examples of strongly self-absorbing $C^*$-algebras. See [265] for the proofs of these and other results concerning strongly self-absorbing $C^*$-algebras.

The following is a useful criterion to determine when a unital, separable $C^*$-algebra absorbs a strongly self-absorbing $C^*$-algebra tensorially. The proof is a straightforward combination of Theorem 2.2 in [265] and Choi-Effros lifting theorem, and we shall omit it.

**Theorem VII.4.2.** Let $A$ be a separable, unital $C^*$-algebra, and let $\mathcal{D}$ be a strongly self-absorbing $C^*$-algebra. Then $A$ is $\mathcal{D}$-stable if and only if for every $\varepsilon > 0$, for every finite subset $F \subseteq A$, and every finite subset $E \subseteq \mathcal{D}$, there exists a unital completely positive map $\varphi\colon \mathcal{D} \to A$ such that

1. $\|a\varphi(d) - \varphi(d)a\| < \varepsilon$ for all $a \in F$ and for all $d \in E$;
2. $\|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$ for every $d, e \in E$.

The following result was obtained as part (1) of Corollary 3.4 in [122], using completely different methods. We include a proof here to illustrate the generality of our approach.

**Theorem VII.4.3.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $\mathcal{D}$ be a strongly self-absorbing $C^*$-algebra and assume that $A$ is $\mathcal{D}$-stable. Then $A^\alpha$ and $A \rtimes_\alpha G$ are $\mathcal{D}$-stable as well.

*Proof.* Since $\mathcal{D}$-stability is preserved under Morita equivalence by Corollary 3.2 in [265], it is enough to prove the result for $A^\alpha$.

Let $\varepsilon > 0$, and let $F \subseteq A^\alpha$ and $E \subseteq \mathcal{D}$ be finite subsets of $A$ and $\mathcal{D}$, respectively. Use Theorem VII.4.2 to choose a unital, completely positive map $\varphi\colon \mathcal{D} \to A$ such that

1. $\|a\varphi(d) - \varphi(d)a\| < \varepsilon$ for all $a \in F$ and for all $d \in E$;

2. $\|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$ for every $d, e \in E$.

Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n\colon A \to A^\alpha$ as in the conclusion of Theorem VII.2.6. Since $\lim\limits_{n\to\infty} \psi_n(a) = a$ for all $a \in F$, we deduce that

$$\limsup_{n\to\infty} \|a\psi_n(\varphi(d)) - \psi_n(\varphi(d))a\| \leq \|a\varphi(d) - \varphi(d)a\| < \varepsilon$$

for all $a \in F$ and all $d \in E$. Likewise,

$$\limsup_{n\to\infty} \|\psi_n(\varphi(de)) - \psi_n(\varphi(d))\psi_n(\varphi(e))\| \leq \|\varphi(de) - \varphi(d)\varphi(e)\| < \varepsilon$$

for all $d, e \in E$. We conclude that for $n$ large enough, the unital completely positive map

$$\psi_n \circ \varphi\colon \mathcal{D} \to A^\alpha$$

satisfies the conclusion of Theorem VII.4.3, showing that $A^\alpha$ is $\mathcal{D}$-stable. □

Similar methods allow one to prove that the property of being approximately divisible is inherited by the crossed product and the fixed point algebra of a compact group action with the Rokhlin property. (This was first obtained by Hirshberg and Winter as part (2) of Corollary 3.4 in [122].) Our proof is completely analogous to that of Theorem VII.4.3 (using a suitable version of Theorem VII.4.2), so for the sake of brevity, we shall not present it here.

Our next goal is to show that Rokhlin actions preserve the property of having tracial rank at most one in the simple, unital case.

We will need a definition of tracial rank zero and one. What we reproduce below are not Lin's original definitions (Definition 2.1 in [169] and Definition 2.1 in [167]). Nevertheless, the notions we define are equivalent in the simple case: for tracial rank zero, this follows from Proposition 3.8 in [167], while the argument in the proof of said proposition can be adapted to show the corresponding result for tracial rank one.

**Definition VII.4.4.** Let $A$ be a simple, unital $C^*$-algebra. We say that $A$ has *tracial rank at most one*, and write $TR(A) \leq 1$, if for every finite subset $F \subseteq A$, for every $\varepsilon > 0$, and for every non-zero positive element $x \in A$, there exist a projection $p \in A$, an AI-algebra $B$, and a unital homomorphism $\varphi\colon B \to A$, such that

1. $\|ap - pa\| < \varepsilon$ for all $a \in F$;
2. $\mathrm{dist}(pap, \varphi(B)) < \varepsilon$ for all $a \in F$;
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.

Additionally, we say that $A$ has *tracial rank zero*, and write $TR(A) = 0$, if the $C^*$-algebra $B$ as above can be chosen to be finite dimensional.

We will need the following notation. For $t \in \left(0, \frac{1}{2}\right)$, we denote by $f_t\colon [0,1] \to [0,1]$ the continuous function that takes the value 0 on $[0,t]$, the value 1 on $[2t,1]$, and is linear on $[t,2t]$.

**Theorem VII.4.5.** Let $A$ be a unital, separable, simple $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ is a unital, separable, simple $C^*$-algebra with $TR(A^\alpha) \leq TR(A)$. If $G$ is finite, then the same holds for the crossed product $A \rtimes_\alpha G$.

When $G$ is not finite (but compact), then $A \rtimes_\alpha G$ is never unital, and the definition of tracial rank zero only applies to unital $C^*$-algebras.

*Proof.* Let $F \subseteq A^\alpha$ be a finite subset, let $\varepsilon > 0$ and let $x \in A^\alpha$ be a non-zero positive element. Without loss of generality, we may assume that $\|a\| \leq 1$ for all $a \in F$, and that $\varepsilon < 1$. Find $t \in \left(0, \frac{1}{2}\right)$ such that $(x-t)_+$ is not zero. Set $y = (x-t)_+$. Then $y$ belongs to $A^\alpha$ and moreover $f_t(x)y = yf_t(x) = y$.

Using that $A$ has tracial rank zero, find a unital AI-algebra $B$, a unital homomorphism $\varphi\colon B \to A$, a projection $q \in \overline{yAy}$ and a partial isometry $s \in A$ such that

- $\|ap - pa\| < \frac{\varepsilon}{9}$ for all $a \in F$;
- $\mathrm{dist}(pap, \varphi(B)) < \frac{\varepsilon}{9}$ for all $a \in F$;
- $1 - p = s^*s$ and $q = ss^*$.

Let $\widetilde{F} \subseteq B$ be a finite subset such that for all $a \in F$, there exists $b \in \widetilde{F}$ with $\|pap - \varphi(b)\| < \frac{\varepsilon}{9}$.

Since $f_t(x)$ is a unit for $\overline{yAy}$, it follows that $q = f_t(x)qf_t(x)$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n \colon A \to A^\alpha$ as in the conclusion of Theorem VII.2.6. We have

(a) $\limsup\limits_{n\to\infty} \|\psi_n(p)a - a\psi_n(p)\| < \frac{\varepsilon}{9}$ for all $a \in F$;

(b) $\limsup\limits_{n\to\infty} \operatorname{dist}(\psi_n(p)a\psi_n(a), (\psi_n \circ \varphi)(B)) < \frac{\varepsilon}{9}$ for all $a \in F$;

(c) $\lim\limits_{n\to\infty} \|\psi_n(p)a\psi_n(p) - \psi_n(pap)\| = 0$;

(d) $\lim\limits_{n\to\infty} \|\psi_n(p)^*\psi_n(p) - \psi_n(p)\| = 0$;

(e) $\lim\limits_{n\to\infty} \|1 - \psi_n(p) - \psi_n(s)^*\psi_n(s)\| = 0$;

(f) $\lim\limits_{n\to\infty} \|\psi_n(q)\psi_n(s)\psi_n(1-p) - \psi_n(s)\| = 0$;

(g) $\lim\limits_{n\to\infty} \|\psi_n(q)^*\psi_n(q) - \psi_n(q)\| = 0$;

(h) $\lim\limits_{n\to\infty} \|\psi_n(q) - \psi_n(s)\psi_n(s)^*\| = 0$;

(i) $\lim\limits_{n\to\infty} \|\psi_n(q) - f_t(x)\psi_n(q)f_t(x)\| = 0$.

With $r_n = f_t(x)\psi_n(q)f_t(x)$ for $n \in \mathbb{N}$, it follows from conditions (g) and (i) that

(j) $\lim\limits_{n\to\infty} \|r_n^*r_n - r_n\| = 0$.

Find $\delta_1 > 0$ such that whenever $e$ is an element in a $C^*$-algebra $C$ such that $\|e^*e - e\| < \delta_1$, then there exists a projection $f$ in $C$ such that $\|e - f\| < \frac{\varepsilon}{9}$. Fix a finite set $\mathcal{G} \subseteq B$ of generators for $B$. Find $\delta_2 > 0$ such that whenever $\rho \colon B \to A^\alpha$ is a unital positive linear map which is $\delta_2$-multiplicative on $\mathcal{G}$, there exists a homomorphism $\pi \colon B \to A^\alpha$ such that $\|\rho(b) - \pi(b)\| < \frac{\varepsilon}{9}$ for all $b \in \widetilde{F}$. Set $\delta = \min\{\delta_1, \delta_2\}$.

Choose $n \in \mathbb{N}$ large enough so that the quantities in conditions (a), (b), (c), (e) and (i) are less than $\frac{\varepsilon}{9}$, the quantities in (d) and (j) are less than $\delta$, the quantities in (e) and (g) are less than $1 - \varepsilon$, and so that $\psi_n \circ \varphi$ is $\delta$-multiplicative on $\mathcal{G}$. Since $r_n$ belongs to $\overline{xA^\alpha x}$ for all $n \in \mathbb{N}$, by the choice of $\delta$ there exist a projection $e$ in $\overline{xA^\alpha x}$ such that $\|e - r_n\| < \frac{\varepsilon}{9}$, and a projection $f \in A^\alpha$ such that $\|f - \psi_n(p)\| < \frac{\varepsilon}{9}$. Let $\pi \colon B \to A^\alpha$ be a homomorphism satisfying

$$\|\pi(b) - (\psi_n \circ \varphi)(b)\| < \frac{\varepsilon}{9}$$

for all $b \in \mathcal{G} \cup \widetilde{F}$.

We claim that the projection $f$ and the homomorphism $\pi\colon B \to A^\alpha$ satisfy the conditions in Definition VII.4.4.

Given $a \in F$, the estimate

$$\|af - fa\| \leq \|a\psi_n(p) - \psi_n(p)a\| + 2\|\psi_n(p) - f\| < \frac{3\varepsilon}{9} < \varepsilon$$

shows that condition (1) is satisfied. In order to check condition (2), given $a \in F$, choose $b \in \widetilde{F}$ such that

$$\|pap - \varphi(b)\| < \frac{\varepsilon}{9}.$$

Then

$$\begin{aligned}
\|faf - \pi(b)\| &\leq \|faf - \psi_n(p)a\psi_n(p)\| + \|\psi_n(p)a\psi_n(p) - \psi_n(\varphi(b))\| \\
&\quad + \|\psi_n(\varphi(b)) - \pi(b)\| \\
&< 2\|f - \psi_n(p)\| + \frac{\varepsilon}{9} + \frac{\varepsilon}{9} < \varepsilon,
\end{aligned}$$

so condition (2) is also satisfied. To check condition (3), it is enough to show that $1-f$ is Murray-von Neumann equivalent (in $A^\alpha$) to $e$. We have

$$\begin{aligned}
\|(1-f) - \psi_n(s)^*\psi_n(s)\| &\leq \|f - \psi_n(p)\| + \|1 - \psi_n(p) - \psi_n(s)^*\psi_n(s)\| \\
&< \frac{\varepsilon}{9} + 1 - \varepsilon = 1 - \frac{8\varepsilon}{9},
\end{aligned}$$

and likewise, $\|e - \psi_n(s)\psi_n(s)^*\| < \frac{\varepsilon}{9} + 1 - \varepsilon$. On the other hand, we use the approximate versions of equation (i) at the second step, and that of equation (f) at the third step, to get

$$\begin{aligned}
\|\psi_n(s) - e\psi_n(s)(1-f)\| &< \frac{2\varepsilon}{9} + \|\psi_n(s) - f_t(x)\psi_n(q)f_t(x)\psi_n(s)\psi_n(1-p)\| \\
&< \frac{3\varepsilon}{9} + \|\psi_n(s) - \psi_n(q)\psi_n(s)\psi_n(1-p)\| \\
&< \frac{4\varepsilon}{9}.
\end{aligned}$$

Now, it is immediate that

$$\begin{aligned}\|(1-f)-(e\psi_n(s)(1-f))^*(e\psi_n(s)(1-f))\| &< 2\|\psi_n(s)-e\psi_n(s)(1-f)\| \\ &\quad + \|(1-f)-\psi_n(s)^*\psi_n(s)\| \\ &< \frac{8\varepsilon}{9}+1-\frac{8\varepsilon}{9}=1.\end{aligned}$$

Likewise,

$$\|e-(e\psi_n(s)(1-f))(e\psi_n(s)(1-f))^*\|<1.$$

By Lemma 2.5.3 in [166] applied to $e\psi_n(s)(1-f)$, we conclude that $1-f$ and $e$ are Murray-von Neumann equivalent in $A^\alpha$, and the proof of the first part of the statement is complete.

It is clear that if $A$ has tracial rank zero and we choose $B$ to be finite dimensional, then the above proof shows that $A^\alpha$ has tracial rank zero as well.

Finally, if $G$ is finite, then the last claim of the statement follows from the fact that $A^\alpha$ and $A \rtimes_\alpha G$ are Morita equivalent. □

We believe that a condition weaker than the Rokhlin property ought to suffice for the conclusion of Theorem VII.4.5 to hold. In view of Theorem 2.8 in [203], we presume that fixed point algebras by a suitable version of the tracial Rokhlin property for compact group actions would preserve the class of simple $C^*$-algebras with tracial rank zero.

We present two consequences of Theorem VII.4.5. The first one is to simple AH-algebras of slow dimension growth and real rank zero, which do not a priori fit into the general framework of Theorem VII.3.10, since the building blocks are not necessarily weakly semiprojective.

**Corollary VII.4.6.** Let $A$ be a simple, unital AH-algebra with slow dimension growth and real rank zero. Let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ is a simple, unital AH-algebra with slow dimension growth and real rank zero.

*Proof.* By Proposition 2.6 in [167], $A$ has tracial rank zero. Thus $A^\alpha$ is a simple $C^*$-algebra with tracial rank zero by Theorem VII.4.5. It is clearly separable, unital, and nuclear. Moreover, it satisfies the Universal Coefficient Theorem by Theorem VII.3.13. Since AH-algebras of slow dimension growth and real rank zero exhaust the Elliott invariant of $C^*$-algebras with tracial rank zero, Theorem 5.2 in [168] implies that $A^\alpha$ is an AH-algebra with slow dimension growth and real rank zero. $\square$

Denote by $\mathcal{Q}$ the universal UHF-algebra. Recall that a simple, separable, unital $C^*$-algebra $A$ is said to be *have rational tracial rank at most one*, if $TR(A \otimes \mathcal{Q}) \leq 1$.

**Corollary VII.4.7.** Let $A$ be a simple, separable, unital $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ has rational tracial rank at most one, then so does $A^\alpha$ (and also $A \rtimes_\alpha G$ if $G$ is finite).

*Proof.* The result is an immediate consequence of Theorem VII.4.5 applied to the action $\alpha \otimes \mathrm{id}_{\mathcal{Q}}\colon G \to \mathrm{Aut}(A \otimes \mathcal{Q})$. $\square$

We now turn to pure infiniteness in the non-simple case. The following is Definition 4.1 in [152]

**Definition VII.4.8.** A $C^*$-algebra $A$ is said to be *purely infinite* if the following conditions are satisfied:

1. There are no non-zero characters (that is, homomorphisms onto the complex numbers) on $A$, and
2. For every pair $a, b$ of positive elements in $A$, with $b$ in the ideal generated by $a$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $A$ such that $\lim\limits_{n\to\infty} \|x_n^* b x_n - a\| = 0$.

The following is Theorem 4.16 in [152] (see also Definition 3.2 in [152]).

**Theorem VII.4.9.** Let $A$ be a $C^*$-algebra. Then $A$ is purely infinite if and only if for every nonzero positive element $a \in A$, we have $a \oplus a \preceq a$.

We use the above result to show that, in the presence of the Rokhlin property, pure infiniteness is inherited by the fixed point algebra and the crossed product.

**Proposition VII.4.10.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ is purely infinite, then so are $A^\alpha$ and $A \rtimes_\alpha G$.

*Proof.* By Proposition VII.2.2 and Theorem 4.23 in [152], it is enough to prove the result for $A^\alpha$. Let $a$ be a nonzero positive element in $A^\alpha$. Since $A$ is purely infinite, by Theorem 4.16 in [152] (here reproduced as Theorem VII.4.9), there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $A$ such that

(a) $\lim\limits_{n\to\infty} \|x_n^* a x_n - a\| = 0$;

(b) $\lim\limits_{n\to\infty} \|x_n^* a y_n\| = 0$;

(c) $\lim\limits_{n\to\infty} \|y_n^* a x_n\| = 0$;

(d) $\lim\limits_{n\to\infty} \|y_n^* a y_n - a\| = 0$.

Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $\psi_n \colon A \to A^\alpha$ as in the conclusion of Theorem VII.2.6. Easy applications of the triangle inequality yield

(a') $\lim\limits_{n\to\infty} \|\psi_n(x_n)^* a \psi_n(x_n) - a\| = 0$;

(b') $\lim\limits_{n\to\infty} \|\psi_n(x_n)^* a \psi_n(y_n)\| = 0$;

(c') $\lim\limits_{n\to\infty} \|\psi_n(y_n)^* a \psi_n(x_n)\| = 0$;

(d') $\lim\limits_{n\to\infty} \|\psi_n(y_n)^* a \psi_n(y_n) - a\| = 0$.

Since $\psi_n(x_n)$ and $\psi_n(y_n)$ belong to $A^\alpha$ for all $n \in \mathbb{N}$, we conclude that $a \oplus a \precsim a$ in $A^\alpha$. It now follows from Theorem 4.16 in [152] (here reproduced as Theorem VII.4.9) that $A^\alpha$ is purely infinite, as desired. □

It is well known (see Corollary 4.6 in [135], here reproduced as part (2) of Theorem II.2.8) that reduced crossed products by pointwise outer actions of discrete groups of purely infinite simple $C^*$-algebras are again purely infinite and simple. In particular, pointwise outer actions of countable amenable discrete groups preserve the class of Kirchberg algebras. The analogous statement for locally compact groups, or even compact groups, is, however, not true. For example, the gauge action $\gamma$ of $\mathbb{T}$ on the Cuntz algebra $\mathcal{O}_\infty$, given by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta$ in $\mathbb{T}$ and all $j$ in $\mathbb{N}$, is pointwise outer by the Theorem in [181], and its crossed product $\mathcal{O}_\infty \rtimes_\gamma \mathbb{T}$ is a non-simple AF-algebra, so it is far from being (simple and) purely infinite.

Hence, even though actions with the Rokhlin property are easily seen to be pointwise outer, this does not imply the following result.

**Corollary VII.4.11.** Let $A$ be a unital Kirchberg algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $A^\alpha$ and $A \rtimes_\alpha G$ are Kirchberg algebras.

*Proof.* It is well-known that $A^\alpha$ and $A \rtimes_\alpha G$ are nuclear and separable. Simplicity follows from Corollary VII.2.9, and pure infiniteness follows from Proposition VII.4.10. □

In the following proposition, the Rokhlin property is surely stronger than necessary for the conclusion to hold, although some condition on the action must be imposed. We do not know, for instance, whether finite Rokhlin dimension with commuting towers preserves real rank zero and stable rank one.

**Proposition VII.4.12.** Let $A$ be a unital, separable $C^*$-algebra, let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

1. If $A$ has real rank zero, then so do $A^\alpha$ and $A \rtimes_\alpha G$.

2. If $A$ has stable rank one, then so do $A^\alpha$ and $A \rtimes_\alpha G$.

*Proof.* By Proposition VII.2.2, Theorem 3.3 in [229], and Theorem 2.5 in [21], it is enough to prove the proposition for $A^\alpha$. Since the proofs of both parts are similar, we only prove the first one.

Let $a$ be a self-adjoint element in $A^\alpha$ and let $\varepsilon > 0$. Since $A$ has real rank zero, there exists an invertible self-adjoint element $b$ in $A$ such that $\|b - a\| < \frac{\varepsilon}{2}$. Let $(\psi_n)_{n\in\mathbb{N}}$ be a sequence of unital completely positive maps $A \to A^\alpha$ as in the conclusion of Theorem VII.2.6. Then $\psi_n(b)$ is self-adjoint for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} \left\|\psi_n(b)\psi_n(b^{-1}) - 1\right\| = \lim_{n\to\infty} \left\|\psi_n(b^{-1})\psi_n(b) - 1\right\| = 0 \ \text{ and}$$
$$\lim_{n\to\infty} \|\psi_n(a) - a\| = 0.$$

Choose $n$ large enough so that

$$\left\|\psi_n(b)\psi_n(b^{-1}) - 1\right\| < 1 \text{ and } \left\|\psi_n(b^{-1})\psi_n(b) - 1\right\| < 1,$$

and also so that $\|\psi_n(a) - a\| < \frac{\varepsilon}{2}$. Then $\psi_n(b)\psi_n(b^{-1})$ and $\psi_n(b^{-1})\psi_n(b)$ are invertible, and hence so is $\psi_n(b)$. Finally,

$$\|a - \psi_n(b)\| \leq \|a - \psi_n(a)\| + \|\psi_n(a) - \psi_n(b)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which shows that $A^\alpha$ has real rank zero. $\square$

We now turn to traces. For a trace $\tau$ on a $C^*$-algebra $A$, we also denote by $\tau$ its amplification to any matrix algebra $M_n(A)$. We denote by $T(A)$ the set of all tracial states on $A$.

The following is one of Blackadar's fundamental comparability questions:

**Definition VII.4.13.** Let $A$ be a simple unital $C^*$-algebra. We say the the *order on projections (in $A$) is determined by traces*, if whenever $p$ and $q$ are projections in $M_\infty(A)$ satisfying $\tau(p) \leq \tau(q)$ for all $\tau \in T(A)$, then $p \precsim_{M-vN} q$.

The following extends, with a simpler proof, Proposition 4.8 in [191].

**Proposition VII.4.14.** Let $A$ be a simple unital $C^*$-algebra, and suppose that the order on its projections is determined by traces. Let $G$ be a second countable compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the order on projections in $A^\alpha$ is determined by traces.

*Proof.* Since $\alpha \otimes \mathrm{id}_{M_n}\colon G \to \mathrm{Aut}(A \otimes M_n)$ has the Rokhlin property and $(A \otimes M_n)^{\alpha\otimes\mathrm{id}_{M_n}} = A^\alpha \otimes M_n$, in Definition VII.4.13 it is enough to consider projections in the algebra.

So let $p$ and $q$ be projections in $A^\alpha$, and suppose that it is not the case that $p \precsim_{M-vN} q$ in $A^\alpha$. We want to show that there exists a tracial state $\tau$ on $A^\alpha$ such that $\tau(p) \geq \tau(q)$. By part (1) in Proposition VI.3.2, it is not the case that $p \precsim_{M-vN} q$ in $A$, so there exists a tracial state $\omega$ on $A$ such that $\omega(p) \geq \omega(q)$. Now take $\tau = \omega|_{A^\alpha}$. $\square$

Finally, we close this section by exploring the extent to which semiprojectivity passes from $A$ to the fixed point algebra and the crossed product by a compact group with the Rokhlin property. Even though we have not been able to answer this question for semiprojectivity, we can provide a satisfying answer for *weak* semiprojectivity.

In order to show this, we introduce the following technical definition, which is inspired in the notion of "corona extendibility" (Definition 1.1 in [174]; we are thankful to Hannes Thiel for providing this reference).

**Definition VII.4.15.** Let $\theta\colon A \to B$ be a homomorphism between $C^*$-algebras $A$ and $B$. We say that $\theta$ is *sequence algebra extendible*, if whenever $E$ is a $C^*$-algebra and $\varphi\colon A \to E_\infty$ is a homomorphism, there exists a homomorphism $\rho\colon B \to E_\infty$ such that $\varphi = \psi \circ \theta$.

In analogy with Lemma 1.4 in [174], we have the following:

**Lemma VII.4.16.** Let $\theta\colon A \to B$ be a sequence algebra extendible homomorphism between $C^*$-algebras $A$ and $B$. If $B$ is weakly semiprojective, then so is $A$.

*Proof.* This is straightforward. □

In the next proposition, we show that weak semiprojectivity passes to fixed point algebras of actions with the Rokhlin property (and to crossed products, whenever the group is finite). Our conclusions seem not to be obtainable with the methods developed in [191], since it is not in general true that a corner of a weakly semiprojective $C^*$-algebra is weakly semiprojective.

**Proposition VII.4.17.** Let $G$ be a second countable compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the canonical inclusion $\iota\colon A^\alpha \to A$ is sequence algebra extendible (Definition VII.4.15).

In particular, if $A$ is weakly semiprojective, then so is $A^\alpha$. If in addition $G$ is finite, then $A \rtimes_\alpha G$ is also weakly semiprojective.

If $G$ is not finite, and even if $A$ is semiprojective, $A \rtimes_\alpha G$ need not be weakly semiprojective. Indeed, $C(\mathbb{T}) \rtimes_{\mathtt{Lt}} \mathbb{T} \cong \mathcal{K}(L^2(\mathbb{T}))$ is not weakly semiprojective.

*Proof.* Use Theorem VII.2.6 to choose a sequence $(\psi_n)_{n\in\mathbb{N}}$ of asymptotically $*$-multiplicative linear maps $\psi_n\colon A \to A^\alpha$ such that $\lim\limits_{n\to\infty} \|\psi_n(a) - a\| = 0$ for all $a \in A^\alpha$. Regard $(\psi_n)_{n\in\mathbb{N}}$ as a homomorphism $\psi\colon A \to (A^\alpha)_\infty$ such that the restriction $\psi|_{A^\alpha}$ agrees with the canonical inclusion $A^\alpha \hookrightarrow (A^\alpha)_\infty$.

Let $E$ be a $C^*$-algebra and let $\varphi\colon A^\alpha \to E_\infty$ be a homomorphism. Denote by

$$\varphi_\infty\colon (A^\alpha)_\infty \to (E_\infty)_\infty \cong E_\infty$$

the homomorphism induced by $\varphi$. (The isomorphism $(E_\infty)_\infty \cong E_\infty$ above is specifically the one obtained by taking diagonal sequences.) There is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] & (A^\alpha)_\infty \arrow[d, "\varphi_\infty"] \\ A^\alpha \arrow[u, "\iota"] \arrow[r, "\varphi"'] & B_\infty. \end{tikzcd}$$

Then the map $\rho = \varphi_\infty \circ \psi\colon A \to B_\infty$ satisfies the conditions in Definition VII.4.15.

The second claim follows form Lemma VII.4.16. Finally, if $G$ is finite, then $A \rtimes_\alpha G$ can be canonically identified with $M_{|G|} \otimes A^\alpha$, and hence it is also weakly semiprojective. $\square$

*The non-unital case*

In view of the results in [243], one may wish to generalize the results in this chapter to actions of compact groups on not necessarily unital $C^*$-algebras. The definition of the Rokhlin property for a compact group action on an arbitrary $C^*$-algebra should be along the lines of Definition 3.1 in [188]. Using the right definition, one should be able to prove a theorem analogous to Theorem VII.2.6, using the techniques from Chapter V and [243]. Once this is achieved, most of the results in this chapter would then carry over to the (separable) non-unital setting as well.

We intend to explore this direction in a future project.

# CHAPTER VIII

# EQUIVARIANT HOMOMORPHISMS, ROKHLIN CONSTRAINTS AND EQUIVARIANT UHF-ABSORPTION

This Chapter is based on joint work with Luis Santiago ([91]).

We classify equivariant homomorphisms between C*-dynamical systems associated to actions of finite groups with the Rokhlin property. In addition, the given actions are classified. An obstruction is obtained for the Cuntz semigroup of a $C^*$-algebra allowing such an action. We also obtain an equivariant UHF-absorption result.

## Introduction

In this chapter, which is based on [91], we extend the classification results of Izumi and Nawata of finite group actions on $C^*$-algebrasi with the Rokhlin property to actions of finite groups with the Rokhlin property on arbitrary separable $C^*$-algebras. This is done by first obtaining a classification result for equivariant homomorphism between C*-dynamical systems associated to actions of finite groups with the Rokhlin property, and then applying Elliott's intertwining argument. In this chapter we also obtain obstructions on the Cuntz semigroup, the Murray-von Neumann semigroup, and the K-groups of a $C^*$-algebra allowing an action of a finite group with the Rokhlin property. These results are used together with the classification result of actions to obtain an equivariant UHF-absorption result.

This chapter is organized as follows. In Section VIII.2, we collect a number of definitions and results that will be used throughout the chapter. In Section VIII.3, we give an abstract classification for equivariant homomorphism between C*-dynamical systems associated to actions of finite groups with the Rokhlin property, as well as, a classification for the given actions. These abstract classification results are used together with known classification results of $C^*$-algebras to obtain specific classification of equivariant homomorphisms and actions of finite groups on $C^*$-algebras that can be written as inductive limits of 1-dimensional NCCW-complexes with trivial $K_1$-groups and for unital simple AH-algebras of no dimension growth.

In Section VIII.4, we obtain obstructions on the Cuntz semigroup, the Murray-von Neumann semigroup, and the $K_*$-groups of a $C^*$-algebra allowing an action of a finite group with the Rokhlin property. Then using the Cuntz semigroup obstruction we show that the Cuntz semigroup of a $C^*$-algebra that admits an action of finite group with the Rokhlin property has certain divisibility property. In this section we also compute the Cuntz semigroup, the Murray-von Neumann semigroup, and the $K_*$-groups of the fixed-point and crossed product $C^*$-algebras associated to an action of a finite group with the Rokhlin property.

In Section VIII.5, we obtain divisibility results for the Cuntz semigroup of certain classes of $C^*$-algebras and use this together with the classification results for actions obtained in Section VIII.3 to prove an equivariant UHF-absorbing result.

## Classification of Actions and Equivariant $*$-homomorphisms

In this section we classify equivariant homomorphisms whose codomain C*-dynamical system have the Rokhlin property. We use this results to classify actions of finite groups on separable $C^*$-algebras with the Rokhlin property. Our results complement and extend those obtained by Izumi in [132] and [133] in the unital setting, and by Nawata in [188] for $C^*$-algebras $A$ that satisfy $A \subseteq \overline{\mathrm{GL}(\widetilde{A})}$.

### *Equivariant homomorphisms*

Let us briefly recall the definition of the Rokhlin property, in the sense of [243, Definition 2], for actions of finite groups on (not necessarily unital) $C^*$-algebras. Actions with the Rokhlin property are the main object of study of this work.

**Definition VIII.2.1.** Let $A$ be a $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on $A$. We say that $\alpha$ has the *Rokhlin property* if for any $\varepsilon > 0$ and any finite subset $F \subseteq A$ there exist mutually orthogonal positive contractions $r_g$ in $A$, for $g \in G$, such that

(1) $\|\alpha_g(r_h) - r_{gh}\| < \varepsilon$ for all $g, h \in G$;

(2) $\|r_g a - a r_g\| < \varepsilon$ for all $a \in F$ and all $g \in G$;

(3) $\left\| \left( \sum\limits_{g\in G} r_g \right) a - a \right\| < \varepsilon$ for all $a \in F$.

The elements $r_g$, for $g \in G$, will be called *Rokhlin elements* for $\alpha$ for the choices of $\varepsilon$ and $F$.

It was shown in [243, Corollary 1] that Definition VIII.2.1 agrees with [132, Definition 3.1] whenever the $C^*$-algebra $A$ is unital. It is also shown in [243, Corollary 2] that Definition VIII.2.1 agrees with [188, Definition 3.1] whenever the $C^*$-algebra $A$ is separable.

The following is a characterization of the Rokhlin property in terms of elements of the sequence algebra $A^\infty$ ([243, Proposition 1]):

**Lemma VIII.2.2.** Let $A$ be a $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group $G$ on $A$. Then the following are equivalent:

(1) $\alpha$ has the Rokhlin property.

(2) For any finite subset $F \subseteq A$ there exist mutually orthogonal positive contractions $r_g$ in $A^\infty \cap F'$, for $g \in G$, such that

  (a) $\alpha_g(r_h) = r_{gh}$ for all $g, h \in G$;

  (b) $\left(\sum_{g\in G} r_g\right) b = b$ for all $b \in F$.

(3) For any separable C*-subalgebra $B \subseteq A$ there are orthogonal positive contractions $r_g$ in $A^\infty \cap B'$ for $g \in G$ such that

  (a) $\alpha_g(r_h) = r_{gh}$ for all $g, h \in G$;

  (b) $\left(\sum_{g\in G} r_g\right) b = b$ for all $b \in B$.

The first part of the following proposition is [243, Theorem 2 (i)]. The second part follows trivially from the definition of the Rokhlin property.

**Proposition VIII.2.3.** Let $G$ be a finite group, let $A$ be a $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property.

1. If $B$ is any $C^*$-algebra and $\beta\colon G \to \mathrm{Aut}(B)$ is any action of $G$ on $B$, then the action
$$\alpha \otimes \beta\colon G \to \mathrm{Aut}(A \otimes_{\min} B)$$
defined by $(\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$ for all $g \in G$, has the Rokhlin property.

2. If $B$ is a $C^*$-algebra and $\varphi\colon A \to B$ is an isomorphism, then the action $g \mapsto \varphi \circ \alpha_g \circ \varphi^{-1}$ of $G$ on $B$ has the Rokhlin property.

The following example may be regarded as the "generating" Rokhlin action for a given finite group $G$. For some classes of $C^*$-algebras, it can be shown that every action of $G$ with the Rokhlin property tensorially absorbs the action we construct below. See [133, Theorems 3.4 and 3.5] and Theorem VIII.4.10 below.

**Example VIII.2.4.** Let $G$ be a finite group. Let $\lambda\colon G \to U(\ell^2(G))$ be the left regular representation, and identify $\ell^2(G)$ with $\mathbb{C}^{|G|}$. Define an action $\mu^G\colon G \to \mathrm{Aut}(M_{|G|^\infty})$ by

$$\mu_g^G = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(\lambda_g)$$

for all $g \in G$. It is easy to check that $\alpha$ has the Rokhlin property. Note that $\mu_g^G$ is approximately inner for all $g \in G$.

It follows from part (1) of Proposition VIII.2.3 that any action of the form $\alpha \otimes \mu^G$ has the Rokhlin property. One of our main results, Theorem VIII.4.10, states that in some circumstances, every action with the Rokhlin property has this form.

Let $A$ and $B$ be $C^*$-algebras and let $G$ be a finite group. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions. Recall that a homomorphism $\phi\colon A \to B$ is said to be *equivariant* if $\phi \circ \alpha_g = \beta_g \circ \phi$ for all $g \in G$.

**Definition VIII.2.5.** Let $A$ and $B$ be $C^*$-algebras and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of a finite group $G$. Let $\phi, \psi\colon A \to B$ be equivariant homomorphisms. We say that $\phi$ and $\psi$ are *equivariantly approximately unitarily equivalent*, and denote this by $\phi \sim_{\mathrm{G-au}} \psi$, if for any finite subset $F \subseteq A$ and for any $\varepsilon > 0$ there exists a unitary $u \in \widetilde{B^\beta}$ such that

$$\|\phi(a) - u^*\psi(a)u\| < \varepsilon,$$

for all $a \in F$.

Note that when $G$ is the trivial group, this definition agrees with the standard definition of approximate unitary equivalence of homomorphisms. In this case we will omit the group $G$ in the notation $\sim_{\mathrm{G-au}}$, and write simply $\sim_{\mathrm{au}}$.

The following lemma can be proved using a standard semiprojectivity argument. Its proof is left to the reader.

**Lemma VIII.2.6.** Let $A$ be a unital $C^*$-algebra and let $u$ be a unitary in $A_\infty$. Given $\varepsilon > 0$ and given a finite subset $F \subseteq A$, there exists a unitary $v \in A$ such that $\|va - av\| < \varepsilon$ for all $a \in F$. If moreover $A$ is separable, then there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $A$ with

$$\lim_{n\to\infty} \|u_n a - a u_n\| = 0$$

for all $a \in A$, such that $\pi_A((u_n)_{n\in\mathbb{N}}) = u$ in $A_\infty$.

**Proposition VIII.2.7.** Let $A$ and $B$ be $C^*$-algebras and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of a finite group $G$ such that $\beta$ has the Rokhlin property. Let $\phi, \psi\colon (A, \alpha) \to (B, \beta)$ be equivariant homomorphisms such that $\phi \sim_{\mathrm{au}} \psi$. Then $\phi \sim_{G-\mathrm{au}} \psi$.

*Proof.* Let $F$ be a finite subset of $A$ and let $\varepsilon > 0$. We have to show that there exists a unitary $w \in \widetilde{B^\beta}$ such that

$$\|\phi(a) - w^*\psi(a)w\| < \varepsilon,$$

for all $a \in F$. Set $F' = \bigcup_{g\in G} \alpha_g(F)$, which is again a finite subset of $A$. Since $\phi \sim_{\mathrm{au}} \psi$, there exists a unitary $u \in \widetilde{B}$ such that

$$\|\phi(b) - u^*\psi(b)u\| < \varepsilon \tag{VIII.1}$$

for all $b \in F'$. Choose $x \in B$ and $\lambda \in \mathbb{C}$ of modulus 1 such that $u = x + \lambda 1_{\widetilde{B}}$. Then equation (VIII.1) above is satisfied if one replaces $u$ with $\overline{\lambda} u$. Thus, we may assume that the unitary $u$ has the form $u = x + 1_{\widetilde{B}}$ for some $x \in B$.

Fix $g \in G$ and $a \in F$. Then $b = \alpha_{g^{-1}}(a)$ belongs to $F'$. Using equation (VIII.1) and the fact that $\phi$ and $\psi$ are equivariant, we get

$$\|\beta_{g^{-1}}(\phi(a)) - u^*\beta_{g^{-1}}(\psi(a))u\| < \varepsilon.$$

By applying $\beta_g$ to the inequality above, we conclude that

$$\|\phi(a) - \beta_g(u)^*\psi(a)\beta_g(u)\| < \varepsilon$$

for all $a \in F$ and $g \in G$

Choose positive orthogonal contractions $(r_g)_{g\in G} \subseteq B_\infty$ as in the definition of the Rokhlin property for $\beta$, and set $v = \sum\limits_{g\in G} \beta_g(x) r_g + 1_{\widetilde{B}}$. Using that $x_g + 1_{\widetilde{B}}$ is a unitary in $\widetilde{B}$, one checks that

$$v^* v = \sum_{g\in G} \big(\beta_g(x^* x) r_g^2 + \beta_g(x) r_g + \beta_g(x) r_g\big) + 1_{\widetilde{B}} = 1_{\widetilde{B}}.$$

Analogously, we have $vv^* = 1_{\widetilde{B}}$, and hence $v$ is a unitary in $\widetilde{B}$. For every $b \in B$, we have

$$v^* b v = \sum_{g\in G} r_g \beta_g(u)^* b \beta_g(u).$$

Therefore,

$$\|\phi(a) - v^*\psi(a)v\| = \left\| \sum_{g\in G} r_g\phi(a) - \sum_{g\in G} r_g \beta_g(u)^*\psi(a)\beta_g(u) \right\| < \varepsilon,$$

for all $a \in F$ (here we are considering $\phi$ and $\psi$ as maps from $A$ to $(\widetilde{B})^\infty$, by composing them with the natural inclusion of $B$ in $(\widetilde{B})^\infty$). Since $v = \sum\limits_{g\in G} \beta_g(x r_e) + 1_{\widetilde{B}}$, we have $v \in (\widetilde{B^\beta})^\infty \subseteq (\widetilde{B})^\infty$. By Lemma VIII.2.6, we can choose a unitary $w \in \widetilde{B^\beta}$ such that

$$\|\phi(a) - w^*\psi(a)w\| < \varepsilon,$$

for all $a \in F$, and the proof is finished. □

**Lemma VIII.2.8.** Let $A$ and $B$ be $C^*$-algebras and let $\psi\colon A \to B$ be a homomorphism. Suppose there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of unitaries in $\widetilde{B}$ such that the sequence $(v_n\phi(x)v_n^*)_{n\in\mathbb{N}}$ converges in $B$ for all $x$ in a dense subset of $A$. Then there exists a homomorphism $\psi\colon A \to B$ such that

$$\lim_{n\to\infty} v_n\phi(x)v_n^* = \psi(x)$$

for all $x \in A$.

*Proof.* Let

$$S = \{x \in A\colon (v_n\phi(x)v_n^*)_{n\in\mathbb{N}} \text{ converges in } B\} \subseteq A.$$

Then $S$ is a dense *-subalgebra of $A$. For each $x \in S$, denote by $\psi_0(x)$ the limit of the sequence $(v_n\phi(x)v_n^*)_{n\in\mathbb{N}}$. The map $\psi_0\colon S \to B$ is linear, multiplicative, preserves the adjoint operation, and is bounded by $\|\phi\|$, so it extends by continuity to a homomorphism $\psi\colon A \to B$. Given $a \in A$ and given $\varepsilon > 0$, use density of $S$ in $A$ to choose $x \in S$ such that $\|a - x\| < \frac{\varepsilon}{3}$. Choose $N \in \mathbb{N}$ such that $\|v_N\phi(x)v_N^* - \psi(x)\| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}\|\psi(a) - v_N\phi(a)v_N^*\| &\leq \|\psi(a-x)\| + \|\psi(x) - v_N\phi(x)v_N^*\| \\ &\quad + \|v_N\phi(x)v_N^* - v_N\phi(a)v_N^*\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

It follows that $\psi(a) = \lim\limits_{n\to\infty} v_n\phi(a)v_n^*$ for all $a \in A$, as desired. □

The unital case of the following proposition is [132, Lemma 5.1]. Our proof for arbitrary C*-dynamical systems follows similar ideas.

**Proposition VIII.2.9.** Let $A$ and $B$ be $C^*$-algebras and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of a finite group $G$. Suppose that $A$ is separable and that $\beta$ has the Rokhlin property. Let $\phi\colon A \to B$ be a homomorphism such that $\beta_g \circ \phi \sim_{\mathrm{au}} \phi \circ \alpha_g$ for all $g \in G$. Then:

(1) For any $\varepsilon > 0$ and for any finite set $F \subseteq A$ there exists a unitary $u \in \widetilde{B}$ such that

$$\begin{aligned}&\|(\beta_g \circ \mathrm{Ad}(w) \circ \phi)(x) - (\mathrm{Ad}(w) \circ \phi \circ \alpha_g)(x)\| < \varepsilon, \quad \forall g \in G, \forall x \in F, \\ &\|(\mathrm{Ad}(w) \circ \phi)(x) - \phi(x)\| < \varepsilon + \sup_{g\in G} \|(\beta_g \circ \phi \circ \alpha_{g^{-1}})(x) - \phi(x)\|, \quad \forall x \in F.\end{aligned} \tag{VIII.2}$$

(2) There exists an equivariant homomorphism $\psi\colon A \to B$ that is approximately unitarily equivalent to $\phi$.

*Proof.* (1) Let $F$ be a finite subset of $A$ and let $\varepsilon > 0$. Set $F' = \bigcup\limits_{g\in G} \alpha_g(F)$, which is a finite subset of $A$. Since $\beta_g \circ \phi \sim_{\mathrm{au}} \phi \circ \alpha_g$ for all $g \in G$, there exist unitaries $(u_g)_{g\in G} \subseteq \widetilde{B}$ such that

$$\|(\beta_g \circ \phi)(a) - (\mathrm{Ad}(u_g) \circ \phi \circ \alpha_g)(a)\| < \frac{\varepsilon}{2},$$

for all $a \in F'$ and $g \in G$. Upon replacing $u_g$ with a scalar multiple of it, one can assume that there are $(x_g)_{g\in G} \subseteq B$ such that $u_g = x_g + 1_{\widetilde{B}}$ for all $g \in G$. For $a \in F$ and $g, h \in G$, we have

$$\begin{aligned}
&\|(\mathrm{Ad}(u_g) \circ \phi \circ \alpha_h)(a) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \phi)(a)\| \\
&\qquad = \big\|(\mathrm{Ad}(u_g) \circ \phi \circ \alpha_g)(\alpha_{g^{-1}h}(a)) \\
&\qquad\qquad -(\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \phi \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(a))\big\| \\
&\qquad \leq \big\|(\mathrm{Ad}(u_g) \circ \phi \circ \alpha_g)(\alpha_{g^{-1}h}(a)) - (\beta_g \circ \phi)(\alpha_{g^{-1}h}(a))\big\| \\
&\qquad\quad + \big\|(\beta_g \circ \phi)(\alpha_{g^{-1}h}(x)) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \phi \circ \alpha_{h^{-1}g})(\alpha_{g^{-1}h}(x))\big\| \\
&\qquad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Choose positive orthogonal contractions $(r_g)_{g\in G} \subseteq B_\infty$ as in the definition of the Rokhlin property for $\beta$, and set

$$u = \sum_{g\in G} r_g x_g + 1_{\widetilde{B}} \in (\widetilde{B})^\infty.$$

Using that $x_g + 1_{\widetilde{B}}$ is a unitary in $\widetilde{B}$, one checks that

$$u^* u = 1_{\widetilde{B}} + \sum_{g\in G} (r_g^2 x_g^* x_g + r_g x_g + r_g x_g^*) = 1_{\widetilde{B}}.$$

Analogously, one also checks that $uu^* = 1_{\widetilde{B}}$, thus showing that $u$ is a unitary in $(\widetilde{B})^\infty$. The map $\mathrm{Ad}(u)$ can be written in terms of the maps $\mathrm{Ad}(u_g)$ and the contractions $(r_g)_{g\in G}$, as follows:

$$(\mathrm{Ad}(u))(x) = uxu^* = \sum_{g\in G} (u_g x u_g^*) r_g = \sum_{g\in G} (\mathrm{Ad}(u_g))(x) r_g,$$

for all $x \in A$. Now for $a \in F$ and considering $\phi$ as a map from $A$ to $(\widetilde{B})^\infty$ by composing it with the natural inclusion of $B$ in $(\widetilde{B})^\infty$, we have the following identities

$$\begin{aligned}
(\beta_h \circ \mathrm{Ad}(u) \circ \phi)(a) &= \sum_{g\in G} r_{hg} (\beta_h \circ \mathrm{Ad}(u_g) \circ \phi)(a) = \sum_{g\in G} r_g (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \phi)(a), \\
(\mathrm{Ad}(u) \circ \phi \circ \alpha_h)(a) &= \sum_{g\in G} r_g (\mathrm{Ad}(u_g) \circ \phi \circ \alpha_h)(a).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(\beta_h \circ \mathrm{Ad}(u) \circ \phi)(a) - (\mathrm{Ad}(u) \circ \phi \circ \alpha_h)(a)\| \\
&\qquad\qquad \leq \sup_{g \in G} \left\|(\mathrm{Ad}(u_g) \circ \phi \circ \alpha_h)(a) - (\beta_h \circ \mathrm{Ad}(u_{h^{-1}g}) \circ \phi)(a)\right\| < \varepsilon.
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
\|(\mathrm{Ad}(u) \circ \phi)(a) - \phi(a)\| &= \left\| \sum_{g \in G} r_g((\mathrm{Ad}(u_g) \circ \phi)(a) - \phi(a)) \right\| \\
&\leq \sup_{g \in G} \|(\mathrm{Ad}(u_g) \circ \phi)(a) - \phi(a)\| \\
&\leq \sup_{g \in G} \big( \left\|(\mathrm{Ad}(u_g) \circ \phi \circ \alpha_g)(\alpha_{g^{-1}}(a)) - (\beta_g \circ \phi)(\alpha_{g^{-1}}(a))\right\| \\
&\quad + \left\|(\beta_g \circ \phi \circ \alpha_{g^{-1}})(a) - \phi(a)\right\| \big) \\
&\leq \varepsilon + \sup_{g \in G} \left\|(\beta_g \circ \phi \circ \alpha_{g^{-1}})(a) - \phi(a)\right\|.
\end{aligned}$$

We have shown that the inequalities in (VIII.2) hold for a unitary $u \in (\widetilde{B})^{\infty}$. By Lemma VIII.2.6, we can replace $u$ with a unitary in $w \in \widetilde{B}$ in such a way that both inequalities still hold for $w$ in place of $u$.

(2) Let $(F_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of $A$ whose union is dense in $A$. Upon replacing each $F_n$ with $\bigcup_{g \in G} \alpha_g(F_n)$, we may assume that $\alpha_g(F_n) = F_n$ for all $g \in G$ and $n \in \mathbb{N}$. Set $\phi_1 = \phi$ and find a unitary $u_1 \in \widetilde{B}$ such that the conclusion of the first part of the proposition is satisfied with $\phi_1$ and $\varepsilon = 1$. Set $\phi_2 = \mathrm{Ad}(u_1) \circ \phi_1$, and find a unitary $u_2 \in \widetilde{B}$ such that the conclusion of the first part of the proposition is satisfied with $\phi_2$ and $\varepsilon = \frac{1}{2}$. Iterating this process, there exist homomorphisms $\phi_n \colon A \to B$ with $\phi_1 = \phi$ and unitaries $(u_n)_{n \in \mathbb{N}}$ in $\widetilde{B}$ such that $\phi_{n+1} = \mathrm{Ad}(u_n) \circ \phi_n$, for all $n \in \mathbb{N}$, which moreover for all $n \in \mathbb{N}$ satisfy

$$\|(\beta_g \circ \phi_n)(x) - (\phi_n \circ \alpha_g)(x)\| < \frac{1}{2^n}$$

for all $g \in G$ and for all $x \in F_n$, and

$$\|\phi_{n+1}(x) - \phi_n(x)\| < \frac{3}{2^n}$$

for all $x \in F_n$. For each $n \in \mathbb{N}$ set $v_n = u_n \cdots u_1$. Then the sequence of unitaries $(v_n)_{n\in\mathbb{N}}$ in $\widetilde{B}$ and the homomorphism $\phi\colon A \to B$ satisfy the hypotheses of Lemma VIII.2.8, so it follows that the sequence $(\phi_n)_{n\in\mathbb{N}}$ converges to a homomorphism $\psi\colon A \to B$ that satisfies $\beta_g \circ \psi = \psi \circ \alpha_g$ for all $g \in G$; that is, $\psi$ is equivariant. Since each $\phi_n$ is unitarily equivalent to $\phi$, we conclude that $\phi$ and $\psi$ are approximately unitarily equivalent. $\square$

*Categories of $C^*$-dynamical systems and abstract classification*

Let $G$ be a second countable compact group and let $\mathbf{A}$ denote the category of separable $C^*$-algebras. Let us denote by $\mathbf{A}_G$ the category whose objects are $G$-C*-dynamical systems $(A, \alpha)$, that is, $A$ is a $C^*$-algebra and $\alpha\colon G \to \mathrm{Aut}(A)$ is a strongly continuous action, and whose morphisms are equivariant homomorphisms. We use the notation $\phi\colon (A, \alpha) \to (B, \beta)$ to denote equivariant homomorphisms $\phi\colon A \to B$. Approximate unitary equivalence of maps in this category is given in Definition VIII.2.5.

If $\mathbf{B}$ is a subcategory of $\mathbf{A}$, we denote by $\mathbf{B}_G$ the full subcategory of $\mathbf{A}_G$ whose objects are C*-dynamical systems $(A, \alpha)$ with $A$ in $\mathbf{B}$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathbf{B}_G}((A, \alpha), (B, \beta)) = \mathrm{Hom}_{\mathbf{A}_G}((A, \alpha), (B, \beta)).$$

**Definition VIII.2.10.** Let $\mathbf{B}$ be a subcategory of $\mathbf{A}$. Let $\mathrm{F}\colon \mathbf{B}_G \to \mathbf{C}$ be a functor from the category $\mathbf{B}_G$ to a category $\mathbf{C}$. We say that the functor F *classifies homomorphisms* if:

(a) For every pair of objects $(A, \alpha)$ and $(B, \beta)$ in $\mathbf{B}_G$ and for every morphism

$$\lambda\colon \mathrm{F}(A, \alpha) \to \mathrm{F}(B, \beta)$$

in $\mathbf{C}$, there exists a homomorphism $\phi\colon (A, \alpha) \to (B, \beta)$ in $\mathbf{B}_G$ such that $\mathrm{F}(\phi) = \lambda$.

(b) For every pair of objects $(A, \alpha)$ and $(B, \beta)$ in $\mathbf{B}_G$ and every pair of homomorphisms

$$\phi, \psi\colon (A, \alpha) \to (B, \beta),$$

one has $\mathrm{F}(\phi) = \mathrm{F}(\psi)$ if and only if $\phi \sim_{\mathrm{G-au}} \psi$.

We say that the functor F *classifies isomorphisms* if it satisfies (a) and (b) above for ismorphisms instead of homomorphisms (such a functor is a *strong classifying functor* in the sense of Elliott (see [59])).

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two categories. Recall that a functor $\mathrm{F}\colon \mathbf{C}_1 \to \mathbf{C}_2$ is said to be *sequentially continuous* if whenever $C = \varinjlim(C_n, \theta_n)$ in $\mathbf{C}_1$ for some sequential direct system $(C_n, \theta_n)_{n\in\mathbb{N}}$ in $\mathbf{C}_1$, then the inductive limit $\varinjlim(\mathrm{F}(C_n), \mathrm{F}(\theta_n))$ exists in $\mathbf{C}_2$, and one has

$$\mathrm{F}(\varinjlim(C_n, \theta_n)) = \varinjlim(\mathrm{F}(C_n), \mathrm{F}(\theta_n)).$$

The following theorem is a consequence of [59, Theorem 3].

**Theorem VIII.2.11.** Let $G$ be a second countable compact group, let $\mathbf{B}$ be a subcategory of $\mathbf{A}$, let $\mathbf{B}_G$ be the associated category of C*-dynamical systems, and let $\mathbf{C}$ be a category in which inductive limits of sequences exist. Let $\mathrm{F}\colon \mathbf{B}_G \to \mathbf{C}$ be a sequentially continuous functor that classifies homomorphisms. Then F classifies isomorphisms.

*Proof.* Let us briefly see that the conditions of [59, Theorem 3] are satisfied for the category $\mathbf{B}_G$. First, using that the algebras in $\mathbf{B}_G$ are separable and that the group is second countable we can see that the set of equivariant homomorphisms between two $C^*$-algebras in $\mathbf{B}_G$ is metrizable. Also, by taking the inner automorphisms of a C*-dynamical systems in $\mathbf{B}_G$ to be conjugation by unitaries in the unitization of the fixed point algebra of the given dynamical system, one can easily see that these automorphisms satisfy the conditions of [59, Theorem 3]. Finally note that the category $\mathbf{D}$ whose objects are objects of $\mathbf{C}$ of the form $\mathrm{F}(A, \alpha)$ for some C*-dynamical system $(A, \alpha)$ in $\mathbf{B}_G$, and whose morphisms between two objects $\mathrm{F}(A, \alpha)$ and $\mathrm{F}(B, \alpha)$ are all the maps of the form $\mathrm{F}(\phi)$ for some equivariant homomorphism $\phi\colon (A, \alpha) \to (B, \beta)$, is just the classifying category of $B_G$ (in the sense of [59]), since F classifies homomorphisms by assumption. Therefore, by [59, Theorem 3] the functor F is a strong classifying functor; in other words, it classifies *-isomorphisms. ☐

**Definition VIII.2.12.** Let $G$ be a compact group. Let $\mathbf{C}$ be a category and let $\mathbf{C}_G$ denote the category whose objects are pairs $(C, \gamma)$, where $C$ is an object in $\mathbf{C}$ and $\gamma\colon G \to \mathrm{Aut}(C)$ is a group homomorphism, also called an action of $G$ on $C$. (We do not require any kind of continuity for this action since $C$ does not a priori have a topology.) The morphisms of $\mathbf{C}_G$ consist of the morphisms of $\mathbf{C}$ that are equivariant.

Let $\mathbf{B}$ be a subcategory of $\mathbf{A}$ and let $\mathbf{B}_G$ be the associated category of C*-dynamical systems. Let $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ be a functor. Then F induces a functor $\mathrm{F}_G\colon \mathbf{B}_G \to \mathbf{C}_G$ as follows:

(1) For an object $(A, \alpha)$ in $\mathbf{B}_G$, define an action $\mathrm{F}(\alpha)\colon G \to \mathrm{Aut}(\mathrm{F}(A))$ by $(\mathrm{F}(\alpha))_g = \mathrm{F}(\alpha_g)$ for all $g \in G$. We then set $\mathrm{F}_G(A, \alpha) = (\mathrm{F}(A), \mathrm{F}(\alpha))$;

(2) For a morphism $\phi \in \mathrm{Hom}_{\mathbf{B}_G}((A, \alpha), (B, \beta))$, we set $\mathrm{F}_G(\phi) = \mathrm{F}(\phi)$.

If $G$ is a finite group, we let $\mathbf{RB}_G$ denote the subcategory of $\mathbf{B}_G$ consisting of those C*-dynamical systems $(A, \alpha)$ in $\mathbf{B}_G$ with the Rokhlin property.

The next theorem is a restatement, in the categorical setting, of Proposition VIII.2.7 and Proposition VIII.2.9 (2).

**Theorem VIII.2.13.** Let $G$ be a finite group. Let $\mathbf{B}$, $\mathbf{B}_G$, $\mathbf{RB}_G$, $\mathbf{C}$, and $\mathbf{C}_G$ be as in Definition VIII.2.12. Let $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ be a functor that classifies homomorphisms.

(1) Let $(A, \alpha)$ be an object in $\mathbf{B}_G$ and let $(B, \beta)$ be an object in $\mathbf{RB}_G$.

  (a) For every morphism $\gamma\colon (\mathrm{F}(A), \mathrm{F}(\alpha)) \to (\mathrm{F}(B), \mathrm{F}(\beta))$ in $\mathbf{C}_G$, there exists a morphism $\phi\colon (A, \alpha) \to (B, \beta)$ in $\mathbf{B}_G$ such that $\mathrm{F}_G(\phi) = \gamma$.

  (b) If $\phi, \psi\colon (A, \alpha) \to (B, \beta)$ are morphisms in $\mathbf{B}_G$ such that $\mathrm{F}_G(\phi) = \mathrm{F}_G(\psi)$, then $\phi \sim_{G-\mathrm{au}} \psi$.

(2) The restriction of the functor $\mathrm{F}_G$ to $\mathbf{RB}_G$ classifies homomorphisms.

*Proof.* (1) Let $(A, \alpha)$ be an object in $\mathbf{B}_G$ and let $(B, \beta)$ be an object in $\mathbf{RB}_G$.

(a) Let $\gamma\colon (\mathrm{F}(A), \mathrm{F}(\alpha)) \to (\mathrm{F}(B), \mathrm{F}(\beta))$ be a morphism in $\mathbf{C}_G$. Using that $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ classifies homomorphisms, choose a homomorphism $\psi\colon A \to B$ such that $\mathrm{F}(\psi) = \gamma$. Note that

$$\mathrm{F}(\psi \circ \alpha_g) = \mathrm{F}(\psi) \circ \mathrm{F}(\alpha_g) = \mathrm{F}(\beta_g) \circ \mathrm{F}(\psi) = \mathrm{F}(\beta_g \circ \psi),$$

for all $g \in G$. Using again that F classifies homomorphisms, we conclude that $\psi \circ \alpha_g$ and $\beta_g \circ \psi$ are approximately unitarily equivalent for all $g \in G$. Therefore, by part (2) of

Proposition VIII.2.9 there exists an equivariant homomorphism $\phi\colon (A,\alpha) \to (B,\beta)$ such that $\phi$ and $\psi$ are approximately unitarily equivalent. Thus $\phi$ is a morphism in $\mathbf{B}_G$ and

$$\mathrm{F}_G(\phi) = \mathrm{F}(\phi) = \mathrm{F}(\psi) = \gamma,$$

as desired.

(b) Let $\phi, \psi\colon (A,\alpha) \to (B,\beta)$ be morphisms in $\mathbf{B}_G$ such that $\mathrm{F}_G(\phi) = \mathrm{F}_G(\psi)$. Then $\phi \sim_{\mathrm{au}} \psi$ because F classifies homomorphisms and F agrees with $\mathrm{F}_G$ on morphisms. It then follows from Proposition VIII.2.7 that $\phi \sim_{G-\mathrm{au}} \psi$.

Part (2) clearly follows from (1). □

**Lemma VIII.2.14.** Let $G$ be a compact group, let $\Lambda$ be a directed set and let $\mathbf{C}$ be a category where inductive limits over $\Lambda$ exist. Let $\mathbf{C}_G$ be the associated category as in Definition VIII.2.12. Then:

(1) Inductive limits over $\Lambda$ exist in $\mathbf{C}_G$.

(2) If $\mathbf{D}$ is a category where inductive limits over $\Lambda$ exist and $\mathrm{F}\colon \mathbf{C} \to \mathbf{D}$ is a functor that preserves direct limits over $\Lambda$, then the associated functor $\mathrm{F}_G\colon \mathbf{C}_G \to \mathbf{D}_G$ also preserves direct limits over $\Lambda$.

*Proof.* (1) Let $((C_\lambda, \alpha_\lambda)_{\lambda\in\Lambda}, (\gamma_{\lambda,\mu})_{\lambda,\mu\in\Lambda,\lambda<\mu})$ be a direct system in $\mathbf{C}_G$ over $\Lambda$, where $\gamma_{\lambda,\mu}\colon (C_\lambda,\alpha_\lambda) \to (C_\mu,\alpha_\mu,)$, for $\lambda < \mu$, is a morphism in $\mathbf{C}_G$. Let $(C, (\gamma_{\lambda,\infty})_{\lambda\in\Lambda})$, with $\gamma_{\lambda,\infty}\colon C_\lambda \to C$, be its direct limit in the category $\mathbf{C}$. Then

$$(\gamma_{\mu,\infty} \circ \alpha_\mu(g)) \circ \gamma_{\lambda,\mu} = \gamma_{\lambda,\infty} \circ \alpha_\lambda(g)$$

for all $\mu \in \Lambda$ with $\lambda < \mu$. Hence, by the universal property of the inductive limit $(C, (\gamma_{\lambda,\infty})_{\lambda\in\Lambda})$, there exists a unique $\mathbf{C}$-morphism $\alpha(g)\colon C \to C$ that satisfies $\alpha(g) \circ \gamma_{\lambda,\infty} = \gamma_{\lambda,\infty} \circ \alpha_\lambda(g)$ for all $\lambda \in \Lambda$. Note that for $g, h \in G$, one has

$$(\alpha(g) \circ \alpha(h)) \circ \gamma_{\lambda,\infty} = \gamma_{\lambda,\infty} \circ \alpha_\lambda(g) \circ \alpha(h) = \gamma_{\lambda,\infty} \circ \alpha_\lambda(gh)$$

for all $\lambda \in \Lambda$. By uniqueness of the morphism $\alpha(gh)$, it follows that $\alpha(g) \circ \alpha(h) = \alpha(gh)$ for all $g, h \in G$. This implies that $\alpha(g)$ is an automorphism of $C$ and that $\alpha \colon G \to \mathrm{Aut}(C)$ is an action. Thus $(C, \alpha)$ is an object in $\mathbf{C}_G$.

We claim that $(C, \alpha)$ is the inductive limit of $((C_\lambda, \alpha_\lambda)_{\lambda\in\Lambda}, (\gamma_{\lambda,\mu})_{\lambda,\mu\in\Lambda,\lambda<\mu})$ in the category $\mathbf{C}_G$. For $\lambda \in \Lambda$, The map $\gamma_{\lambda,\infty}$ is equivariant since $\gamma_{\lambda,\infty} \circ \alpha_\lambda(g) = \alpha(g) \circ \gamma_{\lambda,\infty}$ for all $g \in G$ and $\lambda \in \Lambda$. Let $(D, \beta)$ be an object in $\mathbf{C}_G$ and for $\lambda \in \Lambda$, let $\rho_\lambda \colon (C_\lambda, \alpha_\lambda) \to (D, \beta)$ be an equivariant morphism. By the universal property of the inductive limit $C$, there exists a unique morphism $\rho \colon C \to D$ satisfying $\rho_\lambda = \rho \circ \gamma_{\lambda,\infty}$ for all $\lambda \in \Lambda$. We therefore have

$$
\begin{aligned}
(\beta(g)^{-1} \circ \rho \circ \alpha(g)) \circ \gamma_{\lambda,\infty} &= \beta^{-1}(g) \circ \rho \circ \gamma_{\lambda,\infty} \circ \alpha_\lambda(g) \\
&= \beta^{-1}(g) \circ \rho_{\lambda,\infty} \circ \alpha_\lambda(g) \\
&= \rho_{\lambda,\infty},
\end{aligned}
$$

for all $g \in G$ and $\lambda \in \Lambda$. Hence by uniqueness of $\rho$, we conclude that

$$
\beta^{-1}(g) \circ \rho \circ \alpha(g) = \rho
$$

for all $g \in G$. In other words, $\rho$ is equivariant. We have shown that $(C, \alpha)$ has the universal property of the inductive limit in $\mathbf{C}_G$, thus proving the claim and part (1).

(2) Let $((C_\lambda, \alpha_\lambda)_{\lambda\in\Lambda}, (\gamma_{\lambda,\mu})_{\lambda,\mu\in\Lambda,\lambda<\mu})$ be a direct system in $\mathbf{C}_G$ and let $(C, \alpha)$ be its inductive limit in $\mathbf{C}_G$, which exists by the first part of this lemma. We claim that $(\mathrm{F}(C), \mathrm{F}(\alpha))$ is the inductive limit of

$$
((\mathrm{F}(C_\lambda), \mathrm{F}(\alpha_\lambda))_{\lambda\in\Lambda}, (\mathrm{F}(\gamma_{\lambda,\mu}))_{\lambda,\mu\in\Lambda,\lambda<\mu})
$$

in the category $\mathbf{D}_G$. Let $(D, \delta)$ be an object in $\mathbf{D}_G$ and for $\lambda \in \Lambda$, let

$$
\rho_\lambda \colon (\mathrm{F}(C_\lambda), \mathrm{F}(\alpha_\lambda)) \to (D, \delta)
$$

be an equivariant morphism satisfying $\rho_\mu = \mathrm{F}(\gamma_{\lambda,\mu}) \circ \rho_\lambda$ for all $\mu \in \Lambda$ with $\lambda < \mu$. Since F is continuous by assumption, we have

$$
\mathrm{F}(C) = \varinjlim ((\mathrm{F}(C_\lambda))_{\lambda\in\Lambda}, (\mathrm{F}(\gamma_{\lambda,\mu}))_{\lambda,\mu\in\Lambda,\lambda<\mu})
$$

in $\mathbf{D}$. By the universal property of the inductive limit $\mathrm{F}(C)$ in $\mathbf{D}$, there exits a unique morphism $\rho\colon \mathrm{F}(C) \to D$ in the category $\mathbf{D}$ satisfying $\rho \circ \mathrm{F}(\gamma_{\lambda,\infty}) = \rho_\lambda$. It follows that

$$(\delta(g)^{-1} \circ \rho \circ \mathrm{F}(\alpha(g))) \circ \mathrm{F}(\gamma_{\lambda,\infty}) = \delta(g)^{-1} \circ \rho_\lambda \circ \mathrm{F}(\alpha_\lambda(g)) = \rho_\lambda,$$

for all $g \in G$ and $\lambda \in \Lambda$. By the uniqueness of the morphism $\rho$, we conclude that $\delta(g)^{-1} \circ \rho \circ \mathrm{F}(\alpha)(g) = \rho$ for all $g \in G$. That is, $\rho\colon (\mathrm{F}(C), \mathrm{F}(\alpha)) \to (D, \delta)$ is equivariant. This shows that $(\mathrm{F}(C), \mathrm{F}(\alpha))$ has the universal property of inductive limits in $\mathbf{D}_G$. □

**Theorem VIII.2.15.** Let $G$ be a finite group, let $\mathbf{B}$ be a subcategory of $\mathbf{A}$, and let $\mathbf{C}$ be a category where inductive limits of sequences exist. Let $\mathbf{B}_G$, $\mathbf{RB}_G$, and $\mathbf{C}_G$ be as in Definition VIII.2.12. Let $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ be a sequentially continuous functor that classifies homomorphisms and let $\mathrm{F}_G\colon \mathbf{B}_G \to \mathbf{C}_G$ be the associated functor as in Definition VIII.2.12. Then the restriction of $\mathrm{F}_G$ to $\mathbf{RB}_G$ classifies isomorphisms. In particular, if $(A, \alpha)$ and $(B, \beta)$ are C*-dynamical systems in $\mathbf{RB}_G$, then $\alpha$ and $\beta$ are conjugate if and only if there exists an isomorphism $\rho\colon \mathrm{F}_G(A, \alpha) \to \mathrm{F}_G(B, \beta)$ in $\mathbf{C}_G$.

*Proof.* Since by Theorem VIII.2.13 the restriction of the functor $\mathrm{F}_G$ to $\mathbf{RB}_G$ classifies homomorphisms, it is sufficient to show that the conditions of Theorem VIII.2.11 are satisfied. First note that sequential inductive limits exists in $\mathbf{B}_G$ since $G$ is finite and they exists in $\mathbf{B}$ by assumption. Now by [243, Theorem 2 (v)] the same is true for $\mathbf{RB}_G$. Given that sequential inductive limits exist in the category $\mathbf{C}$ and $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ is sequentially continuous, it follows from Lemma VIII.2.14 applied to $\Lambda = \mathbb{N}$ that sequential inductive limits exist in $\mathbf{C}_G$ and the functor $\mathrm{F}_G\colon \mathbf{B}_G \to \mathbf{C}_G$ is sequentially continuous. In particular, it follows that the restriction of $\mathrm{F}_G$ to $\mathbf{RB}_G$ is sequentially continuous. This shows that the conditions of Theorem VIII.2.11 are met. The last statement of the theorem follows from the definition of a functor that classifies isomorphisms. □

The following result was proved by Izumi in [132, Theorem 3.5] for unital $C^*$-algebras, and more recently by Nawata in [188, Theorem 3.5] for $C^*$-algebras with *almost stable rank one* (that is, $C^*$-algebras $A$ such that $A \subseteq \overline{\mathrm{GL}(\widetilde{A})}$).

**Theorem VIII.2.16.** Let $G$ be a finite group, let $A$ be separable $C^*$-algebra and let $\alpha$ and $\beta$ be actions of $G$ on $A$ with the Rokhlin property. Assume that $\alpha_g \sim_{\mathrm{au}} \beta_g$ for all $g \in G$. Then there exists an approximately inner automorphism $\psi$ of $A$ such that $\psi \circ \alpha_g = \beta_g \circ \psi$ for all $g \in G$.

*Proof.* Let $\mathbf{C}$ be the category whose objects are separable $C^*$-algebras and whose morphisms are given by

$$\mathrm{Hom}(A, B) = \{[\phi]_{\mathrm{au}} \colon \phi \colon A \to B \text{ is a homomorphism}\},$$

where $[\phi]_{\mathrm{au}}$ denotes the approximate unitary equivalence class of $\phi$. (It is easy to check that composition of maps is well defined in $\mathbf{C}$, and thus $\mathbf{C}$ is indeed a category.) Let $\mathrm{F} \colon \mathbf{A} \to \mathbf{C}$ be the functor given by $\mathrm{F}(A) = A$ for any $C^*$-algebra $A$ in $\mathbf{A}$, and $\mathrm{F}(\phi) = [\phi]_{\mathrm{au}}$ for any homomorphism $\phi$ in $\mathbf{A}$. It is straightforward to check that sequential inductive limits exist in $\mathbf{C}$ and that F is sequentially continuous. Moreover, by the construction of $\mathbf{C}$ and F it is clear that F classifies homomorphisms. Therefore, by Theorem VIII.2.15 the restriction of the associated functor $\mathrm{F}_G$ to $\mathbf{RA}_G$ classifies isomorphisms.

Let $A$ be a separable $C^*$-algebra (that is, a $C^*$-algebra in $\mathbf{A}$), and let $\alpha$ and $\beta$ be as in the statement of the theorem. Since $\alpha_g \sim_{\mathrm{au}} \beta_g$ for all $g \in G$, we have

$$\mathrm{F}(\mathrm{id_A}) \circ \mathrm{F}(\alpha_g) = \mathrm{F}(\mathrm{id_A} \circ \alpha_g) = \mathrm{F}(\beta_g \circ \mathrm{id_A}) = \mathrm{F}(\beta_g) \circ \mathrm{F}(\mathrm{id_A}),$$

for all $g \in G$. In other words, the map $[\mathrm{id}_A]_{\mathrm{au}}$ is equivariant. Also, note that this map is an automorphism. Therefore, it is an isomorphism in the category $\mathbf{C}_G$. Since by the previous discussion, the restriction of $\mathrm{F}_G$ to $\mathbf{RA}_G$ classifies isomorphisms, it follows that that there exists an equivariant *-automorphism $\psi \colon (A, \alpha) \to (A, \beta)$ such that $\mathrm{F}_G(\psi) = [\mathrm{id}_A]_{\mathrm{au}}$. In particular, $\alpha$ and $\beta$ are conjugate. Using that $\mathrm{F}(\psi) = \mathrm{F}_G(\psi) = [\mathrm{id}_A]_{\mathrm{au}} = \mathrm{F}(\mathrm{id}_A)$ and that F classifies homomorphisms, we get that $\psi \sim_{\mathrm{au}} \mathrm{id}_A$. In other words, $\psi$ is approximately inner. $\square$

**Remark VIII.2.17.** In view of [188, Remark 3.6], it may be worth pointing out that one can directly modify the proof of [188, Lemma 3.4] to get rid of the assumption that $A$ has almost stable rank one. Indeed, one just needs to replace the element $w$ in the proof by the unitary $w' = \sum\limits_{g \in G} (v_g - \lambda_g 1_{\widetilde{A}}) f_g + 1_{\widetilde{A}}$, where $\lambda_g \in \mathbb{C}$ is such that $v_g - \lambda_g 1_{\widetilde{A}} \in A$.

In this section we apply Theorem VIII.2.13, Theorem VIII.2.15, and Theorem VIII.2.16, and known classification results, to obtain classification of equivariant homomorphisms and finite group actions on certain classes of 1-dimensional NCCW-complexes and AH-algebras.

1-dimensional NCCW-complexes

Let $E$ and $F$ be finite dimensional $C^*$-algebras, and for $x \in [0,1]$, denote by $\mathrm{ev}_x \colon C([0,1],F) \to F$ the evaluation map at the point $x$. Recall that a $C^*$-algebra $A$ is said to be a *one-dimensional non-commutative CW-complex*, abbreviated 1-dimensional NCCW-complex, if $A$ is given by a pullback diagram of the form:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ C([0,1],F) \arrow[r, "\mathrm{ev}_0 \oplus \mathrm{ev}_1"'] & F \oplus F. \end{tikzcd}$$

**Theorem VIII.2.18.** Let $G$ be a finite group. Let $(A,\alpha)$ and $(B,\beta)$ be separable C*-dynamical systems such that $A$ can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial $K_1$-groups and such that $B$ has stable rank one. Assume that $\beta$ has the Rokhlin property.

(1) Fix strictly positive elements $s_A$ and $s_B$ of $A$ and $B$, respectively. Let $\rho\colon \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(B)$ be a morphism in the category **Cu** such that

$$\rho([s_A]) \leq [s_B] \ \text{ and } \ \rho \circ \mathrm{Cu}^\sim(\alpha_g) = \mathrm{Cu}^\sim(\beta_g) \circ \rho$$

for all $g \in G$. Then there exists an equivariant homomorphism

$$\phi\colon (A,\alpha) \to (B,\beta) \ \text{ such that } \ \mathrm{Cu}^\sim(\phi) = \rho.$$

(2) If $\phi, \psi\colon (A,\alpha) \to (B,\beta)$ are equivariant homomorphisms, then $\mathrm{Cu}^\sim(\phi) = \mathrm{Cu}^\sim(\psi)$ if and only if $\phi \sim_{G-\mathrm{au}} \psi$.

Moreover, if $A$ is unital, or if it is simple and has trivial $K_0$-group, or if it can be written as an inductive limit of punctured-trees algebras, then the functor $\mathrm{Cu}^\sim$ can be replaced by the Cuntz functor Cu in the statement of this theorem.

*Proof.* (1) Let $\rho\colon \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(B)$ be as in the statement of the theorem. By [230, Theorem 1], there exists a homomorphism $\psi\colon A \to B$ such that $\mathrm{Cu}^\sim(\psi) = \rho$. Using that $\rho$ is equivariant, we get $\mathrm{Cu}^\sim(\beta_g \circ \psi) = \mathrm{Cu}^\sim(\psi \circ \alpha_g)$ for all $g \in G$. By the uniqueness part of [230, Theorem 1], it follows that $\beta_g \circ \psi \sim_{\mathrm{au}} \psi \circ \alpha_g$ for all $g \in G$. By Proposition VIII.2.9, there exists an equivariant homomorphism $\phi\colon A \to B$ such that $\phi \sim_{\mathrm{au}} \psi$. Since $\mathrm{Cu}^\sim$ is invariant under approximate unitary equivalence, we conclude $\mathrm{Cu}^\sim(\phi) = \mathrm{Cu}^\sim(\psi)$, as desired.

(2) The "if" implication is clear. For the converse, let $\phi$ and $\psi$ be as in the statement of the theorem. By the uniqueness part of [230, Theorem 1], we have $\phi \sim_{\mathrm{au}} \psi$. It now follows from Proposition VIII.2.7 that $\phi \sim_{G-\mathrm{au}} \psi$.

It follows from [230, Remark 3 (2)], and by [230, Corollary 4 (b)], [60, Corollary 6.7], and [261, Corollary 8.6], respectively, that the functors $\mathrm{Cu}^\sim$ and Cu are equivalent when restricted to the class of $C^*$-algebras that are inductive limits of 1-dimensional NCCW-complexes which are either unital or simple and with trivial $K_0$-group. Hence, for these classes of $C^*$-algebras, the theorem holds when $\mathrm{Cu}^\sim$ is replaced by Cu. For $C^*$-algebras that are inductive limits of punctured-trees algebras, one can use [27, Theorem 1.1] instead of [230, Theorem 1] in the proof above to obtain the desired result. Finally, since $B$ has stable rank one, the results in [230] show that $\mathrm{Cu}(B)$ is a subsemigroup of $\mathrm{Cu}^\sim(B)$. In particular, for a homomorphism $\phi\colon A \to B$, the range of the induced map $\mathrm{Cu}^\sim(\phi)\colon \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(B)$ is contained in $\mathrm{Cu}(B) \subseteq \mathrm{Cu}^\sim(B)$. □

**Theorem VIII.2.19.** Let $G$ be a finite group, and let $(A, \alpha)$ and $(B, \beta)$ be separable dynamical systems such that $A$ and $B$ can be written as inductive limits of 1-dimensional NCCW-complexes with trivial $K_1$-groups. Suppose that $\alpha$ and $\beta$ have the Rokhlin property.

(1) Fix strictly positive elements $s_A$ and $s_B$ of $A$ and $B$ respectively. Then the actions $\alpha$ and $\beta$ are conjugate if and only if there exists an isomorphism $\gamma\colon \mathrm{Cu}^\sim(A) \to \mathrm{Cu}^\sim(B)$ with $\gamma([s_A]) = [s_B]$, such that

$$\gamma \circ \mathrm{Cu}^\sim(\alpha_g) = \mathrm{Cu}^\sim(\beta_g) \circ \gamma \text{ for all } g \in G.$$

(2) Assume that $A = B$. Then the actions $\alpha$ and $\beta$ are conjugate by an approximately inner automorphism of $A$ if and only if $\mathrm{Cu}^{\sim}(\alpha_g) = \mathrm{Cu}^{\sim}(\beta_g)$ for all $g \in G$.

Moreover, if both $A$ and $B$ are unital, or if they are simple and have trivial $K_0$-groups, or if they can be written as inductive limits of punctured-trees algebras, then the functor $\mathrm{Cu}^{\sim}$ can be replaced by the Cuntz functor Cu.

*Proof.* Part (2) clearly follows from (1). Let us prove (1). Let $\mathbf{B}$ denote the subcategory of the category $\mathbf{A}$ of $C^*$-algebras consisting of those $C^*$-algebras that can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial $K_1$-groups. By [230, Theorem 1], the functor $(\mathrm{Cu}^{\sim}(\cdot), [s_{\cdot}])$, where $s_{\cdot}$ is a strictly positive element of the given algebra, restricted to $\mathbf{B}$ classifies homomorphisms. Therefore, by Theorem VIII.2.15, the associated functor $(\mathrm{Cu}^{\sim}_G(\cdot), [s_{\cdot}])$ restricted to $\mathbf{RB}_G$ classifies isomorphisms, which implies (1).

The last part of the theorem follows from the same arguments used at the end of the proof of Theorem VIII.2.18. $\square$

Let $G$ be a finite group. Recall that the action $\mu^G \colon G \to \mathrm{Aut}\left(M_{|G|^\infty}\right)$ constructed in Example VIII.2.4 has the Rokhlin property, and that $\mu^G_g$ is approximately inner for all $g \in G$.

In the next corollary, we do not assume that either $\alpha$ or $\beta$ has the Rokhlin property.

**Corollary VIII.2.20.** Let $G$ be a finite group and let $(A, \alpha)$ and $(A, \beta)$ be C*-dynamical systems such that $A$ can be written as an inductive limit of 1-dimensional NCCW-complexes with trivial $K_1$-groups. Suppose that $\mathrm{Cu}^{\sim}(\alpha_g) = \mathrm{Cu}^{\sim}(\beta_g)$ for all $g \in G$. Then $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ are conjugate.

Moreover, if $A$ belongs to one of the classes of $C^*$-algebras described in the last part of Theorem VIII.2.19, then the statement of the corollary holds for the functor Cu in place of the functor $\mathrm{Cu}^{\sim}$.

*Proof.* The actions $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ have the Rokhlin property by part (1) of Proposition VIII.2.3. Note that $\mu^G_g$ is approximately inner for all $g \in G$. Thus,

$$\mathrm{Cu}^{\sim}(\alpha \otimes \mu^G_g) = \mathrm{Cu}^{\sim}(\alpha \otimes \mathrm{id}_{M_{|G|^\infty}}) = \mathrm{Cu}^{\sim}(\beta \otimes \mathrm{id}_{M_{|G|^\infty}}) = \mathrm{Cu}^{\sim}(\beta \otimes \mu^G_g)$$

for all $g \in G$. It follows from Theorem VIII.2.19 (2) that $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ are conjugate. $\square$

AH-algebras

Recall that a $C^*$-algebra $A$ is approximate homogeneous (AH) if it can be written as an inductive limit $A = \varinjlim(A_n, \phi_{n,m})$, with

$$A_n = \oplus_{j=1}^{s(n)} P_{n,j} M_{n,j}(C(X_{n,j})) P_{n,j},$$

where $X_{n,j}$ is a finite dimensional compact metric space, and $P_{n,j} \in M_{n,j}(C(X_{n,j}))$ is a projection for all $n$ and $j$. The $C^*$-algebra $A$ is said to have *no dimension growth* if there exists an inductive limit decomposition of $A$ as an AH-algebra such that

$$\sup_n \max_j \dim X_{n,j} < \infty.$$

Let $A$ be a unital simple separable $C^*$-algebra and let $T(A)$ denote the metrizable compact convex set of tracial states of $A$. Denote by $T$ the induced contravariant functor from the category of unital separable simple $C^*$-algebras to the category of metrizable compact convex sets. It is not difficult to check that $T$ is continuous, meaning that it sends inductive limits to projective limits.

Let $T$ be a metrizable compact convex set and let $\mathrm{Aff}(T)$ denote the set of real-valued continuous affine functions on $T$. Let Aff denote the induced contravariant functor from the category of metrizable compact convex sets to the category of normed vector spaces. Denote by $\rho_A \colon K_0(A) \to \mathrm{Aff}(T(A))$ the map defined by

$$\rho_A([p] - [q])(\tau) = (\tau \otimes \mathrm{Tr}_n)(p) - (\tau \otimes \mathrm{Tr}_n)(q) \tag{VIII.3}$$

for $p, q \in M_n(\mathbb{C})$, where $\mathrm{Tr}_n$ denotes the standard trace on $M_n(\mathbb{C})$.

Let $A$ be a unital $C^*$-algebra. Denote by $\mathcal{U}(A)$ the unitary group of $A$ and by $\mathrm{CU}(A)$ the closure of the normal subgroup generated by the commutators of $\mathcal{U}(A)$. We denote the quotient group by

$$H(A) = \mathcal{U}(A)/CU(A).$$

(see [260] and [190] for properties of this group). The set $H(A)$, endowed with the distance induced by the distance in $\mathcal{U}(A)$, is a complete metric space. We denote by $H$ the induced functor from the category of $C^*$-algebras to the category of complete metric groups. Also, if $A$ is a simple unital AH-algebra of no dimension growth (or more generally, a simple unital $C^*$-algebra of tracial rank no greater than one), then there exists an injection

$$\lambda_A \colon \operatorname{Aff}(T(A))/\overline{\rho_A(K_0(A))} \to H(A). \tag{VIII.4}$$

(See [260] and [171].)

For a $C^*$-algebra $A$, denote by $\underline{K}(A)$ the sum of all K-groups with $\mathbb{Z}/n\mathbb{Z}$ coefficients for all $n \geq 1$. Let $\Lambda$ denote the category generated by the Bockstein operations on $\underline{K}(A)$ (see [43]). Then $\underline{K}(A)$ becomes a $\Lambda$-module and it induces a continuous functor $\underline{K}$ from the category of $C^*$-algebras to the category of $\Lambda$-modules.

Let $A$ and $B$ be unital simple AH-algebras and let $KL(A, B)$ denote the group defined in [234]. By the Universal Coefficient Theorem and the Universal Multicoefficient Theorem (see [43]), the groups $KL(A, B)$ and $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ are naturally isomorphic. Let $KL_e^{++}(A, B)$ be as in [171, Definition 6.4]. By the previous isomorphism, the group $KL_e^{++}(A, B)$ is naturally isomorphic to

$$\{\kappa \in \operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \colon \kappa(K_0(A)_+ \setminus \{0\}) \subseteq K_0(B)_+ \setminus \{0\},\ \kappa([1_A]) = \kappa([1_B])\}.$$

Let us define a functor $\underline{K}^{++}$ from the category of separable, unital, simple, finite $C^*$-algebras to the category whose objects are 4-tuples $(M, N, E, e)$, where $M$ is a $\Lambda$-module, $N$ is a subgroup of $M$, $E$ is a subset of $N$, and $e$ is an element of $N$; and whose morphisms $\kappa \colon (M, N, E, e) \to (M', N', E', e')$ are $\Lambda$-module maps $\kappa \colon M \to M'$ such that $\kappa(N) \subseteq N'$, $\kappa(E) \subseteq E'$, and $\kappa(e) = e'$. The functor $\underline{K}^{++}$ is defined as follows:

$$\underline{K}^{++}(A) = (\underline{K}(A), K_0(A), K_0(A)_+ \setminus \{0\}, [1_A]), \quad \text{and} \quad \underline{K}^{++}(\phi) = \underline{K}(\phi).$$

Note that if $A$ and $B$ are unital AH-algebras then $KL_e^{++}(A, B)$ is isomorphic to $\operatorname{Hom}(\underline{K}^{++}(A), \underline{K}^{++}(B))$.

Let $\mathbf{C}$ denote the category whose objects are tuples

$$((M, N, E, e), T, H, \rho, \lambda),$$

where $(M,N,E,e)$ is as above, $T$ is a metrizable compact convex set, $H$ is a complete metric group, $\rho\colon N \to \mathrm{Aff}(T)$ is a group homomorphism, and $\lambda\colon \mathrm{Aff}(T)/\overline{\rho(N)} \to H$ is an injective continuous group homomorphism. The maps in $\mathbf{C}$ are triples

$$(\kappa,\eta,\mu)\colon\ ((M,N,E,e),T,H,\rho,\lambda) \to ((M',N',E',e'),T',H',\rho',\lambda')\,,$$

where $\kappa\colon (M,N,E,e) \to (M',N',E',e')$, $\eta\colon T' \to T$, and $\mu\colon H \to H'$ are maps in the corresponding categories that satisfy the compatibility conditions:

$$\rho'\circ\kappa|_N = \mathrm{Aff}(\eta)\circ\rho, \quad \text{and} \quad \lambda'\circ\mu = \overline{\mathrm{Aff}}(\eta)\circ\lambda,$$

where

$$\overline{\mathrm{Aff}}(\eta)\colon \mathrm{Aff}(T)/\overline{\rho(N)} \to \mathrm{Aff}(T')/\overline{\rho(N')}$$

is the map induced by $\mathrm{Aff}(\eta)$. Using that inductive limits of sequences exist in each of the categories that form $\mathbf{C}$, it is not difficult to show that $\mathbf{C}$ is also closed under taking inductive limits of sequences. Also, it is easy to see that $\mathrm{F} = (\underline{K}^{++},T,H)$ is a functor from the category of unital, simple, separable, finite $C^*$-algebras to the category $\mathbf{C}$. Moreover, since the functors that form F are continuous, F is also continuous.

**Theorem VIII.2.21.** Let $G$ be a finite group. Let $(A,\alpha)$ and $(B,\beta)$ be dynamical systems such that $A$ and $B$ are unital simple AH-algebras of no dimension growth. Assume that $\beta$ has the Rokhlin property.

(1) Let

$$\kappa\colon \underline{K}^{++}(A) \to \underline{K}^{++}(B), \quad \eta\colon T(B)\to T(A), \quad \text{and} \quad \mu\colon H(A)\to H(B),$$

be maps in the corresponding categories that satisfy the compatibility conditions

$$\rho_B\circ\kappa|_{\mathrm{K_0(A)}} = \mathrm{Aff}(\eta)\circ\rho_A, \quad \text{and} \quad \lambda_B\circ\mu = \overline{\mathrm{Aff}}(\eta)\circ\lambda_A,$$

where $\rho_A$, $\rho_B$, $\lambda_A$, and $\lambda_B$ are as in (VIII.3) and (VIII.4). Suppose that

$$\kappa \circ \underline{K}(\alpha_g) = \underline{K}(\beta_g) \circ \kappa, \quad \eta \circ T(\beta_g) = T(\alpha_g) \circ \eta, \quad \mu \circ H(\alpha_g) = H(\beta_g) \circ \mu,$$

for all $g \in G$. Then there exists an equivariant homomorphism

$$\phi\colon (A, \alpha) \to (B, \beta)$$

such that

$$\underline{K}^{++}(\phi) = \kappa, \quad T(\phi) = \eta, \quad \text{and} \quad H(\phi) = \mu.$$

(2) Let $\phi, \psi\colon A \to B$ be equivariant homomorphisms such that

$$\underline{K}(\phi) = \underline{K}(\psi), \quad T(\phi) = T(\psi), \quad \text{and} \quad H(\phi) = H(\psi).$$

Then $\phi \sim_{G-\mathrm{au}} \psi$.

*Proof.* It is shown in [103] that every unital simple AH-algebra of no dimension growth has tracial rank almost one. By [171, Theorems 5.11 and 6.10] applied to the algebras $A$ and $B$, and using the computations of $KL_e^{++}(A, B)$ given in the paragraphs preceding the theorem, we deduce that the functor F (defined above) restricted to the category of unital simple AH-algebras of no dimension growth classifies homomorphisms. The theorem now follows from Theorem VIII.2.13. ∎

**Theorem VIII.2.22.** Let $G$ be a finite group and let $A$ and $B$ be unital simple AH-algebras of no dimension growth. Let $\alpha$ and $\beta$ be actions of $G$ on $A$ and $B$ with the Rokhlin property.

(1) The actions $\alpha$ and $\beta$ are conjugate if and only if there exist isomorphisms

$$\kappa\colon \underline{K}^{++}(A) \to \underline{K}^{++}(B), \quad \eta\colon T(B) \to T(A), \quad \mu\colon H(A) \to H(B),$$

in the corresponding categories, that satisfy the compatibility conditions of the previous theorem, and such that

$$\kappa \circ \underline{K}(\alpha_g) = \underline{K}(\beta_g) \circ \kappa, \quad \eta \circ T(\beta_g) = T(\alpha_g) \circ \rho, \quad \mu \circ H(\alpha_g) = H(\beta_g) \circ \lambda,$$

for all $g \in G$.

(2) Assume that $A = B$. Then the actions $\alpha$ and $\beta$ are conjugate by an approximately inner automorphism if and only if

$$\underline{K}(\alpha_g) = \underline{K}(\beta_g), \quad T(\alpha_g) = T(\beta_g), \quad \text{and} \quad H(\alpha_g) = H(\beta_g),$$

for all $g \in G$.

*Proof.* Part (2) clearly follows from (1) and part (2) of Theorem VIII.2.21. Let us prove (1). As in the proof of Theorem VIII.2.21, the functor F restricted to the category of unital simple AH-algebras of no dimension growth classifies homomorphisms. The statements of the theorem now follows from Theorem VIII.2.15. □

**Corollary VIII.2.23.** Let $G$ be a finite group and let $A$ be a unital simple AH-algebra of no dimension growth. Let $(A, \alpha)$ and $(A, \beta)$ be C*-dynamical systems. Suppose that

$$\underline{K}^{++}(\alpha_g) = \underline{K}^{++}(\beta_g), \quad T(\alpha_g) = T(\beta_g), \quad \text{and} \quad H(\alpha_g) = H(\beta_g),$$

for all $g \in G$. Then $\alpha \otimes \mu^G$ and $\beta \otimes \mu^G$ are conjugate.

*Proof.* The proof of this corollary follows line by line the proof of Corollary VIII.2.20, using the functor F instead of the functor $\mathrm{Cu}^{\sim}$ and Theorem VIII.2.22 instead of Theorem VIII.2.19. □

### Cuntz Semigroup and $K$-theoretical Constraints

In this section, a Cuntz semigroup obstruction is obtained for a $C^*$-algebra to admit an action with the Rokhlin property. Also, the Cuntz semigroup of the fixed-point $C^*$-algebra and the crossed product $C^*$-algebra associated to an action of a finite group with the Rokhlin property are computed in terms of the Cuntz semigroup of the given algebra. As a corollary, similar results are obtained for the Murray-von Neumann semigroup and the K-groups.

We begin with some preliminaries.

**Definition VIII.3.1.** Let $S$ be a semigroup in the category $\mathbf{Cu}$. Let $I$ be a nonempty set and let $\gamma_i \colon S \to S$ for $i \in I$, be a family of endomorphisms of $S$ in the category $\mathbf{Cu}$. We introduce the following notation:

$$S^\gamma = \left\{ s \in S \colon \exists\ (s_t)_{t\in(0,1]} \text{ in } S \colon \begin{array}{c} s_r \ll s_t \text{ if } r<t,\ s_t = \sup\limits_{r<t} s_r\ \forall\ t \in (0,1], \\ s_1 = s, \text{ and } \gamma_i(s_t) = s_t\ \forall\ t\in(0,1] \text{ and } \forall\ i \in I \end{array} \right\},$$

and

$$S^\gamma_{\mathbb{N}} = \left\{ s \in S \colon \exists\ (s_n)_{n\in\mathbb{N}} \text{ in } S \colon \begin{array}{c} s_n \ll s_{n+1}\ \forall\ n\in\mathbb{N},\ s = \sup\limits_{n\in\mathbb{N}} s_n, \\ \text{and } \gamma_i(s_n) = s_n\ \forall\ n \in \mathbb{N} \text{ and } \forall\ i \in I \end{array} \right\}.$$

**Lemma VIII.3.2.** Let $S$ be a semigroup in the category $\mathbf{Cu}$. Let $I$ be a nonempty set and let $\gamma_i \colon S \to S$ for $i \in I$, be a family of endomorphisms of $S$ in the category $\mathbf{Cu}$. Then

(1) $S^\gamma_{\mathbb{N}}$ is closed under suprema of increasing sequences;

(2) $S^\gamma$ is an object in $\mathbf{Cu}$.

*Proof.* (1). Let $(s_n)_{n\in\mathbb{N}}$ be an increasing sequence in $S^\gamma_{\mathbb{N}}$. For each $n \in \mathbb{N}$, choose a rapidly increasing sequence $(s_{n,m})_{m\in\mathbb{N}}$ in $S$ such that $s_n = \sup\limits_{m\in\mathbb{N}} s_{n,m}$ and $\gamma_i(s_{n,m}) = s_{n,m}$ for all $i \in I$ and $m \in \mathbb{N}$. By the definition of the compact containment relation, there exist increasing sequences $(n_j)_{j\in\mathbb{N}}$ and $(m_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ such that $s_{k,l} \leq s_{n_j,m_j}$ whenever $1 \leq k,l \leq j$, and such that $(s_{n_j,m_j})_{j\in\mathbb{N}}$ is increasing. Let $s$ be the supremum of $(s_{n_j,m_j})_{j\in\mathbb{N}}$ in $S$. Then $s \in S^\gamma_{\mathbb{N}}$, and it is straightforward to check, using a diagonal argument, that $s = \sup\limits_{n\in\mathbb{N}} s_n$, as desired.

(2). It is clear that $S^\gamma$ satisfies O2, O3 and O4. Now let us check that $S^\gamma$ satisfies axiom O1. Let $(s^{(n)})_{n\in\mathbb{N}}$ be an increasing sequence in $S^\gamma$ and let $s$ be its supremum in $S$. It is sufficient to show that $s \in S^\gamma$.

For each $n \in \mathbb{N}$, choose a path $(s^{(n)}_t)_{t\in(0,1]}$ as in the definition of $S^\gamma$ for $s^{(n)}$. Using that $s^{(n)}_t \ll s^{(n+1)}$ for all $n \in \mathbb{N}$ and all $t \in (0,1)$, together with a diagonal argument, choose an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $(0,1]$ converging to 1, such that

$$s^{(n)}_{t_n} \ll s^{(n+1)}_{t_{n+1}} \quad \forall\, n\in\mathbb{N}, \text{ and } s = \sup_{n\in\mathbb{N}} s^{(n)}_{t_n}.$$

This implies, using the definition of the compact containment relation, that for each $n \in \mathbb{N}$ there exists $t'_{n+1}$ such that $t_n < t'_{n+1} < t_{n+1}$ and

$$s^{(n)}_{t_n} \ll s^{(n+1)}_{t} \leq s^{(n+1)}_{t_{n+1}} \text{ for all } t \in (t'_{n+1}, t_{n+1}].$$

Choose an increasing function $f\colon (0,1] \to (0,1]$ such that

$$f\left(\left(1-\frac{1}{n}, 1-\frac{1}{n+1}\right]\right) = (t'_{n+1}, t_{n+1}]$$

for all $n \in \mathbb{N}$. Define a path $(s_t)_{t\in(0,1]}$ in $S$ by taking $s_1 = s$ and

$$s_t = s^{(n+1)}_{f(t)} \text{ for } t \in \left(1-\frac{1}{n}, 1-\frac{1}{n+1}\right].$$

Then $\gamma_i(s_t) = s_t$ for all $t \in (0,1]$ and all $i \in I$, so $s \in S^\gamma$. It is clear that this path satisfies the conditions in the definition of $S^\gamma$ for $s$. □

With the notation of Lemma VIII.3.2, we do not know in general whether $S^\gamma_{\mathbb{N}}$ is an object in **Cu**. However, if $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a finite group $G$ on a $C^*$-algebra $A$ with the Rokhlin property, it will follow from the next theorem that $\mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)}_{\mathbb{N}}$ coincides with $\mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)}$, and with the Cuntz semigroup of $A^\alpha$, so in particular belongs to **Cu**.

We need a lemma.

**Lemma VIII.3.3.** Let $S$ be a semigroup in **Cu**, let $s$ be an element in $S$ and let $(s_n)_{n\in\mathbb{N}}$ be a rapidly increasing sequence in $S$ such that $s = \sup_{n\in\mathbb{N}} s_n$. Let $T$ be a subset of $S$ such that every element of $T$ is the supremum of a rapidly increasing sequence of elements in $T$. Suppose that for every $n \in \mathbb{N}$ there is $t \in T$ such that $s_n \ll t \leq s$. Then there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $T$ such that $s = \sup_{n\in\mathbb{N}} t_n$.

*Proof.* It is sufficient to construct an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers and a sequence $(t_k)_{k\in\mathbb{N}}$ in $T$ such that $s_{n_k} \leq t_k \leq s_{n_{k+1}}$ for all $k \in \mathbb{N}$, since this implies that $s = \sup_{k\in\mathbb{N}} t_k$.

For $k = 1$, set $n_1 = 1$ and $s_{n_1} = 0$. Assume inductively that we have constructed $n_j$ and $t_j$ for all $j \leq k$ and let us construct $n_{k+1}$ and $t_{k+1}$. By the assumptions of the lemma, there exists $t \in T$ such that $s_{n_k} \ll t \leq s$. Also by assumption, $t$ is the supremum of a rapidly increasing sequence of elements of $T$. Hence there exists $t' \in T$ such that $s_{n_k} \leq t' \ll s$. Use that $s = \sup\limits_{n\in\mathbb{N}} s_n$ and $t' \ll s$, to choose $n_{k+1} \in \mathbb{N}$ with $n_{k+1} > n_k$ such that $t' \leq s_{n_{k+1}} \ll s$. Set $t_{k+1} = t'$. Then $s_{n_k} \leq t_{k+1} \leq s_{n_{k+1}}$. This completes the proof of the lemma. $\square$

The following lemma is a restatement of [231, Lemma 4].

**Lemma VIII.3.4.** Let $A$ be a $C^*$-algebra, let $(x_i)_{i=0}^n$ be elements of $\mathrm{Cu}(A)$ such that $x_{i+1} \ll x_i$ for all $i = 0, \ldots, n$, and let $\varepsilon > 0$. Then there exists $a \in (A \otimes \mathcal{K})_+$ such that

$$\begin{aligned} x_n \ll [(a-(n-1)\varepsilon)_+] \ll x_{n-1} \ll [(a-(n-2)\varepsilon)_+] \ll \cdots \\ \cdots \ll x_3 \ll [(a-2\varepsilon)_+] \ll x_2 \ll [(a-\varepsilon)_+] \ll x_1 \ll [a] = x_0. \end{aligned}$$

For use in the proof of the next theorem, if $\phi\colon A \to B$ is a homomorphism between $C^*$-algebras $A$ and $B$, we denote by $\phi^s\colon A\otimes\mathcal{K} \to B\otimes\mathcal{K}$ the stabilized homomorphism $\phi^s = \phi\otimes \mathrm{id}_{\mathcal{K}}$.

**Theorem VIII.3.5.** Let $A$ be a $C^*$-algebra and let $\alpha$ be an action of a finite group $G$ on $A$ with the Rokhlin property. Let $i\colon A^\alpha \to A$ be the inclusion map. Then:

(1) The map $\mathrm{Cu}(\widetilde{i})\colon \mathrm{Cu}(\widetilde{A^\alpha}) \to \mathrm{Cu}(\widetilde{A})$ is an order embedding;

(2) The map $\mathrm{Cu}(i)\colon \mathrm{Cu}(A^\alpha) \to \mathrm{Cu}(A)$ is an order embedding and

$$\mathrm{Im}(\mathrm{Cu}(i)) = \overline{\mathrm{Im}\left(\sum_{g\in G}\mathrm{Cu}(\alpha_g)\right)} = \mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)}_{\mathbb{N}} = \mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)};$$

*Proof.* In the proof of this theorem, we will denote the action induced by $\alpha$ on $\widetilde{A}\otimes\mathcal{K}$ again by $\alpha$.

(1) Let $a, b \in \widetilde{A^\alpha}\otimes\mathcal{K}$ satisfy $a \precsim b$ in $\widetilde{A}\otimes\mathcal{K}$. We want to show that $a \precsim b$ in $\widetilde{A^\alpha}\otimes\mathcal{K}$. Let $\varepsilon > 0$. By Lemma II.6.6, there exists $d \in \widetilde{A}\otimes\mathcal{K}$ such that $(a-\varepsilon)_+ = dbd^*$. Apply $\alpha_g$ to this equation to get $(a-\varepsilon)_+ = \alpha_g(d)b\alpha_g(d^*)$ for all $g \in G$.

Let $\pi\colon \widetilde{A} \to \mathbb{C}$ be the quotient map and let $j\colon \mathbb{C} \to \widetilde{A}$ be the inclusion $j(\lambda) = \lambda 1_{\widetilde{A}}$ for all $\lambda \in \mathbb{C}$. It is clear that $\pi \circ j = \mathrm{id}_{\mathbb{C}}$. Set

$$\begin{aligned} a_1 &= (j^s \circ \pi^s)((a-\varepsilon)_+) \in \mathbb{C}1_{\widetilde{A}} \otimes \mathcal{K}, & a_2 &= (a-\varepsilon)_+ - a_1 \in A^\alpha \otimes \mathcal{K}, \\ b_1 &= (j^s \circ \pi^s)(b) \in \mathbb{C}1_{\widetilde{A}} \otimes \mathcal{K}, & b_2 &= b - b_1 \in A^\alpha \otimes \mathcal{K}, \\ d_1 &= (j^s \circ \pi^s)(d) \in \mathbb{C}1_{\widetilde{A}} \otimes \mathcal{K}, & d_2 &= d - d_1 \in A \otimes \mathcal{K}. \end{aligned}$$

Then $a_1 = d_1 b_1 d_1^*$. Set

$$F = \{\alpha_g(d_2) b \alpha_g(d_2) \colon g \in G\} \cup \{d_1 b \alpha_g(d_2^*) \colon g \in G\} \cup \{a_2 - d_1 b_2 d_1^*\} \subseteq \widetilde{A} \otimes \mathcal{K}.$$

Use Lemma VIII.2.2 (2) to choose orthogonal positive contractions $(r_g)_{g\in G}$ in $(A \otimes \mathcal{K})^\infty \cap F' \subseteq (\widetilde{A} \otimes \mathcal{K})^\infty \cap F'$ such that $\alpha_g(r_g) = r_{gh}$ for all $g, h \in G$, and $\left(\sum\limits_{g\in G} r_g\right) x = x$ for all $x \in F$. Set

$$f = \sum_{g\in G} r_g \alpha_g(d_2) + d_1 \in (\widetilde{A} \otimes \mathcal{K})^\infty.$$

In the following computation, we use in the first step the identities $r_g x = x r_g$ for all $g \in G$ and $x \in F$, $r_g r_h = 0$ for all $g \neq h$, and $(r_g^2 - r_g)x = x$ for all $g \in G$ and $x \in F$; in the second step the definition of $d_2$; in the fourth step that $d_1 \in (\widetilde{A} \otimes \mathcal{K})^\alpha$ and the identity $(a-\varepsilon)_+ = \alpha_g(d) b \alpha_g(d^*)$ for all $g \in G$; in the fifth step the identity $a_1 = d_1 b_1 d_1^*$; and in the last step the identity $(\sum\limits_{g\in G} r_g)x = x$ for all $x \in F$:

$$\begin{aligned}
fbf^* &= \left( \sum_{g,h\in G} r_g\alpha_g(d_2)b\alpha_h(d_2^*)r_h + \sum_{g\in G} r_g\alpha_g(d_2)bd_1^* + \sum_{g\in G} d_1b\alpha_g(d_2^*)r_g \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g\alpha_g(d_2)b\alpha_g(d_2^*) + \sum_{g\in G} r_g\alpha_g(d_2)bd_1^* + \sum_{g\in G} r_g d_1b\alpha_g(d_2^*) \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g \left(\alpha_g(d-d_1)b\alpha_g(d^*-d_1^*) + \alpha_g(d_2)bd_1^* + d_1b\alpha_g(d_2^*)\right) \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g \left(\alpha_g(d)b\alpha_g(d^*) - \alpha_g(d-d_2)bd_1^* - d_1b\alpha_g(d^*-d_2^*) + d_1bd_1^*\right) \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g \left((a-\varepsilon)_+ - d_1bd_1^* - d_1bd_1^* + d_1bd_1^*\right) \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g \left((a-\varepsilon)_+ - d_1bd_1^*\right) \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g \left(a_1 + a_2 - d_1b_1d_1^* - d_1b_2d_1^*\right) \right) + d_1bd_1^* \\
&= \left( \sum_{g\in G} r_g \left(a_2 - d_1b_2d_1^*\right) \right) + d_1bd_1^* \\
&= a_2 + d_1b_1d_1^* \\
&= (a-\varepsilon)_+.
\end{aligned}$$

Shortly, $(a-\varepsilon)_+ = fbf^*$ in $(\widetilde{A}\otimes\mathcal{K})^\infty$. Since

$$f = \sum_{g\in G} r_g\alpha_g(d_2) + d_1 = \sum_{g\in G} \alpha_g(r_e d_2) + d_1,$$

it follows that $\alpha_g(f) = f$ for all $g \in G$. This implies that $f$ is the image of a sequence $(f_n)_{n\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N}, \widetilde{A^\alpha}\otimes\mathcal{K})$, which satisfies

$$\lim_{n\to\infty} f_n b f_n^* = (a-\varepsilon)_+.$$

Thus, $(a-\varepsilon)_+ \precsim b$ in $\widetilde{A^\alpha}\otimes\mathcal{K}$. Since $\varepsilon > 0$ is arbitrary, we conclude that $[a] \leq [b]$ in $\mathrm{Cu}(\widetilde{A^\alpha})$, as desired.

(2) Since $A$ is an ideal in $\widetilde{A}$, the semigroup $\mathrm{Cu}(A)$ can be identified with the subsemigroup of $\mathrm{Cu}(\widetilde{A})$ given by

$$\{[a] \in \mathrm{Cu}(\widetilde{A})\colon a \in (A \otimes \mathcal{K})_+\}.$$

Using this identification, it is clear that the restriction of $\mathrm{Cu}(\widetilde{i})$ to $\mathrm{Cu}(A)$ is $\mathrm{Cu}(i)$. Therefore, it follows from the first part of the theorem that $\mathrm{Cu}(i)$ is an order embedding.

Let us now proceed to prove the equalities stated in the theorem. It is sufficient to show that

$$\mathrm{Im}(\mathrm{Cu}(i)) \subseteq \overline{\mathrm{Im}\left(\sum_{g\in G} \mathrm{Cu}(\alpha_g)\right)} \subseteq \mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)} \subseteq \mathrm{Cu}(A)_{\mathbb{N}}^{\mathrm{Cu}(\alpha)} \subseteq \mathrm{Im}(\mathrm{Cu}(i)). \tag{VIII.5}$$

The third inclusion is immediate and true in full generality. The second inclusion follows using that for $[a] \in \mathrm{Cu}(A)$, the element $\sum\limits_{g\in G} \mathrm{Cu}(\alpha_g)([a])$ is $\mathrm{Cu}(\alpha)$-invariant, that $\sum\limits_{g\in G} \mathrm{Cu}(\alpha_g)([a])$ is the supremum of the path

$$t \mapsto \sum_{g\in G} \mathrm{Cu}(\alpha_g)([(a+t-1)_+]),$$

and that $\mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)} = \overline{\mathrm{Cu}(A)^{\mathrm{Cu}(\alpha)}}$ by part (2) of Lemma VIII.3.2.

We proceed to show the first inclusion. Fix a positive element $a \in A^\alpha \otimes \mathcal{K}$ and let $\varepsilon > 0$. Using the Rokhlin property for $\alpha \otimes \mathrm{id}_{\mathcal{K}}$ with $F = \{a\}$, choose orthogonal positive contractions $(r_g)_{g\in G} \subseteq A \otimes \mathcal{K}$ such that

$$\left\| a - \sum_{g\in G} r_g a r_g \right\| < \varepsilon \quad \text{and} \quad \|\alpha_g(r_e a r_e) - r_g a r_g\| < \varepsilon, \tag{VIII.6}$$

for all $g \in G$. Using the first inequality above and Lemma II.6.6, we obtain

$$\big[(a-4\varepsilon)_+\big] \leq \left[\left(\sum_{g\in G} r_g a r_g - 3\varepsilon\right)_+\right] \leq \left[\left(\sum_{g\in G} r_g a r_g - \varepsilon\right)_+\right] \leq [a].$$

Furthermore, using the second inequality in (VIII.6) and again using Lemma II.6.6, we deduce that

$$\Big[(r_g a r_g - 3\varepsilon)_+\Big] \leq \Big[(\alpha_g(r_e a r_e) - 2\varepsilon)_+\Big] \leq \Big[(r_g a r_g - \varepsilon)_+\Big].$$

Take the sum of the previous inequalities, add them over $g \in G$, and use that $\mathrm{Cu}(\alpha_g)[(r_e a r_e - 2\varepsilon)_+] = [(\alpha_g(r_e a r_e) - 2\varepsilon)_+]$, to conclude that

$$\big[(a-4\varepsilon)_+\big] \ll \sum_{g\in G} \mathrm{Cu}(\alpha_g)\,\big[(r_e a r_e - 2\varepsilon)_+\big] \leq [a].$$

We have shown that for every $\varepsilon > 0$, there is an element $x$ in $\mathrm{Im}\left(\sum_{g\in G} \mathrm{Cu}(\alpha_g)\right)$ such that

$$[(a-\varepsilon)_+] \ll x \leq [a].$$

By Lemma VIII.3.3 applied to $[a] = \sup_{\varepsilon>0}[(a-\varepsilon)_+]$ and to the set $S = \mathrm{Im}\left(\sum_{g\in G} \mathrm{Cu}(\alpha_g)\right)$, it follows that $[a]$ is the supremum of an increasing sequence in $\mathrm{Im}\left(\sum_{g\in G} \mathrm{Cu}(\alpha_g)\right)$, showing that the first inclusion in (VIII.5) holds.

In order to complete the proof, let us show that the fourth inclusion in (VIII.5) is also true. Fix $x \in \mathrm{Cu}(A)_{\mathbb{N}}^{\mathrm{Cu}(\alpha)}$. Choose a rapidly increasing sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathrm{Cu}(A)$ such that $\mathrm{Cu}(\alpha_g)(x_n) = x_n$ for all $n \in \mathbb{N}$ and all for all $g \in G$. Fix $m \in \mathbb{N}$ and consider the elements $x_n$ with $n \geq m$. Note that $x_m \ll x_{m+1} \ll \cdots \ll x$. By Lemma VIII.3.4, there is a positive element $a \in A \otimes \mathcal{K}$ such that

$$x_m \ll [(a-3\varepsilon)_+] \ll x_{m+1} \ll (a-2\varepsilon)_+ \ll x_{m+2} \ll (a-\varepsilon)_+ \ll x = [a].$$

Note that this implies that

$$[\alpha_g(a)] = \mathrm{Cu}(\alpha_g)[a] = \mathrm{Cu}(\alpha_g)(x) = x = [a] \leq [a]$$

and

$$[(a-2\varepsilon)_+] \leq x_{m+2} = \mathrm{Cu}(\alpha_g)(x_{m+2}) \leq \mathrm{Cu}(\alpha_g)[(a-\varepsilon)_+] = [\alpha_g((a-\varepsilon)_+)]$$

for every $g \in G$. By the definition of Cuntz subequivalence, there are elements $f_g, h_g \in A \otimes \mathcal{K}$ for $g \in G$ such that

$$\|\alpha_g(a) - f_g a f_g^*\| < \frac{\varepsilon}{|G|}$$

and

$$\|(a-2\varepsilon)_+ - h_g\alpha_g((a-\varepsilon)_+)h_g^*\| < \frac{\varepsilon}{|G|}.$$

Using the Rokhlin property for $\alpha$, with

$$F = \{\alpha_g(a), \alpha_g((a-\varepsilon)_+), f_g, h_g \colon g \in G\} \cup \{(a-2\varepsilon)_+\},$$

choose positive orthogonal contractions $(r_g)_{g\in G} \subseteq (A \otimes \mathcal{K})^\infty \cap F'$ as in (2) of Lemma VIII.2.2. Set $f = \sum\limits_{g\in G} f_g r_g$ and $h = \sum\limits_{g\in G} h_g r_g$. Then

$$\left\|\sum_{g\in G} r_g\alpha_g(a)r_g - faf^*\right\| = \left\|\sum_{g\in G} r_g(\alpha_g(a) - f_g a f_g^*)\right\| < |G| \cdot \frac{\varepsilon}{|G|} = \varepsilon,$$

in $(A\otimes\mathcal{K})^\infty$. Similarly,

$$\left\|(a-2\varepsilon)_+ - h\left(\sum_{g\in G}\alpha_g((a-\varepsilon)_+)\right)h^*\right\| < \varepsilon.$$

Using that $r_g$ commutes with $\alpha_g(a)$ and that $r_g^2\alpha_g(a) = r_g\alpha_g(a)$ for all $g \in G$, one easily shows that

$$\sum_{g\in G} r_g\alpha_g(a)r_g = \sum_{g\in G}\alpha(r_e a r_e), \quad \text{and} \quad r_g(\alpha_g((a-\varepsilon)_+))r_g = (r_g\alpha_g(a)r_g - \varepsilon)_+,$$

for all $g \in G$. Thus, we have

$$\begin{aligned}
\sum_{g\in G} r_g(\alpha_g((a-\varepsilon)_+))r_g &= \sum_{g\in G} r_g(\alpha_g(a)-\varepsilon)_+ r_g \\
&= \sum_{g\in G}(r_g\alpha_g(a)r_g - \varepsilon)_+ \\
&= \left(\sum_{g\in G} r_g\alpha_g(a)r_g - \varepsilon\right)_+ \\
&= \left(\sum_{g\in G}\alpha_g(r_e a r_e) - \varepsilon\right)_+.
\end{aligned}$$

Therefore, we conclude that
$$\left\| \sum_{g\in G} \alpha_g(r_e a r_e) - faf^* \right\| < \varepsilon,$$
and
$$\left\| (a-2\varepsilon)_+ - h\left( \sum_{g\in G} \alpha_g(r_e a r_e) - \varepsilon \right)_+ h^* \right\| < \varepsilon.$$

Let $(r_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$, and $(h_n)_{n\in\mathbb{N}}$ be representatives of $r_e$, $f$, and $h$ in $\ell^\infty(\mathbb{N}, A\otimes\mathcal{K})$, with $r_n$ positive for all $n\in\mathbb{N}$. By the previous inequalities, there exists $k\in\mathbb{N}$ such that
$$\left\| \sum_{g\in G} \alpha_g(r_k a r_k) - f_k a f_k^* \right\| < \varepsilon,$$
and
$$\left\| (a-2\varepsilon)_+ - h_k\left( \sum_{g\in G} \alpha_g(r_k a r_k) - \varepsilon \right)_+ h_k^* \right\| < \varepsilon$$
hold in $A\otimes\mathcal{K}$. By Lemma II.6.6 applied to the elements $\sum\limits_{g\in G} \alpha_g(r_k a r_k)$ and $f_k a f_k^*$, and to the elements $(a-2\varepsilon)_+$ and $h_k\left( \sum\limits_{g\in G} \alpha(r_k a r_k) - \varepsilon \right)_+ h_k^*$, we deduce that
$$[(a-3\varepsilon)_+] \leq \left[ \left( \sum_{g\in G} \alpha_g(r_k a r_k) - \varepsilon \right)_+ \right] \leq [a].$$
Therefore,
$$x_m \ll \left[ \left( \sum_{g\in G} \alpha_g(r_k a r_k) - \varepsilon \right)_+ \right] \ll x.$$

Note that the element $\left( \sum\limits_{g\in G} \alpha_g(r_k a r_k) - \varepsilon \right)_+$ belongs to $(A\otimes\mathcal{K})^\alpha$ and so it is in the image of the inclusion map $i^s = i\otimes \mathrm{id}_{\mathcal{K}} \colon (A\otimes\mathcal{K})^\alpha \to A\otimes\mathcal{K}$. Since $m$ is arbitrary, we deduce that $x$ is the supremum of an increasing sequence in $\mathrm{Im}(\mathrm{Cu}(i))$ by Lemma VIII.3.3. Choose a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathrm{Cu}(A^\alpha)$ such that $(\mathrm{Cu}(i)(y_n))_{n\in\mathbb{N}}$ is increasing in $\mathrm{Cu}(A)$ and set $x = \sup\limits_{n\in\mathbb{N}}(\mathrm{Cu}(i)(y_n))$. Since $\mathrm{Cu}(i)$ is an order embedding, it follows that $(y_n)_{n\in\mathbb{N}}$ is itself increasing in $\mathrm{Cu}(A^\alpha)$. Set $y = \sup\limits_{n\in\mathbb{N}} y_n$. Then $\mathrm{Cu}(y) = x$ since $\mathrm{Cu}(i)$ preserves suprema of increasing sequences. $\square$

**Corollary VIII.3.6.** Let $A$ be a $C^*$-algebra and let $\alpha$ be an action of a finite group $G$ on $A$ with the Rokhlin property. Then $\mathrm{Cu}(A \rtimes_\alpha G)$ is order isomorphic to the semigroup:

$$\left\{ x \in \mathrm{Cu}(A) \colon \exists\ (x_n)_{n\in\mathbb{N}} \text{ in } \mathrm{Cu}(A) \colon \begin{array}{c} x_n \ll x_{n+1}\ \forall n \in \mathbb{N} \text{ and } x = \sup\limits_{n\in\mathbb{N}} x_n, \\ \mathrm{Cu}(\alpha_g)(x_n) = x_n\ \forall g \in G, \forall n \in \mathbb{N} \end{array} \right\}.$$

*Proof.* Since $\alpha$ has the Rokhlin property, the fixed point algebra $A^\alpha$ is Morita equivalent to the crossed product $A \rtimes_\alpha G$ by [199, Theorem 2.8]. Therefore, there is a natural isomorphism $\mathrm{Cu}(A \rtimes_\alpha G) \cong \mathrm{Cu}(A^\alpha)$. Denote by $i \colon A^\alpha \to A$ the natural embedding. By Theorem VIII.3.5, the semigroup $\mathrm{Cu}(A^\alpha)$ can be naturally identified with its image under the order embedding $\mathrm{Cu}(i)$, which is $\mathrm{Cu}(A)_{\mathbb{N}}^{\mathrm{Cu}(\alpha)}$ again by Theorem VIII.3.5. The result follows. $\square$

**Corollary VIII.3.7.** Let $A$ be a $C^*$-algebra, let $\alpha$ be an action of a finite group $G$ on $A$ with the Rokhlin property, and set $n = |G|$. Suppose that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for every $g \in G$, and that the map multiplication by $n$ on $\mathrm{Cu}(A)$ is an order embedding (in other words, whenever $x, y \in \mathrm{Cu}(A)$ satisfy $nx \leq ny$, one has $x \leq y$.) Then the map multiplication by $n$ in $\mathrm{Cu}(A)$ is an order isomorphism.

*Proof.* It suffices to show that for all $x \in \mathrm{Cu}(A)$, there exists $y \in \mathrm{Cu}(A)$ such that $x = ny$. By Theorem VIII.3.5 (2), we have

$$\overline{\mathrm{Im}\left(\sum_{g\in G} \mathrm{Cu}(\alpha_g)\right)} = \mathrm{Cu}(A)_{\mathbb{N}}^{\mathrm{Cu}(\alpha)}.$$

Since $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$, this identity can be rewritten as

$$\overline{n\mathrm{Cu}(A)} = \mathrm{Cu}(A).$$

In particular, if $x$ is an element in $\mathrm{Cu}(A)$, then there exists a sequence $(y_k)_{k\in\mathbb{N}}$ in $\mathrm{Cu}(A)$ such that $(ny_k)_{k\in\mathbb{N}}$ is increasing and $x = \sup\limits_{k\in\mathbb{N}}(ny_k)$. Since $(ny_k)_{k\in\mathbb{N}}$ is increasing, it follows from our assumptions that $(y_k)_{k\in\mathbb{N}}$ is increasing as well. Set $y = \sup\limits_{k\in\mathbb{N}} y_k$. Then

$$x = \sup_{k\in\mathbb{N}}(ny_k) = n \sup_{k\in\mathbb{N}} y_k = ny,$$

and the claim follows. $\square$

Let $A$ be a $C^*$-algebra and let $p$ and $q$ be projections in $A$. We say that $p$ and $q$ are *Murray-von Neumann equivalent*, and denote this by $p \sim_{\mathrm{MvN}} q$, if there exists $v \in A$ such that $p = v^*v$ and $q = vv^*$. We say that $p$ is *Murray-von Neumann subequivalent* to $q$, and denote this by $p \precsim_{\mathrm{MvN}} q$, if there is a projection $p' \in A$ such that $p \sim_{\mathrm{MvN}} p'$ and $p' \leq q$. The projection $p$ is said to be *finite* if whenever $q$ is a projection in $A$ with $q \leq p$ and $q \sim_{\mathrm{MvN}} p$, then $q = p$.

If $A$ is unital, then $A$ is said to be *finite* if its unit is a finite projection. Moreover, $A$ is said to be *stably finite* if $M_n(A)$ is finite for all $n \in \mathbb{N}$. If $A$ is not unital, we say that $A$ is (stably) finite if so is its unitization $\widetilde{A}$.

**Lemma VIII.3.8.** Let $A$ be a stably finite $C^*$-algebra and let $p \in A \otimes \mathcal{K}$ be a projection. Suppose that there are positive elements $a, b \in A \otimes \mathcal{K}$ such that $[p] = [a] + [b]$ in $\mathrm{Cu}(A)$. Then $a$ and $b$ are Cuntz equivalent to projections in $A \otimes \mathcal{K}$ (see the comments before Lemma 2.4 for the definition of Cuntz equivalence).

*Proof.* Let $a$ and $b$ be elements in $A \otimes \mathcal{K}$ as in the statement. By Remark II.6.7, we have

$$[a] = \sup_{\varepsilon > 0} [(a - \varepsilon)_+] \quad \text{and} \quad [b] = \sup_{\varepsilon > 0} [(b - \varepsilon)_+].$$

Since $[p] \ll [p]$, there exists $\varepsilon > 0$ such that $[p] = [(a - \varepsilon)_+] + [(b - \varepsilon)_+]$. Choose a function $f_\varepsilon \in C_0(0, \infty)$ that is zero on the interval $[\varepsilon, \infty)$, nonzero at every point of $(0, \varepsilon)$ and $\|f_\varepsilon\|_\infty \leq 1$. Then

$$[p] + [f_\varepsilon(a)] + [f_\varepsilon(b)] = [(a - \varepsilon)_+] + [f_\varepsilon(a)] + [(b - \varepsilon)_+] + [f_\varepsilon(b)] \leq [a] + [b] = [p].$$

Hence, $[p] + [f_\varepsilon(a)] + [f_\varepsilon(b)] = [p]$. Choose $c \in (A \otimes \mathcal{K})_+$ such that $[c] = [f_\varepsilon(a)] + [f_\varepsilon(b)]$ and $cp = 0$. Then $p + c \precsim p$. By part (4) of Lemma 2.3 in [152], for every $\delta > 0$ there exists $x \in A \otimes \mathcal{K}$ such that

$$p + (c - \delta)_+ = x^*x, \quad xx^* \in p(A \otimes \mathcal{K})p.$$

Fix $\delta > 0$ and let $x$ be as above. Let $x = v|x|$ be the polar decomposition of $x$ in the bidual of $A \otimes \mathcal{K}$. Set $p' = vpv^*$ and $c' = v(c - \delta)_+ v^*$. Then $p'$ is a projection, $p'$ and $c'$ are orthogonal, $p$ and $p'$ are Murray-von Neumann equivalent, and $p' + c' \in pAp$. Using stable finiteness of $A$ we conclude that $p = p'$ and $c' = 0$. It follows that $(c-\delta)_+ = 0$ for all $\delta > 0$, and thus $c = 0$. Hence, $f_\varepsilon(b) = f_\varepsilon(a) = 0$ and in particular, $a$ and $b$ have a gap in their spectra. Therefore, they are Cuntz equivalent to projections. $\square$

Recall that the *Murray-von Neumann semigroup* of $A$, denoted by $V(A)$, is defined as the quotient of the set of projections of $A \otimes \mathcal{K}$ by the Murray-von Neumann equivalence relation.

Note that $p \precsim_{\mathrm{Cu}} q$ if and only if $p \precsim_{\mathrm{MvN}} q$. On the other hand, $p \precsim_{\mathrm{MvN}} q$ and $q \precsim_{\mathrm{MvN}} p$ do not in general imply that $p \sim_{\mathrm{MvN}} q$, although this is the case whenever $A$ is finite. In particular, if $A$ is finite, then $p \sim_{\mathrm{Cu}} q$ if and only if $p \sim_{\mathrm{MvN}} q$. Hence, if $A$ is stably finite, then the semigroup $\mathrm{V}(A)$ can be identified with the ordered subsemigroup of $\mathrm{Cu}(A)$ consisting of the Cuntz equivalence classes of projections of $A \otimes \mathcal{K}$.

Recall that if $S$ is a semigroup in **Cu** and $x$ and $y$ are elements of $S$, we say that $x$ is *compactly contained* in $y$, and denote this by $x \ll y$, if for every increasing sequence $(y_n)_{n\in\mathbb{N}}$ in $S$ such that $y = \sup\limits_{n\in\mathbb{N}} y_n$, there exists $n_0 \in \mathbb{N}$ such that $x \leq y_n$ for all $n \geq n_0$.

**Definition VIII.3.9.** Let $S$ be a semigroup in **Cu** and let $x$ be an element of $S$. We say that $x$ is *compact* if $x \ll x$. Equivalently, $x$ is compact if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $S$ such that $x = \sup\limits_{n\in\mathbb{N}} x_n$, then there exists $n_0 \in \mathbb{N}$ such that $x_n = x$ for all $n \geq n_0$.

It is easy to check that the Cuntz class $[p] \in \mathrm{Cu}(A)$ of any projection $p$ in a $C^*$-algebra $A$ (or in $A \otimes \mathcal{K}$) is a compact element in $\mathrm{Cu}(A)$. Moreover, when $A$ is stably finite, then every compact element of $\mathrm{Cu}(A)$ is the Cuntz class of a projection in $A \otimes \mathcal{K}$ by [22, Theorem 3.5]. In particular, $V(A)$ can be identified with the semigroup of compact elements of $\mathrm{Cu}(A)$ if $A$ is a stably finite $C^*$-algebra.

When studying stably finite $C^*$-algebras in connection with finite group actions with the Rokhlin property, the following lemma is often times useful. The result may be interesting in its own right, and could have been proved in [191] since it is a direct application of their methods.

**Lemma VIII.3.10.** Let $G$ be a finite group, let $A$ be a unital stably finite $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then the crossed product $A \rtimes_\alpha G$ and the fixed point algebra $A^\alpha$ are stably finite.

*Proof.* The fixed point algebra $A^\alpha$, being a unital subalgebra of $A$, is stably finite. On the other hand, the crossed product $A \rtimes_\alpha G$, being stably isomorphic to $A^\alpha$ by [199, Theorem 2.8], must itself also be stably finite. $\square$

For unital, simple $C^*$-algebras, part (2) of the theorem below was first proved by Izumi in [132]. The proof in our context follows completely different ideas.

**Theorem VIII.3.11.** Let $A$ be a stably finite $C^*$-algebra and let $\alpha$ be an action of a finite group $G$ on $A$ with the Rokhlin property. Let $i\colon A^\alpha \to A$ be the inclusion map.

(1) The map $V(i)\colon V(A^\alpha) \to V(A)$ is an order embedding and

$$\mathrm{Im}(V(i)) = \mathrm{Im}\left(\sum_{g\in G} V(\alpha_g)\right) = \{x \in V(A)\colon V(\alpha_g)(x) = x,\, \forall g \in G\}.$$

(2) If $A$ has an approximate identity consisting of projections, then the map $K_0(i)\colon K_0(A^\alpha) \to K_0(A)$ is an order embedding and

$$\mathrm{Im}(K_0(i)) = \mathrm{Im}\left(\sum_{g\in G} K_0(\alpha_g)\right) = \{x \in K_0(A)\colon K_0(\alpha_g)(x) = x,\, \forall g \in G\}.$$

*Proof.* (1) The fact that $V(i)$ is an order embedding is a consequence of Theorem VIII.3.5 and the remarks before and after Definition VIII.3.9. Let us now show the inclusions

$$\mathrm{Im}(V(i)) \subseteq \mathrm{Im}\left(\sum_{g\in G} V(\alpha_g)\right) \subseteq \{x \in V(A)\colon V(\alpha_g)(x) = x\ \forall\ g \in G\} \subseteq \mathrm{Im}(V(i)). \tag{VIII.7}$$

Let $p \in A^\alpha \otimes \mathcal{K}$ be a projection. By Theorem VIII.3.5, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $(A\otimes\mathcal{K})_+$ such that $\left(\sum\limits_{g\in G} \mathrm{Cu}(\alpha_g)([a_n])\right)_{n\in\mathbb{N}}$ is increasing and

$$[i(p)] = \sup_{n\in\mathbb{N}}\left(\sum_{g\in G} \mathrm{Cu}(\alpha_g)([a_n])\right).$$

Since $[i(p)]$ is a compact element in $\mathrm{Cu}(A)$, it follows that there exists $n_0 \in \mathbb{N}$ such that $[i(p)] = \sum\limits_{g\in G} \mathrm{Cu}(\alpha_g)([a_n])$ for all $n \geq n_0$. Fix $m \geq n_0$. It is easy to check that if $S$ is a semigroup in the category **Cu**, then a sum of elements in $S$ is compact if and only if each summand is compact. It follows that $\mathrm{Cu}(\alpha_g)([a_m])$ is compact for all $g \in G$. In particular, and denoting the unit of $G$ by $e$, we deduce that $[a_m] = \mathrm{Cu}(\alpha_e)([a_m])$ is compact. Since $A$ is stably finite by assumption, there exists a projection $q \in A \otimes \mathcal{K}$ such that $[q] = [a_m]$. Thus

$$V(i)([p]) = \sum_{g \in G} V(\alpha_g)([q]) \in \mathrm{Im}\left(\sum_{g \in G} V(\alpha_g)\right),$$

showing that the first inclusion in (VIII.7) holds.

Using the fact that $\alpha_h \circ \left(\sum_{g \in G} \alpha_g\right) = \sum_{g \in G} \alpha_g$ for all $h \in G$, it is easy to check that

$$\mathrm{Im}\left(\sum_{g \in G} V(\alpha_g)\right) \subseteq \{x \in V(A)\colon V(\alpha_g)(x) = x,\ \forall g \in G\},$$

thus showing that the second inclusion also holds.

We proceed to prove the third inclusion. Let $x \in \mathrm{V}(A)$ be such that $\mathrm{V}(\alpha_g)(x) = x$ for all $g \in G$. Note that $x$ is compact as an element in $\mathrm{Cu}(A)$. It follows that $\mathrm{Cu}(\alpha_g)(x) = x$ for all $g \in G$ and hence by Theorem VIII.3.5 there exists $a \in (A^\alpha \otimes \mathcal{K})_+$ such that $\mathrm{Cu}(i)([a]) = x$. Since the map $\mathrm{Cu}(i)$ is an order embedding again by Theorem VIII.3.5, one concludes that $[a]$ is compact.

Finally, the fixed point algebra $A^\alpha$ is stably finite by Lemma VIII.3.10 and thus there is a projection $p \in A^\alpha \otimes \mathcal{K}$ such that $[p] = [a]$ in $\mathrm{Cu}(A^\alpha)$. It follows that $\mathrm{Cu}(i)([p]) = x$, showing that the third inclusion in (VIII.7) is also true.

(2) Follows using the first part, together with the fact that the $K_0$-group of a $C^*$-algebra containing an approximate identity consisting of projections, agrees with the Grothendieck group of the Murray-von Neumann semigroup of the algebra; see Proposition 5.5.5 in [13]. □

In the following corollary, the picture of $V(A \rtimes_\alpha G)$ is valid for arbitrary $A$.

**Corollary VIII.3.12.** Let $A$ be a stably finite $C^*$-algebra containing an approximate identity consisting of projections, and let $\alpha$ be an action of a finite group $G$ on $A$ with the Rokhlin property. Then there are isomorphisms

$$V(A \rtimes_\alpha G) \cong \{x \in V(A)\colon V(\alpha_g)(x) = x, \forall g \in G\},$$
$$K_*(A \rtimes_\alpha G) \cong \{x \in K_*(A)\colon K_*(\alpha_g)(x) = x, \forall g \in G\}.$$

*Proof.* Recall that if $\alpha$ has the Rokhlin property, then the fixed point algebra $A^\alpha$ and the crossed product $A \rtimes_\alpha G$ are Morita equivalent, and hence have isomorphic $K$-theory and Murray-von Neumann semigroup. The isomorphisms for $V(A \rtimes_\alpha G)$ and $K_0(A \rtimes_\alpha G)$ then follow from Theorem VIII.3.11 above.

Denote $B = A \otimes C(S^1)$ and give $B$ the diagonal action $\beta = \alpha \otimes \mathrm{id}_{C(S^1)}$ of $G$. Note that $B$ is stably finite and has an approximate identity consisting of projections, and that $\beta$ has the Rokhlin property by part (1) of Proposition VIII.2.3. Moreover, there is a natural isomorphism $B \rtimes_\beta G \cong (A \rtimes_\alpha G) \otimes C(S^1)$. Applying the Künneth formula in the first step, together with the conclusion of this proposition for $K_0$ (which was shown to hold in the paragraph above) in the second step, and again the Künneth formula in the fourth step, we obtain

$$\begin{aligned}\{x \in K_*(A)\colon K_*(\alpha_g)(x) = x,\ \forall\, g \in G\} &\cong \{x \in K_0(B)\colon K_0(\beta_g)(x) = x,\ \forall\, g \in G\}\\ &\cong K_0(B \rtimes_\beta G)\\ &\cong K_0((A \rtimes_\alpha G) \otimes C(S^1))\\ &\cong K_*(A \rtimes_\alpha G),\end{aligned}$$

as desired. □

**Equivariant UHF-absorption**

In this section, we study absorption of UHF-algebras in relation to the Rokhlin property. We show that for a certain class of $C^*$-algebras, absorption of a UHF-algebra of infinite type is equivalent to existence of an action with the Rokhlin property that is pointwise approximately inner. (The cardinality of the group is related to the type of the UHF-algebra.) Moreover, in this case, not only the $C^*$-algebra absorbs the corresponding UHF-algebra, but also the action in question absorbs the model action constructed in Example VIII.2.4. Thus, Rokhlin actions allow us to prove that certain algebras are *equivariantly* UHF-absorbing.

*Unique $n$-divisibility.*

The goal of this section is to show that for certain $C^*$-algebras, absorption of the UHF-algebra of type $n^\infty$ is equivalent to its Cuntz semigroup being $n$-divisible. Along the way, we show that for a $C^*$-algebra $A$, the Cuntz semigroups of $A$ and of $A \otimes M_{n^\infty}$ are isomorphic if and only if $\mathrm{Cu}(A)$ is uniquely $n$-divisible.

We point out that some of the results of this section, particularly Theorem VIII.4.5, were independently obtained in the recent preprint [4], as applications of their theory of tensor products of Cuntz semigroups. On the other hand, the proofs we give here are direct and elementary. Additionally, our techniques also apply to other functors, for instance the functor $\mathrm{Cu}^\sim$.

We begin defining the main notion of this section. Recall that if $S$ and $T$ are ordered semigroup and $\varphi\colon S \to T$ is a semigroup homomorphism, we say that $\varphi$ is an *order embedding* if $\varphi(s) \leq \varphi(s')$ implies $s \leq s'$ for all $s, s' \in S$. A semigroup isomorphism is called an *order preserving* semigroup isomorphism if it is an order embedding.

**Definition VIII.4.1.** Let $S$ be an ordered semigroup and let $n$ be a positive integer.

1. We say that $S$ is *$n$-divisible*, if for every $x$ in $S$ there exists $y$ in $S$ such that $x = ny$.
2. We say that $G$ is *uniquely $n$-divisible*, if multiplication by $n$ on $S$ is an order preserving semigroup isomorphism.

Recall that the category **Cu** is closed under sequential inductive limits.

**Lemma VIII.4.2.** Let $n \in \mathbb{N}$ and let $S$ be a semigroup in the category **Cu**. Denote by $\rho\colon S \to S$ the map given by $\rho(s) = ns$ for all $s \in S$. Let $T$ be the semigroup in **Cu** obtained as the inductive limit of the sequence

$$S \xrightarrow{\rho} S \xrightarrow{\rho} S \xrightarrow{\rho} \cdots .$$

Then $T$ is uniquely $n$-divisible.

*Proof.* Let $S$ and $T$ be as in the statement. To avoid any confusion with the notation, we will denote the map between the $k$-th and $(k+1)$-st copies of $S$ by $\rho_k$, so we write $T$ as the direct limit

$$S \xrightarrow{\rho_1} S \xrightarrow{\rho_2} S \xrightarrow{\rho_3} \cdots \longrightarrow T.$$

For $k, m \in \mathbb{N}$ with $m > k$, we let $\rho_{k,m} \colon S \to S$ denote the composition $\rho_{m-1} \circ \rho_{m-2} \circ \cdots \circ \rho_k$, and we let $\rho_{k,\infty} \colon S \to T$ denote the canonical map from the $k$-th copy of $S$ to $T$.

Let $s, t \in T$ satisfy $ns \leq nt$. By part (1) of Proposition II.6.4, there exist sequences $(s_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ in $S$ such that

$$\rho_k(s_k) \ll s_{k+1} \text{ for all } k \in \mathbb{N} \quad \text{and} \quad s = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(s_k)$$

$$\rho_k(t_k) \ll t_{k+1} \text{ for all } k \in \mathbb{N} \quad \text{and} \quad t = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(t_k).$$

It follows that $\rho_{k,\infty}(s_k) \ll \rho_{k+1,\infty}(s_{k+1})$ and $\rho_{k,\infty}(t_k) \ll \rho_{k+1,\infty}(t_{k+1})$ for all $k \in \mathbb{N}$.

Let $k \geq 2$ be fixed. Since

$$\rho_{k,\infty}(ns_k) \ll ns \leq nt = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(nt_k),$$

there exists $l \in \mathbb{N}$ such that $\rho_{k,\infty}(ns_k) \leq \rho_{l,\infty}(nt_l)$. Use part (2) of Proposition II.6.4 and $\rho_{j-1}(s_{j-1}) \ll s_j$ for all $j \in \mathbb{N}$, to choose $m \geq k, l$ such that $\rho_{k-1,m}(ns_{k-1}) \leq \rho_{l,m}(nt_l)$. Therefore,

$$\begin{aligned}
\rho_{k-1,\infty}(s_{k-1}) &= \rho_{m+1,\infty}(\rho_{m+1}(\rho_{k-1,m}(s_{k-1}))) \\
&= \rho_{m+1,\infty}(n\rho_{k-1,m}(s_{k-1})) \\
&= \rho_{m+1,\infty}(\rho_{k-1,m}(ns_{k-1})) \\
&\leq \rho_{m+1,\infty}(\rho_{l,m}(nt_l)) \\
&= \rho_{m+1,\infty}(n\rho_{l,m}(t_l)) \\
&= \rho_{m+1,\infty}(\rho_{m+1}(\rho_{l,m}(t_l))) \\
&= \rho_{l,\infty}(t_l) \\
&\leq t,
\end{aligned}$$

that is, $\rho_{k-1,\infty}(s_{k-1}) \leq t$. Since this holds for all $k \geq 2$, we conclude that

$$s = \sup_{k \geq 2} \rho_{k-1,\infty}(s_{k-1}) \leq t.$$

We have shown that $ns \leq nt$ in $T$ implies $s \leq t$. In other words, multiplication by $n$ on $T$ is an order embedding, as desired.

To conclude the proof, let us show that $T$ is $n$-divisible. Fix $t \in T$ and choose a sequence $(t_k)_{k\in\mathbb{N}}$ in $T$ satisfying

$$\rho_k(t_k) \ll t_{k+1} \text{ for all } k \in \mathbb{N} \quad \text{and} \quad t = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(t_k).$$

For each $k \in \mathbb{N}$ we have

$$\rho_{k,k+2}(t_k) = n^2 t_k = n\rho_{k+1,k+2}(t_k).$$

With $x_k = \rho_{k+1,\infty}(t_k)$, it follows that $\rho_{k,\infty}(t_k) = nx_k$. Since $(\rho_{k,\infty}(t_k))_{k\in\mathbb{N}}$ is an increasing sequence in $T$, we deduce that $(nx_k)_{k\in\mathbb{N}}$ is an increasing sequence in $T$ as well. Since we have shown in the first part of this proof that multiplication by $n$ on $T$ is an order embedding, we conclude that $(x_k)_{k\in\mathbb{N}}$ is also increasing. With $x$ denoting the supremum of $(x_k)_{k\in\mathbb{N}}$, we have

$$t = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(t_k) = \sup nx_k = n \sup_{k\in\mathbb{N}} x_k = nx,$$

which completes the proof. $\square$

We point out that the functor $\mathrm{Cu}^{\sim}$ does not distinguish between homomorphisms that are approximately unitarily equivalent (with unitaries taken in the unitization). On the other hand, the corresponding statement for approximate unitary equivalence with unitaries taken in the multiplier algebra is not known in general. The following proposition, of independent interest, implies that this is the case whenever the codomain has stable rank one. This will be used in the proof of Lemma VIII.4.4 to deduce that certain homomorphisms are trivial at the level of $\mathrm{Cu}^{\sim}$.

**Proposition VIII.4.3.** Let $A$ and $B$ be $C^*$-algebras with $B$ stable, and let $\phi, \psi \colon A \to B$ be homomorphisms. Suppose that $\phi$ and $\psi$ are approximately unitarily equivalent with unitaries taken in the multiplier algebra of $B$. Then $\phi$ and $\psi$ are approximately unitarily equivalent with unitaries taken in the unitization of $B$.

*Proof.* Denote by $\iota\colon B \to M(B)^\infty$ the canonical inclusion as constant sequences. We will identify $B$ with a subalgebra of $M(B)$, and suppress $\iota$ from the notation. Hence we will denote the maps $\iota \circ \phi, \iota \circ \psi\colon A \to M(B)^\infty$ again by $\phi$ and $\psi$, respectively.

Let $F \subseteq A$ be a finite set. Then there exists a unitary $u = \pi_{M(B)}((u_n)_{n\in\mathbb{N}})$ in $M(B)^\infty$ such that $\phi(a) = u\psi(a)u^*$ for all $a \in F$. Choose a sequence $(s_n)_{n\in\mathbb{N}}$ of positive contractions in $B$ such that

$$\lim_{n\to\infty} s_n\psi(a) = \lim_{n\to\infty} \psi(a)s_n = \psi(a)$$

for all $a \in F$. Let $s = \pi_{M(B)}((s_n)_{n\in\mathbb{N}})$ denote the image of $(s_n)_{n\in\mathbb{N}}$ in $B^\infty \subseteq M(B)^\infty$. Then

$$s\phi(a) = \phi(a)s = \phi(a)$$

for all $a \in F$. Since $B$ is stable, we have $B \subseteq \overline{\mathrm{GL}(\widetilde{B})}$ by [15, Lemma 4.3.2]. Hence, elements in $B$ have approximate polar decompositions with unitaries taken in $\widetilde{B}$. This implies that there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of unitaries in $\widetilde{B}$ such that $\lim\limits_{n\to\infty} \|u_n s_n - v_n s_n\| = 0$. Let $v = \pi_{\widetilde{B}}((v_n)_{n\in\mathbb{N}})$ denote the image of $(v_n)_{n\in\mathbb{N}}$ in $(\widetilde{B})^\infty$. Then $us = vs$ and

$$\phi(a) = u\psi(a)u^* = us\psi(a)su^* = vs\psi(a)sv^* = v\psi(a)v^*$$

for all $a \in F$. This implies that $\lim\limits_{n\to\infty} \|\phi(a) - v_n\psi(a)v_n^*\| = 0$ for all $a \in F$. Since $v_n$ is a unitary in $\widetilde{B}$ for all $n \in \mathbb{N}$, we conclude that $\phi$ and $\psi$ are approximately unitarily equivalent with unitaries taken in the unitization of $B$. $\square$

Let $n, k \in \mathbb{N}$. We let $\left(f_{i,j}^{(n^k)}\right)_{i,j=0}^{n^k-1}$ denote the set of matrix units of $M_{n^k}(\mathbb{C})$. Recall that if $A$ and $B$ are $C^*$-algebras and $\phi, \psi\colon A \to B$ are homomorphisms with orthogonal ranges, then $\phi + \psi$ is also a homomorphism and $\mathrm{Cu}(\phi + \psi) = \mathrm{Cu}(\phi) + \mathrm{Cu}(\psi)$.

**Lemma VIII.4.4.** Let $A$ be a $C^*$-algebra and let $n, k \in \mathbb{N}$. Let $\iota_k\colon A \to M_{n^k}(A)$ be the map given by $\iota_k(a) = a \otimes f_{0,0}^{(n^k)}$ for all $a \in A$, and let $j_k\colon M_{n^k}(A) \to M_{n^{k+1}}(A)$ be the map given by

$j_k(a) = a \otimes 1_n$ for all $a \in M_{n^k}(A)$. Then the map

$$\mathrm{Cu}(\iota_{k+1})^{-1} \circ \mathrm{Cu}(j_k) \circ \mathrm{Cu}(\iota_k) \colon \mathrm{Cu}(A) \to \mathrm{Cu}(A),$$

is the map multiplication by $n$.

*Proof.* Since Cu is invariant under stabilization, we may assume that the algebra $A$ is stable.

Fix $k$ in $\mathbb{N}$. For each $0 \leq i \leq n-1$, let $j_{k,i} \colon M_{n^k}(A) \to M_{n^{k+1}(A)}$ be the map defined by $j_{k,i}(b) = b \otimes f_{i,i}^{(n)}$ for all $b \in M_{n^k}(A)$. Then the maps $(j_{k,i})_{i=0}^{n-1}$ have orthogonal ranges and $j_k = \sum\limits_{i=1}^{n-1} j_{k,i}$. By the comments before this lemma, we have

$$\mathrm{Cu}(j_k) = \sum_{i=0}^{n-1} \mathrm{Cu}(j_{k,i}).$$

Since $f_{i,i}^{(n)}$ and $f_{\ell,\ell}^{(n)}$ are unitarily equivalent in $M_n(\mathbb{C})$ for all $i, \ell = 0, \ldots, n-1$, we conclude that the maps $j_{k,i}$ and $j_{k,\ell}$ are unitarily equivalent with unitaries in the multiplier algebra of $M_{n^{k+1}}(A)$. By Proposition VIII.4.3, this implies that the maps $j_{k,i}$ and $j_{k,\ell}$ are approximately unitarily equivalent (with unitaries taken in the unitization of $M_{n^{k+1}}(A)$). Since approximate unitarily equivalent maps yield the same morphism at the level of the Cuntz semigroup, we deduce that $\mathrm{Cu}(j_{k,i}) = \mathrm{Cu}(j_{k,\ell})$ for all $i, \ell = 0, \ldots, n-1$. Given a positive element $a$ in $A \otimes \mathcal{K}$,we have

$$\begin{aligned}
(\mathrm{Cu}(\iota_{k+1})^{-1} \circ \mathrm{Cu}(j_k) \circ \mathrm{Cu}(\iota_k))([a]) &= (\mathrm{Cu}(\iota_{k+1})^{-1} \circ \mathrm{Cu}(j_k)) \left( \left[ a \otimes f_{0,0}^{(n^k)} \right] \right) \\
&= \mathrm{Cu}(\iota_{k+1})^{-1} \left( \sum_{i=0}^{n-1} \mathrm{Cu}(j_{k,i}) \left( \left[ a \otimes f_{0,0}^{(n^k)} \right] \right) \right) \\
&= \mathrm{Cu}(\iota_{k+1})^{-1} \left( n\mathrm{Cu}(j_{k,0}) \left( \left[ a \otimes f_{0,0}^{(n^k)} \right] \right) \right) \\
&= n\mathrm{Cu}(\iota_{k+1})^{-1} \left( \left[ a \otimes f_{0,0}^{(n^k)} \otimes f_{0,0}^{(n)} \right] \right) \\
&= n\mathrm{Cu}(\iota_{k+1})^{-1} \left( \left[ a \otimes f_{0,0}^{(n^{k+1})} \right] \right) \\
&= n[a]
\end{aligned}$$

We conclude that $\mathrm{Cu}(\iota_{k+1})^{-1} \circ \mathrm{Cu}(j_k) \circ \mathrm{Cu}(\iota_k)$ is the map multiplication by $n$. $\square$

**Theorem VIII.4.5.** Let $A$ be a $C^*$-algebra and let $n \in \mathbb{N}$ with $n \geq 2$. Then $\mathrm{Cu}(A)$ is uniquely $n$-divisible if and only if $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes M_{n^\infty})$ as order semigroups.

*Proof.* Assume that there exists an isomorphism $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes M_{n^\infty})$ as ordered semigroups. Using the inductive limit decomposition $M_{n^\infty} = \varinjlim M_{n^k}$ with connecting maps $j_k \colon M_{n^{k-1}}(A) \to M_{n^k}(A)$ is given by $j_k(a) = a \otimes 1_n$ for all $a \in M_{n^{k-1}}(A)$, we can write $A \otimes M_{n^\infty}$ as the inductive limit

$$A \xrightarrow{j_1} M_n(A) \xrightarrow{j_2} M_{n^2}(A) \xrightarrow{j_3} \cdots \longrightarrow A \otimes M_{n^\infty}.$$

By continuity of the functor Cu (see [36, Theorem 2]), the semigroup $\mathrm{Cu}(A \otimes M_{n^\infty})$ is isomorphic to the inductive limit in the category **Cu** of the sequence

$$\mathrm{Cu}(A) \xrightarrow{\mathrm{Cu}(j_0)} \mathrm{Cu}(M_n(A)) \xrightarrow{\mathrm{Cu}(j_1)} \mathrm{Cu}(M_{n^2}(A)) \xrightarrow{\mathrm{Cu}(j_2)} \cdots . \tag{VIII.8}$$

By [36, Appendix], the inclusion $i_k \colon A \to M_{n^k}(A)$ from $A$ into the upper left corner of $M_{n^k}(A)$ induces an isomorphism between the Cuntz semigroup of $A$ and that of $M_{n^k}(A)$. For $k \in \mathbb{N}$, let $\varphi_k \colon \mathrm{Cu}(A) \to \mathrm{Cu}(A)$ be given by

$$\varphi_k = \mathrm{Cu}(i_{k+1})^{-1} \circ \mathrm{Cu}(j_k) \circ \mathrm{Cu}(i_k).$$

The sequence (VIII.8) implies that $\mathrm{Cu}(A \otimes M_{n^\infty})$ is the inductive limit of the sequence

$$\mathrm{Cu}(A) \xrightarrow{\varphi_1} \mathrm{Cu}(A) \xrightarrow{\varphi_2} \mathrm{Cu}(A) \xrightarrow{\varphi_3} \cdots . \tag{VIII.9}$$

By Lemma VIII.4.4, each $\varphi_k$ is the map multiplication by $n$. It follows from Lemma VIII.4.2 that $\mathrm{Cu}(A \otimes M_{n^\infty})$ is uniquely $n$-divisible. This shows the "if" implication.

Conversely, assume that $\mathrm{Cu}(A)$ is uniquely $n$-divisible and adopt the notation used above. The map $\varphi_k$ is the map multiplication by $n$ on $\mathrm{Cu}(A)$ by Lemma VIII.4.4, so it is an order isomorphism by assumption. By the inductive limit expression of $\mathrm{Cu}(A \otimes M_{n^\infty})$ in (VIII.9), we conclude that $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes M_{n^\infty})$, as desired. ∎

**Remark VIII.4.6.** Let $\mathcal{Q}$ denote the universal UHF-algebra. Using the same ideas as in the proof of the previous theorem, one can show that $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes \mathcal{Q})$ if and only if $\mathrm{Cu}(A)$ is uniquely $p$-divisible for every prime number $p$.

We now turn to direct limits of one-dimensional NCCW-complexes. The following lemma will allow us to reduce to the case where the algebra itself is a one-dimensional NCCW-complex when proving that multiplication by $n$ is an order embedding at the level of the Cuntz semigroup.

**Lemma VIII.4.7.** Let $(S_k, \rho_k)_{k\in\mathbb{N}}$ be an inductive system in the category **Cu**, and let $S = \varinjlim(S_k, \rho_k)$ be its inductive limit in **Cu**. Let $n \in \mathbb{N}$. If multiplication by $n$ on $S_k$ is an order embedding for all $k$ in $\mathbb{N}$, then the same holds for $S$.

*Proof.* For $l \geq k$, denote by $\rho_{k,l}\colon S_k \to S_{l+1}$ the composition $\rho_{k,l} = \rho_l \circ \cdots \circ \rho_k$, and denote by $\rho_{k,\infty}\colon S_k \to S$ the canonical map as in the definition of the inductive limit. Let $s, t \in S$ satisfy $ns \leq nt$. By part (1) of Proposition II.6.4, for each $k \in \mathbb{N}$ there exist $s_k, t_k \in S_k$ such that

$$\rho_k(s_k) \ll s_{k+1} \quad \text{and} \quad s = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(s_k),$$
$$\rho_k(t_k) \ll t_{k+1} \quad \text{and} \quad t = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(t_k).$$

Note in particular that $\rho_{k,\infty}(s_k) \ll \rho_{k+1,\infty}(s_{k+1})$ and $\rho_{k,\infty}(t_k) \ll \rho_{k+1,\infty}(t_{k+1})$ for all $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. Then

$$\rho_{k,\infty}(ns_k) \ll \rho_{k+1,\infty}(ns_{k+1}) \ll \sup_{j\in\mathbb{N}} \rho_{j,\infty}(nt_j).$$

By the definition of the compact containment relation, there exists $j \in \mathbb{N}$ such that

$$\rho_{k,\infty}(ns_k) \ll \rho_{k+1,\infty}(ns_{k+1}) \leq \rho_{j,\infty}(nt_j).$$

By part (2) of Proposition II.6.4, there exists $l \in \mathbb{N}$ such that

$$n\rho_{k,l}(s_k) = \rho_{k,l}(ns_k) \leq \rho_{j,l}(nt_j) = n\rho_{j,l}(t_j).$$

Using that multiplication by $n$ on $S_k$ is an order embedding, we obtain $\rho_{k,l}(s_k) \leq \rho_{j,l}(t_j)$. In particular,

$$\rho_{k,\infty}(s_k) \leq \rho_{j,\infty}(t_j) \leq t.$$

Since $k \in \mathbb{N}$ is arbitrary and $s = \sup_{k\in\mathbb{N}} \rho_{k,\infty}(s_k)$, we conclude that $s \leq t$. $\square$

**Proposition VIII.4.8.** Let $A$ be a $C^*$-algebra that can be written as the inductive limit of 1-dimensional NCCW-complexes. Then the endomorphism of $\mathrm{Cu}(A)$ given by multiplication by $n$ is an order embedding.

*Proof.* By Lemma VIII.4.7, it is sufficient to show that the proposition holds when $A$ is a 1-dimensional NCCW-complex. Let $E = \bigoplus_{j=1}^{r} M_{k_j}(\mathbb{C})$ and $F = \bigoplus_{j=1}^{s} M_{l_j}(\mathbb{C})$ be finite dimensional $C^*$-algebras, and for $x \in [0,1]$ denote by $\mathrm{ev}_x \colon C([0,1],F) \to F$ the evaluation map at the point $x$. Assume that $A$ is given by the pullback decomposition

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ C([0,1],F) \arrow[r, "\mathrm{ev}_0 \oplus \mathrm{ev}_1"'] & F \oplus F, \end{tikzcd}$$

By [3, Example 4.2], the Cuntz semigroup of $A$ is order isomorphic to a subsemigroup of

$$\mathrm{Lsc}\left([0,1], \overline{\mathbb{Z}_+}^{s}\right) \oplus (\overline{\mathbb{Z}_+})^r.$$

Since multiplication by $n$ on this semigroup is an order embedding, the same holds for any subsemigroup; in particular, it hold for $\mathrm{Cu}(A)$. $\square$

**Corollary VIII.4.9.** Let $A$ be a $C^*$-algebra in one of the following classes: unital algebras that can written as inductive limits 1-dimensional NCCW-complexes with trivial $K_1$-groups; simple algebras with trivial $K_0$-groups that can be written as inductive limits 1-dimensional NCCW-complexes with trivial $K_1$-groups; and algebras that can written as inductive limits of punctured-tree algebras. Let $n \in \mathbb{N}$. Suppose that the map multiplication by $n$ on $\mathrm{Cu}(A)$ is an order isomorphism. Then $A \cong A \otimes M_{n^\infty}$.

*Proof.* By Proposition VIII.4.8 together with the assumptions in the statement, it follows that the map multiplication by $n$ on $\mathrm{Cu}(A)$ is an order isomorphism. Hence, it is an isomorphism in the category Cu. By part (2) of Theorem VIII.4.5, there is an isomorphism $\mathrm{Cu}(A) \cong \mathrm{Cu}(A \otimes M_{n^\infty})$ in **Cu**.

The same arguments used at the end of the proof of Theorem VIII.2.13 show that the classes of $C^*$-algebras in the statement can be classified up to stable isomorphism by their Cuntz semigroup. Therefore, we deduce that

$$A \otimes \mathcal{K} \cong A \otimes M_{n^\infty} \otimes \mathcal{K}.$$

Using that $M_{n^\infty}$-absorption is inherited by hereditary C*-subalgebras ([265, Corollary 3.1]), we conclude that $A \cong A \otimes M_{n^\infty}$. ☐

*Absorption of the model action*

We now proceed to obtain an equivariant UHF-absorption result (compare with [133, Theorems 3.4 and 3.5]).

**Theorem VIII.4.10.** Let $G$ be a finite group and let $A$ be a $C^*$-algebra belonging to one of the classes of $C^*$-algebras described in Corollary VIII.4.9. Then the following statements are equivalent:

1. The $C^*$-algebra $A$ absorbs the UHF-algebra $M_{|G|^\infty}$.
2. There is an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property such that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$.
3. There are actions of $G$ on $A$ with the Rokhlin property, and for any action $\beta\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property and for any action $\delta\colon G \to \mathrm{Aut}(A)$ such that $\mathrm{Cu}(\beta_g) = \mathrm{Cu}(\delta_g)$ for all $g \in G$, one has
$$(A, \beta) \cong (A \otimes M_{|G|^\infty}, \delta \otimes \mu^G),$$
that is, there is an isomorphism $\varphi\colon A \to A \otimes M_{|G|^\infty}$ such that
$$\varphi \circ \beta_g = (\delta \otimes \mu^G)_g \circ \varphi$$
for all $g$ in $G$.

In particular, if the above statements hold for $A$, and if $\alpha\colon G \to \mathrm{Aut}(A)$ is an action with the Rokhlin property such that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$, then $(A, \alpha) \cong (A \otimes M_{|G|^\infty}, \mathrm{id}_A \otimes \mu^G)$.

*Proof.* (1) implies (2). Fix an isomorphism $\varphi\colon A \to A \otimes M_{|G|^\infty}$ and define an action $\alpha\colon G \to \mathrm{Aut}(A)$ by $\alpha_g = \varphi^{-1} \circ (\mathrm{id}_A \otimes \mu^G)_g \circ \varphi$ for all $g$ in $G$. For a fixed group element $g$ in $G$, the automorphism $\mathrm{id}_A \otimes \mu^G_g$ of $A \otimes M_{|G|^\infty}$ is approximately inner, and hence so is $\alpha_g$. It follows that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g$ in $G$, as desired.

(2) implies (1). Assume that there is an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property such that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$. Then $A \cong A \otimes M_{|G|^\infty}$ by Proposition VIII.4.8, Corollary VIII.4.9 and Corollary VIII.3.7.

(1) and (2) imply (3). Let $\beta$ and $\delta$ be actions of $G$ on $A$ as in the statement. Since $M_{|G|^\infty}$ is a strongly self-absorbing algebra, there exists an isomorphism $\phi\colon A \to A \otimes M_{|G|^\infty}$ that is approximately unitarily equivalent to the map $\iota\colon A \to A \otimes M_{|G|^\infty}$ given by $\iota(a) = a \otimes 1_{M_{|G|^\infty}}$ for $a$ in $A$. In particular, one has $\mathrm{Cu}(\phi) = \mathrm{Cu}(\iota)$. Hence, for every $a \in (A \otimes \mathcal{K})_+$ we have

$$(\mathrm{Cu}(\phi) \circ \mathrm{Cu}(\beta_g))([a]) = \mathrm{Cu}(\iota)[(\beta_g \otimes \mathrm{id}_{\mathcal{K}})(a)] = \left[((\beta_g \otimes \mathrm{id}_{\mathcal{K}})(a)) \otimes 1_{M_{|G|^\infty}}\right]$$

and

$$\begin{aligned}(\mathrm{Cu}(\delta_g \otimes \mu^G) \circ \mathrm{Cu}(\phi))([a]) &= \mathrm{Cu}(\delta_g \otimes \mu^G)\left(\left[a \otimes 1_{M_{|G|^\infty}}\right]\right) \\ &= \left[((\delta_g \otimes \mathrm{id}_{\mathcal{K}})(a)) \otimes 1_{M_{|G|^\infty}}\right].\end{aligned}$$

Since $\mathrm{Cu}(\beta_g) = \mathrm{Cu}(\delta_g)$ for all $g \in G$, it follows that

$$\mathrm{Cu}(\phi) \circ \mathrm{Cu}(\beta_g) = \mathrm{Cu}(\delta_g \otimes \mu_g) \circ \mathrm{Cu}(\phi)$$

for all $g$ in $G$. In other words, the **Cu**-isomorphism $\mathrm{Cu}(\phi)\colon \mathrm{Cu}(A) \to \mathrm{Cu}(A \otimes M_{|G|^\infty})$ is equivariant. Therefore, by the unital case of Theorem VIII.2.19, there exists an isomorphism $\varphi\colon A \to A \otimes M_{|G|^\infty}$ such that $\varphi \circ \beta_g = (\delta \otimes \mu^G)_g \circ \varphi$ for all $g \in G$, showing that $\beta$ and $\delta \otimes \mu^G$ are conjugate.

(3) implies (1). The existence of an action $\beta\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property implies the existence of an isomorphism $A \to A \otimes M_{|G|^\infty}$, simply by taking $\delta = \beta$.

The last claim follows immediately from (3). $\square$

CHAPTER IX

# CLASSIFICATION THEOREMS FOR CIRCLE ACTIONS ON KIRCHBERG ALGEBRAS

We study and classify certain circle actions on Kirchberg algebras. It is shown that the Rokhlin property implies severe $K$-theoretical constraints on the algebra it acts on. We prove that circle actions with the Rokhlin property have an $\mathrm{Ext}(K_*(A^{\mathbb{T}}), K_{*+1}(A^{\mathbb{T}}))$-class naturally associated to them. If $A$ is a Kirchberg algebra satisfying the UCT, this Ext-class is shown to be a complete invariant for such actions up to conjugacy. More generally, circle actions with the Rokhlin property on Kirchberg algebras are uniquely determined by the $KK$-class of their predual automorphisms. We are not able to compute the range of the invariant, but we conjecture that it is an arbitrary invertible $KK$-class in the kernel of the canonical map $KK \to KL$.

We also define a strengthening of the Rokhlin property, asking for a continuous path of unitaries instead of a sequence. Circle actions with the continuous Rokhlin property on Kirchberg algebras are classified by the $KK$-equivalence class of their fixed point algebra, and in the presence of the UCT, by their equivariant $K$-theory. In this case, we are able to completely characterize the $K$-theoretical invariants that arise in this way.

As an important technical result, we show that the continuous Rokhlin property implies the existence of a unital completely positive asymptotic homomorphism from the algebra to its fixed point subalgebra, which is moreover a left inverse for the canonical inclusion. As a consequence, we prove that the UCT is preserved under formation of crossed products and passage to fixed point algebras by such actions.

We exhibit examples that show that the continuous Rokhlin property is really stronger than the Rokhlin property, even on Kirchberg algebras that satisfy the UCT.

## Introduction

It is natural to explore the classification of Rokhlin actions of compact groups on classifiable classes of $C^*$-algebras, generalizing work of Izumi in the finite group case. This chapter focuses on the classification problem for circle actions with the Rokhlin property on purely infinite simple $C^*$-algebras. A $C^*$-algebra is said to be a *Kirchberg algebra* if it is purely infinite, simple, separable and nuclear. (This terminology was first introduced by Rørdam in [235] to recognize the significant contribution of Eberhard Kirchberg to the study and classification of these algebras, which were originally called *pisun*.)

The classification results in [150] and [200] can be thought of as a starting point for this work, so we briefly recall them. If $A$ and $B$ are two unital Kirchberg algebras, then $A$ and $B$ are isomorphic if and only if there is an invertible element in $KK(A, B)$ that respects the classes of the units of $A$ and $B$. If $A$ and $B$ moreover satisfy the Universal Coeffcient Theorem (UCT), then for every $\mathbb{Z}_2$-graded isomorphism $\varphi_*\colon K_*(A) \to K_*(B)$ satisfying $\varphi_0([1_A]) = [1_B]$, there is an isomorphism $\psi\colon A \to B$ such that $K_*(\psi) = \varphi_*$. Furthermore, every pair of countable abelian groups arises as the $K$-groups of a unital Kirchberg algebra that satisfies the UCT, and the class of the unit of the algebra in $K_0$ can be arbitrary.

One of the main results in this chapter, which is based on [80] and [81], asserts that circle actions on Kirchberg algebras are completely classified by the $KK$-class of their predual automorphism. In the presence of the UCT, the invariant reduces to a certain $\mathrm{Ext}(K_*(A^{\mathbb{T}}), K_{*+1}(A^{\mathbb{T}}))$-class naturally associated to them.

The problem of computing the range of the invariant remains open. In an attempt to remedy this, we introduce a strengthening of the Rokhlin property, which we shall call the *continuous* Rokhlin property. The advantage of working with a continuous analog of the Rokhlin property lies in the fact that the fixed point algebra by any such action can be shown to be a *$KK$-retract* of the original algebra, in a sense that will be made precise in Theorem IX.8.3. Roughly speaking, if $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action with the continuous Rokhlin property, then the $KK$-theory of $A^\alpha$ is very closely related to that of $A$. The picture of $KK$-theory that is most suitable for our purposes is the one obtained by Houghton-Larsen and Thomsen in [130], using completely positive contractive asymptotic homomorphisms. With this approach, we are able to prove that the UCT is preserved under formation of crossed products and passage to fixed point algebras by actions with the continuous Rokhlin property, without requiring the algebras to be simple or nuclear (unlike in Theorem VII.3.13).

We point out that the difference between the continuous Rokhlin property and the Rokhlin property is similar to (and, in some sense, "the same" as) the difference between approximate innerness and asymptotic innerness for automorphisms of $C^*$-algebras. In the context of Kirchberg algebras, this amounts to the difference between the functors $KL$ and $KK$.

We show that the $KK$-class of the predual automorphism of a circle action with the continuous Rokhlin property is trivial. In the presence of the UCT, this amounts to saying that the canonical pure extension associated to the circle action is in fact split. The range of the invariant can be completely determined in this case: any split extension arises the invariant of a circle action with the continuous Rokhlin property. Since the continuous Rokhlin property and the Rokhlin property are equivalent for circle actions on Kirchberg algebras with finitely generated $K$-theory, these results can also be applied to many Rokhlin actions of interest.

While working on this project, we learned that Rasmus Bentmann and Ralf Meyer have developed techniques that allow them to classify objects in triangulated categories with projective resolutions of length two. See [9]. Their study applies to circle actions on $C^*$-algebras, the invariant being equivariant $K$-theory, and isomorphism of actions being $KK^{\mathbb{T}}$-equivalence. Their results predict the same outcome that we have obtained, at least up to $KK^{\mathbb{T}}$-equivalence. Some work has to be done to deduce conjugacy from $KK^{\mathbb{T}}$-equivalence for circle actions with the Rokhlin property on Kirchberg algebras. The fact that the corresponding non-equivariant statement is true, as was shown in Corollary 4.2.2 in [200], and also [150], strongly suggests that this ought to be true in the equivariant case as well. In Subsection 9.6.1, we include some comments on how the work of Bentmann-Meyer could be used to obtain $KK^{\mathbb{T}}$-equivalence in the cases we consider.

This chapter is organized as follows. In Section IX.3, we develop an averaging technique using the Rokhlin property that will allow us to define, given a compact subset $F$ of $A$, a linear map $A \to A^\alpha$ which is an approximate $*$-homomorphism on $F$, in a suitable sense. See Theorem IX.3.3. What is done there is really a particular case of what was done in Section V.4. However, we need more general statements because the extra flexibility (particularly in Theorem IX.3.3) will be crucial in Section IX.8, where we will construct homotopies between linear maps $A \to A^\alpha$ coming from different choices of tolerance and compact set.

In Section IX.4, we apply our averaging technique to associate, to each circle action with the Rokhlin property, a pure extension involving the $K$-theory of the underlying algebra and that of the fixed point algebra; see Theorem IX.4.3. In Section IX.5, we specialize to purely infinite simple $C^*$-algebras, and prove classification results for circle actions with the Rokhlin property on them. See Theorem IX.5.4 for the general Kirchberg algebra case, and Theorem IX.5.6 for the Kirchberg UCT case.

In Section IX.6, we motivate some connections with the work of Bentmann-Meyer, motivate some connections with what we do in later sections regarding the continuous Rokhlin property, and also give some indications of the difficulties of adapting the techniques used in this chapter to the classification of actions of other compact Lie groups.

In Section IX.7, we introduce the definition of the continuous Rokhlin property for a circle action on a unital $C^*$-algebra, and develop its basic theory. Our definition of the continuous Rokhlin property is a strengthening of the Rokhlin property of Hirshberg and Winter from [122] that, roughly speaking, asks for a continuous path of unitaries rather than a sequence. The purpose of introducing this definition is to obtain more rigid classifications results than in the case of the Rokhlin property, and to obtain a complete description of the range of the invariant.

Section IX.8 contains our most relevant results for such actions. First, we show in Corollary IX.8.5 that if $A$ is a separable, unital $C^*$-algebra, and $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is a circle action with the continuous Rokhlin property, then there are isomorphisms $K_0(A) \cong K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$. This result should be compared with Theorem IX.4.3, where we only assume that the action has the Rokhlin property, but where some assumtions on $K_*(A)$ are needed. Second, we show in Theorem IX.8.8, that under the same assumptions on the $C^*$-algebra $A$ and the circle action $\alpha$, the $C^*$-algebra $A$ satisfies the UCT if and only if $A^\alpha$ does. Both results will follow from the existence of a unital completely positive asymptotic morphism $A \to A^\alpha$ which is a left inverse for the canonical inclusion of $A^\alpha$ in $A$ at the level of $KK$-theory. See Theorem IX.8.3.

Section IX.9 contains the classification result for circle actions with the continuous Rokhlin property in terms of either the $KK$-equivalence class of the fixed point algebra in the general case, or in terms of the equivariant $K$-theory in the presence of the UCT. We also provide a complete description of the range of the invariant. Subsection 9.9.1 contains some comments and results about existence and non-existence of model actions in this context.

Finally, in Section IX.10, we show that for unital Kirchberg algebras with finitely generated $K$-theory, the continuous Rokhlin property and the Rokhlin property are equivalent for actions of the circle; see Corollary IX.10.2). It is also shown that the continuous Rokhlin property is not equivalent to the Rokhlin property (Example IX.10.3), even on Kirchberg algebras satisfying the UCT (Example IX.10.4).

### The Rokhlin Property for Circle Actions

In this section, we introduce the definition of the Rokhlin property for a circle action, and deduce some of the basic properties that will be needed to prove our classification results.

**Definition IX.2.1.** Let $A$ be a unital $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. We say that $\alpha$ has the *Rokhlin property* if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exists a unitary $u$ in $A$ such that

1. $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$.
2. $\|ua - au\| < \varepsilon$ for all $a \in F$.

**Remark IX.2.2.** Since compact subsets of metric spaces are completely bounded, one gets an equivalent notion if in Definition IX.2.1 above one allows the subset $F$ of $A$ to be norm compact instead of finite. We will make repeated use of this fact without mentioning it further.

We must first check that Definition IX.2.1 is equivalent to Definition VI.2.1 for circle actions.

**Proposition IX.2.3.** Let $A$ be a separable, unital $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. Then $\alpha$ has the Rokhlin property in the sense of Definition VI.2.1 if and only if it has the Rokhlin property in the sense of Definition IX.2.1.

*Proof.* Since unital homomorphisms from $C(\mathbb{T})$ into a unital $C^*$-algebra are in one-to-one correspondence with unitaries in the algebra, it is clear that Definition IX.2.1 implies Definition VI.2.1.

To prove the converse implication, let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Let $u \in A_{\infty,\alpha} \cap A'$ be a unitary inducing a unital homomorphism as in Definition VI.2.1 for $G = \mathbb{T}$. Choose a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $A$ such that

$$\kappa_A((u_n)_{n\in\mathbb{N}}) = u.$$

Since $u$ belongs to the commutant of $A$, we have

$$\lim_{n\to\infty} \|au_n - u_n a\| = 0 \tag{IX.1}$$

for all $a \in A$.

On the other hand, the fact that $(\alpha_\infty)_\zeta(u) = \zeta u$, for all $\zeta \in \mathbb{T}$, shows that

$$\lim_{n\to\infty} \|\alpha_\zeta(u_n) - \zeta u_n\| = 0 \tag{IX.2}$$

for all $\zeta \in \mathbb{T}$. This by itself does not imply that we can choose $n$ large enough so that $\|\alpha_\zeta(u_n) - \zeta u_n\| < \varepsilon$ holds for *all* $\zeta \in \mathbb{T}$. Put in a different way, one needs to show that the sequence of functions $f_n \colon \mathbb{T} \to \mathbb{R}$ given by

$$f_n(\zeta) = \|\alpha_\zeta(u_n) - \zeta u_n\|$$

for $\zeta \in \mathbb{T}$, converges *uniformly* to zero. Equation (IX.2) implies that $(f_n)_{n\in\mathbb{N}}$ converges pointwise to zero. Without losss of generality, we may assume that $f_n(\zeta) \geq f_{n+1}(\zeta)$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$.

Since $u$ belongs to $A_{\alpha,\infty}$ (as opposed to $A_\infty$), it follows that $f_n$ is continuous for all $n \in \mathbb{N}$. Now, by Dini's theorem (Proposition 11 in Chapter 9 of [239]), a decreasing sequence of continuous functions converges uniformly if it converges pointwise to a continuous function. Thus, for the finite set $F \subseteq A$ and the tolerance $\varepsilon > 0$ given, we use this fact together with Equation (IX.1) to find $n_0 \in \mathbb{N}$ such that $\|\alpha_\zeta(u_{n_0}) - \zeta u_{n_0}\| < \varepsilon$ for all $\zeta \in \mathbb{T}$, and $\|au_{n_0} - u_{n_0}a\| < \varepsilon$ for all $a \in F$. $\square$

The result below is an application of the fact that the action of $\mathbb{T}$ on $C(\mathbb{T})$ by left translation is equivariantly semiprojective. Informally speaking, this means that whenever we are given an almost equivariant unital homomorphism from $C(\mathbb{T})$ into another unital $C^*$-algebra with a circle action, then there is a nearby *exactly equivariant* unital homomorphism from $C(\mathbb{T})$ into the same algebra.

**Proposition IX.2.4.** Let $A$ be a unital $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. Then $\alpha$ has the Rokhlin property if and only if for every $\varepsilon > 0$ and every finite subset $F \subseteq A$, there exists a unitary $u$ in $A$ such that

1. $\alpha_\zeta(u) = \zeta u$ for all $\zeta \in \mathbb{T}$.

2. $\|ua - au\| < \varepsilon$ for all $a \in F$.

The definition of the Rokhlin property differs from the conclusion of this proposition in that in condition (1), one only requires $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$.

*Proof.* This is an immediate consequence of Theorem VI.4.6 and the comments after it. $\square$

Since circle actions have an associated 6-term exact sequence for the $K$-theory of the crossed product, Theorem VI.4.2 has strong implications on the $K$-theory of an algebra that admits a circle action with the Rokhlin property (besides the already strong constraints it imposes for general compact groups). Indeed, assume that $A$ is a unital $C^*$-algebra and that $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action with the Rokhlin property. The Pimsner-Voiculescu for $\alpha$ (see Subsection 10.6 in [13]) is

$$\begin{tikzcd}
K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r, "1-K_0(\widehat{\alpha})"] & K_0(A \rtimes_\alpha \mathbb{T}) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(A \rtimes_\alpha \mathbb{T}) \arrow[l] & K_1(A \rtimes_\alpha \mathbb{T}). \arrow[l, "1-K_1(\widehat{\alpha})"]
\end{tikzcd}$$

The automorphism $K_j(\widehat{\alpha})$ is the identity on $K_j(A \rtimes_\alpha \mathbb{T})$ for $j = 0, 1$ by Theorem VI.4.2. Thus the exact sequence above splits into two short exact sequences

$$0 \to K_j(A \rtimes_\alpha \mathbb{T}) \to K_j(A) \to K_{1-j}(A \rtimes_\alpha \mathbb{T}) \to 0$$

for $j = 0, 1$. It follows that if one of the $K$-groups of $A$ is trivial, then so is the other one (this is not true for general compact groups). In particular, there are no circle actions with the Rokhlin property on AF-algebras, AI-algebras, the Jiang-Su algebra $\mathcal{Z}$, or any of the Cuntz algebras $\mathcal{O}_n$ with $n > 2$. On the other hand, there are many such actions on $\mathcal{O}_2$: it is shown in Corollary IX.5.5 that circle actions with the Rokhlin property on $\mathcal{O}_2$ are generic.

There are other restrictions that follow from the short exact sequences above. We list a few of them:

– $K_0(A)$ is finitely generated if and only if $K_1(A)$ is finitely generated.

– $K_0(A)$ is torsion if and only if $K_1(A)$ is torsion.

– If $K_0(A)$ and $K_1(A)$ are free, then $K_0(A) \cong K_1(A)$.

– More generally, the free ranks of the $K$-groups of $A$ must agree, that is, $\mathrm{rk}K_0(A) = \mathrm{rk}K_1(A)$.

It is nevertheless not clear at this point whether $K_0(A) = \mathbb{Z}$ and $K_1(A) = \mathbb{Z} \oplus \mathbb{Z}_2$ can happen. This combination of $K$-groups will be ruled out by Theorem IX.4.3.

**Definition IX.2.5.** Let $B$ be a $C^*$-algebra and let $\beta$ be an automorphism of $B$. Then $\beta$ is said to be *approximately representable* if there exists a unitary $v \in (M(B)^\beta)^\infty$ such that $\beta(b) = vbv^*$ for all $b \in B$.

It is easy to check that the definition above is equivalent to Definition VI.4.1 for $\Gamma = \mathbb{Z}$

Next, we show that every circle action with the Rokhlin property arises as the dual action of an automorphism of the fixed point algebra, and that this automorphism is essentially unique.

**Theorem IX.2.6.** Let $A$ be a $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there exist an approximately representable automorphism $\theta$ of $A^\alpha$ and an equivariant isomorphism

$$\varphi\colon (A^\alpha \rtimes_\theta \mathbb{Z}, \widehat{\theta}) \to (A, \alpha)$$

which is the identity on $A^\alpha$.

Moreover, $\theta$ is unique up to unitary equivalence. This is, if $\theta'$ is another automorphism of $A^\alpha$ and $\varphi'\colon (A^\alpha \rtimes_{\theta'} \mathbb{Z}, \widehat{\theta'}) \to (A, \alpha)$ is another equivariant isomorphism, then there is a unitary $w$ in $A^\alpha$ such that $\theta = \mathrm{Ad}(w) \circ \theta'$.

*Proof.* Existence of $\theta \in \mathrm{Aut}(A^\alpha)$ follows from Corollary VI.4.7. We now turn to uniqueness of $\theta$. Let $\theta'$ and $\varphi'$ be as in the statement. Let $v$ be the canonical unitary in $A^\alpha \rtimes_\theta \mathbb{Z}$ that implements $\widehat{\theta}$, and let $v'$ be the canonical unitary in $A^\alpha \rtimes_{\theta'} \mathbb{Z}$ that implements $\widehat{\theta'}$. Set $w = \varphi(v)\varphi'(v')^*$, which is a unitary in $A$. We claim that $w$ is fixed by $\alpha$. Indeed, for $\zeta \in \mathbb{T}$, we use the facts that $\varphi$ and $\varphi'$ are equivariant, to obtain

$$\alpha_\zeta(w) = \varphi(\widehat{\theta}_\zeta(v))\varphi'(\widehat{\theta'}_\zeta(v'))^*) = \zeta\varphi(v)\overline{\zeta}\varphi'(v') = w.$$

Whence $w$ belongs to $A^\alpha$. Finally, given $a$ in $A^\alpha$, we have

$$\begin{aligned}(\mathrm{Ad}(w)\circ\theta)(a) &= (\varphi(v)\varphi'(v')^*)(\varphi'(v')a\varphi'(v')^*)(\varphi'(v')\varphi(v)^*)\\ &= \varphi(v)a\varphi(v)^* = \theta'(a),\end{aligned}$$

and the result follows. □

If $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action of the circle on a unital $C^*$-algebra $A$ with the Rokhlin property, then we will usually denote an automorphism of $A^\alpha$ as in the conclusion of Theorem IX.2.6, which is unique up to unitary equivalence, by $\check{\alpha}$. Since $\widehat{\check{\alpha}}$ is conjugate to $\alpha$, we will usually refer to $\check{\alpha}$ as the *predual* automorphism of $\alpha$ (hence the notation $\check{\alpha}$).

**Corollary IX.2.7.** Let $A$ be a $C^*$-algebra and let $\alpha\colon \mathbb{T}\to\mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there is a natural isomorphism $A\rtimes_\alpha\mathbb{T}\cong A^\alpha\otimes\mathcal{K}(L^2(\mathbb{T}))$.

*Proof.* This is an immediate consequence of Theorem IX.2.6 together with the natural isomorphism given by Takai duality. □

### An Averaging Technique

The goal of this section is to develop an averaging technique using the Rokhlin property that will allow us to define, given a compact subset $F$ of $A$, a linear map $A\to A^\alpha$ which is an approximate $*$-homomorphism on $F$, in a suitable sense. See Theorem IX.3.3.

What we do here is really a particular case of what was done in Section V.4. However, we need more general statements because the extra flexibility (particularly in Theorem IX.3.3) will be crucial in Section IX.8, where we will construct homotopies between linear maps $A\to A^\alpha$ coming from different choices of tolerance and compact set. We will also take advantage of the fact that homotopies between partitions of unity on $\mathbb{T}$ are easy to construct.

We begin with some farily general observations.

Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G\to\mathrm{Aut}(A)$ be a continuous action. Identify $C(G)\otimes A$ with $C(G,A)$, and denote by $\gamma\colon G\to\mathrm{Aut}(C(G,A))$ the diagonal action, this is, $\gamma_g(a)(h)=\alpha_g(a(g^{-1}h))$ for all $g,h\in G$ and all $a\in C(G,A)$. Define an *averaging process* $\phi\colon C(G,A) \to C(G,A)$ by

$$\phi(a)(g) = \alpha_g(a(1))$$

for all $a$ in $C(G,A)$ and all $g$ in $G$.

We specialize to $G = \mathbb{T}$ now. Let $\varepsilon > 0$ and let $F$ be a compact subset of $C(\mathbb{T},A)$. Set

$$F' = \bigcup_{\zeta\in\mathbb{T}} \gamma_\zeta(F) \quad \text{and} \quad F'' = \{a(\zeta)\colon a \in F', \zeta \in \mathbb{T}\},$$

which are compact subsets of $C(\mathbb{T},A)$ and $A$, respectively. Choose $\delta > 0$ such that whenever $\zeta_1$ and $\zeta_2$ in $\mathbb{T}$ satisfy $|\zeta_1 - \zeta_2| < \delta$, then

$$\|\alpha_{\zeta_1}(a(1)) - \alpha_{\zeta_2}(a(1))\| < \frac{\varepsilon}{2}$$

for all $a$ in $F'$. Let $(f_j)_{j=0}^n$ be a partition of unity of $\mathbb{T}$ and let $\zeta_1, \ldots, \zeta_n$ be group elements in $\mathbb{T}$ such that $f_j(\zeta) \neq 0$ for some $\zeta$ in $\mathbb{T}$ implies $|\zeta - \zeta_j| < \frac{\delta}{2}$. In particular, this implies that $|\zeta_j - \zeta_k| < \delta$ whenever the supports of $f_j$ and $f_k$ are not disjoint. (Such partition of unity and group elements are easy to construct: take, for example, the support of each function $f_j$ to be ani interval of radius $\frac{\delta}{2}$, and let $\zeta_j$ be the center of its support.)

For use in the proof of the following lemma, we recall the following standard fact about self-adjoint elements in a $C^*$-algebra: if $A$ is a unital $C^*$-algebra and $a, b \in A$ with $b^* = b$, then $-\|b\|a^*a \leq a^*ba \leq \|b\|a^*a$, and hence $\|a^*ba\| \leq \|b\|\|a\|^2$.

**Lemma IX.3.1.** Adopt the notation and assumptions of the discussion above. Then $\phi$ is a homomorphism and its range is contained in the fixed point subalgebra of $C(\mathbb{T},A)$. Moreover, for every $\zeta \in \mathbb{T}$ and every $a$ in $F'$, we have

$$\left\| \gamma_\zeta \left( \sum_{j=1}^n f_j \alpha_{\zeta_j}(a(1)) \right) - \sum_{j=1}^n f_j \alpha_{\zeta_j}(a(1)) \right\| < \varepsilon.$$

*Proof.* We begin by showing that the averaging process $\phi\colon C(\mathbb{T}, A) \to C(\mathbb{T}, A)$ is a homomorphism. Let $a, b \in C(\mathbb{T}, A)$, and let $\zeta$ in $\mathbb{T}$. We have

$$(\phi(a)\phi(b))(\zeta) = \alpha_\zeta(a(1))\alpha_\zeta(b(1)) = \alpha_\zeta(ab(1)) = \phi(ab)(\zeta),$$

showing that $\phi$ is multiplicative. It is clearly linear and preserves the involution, so it is a homomorphism.

We will now show that $\gamma_\lambda(\phi(a)) = \phi(a)$ for all $\lambda$ in $\mathbb{T}$ and all $a$ in $C(\mathbb{T}, A)$. Indeed, for $\zeta$ in $\mathbb{T}$, we have

$$\gamma_\lambda(\phi(a))(\zeta) = \alpha_\lambda(\phi(a)(\lambda^{-1}\zeta)) = \alpha_\lambda(\alpha_{\lambda^{-1}\zeta}(a(1))) = \alpha_\zeta(a(1)) = \phi(a)(\zeta),$$

which proves the claim.

Since every element in a $C^*$-algebra is the linear combination of two self-adjoint elements, we may assume without loss of generality that every element of $F$ is self-adjoint, so that the same holds for the elements of $F'$ and $F''$. Given $\zeta$ in $\mathbb{T}$ and $a$ in $F'$, we have

$$\begin{aligned}\phi(a)(\zeta) - \sum_{j=1}^{n} f_j(\zeta)\alpha_{\zeta_j}(a(1)) &= \sum_{j=1}^{n} f_j(\zeta)^{1/2}(\alpha_{\zeta_j}(a(1)) - \alpha_\zeta(a(1)))f_j(\zeta)^{1/2} \\ &\leq \sum_{j=1}^{n} \|\alpha_{\zeta_j}(a(1)) - \alpha_\zeta(a(1))\| f_j(\zeta).\end{aligned}$$

Now, for $j = 1, \ldots, n$, if $f_j(\zeta) \neq 0$, then $|\zeta_j - \zeta| < \delta$, and hence $\|\alpha_{\zeta_j}(a(1)) - \alpha_\zeta(a(1))\| < \frac{\varepsilon}{2}$. In particular, we conclude that

$$-\frac{\varepsilon}{2} < \phi(a)(\zeta) - \sum_{j=1}^{n} f_j(\zeta)\alpha_{\zeta_j}(a(1)) < \frac{\varepsilon}{2}.$$

Since $\zeta$ is arbitrary, we deduce that $\left\|\phi(a) - \sum\limits_{j=1}^{n} f_j\alpha_{\zeta_j}(a(1))\right\| < \frac{\varepsilon}{2}$. Since $\phi(a)$ is fixed by the action $\gamma$, the result follows from an easy application of triangle inequality. $\square$

If we start with a compact subset of $A$, viewed as a compact subset of $C(\mathbb{T}, A)$ consisting of constant functions, then the above lemma shows that any partition of unity with sufficiently small supports provides us with a way to take a discrete average over the group $\mathbb{T}$. We will later see that this discrete averaging has the advantage of being almost multiplicative in an appropriate sense. See Theorem IX.3.3.

We come back to actions with the Rokhlin property in the next lemma.

**Lemma IX.3.2.** Let $A$ be a unital $C^*$-algebra, let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property, let $\varepsilon > 0$, let $F$ be a compact subset of $A$ and let $S$ be a compact subset of $C(\mathbb{T})$ consisting of positive functions. Then there exists a unitary $u$ in $A$ such that $\alpha_\zeta(u) = \zeta u$ for all $\zeta$ in $\mathbb{T}$ and

$$\|af(u) - f(u)a\| < \frac{\varepsilon}{|S|}$$

for all $a$ in $F$ and all $f$ in $S$.

*Proof.* For every $m$ in $\mathbb{N}$, use Proposition IX.2.4 for $\alpha$ to find a unitary $u_m$ in $A$ such that

- $\alpha_\zeta(u_m) = \zeta u_m$ for all $\zeta \in \mathbb{T}$.
- $\|u_m a - a u_m\| < \frac{1}{m}$ for all $a \in F$.

It is clear that

$$\lim_{m\to\infty} \|af(u_m) - f(u_m)a\| = 0$$

for all $a$ in $F$ and all $f$ in $C(\mathbb{T})$. Since $S$ is compact, one can choose $m$ large enough so that, with $u = u_m$, we have

$$\|af(u) - f(u)a\| < \frac{\varepsilon}{|S|}$$

for all $a$ in $F$ and all $f$ in $S$, as desired. □

Let $A$ be a $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. We denote by $E\colon A \to A^\alpha$ the standard conditional expectation. If $\mu$ denotes the normalized Haar measure on $\mathbb{T}$, then $E$ is given by

$$E(a) = \int_{\mathbb{T}} \alpha_\zeta(a)\, d\mu(\zeta)$$

for all $a$ in $A$.

The way the next theorem is formulated will be convenient in the proof of Theorem IX.8.3. See Remark IX.3.4 below.

**Theorem IX.3.3.** Let $A$ be a unital $C^*$-algebra, let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property, let $\varepsilon > 0$ and let $F$ be a compact subset of $A$. Set $F' = \bigcup\limits_{\zeta\in\mathbb{T}} \alpha_\zeta(F)$, which is a compact subset $A$. Let $\delta > 0$ such that whenever $\zeta_1$ and $\zeta_2$ in $\mathbb{T}$ satisfy $|\zeta_1 - \zeta_2| < \delta$, then

$$\|\alpha_{\zeta_1}(a) - \alpha_{\zeta_2}(a)\| < \frac{\varepsilon}{2}$$

for all $a$ in $F'$. Let $(f_j)_{j=0}^n$ be a partition of unity of $\mathbb{T}$ and let $\zeta_1, \ldots, \zeta_n$ be group elements in $\mathbb{T}$ such that $f_j(\zeta) \neq 0$ for some $\zeta$ in $\mathbb{T}$ implies $|\zeta - \zeta_j| < \frac{\delta}{2}$. Let $u$ be a unitary as in the conclusion of Lemma IX.3.2 for the tolerance $\varepsilon$, compact subset $F' \subseteq A$ and compact subset $S = \{f_j, f_j^{1/2} \colon j = 1, \ldots, n\} \subseteq C(\mathbb{T})$.

Define a linear map $\sigma\colon A \to A^\alpha$ by

$$\sigma(a) = E\left(\sum_{j=1}^n f_j(u)^{1/2}\alpha_{\zeta_j}(a) f_j(u)^{1/2}\right)$$

for all $a$ in $A$. Then $\sigma$ is unital and completely positive, and

$$\|\sigma(ab) - \sigma(a)\sigma(b)\| < \varepsilon(\|a\| + \|b\| + 3)$$

for all $a$ and $b$ in $F'$.

We point out that one can always choose a positive number $\delta > 0$, a partition of unity $(f_j)_{j=1}^n$ and a unitary $u$ in $A$ satisfying the hypotheses of this theorem. We will use this fact without further reference in the future, and we will simply say "choose $\delta, (f_j)_{j=1}^n$ and $u$ as in the assumptions of Theorem IX.3.3".

*Proof.* It is clear that $\sigma$ is linear, completely positive, and unital. In particular, it is completely contractive.

We claim that

$$\left\|\sigma(a) - \sum_{j=1}^n f_j(u)^{1/2}\alpha_{\zeta_j}(a) f_j(u)^{1/2}\right\| \leq \varepsilon$$

for all $a$ in $F'$.

Denote by $\psi\colon C(\mathbb{T}, A) \to A$ the equivariant completely positive contractive map which is the identity on $A$, and sends the canonical generator of $C(\mathbb{T})$ to $u$. Let $a$ be in $F'$. For $\zeta$ in $\mathbb{T}$, we use Lemma IX.3.1 at the last step to show that

$$
\begin{aligned}
&\left\| \alpha_\zeta \left( \sum_{j=1}^{n} f_j(u_m)^{1/2} \alpha_{\zeta_j}(a) f_j(u_m)^{1/2} \right) - \sum_{j=1}^{n} f_j(u_m)^{1/2} \alpha_{\zeta_j}(a) f_j(u_m)^{1/2} \right\| \\
&= \left\| \alpha_\zeta \left( \psi \left( \sum_{j=1}^{n} f_j \otimes \alpha_{\zeta_j}(a) \right) \right) - \psi \left( \sum_{j=1}^{n} f_j \otimes \alpha_{\zeta_j}(a) \right) \right\| \\
&= \left\| \psi \left( \gamma_\zeta \left( \sum_{j=1}^{n} f_j \otimes \alpha_{\zeta_j}(a) \right) - \sum_{j=1}^{n} f_j \otimes \alpha_{\zeta_j}(a) \right) \right\| \\
&\leq \left\| \gamma_\zeta \left( \sum_{j=1}^{n} f_j \otimes \alpha_{\zeta_j}(a) \right) - \sum_{j=1}^{n} f_j \otimes \alpha_{\zeta_j}(a) \right\| < \varepsilon,
\end{aligned}
$$

as desired.

Note that $\|[f_j(u)^{1/2}, a]\| < \frac{\varepsilon}{2n}$ for all $a$ in $F'$. Let $a$ and $b$ in $F'$. Using at the third step that $f_j f_k \neq 0$ implies $|\zeta_j - \zeta_k| < \delta$, we have

$$
\begin{aligned}
\sigma(a)\sigma(b) &\approx_{\varepsilon(\|a\|+\|b\|)} \sum_{j,k=1}^{n} f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} f_k(u)^{1/2} \alpha_{\zeta_k}(b) f_k(u)^{1/2} \\
&\approx_\varepsilon \sum_{f_j f_k \neq 0} f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \alpha_{\zeta_k}(b) f_k(u) \\
&\approx_\varepsilon \sum_{f_j f_k \neq 0} f_j(u)^{1/2} \alpha_{\zeta_j}(a) f_j(u)^{1/2} \alpha_{\zeta_j}(b) f_k(u) \\
&\approx_\varepsilon \sum_{f_j f_k \neq 0} f_j(u)^{1/2} \alpha_{\zeta_j}(ab) f_j(u)^{1/2} f_k(u) \\
&= \sum_{j=0}^{n} f_j(u)^{1/2} \alpha_{\zeta_j}(ab) f_j(u)^{1/2} \left( \sum_{k\colon f_k f_j \neq 0} f_k(u) \right) \\
&= \sum_{j=0}^{n} f_j(u)^{1/2} \alpha_{\zeta_j}(ab) f_j(u)^{1/2} = \sigma(ab).
\end{aligned}
$$

Hence $\|\sigma(a)\sigma(b) - \sigma(ab)\| < \varepsilon(\|a\| + \|b\| + 3)$, as desired. $\square$

**Remark IX.3.4.** For the immediate applications of Theorem IX.3.3, it will be enough to choose *any* $\delta, (f_j)_{j=1}^n$ and $u$ satisfying the assumptions of said theorem. However, we will need the more general statement in the proof of Theorem IX.8.3, in which we will need to construct homotopies between the linear maps $\sigma$ obtained from different choices of $\delta, (f_j)_{j=1}^n$ and $u$.

**The Pure Extension of a Circle Action with the Rokhlin Property**

In this section, we show that there are severe obstructions on the $K$-theory of a unital $C^*$-algebra that admits a circle action with the Rokhlin property. Specifically, it will be shown in Theorem IX.4.3, that any circle action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has a canonically associated pure extension

$$0 \to K_0(A^\alpha) \to K_0(A) \to K_1(A^\alpha) \to 0.$$

We begin by recalling the definition of a pure subgroup and a pure extension.

**Definition IX.4.1.** Let $G$ be an abelian group and let $G'$ be a subgroup. We say that $G'$ is *pure* if every torsion element of $G/G'$ lifts to a torsion element of the same order in $G$. Equivalently, $nG' = nG \cap G'$ for all $n$ in $\mathbb{N}$.

An extension $0 \to G' \to G \to G'' \to 0$ is said to be *pure* if $G'$ is a pure subgroup of $G$.

One checks that a subgroup $G'$ of an abelian group $G$ is pure if and only if the following holds: for every finitely generated subgroup $H''$ of $G/G'$, if $H$ denotes the preimage of $H''$ under the canonical quotient map $G \to G/G'$, then the induced extension

$$0 \to G' \to H \to H'' \to 0$$

splits.

It is not true that every pure extension splits. A classical example is the short exact sequence

$$0 \to \mathbb{Z}^\infty \to \mathbb{Z}^\infty \to \mathbb{Q} \to 0$$

associated to a free resolution of $\mathbb{Q}$. Somewhat more surprisingly, there are examples of pure subgroups which are finitely generated, yet not a direct summand (despite being a direct summand in every finitely generated subgroup that contains it). We are thankful to Derek Holt for providing the following example.

**Example IX.4.2.** Let $G_1 = \mathbb{Z}$, which we regard as the free group on the generator $x$, and let $G_2$ be the abelian group on generators $\{x, y_n \colon n \in \mathbb{N}\}$ and relations

$$2y_{n+1} = y_n + x$$

for all $n$ in $\mathbb{N}$. Regard $G_1$ as a subgroup of $G_2$ via the obvious inclusion. We claim that $G_1$ is a pure subgroup of $G_2$.

For $n$ in $\mathbb{N}$, denote by $G_2^{(n)}$ the subgroup of $G_2$ generated by $x$ and $y_n$. Then $G_2^{(n)}$ is a free abelian group of rank 2, and $G_2^{(n)} \subseteq G_2^{(n+1)}$ for all $n$ in $\mathbb{N}$. Let $H$ be a finitely generated subgroup of $G_2$ containing $x$. Then there exists $N$ in $\mathbb{N}$ such that $H \subseteq G_2^{(N)} \cong \mathbb{Z}x \oplus \mathbb{Z}y_N$. Set $H'' = H/G_1$. Then $H''$ is a subgroup of $\mathbb{Z}y_N$, and thus it is free. In particular, the extension

$$0 \to G_1 \to H \to H'' \to 0$$

splits. This shows that $G_1$ is a pure subgroup of $G_2$.

Finally, we claim that $G_1$ is not a direct summand in $G_2$. Assume that it is, and let $G$ be a direct complement of $G_1$ in $G_2$. Denote by $\pi\colon G_2 \to G$ the canonical quotient map, and by $\iota\colon G \to G_2$ the canonical inclusion. For every $n$ in $\mathbb{N}$, set

$$y_n' = (\iota \circ \pi)(y_n),$$

which is an element of $G$. Then $2y_n' = y_{n-1}'$ for $n \geq 2$. Moreover, for every $n$ in $\mathbb{N}$, there exists $k_n$ in $\mathbb{Z}$ such that

$$y_n' = y_n + k_n x.$$

Now, the identities

$$y_{n-1}' = 2y_n' = 2y_n + 2k_n x = y_{n-1} + x + k_n x = y_{n-1} + (2k_n + 1)x,$$

imply that $k_{n-1} = 2k_n + 1$ for all $n \geq 2$. Thus $k_1 = 2^n k_{n+1} + 1$, and hence $k_1 - 1$ is divisible by $2^n$ for all $n$ in $\mathbb{N}$. This is a contradiction, which shows that $G_1$ does not have a direct complement in $G_2$.

The example constructed above will be relevant in Section IX.10, where we will show that there exist circle actions with the Rokhlin property that do not have the continuous Rokhlin property, even on Kirchberg algebras that satisfy the UCT.

**Theorem IX.4.3.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action of the circle on a unital $C^*$-algebra $A$ with the Rokhlin property. Then its 6-term Pimsner-Voiculescu exact sequence induces the pure extension

$$0 \longrightarrow K_*(A^\alpha) \xrightarrow{K_*(\iota)} K_*(A) \longrightarrow K_{*+1}(A^\alpha) \longrightarrow 0.$$

In particular, if either $K_0(A)$ or $K_1(A)$ is finitely generated, then both $K_0(A)$ and $K_1(A)$ are finitely generated, and there are isomorphisms

$$K_0(A) \cong K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$$

such that the class of the unit $[1_A] \in K_0(A)$ is sent to $[(1_{A^\alpha}, 0)] \in K_0(A^\alpha) \oplus K_1(A^\alpha)$.

*Proof.* Since $\check{\alpha}$ is approximately inner, the 6-term Pimsner-Voiculescu exact sequence for $\alpha$ reduces to the exact sequence

$$0 \longrightarrow K_*(A^\alpha) \xrightarrow{K_*(\iota)} K_*(A) \longrightarrow K_{*+1}(A^\alpha) \longrightarrow 0.$$

Now, it follows from Theorem VI.3.3 that this extension is pure.

To prove the second pat of the statement, assume, without loss of generality,, that $K_0(A)$ is finitely generated. It follows from Theorem VI.3.3 that $K_0(A^\alpha)$ is a direct summand of $K_0(A)$, where the inclusion $K_0(A^\alpha) \to K_0(A)$ is induced by the canonical inclusion $A^\alpha \to A$. Since the factor $K_0(A)/K_0(A^\alpha)$ (with the above mentioned embedding) is isomorphic to $K_1(A^\alpha)$ by the Pimsner-Voiculescu exact sequence, it follows that there is an isomorphism $K_0(A^\alpha) \oplus K_1(A^\alpha) \cong K_0(A)$. In particular, the groups $K_0(A^\alpha)$ and $K_1(A^\alpha)$ are finitely generated. The short exact sequence

$$0 \to K_1(A^\alpha) \to K_1(A) \to K_0(A^\alpha) \to 0$$

forces $K_1(A)$ to be finitely generated as well. Another application of Theorem VI.3.3, together with the short exact sequence above, yields an isomorphism $K_1(A) \cong K_1(A^\alpha) \oplus K_0(A^\alpha)$, as desired.

Finally, since the unit of $A$ belongs to $A^\alpha$, it is clear that the isomorphism $K_0(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$ sends the class of the unit of $A$ in $K_0(A)$ to $([1_{A^\alpha}], 0)$ in $K_0(A^\alpha) \oplus K_1(A^\alpha)$. $\square$

**Classification of Rokhlin Actions on Kirchberg Algebras**

This section contains our main results concerning the classification of circle actions with the Rokhlin property on Kirchberg algebras.

**Definition IX.5.1.** If $A$ is a $C^*$-algebra and $\varphi$ is an automorphism of $A$, we say that $\varphi$ is *aperiodic*, if $\varphi^n$ is not inner for all $n$ in $\mathbb{N}$.

Recall that the center of a simple unital $C^*$-algebra is trivial.

**Proposition IX.5.2.** Let $A$ be a simple unital $C^*$-algebra, and let $\varphi$ be an automorphism of $A$. Then $\varphi$ is aperiodic if and only if $A \rtimes_\varphi \mathbb{Z}$ is simple.

*Proof.* If $\varphi$ is aperiodic, it follows from Theorem 3.1 in [156] that the crossed product $A \rtimes_\varphi \mathbb{Z}$ is simple.

Conversely, suppose that there exist $n$ in $\mathbb{N}$ and a unitary $v$ in $A$ such that $\varphi^n = \mathrm{Ad}(v)$. Set

$$w = v\varphi(v)\cdots\varphi^{n-1}(v).$$

Then $\varphi^{n^2} = \mathrm{Ad}(w)$, so $\varphi^{n^2}$ is also inner. Moreover, it follows from the fact that $v$ is $\varphi^n$-invariant that $w$ is $\varphi$-invariant. With $u$ denoting the canonical unitary in $A \rtimes_\varphi \mathbb{Z}$ that implements $\varphi$, we have $uwu^* = w$ in $A \rtimes_\varphi \mathbb{Z}$. Set $z = u^{n^2}w^*$. We claim that $z$ is a unitary in the center of $A \rtimes_\alpha \mathbb{Z}$.

It is clear that $z$ commutes with $u$, and for $a$ in $A$ we have

$$zaz^* = u^{n^2}w^*aw\left(u^{n^2}\right)^* = u^{n^2}\alpha^{-n^2}(a)\left(u^{n^2}\right)^* = a,$$

so the claim follows.

Since the center of $A \rtimes_\alpha \mathbb{Z}$ is trivial, there is a complex number $\lambda$ with $|\lambda| = 1$ such that $u^{n^2} = \lambda w$. In particular, $u^{n^2}$ belongs to $A$, which is a contradiction. This shows that $\varphi$ is aperiodic. $\square$

**Definition IX.5.3.** Let $A$ and $B$ be $C^*$-algebras, and let $\varphi$ and $\psi$ be automorphisms of $A$ and $B$ respectively. We say that $\varphi$ and $\psi$ are *$KK$-conjugate*, if there exists an invertible element $x$ in $KK(A,B)$ such that $[1_A] \times x = [1_B]$ and $KK(\psi)\cdot x = x \cdot KK(\varphi)$.

**Theorem IX.5.4.** Let $A$ and $B$ be unital Kirchberg algebras, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be actions with the Rokhlin property. Denote by $\check{\alpha}$ and $\check{\beta}$ the predual automorphisms of $\alpha$ and $\beta$ respectively. (See Theorem IX.2.6.) Then $\alpha$ and $\beta$ are conjugate if and only if $\check{\alpha}$ and $\check{\beta}$ are $KK$-conjugate.

*Proof.* Assume that $\alpha$ and $\beta$ are conjugate, and let $\theta\colon A \to B$ be an isomorphism such that $\theta \circ \alpha_\zeta = \beta_\zeta \circ \theta$ for all $\zeta$ in $\mathbb{T}$. Then $\theta$ maps $A^\alpha$ onto $B^\beta$. Denote by $\phi\colon A^\alpha \to B^\beta$ the restriction of $\theta$ to $A^\alpha$. Then $\phi$ is an isomorphism, and $KK(\phi)$ is invertible in $KK(A^\alpha, B^\beta)$.

Denote by $u$ the canonical unitary in $A \cong A^\alpha \rtimes_{\check{\alpha}} \mathbb{Z}$ that implements $\check{\alpha}$, and likewise, denote by $v$ the canonical unitary in $B \cong B^\beta \rtimes_{\check{\beta}} \mathbb{Z}$ that implements $\check{\beta}$. Set $w = v\theta(u)^*$, which is a unitary in $B$. We claim that $w$ belongs to $B^\beta$. To see this, note that if $\zeta$ belongs to $\mathbb{T}$, then

$$\beta_\zeta(w) = \beta_\zeta(v\theta(u)^*) = \beta_\zeta(v)\theta(\alpha_\zeta(u))^* = \zeta v\overline{\zeta}\theta(u)^* = v\theta(u)^* = w,$$

which proves the claim.

Given $a$ in $A^\alpha$, we have

$$\begin{aligned}
(\mathrm{Ad}(w) \circ \phi \circ \check{\alpha})(a) &= w(\theta(uau^*))w^* \\
&= w\theta(u)\phi(a)\theta(u)^*w^* \\
&= v\phi(a)v^* \\
&= (\mathrm{Ad}(v) \circ \phi)(a) \\
&= (\check{\beta} \circ \phi)(a).
\end{aligned}$$

In particular, $\phi \circ \check{\alpha}$ and $\check{\beta} \circ \phi$ are unitarily equivalent, and thus $KK(\phi)$ is a $KK$-equivalence between $A^\alpha$ and $B^\beta$ intertwining $KK(\check{\alpha})$ and $KK(\check{\beta})$. This shows the "only if" implication.

Conversely, assume that $\check{\alpha}$ and $\check{\beta}$ are $KK$-conjugate, and let $x \in KK(A^\alpha, B^\beta)$ be an invertible element implementing the equivalence. Since $A^\alpha$ and $B^\beta$ are Kirchberg algebras by Corollary VII.4.11, it follows from Theorem 4.2.1 in [200] that there exists an isomorphism $\phi\colon A^\alpha \to B^\beta$ such that $KK(\phi) = x$. Thus, $\phi \circ \check{\alpha} \circ \phi^{-1}$ and $\check{\beta}$ determine the same class in $KK(B^\beta, B^\beta)$. Now, since $A$ and $B$ are simple, it follows from Proposition IX.5.2 that $\check{\alpha}$ and $\check{\beta}$ are aperiodic, and, consequently, they are cocycle conjugate by Theorem 5 in [187]. In particular,

$\check{\alpha}$ and $\check{\beta}$ are exterior conjugate. Finally, Proposition II.3.6 implies that the dual actions of $\check{\alpha}$ and $\check{\beta}$, which are themselves conjugate to $\alpha$ and $\beta$, respectively, are conjugate. This finishes the proof. $\square$

As a consequence, we can show that any two circle actions with the Rokhlin property on $\mathcal{O}_2$ are conjugate.

**Corollary IX.5.5.** Let $\alpha$ and $\beta$ be circle actions with the Rokhlin property on $\mathcal{O}_2$. Then $\alpha$ and $\beta$ are conjugate.

*Proof.* The fixed point algebras $(\mathcal{O}_2)^\alpha$ and $(\mathcal{O}_2)^\beta$ are Kirchberg algebras by Corollary VII.4.11, have trivial $K$-theory by Theorem VI.3.3. They satisfy the UCT by Theorem VII.3.13, so we conclude that they are isomorphic to $\mathcal{O}_2$ by classification. Moreover, $KK(\check{\alpha})$ and $KK(\check{\beta})$ are both trivial since $KK(\mathcal{O}_2, \mathcal{O}_2) = 0$. It follows from Theorem IX.5.4 that $\alpha$ and $\beta$ are conjugate. $\square$

In the presence of the UCT, the invariant takes a more manageable form.

**Theorem IX.5.6.** Let $A$ and $B$ be unital Kirchberg algebras, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be actions with the Rokhlin property. Denote by $\iota_A\colon A^\alpha \to A$ and $\iota_B\colon B^\beta \to B$ the canonical inclusions. Then $\alpha$ and $\beta$ are conjugate if and only if there are $\mathbb{Z}_2$-graded group isomorphisms

$$\varphi_*\colon K_*(A) \to K_*(B) \quad \text{and} \quad \psi_*\colon K_*(A^\alpha) \to K_*(B^\beta),$$

with $\varphi_0([1_A]) = [1_B]$ and $\psi_0([1_{A^\alpha}]) = [1_{B^\beta}]$, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_j(A^\alpha) \arrow[r, "K_j(\iota_A)"] \arrow[d, "\psi_j"] & K_j(A) \arrow[r] \arrow[d, "\varphi_j"] & K_{1-j}(A^\alpha) \arrow[r] \arrow[d, "\psi_{1-j}"] & 0 \\
0 \arrow[r] & K_j(B^\beta) \arrow[r, "K_j(\iota_B)"'] & K_j(B) \arrow[r] & K_{1-j}(B^\beta) \arrow[r] & 0
\end{tikzcd}$$

is commutative for $j = 0, 1$.

*Proof.* We will check that the assumptions of this theorem imply the hypotheses of Theorem IX.5.4.

Note that $A^\alpha$ and $A^\beta$ satisfy the UCT by Theorem VII.3.13. Denote by $\check{\alpha}$ and $\check{\beta}$ the predual automorphisms of $\alpha$ and $\beta$ respectively. Then $\check{\alpha}$ is approximately representable by

Theorem IX.2.6, so it induces the identity map on $K$-theory. Consider the short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_*(A^\alpha), K_{*+1}(A^\alpha)) \xrightarrow{\varepsilon} KK(A^\alpha, A^\alpha) \xrightarrow{\tau} \mathrm{Hom}(K_*(A^\alpha), K_*(A^\alpha)) \longrightarrow 0,$$

coming from the UCT for the pair $(A^\alpha, A^\alpha)$. Then $\tau(1 - KK(\check{\alpha})) = 0$, and thus $1 - KK(\check{\alpha})$ is represented by a class in $\mathrm{Ext}(K_*(A^\alpha), K_{*+1}(A^\alpha))$. This extension is precisely the sum of the two short exact sequences arising from the Pimsner-Voiculescu 6-term exact sequence (that one really gets two short exact sequences follows from the fact that $K_*(\check{\alpha}) = 1$). An analogous statement holds for $B^\beta$ and $\check{\beta}$.

Using the UCT for $A^\alpha$ and $B^\beta$, choose an invertible element $x$ in $KK(A^\alpha, B^\beta)$ such that $\tau(x) = \psi_*$. The assumptions on the maps $\varphi_0$ and $\varphi_1$ imply that $x$ implements a $KK$-equivalence between $1 - KK(\check{\alpha})$ and $1 - KK(\check{\beta})$. Hence it also implements a $KK$-equivalence between $KK(\check{\alpha})$ and $KK(\check{\beta})$, and thus the result follows from Theorem IX.5.4 above. □

## Some Remarks

In this section, we describe a possible alternative approach to Theorem IX.5.4, bases on the work of Bentmann-Meyer. We also motivate some connections with the second part of this work, where we will study the continuous Rokhlin property for circle actions. Finally, we give some indications of the difficulties of extending the results in this chapter to actions of other compact Lie groups.

### *An alternative approach using Bentmann-Meyer's work*

In [9], Bentmann and Meyer use homological algebra to classify objects in triangulated categories that have a projective resolution of length two. Starting with a certain homological invariant, their results show that two objects with a projective resolution of length two can be classified by the invariant together with a certain obstruction class in an $\mathrm{Ext}^2$-group computed from the given invariant. Their methods apply to the triangulated category $\mathcal{KK}^{\mathbb{T}}$ of $C^*$-algebras with a circle action, where morphisms are given by elements of the equivariant $KK$-theory, and the homological invariant is equivariant $K$-theory. ($R(\mathbb{T})$-modules have projective resolutions of length two, since the circle group has dimension one. In general, if $G$ is a Lie group and $T$ is any maximal torus, then the cohomological dimension of $R(G)$ is $\mathrm{rank}(T)+1$. See the comments below Proposition 3.1 in [9].)

We compute the equivariant $K$-theory of a circle action with the Rokhlin property in the proposition below. We show that for such actions, equivariant $K$-theory and $K$-theory of the fixed point algebra are isomorphic as $R(\mathbb{T})$-modules (the latter carrying the trivial $R(\mathbb{T})$-module structure), thus placing our results (particularly Theorem IX.5.6) in the homological algebra context of Bentmann-Meyer's work.

**Proposition IX.6.1.** Let $A$ be a unital $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then there is a natural $R(\mathbb{T})$-module isomorphism

$$K_*^\alpha(A) \cong K_*(A^\alpha),$$

where the $R(\mathbb{T})$-module structure on $K_*(A^\alpha)$ is the trivial one.

*Proof.* Recall that $R(\mathbb{T}) \cong \mathbb{Z}[x, x^{-1}]$. By Julg's Theorem (here reproduced as Theorem II.3.3), there is a natural isomorphism $K_*^\alpha(A) \cong K_*(A \rtimes_\alpha \mathbb{T})$, where the $\mathbb{Z}[x, x^{-1}]$-module structure on $K_*(A \rtimes_\alpha \mathbb{T})$ is determined by the dual action $\widehat{\alpha}$, meaning that the action of $x$ agrees with the action of $K_*(\widehat{\alpha})$. Now, $\widehat{\alpha}$ is approximately inner by Theorem IX.2.6, so it induces the trivial automorphism of the $K$-theory. This shows that the $R(\mathbb{T})$-module structure on $K_*^\alpha(A)$ is the trivial one.

Finally, there is a natural isomorphism $K_*(A \rtimes_\alpha \mathbb{T}) \cong K_*(A^\alpha)$ by Corollary IX.2.7. □

Let $A$ and $B$ be unital $C^*$-algebras (not necessarily Kirchberg algebras), and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be circle actions with the Rokhlin property. Assume that $\alpha$ and $\beta$ belong to the equivariant bootstrap class. Bentmann and Meyer show (see Subsection 3.2 in [9]) that in this context, the actions $\alpha$ and $\beta$ are $KK^{\mathbb{T}}$-equivalent if and only if there is an isomorphism $K_*^\alpha(A) \cong K_*^\beta(B)$ that respects the elements in $\mathrm{Ext}_{\mathbb{Z}}(K_*^\alpha(A), K_{*+1}^\alpha(A))$ and $\mathrm{Ext}_{\mathbb{Z}}(K_*^\beta(B), K_{*+1}^\beta(B))$ determined by $\alpha$ and $\beta$ respecively.

It is shown in Proposition 3.1 in [9] that a circle action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ on a unital $C^*$-algebra $A$ belongs to the equivariant bootstrap class if and only if $A$ and $A \rtimes_\alpha \mathbb{T}$ satisfy the UCT. When $\alpha$ has the Rokhlin property, this is equivalent to $A^\alpha$ satisfying the UCT. Thus, the

UCT assumptions in Theorem IX.5.6 amount to requiring the actions $\alpha$ and $\beta$ there to be in the equivariant bootstrap class.

We have not been able to identify the element in

$$\mathrm{Ext}_{\mathbb{Z}}(K_*^\alpha(A), K_{*+1}^\alpha(A)) \cong \mathrm{Ext}_{\mathbb{Z}}(K_*(A^\alpha), K_{*+1}(A^\alpha))$$

determined by $\alpha$. However, we suspect that under the natural identifications, and up to a sign, it must agree with the Ext class of its predual automorphism $\check{\alpha}$. If this were true, we would have proved the following.

**Conjecture IX.6.2.** Let $A$ and $B$ be separable, unital $C^*$-algebras and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be circle actions with the Rokhlin property. Assume that $A^\alpha$ and $B^\beta$ satisfy the UCT. Then the following statements are equivalent:

1. The actions $\alpha$ and $\beta$ are $KK^{\mathbb{T}}$-equivalent;
2. The automorphisms $\check{\alpha}$ and $\check{\beta}$ are $KK$-conjugate;
3. There are group isomorphisms

$$\varphi_*\colon K_*(A) \to K_*(B) \quad \text{and} \quad \psi_*\colon K_*(A^\alpha) \to K_*(B^\beta),$$

with $\varphi_0([1_A]) = [1_B]$ and $\psi_0([1_{A^\alpha}]) = [1_{B^\beta}]$, such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & K_j(A^\alpha) \arrow[r, "K_j(\iota_A)"] \arrow[d, "\psi_j"] & K_j(A) \arrow[r] \arrow[d, "\varphi_j"] & K_{1-j}(A^\alpha) \arrow[r] \arrow[d, "\psi_{1-j}"] & 0 \\ 0 \arrow[r] & K_j(B^\beta) \arrow[r, "K_j(\iota_B)"'] & K_j(B) \arrow[r] & K_{1-j}(B^\beta) \arrow[r] & 0 \end{tikzcd}$$

is commutative for $j = 0, 1$.

If in the conjecture above, $A$ and $B$ are simple and nuclear, then the assumption that $A^\alpha$ and $B^\beta$ satisfy the UCT is automatic by Theorem VII.3.13. In the general case, however, we do not know whether this is the case. We formally raise this as a question:

**Question IX.6.3.** Let $A$ be a separable, unital $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $A$ satisfies the UCT, does it follow that $A^\alpha$ satisfies the UCT as well?

**Remark IX.6.4.** If $\alpha$ is assumed to have the continuous Rokhlin property, then the result will be shown to be true in Theorem IX.8.8. If one replaces the circle group $\mathbb{T}$ with a finite group, then the resulting question has a positive answer in the *nuclear* case, as was shown in Corollary 3.9 in [191].

A positive answer to Conjecture IX.6.2 would have the advantage of holding for arbitrary separable, unital $C^*$-algebras $A$ and $B$ satisfying the UCT (not necessarily purely infinite, or even simple). In order to recover Theorem IX.5.6 from it, it would be enough to show that under the assumptions of Conjecture IX.6.2, if moreover $A$ and $B$ are Kirchberg algebras, then $\alpha$ and $\beta$ are conjugate if and only if they are $KK^{\mathbb{T}}$-equivalent via a $KK^{\mathbb{T}}$-equivalence that respects the classes of the units. Corollary 4.2.2 in [200] (see also [150]) suggests that one may be able to prove this directly, and maybe without even assuming that the actions have the Rokhlin property. We have, however, not explored this direction any further.

*Range of the invariant and related questions*

We have shown in Theorem IX.5.4 that circle actions with the Rokhlin property on Kirchberg algebras are completely determined, up to conjugacy, by the pair $(A^\alpha, KK(\check{\alpha}))$, consisting of the fixed point algebra $A^\alpha$ together with the $KK$-class $KK(\check{\alpha})$ of the predual automorphism. However, we have not said anything about what pairs $(B, x)$, consisting of a Kirchberg algebra $B$ and an invertible element $x$ in $KK(B, B)$, arise from circle actions with the Rokhlin property as described above. There are no obvious restrictions on the $C^*$-algebra $B$, while Theorem IX.2.6 shows that $x$ must belong to the kernel of the natural map $KK(B, B) \to KL(B, B)$. We do not know whether all such pairs are realized by a circle action with the Rokhlin property.

Section IX.8 addresses this question, and provides a complete answer under the additional assumption that the action have the continuous Rokhlin property. In Theorem IX.9.3, we show that every pair $(B, x)$ as above arises from a circle action with the continuous Rokhlin property if and only if $x = 1$. In other words, the fixed point algebra is arbitrary and the predual automorphism is an arbitrary $KK$-trivial aperiodic automorphism.

We also show in Proposition IX.9.6, that all circle actions with the continuous Rokhlin property on Kirchberg algebras are "tensorially generated" by a specific one (which necessarily has the continuous Rokhlin property).

Finally, in Section IX.8, we also provide a partial answer to Question IX.6.3, answering it affirmatively whenever $\alpha$ has the continous Rokhlin property. See Theorem IX.8.8.

*Beyond circle actions*

We close this article by explaining what difficulties one may encounter when trying to generalize the methods exhibited here to more general compact Lie group actions.

Bentmann-Meyer's techniques depend heavily on the fact that $R(\mathbb{T})$-modules have projective resolutions of length two, essentially because the circle has dimension one. While this is also true for $SU(2)$, it fails for other natural examples of compact Lie groups like the two-torus $\mathbb{T}^2$, so their methods break down already in this case.

Our methods are no less dependent on low-dimensionality of the circle, though the dependence is slightly more subtle. For example, already in dimension two, $(C(\mathbb{T}^2), \mathtt{Lt})$ is not equivariantly semiprojective (because $C(\mathbb{T}^2)$ is not semiprojective), so Proposition IX.2.4, and hence Theorem IX.2.6, will not be true in general. There is another instance where one-dimensionality of the circle (or rather, the fact that its dual group $\mathbb{Z}$ has rank one) was used, namely in the proof of Proposition IX.5.2. In fact, the corresponding statement for arbitrary discrete abelian groups is not true: Example 4.2.3 of [199] shows that there exists an action $\varphi$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $M_2$, with $M_2 \rtimes_\varphi (\mathbb{Z}_2 \times \mathbb{Z}_2) \cong M_4$, and such that $\varphi|_{\mathbb{Z}_2 \times \{1\}}$ is inner. (For an example on a UCT Kirchberg algebra, simply tensor with $\mathcal{O}_\infty$ with the trivial action.) Moreover, the classification of not necessarily pointwise outer actions of discrete groups (or even finite cyclic groups!) on Kirchberg algebras is probably a very challenging task.

The conclusion seems to be that neither approach is likely to work for general compact Lie groups, and that an eventual classification would require a rather different approach and machinery.

## Circle Actions with the Continuous Rokhlin Property

The Rokhlin property, as in Definition IX.2.1, should be thought of as a "sequential" Rokhlin property. Indeed, assume that $A$ is a separable unital $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $(F_n)_{n\in\mathbb{N}}$ be an increasing family of finite subsets of $A$ whose union is dense in $A$, and for every $\varepsilon_n = \frac{1}{n}$, choose a unitary $u_n \in \mathcal{U}(A)$ such that

- $\|\alpha_\zeta(u_n) - \zeta u_n\| < \frac{1}{n}$ for all $\zeta \in \mathbb{T}$, and
- $\|u_n a - a u_n\| < \frac{1}{n}$ for all $a \in F_n$.

We thus obtain a sequence $(u_n)_{n\in\mathbb{N}}$ of unitaries in $A$ such that

1. $\lim\limits_{n\to\infty} \|\alpha_\zeta(u_n) - \zeta u_n\| = 0$ uniformly on $\zeta \in \mathbb{T}$,
2. $\lim\limits_{n\to\infty} \|u_n a - a u_n\| = 0$ for all $a \in A$.

In fact, it is easy to show that if $A$ is separable, then the Rokhlin property for $\alpha$ is equivalent to the existence of a sequence of unitaries in $A$ satisfying (1) and (2) above.

We will consider a strengthening of the Rokhlin property in which one asks for a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries satisfying conditions analogous to (1) and (2) above. We call it the *continuous* Rokhlin property, and present its precise definition below.

**Definition IX.7.1.** Let $A$ be a unital $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action of $\mathbb{T}$ on $A$. We say that $\alpha$ has the *continuous Rokhlin property* if there exists a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $A$ such that

1. $\lim\limits_{t\to\infty} \|\alpha_\zeta(u_t) - \zeta u_t\| = 0$ uniformly on $\zeta \in \mathbb{T}$,
2. $\lim\limits_{t\to\infty} \|u_t a - a u_t\| = 0$ for all $a \in A$.

**Remarks IX.7.2.** We have the following easy observations:

1. It is immediate that if a circle action has the continuous Rokhlin property, then it has the Rokhlin property.
2. It is also clear that condition (2) in Definition IX.7.1 is satisfied for all $a$ in $A$ if and only if it is satisfied for all elements of some generating set. This easy observation will be used repeatedly and without reference.

In view of the first of the remarks above, an obvious question is whether the continuous Rokhlin property is actually equivalent to the Rokhlin property. We address this question in detail in Section IX.10. There, it is shown that, while this is indeed the case for certain classes of $C^*$-algebras (see Corollary IX.10.2 and Proposition IX.10.5), it is not true in full generality, even on Kirchberg algebras that satisfy the UCT (see Example IX.10.3 and Example IX.10.4).

We begin by developing some of the basic theory for actions satisfying the continuous Rokhlin property.

**Proposition IX.7.3.** Let $A$ be a unital $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property. If $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ is any action of $\mathbb{T}$ on a unital $C^*$-algebra $B$, then the tensor product action $\zeta \mapsto \alpha_\zeta \otimes \beta_\zeta$ of $\mathbb{T}$ on $A \otimes B$, for any $C^*$-tensor product on which it is defined, has the continuous Rokhlin property.

*Proof.* Choose a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $A$ as in the definition of continuous Rokhlin property for $\alpha$. For $t \in [1,\infty)$, set $v_t = u_t \otimes 1$, which is a unitary in $A \otimes B$. For $\zeta \in \mathbb{T}$, we have

$$\|(\alpha \otimes \beta)_\zeta(v_t) - \zeta v_t\| = \|(\alpha_\zeta(u_t) \otimes \beta_\zeta(1)) - \zeta(u_t \otimes 1)\| = \|\alpha_\zeta(u_t) - \zeta u_t\|,$$

and thus $\lim\limits_{t\to\infty} \|(\alpha \otimes \beta)_\zeta(v_t) - \zeta v_t\| = 0$ uniformly on $\zeta \in \mathbb{T}$, and condition (1) of Definition IX.7.1 is satisfied. To check condition (2), let $x \in A \otimes B$ and assume that $x = a \otimes b$ for some $a$ in $A$ and some $b$ in $B$. (Note that such elements generate $A \otimes B$.) Then

$$\|v_t x - x v_t\| = \|(u_t a - a u_t) \otimes b\| \leq \|u_t a - a u_t\|\|b\| \to 0$$

as $t \to \infty$. This finishes the proof. □

Although we will not make use of the next proposition here, we present it to illustrate the difference between the Rokhlin property and the continuous Rokhlin property. Some technical condition seems to be necessary to show that the continuous Rokhlin property passes to direct limits, although we do not have an example that shows that the result may fail otherwise. The main difference with part (4) of Theorem VI.2.3, is that one cannot in general get a continuous path of unitaries using a diagonal argument.

**Proposition IX.7.4.** Let $A$ be a unital $C^*$-algebra. Suppose that $A = \varinjlim(A_n, \iota_n)$ is a direct limit of unital $C^*$-algebras with unital maps, and that $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action obtained as the direct limit of actions $\alpha^{(n)}\colon \mathbb{T} \to \mathrm{Aut}(A_n)$, such that $\alpha^{(n)}$ has the continuous Rokhlin property for all $n$. For every $n \in \mathbb{N}$, let $(u_t^{(n)})_{t\in[1,\infty)}$ be a continuous path of unitaries as in the definition of continuous Rokhlin property for $\alpha^{(n)}$. Assume that there exists a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[1,\infty)$ with $\lim\limits_{n\to\infty} t_n = \infty$ such that $\iota_n(u_{t_n}^{(n)}) = u_{t_{n+1}}^{(n+1)}$ for all $n \in \mathbb{N}$. Then $\alpha$ has the continuous Rokhlin property.

*Proof.* We define a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $A$ via

$$u_t = \begin{cases} \iota_{1,\infty}(u_t^{(1)}), & \text{for } t \in [0, t_1] \\ \iota_{2,\infty}(u_t^{(2)}), & \text{for } t \in [t_1, t_2] \\ \vdots \end{cases}$$

We claim that $(u_t)_{t\in[1,\infty)}$ is a continuous path of Rokhlin unitaries for $\alpha$. It is easy to see that for $\zeta \in \mathbb{T}$, we have

$$\lim_{t\to\infty} \|\alpha_\zeta(u_t) - \zeta u_t\| = 0$$

and that the convergence is uniform on $\zeta \in \mathbb{T}$. On the other hand, given $a \in A$ and $\varepsilon > 0$, find $N \in \mathbb{N}$ and $b \in A_N$ such that $\|\iota_{N,\infty}(b) - a\| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned} \|u_t a - a u_t\| &\le \|u_t a - u_t \iota_{N,\infty}(b)\| + \|u_t \iota_{N,\infty}(b) - \iota_{N,\infty}(b) u_t\| + \|\iota_{N,\infty}(b) u_t - a u_t\| \\ &< \varepsilon + \|u_t \iota_{N,\infty}(b) - \iota_{N,\infty}(b) u_t\| \end{aligned}$$

and hence $\lim\limits_{t\to\infty} \|u_t a - a u_t\| \le \varepsilon$. Since $\varepsilon$ is arbitrary, this proves the claim. This finishes the proof of the proposition. $\square$

**Example IX.7.5.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(C(\mathbb{T}))$ be the action by left translation. Then $\alpha$ has the continuous Rokhlin property. Simply take $u_t(\zeta) = \zeta$ for all $t \in [1,\infty)$ and all $\zeta \in \mathbb{T}$.

The following example is similar to Example VI.2.8. Showing that it has the continuous Rokhlin property requires some work. Unfortunately, we cannot apply Proposition IX.7.4 in this example since it is not clear how to choose the continuous paths of Rokhlin unitaries to satisfy its hipotheses.

**Example IX.7.6.** For $n \in \mathbb{N}$, let $A_n = C(\mathbb{T}) \otimes M_{2^n}$, which we identify with $C(\mathbb{T}, M_n)$ when necessary. Consider the action $\alpha^{(n)} \colon \mathbb{T} \to \mathrm{Aut}(A_n)$ given by $\alpha^{(n)}(\zeta)(f)(w) = f(\zeta^{-1}w)$ for $\zeta, w \in \mathbb{T}$ and $f$ in $C(\mathbb{T}, M_{2^n})$. In other words, $\alpha^{(n)}$ is the tensor product of the regular representation of $\mathbb{T}$ with the trivial action on $M_{2^n}$. Then $\alpha^{(n)}$ has the continuous Rokhlin property by Proposition IX.7.3 and Example IX.7.5.

We construct a direct limit algebra $A = \varinjlim(A_n, \iota_n)$ as follows. Fix a countable dense subset $X = \{x_1, x_2, x_3, \ldots\} \subseteq \mathbb{T}$. With $f_x(\zeta) = f(x^{-1}\zeta)$ for $f \in A_n$, $x \in X$ and $\zeta \in \mathbb{T}$, define maps $\iota_n \colon A_n \to A_{n+1}$ for $n \in \mathbb{N}$, by

$$\iota_n(f) = \begin{pmatrix} f & 0 \\ 0 & f_{x_n} \end{pmatrix} \quad f \in A_n.$$

The limit algebra $A = \varinjlim(A_n, \iota_n)$ is a simple unital A$\mathbb{T}$-algebra.

Moreover, it is clear that $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $\mathbb{T}$ on $A$. We claim that $\alpha$ has the continuous Rokhlin property. For each $n \in \mathbb{N}$, write $x_n = e^{2\pi i s_n}$ for some $s_n \in \mathbb{R}$. Define a homotopy $H^{(n)} \colon [0,1] \to M_{2^{n+1}}(C(\mathbb{T}))$ between $\iota_n(z \otimes 1_{M_{2^n}})$ and $z \otimes 1_{M_{2^{n+1}}}$ by

$$H_t^{(n)}(\zeta) = \begin{pmatrix} \zeta & 0 \\ 0 & x_n^{-1} e^{2\pi i t s_n} \zeta \end{pmatrix} \otimes 1_{M_{2^n}}$$

for all $t \in [0,1]$ and all $\zeta \in \mathbb{T}$. Note that $H_t^{(n)}$ commutes with $\iota_n(A_n)$ for all $t \in [0,1]$ and all $n \in \mathbb{N}$.

Define a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $A$ by

$$u_t = \begin{cases} \iota_{1,\infty}(u_t^{(1)}), \text{ for } t \in [0, \frac{1}{2}] \\ \iota_{2,\infty}(H_{2t-1}^{(1)}), \text{ for } t \in [\frac{1}{2}, 1] \\ \iota_{2,\infty}(u_t^{(2)}), \text{ for } t \in [1, 3/2] \\ \iota_{3,\infty}(H_{2t-3}^{(2)}), \text{ for } t \in [3/2, 2] \\ \vdots \end{cases}$$

We claim that $(u_t)_{t\in[1,\infty)}$ is a continuous path of Rokhlin unitaries for $\alpha$. It is easy to check that $\alpha_\zeta(u_t) = \zeta u_t$ for all $\zeta \in \mathbb{T}$ and all $t \in [1, \infty)$, since this is true for each of the paths $\left(u_t^{(n)}\right)_{t\in[1,\infty)}$ and for each of the homotopies $H^{(n)}$, for $n$ in $\mathbb{N}$. On the other hand, given $a \in A$ and $\varepsilon > 0$, find $N \in \mathbb{N}$ and $b \in A_N$ such that $\|\iota_{N,\infty}(b) - a\| < \frac{\varepsilon}{2}$. Then

$$\|u_t\iota_{N,\infty}(b) - \iota_{N,\infty}(b)u_t\| = 0$$

for every $t \geq N + 1$, since all of the images of the homotopies $H^{(n)}$ and all of the unitaries $(u_t^{(n)})_{t\in[1,\infty)}$, for $n \geq N + 1$, commute with the image of $A_N$ in $A_n$. The rest is a routine application of the triangle inequality:

$$\begin{aligned} \|u_t a - a u_t\| &\leq \|u_t a - u_t\iota_{N,\infty}(b)\| + \|u_t\iota_{N,\infty}(b) - \iota_{N,\infty}(b)u_t\| + \|\iota_{N,\infty}(b)u_t - a u_t\| \\ &< \varepsilon + \|u_t\iota_{N,\infty}(b) - \iota_{N,\infty}(b)u_t\| \end{aligned}$$

and hence $\lim\limits_{t\to\infty} \|u_t a - a u_t\| \leq \varepsilon$. Since $\varepsilon$ is arbitrary, this proves the claim.

The next result should be thought of as asserting that the action of $\mathbb{T}$ on $C(\mathbb{T})$ by left translation is *continuously* equivariantly semiprojective, in an appropriate sense which we do not make explicit here.

**Proposition IX.7.7.** Let $A$ be a unital $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. Then $\alpha$ has the continuous Rokhlin property if and only if there exists a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $A$ such that

1. $\alpha_\zeta(u_t) = \zeta u_t$ for all $\zeta \in \mathbb{T}$ and all $t \in [1, \infty)$.

2. $\lim\limits_{t\to\infty} \|u_t a - a u_t\| = 0$ for all $a \in A$.

The definition of the continuous Rokhlin property differs in that in condition (1), one only requires $\lim\limits_{t\to\infty} \|\alpha_\zeta(u_t) - \zeta u_t\| = 0$ uniformly on $\zeta \in \mathbb{T}$.

*Proof.* Choose a path $(v_t)_{t\in[1,\infty)}$ of unitaries in $A$ as in the definition of the continuous Rokhlin property. Without loss of generality, we may assume that $\|\alpha_\zeta(v_t) - \zeta v_t\| < \frac{1}{3}$ for all $\zeta$ in $\mathbb{T}$ and all $t$ in $[1,\infty)$. Denote by $\mu$ the normalized Haar measure on $\mathbb{T}$, and for $t$ in $[1,\infty)$, set

$$x_t = \int_{\mathbb{T}} \overline{\zeta}\alpha_\zeta(v_t)\ d\mu(\zeta).$$

Given $t$ in $[1,\infty)$, one checks that $\|x_t\| \leq 1$ and $\|x_t - v_t\| \leq \frac{1}{3}$. Thus $\|x_t^* x_t - 1\| < 1$, so $x_t^* x_t$ is invertible. Set $u_t = x_t(x_t^* x_t)^{-\frac{1}{2}}$, which is a unitary in $A$.

For $\zeta$ in $\mathbb{T}$ and $t$ in $[1,\infty)$, it is immediate to check that $\alpha_\zeta(x_t) = \zeta x_t$, and thus $\alpha_\zeta(u_t) = \zeta u_t$. An application of the triangle inequality shows that $\lim\limits_{t\to\infty} \|u_t a - a u_t\| = 0$ for all $a$ in $A$. Finally,

$$\|x_t - x_s\| = \left\| \int_{\mathbb{T}} \overline{\zeta}\alpha_\zeta(v_t - v_s)\ d\mu(\zeta) \right\| \leq \|v_t - v_s\|$$

for all $t$ and $s$ in $[1,\infty)$, which shows that the map $t \mapsto x_t$ is continuous. This proves that $t \mapsto u_t$ is also continuous, and hence the path $(u_t)_{t\in[1,\infty)}$ satisfies conditions (1) and (2) of the statement. □

We turn to duality.

**Definition IX.7.8.** Let $B$ be a $C^*$-algebra and let $\beta$ be an automorphism of $B$. We say that $\beta$ is *asymptotically representable* if there exists a continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $M(B)$ such that

1. $\lim\limits_{t\to\infty} \|\beta(b) - u_t b u_t^*\| = 0$ for all $b \in B$, and
2. $\lim\limits_{t\to\infty} \|\beta(u_t) - u_t\| = 0$.

**Remarks IX.7.9.** We have the following easy observations:

1. It is immediate that if an automorphism is asymptotically representable, then it is asymptotically inner and approximately representable.

2. It is also clear that condition (2) in Definition IX.7.8 is satisfied for all $b$ in $B$ if and only if it is satisfied for all elements of some generating set. This easy observation will be used repeatedly and without reference.

**Remark IX.7.10.** Let $B$ be a $C^*$-algebra and let $\beta$ be an automorphism of $B$. One can easily show that $\beta$ is asymptotically representable if and only if there exists a unitary $v$ in

$$C_b([1,\infty), M(B)^\beta)/C_0([1,\infty), M(B)^\beta)$$

such that $\beta(b) = vbv^*$ for all $b$ in $B$. We leave the proof as an exercise for the reader. We point out that one does not need to assume the $C^*$-algebra $B$ to be separable.

We proceed to show that asymptotic representability is the notion dual to the continuous Rokhlin property, in complete analogy with the duality between the Rokhlin property and approximate representability.

**Proposition IX.7.11.** Let $B$ be a unital $C^*$-algebra and let $\beta$ be an automorphism of $B$. Consider the dual action $\widehat{\beta}\colon \mathbb{T} \to \mathrm{Aut}(B \rtimes_\varphi \mathbb{Z})$ of $\mathbb{T}$ on the crossed product. Then $\beta$ is asymptotically representable if and only if $\widehat{\beta}$ has the continuous Rokhlin property.

*Proof.* Assume that $\beta$ is asymptotically representable. Let $(u_t)_{t\in[1,\infty)}$ be a continuous path of unitaries in $B$ satisfying

$$\lim_{t\to\infty} \|\beta(b) - u_t b u_t^*\| = 0 \ \text{ for all } b \in B \ \text{ and } \ \lim_{t\to\infty} \|\beta(u_t) - u_t\| = 0.$$

Denote by $v$ the canonical unitary in $B \rtimes_\beta \mathbb{Z}$ that implements $\beta$. For $t \in [1,\infty)$, set $w_t = u_t^* v$, which is a unitary in $B \rtimes_\beta \mathbb{Z}$. Moreover, for $\zeta$ in $\mathbb{T}$ we have $\widehat{\beta}_\zeta(w_t) = \zeta w_t$, so condition (1) of Definition IX.7.1 is satisfied for $\widehat{\beta}$ with $(w_t)_{t\in[1,\infty)}$. To check condition (2), it is enough to consider $a \in B \cup \{v\}$. For $a$ in $B$, we have

$$w_t a w_t^* = u_t^* v a v^* u_t = u_t^* \beta(a) u_t \to_{t\to\infty} \beta^{-1}(\beta(a)) = a,$$

and hence $\lim\limits_{t\to\infty} \|w_t a - a w_t\| = 0$, as desired. Finally,

$$\|w_t v w_t^* - v\| = \|u_t^* v u_t - v\| = \|v u_t v^* - u_t\| = \|\beta(u_t) - u_t\| \to_{t\to\infty} 0.$$

We conclude that $\widehat{\beta}$ has the continuous Rokhlin property.

Conversely, assume that $\widehat{\beta}$ has the continuous Rokhlin property. Use Proposition IX.7.7 to choose a continuous path $(w_t)_{t\in[1,\infty)}$ of unitaries in $B \rtimes_\beta \mathbb{Z}$ such that

- $\widehat{\beta}_\zeta(w_t) = w_t$ for all $\zeta$ in $\mathbb{T}$ and all $t$ in $[1,\infty)$;
- $\lim\limits_{t\to\infty} \|w_t a - a w_t\| = 0$ for all $a$ in $B \rtimes_\beta \mathbb{Z}$.

For $t$ in $[1,\infty)$, set $u_t = v w_t^*$, which is a unitary in $B \rtimes_\beta \mathbb{Z}$. We claim that $u_t$ belongs to $B$. For $\zeta$ in $\mathbb{T}$, we have $\widehat{\beta}_\zeta(u_t) = u_t$, so $u_t$ belongs to $(B \rtimes_\beta \mathbb{Z})^{\widehat{\beta}} = B$, as desired.

For $b$ in $B$, we have

$$\|u_t b u_t^* - \beta(b)\| = \|v w_t^* b w_t v^* - v b v^*\| = \|w_t^* b w_t - b\| \to_{t\to\infty} 0.$$

It follows that the continuous path $(u_t)_{t\in[1,\infty)}$ of unitaries in $B$ satisfies the conditions of Definition IX.7.8, and hence $\beta$ is asymptotically representable. □

**Proposition IX.7.12.** Let $A$ be a unital $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be a continuous action. Then $\alpha$ has the continuous Rokhlin property if and only if $\widehat{\alpha}$ is asymptotically representable.

We point out thet we do not require $A$ to be separable, unlike in Theorem VI.4.2.

*Proof.* Assume that $\alpha$ has the continuous Rokhlin property. By Theorem IX.2.6, there are an approximately representable automorphism $\theta$ of $A^\alpha$ and an isomorphism between $A \rtimes_\theta \mathbb{Z}$ and $A$ that intertwines the dual action of $\theta$ and $\alpha$. Denote by $\lambda\colon \mathbb{T} \to \mathcal{U}(\mathcal{B}(L^2(\mathbb{T})))$ the left regular representation of $\mathbb{T}$. Since $\widehat{\widehat{\theta}}$ is conjugate to $\theta \otimes (\mathrm{Ad} \circ \lambda)$, it follows that $\widehat{\widehat{\theta}}$ is asymptotically representable as well. The result now follows since $\widehat{\widehat{\widehat{\theta}}}$ is conjugate to $\widehat{\alpha}$.

The converse is analogous, and is left to the reader. □

The next proposition will not be needed until the following section. In contrast to Lemma IX.7.14, the analogous result for an infinite tensor product is probably not true, although we do not have a counterexample.

**Proposition IX.7.13.** Let $A$ and $B$ be unital $C^*$-algebras, let $\varphi \in \mathrm{Aut}(A)$ and $\psi \in \mathrm{Aut}(B)$ be asymptotically representable automorphisms of $A$ and $B$, respectively. Then the automorphism $\varphi \otimes \psi$ of $A \otimes B$, for any tensor product on which it is defined, is asymptotically representable.

*Proof.* Let $(u_t)_{t\in[1,\infty)}$ and $(v_t)_{t\in[1,\infty)}$ be two continuous paths of unitaries in $A$ and $B$ satisfying the conditions in Definition IX.7.8 for $\varphi$ and $\psi$ respectively. For each $t \in [1,\infty)$, set $w_t = u_t \otimes v_t$. Then $w_t$ is a unitary in $A \otimes B$ for all $t$, and moreover $t \mapsto w_t$ is continuous. We claim that $(w_t)_{t\in[1,\infty)}$ is the desired path of unitaries for $\varphi \otimes \psi$. We have

$$\limsup_{t\to\infty} \|(\varphi \otimes \psi)(u_t \otimes v_t) - u_t \otimes v_t\| \leq \lim_{t\to\infty} (\|\varphi(u_t) - u_t\| + \|\psi(v_t) - v_t\|) = 0,$$

so condition (1) is satisfied. In order to check condition (2), let $x \in A \otimes B$. Since $A$ and $B$ are unital, it follows that $A \otimes B$ is generated by the set

$$\{a \otimes 1 \colon a \in A\} \cup \{1 \otimes b \colon b \in B\}.$$

We may therefore assume that $x = a \otimes 1$ for some $a$ in $A$. Then

$$\|(\varphi \otimes \psi)(x) - w_t x w_t^*\| = \|\varphi(a) \otimes 1 - u_t a u_t^* \otimes 1\| = \|\varphi(a) - u_t a u_t^*\| \to_{t\to\infty} 0,$$

which completes the proof. $\square$

For the sake of comparison and for later use, we show next that the tensor product of countably many approximately representable automorphisms is again approximately representable.

**Lemma IX.7.14.** Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of unital $C^*$-algebras. For each $n \in \mathbb{N}$, let $\varphi_n \in \mathrm{Aut}(A_n)$ be an approximately representable automorphism, and let $\varphi$ be the product type automorphism $\varphi = \bigotimes_{n=1}^{\infty} \varphi_n$ of $A = \bigotimes_{n=1}^{\infty} A_n$, for any tensor product on which it is defined. Then $\varphi$ is approximately representable.

*Proof.* Let $\varepsilon > 0$ and let $F \subseteq A$ be a finite set. With $m = \mathrm{card}(F)$, write $F = \{a_1, \ldots, a_m\}$. Find $N \in \mathbb{N}$ and $a_1', \ldots, a_m'$ in the finite tensor product $\bigotimes_{j=1}^{N} A_n$ such that $\|a_k - a_k'\| < \frac{\varepsilon}{4}$ for all $k = 1, \ldots, m$. Moreover, for each $k = 1, \ldots, m$, find a positive integer $L_k \in \mathbb{N}$ and $a_1^{(k,j)}, \ldots, a_{L_k}^{(k,j)} \in A_j$ for $j = 1, \ldots, N$, satisfying

$$\left\| a_k' - \sum_{\ell=1}^{L_k} a_\ell^{(k,1)} \otimes a_\ell^{(k,2)} \otimes \cdots \otimes a_\ell^{(k,N)} \right\| < \varepsilon.$$

Set

$$K = \max\left\{ \left\| a_\ell^{(k,j)} \right\| : k = 1, \ldots, m, j = 1, \ldots, N, \ell = 1, \ldots, L_k \right\}.$$

For each $j = 1, \ldots, N$, choose a unitary $u_j \in \mathcal{U}(A_j)$ such that

– $\left\| \varphi_j(a_\ell^{(k,1)}) - u_j a_\ell^{(k,1)} u_j^* \right\| < \frac{\varepsilon}{2NK^{N-1}+2}$ for all $\ell = 1, \ldots, L_k$ and for all $k = 1, \ldots, m$,

– $\|\varphi_j(u_j) - u_j\| < \frac{\varepsilon}{2NK^{N-1}+2}$.

Set $u = u_1 \otimes \cdots \otimes u_N \otimes 1 \otimes \cdots \in \mathcal{U}(A)$. For $j = 1, \ldots, m$ we have

$$\begin{aligned}
\|\varphi(a_j) - u a_j u^*\| &\leq \left\|\varphi(a_j) - \varphi(a_j')\right\| + \left\|\varphi(a_j') - u a_j' u^*\right\| + \left\|u a_j' u^* - u a_j u^*\right\| \\
&\leq \frac{\varepsilon}{2} + \left\| \varphi(a_j') - \varphi\left( \sum_{\ell=1}^{L_k} a_\ell^{(k,1)} \otimes a_\ell^{(k,2)} \otimes \cdots \otimes a_\ell^{(k,N)} \right) \right\| \\
&\quad + \left\| \varphi\left( \sum_{\ell=1}^{L_k} a_\ell^{(k,1)} \otimes a_\ell^{(k,2)} \otimes \cdots \otimes a_\ell^{(k,N)} \right) \right. \\
&\qquad \left. - u \left( \sum_{\ell=1}^{L_k} a_\ell^{(k,1)} \otimes a_\ell^{(k,2)} \otimes \cdots \otimes a_\ell^{(k,N)} \right) u^* \right\| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2NK^{N-1}+2} + NK^{N-1} \frac{\varepsilon}{2NK^{N-1}+2} \\
&= \varepsilon.
\end{aligned}$$

Moreover, a repeated use of the triangle inequality yields

$$\begin{aligned}
\|\varphi(u) - u\| &= \|\varphi_1(u_1) \otimes \cdots \otimes \varphi_N(u_N) - u_1 \otimes \cdots \otimes u_N\| \\
&< N \frac{\varepsilon}{2NK^{N-1}+2} < \varepsilon.
\end{aligned}$$

Hence, $u$ is an approximately fixed implementing unitary for $\varphi$, and thus $\varphi$ is approximately representable. □

**Corollary IX.7.15.** Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of unital $C^*$-algebras. For each $n \in \mathbb{N}$, let $u_n$ be a unitary in $A_n$. Let $\varphi$ be the product type automorphism $\varphi = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(u_n)$ of $A = \bigotimes_{n=1}^{\infty} A_n$, for any tensor product on which it is defined. Then $\varphi$ is approximately representable.

*Proof.* This follows from Lemma IX.7.14 and the fact that inner automorphisms are approximately representable. □

We return to the main development of the section. The example constructed in Theorem IX.7.17 below will be needed in the proof of Theorem IX.9.3 to show that every possible value of the equivariant $K$-theory can be realized by an action with the continuous Rokhlin property. We introduce a definition from [187] first.

**Definition IX.7.16.** (See Theorem 1 in [187].) Let $A$ be a unital $C^*$-algebra and let $\varphi$ be an automorphism of $A$. We say that $\varphi$ has the *Rokhlin property* if for every $\varepsilon > 0$, for every finite subset $F \subseteq A$ and for every $N \in \mathbb{N}$, there exist projections $e_0, \ldots, e_{N-1}$ and $f_0, \ldots, f_N$ in $A$ such that

1. $\sum\limits_{j=0}^{N-1} e_j + \sum\limits_{k=0}^{N} f_j = 1$
2. $\|e_j a - a e_j\| < \varepsilon$ and $\|f_k a - a f_k\| < \varepsilon$ for all $j = 0, \ldots, N-1$, for all $k = 0, \ldots, N$ and for all $a \in F$.
3. $\|\varphi(e_j) - e_{j+1}\| < \varepsilon$ and $\|\varphi(f_k) - f_{k+1}\| < \varepsilon$ for all $j = 0, \ldots, N-1$ and for all $k = 0, \ldots, N$, where $e_N$ is taken to be $e_0$ and $f_{N+1}$ is taken to be $f_0$.

It is easy to show that an automorphism with the Rokhlin property is aperiodic, meaning that none of its powers is inner. In [187], Nakamura showed that an automorphism of a unital Kirchberg algebra is aperiodic *if and only if* it has the Rokhlin property.

**Theorem IX.7.17.** There is an approximately representable automorphism $\psi$ of $\mathcal{O}_\infty$ with the Rokhlin property (and in particular, aperiodic by the comments above). Moreover, the automorphism $\psi$ can be chosen to be asymptotically representable.

*Proof.* For every $n \in \mathbb{N}$, choose a unital embedding $\varphi_n \colon M_n \oplus M_{n+1} \hookrightarrow \mathcal{O}_\infty$ such that if $e \in M_n$ and $f \in M_{n+1}$ are rank one projections, then $\varphi_n((e, 0)) = p$ with $[p] = 1$ in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ and $\varphi_n((0, f)) = q$ with $[q] = -1$ in $K_0(\mathcal{O}_\infty)$. Consider the permutation unitary

$$u_n^{(1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_n.$$

Then $u_n = \varphi_n\left(u_n^{(1)}, u_{n+1}^{(1)}\right)$ is a unitary in $\mathcal{O}_\infty$ with the property that there are two towers $e_0, \ldots, e_{n-1}$ and $f_0, \ldots, f_n$ of projections in $\varphi_n(M_n \oplus 0) \subseteq \mathcal{O}_\infty$ and $\varphi_n(0 \oplus M_{n+1}) \subseteq \mathcal{O}_\infty$, respectively, such that

1. $\sum\limits_{j=0}^{n-1} e_j + \sum\limits_{k=0}^{n} f_k = 1$,
2. with $e_n = e_0$, we have $\mathrm{Ad}(u_n)(e_j) = e_{j+1}$ for all $j = 0, \ldots, n-1$, and
3. with $f_{n+1} = f_0$, we have $\mathrm{Ad}(u_n)(f_k) = f_{k+1}$ for all $k = 0, \ldots, n$.

Set $\varphi = \bigotimes\limits_{n=1}^{\infty} \mathrm{Ad}(u_n)$, which defines an automorphism of $\bigotimes\limits_{n=1}^{\infty} \mathcal{O}_\infty \cong \mathcal{O}_\infty$. We claim that $\varphi$ is aperiodic (and hence has the Rokhlin property thanks to Nakamura's result). Assume that $\varphi^m = \mathrm{Ad}(u)$ for some $m \in \mathbb{N}$ and some $u \in \mathcal{U}\left(\bigotimes\limits_{n=1}^{\infty} \mathcal{O}_\infty\right)$. Given $\varepsilon < 1$, let $M \in \mathbb{N}$ and $v \in \mathcal{U}\left(\mathcal{O}_\infty^{\otimes M}\right)$ such that $\|u - v\| < \varepsilon$. Choose $N > \max\{m, M\}$ and find towers $e_0, \ldots, e_{N-1}$ and $f_0, \ldots, f_N$ of nonzero projections in $\mathcal{O}_\infty$ such that, with

$$e_j' = 1 \otimes \cdots \otimes 1 \otimes e_j \otimes 1 \otimes \cdots \in \bigotimes_{n=1}^{\infty} \mathcal{O}_\infty \quad \text{and} \quad f_k' = 1 \otimes \cdots \otimes 1 \otimes f_k \otimes 1 \otimes \cdots \in \bigotimes_{n=1}^{\infty} \mathcal{O}_\infty$$

for $j = 0, \ldots, N-1$ and for $k = 0, \ldots, N$, the following hold:

1. $\sum\limits_{j=0}^{n-1} e_j' + \sum\limits_{k=0}^{n} f_k' = 1$,
2. with $e_n' = e_0'$, we have $\varphi(e_j') = e_{j+1}'$ for all $j = 0, \ldots, n-1$, and
3. with $f_{n+1}' = f_0'$, we have $\varphi(f_k') = f_{k+1}'$ for all $k = 0, \ldots, n$.

In particular,

$$\varphi^m(e_0') = e_{m-1}' \quad \text{and} \quad e_0' e_{m-1}' = 0.$$

It moreover follows that $e_0'$ and $e_{m-1}'$ commute with $v$, and hence

$$2 = \|e_0' - e_{m-1}'\| = \|e_0' - u e_0' u^*\| \leq \|e_0' - v e_0' v^*\| + 2\|u - v\| = 2\varepsilon < 2,$$

which is a contradiction. This shows that $\varphi^m$ is not inner. Since $m$ is arbitrary, it follows that $\varphi$ is aperiodic.

Since $\varphi$ is the direct limit of the inner automorphisms $\varphi_k = \mathrm{Ad}\left(\bigotimes\limits_{n=1}^{k} u_n\right)$ for $k$ in $\mathbb{N}$, it follows from Corollary IX.7.15 that it is approximately representable.

Finally, we claim that $\varphi$ is asymptotically representable. For $n \in \mathbb{N}$, set $\widetilde{u}_n = u_1 \otimes \cdots \otimes u_n$, which is a unitary in $\mathcal{O}_\infty$. Note that $\varphi = \varinjlim \mathrm{Ad}(\widetilde{u}_n)$ and that $\varphi(\widetilde{u}_n) = \widetilde{u}_n$ for all $n \in \mathbb{N}$. Thus, in order to show that $\varphi$ is asymptotically representable, it will be enough to show that for each $n \in \mathbb{N}$, the unitary $\widetilde{u}_n$ can be connected to $\widetilde{u}_{n+1}$ by a path of unitaries within the fixed point algebra of $\varphi$. For this, it will be enough to show that each $u_n$ is connected to the unit of $\mathcal{O}_\infty$ *within* $\varphi_n(M_n \oplus M_{n+1})$ by a path of unitaries that are fixed by $\mathrm{Ad}(u_n)$, that is, a path of unitaries that commute with $u_n$. It is easily seen that the set of elements in $M_n \oplus M_{n+1}$ which commute with $\left(u_n^{(1)}, u_{n+1}^{(1)}\right)$ is isomorphic to $\mathbb{C}^n \oplus \mathbb{C}^{n+1}$. Since the unitary group of this $C^*$-algebra is connected, this shows that $\varphi$ is asymptotically representable. $\square$

**Remark IX.7.18.** Adopt the notation of Theorem IX.7.17 above. Using the Pimsner-Voiculescu exact sequence for $\psi$, the $K$-theory of $\mathcal{O}_\infty \rtimes_\psi \mathbb{Z}$ is easily seen to be

$$K_0(\mathcal{O}_\infty \rtimes_\psi \mathbb{Z}) \cong K_1(\mathcal{O}_\infty \rtimes_\psi \mathbb{Z}) \cong \mathbb{Z},$$

with $[1_{\mathcal{O}_\infty \rtimes_\psi \mathbb{Z}}] = 1$ in $K_0(\mathcal{O}_\infty \rtimes_\psi \mathbb{Z})$.

**Corollary IX.7.19.** Let $A$ be a unital $C^*$-algebra such that $A \otimes \mathcal{O}_\infty \cong A$. Then there exists a asymptotically representable, aperiodic automorphism of $A$.

*Proof.* Let $\phi\colon A \otimes \mathcal{O}_\infty \to A$ be an isomorphism. Use Theorem IX.7.17 to choose an asymptotically representable, aperiodic automorphism $\psi$ of $\mathcal{O}_\infty$. Proposition IX.7.13 then shows that $\phi \circ (\mathrm{id}_A \otimes \psi) \circ \phi^{-1}$ is an asymptotically representable automorphism of $A$, and it is clearly aperiodic. $\square$

**Proposition IX.7.20.** Let $A$ be a unital Kirchberg algebra, and let $\varphi$ be an aperiodic, $KK$-trivial automorphism of $A$. Then $\varphi$ is asymptotically representable and its dual action is an action of the circle on a unital Kirchberg algebra with the continuous Rokhlin property. Moreover, any two such automorphisms are cocycle conjugate.

*Proof.* Most of the work has already been done. Since $A$ absorbs $\mathcal{O}_\infty$ by Theorem 3.15 in [151], we can use Corollary IX.7.19 to choose an asymptotically representable, aperiodic automorphism $\psi$ of $A$. It follows from Theorem 5 in [187] that $\varphi$ and $\psi$ are cocycle conjugate, and hence $\varphi$ is asymptotically representable. It follows from Proposition IX.7.11 that the dual action of $\varphi$ has the continuous Rokhlin property. Finally, it is well-known that crossed products by aperiodic automorphisms preserve unital Kirchberg algebras. (See Corollary 4.6 in [135], here reproduced as part (2) of Theorem II.2.8, for preservation of pure infiniteness in the simple case.)

Uniqueness up to conjugacy follows from Theorem 5 in [187]. □

## Asymptotic Homomorphisms, $K$-theoretical Obstructions and the Universal Coefficient Theorem

The goal of this section is to prove two crucial results. First, we will show in Corollary IX.8.5 that if $A$ is a separable, unital $C^*$-algebra, and $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action with the continuous Rokhlin property, then there are isomorphisms

$$K_0(A) \cong K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha).$$

(Compare with Theorem IX.4.3, where we only assumed that the action has the Rokhlin property, but where some assumptions on $K_*(A)$ were needed.) Second, we will show that, under the same assumptions, the $C^*$-algebra $A$ satisfies the UCT if and only if $A^\alpha$ satisfies the UCT; see Theorem IX.8.8. Both results will follow from the existence of an asymptotic morphism $A \to A^\alpha$ which is a left inverse for the canonical inclusion $A^\alpha \to A$ at the level of $KK$-theory. See Theorem IX.8.3 below. We therefore begin by recalling the definition of asymptotic morphisms from [31], and that of a completely positive contractive asymptotic morphism from [130].

**Definition IX.8.1.** Let $A$ and $B$ be $C^*$-algebras. An *asymptotic morphism* from $A$ to $B$, is a family $\psi = (\psi_t)_{t\in[1,\infty)}$ of maps $\psi_t\colon A \to B$, satisfying the following conditions:

1. For every $a$ in $A$, the map $[1,\infty) \to B$ given by $t \mapsto \psi_t(a)$ is continuous.

2. For every $\lambda$ in $\mathbb{C}$ and every $a$ and $b$ in $A$, we have

$$\lim_{t\to\infty} \|\psi_t(\lambda a + b) - \lambda\psi_t(a) - \psi_t(b)\| = 0,$$

$$\lim_{t\to\infty} \|\psi_t(ab) - \psi_t(a)\psi_t(b)\| = 0, \quad \text{and} \quad \lim_{t\to\infty} \|\psi_t(a^*) - \psi_t(a)^*\| = 0.$$

Let $\psi = (\psi_t)_{t\in[0,\infty)}: A \to B$ be an asymptotic morphism. We say that $\psi$ is *completely positive* (respectively, *unital*, or *contractive*), if there exists $t_0 \in [0,\infty)$ such that $\psi_t$ is completely positive (respectively, unital, or contractive) for all $t \geq t_0$.

It is clear that a unital, completely positive asymptotic morphism is contractive.

**Remark IX.8.2.** $E$-theory was introduced by Connes and Higson in [31], using a suitable equivalence between asymptotic morphisms between $C^*$-algebras. Despite coinciding when the first variable is nuclear, $E$-theory and $KK$-theory do not in general agree. Even more, there are $C^*$-algebras that satisfy the UCT in $E$-theory, but do not satisfy the UCT (in $KK$-theory); see [252] (we are thankful to Rasmus Bentmann for providing this reference).

On the other hand, Theorem 4.2 in [130] asserts that if one only considers *completely positive contractive* asymptotic morphisms, and carries out the construction used to define $E$-theory, the object one obtains is canonically isomorphic to $KK$-theory. We will use this fact in Corollary IX.8.4.

It should be pointed out that the arguments in this section can be simplified if one is only interested in *nuclear* $C^*$-algebras, since in this case $E$-theory agrees with $KK$-theory. In fact, in [257], Szábo has provided a shorter proof of the implication (1) $\Rightarrow$ (2) of Theorem IX.8.8 under the additional assumption that the algebra $A$ be nuclear. (The extra condition is needed to deduce the UCT from the $E$-theoretic version of the UCT.)

Our approach, despite being more technical, requires only minimal assumptions. Moreover, some of the arguments are needed elsewhere.

Given a real number $t$, we denote

$$\lfloor t \rfloor = \max\{n \in \mathbb{Z}: n \leq t\}.$$

**Theorem IX.8.3.** Let $A$ be a unital separable $C^*$-algebra and let $\alpha: \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property. Denote by $\iota: A^\alpha \to A$ the canonical inclusion. Then there exists a unital completely positive asymptotic morphism $\psi = (\psi_t)_{t\in[0,\infty)}: A \to A^\alpha$ satisfying

$$\lim_{t\to\infty} \|(\psi_t \circ \iota)(a) - a\| = 0$$

for all $a$ in $A^\alpha$.

*Proof.* Let $(F_n)_{n\in\mathbb{N}}$ be an increasing family of compact subsets of $A$ such that $\bigcup\limits_{n\in\mathbb{N}} F_n$ is dense in $A$. Upon replacing $F_n$ with $\bigcup\limits_{\zeta\in\mathbb{T}} \alpha_\zeta(F_n)$, we may assume that $F_n$ is invariant under $\alpha$ for all $n$ in $\mathbb{N}$. Fix $t$ in $[0,\infty)$, and choose $\delta_t > 0$ such that whenever $\zeta_1$ and $\zeta_2$ in $\mathbb{T}$ satisfy $|\zeta_1 - \zeta_2| < \delta_t$, then

$$\|\alpha_{\zeta_1}(a) - \alpha_{\zeta_2}(a)\| < \frac{1}{2t}$$

for all $a$ in $F_{\lfloor t\rfloor}$. Let $N_t$ be a positive integer such that $N_t > \frac{2}{\delta_t}$ and let $f^{(t)}$ be a continuous function on $\mathbb{T}$ whose support is contained in the interval of radius $\frac{1}{N_t}$ around 1 and such that

- $0 \leq f^{(t)} \leq 1$;
- With $\zeta_j^{(t)} = e^{\frac{2\pi i j}{N_t}}$ for $j = 0, \ldots, N_t - 1$, set $f_j^{(t)} = \mathtt{Lt}_{\zeta_j^{(t)}}(f^{(t)})$. Then the family $\left\{f_j^{(t)}\right\}_{j=0}^{N_t-1}$ is a partition of unity on $\mathbb{T}$.

We assume further that the graph of the function $f^{(t)}$ is a symmetric triangle whose base is centered at 1. This assumption is not strictly necessary, but it is made to give an explicit description of the relevant homotopy below.

Use Proposition IX.7.7 to find a continuous path $(u_s)_{s\in[1,\infty)}$ of unitaries in $A$ such that

- $\alpha_\zeta(u_s) = \zeta u_s$ for all $\zeta$ in $\mathbb{T}$ and all $s$ in $[0,\infty)$;
- $\lim\limits_{s\to\infty} \|u_s a - a u_s\| = 0$ for all $a$ in $A$.

Fix $t$ in $[1,\infty)$, and choose $s_t$ in $[0,\infty)$ satisfying

$$\left\|f^{(t)}(u_{s_t})a - af^{(t)}(u_{s_t})\right\| < \frac{1}{t} \quad \text{and} \quad \left\|(f^{(t)})^{\frac{1}{2}}(u_{s_t})a - a(f^{(t)})^{\frac{1}{2}}(u_{s_t})\right\| < \frac{1}{t}$$

for all $a$ in $F_{\lfloor t\rfloor}$. Abbreviate $u_{s_t}$ to $u^{(t)} = u_{s_t}$.

Denote by $\mu$ the normalized Lebesgue measure on $\mathbb{T}$, and by $E\colon A \to A^\alpha$ the standard conditional expectation, which is given by

$$E(a) = \int_{\mathbb{T}} \alpha_\zeta(a)\; d\mu(\zeta)$$

for all $a$ in $A$. Define a unital completelyi positive linear map $\sigma_t \colon A \to A^\alpha$ by

$$\sigma_t(a) = E\left(\sum_{j=0}^{N_t-1} f_j^{(t)}(u^{(t)})^{\frac{1}{2}} \alpha_{\zeta_j^{(t)}}(a) f_j^{(t)}(u^{(t)})^{\frac{1}{2}}\right)$$

for all $a$ in $A$. By Theorem IX.3.3, we have

$$\|\sigma_t(ab) - \sigma_t(a)\sigma_t(b)\| < \frac{1}{t}(\|a\| + \|b\| + 3) \tag{IX.3}$$

for all $a$ and $b$ in $F_{\lfloor t \rfloor}$.

Note that for all $\lambda$ in $\mathbb{C}$ and for all $a$ and $b$ in $A$, we have

$$\sigma_t(\lambda a + b) = \lambda\sigma_t(a) + \sigma_t(b) \quad \text{and} \quad \sigma_t(a^*) = \sigma_t(a)^*$$

for all $t$ in $[0, \infty)$, and

$$\lim_{t\to\infty} \|\sigma_t(ab) - \sigma_t(a)\sigma_t(b)\| = 0.$$

Hence, condition (2) in Definition IX.8.1 is satisfied. However, condition (1) is not in general satisfied for the family $(\sigma_t)_{t\in[0,\infty)}$, so we cannot conclude that $\sigma$ is an asymptotic homomorphism.

The strategy will be to "keep" $\sigma_t$ for integer $t$, which will be denoted by $\psi_n$ for $n$ in $\mathbb{N}$, and connect these by taking homotopies of the corresponding partitions of unity $(f_j^{(n)})_{j=0}^{N_n-1}$ in such a way that the multiplicativity of the intermediate averages is controlled by the multiplicativity of the averages at the endpoints.

We make this argument rigorous as follows. Note that for each $n$ in $\mathbb{N}$, if one replaces $N_n$ by a larger integer and takes another partition of unity as above for the larger integer, then the resulting positive linear map satisfies the inequality in (Equation IX.3) for all $a$ in $F_{\lfloor t \rfloor}$, by Theorem IX.3.3. We may therefore assume, for simplicity of the argument, that $N_n$ is a power of two, and that $N_n$ divides $N_{n+1}$ for every $n$ in $\mathbb{N}$.

Fix $n$ in $\mathbb{N}$. We will construct a homotopy between the linear maps $\psi_n$ and $\psi_{n+1}$. Since $N_{n+1}/N_n$ is a multiple of 2, we may assume, without loss of generality, that $N_{n+1} = 2N_n$. In the general case, if $N_{n+1}/N_n = 2^k$, then one divides the interval $[0, 1]$ in $k - 1$ intervals, and performs $k$ homotopies of the same kind as the one we will perform below.

The partition of unity of $\mathbb{T}$ corresponding to $\psi_{n+1}$ has $2N_n$ functions, so the idea will be to construct a homotopy that "splits", in a controlled way, each of the $N_n$ functions appearing in the formula for $\psi_n$, into two of the functions that appear in the formula of $\psi_{n+1}$. To be more precise, for $j = 0, \ldots, 2N_n - 1$, set $\zeta_j = e^{\frac{2\pi i j}{N_n}}$, and for $k = 0, \ldots, 2N_n - 1$, set $\zeta'_k = e^{\frac{2\pi i j}{2N_n}}$. (We will not include $n$ explicitly in the notation for the circle elements $\zeta_j$ and $\zeta'_k$ because $n$ is fixed.) With

$$\left\{f_0^{(n)}, f_1^{(n)}, \ldots, f_{N_n-1}^{(n)}\right\} \quad \text{and} \quad \left\{f_0^{(n+1)}, f_1^{(n+1)}, \ldots, f_{2N_n-1}^{(n+1)}\right\}$$

denoting the partitions of unity corresponding to $\psi_n$ and $\psi_{n+1}$, respectively, we will construct "controlled" homotopies

$$f_0^{(n)}(u^{(n)})\alpha_{\zeta_0}(a) \sim \left(f_0^{(n+1)}(u^{(n+1)})\alpha_{\zeta'_0}(a) + f_1^{(n+1)}(u^{(n+1)})\alpha_{\zeta'_1}(a)\right),$$

and similarly with the other functions appearing in the formula for $\psi_n$. What we mean by "controlled" is that the resulting path $t \mapsto \psi_t$, for $t$ in $[1, \infty)$, will be an asymptotic morphism.

It is enough to find homotopies

$$\begin{aligned}
&\frac{1}{2} f_0^{(n)}(u^{(n)})\alpha_{\zeta_0}(a) \sim_h f_0^{(n+1)}(u^{(n+1)})\alpha_{\zeta'_0}(a) \ \text{ and} \\
&\frac{1}{2} f_0^{(n)}(u^{(n)})\alpha_{\zeta_0}(a) \sim_h f_1^{(n+1)}(u^{(n+1)})\alpha_{\zeta'_1}(a),
\end{aligned}$$

since the other ones will be obtained by translating appropriately. The assumption that the graphs of the functions $f^{(n)}$ and $f^{(n+1)}$ are symmetric triangles centered at 1 implies that the identity $f^{(n+1)}(\zeta) = f^{(n)}(\zeta^2)$ holds for all $\zeta$ in $\mathbb{T}$. We define a homotopy $H\colon [0,1] \times \mathbb{T} \to \mathbb{T}$ with $H(0, \zeta) = \frac{1}{2} f^{(n)}(\zeta)$ and $H(1, \zeta) = f^{(n+1)}(\zeta)$ for $\zeta$ in $\mathbb{T}$, by

$$H(s, \zeta) = \frac{s+1}{2} f^{(n)}(\zeta^{s+1})$$

for $s$ in $[0,1]$ and $\zeta$ in $\mathbb{T}$. Let $G\colon [0,1] \times \mathbb{T} \to \mathbb{T}$ satisfying $G(0, \zeta) = \frac{1}{2} f^{(n)}(\zeta)$ and $G(1, \zeta) = f_1^{(n+1)}(\zeta)$ for $\zeta$ in $\mathbb{T}$, be constructed analogously (for fixed $s$, the function $\zeta \mapsto G(s, \zeta)$ will be an appropriate translate of $\zeta \mapsto H(s, \zeta)$). Now, for $t$ in $[n, n+1]$, denote by $H_t\colon \mathbb{T} \to \mathbb{T}$ the function given by $H_t(\zeta) = H(t - n, \zeta)$ for $\zeta$ in $\mathbb{T}$, and similarly for $G_t$. For $j = 0, \ldots, 2N_n - 1$, denote by $\zeta_j^{(t)}$ the center of the support of $\mathtt{Lt}_{\zeta_j} G_t$, and note that $t \mapsto \zeta_j^{(t)}$ is a path from $\zeta_j$ to $\zeta'_{2j+1}$. For $t$ in

$[n, n+1]$ and $a$ in $A$, define

$$\psi_t(a) = E\left(\sum_{j=0}^{2N_n-1} (\mathtt{Lt}_{\zeta_j} H_t)(u^{(t)})^{\frac{1}{2}} \alpha_{\zeta_j}(a) (\mathtt{Lt}_{\zeta_j} H_t)(u^{(t)})^{\frac{1}{2}} \right.$$
$$\left. + \sum_{j=0}^{2N_n-1} (\mathtt{Lt}_{\zeta_j} G_t)(u^{(t)})^{\frac{1}{2}} \alpha_{\zeta_j^{(t)}}(a) (\mathtt{Lt}_{\zeta_j} G_t)(u^{(t)})^{\frac{1}{2}} \right).$$

It is clear that each $\psi_t$ is unital and completely positive.

We claim that the family $\psi = (\psi_t)_{t \in [1,\infty)}$ is an asymptotic homomorphism in the sense of Definition IX.8.1.

Fix $t$ in $[n, n+1]$. To check approximate multiplicativity, we will verify the hypotheses of Theorem IX.3.3. Note that the family

$$\left\{\mathtt{Lt}_{\zeta_j} H_t \colon j = 0, \ldots, 2N_n - 1\right\} \cup \left\{\mathtt{Lt}_{\zeta_j} G_t \colon j = 0, \ldots, 2N_n - 1\right\}$$

is a partition of unity of $\mathbb{T}$. Moreover, $(\mathtt{Lt}_{\zeta_j} H_t)(\zeta) \neq 0$ implies $|\zeta - \zeta_j| < \frac{2}{\delta_n}$, and similarly $(\mathtt{Lt}_{\zeta_j} G_t)(\zeta) \neq 0$ implies $|\zeta - \zeta_j^{(t)}| < \frac{2}{\delta_n}$. It follows from the choices of $\delta_n$ and $u^{(t)}$, and from Theorem IX.3.3, that

$$\|\psi_t(ab) - \psi_t(a)\psi_t(b)\| < \frac{1}{n}(\|a\| + \|b\| + 3)$$

for all $a$ and $b$ in $F_n$. In particular,

$$\lim_{t\to\infty} \|\psi_t(ab) - \psi_t(a)\psi_t(b)\| = 0$$

for all $a$ and $b$ in $A$. We conclude that $\psi = (\psi_t)_{t \in [0,\infty)}$ is an asymptotic homomorphism.

It remains to check that $\psi$ is a left inverse of the canonical inclusion of $A^\alpha$ into $A$. Note first that

$$\lim_{t\to\infty} \|au^{(t)} - u^{(t)}a\| = 0$$

for all $a$ in $A$. Hence, the difference in norm between $\psi_t(a)$ and

$$E\left(\sum_{j=0}^{2N_n-1} (\mathtt{Lt}_{\zeta_j} H_t)(u^{(t)}) \alpha_{\zeta_j}(a) + \sum_{j=0}^{2N_n-1} (\mathtt{Lt}_{\zeta_j} G_t)(u^{(t)}) \alpha_{\zeta_j^{(t)}}(a)\right)$$

is negligible as $t$ becomes arbitrarily large, for any $a$ in $A$. Since the expresion above equals $a$ whenever $a$ is fixed by $\alpha$, it follows that

$$\lim_{t\to\infty} \|(\psi \circ \iota)(a) - a\| = 0$$

for $a$ in $A^\alpha$, which concludes the proof. $\square$

**Corollary IX.8.4.** Let $A$ be a unital separable $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property. Let $B$ be any separable $C^*$-algebra, and denote by $\iota^*\colon KK(A,B) \to KK(A^\alpha, B)$ the group homomorphism induced by the canonical inclusion $\iota\colon A^\alpha \to A$. Then there exists $\psi^*\colon KK(A^\alpha, B) \to KK(A,B)$ such that $\iota^* \circ \psi^* = \mathrm{id}_{KK(A^\alpha,B)}$. In particular,

$$KK(A,B) \cong KK(A^\alpha, B) \oplus \ker(\psi^*).$$

*Proof.* Recall (see Theorem 4.2 in [130]) that given separable $C^*$-algebras $A$ and $B$, the $KK$-group $KK(A,B)$ is canonically isomorphic to the group of homotopy classes of completely positive asymptotic morphisms $SA \to SB \otimes \mathcal{K}$. The unital completely positive asymptotic morphism $A \to A^\alpha$ constructed in Theorem IX.8.3 induces a group homomorphism $\psi^*\colon KK(A^\alpha, B) \to KK(A,B)$ which satisfies

$$\iota^* \circ \psi^* = \mathrm{id}_{KK(A^\alpha,B)},$$

since $\psi \circ \iota$ is in fact asymptotically equal to the identity on $A^\alpha$ (not just homotopic). This proves the first claim. The existence of an isomorphism $KK(A,B) \cong KK(A^\alpha, B) \oplus \ker(\psi^*)$ is a standard fact in group theory. $\square$

Using these results, we can show that the ismorphisms $K_0(A) \cong K_1(A) \cong K_0(A^\alpha) \oplus K_1(A^\alpha)$, which were shown to exist when $\alpha$ has the Rokhlin property and either $K_0(A)$ or $K_1(A)$ is finitely generated in Theorem IX.4.3, exist in full generality if $\alpha$ is assumed to have the continuous Rokhlin property.

**Corollary IX.8.5.** Let $A$ be a unital separable $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property.

1. There is an isomorphism $\varphi\colon K_0(A^\alpha) \oplus K_1(A^\alpha) \cong K_0(A)$ such that
$$\varphi([1_{A^\alpha}], 0) = [1_A].$$
2. There is an isomorphism $K_0(A) \cong K_1(A)$.

*Proof.* (1). By taking $\mathbb{C}$ as the first coordinate and $A$ as the second in the conclusion of Corollary IX.8.4, we deduce that $K_0(A) \cong K_0(A^\alpha) \oplus \ker(\psi^*)$, where $\psi^*\colon K_0(A) \to K_0(A^\alpha)$ is the group homomorphism induced by the asymptotic morphism $\psi\colon A \to A^\alpha$ given by Theorem IX.8.3. Moreover, the canonical inclusion $A^\alpha \to A$ induces the above splitting of $K_0(A)$.

Consider the Pimsner-Voiculescu exact sequence for $\check{\alpha}\colon \mathbb{Z} \to \mathrm{Aut}(A^\alpha)$:

$$\begin{tikzcd}
K_0(A^\alpha) \arrow[r, "1-K_0(\check{\alpha})"] & K_0(A^\alpha) \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(A^\alpha) \arrow[l] & K_1(A^\alpha). \arrow[l, "1-K_1(\check{\alpha})"]
\end{tikzcd}$$

Since $\check{\alpha}$ acts trivially on $K$-theory, the above sequence splits into two short exact sequences

$$0 \to K_j(A^\alpha) \to K_j(A) \to K_{1-j}(A^\alpha) \to 0$$

for $j = 0, 1$. Since $\psi^*\colon K_0(A) \to K_0(A^\alpha)$ is a splitting for the map on $K_0$, and hence it follows that $K_0(A)/K_0(A^\alpha)$ is isomorphic to $K_1(A^\alpha)$. This shows that there is an isomorphism $K_0(A^\alpha) \oplus K_1(A^\alpha) \cong K_0(A)$, which clearly maps $([1_{A^\alpha}], 0)$ to $[1_A]$.

(2). An analogous argument, taking suspensions, shows that the inclusion $A^\alpha \to A$ induces a direct sum decomposition $K_1(A) \cong K_1(A^\alpha) \oplus K_0(A^\alpha)$. Using the first part of this corollary, we conclude that $K_0(A) \cong K_1(A)$. □

**Remark IX.8.6.** The argument used in the proof of Corollary IX.8.5 can be modified in a straightforward manner to show something slightly stronger: under the assumptions there, the $C^*$-algebra $A$ is $KK$-equivalent to $A \oplus SA$. Since we do not need this, and for the sake of brevity, we do not present the proof here. (The nuclear case follows from Theorem 3.1 in [257].)

We can also show that in the presence of the continuous Rokhlin property, the UCT for the underlying algebra is equivalent to the UCT for the fixed point algebra. We begin by defining what exactly it means for a $C^*$-algebra to "satisfy the UCT".

**Definition IX.8.7.** Let $A$ and $B$ be separable $C^*$-algebras. We say that the pair $(A, B)$ *satisfies the UCT* if the following conditions are satisfied:

1. The natural map $\tau_{A,B}\colon KK(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$ defined by Kasparov in [145], is surjective.
2. The natural map $\mu_{A,B}\colon \ker(\tau_{A,B}) \to \mathrm{Ext}(K_*(A), K_{*+1}(B))$ is an isomorphism.

If this is the case, by setting $\varepsilon_{A,B} = \mu_{A,B}^{-1}\colon \mathrm{Ext}(K_*(A), K_{*+1}(B)) \to KK(A,B)$, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \xrightarrow{\varepsilon_{A,B}} KK(A,B) \xrightarrow{\tau_{A,B}} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0,$$

which is natural on both variables because so are $\tau_{A,B}$ and $\mu_{A,B}$.

We further say that $A$ *satisfies the UCT*, if $(A, B)$ satisfies the UCT for every separable $C^*$-algebra $B$.

**Theorem IX.8.8.** Let $A$ be a unital separable $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property. Then the following are equivalent

1. $A$ satisfies the UCT;
2. The crossed product $A \rtimes_\alpha \mathbb{T}$ satisfies the UCT;
3. The fixed point algebra $A^\alpha$ satisfied the UCT.

Note that, unlike in Corollary 3.9 in [191], we do not assume the algebra $A$ to be nuclear.

*Proof.* The equivalence between assertions (2) and (3) follows from the fact that $A \rtimes_\alpha \mathbb{T} \cong A^\alpha \otimes \mathcal{K}(L^2(\mathbb{T}))$ by Corollary IX.2.7. That (3) implies (1) follows from the fact that $A^\alpha \rtimes_{\check{\alpha}} \mathbb{Z} \cong A$ by Theorem IX.2.6.

We will prove that (1) implies (3), so assume that $A$ satisfies the UCT. Let $B$ be a separable $C^*$-algebra. Since $A$ satisfies the UCT, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}(K_*(A), K_{*+1}(B)) \xrightarrow{\varepsilon_{A,B}} KK(A,B) \xrightarrow{\tau_{A,B}} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0,$$

which is natural on both variables. Denote by $\iota\colon A^\alpha \to A$ the canonical inclusion. Use Theorem IX.8.3 to choose a unital completely positive asymptotic morphism $\psi = (\psi_t)_{t\in[1,\infty)}\colon A \to A^\alpha$ with

$$\lim_{t\to\infty} \|(\psi_t \circ \iota)(a) - a\| = 0$$

for all $a$ in $A^\alpha$. Then $\psi$ induces group homomorphisms

$$\mathrm{Ext}(K_*(A^\alpha), K_{*+1}(B)) \to \mathrm{Ext}(K_*(A), K_{*+1}(B))$$

$$KK(A^\alpha, B) \to KK(A,B)$$

$$\mathrm{Hom}(K_*(A^\alpha), K_*(B) \to \mathrm{Hom}(K_*(A), K_*(B),$$

which we will all denote by $\psi^*$, that are right inverses of the canonical homomorphisms induced by $\iota$ (which we will all denote by $\iota^*$).

The diagrams

$$\begin{tikzcd}
\mathrm{Ext}(K_*(A), K_{*+1}(B)) \arrow[d, bend left, "\iota^*"] & \ker(\tau_{A,B}) \arrow[l, "\mu_{A,B}"'] \arrow[d, bend left, "\iota^*"] \\
\mathrm{Ext}(K_*(A^\alpha), K_{*+1}(B)) \arrow[u, bend left, "\psi^*"] & \ker(\tau_{A^\alpha,B}) \arrow[l, "\mu_{A^\alpha,B}"] \arrow[u, bend left, "\psi^*"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
KK(A,B) \arrow[r, "\tau_{A,B}"] \arrow[d, bend left, "\iota^*"] & \mathrm{Hom}(K_*(A), K_*(B)) \arrow[d, bend left, "\iota^*"] \\
KK(A^\alpha, B) \arrow[r, "\tau_{A^\alpha,B}"] \arrow[u, bend left, "\psi^*"] & \mathrm{Hom}(K_*(A^\alpha), K_*(B)) \arrow[u, bend left, "\psi^*"]
\end{tikzcd}$$

are easily seen to be commutative, using naturality of all the horizontal maps involved.

We claim that $\mu_{A^\alpha,B}$ is an isomorphism. Since

$$\psi^* \circ \mu_{A^\alpha,B} = \mu_{A,B} \circ \psi^*$$

and $\psi^*, \mu_{A,B}$ and $\psi^*$ are injective, it follows that $\mu_{A^\alpha,B}$ is injective. Surjectivity follows similarly from the identity

$$\mu_{A^\alpha,B} \circ \iota^* = \iota^* \circ \mu_{A,B}$$

and the fact that $\iota^*, \mu_{A,B}$ and $\iota^*$ are surjective. The claim is proved.

We now claim that $\tau_{A^\alpha,B}$ is surjective. Given $x$ in $\mathrm{Hom}(K_*(A^\alpha), K_*(B))$, use surjectivity of $\tau_{A,B}$ to choose $y$ in $KK(A,B)$ such that $\tau_{A,B}(y) = \psi^*(x)$. Then

$$(\tau_{A^\alpha,B} \circ \iota^*)(y) = (\iota^* \circ \tau_{A,B})(y) = x,$$

showing that $\tau_{A^\alpha,B}$ is surjective. This proves the claim.

We conclude that $(A,B)$ satisfies the UCT. Since $B$ is arbitrary, it follows that $A$ satisfies the UCT. $\square$

**Remark IX.8.9.** Adopt the notation of the theorem above. It is clear that the same argument, verbatim, shows that if $A$ satisfies the $E$-theoretic version of the UCT, then so do $A^\alpha$ and $A \rtimes_\alpha \mathbb{T}$.

*More general compact groups*

In this subsection, we give some indication of how to generalize Theorem IX.8.8 to actions of more general compact groups with the continuous Rokhlin property, with focus on finite groups. We begin by defining the latter in a way which is convenient for our purposes.

**Definition IX.8.10.** Let $G$ be a second countable compact group, let $A$ be a separable unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. We say that $\alpha$ has the *continuous Rokhlin property* if there exists a unital asymptotic morphism

$$\varphi = (\varphi_t)_{t\in[0,\infty)} \colon C(G) \to A$$

such that

1. $\lim\limits_{t\to\infty}\sup\limits_{g\in G}\|\varphi_t(\mathtt{Lt}_g(f))-\alpha_g(\varphi_t(f))\|=0$ for all $f\in C(G)$.

2. $\lim\limits_{t\to\infty}\|\varphi_t(f)a-a\varphi_t(a)\|=0$ for all $a\in A$.

The techniques used in the first part of this section can be adapted to deal with arbitrary second countable compact groups:

**Theorem IX.8.11.** Let $G$ be a second countable compact group, let $A$ be a separable $C^*$-algebra, and let $\alpha\colon G\to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property. If $A$ satisfies the UCT (or its $E$-theoretic analog), then so do $A^\alpha$ and $A$.

Again, note that we do not assume the algebra $A$ to be nuclear in the theorem above. We point out that Szabo's argument in [257] also works for metrizable compact groups, and Theorem 2.5 in [257] provides an alternative proof for the $E$-theory part of our Theorem IX.8.11.)

The proof Theorem IX.8.11 is more technical than that of Theorem IX.8.8, but the argument is identical. Since we do not have any immediate application for it, we omit the proof. For finite groups, however, the proof of Theorem IX.8.11 takes a much simpler form, which we proceed to sketch.

Assume that $\alpha\colon G\to \mathrm{Aut}(A)$ is an action of a finite group $G$ on a separable, unital $C^*$-algebra $A$ with the continuous Rokhlin property. Use Definition IX.8.10, and equivariant semiprojectivity of $(C(G),\mathtt{Lt})$ (see [205]), to choose continuous paths $t\mapsto e_g^{(t)}$ of projections in $A$, for $g\in G$, satisfying

1. $\alpha_g(e_h^{(t)})=e_{gh}^{(t)}$ for all $g,h\in G$ and all $t\in[0,\infty)$;

2. $\lim\limits_{t\to\infty}\left\|e_g^{(t)}a-ae_g^{(t)}\right\|=0$ for all $a\in A$;

3. $\sum\limits_{g\in G}e_g^{(t)}=1$ for all $t\in[0,\infty)$.

Fix $t\in[0,\infty)$, and consider the linear map $\psi_t\colon A\to A^\alpha$ given by

$$\psi_t(a)=\sum_{g\in G}e_g^{(t)}\alpha_g(a)e_g^{(t)}$$

for $a\in A$. It is easy to check that the range of $\psi_t$ is really contained in $A^\alpha$, and that $\psi_t$ is unital and completely positive. It is also readily verified that $(\psi_t)_{t\in[0,\infty)}$ is an asymptotic morphism $A\to A^\alpha$, and that

$$\lim_{t\to\infty}\|\psi_t(a)-a\|=0$$

for all $a \in A^\alpha$. This proves the analog of Theorem IX.8.3 in the case of a finite group, and the proof of Theorem IX.8.11 follows the same argument as that of Theorem IX.8.8. We omit the details.

### Existence and Uniqueness Results for Circle Actions

In the following theorem, we use equivariant $K$-theory as the invariant. See Subsection 2.2.

**Theorem IX.9.1.** Let $A$ and $B$ be unital Kirchberg algebras, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ and $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be actions with the continuous Rokhlin property.

1. The actions $\alpha$ and $\beta$ are conjugate if and only if $A^\alpha$ and $B^\beta$ are $KK$-equivalent.
2. Assume that $A$ and $B$ satisfy the UCT. Then the actions $\alpha$ and $\beta$ are conjugate if and only if $\alpha$ and $\beta$ have isomorphic equivariant $K$-theory, that is, if and only if there is a $\mathbb{Z}_2$-graded $R(\mathbb{T})$-module (with distinguished element) isomorphism

$$(K_0^\alpha(A), [1_A], K_1^\alpha(A)) \cong (K_0^\beta(B), [1_B], K_1^\beta(B)).$$

*Proof.* (1). It is immediate to verify that if $\alpha$ and $\beta$ are conjugate via an isomorphism $\theta\colon A \to B$, then $\theta$ restricts to an isomorphism between $A^\alpha$ and $B^\beta$, and hence these algebras are $KK$-equivalent.

Conversely, assume that $A^\alpha$ and $B^\beta$ are $KK$-equivalent. Use Theorem 4.2.1 in [200] to choose an isomorphism $\phi\colon A^\alpha \to B^\beta$ implementing the equivalence. Denote by $\check{\alpha}$ and $\check{\beta}$ the predual automorphisms of $\alpha$ and $\beta$, respectively, given by Theorem IX.2.6. It follows from Proposition IX.7.11 that $\check{\alpha}$ and $\check{\beta}$ represent the trivial $KK$-elements on $A^\alpha$ and $B^\beta$ respectively. Thus, $\phi \circ \check{\alpha} \circ \phi^{-1}$ and $\check{\beta}$ determine the same $KK$-class on $B^\beta$. Now, $A^\alpha$ and $B^\beta$ are Kirchberg algebras and $\check{\alpha}$ and $\check{\beta}$ are aperiodic by Proposition IX.5.2, so it follows from Theorem 5 in [187] that $\phi \circ \check{\alpha} \circ \phi^{-1}$ and $\check{\beta}$ are cocycle conjugate. In other words, $\check{\alpha}$ and $\check{\beta}$ are exterior equivalent, and thus $\alpha$ and $\beta$ are conjugate by Proposition II.3.6.

(2). It is immediate to verify that if $\alpha$ and $\beta$ are conjugate, then their equivariant $K$-theories are isomorphic as $R(\mathbb{T})$-modules.

Conversely, suppose that there is a $\mathbb{Z}_2$-graded $R(\mathbb{T})$-module isomorphism

$$\psi\colon (K_0^\alpha(A), [1_A], K_1^\alpha(A)) \to (K_0^\beta(B), [1_B], K_1^\beta(B)).$$

By Julg's Theorem, there is a natural group isomorphism $K_*^\alpha(A) \cong K_*(A \rtimes_\alpha \mathbb{T})$. In addition to this, there is a natural isomorphism $A \rtimes_\alpha \mathbb{T} \cong A^\alpha \otimes \mathcal{K}(L^2(\mathbb{T}))$ by Corollary IX.2.7. It follows that there is a natural group isomorphism $K_*^\alpha(A) \cong K_*(A^\alpha)$, that is, the equivariant $K$-theory for $\alpha$ and the $K$-theory of its fixed point algebra agree. It is clear that this isomorphism maps $[1_A]$ in $K_0^\alpha(A)$ to $[1_{A^\alpha}]$ in $K_0(A^\alpha)$. Similarly, $K_*^\beta(B) \cong K_*(B^\beta)$ via an isomorphism that sends $[1_B]$ in $K_0^\beta(B)$ to $[1_{B^\beta}]$ in $K_0(B^\beta)$. Since $A^\alpha$ and $B^\beta$ satisfy the UCT by Theorem IX.8.8, it follows from Kirchberg-Phillips classification theorem (see, for example, Theorem 4.2.4 in [200]), that $A^\alpha$ and $B^\beta$ are isomorphic. The result now follows from part (1) of this theorem. This finishes the proof. □

**Remark IX.9.2.** It should be noted that the $R(\mathbb{T})$-module structure of the equivariant $K$-theory was not used in the proof of part (2) of Theorem IX.9.1. In fact, this module structure is trivial. Indeed, $R(\mathbb{T})$ is isomorphic to $\mathbb{Z}[x, x^{-1}]$, where the action of $x$ on $K_*^\alpha(A) \cong K_*(A \rtimes_\alpha \mathbb{T})$ is given by $K_*(\widehat{\alpha})$. The automorphism $\widehat{\alpha}$ is approximately inner by Theorem VI.4.2, and hence it is trivial on $K$-theory.

Theorem IX.9.1 may be regarded as a *uniqueness* theorem. It states that whenever two circle actions with the continuous Rokhlin property on a unital Kirchberg algebra that satisfies the UCT, are conjugate whenever they have the same equivariant $K$-theory. It is natural to look for existence results, that is, try to answer the following question. What pairs of triples

$$((G_0, g_0, G_1), (H_0, h_0, H_1))$$

where $G_0$ and $G_1$ are abelian groups, $H_0$ and $H_1$ are $R(\mathbb{T})$-modules, and $g_0 \in G_0$ and $h_0 \in H_0$ are distinguished elements, arise as the $K$-theory and equivariant $K$-theory of an action of the circle on a unital Kirchberg algebra with the continuous Rokhlin property?

We address this existence question in the remainder of the present section, and show that the only possible restrictions are the ones that were already discovered in Corollary IX.8.5 and Remark IX.9.2.

**Theorem IX.9.3.** Let $A$ be unital Kirchberg algebra that satisfies the UCT. A triple $(H_0, h_0, H_1)$ consisting of $R(\mathbb{T})$-modules $H_0$ and $H_1$, together with a distinguished element $h_0 \in H_0$, is the equivariant $K$-theory of a circle action on $A$ with the continuous Rokhlin property, if and only if the following conditions hold:

1. The $R(\mathbb{T})$-module structures on $H_0$ and $H_1$ are trivial,
2. There is an isomorphism $K_0(A) \cong K_1(A)$, and
3. There exists a group isomorphism $\varphi\colon H_0 \oplus H_1 \to K_0(A)$ such that $\varphi(h_0, 0) = [1_A]$.

*Proof.* Necessity of condition (1) follows from Remark IX.9.2, and necessity of conditions (2) and (3) follows from Corollary IX.8.5.

Conversely, assume that $K_0(A) \cong K_1(A)$, and suppose that $(H_0, h_0, H_1)$ satisfies $H_0 \oplus H_1 \cong K_0(A)$ via an isomorphism that sends $(h_0, 0)$ to $[1_A]$. Use Theorem 4.2.5 in [200] to choose a unital Kirchberg algebra $B$ satisfying the UCT such that

$$(K_0(B), [1_B], K_1(B)) \cong (H_0, h_0, H_1).$$

Let $\theta\colon B \otimes \mathcal{O}_\infty \to B$ be an isomorphism, and let $\psi \in \mathrm{Aut}(\mathcal{O}_\infty)$ be the automorphism constructed in Theorem IX.7.17. Define an automorphism $\varphi$ of $B$ by $\varphi = \phi \circ (\mathrm{id}_B \otimes \psi) \circ \phi^{-1}$, and note that $\varphi$ is asymptotically representable. The crossed product

$$B \rtimes_\varphi \mathbb{Z} \cong (\mathcal{O}_\infty \rtimes_\psi \mathbb{Z}) \otimes B$$

is a unital Kirchberg algebra satisfying the UCT, and the Künneth formula together with Remark IX.7.18 yield

$$K_0(B \rtimes_\varphi \mathbb{Z}) \cong K_1(B \rtimes_\varphi \mathbb{Z}) \cong K_0(B) \oplus K_1(B) \cong H_0 \oplus H_1,$$

in such a way that the class of the unit in $K_0(B \rtimes_\varphi \mathbb{Z})$ is sent to $(h_0, 0) \in H_0 \oplus H_1$. It follows from the classification of Kirchberg algebras satisfying the UCT that there exists an isomorphism $\gamma\colon B \rtimes_\varphi \mathbb{Z} \to A$.

Denote by $\beta\colon \mathbb{T} \to \mathrm{Aut}(B \rtimes_\varphi \mathbb{Z})$ the dual action of $\varphi$. It follows from Proposition IX.7.11 that $\beta$ has the continuous Rokhlin property. Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be given by

$$\alpha_\zeta = \gamma \circ \beta_\zeta \circ \gamma^{-1}$$

for $\zeta$ in $\mathbb{T}$. Then $\alpha$ has the continuous Rokhlin property as well. Moreover,

$$\begin{aligned}(K_0^\alpha(A), [1_A], K_1^\alpha(A)) &\cong (K_0(A^\alpha), [1_{A^\alpha}], K_1(A^\alpha)) \\ &\cong (K_0(B), [1_B], K_1(B)) \\ &\cong (H_0, h_0, H_1).\end{aligned}$$

Therefore $\alpha$ is the desired action on $A$ with the continuous Rokhlin property, and the proof is complete. □

As a simple application, we show how we can use Theorem IX.9.3 to compute the number of conjugacy classes of circle actions with the continuous Rokhlin property that a given Kirchberg algebra has.

**Example IX.9.4.** Let $A$ be a unital Kirchberg algebra satisfying the UCT, with $K$-theory given by

$$K_0(A) \cong K_1(A) \cong \mathbb{Z} \oplus \mathbb{Z}_6,$$

such that $[1_A]$ corresponds to $(1,0)$ in $K_0(A)$. We will compute how many conjugacy classes of circle actions with the continuous Rokhlin property on $A$ there are. By Theorem IX.9.3, conjugacy classes are in bijection with direct sum decompositions of the form $\mathbb{Z} \oplus \mathbb{Z}_6 \cong H_0 \oplus H_1$ that satisfy $(1,0) \mapsto (h_0, 0)$ for some $h_0$ in $H_0$. There are only 4 such direct sum decompositions, namely:

$$\mathbb{Z} \oplus \mathbb{Z}_6 \cong (\mathbb{Z} \oplus \mathbb{Z}_6) \oplus \{0\} \cong (\mathbb{Z} \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_3 \cong (\mathbb{Z} \oplus \mathbb{Z}_3) \oplus \mathbb{Z}_2.$$

(The direct sum decompositions $\{0\} \oplus (\mathbb{Z} \oplus \mathbb{Z}_6)$, $\mathbb{Z}_2 \oplus (\mathbb{Z} \oplus \mathbb{Z}_3)$, $\mathbb{Z}_3 \oplus (\mathbb{Z} \oplus \mathbb{Z}_2)$ and $\mathbb{Z}_6 \oplus \mathbb{Z}$ do not satisfy condition (3) in Theorem IX.9.3.) We conclude that there are exactly 4 conjugacy classes.

It will be shown in Corollary IX.10.2 that the continuous Rokhlin property agrees with the Rokhlin property for circle actions on Kirchberg algebras whose $K$-theory is finitely generated.

In particular, in the example above we can omit the word "continuous" everywhere, and the conclusion is that there are exactly 4 conjugacy classes of circle actions with the Rokhlin property on the algebra considered.

We now give a complete answer to the question stated before Theorem IX.9.3.

**Corollary IX.9.5.** Any unital Kirchberg algebra (not necessarily satisfying the UCT) arises as the fixed point algebra of a circle action with the continuous Rokhlin property on some other Kirchberg algebra. In particular, a pair of triples

$$((G_0, g_0, G_1), (H_0, h_0, H_1))$$

where $G_0$ and $G_1$ are abelian groups, $H_0$ and $H_1$ are $R(\mathbb{T})$-modules, and $g_0 \in G_0$ and $h_0 \in H_0$ are distinguished elements, arises as the $K$-theory and equivariant $K$-theory of an action of the circle on a unital Kirchberg algebra with the continuous Rokhlin property if and only if the $R(\mathbb{T})$-module structures on $H_0$ and $H_1$ are trivial, and there are isomorphisms $H_0 \oplus H_1 \cong G_0 \cong G_1$, the first one of which maps $(h_0, 0)$ to $g_0$.

*Comments on (non-)existence of model actions.*

As an application of Theorem IX.9.3, we explain why no obvious generalization of Theorem 3.4 in [133] is possible for circle actions with the Rokhlin property.

It is already clear from the family of examples constructed in Example VI.2.8 that, even within the class of unital Kirchberg algebras, there is *no* $C^*$-algebra $D$ with a circle action $\delta\colon \mathbb{T} \to \mathrm{Aut}(D)$ that has the Rokhlin property, and such that whenever $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action on a unital Kirchberg algebra $A$ with the Rokhlin property (or even the continuous Rokhlin property), then $(A, \alpha) \cong (A \otimes D, \mathrm{id}_A \otimes \delta)$. (This is the straightforward and also quite naive generalization of Theorem 3.4 in [133], since arbitrary actions of the circle are always trivial on $K$-theory by Proposition XI.3.6.) To see this, construct simple unital A$\mathbb{T}$-algebras $A_1$ and $A_2$ with Rokhlin actions of the circle as in Example VI.2.8, using the $2^\infty$ and $3^\infty$ UHF-patterns instead of the rational UHF-pattern. Tensor these algebras with $\mathcal{O}_\infty$, and take the trivial circle action on $\mathcal{O}_\infty$, to obtain two unital Kirchberg algebras $B_1$ and $B_2$ such that $K_0(B_1) \cong K_1(B_1) \cong \mathbb{Z}\left[\frac{1}{2}\right]$ and $K_0(B_2) \cong K_1(B_2) \cong \mathbb{Z}\left[\frac{1}{3}\right]$. $K$-theoretical considerations show that the only possible $K$-groups of a unital $C^*$-algebra $D$ that is absorbed both by $B_1$ and $B_2$, are either $(\mathbb{Z}, 0)$ or $(0, \mathbb{Z})$. However, none of these groups arises as the $K$-groups of a unital $C^*$-algebra that admits a circle action with the Rokhlin property by Theorem IX.4.3.

The conclusion is that there is no "absorbing", or model action, for circle actions with the Rokhlin property. Nevertheless, with a weaker notion of "model action", an analogous result for circle actions on Kirchberg algebras does in fact hold, at least for actions with the continuous Rokhlin property. Denote by $\psi$ the automorphism of $\mathcal{O}_\infty$ constructed in Theorem IX.7.17, and set $D = \mathcal{O}_\infty \rtimes_\psi \mathbb{Z}$. Denote by $\delta\colon \mathbb{T} \to \mathrm{Aut}(D)$ the dual action of $\psi$. Then $\delta$ has the continuous Rokhlin property. It is clear that $\delta$ will not be absorbed by an arbitrary action with the (continuous) Rokhlin property on a Kirchberg algebra, for example, if the algebra has $K$-theory $(\mathbb{Z}_2, \mathbb{Z}_2)$. However, $\delta$ is a *generating* action, in the sense of the following proposition.

**Proposition IX.9.6.** Adopt the notation of the comments above. Let $A$ be a unital Kirchberg algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the continuous Rokhlin property. Then there is an equivariant isomorphism

$$\theta\colon (A, \alpha) \to (A^\alpha \otimes D, \mathrm{id}_{A^\alpha} \otimes \delta).$$

*Proof.* It is easy to check, using that $D^\delta = \mathcal{O}_\infty$, that the fixed point algebra of $A^\alpha \otimes D$ is isomorphic to $A^\alpha$. The result follows from part (1) of Theorem IX.9.1. □

We point out that we did not need to assume that the algebra $A$ in the proposition above satisfies the UCT, unlike in Theorem 3.4 in [133].

### Comparison Between the Rokhlin Property and the Continuous Rokhlin Property

The goal of this section is to show that for unital Kirchberg algebras with finitely generated $K$-theory, the Rokhlin property and the continuous Rokhlin property are equivalent; see Corollary IX.10.2. A similar result is proved for the class of commutative unital $C^*$-algebras; see Proposition IX.10.5. We also show in Example IX.10.3 and Example IX.10.4, that the two notions are not in general equivalent, even on Kirchberg algebras satisfying the UCT.

In the next result, we characterize those circle actions with the Rokhlin property on Kirchberg algebras that have the continuous Rokhlin property. We point out that no UCT assumptions are needed.

**Theorem IX.10.1.** Let $A$ be a unital Kirchberg $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Denote by $\check{\alpha}$ the predual automorphism of $\alpha$. Then $\alpha$ has the continuous Rokhlin property if and only if $KK(\check{\alpha}) = 1$.

*Proof.* If $\alpha$ has the continuous Rokhlin property, then $\check{\alpha}$ is asymptotically representable by Proposition IX.7.11. Hence it is asymptotically inner, and $KK(\check{\alpha}) = 1$.

Conversely, assume that $KK(\check{\alpha}) = 1$. Since $\check{\alpha}$ is aperiodic by Proposition IX.5.2, it follows from Proposition IX.7.20 that it is asymptotically representable. $\square$

We recall the construction of the PExt-group. Given abelian groups $G_1$ and $G_2$, the group $\mathrm{PExt}(G_2, G_1)$ is the subgroup of $\mathrm{Ext}(G_2, G_1)$ consisting of the pure extensions of $G_1$ by $G_2$ (Definition IX.4.1). See [248] for more about the PExt-group. We refer the reader to Example 8.4.14 in [235] for the definition of the $KL$-class of an automorphism.

**Corollary IX.10.2.** Let $A$ be a unital Kirchberg $C^*$-algebra and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Assume that $\mathrm{PExt}(K_*(A^\alpha), K_{*+1}(A^\alpha)) = 0$. Then $\alpha$ has the continuous Rokhlin property. In particular, if $A$ has finitely generated $K$-theory, then every circle action on $A$ with the Rokhlin property has the continuous Rokhlin property.

*Proof.* Since $\mathrm{PExt}(K_*(A^\alpha), K_{*+1}(A^\alpha)) = 0$, it follows that an automorphism of $A^\alpha$ is $KK$-trivial if and only if it is $KL$-trivial. Let $\check{\alpha}$ be the predual automorphism of $\alpha$. Then $\check{\alpha}$ is approximately representable by Theorem IX.2.6, and in particular $KL$-trivial. The first part of the corollary then follows from Theorem IX.10.1 above.

If the $K$-groups of $A$ are finitely generated, then the condition

$$\mathrm{PExt}(K_*(A^\alpha), K_{*+1}(A^\alpha)) = 0$$

is automatically satisfied, since the $K$-groups of $A^\alpha$ are also finitely generated by Theorem VI.3.3. This finishes the proof. $\square$

In the corollary above, the condition $\mathrm{PExt}(K_*(A^\alpha), K_{*+1}(A^\alpha)) = 0$ will also be satisfied if the $K$-groups of $A$ are (possibly infinite) direct sums of cyclic groups. However, it is in general unclear whether one can replace said condition with

$$\mathrm{PExt}(K_*(A), K_{*+1}(A)) = 0,$$

which is significantly easier to check in practice. The problem is that $K_*(A^\alpha)$ is not in general a direct summand in $K_*(A)$, just a subgroup.

As promised after Definition IX.7.1, we will exhibit an example of a circle action that has the Rokhlin property but not the continuous Rokhlin property, showing that these two notions are not equivalent in general. This can happen even on Kirchberg algebras that satisfy the UCT (although their $K$-theory must be infinitely generated, by Corollary IX.10.2).

We need to introduce some notation first. Let $A$ be a unital $C^*$-algebra and let $\varphi$ be an approximately inner automorphism of $A$. With $\iota\colon A \to A \rtimes_\varphi \mathbb{Z}$ denoting the canonical inclusion, the Pimsner-Voiculescu exact sequence for $\varphi$ reduces to the short exact sequences

$$0 \longrightarrow K_j(A) \xrightarrow{K_j(\iota)} K_j(A \rtimes_\varphi \mathbb{Z}) \longrightarrow K_{1-j}(A) \longrightarrow 0 \quad j = 0, 1.$$

We denote the class of the above extensions by $\eta_j(\varphi)$ for $j = 0, 1$, and by

$$\eta\colon \overline{\mathrm{Inn}}(A) \to \mathrm{Ext}(K_1(A), K_0(A)) \oplus \mathrm{Ext}(K_0(A), K_1(A))$$

the map $\eta(\varphi) = (\eta_0(\varphi), \eta_1(\varphi))$ for $\varphi \in \overline{\mathrm{Inn}}(A)$. It is well known that $\eta$ is a group homomorphism when $A$ satisfies the UCT (the operation on $\overline{\mathrm{Inn}}(A)$ is composition), but we shall not make use of this fact here.

**Example IX.10.3.** Let $G_1 = \mathbb{Z}\left[\frac{1}{2}\right]$, which we will regard as the abelian group generated by elements $y_n$ with $n$ in $\mathbb{N}$, subject to the relations

$$2y_{n+1} = y_n$$

for all $n$ in $\mathbb{N}$. It is clear that $G_1$ is torsion free. Let $G_0 = \mathbb{Z}$, and let $E$ be the abelian group generated by the set $\{x, y_n : n \in \mathbb{N}\}$, subject to the relations

$$2y_{n+1} = y_n + x$$

for all $n$ in $\mathbb{N}$. There is an extension

$$0 \to G_0 \to E \to G_1 \to 0,$$

where the map $G_0 \to E$ is determined by $1 \mapsto x$, and the map $E \to G_1$ is the corresponding quotient map. It was shown in Example IX.4.2 that this extension is pure but not trivial (that is, it does not split). (We warn the reader that the notation we are using here differs slightly from the one used in Example IX.4.2.) Denote by $\xi \in \mathrm{Ext}(G_1, G_0)$ the extension class determined by $E$, and note that $\xi \neq 0$.

Use Elliott's classification of A$\mathbb{T}$-algebras (see [58]), or the comments before Proposition 3.2.7 in [235]) to find a simple, unital A$\mathbb{T}$-algebra $A$ with real rank zero, such that $K_j(A) \cong G_j$ for $j = 0, 1$. Use Theorem 3.1 in [159] in the case $i = 1$ to find an approximately inner automorphism $\varphi$ of $A$ such that $\eta(\varphi) = (0, \xi)$. The proof of Theorem 3.1 in [159] is constructive, and the case $i = 1$ (which is presented in Subsection 3.11 in [159]) shows that for $n$ in $\mathbb{N}$, there are a circle algebra $A_n$, an embedding $\psi_n : A_n \to A_{n+1}$ and a unitary $u_n$ in $A_n$ such that

$$\mathrm{Ad}(u_{n+1}) \circ \psi_n = \psi_n \circ \mathrm{Ad}(u_n)$$

and $\varinjlim \mathrm{Ad}(u_n) = \varphi$. It is immediate to check that such a direct limit action is approximately representable in the sense of Definition IX.2.5.

Denote by $\alpha : \mathbb{T} \to \mathrm{Aut}(A \rtimes_\varphi \mathbb{Z})$ the dual action of $\varphi$. Then $\alpha$ has the Rokhlin property by Theorem VI.4.2. On the other hand, since $\eta(\varphi)$ is not the trivial class, we conclude that $\varphi$ is not asymptotically inner (let alone asymptotically representable), and hence $\alpha$ does not have the continuous Rokhlin property.

The example above can be adapted to construct a circle action on a Kirchberg algebra satisfying the UCT that has the Rokhlin property but not the continuous Rokhlin property.

**Example IX.10.4.** Adopt the notation of the previous example. The automorphism $\varphi$ of $A$ obtained there is easily seen to be aperiodic, so the crossed product $A \rtimes_\varphi \mathbb{Z}$ is simple by Theorem 3.1 in [156]. Set $B = A \otimes \mathcal{O}_\infty$ and $\phi = \varphi \otimes \mathrm{id}_{\mathcal{O}_\infty}$, which is an automorphism of $B$. Moreover, $K_*(A) \cong K_*(B)$ by the Künneth formula, and clearly $\eta(\phi) = \eta(\varphi)$. With $\beta \colon \mathbb{T} \to \mathrm{Aut}(B \rtimes_\phi \mathbb{Z})$ denoting the dual action of $\phi$, the same argument used in Example IX.10.3 shows that $\beta$ has the Rokhlin property and does not have the continuous Rokhlin property. Finally, note that $B \rtimes_\phi \mathbb{Z} \cong (A \rtimes_\varphi \mathbb{Z}) \otimes \mathcal{O}_\infty$ is a Kirchberg algebra, and it satisfies the UCT because $A$ does, since crossed products by $\mathbb{Z}$ preserve the UCT.

M. Izumi has found (see [134]) examples of $\mathbb{Z}_2$-actions on $\mathcal{O}_\infty$ that are approximately representable (see Definition 3.6 in [132]) but not asymptotically representable (in the obvious sense for actions of $\mathbb{Z}_2$). The crossed products by the actions he constructed are Kirchberg algebras satisfying the UCT, so by taking the dual actions of his examples, one obtains actions of $\mathbb{Z}_2$ on Kirchberg algebras satisfying the UCT that have the Rokhlin property but not the continuous Rokhlin property. We point out that in all these examples, the Kirchberg algebra in question has infinitely generated $K$-theory, and he also shows that his method cannot produce similar examples with finitely generated $K$-groups. It seems plausible, then, that a result similar to Corollary IX.10.2 holds for finite abelian group actions as well, and possibly even more generally. We have, nevertheless, not explored this direction any further.

We conclude this work by proving that the Rokhlin property agrees with the continuous Rokhlin property on commutative $C^*$-algebras.

**Proposition IX.10.5.** Let $A$ be a commutative unital $C^*$-algebra and let $\alpha \colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $\alpha$ has the continuous Rokhlin property.

*Proof.* By Theorem VI.4.9, there is an equivariant isomorphism $A \cong A^\alpha \otimes C(\mathbb{T})$, where $\mathbb{T}$ acts trivially on $A^\alpha$ and via $\mathtt{Lt}$ on $C(\mathbb{T})$. Since $\mathtt{Lt}$ has the continuous Rokhlin property by Example IX.7.5, the result follows from Proposition IX.7.3. $\square$

CHAPTER X

# AUTOMATIC TOTAL DISCONNECTEDNESS FOR GROUPS ACTING WITH THE ROKHLIN PROPERTY

We study compact group actions with the Rokhlin property on $C^*$-algebras with exactly one vanishing $K$-group. One of our main results is that the group must be totally disconnected, which can be regarded as the noncommutative counterpart of the fact that if a compact group acts freely on a totally disconnected metric compact space, then the group itself must be totally disconnected. Along the way, we develop further properties of arbitrary compact group actions with the Rokhlin property. The main tools are the interactions between subgroups or quotients of the acting group with the $K$-theory of the algebra.

Generalizing techniques from the finite group case, we prove classification results for equivariant homomorphisms between Rokhlin dynamical systems with totally disconnected groups. As an application, we classify compact group actions with the Rokhlin property on AF-algebras and on certain direct limits of one-dimensional noncommutative CW-complexes.

Given a totally disconnected compact group $G$, we construct a model action of $G$ on a certain UHF-algebra naturally associated with $G$. This action is shown to have the Rokhlin property, and it is moreover proved that it tensorially generates (and is tensorially absorbed by) all actions of $G$ with the Rokhlin property on a class of UHF-absorbing $C^*$-algebras.

## Introduction

The Rokhlin property for finite groups, formally defined by Izumi in [132], has been extensively studied by a number of authors; see, for example, [133], [191], [194], [91], [188], and [243]. The Rokhlin property is rare, but there exist interesting examples of finite group actions with the Rokhlin property. For example, Izumi constructed ([132]) nontrivial Rokhlin actions of cyclic groups on Cuntz algebras, and Phillips and Viola ([214]) used the Rokhlin property of a certain $\mathbb{Z}_3$-action to construct a separable $C^*$-algebra not isomorphic to its opposite.

Hirshberg and Winter extended in [122] Izumi's definition to actions of compact groups, where they began a study of the structure of their crossed products. Examples of non-finite compact group actions with the Rokhlin property were constructed in [81] and [85], but it would be desirable to have more examples. Some obstructions were obtained in [79] and [80] for circle actions, and in [83] for Lie groups.

In this work, we show that any $C^*$-algebra with exactly one vanishing $K$-group does not admit an action with the Rokhlin property of a compact group which is not totally disconnected. This class of $C^*$-algebras contains all Cuntz algebras on at least three generators, as well as all AF- and AI-algebras. The motivation for this result was the fact, proved in [79], that the circle action on a UHF-algebra $\bigotimes\limits_{k=1}^{\infty} M_{k_n}$, given by

$$\zeta \mapsto \mathrm{Ad}\left(\mathrm{diag}\left(1, e^{\frac{2\pi i}{n_k}}, \ldots, e^{\frac{2\pi i(n_k-1)}{n_k}}\right)\right),$$

does not have the Rokhlin property (even though its restrictions to cyclic groups usually do).

We provide the necessary generalizations of Izumi's techniques that allow us to classify equivariant homomorphisms between totally disconnected group actions with the Rokhlin property. For every such group $G$, we construct a model action $\mu^G$ on a UHF-algebra $D_G$ whose type is naturally associated to $G$, and prove that this action generates all Rokhlin actions of $G$ on algebras that absorb $D_G$. We also establish conditions under which absorption of $D_G$ is automatic.

This chapter, which is based on [84], is organized as follows. In Section X.2, we show that the Rokhlin property is preserved under passing to a subgroup in a special case (Lemma X.2.1), and that it is always preserved under taking the induced action of the quotient group on the corresponding fixed point algebra (Proposition X.2.2). In Section X.3, we prove our main result, Theorem X.3.3, which asserts that any $C^*$-algebra with exactly one vanishing $K$-group does not admit an action with the Rokhlin property of a compact group which is not totally disconnected. We devote the rest of that section to constructing a model action $\mu^G$ of a totally disconnected group $G$ on a UHF-algebra $D_G$. (Uniqueness of $\mu^G$ will not be proved until Section X.5.) Section X.4 contains our classification result (Theorem X.4.7) for Rokhlin actions of totally disconnected groups, using techniques from [132] and [91]. Finally, in Section X.5 we prove that the action $\mu^G$ constructed in Section X.3 is unique up to equivariant isomorphism (Proposition X.5.8), and that it tensorially generates all Rokhlin actions of a given totally disconnected group on certain stably finite $C^*$-algebras with trivial $K_1$-groups.

## Subgroups and Quotient Groups Acting with The Rokhlin Property

In this section, we complement the results in Chapter VI about compact groups actions with the Rokhlin property.

While the restriction of an action with the Rokhlin property rarely has the Rokhlin property, this is true in some special cases; see Proposition VI.2.4. The next observation will be used in the proof of Theorem X.3.3 in the following section.

**Lemma X.2.1.** Let $(G_n, \pi_n)_{n\in\mathbb{N}}$ be an inverse limit of compact groups with quotient maps $\pi_n\colon G_n \to G_{n+1}$ for $n \in \mathbb{N}$. Denote its inverse limit by $G = \varprojlim (G_n, \pi_n)_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$, let $H_n$ be a subgroup of $G_n$ satisfying $\pi_n(H_n) \subseteq H_{n+1}$. Assume moreover that $H_n$ is a direct summand in $G_n$. Set $H = \varprojlim (H_n, \pi_n|_{H_n})_{n\in\mathbb{N}}$, which is a closed subgroup of $G$.

Let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $\alpha|_H\colon H \to \mathrm{Aut}(A)$ has the Rokhlin property.

*Proof.* We will construct an $H$-equivariant unital homomorphism $\rho\colon C(H) \to C(G)$. Assume we have accomplished this, and let $\varphi\colon C(G) \to A_{\infty,\alpha} \cap A'$ be a unital $G$-equivariant homomorphism as in Definition VI.2.1 for $\alpha$. Then $\varphi \circ \rho\colon C(H) \to A_{\infty,\alpha} \cap A'$ is a a unital $H$-equivariant homomorphism, showing that $\alpha|_H$ has the Rokhlin property.

For $n \in \mathbb{N}$, denote by $\psi_n\colon C(G_n) \to C(G_{n+1})$ the unital homomorphism induced by $\pi_n$; denote by $\phi_n\colon C(H_n) \to C(H_{n+1})$ the unital homomorphism induced by $\pi_n|_{H_n}$; and denote by $\rho_n\colon C(H_n) \to C(G_n)$ the unital homomorphism induced by the canonical quotient map $G_n \to H_n$. Since $H$ is the inverse limit of $H_n$ with the restricted quotient maps, it follows that the diagram

$$\begin{tikzcd}
C(G_1) \arrow[r, "\psi_1"] & \cdots \arrow[r] & C(G_n) \arrow[r, "\psi_n"] & \cdots \arrow[r] & C(G) \\
C(H_1) \arrow[u, "\rho_1"] \arrow[r, "\phi_1"'] & \cdots \arrow[r] & C(H_n) \arrow[u, "\rho_n"] \arrow[r, "\phi_n"'] & \cdots \arrow[r] & C(H)
\end{tikzcd}$$

is commutative. By the universal property of the inductive limit in the category of $C^*$-algebras, there exists a unital homomorphism $\rho\colon C(H) \to C(G)$. It is straightforward to check that $\rho$ is $H$-equivariant, and the result follows. □

On the other hand, the Rokhlin property is preserved under taking the induced action of a quotient group on the corresponding fixed point algebra, as the next proposition shows.

**Proposition X.2.2.** Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $H$ be a normal closed subgroup of $G$. Then the induced action $\overline{\alpha}\colon G/H \to \mathrm{Aut}(A^H)$ has the Rokhlin property.

*Proof.* Let $\varphi\colon C(G) \to A_{\infty,\alpha} \cap A'$ be a unital equivariant homomorphism as in the definition of the Rokhlin property for $\alpha$. Let $\phi\colon C(G/H) \to C(G)$ be the $H$-equivariant unital homomorphism associated with the quotient map $G \to G/H$. Since $\phi \circ \varphi$ is $H$-equivariant, its image is contained in the $H$-fixed point algebra $(A_{\infty,\alpha} \cap A')^H$. We claim that

$$(A_{\infty,\alpha} \cap A')^H = (A^H)_{\infty,\alpha} \cap A'.$$

It is immediate that $(A_{\infty,\alpha} \cap A')^H = (A_{\infty,\alpha})^H \cap A'$. It therefore suffices to check that $(A_{\infty,\alpha})^H = (A^H)_{\infty,\alpha}$. The inclusion of the right-hand side in the left-hand side is immediate. Conversely, let $a = \kappa_A((a_n)_{n\in\mathbb{N}})$ be an element of $(A_{\infty,\alpha})^H$. For every $n \in \mathbb{N}$, let

$$b_n = \int_H \alpha_h(a_n)dh,$$

which is an element in $A^H$. Thus $\kappa_A((b_n)_{n\in\mathbb{N}})$ belongs to $(A^H)_\infty$. By compactness of $H$, for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|\alpha_h(a_n) - a_n\| < \varepsilon$$

for all $h \in H$ and all $n \geq n_0$. Denote by $dh$ the normalized Haar measure on $H$. Moreover,

$$\|b_n - a_n\| \leq \int_H \|\alpha_h(a_n) - a_n\| dh \leq \varepsilon,$$

and hence $\kappa_A((b_n)_{n\in\mathbb{N}}) = a$ in $A_\infty$, showing that $a \in (A^H)_\infty$. It remains to check that

$$g \mapsto (\alpha_\infty)_g((b_n)_{n\in\mathbb{N}})$$

is continuous as a map $G \to A_\infty$. This is immediate, since

$$(\alpha_\infty)_g((b_n)_{n\in\mathbb{N}}) = (\alpha_\infty)_g((a_n)_{n\in\mathbb{N}})$$

for all $g$ in $G$, and $(a_n)_{n\in\mathbb{N}}$ belongs to $A_{\infty,\alpha}$. This proves the claim.

Denote by $\overline{\alpha}\colon G/H \to \mathrm{Aut}(A^H)$ the induced action. We then get a diagram

$$\begin{tikzcd}
C(G) \arrow[r, "\varphi"] & A_{\infty,\alpha} \cap A' & \\
C(G/H) \arrow[u, "\phi"] \arrow[r, dashed] & (A^H)_{\infty,\alpha} \cap A' \arrow[u] \arrow[r] & (A^H)_{\infty,\overline{\alpha}} \cap (A^H)'.
\end{tikzcd}$$

The map $C(G/H) \to (A^H)_{\infty,\overline{\alpha}} \cap (A^H)'$ is easily seen to be a unital, $G/H$-equivariant homomorphism, thus showing that $\overline{\alpha}$ has the Rokhlin property. $\square$

The proof of the lemma above shows something even stronger, namely the following. If $\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property and $H$ is a normal closed subgroup, then there is an equivariant unital embedding $C(G/H) \to (A^H)_{\infty,\overline{\alpha}} \cap A'$. In particular, the asymptotical embedding of $C(G/H)$ into $A^H$ can be chosen to approximately commute with arbitrary finite subsets of $A$, rather than finite subsets of just $A^H$. The following lemma is a convenient formulation of this fact incorporating the passage to a finite subgroup of the quotient. It will be crucial in the proof of Lemma X.4.2, which is rather technical.

**Lemma X.2.3.** Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $H$ be a normal subgroup of $G$ with finite index (and hence automatically closed). Denote by $\overline{\alpha}\colon G/H \to \mathrm{Aut}(A^H)$ the induced action. Let $K$ be any subgroup of $G/H$. Then for any $\varepsilon > 0$ and any finite subset $F \subseteq A$, there exist orthogonal projections $e_k$ in $A^H$, for $k$ in $K$, such that

1. $\|\overline{\alpha}_k(e_h) - e_{kh}\| < \varepsilon$ for $k, h$ in $K$,
2. $\|e_k a - a e_k\| < \varepsilon$ for all $k$ in $K$ and all $a$ in $F$, and
3. $\sum\limits_{k\in K} e_k = 1$.

*Proof.* It follows from (the proof of) Proposition X.2.2 (see also the comments above), that there is an equivariant unital embedding

$$C(G/H) \to (A^H)_{\infty,\overline{\alpha}} \cap A'.$$

Set $n = |G/H|$. The existence of the above embedding implies the existence of projections $p_{\overline{g}}$ in $A^H$, for $\overline{g}$ in $G/H$, such that

- $\|\overline{\alpha}_{\overline{h}}(p_{\overline{h}}) - p_{\overline{gh}}\| < \frac{\varepsilon}{n}$ for $\overline{g}, \overline{h}$ in $G/H$,
- $\|p_{\overline{g}}a - ap_{\overline{g}}\| < \frac{\varepsilon}{n}$ for all $\overline{g}$ in $G/H$ and all $a$ in $F$, and
- $\sum\limits_{\overline{g}\in G/H} p_{\overline{g}} = 1$.

Choose a set $R$ of right coset representatives of $K$ in $G/H$. For $k$ in $K$, set $e_k = \sum\limits_{r\in R} p_{rk}$. It is immediate that the projections $e_k$ for $k$ in $K$, satisfying conditions (1) and (3) in the statement of the lemma (with $\frac{\varepsilon}{n}$ in place of $\varepsilon$). For the second one, we have

$$\|e_k a - a e_k\| \leq \sum_{r\in R} \|p_{rk}a - ap_{rk}\| \leq \varepsilon.$$

This finishes the proof. □

Next, we recall results from [80] and [81] concerning circle actions with the Rokhlin property, and combine them in a form that is convenient for our purposes.

**Theorem X.2.4.** Let $A$ be a unital $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Let $j \in \{0,1\}$. If $H$ is a finitely generated subgroup of $K_j(A)$, then there exists an injective group homomorphism

$$\psi\colon H \to K_{1-j}(A).$$

In particular, every finitely generated subgroup of $K_j(A)$ is isomorphic to a finitely generated subgroup of $K_{1-j}(A)$.

Moreover, this homomorphism is natural in the following sense. Let $\beta\colon \mathbb{T} \to \mathrm{Aut}(B)$ be another circle action on a unital $C^*$-algebra $B$, let $s \in \mathbb{N}$, and let $\iota\colon A \to B$ be a unital homomorphism satisfying

$$\beta_\zeta(\iota(a)) = \iota(\alpha_{\zeta^s}(a))$$

for all $\zeta \in \mathbb{T}$ and for all $a \in A$. Suppose that given $\varepsilon > 0$ and a finite subset $F \subseteq B$, there exists a unitary $u \in \mathcal{U}(A)$ such that

1. $\|\beta_\zeta(\iota(u)) - \zeta\iota(u)\| < \varepsilon$ for all $\zeta \in \mathbb{T}$; and

2. $\|\iota(u)b - b\iota(u)\| < \varepsilon$ for all $b \in F$.

(This, in particular, implies that $\beta$ has the Rokhlin property.)

Let $j \in \{0, 1\}$ and let $H$ be a finitely generated subgroup of $K_j(A)$. Then there exist injective group homomorphisms

$$\psi_A \colon H \to K_{1-j}(A) \quad \text{and} \quad \psi_B \colon K_j(\iota)(H) \to K_{1-j}(B)$$

making the following diagram commute:

$$\begin{tikzcd} H \arrow[r, "K_0(\iota)"] \arrow[d, "\psi_A"'] & K_0(\iota)(H) \arrow[d, "\psi_B"] \\ K_1(A) \arrow[r, "K_1(\iota)"'] & K_1(B). \end{tikzcd}$$

*Proof.* Without loss of generality, we may assume that $j = 0$. The first part is what is actually proved in part (2) of Theorem 5.2 in [80]. We review its proof since it will be needed to prove the second claim. We adopt the notation used in Theorem 5.2 of [80]. Let $h_1, \ldots, h_n$ be generators of $H$. Without loss of generality, we can assume that there exist projections $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ in $A$ (rather than in matrices over $A$) such that $h_j = [p_j] - [q_j]$ for $j = 1, \ldots, n$. Denote by $u \in A$ any unitary as in the definition of the Rokhlin property, for an appropriately chosen finite subset (and tolerance). For a partition of unity $(f_k)_{k=1}^m$ in $C(\mathbb{T})$ with sufficiently small supports, and group elements $g_k \in \operatorname{supp}(f_k)$, we define a completely positive contractive map $\sigma \colon A \to A^\alpha$ by

$$\sigma(a) = E\left(\sum_{k=1}^{m} f_k(u)^{\frac{1}{2}} \alpha_{g_k}(a) f_k(u)^{\frac{1}{2}}\right)$$

for $a \in A$. If the finite set and the tolerances are chosen appropriately, then one can show that $\sigma$ induces a local splitting $\psi \colon H \to K_0(A)$.

Naturality can be proved in a straightforward manner, by choosing a partition of unity $(f_k)_{k=1}^m$, and group elements in their supports, such that both $f_k$ and the function $\zeta \mapsto f_k(\zeta^s)$ has sufficiently small support. We omit the details. $\square$

We review here the definition of discrete $K$-theory for a compact group action on a $C^*$-algebra. (We will use equivariant $K$-theory as in Chapter 2 of [199].) This notion will be crucial in the proof of Theorem X.3.3.

**Definition X.2.5.** If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of a compact group $G$ on a $C^*$-algebra $A$, we say that $\alpha$ has *discrete $K$-theory* if there exists $n \in \mathbb{N}$ such that $I_G^n \cdot K_*^G(A) = 0$.

When $G$ is abelian, discrete $K$-theory can be expressed in a more manageable way using the dual action $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes_\alpha G$. We will only need this in the case when $G$ is the circle; see Lemma X.2.7.

**Notation X.2.6.** If $\beta$ is an automorphism of a $C^*$-algebra $B$, we usually identify it with the integer action it generates, and use the symbol $\beta$ to denote both the automorphism and the action $n \mapsto \beta^n$ of $\mathbb{Z}$ on $B$.

**Lemma X.2.7.** Let $A$ be a unital C*-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action of the circle group $\mathbb{T}$ on $A$. Then $\alpha$ has discrete $K$-theory if and only if there exists $n$ in $\mathbb{N}$ such that

$$(\mathrm{id}_{K_*(A\rtimes_\alpha \mathbb{T})} - K_*(\widehat{\alpha}))^n = 0$$

as a homomorphism $K_*(A \rtimes_\alpha \mathbb{T}) \to K_*(A \rtimes_\alpha \mathbb{T})$.

*Proof.* Recall that $R(\mathbb{T}) \cong \mathbb{Z}[t, t^{-1}]$, and that under this identification, $I_{\mathbb{T}}$ is the ideal generated by $1-t$. Moreover, under the natural identification $K_*^{\mathbb{T}}(A) \cong K_*(A \rtimes_\alpha \mathbb{T})$ given by Julg's isomorphism, the action of $t$ on $K_*(A \rtimes_\alpha \mathbb{T})$ is given by $K_*(\widehat{\alpha})$. It follows that

$$\mathrm{Im}\left((\mathrm{id}_{K_*(A\rtimes_\alpha \mathbb{T})} - K_*(\widehat{\alpha}))^m\right) = I_{\mathbb{T}}^m \cdot K_*(A \rtimes_\alpha \mathbb{T})$$

for every $m \in \mathbb{N}$. For a given $m \in \mathbb{N}$, it follows that $I_{\mathbb{T}}^m \cdot K_*^{\mathbb{T}}(A) = 0$ if and only if $(\mathrm{id}_{K_*(A\rtimes_\alpha \mathbb{T})} - K_*(\widehat{\alpha}))^m = 0$, so the result follows. $\square$

## Totally Disconnected Compact Groups

In this section, we show that if $A$ is a $C^*$-algebra such that exactly one of either $K_0(A)$ or $K_1(A)$ vanishes, and if $G$ is a compact group acting on $A$ with the Rokhlin property, then $G$ must be totally disconnected. See Theorem X.3.3 below. We spend the rest of the section constructing examples of actions of totally disconnected groups with the Rokhlin property on UHF-algebras; see Example X.3.8. These actions will later be shown to be universal in some sense; see Theorem X.5.13.

The following result is well known.

**Theorem X.3.1.** (von Neumann) Let $G$ be a second countable compact group. Then there exists a decreasing sequence $(H_n)_{n\in\mathbb{N}}$ of closed normal subgroups of $G$ such that $G/H_n$ is a Lie group and $\bigcap\limits_{n\in\mathbb{N}} H_n = \{e\}$. In other words, $G$ is an inverse limit of Lie groups.

**Corollary X.3.2.** Let $G$ be a totally disconnected compact group. Then there exists a decreasing sequence $(H_n)_{n\in\mathbb{N}}$ of closed normal subgroups of $G$ such that $G/H_n$ is a finite group and $\bigcap\limits_{n\in\mathbb{N}} H_n = \{e\}$.

*Proof.* It is immediate to check that the quotient of a totally disconnected group by a normal subgroup is again totally disconnected, and that a compact Lie group is totally disconnected if and only if it is finite. This, together with Theorem X.3.1, implies the result. $\square$

The argument in the following theorem would be much simpler if every compact connected group had one-parameter subgroups. Unfortunately, this is not the case: the compact group $G$ obtained as the inverse limit of the inverse system $G \to \cdots \mathbb{T} \to \mathbb{T} \to \cdots$ with stationary maps $\mathbb{T} \to \mathbb{T}$ given by $\zeta \mapsto \zeta^2$, is connected but does not have a subgroup isomorphic to $\mathbb{T}$.

**Theorem X.3.3.** Let $G$ be a compact group, let $A$ be a unital $C^*$-algebra such that exacly one of either $K_0(A)$ or $K_1(A)$ is zero, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then $G$ is totally disconnected.

*Proof.* Assume first that $K_1(A) = 0$, and assume that $G$ is connected. Denote by $\mathcal{Z}_0(G)$ the connected component of the unit in the center of $G$. By Corollary 12.37 in [129], there are a (possibly empty) indexing family $J$, simply connected compact Lie groups $S_j$ for $j \in J$, a central totally disconnected compact subgroup $N$ of $\mathcal{Z}_0(G) \times \prod_{j\in J} S_j$, and an isomorphism

$$G \cong \frac{\mathcal{Z}_0(G) \times \prod_{j\in J} S_j}{N}.$$

Now, assume the family $J$ is non-empty. We claim that $G$ has a closed subgroup isomorphic to $\mathbb{T}$. Indeed, choose $j \in J$. Then $S_j$ has a closed subgroup $H$ isomorphic to $\mathbb{T}$. Then $H/(H \cap N)$ is a closed subgroup of $G$. Since $N$ is zero dimensional and $H \cap N$ is a Lie group (being a closed subgroup of the Lie group $H$), we deduce that $H \cap N$ is finite. Hence, the image of $H$ in $G$ is also isomorphic to $\mathbb{T}$. The claim follows.

Denote by $\beta\colon \mathbb{T} \to \mathrm{Aut}(A)$ the restriction of $\alpha$ to $\mathbb{T}$. Then $\beta$ has finite Rokhlin dimension with commuting towers by Theorem 4.5 in [83] (see also the comments after the theorem). By Corollary 4.6 in [83], $\beta$ has discrete $K$-theory (Definition X.2.5). By Lemma X.2.7, there exists $n \in \mathbb{N}$ such that

$$\ker((\mathrm{id}_{K_*(A\rtimes_\beta \mathbb{T})} - K_*(\widehat{\beta}))^n) = K_*(A \rtimes_\beta \mathbb{T}).$$

Using that $K_1(A) = 0$, it follows from the Pimsner-Voiculescu exact sequence associated to $\beta$,

$$\begin{array}{ccccc} K_0(A \rtimes_\beta \mathbb{T}) & \xrightarrow{1-K_0(\widehat{\beta})} & K_0(A \rtimes_\beta \mathbb{T}) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(A \rtimes_\beta \mathbb{T}) & \xleftarrow[1-K_1(\widehat{\beta})]{} & K_1(A \rtimes_\beta \mathbb{T}), \end{array}$$

that the map $\mathrm{id}_{K_0(A\rtimes_\beta\mathbb{T})} - K_0(\widehat{\beta})$ is injective. It follows that $(\mathrm{id}_{K_0(A\rtimes_\beta\mathbb{T})} - K_0(\widehat{\beta}))^n$ is also injective, and from this we conclude that $K_0(A \rtimes_\beta \mathbb{T}) = 0$. The remaining potentially non-zero terms in the Pismner-Voiculescu exact sequence yield the short exact sequence

$$0 \to K_0(A) \to K_1(A \rtimes_\beta \mathbb{T}) \to K_1(A \rtimes_\beta \mathbb{T}) \to 0,$$

where the last map is $\mathrm{id}_{K_1(A\rtimes_\beta\mathbb{T})} - K_1(\widehat{\beta})$. Since said map is surjective, every power of it is surjective as well, and hence the identity

$$\left(\mathrm{id}_{K_1(A\rtimes_\beta\mathbb{T})} - K_1(\widehat{\beta})\right)^n = 0$$

forces $K_1(A \rtimes_\beta \mathbb{T}) = 0$. In this case, it must be $K_0(A) = 0$ as well, which contradicts the fact that $K_0(A)$ is not zero. The contradiction implies that $J$ must be empty.

If $J$ is empty, then $G$ is abelian. Since any compact abelian connected Lie group is isomorphic to $\mathbb{T}^m$ for some $m \in \mathbb{N}$, it follows that $G$ is an inverse limit of tori. Choose a decreasing sequence $(H_n)_{n\in\mathbb{N}}$ of closed subgroups of $G$ such that $\bigcap\limits_{n\in\mathbb{N}} H_n = \{e\}$ and $G/H_n \cong \mathbb{T}^{m_n}$ for some $m_n \in \mathbb{N}$, for all $n \in \mathbb{N}$. By Lemma X.2.1, we may assume that $m_n = 1$ for all $n \in \mathbb{N}$, so that $G$ is a so-called solenoid. Observe that for $n \in \mathbb{N}$, there exists $k_n \in \mathbb{Z}$ with $k_n \neq 0$, such that the induced group homomorphism

$$G/H_n \cong \mathbb{T} \to G/H_{n+1} \cong \mathbb{T}$$

is given by $\zeta \mapsto \zeta^{k_n}$.

For $n \in \mathbb{N}$, denote by $\iota_n \colon A^{H_n} \to A^{H_{n+1}}$ the canonical inclusion. Then $A$ is isomorphic to the direct limit

$$A^{H_1} \xrightarrow{\iota_1} A^{H_2} \xrightarrow{\iota_2} \cdots \longrightarrow A.$$

Moreover, the induced action of $G/H_n \cong \mathbb{T}$ on $A^{H_n}$ has the Rokhlin property by Proposition X.2.2.

*Claim:* $K_0(A) \cong 0$. Once we have proved the claim, we will have contradicted our assumptions, from which it will follow that $G$ must be trivial (since it was assumed to be connected).

Since $K_0(A)$ is isomorphic to the direct limit $\varinjlim(K_0(A^{H_n}), K_0(\iota_n))_{n\in\mathbb{N}}$, it is enough to show that for $\ell \in \mathbb{N}$ and $x \in K_0(A^{H_\ell})$, there exists $m > \ell$ such that

$$(K_0(\iota_{m-1}) \circ \cdots \circ K_0(\iota_\ell))(x) = 0$$

in $K_0(A^{H_m})$. Let $\ell \in \mathbb{N}$ and $x \in K_0(A^{H_\ell})$. By dropping the first $\ell - 1$ terms in the direct system, we may assume, without loss of generality, that $\ell = 1$. Set $x_1 = x$, and for $n \geq 2$, set

$$x_n = K_0(\iota_{n-1})(x_{n-1}) \in K_0(A^{H_n}).$$

For $n \in \mathbb{N}$, denote by $\langle x_n \rangle$ the subgroup of $K_0(A^{H_n})$ generated by $x_n$, which is obviously finitely generated. By Theorem X.2.4, there exist injective group homomorphisms $\psi_n \colon \langle x_n \rangle \to K_1(A^{H_n})$, for $n \in \mathbb{N}$, satisfying

$$K_1(\iota_n) \circ \psi_n = \psi_{n+1} \circ K_0(\iota_n).$$

Set $y_n = \psi_n(x_n) \in K_1(A^{H_n})$. Since $K_1(A) \cong \varinjlim(K_1(A^{H_n}), K_1(\iota_n))_{n \in \mathbb{N}}$ is the trivial group, there exists $m \in \mathbb{N}$ such that $y_m = 0$. Using commutativity of the diagram

$$\begin{tikzcd}
\langle x_1 \rangle \arrow[r, "K_0(\iota_n)|_{\langle x_1 \rangle}"] \arrow[d, "\psi_1"'] & \langle x_1 \rangle \arrow[d, "\psi_1"'] \\
K_1(A^{H_1}) \arrow[r, "K_1(\iota_n)"'] & K_1(A^{H_m}),
\end{tikzcd}$$

and injectivity of $\psi_m$, we deduce that $x_m = 0$. We conclude that $K_0(A) \cong 0$, as desired.

In the general case, the restriction of $\alpha$ to the connected component of the unit $G_0$ has the Rokhlin property by part (3) of Proposition VI.2.4. The argument above implies that $G_0$ is the trivial group, and thus $G$ is totally disconected.

Assume now that $K_0(A) = 0$. Let $P$ be any unital $C^*$-algebra with $K_0(P) = 0$ and $K_1(P) \cong \mathbb{Z}$. Set $B = A \otimes P$ and let $\beta \colon G \to \mathrm{Aut}(B)$ be the diagonal action, that is, $\beta_g = \alpha_g \otimes \mathrm{id}_P$ for all $g$ in $G$. Then $\beta$ has the Rokhlin property by part (1) of Proposition VI.2.4. Moreover, $K_0(B) \cong K_1(A)$ and $K_1(B) \cong K_0(A)$ by the Künneth formula. It follows from the first case of this proof that $G$ must be totally disconnected. $\square$

As a consequence of the above theorem, we show that no non-trivial compact group acts on the Cuntz algebra $\mathcal{O}_\infty$ or the Jiang-Su algebra $\mathcal{Z}$ with the Rokhlin property. Some time after we proved Theorem X.3.4 below, we learned that Hirshberg and Phillips proved a stronger result under the additional assumption that $G$ be a Lie group: there are no non-trivial compact Lie group actions on $\mathcal{O}_\infty$ or $\mathcal{Z}$ with the $X$-Rokhlin property for *any* free $G$-space $X$. See [120]. Using Theorem 4.5 in [83], we deduce that there are no non-trivial compact Lie group actions on $\mathcal{O}_\infty$ or $\mathcal{Z}$ with finite Rokhlin dimension with commuting towers. Our techniques are, nevertheless, different from those used by Hirshberg and Phillips.

**Theorem X.3.4.** There are no non-trivial compact group actions with the Rokhlin property on either the Cuntz algebra $\mathcal{O}_\infty$ or the Jiang-Su algebra $\mathcal{Z}$.

*Proof.* We claim that it is enough to prove the result for $\mathcal{O}_\infty$. Indeed, there is an isomorphism $\mathcal{O}_\infty \otimes \mathcal{Z} \cong \mathcal{O}_\infty$, and hence if there were a non-trivial compact group $G$ acting on $\mathcal{Z}$ with the Rokhlin property, then by tensoring such action with the trivial action on $\mathcal{O}_\infty$ and using part (1) of Proposition VI.2.4, we would conclude that $G$ also acts on $\mathcal{O}_\infty$ with the Rokhlin property. This proves the claim.

Assume that $G$ is a compact group acting on $\mathcal{O}_\infty$ with the Rokhlin property, and let $\alpha$ be one such action. Since $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) \cong \{0\}$, Theorem X.3.3 above implies that $G$ must be totally disconnected. Let $N$ be a normal subgroup of $G$ such that $G/N$ is finite. Since $G/N$ is finite, the restriction of $\alpha$ to $N$ has the Rokhlin property by part (1) of Proposition VI.2.4. It follows from part (1) in Theorem 3.3 of [85] that the canonical inclusion $\iota\colon \mathcal{O}_\infty^N \to \mathcal{O}_\infty$ induces an injective group homomorphism $K_0(\iota)\colon K_0(\mathcal{O}_\infty^N) \to K_0(\mathcal{O}_\infty)$. Since the unit of $\mathcal{O}_\infty^N$ is the unit of $\mathcal{O}_\infty$, and $[1_{\mathcal{O}_\infty}]$ generates $K_0(\mathcal{O}_\infty)$, it follows that $K_0(\iota)$ is surjective and thus an automorphism. We deduce that $K_0(\mathcal{O}_\infty^N) \cong \mathbb{Z}$, with the class of the unit corresponding to $1 \in \mathbb{Z}$.

Set $H = G/N$ and denote by $\overline{\alpha}\colon H \to \mathrm{Aut}(\mathcal{O}_\infty^N)$ the action induced by $\alpha$. Being unital, the automorphism $\overline{\alpha}_h$ induces the identity map on $K_0(\mathcal{O}_\infty^N)$, for every $h \in H$. Let $\varepsilon = 1$ and $F = \emptyset$. Using the definition of the Rokhlin property for $\overline{\alpha}$, which is a finite group action, choose projections $e_h$ in $\mathcal{O}_\infty^N$ for $h \in H$ such that

1. $\|\overline{\alpha}_h(e_1) - e_h\| < 1$ for all $k, h \in H$.

2. $\sum\limits_{h\in H} e_h = 1$.

One concludes from condition (1) that $\overline{\alpha}_h(e_1)$ is unitarily equivalent (in ${\mathcal{O}_\infty}^N$) to $e_h$ for all $h \in H$, and hence they determine the same element in $K_0(\mathcal{O}_\infty^N)$. Since $K_0(\overline{\alpha}_h) = \mathrm{id}_{K_0(\mathcal{O}_\infty^N)}$, it follows that $[e_h] = [e_1]$ for all $h \in H$. In particular, condition (2) implies that the class of the unit in $K_0(\mathcal{O}_\infty^N) \cong \mathbb{Z}$ is divisible by $|H|$, forcing $H$ to be the trivial group.

It follows that the only finite quotient of $G$ is the trivial group. Since $G$ is the inverse limit of its finite quotients, it follows that $G$ itself is trivial. ☐

**Remark X.3.5.** The above argument works for any unital $C^*$-algebra $A$ with $K_0(A) = \mathbb{Z}$ with $[1_A] = 1$ and $K_1(A) = \{1\}$. (It does not depend on $A$ being strongly self-absorbing, unlike the argument in Theorem 4.6 of [120].)

It is straightforward to show that a compact group admits a free action on a compact, totally disconnected space if and only if it is totally disconnected; this follows from the fact that each orbit is homeomorphic to $G$. The following is its non-commutative analog.

**Corollary X.3.6.** Let $G$ be a compact group. Then $G$ admits an action on a unital AF-algebra with the Rokhlin property if and only if it is totally disconnected.

*Proof.* If $G$ is totally disconnected, then $C(G)$ is a unital AF-algebra and the action of $G$ by left translation has the Rokhlin property. The converse follows from Theorem X.3.3 since AF-algebras have trivial $K_1$-group. ☐

More interesting examples of totally disconnected group actions with the Rokhlin property on AF-algebras (in particular, on simple AF-algebras), were constructed in Example 2.11 in [85]. In general, the AF-algebras constructed there will not be UHF-algebras, even if a UHF-pattern is followed (unless $G$ is the trivial group).

Example X.3.8 provides an example of a totally disconnected group action with the Rokhlin property on a UHF-algebra. It requires some preparation.

**Lemma X.3.7.** Let $A$ be a unital $C^*$-algebra and let $G$ be a locally compact group. Suppose that

(a) $A = \varinjlim(A_n, \iota_n)$ is a direct limit of unital $C^*$-algebras with unital connecting maps $\iota_n \colon A_n \to A_{n+1}$ for $n$ in $\mathbb{N}$,

(b) $G = \varprojlim(G_n, \pi_n)$ is an inverse limit of locally compact groups with quotient maps $\pi_n \colon G_n \to G_{n-1}$ for $n$ in $\mathbb{N}$,

(c) There are continuous actions $\alpha^{(n)}\colon G_n \to \mathrm{Aut}(A_n)$ satifying

$$\alpha_g^{(n+1)} \circ \iota_n = \iota_n \circ \alpha_{\pi_{n+1}(g)}^{(n)}$$

for all $n \in \mathbb{N}$ and all $g \in G_{n+1}$.

Then there is a continuous action $\alpha\colon G \to \mathrm{Aut}(A)$ such that for $g \in G_k$ and $a \in A_n$, one has

$$\alpha_g(a) = \iota_{\infty,m}\left(\alpha_{\pi_{m,k}(g)}^{(m)}(\iota_{m-1,n}(a))\right) \tag{X.1}$$

for any $m \geq k, n$.

*Proof.* We start by checking that for $g \in G_k$ and $a \in A_n$, the expression in Equation X.1 does not depend on $m$ as long as $m \geq n, k$. This follows from the following computation:

$$\begin{aligned}
\iota_{\infty,m+1}\left(\alpha_{\pi_{m,k}(g)}^{(m+1)}(\iota_{m,n}(a))\right) &= \left(\iota_{m+1} \circ \alpha_{\pi_{m,k}(g)}^{(m+1)} \circ \iota_{m,n}\right)(a) \\
&= \left(\iota_{m+1} \circ \iota_m \circ \alpha_{\pi_{m-1,k}(g)}^{(m)} \circ \iota_{m-1,n}\right)(a) \\
&= \iota_{\infty,m}\left(\alpha_{\pi_{m-1,k}(g)}^{(m)}(\iota_{m-1,n}(a))\right).
\end{aligned}$$

We conclude that for each $g$ in $G$, Equation X.1 defines an automorphism $\alpha_g$ of $A$. One easily checks that the assignment $g \mapsto \alpha_g$ determines a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$. We claim that $\alpha$ is continuous.

Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, write $F = \{a_1, \ldots, a_N\}$ for some $N \in \mathbb{N}$ and some $a_1, \ldots, a_N \in A$. Choose $n \in \mathbb{N}$ and $b_1, \ldots, b_N \in A_n$ such that $\|a_j - b_j\| < \frac{\varepsilon}{3}$ for all $j = 1, \ldots, N$. Set $F' = \{b_1, \ldots, b_N\} \subseteq A_n$. Since $\alpha^{(n)}$ is continuous, there exists $\delta > 0$ such that whenever $h$ and $h'$ are elements of $G_n$ such that $d(h, h') < \delta$, it follows that $\|\alpha_h^{(n)}(b) - \alpha_{h'}^{(n)}(b)\| < \frac{\varepsilon}{3}$ for all $b$ in $F'$. Given $g$ and $g'$ in $G$ such that $d(g, g') < \delta$ and given $j = 1, \ldots, N$, we have

$$\begin{aligned}
\|\alpha_g(a_j) - \alpha_{g'}(a_j)\| \leq \|\alpha_g(a_j) - \alpha_g(\iota_{\infty,n}(b_j))\| &+ \|\alpha_g(\iota_{\infty,n}(b_j)) - \alpha_{g'}(\iota_{\infty,n}(b_j))\| \\
&+ \|\alpha_{g'}(\iota_{\infty,n}(b_j)) - \alpha_{g'}(a_j)\| \\
< 2\|a_j - b_j\| + \frac{\varepsilon}{3} < \varepsilon,
\end{aligned}$$

thus showing that $\alpha$ is a continuous action. $\square$

The following example will play a crucial role in Theorem X.5.13.

**Example X.3.8.** Let $G$ be a totally disconnected group. We construct an action of $G$ with the Rokhlin property on a UHF-algebra as follows. Let $(H_k)_{k\in\mathbb{N}}$ be a decreasing sequence of normal (closed) subgroups of $G$ with $\bigcap\limits_{k\in\mathbb{N}} H_k = \{e\}$, such that $G_k = G/H_k$ is finite for all $k \in \mathbb{N}$. (Closedness of $H_k$ is redundant: any subgroup of finite index is automatically closed.) For each $k \in \mathbb{N}$, set $A_k = \mathcal{B}(\ell^2(G_k))$ and let $\alpha^{(k)} \colon G_k \to \mathrm{Aut}(A_k)$ be conjugation by the left regular representation. Since $G_k$ is a quotient of $G_{k+1}$, it follows that $|G_k|$ divides $|G_{k+1}|$. Set $d_k = |G_{k+1}|/|G_k|$ and let $\iota_k \colon A_k \to A_{k+1}$ be given by $a \mapsto \mathrm{diag}(a, \ldots, a)$, where $a$ is repeated $d_k$ times. It is easy to check that $\iota_k(A_k)$ is precisely the fixed point algebra of $A_{k+1}$ by the action $\alpha^{(k+1)}$ restricted to $H_k/H_{k+1}$, and hence $\iota_k$ is equivariant with respect to the actions $\alpha^{(k)}$ and $\alpha^{(k+1)}$.

Define $A$ as the direct limit

$$A_1 \xrightarrow{\iota^{(1)}} A_2 \xrightarrow{\iota^{(2)}} \cdots \longrightarrow A,$$

and let $\alpha \colon G \to \mathrm{Aut}(A)$ be the limit action given by Lemma X.3.7 above. It is clear that $A$ is a UHF-algebra.

There is a commutative diagram

$$\begin{tikzcd}
C(G_1) \arrow[r] \arrow[d, "\varphi_1"] & C(G_2) \arrow[r] \arrow[d, "\varphi_2"] & \cdots \arrow[r] & C(G) \arrow[d, "\varphi"] \\
A_1 \arrow[r, "\iota^{(1)}"'] & A_2 \arrow[r, "\iota^{(2)}"'] & \cdots \arrow[r] & A,
\end{tikzcd}$$

where all the maps in the finite stages are unital and injective. For $k \in \mathbb{N}$, let $G_k$ act on $C(G_k)$ by left translation, and let it act on $A_k$ via $\alpha^{(k)}$. Then all the maps are also equivariant. It follows that $\varphi \colon C(G) \to A$ is unital and equivariant. Note that this does not a imply that $\alpha$ has the Rokhlin property, since the homomorphism $\varphi$ is not necessarily approximately central. (The action $\alpha$ will in fact almost never have the Rokhlin property.) We fix this by taking the infinite tensor product of copies of $\alpha$.

Set $B = \bigotimes\limits_{n\in\mathbb{N}} A$ and let $\beta \colon G \to \mathrm{Aut}(B)$ be the diagonal action, that is, $\beta_g = \bigotimes\limits_{n\in\mathbb{N}} \alpha_g$. Again, $B$ is a UHF-algebra, and we moreover claim that $\beta$ has the Rokhlin property.

Let $F \subseteq B$ be a finite set and $\varepsilon > 0$. Write $F = \{b_1, \ldots, b_N\}$ for some $N \in \mathbb{N}$ and some $b_1, \ldots, b_N \in B$. Find $M \in \mathbb{N}$ and $c_1, \ldots, c_N \in \bigotimes_{n=1}^{M} A$ such that $\|b_j - c_j\| < \frac{\varepsilon}{2}$ for all $j = 1, \ldots, N$. Let $\psi\colon C(G) \to B$ be the composition of the map $\varphi$ with the inclusion of $A$ in $B$ as the $(M+1)$-st factor. Then $\psi$ is a unital equivariant homomorphism, and it remains to check that its image approximately commutes with the elements of $F$. It is clear that $\psi(C(G))$ exactly commutes with $c_j$ for all $j = 1, \ldots, N$. The rest is just an $\frac{\varepsilon}{2}$ argument: for $j = 1, \ldots, N$ and for $f \in C(G)$ of norm at most one, we have

$$\begin{aligned}\|\psi(f)b_j - b_j\psi(f)\| &\leq \|\psi(f)(b_j - c_j)\| + \|\psi(f)c_j - c_j\psi(f)\| + \|(b_j - c_j)\psi(f)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

and the result follows.

**Classification and Consequences**

Finite group actions with the Rokhlin property are considerably easier to handle than general compact group actions with the Rokhlin property. Nevertheless, in some cases, if one starts with a compact group action with the Rokhlin property, then the restriction to a finite subgroup again has the Rokhlin property. If one is able to classify these restrictions, then one can use an intertwining argument applied to an exhausting increasing sequence of finite subgroups to obtain classification of the actions.

Unfortunately, this idea has no hope of working beyond the compact Lie group case, since compact groups need not have *any* torsion elements. For example, give $\mathbb{Q}$ the discrete topology, and set $G = \widehat{\mathbb{Q}}$. Then $G$ is a compact, connected, second countable abelian group with no torsion elements. For a totally disconnected example, consider the $p$-adic integers $\mathbb{Z}_p$.

Nevertheless, for our purposes, it will be enough to consider "approximate" subgroups, in the sense of the following lemma.

**Lemma X.4.1.** Let $G$ be a compact group, let $\varepsilon > 0$ and let $g \in G$. Then there exists $n \in \mathbb{N}$ such that $d(g^n, 1) < \varepsilon$.

*Proof.* Consider the sequence $(g^m)_{m\in\mathbb{N}}$ in $G$. Since $G$ is compact, there exist a subsequence $(m_k)_{k\in\mathbb{N}}$ and a group element $g_0 \in G$ such that $g^{m_k} \to g_0$ in $G$ as $k \to \infty$. In particular, for the $\varepsilon > 0$ given in the statement, and using translation invariance of the metric $d$ in the second step, we conclude that there exists $k_0 \in \mathbb{N}$ such that

$$\varepsilon > d(g^{m_{k_0+1}}, g^{m_{k_0}}) = d(g^{m_{k_0+1}-m_{k_0}}, 1).$$

Now $n = m_{k_0+1} - m_{k_0}$ is the desired positive integer. $\square$

With the notation of the above lemma, we may regard the set $\{1, g, \ldots, g^{n-1}\}$ as an $\varepsilon$-approximate subgroup of $G$.

**Lemma X.4.2.** Let $G$ be a totally disconnected compact group, let $A$ and $B$ be unital $C^*$-algebras, let $\alpha$ and $\beta$ be actions of $G$ on $A$ and $B$ respectively, such that $\beta$ has the Rokhlin property, and let $\psi\colon A \to B$ be a homomorphism such that $\psi \circ \alpha_g$ is approximately unitarily equivalent to $\beta_g \circ \psi$ for all $g \in G$. Then for all $\varepsilon > 0$, for all finite subsets $F \subseteq A$ and for all $g_0 \in G$, there exist a finite subset $S \subseteq G$ containing $g_0$ and a unitary $w$ in $\mathcal{U}(B)$ such that

$$\|(\mathrm{Ad}(w) \circ \beta_g \circ \mathrm{Ad}(w^*) \circ \psi)(a) - (\psi \circ \alpha_g)(a)\| < \varepsilon$$

for all $g \in S$ and all $a \in F$, and

$$\|w\psi(a) - \psi(a)w\| < \varepsilon + \sup_{g \in S} \|(\beta_g \circ \psi)(a) - (\psi \circ \alpha_g)(a)\|$$

for all $a \in F$. Moreover, the finite set $S \subseteq G$ can be chosen to have the form $S = \{1, g_0, \ldots, g_0^N\}$ for some $N \in \mathbb{N}$.

*Proof.* Let $\varepsilon > 0$, let $F \subseteq A$ be a finite set, and let $g_0 \in G$. Upon normalizing the elements of $F$, we may and will assume that $\|a\| \leq 1$ for all $a$ in $F$. Observe that if $g_0$ is the unit of $G$, then the result can be obtained by simply setting $w = 1$ and $S = \{g_0\}$. We may therefore assume that $g_0$ is not the unit of $G$.

Set $F' = \bigcup\limits_{g \in G} \alpha_g(F)$, which is a compact subset of $A$. Choose $\delta > 0$ such that for $g$ and $g'$ in $G$ with $d(g, g') < \delta$, one has

$$\|\alpha_g(x) - \alpha_{g'}(x)\| < \frac{\varepsilon}{14} \quad \text{and} \quad \|\beta_g(\psi(x)) - \beta_{g'}(\psi(x))\| < \frac{\varepsilon}{14}$$

for all $x \in F'$. For $g$ in $G$, use the fact that $\psi \circ \alpha_g$ is approximately unitarily equivalent to $\beta_g \circ \psi$ to choose a unitary $v_g$ in $B$ such that

$$\|(\psi \circ \alpha_g)(x) - (\mathrm{Ad}(v_g) \circ \beta_g \circ \psi)(x)\| < \varepsilon$$

for all $x$ in $F'$.

Using Lemma X.4.1, denote by $n \in \mathbb{N}$ the smallest positive integer such that $d(g_0^n, 1) < \delta$. By Corollary X.3.2, there is a normal closed subgroup $H$ of $G$ such that $G/H$ is finite and the group elements $1, g_0, \ldots, g^{n-1}$ are sent to pairwise distinct elements in $G/H$. By Proposition X.2.2, the action $\overline{\beta}\colon G/H \to \mathrm{Aut}(B^H)$ has the Rokhlin property. Since the Rokhlin property for finite groups passes to arbitrary subgroups, we may assume that $G/H$ is the cyclic group generated be the image of $g_0$ in $G/H$, so that there exists $N \in \mathbb{N}$ such that $G/H = \{1, \overline{g}_0, \ldots, \overline{g}_0^N\}$. (Note that we must have $N \geq n-1$.)

Choose $\delta_0 > 0$ such that whenever $s \in A$ satisfies $\|s^*s - 1\| \leq \delta_0$ and $\|ss^* - 1\| \leq \delta_0$, then there exists a unitary $u \in \mathcal{U}(A)$ such that $\|u - s\| \leq \frac{\varepsilon}{14}$. Set $\varepsilon_0 = \min\{\frac{\varepsilon}{14}, \delta_0\}$.

Use Lemma X.2.3 and Proposition 5.26 in [205] to choose projections $e_0, \ldots, e_N$ in $B^H \subseteq B$ such that

1. $\beta_{\overline{g_0}^j}(e_k) = e_{j+k}$ for all $j, k = 0, \ldots, N$, where the indices are taken modulo $N+1$.
2. $\|e_j y - y e_j\| < \frac{\varepsilon_0}{2}$ for all $j = 0, \ldots, N$ and all
$$y \in \{v_{g_0^k} \colon k = 0, \ldots, N\} \cup \{\psi(a) \colon a \in F\}.$$
3. $\sum\limits_{j=0}^{N} e_j = 1$.

For $x$ in $F'$ and $j, k$ in $\{0, \ldots, N\}$, we have $d(g_0^j g_0^k, g_0^{j+k}) < \delta$, and consequently

$$\left\| \left(\beta_{g_0^j} \circ \beta_{g_0^k} \circ \psi\right)(x) - \left(\beta_{g_0^{j+k}} \circ \psi\right)(x) \right\| < \frac{\varepsilon}{14},$$

and similarly with $\alpha$:

$$\left\| \left(\alpha_{g_0^j} \circ \alpha_{g_0^k}\right)(x) - \left(\alpha_{g_0^{j+k}}\right)(x) \right\| < \frac{\varepsilon}{14}.$$

Set

$$w = \sum_{j=0}^{N} e_j v_{g_0^j}.$$

Then $w$ is an almost unitary. Indeed, we have

$$\|w^*w - 1\| = \left\| \sum_{j,k=0}^{N} v_{g_0^j}^* e_j e_k v_{g_0^k} - 1 \right\| = \left\| \sum_{j=0}^{N} v_{g_0^j}^* e_j v_{g_0^j} - 1 \right\| \leq \frac{\varepsilon_0}{2} < \delta_0.$$

Likewise,

$$\|ww^* - 1\| = \left\| \sum_{j,k=0}^{N} e_j v_{g_0^j} v_{g_0^k}^* e_k - 1 \right\| = 2\frac{\varepsilon_0}{2} + \left\| \sum_{j=0}^{N} e_j v_{g_0^j} v_{g_0^j}^* - 1 \right\| = \varepsilon_0 \leq \delta_0.$$

It follows that there exists a unitary $u$ in $B$ such that $\|u - w\| \leq \frac{\varepsilon}{14}$.

From now on and until the end of this proof, whenever $c$ and $d$ are elements of $B$ and $t \in \mathbb{R}_{>0}$, the symbol $c =_t d$ will mean $\|c - d\| \leq t$.

For $a$ in $F$ and $k \in \{0, \ldots, N\}$, we have

$$\begin{aligned}
&\left(\mathrm{Ad}(u) \circ \beta_{g_0^k} \circ \mathrm{Ad}(u^*) \circ \psi\right)(a) \\
&\qquad =_{\frac{4\varepsilon}{14}} \left(\mathrm{Ad}(w) \circ \beta_{g_0^k} \circ \mathrm{Ad}(w^*) \circ \psi\right)(a) \\
&\qquad =_{\frac{\varepsilon_0}{2}} \sum_{j=0}^{N} \left(\mathrm{Ad}(w) \circ \beta_{g_0^k}\right) \left(e_j v_{g_0^j}^* \psi(a) v_{g_0^j}\right) \\
&\qquad =_{\frac{\varepsilon_0}{2}} \sum_{j=0}^{N} e_{j+k} \left(\mathrm{Ad}(w) \circ \beta_{g_0^k} \circ \mathrm{Ad}\left(v_{g_0^j}^*\right) \circ \psi\right)(a) \\
&\qquad = \sum_{j=0}^{N} e_{j+k} \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{g_0^k} \circ \mathrm{Ad}\left(v_{g_0^j}^*\right) \circ \psi\right)(a) \\
&\qquad = \sum_{j=0}^{N} e_{j+k} \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{g_0^k} \circ \beta_{g_0^j} \circ \left(\mathrm{Ad}\left(v_{g_0^j}\right) \circ \beta_{g_0^j}\right)^{-1} \circ \psi\right)(a).
\end{aligned}$$

For $k \in \{0, \ldots, N\}$, we have the following computation, where in the fourth step we set $x = \alpha_{g_0^j}^{-1}(a)$, which belongs to $F'$:

$$
\begin{aligned}
&\left\|\left(\psi \circ \alpha_{g_0^k}\right)(a) - \left(\mathrm{Ad}(u) \circ \beta_{g_0^k} \circ \mathrm{Ad}(u^*) \circ \psi\right)(a)\right\| \\
&\quad \leq \frac{4\varepsilon}{14} + \left\|\left(\psi \circ \alpha_{g_0^k}\right)(a) - \left(\mathrm{Ad}(w) \circ \beta_{g_0^k} \circ \mathrm{Ad}(w^*) \circ \psi\right)(a)\right\| \\
&\quad \leq \frac{8\varepsilon}{14} + \varepsilon_0 + \left\|\sum_{j=0}^{N} e_{j+k}\left[\left(\psi \circ \alpha_{g_0^k}\right)(a) - \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{g_0^k} \circ \beta_{g_0^j} \circ \left(\mathrm{Ad}\left(v_{g_0^j}\right) \circ \beta_{g_0^j}\right)^{-1} \circ \psi\right)(a)\right]\right\| \\
&\quad \leq \frac{8\varepsilon}{14} + \varepsilon_0 + \sup_{j=0,\ldots,N}\left\|\left(\psi \circ \alpha_{g_0^k}\right)(a) - \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{g_0^k} \circ \beta_{g_0^j} \circ \left(\mathrm{Ad}\left(v_{g_0^j}\right) \circ \beta_{g_0^j}\right)^{-1} \circ \psi\right)(a)\right\| \\
&\quad = \frac{8\varepsilon}{14} + \varepsilon_0 + \sup_{j=0,\ldots,N}\left\|\left(\psi \circ \alpha_{g_0^k} \circ \alpha_{g_0^j}\right)(x) - \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{g_0^k} \circ \beta_{g_0^j} \circ \left(\mathrm{Ad}\left(v_{g_0^j}\right) \circ \beta_{g_0^j}\right)^{-1} \circ \psi \circ \alpha_{g_0^j}\right)(x)\right\| \\
&\quad \leq \frac{8\varepsilon}{14} + \varepsilon_0 + \sup_{j=0,\ldots,N}\left\|\left(\psi \circ \alpha_{g_0^k} \circ \alpha_{g_0^j}\right)(x) - \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{g_0^k} \circ \beta_{g_0^j} \circ \psi\right)(x)\right\| \\
&\qquad\qquad + \left\|\psi(x) - \left(\left(\mathrm{Ad}\left(v_{g_0^j}\right) \circ \beta_{g_0^j}\right)^{-1} \circ \psi \circ \alpha_{g_0^j}\right)(x)\right\| \\
&\quad \leq \frac{10\varepsilon}{14} + \varepsilon_0 + \sup_{j=0,\ldots,N}\left\|\left(\psi \circ \alpha_{g_0^{k+j}}\right)(x) - \left(\mathrm{Ad}\left(v_{g_0^{j+k}}\right) \circ \beta_{{g_0^{k+j}}_k} \circ \psi\right)(x)\right\| \\
&\qquad\qquad + \left\|\left(\mathrm{Ad}\left(v_{g_0^j}\right) \circ \beta_{g_0^j} \circ \psi\right)(x) - \left(\psi \circ \alpha_{g_0^j}\right)(x)\right\| \\
&\quad \leq \frac{12\varepsilon}{14} + \varepsilon_0 \\
&\quad < \varepsilon.
\end{aligned}
$$

Finally, if $a \in F$, and again by setting $x = \alpha_{g_0^j}^{-1}(a) \in F'$ in the fifth step, we have

$$\begin{aligned}
\|\psi(a)u - u\psi(a)\| &= \|u^*\psi(a)u - \psi(a)\| \\
&\leq \frac{2\varepsilon}{14} + \|w^*\psi(a)w - \psi(a)\| \\
&\leq \frac{5\varepsilon}{14} + \left\| \sum_{j=0}^{N} e_j \left( v_{g_0^j}^* \psi(a) v_{g_0^j} - \psi(a) \right) \right\| \\
&\leq \frac{5\varepsilon}{14} + \sup_{j=0,\ldots,N} \left\| \left( \mathrm{Ad}\left( v_{g_0^j}^* \right) \circ \psi \circ \alpha_{g_0^j} \circ \alpha_{g_0^j}^{-1} \right)(a) - \psi(a) \right\| \\
&\leq \frac{5\varepsilon}{14} + \sup_{j=0,\ldots,N} \left\| \left( \mathrm{Ad}\left( v_{g_0^j}^* \right) \circ \psi \circ \alpha_{g_0^j} \right)(x) - \left( \beta_{g_0^j} \circ \psi \right)(x) \right\| \\
&\qquad + \sup_{j=0,\ldots,N} \left\| \left( \beta_{g_0^j} \circ \psi \circ \alpha_{g_0^{-j}} \right)(a) - \psi(a) \right\| \\
&\leq \frac{6\varepsilon}{14} + \sup_{j=0,\ldots,N} \left\| \left( \psi \circ \alpha_{g_0^{-j}} \right)(a) - \left( \beta_{g_0^{-j}} \circ \psi \right)(a) \right\| \\
&< \varepsilon + \sup_{j=0,\ldots,N} \left\| \left( \psi \circ \alpha_{g_0^{-j}} \right)(a) - \left( \beta_{g_0^{-j}} \circ \psi \right)(a) \right\|,
\end{aligned}$$

and the result follows.

$\square$

We will need the following technical lemma.

**Lemma X.4.3.** Let $A$ be a separable $C^*$-algebra, let $G$ be a compact group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum\limits_{n\in\mathbb{N}} \varepsilon_n < \infty$, and choose an increasing family of compact subsets $F_n \subseteq A$ for $n \in \mathbb{N}$ whose union is dense in $A$. Assume that we can inductively choose a unitary $u_n$ in $A$ for $n \in \mathbb{N}$ such that, with $\alpha^{(1)} = \alpha$ and $F_1' = \bigcup\limits_{g\in G} \alpha_g(F_1)$, if we let

$$\alpha_g^{(n-1)} = \mathrm{Ad}(u_{n-1}) \circ \alpha_g^{(n-2)} \circ \mathrm{Ad}(u_{n-1}^*)$$

for $n \geq 2$, then $\|u_n a - a u_n\| < \varepsilon_n$ for all $a$ in a compact set $F_n'$ that contains

$$\bigcup_{g\in G} \alpha_g^{(n-1)} \left( F_{n-1}' \cup F_n \cup \mathrm{Ad}(u_{n-1} \cdots u_1)(F_{n-1}') \right).$$

Then $\lim\limits_{n\to\infty} \mathrm{Ad}(u_n \cdots u_1)$ exists in the topology of pointwise norm convergence in $\mathrm{Aut}(A)$ and defines an approximately inner automorphism $\mu$ of $A$. Moreover, for every $g$ in $G$ and for every $a$

in $A$, the sequence $\left(\alpha_g^{(n)}(a)\right)_{n\in\mathbb{N}}$ converges and

$$\lim_{n\to\infty} \alpha_g^{(n)}(a) = \mu \circ \alpha_g(a) \circ \mu^{-1}.$$

In particular, $g \mapsto \lim\limits_{n\to\infty} \alpha_g^{(n)}$ is a continuous action of $G$ on $A$.

*Proof.* We will first show that $\lim\limits_{n\to\infty} \mathrm{Ad}(u_n \cdots u_1)$ exists and defines an automorphism of $A$. Such an automorphism will clearly be approximately inner.

For $n \geq 1$, set $v_n = u_n \cdots u_1$. Let

$$S = \{a \in A \colon (v_n a v_n^*)_{n\in\mathbb{N}} \text{ converges in } A\}.$$

We claim that $S$ is dense in $A$. Indeed, $S$ contains the set $F_n$ for all $n$, and since $\bigcup\limits_{n\in\mathbb{N}} F_n$ is dense in $A$, the claim follows.

In particular, $S$ is a dense $*$-subalgebra of $A$. For each $a$ in $S$, denote by $\mu_0(a)$ the limit of the sequence $(v_n a v_n^*)_{n\in\mathbb{N}}$. The map $\mu_0 \colon S \to A$ is linear, multiplicative, preserves the adjoint, and is isometric, and therefore extends by continuity to a homomorphism $\mu \colon A \to A$ with $\|\mu(a)\| = \|a\|$ for all $a$ in $A$. Given $a$ in $A$ and given $\varepsilon > 0$, choose $b$ in $S$ such that $\|a - b\| < \frac{\varepsilon}{3}$, using density of $S$ in $A$. Choose $N \in \mathbb{N}$ such that $\|v_N b v_N^* - \mu(b)\| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}\|\mu(a) - v_N a v_N^*\| &\leq \|\mu(a-b)\| + \|\mu(b) - v_N b v_N^*\| + \|v_N a v_N^* - v_N b v_N^*\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

It follows that $\mu = \lim\limits_{n\to\infty} \mathrm{Ad}(u_n \cdots u_1)$ is an endomorphism of $A$, and it is isometric since it is a norm-pointwise limit of isometric maps. In particular, it is injective. We still need show that it is an automorphism. For this, we construct its inverse. By a similar reasoning, the norm-pointwise limit $\lim\limits_{n\to\infty} \mathrm{Ad}(u_1^* \cdots u_n^*)$ defines another injective endomorphism of $A$, since $\|u_n^* a u_n - a\| < \varepsilon_n$ for all $n \in \mathbb{N}$ and for all $a$ in $F_n'$. We denote this endomorphism by $\nu$. We claim that $\nu$ is a right inverse for $\mu$, which will imply that $\mu$ is an automorphism since it is injective. For $a$ in $F_k$ and $k \leq n \leq m$, and using that $\mathrm{Ad}(u_j \cdots u_n)(a) \in F_j'$ and that $u_{j+1}$ commutes with the elements of $F_j'$

up to $\varepsilon_j$ in the last step, we have

$$
\begin{aligned}
\|a - \left(\mu_m \circ \mu_n^{-1}\right)(a)\| &= \|u_m u_{m-1} \cdots u_{n+1} a u_{n+1}^* \cdots u_m^* - a\| \\
&\leq \sum_{j=n}^{m-1} \|\mathrm{Ad}(u_{j+1} \cdots u_n)(a) - \mathrm{Ad}(u_j \cdots u_n)(a)\| \\
&< \sum_{j=n}^{m-1} \varepsilon_j.
\end{aligned}
$$

The estimate above implies that $\mu(\nu(a)) = a$ since $\sum_{j=1}^{\infty} \varepsilon_j$ converges. Now, the union of the sets $F_k$ with $k \in \mathbb{N}$ is dense in $A$, so this proves the first part of the statement.

We now show that given $g$ in $G$ and $a$ in $A$, the sequence $\left(\alpha_g^{(n)}(a)\right)_{n\in\mathbb{N}}$ converges by showing that it is Cauchy. Since $\bigcup_{n=1}^{\infty} F_n$ is dense in $A$, it suffices to consider elements in this union. Let $n \in \mathbb{N}$ and choose $a$ in $F_n \subseteq F_n'$. If $g \in G$ and $m \geq k \geq n$, then

$$
\begin{aligned}
&\|\alpha_g^{(k)}(a) - \alpha_g^{(m)}(a)\| \\
&= \|\alpha_g^{(k)}(a) - \mathrm{Ad}(u_m) \circ \cdots \circ \mathrm{Ad}(u_{k+1}) \circ \alpha_g^{(k)} \circ \mathrm{Ad}(u_{k+1}^*) \circ \cdots \circ \mathrm{Ad}(u_m^*)(a)\| \\
&\leq \sum_{j=k}^{m-1} \big(\|\mathrm{Ad}(u_{j+1} \cdots u_k) \circ \alpha_g^{(k)} \circ \mathrm{Ad}(u_k^* \cdots u_{j+1}^*)(a) \\
&\qquad\qquad - \mathrm{Ad}(u_j \cdots u_k) \circ \alpha_g^{(k)} \circ \mathrm{Ad}(u_k^* \cdots u_j^*)(a)\|\big) \\
&< \sum_{j=k}^{m-1} \varepsilon_j,
\end{aligned}
$$

where in the last step we use that

$$
\left(\mathrm{Ad}(u_j \cdots u_k) \circ \alpha_g^{(k)} \circ \mathrm{Ad}(u_k^* \cdots u_j^*)\right)(a) \in F_j'
$$

and that $u_{j+1}$ commutes with the elements of $F_j'$ up to $\varepsilon_j$. Since $\sum_{j\in\mathbb{N}} \varepsilon_j < \infty$, the claim follows.

Fix $g \in G$ and $a \in A$. We claim that

$$
\lim_{n\to\infty} (\mathrm{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mathrm{Ad}(u_1^* \cdots u_n^*))(a) = (\mu \circ \alpha_g \circ \mu^{-1})(a).
$$

Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\|\mathrm{Ad}(u_n \cdots u_1)(x) - \mu(x)\| < \frac{\varepsilon}{2}$$

for all $x \in \{a\} \cup \bigcup\limits_{g \in G} \alpha_g(\mu^{-1}(a))$. Then

$$\begin{aligned}
&\|(\mathrm{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mathrm{Ad}(u_1^* \cdots u_n^*))(a) - (\mu \circ \alpha_g \circ \mu^{-1})(a)\| \\
&\quad \leq \|(\mathrm{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mathrm{Ad}(u_1^* \cdots u_n^*))(a) - (\mathrm{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mu^{-1})(a)\| \\
&\qquad + \|(\mathrm{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mu^{-1})(a) - (\mu \circ \alpha_g \circ \mu^{-1})(a)\| \\
&\quad = \|\mathrm{Ad}(u_1^* \cdots u_n^*)(a) - \mu^{-1}(a)\| \\
&\qquad + \|(\mathrm{Ad}(u_n \cdots u_1) \circ \alpha_g \circ \mu^{-1})(a) - (\mu \circ \alpha_g \circ \mu^{-1})(a)\| \\
&\quad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This finishes the proof. $\square$

The following is a variant of an argument used by Evans and Kishimoto. Some of the hypotheses of the theorem can be relaxed, but this version is good enough for our purposes. In particular, for the result to hold, the connecting maps need not be injective, and conditions (5) and (6) can assumed to hold only approximately on the sets $X_n$ and $Y_n$ respectively. On the other hand, we do not assume that the actions $\alpha$ and $\beta$ are direct limit actions, but only limit actions.

**Theorem X.4.4.** Let $G$ be a locally compact group. Let $(A_n, i_n)$ and $(B_n, j_n)$ be direct systems in which $i_n \colon A_n \to A_{n+1}$ and $j_n \colon B_n \to B_{n+1}$ are inclusions for all $n \in \mathbb{N}$. Let $\alpha^{(n)} \colon G \to \mathrm{Aut}(A_n)$ and $\beta^{(n)} \colon G \to \mathrm{Aut}(B_n)$ be continuous actions that induce norm-pointwise limit actions $\alpha = \lim\limits_n \alpha^{(n)}$ and $\beta = \lim\limits_n \beta^{(n)}$ of $G$ on $A = \varinjlim A_n$ and $B = \varinjlim B_n$ respectively. This is, for every $k$ in $\mathbb{N}$, we have that $\alpha_g(a) = \lim\limits_{n \to \infty, n \geq k} \alpha_g^{(n)}(a)$ exists for every $g \in G$ and every $a \in A_k$, and $g \mapsto \alpha_g$ defines a continuous action of $G$ on $\varinjlim A_n$, and similarly with $\beta$.

Suppose there are a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers, maps $\varphi_n \colon A_n \to B_n$, increasing compact subsets $X_n \subseteq A_n$ and $Y_n \subseteq B_n$, and a family of subsets $G_n \subseteq G$ such that

1. $\sum\limits_{n \in \mathbb{N}} \varepsilon_n < \infty$;
2. $\left( \bigcup\limits_{m=n}^{\infty} X_m \right) \cap A_n$ is dense in $A_n$ and $\left( \bigcup\limits_{m=n}^{\infty} Y_m \right) \cap B_n$ is dense in $B_n$ for all $n \in \mathbb{N}$;

3. $\varphi_n(X_n) \subseteq Y_n$ for all $n \in \mathbb{N}$;

4. $\alpha_g^{(n)}(X_n) \subseteq X_n$ and $\beta_g^{(n)}(Y_n) \subseteq Y_n$ for all $g \in G$ and all $n \in \mathbb{N}$;

5. $\|(j_n \circ \varphi_n)(a) - (\varphi_{n+1} \circ i_n)(a)\| < \varepsilon_n$ for all $a \in X_n$ and all $n \in \mathbb{N}$;

6. $\overline{\bigcup\limits_{n \geq N} G_n} = G$ for all $N \in \mathbb{N}$;

7. $\|(\varphi_n \circ \alpha_g^{(n)})(a) - (\beta_g^{(n)} \circ \varphi_n)(a)\| < \varepsilon_n$ for all $a \in X_n$ and for all $g \in G_n$;

8. $\|(i_n \circ \alpha_g^{(n)})(a) - (\alpha_g^{(n+1)} \circ i_n)(a)\| < \epsilon_n$ for all $a \in X_n$ and $g \in G_n$; and

9. $\|(j_n \circ \beta_g^{(n)})(b) - (\beta_g^{(n+1)} \circ j_n)(b)\| < \epsilon_n$ for all $b \in Y_n$ and $g \in G_n$.

Then the map $\varphi_0 \colon \bigcup\limits_{n \in \mathbb{N}} A_n \to \bigcup\limits_{n \in \mathbb{N}} B_n$, given by

$$\varphi_0(a) = \lim_{n \to \infty} \varphi_n(a)$$

is well defined and extends by continuity to a homomorphism $\varphi \colon A \to B$ such that

$$\varphi \circ \alpha_g = \beta_g \circ \varphi$$

for all $g \in G$.

*Proof.* Consider the map $\varphi_0 \colon \bigcup\limits_{n \in \mathbb{N}} A_n \to \bigcup\limits_{n \in \mathbb{N}} B_n$ defined above. It is straightforward to show that $\varphi_0$ is well-defined and norm-decreasing, and hence it extends by continuity to a homomorphism $\varphi \colon A \to B$. It remains to check that it intertwines the actions $\alpha$ and $\beta$.

Fix $n \in \mathbb{N}$ and choose $g$ in $G_n$ and $a$ in $X_n$. We claim that

$$(\varphi \circ \alpha_g)(a) = (\beta_g \circ \varphi)(a).$$

For $m, k \in \mathbb{N}$ with $m \geq k$, denote by $i_{m-1,k} \colon A_k \to A_m$ and $j_{m-1,k} \colon B_k \to B_m$ the compositions $i_{m-1,k} = i_{m-1} \circ \cdots \circ i_k$ and $j_{m-1,k} = j_{m-1} \circ \cdots \circ j_k$, respectively. We have

$$(\varphi \circ \alpha_g)(a) = \lim_{m \to \infty} (\varphi \circ \alpha_g^{(m)} \circ i_{m-1,n})(a) = \lim_{m \to \infty} (\varphi_m \circ \alpha_g^{(m)} \circ i_{m-1,n})(a)$$

and likewise,

$$(\beta_g \circ \varphi)(a) = \lim_{m\to\infty} (\beta_g \circ \varphi_m \circ i_{m-1,n})(a) = \lim_{m\to\infty} (\beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a).$$

Moreover, if $r \geq m$, then

$$\begin{aligned}
&\|(\varphi_r \circ \alpha_g^{(r)} \circ i_{r-1,n})(a) - (j_{r-1,m} \circ \beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a)\| \\
&\quad \leq \|(\varphi_r \circ \alpha_g^{(r)} \circ i_{r-1,n})(a) - (\beta_g^{(r)} \circ \varphi_r \circ i_{r-1,n})(a)\| \\
&\qquad + \|(\beta_g^{(r)} \circ \varphi_r \circ i_{r-1,n})(a) - (j_{r-1,m} \circ \beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a)\| \\
&\quad < \varepsilon_r + \sum_{k=m}^{r-1} \|(\beta_g^{(k+1)} \circ \varphi_{k+1} \circ i_{k,n})(a) - (j_k \circ \beta_g^{(k)} \circ \varphi_k \circ i_{k-1,n})(a)\| \\
&\quad = \varepsilon_r + \sum_{k=m}^{r-1} \|(\beta_g^{(k+1)} \circ \varphi_{k+1} \circ i_k(i_{k-1,n})(a)) - (j_k \circ \beta_g^{(k)} \circ \varphi_k(i_{k-1,n})(a))\| \\
&\quad < \varepsilon_r + \sum_{k=m}^{r-1} \left\|(\beta_g^{(k+1)} \circ \varphi_{k+1} \circ i_k(i_{k-1,n})(a)) - (j_k \circ \beta_g^{(k)} \circ \varphi_{k+1} \circ i_k(i_{k-1,n})(a))\right\| \\
&\quad < \sum_{k=m}^{r} \varepsilon_k.
\end{aligned}$$

Since $\sum_{j\in\mathbb{N}} \varepsilon_j < \infty$, we conclude that

$$\lim_{m\to\infty} (\varphi_m \circ \alpha_g^{(m)} \circ i_{m-1,n})(a) = \lim_{m\to\infty} (\beta_g^{(m)} \circ \varphi_m \circ i_{m-1,n})(a)$$

and the claim follows.

For fixed $a \in X_n$, the conclusion is that the identity

$$(\varphi \circ \alpha_g)(a) = (\beta_g \circ \varphi)(a)$$

holds for all $g$ in $G_n$. Since the family $(X_n)_{n\in\mathbb{N}}$ is increasing, the identity above holds for all $g$ in $\bigcup_{m\geq n} G_m$. Since $\bigcup_{m\geq n} G_m$ is dense in $G$, the identity $(\varphi \circ \alpha_g)(a) = (\beta_g \circ \varphi)(a)$ holds for all $g$ in $G$, and for all $a$ in $X_n$. Therefore it holds for all $a$ in $\bigcup_{n\in\mathbb{N}} X_n$, and by continuity, it holds for all $a$ in $A$. The result follows. $\square$

With the aid of Lemma X.4.2, we can show that the results in [132] (specifically Theorem 3.5 there) hold for totally disconnected compact groups. We begin proving an existence result for equivariant homomorphisms.

**Proposition X.4.5.** (Compare with Proposition 4.2 of [91].) Let $G$ be a totally disconected compact group, let $A$ and $B$ be unital $C^*$-algebras, with $A$ separable, and let $\alpha$ and $\beta$ be actions of $G$ on $A$ and $B$ respectively, such that $\beta$ has the Rokhlin property. Let $\psi\colon A \to B$ be a homomorphism such that $\beta_g \circ \psi$ and $\psi \circ \alpha_g$ are approximately unitarily equivalent for all $g \in G$. Then there exists an equivariant homomorphism $\theta\colon A \to B$ that is approximately unitarily equivalent to $\psi$.

*Proof.* Choose an increasing family of finite subsets $F_n \subseteq A$ for $n \in \mathbb{N}$ whose union is dense in $A$. Choose a countable subset $\{g_1, g_2, \ldots\} \subseteq G$ such that $\{g_n, g_{n+1}, \ldots\}$ is dense in $G$ for all $n \in \mathbb{N}$. Set $\alpha^{(1)} = \alpha$, $\beta^{(1)} = \beta$ and $\psi^{(1)} = \psi$. Choose a unitary $u_1 \in \mathcal{U}(A)$ and a finite subset $S_1$ of $G$ such that the conclusion of Lemma X.4.2 holds for $\alpha^{(1)}$, $\beta^{(1)}$ and $\psi^{(1)}$, with the choices $g_0 = g_1$, with $F_1' = \bigcup\limits_{g\in G} (\alpha_g(F_1) \cup \beta_g(F_1))$ and $\varepsilon = 1$. For $g \in G$, let $\alpha_g^{(2)} = \mathrm{Ad}(u_1) \circ \alpha^{(1)} \circ \mathrm{Ad}(u_1^*)$, let $\psi^{(2)} = \mathrm{Ad}(u_1) \circ \psi^{(1)}$, and let

$$F_2' = \bigcup_{g\in G} \alpha_g^{(2)}(F_2 \cup F_1' \cup \mathrm{Ad}(u_1)(F_1')) \cup \bigcup_{g\in G} \beta_g^{(1)}(F_2 \cup F_1' \cup \mathrm{Ad}(u_1)(F_1')).$$

Choose a unitary $v_1 \in \mathcal{U}(A)$ such that the conclusion of Lemma X.4.2 is satisfied with $\alpha^{(2)}$, $\beta^{(1)}$ and $\psi^{(2)}$, with the choices $g_0 = g_2$, with $F = F_2'$ and $\varepsilon = \frac{1}{2}$. Analogously, for $g \in G$, set $\beta_g^{(2)} = \mathrm{Ad}(v_1) \circ \beta_g \circ \mathrm{Ad}(v_1^*)$ and $\psi^{(3)} = \mathrm{Ad}(v_1) \circ \psi^{(2)}$, and let

$$F_3' = \bigcup_{g\in G} \alpha_g^{(2)}(F_2 \cup F_1' \cup \mathrm{Ad}(v_1)(F_1')) \cup \bigcup_{g\in G} \beta_g^{(2)}(F_2 \cup F_1' \cup \mathrm{Ad}(v_1)(F_1')).$$

Iterating this process, we obtain a family of compact subsets $F_n' \subseteq A$ for $n \in \mathbb{N}$ whose union is dense in $A$, a family of finite subsets $S_n \subseteq G$ such that $\bigcup\limits_{n\geq N} S_n$ is dense in $G$ for all $N \in \mathbb{N}$, two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ of unitaries in $A$, two families $(\alpha^{(n)})_{n\in\mathbb{N}}$ and $(\beta^{(n)})_{n\in\mathbb{N}}$ of actions of $G$ on $A$ and a family $(\psi^{(n)})_{n\in\mathbb{N}}$ of endomorphisms of $A$, that with $\alpha^{(1)} = \alpha$, $\beta^{(1)} = \beta$ and $\psi^{(1)} = \psi$, are given by

$$\alpha_g^{(n+1)} = \mathrm{Ad}(u_n) \circ \alpha_g^{(n)} \circ \mathrm{Ad}(u_n^*) \quad , \quad \beta_g^{(n+1)} = \mathrm{Ad}(v_n) \circ \beta_g^{(n)} \circ \mathrm{Ad}(v_n^*),$$

$$\psi^{(2n+1)} = \mathrm{Ad}(u_n) \circ \psi^{(2n)} \quad \text{and} \quad \psi^{(2n)} = \mathrm{Ad}(v_n) \circ \psi^{(2n-1)}$$

for all $g \in G$, and such that, with $F_1' = \bigcup_{g \in G} (\alpha_g(F_1) \cup \beta_g(F_1))$, the compact sets $F_n'$ are given by

$$\begin{aligned} F_{2n+1}' &= \bigcup_{g \in G} \alpha_g^{(n+1)}(F_{2n+1} \cup F_{2n}' \cup \mathrm{Ad}(v_n v_{n-1} \cdots v_1)(F_{2n}')) \\ &\quad \cup \bigcup_{g \in G} \beta_g^{(n)}(F_{2n+1} \cup F_{2n}' \cup \mathrm{Ad}(v_n v_{n-1} \cdots v_1)(F_{2n}')) \\ F_{2n+2}' &= \bigcup_{g \in G} \alpha_g^{(n+1)}(F_{2n+2} \cup F_{2n+1}' \cup \mathrm{Ad}(u_n u_{n-1} \cdots u_1)(F_{2n}')) \\ &\quad \cup \bigcup_{g \in G} \beta_g^{(n+1)}(F_{2n+2} \cup F_{2n+1}' \cup \mathrm{Ad}(u_n u_{n-1} \cdots u_1)(F_{2n}')); \end{aligned}$$

the actions $\alpha^{(n)}$ and $\beta^{(n)}$ become closer to being intertwined by the homomorphism $\psi^{(n)}$:

$$\left\| \left(\beta_g^{(n)} \circ \psi^{(2n)}\right)(a) - \left(\psi^{(2n)} \circ \alpha_g^{(n+1)}\right)(a) \right\| < \frac{1}{2^n}$$

for $a$ in $F_{2n}'$ and $g$ in $S_{n+1}$, and

$$\left\| \left(\beta_g^{(n+1)} \circ \psi^{(2n+1)}\right)(a) - \left(\psi^{(2n+1)} \circ \alpha_g^{(n+1)}\right)(a) \right\| < \frac{1}{2^{n+1}}$$

for $a$ in $F_{2n+1}'$ and $g$ in $S_{n+1}$. The unitaries $u_n$ and $v_n$ satisfy the following approximate commutation condition:

$$\begin{aligned} &\left\| v_n \left(\psi^{(2n+1)}(a)\right) - \left(\psi^{(2n+1)}(a)\right) v_n \right\| \\ &\qquad < \frac{1}{2^n} + \sup_{g \in S_n} \left\| \left(\beta_g^{(n)} \circ \psi^{(2n)}\right)(a) - \left(\psi^{(2n)} \circ \alpha_g^{(n+1)}\right)(a) \right\| < \frac{1}{2^{n-1}} \end{aligned}$$

for $a \in F_{2n}'$, and

$$\begin{aligned} &\left\| u_{n+1} \left(\psi^{(2n)}(a)\right) - \left(\psi^{(2n)}(a)\right) u_{n+1} \right\| \\ &\qquad < \frac{1}{2^{n+1}} + \sup_{g \in S_{n+1}} \left\| \left(\beta_g^{(n+1)} \circ \psi^{(2n+1)}\right)(a) - \left(\psi^{(2n+1)} \circ \alpha_g^{(n+1)}\right)(a) \right\| < \frac{1}{2^n} \end{aligned}$$

for $a \in F_{2n+1}'$ for $n \in \mathbb{N}$.

By Lemma X.4.3, there exist approximately inner automorphisms $\mu$ and $\nu$ of $A$ and $B$ respectively, such that for all $g \in G$,

$$\lim_{n\to\infty} \alpha_g^{(n)} = \mu \circ \alpha_g \circ \mu^{-1} \quad \text{and} \quad \lim_{n\to\infty} \beta_g^{(n)} = \nu \circ \beta_g \circ \nu^{-1}.$$

The conditions of Theorem X.4.4 are satisfied with the direct limit decompositions $A_n = A$ and $B_n = B$, connecting maps $i_n = \mathrm{id}_A$ and $j_n = \mathrm{id}_B$, morphisms $\varphi_n = \psi^{(n)}$, and subsets $G_n = S_n$ for all $n \in \mathbb{N}$. Denote by $\varphi\colon A \to B$ the homomorphism provided by Theorem X.4.4. It follows that

$$\varphi \circ \mu \circ \alpha_g \circ \mu^{-1} = \nu \circ \beta_g \circ \nu^{-1} \circ \varphi$$

for all $g \in G$, and by setting $\theta = \nu^{-1} \circ \varphi \circ \mu$, the claim follows.

$\square$

In order to state our results in full generality, we recall some terminology and notation used in [91].

**Notation X.4.6.** Let $G$ be a locally compact group. Let $\mathbf{B}$ be a subcategory of the category $\mathbf{A}$ of $C^*$-algebras, let $\mathbf{C}$ be a category and let $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ be a functor.

1. Let $\mathbf{B}_G$ denote the category whose objects are dynamical systems $(G, A, \alpha)$ with $A$ in $\mathbf{B}$, and its morphisms are equivariant homomorphisms of $C^*$-algebras in $\mathbf{B}$.

2. Let $\mathbf{C}_G$ denote the category whose objects are triples $(G, C, \gamma)$, where $C$ is an object in $\mathbf{C}$ and $\gamma\colon G \to \mathrm{Aut}(C)$ is a group homomorphism, and whose morphisms are the morphisms of $\mathbf{C}$ that are equivariant.

3. Let $\mathrm{F}_G\colon \mathbf{B}_G \to \mathbf{C}_G$ denote the functor defined as follows:

   (a) For an object $(G, A, \alpha)$ in $\mathbf{B}_G$, define an action $\mathrm{F}(\alpha)\colon G \to \mathrm{Aut}(\mathrm{F}(A))$ by $(\mathrm{F}(\alpha))_g = \mathrm{F}(\alpha_g)$ for $g \in G$. We then set $\mathrm{F}_G(A, \alpha) = (\mathrm{F}(A), \mathrm{F}(\alpha))$;

   (b) For an equivariant morphism $\phi\colon A \to B$, we set $\mathrm{F}_G(\phi) = \mathrm{F}(\phi)$.

When $G$ is compact, we let $\mathbf{RB}_G$ denote the subcategory of $\mathbf{B}_G$ consisting of those $C^*$-dynamical systems $(G, A, \alpha)$ such that $A$ is separable and $\alpha$ has the Rokhlin property.

The next theorem asserts that the functor on $\mathbf{RB}_G$ induced by a functor that classifies homomorphisms on a subcategory $\mathbf{B}$ of $C^*$-algebras, again classifies homomorphisms. Its proof is identical to that of Theorem 3.2 in [91], using Proposition X.4.5 above instead of Proposition 3.2 in [91]. We omit the details. See Definition 4.2 in [91] for the definition of *functor that classifies homomorphisms.*

**Theorem X.4.7.** Let $G$ be a totally disconnected compact group, let $\mathbf{B}$ be a subcategory of $\mathbf{A}$ that is closed under countable direct limits, and let $\mathbf{C}$ be a category where inductive limits of sequences exist. Let $\mathrm{F}\colon \mathbf{B} \to \mathbf{C}$ be a functor that classifies homomorphisms. Assume that F preserves countable direct limits.

1. Let $(G, A, \alpha)$ be an object in $\mathbf{B}_G$ and let $(G, B, \beta)$ be an object in $\mathbf{RB}_G$. Assume that $A$ and $B$ are separable. Then
   (a) For every morphism $\gamma\colon (G, \mathrm{F}(A), \mathrm{F}(\alpha)) \to (G, \mathrm{F}(B), \mathrm{F}(\beta))$ in $\mathbf{C}_G$ there exists a morphism $\phi\colon (G, A, \alpha) \to (G, B, \beta)$ in $\mathbf{B}_G$ such that $\mathrm{F}_G(\phi) = \gamma$.
   (b) If $\phi, \psi\colon (G, A, \alpha) \to (G, B, \beta)$ are morphisms in $\mathbf{B}_G$ such that $\mathrm{F}_G(\phi) = \mathrm{F}_G(\psi)$, then $\phi$ and $\psi$ are equivariantly unitarily approximately equivalent.
2. The restriction of the functor $\mathrm{F}_G$ to $\mathbf{RB}_G$ classifies homomorphisms.

Analogously, one can show that the functor on $\mathbf{RB}_G$, induced by a functor that classifies isomorphisms on a subcategory $\mathbf{B}$ of $C^*$-algebras, again classifies isomorphisms. (Compare with Theorem

We give two examples of applications of Theorem X.4.7 to concrete classes of $C^*$-algebras.

**Theorem X.4.8.** Let $G$ be a compact group, let $A = \varinjlim(A_n, \iota_n)$ and $B = \varinjlim(B_n, j_n)$ be direct limits of unital 1-dimensional NCCW-complexes $A_n$ and $B_n$ with trivial $K_1$-groups, and unital connecting maps $\iota_n\colon A_n \to A_{n+1}$ and $j\colon B_n \to B_{n+1}$, and let $\alpha$ and $\beta$ be actions of $G$ with the Rokhlin property on $A$ and $B$ respectively.

Then for every morphism $\phi\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ such that

$$\phi([1_A]) \leq [1_B] \quad \text{and} \quad \phi \circ \mathrm{Cu}(\alpha_g) = \mathrm{Cu}(\beta_g) \circ \phi \quad \text{for all } g \in G,$$

there exists a unital homomorphism $\theta\colon A \to B$ such that $\theta \circ \alpha_g = \beta_g \circ \theta$ for all $g \in G$. Moreover,

1. The homomorphism $\theta$ is unique up to equivariant approximate unitary equivalence.

2. The homomorphism $\theta$ is unital if and only if $\phi([1_A]) = [1_B]$.

3. The actions $\alpha$ and $\beta$ are conjugate if and only if there exists a unit-preserving isomorphism $\phi\colon \mathrm{Cu}(A) \to \mathrm{Cu}(B)$ such that $\phi^{-1} \circ \mathrm{Cu}(\beta_g) \circ \phi = \mathrm{Cu}(\alpha_g)$ for all $g \in G$.

4. If $A = B$, then the actions $\alpha$ and $\beta$ are approximately inner conjugate if and only if $\mathrm{Cu}(\alpha_g) = \mathrm{Cu}(\beta_g)$ for all $g \in G$.

*Proof.* Let $\mathbf{B}$ denote the class of $C^*$-algebras consisting of all $C^*$-algebras $A$ such that $A$ is isomorphic to a direct limit of the form $\varinjlim(A_n, \iota_n)$, where each of the $A_n$ is a 1-dimensional NCCW-complex with $K_1(A_n) = 0$, and each of the $\iota_n\colon A_n \to A_{n+1}$ is unital. By Theorem 1 in [230], the functor consisting of the Cuntz semigroup and the class of the unit classifies homomorphisms in $\mathbf{B}$. This theorem is then a consequence of Theorem X.4.7 above. □

Likewise, since the ordered $K_0$ group classifies homomorphisms of AF-algebras, we conclude the following.

**Theorem X.4.9.** Let $G$ be a compact group, let $A$ and $B$ be unital AF-algebras, and let $\alpha$ and $\beta$ be actions of $G$ on $A$ and $B$ respectively with the Rokhlin property. Then for every morphism $\phi\colon K_0(A) \to K_0(B)$ of partially ordered groups such that

$$\phi([1_A]) \leq [1_B] \quad \text{and} \quad \phi \circ K_0(\alpha_g) = K_0(\beta_g) \circ \phi \quad \text{for all } g \in G,$$

there exists a homomorphism $\theta\colon A \to B$ such that $\theta \circ \alpha_g = \beta_g \circ \theta$ for all $g \in G$. Moreover,

1. The homomorphism $\theta$ is unique up to equivariant approximate unitary equivalence.

2. The actions $\alpha$ and $\beta$ are conjugate if and only if there exists a positive isomorphism $\phi\colon K_0(A) \to K_0(B)$ such that $\phi^{-1} \circ K_0(\beta_g) \circ \phi = K_0(\alpha_g)$ for all $g \in G$.

3. When $A = B$, the actions $\alpha$ and $\beta$ are approximately inner conjugate if and only if $K_0(\alpha_g) = K_0(\beta_g)$ for all $g \in G$.

## Model Actions

We return to Example X.3.8 and prove that the action constructed there is unique up to equivariant isomorphism. We need some results on profinite groups first.

Compare the following definition with the usual definitions of subnormal and composition series, which are always taken to have finite length.

**Definition X.5.1.** Let $G$ be a totally disconnected compact group.

1. A *profinite subnormal series* is a sequence

$$G = H_1 \geq H_2 \geq \cdots$$

of subgroups of $G$ satisfying $\bigcap\limits_{n \in \mathbb{N}} H_n = \{e\}$ such that for all $n$ in $\mathbb{N}$, $H_{n+1}$ is normal in $H_n$ and $H_n/H_{n+1}$ is finite.

2. A *refinement* of a profinite subnormal series

$$G = H_1 \geq H_2 \geq \cdots$$

is another profinite subnormal series

$$G = K_1 \geq K_2 \geq \cdots$$

such that for all $n$ in $\mathbb{N}$ there exists $m$ in $\mathbb{N}$ such that $H_n = K_m$.

3. A *profinite composition series* is a profinite subnormal series

$$G = H_1 \geq H_2 \geq \cdots$$

where $H_n/H_{n+1}$ is simple for all $n$ in $\mathbb{N}$. The factors $H_n/H_{n+1}$ are called *composition factors* of $G$.

It is clear that only finite groups can have profinite subnormal series of finite length. In particular, this notion is really different from the usual one used in group theory. Nevertheless, we will show that profinite composition series have the same uniqueness property that composition series have.

**Remark X.5.2.** Using that the factors in a profinite subnormal series are finite, it is easy to see that any profinite subnormal series has a refinement which is a profinite composition series.

We will prove analogs of the classical results for composition series for profinite composition series. Our main technical device will be Zassenhaus lemma, whose statement we recall below.

**Lemma X.5.3.** (Zassenhaus) Let $G$ be a group and let $H_1, H_2, K_1, K_2$ be subgroups of $G$ such that $H_2$ is normal in $H_1$ and $K_2$ is normal in $K_1$. Then there is an isomorphism

$$\frac{H_2(H_1 \cap K_1)}{H_2(H_1 \cap K_2)} \cong \frac{K_2(H_1 \cap K_1)}{K_2(H_1 \cap K_2)}.$$

**Definition X.5.4.** Let $G$ be a totally disconnected compact group. Given two profinite subnormal sequences

$$G = H_1 \geq H_2 \geq \cdots \quad \text{and} \quad G = K_1 \geq K_2 \geq \cdots,$$

set $H^{(n)} = H_n/H_{n+1}$ and $K^{(n)} = K_n/K_{n+1}$ for all $n$ in $\mathbb{N}$. The profinite subnormal sequences are said to be *equivalent* if there is a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $H^{(n)} \cong K^{(\sigma(n))}$ for all $n$ in $\mathbb{N}$.

The following is the promised generalization of the Jordan-Hölder Theorem.

**Theorem X.5.5.** Let $G$ be a totally disconnected compact group.

1. Any two profinite subnormal series for $G$ have equivalent refinements.
2. Any two profinite composition series for $G$ are equivalent.

*Proof.* (1). Suppose that

$$G = H_1 \geq H_2 \geq \cdots \quad \text{and} \quad G = K_1 \geq K_2 \geq \cdots,$$

are two profinite subnormal series for $G$. Given $n$ in $\mathbb{N}$, we claim that there exists $m_n$ in $\mathbb{N}$ such that

$$H_{n+1}(H_n \cap K_{m_n}) = H_{n+1}.$$

To see this, notice that

$$H_{n+1} \leq \cdots \leq H_{n+1}(H_n \cap K_2) \leq H_{n+1}(H_n \cap K_1) = H_n$$

is a chain of subgroups, each of which is normal in the next one by the Second Isomorphism Theorem. Now, $H_n/H_{n+1}$ is a finite group and hence the sequence must eventually stabilize at $H_{n+1}$. This proves the claim.

For each $n$ in $\mathbb{N}$, we replace the pair $H_n \geq H_{n+1}$ by the finite subnormal series

$$H_n = H_{n+1}(H_n \cap K_1) \geq H_{n+1}(H_n \cap K_2) \geq \cdots \geq H_{n+1}(H_n \cap K_{m_n}) = H_{n+1}.$$

The resulting series is clearly a refinement of $G = H_1 \geq H_2 \geq \cdots$. This refinement has quotients of the form

$$\frac{H_{n+1}(H_n \cap K_m)}{H_{n+1}(H_n \cap K_{m+1})}$$

for $n$ and $m$ in $\mathbb{N}$.

Perform the analogous operation to the series $G = K_1 \geq K_2 \geq \cdots$ to obtain the corresponding refinement, whose quotients have the form

$$\frac{K_{m+1}(K_m \cap H_n)}{K_{m+1}(K_m \cap H_{n+1})}$$

for $n$ and $m$ in $\mathbb{N}$. It follows by Zassenhaus lemma that the quotients associated to these two refinements are isomorphic, and hence the refinements are equivalent.

(2). Given two profinite composition series for $G$, use (1) to find equivalent refinements. Since profinite composition series have no proper refinements, this shows that the two profinite composition series we started with are themselves equivalent. $\square$

**Notation X.5.6.** Let $G$ be a totally disconnected group and let $G = H_1 \geq H_2 \geq \cdots$ be a profinite subnormal series. Denote by $\mathcal{P}$ the set of all prime numbers. Define its associated supernatural number $S_{\{H_n\}_{n\in\mathbb{N}}} \colon \mathcal{P} \to \{0,\ldots,\infty\}$ by

$$S_{\{H_n\}_{n\in\mathbb{N}}}(p) = \begin{cases} \infty, & \text{if } p \text{ divides the order of } H_n/H_{n+1} \text{ for some } n \text{ in } \mathbb{N}; \\ 0, & \text{otherwise} \end{cases}$$

for $p \in \mathcal{P}$.

**Lemma X.5.7.** Let $G$ be a totally disconnected group.

1. Let $G = H_1 \geq H_2 \geq \cdots$ be a profinite subnormal series for $G$ and let $G = K_1 \geq K_2 \geq \cdots$ be a refinement. Then
$$S_{\{H_n\}_{n\in\mathbb{N}}} = S_{\{K_n\}_{n\in\mathbb{N}}}.$$
2. Any two profinite subnormal series have the same associated supernatural number.

In particular, the supernatural number associated to a profinite subnormal series is independent of the series, and depends only on the group $G$.

*Proof.* (1). For every $n$ in $\mathbb{N}$ there exist $m$ and $k$ in $\mathbb{N}$ such that

$$H_n \geq K_m \geq K_{m+1} \geq \cdots \geq K_{m+k} \geq H_{m+1}$$

is subnormal. It follows that

$$|H_n/H_{n+1}| = |K_m/K_{m+1}| \cdots |K_{m+k-1}/K_{m+k}| \,.$$

Thus, a prime number $p$ divides the order of $H_n/H_{n+1}$ for some $n$ in $\mathbb{N}$ if and only if it divides the order of $K_m/K_{m+1}$ for some $m$ in $\mathbb{N}$, showing that $S_{\{H_n\}_{n\in\mathbb{N}}} = S_{\{K_n\}_{n\in\mathbb{N}}}$.

(2). Given two profinite subnormal series, use part (2) of Theorem X.5.5 to find equivalent refinements, which by part (1) have the same associated supernatural number as the original series. Since equivalent profinite composition series have the same associated supernatural number, we conclude that the two profinite subnormal series have the same associated supernatural number.

The last claim follows immediately from (2). □

The following proposition implies that the algebra and the action constructed in Example X.3.8 are unique up to equivariant isomorphism.

**Proposition X.5.8.** Let $G$ be a totally disconnected compact group. Given two decreasing sequences $\{N_k\}_{k\in\mathbb{N}}$ and $\{N_k'\}_{k\in\mathbb{N}}$ of normal subgroups of $G$ with $\bigcap\limits_{k\in\mathbb{N}} N_k = \bigcap\limits_{k\in\mathbb{N}} N_k' = \{e\}$ and such that $G/N_k$ and $G/N_k'$ are finite for all $k \in \mathbb{N}$, denote by $(B, \beta)$ and $(B', \beta')$ the $G$-dynamical systems with the Rokhlin property obtained by applying the construction of Example X.3.8 to both sequences of subgroups.

Then there is an isomorphism $\theta \colon B \to B'$ such that $\theta \circ \beta_g = \beta_g' \circ \theta$ for all $g \in G$.

*Proof.* It follows from part (2) of Lemma X.5.7 above that the corresponding profinite subnormal series associated to $\{N_k\}_{k\in\mathbb{N}}$ and $\{N_k'\}_{k\in\mathbb{N}}$ have the same associated supernatural number. Since these are the supernatural numbers associated to $B$ and $B'$ respectively, it follows that there is an isomorphism $\varphi\colon B \to B'$. The order isomorphism $K_0(\varphi)\colon K_0(B) \to K_0(B')$ intertwines $K_0(\beta_g)$ and $K_0(\beta_g')$ for all $g \in G$, since these maps are the identity on the respective $K_0$-groups. By Theorem X.4.9, we conclude that $\beta$ and $\beta'$ are conjugate, as desired. □

If $G$ is a totally disconnected compact group, we denote by $D_G$ and $\mu^G \colon G \to \mathrm{Aut}(D_G)$ the (unique up to equivariant isomorphism) UHF-algebra and Rokhlin action constructed in Example X.3.8. Notice that $\mu_g^G$ is approximately inner for all $g \in G$.

**Remark X.5.9.** Let $G$ be a compact, totally disconnected group. Denote by $\mathcal{P}$ the set of all prime numbers. Then it is easy to check that the supernatural number $S_G \colon \mathcal{P} \to \{0, \ldots, \infty\}$ associated to the UHF-algebra $D_G$ is

$$S_G(p) = \begin{cases} \infty, & \text{if there is a finite quotient of } G \text{ whose order is divisible by } p; \\ 0, & \text{otherwise} \end{cases}$$

for $p \in \mathcal{P}$. In particular, it follows that $D_G$ is strongly self-absorbing; see [265].

As a consequence of our results, we are able to describe all Rokhlin actions of a compact group on a unital direct limit of 1-dimensional NCCW-complexes with trivial $K_1$-groups, whose induced action at the level of the Cuntz semigroup is trivial: they are all conjugate to the tensor product of the identity on the algebra with the model action $\mu^G$. See Theorem X.5.13.

Analogous results are available for finite groups (see Chapter VIII), and it was shown in Chapter IX that nothing like this can be true for circle actions with the Rokhlin property. Similar arguments can be used to show that there is no "generating" or model Rokhlin action (in the sense of Theorem X.5.13 below) for a compact group $G$ that is not totally disconnected, using that the restriction of a Rokhlin action of $G$ to the connected component of the unit $G_0$ must again have the Rokhlin property. In this sense, our results on model actions are the best possible: only totally disconnected compact groups admit model Rokhlin actions.

**Definition X.5.10.** Let $S$ be an abelian partially ordered semigroup and let $n \in \mathbb{N}$.

1. We say that $S$ is *$n$-divisible*, if multiplication by $n$, as a map $S \to S$ of partially ordered semigroups, is surjective.

2. We say that $S$ is *uniquely $n$-divisible*, if multiplication by $n$, as a map $S \to S$ of partially ordered semigroups, is an isomorphism.

**Remark X.5.11.** Let $S$ be an abelian partially ordered semigroup and let $n \in \mathbb{N}$. If $S$ is (uniquely) $n$-divisible and $d$ divides $n$, then $S$ is (uniquely) $d$-divisible. Indeed, write $n = dk$. Given $s$ in $S$, choose $t$ in $S$ such that $nt = s$. Then $d(kt)t = s$, showing that $S$ is $d$-divisible. Now suppose $S$ is uniquely $n$-divisible, and let $s, t$ in $S$ satisfy $ds \leq dt$. Then $ns = k(ds) \leq k(dt) = nt$, and thus $s \leq t$.

**Lemma X.5.12.** Let $S$ be an abelian partially ordered semigroup and that can be written as a direct limit $S = \varinjlim(S_k, \iota_k)$ in the category **Cu**, and let $n \in \mathbb{N}$. Assume that there exists $k_0 \in \mathbb{N}$ such that $S_k$ is uniquely $n$-divisible for all $k \geq k_0$. Then $S$ is uniquely $n$-divisible.

*Proof.* By dropping the first $k_0 - 1$ terms of the sequence of semigroups if necessary, we can assume that $k_0 = 1$. It was shown in Lemma 2.9 in [91] that multiplication by $n$ on $S$ is injective, so we just need to show that this map is surjective, which is easier. Let $s$ in $S$ and choose semigroup elements $s_k$ in $S_k$ for $k \in \mathbb{N}$ such that

$$\iota_k(s_k) \ll s_{k+1}, \quad s = \sup_{k \in \mathbb{N}} \iota_{k,\infty}(s_k).$$

Since multiplication by $n$ on $S_k$ is a surjection, it follows that there exist $t_k \in S_k$ for $k \in \mathbb{N}$ such that $nt_k = s_k$. It then follows that $(\iota_{k,\infty}(t_k))_{k\in\mathbb{N}}$ is increasing in $S$, and hence it has a limit

$$t = \sup_{k\in\mathbb{N}} \iota_{k,\infty}(t_k).$$

Now,

$$nt = n \sup_{k\in\mathbb{N}} \iota_{k,\infty}(t_k) = \sup_{k\in\mathbb{N}} \iota_{k,\infty}(nt_k) = \sup_{k\in\mathbb{N}} \iota_{k,\infty}(s_k) = s,$$

showing that $S$ is $n$-divisible. □

Compare the following with Theorem 3.5 and Theorem 3.6 in [133], and with Theorem 4.26 in [91].

**Theorem X.5.13.** Let $G$ be a totally disconnected compact group and let $A$ be a unital $C^*$-algebra that can be written as direct limit of a sequence of unital 1-dimensional NCCW-complexes with trivial $K_1$-groups. Then the following statements are equivalent:

1. There is an isomorphism $A \to A \otimes D_G$.

2. There is an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property such that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$.

3. There are actions of $G$ on $A$ with the Rokhlin property, and for any action $\beta\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property and for any action $\delta\colon G \to \mathrm{Aut}(A)$ such that $\mathrm{Cu}(\beta_g) = \mathrm{Cu}(\delta_g)$ for all $g$ in $G$, one has $(A, \beta) \cong (A\otimes D_G, \delta\otimes\mu^G)$, that is, there is an isomorphism $\varphi\colon A \to A\otimes D_G$ such that

$$\varphi \circ \beta_g = (\delta \otimes \mu^G)_g \circ \varphi$$

for all $g$ in $G$.

In particular, if the above statements hold for $A$ and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action with the Rokhlin property such that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$, then $(A, \alpha) \cong (A \otimes D_G, \mathrm{id}_A \otimes \mu^G)$.

*Proof.* The proofs of (1) implies (2), (1)+(2) imply (3), and (3) implies (1) are similar to the ones exhibited in the proof of Theorem 4.26 in [91].

(2) implies (1). Assume that there is an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property such that $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ for all $g \in G$. Choose a decreasing sequence $(H_n)_{n\in\mathbb{N}}$ of normal

subgroups of $G$ with $\bigcap_{n\in\mathbb{N}} H_n = \{e\}$ and such that the factor $G_n = G/H_n$ is finite for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$, denote by $d_n$ the cardinality of $G_n$. In order to show that $A$ absorbs the UHF-algebra $D_G$, it is enough, by Remark X.5.9, to show that $A$ absorbs the UHF-algebra of type $d_n^\infty$ for all $n \in \mathbb{N}$. Using Corollary 2.13 in [91] and the fact that $A$ can be written as direct limit of a sequence of 1-dimensional NCCW-complexes with trivial $K_1$-group, it is enough to show that $\mathrm{Cu}(A)$ is $d_n$-divisible. (In other words, $d_n$-divisibility implies *unique* $d_n$-divisibility in this context.)

We claim that $\mathrm{Cu}(\overline{\alpha}_{gH_n}) = \mathrm{id}_{\mathrm{Cu}(A^{H_n})}$ for all $gH_n$ in $G/H_n$ and for all $n$ in $\mathbb{N}$. To see this, given $g$ in $G$ and $n$ in $\mathbb{N}$, consider the commutative diagram in **Cu**

$$\begin{tikzcd}
\mathrm{Cu}(A^{H_n}) \arrow[r, "\mathrm{Cu}(\iota^{H_n})"] \arrow[d, "\mathrm{Cu}(\overline{\alpha}_{gH_n})"'] & \mathrm{Cu}(A) \arrow[d, "\mathrm{Cu}(\alpha_g)"] \\
\mathrm{Cu}(A^{H_n}) \arrow[r, "\mathrm{Cu}(\iota^{H_n})"'] & \mathrm{Cu}(A).
\end{tikzcd}$$

Since $G/H_n$ is finite, the restriction of $\alpha$ to $H_n$ has the Rokhlin property by part (1) of Proposition VI.2.4. It follows from part (1) of Theorem 3.14 in [85] that the horizontal arrows are order embeddings, and $\mathrm{Cu}(\alpha_g) = \mathrm{id}_{\mathrm{Cu}(A)}$ by assumption. It follows that $\mathrm{Cu}(\overline{\alpha}_{gH_n}) = \mathrm{id}_{\mathrm{Cu}(A^{H_n})}$, as desired. This proves the claim.

Fix $n \in \mathbb{N}$. For any $k \in \mathbb{N}$, it is true that $G_n$ is a factor of $G_{n+k}$, and thus it follows that $d_n$ divides $d_{n+k}$. The induced action $\overline{\alpha}\colon G_{n+k} \to \mathrm{Aut}\left(A^{H_{n+k}}\right)$ has the Rokhlin property by Proposition X.2.2. Since $G_{n+k}$ is a finite group and $\mathrm{Cu}(\overline{\alpha}_{gH_{n+k}}) = \mathrm{id}_{\mathrm{Cu}(A^{H_{n+k}})}$ by the claim above, it follows from Corollary 4.25 in [91] that $\mathrm{Cu}(A^{H_{n+k}})$ is $d_{n+k}$-divisible. Thus it is uniquely $d_{n+k}$-divisible, and therefore also uniquely $d_n$-divisible by Remark X.5.11.

Recall that if we denote by $\iota_n\colon A^{H_n} \to A^{H_{n+1}}$ the inclusion map, then $A$ can be written as the direct limit $A = \varinjlim(A^{H_n}, \iota_n)$. By continuity of the functor Cu, it follows that $\mathrm{Cu}(A)$ can be written as the direct limit

$$\mathrm{Cu}(A^G) \xrightarrow{\mathrm{Cu}(\iota_1)} \mathrm{Cu}(A^{H_2}) \xrightarrow{\mathrm{Cu}(\iota_2)} \cdots \longrightarrow \mathrm{Cu}(A)$$

in the category **Cu**. Using that $\mathrm{Cu}(A^{H_{n+k}})$ is uniquely $d_n$-divisible for all $k \in \mathbb{N}$ together with Lemma X.5.12, it follows that $\mathrm{Cu}(A)$ is uniquely $d_n$-divisible as well. This shows that $A$ absorbs the UHF-algebra $M_{d_n^\infty}$ for all $n$ in $\mathbb{N}$, and hence also $D_G \cong \bigotimes_{n\in\mathbb{N}} M_{d_n^\infty}$.

The last claim follows immediately from (3). $\square$

**Remark X.5.14.** Let $G$ be a totally disconnected group and $A$ be a unital $C^*$-algebra as in the theorem above. Assume that $A$ absorbs $D_G$ and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. It follows from Theorem X.5.13 that $\alpha$ absorbs the model action $\mu^G$ tensorially.

For AF-algebras, one can prove a result analogous to Theorem X.5.13 using $K$-theory instead of the Cuntz semigroup, and Elliott's classification results instead of Robert's. We present the statement but omit the proof.

**Theorem X.5.15.** Let $G$ be a totally disconnected compact group and let $A$ be a unital AF-algebra. Then the following statements are equivalent:

1. There is an isomorphism $A \to A \otimes D_G$.
2. There is an action $\alpha\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property such that $K_0(\alpha_g) = \mathrm{id}_{K_0(A)}$ for all $g \in G$.
3. For any action $\beta\colon G \to \mathrm{Aut}(A)$ with the Rokhlin property and for any action $\delta\colon G \to \mathrm{Aut}(A)$ such that $K_0(\beta_g) = K_0(\delta_g)$ for all $g$ in $G$, one has $(A, \beta) \cong (A \otimes D_G, \delta \otimes \mu^G)$, that is, there is an isomorphism $\varphi\colon A \to A \otimes D_G$ such that

$$\varphi \circ \beta_g = (\delta \otimes \mu^G)_g \circ \varphi$$

for all $g$ in $G$.

In particular, if the above statements hold for $A$ and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action with the Rokhlin property such that $K_0(\alpha_g) = \mathrm{id}_{K_0(A)}$ for all $g \in G$, then $(A, \alpha) \cong (A \otimes D_G, \mathrm{id}_A \otimes \mu^G)$.

# CHAPTER XI

# CIRCLE ACTIONS ON UHF-ABSORBING $C^*$-ALGEBRAS

We study circle actions with the Rokhlin property, in relation to their restrictions to finite subgroups. We construct examples showing the following: the restriction of a circle action with the Rokhlin property (even on a real rank zero $C^*$-algebra), need not have the Rokhlin property; and even if every restriction of a given circle action has the Rokhlin property, the circle action itself need not have it. As a positive result, we show that the restriction of a circle action with the Rokhlin property to the subgroup $\mathbb{Z}_n$ has the Rokhlin property if the underlying algebra absorbs $M_{n^\infty}$. The condition on the algebra is also necessary in most cases of interest.

## Introduction

This chapter, which is based on [79], is devoted to the study of restrictions of circle actions with the Rokhlin property. Our main result is as follows: if $A$ is a separable, unital $C^*$-algebra that absorbs the UHF-algebra $M_{n^\infty}$, and if $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ is an action with the Rokhlin property, then the restriction of $\alpha$ to the finite cyclic group $\mathbb{Z}_n \subseteq \mathbb{T}$ has the Rokhlin property. See Theorem XI.2.17. The condition that $A$ absorb $M_{n^\infty}$ is shown to be necessary in most cases of interest. We also give examples of circle actions with the Rokhlin property such that *no* restriction to any finite cyclic group has the Rokhlin property; see Example XI.3.5 and Example XI.3.7. Additionally, Example XI.3.8 and Example XI.3.9 show that even if a circle action has the property that *every* restriction to a finite subgroup has the Rokhlin property, the action itself need not have the Rokhlin property, even on Kirchberg algebras satisfying the UCT.

Theorem XI.2.17, together with Izumi's classification of finite group actions with the Rokhlin property, will be used in subsequent work to classify circle actions on UHF-absorbing $C^*$-algebras.

## Restrictions of Circle Actions with the Rokhlin Property

This section is devoted to proving that for $n \in \mathbb{N}$, the restriction of a circle action with the Rokhlin property on a $M_{n^\infty}$-absorbing $C^*$-algebra to the finite cyclic group $\mathbb{Z}_n$ again has the Rokhlin property; see Theorem XI.2.17.

We give a rough outline of what our strategy will be. We will first focus on cyclic group actions which are restrictions of circle actions with the Rokhlin property. These have what we call the "unitary Rokhlin property", which is a weakening of the Rokhlin property of Definition XI.2.1, that asks for a unitary instead of projections; see Definition XI.2.6. Dual actions of actions with the unitary Rokhlin property can be completely characterized, and we do so in Proposition XI.2.11. The relevant notion is that of "strong approximate innerness"; see Definition XI.2.3. We will later show in (the proof of) Theorem XI.2.17 that, under a number of assumptions, every strongly approximately inner action of $\mathbb{Z}_n$ is approximately representable, which is the notion dual to the Rokhlin property, as was shown by Izumi in [132]. The conclusion is then that the original restriction, which a priori had the unitary Rokhlin property, actually has the Rokhlin property.

We begin this section by recalling the definition of the Rokhlin property for a finite group action on a unital $C^*$-algebra.

**Definition XI.2.1.** Let $A$ be a unital $C^*$-algebra, let $G$ be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that $\alpha$ has the *Rokhlin property* if for every $\varepsilon > 0$ and for every finite set $F \subseteq A$ there exist orthogonal projections $e_g$ in $A$ for $g$ in $G$ such that

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g$ and $h$ in $G$
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g$ in $G$ and all $a$ in $F$
3. $\sum\limits_{g\in G} e_g = 1$.

The definition of the Rokhlin property for finite group actions on $C^*$-algebras was originally introduced by Izumi in [132], although a similar notion has been studied by Herman and Jones in [113] for $\mathbb{Z}_2$ actions on UHF-algebras, and by Herman and Ocneanu in [114] for integer actions. The Rokhlin property also played a crucial role in the classification of finite group actions on von Neumann algebras.

The following is part of Proposition 2.14 in [202], and we include the proof for the convenience of the reader. This result should be compared with Example XI.3.5 and Example XI.3.7.

**Proposition XI.2.2.** Let $A$ be a unital $C^*$-algebra, let $G$ be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. If $H \subseteq G$ is a subgroup, then $\alpha|_H$ has the Rokhlin property.

*Proof.* Set $n = \mathrm{card}(G/H)$. Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, choose projections $e_g$ for $g$ in $G$ as in the definition of the Rokhlin property for $F$ and $\frac{\varepsilon}{n}$. We claim that the projections $f_h = \sum\limits_{\overline{x}\in G/H} e_{hx}$ for $h$ in $H$, form a family of Rokhlin projections for the action $\alpha|_H$, the finite set $F$ and tolerance $\varepsilon$.

Given $h$ and $k$ in $H$, we have

$$\begin{aligned}\|\alpha_k(f_h) - f_{kh}\| &= \left\| \sum_{\overline{x}\in G/H} \alpha_k(e_{hx}) - e_{khx} \right\| \\ &\leq \sum_{\overline{x}\in G/H} \|\alpha_k(e_{hx}) - e_{khx}\| \leq \mathrm{card}(G/H)\frac{\varepsilon}{n} = \varepsilon.\end{aligned}$$

Finally, for $a$ in $F$ and $h$ in $H$, we have

$$\|af_h - f_h a\| \leq \sum_{\overline{x}\in G/H} \|ae_{hx} - e_{hx}a\| < \varepsilon.$$

□

The following is Definition 3.6 in [132].

**Definition XI.2.3.** Let $B$ be a unital $C^*$-algebra, and let $\beta$ be an action of a finite abelian group $G$ on $B$.

1. We say that $\beta$ is *strongly approximately inner* if there exist unitaries $u(g)$ in $(B^\beta)^\infty$, for $g$ in $G$, such that
$$\beta_g(b) = u(g)bu(g)^*$$
for $b$ in $B$ and $g$ in $G$.

2. We say that $\beta$ is *approximately representable* if $\beta$ is strongly approximately inner and the unitaries $u(g)$ for $g \in G$ as in (1) above, can be chosen to form a representation of $G$ in $(B^\beta)^\infty$.

**Notation XI.2.4.** Let $B$ be a $C^*$-algebra, let $G$ be a cyclic group (that is, either $\mathbb{Z}$ or $\mathbb{Z}_n$ for some $n$ in $\mathbb{N}$), and let $\beta\colon G \to \mathrm{Aut}(B)$ be action of $G$ on $B$. We will usually make a slight abuse of notation and also denote by $\beta$ the generating automorphism $\beta_1$.

If $G$ is a finite cyclic group, we have the following characterization of strong approximate innerness in terms of elements in $B$, rather than in $(B^\beta)^\infty$.

**Lemma XI.2.5.** Let $B$ be a separable, unital $C^*$-algebra, let $n \in \mathbb{N}$, and let $\beta$ be an action of $\mathbb{Z}_n$ on $B$. Then $\beta$ is strongly approximately inner if and only if for every finite subset $F \subseteq B$ and every $\varepsilon > 0$, there is a unitary $w$ in $\mathcal{U}(B)$ such that $\|\beta(w) - w\| < \varepsilon$ and $\|\beta(b) - wbw^*\| < \varepsilon$ for all $b$ in $F$. Moreover, $\beta$ is approximately representable if and only if the unitary $w$ above can be chosen so that $w^n = 1$.

*Proof.* Assume that $\beta$ is strongly approximately inner. Use a standard perturbation argument to choose a sequence $(u_m)_{m\in\mathbb{N}}$ of unitaries in $B^\beta$ that represents $u(1)$ in $(B^\beta)^\infty$. Then $\lim\limits_{m\to\infty} \|\beta(u_m) - u_m\| = 0$, and for $b$ in $B$, we have $\lim\limits_{m\to\infty} \|\beta(b) - u_m b u_m^*\| = 0$.

Given a finite set $F \subseteq B$ and $\varepsilon > 0$, choose $M \in \mathbb{N}$ such that $\|\beta(u_M) - u_M\| < \varepsilon$ and $\|\beta(b) - u_M b u_M^*\| < \varepsilon$ for all $b$ in $F$, and set $w = u_M$.

Conversely, for $m$ in $\mathbb{N}$, set $\varepsilon = \frac{1}{m}$ and let $w_m$ be as in the statement. Then

$$u = \overline{(w_m)_{m\in\mathbb{N}}} \in (B^\beta)^\infty$$

satisfies $\beta(b) = ubu^*$ for all $b$ in $F$, and hence $\beta$ is strongly approximately inner.

For the second statement, observe that a unitary of order $n$ in $(B^\beta)^\infty$ can be lifted to a sequence unitaries of order $n$ in $B^\beta$. Indeed, a standard functional calculus argument shows that if $v$ is a unitary in $B^\beta$ such that $\|v^n - 1\|$ is small, then $v$ is close to a unitary $\widetilde{v}$ in $B^\beta$ such that $\widetilde{v}^n = 1$. □

Theorem VI.4.2 asserts that the Rokhlin property and approximate representability are dual notions. It is natural to ask what condition on $\beta$ is equivalent to its dual action being strongly approximately inner. Such a condition will necessarily be weaker than the Rokhlin property. We define the relevant notion below.

**Definition XI.2.6.** Let $B$ be a unital $C^*$-algebra, let $n \in \mathbb{N}$ and let $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ be an action. We say that $\beta$ has the *unitary Rokhlin property* if for every $\varepsilon > 0$ and for every finite subset $F \subseteq B$, there exists $u \in \mathcal{U}(B)$ such that $\|ub - bu\| < \varepsilon$ for all $b$ in $F$ and $\left\|\beta_k(u) - e^{2\pi ik/n}u\right\| < \varepsilon$ for all $k \in \mathbb{Z}_n$.

Let $A$ be a unital $C^*$-algebra. Given a continuous action $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$, and $n \in \mathbb{N}$, we denote by $\alpha|_n$ the restriction $\alpha|_{\mathbb{Z}_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ of $\alpha$ to

$$\{1, e^{2\pi i/n}, \ldots, e^{2\pi i(n-1)/n}\} \cong \mathbb{Z}_n.$$

Recall that if $v$ is the canonical unitary in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ implementing $\alpha|_n$, then the dual action

$$\widehat{\alpha|_n}\colon \mathbb{Z}_n \cong \widehat{\mathbb{Z}_n} \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$$

of $\alpha|_n$ is given by $\left(\widehat{\alpha|_n}\right)_k (a) = a$ for all $a$ in $A$ and $\left(\widehat{\alpha|_n}\right)_k (v) = e^{2\pi ik/n}v$ for all $k \in \mathbb{Z}_n$.

The following easy lemmas provide us with many examples of cyclic group actions with the unitary Rokhlin property.

**Lemma XI.2.7.** If $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ has the Rokhlin property, then $\alpha|_n$ has the unitary Rokhlin property for all $n \in \mathbb{N}$.

*Proof.* Given $\varepsilon > 0$ and a finite subset $F \subseteq A$, choose a unitary $u$ in $\mathcal{U}(A)$ such that $\|ua - au\| < \varepsilon$ for all $a$ in $F$ and $\|\alpha_\zeta(u) - \zeta u\| < \varepsilon$ for all $\zeta \in \mathbb{T}$. If $n \in \mathbb{N}$, then

$$\left\|(\alpha|_n)_k(u) - e^{2\pi ik/n}u\right\| = \left\|\alpha_{e^{2\pi ik/n}}(u) - e^{2\pi ik/n}u\right\| < \varepsilon$$

for all $k \in \mathbb{Z}_n$, as desired. □

**Lemma XI.2.8.** If $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ has the Rokhlin property, then it has the unitary Rokhlin property.

*Proof.* Given $\varepsilon > 0$ and a finite subset $F \subseteq B$, choose projections $e_0, \ldots, e_{n-1}$ as in the definition of the Rokhlin property for the tolerance $\frac{\varepsilon}{n}$ and the finite set $F$, and set $u = \sum\limits_{j=0}^{n-1} e^{-2\pi ij/n}e_j$. Then $u$ is a unitary in $B$. Moreover, $\|ub - bu\| < \varepsilon$ for all $b$ in $F$ and

$$\left\|\beta_k(u) - e^{2\pi ik/n}u\right\| = \left\|\sum_{j=0}^{n-1} e^{2\pi ij/n}\beta_k(e_j) - e^{2\pi ik/n}\sum_{j=0}^{n-1} e^{-2\pi ij/n}e_j\right\| < \varepsilon$$

since $\|\beta_k(e_j) - e_{j+k}\| < \frac{\varepsilon}{n}$ for all $j, k \in \mathbb{Z}_n$, and the projections $e_0, \ldots, e_{n-1}$ are pairwise orthogonal. $\square$

The converse of the preceding lemma is not in general true, since the unitary Rokhlin property does not ensure the existence of any non-trivial projections on the algebra. We present examples of this situation in Section XI.3.

We will nevertheless show that the restriction of an action of the circle with the Rokhlin property to any finite cyclic subgroup again has the Rokhlin property if the algebra is separable and absorbs the universal UHF-algebra $\mathcal{Q}$. See Corollary XI.2.18 below.

**Lemma XI.2.9.** Let $A$ be a separable unital $C^*$-algebra, let $n \in \mathbb{N}$ and let $\alpha\colon \mathbb{Z}_n \to \mathrm{Aut}(A)$ be an action of $\mathbb{Z}_n$ on $A$. Regard $\mathbb{Z}_n \subseteq \mathbb{T}$ as the $n$-th roots of unitry, and let $\gamma\colon \mathbb{Z}_n \to \mathrm{Aut}(C(\mathbb{T}))$ be the restriction of the action by left translation of $\mathbb{T}$ on $C(\mathbb{T})$. Let $\alpha_\infty\colon \mathbb{Z}_n \to \mathrm{Aut}(A_\infty \cap A')$ be the action on $A_\infty \cap A'$ induced by $\alpha$. Then $\alpha$ has the unitary Rokhlin property if and only if there exists a unital equivariant homomorphism

$$\varphi\colon (C(\mathbb{T}), \gamma) \to (A_\infty \cap A', \alpha_\infty).$$

*Proof.* Choose an increasing sequence $(F_m)_{m\in\mathbb{N}}$ of finite subsets of $A$ such that $\overline{\bigcup_{m\in\mathbb{N}} F_m} = A$. For each $m \in \mathbb{N}$, there exists a unitary $u_m$ in $A$ such that

$$\|u_m a - a u_m\| < \frac{1}{m} \quad \text{and} \quad \left\|\alpha_j(u_m) - e^{2\pi i j/n} u_m\right\| < \frac{1}{m}$$

for every $a$ in $F_m$ and for every $j$ in $\mathbb{Z}_n$. Denote by $u = \overline{(u_m)_{m\in\mathbb{N}}}$ the image of the sequence of unitaries $(u_m)_{m\in\mathbb{N}}$ in $A_\infty$. Then $u$ belongs to the relative commutant of $A$ in $A_\infty$. Consider the unital map $\varphi\colon C(\mathbb{T}) \to A_\infty \cap A'$ given by $\varphi(f) = f(u)$ for $f$ in $C(\mathbb{T})$. One checks that

$$\alpha_j(\varphi(f)) = \varphi(\gamma_{e^{2\pi i j/n}}(f))$$

for all $j \in \mathbb{Z}_n$ and all $f$ in $C(\mathbb{T})$, so $\varphi$ is equivariant.

Conversely, assume that there is an equivariant unital homomorphism

$$\varphi\colon C(\mathbb{T}) \to A_\infty \cap A'.$$

Let $z \in C(\mathbb{T})$ be the unitary given by $z(\zeta) = \zeta$ for all $\zeta$ in $\mathbb{T}$, and let $v = \varphi(z)$. By semiprojectivity of $C(\mathbb{T})$, we can choose a representing sequence $(v_m)_{m\in\mathbb{N}}$ in $\ell^\infty(\mathbb{N}, A)$ consisting of unitaries. It follows that

$$\lim_{m\to\infty} \left\| \alpha_j(v_m) - e^{2\pi i j/n} v_m \right\| = 0 = \lim_{m\to\infty} \|v_m a - a v_m\|$$

for every $a$ in $A$, and this is clearly equivalent to $\alpha$ having the unitary Rokhlin property. □

The following result is analogous to Proposition IX.2.4, and so is its proof.

**Proposition XI.2.10.** Let $B$ be a separable, unital $C^*$-algebra, let $n \in \mathbb{N}$ and let $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ be an action on $B$. Then $\beta$ has the unitary Rokhlin property if and only if for every finite set $F \subseteq B$ and every $\varepsilon > 0$, there is a unitary $u \in \mathcal{U}(B)$ such that

1. $\beta_k(u) = e^{2\pi i k/n} u$ for all $k \in \mathbb{Z}_n$;
2. $\|ub - bu\| < \varepsilon$ for all $b$ in $F$.

Similarly to what was pointed out after the statement of Proposition IX.2.4, the definition of the unitary Rokhlin property differs in that in condition (1), one only requires $\|\beta_k(u) - e^{2\pi i k/n} u\| < \varepsilon$ for all $k \in \mathbb{Z}_n$.

*Proof.* Recall that $(C(\mathbb{T}), \mathbb{T}, \mathtt{Lt})$ is equivariantly semiprojective. Since the quotient $\mathbb{T}/\mathbb{Z}_n$ is compact, it follows from Theorem 3.11 in [213] that the restriction $(C(\mathbb{T}), \mathbb{Z}_n, \mathtt{Lt})$ is equivariantly semiprojective as well. The result now follows using an argument similar to the one used in the proof of Proposition IX.2.4. The details are left to the reader. □

**Proposition XI.2.11.** Let $n$ in $\mathbb{N}$ and let $\beta\colon \mathbb{Z}_n \to \mathrm{Aut}(B)$ be an action of $\mathbb{Z}_n$ on a unital separable $C^*$-algebra $B$.

1. The action $\beta$ has the unitary Rokhlin property if and only if its dual action $\widehat{\beta}$ is strongly approximately inner.
2. The action $\beta$ is strongly approximately inner if and only if its dual action $\widehat{\beta}$ has the unitary Rokhlin property.

*Proof.* Part (a). Assume that $\beta$ has the unitary Rokhlin property. Use Lemma XI.2.9 to choose a unital equivariant homomorphism $\varphi\colon C(\mathbb{T}) \to B_\infty \cap B'$. Denote by $u \in B_\infty \cap B'$ the image under this homomorphism of the unitary $z \in C(\mathbb{T})$ given by $z(\zeta) = \zeta$ for $\zeta \in \mathbb{T}$, and denote by $\lambda$ the implementing unitary representation of $\mathbb{Z}_n$ in $B \rtimes_\beta \mathbb{Z}_n$ for $\beta$. In $(B \rtimes_\beta \mathbb{Z}_n)_\infty$, we have $u^*\lambda_j u = e^{2\pi ij/n}\lambda_j$ for all $j \in \mathbb{Z}_n$, and $ub = bu$ (that is, $ubu^* = b$) for all $b$ in $B$. Therefore, if $\beta$ has the unitary Rokhlin property, then $\widehat{\beta}$ is implemented by $u^*$, and thus it is approximately representable. The converse follows from the same computation, as we have $(B \rtimes_\beta \mathbb{Z}_n)^{\widehat{\beta}} = B$.

Part (b). Denote by $v$ the canonical unitary in the crossed product, and assume that $\beta$ is strongly approximately inner. Let $F \subseteq B \rtimes_\beta \mathbb{Z}_n$ be a finite subset, and let $\varepsilon > 0$. Since $B$ and $v$ generate $B \rtimes_\beta \mathbb{Z}_n$, we can assume that there is a finite subset $F' \subseteq B$ such that $F = F' \cup \{v\}$. Choose $w \in \mathcal{U}(B)$ such that $\|\beta(w) - w\| < \varepsilon$ and $\|\beta(b) - wbw^*\| < \varepsilon$ for all $b$ in $F'$. Since $\beta(b) = vbv^*$ for every $b$ in $B$, if we let $u = w^*v$, the first of these conditions is equivalent to $\|vu - uv\| < \varepsilon$, while the second one is equivalent to $\|ub - bu\| < \varepsilon$ for all $b$ in $F'$. On the other hand, $\widehat{\beta}_k(u) = \widehat{\beta}_k(w^*v) = w^*(e^{2\pi ik/n}v) = e^{2\pi ik/n}u$ for $k \in \mathbb{Z}_n$. Thus, $u$ is the desired unitary, and $\widehat{\beta}$ has the unitary Rokhlin property.

Conversely, assume that $\widehat{\beta}$ has the unitary Rokhlin property. Let $F' \subseteq B$ be a finite subset, and let $\varepsilon > 0$. Set $F = F' \cup \{v\}$. Use Proposition XI.2.10 to choose $u$ in the unitary group of $A \rtimes_\beta \mathbb{Z}_n$ such that $\|ub - bu\| < \varepsilon$ for all $b$ in $F$, and $\widehat{\beta}_k(u) = e^{2\pi ik/n}u$ for all $k \in \mathbb{Z}_n$. Set $w = vu^*$. Then $w \in B$ since

$$\widehat{\beta}_k(w) = e^{2\pi ik/n}v\overline{e^{2\pi ik/n}}u^* = vu^* = w$$

for all $k \in \mathbb{Z}_n$ and $(B \rtimes_\beta \mathbb{Z}_n)^{\mathbb{Z}_n} = B$. On the other hand,

$$\|\beta(b) - wbw^*\| = \|vbv^* - vu^*buv^*\| = \|b - u^*bu\| = \|ub - bu\| < \varepsilon,$$

for all $b$ in $F$. Hence $w$ is an implementing unitary for $F'$ and $\varepsilon$, and $\beta$ is strongly approximately inner. ∎

**Corollary XI.2.12.** Let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property, and let $n \in \mathbb{N}$. Then $\widehat{\alpha|_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ is strongly approximately inner.

*Proof.* The restriction $\alpha|_n$ has the unitary Rokhlin property by Lemma XI.2.7, and by part (a) of Proposition XI.2.11, its dual $\widehat{\alpha|_n}$ is strongly approximately inner. ∎

The next ingredient needed is showing that crossed products by restrictions of Rokhlin actions of compact groups preserve absorption of strongly self-absorbing $C^*$-algebras. For Rokhlin actions, this was shown by Hirshberg and Winter in [122]. The more general statement is proved using similar ideas.

**Proposition XI.2.13.** Let $A$ be a separable unital $C^*$-algebra, let $G$ be a compact Hausdorff second countable group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action satisfying the Rokhlin property. Let $H$ be a closed subgroup of $G$. If $B$ is a unital separable $C^*$-algebra which admits a central sequence of unital homomorphisms into $A$, then $B$ admits a unital homomorphism into the fixed point subalgebra of $\alpha|_H$ in $A_\infty \cap A'$.

*Proof.* Notice that $(A_\infty \cap A')^\alpha$ is a subalgebra of $(A_\infty \cap A')^{\alpha|_H}$. The result now follows from Theorem 3.3 in [122]. □

**Remark XI.2.14.** In the proposition above, if $B$ is moreover assumed to be simple, for example if it is strongly self-absorbing, it follows that the unital homomorphism obtained is actually an embedding, since it is not zero.

Recall the following result by Hirshberg and Winter.

**Lemma XI.2.15.** (Lemma 2.3 of [122].) Let $A$ and $\mathcal{D}$ be unital separable $C^*$-algebras. Let $G$ be a compact group and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action. If there is a unital homomorphism $\mathcal{D} \to (A_\infty \cap A')^G$, then there is a unital homomorphism

$$\mathcal{D} \to (M(A \rtimes_\alpha G))_\infty \cap (A \rtimes_\alpha G)'.$$

**Theorem XI.2.16.** Let $\mathcal{D}$ be a strongly self-absorbing $C^*$-algebra, let $A$ be a $\mathcal{D}$-absorbing, separable unital $C^*$-algebra, and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Then, for every $n \in \mathbb{N}$, the crossed product $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ is a unital separable $\mathcal{D}$-absorbing $C^*$-algebra.

*Proof.* By Theorem 7.2.2 in [235], there exists a unital embedding of $\mathcal{D}$ into $A_\infty \cap A'$, which is equivalent to the existence of a central sequence of unital embeddings of $\mathcal{D}$ into $A$. Use Proposition XI.2.13 to obtain a unital homomorphism of $\mathcal{D}$ into the fixed point subalgebra of $\alpha|_{\mathbb{Z}_n}$ in $A_\infty \cap A'$. It follows that this homomorphism is actually an embedding, since it is not zero and

$\mathcal{D}$ is simple, by Theorem 1.6 in [265]. Lemma 2.3 in [122] (here reproduced as Lemma XI.2.15) provides us with a unital embedding of $\mathcal{D}$ into $(A \rtimes_\alpha \mathbb{Z}_n)_\infty \cap (A \rtimes_\alpha \mathbb{Z}_n)'$, which again by Theorem 7.2.2 in [235] is equivalent to $A \rtimes_\alpha \mathbb{Z}_n$ being $\mathcal{D}$-absorbing, since $\mathcal{D}$ is strongly self-absorbing. $\square$

The following is the main theorem of this section.

**Theorem XI.2.17.** Let $A$ be a separable unital $C^*$-algebra, let $n \in \mathbb{N}$ and let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property. Suppose that $A$ absorbs $M_{n^\infty}$. Then $\alpha|_n$ has the Rokhlin property.

*Proof.* By Theorem VI.4.2, it is enough to show that $\widehat{\alpha|_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ is approximately representable. Recall that by Corollary XI.2.12, the action $\widehat{\alpha|_n}$ is strongly approximately inner. In view of Lemma 3.10 in [132], to show that it is in fact approximately representable, it is enough to show that there is a unital map

$$M_n \to \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}}\right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)',$$

where the relative commutant is taken in $(A \rtimes_{\alpha|_n} \mathbb{Z}_n)_\infty$.

<u>**Claim:**</u> $\left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}}\right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' = (A_\infty \cap A')^{(\alpha|_n)_\infty}$.
Since $(A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}} = A$, we have

$$\begin{aligned}&\left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}}\right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' = A_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)' \\ &= \left\{\overline{(a_m)}_{m\in\mathbb{N}} \in A_\infty \colon \lim_{m\to\infty} \|a_m x - x a_m\| = 0 \text{ for all } x \in A \rtimes_{\alpha|_n} \mathbb{Z}_n\right\}.\end{aligned}$$

Let $v$ be the canonical unitary in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ that implements $\alpha|_n$ in the crossed product. Then for a bounded sequence $(a_m)_{m\in\mathbb{N}}$ in $A$, the condition $\lim\limits_{m\to\infty} \|a_m x - x a_m\| = 0$ for all $x$ in $A \rtimes_{\alpha|_n} \mathbb{Z}_n$ is equivalent to $\lim\limits_{m\to\infty} \|a_m a - a a_m\| = 0$ for all $a$ in $A$ and $\lim\limits_{m\to\infty} \|a_m v - v a_m\| = 0$. The above set is therefore equal to

$$\left\{\overline{(a_m)}_{m\in\mathbb{N}} \in A_\infty \colon \begin{array}{c} \lim\limits_{m\to\infty} \|a_m a - a a_m\| = 0 \text{ for all } a \in A \text{ and} \\ \lim\limits_{m\to\infty} \|(\alpha|_n)(a_m) - a_m\| = 0 \end{array}\right\},$$

which is precisely the same as the subset of $A_\infty \cap A'$ that is fixed under the action on $A_\infty \cap A'$ induced by $\alpha|_n$. This proves the claim.

Since $A$ absorbs the UHF-algebra $M_{n^\infty}$, it follows that there is a unital embedding $\iota\colon M_n \to A_\infty \cap A'$. By Proposition XI.2.13, there is a unital homomorphism $M_n \to (A_\infty \cap A')^{(\alpha|_n)_\infty}$. Using the claim above, we conclude that there is a unital homomorphism

$$M_n \to \left((A \rtimes_{\alpha|_n} \mathbb{Z}_n)^{\widehat{\alpha|_n}}\right)_\infty \cap (A \rtimes_{\alpha|_n} \mathbb{Z}_n)'.$$

This homomorphism is necessarily an embedding, since it is not zero. Apply Lemma 3.10 in [132] to the action $\widehat{\alpha|_n}\colon \mathbb{Z}_n \to \mathrm{Aut}(A \rtimes_{\alpha|_n} \mathbb{Z}_n)$ to conclude that $\widehat{\alpha|_n}$ is approximately representable, and hence that $\alpha|_n$ has the Rokhlin property, by Theorem VI.4.2. $\square$

Denote by $\mathcal{Q}$ the universal UHF-algebra, that is, the unique, up to isomorphism, UHF-algebra with $K$-theory

$$(K_0(\mathcal{Q}), [1_\mathcal{Q}]) \cong (\mathbb{Q}, 1).$$

It is well-known that $\mathcal{Q} \otimes M_{n^\infty} \cong \mathcal{Q}$ for all $n$ in $\mathbb{N}$, and that $\mathcal{Q} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

**Corollary XI.2.18.** Let $A$ be a separable, $\mathcal{Q}$-absorbing unital $C^*$-algebra, let $\alpha\colon \mathbb{T} \to \mathrm{Aut}(A)$ be an action with the Rokhlin property and let $n \in \mathbb{N}$. Then $\alpha|_n$ has the Rokhlin property. In particular, restrictions of circle actions with the Rokhlin property on separable, unital $\mathcal{O}_2$-absorbing $C^*$-algebras, again have the Rokhlin property.

## Counterexamples

In this section, we present some examples related to Theorem XI.2.17.

### *Nonexistence of actions with the Rokhlin property*

The goal of this subsection is to prove that UHF-algebras do not admit any direct limit action of the circle with the Rokhlin property; see Theorem XI.3.4. We need some preliminary results.

**Notation XI.3.1.** Let $n \in \mathbb{N}$. We denote by $\mathcal{U}_n(\mathbb{C})$ the unitary group of $M_n$. Identify $\mathbb{T}$ with the center $\mathcal{Z}(\mathcal{U}_n(\mathbb{C}))$ of $\mathcal{U}_n(\mathbb{C})$ via the map $\zeta \mapsto \mathrm{diag}(\zeta, \ldots, \zeta)$, and denote by $P\mathcal{U}_n(\mathbb{C})$ the quotient group $P\mathcal{U}_n(\mathbb{C}) = \mathcal{U}_n(\mathbb{C})/\mathbb{T}$.

**Proposition XI.3.2.** Let $n \in \mathbb{N}$ and let $\gamma \colon \mathbb{T} \to \mathrm{Aut}(M_n)$ be a continuous action. Then there exists a continuous map $v \colon \mathbb{T} \to \mathcal{U}_n(\mathbb{C})$ such that $\gamma_\zeta = \mathrm{Ad}(v(\zeta))$ for all $\zeta$ in $\mathbb{T}$.

*Proof.* Recall that every automorphism of $M_n$ is inner, so that for every $\zeta \in \mathbb{T}$ there exists a unitary $u(\zeta) \in \mathcal{U}_n(\mathbb{C})$ such that $\alpha_\zeta = \mathrm{Ad}(u(\zeta))$. Moreover, $u(\zeta)$ is uniquely determined up to multiplication by elements of $\mathbb{T} = \mathcal{Z}(\mathcal{U}_n(\mathbb{C}))$ and hence $\gamma_\zeta$ determines a continuous group homomorphism $u \colon \mathbb{T} \to P\mathcal{U}_n(\mathbb{C})$. Denote by $\rho \colon \mathcal{U}_n(\mathbb{C}) \to P\mathcal{U}_n(\mathbb{C})$ the canonical projection. We want to solve the following lifting problem:

$$\begin{tikzcd} & \mathcal{U}_n(\mathbb{C}) \arrow[d, "\rho"] \\ \mathbb{T} \arrow[ur, dashed, "v"] \arrow[r, "u"'] & P\mathcal{U}_n(\mathbb{C}). \end{tikzcd}$$

The map $u$ determines an element $[u] \in \pi_1(P\mathcal{U}_n(\mathbb{C}))$ and $\rho$ induces a group homomorphism $\pi_1(\rho) \colon \pi_1(\mathcal{U}_n(\mathbb{C})) \to \pi_1(P\mathcal{U}_n(\mathbb{C}))$. The quotient map $\rho \colon \mathcal{U}_n(\mathbb{C}) \to P\mathcal{U}_n(\mathbb{C})$ is actually a fiber bundle, since $\mathcal{U}_n(\mathbb{C})$ is a manifold and the action of $\mathbb{T}$ on $\mathcal{U}_n(\mathbb{C})$ is free. See the Theorem in Section 4.1 of [192]. The long exact sequence in homotopy for this fiber bundle is

$$\cdots \longrightarrow \pi_1(\mathbb{T}) \longrightarrow \pi_1(\mathcal{U}_n(\mathbb{C})) \xrightarrow{\pi_1(\rho)} \pi_1(P\mathcal{U}_n(\mathbb{C})) \longrightarrow \pi_0(\mathbb{T}).$$

Recall that $\pi_1(\mathcal{U}_n(\mathbb{C})) \cong \mathbb{Z}$, and that $\pi_0(\mathbb{T}) \cong 0$. The map $\pi_1(\mathbb{T}) \to \pi_1(\mathcal{U}_n(\mathbb{C}))$ is induced by $\zeta \mapsto \mathrm{diag}(\zeta, \ldots, \zeta)$, which on $\pi_1$ corresponds to multiplication by $n$. In other words, the above exact sequence is

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\pi_1(\rho)} \pi_1(P\mathcal{U}_n(\mathbb{C})) \longrightarrow 0,$$

which implies that $\pi_1(P\mathcal{U}_n(\mathbb{C})) \cong \mathbb{Z}_n$ and that the map $\pi_1(\rho)$ is surjective. It follows that $u$ is homotopic to a map $\widehat{u} \colon \mathbb{T} \to P\mathcal{U}_n(\mathbb{C})$ that is liftable. The homotopy lifting property for fiber bundles implies that $u$ itself is liftable, that is, there exists a continuous map $v \colon \mathbb{T} \to \mathcal{U}_n(\mathbb{C})$

such that $u(\zeta) = \rho(v(\zeta))$ for all $\zeta \in \mathbb{T}$. (See the paragraph below Theorem 4.41 in [111] for the definition of the homotopy lifting property. Proposition 4.48 in [111] shows that every fiber bundle has this property.) This concludes the proof. $\square$

**Lemma XI.3.3.** Let $n \in \mathbb{N}$ and let $v\colon \mathbb{T} \to \mathcal{U}_n(\mathbb{C})$ be a continuous map. Then for every $u$ in $\mathcal{U}_n(\mathbb{C})$, there exists $\zeta$ in $\mathbb{T}$ such that

$$\|v(\zeta)uv(\zeta)^* - \zeta u\| \geq 2.$$

*Proof.* Assume that there exists $u \in \mathcal{U}_n(\mathbb{C})$ such that $\|v(\zeta)uv(\zeta)^* - \zeta u\| < 2$ for all $\zeta \in \mathbb{T}$. Define $w \in C(\mathbb{T}, M_n)$ by $w(\zeta) = \overline{\zeta}v(\zeta)uv(\zeta)^*u^*$ for all $\zeta$ in $\mathbb{T}$. Then $w$ is a unitary in $C(\mathbb{T}, M_n)$ and $\left\|w - 1_{C(\mathbb{T},M_n)}\right\| < 2$. It follows that $w$ is homotopic to $1_{C(\mathbb{T},M_n)}$. Define a continuous functions $f\colon \mathbb{T} \to \mathbb{T}$ by $f = \det \circ w$. Then $f$ is homotopic to the constant map, and thus its winding number is zero.

On the other hand,

$$f(\zeta) = \det(w(\zeta)) = \det(\overline{\zeta}v(\zeta)uv(\zeta)^*u^*) = \overline{\zeta}^n,$$

so the winding number is actually $-n$. This is a contradiction, and the result follows. $\square$

**Theorem XI.3.4.** Assume that $A = \varinjlim(M_{k_n}, \iota_n)$ is an unital UHF-algebra with unital connecting maps. If $\alpha = \varinjlim \alpha^{(n)}$ is a direct limit action of the circle on $A$, then $\alpha$ does not have the Rokhlin property.

*Proof.* Assume that $\alpha$ has the Rokhlin property. Let $F \subseteq A$ be a finite set, and let $\varepsilon = 2$. A standard approximation argument shows that there exist $n \in \mathbb{N}$ and $u \in \mathcal{U}_{k_n}(\mathbb{C})$ such that

$$\|ua - au\| < 2 \quad \text{and} \quad \left\|\alpha_\zeta^{(n)}(u) - \zeta u\right\| < 2$$

for all $a$ in $F$ and for all $\zeta$ in $\mathbb{T}$. By Proposition XI.3.2, there is a continuous map $v\colon \mathbb{T} \to \mathcal{U}_{k_n}(\mathbb{C})$ such that $\alpha_\zeta^{(n)} = \mathrm{Ad}(v(\zeta))$ for all $\zeta \in \mathbb{T}$. Now, Lemma XI.3.3 implies that $\|v(\zeta)uv(\zeta)^* - \zeta u\| \geq 2$ for all $\zeta \in \mathbb{T}$. Therefore, $2 > \left\|\alpha_\zeta^{(n)}(u) - \zeta u\right\| \geq 2$, which is a contradiction. Thus, $\alpha$ does not have the Rokhlin property. $\square$

We begin by exhibiting examples of circle actions with the Rokhlin property whose restrictions to cyclic subgroups do not have the Rokhlin property. The first one is a rather trivial one:

**Example XI.3.5.** Consider the action of left translation of $\mathbb{T}$ on $C(\mathbb{T})$. It has the Rokhlin property, so its restriction to any $\mathbb{Z}_n \subseteq \mathbb{T}$ has the unitary Rokhlin property. However, no non-trivial finite group action on $C(\mathbb{T})$ can have the Rokhlin property since $C(\mathbb{T})$ has no non-trivial projections.

Besides merely the lack of projections, there are less obvious $K$-theoretic obstructions for the restrictions of an action with the Rokhlin property to have the Rokhlin property. See Example XI.3.7.

We need a lemma first.

**Proposition XI.3.6.** Let $G$ be a connected metric group, let $A$ be a unital $C^*$-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a continuous action (not necessarily with the Rokhlin property). Then $K_*(\alpha_g) = \mathrm{id}_{K_*(A)}$ for all $g$ in $G$.

*Proof.* We just prove it for $K_0$; the proof for $K_1$ is similar, or follows by replacing $(A, \alpha)$ with $(A \otimes B, \alpha \otimes \mathrm{id}_B)$, where $B$ is any $C^*$-algebra satisfying the UCT such that $K_0(B) = 0$ and $K_1(B) = \mathbb{Z}$, and using the Künneth formula. (For example, $B = C_0(\mathbb{R})$ will do.)

Denote the metric on $G$ by $d$. Let $n \in \mathbb{N}$ and let $p$ be a projection in $M_n(A)$. Set $\alpha^{(n)} = \alpha \otimes \mathrm{id}_{M_n}$, the augmentation of $\alpha$ to $M_n(A)$. Since $\alpha^{(n)}$ is continuous, there exists $\delta > 0$ such that $\left\|\alpha^{(n)}_g(p) - \alpha^{(n)}_h(p)\right\| < 1$ whenever $g$ and $h$ in $G$ satisfy $d(g,h) < \delta$. Since $\alpha^{(n)}_g(p)$ and $\alpha^{(n)}_h(p)$ are projections in $M_n(A)$, it follows that $\alpha^{(n)}_g(p)$ and $\alpha^{(n)}_h(p)$ are homotopic, and hence their classes in $K_0(A)$ agree, that is, $K_0(\alpha_g)([p]_0) = K_0(\alpha_h)([p]_0)$. Denote by $e$ the unit of $G$. Since $g$ and $h$ satisfying $d(g,h) < \delta$ are arbitrary, and since $G$ is connected, it follows that

$$K_0(\alpha_g)([p]_0) = K_0(\alpha_e)([p]_0) = [p]_0$$

for any $g$ in $G$. Since $p$ is an arbitrary projection in $A \otimes \mathcal{K}$, it follows that $K_0(\alpha_g) = \mathrm{id}_{K_0(A)}$ for all $g$ in $G$, as desired. □

**Example XI.3.7.** This is an example of a purely infinite simple separable nuclear unital $C^*$-algebra (in particular, with many projections), and an action of the circle on it satisfying the Rokhlin property, such that no restriction to a finite subgroup of $\mathbb{T}$ has the Rokhlin property.

Let $\{p_n\}_{n\in\mathbb{N}}$ be an enumeration of the prime numbers, and for every $n$ in $\mathbb{N}$, set $q_n = p_1 \cdots p_n$. Fix a countable dense subset $X = \{x_1, x_2, x_3, \ldots\}$ of $\mathbb{T}$ with $x_1 = 1$. For $x$ in $X$ and $f$ in $C(\mathbb{T})$, denote by $f_x$ the function in $C(\mathbb{T})$ given by $f_x(\zeta) = f(x^{-1}\zeta)$ for $\zeta \in \mathbb{T}$. For $n$ in $\mathbb{N}$, define a unital injective map

$$\iota_n \colon M_{q_n}(C(\mathbb{T})) \to M_{q_{n+1}}(C(\mathbb{T}))$$

by $\iota_n(f) = \operatorname{diag}\left(f_1, f_{x_2}, \ldots, f_{x_{p_n}}\right)$ for $f$ in $M_{q_n}(C(\mathbb{T}))$. The direct limit $A = \varinjlim(M_{q_n}(C(\mathbb{T})), \iota_n)$ is a simple unital A$\mathbb{T}$-algebra. For $n \in \mathbb{N}$, let $\alpha^{(n)} \colon \mathbb{T} \to \operatorname{Aut}(M_{q_n}(C(\mathbb{T})))$ be the tensor product of the trivial action on $M_{q_n}$ with the action coming from left translation on $C(\mathbb{T})$. Then $\alpha^{(n)}$ has the Rokhlin property by part (1) of Theorem VI.2.3. Since $\iota_n \circ \alpha^{(n)}_\zeta = \alpha^{(n+1)}_\zeta \circ \iota_n$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, the sequence $\left(\alpha^{(n)}\right)_{n\in\mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $\mathbb{T}$ on $A$, which has the Rokhlin property by part (4) of Theorem VI.2.3.

Now set $B = A \otimes \mathcal{O}_\infty$ and define $\beta \colon \mathbb{T} \to \operatorname{Aut}(B)$ by $\beta = \alpha \otimes \mathrm{id}_{\mathcal{O}_\infty}$. Then $B$ is a purely infinite, simple, separable, nuclear unital $C^*$-algebra, and $\beta$ has the Rokhlin property, by part (1) of Theorem VI.2.3. We claim that for every $m > 1$, the restriction $\beta|_m \colon \mathbb{Z}_m \to \operatorname{Aut}(B)$ does not have the Rokhlin property.

Fix $m > 1$, and assume that $\beta|_m$ has the Rokhlin property. By Proposition XI.3.6, we have $K_*(\beta_\zeta) = \mathrm{id}_{K_*(B)}$ for all $\zeta \in \mathbb{T}$. By Theorem 3.4 in [133], it follows that every element of $K_0(B)$ is divisible by $m$. On the other hand,

$$(K_0(B), [1_B]) \cong (K_0(A), [1_A])$$

$$\cong \left(\left\{\frac{a}{b} \colon a \in \mathbb{Z}, b = p_{k_1} \cdots p_{k_n} \colon n, k_1, \ldots, k_n \in \mathbb{N}, k_j \neq k_\ell \text{ for } j \neq \ell\right\}, 1\right),$$

where not every element is divisible by $m$. This is a contradiction.

We finish this work by showing that the Rokhlin property for a circle action cannot in general be determined just by looking at its restrictions to finite subgroups.

**Example XI.3.8.** There are a unital $C^*$-algebra $A$ and a circle action on $A$ such that its restriction to every proper subgroup has the Rokhlin property, but the action itself does not.

Let $A$ be the universal UHF-algebra, that is, $A = \varinjlim(M_{n!}, \iota_n)$ where $\iota_n \colon M_{n!} \to M_{(n+1)!}$ is given by $\iota_n(a) = \mathrm{diag}(a, \ldots, a)$ for all $a$ in $M_{n!}$. For every $n \in \mathbb{N}$, let $\alpha^{(n)} \colon \mathbb{T} \to \mathrm{Aut}(M_{n!})$ be given by

$$\alpha^{(n)}_\zeta = \mathrm{Ad}(\mathrm{diag}(1, \zeta, \ldots, \zeta^{n!-1}))$$

for all $\zeta \in \mathbb{T}$. Then $\iota_n \circ \alpha^{(n)}_\zeta = \alpha^{(n+1)}_\zeta \circ \iota_n$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, and hence there is a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of $\mathbb{T}$ on $A$. This action does not have the Rokhlin property by Theorem XI.3.4.

On the other hand, we claim that given $m \in \mathbb{N}$, the restriction $\alpha|_m \colon \mathbb{Z}_m \to \mathrm{Aut}(A)$ has the Rokhlin property. So fix $m \in \mathbb{N}$. Then $\alpha|_m$ is the direct limit of the actions $\left(\alpha^{(n)}|_m\right)_{n\in\mathbb{N}}$, whose generating automorphisms are

$$\alpha^{(n)}_{e^{2\pi i/m}} = \mathrm{Ad}(\mathrm{diag}(1, e^{2\pi i/m}, \ldots, e^{2\pi i(n!-1)/m})).$$

Let $F \subseteq A$ be a finite subset and let $\varepsilon > 0$. Write $F = \{a_1, \ldots, a_N\}$. Since $\bigcup\limits_{n\in\mathbb{N}} M_{n!}$ is dense in $A$, there are $k \in \mathbb{N}$ and a finite subset $F' = \{b_1, \ldots, b_N\} \subseteq M_{k!}$ such that $\|a_j - b_j\| < \frac{\varepsilon}{2}$ for all $j = 1, \ldots, N$.

Let $n \geq \max\{k, m\}$. Then the $\mathbb{Z}_m$-action $\alpha^{(n)}|_m$ on $M_{n!}$ is generated by the automorphism

$$\alpha^{(n)}_{e^{2\pi i/m}} = \mathrm{Ad}(1, e^{2\pi i/m}, \ldots, e^{2\pi i(m-1)/m}, \ldots, 1, e^{2\pi i/m}, \ldots, e^{2\pi i(m-1)/m}).$$

(There are $n!/m$ repetitions.) Denote by $e_0$ the projection

$$1_{M_{(n-1)!}} \otimes \begin{pmatrix} \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{m} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{m} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{1}{m} & \frac{1}{m} & \cdots & \frac{1}{m} \end{pmatrix}$$

in $M_{n!} \subseteq A$, and for $j = 1, \ldots, m-1$, set $e_j = \alpha^{(n)}_{e^{2\pi i j/m}}(e_0) \in A$. One checks that $e_0, \ldots, e_{m-1}$ are orthogonal projections with $\sum\limits_{j=0}^{m-1} e_j = 1$, and moreover that $\alpha^{(n)}_{e^{2\pi i/m}}(e_{m-1}) = e_0$.

By construction, these projections are cyclically permuted by the action $\alpha|_m$ and they sum up to one, so we only need to check that they almost commute with the given finite set. The projections $e_0, \ldots, e_{m-1}$ exactly commute with the elements of $F'$. Thus, if $k \in \{1, \ldots, N\}$ and $j \in \{0, \ldots, m-1\}$, then

$$\begin{aligned} \|a_k e_j - e_j a_k\| &\leq \|a_k e_j - b_k e_j\| + \|b_k e_j - e_j b_k\| + \|e_j b_k - e_j a_k\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and hence $\alpha|_m$ has the Rokhlin property.

The phenomenon exhibited in the example above is not special to UHF-algebras:

**Example XI.3.9.** If $A$ and $\alpha$ are as in Example XI.3.8, set $B = A \otimes \mathcal{O}_\infty$ and let $\beta \colon \mathbb{T} \to \mathrm{Aut}(B)$ be given by $\beta_\zeta = \alpha_\zeta \otimes \mathrm{id}_{\mathcal{O}_\infty}$ for all $\zeta \in \mathbb{T}$. Then $B$ is a unital Kirchberg algebra satisfying the UCT, the action $\beta$ does not have the Rokhlin property, and for every $m \in \mathbb{N}$, the restriction $\beta|_m \colon \mathbb{Z}_m \to \mathrm{Aut}(B)$ has the Rokhlin property.

Example XI.3.8 and Example XI.3.9 and should be contrasted with the following fact.

**Proposition XI.3.10.** Let a compact Lie group $G$ act on a locally compact Hausdorff space $X$. Then the action is free if and only if its restriction to every finite cyclic subgroup of $G$ of *prime order* is free.

*Proof.* The "only if" implication is immediate. For the "if" implication, let $g \in G \setminus \{1\}$ and assume that there exists $x$ in $X$ with $gx = x$. The stabilizer subgroup

$$S_x = \{h \in G \colon hx = x\}$$

of $x$ is therefore non-trivial. Being a closed subgroup of $G$, it is a Lie group by Cartan's theorem. It follows that $S_x$ has a finite cyclic group of prime order: this is immediate if $S_x$ is finite, while if $S_x$ is infinite, it must contain a (maximal) torus. Now, the restriction of the action to any such subgroup is trivial, contradicting the assumption. It follows that the action of $G$ on $X$ is free. $\square$

# CHAPTER XII

# NONCLASSIFIABILITY OF AUTOMORPHISMS OF $\mathcal{O}_2$

This chapter is based on joint work with Martino Lupini ([88]).

The group of automorphisms of the Cuntz algebra $\mathcal{O}_2$ is a Polish group with respect to the topology of pointwise convergence in norm. Our main result is that the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel. Moreover, we show that from the point of view of invariant complexity theory, classifying automorphisms of $\mathcal{O}_2$ up to conjugacy or cocycle conjugacy is strictly more difficult than classifying up to isomorphism any class of countable structures with Borel isomorphism relation. In fact the same conclusions hold even if one only considers automorphisms of $\mathcal{O}_2$ of a fixed finite order. We moreover show that for any prime number $p$ the relation of isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_p$ (with trivial $K_1$-group and satisfying the Universal Coefficient Theorem) is not Borel. Moreover, it is strictly more difficult to classify such crossed products than classifying up to isomorphism any class of countable structures with Borel isomorphism relation.

## Introduction

The *Cuntz algebra* $\mathcal{O}_2$ can be described as the universal unital $C^*$-algebra generated by two isometries $s_1$ and $s_2$ subject to the relation

$$s_1 s_1^* + s_2 s_2^* = 1.$$

It was defined and studied by Cuntz in the groundbreaking paper [39]. Since then, a stream of results has made clear the key role of $\mathcal{O}_2$ in the classification program of $C^*$-algebras; see [235, Chapter 2] for a complete account and more references. This has served as motivation for an intensive study of the structural properties of $\mathcal{O}_2$ and its automorphism group, as in [181], [266], [32], [34], and [33]. In particular, considerable effort has been put into trying to classify several important classes of automorphisms; see for example [132] and [133].

The main result of this chapter, which is based on [88], asserts that the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are complete analytic sets when regarded as subsets of $\mathrm{Aut}(\mathcal{O}_2) \times \mathrm{Aut}(\mathcal{O}_2)$, and in particular not Borel.

The fact that conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are not Borel should be compared with the fact that for any separable $C^*$-algebra $A$, the relation of unitary equivalence of automorphisms of $A$ is Borel. This is because the relation of coset equivalence modulo the Borel subgroup $\mathrm{Inn}(A)$ of $\mathrm{Aut}(A)$. (This does not necessarily mean that the problem of classifying the automorphisms of $A$ up to unitary equivalence is more tractable: It is shown in [177] that whenever $A$ is simple –or just does not have continuous trace– then the automorphisms of $A$ cannot be classified up to unitary equivalence using countable structures as invariants.) Similarly, the spectral theorem for unitary operators on the Hilbert space shows that the relation of conjugacy of unitary operators is Borel; more details can be found in [74, Example 55]. On the other hand, the main result of [74] asserts that the relation of conjugacy for ergodic measure-preserving transformations on the Lebesgue space is also a complete analytic set.

We will moreover show that classifying automorphisms of $\mathcal{O}_2$ up to either conjugacy or cocycle conjugacy is strictly more difficult than classifying any class of countable structures with Borel isomorphism relation.

All of these results will be shown to hold even if one only considers automorphisms of a fixed finite order. Moreover, it will follow from our proof that the same assertions hold for the relation of isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_p$ (with trivial $K_0$-group and satisfying the Universal Coefficient Theorem), where $p$ is any prime number.

It should be mentioned that by the main result of [149] the automorphisms of $\mathcal{O}_2$ are not classifiable up to conjugacy by countable structures. This means that there is no explicit way to assign a countable structure to every automorphism of $\mathcal{O}_2$, in such a way that two automorphisms are conjugate if and only if the corresponding structures are isomorphic. More precisely, for no class $\mathcal{C}$ of countable structures, is the relation of conjugacy of automorphisms of $\mathcal{O}_2$ Borel reducible to the relation of isomorphisms of elements of $\mathcal{C}$. Moreover the same conclusions hold for any set of automorphisms of $\mathcal{O}_2$ which is not meager in the topology of pointwise convergence. Similar conclusions hold for automorphisms of any separable $C^*$-algebra absorbing the Jiang-Su algebra tensorially.

The strategy of the proof of the main theorem is as follows. Using techniques from [126] and [52], we show that the relation of isomorphism of countable 2-divisible torsion free abelian groups is a complete analytic set, and it is strictly more complicated than the relation of isomorphism of any class of countable structures with Borel isomorphism relation. We then show that the relation of isomorphism of 2-divisible abelian groups is Borel reducible to the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ of order 2. This is achieved by showing that there is a Borel way to assign to a countable abelian group $G$ to assign to a countable abelian group a Kirchberg algebra $A_G$ with trivial $K_1$-group, $K_0$-group isomorphic to $G$, and with the class of the unit in $K_0$ being the zero element. We then use a result of Izumi from [132] asserting that there is an automorphism $\nu$ of $\mathcal{O}_2$ of order 2 with the following property: Tensoring the identity automorphism of $A_G$ by $\nu$, and identifying $A_G \otimes \mathcal{O}_2$ with $\mathcal{O}_2$ by Kirchberg's absorption theorem, gives a reduction of isomorphism of Kirchberg algebras with 2-divisible $K_0$-group and with the class of the unit being the zero element in $K_0$, to conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ of order 2. The proof is concluded by showing –using results from [71]– that such reduction is implemented by a Borel map. A suitable modification of this argument yields the same result where 2 is replaced by an arbitrary prime number.

The present chapter is organized as follows. Section XII.2 contains some background notions on model theory. section XII.3 presents a functorial version of the notion of standard Borel parametrization of a category as defined in [71]. Several functorial parametrizations for the category are then presented and shown to be equivalent. Finally, many standard constructions in $C^*$-algebra theory are shown to be computable by Borel maps in these parametrizations. The main result of Section XII.4 asserts that the reduced crossed product of a $C^*$-algebra by an action of a countable group can be computed in a Borel way. The same conclusion holds for crossed products by a corner endomorphism in the sense of [18]. section XII.5 provides a Borel version of the correspondence between unital AF-algebras and dimension groups established in [57] and [54]. We show that there is a Borel map that assigns to a dimension group $D$, a unital AF-algebra $B_D$ such that $D$ is isomorphic to the $K_0$-group of $B_D$. Moreover, given an endomorphism $\beta$ of $D$, one can select in a Borel fashion an endomorphism $\rho_{D,\beta}$ of $B_D$ whose induced endomorphism of $K_0(B_D)$ is conjugate to $\beta$. Finally, section XII.6 contains the proof of the main results.

In the following, all $C^*$-algebras and Hilbert spaces are assumed to be *separable*, and all discrete groups are assumed to be *countable*. We denote by $\omega$ the set of natural numbers *including* 0. An element $n \in \omega$ will be identified with the set $\{0, 1, \ldots, n-1\}$ of its predecessors. (In particular 0 is identified with the empty set.) We will therefore write $i \in n$ to mean that $i$ is a natural number and $i < n$.

If $X$ is a Polish space and $D$ is a countable set, we endow the set $X^D$ of $D$-indexed sequences of elements of $X$ with the product topology. Likewise, if $X$ is a standard Borel space, then we give $X^D$ the product Borel structure. In the particular case where $X = 2 = \{0, 1\}$, we identify $2^D$ with the set of subsets of $D$ with its Cantor set topology, and the corresponding standard Borel structure. In the following we will often make use –without explicit mention– of the following basic principle: Suppose that $X$ is a standard Borel space, $D$ is a countable set, and $B$ is a Borel subset of $X \times D$ such that for every $x \in X$ there is $y \in D$ such that $(x, y) \in B$. Then there is a Borel selector for $B$, that is, a function $f$ from $X$ to $D$ such that $(x, f(x)) \in B$ for every $x \in X$. To see this one can just fix a well order $<$ of $D$ and define $f(x)$ to be the $<$-minimum of the set of $y \in D$ such that $(x, y) \in B$.

We will use throughout the chapter the fact that a $G_\delta$ subspace of a Polish space is Polish in the subspace topology [147, Theorem 3.11], and that a Borel subspace of a standard Borel space is standard with the inherited Borel structure [147, Proposition 12.1].

### Preliminaries on Borel Complexity

Recall that a topological space is said to be *Polish* if it is separable and its topology is induced by a complete metric. A *Polish group* is a topological group whose topology is Polish. A *standard Borel space* is a set endowed with a $\sigma$-algebra which is the $\sigma$-algebra of Borel sets for some Polish topology on the space. It is not difficult to verify that, under the assumption that $A$ is separable, its automorphism group $\mathrm{Aut}(A)$ is a Polish group with respect to the topology of pointwise convergence in norm.

**Definition XII.2.1.** A subset $B$ of a standard Borel space $X$ is said to be *analytic* if it is the image of a standard Borel space under a Borel function.

If $B$ and $C$ are analytic subsets of the standard Borel spaces $X$ and $Y$, then $B$ is said to be *Wadge reducible* to $C$ if there is a Borel map $f \colon X \to Y$ such that $B$ is the inverse image of $C$

under $f$; see [147, Section 2.E]. An analytic set which is moreover a maximal element in the class of analytic sets under Wadge reducibility is called a *complete analytic set*; more information can be found in [147, Section 26.C].

It is a classical result of Souslin from the early beginnings of descriptive set theory, that there are analytic sets which are not Borel. In particular, a complete analytic set is not Borel, since a set that is Wadge reducible to a Borel set is Borel.

Informally speaking, a set (or function) is Borel whenever it can be computed by a countable protocol whose basic bit of information is membership in open sets. The fact that a set $X$ is not Borel can be interpreted as the assertion that the problem of membership in $X$ can not be decided by such a countable protocol, and it is therefore highly intractable. We can therefore reformulate the main result of this chapter as follows: There does not exist any countable protocol able to determine whether a given pair of automorphisms of $\mathcal{O}_2$ are conjugate or cocycle conjugate by only evaluating, at any given stage of the computation, the given automorphisms in some arbitrarily large finite set of elements of $\mathcal{O}_2$ up to some arbitrarily small strictly positive error.

We will work in framework of invariant complexity theory. In this context, classification problems are regarded as equivalence relations on standard Borel spaces. Virtually any concrete classification problem in mathematics can be regarded –possibly after a suitable parametrization– as the problem of classifying the elements of some standard Borel space up to some equivalence relation. The key notion of comparison between equivalence relations is the notion of Borel reduction.

**Definition XII.2.2.** Suppose that $E$ and $F$ are equivalence relation on standard Borel spaces $X$ and $Y$. A *Borel reduction* from $E$ to $F$ is a Borel function $f\colon X \to Y$ such that

$$xEx' \quad \text{if and only if} \quad f(x)Ff(x').$$

A Borel reduction from $E$ to $F$ can be regarded as a way to assign –in a constructive way– to the objects of $X$, equivalence classes of $F$ as complete invariants for $E$.

**Definition XII.2.3.** The equivalence relation $E$ is said to be *Borel reducible* to $F$, in symbol $E \leq_B F$, if there is a Borel reduction from $E$ to $F$.

In this case, the equivalence relation $F$ can be thought of as being more complicated than $E$, since any Borel classification of the objects of $Y$ up to $F$ entails –by precomposing with a Borel reduction from $E$ to $F$– a Borel classification of objects of $X$ up to $E$. It is immediate to check that if $E$ is Borel reducible to $F$, then $E$ (as a subset of $X \times X$) is Wadge reducible to $F$ (as a subset of $Y \times Y$). In particular, if $E$ is a complete analytic set and $E \leq_B F$, then $F$ is a complete analytic set. Observe that if $F$ is an equivalence relation on $Y$, and $X$ is an $F$-invariant Borel subset of $Y$, then the restriction of $F$ to $X$ is Borel reducible to $F$.

Using this terminology, we can reformulate the assertion about the complexity of the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ made in the introduction, as follows. If $\mathcal{C}$ is any class of countable structures such that the corresponding isomorphism relation $\cong_{\mathcal{C}}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to both conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$. Furthermore, if $E$ is any Borel equivalence relation, then the relations of conjugacy and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are *not* Borel reducible to $E$. In particular this rules out any classification that uses as invariant Borel measures on a Polish space (up to measure equivalence) or unitary operators on the Hilbert space (up to conjugacy). In fact, as observed before, the relations of measure equivalence and, by the spectral theorem, the relation of conjugacy of unitary operators are Borel; see [74, Example 55].

### Parametrizing the Category of $C^*$-algebras

*Functorial parametrization*

Recall that a *(small) semigroupoid* is a quintuple $(X, \mathcal{C}_X, s, r, \cdot)$, where $X$ and $\mathcal{C}_X$ are sets, $s, r$ are functions from $\mathcal{C}_X$ to $X$, and $\cdot$ is an associative partially defined binary operation on $\mathcal{C}_X$ with domain

$$\{(x, y) \in \mathcal{C}_X \times \mathcal{C}_X : s(x) = r(y)\}$$

such that $r(x \cdot y) = r(x)$ and $s(x \cdot y) = s(y)$ for all $x$ and $y$ in $X$. The elements of $X$ are called objects, the elements of $\mathcal{C}_X$ morphisms, the map $\cdot$ composition, and the maps $s$ and $r$ source and range map. In the following, a semigroupoid $(X, \mathcal{C}_X, s, r, \cdot)$ will be denoted simply by $\mathcal{C}_X$. Note that a (small) category is precisely a (small) semigroupoid, where moreover the identity arrow

$\mathrm{id}_x \in \mathcal{C}_X$ is associated with the element $x$ of $X$. A morphism between semigroupoids $\mathcal{C}_X$ and $\mathcal{C}_{X'}$ is a pair $(f, F)$ of functions $f\colon X \to X'$ and $F\colon \mathcal{C}_X \to \mathcal{C}_{X'}$ such that

- $s \circ F = f \circ s$,
- $r \circ F = f \circ r$, and
- $F(a \cdot b) = F(a) \cdot F(b)$ for every $a$ and $b \in \mathcal{C}_X$.

In the case of categories, a morphism of semigroupoids is just a functor.

A *standard Borel semigroupoid* is a semigroupoid $\mathcal{C}_X$ such that $X$ and $\mathcal{C}_X$ are endowed with standard Borel structures making the composition function $\cdot$ and the source and range functions $s$ and $r$ Borel.

**Definition XII.3.1.** Let $\mathcal{D}$ be a category, let $\mathcal{C}_X$ be a standard Borel semigroupoid, and let $(f, F)$ be a morphism from $\mathcal{C}_X$ to $\mathcal{D}$. We say that $(\mathcal{C}_X, f, F)$ is a *good parametrization* of $\mathcal{D}$ if

- $(f, F)$ is *essentially surjective*, that is, if every object of $\mathcal{D}$ is isomorphic to an object in the range of $f$,
- $(f, F)$ is *full*, that is, if for every $x, y \in X$ the set $\mathrm{Hom}(f(x), f(y))$ is contained in the range of $F$, and
- the set $\mathrm{Iso}_X$ of elements of $\mathcal{C}_X$ that are mapped by $F$ to isomorphisms of $\mathcal{D}$, is Borel.

Observe that if $(\mathcal{C}_X, f, F)$ is a *good parametrization* of $\mathcal{D}$, then $(X, f)$ is a good parametrization of $\mathcal{C}$ in the sense of [71, Definition 2.1].

**Definition XII.3.2.** Let $\mathcal{D}$ be a category and let $(\mathcal{C}_X, f, F)$ and $(\mathcal{C}_{X'}, f', F')$ be good parametrizations of $\mathcal{D}$. A *morphism* from $(\mathcal{C}_X, f, F)$ to $(\mathcal{C}_{X'}, f', F')$ is a triple $(g, G, \eta)$ of maps $g\colon X \to X'$, $G\colon \mathcal{C}_X \to \mathcal{C}_{X'}$, and $\eta\colon X \to \mathcal{D}$, satisfying the following conditions:

1. The functions $g$ and $G$ are Borel;
2. $\eta(x)$ is an isomorphism from $f(x)$ to $(f' \circ g)(x)$ for every $x \in X$;
3. The pair $(f' \circ g, F' \circ G)$ is a semigroupoid morphism $\mathcal{C}_X \to \mathcal{D}$;
4. We have $s_{X'} \circ G = g \circ s_X$ and $r_{X'} \circ G = g \circ r_X$;

5. For every $a \in \mathcal{C}_X$

$$F'(G(a)) \circ \eta(s(a)) = \eta(r(a)) \circ F(a).$$

Two good parametrizations $(\mathcal{C}_X, f, F)$ and $(\mathcal{C}_{X'}, f', F')$ of $\mathcal{C}$ are said to be *equivalent* if there are isomorphisms from $(\mathcal{C}_X, f, F)$ to $(\mathcal{C}_{X'}, f', F')$ and viceversa. It is not difficult to verify that if $(\mathcal{C}_X, f, F)$ and $(\mathcal{C}_{X'}, f', F')$ are equivalent parametrizations of $\mathcal{D}$, then $(X, f)$ and $(X', f')$ are weakly equivalent parametrizations of $\mathcal{D}$ in the sense of [71, Definition 2.1].

In the following, a good parametrization $(\mathcal{C}_X, f, F)$ of $\mathcal{D}$ will be denoted by $\mathcal{C}_X$ for short.

*The space* $\mathcal{C}_{\widehat{\Xi}}$

We follow the notation in [71, Section 2.2], and denote by $\mathbb{Q}(i)$ the field of complex rationals. A $\mathbb{Q}(i)$-$*$-algebra is an algebra over the field $\mathbb{Q}(i)$ endowed with an involution $x \mapsto x^*$. We define $\mathcal{U}$ to be the $\mathbb{Q}(i)$-$*$-algebra of noncommutative $*$-polynomials with coefficients in $\mathbb{Q}(i)$ and without constant term in the formal variables $X_k$ for $k \in \omega$. If $A$ is a $C^*$-algebra, $\gamma = (\gamma_n)_{n\in\omega}$ is a sequence of elements of $A$, and $p \in \mathcal{U}$, we define $p(\gamma)$ to be the element of $A$ obtained by evaluating $p$ in $A$, where for every $k \in \omega$, the formal variables $X_k$ and $X_k^*$ are replaced by the elements $\gamma_k$ and $\gamma_k^*$ of $A$.

We denote by $\widehat{\Xi}$ the set of elements

$$A = (f, g, h, k, r) \in \omega^{\omega\times\omega} \times \omega^{\mathbb{Q}(i)\times\omega} \times \omega^{\omega\times\omega} \times \omega^{\omega} \times \mathbb{R}^{\omega}$$

that code on $\omega$ a structure of $\mathbb{Q}(i)$-$*$-algebra $A$ endowed with a norm satisfying the C*-identity. The completion $\widehat{A}$ of $\omega$ with respect to such norm is a $C^*$-algebra (denoted by $B(A)$ in [71, Subection 2.4]). It is not hard to check that $\widehat{\Xi}$ is a Borel subset of $\omega^{\omega\times\omega} \times \omega^{\mathbb{Q}(i)\times\omega} \times \omega^{\omega\times\omega} \times \omega^{\omega} \times \mathbb{R}^{\omega}$. As observed in [71, Subection 2.4], $\widehat{\Xi}$ can be thought of as a natural parametrization for *abstract $C^*$-algebras*. We use the notation of [71, Subsection 2.4] to denote the operations on $\omega$ coded by an element $A = (f, g, h, k, r)$ of $\widehat{\Xi}$. We denote by $d_A$ the metric on $\omega$ coded by $A$, which is given by

$$d_A(n, m) = \|n +_f (-1) \cdot_g m\|_r$$

for $n, m \in \omega$. We will also write $n +_A m$ for $n +_f m$, and similarly for $g, h, k, r$.

**Definition XII.3.3.** Suppose that $A = (f, g, h, k, r)$ and $A' = (f', g', h', k', r')$ are elements of $\widehat{\Xi}$, and that $\Phi = (\Phi_n)_{n\in\omega} \in (\omega^\omega)^\omega$ is a sequence of functions from $\omega$ to $\omega$. We say that $\Phi$ is a *code for a homomorphism* from $\widehat{A}$ to $\widehat{A}'$ if the following conditions hold:

1. The sequence $(\Phi_n(k))_{n\in\omega}$ is Cauchy uniformly in $k \in \omega$ with respect to the metric $d_A$, and in particular converges to an element $\widehat{\Phi}(k)$ of $\widehat{A}$;
2. The map $k \mapsto \widehat{\Phi}(k)$ is a contractive homomorphism of $\mathbb{Q}(i)$-$*$-algebras, and hence it induces a homomorphism $\widehat{\Phi}$ from $\widehat{A}$ to $\widehat{A}'$.

We say that $\Phi$ is a *code for an isomorphism* from $\widehat{A}$ to $\widehat{A}'$ if $\Phi$ is a code for a homomorphism from $\widehat{A}$ to $\widehat{A}'$, and $\widehat{\Phi}$ is an isomorphism. If $\Phi$ and $\Phi'$ are codes for homomorphisms from $\widehat{A}$ to $\widehat{A}'$ and from $\widehat{A}'$ to $\widehat{A}''$ respectively, we define their composition $\Phi' \circ \Phi$ by $(\Phi' \circ \Phi)_n = \Phi'_n \circ \Phi_n$ for $n \in \omega$.

It is easily checked that $\Phi' \circ \Phi \in (\omega^\omega)^\omega$ is a code for the homomorphism $\widehat{\Phi}' \circ \widehat{\Phi}$ from $\widehat{A}$ to $\widehat{A}''$ One can verify that the set $\mathcal{C}_{\widehat{\Xi}}$ of triples $(A, A', \Phi) \in \widehat{\Xi} \times \widehat{\Xi} \times (\omega^\omega)^\omega$ such that $\Phi$ is a code for a homomorphism from $\widehat{A}$ to $\widehat{A}'$, is Borel. We can regard $\mathcal{C}_{\widehat{\Xi}}$ as a standard semigroupoid having $\widehat{\Xi}$ as set of objects, where the composition of $(A, A', \Phi)$ and $(A', A'', \Phi')$ is $(A', A'', \Phi' \circ \Phi)$, and the source and range of $(A, A', \Phi)$ are $A$ and $A'$ respectively. The semigroupoid morphism $(A, A', \Phi) \mapsto \left(\widehat{A}, \widehat{A}', \widehat{\Phi}\right)$ defines a parametrization of the category of $C^*$-algebras with homomorphisms. It is easy to see that this is a good parametrization in the sense of Definition XII.3.2. In particular, the set $\mathrm{Iso}_{\widehat{\Xi}}$ of elements $(A, A', \Phi)$ of $\widehat{\Xi} \times \widehat{\Xi} \times (\omega^\omega)^\omega$ such that $\Phi$ is a code for an isomorphism from $\widehat{A}$ to $\widehat{A}'$, is Borel.

*The space $\mathcal{C}_\Xi$*

We denote by $\Xi$ the $G_\delta$ subset of $\mathbb{R}^{\mathcal{U}}$ consisting of the nonzero functions $\delta\colon \mathcal{U} \to \mathbb{R}$ such that there exists a $C^*$-algebra $A$ and a dense subset $\gamma = (\gamma_n)_{n\in\omega}$ of $A$, such that

$$\delta(p) = \|p(\gamma)\| .$$

It could be observed that, differently from [71, Subsection 2.3], we are not considering the function constantly equal to zero as an element of $\Xi$; this choice is just for convenience and will play no role in the rest of the discussion. Observe that any element $\delta$ of $\Xi$ determines a seminorm on the $\mathbb{Q}(i)$-$*$-algebra $\mathcal{U}$; therefore one can consider the corresponding Hausdorff completion of $\mathcal{U}$. Denote by $I_\delta$ the ideal of $\mathcal{U}$ given by

$$I_\delta = \{p \in \mathcal{U} \colon \delta(p) = 0\}.$$

Then $\mathcal{U}/I_\delta$ is a normed $\mathbb{Q}(i)$-$*$-algebra. Its completion is a $C^*$-algebra, which we shall denote by $\widehat{\delta}$. (What we denote by $\widehat{\delta}$ is denoted by $B(\delta)$ in [71, Subsection 2.3].)

**Definition XII.3.4.** Let $\delta$ and $\delta'$ be elements in $\Xi$, and let $\Phi = (\Phi_n)_{n\in\omega} \in \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ be a sequence of functions from $\mathcal{U}$ to $\mathcal{U}$. We say that $\Phi$ is a *code for a homomorphism* from $\widehat{\delta}$ to $\widehat{\delta}'$, if

1. for every $p \in \mathcal{U}$, the sequence $(\Phi_n(p))_{n\in\omega}$ is Cauchy uniformly in $p \in \mathcal{U}$, with respect to the pseudometric $(q, q') \mapsto \delta\,(q - q')$ on $\mathcal{U}$, and in particular converges in $\widehat{\delta}$ to an element $\widehat{\Phi}(p)$, and
2. $p \mapsto \widehat{\Phi}(p)$ is a morphism of $\mathbb{Q}(i)$-$*$-algebras such that $\left\|\widehat{\Phi}(p)\right\| \leq \delta(p)$, and hence induces a homomorphism from $\widehat{\delta}$ to $\widehat{\delta}'$.

Writing down explicit formulas defining a code for a homomorphism makes it clear that the set $\mathcal{C}_\Xi$ of triples $(\delta, \delta', \Phi) \in \Xi \times \Xi \times \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ such that $\Phi$ is a code for a homomorphism from $\widehat{\delta}$ to $\widehat{\delta'}$ is Borel. Suppose that $\Phi, \Phi'$ are code for homomorphisms from $\delta$ to $\delta'$ and from $\delta'$ to $\delta''$. Similarly as in Subsection XII.3, it is easy to check that defining

$$(\Phi' \circ \Phi)_n = \Phi'_n \circ \Phi_n$$

for $n \in \omega$ gives a code for a homomorphism from $\delta$ to $\delta''$. This defines a standard Borel semigroupoid structure on $\mathcal{C}_\Xi$, such that the map $(\delta, \delta', \Phi) \mapsto \left(\widehat{\delta}, \widehat{\delta}', \widehat{\Phi}\right)$ is a good standard Borel parametrization of the category of $C^*$-algebras.

*The space $\mathcal{C}_{\Gamma(H)}$*

Denote by $B_1(H)$ the unit ball of $B(H)$ with respect to the operator norm. Recall that $B_1(H)$ is a compact Hausdorff space when endowed with the weak operator topology. The standard Borel structure generated by the weak operator topology on $B_1(H)$ coincide with the Borel structure generated by several other operator topologies on $B_1(H)$, such as the $\sigma$-weak, strong, $\sigma$-strong, strong-*, and $\sigma$-strong-* operator topology; see [14, I.3.1.1]. Denote by $B_1(H)^\omega$ the product of countably many copies of $B_1(H)$, endowed with the product topology, and define $\Gamma(H)$ to be the Polish space obtained by removing from $B_1(H)^\omega$ the sequence constantly equal to 0. (The space $\Gamma(H)$ is defined similarly in [71, Subection 2.1]; the only difference is that here the sequence constantly equal to 0 is excluded for convenience.) Given an element $\gamma$ in $\Gamma(H)$, denote by $C^*(\gamma)$ the C*-subalgebra of $B(H)$ generated by $\{\gamma_n \colon n \in \omega\}$. As explained in [71, Subsection 2.1 and Remark 2.3], the space $\Gamma(H)$ can be thought of as a natural parametrization of *concrete $C^*$-algebras*.

**Definition XII.3.5.** Let $\gamma$ and $\gamma'$ be elements in $\Gamma(H)$, and let $\Phi = (\Phi_n)_{n\in\omega} \in \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ be a sequence of functions from $\mathcal{U}$ to $\mathcal{U}$. We say that $\Phi$ is a *code for a homomorphism* from $C^*(\gamma)$ to $C^*(\gamma')$, if

1. the sequence $(\Phi_n(p)(\gamma'))_{n\in\omega}$ of elements of $C^*(\gamma')$ is Cauchy uniformly in $p$, and hence converges to an element $\widehat{\Phi}(p(\gamma))$ of $C^*(\gamma')$, and
2. the function $p(\gamma) \mapsto \widehat{\Phi}(p(\gamma))$ extends to a homomorphism from $C^*(\gamma)$ to $C^*(\gamma')$.

Again, it is easily checked that the set $\mathcal{C}_{\Gamma(H)}$ of triples $(\gamma, \gamma', \Phi)$ such that $\Phi$ is a code for a homomorphism from $C^*(\gamma)$ to $C^*(\gamma')$, is Borel. Moreover, one can define a standard Borel semigroupoid structure on $\mathcal{C}_{\Gamma(H)}$, in such a way that the map $(\gamma, \gamma', \Phi) \mapsto \left(C^*(\gamma), C^*(\gamma'), \widehat{\Phi}\right)$ is a good parametrization of the category of $C^*$-algebras.

For future reference, we show in Lemma XII.3.6 below that in the parametrization $\mathcal{C}_{\Gamma(H)}$ one can compute a code for the inverse of an isomorphism in a Borel way. Recall that, consistently with Definition XII.3.1, $\mathrm{Iso}_{\Gamma(H)}$ denotes the Borel set of $(\gamma, \gamma', \Phi) \in \mathcal{C}_{\Gamma(H)}$ that code an isomorphism.

**Lemma XII.3.6.** There is a Borel map from $\mathrm{Iso}_{\Gamma(H)}$ to $\left(\mathcal{U}^{\mathcal{U}}\right)^\omega$, assigning to an element $(\gamma, \gamma', \Phi)$ of $\mathrm{Iso}_{\Gamma(H)}$ a code $\mathrm{Inv}(\gamma, \gamma', \Phi)$ for an isomorphism from $C^*(\gamma')$ to $C^*(\gamma)$ such that $\mathrm{Inv}\widehat{(\gamma, \gamma', \Phi)} = \widehat{\Phi}^{-1}$.

*Proof.* Observe that the set $\mathcal{E}$ of tuples

$$((\gamma, \gamma', \Phi), p, n, q, N) \in \mathrm{Iso}_{\Gamma(H)} \times \mathcal{U} \times \omega \times \mathcal{U} \times \omega$$

such that

$$\|q(\gamma') - \Phi_M(p)(\gamma')\| < \frac{1}{n},$$

and

$$\|\Phi_{M'}(p)(\gamma') - \Phi_M(p)(\gamma')\| < \frac{1}{n},$$

for every $M, M' \geq N$ is Borel. Therefore one can find Borel functions $(\xi, q, n) \mapsto p_{(\xi,p,n)}$ and $(\xi, p, n) \mapsto N_{(\xi,p,n)}$ from $\mathrm{Iso}_{\Gamma(H)} \times \mathcal{U} \times \omega$ to $\mathcal{U}$ and $\omega$ respectively such that

$$\left(\xi, q, n, p_{(\xi,q,n)}, N_{(\xi,q,n)}\right) \in \mathcal{E}$$

for every $(\xi, q, n) \in \mathrm{Iso}_{\Gamma(H)} \times \mathcal{U} \times \omega$. Defining now $\mathrm{Inv}(\xi)_n(q) = p_{(\xi,q,n)}$ for every $n \in \omega$ and $q \in \mathcal{U}$ one obtains a Borel map $\xi \mapsto \mathrm{Inv}(\xi)$. Moreover,

$$\begin{aligned}
&\left\|\mathrm{Inv}(\xi)_n(q)(\gamma) - \widehat{\Phi}^{-1}(q(\gamma'))\right\| \\
&\leq \left\|p_{(\xi,q,n)}(\gamma) - \widehat{\Phi}^{-1}(\Phi_{N_{(\xi,q,n)}}(p)(\gamma'))\right\| + \frac{1}{n} \\
&\leq \left\|\widehat{\Phi}(p_{(\xi,k,n)}(\gamma)) - \Phi_{N_{(\xi,q,n)}}(p)(\gamma')\right\| + \frac{1}{n} \\
&\leq \frac{1}{2n}.
\end{aligned}$$

This shows that $\mathrm{Inv}(\xi)$ is a code for the inverse of $\widehat{\Phi}$. □

*Equivalence of $\mathcal{C}_{\widehat{\Xi}}$, $\mathcal{C}_{\Xi}$ and $\mathcal{C}_{\Gamma}$.*

Recall that given an element $\delta$ of $\Xi$, we denote by $I_\delta$ the ideal of $\mathcal{U}$ given by

$$I_\delta = \{p \in \mathcal{U} : \delta(p) = 0\}.$$

**Theorem XII.3.7.** The good parametrizations $\mathcal{C}_{\widehat{\Xi}}$, $\mathcal{C}_{\Xi}$, and $\mathcal{C}_{\Gamma}$, of the category of $C^*$-algebras with homomorphisms, are equivalent in the sense of Definition XII.3.2.

*Proof.* We will show first that $\mathcal{C}_{\widehat{\Xi}}$ and $\mathcal{C}_{\Xi}$ are equivalent.

We start by constructing a morphism from $\mathcal{C}_{\Xi}$ to $\mathcal{C}_{\widehat{\Xi}}$ as in Definition XII.3.2 as follows. As in the proof of [71, Proposition 2.6], for every $n \in \omega$ define a Borel map $p_n : \Xi \to \mathcal{U}$, denoted

$\delta \mapsto p_n^\delta$ for $\delta$ in $\Xi$, such that

$$\left\{p_n^\delta + I_\delta \colon n \in \omega\right\}$$

is an enumeration of $\mathcal{U}/I_\delta$ for every $\delta \in \Xi$ . For $\delta \in \Xi$, define a structure of C*-normed $\mathbb{Q}(i)$-$*$-algebra $A_\delta = (f_\delta, g_\delta, h_\delta, k_\delta, r_\delta)$ on $\omega$ by:

- $m +_{f_\delta} n = t$ whenever $p_m^\delta + p_n^\delta + I_\delta = p_t^\delta + I_\delta$;

- $q \cdot_{g_\delta} m = t$ whenever $q \cdot p_m^\delta + I_\delta = p_t^\delta + I_\delta$;

- $m \cdot_{h_\delta} n = t$ whenever $q_m^\delta q_n^\delta + I_\delta = q_t^\delta + I_\delta$;

- $m^{*k_\delta} = t$ whenever $\left(q_m^\delta\right)^* + I_\delta = q_t^\delta + I_\delta$;

- $\|m\|_{r_\delta} = \delta\left(q_m^\delta\right)$.

It is clear that the map $\delta \mapsto A_\delta$ is Borel. Moreover, for fixed $\delta$ in $\Xi$, the map $n \mapsto p_n^\delta + I_\delta$ is an isomorphism of normed $\mathbb{Q}(i)$-$*$-algebras from $A_\delta$ onto $\mathcal{U}/I_\delta$. We denote by $\eta_\delta \colon \widehat{A}_\delta \to \widehat{\delta}$ the induced isomorphism of $C^*$-algebras.

Now, if $\xi = (\delta, \delta', \Phi)$ belongs to $\mathcal{C}_\Xi$, define $\Psi_\xi \in \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ by

$$(\Psi_\xi)_n(m) = k \text{ whenever } \Phi_n\left(p_m^\delta\right) + I_\delta = p_k^\delta + I_\delta,$$

for $n, m$ and $k$ in $\omega$. It is not difficult to check that $\Psi_\xi$ is a code for a homomorphism from $A_\delta$ to $A_{\delta'}$, and that the assignment $\xi \mapsto \Psi_\xi$ is Borel. Thus, the map from $\mathcal{C}_\Xi$ to $\mathcal{C}_{\widehat{\Xi}}$ that assigns to the element $\xi = (\delta, \delta', \Phi)$ in $\mathcal{C}_\Xi$, the element $(A_\delta, A_{\delta'}, \Psi_\xi)$ of $\mathcal{C}_{\widehat{\Xi}}$, is Borel. Finally, it is easily verified that the map

$$\xi = (\delta, \delta', \Phi) \mapsto \left(\widehat{A}_\delta, \widehat{A}_{\delta'}, \widehat{\Psi}_\xi\right)$$

is a functor from $\mathcal{C}_\Xi$ to the category of $C^*$-algebras. Moreover, if $\xi = (\delta, \delta', \Phi) \in \Xi$, then it follows from the construction that

$$\widehat{\Phi} \circ \eta_\delta = \eta_{\delta'} \circ \widehat{\Psi}_\xi.$$

We now proceed to construct morphism from $\mathcal{C}_{\widehat{\Xi}}$ to $\mathcal{C}_\Xi$. This will conclude the proof that $\mathcal{C}_{\widehat{\Xi}}$ and $\mathcal{C}_\Xi$ are equivalent parametrizations according to Definition XII.3.2.

For $A \in \widehat{\Xi}$ and $p \in \mathcal{U}$, denote by $p_A$ the evaluation of $p$ in the $\mathbb{Q}(i)$-$*$-algebra on $\omega$ coded by $A$, where the formal variable $X_j$ is replaced by $j$ for every $j \in \omega$. Write $A = (f, g, h, k, r)$, and define an element $\delta_A$ of $\Xi$ by

$$\delta_A(p) = \|p_A\|_r \,,$$

for all $p$ in $\mathcal{U}$. It is easily checked that the map $A \mapsto \delta_A$ is a Borel function from $\widehat{\Xi}$ to $\Xi$. For every $n \in \omega$, define a Borel map $p_n \colon \widehat{\Xi} \to \mathcal{U}$, denoted $A \mapsto p_n^A$ for $A$ in $\widehat{\Xi}$, such that

$$\left\{p_n^A + I_{\delta_A} : n \in \omega\right\}$$

is an enumeration of $\mathcal{U}/I_{\delta_A}$. The function $n \mapsto p_n^A + I_{\delta_A}$ induces an isomorphism of normed $\mathbb{Q}(i)$-$*$-algebras, from $\omega$ with the structure coded by $A$, and $\mathcal{U}I_{\delta_A}$. One checks that this isomorphism induces a $C^*$-algebra isomorphism between $\widehat{A}$ and $\widehat{\delta}_A$.

For $\xi = (A, A', \Psi) \in \mathcal{C}_{\widehat{\Xi}}$, define $\Psi_\xi \in \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ by

$$\left(\Psi_\xi\right)_n(p) = q_m^A \text{ whenever } p + I_{\delta_A} = p_k^A + I_{\delta_A} \text{ and } \Psi_n(k) = m.$$

It can easily be checked that

- $\Psi_\xi$ is a code for a homomorphism from $\widehat{\delta}^A$ to $\widehat{\delta}_{A'}$,
- the map $\xi \mapsto \Psi_\xi$ is Borel, and
- $\widehat{\Psi}_\xi \circ \eta_A = \eta_{A'} \circ \widehat{\Psi}_\xi$.

This concludes the proof that $\mathcal{C}_\Xi$ and $\mathcal{C}_{\widehat{\Xi}}$ are equivalent good parametrizations of the category of $C^*$-algebras.

We proceed to show that $\mathcal{C}_\Xi$ and $\mathcal{C}_\Gamma$ are equivalent parametrizations.

Denote by $\delta \colon \Gamma(H) \to \Xi$ and $\gamma \colon \Xi \to \Gamma(H)$ the Borel maps defined in the proof of [71, Proposition 2.7] witnessing the fact that $\Xi$ and $\Gamma(H)$ are weakly equivalent parametrizations in the sense of [71, Definition 2.1]. It is straightforward to check that the maps $\Delta \colon \mathcal{C}_{\Gamma(H)} \to \mathcal{C}_\Xi$ and $\Gamma \colon \mathcal{C}_\Xi \to \mathcal{C}_{\Gamma(H)}$ given by

$$\Delta(\gamma, \gamma', \Phi) = (\delta_\gamma, \delta_{\gamma'}, \Phi) \quad \text{ and } \quad \Gamma(\delta, \delta', \Psi) = (\gamma_\delta, \gamma_{\delta'}, \Psi)$$

are morphisms of good parametrizations, witnessing the facts that $\mathcal{C}_{\Gamma(H)}$ and $\mathcal{C}_{\Xi}$ are equivalent. $\square$

*Direct limits of $C^*$-algebras*

An *inductive system* in the category of $C^*$-algebras is a sequence $(A_n, \varphi_n)_{n\in\omega}$, where for every $n$ in $\omega$, $A_n$ is a $C^*$-algebra, and $\varphi_n \colon A_n \to A_{n+1}$ is a homomorphism. The *inductive limit* of the inductive system $(A_n, \varphi_n)_{n\in\omega}$ is the $C^*$-algebra $\varinjlim (A_n, \varphi_n)$ defined as in [14, II.8.2]. It is verified in [71, Subsection 3.2] that the inductive limit of an inductive system of $C^*$-algebras can be computed in a Borel way. We report here, for the sake of completeness, a different proof.

We will work in the parametrization $\mathcal{C}_{\Xi}$ of the category of $C^*$-algebras. In view of the equivalence of the parametrizations $\mathcal{C}_{\Xi}$, $\mathcal{C}_{\widehat{\Xi}}$, and $\mathcal{C}_{\Gamma(H)}$, the same result holds if one instead considers either one of the parametrizations $\mathcal{C}_{\widehat{\Xi}}$ or $\mathcal{C}_{\Gamma(H)}$.

Denote by $R_{dir}(\Xi)$ the set of sequences $(\delta_n, \Phi_n)_{n\in\omega} \in \left(\Xi \times \left(\mathcal{U}^{\mathcal{U}}\right)^{\omega}\right)^{\omega}$ such that $\Phi_n$ is a code for a homomorphism $\widehat{\delta}_n \to \widehat{\delta}_{n+1}$ for every $n \in \omega$. We can regard $R_{dir}(\Xi)$ as the standard Borel space parametrizing inductive systems of $C^*$-algebras. (The subscript in $R_{dir}$ stands for "direct system". There is essentially no difference in considering inductive systems or more general *countable* direct systems. This justifies the notation $R_{dir}$, which is chosen for consistency with [70]. )

**Proposition XII.3.8.** There is a Borel map from $R_{dir}(\Xi)$ to $\Xi$ that assigns to an element $(\delta_n, \Phi_n)_{n\in\omega}$ of $R_{dir}(\Xi)$ an element $\lambda_{(\delta_n,\Phi_n)_{n\in\omega}}$ of $\Xi$ such that $\widehat{\lambda}_{(\delta_n,\Phi_n)_{n\in\omega}}$ is isomorphic to the inductive limit of the inductive system $\left(\widehat{\delta}_n, \widehat{\Phi}_n\right)_{n\in\omega}$. Moreover, for every $k \in \omega$ there is a Borel map from $R_{dir}(\Xi)$ to $\left(\mathcal{U}^{\mathcal{U}}\right)^{\omega}$ that assigns to $(\delta_n, \Phi_n)_{n\in\omega}$ a code $I_k$ for the canonical homomorphism from $\widehat{\delta}_k$ to the inductive limit $\widehat{\lambda}_{(\delta_n,\Phi_n)_{n\in\omega}}$.

*Proof.* Denote for $n \in \omega$ by $\mathcal{U}_n$ the $\mathbb{Q}(i)$-$*$-algebra of $*$-polynomials in the pairwise distinct noncommutative variables $\left(X_i^{(n)}\right)_{i\in\omega}$. Similarly define $\mathcal{U}_\infty$ to be the $\mathbb{Q}(i)$-$*$-algebra of $*$-polynomials in the noncommutative variables $\left(X_i^{(n)}\right)_{(i,n)\in\omega\times\omega}$. We will naturally identify $\mathcal{U}_n$ as a $\mathbb{Q}(i)$-$*$-subalgebra of $\mathcal{U}_\infty$, and define $\mathcal{V}_n$ to be the $\mathbb{Q}(i)$-$*$-subalgebra of $\mathcal{U}_\infty$ generated by

$$\bigcup_{i\in n} \mathcal{U}_i$$

inside $\mathcal{U}_\infty$. Fix an element $(\delta_n, \Phi_n)_{k\in\omega}$ of $R_{dir}(\Xi)$. To simplify the notation we will assume that $\delta_n\colon \mathcal{U}_n \to \mathbb{R}$ for every $n \in \omega$, and $\Phi_n \in \left(\mathcal{U}_{n+1}^{\mathcal{U}_n}\right)^\omega$. Correspondingly we will define a function $\lambda_{(\delta_n,\Phi_n)_{n\in\omega}}\colon \mathcal{U}_\infty \to \mathbb{R}$. Fix $n \in n' \in \omega$ and $k \in \omega$. Define

$$\Phi_{n',n,k}\colon \mathcal{V}_n \to \mathcal{U}_{n'}$$

to be the function obtained by freely extending the maps

$$\left(\Phi_{n'-1} \circ \cdots \circ \Phi_i\right)_k : \mathcal{U}_i \to \mathcal{U}_{n'}$$

for $i \in n$. Finally define for every $N \in \omega$ and $p \in \mathcal{V}_N \subset \mathcal{U}_\infty$

$$\lambda_{(\delta_n,\Phi_n)_{n\in\omega}}(p) = \lim_{n'>N} \lim_{k\to\infty} \delta_{n'}\left(\Phi_{n',N,k}(p)\right).$$

It is immediate to verify that the definition does not depend on $N$. Moreover $\lambda_{(\delta_n,\Phi_n)_{n\in\omega}} \to \mathbb{R}$ define a seminorm on $\mathcal{U}_\infty$ such that $\widehat{\lambda}_{(\delta_n,\Phi_n)_{n\in\omega}}$ is isomorphic to the direct limit of the inductive system $\left(\widehat{\delta}_n, \widehat{\Phi}_n\right)_{n\in\omega}$. If $N \in \omega$ and $\iota_N\colon \mathcal{U}_N \to \mathcal{U}_\infty$ denotes the inclusion map, and $I_N \in \left(\mathcal{U}_\infty^{\mathcal{U}_N}\right)^\omega$ denotes the sequence constantly equal to $\iota_N$, then $I_N$ is a code for the canonical homomorphism from $\widehat{\delta}_k$ to the direct limit $\widehat{\lambda}_{(\delta_n,\Phi_n)_{n\in\omega}}$. $\square$

*One sided intertwinings*

**Definition XII.3.9.** Let $(A_n, \varphi_n)_{n\in\omega}$ and $(A'_n, \varphi'_n)_{n\in\omega}$ be inductive systems of $C^*$-algebras. A sequence $(\psi_n)_{n\in\omega}$ of homomorphisms $\psi_n\colon A_n \to A'_n$ is said to be a *one sided intertwining* between $(A_n, \varphi_n)_{n\in\omega}$ and $(A'_n, \varphi'_n)_{n\in\omega}$, if the diagram

$$\begin{tikzcd}
A_0 \arrow[r, "\varphi_0"] \arrow[d, "\psi_0"'] & A_1 \arrow[r, "\varphi_1"] \arrow[d, "\psi_1"'] & A_2 \arrow[r, "\varphi_2"] \arrow[d, "\psi_2"'] & \cdots \\
A'_0 \arrow[r, "\varphi'_0"'] & A'_1 \arrow[r, "\varphi'_1"'] & A'_2 \arrow[r, "\varphi'_2"'] & \cdots
\end{tikzcd}$$

is commutative.

If $(\psi_n)_{n\in\omega}$ is a one sided intertwining between $(A_n, \varphi_n)_{n\in\omega}$ and $(A'_n, \varphi'_n)_{n\in\omega}$, then there is an inductive limit homomorphism

$$\psi = \varinjlim \psi_n \colon \varinjlim (A_n, \varphi_n) \to \varinjlim (A'_n, \varphi'_n)$$

that makes the diagram

$$\begin{array}{ccccccccc} A_0 & \xrightarrow{\varphi_0} & A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & \cdots & \longrightarrow & \varinjlim (A_n, \varphi_n) \\ {\scriptstyle \psi_0}\downarrow & & {\scriptstyle \psi_1}\downarrow & & {\scriptstyle \psi_2}\downarrow & & & & \downarrow{\scriptstyle \psi} \\ A'_0 & \xrightarrow[\varphi'_0]{} & A'_1 & \xrightarrow[\varphi'_1]{} & A'_2 & \xrightarrow[\varphi'_2]{} & \cdots & \longrightarrow & \varinjlim (A'_n, \varphi'_n) \end{array}$$

commutative.

In this subsection, we verify that the inductive limit homomorphism $\varinjlim \psi_n$ can be computed in a Borel way. We will work in the parametrization $\mathcal{C}_\Xi$ of $C^*$-algebras. In view of the equivalence between the parametrizations $\mathcal{C}_\Xi$, $\mathcal{C}_{\widehat{\Xi}}$ and $\mathcal{C}_{\Gamma(H)}$, the same result holds if one instead uses $\mathcal{C}_{\widehat{\Xi}}$ or $\mathcal{C}_{\Gamma(H)}$.

Define $R_{int}(\Xi)$ to be the Borel set of all elements

$$\left( (\delta_n, \Phi_n)_{n\in\omega}, (\delta'_n, \Phi'_n)_{n\in\omega}, (\Psi_n)_{n\in\omega} \right) \in R_{dir}(\Xi) \times R_{dir}(\Xi) \times \left( \left( \mathcal{U}^{\mathcal{U}} \right)^\omega \right)^\omega$$

such that $\Psi_n$ is a code for a homomorphism from $\widehat{\delta}_n$ to $\widehat{\delta}'_{n+1}$ satisfying

$$\widehat{\Psi}_{n+1} \circ \widehat{\Phi}_n = \widehat{\Phi}'_n \circ \widehat{\Psi}_n$$

for every $n \in \omega$. In other words, $(\Psi_n)_{n\in\omega}$ is a sequence of codes for a one sided intertwining between the inductive systems coded by $(\delta_n, \Phi_n)_{n\in\omega}$ and $(\delta'_n, \Phi'_n)_{n\in\omega}$.

**Proposition XII.3.10.** There is a Borel map from $R_{int}(\Xi)$ to $\left( \mathcal{U}^{\mathcal{U}} \right)^\omega$ assigning to an element

$$\left( (\delta_n, \Phi_n)_{n\in\omega}, (\delta'_n, \Phi'_n)_{n\in\omega}, (\Psi_n)_{n\in\omega} \right)$$

of $R_{int}(\Xi)$, a code $\Lambda$ for the corresponding inductive limit homomorphism between the inductive limits of the systems coded by $(\delta_n, \Phi_n)_{n\in\omega}$ and $(\delta'_n, \Phi'_n)_{n\in\omega}$.

*Proof.* We will use the same notation as in the proof of Proposition XII.3.8. Fix an element $\left((\delta_n,\Phi_n)_{n\in\omega},(\delta'_n,\Phi'_n)_{n\in\omega},(\Psi_n)_{n\in\omega}\right)$ of $R_{int}(\Xi)$. As in the proof of Proposition XII.3.8 we will assume that for every $n\in\omega$

$$\delta_n\colon \mathcal{U}_n\to\mathbb{R},$$

$$\delta'_n\colon \mathcal{U}_n\to\mathbb{R},$$

$$\Phi_n\in\left(\mathcal{U}_{n+1}^{\mathcal{U}_n}\right)^{\omega}$$

and

$$\Phi'_n\in\left(\mathcal{U}_{n+1}^{\mathcal{U}_n}\right)^{\omega}.$$

Therefore

$$\Psi_n\in\left(\mathcal{U}_n^{\mathcal{U}_n}\right)^{\omega}$$

for every $n\in\omega$. Similarly the codes $\lambda_{(\delta_n,\Phi_n)_{n\in\omega}}$ and $\lambda_{(\delta'_n,\Phi'_n)_{n\in\omega}}$ for the direct limits of the systems coded by $(\delta_n,\Phi_n)_{n\in\omega}$ and $(\delta'_n,\Phi'_n)_{n\in\omega}$ will be supposed to be functions from $\mathcal{U}_\infty$ to $\mathbb{R}$. We will therefore define a code $\Lambda\in\left(\mathcal{U}_\infty^{\mathcal{U}_\infty}\right)^{\omega}$ for the homomorphism coded by $(\Psi_n)_{n\in\omega}$. Recall from the proof of Proposition XII.3.8 the definition of $\mathcal{V}_N$ and $\Phi_{n',N,k}\colon\mathcal{V}_N\to\mathcal{U}_{n'}$ for $N\in n'\in\omega$ and $k\in\omega$. Fix functions $\sigma_0,\sigma_1,\sigma_2\colon\omega\to\omega$ such that

$$n\mapsto(\sigma_0(n),\sigma_1(n),\sigma_2(n))$$

is a bijection from $\omega$ to $\omega\times\omega\times\omega$. Fix $N\in\omega$ and define for $p\in\mathcal{V}_{\sigma_0(N)}$

$$\Lambda_N(p)=\left(\Psi_{\sigma_1(N),\sigma_2(N)}\circ\Phi_{\sigma_1(N),\sigma_0(N),\sigma_2(N)}\right)(p)$$

and

$$\Lambda_N(p)=0$$

for $p\notin\mathcal{V}_{\sigma_0(N)}$. It is not difficult to check that the sequence $(\Lambda_N)_{N\in\omega}\in\left(\mathcal{U}_\infty^{\mathcal{U}_\infty}\right)^{\omega}$ indeed defines a code for the inductive limit homomorphism defined by the sequence $\left(\widehat{\Psi}_n\right)_{n\in\omega}$. $\square$

We consider as *standard Borel space* of infinite countable groups the set $\mathcal{G}$ of functions $f\colon \omega\times\omega\to\omega$ such that the identity $n\cdot_f m = f(n,m)$ for $n,m\in\omega$, defines a group structure on $\omega$. We consider $\mathcal{G}$ as a Borel space with respect to the Borel structure inherited from $\omega^{\omega\times\omega}$; such Borel structure is standard, since $\mathcal{G}$ is a Borel subset of $\omega^{\omega\times\omega}$. In the following, we will identify a group $G$ and its code as an element of $\omega^{\omega\times\omega}$.

It is not difficult to check that most commonly studied classes of groups correspond to Borel subsets of $\mathcal{G}$. In particular we will denote by $\mathcal{AG}$ the Borel set of abelian groups, and by $\mathcal{AG}_{TF}$ the Borel set of torsion free abelian groups.

Let $G$ be a countable group and let $\alpha$ be an endomorphism of $G$. We will denote by $G_\infty = \varinjlim(G,\alpha)$ the inductive limit of the inductive system

$$G \xrightarrow{\alpha} G \xrightarrow{\alpha} \cdots \longrightarrow G_\infty\ .$$

For $n$ in $\omega$, denote by $\varphi_n\colon G\to G_\infty$ the canonical group homomorphism obtained by regarding $G$ as the $n$-th stage of the inductive system above. Denote by $\alpha_\infty$ the unique automorphism of $G_\infty$ such that $\alpha_\infty\circ\varphi_{n+1}=\varphi_n$ for every $n\in\omega$.

Denote by $\mathrm{End}_{\mathcal{G}}$ the set of all pairs $(G,\alpha)\ \in\ \mathcal{G}\times\omega^\omega$, such that $\alpha$ is an injective endomorphism of $G$ with respect to the group structure on $\omega$ coded by $G$, and note that $\mathrm{End}_{\mathcal{G}}$ is Borel. Similarly define $\mathrm{DLim}_{\mathcal{G}}$ to be the set of pairs $(G,\alpha)\in\mathrm{End}_{\mathcal{G}}$ such that the direct limit $\varinjlim(G,\alpha)$ is infinite.

**Proposition XII.3.11.** The set $\mathrm{DLim}_{\mathcal{G}}$ is a Borel subset of $\mathrm{End}_{\mathcal{G}}$. Moreover there is a Borel map from $\mathrm{DLim}_{\mathcal{G}}$ to $\mathrm{End}_{\mathcal{G}}$ that assigns to $(G,\alpha)\in\mathrm{DLim}_{\mathcal{G}}$ the pair $\left(\varinjlim(G,\alpha),\alpha_\infty\right)$.

*Proof.* Let $(G,\alpha)$ be an element in $\mathrm{End}_{\mathcal{G}}$. Consider the equivalence relation $\sim_\alpha$ on $\omega\times\omega$ defined by

$$(x,i)\sim_\alpha (y,j)\ \text{ iff there exists } k\geq\max\{i,j\}\ \text{ with } \alpha^{k-i}(x)=\alpha^{k-j}(y).$$

Observe that $(G,\alpha)\ \in\ \mathrm{DLim}_{\mathcal{G}}$ iff $\sim_\alpha$ has infinitely many classes. Therefore $\mathrm{DLim}_{\mathcal{G}}$ is a Borel subset of $\mathcal{G}$ by [147, Theorem 18.10]. Suppose now that $(G,\alpha)\ \in\ \mathrm{DLim}_{\mathcal{G}}$. Consider the lexicographic order $<_{lex}$ on $\omega\times\omega$, and define the injective function $\eta_\alpha\colon\omega\to\omega\times\omega$ recursively on $n$ as follows. Set $\eta_\alpha(0) = (0,0)$, and for $n > 0$, define $\eta_\alpha(n)$ to be the $<_{lex}$-minimum element $(m,i)$ of $\omega \times \omega$ such that for every $k \in n$, we have $\eta_\alpha(k) \not\sim_\alpha (m,i)$. (Observe that the set of such elements is nonempty since we are assuming that $\sim_\alpha$ has infinitely many classes.) Define the group operation on $\omega$ by

$$n_0 \cdot_{G_\infty} n_1 = n$$

whenever there are $m_0, m_1, m, i_0, i_1, i, \widetilde{i} \in \omega$ satisfying:

- $\eta_\alpha(n_0) = (m_0, i_0)$;
- $\eta_\alpha(n_1) = (m_1, i_1)$;
- $\eta(n) = (m,i)$;
- $\max\{i_0, i_1\} = \widetilde{i}$;
- $\left(\alpha^{\widetilde{i}-i_0}(m_0) \cdot_G \alpha^{\widetilde{i}-i_1}(m_1), \widetilde{i}\right) \sim (m,i)$.

Define the function $\alpha_\infty \colon \omega \to \omega$ by $\alpha_\infty(n) = n'$ if and only if there are $m, i, m', i' \in \omega$ such that:

- $\eta_\alpha(n) = (m,i)$;
- $\eta_\alpha(n') = (m', i')$;
- $(\alpha(m), i) \sim (m', i')$.

It is not difficult to check that $G_\infty$ is the direct limit $\varinjlim(G,\alpha)$, and $\alpha_\infty$ is the automorphism of $\varinjlim(G,\alpha)$ corresponding to the endomorphism $\alpha$ of $G$. Moreover the function $(G,\alpha) \mapsto (G_\infty, \alpha_\infty)$ is Borel by construction. $\square$

*Borel version of the Nielsen-Schreier theorem*

The celebrated Nielsen-Schreier theorem asserts that a subgroup of a countable free group is free. In this subsection we will prove a *Borel version* of such theorem, to be used in the proof of Lemma XII.3.14. This will be obtained by analyzing Schreier's proof of the theorem, as presented in [138, Chapter 2].

Denote by $F$ the (countable) set of *reduced* words in the indeterminates $x_n$ for $n \in \omega$ ordered lexicographically. We can identify the free group on countable many generators with $F$

with the operation of *reduced concatenation* of words. It is immediate to check that the set $\mathcal{S}(F)$ of $H \in 2^{\omega}$ such that $H$ is a subgroup of $F$ is Borel.

**Lemma XII.3.12.** There is a Borel function $H \mapsto B_H$ from $\mathcal{S}(F)$ to $2^F$ such that $L_H$ is a set of free generators of $H$ for every $H \in \mathcal{S}(F)$.

*Proof.* Suppose that $H \in \mathcal{S}(F)$. If $a \in F$ denote by $\phi_H(a)$ the $<$-minimal element of the coset $Ha$, where $<$ is the lexicographic order of $F$. Observe that $\phi_H(a) \leq b$ iff there is $b' \leq b$ such that $b'a^{-1} \in H$. This shows that the map

$$\begin{aligned} \mathcal{S}(F) &\to F^F \\ H &\mapsto \phi_H \end{aligned}$$

is Borel. Define $B_H$ to be the set containing

$$\phi_H(a)\, x_n \phi_H\left(\phi_H(a)\, x_n\right)^{-1}$$

for every $(n, a) \in \omega \times F$ with the property that $\phi_H(a)\, x_n \neq \phi_H(c)$ for every $c \in F$. It is clear that the map $H \mapsto B_H$ is Borel. Moreover it can be shown as in [138, Chapter 2, Lemmas 3,4,5] that $B_H$ is a free set of generators of $H$. $\square$

Suppose now that $\mathbb{F}_{\omega}$ is an element of $\mathcal{G}$ representing the group of countably many generators, and $\mathcal{S}(F)$ is the Borel set of $H \in 2^{\omega}$ such that $H$ is a subgroup of $\mathbb{F}_{\infty}$. Proposition can be seen as just a reformulation of Lemma

**Proposition XII.3.13.** There is a Borel map $H \mapsto B_H$ from $\mathcal{S}(F)$ to $2^{\omega}$ that assigns to $H \in \mathcal{S}(F)$ a free set of generators of $H$.

*An exact sequence*

The following lemma asserts that the construction of [233, Proposition 3.5] can be made in a Borel way.

**Lemma XII.3.14.** There is a Borel function from $\mathcal{AG}$ to $\mathcal{AG}_{TF} \times \omega^{\omega}$ that assigns to an infinite abelian group $G$, a pair $(H, \alpha)$, where $H$ is an infinite torsion free abelian group, and $\alpha$ is an automorphism of $H$ such that

$$H/(id_H - \alpha)[H] \cong G.$$

*Proof.* Denote by $\mathbb{F}_{\omega\times\omega}$ the free group with generators $x_{n,m}$ for $(n,m) \in \omega\times\omega$, suitably coded as an element of the standard Borel spaces of countable groups $\mathcal{G}$. Given an element $G \in \mathcal{AG}$, denote by $N_G$ the subset of $\omega$ coding the kernel of the homomorphism from $\mathbb{F}_{\omega\times\omega}$ to $G$ obtained by sending $x_{n,m}$ to $n$ if $m = 0$, and to zero otherwise. In view of Proposition XII.3.13 one can find a Borel map

$$\begin{aligned} \mathcal{AG} &\to \omega^\omega \\ G &\mapsto x^G \end{aligned}$$

such that $x^G = \left(x_n^G\right)_{n\in\omega}$ is an enumeration of a free set of generators of $N_G$. Define an injective endomorphism $\delta_G$ of $\mathbb{F}_{\omega\times\omega}$ by

$$\delta_G(x_{n,m}) = \begin{cases} x_{n,m+1} & \text{if } m \neq -1, \\ x_n^G & \text{otherwise.} \end{cases}$$

Let $\beta_G \colon \mathbb{F}_{\omega\times\omega} \to \mathbb{F}_{\omega\times\omega}$ be $\beta_G = \mathrm{id}_{\mathbb{F}_{\omega\times\omega}} - \delta_G$. By construction, the map $G \mapsto \beta_G$ is Borel. From now on we fix a group $G$, and abbreviate $\beta_G$ to just $\beta$.

By Proposition XII.3.11, the inductive limit group $G_\infty = \varinjlim(G,\beta)$ and the automorphism $\beta_\infty = \varinjlim \beta$ can be constructed in a Borel way from $G$ and $\beta$. We take $H = G_\infty$ and $\alpha = \beta_\infty$. It can now be verified, as in the proof of [233, Proposition 3.5], that $G$ is isomorphic to the quotient of $H$ by the image of $\mathrm{id}_H - \alpha$. Moreover, it follows that the map $G \mapsto (H,\alpha)$ is Borel. This finishes the proof. □

### Computing Reduced Crossed Products

The goal of this section is to show that the reduced crossed product of a $C^*$-algebra by an action of a countable group can be computed in a Borel way.

*Parametrizing actions of countable groups on $C^*$-algebras*

We proceed to construct a standard Borel parametrization of the space of all actions of countable groups on $C^*$-algebras. For convenience, we will work using the parametrization $\Gamma(H)$ of $C^*$-algebras. In view of the weak equivalence of $\Xi$, $\widehat{\Xi}$, and $\Gamma(H)$, similar statements will hold for the parametrizations $\Xi$ and $\widehat{\Xi}$.

**Definition XII.4.1.** Let $\gamma$ be an element of $\Gamma(H)$, and $G$ be an element of $\mathcal{G}$. Suppose that $\Phi = (\Phi_{m,n})_{(m,n)\in\omega\times\omega} \in \left(\mathcal{U}^{\mathcal{U}}\right)^{\omega\times\omega}$ is an $(\omega\times\omega)$-sequence of functions from $\mathcal{U}$ to $\mathcal{U}$. We say that $\Phi$ is a *code for an action of* $G$ *on* $C^*(\gamma)$, if the following conditions hold:

1. for every $m \in \omega$, the sequence $(\Phi_{m,n})_{n\in\omega} \in \left(\mathcal{U}^{\mathcal{U}}\right)^{\omega}$ is a code for an automorphism $\widehat{\Phi}_m$ of $C^*(\gamma)$,
2. $\Phi_{0,n}(m) = m$ for every $n, m \in \omega$, and
3. the function $m \mapsto \widehat{\Phi}_m$ is an action of $G$ on $C^*(\gamma)$, that is,

$$\widehat{\Phi}_m \circ \widehat{\Phi}_k = \widehat{\Phi}_n$$

   whenever $(m,k,n) \in G$.

It is easy to verify that any action of $G$ on $C^*(\gamma)$ can be coded in such way. Moreover, the set $\mathrm{Act}_{\Gamma(H)}$ of triples $(G,\gamma,\Phi) \in \mathcal{G}\times\Gamma(H)\times(\mathcal{U}^{\omega})^{\omega\times\omega}$ such that $\Phi$ is a code for an action of $G$ on $C^*(\gamma)$, is a Borel subset of $\mathcal{G}\times\Gamma(H)\times(\mathcal{U}^{\omega})^{\omega\times\omega}$. We will regard $\mathrm{Act}_{\Gamma(H)}$ as the standard Borel space of actions of countable groups on $C^*$-algebras.

*Computing the reduced crossed product*

We are now ready to prove that the reduced crossed product of a $C^*$-algebra by an action of a countable group can be computed in a Borel way.

**Proposition XII.4.2.** Let $H$ be a separable Hilbert space. Then there is a Borel map $(G,\gamma,\Phi) \mapsto \delta_{(G,\gamma,\Phi)}$ from $\mathrm{Act}_{\Gamma(H)}$ to $\Gamma(H)$ such that $C^*\left(\delta_{(G,\gamma,\Phi)}\right) \cong C^*(\gamma) \rtimes^r_{\widehat{\Phi}} G$. In other words, there is a Borel way to compute the code of the reduced crossed product of separable $C^*$-algebras by countable groups.

*Proof.* Denote by $\{e_k : k \in \omega\}$ the canonical basis of $\ell_2$. Let $(G, \gamma, \Phi)$ be an element of $\mathrm{Act}_{\Gamma(H)}$. Define the element $\delta_{(G,A,\Phi)}$ of $\Gamma(H)$ as follows. Given $m$ in $\omega$, denote by $m' \in \omega$ the inverse of $m$ in $G$. Now set

$$(\delta_{(G,\gamma,\Phi)})_n(\xi \otimes m) = \begin{cases} \lim_{k\to+\infty} \gamma_{\Phi_{(m',k)}(r)} & \text{if } n = 2r, \text{ where } (n, m, k) \in G, \\ \xi \otimes k & \text{otherwise,} \end{cases}$$

for all $\xi$ in $H$ and all $m$ in $\omega$. The fact that $C^*\left(\delta_{(G,\gamma,\Phi)}\right) \cong C^*(\gamma) \rtimes_{\widehat{\Phi},r} G$ follows from [197, Theorem 7.7.5]. Moreover, the map $(G, \gamma, \Phi) \mapsto \delta_{(G,\gamma,\Phi)}$ is Borel by construction and by [71, Lemma 3.4]. $\square$

Proposition XII.4.2 above answers half of [71, Problem 9.5(2)]. It is not clear how to treat the case of full crossed products, even in the special case when the algebra is $\mathbb{C}$.

*Crossed products by a single automorphism*

In this subsection, we want to show that the crossed product of a $C^*$-algebra by a single automorphism, when regarded as an action of $\mathbb{Z}$, can be computed in a Borel way. In view of the equivalence of the good parametrizations $\mathcal{C}_\Xi$, $\mathcal{C}_{\widehat{\Xi}}$, and $\mathcal{C}_{\Gamma(H)}$, we can work in any of these. For convenience, we consider the parametrization $\mathcal{C}_{\Gamma(H)}$.

Let us denote by $\mathrm{Aut}_{\Gamma(H)}$ the set of pairs $(\gamma, \Phi)$ in $\Gamma(H) \times \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ such that $\Phi$ is a code for an automorphism of $C^*(\gamma)$. It is immediate to check that such set is Borel. We can regard $\mathrm{Aut}_{\Gamma(H)}$ as the standard Borel space of automorphisms of $C^*$-algebras.

**Lemma XII.4.3.** There is a Borel map from $\mathrm{Aut}_{\Gamma(H)}$ to $\mathrm{Act}_{\Gamma(H)}$ that assigns to an element $(\gamma, \Phi)$ in $\mathrm{Aut}_{\Gamma(H)}$, a code for the action of $\mathbb{Z}$ on $C^*(\gamma)$ associated with the automorphism coded by $\Phi$.

*Proof.* In the parametrization $\mathcal{G}$ of countable infinite groups described before, the group of integers $\mathbb{Z}$ is coded, for example, by the element $f_{\mathbb{Z}}$ of $\omega^{\omega\times\omega}$ given by

$$
\begin{aligned}
f_{\mathbb{Z}}(2n, 2m) &= 2(n+m) \\
f_{\mathbb{Z}}(2n-1, 2m-1) &= 2(n+m)-1 \\
f_{\mathbb{Z}}(2n-1, m) &= f_{\mathbb{Z}}(m, 2n-1) = 2(n-m)-1 \\
f_{\mathbb{Z}}(2m-1, n) &= f_{\mathbb{Z}}(n, 2m-1) = 2(n-m) \\
f_{\mathbb{Z}}(k, 0) &= f_{\mathbb{Z}}(0, k) = k
\end{aligned}
$$

for $n, m, k \in \omega$ with $n, m \geq 1$. Recall that by Lemma XII.3.6 there is a Borel map $\xi \mapsto \mathrm{Inv}\,(\xi)$ from $\mathrm{Iso}_{\Gamma(H)}$ to $\left(\mathcal{U}^{\mathcal{U}}\right)^{\omega}$ such that if $\xi = (\gamma, \gamma', \Phi)$, then $\mathrm{Inv}\,(\xi)$ is a code for the inverse of the *-isomorphism coded by $\Phi$. Suppose now that $(\gamma, \Phi) \in \mathrm{Aut}_{\Gamma(H)}$. We want to define a code $\Psi$ for the action of $\mathbb{Z}$ on $C^*(\gamma)$ induced by $\widehat{\Phi}$. For $n, m \in \omega$ with $m \geq 1$ define

$$
\Psi_{0,n}(k) = k,
$$

$$
\Psi_{2m,n} = \overbrace{\Phi_n \circ \Phi_n \cdots \circ \Phi_n}^{m \text{ times}},
$$

and

$$
\Psi_{2m+1,n} = \mathrm{Inv}\,(\gamma, \gamma, \Psi_m)\,.
$$

Observe that $(f_{\mathbb{Z}}, A, \Psi)$ is a code for the action of $\mathbb{Z}$ associated with the automorphism $\widehat{\Phi}$ of $C^*(\gamma)$. It is not difficult to verify that the map assigning $(f_{\mathbb{Z}}, A, \Psi)$ to $(A, \Phi)$ is Borel. We omit the details. □

**Corollary XII.4.4.** Given a $C^*$-algebra $A$ and an automorphism $\alpha$ of $A$, there is a Borel way to compute the crossed product $A \rtimes_\alpha \mathbb{Z}$.

*Proof.* Note that the group of integers $\mathbb{Z}$ is amenable, so full and reduced crossed products coincide. The result now follows immediately from Lemma XII.4.3 together with Proposition XII.4.2. □

We now turn to crossed products by injective, corner endomorphisms, as introduced by Paschke in [193], building on previous work of Cuntz from [39]. Although there are more general theories for such crossed products allowing arbitrary endomorphisms of $C^*$-algebras (see, for example, [65]), the endomorphisms considered by Paschke will suffice for our purposes. We begin by presenting the precise definition of a corner endomorphism. Throughout this subsection, we fix a unital $C^*$-algebra $A$.

**Definition XII.4.5.** An endomorphism $\rho\colon A \to A$ is said to be a *corner* endomorphism if $\rho(A)$ is a corner of $A$, that is, if there exists a projection $p$ in $A$ such that $\rho(A) = pAp$.

Since $A$ is unital, if $\rho\colon A \to A$ is a corner endomorphism and $\rho(A) = pAp$ for some projection $p$ in $A$, then we must have $p = \rho(1)$. Let us observe for future reference that the set $CorEnd_\Gamma$ of pairs $(\gamma, \Phi) \in \Gamma \times \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$ such that $C^*(\gamma)$ is unital and $\Phi$ is a code for an injective corner endomorphism of $C^*(\gamma)$ is Borel. By [71, Lemma 3.14] the set $\Gamma_u$ of $\gamma \in \Gamma$ such that $C^*(\gamma)$ is unital is Borel. Moreover, there is a Borel map $\mathrm{Un}\colon \Gamma_u \to B_1(H)$ such that $\mathrm{Un}(\gamma)$ is the unit of $C^*(\gamma)$ for every $\gamma \in C^*(\gamma)2$. If now $\gamma \in \Gamma_u$ and $\Phi \in \left(\mathcal{U}^{\mathcal{U}}\right)^\omega$, then $\Phi$ is a code for an injective corner endomorphism of $C^*(\gamma)$ if and only if $\Phi$ is a code for an endomorphism of $A$ (which is a Borel condition, as observed in Subsection XII.3), and for every $p \in \mathcal{U}$ and $n \in \omega$ there is $m_0 \in \omega$ and $q \in \mathcal{U}$ such that for every $m \geq m_0$

$$\|\Phi_m(p)(\gamma)\| \geq \|p(\gamma)\| - \frac{1}{n}$$

and

$$\|\mathrm{Un}(\gamma)p(\gamma)\mathrm{Un}\,(\gamma) - \Phi_m\,(q)\,(\gamma)\| \leq \frac{1}{n}.$$

Let $\rho$ be an injective corner endomorphism of $A$. The crossed product $A \rtimes_\rho \mathbb{N}$ of $A$ by $\rho$ is implicitly defined in [193] as the universal $C^*$-algebra generated by a unital copy of $A$ together with an isometry $s$, subject to the relation

$$sas^* = \rho(a)$$

for all $a$ in $A$. Suppose that $s$ is an isometry of $A$. Notice that the endomorphisms $a \mapsto sas^*$ is injective and its range is the corner $(ss^*)A(ss^*)$ of $A$.

Instead of using this construction, which involves universal $C^*$-algebras on generators and relations, we will use the construction of the endomorphism crossed product described by Stacey in [255]. Stacey's picture has the advantage that, given what we have proved so far, it will be relatively easy to conclude that crossed products by injective corner endomorphisms can be computed in a Borel way.

We proceed to describe Stacey's construction.

**Definition XII.4.6.** Let $\rho\colon A \to A$ be an injective corner endomorphism. Consider the inductive system $(A, \alpha)_{n\in\omega}$ (the same algebra and same connecting maps throughout the sequence). Denote by $A_\infty$ its inductive limit, and by $\iota_{n,\infty}\colon A \to A_\infty$ the canonical map into the inductive limit. The commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & A \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & \cdots \arrow[r] & A_\infty \arrow[d, dashed, "\alpha_\infty"] \\ A \arrow[r, "\alpha"'] & A \arrow[r, "\alpha"'] & \cdots \arrow[r] & A_\infty. \end{tikzcd}$$

gives rise in the limit to an endomorphism $\alpha_\infty\colon A_\infty \to A_\infty$, which is in fact an automorphism of $A_\infty$. Denote by $e$ the projection of $A_\infty$ corresponding to the unit of $A$. The *(endomorphism) crossed product* of $A$ by $\rho$ is the corner $e(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z})e$ of the (automorphism) crossed product $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$.

As mentioned before, this construction of the crossed product of a $C^*$-algebra by an endomorphism makes it apparent that it can be computed in a Borel way. In fact, we have verified in Proposition XII.3.10, that the limit of a one sided intertwining can be computed in a Borel way, and in Corollary XII.4.4, that the crossed product of a $C^*$-algebra by an automorphism can be computed in a Borel way. Moreover, it is shown in [71, Lemma 3.14], that one can select in a Borel way the unit of a unital $C^*$-algebra. The only missing ingredient in the construction is taking a corner by a projection, which is shown to be Borel in the following lemma. We will work, for convenience, in the parametrization $\Gamma(H)$ of $C^*$-algebras.

**Lemma XII.4.7.** The set $\Gamma_{proj}(H)$ of pairs $(\gamma, e)$ in $\Gamma(H) \times B(H)$ such that $e$ is a nonzero projection in $C^*(\gamma)$, is Borel. Moreover, there is a Borel map $(\gamma, e) \mapsto c_{\gamma,e}$ from $\Gamma_{proj}(H)$ to $\Gamma(H)$ such that $C^*(c_{\gamma,e})$ is the corner $eC^*(\gamma)e$ of $C^*(\gamma)$.

*Proof.* Enumerate a dense subset $\{\xi_n : n \in \omega\}$ of $H$, and let $(\gamma, e)$ be an element in $\Gamma(H) \times B(H)$. Observe that $(\gamma, e)$ belongs to $\Gamma_{proj}(H)$ if and only if the following conditions hold:

1. The element $e$ is a projection, that is, for every $n, k \in \omega$,

$$\|(e - e^*)\,\xi_k\| < \frac{1}{n+1} \quad \text{and} \quad \left\|\left(e^2 - e\right)\xi_k\right\| < \frac{1}{n+1};$$

2. The element $e$ is non-zero, that is, there are $k, n \in \omega$ such that

$$\|e\xi_k\| > \frac{1}{n+1};$$

3. The element $e$ is in $C^*(\gamma)$, that is, for every $n \in \omega$ there is $p \in \mathcal{U}$ such that

$$\|(p(\gamma) - e)\,\xi_m\| < \frac{1}{n+1},$$

   for every $m \in \omega$.

This shows that $\Gamma_{proj}(H)$ is a Borel subset of $\Gamma(H) \times B(H)$. Observe now that by setting

$$\left(c_{\gamma,e}\right)_n = e\gamma_n e$$

for every $n \in \omega$, one obtains an element $c_{\gamma,e}$ of $\Gamma(H)$ such that $C^*\left(c_{\gamma,e}\right) = eC^*(\gamma)e$. It is immediate to check that the map $(\gamma, e) \mapsto c_{\gamma,e}$ is Borel. □

We have thus proved the following.

**Corollary XII.4.8.** Given a unital $C^*$-algebra $A$ and an injective corner endomorphism $\rho$ of $A$, there is a Borel way to compute a code for the crossed product of $A$ by $\rho$.

More precisely, there is a Borel map

$$\begin{aligned} CorEnd_\Gamma &\to \Gamma \\ (\gamma, \Phi) &\mapsto \delta_{\gamma,\Phi} \end{aligned}$$

such that $C^*(\delta_{\gamma,\Phi}) \cong C^*(\gamma) \rtimes_{\widehat{\Phi}} \mathbb{N}$, where, as before, $CorEnd_\Gamma$ is the Borel space of pairs $(\gamma, \Phi)$ in $\Gamma \times (\mathcal{U}^{\mathcal{U}})^\omega$ such that $C^*(\gamma)$ is unital and $\Phi$ is a code for an injective corner endomorphism of $C^*(\gamma)$ .

*Proof.* Combine Lemma XII.4.7 with Proposition and Corollary XII.4.4. □

**Borel Selection of AF-algebras**

*Bratteli diagrams*

We refer the reader to [63] for the standard definition of a Bratteli diagram. We will identify Bratteli diagrams with elements

$$\left(l, (w_n)_{n\in\omega}, (m_n)_{n\in\omega}\right) \in \omega^\omega \times (\omega^\omega)^\omega \times \left(\omega^{\omega\times\omega}\right)^\omega$$

such that for every $i, j, n, m \in \omega$, the following conditions hold:

1. $l(0) = 1$;
2. $w_0(0) = 1$;
3. $w_n(i) > 0$ if and only if $i \in l(n)$;
4. $m_n(i, j) = 0$ whenever $i \geq l(n)$ or $j \geq l(n+1)$
5. Setting $k_n = i$,
$$w_n(i) = \sum_{(k_j)_{j\in n} \in l(j)^n} \prod_{t\in n} m_t(k_t, k_{t+1}).$$

We denote by $\mathcal{BD}$ the Borel set of all elements $(l, w, m)$ in $\omega^\omega \times (\omega^\omega)^\omega \times (\omega^{\omega\times\omega})^\omega$ that satisfy conditions 1–5 above. An element $(l, w, m)$ of $\mathcal{BD}$ codes the Bratteli diagram with $l(n)$ vertices at the $n$-th level of weight $w_n(0), \ldots, w(l(n) - 1)$ and with $m_n(i, j)$ arrows from the $i$-th vertex at the $n$-th level to the $j$-th vertex and the $(n+1)$-st level for $n \in \omega$, $i \in l(n)$, and $j \in l(n+1)$. We call the elements of $\mathcal{BD}$ simply "Bratteli diagrams".

*Dimension groups*

**Definition XII.5.1.** An *ordered abelian group* is a pair $(G, G^+)$, where $G$ is an abelian group and $G^+$ is a subset of $G$ satisfying

1. $G^+ + G^+ \subseteq G^+$;

2. $0 \in G^+$;

3. $G^+ \cap (-G^+) = \{0\}$;

4. $G^+ - G^+ = G$.

We call $G^+$ the *positive cone* of $G$. It defines an order on $G$ by declaring that $x \leq y$ whenever $y - x \in G^+$. An element $u$ of $G^+$ is said to be an *order unit* for $(G, G^+)$, if for every $x$ in $G$, there exists a positive integer $n$ such that

$$-nu \leq x \leq nu.$$

An ordered abelian group $(G, G^+)$ is said to be *unperforated* if whenever a positive integer $n$ and $a \in A$ satisfy $na \geq 0$, then $a \geq 0$. Equivalently, $G^+$ is divisible.

An ordered abelian group is said to have the *Riesz interpolation property* if for every $x_0, x_1, y_0, y_1 \in G$ such that $x_i \leq y_j$ for $i, j \in 2$, there is $z \in G$ such that

$$x_i \leq z \leq y_j$$

for $i, j \in 2$.

**Definition XII.5.2.** A *dimension group* is an unperforated ordered abelian group $(G, G^+, u)$ with the Riesz interpolation property and a distinguished order unit $u$.

Let $(G, G^+, u)$ and $(H, H^+, w)$ be dimension groups, and let $\phi\colon G \to H$ be a group homomorphism. We say that $\phi$ is

1. is *positive* if $\phi(G^+) \subseteq H^+$, and

2. *preserves the unit* if $\phi(u) = w$.

Notice that positivity for a homomorphism between ordered groups is equivalent to being increasing with respect to the natural order.

**Example XII.5.3.** If $l \in \omega$ and $w_0, \dots, w_{l-1} \in \mathbb{N}$, then $\mathbb{Z}^l$ with $\mathbb{N}^l$ as its positive cone, and $(w_0, \dots, w_{l-1})$ as order unit, is a dimension group. We denote by $e_0^{(l)}, \dots, e_{l-1}^{(l)}$ the canonical basis of $\mathbb{Z}^l$.

We refer the reader to [235, Section 1.4] for a more complete exposition on dimension groups.

A dimension group can be coded in a natural way as an element of $\omega^{\omega\times\omega} \times 2^\omega \times \omega$. The set $\mathcal{DG}$ of codes for dimension groups is a Borel subset of $\omega^{\omega\times\omega} \times 2^\omega \times \omega$, which can be regarded as the standard Borel space of dimension groups.

One can associate to a Bratteli diagram $(l, w, m)$ the dimension group $G_{(l,w,m)}$ obtained as follows. For $n$ in $\omega$, denote by

$$\varphi_n \colon \mathbb{Z}^{l(n)} \to \mathbb{Z}^{l(n+1)}$$

the homomorphism given on the canonical bases of $\mathbb{Z}^{l(n)}$ by

$$\varphi_n\left(e_k^{(l(n))}\right) = \sum_{i \in l(n+1)} m_n(i,j) e_j^{(l(n+1))},$$

for all $k$ in $l(n)$. Then $G_{(l,w,m)}$ is defined as the inductive limit of the inductive system

$$\left(\mathbb{Z}^{l(n)}, \left(w_n\left(0\right), \dots, w_n\left(l(n)-1\right)\right), \varphi_n\right)_{n\in\omega}.$$

Theorem 2.2 in [54] asserts that any dimension group is in fact isomorphic to one of the form $G_{(l,w,m)}$ for some Bratteli diagram $(l, w, m)$. The key ingredient in the proof of [54, Theorem 2.2] is a lemma due to Shen, see [54, Lemma 2.1] and also [251, Theorem 3.1]. We reproduce here the statement of the lemma, for convenience of the reader.

**Lemma XII.5.4.** Suppose that $(G, G^+, u)$ is a dimension group. If $n \in \omega$ and $\theta \colon n \to G$ is any function, then there are $N \in \omega$, and functions $\Phi \colon N \to G^+$ and $g \colon n \times N \to \omega$, satisfying the following conditions:

1. For all $i \in n$,
$$\theta(i) = \sum_{j \in N} g(i,j)\Phi(j).$$
2. Whenever $(k_i)_{i \in n} \in \mathbb{Z}^n$ is such that $\sum_{i \in n} k_i \theta(i) = 0$, then
$$\sum_{i \in l^G(n)} k_i g(i,j) = 0$$
for every $j \in N$.

It is immediate to note that the set of tuples $((G, G^+, u), n, \theta, N, \Phi, g)$ satisfying 1 and 2 of Lemma XII.5.4 is Borel. It follows that in Lemma XII.5.4 the number $N$ and the maps $\Phi$ and $g$ can be computed from $(G, G^+, u)$, $n$, and $\theta$ is a Borel way. This will be used to show that if we start with a dimension group $(G, G^+, u)$, then we can choose in a Borel way a Bratteli diagram $(l^G, w^G, m^G)$ such that $G_{l^G, w^G, m^G}$ is isomorphic to $G$ as dimension group with order unit. (The existence of such Bratteli diagram is established in [54, Theorem 2.2].) This is the content of Proposition XII.5.5 below. A Borel version of [54, Theorem 2.2] is also proved in [63, Theorem 5.3]. We present here a proof, for the convenience of the reader, and to introduce ideas and notations to be used in the proof of Proposition XII.5.12.

**Proposition XII.5.5.** There is a Borel function that associates to a dimension group $G = (G, G^+, u) \in \mathcal{DG}$ a Bratteli diagram $\left(l^G, w^G, m^G\right) \in \mathcal{BD}$ such that the dimension group associated with $\left(l^G, w^G, m^G\right)$ is isomorphic to $G$.

*Proof.* It is enough to construct in a Borel way a Bratteli diagram $\left(l^G, w^G, m^G\right)$ and maps $\theta_n^G \colon l^G(n) \to G$ satisfying the following conditions:

1. For all $i \in l^G(n)$,
$$\theta_n^G(i) = \sum_{j \in l^G(n+1)} m_n^G(i,j)\theta_n^G(j);$$
2. For any $k_0, \ldots, k_{l^G(n)-1} \in \mathbb{Z}$ such that
$$\sum_{i \in l^G(n)} k_i \theta_n^G(i) = 0,$$

we have that

$$\sum_{i \in l^G(n)} k_i m_n^G(i,j) = 0,$$

for every $j \in l^G(n+1)$;

3. For every $x \in G^+$, there are $n \in \omega$ and $i \in l^G(n)$, such that $\theta_n^G(i) = x$.

It is not difficult to verify that conditions (1), (2) and (3) ensure that the dimension group coded by the Bratteli diagram $\left(l^G, w^G, m^G\right)$ is isomorphic to $G$, via the isomorphism coded by $\left(\theta_n^G\right)_{n\in\omega}$.

We define $\theta_n^G$, $l^G(n)$, $w_n^G$ and $m_n^g$ by recursion on $n$. Define $l^G(0) = 1$ and $\theta^G(0) = u$. Suppose that $l^G(k)$, $w_k^G$, $m_k^G$, and $\theta_k^G$ have been defined for $k \leq n$. Define $\theta' \colon l^G(n) + 1 = \{0, \dots, l^G(n)\} \to G$ by

$$\theta'(i) = \begin{cases} \theta(i), & \text{if } i \in l^G(n), \\ n, & \text{if } i = l^G(n) \text{ and } n \in G^+, \\ u, & \text{otherwise.} \end{cases}$$

Suppose that the positive integer $N$ and the functions $\Phi \colon N \to G^+$ and $g \colon n \times N \to \omega$ are obtained from $l^G(n)$ and $\theta'$ via Lemma XII.5.4. Define now:

$$l^G(n+1) = N$$

$$m_n(i,j) = \begin{cases} g(i,j), & \text{if } i \in l^G(n) \text{ and } j \in l^G(n+1), \\ 0, & \text{otherwise.} \end{cases}$$

$$w_{n+1}(j) = \sum_{i \in l^G(n)} w_n(i,n) m_n(i,j).$$

It is left as an exercise to check that with these choices, conditions (1), (2) and (3) are satisfied. This finishes the proof. □

*Approximately finite dimensional $C^*$-algebras*

A unital $C^*$-algebra $A$ is said to be *approximately finite dimensional*, or *AF-algebra* if it is isomorphic to a direct limit of a direct system of finite dimensional $C^*$-algebras with unital connecting maps.

It is a standard result in the theory of $C^*$-algebras, that any finite dimensional $C^*$-algebra is isomorphic to a direct sum of matrix algebras over the complex numbers [47, Theorem III.l.l]. A fundamental result due to Bratteli (building on previous work of Glimm) asserts that unital AF-algebras are precisely the unital $C^*$-algebras that can be locally approximated by finite dimensional $C^*$-algebras.

**Theorem XII.5.6** (Bratteli-Glimm [19],[102])**.** Let $A$ be a separable $C^*$-algebra. Then the following are equivalent:

1. $A$ is a unital AF-algebra;
2. For every finite subset $F$ of $A$ and every $\varepsilon > 0$, there exists a finite dimensional C*-subalgebra $B$ of $A$, such that for every $a \in F$ there is $b \in B$ such that $\|a - b\| < \varepsilon$.

A modern presentation of the proof of Theorem XII.5.6 can be found in [235, Proposition 1.2.2].

A distinguished class of unital AF-algebras are the so called unital *UHF-algebras*. These are the unital AF-algebras that are isomorphic to a direct limit of *full matrix algebras*. Of particular importance are the UHF-algebras *of infinite type*. These can be described as follows: Fix a strictly positive integer $n$. Denote by $M_{n^\infty}$ the $C^*$-algebra obtained as a limit of the inductive system

$$M_n \to M_{n^2} \to M_{n^3} \to \cdots$$

where the inclusion of $M_{n^k}$ into $M_{n^{k+1}}$ is given by the diagonal embedding $a \mapsto \operatorname{diag}(a, \ldots, a)$. The UHF-algebras of infinite type are precisely those ones of the form $\mathbb{M}_{n^\infty}$ for some $n \in \mathbb{N}$.

A celebrated theorem of Elliott asserts that unital AF-algebras are classified up to isomorphism by their ordered $K_0$-group. Moreover, dimension groups can be characterized within the class of ordered abelian groups with a distinguished order unit as the $K_0$-groups of AF-algebras.

**Theorem XII.5.7** (Elliott [57])**.** Let $A$ and $B$ be unital AF-algebras.

1. For every positive morphism

$$\phi\colon (K_0(A), K_0(A)^+) \to (K_0(B), K_0(B)^+),$$

such that $\phi([1_A]) \leq [1_B]$, there exists a homomorphism $\rho\colon A \to B$ such that $K_0(\rho) = \phi$.

2. $A$ and $B$ are isomorphic if and only if there is an isomorphism

$$(K_0(A), K_0(A)^+, [1_A]) \cong (K_0(B), K_0(B)^+, [1_B])$$

as dimension groups with order units.

In (1), the range of $\rho$ is a corner of $B$ if and only if the set

$$\{x = [p] - [q] \in K_0(B)^+ \colon p, q \in B \text{ and } x \leq \phi([1_A])\}$$

is contained in $\phi(K_0(A)^+)$.

Let $(l, w, m)$ be a Bratteli diagram. We will describe how to canonically associate to it a unital AF-algebra, which we will denote by $A_{(l,w,m)}$. For each $n$ in $\omega$, define a finite dimensional $C^*$-algebra $F_n$ by

$$F_n = \bigoplus_{i \in l(n)} M_{w_n(i)}.$$

Denote by $\varphi_n\colon F_n \to F_{n+1}$ the unital injective homomorphism determined as follows. For every $i \in l(n)$ and $j \in l(n+1)$, the restriction of $\varphi_n$ to the $i$-th direct summand of $F_n$ and the $j$-th direct summand of $F_{n+1}$ is a diagonal embedding of $m_n(i,j)$ copies of $M_{w_n(i)}$ in $M_{w_{n+1}(j)}$. Then $A_{(l,w,m)}$ is the inductive limit of the inductive system $(F_n, \varphi_n)_{n\in\omega}$.

The $K_0$-group of $A_{(l,w,m)}$ is isomorphic to the dimension group $G_{l,w,m}$ associated with $(l, w, m)$.

The main result of [19] asserts that any unital AF-algebra is isomorphic to the $C^*$-algebra associated with a Bratteli diagram. We show below that the code for such an AF-algebra can be computed in a Borel way.

**Proposition XII.5.8.** Given a Bratteli diagram, there is a Borel way to compute the code for its associated unital AF-algebra.

*Proof.* By Proposition XII.3.8, the inductive limit of an inductive system of $C^*$-algebras can be computed in a Borel way. It is therefore enough to show that there is a Borel map that assigns to each Bratteli diagram, a code for the corresponding inductive system of $C^*$-algebras. We will work, for convenience, with the parametrization $\Gamma(H)$ of $C^*$-algebras.

Let $\left\{\xi_n^k\colon (k,n)\in\omega\times\omega\right\}$ be an orthonormal basis of $H$. For $n,m,k\in\omega$, denote by $e_{n,m}^{(k)}$ the rank 1 operator in $B(H)$ sending $\xi_n^k$ to $\xi_m^k$. For convenience, we will also identify $\Gamma(H)$ with the space of nonzero functions from $\omega\times\omega\times\omega$ to the unit ball of $B(H)$. For $n\in\omega$, define $\gamma^{(n)}\in\Gamma(H)$ by

$$\gamma_{i,j,k}^{(n)}=\begin{cases} e_{i,j}^{(k)}, & \text{if } k\in l(n) \text{ and } i,j\in w_n(k),\\ 0, & \text{otherwise.}\end{cases}$$

Denote by $A_{i,j,k}^n$ the set of triples in $\omega\times\omega\times\omega$ of the form

$$\left(\sum_{k'\in k} w_n(k')m_n(k',t,n)+dw_n(k)+i,\sum_{k'\in k} w_n(k')m_n(k',t)+dw_n(n),t\right)$$

such that $d\in m_n(k,t)$, $i,j\in w_n(k)$ and $t\in l(n+1)$. It is clear that $C^*\left(\gamma^{(n)}\right)$ is a finite dimensional $C^*$-algebra isomorphic to

$$\bigoplus_{i\in l(n)} M_{w(i,n)}.$$

For $n$ in $\omega$, let $\Phi^{(n)}\colon\mathcal{U}\to\mathcal{U}$ be the unique morphism of $\mathbb{Q}(i)$-$*$-algebras satisfying

$$\Phi^{(n)}(X_{ijk})=\sum_{(a,b,t)\in A_{ijk}^n} X_{a,b,t},$$

for $i,j,k$ in $\omega$. Then $\Phi^{(n)}$ is a code for the unital injective homomorphism from $C^*\left(\gamma^{(n)}\right)$ to $C^*\left(\gamma^{(n+1)}\right)$ given by the diagonal embedding of $m_n(k,t,n)$ copies of $M_{w_n(k)}$ in $M_{w_n(t)}$ for every $k\in l(n)$ and $t\in l(n+1)$. By construction, the map $\mathcal{BD}\to R_{dir}\left(\Gamma(H)\right)$ that assigns to every Bratteli diagram $(l,w,m)$, the code $\left(\gamma^{(n)},\Phi^{(n)}\right)_{n\in\omega}$, is Borel. This finishes the proof. $\square$

Proposition XII.5.5 together with Proposition XII.5.8 imply the following corollary.

**Corollary XII.5.9.** There is a Borel map that assigns to a dimension group $D$ a unital AF-algebra $A_D$ whose ordered $K_0$-group is isomorphic to $D$ as dimension group.

Since by [70, Proposition 3.4] the $K_0$-group of a $C^*$-algebra can be computed in a Borel way, one can conclude that if $\mathcal{A}$ is any Borel set of dimension groups, then the relation of isomorphisms restricted to $\mathcal{A}$ is Borel bireducible with the relation of isomorphism unital AF-algebras whose $K_0$-group is isomorphic to an element of $\mathcal{A}$.

Fix $n \in \mathbb{N}$. A dimension group has *rank* $n$ if $n$ is the largest size of a linearly independent subset. Let us denote by $\cong_n^+$ the relation of isomorphisms of dimension groups of rank $n$, and by $\cong_n^{AF}$ the relation of isomorphisms of AF-algebras whose dimension group has rank $n$. By the previous discussion and the fact that the computation of the $K_0$-group is given by a Borel function [70, Corollary 3.7], the relations $\cong_n^+$ and $\cong_n^{AF}$ are Borel bireducible. Moreover [63, Theorem 1.11] asserts that

$$\cong_n^+ <_B \cong_{n+1}^+$$

for every $n \in \mathbb{N}$. This means that $\cong_n^+$ is Borel reducible to $\cong_{n+1}^+$, but $\cong_{n+1}^+$ is *not* Borel reducible to $\cong_n^+$. It follows that the same conclusions hold for the relations $\cong_n^{AF}$: For every $n \in \mathbb{N}$

$$\cong_n^{AF} <_B \cong_{n+1}^{AF} .$$

This amounts to saying that it is strictly more difficult to classify AF-algebras with $K_0$-group of rank $n+1$ than classifying AF-algebras with $K_0$-group of rank $n$.

*Enomorphisms of Bratteli diagrams*

**Definition XII.5.10.** Let $T = (l, w, m)$ be a Bratteli diagram. We say that an element $q = (q_n)_{n\in\omega} \in \left(\omega^{\omega\times\omega}\right)^{\omega}$ is an *endomorphism* of $T$, if for every $n \in \omega$, $i \in l(n)$ and $t' \in l(n+1)$, the following identity holds

$$\sum_{t\in l(n+1)} m_n(i,t) q_{n+1}(t,t') = \sum_{t\in l(n+1)} q_n(i,t) m_{n+1}(t,t'). \tag{XII.1}$$

The set $End_{\mathcal{BD}}$ of pairs $(T, q) \in \mathcal{BD} \times \left(\omega^{\omega\times\omega}\right)^{\omega}$ such that $T$ is a Bratteli diagram and $q$ is an endomorphism of $T$, is Borel.

We proceed to describe how an endomorphism of a Bratteli diagram, in the sense of the definition above, gives rise to an endomorphism of the unital AF-algebra associated with it. Let $(F_n, \varphi_n)_{n\in\omega}$ be the inductive system of finite dimensional $C^*$-algebras associated with $T$, and denote by $A_T$ its inductive limit. By repeatedly applying [47, Lemma III.2.1], one can define unital homomorphisms $\psi_n\colon F_n \to F_{n+1}$ for $n$ in $\omega$, satisfying the following conditions:

1. $\psi_n$ is unitarily equivalent to the homomorphism from $F_n$ to $F_{n+1}$ such that for every $i \in l(n)$ and $j \in l\,(n+1)$ the restriction of $\psi_n$ to the $i$-th direct summand of $F_n$ and the $j$-th direct summand of $F_{n+1}$ is a diagonal embedding of $q_n(i,j)$ copies of $M_{w_n(i)}$ in $M_{w_{n+1}(j)}$;
2. $\psi_n \circ \varphi_{n-1} = \varphi_n \circ \psi_{n-1}$ whenever $n \geq 1$.

(Notice in particular that $\psi_0$ is determined solely by condition (1).) One thus obtains a one sided intertwining $(\psi_n)_{n\in\omega}$ from $(F_n, \varphi_n)_{n\in\omega}$ to itself. We denote by $\psi_{T,q}\colon A_T \to A_T$ the corresponding inductive limit endomorphism.

**Proposition XII.5.11.** Given a Bratteli diagram $T$ and an endomorphism $q$ of $T$, there is a Borel way to compute a code for the endomorphism $\psi_{T,q}$ of $A_T$ associated with $q$.

*Proof.* By Proposition XII.3.10, a code for the limit homomorphism between two inductive limits of $C^*$-algebras can be computed in a Borel way. It is therefore enough to show that there is a Borel function from $\mathrm{End}_{\mathcal{BD}}$ to $R_{int}\left(\Gamma(H)\right)$ assigning to an element $(T,q)$ of $\mathrm{End}_{\mathcal{BD}}$, a code for the corresponding one sided intertwining system.

Let $((l,w,m),q)$ be an element in $\mathrm{End}_{\mathcal{BD}}$, and let $\{\xi^n_m\colon\, (n,m) \in \omega\times\omega\}$ be an orthonormal basis of $H$. Denote by $\left(\gamma^{(n)}, \Phi^{(n)}\right)_{n\in\omega}$ the element of $R_{dir}\left(\Gamma(H)\right)$ associated with the Bratteli diagram $(l,w,m)$ as in the proof of Proposition XII.5.8. For $n$ in $\omega$, define

$$u^{(n)} = \sum_{k\in l(n)} \sum_{i\in w_n(k)} e^{(k)}_{ii},$$

which is an element of $B(H)$. Observe that $u^{(n)}$ is the unit of $C^*\left(\gamma^{(n)}\right)$. We define the sequence $\left(\Psi^{(n)}\right)_{n\in\omega}$ in $\left(\mathcal{U}^{\mathcal{U}}\right)^{\omega}$ as follows. Let $A^n_{i,j,k}$ denote the set of triples

$$\left(\sum_{k'\in k} w_n\left(k'\right) q_n\left(k',t,n\right) + dw_n(k) + i, \sum_{k'\in k} w_n\left(k'\right) q_n\left(k',t\right) + dw_n(k), t\right)$$

such that $d$ belongs to $q_n(k,t)$, $i$ and $j$ belong to $w_n(k)$, and $t$ belongs to $l(n+1)$. Let $\psi^{(n)}\colon \mathcal{U} \to \mathcal{U}$ be the unique homomorphism of $\mathbb{Q}(i)$-$*$-algebras satisfying

$$\psi^{(n)}\left(X_{ijk}\right) = \sum_{(a,b,t)\in A^n_{ijk}} X_{a,b,t}$$

for $i, j$ and $k$ in $\omega$. For $p \in \mathcal{U}$, set

$$\Psi^{(n)}_0(p) = \psi^{(n)}(p).$$

By construction, the elements

$$\left(\Psi^{(n)}_0 \circ \Phi^{(n-1)}_k\right)(p)\left(\gamma^{(n-1)}\right) \quad \text{and} \quad \left(\Phi_k \circ \Psi^{(n-1)}_0\right)(p)\left(\gamma^{(n-1)}\right)$$

are unitarily equivalent in $C^*\left(\gamma^{(n+1)}\right)$ for every $k \in \omega$. Using that $\gamma^{(n)}$ is a unitary for all $n$ in $\omega$, choose elements $p^{(n)}_k$ in $\mathcal{U}$, for $k$ in $\omega$, satisfying the following conditions:

$$\begin{aligned}
\left\|p^{(n)}_k\left(\gamma^{(n+1)}\right)p^{(n)}_k\left(\gamma^{(n+1)}\right)^* - 1\right\| &< \frac{1}{k+1}\\
\left\|p^{(n)}_k\left(\gamma^{(n+1)}\right)^* p^{(n)}_k\left(\gamma^{(n+1)}\right) - 1\right\| &< \frac{1}{k+1}\\
\left\|p^{(n)}_k\left(\gamma^{(n+1)}\right) - p^{(n)}_m\left(\gamma^{(n+1)}\right)\right\| &< \frac{1}{\min\{k,m\}+1}
\end{aligned}$$

and

$$\begin{aligned}
\Big\|p^{(n)}_k\left(\gamma^{(n+1)}\right)&\left(\left(\Psi^n_0 \circ \Phi^{n-1}_k\right)(p)\left(\gamma^{(n-1)}\right)\right)p^{(n)}_k\left(\gamma^{(n+1)}\right)^*\\
&- \left(\Phi_k \circ \Psi^{(n-1)}_0\right)(p)\left(\gamma^{(n-1)}\right)\Big\| < \frac{1}{k+1}.
\end{aligned}$$

Finally, define

$$\Psi^{(n)}_k(p) = p^{(n)}_k \Psi^{(n)}_0(p)\left(p^{(n)}_k\right)^*$$

for all $p$ in $\mathcal{U}$. It is clear that for fixed $n$ in $\omega$, the sequence $\Psi^{(n)} = (\Psi^{(n)}_k)_{k\in\omega}$ is a code for a homomorphism $\widehat{\psi^{(n)}}\colon \widehat{\gamma}^{(n)} \to \widehat{\gamma}^{(n+1)}$ that moreover satisfies

$$\widehat{\Psi}^{(n)} \circ \widehat{\Phi}^{(n-1)} = \widehat{\Phi}^{(n)} \circ \widehat{\Psi}^{(n-1)}.$$

Thus,

$$\left(\left(\gamma^{(n)}, \Phi^{(n)}\right)_{n\in\omega}, \left(\gamma^{(n)}, \Phi^{(n)}\right)_{n\in\omega}, \left(\Psi^{(n)}\right)_{n\in\omega}\right)$$

is an element in $R_{int}\left(\Gamma(H)\right)$. It is clear that this is a code for the one sided intertwining system associated with $((l,m,w),q)$, and that it can be computed in a Borel fashion. □

*Endomorphisms of dimension groups*

Let $(G, G^{+}, u)$ be a dimension group. Let us denote by $\mathrm{End}_{\mathcal{DG}}$ the set of pairs $(G,\phi) \in \mathcal{DG} \times \omega^{\omega}$ such that $G$ is a dimension group and $\phi$ is an endomorphism of $G$.

Let $(l,w,m)$ be a Bratteli diagram, and let

$$\left(\mathbb{Z}^{l(n)}, \left(w_n\left(0\right), \ldots, w_n\left(l(n)-1\right)\right), \varphi_n\right)_{n\in\omega}$$

be the inductive system of dimension groups whose inductive limit is the dimension group $G_{l,w,m}$ associated with $(l,w,m)$. Fix an endomorphism $q$ of $(l,w,m)$, and for $n \in \omega$, define a positive homomorphism $\psi_n \colon \mathbb{Z}^{l(n)} \to \mathbb{Z}^{l(n+1)}$ by

$$\psi_n\left(e_i^{(l(n))}\right) = \sum_{j\in l(n+1)} q_n(i,j) e_j^{(l(n+1))}.$$

Observe that the sequence $\left(\psi_n\right)_{n\in\omega}$ induces an inductive limit endomorphism

$$\phi_{((l,w,m),q)} \colon G_{(l,w,m)} \to G_{(l,w,m)}.$$

**Proposition XII.5.12.** There is a Borel map

$$\begin{aligned} \mathrm{End}_{\mathcal{DG}} &\to \mathrm{End}_{\mathcal{BD}} \\ (G,\phi) &\mapsto \left(T^{G}, q^{G,\phi}\right), \end{aligned}$$

such that the dimension group associated with $T^{G}$ is isomorphic to $G$, and the endomorphism of the dimension group associated with $T^{G}$ corresponding to $q^{G,\phi}$ is conjugate to $\phi$.

*Proof.* It is enough to construct, in a Borel way,

– a Bratteli diagram $\left(l^{G,\phi}, w^{G,\phi}, m^{G,\phi}\right)$,

– an endomorphism $q^{G,\phi}$ of $\left(l^{G,\phi}, w^{G,\phi}, m^{G,\phi}\right)$, and

– functions $\theta_n^{G,\phi}\colon l^{G,\phi}(n) \to G^+$ for $n$ in $\omega$,

such that the following conditions hold:

1. For every $i \in l^{G,\phi}(n)$,
$$\theta_n^{G,\phi}(i) = \sum_{j \in l^G(n+1)} m_n^{G,\phi}(i,j)\theta_n^{G,\phi}(j);$$

2. For any $k_0, \ldots, k_{l(n)-1} \in \mathbb{Z}$ such that
$$\sum_{i \in l^G(n)} k_i \theta_n^{G,\phi}(i) = 0,$$
we have that
$$\sum_{i \in l^G(n)} k_i m_n^{G,\phi}(i,j) = 0,$$
for every $j \in l^{G,\phi}(n+1)$;

3. For every $x \in G^+$, there are $n \in \omega$ and $i \in l^{G,\phi}(n)$ such that $\theta_n^{G,\phi}(i) = x$;

4. $\phi\left(\theta_n^{G,\phi}(i)\right) = q^G(i)$ for $n \in \omega$ and $i \in l^{G,\phi}(n)$.

In fact, it is not difficult to see that Conditions (1), (2), (3), and (4) ensure that $\left(\theta_n^{G,\phi}\right)_{n\in\omega}$ defines an isomorphism from the dimension group associated with $\left(l^{G,\phi}, w^{G,\phi}, m^{G,\phi}\right)$ to $G$ that conjugates the endomorphism associated with $q^{G,\phi}$ and $\phi$.

We define $l_n^{G,\phi}, w_n^{G,\phi}, m_n^{G,\phi}, q_n^{G,\phi}$ and $\theta_n^{G,\phi}$ satisfying conditions 1–4 by recursion on $n$. Define $l^{G,\phi}(0) = 1$ and $\theta_0^{G,\phi}(0) = u$. Suppose that $l^{G,\phi}(k)$, $w_k^{G,\phi}$, $m_{k-1}^{G,\phi}$, and $\theta_k^{G,\phi}$ have been defined for $k \le n$. Define
$$\theta'\colon 2l^{G,\phi}(n) + 1 = \{0, \ldots, 2l^{G,\phi}(n)\} \to \omega$$

by

$$\theta'(i) = \begin{cases} \theta^{G,\phi}(i) & \text{if } 0 \leq i < l^{G,\phi}(n), \\ \phi\left(\theta^{G,\phi}(i - l^{G,\phi}(n))\right) & \text{if } l^{G,\phi}(n) \leq i < 2l^{G,\phi}(n), \\ n & \text{if } i = 2l^{G,\phi}(n) \text{ and } n \in G^+, \\ u & \text{otherwise.} \end{cases}$$

Suppose the integer $N$, and the functions $\Phi\colon N \to G^+$ and $g\colon (2l^{G,\phi}(n)+1) \times N \to \omega$ are obtained via Lemma XII.5.4 from $2l^{G,\phi}(n)+1$ and $\theta'$, and let $N' \in \omega$, $\Phi'\colon N' \to \omega$, and $g'\colon N \times N' \to \omega$ satisfy the conclusion of Lemma XII.5.4 for the choices $N$ and $\Phi$. Define now:

$$\begin{aligned} l^{G,\phi}(n+1) &= N'; \\ w_n^{G,\phi}(j) &= \sum_{i \in l^G(n)} w_n^{G,\phi}(i) m_n^{G,\phi}(i,j) \\ m_n^{G,\phi}(i,j) &= \begin{cases} \sum_{t \in N} g(i,t)g'(t,j) & \text{if } i \in l^{G,\phi}(n) \text{ and } j \in l^{G,\phi}(n+1), \\ 0 & \text{otherwise.} \end{cases} \\ q_n^{G,\phi}(i,j) &= \sum_{t \in N} g(2i,t)g'(t,j) \\ \theta_{n+1}^{G,\phi} &= \Phi. \end{aligned}$$

It is not difficult to check that this recursive construction gives maps satisfying conditions 1–4. $\square$

**Corollary XII.5.13.** There is a Borel map that assigns to a dimension group $G$ with a distinguished endomorphism $\phi$, a code for a unital AF-algebra $A$ and a code for an endomorphism $\rho$ of $A$, such that the $K_0$-group of $A$ is isomorphic to $G$ as dimension groups with order units, and the endomorphism of the $K_0$-group of $A$ corresponding to $\rho$ is conjugate to $\phi$.

*Proof.* Let $G$ be a dimension group, and let $\phi$ be an endomorphism of $G$. Using Proposition XII.5.12, choose in a Borel way a Bratteli diagram $(l, m, w)$ and an endomorphism $q$ of $(l, m, w)$ such that $G$ is isomorphic to the dimension group associated with $(l, m, w)$, and $\rho$ is conjugate to the endomorphism associated with $q$. Use Proposition XII.5.8 to choose in a Borel way, a unital AF-algebra $A$ whose Bratteli diagram is $(l, m, w)$. Apply Proposition XII.5.11 to choose in a Borel way an endomorphism $\rho$ of $A$ whose induced endomorphism of the Bratteli diagram is $q$. It is clear from the construction that the $K_0$-group of $A$ is isomorphic to $G$. Moreover, $\phi$ is conjugate to the endomorphism of the $K_0$-group of $A$ corresponding to $\rho$. Therefore, the result follows from Proposition XII.5.12 and Proposition XII.5.11. ☐

### Conjugacy and Cocycle Conjugacy of Automorphisms of $\mathcal{O}_2$

*Strongly self-absorbing $C^*$-algebras*

Upon studying the literature around Elliott's classification program, it is clear that certain $C^*$-algebras play a central role in major stages of the program: UHF-algebras (particularly those of infinite type), the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ [39], and, more recently, the Jiang-Su algebra $\mathcal{Z}$ [136]. In [265], Toms and Winter were able to pin down the abstract property that singles out these algebras. The relevant notion is that of strongly self-absorbing $C^*$-algebra, which we define below; see also [265, Definition 1.3].

**Definition XII.6.1.** Let $\mathcal{D}$ be a separable, unital, infinite dimensional $C^*$-algebra. Denote by $\mathcal{D} \otimes \mathcal{D}$ the completion of the algebraic tensor product $\mathcal{D} \odot \mathcal{D}$ with respect to any compatible C*-norm on $\mathcal{D} \odot \mathcal{D}$. We say that $\mathcal{D}$ is *strongly self-absorbing* if there exists an isomorphism $\varphi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the map $a \mapsto a \otimes 1_{\mathcal{D}}$.

It is shown in [265] that a $C^*$-algebra $\mathcal{D}$ satisfying Definition XII.6.1 is automatically nuclear. In particular the choice of the tensor product norm on $\mathcal{D} \odot \mathcal{D}$ is irrelevant. By [265, Examples 1.14] the following $C^*$-algebras are strongly-self-absorbing: UHF-algebras of infinite type, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the tensor product of a UHF-algebra of infinite type and $\mathcal{O}_\infty$, and the Jiang-Su algebra. No other strongly self-absorbing $C^*$-algebra is currently known.

**Definition XII.6.2.** Suppose that $\mathcal{D}$ is a nuclear $C^*$-algebra. A $C^*$-algebra $A$ absorbs $\mathcal{D}$ tensorially –or is $\mathcal{D}$-*absorbing*– if the tensor product $A \otimes \mathcal{D}$ is isomorphic to $A$.

The particular case of Theorem XII.6.3 when $\mathcal{D}$ is the Jiang-Su algebra $\mathcal{Z}$ has been proved in [70, Theorem A.1].

**Theorem XII.6.3.** Suppose that $\mathcal{D}$ is a strongly self-absorbing $C^*$-algebra. The set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a $\mathcal{D}$-absorbing unital $C^*$-algebra is Borel.

*Proof.* By [71, Lemma 3.14], the set $\Gamma_u(H)$ of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is unital, is Borel. Moreover, there is a Borel function $Un\colon \Gamma_u(H) \to B(H)$ such that $Un(\gamma)$ is the unit of $C^*(\gamma)$

for every $\gamma \in \Gamma_u(H)$. Denote as in Subsection XII.3 by $\mathcal{U}$ the $\mathbb{Q}(i)$-$*$-algebra of polynomials with coefficients in $\mathbb{Q}(i)$ and without constant term in the formal variables $X_k$ for $k \in \omega$ Let $\{d_n : n \in \omega\}$ be an enumeration of a dense subset of $\mathcal{D}$ such that $d_0 = 1$, and let $\{p_n : n \in \omega\}$ be an enumeration of $\mathcal{U}$. By [265, Theorem 2.2], or [235, Theorem 7.2.2], a unital $C^*$-algebra $A$ is $\mathcal{D}$-absorbing if and only if for every $n, m \in \mathbb{N}$ and every finite subset $F$ of $A$, there are $a_0, a_1, \ldots, a_n \in A$ such that$\mathbb{Q}(i)$-$*$-algebra of noncommutative $*$-polynomials with coefficients

- $a_0$ is the unit of $A$,
- $\|xa_i - a_i x\| < \frac{1}{m}$ for every $i \in n$ and $x \in F$, and
- $\|p_i(a_0, \ldots, a_n) - p_i(d_0, \ldots, d_n)\| < \frac{1}{m}$ for every $i \in m$.

Let $\gamma \in \Gamma(H)$ be such that $C^*(\gamma)$ is unital. Then $C^*(\gamma)$ is $\mathcal{D}$-absorbing if and only if for every $n, m \in \mathbb{N}$ there are $k_1, \ldots, k_n \in \omega$ such that

- $\left\|\gamma_i \gamma_{k_j} - \gamma_{k_j} \gamma_i\right\| < \frac{1}{m}$ for $i \in m$ and $1 \leq j \leq n$,
- $\|p_i(Un(\gamma), \gamma_{k_1}, \ldots, \gamma_{k_n}) - p_i(d_0, \ldots, d_n)\| < \frac{1}{m}$ for every $i \in m$.

This shows that the set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is unital and $\mathcal{D}$-absorbing, is Borel.

□

*Borel spaces of Kirchberg algebras*

We will denote by $\Gamma_{uKir}(H)$ the set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a *unital* Kirchberg algebra.

**Proposition XII.6.4.** The set $\Gamma_{uKir}(H)$ is Borel.

*Proof.* Corollary 7.5 of [71] asserts that the set $\Gamma_{uns}(H)$ of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is unital, nuclear, and simple is Borel. The result then follows from this fact together with Theorem XII.6.3.

□

**Definition XII.6.5.** Fix a projection $p$ in $\mathcal{O}_\infty$ such that $[p] = 0$ in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. Define the *standard Cuntz algebra* $\mathcal{O}_\infty^{st}$ to be the corner $p\mathcal{O}_\infty p$.

The $C^*$-algebra $\mathcal{O}_\infty^{st}$ is a unital Kirchberg algebra that satisfies the UCT, with $K$-theory given by

$$(K_0(\mathcal{O}_\infty^{st}), \left[1_{\mathcal{O}_\infty^{st}}\right], K_1(\mathcal{O}_\infty^{st})) \cong (\mathbb{Z}, 0, 0).$$

Moreover, it is the unique, up to isomorphism, unital Kirchberg algebra satisfying the UCT with said $K$-theory. In particular, a different choice of the projection $p$ in Definition XII.6.5 (as long as its class on $K$-theory is 0), would yield an isomorphic $C^*$-algebra.

We point out that, even though there is an isomorphism $\mathcal{O}_\infty^{st} \otimes \mathcal{O}_\infty^{st} \cong \mathcal{O}_\infty^{st}$ (see comments on page 262 of [132]), the $C^*$-algebra $\mathcal{O}_\infty^{st}$ is not strongly self-absorbing. Indeed, if $\mathcal{D}$ is a strongly self-absorbing $C^*$-algebra, then the infinite tensor product $\bigotimes_{n=1}^{\infty} \mathcal{D}$ of $\mathcal{D}$ with itself, is isomorphic to $\mathcal{D}$. However, $\bigotimes_{n=1}^{\infty} \mathcal{O}_\infty^{st}$ is isomorphic to $\mathcal{O}_2$, and thus $\mathcal{O}_\infty^{st}$ is not strongly self-absorbing.

We proceed to give a $K$-theoretic characterization of those unital Kirchberg algebras that absorb $\mathcal{O}_\infty^{st}$. Our characterization will be used to show that the set of all $\mathcal{O}_\infty^{st}$-absorbing unital Kirchberg algebras is Borel.

For use in the proof of the following lemma, we recall here that if $A$ and $B$ are nuclear separable $C^*$-algebras, and at least one of them satisfies the UCT, then the $K$-groups of their tensor product $A \otimes B$ are "essentially" determined by the $K$-groups of $A$ and $B$, up to an extension problem. This is the content of the Künneth formula, which will be needed in the next proof.

**Lemma XII.6.6.** Let $A$ be a unital Kirchberg algebra. Then the following are equivalent:

1. $A$ is $\mathcal{O}_\infty^{st}$-absorbing.

2. The class $[1_A]$ of the unit of $A$ in $K_0(A)$ is zero.

*Proof.* We first show that (1) implies (2). Since $\mathcal{O}_\infty^{st}$ satisfies the UCT, the Knneth formula applied to $A \otimes \mathcal{O}_\infty^{st}$ gives

$$K_0(A \otimes \mathcal{O}_\infty^{st}) \cong K_0(A) \quad \text{and} \quad K_1(A \otimes \mathcal{O}_\infty^{st}) \cong K_1(A),$$

with $\left[1_{A \otimes \mathcal{O}_\infty^{st}}\right] = 0$ as an element in $K_0(A)$. The claim follows since any isomorphism $A \otimes \mathcal{O}_\infty^{st} \cong A$ must map the unit of $A \otimes \mathcal{O}_\infty^{st}$ to the unit of $A$.

Let us now show that (2) implies (1). Fix a non-zero projection $p$ in $\mathcal{O}_\infty$ such that $[p] = 0$ as an element of $K_0(\mathcal{O}_\infty)$. Then $1_A \otimes 1_{\mathcal{O}_\infty}$, which is an element of $A \otimes \mathcal{O}_\infty$, represents the zero element in $K_0(A \otimes \mathcal{O}_\infty)$. Likewise, $1_A \otimes p$ also represents the zero element in $K_0(A \otimes \mathcal{O}_\infty)$. Since any two non-zero projections in a Kirchberg algebra are Murray-von Neumann equivalent if and only if they determine the same class in $K$-theory (see [41]), it follows that there is an isometry $v$ in $A \otimes \mathcal{O}_\infty$ such that

$$vv^* = 1_A \otimes p.$$

The universal property of the algebraic tensor product yields a linear map

$$\varphi_0 \colon A \odot \mathcal{O}_\infty \to (1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p) \cong A \otimes \mathcal{O}_\infty^{st}$$

such that $\varphi_0(a \otimes b) = v(a \otimes b)v^*$ for $a$ in $A$ and $b$ in $\mathcal{O}_\infty$. It is straightforward to check that $\varphi_0$ extends to a homomorphism $\varphi \colon A \otimes \mathcal{O}_\infty \to A \otimes \mathcal{O}_\infty^{st}$. We claim that $\varphi$ is an isomorphism. For this, it is enough to check that the homomorphism

$$\psi \colon (1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p) \to A \otimes \mathcal{O}_\infty$$

given by $\psi(x) = v^*xv$ for all $x$ in $(1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p)$, is an inverse for $\varphi$. This is immediate since $(1_A \otimes p)x(1_A \otimes p) = x$ for all $x$ in $(1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p)$.

Once we have $A \otimes \mathcal{O}_\infty^{st} \cong A \otimes \mathcal{O}_\infty$, the result follows from the fact that there is an isomorphism $A \cong A \otimes \mathcal{O}_\infty$ by Kirchberg's $\mathcal{O}_\infty$-isomorphism Theorem (Theorem 3.15 in [151]). This finishes the proof of the lemma. □

**Corollary XII.6.7.** The set of all $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a $\mathcal{O}_\infty^{st}$-absorbing unital Kirchberg algebra, is Borel.

*Proof.* This follows from Lemma XII.6.6, together with the fact that the $K$-theory of a $C^*$-algebra and the class of its unit in $K_0$ can be computed in a Borel fashion; see [70, Section 3.3]. □

We will denote by $D_2$ the tensor product of $\mathcal{O}_\infty^{st}$ with the UHF algebra $M_{2^\infty}$ of type $2^\infty$. Since $M_{2^\infty}$ is strongly self-absorbing, it follows from Theorem XII.6.3 and Corollary XII.6.7, that the set $\mathcal{D}$ of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a unital $D_2$-absorbing Kirchberg $C^*$-algebra not isomorphic to $\mathcal{O}_2$ is a Borel subset of $\Gamma(H)$.

*Automorphisms of $\mathcal{O}_2$*

Denote by $\mathrm{Aut}(\mathcal{O}_2)$ the Polish group of automorphisms of $\mathcal{O}_2$ with respect to the topology of pointwise convergence. Given a positive integer $n$, the closed subspace $\mathrm{Aut}_2(\mathcal{O}_2)$, of automorphisms of $\mathcal{O}_2$ of order 2 can be identified with the space of actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$.

**Definition XII.6.8.** An action $\alpha$ of $\mathbb{Z}_2$ on $\mathcal{O}_2$ is said to be *approximately representable* if for every $\varepsilon > 0$ and for every finite subset $F$ of $\mathcal{O}_2$, there exists a unitary $u$ of $\mathcal{O}_2$ such that

1. $\left\|u^2 - 1\right\| < \varepsilon$,
2. $\|\alpha(u) - u\| < \varepsilon$, and
3. $\|\alpha(a) - uau^*\| < \varepsilon$ for every $a \in F$.

It is clear that the set of approximately representable automorphisms of order 2 of $\mathcal{O}_2$ is a $G_\delta$ subset of $\mathrm{Aut}_2(\mathcal{O}_2)$.

We now recall a construction of a *model action* of $\mathbb{Z}_2$ on $\mathcal{O}_2$ from [132, page 262]. Fix a projection $e$ of $\mathcal{O}_\infty^{st}$ such that $[e]$ is a generator of $K_0\left(\mathcal{O}_\infty^{st}\right)$ and set $u$ to be the order 2 unitary $u = 2e - 1$ of $\mathcal{O}_\infty^{st}$. Identifying $\mathcal{O}_2$ with the infinite tensor product $\bigotimes_{n\in\omega} \mathcal{O}_\infty^{st}$ one can define the approximately representable action

$$\nu = \bigotimes_{n\in\omega} \mathrm{Ad}\,(u)$$

of $\mathbb{Z}_2$ on $\mathcal{O}_2$. In view of the classification results in [150] and [200], Lemma 4.7 of [132] asserts that the crossed product

$$\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2$$

is isomorphic to the algebra $D_2 = \mathcal{O}_\infty^{st} \otimes M_{2^\infty}$.

For a simple nuclear unital $C^*$-algebra $A$, denote by $\widetilde{\alpha}_A$ the automorphism of $A \otimes \mathcal{O}_2$ defined by $\mathrm{id}_A \otimes \nu$. By Kirchberg's $\mathcal{O}_2$-isomorphism theorem [151, Theorem 3.8], there is an isomorphism $\varphi\colon A \otimes \mathcal{O}_2 \to \mathcal{O}_2$. Denote by $\alpha_A$ the automorphism of $\mathcal{O}_2$ given by

$$\alpha_A = \varphi \circ \widetilde{\alpha}_A \circ \varphi^{-1}.$$

It is immediate to check that $\alpha_A$ is approximately representable, using that $\nu$ is approximately representable.

**Remark XII.6.9.** If $A$ is a simple nuclear unital $C^*$-algebra, then $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2 \cong A \otimes D$.

*Proof.* By Kirchberg's $\mathcal{O}_2$-isomorphism [151, Theorem 3.8], there are isomorphisms

$$\begin{aligned}\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2 &\cong (A \otimes \mathcal{O}_2) \rtimes_{\mathrm{id}_A \otimes \nu} \mathbb{Z}_2 \\ &\cong A \otimes (\mathcal{O}_2 \rtimes_\nu \mathbb{Z}_2) \\ &\cong A \otimes D_2.\end{aligned}$$

□

**Proposition XII.6.10.** Let $A$ and $B$ be simple nuclear unital $C^*$-algebras. The following statements are in decreasing order of strength:

1. $A$ and $B$ are isomorphic;
2. The actions $\alpha_A$ and $\alpha_B$ are conjugate;
3. The actions $\alpha_A$ and $\alpha_B$ are cocycle conjugate;
4. The crossed products $\mathcal{O}_2 \rtimes_{\alpha_A} \mathbb{Z}_2$ and $\mathcal{O}_2 \rtimes_{\alpha_B} \mathbb{Z}_2$ are isomorphic;
5. $A \otimes D_2$ and $B \otimes D_2$ are isomorphic.

In particular if $A$ and $B$ are $D_2$-absorbing unital Kirchberg algebras, then all the statements above are equivalent.

*Proof.* If $\psi\colon A \to B$ is an isomorphism, then $\psi \otimes \mathrm{id}_{\mathcal{O}_2}\colon A \otimes \mathcal{O}_2 \to B \otimes \mathcal{O}_2$ conjugates $\mathrm{id}_A \otimes \nu_p$ and $\mathrm{id}_B \otimes \nu_p$, and hence $\alpha_{A,p}$ and $\alpha_{B,p}$ are conjugate. This shows that (1) implies (2). It is well known that (2) implies (3) and (3) implies (4). Remark XII.6.9 shows immediately that (4) implies (5). □

We moreover have the following.

**Proposition XII.6.11.** Let $A$ be a $D_2$-absorbing unital Kirchberg algebra not isomorphic to $\mathcal{O}_2$. The action $\alpha_A$ of $\mathbb{Z}_2$ on $\mathcal{O}_2$ defined above has Rokhlin dimension 1 (see Definition IV.2.2).

*Proof.* Lemma 2.1 of [7] shows that the action $\nu$ of $\mathbb{Z}_2$ on $\mathcal{O}_2$ has Rokhlin dimension at most 1. Since $\alpha_A$ is conjugate to $id_A \otimes \nu$, it follows from part (1) in Theorem IV.2.8 that $\alpha_A$ has Rokhlin dimension at most 1. Assume now by contradiction that $\alpha_A$ has Rokhlin dimension 0, that is, it has the Rokhlin property. Corollary 3.4 in [122] asserts that the crossed product $\mathcal{O}_2 \rtimes_{\alpha_A} \mathbb{Z}_2$ is isomorphic to $\mathcal{O}_2$. At the same time $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to $A$ by Remark XII.6.9. Therefore $A \cong \mathcal{O}_2$, contradicting our assumption. $\square$

*Isomorphism of p-divisible torsion free abelian groups*

**Definition XII.6.12.** Let $G$ be an abelian group and let $n$ be a positive integer.

1. We say that $G$ is *$n$-divisible*, if for every $x$ in $G$ there exists $y$ in $G$ such that $x = ny$.
2. We say that $G$ is *uniquely $n$-divisible*, if for every $x$ in $G$ there exists a unique $y$ in $G$ such that $x = ny$.

Given a set $S$ of positive integers, we say that $G$ is (uniquely) $S$-divisible, if $G$ is (uniquely) $n$-divisible for every $n$ in $S$.

It is clear that if $n$ is a positive integer, then any $n$-divisible torsion free abelian group is uniquely $n$-divisible.

It is easily checked that the following classes of abelian groups are Borel subsets of the standard Borel space of countable infinite groups $\mathcal{G}$:

- Torsion free groups;
- $n$-divisible groups, for any positive integer $n$;
- Uniquely $n$-divisible groups.

The main result of [126] asserts that if $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphisms of elements of $\mathcal{C}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation $\cong_{TFA}$ of isomorphism of torsion free abelian groups. Moreover, [52, Theorem 1.1] asserts that $\cong_{TFA}$ is a complete analytic set and, in particular, not Borel.

**Proposition XII.6.13.** Suppose that $\mathcal{P}$ is a set of prime numbers which is coinfinite in the set of all primes. If $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel, then $\mathcal{C}$ is Borel reducible to the relation of isomorphism of torsion free $\mathcal{P}$-divisible countable infinite groups. Moreover, the latter equivalence relation is a complete analytic set and, in particular, not Borel.

*Proof.* A variant of the argument used in the proof of the main result of [126] can be used to prove the first assertion. Indeed, the only modification needed is in the definition of the *group eplag* associated with an *excellent prime labeled graph* as in [126, Section 2] (we refer to [126] for the definitions of these notions). Suppose that $(V, E, f)$ is an excellent prime labeled graph such that the range of $f$ is disjoint from $\mathcal{P}$. Denote by $\mathbb{Q}^{(V)}$ the direct sum

$$\mathbb{Q}^{(V)} = \bigoplus_{v \in V} \mathbb{Q}$$

of copies of $\mathbb{Q}$ indexed by $V$, and identify an element $v$ of $V$ with the corresponding copy of $\mathbb{Q}$ in $\mathbb{Q}^{(V)}$. We define the $\mathcal{P}$-divisible group eplag $\mathcal{G}_{\mathcal{P}}(V, E, f)$ associated with $(V, E, f)$, to be the subgroup of $\mathbb{Q}^{(V)}$ generated by

$$\left\{ \frac{v}{p^n f(v)^m}, \frac{v+w}{p^n f(\{v, w\})} : v \in V, \{v, w\} \in E, n, m \in \omega, p \in \mathcal{P} \right\}.$$

It is easy to check that $\mathcal{G}_{\mathcal{P}}(V, E, f)$ is indeed a torsion-free $\mathcal{P}$-divisible abelian group. The group eplag $\mathcal{G}(V, E, f)$ as defined in [126, Section 2], is the particular case of this definition with $\mathcal{P} = \varnothing$. The same argument as in [126], where

1. the group eplag $\mathcal{G}(V, E, f)$ is replaced everywhere by $\mathcal{G}_{\mathcal{P}}(V, E, f)$, and
2. all the primes are chosen from the *complement* of $\mathcal{P}$,

gives a proof of the first claim of this proposition.

The second claim follows by modifying the argument in [52] and, in particular, the construction of the torsion-free abelian group associated with a tree on $\omega$ as in [52, Theorem 2.1]. Choose injective enumerations $(p_n)_{n \in \omega}$ and $(q_n)_{n \in \omega}$ of disjoint subsets of the complement of $\mathcal{P}$ in the set of all primes, and let $T$ be a tree on $\omega$. Define the excellent prime labeled graph $(V_T, E_T, f_T)$ as follows. The graph $(V_T, E_T)$ is just the tree $T$, and

$$f \colon V_T \cup E_T \to \{p_n, q_n : n \in \omega\}$$

is defined by

$$f(x)=\begin{cases} q_n & \text{if } x \text{ is a vertex in the } n\text{-th level of } T; \\ p_n & \text{if } x \text{ is an edge between the } n\text{-th and } n+1\text{-th levels of } T. \end{cases}$$

Define the $\mathcal{P}$-divisible torsion free abelian group $G_{\mathcal{P}}(T)$ to be the group eplag $\mathcal{G}_{\mathcal{P}}(V_T, E_T, f_T)$. The same proof as that of [52, Theorem 2.1] shows the following facts: If $T$ and $T'$ are isomorphic trees, then the groups $G_{\mathcal{P}}(T)$ and $G_{\mathcal{P}}(T')$ are isomorphic. On the other hand, if $T$ is well-founded and $T'$ is ill-founded, then $G_{\mathcal{P}}(T)$ and $G_{\mathcal{P}}(T')$ are not isomorphic. The second claim of this proposition can now be proved as [52, Theorem 1.1]. ☐

*Constructing Kirchberg algebras with a given $K_0$-group*

The following is the main result of this section.

**Theorem XII.6.14.** There is a Borel map from the Borel space $\mathcal{G}$ of countable infinite groups to the Borel space $\Gamma_{uKir}(H)$ parametrizing unital Kirchberg algebras, which assigns to every infinite countable abelian group $G$, a code $\gamma$ for a unital Kirchberg algebra $C^*(\gamma)$ that satisfies the UCT, and with $K$-theory given by

$$(K_0(C^*(\gamma)), \left[1_{C^*(\gamma)}\right], K_1(C^*(\gamma))) \cong (G, 0, \{0\}).$$

Moreover $C^*(\gamma)$ is $D_2$-absorbing if and only if $G$ is uniquely 2-divisible.

*Proof.* Use Lemma XII.3.14 to choose, in a Borel way from $G$, a torsion free abelian group $H$ and an automorphism $\alpha$ of $H$ such that

$$H/\mathrm{Im}(\mathrm{id}_H - \alpha) \cong G.$$

Denote by $L$ the dimension group given by

$$L = \mathbb{Z}\left[\frac{1}{2}\right] \oplus H$$

with positive cone

$$L^+ = \{(t,h) \in D \colon t > 0\} \cup \{(0,0)\},$$

and order unit $(1,0)$. Consider the endomorphism $\rho$ of $L$ defined by

$$\beta(t,h)=\left(\frac{t}{2},\alpha(h)\right)$$

for $(t,h)$ in $L$. It is clear that $L$ and $\beta$ can be computed in a Borel way from $H$ and $\alpha$. By Corollary XII.5.13, one can obtain in a Borel way from $H$ and $\beta$, a code for a unital AF-algebra $B$ and a code for an injective corner endomorphism $\rho$ of $B$ such that the $K_0$-group of $B$ is isomorphic to $L$, and the endomorphism of the $K_0$-group of $B$ induced by $\rho$ is conjugate to $\beta$. By Corollary XII.4.8, one can obtain in a Borel way a code $\gamma_G\in\Gamma(H)$ for the crossed product $B\rtimes_\rho\mathbb{N}$ of $B$ by the endomorphism $\rho$. It can be shown, as in the proof of [233, Theorem 3.6], that $C^*(\gamma_G)$ is a unital Kirchberg algebra satisfying the UCT, with trivial $K_1$-group, $K_0$-group isomorphic to $G$, and $\left[1_{C^*(\gamma_G)}\right]=0$ in $K_0\left(C^*\left(\gamma\right)\right)$. An easy application of the Pimsner-Voiculescu exact sequence (see Theorem 10.2.1 in [13]) gives the computation of the $K$-theory; see [233, Corollary 2.2]. Pure infiniteness of $C^*(\gamma_G)$ is proved in [233, Theorem 3.1]. The map

$$\begin{aligned}\mathcal{G}&\to\Gamma_{uKir}(H)\\ G&\mapsto\gamma_G\end{aligned}$$

is Borel by construction. □

*Nonclassification of automorphisms of $\mathcal{O}_2$ of order 2*

Denote as before by $\mathcal{D}$ the Borel set of all elements $\gamma$ in $\Gamma(H)$ such that $C^*(\gamma)$ is a unital $D_2$-absorbing Kirchberg algebra not isomorphic to $\mathcal{O}_2$. (Recall that $D_2$ is the $C^*$-algebra $\mathcal{O}_\infty^{st}\otimes M_{2^\infty}$ where $M_{2^\infty}$ denote the UHF algebra of type $2^\infty$.) One can regard $\mathcal{D}$ as the standard Borel space parametrizing $D_2$-absorbing unital Kirchberg algebras not isomorphic to $\mathcal{O}_2$. Thus, the equivalence relation $E$ on $\mathcal{D}$ defined by

$$\gamma E\gamma'\quad\text{if and only if}\quad C^*(\gamma)\cong C^*\left(\gamma'\right),$$

can be identified with the relation of isomorphism of unital $D_2$-absorbing Kirchberg algebras not isomorphic to $\mathcal{O}_2$.

**Theorem XII.6.15.** There are Borel reductions:

1. From the relation of isomorphism of $D_2$-absorbing unital Kirchberg algebras not isomorphic to $\mathcal{O}_2$, to the relation of cocycle conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension 1.

2. From the relation of isomorphism of $D_2$-absorbing unital Kirchberg algebras, to the relation of conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension 1.

*Proof.* In view of Proposition XII.6.10, Proposition XII.6.11 and Elliott's theorem $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, it is enough to show that there is a Borel function from $\Gamma_{uKir}(H)$ to $\mathrm{Aut}_2\left(\mathcal{O}_2 \otimes \mathcal{O}_2\right)$ that assigns to every $\gamma \in \Gamma_{uKir}(H)$, an automorphism $\alpha_\gamma$ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ which is conjugate to $\mathrm{id}_{C^*(\gamma)} \otimes \nu$.

We follow the notation of [71, Section 6.1], and denote by $SA(\mathcal{O}_2)$ the space of C*-subalgebras of $\mathcal{O}_2$. Then $SA(\mathcal{O}_2)$ is a Borel subset of the Effros Borel space of closed subsets of $\mathcal{O}_2$, as defined in [147, Section 12.C]. It follows from [71, Theorem 6.5] that the set $SA_{uKir}\left(\mathcal{O}_2\right)$ of C*-subalgebras of $\mathcal{O}_2$ isomorphic to a unital Kirchberg algebra is Borel. Moreover, again by [71, Theorem 6.5], there is a Borel function from $\Gamma_{uKir}(\mathcal{O}_2)$ to $SA_{uKir}\left(\mathcal{O}_2\right)$ that assigns to an element $\gamma$ of $\Gamma_{uKir}\left(\mathcal{O}_2\right)$ a subalgebra of $\mathcal{O}_2$ isomorphic to $C^*(\gamma)$. It is therefore enough to show that there is a Borel function from $SA_{uKir}(\mathcal{O}_2)$ to $\mathrm{Aut}_2\left(\mathcal{O}_2 \otimes \mathcal{O}_2\right)$ that assigns to $A \in SA_{uKir}(\mathcal{O}_2)$ an automorphism $\alpha_A$ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ conjugate to $\mathrm{id}_A \otimes \nu$.

Denote by $\mathrm{End}\left(\mathcal{O}_2 \otimes \mathcal{O}_2\right)$ the space of endomorphism of $\mathcal{O}_2 \otimes \mathcal{O}_2$. By [71, Theorem 7.6], there is a Borel map from $SA_{uKir}(\mathcal{O}_2)$ to $\mathrm{End}\left(\mathcal{O}_2 \otimes \mathcal{O}_2\right)$ that assigns to an element $A$ in $SA_{uKir}(\mathcal{O}_2)$ a unital injective endomorphism $\eta_A$ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with range $A \otimes \mathcal{O}_2$. In particular, $\eta_A$ is an isomorphism between $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $A \otimes \mathcal{O}_2$. For $A$ in $SA_{uKir}(\mathcal{O}_2)$, define

$$\alpha_A = \eta_A^{-1} \circ \left(\mathrm{id}_A \otimes \nu\right) \circ \eta_A,$$

and note that the map $A \mapsto \alpha_A$ is Borel.

It is enough to show that for every $x, y \in \mathcal{O}_2$ and every $\varepsilon > 0$, the set of $C^*$-algebras $A$ in $SA_{uKir}\left(\mathcal{O}_2\right)$ such that

$$\|\alpha_A(x) - y\| < \varepsilon,$$

is Borel. Fix $x$ and $y$ in $\mathcal{O}_2$.

By [147, Theorem 12.13], there is a sequence

$$a_n \colon SA_{uKir}(\mathcal{O}_2) \to \mathcal{O}_2,$$

with $n \in \omega$, of Borel functions, such that for $A$ in $SA_{uKir}(\mathcal{O}_2)$, the set $\{a_n^A \colon n \in \omega\}$ is an enumeration of a dense subset of $A$.

Fix a countable dense subset $\{b_n \colon n \in \omega\}$ of $\mathcal{O}_2$. Then

$$\|\alpha_A(x) - y\| = \|(\mathrm{id}_A \otimes \nu)(\eta_A(x)) - \eta_A(y)\|,$$

and thus $\|\alpha_A(x) - y\| < \varepsilon$ if and only if there are positive integers $k \in \omega$ and $n_0, \ldots, n_{k-1}, m_0, \ldots, m_{k-1} \in \omega$, and scalars $\lambda_0, \ldots, \lambda_{k-1} \in \mathbb{Q}(i)$, such that

$$\left\| \eta_A(x) - \sum_{i \in k} \lambda_i a_{n_i}^G \otimes b_{m_i} \right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \sum_{i \in k} \lambda_i a_{n_i}^G \otimes \nu(b_{m_i}) - \eta_A(y) \right\| < \frac{\varepsilon}{2}.$$

Since the map $A \mapsto \eta_A$ is Borel, it follows that the set of all $C^*$-algebras $A$ in $SA_{uKir}(\mathcal{O}_2)$ such that $\|\alpha_A(x) - y\| < \varepsilon$ is Borel. The result follows. $\square$

**Theorem XII.6.16.** There are Borel reductions:

1. From the relation of isomorphism of infinite countable abelian groups, to the relation of isomorphism of $\mathcal{O}_\infty^{st}$-absorbing unital Kirchberg algebras satisfying the UCT with infinite $K_0$-group and with trivial $K_1$-group.

2. From the relation of isomorphism of uniquely 2-divisible infinite countable abelian groups, to the relation of isomorphism of $D_2$-absorbing unital Kirchberg algebras satisfying the UCT with infinite $K_0$-group and with trivial $K_1$-group that are a crossed product of $\mathcal{O}_2$ by an action of $\mathbb{Z}_2$ of Rokhlin dimension 1.

*Proof.* Both results follow from Remark XII.6.9, Proposition XII.6.11, Theorem XII.6.15, and Theorem XII.6.14, together with the Kirchberg-Phillips classification theorem ([150] and [200]). $\square$

**Corollary XII.6.17.** Let $\mathcal{C}$ be any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel. Assume that $F$ be any of the following equivalence relations:

– isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_2$ (with infinite $K_0$-group and trivial $K_1$-group);

– conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension 1,

– cocycle conjugacy of approximately representable actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$ that have Rokhlin dimension 1.

Then $\cong_{\mathcal{C}}$ is Borel reducible to $F$, and moreover $F$ is a complete analytic set.

In the case when $G$ is the group of integers $\mathbb{Z}$, actions of $\mathbb{Z}$ on $A$ naturally correspond to single automorphisms of $A$. Similarly, if $G$ is the group $\mathbb{Z}_n$, then actions of $\mathbb{Z}_n$ on $A$ correspond to automorphisms of $A$ whose order divides $n$. We show in Lemma XII.6.18 below that the notions of conjugacy and cocycle conjugacy (Definition III.5.4) for finite cyclic group actions and automorphisms are respected by this correspondence when $A$ has trivial center.

**Lemma XII.6.18.** Suppose that $\alpha$ and $\beta$ are automorphisms of a unital $C^*$-algebra $A$.

1. Then the following statements are equivalent:

   (a) The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of $\mathbb{Z}$ on $A$ are cocycle conjugate;

   (b) There are an automorphism $\gamma$ of $A$ and a unitary $u$ of $A$ such that $\mathrm{Ad}(u) \circ \alpha = \gamma \circ \beta \circ \gamma^{-1}$.

2. Assume moreover that $\alpha$ and $\beta$ have order $k \geq 2$ and that $A$ has trivial center (for example, if $A$ is simple). Then the following statements are equivalent:

   (a) The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of $\mathbb{Z}_k$ on $A$ are cocycle conjugate;

   (b) The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of $\mathbb{Z}$ on $A$ are cocycle conjugate.

*Proof.* (1). To show that (a) implies (b), simply take the unitary $u = u_1$ coming from the $\alpha$-cocycle $u\colon \mathbb{Z} \to U(A)$. Conversely, if $u$ is a unitary in $A$ as in the statement, we define an $\alpha$-cocycle as follows. Set $u_0 = 1$ and $u_1 = u$, and for $n \geq 2$ define $u_n$ inductively by $u_n = u_1\alpha(u_{n-1})$. Set $u_{-1} = \alpha^{-1}(u_1^*)$, and for $n \leq -2$, define $u_n$ inductively by $u_n = u_{-1}\alpha^{-1}(u_{n+1})$. It is straightforward to check that $n \mapsto u_n$ is an $\alpha$-cocycle, and that the automorphism $\gamma$ in the statement implements the conjugacy between $\alpha^u$ and $\beta$.

(2). To show that (a) implies (b), it is enough to note that if $u\colon \mathbb{Z}_k \to U(A)$ is an $\alpha$-cocycle, when we regard $\alpha$ as a $\mathbb{Z}_k$ action, then the sequence $(v_m)_{m\in\mathbb{N}}$ of unitaries in $A$ given by $v_m = u_n$ if $m = n \bmod k$, is an $\alpha$-cocycle, when we regard $\alpha$ as a $\mathbb{Z}$ action. Assume that $\alpha$ and $\beta$ are cocycle conjugate as automorphisms of $A$. Let $(u_n)_{n\in\mathbb{N}}$ be an $\alpha$-cocycle and let $\gamma$ be an automorphism implementing the conjugacy. Fix $n$ in $\mathbb{N}$, and write $n = km + r$ for uniquely determined $k \in \mathbb{Z}$ and $r \in k$. Since $\alpha$ and $\beta$ have order $k$, we have

$$\begin{aligned}
\mathrm{Ad}(u_{km+r}) \circ \alpha^r &= \mathrm{Ad}(u_{km+r}) \circ \alpha^{km+r} \\
&= \gamma \circ \beta^{km+r} \circ \gamma^{-1} \\
&= \gamma \circ \beta^r \circ \gamma^{-1} \\
&= \mathrm{Ad}(u_r) \circ \alpha^r.
\end{aligned}$$

In particular, $\mathrm{Ad}(u_{n+mk}) = \mathrm{Ad}(u_n)$, so $u_{n+mk}$ and $u_n$ differ by a central unitary. Since the center of $A$ is trivial, upon correcting by a scalar, we may assume that $u_{n+mk} = u_n$. Thus, the assignment $v\colon \mathbb{Z}_k \to U(A)$ given by $n \mapsto u_n$ is an $\alpha$-cocycle, when we regard $\alpha$ as a $\mathbb{Z}_k$ action, and $\gamma$ implements an conjugacy between the $\mathbb{Z}_k$ actions $\alpha^v$ and $\beta$. This finishes the proof. $\square$

**Corollary XII.6.19.** The relations of isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_2$ satisfying the UCT, conjugacy of automorphisms of $\mathcal{O}_2$, and cocycle conjugacy of automorphisms of $\mathcal{O}_2$ are complete analytic sets, and in particular, not Borel.

Recall that by Theorem 3.5 in [133], if $G$ is a finite group, then any two actions of $G$ on $\mathcal{O}_2$ with the Rokhlin property are conjugate. On the other hand there are at the moment no classification results for actions of $\mathbb{Z}_2$ on $\mathcal{O}_2$, even in the case of Rokhlin dimension 1. Corollary XII.6.17 shows that such classification problem is rather complicate.

*Nonclassification of automorphisms of $\mathcal{O}_2$ of order $p$*

Corollary XII.6.17 can be in fact generalized to automorphisms of order $p$ for any prime number $p$. This is the content of the following theorem.

**Theorem XII.6.20.** Fix a prime number $p$. Let $\mathcal{C}$ be any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel. Let $F$ be any of the following equivalence relations:

- isomorphism of simple purely infinite crossed products $\mathcal{O}_2 \rtimes \mathbb{Z}_p$ (with infinite $K_0$-group and trivial $K_1$-group);
- conjugacy of approximately representable actions of $\mathbb{Z}_p$ on $\mathcal{O}_2$ that have Rokhlin dimension 1,
- cocycle conjugacy of approximately representable actions of $\mathbb{Z}_p$ on $\mathcal{O}_2$ that have Rokhlin dimension 1.

Then $\cong_{\mathcal{C}}$ is Borel reducible to $F$. Moreover, $F$ is a complete analytic set.

We explain here how to adapt the arguments in Subsections XII.6,XII.6, and XII.6 to obtain Theorem XII.6.20. In the rest of this subsection we suppose that $p$ is a fixed prime number.

Definition XII.6.8 easily generalizes to actions of $\mathbb{Z}_p$. (General approximately representable actions of finite abelian groups on $C^*$-algebras have been defined in [132, Definition 3.6].)

An analog of the model action $\nu\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{O}_2)$, for actions of $\mathbb{Z}_p$, has been obtained in [8, Proposition 4.15]. We recall its construction in a form which is suitable for our purposes. Denote by $D_p$ the unique (up to isomorphism) unital Kirchberg algebra satisfying the UCT whose $K$-theory is given by

$$\left(K_0(D_p), \left[1_{D_p}\right], K_1(D_p)\right) \cong \left(\mathbb{Z}\left[\frac{1}{p}\right] \oplus \cdots \oplus \mathbb{Z}\left[\frac{1}{p}\right], 0, \{0\}\right).$$

where the non-trivial group on the right-hand side has $p-1$ direct summands. Note that when $p = 2$, the algebra $D_p$ is isomorphic to the algebra $D_2 = M_{2^\infty} \otimes \mathcal{O}_\infty^{st}$ previously defined.

**Proposition XII.6.21.** Let $p$ be a prime number. Then there exists an approximately representable action $\nu_p\colon \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{O}_2)$ such that

$$\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_2 \cong D_p.$$

*Proof.* The proof is essentially the same as that of [8, Proposition 4.15]. Set $\zeta_p$ to be a primitive $p$-th rooth of unity. One may identify the group $K_0(D_p)$ with the additive group of the ring $\mathbb{Z}\left[\frac{1}{p}, \zeta_p\right]$. Under this identification, the automorphism of $K_0(D_p)$ determined by multiplication by $\zeta_p$, can be lifted to an action $\mu_p\colon \mathbb{Z}_p \to \mathrm{Aut}(D_p)$ with the Rokhlin property, by [8, Theorem 2.10]. The crossed product of such action is easily seen to be a Kirchberg algebra with trivial $K$-theory. Since it satisfies the UCT by [191, Proposition 3.7], it follows from the classification of Kirchberg algebras ([150], [200]) that

$$D_p \rtimes_{\mu_p} \mathbb{Z}_p \cong \mathcal{O}_2.$$

The desired action is then the dual action $\nu_p = \widehat{\mu}_p\colon \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{O}_2)$. □

If $A$ is a simple nuclear unital $C^*$-algebra, then one can define the action $\alpha_{A,p}$ of $\mathbb{Z}_p$ on $A$ analogously as in Subsection XII.6, after replacing $\nu$ with $\nu_p$. The proof of Proposition XII.4.2 goes through without change when $\alpha_A$ is replaced with $\alpha_{A,p}$. Similarly, the proof of Proposition XII.6.11 can be easily generalized to actions of $\mathbb{Z}_p$. (See [123, Definition 1.1] for the definition of Rokhlin dimension of an action of an arbitrary finite group on a unital $C^*$-algebra.) Theorem XII.6.15 is then generalized after replacing $D_2$ with $D_p$ and $\mathbb{Z}_2$ with $\mathbb{Z}_p$.

Let us say that a countable infinite group $G$ is self-absorbing if $G \oplus G \cong G$. It follows from the classification of Kirchberg algebras ([150], [200]) that, if $A$ is a unital Kirchberg algebra with $K$-theory $(G, 0, \{0\})$, and $G$ is a *self-absorbing* torsion-free $p$-divisible abelian group, then $A$ is $D_p$-absorbing. Thus, Theorem XII.6.16 still holds after replacing $D_2$ with $D_p$, and uniquely 2-divisible infinite countable abelian groups with *self-absorbing* uniquely $p$-divisible infinite countable abelian groups.

Finally, one needs to modify the construction in Proposition XII.6.13 to obtain *self-absorbing* $p$-divisible torsion-free abelian groups. This is accomplished by considering the following modification in the definition of the group eplag associated with an excellent prime labeled graph.

Suppose that $(V, E, f)$ is an excellent prime labeled graph such that the range of $f$ is disjoint from $\mathcal{P}$. Denote by $\mathbb{Q}^{(V\times\omega)}$ the direct sum

$$\mathbb{Q}^{(V\times\omega)} = \bigoplus_{(v,n)\in V\times\omega} \mathbb{Q}$$

of copies of $\mathbb{Q}$ indexed by $V \times \omega$, and identify an element $(v, n)$ of $V \times \omega$ with the corresponding copy of $\mathbb{Q}$ in $\mathbb{Q}^{(V)}$. We define the $p$-divisible *self-absorbing* group eplag $\mathcal{G}_p^{sa}(V, E, f)$ associated with $(V, E, f)$, to be the subgroup of $\mathbb{Q}^{(V\times\omega)}$ generated by

$$\left\{\frac{(v,k)}{p^n f(v)^m}, \frac{(v,k)+(w,k)}{p^n f(\{v,w\})} \colon v, w \in V, \{v,w\} \in E, n, m, k \in \omega, p \in \mathcal{P}\right\}.$$

It is easy to check that $\mathcal{G}_p^{sa}(V, E, f)$ is indeed a self-absorbing $p$-divisible torsion-free abelian group. The proof of Theorem XII.6.20 is thus complete.

*Actions of countable groups on $\mathcal{O}_2$*

Let $G$ be a countable (discrete) group. Denote by $\mathrm{Act}(G, A)$ the space of actions of $G$ on $A$ endowed with the topology of pointwise convergence in norm. It is clear that $\mathrm{Act}(G, A)$ is homeomorphic to a $G_\delta$ subspace of the product of countably many copies of $A$ and, in particular, is a Polish space.

Let $G$ and $H$ be countable groups, and let $\pi\colon G \to H$ be a surjective homomorphism from $G$ to $H$. Define the Borel map $\pi^*\colon \mathrm{Act}(H, A) \to \mathrm{Act}(G, A)$ by $\pi^*(\alpha) = \alpha \circ \pi$ for $\alpha$ in $\mathrm{Act}(H, A)$. It is easy to check that $\pi^*$ is a Borel reduction from the relation of conjugacy of actions of $H$ to the relation of conjugacy of actions of $G$. The following proposition is then an immediate consequence of this observation together with Theorem XII.6.20.

**Proposition XII.6.22.** Let $G$ be a countable group with a nontrivial cyclic quotient. If $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation of conjugacy of actions of $G$ on $\mathcal{O}_2$.

Moreover, the latter equivalence relation is a complete analytic set as a subset of $\mathrm{Act}(G, A) \times \mathrm{Act}(G, A)$ and, in particular, is not Borel.

The situation for cocycle conjugacy is not as clear. It is not hard to verify that if $G = H \times N$ and $\pi\colon G \to H$ is the canonical projection, then $\pi^*$, as defined before, is a Borel reduction from the relation of cocycle conjugacy in $\mathrm{Act}(H, A)$ to the relation of cocycle conjugacy in $\mathrm{Act}(G, A)$. Using this observation and the structure theorem for finitely generated abelian groups, one obtains the following fact as a consequence of Theorem XII.6.20 :

**Proposition XII.6.23.** Let $G$ be any finitely generated abelian group. If $\mathcal{C}$ is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of $\mathcal{C}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation of conjugacy of actions of $G$ on $\mathcal{O}_2$.

Moreover, the latter equivalence relation is a complete analytic set as a subset of $\mathrm{Act}(G, A) \times \mathrm{Act}(G, A)$ and, in particular, not Borel.

### Final Comments and Remarks

Recall that an automorphism of a $C^*$-algebra $A$ is said to be *pointwise outer* (or *aperiodic*) if none of its nonzero powers is inner. By [187, Theorem 1], an automorphism of a Kirchberg algebra is pointwise outer if and only if it has the Rokhlin property. Moreover, it follows from this fact, together with [206, Corollary 5.14], that the set $\mathrm{Rok}(A)$ of pointwise outer automorphisms of a Kirchberg algebra $A$ is a dense $G_\delta$ subset of $\mathrm{Aut}(A)$, which is moreover easily seen to be invariant by cocycle conjugacy.

It is an immediate consequence of [187, Theorem 9], that aperiodic automorphisms of $\mathcal{O}_2$ form a single cocycle conjugacy class. In particular, and despite the fact that the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ is not Borel, its restriction to the comeager subset $\mathrm{APer}(\mathcal{O}_2)$ of $\mathrm{Aut}(\mathcal{O}_2)$ consisting of aperiodic automorphisms, has only one class and, in particular, is Borel. This can be compared with the analogous situation for the group of ergodic measure preserving transformations of the Lebesgue space: The main result of [75] asserts that the relation of conjugacy of ergodic measure preserving transformations of the Lebesgue space is a complete analytic set. On the other hand, the restriction of such relation to the comeager set of ergodic *rank one* measure preserving transformations is Borel.

It is conceivable that similar conclusions might hold for the relation of conjugacy of automorphisms of $\mathcal{O}_2$. We therefore suggest the following problem:

**Question XII.7.1.** Consider the relation of conjugacy of automorphisms of $\mathcal{O}_2$, and restrict it to the invariant dense $G_\delta$ set of aperiodic automorphisms. Is this equivalence relation Borel?

By [149, Theorem 4.5], the automorphisms of $\mathcal{O}_2$ are not classifiable up to conjugacy by countable structures. This means that if $\mathcal{C}$ is any class of countable structures, then the relation of conjugacy of automorphisms of $\mathcal{O}_2$ is *not* Borel reducible to the relation of isomorphisms of structures from $\mathcal{C}$. It would be interesting to know if one can obtain a similar result for the relation of cocycle conjugacy.

**Question XII.7.2.** Is the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ classifiable by countable structures?

Theorem 4.5 of [149] shows that the relation of conjugacy of automorphisms is not classifiable for a large class of $C^*$-algebras, including all $C^*$-algebras that are classifiable according to the Elliott classification program [235, Section 2.2]. It would be interesting to know if the same holds for the relation of cocycle conjugacy. More generally, it would be interesting to draw similar conclusions about the complexity of the relation of cocycle conjugacy for automorphisms of other simple $C^*$-algebras. This problem seems to be currently wide open.

**Problem XII.7.3.** Find an example of a simple unital nuclear separable $C^*$-algebra for which the relation of cocycle conjugacy of automorphisms is not classifiable by countable structures.

Recall that an equivalence relation on a standard Borel space is said to be *smooth*, or *concretely classifiable*, if it is Borel reducible to the relation of equality in some standard Borel space. A smooth equivalence relation is in particular Borel and classifiable by countable structures. Thus, it is a consequence of Corollary XII.6.19 that the relation of cocycle conjugacy of automorphisms of $\mathcal{O}_2$ is not smooth.

If $X$ is a compact Hausdorff space, we denote by $C(X)$ the unital commutative $C^*$-algebra of complex-valued continuous functions on $X$. It is a classical result of Gelfand and Naimark that any unital commutative $C^*$-algebra is of this form; see [14, Theorem II.2.2.4]. Moreover, by [14, II.2.2.5], the group $\mathrm{Aut}(C(X))$ of automorphisms of $C(X)$ is canonically isomorphic to the group $\mathrm{Homeo}(X)$ of homeomorphisms of $X$. It is clear that in this case the relations of conjugacy and cocycle conjugacy of automorphisms coincide. By [25, Theorem 5], if $X$ is the Cantor set, then the relation of (cocycle) conjugacy of automorphisms of $C(X)$ is not smooth (but classifiable by countable structures). On the other hand, when $X$ is the unit square $[0,1]^2$, then the relation of cocycle conjugacy of automorphisms of $C(X)$ is not classifiable by countable structures in view of [125, Theorem 4.17]. This addresses Problem XII.7.3 in the case of abelian unital $C^*$-algebras. No similar examples are currently known for simple unital $C^*$-algebras.

It is worth mentioning here that if one considers instead the relation of *unitary conjugacy* of automorphisms, then there is a *strong dichotomy* in the complexity. Recall that two automorphisms $\alpha, \beta$ of a unital $C^*$-algebra are unitarily conjugate if $\alpha \circ \beta^{-1}$ is an inner automorphism, that is, implemented by a unitary element of $A$. Theorem 1.2 in [177] shows that whenever this relation is not smooth, then it is even not classifiable by countable structures. The same phenomenon is shown to hold for unitary conjugacy of *irreducible representations* in [148, Theorem 2.8.]; see also [197, Section 6.8]. It is possible that similar conclusions might hold for the relation of conjugacy or cocycle conjugacy of automorphisms of simple $C^*$-algebras.

**Question XII.7.4.** Is it true that, whenever the relation of (cocycle) conjugacy of automorphisms of a simple unital $C^*$-algebra $A$ is not smooth, then it is not even classifiable by countable structures?

The Kirchberg-Phillips classification theorem asserts that Kirchberg algebras satisfying the UCT are classified up to isomorphism by their $K$-theory. By [70, Section 3.3], the $K$-theory of a $C^*$-algebra can be computed in a Borel way. It follows that Kirchberg algebras satisfying the UCT are classifiable up to isomorphism by countable structures. Conversely, by Corollary XII.6.17, if $\mathcal{C}$ is any class of countable structure *with Borel isomorphism relation*, then the relation of isomorphism of elements of $\mathcal{C}$ is Borel reducible to the relation of isomorphism of Kirchberg algebras satisfying the UCT. It is natural to ask whether the same conclusion holds for *any* class of countable structures $\mathcal{C}$.

**Question XII.7.5.** Suppose that $\mathcal{C}$ is a class of countable structures. Is the relation of isomorphism of elements of $\mathcal{C}$ Borel reducible to the relation of isomorphism of Kirchberg algebras with the UCT?

A class $\mathcal{D}$ of countable structures is *Borel complete* if the following holds: For any class of countable structures $\mathcal{C}$ the relation of isomorphism of elements of $\mathcal{C}$ is Borel reducible to the relation of isomorphism of elements of $\mathcal{D}$. Theorem 1, Theorem 3, and Theorem 10 of [78] assert that the classes of countable trees, countable linear orders, and countable fields of any fixed characteristic are Borel complete. Theorem 7 of [78] shows that the relation of isomorphism of countable groups is Borel complete. A long standing open problem –first suggested in [78]– asks whether the class of (torsion-free) abelian groups is Borel complete. In view of Theorem XII.6.16, a positive answer to such problem would settle Question XII.7.5 affirmatively.

For example, it would be interesting to determine for other "interesting" $C^*$-algebras (maybe one should start with strongly self-absorbing $C^*$-algebras, or $C^*$-algebras in a class that is well understood), whether the relation of cocycle conjugacy of their automorphisms is or not complete analytic, Borel, countable, etc.

Getting partial results for Kirchberg algebras should not be hard, at least when they satisfy the UCT. Indeed, cocycle conjugacy classes of aperiodic automorphisms of a Kirchberg algebra $A$ are in bijection with certain elements of $KK(A,A)$. Now, $KK(A,A)$ is a countable group. Thus, if one is able to show that the assignment $\varphi \mapsto KK(\varphi)$ as a map $\mathrm{Aut}(A) \to KK(A,A)$, is Borel, then if would follow that its restriction to the $G_\delta$ subset of aperiodic automorphisms is a Borel reduction, and hence said class of automorphisms would be classifiable by countable structures up to cocycle conjugacy.

# CHAPTER XIII

# INTRODUCTION

Associated to any locally compact group $G$, there are three fundamentally important operator algebras: its reduced group $C^*$-algebra $C^*_\lambda(G)$, its full group $C^*$-algebra $C^*(G)$, and its group von Neumann algebra $L(G)$. These are, respectively, the $C^*$-algebra generated by the left regular representation of $G$ in $\mathcal{B}(L^2(G))$; the universal $C^*$-algebra with respect to unitary representations of $G$ on Hilbert spaces; and the weak-$*$ closure (also called ultraweak closure) of $C^*_\lambda(G)$ in $\mathcal{B}(L^2(G))$. (We identify $\mathcal{B}(L^2(G))$ with the dual of the projective tensor product $L^2(G)\widehat{\otimes}L^2(G)$ canonically.) Equivalently, $L(G)$ is the double commutant of $C^*_\lambda(G)$ in $\mathcal{B}(L^2(G))$.

These operator algebras admit generalizations to representations of $G$ on $L^p$-spaces, for $p \in [1,\infty)$. The analog of $C^*_\lambda(G)$ is the the algebra $PF_p(G)$ of $p$-pseudofunctions on $G$, first introduced by Herz in [116]. (Phillips also considered this algebra in [207], where he called it the reduced group $L^p$-operator algebra of $G$, and denoted it $F^p_\nabla$. We will use the notation $F^p_\lambda(G)$. ) The analog of $C^*(G)$ is the full group $L^p$-operator algebra $F^p(G)$, defined by Phillips in [207]. Finally, the von Neumann algebra $L(G)$ has two analogs: the algebra of $p$-pseudomeasures $PM_p(G)$, which is the weak-$*$ closure of $F^p_\lambda(G)$ in $\mathcal{B}(L^p(G))$ (where we identify $\mathcal{B}(L^p(G))$ with the dual of the projective tensor product $L^p(G)\widehat{\otimes}(L^p(G))^*$ canonically); and the algebra of $p$-convolvers $CV_p(G)$, which is the double commutant of $PF_p(G)$ in $\mathcal{B}(L^p(G))$ (it is also the commutant of the right regular representation). Both these algebras were also introduced by Herz in [116]. The algebras $F^p_\lambda(G)$, $PM_p(G)$ and $CV_p(G)$ are sometimes referred to as "convolution algebras (acting on $L^p$-spaces)".

The algebras $F^p_\lambda(G)$, $F^p(G)$, $PM_p(G)$ and $CV_p(G)$ are examples of algebras of operators on $L^p$-spaces (from now on, $L^p$-operator algebras). This second part of the dissertation is devoted to the study of these algebras (with focus on $F^p_\lambda(G)$ and $F^p(G)$). In the last two chapters, we study a generalization of these objects, where the group is replaced by a groupoid.

For a long time, $L^p$-operator algebras had mainly arisen in the context of harmonic analysis. However, these objects have received more attention since Phillips introduced and studied in [204], [208] and [209], certain $L^p$-analogs of Cuntz algebras (denoted $\mathcal{O}^p_d$, for $d \in \mathbb{N}$) and of UHF-algebras. A more abstract study of $L^p$-operator algebras was initiated in [207]. The motivation for looking at $L^p$-operator algebras is, in retrospect, rather unexpected: the algebras $\mathcal{O}_d^p$ were defined to show that the methods in [35] can be applied to them, to show that the comparison map

$$K_*(\mathcal{O}_d^p) \to K_*^{\mathrm{top}}(\mathcal{O}_d^p),$$

from algebraic to topological $K$-theory, is an isomorphism in all degrees.

There is, by now, a decent collection of examples of $L^p$-operator algebras: $L^p$-Cuntz algebras ([204], (full and reduced) group $L^p$-operator algebras (and the analogs of the von Neumann algebra) ([116], [49], [207], [217], [98]), $L^p$-AF-algebras ([215]), groupoid $L^p$-operator algebras ([89], crossed products of $L^p$-operator algebras ([207]), and $L^p$-analogs of the irrational rotation algebras ([93])

On the other hand, and by comparison with the case $p = 2$, very little is know about general $L^p$-operator algebras, although there seems to be an interesting theory to discover. Despite the absence of an involution, much of the theory so far developed resembles and is inspired by the theory of $C^*$-algebras. Nevertheless, it is becoming apparent that $L^p$-operator algebras, for $p$ different from 2, are far more rigid objects than $C^*$-algebras. The most clear expression of this rigidity, at least so far, is studied in Chapter XVI, where we show that for $p \in [1, \infty) \setminus \{2\}$, a locally compact group $G$ can be recovered from any of the algebras $F_\lambda^p(G)$, $PM_p(G)$, or $CV_p(G)$. This is in stark contrast with the case $p = 2$:

- The groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ have isomorphic group $C^*$-algebras; and
- it is a long standing open problem whether the von Neumann algebras $L(\mathbb{F}_2)$ and $L(\mathbb{F}_3)$ of the free groups $\mathbb{F}_2$ and $\mathbb{F}_3$ on two and three generators, are isomorphic. (The $C^*$-algebras are known not to be isomorphic; they have different $K_0$.)

Although we will not say much here, we mention that recent work, joint with Hannes Thiel and Chris Phillips, seems to suggest that $L^p$-crossed products of minimal homeomorphisms, for $p \neq 2$, may remember the dynamical system up to flip conjugacy.

The chapters in this second part of the dissertation are all based on joint works: Chapters XIV through XIX are joint with Hannes Thiel, and Chapters XX and XXI are joint with Martino Lupini. The main results are, quite possibly, the ones in Chapters XVI and XIX, which we describe below.

The main result of Chapter XVI is as follows: Let $G$ and $H$ be two locally compact groups, let $p, q \in [1, \infty)$, and suppose that there exists a contractive isomorphism

$$F^p_\lambda(G) \cong F^q_\lambda(H) \text{ or } PM_p(G) \cong PM_q(H) \text{ or } CV_p(G) \cong CV_q(H).$$

Then $p$ and $q$ are either equal or conjugate, and $G$ is isomorphic to $H$. When $p = q = 1$, this recovers, with a different proof, a classical result of Wendel from the 60's.

The main result of Chapter XIX is an answer to a 20 year old question of Le Merdy and Marius Junge: $L^p$-operator algebras are not closed under quotients. The algebra and the ideal are not difficult to construct: the algebra is $F^p_\lambda(\mathbb{Z})$, and the ideal is the one associated to the upper open half-semicircle. However, proving that the quotient is not representable on an $L^p$-space is rather involved, and it requires the full strength of the results in Chapter XVIII. (The relevant quotient is in fact not representable on any $L^q$-space, for $q \in [1, \infty)$.)

**Notation and Preliminaries**

All locally compact groups will be endowed with their left Haar measure (see section II.1). We write $\mathbb{N}$ for $\{1, 2, \ldots\}$; we write $\overline{\mathbb{N}}$ for $\mathbb{N} \cup \{\infty\}$; we write $\mathbb{Z}_{\geq 0}$ for $\{0, 1, 2, \ldots\}$; and we write $\overline{\mathbb{Z}_{\geq 0}}$ for $\mathbb{Z}_{\geq 0} \cup \{\infty\}$.

For $n \in \mathbb{N}$, the finite cyclic group of order $n$ will be denote by $\mathbb{Z}_n$. For $n \in \mathbb{N}$ and $p \in [1, \infty]$, we write $\ell^p_n$ in place of $\ell^p(\{1, \ldots, n\})$, and we write $\ell^p$ in place of $\ell^p(\mathbb{Z})$.

Let $E$ be a Banach space. We write $E_1$ for the unit ball of $E$ and $E'$ for its dual space. If $F$ is another Banach space, we denote by $\mathcal{B}(E, F)$ the Banach space of all bounded linear operators $E \to F$, and write $\mathcal{B}(E)$ in place of $\mathcal{B}(E, E)$. For a bounded linear map $\varphi \colon E \to F$, we will write $\varphi' \colon F' \to E'$ for its dual map. For $p \in (1, \infty)$, we denote by $p'$ its conjugate (Hölder) exponent, which satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

Recall that a measure space $(X, \mathcal{A}, \mu)$ is said to be *complete* if whenever $Y \in \mathcal{A}$ satisfies $\mu(Y) = 0$ and $Z$ is a subset of $Y$, then $Z \in \mathcal{A}$. If $(X, \mathcal{A}, \mu)$ is an arbitrary measure space, we denote by $(X, \overline{\mathcal{A}}, \overline{\mu})$ its completion. It is easy to show that if $p \in [1, \infty)$, then there is a canonical isometric isomorphism

$$L^p(X, \mathcal{A}, \mu) \cong L^p(X, \overline{\mathcal{A}}, \overline{\mu}).$$

We will usually omit the $\sigma$-algebra in the notation for $L^p$-spaces and measure spaces (unless they are relevant, such as in Chapter XIV, and specifically in Theorem XIV.2.21).

We recall some standard definitions and facts about measure spaces. A measurable space is called a *standard Borel space* when it is endowed with the $\sigma$-algebra of Borel sets with respect to some Polish topology on the space. If $(Z,\lambda)$ is a measure space for which $L^p(Z,\lambda)$ is separable, then there exists a complete $\sigma$-finite measure $\mu$ on a standard Borel space $X$ such that $L^p(Z,\lambda)$ is isometrically isomorphic to $L^p(X,\mu)$.

**Isometric Isomorphisms Between $L^p$-spaces.**

In this section, we review Lamperti's theorem characterizing isometries between $L^p$-spaces, for $p \in [1,\infty) \setminus 2$, and present it in a form that is convenient for our uses. The characterization can be roughly described as follows: invertible isometries between $L^p$-spaces, for $p \neq 2$, are all a combination of a bimeasurable bijective *measure class preserving transformation* (a suitable flexibilization of the notion of measure preserving bimeasurable bijection; see Definition XIII.2.1) on the underlying measure space, and a multiplication operator by a function from the space into the scalars of modulus one.

For example, Lamperti's theorem implies that for $p \neq 2$, the only invertible isometries of $M_n$, when identified with $\mathcal{B}(\ell^p_n)$ (we will usually denote this Banach algebra by $M^p_n$), are the complex permutation matrices; these are the matrices that have only one nonzero entry in each column and row, and this entry is a complex number of modulus one. In particular, the group of invertible isometries of $M^p_n$ is not connected for $p \neq 2$, and it is much smaller than the group of unitary matrices.

For most of what we do in this second part of the dissertation, Lamperti's theorem is extremely crucial, and many of our results ultimately depend on it. Roughly speaking, Lamperti's

The basic reference is [161]. We point out that Lamperti's result is more general than what we reproduce here as Theorem XIII.2.4. Finally, it should be mentioned that the statement in Theorem XIII.2.4 actually contains a mistake, and a corrected version can be found in [72].

If $X$ is a set $E$ and $F$ are subsets of a set $X$, we denote by $E\triangle F$ their symmetric difference, that is, $E\triangle F = (E\cap F^c)\cup(E^c\cap F)$.

**Definition XIII.2.1.** Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces. A *measure class preserving transformation* from $X$ to $Y$ is a measurable function $T\colon X \to Y$ satisfying

$$\mu(T^{-1}(F)) = 0 \text{ if and only if } \nu(F) = 0$$

for every measurable set $F \subseteq Y$.

A measure class preserving transformation $T\colon X \to Y$ is said to be *invertible*, if it is invertible as a function from $X$ to $Y$ and its inverse is measurable.

Note that we do not require measure class preserving transformations to preserve the measure; just the null-sets. Also, the inverse of a measure class preserving transformation is automatically a measure class preserving transformation.

**Remark XIII.2.2.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces. Set

$$\mathcal{N}(\mu) = \{E \in \mathcal{A} \colon \mu(E) = 0\}$$

and let $\mathcal{A}/\mathcal{N}(\mu)$ be the quotient of $\mathcal{A}$ by the relation $E \sim F$ if $E \triangle F$ belongs to $\mathcal{N}(\mu)$. Define $\mathcal{B}/\mathcal{N}(\nu)$ analogously. If $T\colon X \to Y$ is a measure class preserving transformation, then $T$ induces a function

$$T^*\colon \mathcal{B}/\mathcal{N}(\nu) \to \mathcal{A}/\mathcal{N}(\mu)$$

given by $T^*(F + \mathcal{N}(\nu)) = T^{-1}(F) + \mathcal{N}(\mu)$ for all $F$ in $\mathcal{B}$. Moreover, if $T$ is invertible then $T^*$ is invertible, and one has $(T^*)^{-1} = (T^{-1})^*$.

The induced map $T^*\colon \mathcal{B}/\mathcal{N}(\nu) \to \mathcal{A}/\mathcal{N}(\mu)$ is an example of an *order continuous Boolean homomorphism*. See Definition 313H in [77] and also Definition 4.9 in [204], where they are called $\sigma$-homomorphisms. Under fairly general assumptions on the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, every such homomorphism $\mathcal{B}/\mathcal{N}(\nu) \to \mathcal{A}/\mathcal{N}(\mu)$ lifts to a measure class preserving transformation $X \to Y$. See Theorem 343B in [77].

We now present some examples of isometric isomorphisms between $L^p$-spaces.

**Examples XIII.2.3.** Let $(X, \mu)$ and $(Y, \nu)$ be measure spaces and let $p \in [1, \infty)$.

1. For a measurable function $h\colon Y \to S^1$, the associated multiplication operator $m_h\colon L^p(Y,\nu) \to L^p(Y,\nu)$, which is given by

$$m_h(f)(y) = h(y)f(y)$$

for all $f$ in $L^p(Y,\nu)$ and all $y$ in $Y$, is an isometric isomorphism. Indeed, its inverse is easily seen to be $m_{\overline{h}}$.

2. Let $T\colon X \to Y$ be an invertible measure class preserving transformation. Then the linear map $u_T\colon L^p(X,\mu) \to L^p(Y,\nu)$ given by

$$u_T(f)(y) = \left(\left[\frac{d(\mu \circ T^{-1})}{d\nu}\right](y)\right)^{\frac{1}{p}} f(T^{-1}(y))$$

for all $f$ in $L^p(X,\mu)$ and all $y$ in $Y$, is an isometric isomorphism. Indeed, its inverse is $u_{T^{-1}}$.

3. If $h\colon Y \to S^1$ and $T\colon X \to Y$ are as in (1) and (2), respectively, then

$$m_h \circ u_T\colon L^p(X,\mu) \to L^p(Y,\nu)$$

is an isometric isomorphism with inverse $u_{T^{-1}} \circ m_{\overline{h}}$.

The following result is a particular case of Lamperti's Theorem (see the Theorem in [161]; see also Theorem 6.9 in [204]), and it can be regarded as a structure theorem for isometric isomorphisms between $L^p$-spaces. It states that if $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ are complete $\sigma$-finite measure spaces, then the linear operators of the form $m_h \circ u_T$ are the only isometric isomorphisms between $L^p(X,\mu)$ and $L^p(Y,\nu)$ for $p \in [1,\infty) \setminus \{2\}$. (Although Lamperti's Theorem was originally stated and proved assuming $(X,\mathcal{A},\mu) = (Y,\mathcal{B},\nu)$, this assumption was never actually used in his proof.)

The version we exhibit here is not the most general possible, but it is enough for our purposes.

**Theorem XIII.2.4.** Let $p \in [1,\infty) \setminus \{2\}$. Let $(X,\mathcal{A},\mu)$ and $(Y,\mathcal{B},\nu)$ be complete $\sigma$-finite standard Borel spaces, and let $\varphi\colon L^p(X,\mu) \to L^p(Y,\nu)$ be an isometric isomorphism. Then there exist a measurable function $h\colon Y \to S^1$ and an invertible measure class preserving transformation

$T\colon X \to Y$ such that

$$\varphi(f)(y) = h(y) \left( \left[ \frac{d(\mu \circ T^{-1})}{d\nu} \right] (y) \right)^{\frac{1}{p}} f(T^{-1}(y))$$

for all $f$ in $L^p(X, \mu)$ and all $y$ in $Y$. In other words, $\varphi = m_h \circ u_T$.

Moreover, the pair $(h, T)$ is unique in the following sense: if $\widetilde{h}\colon Y \to S^1$ and $\widetilde{T}\colon X \to Y$ are, respectively, a measurable function and an invertible measure class preserving transformation satisfying $v = m_{\widetilde{h}} \circ u_{\widetilde{T}}$, then $h(y) = \widetilde{h}(y)$ for $\nu$-almost every $y$ in $Y$, and

$$\nu \left( T(E) \triangle \widetilde{T}(E) \right) = 0$$

for every measurable subset $E \subseteq X$.

*Proof.* We will use the language and notation from Sections 5 and 6 in [204], where Phillips develops the material needed to make effective use of Lamperti's Theorem (Theorem 3.1 in [161]).

By Theorem 6.9 in [204], the invertible isometry $v$ is spatial (see Definition 6.4 in [204]). Let $(E, F, \phi, h)$ be a spatial system for $\varphi$. By Lemma 6.12 in [204] together with the fact that $\varphi$ is bijective, we deduce that $\mu(X \setminus E) = 0$ and $\nu(Y \setminus F) = 0$.

The assumptions on the measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ ensure that Theorem 343B in [77] applies, so the order continuous Boolean homomorphisms $\phi$ and $\phi^{-1}$ can be lifted to measure class preserving transformations $f_\phi\colon X \to Y$ and $f_{\phi^{-1}}\colon Y \to X$. By Corollary 343G in [77], the identities

$$f_\phi \circ f_{\phi^{-1}} = \mathrm{id}_Y \quad \text{and} \quad f_{\phi^{-1}} \circ f_\phi = \mathrm{id}_X,$$

hold up to $\nu$- and $\mu$-null sets, respectively. Upon redifining them on sets of measure zero (see, for example, Theorem 344B and Corollary 344C in [77]), we may take $T = f_\phi$ and $T^{-1} = f_{\phi^{-1}}$.

We now turn to uniqueness of the pair $(h, T)$. Denote by $\mathcal{A}$ and $\mathcal{B}$ the domains of $\mu$ and $\nu$, respectively, and by $\mathcal{N}(\mu)$ and $\mathcal{N}(\nu)$ the subsets of $\mathcal{A}$ and $\mathcal{B}$ consisting of the $\mu$ and $\nu$-null sets, respectively. Uniqueness of $h$ up to null sets, and uniqueness of $T^*\colon \mathcal{A}/\mathcal{N}(\mu) \to \mathcal{B}/\mathcal{N}(\nu)$ (see Remark XIII.2.2 for the definition of $T^*$) were established in Lemma 6.6 in [204]. It follows from Corollary 343G in [77] that $T$ is unique up to changes on sets of measure zero, which is equivalent to the formulation in the statement. $\square$

Theorem XIII.2.4 will be crucial in our study of $L^p$-operator algebras generated by invertible isometries. Since the invertible isometries of a given $L^p$-space are the main object of study of this work, we take a closer look at their algebraic structure in the remainder of this section.

**Definition XIII.2.5.** Let $(X, \mu)$ be a measure space.

1. We denote by $L^0(X, S^1)$ the Abelian group (under pointwise multiplication) of all measurable functions $X \to S^1$, where two such functions that agree $\mu$-almost everywhere on $X$ are considered to be the same element in $L^0(X, S^1)$.

2. We denote by $\mathrm{Aut}_*(X, \mu)$ the group (under composition) of all invertible measure class preserving transformations $X \to X$.

3. For $p \in [1, \infty)$, we denote by $\mathrm{Aut}(L^p(X, \mu))$ the group (under composition) of isometric automorphisms of $L^p(X, \mu)$. Equivalently, $\mathrm{Aut}(L^p(X, \mu))$ is the group of all invertible isometries of $L^p(X, \mu)$.

We point out that similar definitions and notation were introduced in [198].

**Remark XIII.2.6.** Let $(X, \mu)$ be a measure space and let $p \in [1, \infty)$. Then the maps

$$m\colon L^0(X, S^1) \to \mathrm{Aut}(L^p(X, \mu) \quad \text{and} \quad u\colon \mathrm{Aut}_*(X, \mu) \to \mathrm{Aut}(L^p(X, \mu)$$

given by $h \mapsto m_h$ and $T \mapsto u_T$, respectively, are injective group homomorphisms via which we may identify $L^0(X, S^1)$ and $\mathrm{Aut}_*(X, \mu)$ with subgroups of $\mathrm{Aut}(L^p(X, \mu)$.

**Proposition XIII.2.7.** Let $(X, \mathcal{A}, \mu)$ be a complete $\sigma$-finite standard Borel space, and let $p \in [1, \infty)$. Then $L^0(X, S^1)$ is a normal subgroup in $\mathrm{Aut}(L^p(X, \mu))$, and there is a canonical algebraic isomorphism

$$\mathrm{Aut}(L^p(X, \mu)) \cong \mathrm{Aut}_*(X, \mu) \rtimes L^0(X, S^1).$$

*Proof.* Since every element in $\mathrm{Aut}(L^p(X, \mu))$ is of the form $m_h \circ u_T$ for some $h \in L^0(X, S^1)$ and some $T \in \mathrm{Aut}_*(X, \mu)$ by Theorem XIII.2.4, it is enough to check that conjugation by $u_T$ leaves $L^0(X, S^1)$ invariant.

Given $h \in L^0(X, S^1)$ and $T \in \mathrm{Aut}_*(X, \mu)$, let $f \in L^p(X, \mu)$ and $x$ in $X$. We have

$$\begin{aligned}
(m_h \circ u_T)(f)(x) &= h(x)\left(\left[\frac{d(\mu \circ T^{-1})}{d\mu}\right](x)\right)^{\frac{1}{p}} f(T^{-1}(x)) \\
&= \left(\left[\frac{d(\mu \circ T^{-1})}{d\mu}\right](x)\right)^{\frac{1}{p}} (m_{h\circ T} f)(T^{-1}(x)) \\
&= (u_T \circ m_{h\circ T})(f)(x).
\end{aligned}$$

We conclude that $u_{T^{-1}} \circ m_h \circ u_T = m_{h\circ T}$, which shows that $L^0(X, S^1)$ is a normal subgroup in $\mathrm{Aut}(L^p(X, \mu))$.

The algebraic isomorphism $\mathrm{Aut}(L^p(X, \mu)) \cong \mathrm{Aut}_*(X, \mu) \rtimes L^0(X, S^1)$ now follows from Lamperti's Theorem XIII.2.4. $\square$

Adopt the notation of the proposition above. Endow $L^0(X, S^1)$ with the topology of convergence in measure, endow $\mathrm{Aut}_*(X, \mu)$ with the weak topology, and endow $\mathrm{Aut}(L^p(X, \mu))$ with the strong operator topology. It is shown in [198] that these topologies turn these groups into Polish groups. Using this technical fact, it is shown in Corollary 3.3.46 of [198] that the algebraic isomorphism of the above proposition is in fact an isomorphism of topological groups.

# CHAPTER XIV

# GROUP ALGEBRAS ACTING ON $L^P$-SPACES

This chapter is based on joint work with Hannes Thiel ([98]).

For $p \in [1, \infty)$ we study representations of a locally compact group $G$ on $L^p$-spaces and $QSL^p$-spaces. The universal completions $F^p(G)$ and $F^p_{\mathrm{QS}}(G)$ of $L^1(G)$ with respect to these classes of representations (which were first considered by Phillips and Runde, respectively), can be regarded as analogs of the full group $C^*$-algebra of $G$ (which is the case $p = 2$). We study these completions of $L^1(G)$ in relation to the algebra $F^p_\lambda(G)$ of $p$-pseudofunctions. We prove a characterization of group amenability in terms of certain canonical maps between these universal Banach algebras. In particular, $G$ is amenable if and only if $F^p_{\mathrm{QS}}(G) = F^p(G) = F^p_\lambda(G)$.

One of our main results is that for $1 \leq p < q \leq 2$, there is a canonical map $\gamma_{p,q} \colon F^p(G) \to F^q(G)$ which is contractive and has dense range. When $G$ is amenable, $\gamma_{p,q}$ is injective, and it is never surjective unless $G$ is finite. We use the maps $\gamma_{p,q}$ to show that when $G$ is discrete, all (or one) of the universal completions of $L^1(G)$ are amenable as a Banach algebras if and only if $G$ is amenable.

Finally, we exhibit a family of examples showing that the characterizations of group amenability mentioned above cannot be extended to $L^p$-operator crossed products of topological spaces.

## Introduction

In this chapter, which is based on [98], we study representations of a locally compact group $G$ on certain classes of Banach spaces, as well as the corresponding completions of the group algebra $L^1(G)$. More specifically, for each $p \in [1, \infty)$, we consider the class $L^p$ of $L^p$-spaces; the class $SL^p$ of closed subspaces of Banach spaces in $L^p$; the class $QL^p$ of quotients of Banach spaces in $L^p$ by closed subspaces; and the class $QSL^p$ of quotients of Banach spaces in $SL^p$ by closed subspaces (Definition XIV.2.5). We denote the corresponding universal completions of $L^1(G)$ by $F^p(G), F^p_{\mathrm{S}}(G), F^p_{\mathrm{Q}}(G)$ and $F^p_{\mathrm{QS}}(G)$, respectively; see Notation XIV.2.8.

We also study the algebra $F^p_\lambda(G)$ of $p$-pseudofunctions on $G$. This is the Banach subalgebra of $\mathcal{B}(L^p(G))$ generated by all the operators of left convolution by functions in $L^1(G)$. Equivalently, $F^p_\lambda(G)$ is the closure of the image of the left regular representation $\lambda_p\colon L^1(G) \to \mathcal{B}(L^p(G))$. This algebra was introduced by Herz in [116], where $F^p_\lambda(G)$ was denoted $PF_p(G)$ (see also [189]).

The fact that $L^1(G)$ has a contractive approximate identity implies that there exist canonical isometric isomorphisms

$$L^1(G) \cong F^1_\lambda(G) \cong F^1(G) \cong F^1_{\mathrm{S}}(G) \cong F^1_{\mathrm{Q}}(G) \cong F^1_{\mathrm{QS}}(G);$$

see Proposition XIV.2.11 and Remark XIV.2.12. However, for $p \neq 1$, the existence of a canonical (not necessarily isometric) isomorphism between any of the algebras $F^p(G)$, $F^p_{\mathrm{S}}(G)$, $F^p_{\mathrm{Q}}(G)$ or $F^p_{\mathrm{QS}}(G)$, and the algebra $F^p_\lambda(G)$, is equivalent to amenability of $G$; see Theorem XIV.3.7. For $p = 2$, the algebra $F^2(G)$ is the full group $C^*$-algebra of $G$, usually denoted $C^*(G)$.

Using an extension theorem of Hardin, we show that for $p \notin \{4, 6, 8, \ldots\}$ and $q \notin \{\frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots\}$, and regardless of $G$, there are canonical isometric isomorphisms

$$F^p_{\mathrm{S}}(G) \cong F^p(G) \quad \text{and} \quad F^q_{\mathrm{Q}}(G) \cong F^q(G).$$

A consequence of the existence of such isomorphisms is that, for $1 \leq p \leq q \leq 2$ and for $2 \leq r \leq s < \infty$, there are canonical, contractive homomorphisms

$$\gamma_{p,q}\colon F^p(G) \to F^q(G) \quad \text{and} \quad \gamma_{s,r}\colon F^s(G) \to F^r(G)$$

with dense range; see Theorem XIV.2.30. This can be interpreted as saying that the algebras $F^p(G)$ form a 'continuous interpolating family' of Banach algebras between the group algebra $F^1(G) = L^1(G)$ and the full group $C^*$-algebra $F^2(G) = C^*(G)$.

When $G$ is amenable, our results recover, using different methods, a result announced by Herz as Theorem C in [115], and whose proof appears in the corollary on page 512 of [118]. Furthermore, in this case, we show that for $1 \leq p < q \leq 2$ or $2 \leq q < p < \infty$, the map $\gamma_{p,q}\colon F^p(G) \to F^q(G)$ is injective, and that it is never surjective unless $G$ is finite; see Corollary XIV.3.20.

Another application of Theorem XIV.2.30 is as follows: when $G$ is discrete, amenability of any of the Banach algebras $F^p(G)$, $F^p_{\mathrm{S}}(G)$, $F^p_{\mathrm{Q}}(G)$ or $F^p_{\mathrm{QS}}(G)$, is equivalent to amenability of $G$; see Theorem XIV.3.11. The cases $p = 1$ and $p = 2$ of this theorem are well-known, the first one being due to B. Johnson [137], and holding even if $G$ is not discrete.

A partial summary of our results on characterization of group amenability is as follows. (The equivalence between (1) and (2) below, specifically for $L^p$-spaces, was independently obtained by Phillips, whose methods are inspired in $C^*$-algebraic techniques.)

**Theorem.** Let $G$ be a locally compact group and let $p \in (1, \infty)$. Consider the following statements:

1. The group $G$ is amenable.
2. The canonical map from each of the algebras $F^p(G)$, $F^p_{\mathrm{S}}(G)$, $F^p_{\mathrm{Q}}(G)$, or $F^p_{\mathrm{QS}}(G)$, to $F^p_\lambda(G)$, is an isometric isomorphism.
3. The canonical map from any of the algebras $F^p(G)$, $F^p_{\mathrm{S}}(G)$, $F^p_{\mathrm{Q}}(G)$, or $F^p_{\mathrm{QS}}(G)$, to $F^p_\lambda(G)$, is a (not necessarily isometric) isomorphism.
4. The algebras $F^p(G), F^p_{\mathrm{S}}(G), F^p_{\mathrm{Q}}(G)$, and $F^p_{\mathrm{QS}}(G)$, are amenable.
5. At least one of $F^p(G), F^p_{\mathrm{S}}(G), F^p_{\mathrm{Q}}(G)$, or $F^p_{\mathrm{QS}}(G)$, is amenable.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). If $G$ is discrete, then all five statements are equivalent.

Finally, we show in Theorem XIV.4.3 that the theorem above, particularly the implications '(2) $\Rightarrow$ (1)' and '(4) $\Rightarrow$ (1)', cannot be generalized to $L^p$-crossed products of discrete groups acting on algebras of the form $C_0(X)$, for some locally compact Hausdorff space $X$.

Further connections between $G$ and $F^p_\lambda(G)$ will be explored in Chapter XVI; functoriality properties of $F^p_\lambda(G)$ (and those of the universal completions of $L^1(G)$ discussed above) are studied in Chapter XV; and applications of the results in this chapter are given in Chapters XVII and XVIII.

We have made the effort of not adopting any cardinality assumptions ($\sigma$-finiteness of measures, second-countability or $\sigma$-compactness of groups, or separability of Banach spaces) whenever possible. This implies considerable additional work when showing the existence of the maps $\gamma_{p,q} \colon F^p(G) \to F^q(G)$; see Theorem XIV.2.30. Indeed, some of the techniques for dealing with $L^p$-spaces require the involved measure spaces to be $\sigma$-finite (as is the case with Hardin's theorem). In order to directly apply such techniques, one has to restrict to second countable (or sometimes $\sigma$-compact) locally compact groups. However, one can often reduce a problem about a locally compact group $G$ to second countable locally compact groups by applying the following two steps: First, $G$ is the union of open, $\sigma$-compact subgroups. Second, by the Kakutani-Kodaira Theorem, every $\sigma$-compact, locally compact group $H$ contains an arbitrarily small compact, normal subgroup $N$ such that $H/N$ is second countable. This technique is for instance used to prove Theorem XIV.2.26.

For a Banach space $E$, we denote by $\mathrm{Isom}(E)$ the subset of $\mathcal{B}(E)$ consisting of invertible isometries. (In this dissertation, invertible isometries will always be assumed to be surjective.) The group $\mathrm{Isom}(E)$ will be endowed with the strong operator topology. It is a standard fact that $\mathrm{Isom}(E)$ is a Polish group whenever $E$ is a separable Banach space. Moreover, it is easy to check that if $X$ is a topological (measurable) space, a function $u\colon X \to \mathrm{Isom}(E)$ is continuous (measurable) if and only if for every $\xi \in E$, the map $X \to E$ given by $x \mapsto u(x)\xi$ for $x \in X$, is continuous (measurable).

### Universal Completions of $L^1(G)$

Let $G$ be a locally compact group. We let $L^1(G)$ denote the Banach algebra of complex-valued functions on $G$ that are integrable (with respect to the left Haar measure), with product given by convolution.

A representation of a Banach algebra $A$ on a Banach space $E$ is a contractive homomorphism $A \to \mathcal{B}(E)$.

**Definition XIV.2.1.** Let $\mathcal{E}$ be a class of Banach spaces. We denote by $\mathrm{Rep}_{\mathcal{E}}(G)$ the class of all non-degenerate representations of $L^1(G)$ on Banach spaces in $\mathcal{E}$. Given $f$ in $L^1(G)$, set

$$\|f\|_{\mathcal{E}} = \sup\left\{\|\pi(f)\|\colon \pi \in \mathrm{Rep}_{\mathcal{E}}(G)\right\}.$$

(Note that the supremum exists, even if $\mathcal{E}$ is not a set.) Then $\|\cdot\|_{\mathcal{E}}$ defines a seminorm on $L^1(G)$. Set

$$I_{\mathcal{E}} = \left\{f \in L^1(G)\colon \|f\|_{\mathcal{E}} = 0\right\} = \bigcap_{\pi \in \mathrm{Rep}_{\mathcal{E}}(G)} \ker(\pi),$$

which is a closed ideal in $L^1(G)$. We write $F_{\mathcal{E}}(G)$ for the completion of $L^1(G)/I_{\mathcal{E}}$ in the induced norm.

**Remark XIV.2.2.** Let $G$ be a locally compact group, and let $\mathcal{E}$ be a class of Banach spaces. Then the canonical map $L^1(G) \to F_{\mathcal{E}}(G)$ is contractive and has dense range. Moreover, it is injective if and only if the elements of $\mathrm{Rep}_{\mathcal{E}}(G)$ separate the points of $L^1(G)$, meaning that for every $f \in L^1(G)$ with $f \neq 0$, there exists $\pi \in \mathrm{Rep}_{\mathcal{E}}(G)$ such that $\pi(f) \neq 0$.

**Definition XIV.2.3.** Let $G$ be a locally compact group. An *isometric representation* of $G$ on a Banach space $E$ is a continuous group homomorphism of $G$ into the group $\mathrm{Isom}(E)$ of invertible isometries of $E$, where $\mathrm{Isom}(E)$ is equipped with the strong operator topology. Equivalently, an isometric representation of $G$ is a strongly continuous action of $G$ on $E$ via isometries.

The following result is folklore, and we omit its proof.

**Proposition XIV.2.4.** Let $G$ be a locally compact group and let $E$ be a Banach space. Then there is a natural bijective correspondence between nondegenerate representations $L^1(G) \to \mathcal{B}(E)$ and isometric representations of $G$ on $E$.

If $\rho\colon G \to \mathrm{Isom}(E)$ is an isometric representation, then the induced nondegenerate representation $\pi_\rho\colon L^1(G) \to \mathcal{B}(E)$ is given by

$$\pi_\rho(f)(\xi) = \int_G f(s)\rho_s(\xi)\ ds$$

for all $f \in L^1(G)$ and all $\xi \in E$, and it is called the *integrated form* of $\rho$.

The following are the classes of Banach spaces that we are mostly interested in.

**Definition XIV.2.5.** Let $p \in [1, \infty)$, and let $E$ be a Banach space.

1. We say that $E$ is an *$L^p$-space* if there exists a measure space $(X, \mu)$ such that $E$ is isometrically isomorphic to $L^p(X, \mu)$.

2. We say that $E$ is an *$SL^p$-space* if there is an $L^p$-space $F$ such that $E$ is isometrically isomorphic to a closed subspace of $F$.

3. We say that $E$ is a *$QL^p$-space* if there is an $L^p$-space $F$ such that $E$ is isometrically isomorphic to a quotient of $F$ by a closed subspace.

4. We say that $E$ is a *$QSL^p$-space* if there is an $SL^p$-space $F$ such that $E$ is isometrically isomorphic to a quotient of $F$ by a closed subspace.

We let $\mathcal{L}^p$ (respectively, $\mathcal{SL}^p$, $\mathcal{QL}^p$, $\mathcal{QSL}^p$) denote the class of all $L^p$-spaces (respectively, $SL^p$-spaces, $QL^p$-spaces, $QSL^p$-spaces).

**Remarks XIV.2.6.** Let $p \in [1, \infty)$.

(1) If $E$ is a separable $L^p$-space, then there exists a finite measure space $(X, \mu)$ such that $E$ is isometrically isomorphic to $L^p(X, \mu)$. One can also show that every separable $SL^p$-space is a subspace of a separable $L^p$-space, and analogously for separable $QL^p$-spaces and $QSL^p$-spaces. See [100].

(2) It is a routine exercise to check that if $F$ is a closed subspace of a quotient of an $L^p$-space $E$, then $F$ is (isometrically isomorphic to) a quotient of a subspace of $E$. It follows that the class $\mathcal{QSL}^p$ coincides with the class of Banach spaces that are isometrically isomorphic to a closed subspace of a $QL^p$-space. (The latter is the class that one would denote by $\mathcal{SQL}^p$.)

(3) Examples of $L^p$-spaces are $L^p([0, 1])$ with Lebesgue measure; $\ell^p$ with counting measure; and $\ell^p_n$ with counting measure, for $n \in \mathbb{N}$. It is well-known that every separable $L^p$-space is isometrically isomorphic to a countable $p$-direct sum of these. In particular, up to isometric isomorphism, there are only countably many separable Banach space in the class $\mathcal{L}^p$. The classes $\mathcal{SL}^p$ and $\mathcal{QL}^p$ are much larger, as the following result shows.

**Proposition XIV.2.7.** Let $p, q \in [1, \infty)$.

1. If $p \leq q \leq 2$, then $\mathcal{SL}^p \supseteq \mathcal{L}^q$.
2. If $2 \leq r \leq s$, then $\mathcal{L}^r \subseteq \mathcal{QL}^s$.

*Proof.* Let $1 \leq p \leq q \leq 2$. In Proposition 11.1.9 in [1] it is shown that the space $L^p([0, 1])$ isometrically embeds into $L^q([0, 1])$. Essentially the same argument can be used to prove part (1) as follows: Given an $L^q$-space $E$, it is clear that $E$ is finitely representable in $\ell^q$. The assumptions on $p$ and $q$ ensure that $\ell^q$ is finitely representable in $\ell^p$. It follows that $E$ is finitely representable on $\ell^p$. Therefore, $E$ is isometrically isomorphic to a subspace of some ultrapower of $\ell^p$. The result follows since $\mathcal{L}^p$ is closed under ultrapowers. (See [112] and [100] for details.)

Part (2) can be deduced from part (1) using duality. $\square$

Since we are mostly interested in the completions of $L^1(G)$ with respect to the classes from Definition XIV.2.5, we will use special notation for these.

**Notation XIV.2.8.** Let $p \in [1, \infty)$, and let $G$ be a locally compact group. We make the following abbreviations:

$$F^p(G) = F_{L^p}(G), \quad F^p_{\mathrm{Q}}(G) = F_{QL^p}(G),$$
$$F^p_{\mathrm{S}}(G) = F_{SL^p}(G), \quad F^p_{\mathrm{QS}}(G) = F_{QSL^p}(G).$$

**Remarks XIV.2.9.** Let $p \in [1, \infty)$, and let $G$ be a locally compact group.

(1) It is well known that every locally compact group $G$ admits a faithful isometric representation on an $L^p$-space (namely, the left regular representation). It follows that the natural map from $L^1(G)$ to any of $F^p(G), F^p_{\mathrm{S}}(G), F^p_{\mathrm{Q}}(G)$ and $F^p_{\mathrm{QS}}(G)$ is injective.

(2) Let $\mathcal{E}$ denote any of the classes $L^p, SL^p, QL^p$ or $QSL^p$. Since $\mathcal{E}$ is closed under $p$-direct sums, it follows that that $F_{\mathcal{E}}(G)$ is an $\mathcal{E}$-operator algebra by the results in [100]. This means that there exists a Banach space $E$ in $\mathcal{E}$ and an isometric representation $F_{\mathcal{E}}(G) \to \mathcal{B}(E)$. Moreover, $E$ can be chosen such that the density character of $E$ is dominated by that of $A$; see [100]. In particular, if $G$ is second countable, then $L^1(G)$ is a separable Banach algebra and consequently $F_{\mathcal{E}}(G)$ can be isometrically represented on a separable Banach space in $\mathcal{E}$.

(3) Group representations on $QSL^p$-spaces have been studied by Volker Runde in [242]. For a locally compact group $G$, he defined the algebra $\mathrm{UPF}_p(G)$ of *universal $p$-pseudofunctions* as follows. With $\pi\colon L^1(G) \to \mathcal{B}(E)$ denoting a contractive, nondegenerate representation on a $QSL^p$-space $E$ with the property that any other contractive, nondegenerate representation of $L^1(G)$ on a $QSL^p$-space is isometrically conjugate to a subrepresentation of $\pi$, the algebra $\mathrm{UPF}_p(G)$ is the closure of $\pi(L^1(G))$ in $\mathcal{B}(E)$ (Definition 6.1 in [242]).

It is straightforward to check that there is a canonical isometric isomorphism $\mathrm{UPF}_p(G) \cong F^p_{\mathrm{QS}}(G)$.

Note that $L^2$ is precisely the class of all Hilbert spaces. Since subspaces and quotients of Hilbert spaces are again Hilbert spaces, we have $QSL^2 = QL^2 = SL^2 = L^2$. This easily implies the following result.

**Lemma XIV.2.10.** Let $G$ be a locally compact group, and let $f \in L^1(G)$. Then

$$\|f\|_{QSL^2} = \|f\|_{QL^2} = \|f\|_{SL^2} = \|f\|_{L^2}.$$

It follows that there are canonical, isometric isomorphisms

$$F^2_{\mathrm{QS}}(G) \cong F^2_{\mathrm{Q}}(G) \cong F^2_{\mathrm{S}}(G) \cong F^2(G).$$

The algebra $F^2(G)$ is in fact a $C^*$-algebra, called the *full group $C^*$-algebra* of $G$, and it is usually denoted $C^*(G)$.

The universal completions from Notation XIV.3.10 also agree when $p = 1$, in which case they are all equal to $L^1(G)$:

**Proposition XIV.2.11.** Let $G$ be a locally compact group, and let $f \in L^1(G)$. Then

$$\|f\|_{QSL^1} = \|f\|_{QL^1} = \|f\|_{SL^1} = \|f\|_{L^1} = \|f\|_1.$$

It follows that there are canonical, isometric isomorphisms

$$F^1_{\mathrm{QS}}(G) \cong F^1_{\mathrm{Q}}(G) \cong F^1_{\mathrm{S}}(G) \cong F^1(G) \cong L^1(G).$$

*Proof.* Let us denote by $\lambda_1 \colon L^1(G) \to \mathcal{B}(L^1(G))$ the integrated form of the left regular representation. Then $\lambda_1$ is the action of $L^1(G)$ on itself by left convolution.

Given $f$ in $L^1(G)$, it is clear that $\|f\|_{L^1} \leq \|f\|_1$. For the reverse inequality, let $(e_d)_{d\in\Lambda}$ be a contractive approximate identity for $L^1(G)$. Then

$$\|\lambda_1(f)\| = \sup\left\{\tfrac{\|f*\xi\|_1}{\|\xi\|_1} \colon \xi \in L^1(G), \xi \neq 0\right\} \geq \sup_{d\in\Lambda} \|f * e_d\|_1 = \|f\|_1,$$

so $\|f\|_1 = \|\lambda_1(f)\| \leq \|f\|_{L^1} \leq \|f\|_1$. It follows that the norms $\|\cdot\|_1$ and $\|\cdot\|_{L^1}$ agree, and thus $F^1(G) = L^1(G)$.

Let $\mathcal{E}$ be any of the classes $QSL^1$, $SL^1$, or $QL^1$. It follows from the paragraph above that the composition $L^1(G) \to F_{\mathcal{E}}(G) \to F^1(G)$ equals the identity map on $L^1(G)$. The canonical map $F_{\mathcal{E}}(G) \to F^1(G)$ is therefore isometric, and the result follows. $\square$

We will later show that for most values of $p \in [1,\infty)$, the algebras $F^p_{\mathrm{S}}(G)$, $F^p_{\mathrm{Q}}(G)$, and $F^p(G)$ are canonically isometrically isomorphic, regardless of $G$; see Corollary XIV.2.27. When $G$ is amenable, we have $F^p_{\mathrm{QS}}(G) = F^p_{\mathrm{S}}(G) = F^p_{\mathrm{Q}}(G) = F^p(G)$, regardless of $p$; see Theorem XIV.3.7.

**Remark XIV.2.12.** The proof of Proposition XIV.2.11 also shows that the algebra of 1-pseudofunctions $F^1_\lambda(G)$ on $G$ (see Definition XIV.3.1) is canonically isometrically isomorphic to $L^1(G)$ as well. However, for $p > 1$, the analogous result holds if and only if $G$ is amenable; see Theorem XIV.3.7.

The following observation will allow us to define natural maps between the different universal completions.

**Remark XIV.2.13.** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be classes of Banach spaces, and suppose that $\mathcal{E}_1 \subseteq \mathcal{E}_2$. Then

$$\|f\|_{\mathcal{E}_1} \leq \|f\|_{\mathcal{E}_2}$$

for all $f \in L^1(G)$, and hence the identity map on $L^1(G)$ induces a canonical contractive homomorphism $F_{\mathcal{E}_2}(G) \to F_{\mathcal{E}_1}(G)$.

**Notation XIV.2.14.** Let $G$ be a locally compact group, and let $p \in [1,\infty)$. By Remark XIV.2.13, the inclusions $L^p \subseteq SL^p \subseteq QSL^p$ and $L^p \subseteq QL^p \subseteq QSL^p$ induce canonical contractive homomorphisms between the corresponding universal completions. We summarize the induced maps in the following commutative diagram:

$$\begin{tikzcd} & F^p_{\mathrm{S}}(G) \arrow[dr, "\kappa^p_{\mathrm{S}}"] & \\ F^p_{\mathrm{QS}}(G) \arrow[ur, "\kappa^p_{\mathrm{QS,S}}"] \arrow[dr, "\kappa^p_{\mathrm{QS,Q}}"'] & & F^p(G). \\ & F^p_{\mathrm{Q}}(G) \arrow[ur, "\kappa^p_{\mathrm{Q}}"'] & \end{tikzcd}$$

We write $\kappa^p_{\mathrm{QS}}$ for the composition $\kappa^p_{\mathrm{S}} \circ \kappa^p_{\mathrm{QS,S}}$, which also equals $\kappa^p_{\mathrm{Q}} \circ \kappa^p_{\mathrm{QS,Q}}$. Finally, when the Hölder exponent $p$ is clear from the context, we will drop it from the notation in the natural maps.

**Notation XIV.2.15.** Let $G$ be a locally compact group, and let $p, q \in [1, \infty)$. If $p \leq q \leq 2$, Proposition XIV.2.7 implies that there is a canonical contractive homomorphism

$$\kappa_{SL^p, L^q} \colon F^p_{\mathrm{S}}(G) \to F^q(G)$$

with dense range. Likewise, for $2 \leq r \leq s$, there is a canonical contractive homomorphism

$$\kappa_{QL^s, L^r} \colon F^s_{\mathrm{Q}}(G) \to F^r(G)$$

with dense range.

*Duality*

Recall that if $A$ is a complex algebra, its *opposite algebra*, denoted by $A^{op}$, is the complex algebra whose underlying vector space structure is the same as for $A$, and the product of two elements $a$ and $b$ in $A^{op}$ is equal to $ba$. When $A$ is moreover a Banach ($*$-)algebra, we take $A^{op}$ to carry the same norm (and involution) as $A$.

An *anti-homomorphism* $\varphi\colon A \to B$ between algebras $A$ and $B$ is a linear map satisfying $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a$ and $b$ in $A$. Equivalently, $\varphi$ is a homomorphism $A \to B^{op}$. Similar terminology and definitions apply to other objects such as topological groups.

**Lemma XIV.2.16.** Let $G$ be a locally compact group. Then there is a canonical isometric isomorphism $L^1(G^{op}) \cong L^1(G)^{op}$. Furthermore, if $\mathcal{E}$ is a class of Banach spaces, then there is a canonical isometric isomorphism $F_{\mathcal{E}}(G^{op}) \cong F_{\mathcal{E}}(G)^{op}$.

*Proof.* We fix a left Haar measure $\mu$ on $G$, and we denote by $\nu$ the right Haar measure on $G$ given by $\nu(E) = \mu(E^{-1})$ for every Borel set $E \subseteq G$. Then $\nu$ is a left Haar measure for $G^{op}$. It is immediate that inversion on $G$, when regarded as a map $(G, \mu) \to (G^{op}, \nu)$, is a measure-preserving group isomorphism. We denote by $*$ the operation of convolution on $L^1(G)$, and by $*_{op}$ the operation of convolution on $L^1(G^{op})$ (which is performed with respect to $\nu$). Similarly, we denote by $\cdot$ the product in $G$, and by $\cdot_{op}$ the product in $G^{op}$.

Given $f$ and $g$ in $L^1(G^{op})$, and given $s \in G^{op}$, we have

$$\begin{aligned}(f *_{op} g)(s) &= \int_G f(t) g(t^{-1} \cdot_{op} s)\, d\nu(t) \\ &= \int_G f(t) g(s \cdot t^{-1})\, d\nu(t) \\ &= \int_G f(t^{-1}) g(s \cdot t)\, d\mu(t) \\ &= (g * f)(s).\end{aligned}$$

It follows that the identity map on $L^1(G)$ induces a canonical isometric anti-isomorphism $L^1(G^{op}) \to L^1(G)$, and the result follows.

The last claim is straightforward. □

**Remark XIV.2.17.** We mention, without proof, two easy facts that will be used in the proof of Proposition XIV.2.18. First, if $\mathcal{E}$ is a class of Banach spaces, and $G$ and $H$ are locally compact groups, then any isomorphism $G \to H$ induces an isometric isomorphism $F_{\mathcal{E}}(H) \to F_{\mathcal{E}}(G)$. Second, with the same notation, inversion on $G$ defines an isomorphism $G \to G^{op}$, so there are canonical isomorphisms

$$F_{\mathcal{E}}(G) \cong F_{\mathcal{E}}(G^{op}) \cong F_{\mathcal{E}}(G)^{op},$$

where the second isomorphism is the one given by Lemma XIV.2.16.

**Proposition XIV.2.18.** Let $G$ be a locally compact group, let $p \in (1, \infty)$, and let $p'$ be its conjugate exponent. Then there are canonical isometric isomorphisms

$$F^p_{\mathrm{QS}}(G) \cong F^{p'}_{\mathrm{QS}}(G), \quad F^p_{\mathrm{Q}}(G) \cong F^{p'}_{\mathrm{S}}(G), \quad \text{and} \quad F^p(G) \cong F^{p'}(G).$$

*Proof.* Let $\mathcal{E}$ be any class of Banach spaces, and denote by $\mathcal{E}'$ the class of those Banach spaces that are dual to a Banach space in $\mathcal{E}$. For $\pi$ in $\mathrm{Rep}_{\mathcal{E}}(G)$ and $f$ in $L^1(G)$, we have

$$\|(\pi(f))'\| = \|\pi(f)\|.$$

Since taking adjoints reverses multiplication of operators, it follows that the identity map on $L^1(G)$ induces a canonical isometric isomorphism $F_{\mathcal{E}}(G) \cong F_{\mathcal{E}'}(G)^{op}$. Upon composing this isomorphism with the isomorphism $F_{\mathcal{E}'}(G) \cong F_{\mathcal{E}'}(G)^{op}$ described in Remark XIV.2.17, we obtain a canonical isometric isomorphism

$$F_{\mathcal{E}}(G) \cong F_{\mathcal{E}'}(G).$$

To finish the proof, it is enough to observe that for $p \in (1, \infty)$, there are natural identifications

$$(QSL^p)' = SQL^{p'} = QSL^{p'}, \ (QL^p)' = SL^{p'}, \ (SL^p)' = QL^{p'}, \ \text{ and } \ (L^p)' = L^{p'}.$$

$\square$

*Canonical maps $F^p(G) \to F^q(G)$*

In this subsection, we will construct, for any locally compact group $G$, a natural map $F^p(G) \to F^q(G)$ whenever $1 \leq p \leq q \leq 2$ or $2 \leq q \leq p < \infty$.

The construction of these maps takes considerable work, since $L^q$-spaces are never $L^p$-spaces, except in trivial cases. However, it is often the case that an $L^q$-space is a subspace of an $L^p$-space (see Proposition XIV.2.7). To take advantage of this fact, we need to study extensions of isometries from subspaces of $L^p$-spaces to $L^p$-spaces; see Theorem XIV.2.21. Our argument is based on ideas used by Hardin ([110]).

The following definition is due to Hardin.

**Definition XIV.2.19.** Let $(X, \mathfrak{A}, \mu)$ be a $\sigma$-finite measure space, let $p \in [1, \infty)$ and let $f_0 \in L^p(X, \mu)$. Define the *support* of $f_0$ to be

$$\mathrm{supp}(f_0) = \{x \in X \colon f_0(x) \neq 0\}.$$

Note that $\mathrm{supp}(f_0)$ is well-defined up to null sets. If $F$ is a closed subspace of $L^p(X, \mu)$, we say that $f_0$ has *full support in $F$* if

$$\mu\left(\mathrm{supp}(f) \setminus \mathrm{supp}(f_0)\right) = 0$$

for all $f \in F$.

The following terminology and notations will be convenient.

**Definition XIV.2.20.** Let $F$ be a Banach space. We denote by $\mathrm{Isom}(F)$ the group of surjective, linear isometries of $F$, and we equip it with the strong operator topology. In this topology, a net $(u_d)_{d\in\Lambda}$ in $\mathrm{Isom}(F)$ converges to $u \in \mathrm{Isom}(F)$ if and only if $\lim\limits_{d\in\Lambda} \|u_d(\xi) - u(\xi)\| = 0$ for every $\xi \in F$. We call $\mathrm{Isom}(F)$ the *isometry group* of $F$.

If $F$ is a closed subspace of another Banach space $\widetilde{F}$, we let $\mathrm{Isom}(\widetilde{F}, F)$ denote the subgroup of $\mathrm{Isom}(\widetilde{F})$ consisting of those isometries that leave $F$ invariant.

The next result asserts that if $p$ is not a multiple of 2 greater than 2, then for every separable $SL^p$-space $F$, there exists an $L^p$-space $\widetilde{F}$ containing $F$, such that every invertible isometry on $F$ can be extended to an invertible isometry on $\widetilde{F}$. Note that this is stronger than the statement that every isometry on a separable $SL^p$-space can be extended to an isometry on *some* $L^p$-space.

**Theorem XIV.2.21.** Let $p \in [1, \infty) \setminus \{4, 6, \ldots\}$, and let $F$ be a separable $SL^p$-space. Then there exists a separable $L^p$-space $\widetilde{F}$ such that $F$ is isometrically isomorphic to a subspace of $\widetilde{F}$, and such that every surjective, linear isometry on $F$ can be extended to a surjective, linear isometry on $\widetilde{F}$. Moreover, the restriction map

$$\varphi\colon \mathrm{Isom}(\widetilde{F}, F) \to \mathrm{Isom}(F),$$

is a surjective homeomorphism.

*Proof.* The statement is trivial for $p = 2$, since a closed subspace of a Hilbert space is a Hilbert space itself. We may therefore assume that $p \in [1, \infty) \setminus \{2, 4, 6, \ldots\}$.

The proof in this case is based on the proof of Theorem 4.2 in [110], and we use the same notation when possible. Since $\sigma$-algebras will play an important role, they will not be omitted from the notation.

Let $(X, \mathfrak{A}, \mu)$ be a $\sigma$-finite measure space such that $F$ can be identified with a closed subspace of $L^p(X, \mathfrak{A}, \mu)$. We let $\mathrm{Full}(F)$ denote the set of elements in $F$ that have full support in $F$. It follows from Lemma 3.2 in [110] that $\mathrm{Full}(F)$ is nonempty. Without loss of generality, we may assume that $\mathrm{supp}(f_0) = X$ for every $f_0 \in \mathrm{Full}(F)$.

Let $f_0 \in \mathrm{Full}(F)$. Set

$$Q(f_0) = \left\{ \tfrac{f}{f_0} \colon f \in F \right\},$$

and let $\sigma(Q(f_0))$ denote the smallest $\sigma$-algebra on $X$ such that every function in $Q(f_0)$ is measurable with respect to $\sigma(Q(f_0))$. Hardin showed in the proof of Lemma 3.4 in [110] that $\sigma(Q(f_0))$ does not depend on the choice of $f_0$ in $\mathrm{Full}(F)$. We will therefore write $\mathfrak{B}$ for $\sigma(Q(f_0))$, which equals $\sigma(Q(g_0))$ for any other $g_0 \in \mathrm{Full}(F)$. Since every element of $Q(f_0)$ is $\mathfrak{A}$-measurable, we clearly have $\mathfrak{B} \subseteq \mathfrak{A}$.

It is not in general the case that the elements of $Q(f_0)$ are in $L^p(X, \mathfrak{B}, \mu)$. Instead, we shall consider the measure $|f_0|^p\mu$ on $(X, \mathfrak{B})$. If $f \in F$, then

$$\|f\|_p = \left(\int_X |f|^p \; d\mu\right)^{\frac{1}{p}} = \left(\int_X \left|\frac{f}{f_0}\right|^p |f_0|^p \; d\mu\right)^{\frac{1}{p}} < \infty.$$

It follows that $Q(f_0)$ is a subset of $L^p\left(X, \mathfrak{B}, |f_0|^p\mu\right)$, and that the map

$$D_{f_0} \colon F \to L^p\left(X, \mathfrak{B}, |f_0|^p\mu\right)$$

given by $D_{f_0}(f) = \frac{f}{f_0}$ for $f \in F$, is an isometry. Set

$$\widetilde{F} = \left\{f_0 h \colon h \in L^p\left(X, \mathfrak{B}, |f_0|^p\mu\right)\right\}.$$

By the proof of Theorem 4.2 in [110], the space $\widetilde{F}$ does not depend on the choice of the element $f_0$ of full support. Moreover, $F$ is a subspace of $\widetilde{F}$ and $D_{f_0}$ extends to an isometric isomorphism $\widetilde{F} \to L^p\left(X, \mathfrak{B}, |f_0|^p\mu\right)$, which we also denote by $D_{f_0}$.

Let $u \in \mathrm{Isom}(F)$. We claim that $u$ can be canonically extended to a surjective, linear isometry $\widetilde{u}$ on $\widetilde{F}$.

Set $g_0 = u(f_0)$, which belongs to $\mathrm{Full}(F)$ by Lemma 3.4 in [110]. Then $u$ induces a linear isometry $v$ from the subspace $Q(f_0)$ of $L^p\left(X, \mathfrak{B}, |f_0|^p\mu\right)$ to the subspace $Q(g_0)$ of $L^p\left(X, \mathfrak{B}, |g_0|^p\mu\right)$. The maps to be constructed are shown in the following commutative diagram:

$$
\begin{tikzcd}
F \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, "u"'] & & Q(f_0) \arrow[dr, hook] \arrow[dd, "v"'] & \\
& \widetilde{F} \arrow[rr, "D_{f_0}", "\cong"'] \arrow[dd, dashed, "\widetilde{u}"'] & & L^p(\mathfrak{B}, |f_0|^p\mu) \arrow[dd, dashed, "\widetilde{v}"] \\
F \arrow[rr, hook] \arrow[dr, hook] & & Q(g_0) \arrow[dr, hook] & \\
& \widetilde{F} \arrow[rr, "D_{g_0}", "\cong"'] & & L^p(\mathfrak{B}, |g_0|^p\mu)
\end{tikzcd}
$$

The constant function $1 = \frac{f_0}{f_0}$ belongs to $Q(f_0)$, and also to $Q(g_0)$. It is easy to see that $v$ satisfies $v(1) = 1$. Apply Theorem 2.2 in [110] to extend $v$ uniquely to a surjective, linear isometry $\widetilde{v}$ on $\widetilde{F}$. The desired isometry $\widetilde{u}$ is then given by

$$\widetilde{u} = D_{g_0}^{-1} \circ \widetilde{v} \circ D_{f_0}.$$

The claim is proved.

The assignment $u \mapsto \widetilde{u}$ defines an inverse to the restriction map $\varphi$, so $\varphi$ is surjective. It is clear that $\varphi$ is a group homomorphism, and it is easy to see that $\varphi$ is continuous. It remains to show that $\varphi^{-1}$ is continuous.

We first describe the elements in $\widetilde{F}$. Let $n \in \mathbb{N}$ and let $\xi\colon \mathbb{C}^n \to \mathbb{C}$ be a measurable map. If $f_1, \ldots, f_n$ are elements of $F$, then the function

$$h = \xi\left(\tfrac{f_1}{f_0}, \ldots, \tfrac{f_n}{f_0}\right)$$

is $\mathfrak{B}$-measurable. Therefore, $f_0 h$ is an element of $\widetilde{F}$ whenever

$$\int |f_0(x)h(x)|^p \, d\mu(x) < \infty.$$

It is not hard to check that elements of this form are dense in $\widetilde{F}$. Finally, if $u \in \mathrm{Isom}(F)$, then its unique extension $\widetilde{u}$ to $\widetilde{F}$ satisfies

$$\widetilde{u}(f_0 h) = u(f_0)\xi\left(\tfrac{u(f_1)}{u(f_0)}, \ldots, \tfrac{u(f_n)}{u(f_0)}\right).$$

It follows that the assignment $u \mapsto \widetilde{u}(f_0 h)$, as a map $\mathrm{Isom}(F) \to \widetilde{F}$, is measurable, and hence $\varphi^{-1}$ is measurable.

Note that $\mathrm{Isom}(F)$ and $\mathrm{Isom}(\widetilde{F})$ are Polish groups, since $F$ and $\widetilde{F}$ are separable Banach spaces. It follows that $\mathrm{Isom}(\widetilde{F}, F)$ is a Polish group, being a closed subgroup of $\mathrm{Isom}(\widetilde{F})$. Since $\varphi^{-1}$ is a measurable group homomorphism between Polish groups, it follows that $\varphi^{-1}$ is continuous (see, for example, Theorem 9.10 in [147]). The proof is finished. □

**Corollary XIV.2.22.** Let $p \in [1, \infty) \setminus \{4, 6, \ldots\}$. Then, the isometry group of every separable $SL^p$-space is topologically isomorphic to a closed subgroup of the isometry group of a separable $L^p$-space.

**Question XIV.2.23.** Does Corollary XIV.2.22 also hold in the nonseparable case? That is, is the the isometry group of every $SL^p$-space topologically isomorphic to a closed subgroup of the isometry group of an $L^p$-space?

We now turn to the construction of the maps $F^p(G) \to F^q(G)$. When $G$ is second countable, their existence follows easily from the next corollary. For general $G$, however, said corollary will be used as the base case of an induction argument, the inductive step being Lemma XIV.2.25.

**Corollary XIV.2.24.** Let $p \in [1, \infty) \setminus \{4, 6, \ldots\}$, let $G$ be a second countable, locally compact group, and let $f \in L^1(G)$. Then $\|f\|_{SL^p} = \|f\|_{L^p}$.

*Proof.* Since $L^p \subseteq SL^p$, we have $\|f\|_{L^p} \leq \|f\|_{SL^p}$; see also Remark XIV.2.13. By (2) in Remarks XIV.2.9, there exist a separable $SL^p$-space $E$ and an isometric group representation $\rho\colon G \to \mathrm{Isom}(E)$ such that, with $\pi\colon L^1(G) \to \mathcal{B}(E)$ denoting the integrated form of $\rho$, we have

$$\|f\|_{SL^p} = \|\pi(f)\|.$$

By Theorem XIV.2.21, there exist an $L^p$-space $\widetilde{E}$ containing $E$ as a closed subspace, and a topological group isomorphism

$$\varphi\colon \mathrm{Isom}(E) \to \mathrm{Isom}(\widetilde{E}, E).$$

Consider the isometric group representation of $G$ on $\widetilde{E}$ given by

$$\widetilde{\rho} = \varphi \circ \rho \colon G \to \mathrm{Isom}(\widetilde{E}, E) \subseteq \mathrm{Isom}(\widetilde{E}).$$

Let $\widetilde{\pi} \colon L^1(G) \to \mathcal{B}(\widetilde{E})$ be the integrated form of $\widetilde{\rho}$.

It is easy to see, using that $\widetilde{\rho}(s)$ leaves $E$ invariant for every $s \in G$, that $\widetilde{\pi}(f)$ leaves $E$ invariant as well. Using this at the second step, we get

$$\begin{aligned}\|\pi(f)\| &= \sup\left\{\|\pi(f)\xi\|_p \colon \xi \in E, \|\xi\|_p \leq 1\right\} \\ &\leq \sup\left\{\|\widetilde{\pi}(f)\xi\|_p \colon \xi \in \widetilde{E}, \|\xi\|_p \leq 1\right\} \\ &= \|\widetilde{\pi}(f)\|.\end{aligned}$$

Therefore,

$$\|f\|_{SL^p} = \|\pi(f)\| \leq \|\widetilde{\pi}(f)\| \leq \sup\left\{\|\tau(f)\| \colon \tau \in \mathrm{Rep}_G(L^p)\right\} = \|f\|_{L^p},$$

as desired. $\square$

**Lemma XIV.2.25.** Let $p \in [1,\infty) \setminus \{4,6,\ldots\}$, let $G$ be a $\sigma$-compact, locally compact group, and let $f \in L^1(G)$. Then $\|f\|_{SL^p} = \|f\|_{L^p}$.

*Proof.* We may assume that $\|f\|_1 \leq 1$. As in the proof of Corollary XIV.2.24, we have $\|f\|_{L^p} \leq \|f\|_{SL^p}$. Let us show the reverse inequality. By (2) in Remarks XIV.2.9, there exists a $SL^p$-space $E$ and an isometric group representation $\rho \colon G \to \mathrm{Isom}(E)$ whose integrated form $\pi \colon L^1(G) \to \mathcal{B}(E)$ satisfies

$$\|f\|_{SL^p} = \|\pi(f)\|.$$

Let $\varepsilon > 0$. Choose $\xi_0$ in $E$ with $\|\xi_0\| = 1$ such that

$$\|\pi(f)\xi_0\| \geq \|\pi(f)\| - \tfrac{\varepsilon}{2}. \tag{XIV.1}$$

Since $\rho$ is continuous, there exists an open neighborhood $V$ of the identity element $e$ in $G$ such that $\|\rho(s) - \mathrm{id}_E\| < \frac{\varepsilon}{2}$ for every $s \in V$. By the Kakutani-Kodaira Theorem (see e.g. [119, Theorem 8.7, p.71]), there exists a compact, normal subgroup $N$ of $G$ such that $N \subseteq V$ and such that $G/N$ is second countable.

We consider the fixed point subspace of $E$ for the (restricted) action of $N$ on $E$:

$$E^N = \{\xi \in E \colon \rho(s)\xi = \xi \text{ for all } s \in N\}.$$

With $\mu$ denoting the normalized Haar measure on $N$, define an averaging map $P_N \colon E \to E^N$ by

$$P_N(\xi) = \int_N \rho(s)\xi \ d\mu(s)$$

for all $\xi \in E$. Then $P_N$ is contractive and linear.

For every $\eta$ in $E$ with $\|\eta\| \leq 1$, we have

$$\|P_N(\eta) - \eta\| = \left\| \int_N (\rho(s)\eta - \eta) \ \ d\mu(s) \right\| \leq \int_N \|\rho(s) - \mathrm{id}_E\| \cdot \|\eta\| \ d\mu(s) < \tfrac{\varepsilon}{2}.$$

Since $\|\pi(f)\xi_0\| \leq 1$, we deduce that

$$\|P_N(\pi(f)\xi_0)\| \geq \|\pi(f)\xi_0\| - \frac{\varepsilon}{2}. \tag{XIV.2}$$

The isometric representation $\rho \colon G \to \mathrm{Isom}(E)$ induces an isometric representation $\rho_N \colon G/N \to \mathrm{Isom}(E^N)$ given by

$$\rho(sN)\xi = \rho(s)\xi$$

for all $s \in G$ and $\xi \in E$. Let $\pi_N \colon L^1(G/N) \to \mathcal{B}(E^N)$ denote the integrated form of $\rho_N$. Since $E^N$ is a closed subspace of $E$, it follows that $E^N$ is a $SL^p$-space.

The map $P_N$ induces a linear map $Q_N \colon \mathcal{B}(E) \to \mathcal{B}(E^N)$, which sends an operator $a \in \mathcal{B}(E)$ to the operator $Q_N(a) \colon E^N \to E^N$ given by

$$Q_N(a)\xi = P_N(a\xi)$$

for $\xi \in E^N \subseteq E$.

Consider the map $T_N \colon L^1(G) \to L^1(G/N)$ given by

$$T_N(g)(sN) = \int_N g(sn) \ d\mu(n).$$

for $s \in G$. It is well-known that $T_N$ is a contractive homomorphism; see [224, Theorem 3.5.4, p.106]. It is straightforward to check that the following diagram is commutative:

$$\begin{tikzcd} L^1(G) \arrow[r, "\pi"] \arrow[d, "T_N"'] & \mathcal{B}(E) \arrow[d, "Q_N"] \\ L^1(G/N) \arrow[r, "\pi_N"'] & \mathcal{B}(E^N). \end{tikzcd}$$

In particular,

$$\pi_N(T_N(f))P_N(\xi) = P_N(\pi(f)\xi)$$

for all $\xi \in E$.

Using that $\|P_N(\xi_0)\| \leq 1$ at the second step, using (XIV.2) at the fourth step, and using (XIV.1) at the last step, we conclude that

$$\begin{aligned} \|T_N(f)\|_{SL^p} &\geq \|\pi_N(T_N(f))\| \\ &\geq \|\pi_N(T_N(f))P_N(\xi_0)\| \\ &= \|P_N(\pi(f)\xi_0)\| \\ &\geq \|\pi(f)\xi_0\| - \tfrac{\varepsilon}{2} \\ &\geq \|\pi(f)\| - \varepsilon. \end{aligned}$$

Since $G/N$ is second countable, we may apply Corollary XIV.2.24 at the second step to obtain

$$\|f\|_{L^p} \geq \|T_N(f)\|_{L^p} = \|T_N(f)\|_{SL^p} \geq \|\pi(f)\| - \varepsilon = \|f\|_{SL^p} - \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have shown that $\|f\|_{SL^p} \leq \|f\|_{L^p}$, as desired. $\square$

**Theorem XIV.2.26.** Let $p \in [1, \infty) \setminus \{4, 6, \ldots\}$, let $G$ be a locally compact group, and let $f \in L^1(G)$. Then $\|f\|_{SL^p} = \|f\|_{L^p}$.

*Proof.* It is enough to prove the statement under the additional assumption that $f$ belongs to $C_c(G)$, the algebra of continuous functions $G \to \mathbb{C}$ that have compact support. In this case, there exists an open (and hence closed) subgroup $H$ of $G$ such that $H$ is $\sigma$-compact and such that the support of $f$ is contained in $H$.

Given a function $g$ in $L^1(H)$, we extend $g$ to a function in $L^1(G)$, also denoted by $g$, by setting $g(s) = 0$ for all $s \in G \setminus H$. This defines an isometric homomorphism $\iota\colon L^1(H) \to L^1(G)$. Considering the universal completions for representations on $SL^p$- and $L^p$-spaces, we obtain two contractive homomorphisms

$$\alpha_{\mathrm{S}}\colon F^p_{\mathrm{S}}(H) \to F^p_{\mathrm{S}}(G), \quad \alpha\colon F^p(H) \to F^p(G).$$

Together with the maps $\kappa_{\mathrm{S}}^H$ and $\kappa_{\mathrm{S}}^G$ from Notation XIV.2.14, we obtain the following commutative diagram:

$$\begin{tikzcd}
F^p_{\mathrm{S}}(H) \arrow[r, "\kappa_{\mathrm{S}}^H"] \arrow[d, "\alpha_{\mathrm{S}}"'] & F^p(H) \arrow[d, "\alpha"] \\
F^p_{\mathrm{S}}(G) \arrow[r, "\kappa_{\mathrm{S}}^G"'] & F^p(G).
\end{tikzcd}$$

We can consider $f$ as a function in $L^1(H)$ or in $L^1(G)$, and we will denote the norms in the corresponding universal completions by $\|f\|^H_{L^p}$, $\|f\|^H_{SL^p}$, $\|f\|^G_{L^p}$ and $\|f\|^G_{SL^p}$.

As in the proof of Corollary XIV.2.24, we have $\|f\|^G_{L^p} \leq \|f\|^G_{SL^p}$. Using that $\alpha_S$ is contractive at the first step, and applying Lemma XIV.2.25 for $H$ at the second step, we deduce that

$$\|f\|^G_{SL^p} \leq \|f\|^H_{SL^p} = \|f\|^H_{L^p}.$$

Thus, in order to obtain the desired inequality $\|f\|^G_{SL^p} \leq \|f\|^G_{L^p}$, it is enough to show that

$$\|f\|^H_{L^p} \leq \|f\|^G_{L^p}.$$

By (2) in Remarks XIV.2.9, there exist a $L^p$-space $E$ and an isometric group representation $\rho\colon H \to \mathrm{Isom}(E)$ whose integrated form $\pi\colon L^1(H) \to \mathcal{B}(E)$ satisfies

$$\|f\|^H_{L^p} = \|\pi(f)\|.$$

We can induce $\rho$ to an isometric representation of $G$ as follows: Consider the space

$$\operatorname{Ind}_H^G(E) = \left\{ \omega \in \ell^\infty(G,E) \ : \ \begin{array}{c} \rho(h)(\omega(sh)) = \omega(s) \text{ for all } s \in G \text{ and } h \in H \\ \text{and } (sH \mapsto \|\omega(s)\|) \text{ is in } \ell^p(G/H) \end{array} \right\},$$

with the norm of an element $\omega \in \operatorname{Ind}_H^G(E)$ given by

$$\|\omega\| = \left( \sum_{sH \in g/H} \|\omega(s)\|^p \right)^{1/p}.$$

(The covariance condition $\rho(h)(\omega(sh)) = \omega(s)$ ensures that for each $s$ in $S$ the norm $\|\omega(s)\|$ depends only on the class of $s$ in $G/H$.) Since $G/H$ is discrete, we can choose a section $\sigma\colon G/H \to G$. By assigning to an element $\omega \in \operatorname{Ind}_H^G(E) \subseteq \ell^\infty(G,E)$ the function $\omega\circ\sigma\colon G/H \to E$, we obtain an isometric isomorphism

$$\operatorname{Ind}_H^G(E) \xrightarrow{\cong} \ell^P(G/H,E) \cong \bigoplus_{G/H}^{p} E,$$

which shows that $\operatorname{Ind}_H^G(E)$ is an $L^p$-space.

The induced representation $\widetilde{\rho} = \operatorname{Ind}_H^G(\rho)\colon G \to \operatorname{Isom}(\operatorname{Ind}_H^g(E))$ is given by

$$(\widetilde{\rho}(s)\omega)(t) = \omega(s^{-1}t),$$

for $\omega \in \widetilde{E}$, and $s,t \in G$. We let $\widetilde{\pi}\colon L^1(G) \to \mathcal{B}(\operatorname{Ind}_H^G(E))$ denote the integrated form of $\widetilde{\rho}$.

Consider the map $\varepsilon\colon E \to \operatorname{Ind}_H^G(E)$ given by

$$\varepsilon(\xi)(s) = \begin{cases} \rho(s^{-1})\xi, & s \in H \\ 0, & s \notin H \end{cases}$$

for $\xi \in E$. Let $e$ denote the unit element in $G$, and consider the evaluation map $\mathrm{ev}_e\colon \operatorname{Ind}_H^G(E) \to E$. We have that $\varepsilon$ and $\mathrm{ev}_e$ are linear and contractive, and that $\mathrm{ev}_e \circ \varepsilon = \mathrm{id}_E$. It follows in particular that $\varepsilon$ defines an isometric embedding of $E$ into $\operatorname{Ind}_H^G(E)$. We can use $\varepsilon$ and $\mathrm{ev}_e$ to define an isometric homomorphism $Q\colon \mathcal{B}(E) \to \mathcal{B}(\mathrm{Ind}_H^G(E))$ by

$$Q(a)\omega = \varepsilon(a\mathrm{ev}_e(\omega)) = \varepsilon(a\omega(e))$$

for $a \in \mathcal{B}(E)$ and $\omega \in \mathrm{Ind}_H^G(E)$. It is then straightforward to check that the following diagram is commutative:

$$\begin{tikzcd} L^1(H) \arrow[r, "\pi"] \arrow[d, "\iota"] & \mathcal{B}(E) \arrow[d, "Q"] \\ L^1(G) \arrow[r, "\widetilde{\pi}"] & \mathcal{B}(\widetilde{E}). \end{tikzcd}$$

We a slight abuse of notation, we write $f = \iota(f)$. We conclude that

$$\|f\|_{L^p}^G \leq \|\widetilde{\pi}(f)\| = \|Q(\pi(f))\| = \|\pi(f)\| = \|f\|_{L^p}^H,$$

which is the desired inequality. □

The above theorem can be used to show that, for some values of $p \in [1, \infty)$, the canonical maps between certain universal completions are always isometric.

**Corollary XIV.2.27.** Let $p \in [1, \infty)$ and let $G$ be a locally compact group.

1. If $p \notin \{4, 6, 8, \ldots\}$, then the canonical map

$$\kappa_{\mathrm{S}}\colon F_{\mathrm{S}}^p(G) \to F^p(G),$$

   is an isometric isomorphism.

2. If $p \notin \{\frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \ldots\}$, then the canonical map

$$\kappa_{\mathrm{Q}}\colon F_{\mathrm{Q}}^p(G) \to F^p(G),$$

   is an isometric isomorphism.

*Proof.* The first statement is an immediate consequence of Theorem XIV.2.26. The second claim follows from part (1), using duality. We omit the details. □

**Remark XIV.2.28.** It is known that the conclusion of Theorem XIV.2.21 fails for $p \in \{4, 6, 8, \ldots\}$. Indeed, in [178], Lusky shows that the Hardin's extension theorem (Theorem 4.2 in [110]) fails for all even integers greater than 2, providing concrete counterexamples that are based on computations of Rudin in Example 3.6 of [240]. However, we do not know whether the restrictions on the Hölder exponent are necessary in Theorem XIV.2.26 and Corollary XIV.2.27.

Corollary XIV.2.27 suggests the following:

**Question XIV.2.29.** Let $G$ be a locally compact group and let

$$p \in [1, \infty) \setminus \left\{2n, \frac{2n}{2n-1} : n \geq 2\right\}.$$

Then there are canonical isometric isomorphisms $F^p(G) \cong F^p_{\mathrm{S}}(G) \cong F^p_{\mathrm{Q}}(G)$. Is the canonical map $F^p_{\mathrm{QS}}(G) \to F^p(G)$ an isometric isomorphism?

Again, the answer to the above question is yes if $p = 1, 2$, and also if $G$ is amenable (for arbitrary $p$). One would hope to be able to combine the facts that $F^p(G) \cong F^p_{\mathrm{S}}(G)$ and $F^p(G) \cong F^p_{\mathrm{Q}}(G)$ to say something about $F^p_{\mathrm{QS}}(G)$ in relation to $F^p(G)$, but this is not clear. Question XIV.2.29 may well have a negative answer.

The following is the main result of this subsection.

**Theorem XIV.2.30.** Let $G$ be a locally compact group.

1. If $1 \leq p \leq q \leq 2$, then the identity map on $L^1(G)$ extends to a contractive homomorphism

$$\gamma_{p,q} \colon F^p(G) \to F^q(G)$$

with dense range.

2. If $2 \leq r \leq s$, then the identity map on $L^1(G)$ extends to a contractive homomorphism

$$\gamma_{s,r} \colon F^s(G) \to F^r(G)$$

with dense range.

Moreover, the following diagram is commutative:

$$\begin{tikzcd}
F^p_{\mathrm{S}}(G) \arrow[d, "\kappa^p_{\mathrm{S}}", "\cong"'] \arrow[dr] & F^q_{\mathrm{S}}(G) \arrow[d, "\kappa^q_{\mathrm{S}}", "\cong"'] \arrow[dr] & F^2_{\mathrm{S}}(G) \arrow[d, equal] & & \\
F^p(G) \arrow[r, dashed, "\gamma_{p,q}"'] & F^q(G) \arrow[r, dashed, "\gamma_{q,2}"'] & C^*(G) \arrow[d, equal] & F^r(G) \arrow[l, dashed, "\gamma_{r,2}"'] & F^s(G) \arrow[l, dashed, "\gamma_{s,r}"'] \\
 & & F^2_{\mathrm{Q}}(G) & F^r_{\mathrm{Q}}(G) \arrow[u, "\kappa^r_{\mathrm{Q}}", "\cong"'] \arrow[ul] & F^s_{\mathrm{Q}}(G) \arrow[u, "\kappa^s_{\mathrm{Q}}", "\cong"'] \arrow[ul]
\end{tikzcd}$$

*Proof.* We only prove the first part, since the second one follows from the first using duality. Let $f \in L^1(G)$. We use Proposition XIV.2.7 at the first step and Corollary XIV.2.27 at the second to get

$$\|f\|_{L^q} \leq \|f\|_{SL^p} = \|f\|_{L^p}.$$

We conclude that the identity map on $L^1(G)$ extends to a contractive homomorphism $\gamma_{p,q}\colon F^p(G) \to F^q(G)$ with dense range. Commutativity of the diagram depicted in the statement follows from the fact that all the maps involved are the identity on the respective copies of $L^1(G)$. $\square$

We point out that the statement analogous to Theorem XIV.2.30 is in general false for etale groupoids. (Etale groupoid $L^p$-operator algebras are introduced and studied in Chapter XX.) Indeed, the analogs of Cuntz algebras $\mathcal{O}^p_n$ on $L^p$-spaces (introduced by Phillips in [204]), are groupoid $L^p$-operator algebras by Theorem XX.7.7. On the other hand, if $p$ and $q$ are different Hölder exponents and $n \geq 2$ is an integer, then there is no non-zero continuous homomorphism $\mathcal{O}^p_n \to \mathcal{O}^q_n$ by Corollary 9.3 in [204] (see also the comments after it), which rules out any reasonable generalization of Theorem XIV.2.30 to groupoids. In particular, there seems to be no analog of Hardin's results (or our Theorem XIV.2.21) for groupoids.

### Algebras of $p$-pseudofunctions and Amenability

In this section, we recall the construction of the algebra $F^p_\lambda(G)$ of $p$-pseudofunctions on a locally compact group $G$, for $p \in [1, \infty)$, as introduced by Herz in [116]. (Our notation differs from the one used by Herz.) There are natural contractive homomorphisms with dense range from any of the universal completions studied in the previous section, to the algebra of $p$-pseudofunctions.

In Theorem XIV.3.7, we characterize amenability of a locally compact group $G$ in terms of these maps. As an application, we show in Corollary XIV.3.20 that for an amenable group $G$, and for $1 \leq p \leq q \leq 2$ or for $2 \leq q \leq p < \infty$, the canonical map $\gamma_{p,q} \colon F^p(G) \to F^q(G)$ constructed in Theorem XIV.2.30 is always injective, and that it is surjective if and only if $G$ is finite.

*Algebras of p-pseudofunctions*

We denote by $p$ a fixed Hölder exponent in $[1, \infty)$. For a locally compact group $G$, let $\lambda_p \colon L^1(G) \to \mathcal{B}(L^p(G))$ denote the (integrated form of the) left regular representation of $G$ on $L^p(G)$. For $f$ in $L^1(G)$, we have $\lambda_p(f)\xi = f * \xi$ for all $\xi \in L^p(G)$.

**Definition XIV.3.1.** Let $G$ be a locally compact group. The *algebra of p-pseudofunctions* on $G$, here denoted by $F^p_\lambda(G)$, is the completion of $L^1(G)$ with respect to the norm induced by the left regular representation of $L^1(G)$ on $L^p(G)$.

**Remark XIV.3.2.** Let $G$ be a locally compact group. The left regular representation $\lambda_p \colon L^1(G) \to \mathcal{B}(L^p(G))$ induces an isometric embedding $F^p_\lambda(G) \to \mathcal{B}(L^p(G))$ under which we regard $F^p_\lambda(G)$ as represented on $L^p(G)$. In particular, $F^p_\lambda(G)$ is an $L^p$-operator algebra.

In the literature, the elements of $F^p_\lambda(G)$ have been called *p-pseudofunctions*, and the Banach algebra $F^p_\lambda(G)$ is usually denoted by $\mathrm{PF}_p(G)$. This terminology goes back to Herz; see Section 8 in [116]. (We are thankful to Y. Choi and M. Daws for providing this reference.) Our notation follows one of the standard conventions in $C^*$-algebra theory ([24]). We also warn the reader that $F^p_\lambda(G)$ has also been called the *reduced group $L^p$-operator algebra* of $G$, and is sometimes denoted $F^p_{\mathrm{r}}(G)$, for example in [207].

It is immediate to check that when $p = 2$, the algebra $F^2_\lambda(G)$ agrees with the *reduced group $C^*$-algebra* of $G$, which is usually denoted $C^*_\lambda(G)$.

The proof of the following proposition is straightforward and is left to the reader.

**Proposition XIV.3.3.** Let $G$ be a locally compact group. The left regular representation of $L^1(G)$ on $L^p(G)$ is a representation in $\mathrm{Rep}_G(L^p)$. Therefore, the identity map on $L^1(G)$ induces a canonical contractive homomorphism

$$\kappa \colon F^p(G) \to F^p_\lambda(G)$$

with dense range.

We now turn to duality. Let $G$ be a locally compact group, and denote by $\Delta\colon G\to\mathbb{R}$ its modular function. For $f\in L^1(G)$, let $f^\sharp\colon G\to\mathbb{C}$ be given by $f^\sharp(s)=\Delta(s^{-1})f(s^{-1})$ for all $s$ in $G$.

**Remark XIV.3.4.** For $f$ and $g$ in $L^1(G)$, it is straightforward to check that

1. $(f^\sharp)^\sharp=f$;
2. $(f*g)^\sharp=g^\sharp*f^\sharp$; and
3. $\|f^\sharp\|_1=\|f\|_1$ for all $f$ in $L^1(G)$.

In other words, the map $f\mapsto f^\sharp$ defines an isometric anti-automorphism of $L^1(G)$. It is also immediate to check that if $G$ is unimodular, then $f\mapsto f^\sharp$ also defines an isometric anti-automorphism of $L^p(G)$ for every $p\in[1,\infty]$.

Let $p\in(1,\infty)$, and denote by $p'$ its conjugate exponent. Consider the bilinear paring $L^p(G)\times L^{p'}(G)\to\mathbb{C}$ given by

$$\langle\xi,\eta\rangle=\int_G\xi(s)\eta(s)ds$$

for all $\xi\in L^p(G)$ and all $\eta\in L^{p'}(G)$. It is a standard fact that

$$\|\xi\|_p=\sup\left\{|\langle\xi,\eta\rangle|:\eta\in L^{p'}(G),\|\eta\|_{p'}=1\right\}$$

for every $\xi\in L^p(G)$.

**Proposition XIV.3.5.** Let $G$ be a locally compact group, let $p\in(1,\infty)$, and let $p'$ be its conjugate exponent. Then there is a canonical isometric anti-isomorphism $F^p_\lambda(G)\cong F^{p'}_\lambda(G)$, which is induced by the map $\sharp\colon L^1(G)\to L^1(G)$.

*Proof.* Given $f$ in $L^1(G)$, we claim that $\lambda_p(f^\sharp)' = \lambda_{p'}(f)$. Fix $\xi \in L^p(G)$ and $\eta \in L^{p'}(G)$. Then

$$\begin{aligned}
\langle f^\sharp * \xi, \eta\rangle &= \int_G (f^\sharp * \xi)(s)\eta(s)ds \\
&= \int_G \left(\int_G \Delta(t^{-1}) f^\sharp(st^{-1})\xi(t)dt\right) \eta(s)ds \\
&= \int_G \int_G \Delta(t^{-1})\Delta(ts^{-1}) f(ts^{-1})\xi(t)\eta(s)dtds \\
&= \int_G \left(\int_G \Delta(s^{-1}) f(ts^{-1})\eta(s)ds\right) \xi(t)dt \\
&= \int_G (f * \eta)(t)\xi(t)dt \\
&= \langle \xi, f * \eta\rangle,
\end{aligned}$$

so the claim follows.

It follows that $\|\lambda_p(f^\sharp)\| = \|\lambda_{p'}(f)\|$, and hence the map $\sharp\colon L^1(G) \to L^1(G)$ induces a canonical isometric anti-isomorphism $F^p_\lambda(G) \to F^{p'}_\lambda(G)$, as desired. □

With the notation of the proposition above, we point out that when $p \neq 2$, we do not seem to get the existence of a canonical isometric isomorphism $F^p_\lambda(G) \cong F^{p'}_\lambda(G)$, since $\|\lambda_p(f)\|$ and $\|\lambda_p(f^\sharp)\|$ are not in general equal, unless $G$ is abelian (see Proposition XIV.3.22). In fact, Herz proved in Corollary 1 of [117], that for *every* finite non-abelian group $G$, and for *every* $p \in (1, \infty) \setminus \{2\}$, there exists $f \in \ell^1(G)$ such that $\|\lambda_p(f)\|_p \neq \|\lambda_{p'}(f)\|_{p'}$.

*Group and Banach algebra amenability*

Let us recall some facts from functional analysis. If $E$ is a Banach space and $\xi \in E$, then

$$\|\xi\| = \sup\{|f(\xi)|\colon f \in E', \|f\| = 1\},$$

This can be used to easily prove the following result.

**Lemma XIV.3.6.** Let $E$ and $F$ be two Banach spaces, and let $\varphi\colon E \to F$ be a bounded linear map.

1. The map $\varphi$ has dense image if and only if $\varphi'$ is injective.

2. The map $\varphi$ is an isometric isomorphism if and only if $\varphi'$ is.

The next theorem characterizes amenability of a locally compact group in terms of the canonical maps between its enveloping operator algebras. The case $p = 2$ of the result below is a standard fact in $C^*$-algebra theory; see Theorem 2.6.8 in [24].

When $\mathcal{E} = L^p$, the equivalence between (1) and (2) in the theorem below was independently obtained by Phillips, whose methods are inspired in $C^*$-algebraic techniques; see [211].

We denote by $\kappa_u \colon F^p_{\mathrm{QS}}(G) \to F^p_\lambda(G)$ the composition $\kappa \circ \kappa_{\mathrm{QS}}$.

**Theorem XIV.3.7.** Let $G$ be a locally compact group, and let $p \in (1, \infty)$. Then the following are equivalent:

1. The group $G$ is amenable.
2. With $\mathcal{E}$ denoting any of the classes $QSL^p$, $SL^p$, $QL^p$ or $L^p$, the canonical map $F_{\mathcal{E}}(G) \to F^p_\lambda(G)$ is an isometric isomorphism.
3. With $\mathcal{E}$ denoting any of the classes $QSL^p$, $SL^p$, $QL^p$ or $L^p$, the canonical map $F_{\mathcal{E}}(G) \to F^p_\lambda(G)$ is a (not necessarily isometric) isomorphism.

*Proof.* We begin by introducing the notation that will be used to prove the equivalences.

Let $p'$ be the dual exponent of $p$. Let $B_{p'}(G)$ be the $p'$-analog of the Fourier-Stieltjes algebra, as introduced in [242]. By definition, $B_{p'}(G)$ is the set of coefficient functions of representations of $G$ on $QSL^p$-spaces. We may think of $B_{p'}(G)$ as a subalgebra of the algebra $C_b(G)$ of bounded continuous functions on $G$, except that the norm of $B_{p'}(G)$ is not induced by the norm of $C_b(G)$.

Under the canonical identification of $\mathrm{UPF}_p(G)$ and $F^p_{\mathrm{QS}}(G)$ (see Remarks XIV.2.9), Theorem 6.6 in [242] provides a canonical isometric isomorphism

$$F^p_{\mathrm{QS}}(G)' \cong B_{p'}(G) \subseteq C_b(G).$$

We now turn to the equivalence between the statements.

(1) implies (2). It is enough to show that the map $\kappa_u \colon F^p_{\mathrm{QS}}(G) \to F^p_\lambda(G)$ is isometric. Under the assumption that $G$ is amenable, it follows from Theorems 6.7 in [242] that the dual map $\kappa' \colon F^p_\lambda(G)' \to F^p_{\mathrm{QS}}(G)'$ of $\kappa$, is an isometric isomorphism. Indeed, with the notation used

there, and writing $\cong$ for isometric isomorphism, we have

$$F^p_\lambda(G)' \cong \mathrm{PF}_p(G)' \xrightarrow{\cong} B_{p'}(G) \cong F^p_{\mathrm{QS}}(G)'.$$

It thus follows from Lemma XIV.3.6 that $\kappa_u$ is an isometric isomorphism, as desired.

(2) implies (3). Clear.

(3) implies (1). It is enough to show the result assuming that $\kappa\colon F^p(G) \to F^p_\lambda(G)$ is an isomorphism.

We regard the dual of $\kappa_u$ as a map $\kappa_u'\colon F^p_\lambda(G)' \to B_{p'}(G) \subseteq C_b(G)$. By Theorem 4.1 in [189], a locally compact group $G$ is amenable if and only if the constant function 1 on $G$ belongs to the image of $F^p_\lambda(G)'$ in $B_{p'}(G) \subseteq C_b(G)$. Note that 1 always belongs to the image of $F^p(G)'$ in $B_{p'}(G)$, since it is a coefficient function of the trivial representation (on an $L^{p'}$-space). Now, if $\kappa\colon F^p(G) \to F^p_\lambda(G)$ is an isomorphism, then so is the dual map $\kappa'\colon F^p_\lambda(G)' \to F^p(G)'$. It follows that 1 is in the image of $F^p_\lambda(G)'$ in $B_p(G) \subseteq C_b(G)$, and hence $G$ is amenable. $\square$

One must exclude $p = 1$ in Theorem XIV.3.7, since the canonical maps are *always* isometric in this case, as was shown in Proposition XIV.2.11 and Remark XIV.2.12.

Theorem XIV.3.7 raises the following natural questions:

**Question XIV.3.8.** Let $G$ be a locally compact group and let $p \in (1, \infty)$. Is the canonical map $\kappa\colon F^p(G) \to F^p_\lambda(G)$ always surjective?

The question above has an affirmative answer if either $p = 2$ or $G$ is amenable, and there are no known counterexamples.

If Question XIV.3.8 has a negative answer, it would be interesting to explore the following:

**Problem XIV.3.9.** Let $p \in (1, \infty) \setminus \{2\}$. Characterize those locally compact groups $G$ for which the canonical map $\kappa\colon F^p(G) \to F^p_\lambda(G)$ is injective.

If the answer to Question XIV.3.8 is 'yes', then the problem above would have the expected solution: injectivity of the canonical map $\kappa\colon F^p(G) \to F^p_\lambda(G)$ would be equivalent to amenability, by the equivalence between (1) and (3) in Theorem XIV.3.7, by the Open Mapping Theorem.

Although unlikely, the answer to Problem XIV.3.9 could in principle depend on $p$.

**Notation XIV.3.10.** If $G$ is a locally compact amenable group and $p \in (1,\infty)$, or if $G$ is any locally compact group and $p = 1$, we will write $F^p(G)$ instead of any of $F^p_{\mathrm{QS}}(G), F^p_{\mathrm{S}}(G), F^p_{\mathrm{Q}}(G)$ or $F^p_\lambda(G)$, since they are isometrically isomorphic by Theorem XIV.3.7, Proposition XIV.2.11 and Remark XIV.2.12.

In the discrete case, amenability of the group is characterized by amenability of any of its associated universal enveloping algebras.

**Theorem XIV.3.11.** Let $G$ be a locally compact group and let $p \in [1,\infty)$. Let $\mathcal{E}$ be any of the classes $QSL^p$, $QL^p$, $SL^p$, or $L^p$. If $G$ is amenable, then so is $F_{\mathcal{E}}(G)$. The converse is true if $G$ is discrete.

*Proof.* It is a well known result due to B. Johnson (see [137]) that $G$ is amenable if and only if the group algebra $L^1(G)$ is amenable (even if $G$ is not discrete). If $G$ is amenable, then so is $L^1(G)$, and hence also $F_{\mathcal{E}}(G)$ by Proposition 2.3.1 in [241], since the image of $L^1(G)$ in $F_{\mathcal{E}}(G)$ is dense.

Conversely, suppose that $F_{\mathcal{E}}(G)$ is amenable and that $G$ is discrete. Then $F^p(G)$ is amenable again by Proposition 2.3.1 in [241], because there is a contractive homomorphism $F_{\mathcal{E}}(G) \to F^p(G)$ with dense range. Another use of Proposition 2.3.1 in [241], this time with the map $\gamma_{p,2}\colon F^p(G) \to F^2(G) = C^*(G)$ constructed in Theorem XIV.2.30, shows that $C^*(G)$ must be amenable in this case. Now Theorem 2.6.8 in [24] implies that $G$ is amenable. □

The following question naturally arises:

**Question XIV.3.12.** Does amenability of $F^p(G)$, for $p \neq 2$, characterize amenability of $G$ in full generality?

For $p = 1$, the answer is yes, by Johnson's Theorem ([137]). The result is known to be false for $p = 2$. Indeed, Connes proved in [29] that if $G$ is a connected Lie group, then $C^*(G)$ (and hence $C^*_\lambda(G)$) is amenable. However, there are non-amenable connected Lie groups, such as $\mathrm{SL}_2(\mathbb{R})$ (whose group $C^*$-algebra is even type I). We do not know, however, whether $F^p(\mathrm{SL}_2(\mathbb{R}))$ is amenable for $p \neq 1, 2$.

We close this subsection with the computation of the maximal ideal space of $F^p(G)$ when $G$ is an abelian locally compact group. This result, in this form, will be crucial in the proof of Theorem XV.3.7. The result is almost certainly well-known, but we have not been able to find a reference in the literature.

**Proposition XIV.3.13.** Let $G$ be an abelian locally compact group, and let $p \in [1, \infty)$. Then there is a canonical homeomorphism $\phi\colon \mathrm{Max}(F^p(G)) \to \widehat{G}$. Moreover, the Gelfand transform

$$\Gamma_p \colon F^p(G) \to C_0(\widehat{G})$$

is injective, contractive and has dense range.

*Proof.* It is well-known that $\mathrm{Max}(L^1(G))$ is canonically homeomorphic to $\widehat{G}$, and that the following diagram is commutative:

$$\begin{tikzcd} L^1(G) \arrow[r] \arrow[d, "\Gamma_1"'] & C^*(G) \arrow[d, "\Gamma_2"] \\ C_0(\widehat{G}) & C_0(\widehat{G}). \arrow[l, "\mathrm{id}_{C_0(\widehat{G})}"] \end{tikzcd}$$

Set $X = \mathrm{Max}(F^p(G))$. The canonical map $\gamma_{p,2}\colon F^p(G) \to C^*(G)$ induces a continuous function $\phi\colon \widehat{G} \to X$. We claim that $\phi$ is injective. Indeed, $\phi$ is the restriction of $\gamma_{p,2}'\colon C^*(G)' \to F^p(G)'$ to the multiplicative linear functionals of norm one. Since $\gamma_{p,2}'$ is injective by part (1) of Lemma XIV.3.6, the claim follows.

A similar argument shows that the canonical inclusion $L^1(G) \to F^p(G)$ induces an injective continuous function $\psi\colon X \to \widehat{G}$. By naturality of the Gelfand transform, we must have $\psi \circ \phi = \mathrm{id}_{\widehat{G}}$, showing that $\widehat{G}$ and $X$ are homeomorphic.

For the last claim, it only remains to show that $\Gamma_p$ is injective and has dense range. Injectivity follows from Theorem 4 in Section 1.5 of [50]. Density of its range follows from the facts that $\gamma_{p,2}$ has dense range by Theorem XIV.2.30, that $\Gamma_2$ is an isometric isomorphism, and that the diagram below commutes:

$$\begin{tikzcd} L^1(G) \arrow[r] \arrow[d, "\Gamma_1"'] & F^p(G) \arrow[r, "\gamma_{p,2}"] \arrow[d, "\Gamma_p"] & C^*(G) \arrow[d, "\Gamma_2"] \\ C_0(\widehat{G}) & C_0(\widehat{G}) \arrow[l, "\cong"] & C_0(\widehat{G}). \arrow[l, "\cong"] \end{tikzcd}$$

□

*Canonical maps $F^p(G) \to F^q(G)$ revisted*

In this subsection, we will use Theorem XIV.3.7 to obtain further information about the maps $\gamma_{p,q}$ constructed in Theorem XIV.2.30, by regarding them as maps between the algebras of $p$- and $q$-pseudofunctions; see Corollary XIV.3.20.

We begin with a general discussion (see also Section 8 in [116]).

**Definition XIV.3.14.** Let $E$ be a reflexive Banach space. It is well-known that the Banach algebra $\mathcal{B}(E)$ is the Banach space dual of the projective tensor product $E\widehat{\otimes}E'$. The weak*-topology inherited by $\mathcal{B}(E)$ from this identification is usually called the *ultraweak topology* on $\mathcal{B}(E)$.

Given a Banach space $E$, we write $\langle\cdot,\cdot\rangle_{E,E'}$ for the canonical pairing $E \times E' \to \mathbb{C}$. Given $\xi \in E$ and given $\eta \in E'$, we write $\xi \otimes \eta$ for the simple tensor product in $E\widehat{\otimes}E'$. Regarding an operator $a \in \mathcal{B}(E)$ as a functional on $E\widehat{\otimes}E'$, the action of $a$ on $\xi \otimes \eta$ is given by

$$\langle a, \xi \otimes \eta\rangle_{(E\widehat{\otimes}E')',E\widehat{\otimes}E'} = \langle a\xi, \eta\rangle_{E,E'}. \tag{XIV.3}$$

**Definition XIV.3.15.** Let $E$ be a Banach space. Let $(a_j)_{j\in J}$ be a net of operators in $\mathcal{B}(E)$, and let $a \in \mathcal{B}(E)$ be another operator. We say that $(a_j)_{j\in J}$ *converges ultraweakly* to $a$, if for every $x \in E\widehat{\otimes}E'$ we have

$$\lim_{j\in J} |\langle a_j, x\rangle - \langle a, x\rangle| = 0.$$

The ultraweak topology on $\mathcal{B}(E)$ should not be confused with its weak operator topology. By definition, a net $(a_j)_{j\in J}$ in $\mathcal{B}(E)$ converges in the weak operator topology to an operator $a \in \mathcal{B}(E)$ if for every $\xi \in E$ and every $\eta \in E'$, we have

$$\lim_{j\in J} |\langle a_j\xi, \eta\rangle - \langle a\xi, \eta\rangle| .$$

**Remark XIV.3.16.** Since a pair $(\xi, \eta) \in E \times E'$ defines an element $x = \xi \otimes \eta$ in $E\widehat{\otimes}E'$, it follows from (XIV.3), that the ultraweak topology is stronger than the weak operator topology. On the other hand, it is well-known that the ultraweak topology and the weak operator topology agree on (norm) bounded subsets of $\mathcal{B}(E)$.

The following class of Banach algebras will be needed in the proof of Theorem XIV.3.18.

**Definition XIV.3.17.** Let $G$ be a locally compact group and let $p \in [1, \infty)$.

1. The algebra of *p-pseudomeasures* on $G$, denoted $PM_p(G)$, is the ultraweak closure of $F^p_\lambda(G)$ in $\mathcal{B}(L^p(G))$.

2. The algebra of *p-convolvers* on $G$, denoted $CV_p(G)$, is the bicommutant of $F^p_\lambda(G)$ in $\mathcal{B}(L^p(G))$.

Algebras of pseudomeasures and convolvers on groups have been thoroughly studied since their introduction by Herz in Section 8 in [116]. It is clear that $PM_p(G) \subseteq CV_p(G)$, and it is conjectured that they are equal for every locally compact group and every Hölder exponent $p \in [1, \infty)$. The reader is referred to [49] for a more thorough description of the problem, as well as for the available partial results.

**Theorem XIV.3.18.** Let $G$ be a locally compact group, and let $p, q \in [1, \infty)$ with either $p \leq q \leq 2$ or $2 \leq q \leq p$. Assume that

$$\|\lambda_q(f)\|_q \leq \|\lambda_p(f)\|_p \tag{XIV.4}$$

for every $f \in L^1(G)$. Then the identity map on $L^1(G)$ extends to a contractive map

$$\gamma^\lambda_{p,q} \colon F^p_\lambda(G) \to F^q_\lambda(G),$$

with dense range. Moreover,

1. The map $\gamma^\lambda_{p,q}$ is injective.

2. Suppose that $CV_p(G) = PM_p(G)$ and $CV_q(G) = PM_q(G)$. If $p \neq q$, then $\gamma^\lambda_{p,q}$ is not surjective unless $G$ is finite.

We emphasize that the assumptions $CV_p(G) = PM_p(G)$ and $CV_q(G) = PM_q(G)$ in part (2) of this theorem, are conjecturally not a restriction; see [49].

*Proof.* We show first the existence of $\gamma^\lambda_{p,q}$. Let $a$ be an operator in $F^p_\lambda(G)$ and choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(G)$ such that $\lim\limits_{n\to\infty} \|a - \lambda_p(f_n)\| = 0$. Then $(\lambda_p(f_n))_{n\in\mathbb{N}}$ is a Cauchy-sequence in

$\mathcal{B}(L^p(G))$, and hence $(\lambda_q(f_n))_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\mathcal{B}(L^q(G))$ as well, by the inequality in (XIV.4). Set

$$\gamma^{\lambda}_{p,q}(a) = \lim_{n\to\infty} \lambda_q(f_n).$$

It is straightforward to check that this definition is independent of the choice of the sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(G)$. Moreover, it is clear that the resulting homomorphism $\gamma^{\lambda}_{p,q}\colon F^p_\lambda(G) \to F^q_\lambda(G)$ is contractive and has dense range.

(1). Let us show that $\gamma^{\lambda}_{p,q}$ is injective. Fix $a \in F^p_\lambda(G)$ and choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(G)$ such that $\lambda_p(f_n)$ converges to $a$ in $\mathcal{B}(L^p(G))$. Assume that $\gamma^{\lambda}_{p,q}(a) = 0$, so that $(\lambda_q(f_n))_{n\in\mathbb{N}}$ converges to the zero operator on $L^q(G)$. In order to arrive at a contradiction, suppose that $a \neq 0$. Choose $\xi \in C_c(G)$ such that $a\xi \neq 0$. Set $\eta = a\xi$. Since

$$\lim_{n\to\infty} \|f_n * \xi - \eta\|_p = 0,$$

upon passing to a subsequence, we may assume that $(f_n * \xi)_{n\in\mathbb{N}}$ converges pointwise almost everywhere to $\eta$.

Since $(\lambda_q(f_n))_{n\in\mathbb{N}}$ converges to zero in $\mathcal{B}(L^q(G))$, it follows that $(\lambda_q(f_n)\xi)_{n\in\mathbb{N}}$ converges to zero in $L^q(G)$. Again, upon passing to a subsequence, we may assume that $f_n * \xi$ converges pointwise almost everywhere to zero. This clearly implies that $\eta = 0$ almost everywhere on $G$, which is a contradiction. This implies that $a = 0$, and hence $\gamma^{\lambda}_{p,q}$ is injective.

Before proving part (2), let us show that $\gamma^{\lambda}_{p,q}$ extends to a map

$$\delta_{p,q}\colon PM_p(G) \to PM_q(G)$$

between the ultraweak closures of $F^p_\lambda(G)$ and $F^q_\lambda(G)$ in $\mathcal{B}(L^p(G))$ and $\mathcal{B}(L^q(G))$, respectively. The existence of such a map is well-known to the experts, so we only sketch its construction.

Let $a$ be an operator in $PM_p(G)$, and choose a net $(f_j)_{j\in J}$ in $C_c(G)$ such that $(\lambda_p(f_j))_{j\in J}$ converges to $a$ in the ultraweak topology. Since the sequence $(\lambda_p(f_j))_{j\in J}$ converges ultraweakly, it follows that it is norm-bounded. Since $L^p(G)$ is a separable Banach space, the ultraweak topology is metrizable on bounded subsets of $\mathcal{B}(L^p(G))$, and hence there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c(G)$ such that $\lambda_p(f_n)$ converges ultraweakly to $a$.

By the inquality in (XIV.4), the sequence $(\lambda_q(f_n))_{n\in\mathbb{N}}$ is norm-bounded in $\mathcal{B}(L^q(G))$. By the Banach-Alaoglu Theorem, the sequence $(\lambda_q(f_n))_{n\in\mathbb{N}}$ has an ultraweak limit point, so there is a subsequence $(\lambda_q(f_{n_k}))_{k\in\mathbb{N}}$ that converges ultraweakly to an operator $b \in \mathcal{B}(L^q(G))$. By construction, $b$ belongs to $PM_q(G)$. One can show that $b$ does not depend on the choices made, so we set $\delta_{p,q}(a) = b$.

The resulting homomorphism $\delta_{p,q}\colon PM_p(G) \to PM_q(G)$ is easily seen to be contractive and to have dense range.

(2). If $G$ is finite, then $\gamma^\lambda_{p,q}$ is clearly surjective, since it has dense range and $F^q_\lambda(G)$ is finite dimensional. Conversely, assume that $G$ is infinite. We will show that $\gamma^\lambda_{p,q}$ is not surjective using results from [37].

To reach a contradiction, assume that $\gamma^\lambda_{p,q}$ is surjective. It follows from the Open Mapping Theorem and part (1) of this theorem, that $\gamma^\lambda_{p,q}$ is an isomorphism (although not necessarily isometric). This means that there is a constant $K > 0$ such that

$$\|\lambda_p(f)\|_p \leq K\|\lambda_q(f)\|_q,$$

for every $f \in L^1(G)$.

We claim that $\delta_{p,q}$ is also surjective. Given $b \in PM_q(G)$, choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $C_c(G)$ such that the sequence of operators $(\lambda_q(f_n))_{n\in\mathbb{N}}$ in $\mathcal{B}(L^q(G))$ converges ultraweakly to $b$. Then the sequence $(\lambda_q(f_n))_{n\in\mathbb{N}}$ is norm-bounded in $\mathcal{B}(L^q(G))$. It follows that

$$\sup_{n\in\mathbb{N}} \|\lambda_p(f_n)\| \leq K \sup_{n\in\mathbb{N}} \|\lambda_q(f_n)\| < \infty,$$

so $(\lambda_p(f_n))_{n\in\mathbb{N}}$ is norm-bounded in $\mathcal{B}(L^p(G))$ as well. By the Banach-Alaoglu Theorem, there exists a subsequence $(\lambda_p(f_{n_k}))_{k\in\mathbb{N}}$ that converges ultraweakly to an operator $a \in \mathcal{B}(L^p(G))$. By construction, $a$ belongs to $PM_p(G)$.

We claim that $\delta_{p,q}(a) = b$. For $\xi, \eta, f \in C_c(G)$, we have

$$\langle \lambda_p(f)\xi, \eta\rangle_{L^p(G),L^{p'}(G)} = \int_G\int_G f(st^{-1})\xi(t)\eta(s)\; dsdt = \langle \lambda_q(f)\xi, \eta\rangle_{L^q(G),L^{q'}(G)}.$$

We deduce that

$$\langle a\xi, \eta\rangle = \lim_{k\to\infty} \langle \lambda_p(f_{n_k})\xi, \eta\rangle = \lim_{k\to\infty} \langle \lambda_q(f_{n_k})\xi, \eta\rangle = \lim_{n\to\infty} \langle \lambda_q(f_n)\xi, \eta\rangle = \langle b\xi, \eta\rangle.$$

Using that on bounded sets the weak operator topology agrees with the ultraweak topology, it follows from the definition that $\delta_{p,q}(a) = b$, as desired.

By our assumption $PM_p(G) = CV_p(G)$ and $PM_q(G) = CV_q(G)$, we can regard $\delta_{p,q}$ as a map between the respective $p$- and $q$-convolvers on $G$. Now, the fact that $\delta_{p,q}\colon CV_p(G) \to CV_q(G)$ is surjective contradicts Theorem 2 in [37] (where $CV_p(G)$ and $CV_q(G)$ are denoted by $L_p^p(G)$ and $L_q^q(G)$, respectively). This contradiction implies that $\gamma_{p,q}^{\lambda}$ is not surjective, as desired. □

**Remark XIV.3.19.** Let $G$ be a locally compact group and let $p, q \in [1, \infty)$ satisfy $\left|\frac{1}{q} - \frac{1}{2}\right| < \left|\frac{1}{p} - \frac{1}{2}\right|$. We point out that even though the map $\gamma_{p,q}\colon F^p(G) \to F^q(G)$ constructed in Theorem XIV.2.30 exists in full generality, the map $\gamma_{p,q}^{\lambda}\colon F_\lambda^p(G) \to F_\lambda^q(G)$ may fail to exist for some groups and some exponents $p$ and $q$, since there may be no constant $M > 0$ such that

$$\|\lambda_q(f)\|_q \le M\|\lambda_p(f)\|_p$$

holds for all $f \in L^1(G)$. Indeed, as it is shown in [219] and [220], this is the case for any noncommutative free group, and for any exponents $p, q \in (1, \infty)$ with $p \neq q$.

In contrast to what was pointed out in Remark XIV.3.19, we have that the assumptions of Theorem XIV.3.18 are satisfied in a number of situations, particularly (but not only) if $G$ is amenable. We state this explicit in the following corollary.

**Corollary XIV.3.20.** Let $G$ be an amenable locally compact group, and let $p, q \in [1, \infty)$. Denote by $\lambda_p\colon L^1(G) \to \mathcal{B}(L^p(G))$ and $\lambda_q\colon L^1(G) \to \mathcal{B}(L^q(G))$ the corresponding left regular representations of $G$. If either $p \le q \le 2$ or $2 \le q \le p$, then

$$\|\lambda_q(f)\|_q \le \|\lambda_p(f)\|_p$$

for every $f$ in $L^1(G)$. Moreover,

1. The map $\gamma_{p,q}$ constructed in Theorem XIV.2.30 is injective.

2. If $p \neq q$, then $\gamma_{p,q}$ is surjective if and only if $G$ is finite.

*Proof.* It is enough to assume that $1 \leq p \leq q \leq 2$, since the other case can then be deduced by using duality. Given $f \in L^1(G)$, we use Theorem XIV.3.7 at the first and third step, and Proposition XIV.2.7 at the second step, to get

$$\|\lambda_q(f)\|_q = \|f\|_{L^q} \leq \|f\|_{L^p} = \|\lambda_p(f)\|_p,$$

as desired.

Part (1) follows immediately from part (1) of Theorem XIV.3.18, since $\gamma_{p,q} = \gamma_{p,q}^{\lambda}$ by amenability of $G$. Likewise, part (2) follows from part (2) of Theorem XIV.3.18, together with Herz's result (Theorem 5 in [116]) that $PM_r(G) = CV_r(G)$ for all $r \in [1,\infty)$ whenever $G$ is amenable. $\square$

**Remark XIV.3.21.** Adopt the notation from the statement of Corollary XIV.3.20. The fact that $\|\lambda_q(f)\|_q \leq \|\lambda_p(f)\|_p$ was announced as Theorem C in [115] (see the corollary on page 512 of [118] for a proof).

It is a consequence of Corollary XIV.3.20 that the Banach algebras $F^p(\mathbb{Z})$ for $p \in [1,2]$, are pairwise not *canonically* isometrically isomorphic. (Here 'canonical' means via a map which is the identity on $\ell^1(\mathbb{Z})$.) In Theorem XV.3.7, we show that for $p, q \in [1,2]$, the algebras $F^p(\mathbb{Z})$ and $F^q(\mathbb{Z})$ are not even *abstractly* isometrically isomorphic.

We point out that when $G$ is abelian, the unnumbered claim in Corollary XIV.3.20 (see also Remark XIV.3.21) can be proved much more directly. We do so in the proposition below, which is well-known to the experts and appears implicitly in the literature.

**Proposition XIV.3.22.** Let $G$ be an abelian locally compact group, and let $f \in L^1(G)$.

1. Let $p \in (1,\infty)$. Then $\|\lambda_p(f)\|_p = \|\lambda_{p'}(f)\|_{p'}$.

2. Let $p, q \in [1,\infty)$, and suppose that either $p \leq q \leq 2$ or $2 \leq q \leq p$. Then $\|\lambda_q(f)\|_q \leq \|\lambda_p(f)\|_p$.

*Proof.* (1). Let $f \in L^1(G)$. It was shown in the proof of Proposition XIV.3.5 that $\|\lambda_p(f^\sharp)\|_p = \|\lambda_{p'}(f)\|_{p'}$. Since $G$ is unimodular, we have $\|\xi^\sharp\|_p = \|\xi\|_p$ for all $\xi \in L^p(G)$. Using this fact at the third step, we get

$$\begin{aligned}\|\lambda_p(f^\sharp)\|_p &= \sup_{\xi\in L^p(G),\xi\neq 0} \frac{\|f^\sharp*\xi\|_p}{\|\xi\|_p}\\ &= \sup_{\xi\in L^p(G),\xi\neq 0} \frac{\|(\xi^\sharp*f)^\sharp\|_p}{\|\xi\|_p}\\ &= \sup_{\xi\in L^p(G),\xi\neq 0} \frac{\|\xi*f\|_p}{\|\xi\|_p}.\end{aligned}$$

Since $G$ is abelian, we have $\xi * f = f * \xi$ for all $\xi \in L^p(G)$, and the result follows.

(2). Denote by $\gamma\colon (1,\infty) \to \mathbb{R}$ the function given by $\gamma(r) = \|\lambda_r(f)\|_r$ for all $r \in (1,\infty)$. Then $\gamma(r) = \gamma(r')$ for all $r \in (1,\infty)$ by part (1), and $\gamma$ is log-convex by the Riesz-Thorin Interpolation Theorem. It follows that $\gamma$ has a minimum at $r = 2$, and that it is decreasing on $[1,2]$ and increasing on $[2,\infty)$. This finishes the proof. □

### $L^p$-crossed Products and Amenability

N. C. Phillips has announced that for $p \in [1,\infty)$, if $G$ is an amenable discrete group and $\alpha\colon G \to \mathrm{Aut}(A)$ is an isometric action of $G$ on an $L^p$-operator algebra $A$, then the canonical contractive map $\kappa\colon F^p(G,A,\alpha) \to F^p_\lambda(G,A,\alpha)$ is an isometric isomorphism. (The reader is referred to [207] for the construction of full and reduced crossed products, as well as for the definition of the canonical map.) The proof is analogous to the $C^*$-algebra case, and is likely to appear in a second version of the preprint [207].

In analogy with Theorem XIV.3.7 and Theorem XIV.3.11, one may be tempted to conjecture that for $p > 1$, amenability of a discrete group $G$ may be equivalent to the canonical $\kappa\colon F^p(G,X) \to F^p_\lambda(G,X)$ being an isometric isomorphism for every locally compact Hausdorff $G$-space $X$, and that this in turn should be equivalent to amenability of $F^p(G,X)$. (Theorem XIV.3.7 and Theorem XIV.3.11 are the case $X = *$ of this statement.) There are many examples that show that this is not true when $p = 2$, but one may hope that this holds in all other cases, because of the extra rigidity of $L^p$-operator crossed (see also Question XIV.3.12).

However, the statement fails for every $p \in (1,\infty)\setminus\{2\}$, and we will devote this subsection to the construction of a family of counterexamples.

A crucial notion in our proof of Theorem XIV.4.3 is that of incompressible Banach algebras, which is due to Phillips ([210]).

**Definition XIV.4.1.** Let $p \in [1,\infty)$ and let $A$ be a Banach algebra. We say that $A$ is *$p$-incompressible* if for every $L^p$-space $E$, every contractive, injective homomorphism $\rho\colon A \to \mathcal{B}(E)$ is isometric.

If the Hölder exponent $p$ is clear from the context, we will just say that $A$ is incompressible.

Examples of $p$-incompressible Banach algebras include $\mathcal{B}(\ell^p_n)$ for $n \in \mathbb{N}$ (see Theorem 7.2 in [204], where $\mathcal{B}(\ell^p_n)$ is denoted $M^p_n$), the analogs $\mathcal{O}^p_d$ of Cuntz algebras on $L^p$-spaces (see Corollary 8.10 in [204]), and $C^*$-algebras.

The next lemma asserts that a direct limit of $p$-incompressible Banach algebras is again $p$-incompressible.

**Lemma XIV.4.2.** Let $((A_\mu)_{\mu\in\Lambda}, (\varphi_{\mu,\nu})_{\mu,\nu\in\Lambda})$ be a direct limit of Banach algebras with injective, contractive maps $\varphi_{\mu,\nu}\colon A_\mu \to A_\nu$ for all $\mu$ and $\nu$ in $\Lambda$ with $\mu \leq \nu$, and denote by $A$ its direct limit. Let $p \in [1,\infty)$. If $A_\mu$ is $p$-incompressible for all $\mu$ in $\Lambda$, then so is $A$.

*Proof.* For $\mu$ in $\Lambda$, denote by $\varphi_{\mu,\infty}\colon A_\mu \to A$ the canonical contractive homomorphism into the direct limit algebra. Let $E$ be a separable $L^p$-space, and let $\rho\colon A \to \mathcal{B}(E)$ be a contractive injective representation. Given $\mu$ in $\Lambda$, the representation $\rho \circ \varphi_{\lambda,\infty}\colon A_\mu \to \mathcal{B}(E)$ is injective and contractive. Since $A_\mu$ is incompressible, it follows that $\rho\circ\varphi_{\mu,\infty}$ is isometric. Hence, for $a$ in $A_\mu$, we have

$$\|\varphi_{\mu,\infty}(a)\| \leq \|a\|_{A_\mu} = \|(\rho\circ\varphi_{\mu,\infty})(a)\| \leq \|\varphi_{\mu,\infty}(a)\|,$$

ant thus $\|(\rho\circ\varphi_{\mu,\infty})(a)\| = \|\varphi_{\mu,\infty}(a)\|$. We conclude that $\rho|_{\varphi_{\mu,\infty}(A_\mu)}$ is isometric for all $\mu$ in $\Lambda$. Hence $\rho$ is isometric, and $A$ is incompressible. $\square$

We are now ready to show that for any discrete group $G$ (amenable or not) and for any $p \in [1,\infty)$, there exists a locally compact Hausdorff $G$-space $X$ such that the canonical map $\kappa\colon F^p(G,X) \to F^p_\lambda(G,X)$ is isometric and such that $F^p(G,X)$ is amenable. In particular, the analog of Theorem XIV.3.7, where one considers actions of $G$ on arbitrary topological spaces other than the one point space, is false.

Recall (see, for example, [10]) that if $A$ is a Banach algebra, an element $a \in A$ is said to be *hermitian* if $\|e^{ita}\| = 1$ for all $t \in \mathbb{R}$. If $X$ is a locally compact Hausdorff space, then every idempotent in $C_0(X)$ is hermitian.

The terminology "spatial partial isometry" and the notion of spatial system are borrowed from Definition 6.4 in [204], and the notation $\overline{M}_G^p$ is taken from Example 1.6 in [204].

**Theorem XIV.4.3.** Let $G$ be a discrete group, and let $\alpha\colon G \to \mathrm{Aut}(C_0(G))$ be the isometric action induced by left translation of $G$ on itself. Let $p \in [1, \infty)$. Then there are natural isometric isomorphisms

$$F^p(G, C_0(G), \alpha) \xrightarrow{\ \kappa\ } F^p_\lambda(G, C_0(G), \alpha) \longrightarrow \overline{M}_G^p\ .$$

Moreover, the right-hand side equals $\mathcal{K}(\ell^p(G))$ when $p > 1$, and is strictly smaller than $\mathcal{K}(\ell^1(G))$ when $p = 1$.

Note that $\overline{M}_G^p$, being the direct limit of matrix algebras, is amenable even if $G$ is not.

*Proof.* The last claim follows from Corollary 1.9 and Example 1.10 in [207].

We begin by showing that there is a natural isometric isomorphism

$$F^p(G, C_0(G), \alpha) \cong \overline{M}_G^p.$$

For $s \in G$, let $u_s$ be the standard invertible isometry implementing $\alpha_s$ in the crossed product, and let $\delta_s \in C_0(G)$ be the function $\chi_{\{s\}}$. Then $\alpha_s(\delta_t) = \delta_{st}$ for all $s, t \in G$, and $\mathrm{span}\left(\{\delta_s \colon s \in G\}\right)$ is dense in $C_0(G)$.

For $s, t \in G$, set $a_{s,t} = \delta_s u_{st^{-1}} \in \ell^1(G, C_0(G), \alpha)$. For $s_1, s_2, t_1, t_2 \in G$, we have

$$\begin{aligned} a_{s_1,t_1} a_{s_2,t_2} &= \delta_{s_1} u_{s_1 t_1^{-1}} \delta_{s_2} u_{s_2 t_2^{-1}} \\ &= \delta_{s_1} \alpha_{s_1 t_1^{-1}}(\delta_{s_2}) u_{s_1 t_1^{-1}} u_{s_2 t_2^{-1}} \\ &= \delta_{s_1} \delta_{s_1 t_1^{-1} s_2} u_{s_1 t_1^{-1} s_2 t_2^{-1}}. \end{aligned}$$

Thus, if $s_2 \neq t_1$, then $a_{s_1,t_1} a_{s_2,t_2} = 0$, because in this case $\delta_{s_1} \delta_{s_1 t_1^{-1} s_2} = 0$. Taking $s_2 = t_1$, we get $a_{s_1,t_1} a_{t_1,t_2} = a_{s_1,t_2}$. Hence the elements $\{a_{s,t} \colon s, t \in G\}$ satisfy the relations for a system of matrix units indexed by $G$. Also, $\mathrm{span}\left(\{a_{s,t} \colon s, t \in G\}\right)$ is dense in $\ell^1(G, C_0(G), \alpha)$, and hence also in $F^p(G, C_0(G), \alpha)$.

Let $S$ be a finite subset of $G$. Then $\{a_{s,t} \colon s, t \in S\}$ is a standard system of matrix units for $M_{|S|}$, so the subalgebra $M_S$ of $F^p(G, C_0(G), \alpha)$ they generate is canonically isomorphic, as a

Banach algebra, to $M^p_{|S|}$. We claim that this isomorphism is isometric, that is, that the norm that $M_S$ inherits as a subalgebra of $F^p(G, C_0(G), \alpha)$ is the standard norm of $M^p_{|S|}$. To check this, it will be enough to show that if

$$\rho\colon \ell^1(G, C_0(G), \alpha) \to \mathcal{B}(L^p(X, \mu))$$

is a nondegenerate contractive representation on a $\sigma$-finite measure space $(X, \mu)$, then the restriction

$$\rho|_{M_S}\colon M_S \to \mathcal{B}(L^p(X, \mu))$$

is spatial in the sense of Definition 7.1 in [204].

Given such a representation $\rho\colon \ell^1(G, C_0(G), \alpha) \to \mathcal{B}(L^p(X, \mu))$, let $\pi\colon C_0(G) \to \mathcal{B}(L^p(X, \mu))$ be the nondegenerate contractive representation, and let $v\colon G \to \mathrm{Isom}(L^p(X, \mu))$ be the isometric group representation such that $(\pi, v)$ is the covariant representation of $(G, C_0(G), \alpha)$ whose integrated form is $\rho$. For $s$ and $t$ in $S$, one has

$$\rho(a_{s,t}) = \pi(\chi_{\{s\}})v_{st^{-1}}.$$

Since $\chi_{\{s\}}$ is a hermitian idempotent in $C_0(G)$, it follows that $\pi(\chi_{\{s\}})$ is also hermitian. Use Example 1.1 in [10] to choose a measurable subset $F$ of $X$ such that $\pi(\chi_{\{s\}}) = \chi_F$. Since $v_{st^{-1}}$ is a bijective isometry, it is spatial. If $(X, X, T, g)$ is a spatial system for $v_{st^{-1}}$ (see Definition 6.1 in [204]), then it is easy to check that $(F, X, T, g)$ is a spatial system for $\rho(a_{s,t})$. This shows that $\rho(a_{s,t})$ is a spatial partial isometry, and hence $\rho|_{M_S}$ is a spatial representation.

Denote by $\mathcal{F}$ the upward directed family of all finite subsets $S$ of $G$. For each $S$ in $\mathcal{F}$, let $\varphi_S\colon M^p_{|S|} \to F^p(G, C_0(G), \alpha)$ be the canonical isometric isomorphism that sends the standard matrix units of $M_{|S|}$ to the set of matrix units $\{a_{s,t}\colon s, t \in S\}$. It is clear that there is an isometric homomorphism

$$\varphi_0\colon \bigcup_{S\in\mathcal{F}} M^p_{|S|} \to F^p(G, C_0(G), \alpha)$$

whose range contains $\{a_{s,t}\colon s,t\in G\}$. Note that $\bigcup\limits_{S\in\mathcal{F}} M^p_{|S|}$ is a subalgebra of $\mathcal{K}(\ell^p(G))$. Since $\varphi_0$ is isometric, it extends by continuity to an isometric homomorphism

$$\varphi\colon \overline{\bigcup_{S\in\mathcal{F}} M^p_{|S|}} \to F^p(G, C_0(G), \alpha),$$

which must be surjective since its range is dense. This is the desired isometric isomorphism.

We will now show that the canonical map

$$\kappa\colon F^p(G, C_0(G), \alpha) \to F^p_\lambda(G, C_0(G), \alpha)$$

is an isometric isomorphism. The usual argument for $C^*$-algebras is that $\kappa$ is surjective, and since $F^2(G, C_0(G), \alpha) \cong \mathcal{K}(\ell^2(G))$ is simple, $\kappa$ must be an isomorphism. However, when $p \neq 2$, we do not know whether $\kappa$ has closed range. Here is where incompressibility comes into play.

The full crossed product $F^p(G, C_0(G), \alpha)$ is $p$-incompressible by Lemma XIV.4.2, because it is the direct limit of the $p$-incompressible Banach algebras $M^p_S$ (see Theorem 7.2 in [204]). Since $F^p_\lambda(G, C_0(G), \alpha)$ can be isometrically represented on an $L^p$-space, it follows that $\kappa$ must be isometric. Finally, having dense range, $\kappa$ is an isometric isomorphism. This finishes the proof. $\square$

# CHAPTER XV

# FUNCTORIALITY OF GROUP ALGEBRAS ACTING ON $L^P$-SPACES

This chapter is based on joint work with Hannes Thiel ([94]).

We continue our study of group algebras acting on $L^p$-spaces, particularly of algebras of $p$-pseudofunctions of locally compact groups. We focus on the functoriality properties of these objects. We show that $p$-pseudofunctions are functorial with respect to homomorphisms that are either injective, or whose kernel is amenable and has finite index. We also show that the universal completion of the group algebra with respect to representations on $L^p$-spaces, is functorial with respect to quotient maps.

As an application, we show that the algebras of $p$- and $q$-pseudofunctions on $\mathbb{Z}$ are isometrically isomorphic as Banach algebras if and only if $p$ and $q$ are either equal or conjugate.

## Introduction

Associated to a locally compact group, there are several Banach algebras that capture different aspects of its structure and representation theory. For instance, in [116], Herz introduced the Banach algebra of $p$-pseudofunctions of a locally compact group $G$, for a fixed Hölder exponent $p \in [1, \infty)$. (We are thankful to Yemon Choi and Matthew Daws for providing this reference.) This Banach algebra is defined as the completion of the group algebra $L^1(G)$ with respect to the norm induced by the left regular representation $\lambda_p$ of $G$ on $L^p(G)$. We denote this algebra by $F^p_\lambda(G)$, so that

$$F^p_\lambda(G) = \overline{\lambda_p(L^1(G))} \subseteq \mathcal{B}(L^p(G)).$$

In Chapter XIV, we studied the universal completion of $L^1(G)$ for representations of $G$ on $L^p$-spaces, which we denote by $F^p(G)$ (this algebra first appeared in [207], as the crossed product of $G$ on the $L^p$-operator algebra $\mathbb{C}$).

For $p = 2$, the Banach algebra $F^2(G)$ is the full group $C^*$-algebra of $G$, usually denoted $C^*(G)$, and $F^2_\lambda(G)$ is the reduced group $C^*$-algebra of $G$, usually denoted $C^*_\lambda(G)$. The functoriality properties of the full and reduced group $C^*$-algebras are well-understood. Given a locally compact group $G$, a normal subgroup $N$ of $G$, and a closed subgroup $H$ of $G$, the following results can be found in [24]:

(a) If $G$ is discrete, then the inclusion map $H \to G$ induces natural isometric, unital homomorphism $C^*_\lambda(H) \to C^*_\lambda(G)$;

(b) The quotient map $G \to G/N$ induces a natural quotient homomorphism $C^*(G) \to C^*(G/N)$;

(c) If $N$ is amenable, then the quotient map $G \to G/N$ induces a natural homomorphism $C^*_\lambda(G) \to C^*_\lambda(G/N)$.

In this chapter, which is based on [94], we explore the extent to which these results generalize to the case $p \neq 2$. Many techniques from $C^*$-algebra theory, such as positivity, are no longer available for Banach algebras acting on $L^p$-spaces. In particular, some standard facts in $C^*$-algebras fail for the classes of Banach algebras here considered. For example, a contractive homomorphism with dense range is not necessarily surjective, and an injective homomorphism need not be isometric.

The results in Section XV.2 are as follows (the second one is proved in greater generality than what is reproduced below):

1. If $H$ is a subgroup of a discrete group $G$, then there is a natural isometric unital map $F^p_\lambda(H) \to F^p_\lambda(G)$ (Proposition XV.2.1);

2. If $N$ is a closed normal subgroup of a locally compact group $G$, then there is a natural contractive map $F^p(G) \to F^p(G/N)$ with dense range (Proposition XV.2.2);

3. If $N$ is an amenable normal subgroup of a discrete group $G$, and $G/N$ is finite, then the natural map $F^p(G) \to F^p(G/N)$ is a quotient map (Theorem XV.2.3).

We point out that the assumption that $G/N$ be finite in (3) above is likely to be unnecessary. On the other hand, we show in Example XV.2.4, using a result of Pooya-Hejazian in [217], that amenability of $N$ is necessary.

In Section XV.3, we apply the above results to study the isomorphism type of the Banach algebras $F^p_\lambda(\mathbb{Z})$, with focus on its dependence on the Hölder exponent $p$. We show that for $p, q \in [1, 2]$, there is an isometric isomorphism between $F^p_\lambda(\mathbb{Z})$ and $F^q_\lambda(\mathbb{Z})$ if and only if $p = q$.

Further applications of the results in this chapter appear in Chapers XVII and XVIII .

Throughout, we will assume that all measure spaces are $\sigma$-finite, and that all Banach spaces are separable. Consistently, all locally compact groups will be assumed to be second countable, and will be endowed with a left Haar measure.

### Functoriality Properties

In this section, we study the extent to which group homomorphisms induce Banach algebra homomorphisms between the respective group operator algebras we studied in Chapter XIV. As in the case of group $C^*$-algebras, these completions are not functorial with respect to arbitrary group homomorphisms. Section XV.3 contains an application of these results, particularly of Theorem XV.2.3: the Banach algebras $F^p_\lambda(\mathbb{Z})$ and $F^q_\lambda(\mathbb{Z})$ are isometrically isomorphic if and only if either $p = q$ or $p = q'$; see Theorem XV.3.7.

The case $p = 2$ of the following result is proved, for example, as Proposition 2.5.9 in [24].

**Proposition XV.2.1.** Let $p \in [1, \infty)$, let $G$ be a discrete group and let $H$ be a subgroup of $G$. Then the canonical inclusion $\iota\colon H \hookrightarrow G$ induces an isometric embedding $F^p_\lambda(H) \to F^p_\lambda(G)$.

*Proof.* We denote also by $\iota\colon \mathbb{C}[H] \to \mathbb{C}[G]$ the induced algebra homomorphism. Let $\lambda^G_p\colon \mathbb{C}[G] \to \mathcal{B}(\ell^p(G))$ and $\lambda^H_p\colon \mathbb{C}[H] \to \mathcal{B}(\ell^p(H))$ denote the left regular representations of $G$ and $H$, respectively. Then $\lambda^G_p \circ \iota$ is conjugate, via an invertible isometry, to a multiple of $\lambda^H_p$. More precisely, let $Q$ be a subset of $G$ containing exactly one element from each coset in $G/H$. Then there is a canonical isometric isomorphism

$$\ell^p(G) \cong \bigoplus_{x\in Q} \ell^p(xH).$$

The representation $\lambda^G_p \circ \iota\colon \mathbb{C}[H] \to \mathcal{B}(\ell^p(G))$ leaves each of the subspaces $\ell^p(xH) \subseteq \ell^p(G)$ invariant, and hence

$$\lambda^G_p \circ \iota \cong \bigoplus_{x\in Q} \lambda^H_p.$$

It follows that

$$\|\iota(f)\|_{F^p_\lambda(G)} = \|(\lambda^G_p \circ \iota)(f)\| = \left\| \bigoplus_{x\in Q} \lambda^H_p(f) \right\| = \max_{x\in Q} \|\lambda^H_p(f)\| = \|f\|_{F^p_\lambda(H)}$$

for every $f \in \mathbb{C}[H]$. Thus, the canonical map $\iota\colon F^p_\lambda(H) \to F^p_\lambda(G)$ is isometric, as desired. $\square$

We need some notation for the next result. If $G$ is a locally compact group and $N$ is a closed normal subgroup, then there is a canonical surjective contractive homomorphism $\psi_N \colon L^1(G) \to L^1(G/N)$ which satisfies

$$\int_{G/N} \psi_N(f)(sN)\ d(sN) = \int_G f(s)\ ds$$

for all $f$ in $L^1(G)$; see Theorem 3.5.4 in [224].

**Proposition XV.2.2.** Let $p \in [1, \infty)$, let $G$ be a locally compact group, let $N$ be a closed normal subgroup of $G$, and let $\pi \colon G \to G/N$ be the canonical quotient map. If $\mathcal{E}$ denotes any of the classes $QSL^p$, $SL^p$, $QL^p$, or $L^p$, then $\pi$ induces a natural contractive map $F_{\mathcal{E}}(G) \to F_{\mathcal{E}}(G/N)$ with dense range.

*Proof.* Let $\mathcal{E}$ denote any of the classes $QSL^p$, $SL^p$, $QL^p$, or $L^p$. Denote by $\psi_N \colon L^1(G) \to L^1(G/N)$ the surjective contractive homomorphism described in the comments above. Given $f \in L^1(G)$, we have

$$\begin{aligned}
\|\psi_N(f)\|_{\mathcal{E}} &= \sup\{\|(\omega \circ \psi_N)(f)\| \colon \omega \in \mathrm{Rep}_H(\mathcal{E})\} \\
&\leq \sup\{\|\rho(f)\| \colon \rho \in \mathrm{Rep}_G(\mathcal{E})\} \\
&= \|f\|_{\mathcal{E}}.
\end{aligned}$$

It follows that $\psi_N$ extends to a contractive homomorphism $F_{\mathcal{E}}(G) \to F_{\mathcal{E}}(G/N)$ with dense range. $\square$

The above proposition shows that the universal completions of $L^1(G)$ are functorial with respect to surjective group homomorphisms. When $p$ is not equal to 1 or 2, it is not clear whether the resulting homomorphism $F^p(G) \to F^p(G/N)$ is a quotient map, or even if it is surjective. In the following theorem, we prove that this is indeed the case whenever $N$ is amenable and $G/N$ is finite.

**Theorem XV.2.3.** Let $G$ be a discrete group, let $p \in [1, \infty)$, and let $N$ be an amenable normal subgroup of $G$ such that $G/N$ is finite. Then the canonical map $G \to G/N$ induces a natural quotient homomorphism $F^p_\lambda(G) \to F^p_\lambda(G/N)$.

*Proof.* We establish some notation first:

- For $s \in G$, we write $u_s$ for the corresponding element in $\mathbb{C}[G]$, and $\delta_s \in \ell^p(G)$ for the corresponding basis element;
- For $s \in G$, we write $v_{sN}$ for the corresponding element in $\mathbb{C}[G/N]$, and $\delta_{sN} \in \ell^p(G/N)$ for the corresponding basis element;
- For $n \in N$, we write $w_n$ for the corresponding element in $\mathbb{C}[N]$, and $\delta_n \in \ell^p(N)$ for the corresponding basis element;
- We write $\pi \colon \mathbb{C}[G] \to \mathbb{C}[G/N]$ for the map given by $u_s \mapsto v_{sN}$ for $s \in G$.

Fix a section $\sigma \colon G/N \to G$, and define an isometric isomorphism

$$\varphi \colon \ell^p(G/N) \otimes \ell^p(N) \to \ell^p(G)$$

by $\varphi(\delta_{sN} \otimes \delta_n) = \delta_{\sigma(sN)n}$ for $s \in G$ and $n \in N$. Let

$$\Phi \colon \mathcal{B}(\ell^p(G/N)) \otimes \mathcal{B}(\ell^p(N)) \to \mathcal{B}(\ell^p(G))$$

be the isometric isomorphism given by $\Phi(x) = \varphi \circ x \circ \varphi^{-1}$ for $x \in \mathcal{B}(\ell^p(G/N)) \otimes \mathcal{B}(\ell^p(N))$. It is a routine exercise to check that

$$\Phi(v_{sN} \otimes w_n)(\delta_{\sigma(tN)m}) = \delta_{\sigma(stN)nm}$$

for all $s, t \in G$ and all $n, m \in N$.

Let $f$ be an element in $\mathbb{C}[G/N]$. We want to show that

$$\|f\| = \inf\{\|\widetilde{f}\| \colon \widetilde{f} \in \mathbb{C}[G], \pi(\widetilde{f}) = f\}.$$

For this, it is enough to find sequences $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}(\ell^p(G))$ (but not necessarily in $\mathbb{C}[G]$) and $(\widetilde{f}_k)_{k\in\mathbb{N}}$ in $\mathbb{C}[G] \subseteq \mathcal{B}(\ell^p(G))$, such that

1. $\|f_k\| \leq \|f\|$ for all $k \in \mathbb{N}$;
2. $\pi(\widetilde{f}_k) = f$ for all $k \in \mathbb{N}$; and

3. $\lim\limits_{k\to\infty} \|\widetilde{f}_k - f_k\| = 0.$

Let $S \subseteq G$ be a finite set such that $f$ can be written as a finite linear combination $f = \sum\limits_{s\in S} a_{sN} v_{sN}$, where $a_{sN}$ is a complex number for $s \in S$. Using amenability of $N$, choose a Følner sequence $(F_k)_{k\in\mathbb{N}}$ of finite subsets of $N$ satisfying

$$\lim_{k\to\infty} \frac{|F_k \triangle F_k x|}{|F_k|} = 0$$

for all $x \in N$. For $k \in \mathbb{N}$, set $T_k = \frac{1}{|F_k|} \sum\limits_{n\in F_k} w_n$, which is an element in $\mathbb{C}[N]$.

Let $k \in \mathbb{N}$. We claim that $\|T_k\|_{F^p(N)} = 1$.

Note that $T_k$ is a linear combination of the canonical generating invertible isometries with positive coefficients (the coefficients are all either $\frac{1}{|F_k|}$ or 0). It follows from Theorem 4.19 in [195] that $\|T_k\|_p = \|T_k\|_2$. Furthermore, the equivalence between (1) and (8) in Theorem 2.6.8 in [24] shows that $\|T_k\|_2 = 1$. The claim is proved.

Fix $k \in \mathbb{N}$, and set

$$f_k = T_k \circ \Phi(f \otimes 1),$$

which is an element in $\mathcal{B}(\ell^p(G))$. (Note that $f_k$ will not in general belong to the group algebra $\mathbb{C}[G]$.) Basic properties of $p$-tensor products give $\|\Phi(f\otimes 1)\| = \|f\|$, and hence $\|f_k\| \leq \|T_k\| \cdot \|f\| = \|f\|$, so condition (1) above is satisfied. Set

$$\widetilde{f}_k = \frac{1}{|F_k|} \sum_{s\in S} \sum_{n\in F_k} a_{sN} u_{n\sigma(sN)},$$

which is an element in $\mathbb{C}[G] \subseteq \mathcal{B}(\ell^p(G))$. It is clear that $\pi(\widetilde{f}_k) = f$, so condition (2) above is also satisfied. We need to check (3). With $M = \max\limits_{s\in S} |a_{sN}|$, we have

$$\begin{aligned} \|\widetilde{f}_k - f_k\|_p &= \frac{1}{|F_k|} \left\| \sum_{s\in S} a_{sN} \sum_{n\in F_k} u_{n\sigma(sN)} - u_n \Phi(v_{sN} \otimes 1) \right\|_p \\ &\leq M \left( \frac{1}{|F_k|} \left\| \sum_{n\in F_k} u_{n\sigma(sN)} - u_n \Phi(v_{sN}\otimes 1) \right\|_p \right). \end{aligned}$$

Given $s$ in $G$, it is therefore enough to show that

$$\lim_{k\to\infty} \frac{1}{|F_k|} \left\| \sum_{n\in F_k} u_{n\sigma(sN)} - u_n\Phi(v_{sN}\otimes 1) \right\|_p = 0.$$

Fix $s$ in $G$ and set

$$\theta_k = \frac{1}{|F_k|} \sum_{n\in F_k} u_{n\sigma(sN)} - u_n\Phi(v_{sN}\otimes 1),$$

regarded as an operator on $c_c(G)$. It is immediate that for $q \in [1,\infty]$, the operator $\theta$ extends to a bounded operator $\theta_k^{(q)}$ on $\ell^q(G)$ with $\left\|\theta_k^{(q)}\right\|_q \le 2$, and the Riesz-Thorin Interpolation Theorem gives

$$\left\|\theta_k^{(p)}\right\|_p \le \left\|\theta_k^{(1)}\right\|_1^{\frac{1}{p}} \left\|\theta_k^{(\infty)}\right\|_\infty^{\frac{1}{p'}} \le 2\left\|\theta_k^{(1)}\right\|_1^{\frac{1}{p}}.$$

It therefore suffices to show that $\lim\limits_{k\to\infty}\left\|\theta_k^{(1)}\right\|_1 = 0$.

Let $c\colon G\times G \to N$ be the 2-cocycle given by

$$c(t,r)\sigma(tN)\sigma(rN) = \sigma(trN)$$

for all $t$ and $r$ in $G$. Since $G/N$ is finite, the image $\mathrm{Im}(c)$ of the 2-cocycle $c$ is a finite subset of $N$. Given $t \in G$ and $m \in N$, we have

$$\begin{aligned}
\theta_k^{(1)}(\delta_{\sigma(tN)m}) &= \frac{1}{|F_k|} \sum_{n\in F_k} \left(\delta_{n\sigma(sN)\sigma(tN)m} - \delta_{n\sigma(stN)m}\right) \\
&= \frac{1}{|F_k|} \sum_{n\in F_k} \left(\delta_{n\sigma(sN)\sigma(tN)m} - \delta_{nc(s,t)\sigma(sN)\sigma(tN)m}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\|\theta_k^{(1)}\right\|_1 &= \sup_{t\in G}\sup_{m\in N}\left\|\theta_k^{(1)}(\delta_{\sigma(tN)m})\right\|_1\\
&= \sup_{x\in\mathrm{Im}(c)}\sup_{\substack{t\in G:\\ c(s,t)=x}}\sup_{m\in N}\frac{1}{|F_k|}\left\|\sum_{n\in F_k}\delta_{n\sigma(sN)\sigma(tN)m}-\delta_{nx\sigma(sN)\sigma(tN)m}\right\|_1\\
&= \sup_{x\in\mathrm{Im}(c)}\sup_{\substack{t\in G:\\ c(s,t)=x}}\sup_{m\in N}\frac{|F_k\sigma(sN)\sigma(tN)m \bigtriangleup F_kx\sigma(sN)\sigma(tN)m|}{|F_k|}\\
&= \sup_{x\in\mathrm{Im}(c)}\sup_{\substack{t\in G:\\ c(s,t)=x}}\sup_{m\in N}\frac{|F_k\triangle F_kx|}{|F_k|}\\
&= \sup_{x\in\mathrm{Im}(c)}\frac{|F_k\triangle F_kx|}{|F_k|}
\end{aligned}$$

Since $(F_k)_k$ is a Følner sequence and $\mathrm{Im}(c)$ is finite, the above computation implies that $\lim\limits_{k\to\infty}\left\|\theta_k^{(1)}\right\|_1=0$, as desired. This finishes the proof. $\square$

We point out that the assumption that $N$ be amenable is necessary in the theorem above, at least when $p\neq 1$, as the next example shows.

**Example XV.2.4.** Fix $p\in(1,\infty)$. Let $\mathbb{F}_2$ denote the free group on two generators, and let $N$ be a normal subgroup of $\mathbb{F}_2$ such that $\mathbb{F}_2/N$ is isomorphic to $\mathbb{Z}_2$. The quotient map $\pi\colon\mathbb{F}_2\to\mathbb{Z}_2$ does not induce a quotient map $F^p_\lambda(\mathbb{F}_2)\to F^p_\lambda(\mathbb{Z}_2)$, since $F^p_\lambda(\mathbb{F}_2)$ is simple by Corollary 3.11 in [217].

On the other hand, we suspect that no condition on $G/N$ is needed for the conclusion of Theorem XV.2.3 to hold (and that, in particular, the group $G$ need not be amenable), but we have not been able to prove the more general statement. For $p=2$, this can be proved as follows. Since $N$ is amenable, its trivial representation is weakly contained in its left regular representation (see Theorem 2.6.8 in [24]). Using the fact that the induction functor preserves weak containment of representations, this shows that the left regular representation of $G/N$ is weakly contained in the left regular representation of $G$. By the comments at the beginning of Appendix D in [24], this implies that there is a homomorphism $C^*_\lambda(G)\to C^*_\lambda(G/N)$ with dense range. Finally, basic $C^*$-algebra theory (for example, the fact that homomorphisms have closed range) shows that this map is indeed a quotient map.

There is an alternative proof of this fact using Følner sets, similarly to what we did in the proof of Theorem XV.2.3, but the argument also involves the GNS construction, which so far has no analog in the context of $L^p$-operator algebras.

### An Application: When is $F^p(\mathbb{Z})$ Isomorphic to $F^q(\mathbb{Z})$?

The goal of this section is to show that for $p$ and $q$ in $[1, \infty)$, there is an isometric isomorphism between $F^p(\mathbb{Z})$ and $F^q(\mathbb{Z})$ if and only if either $p = q$ or $\frac{1}{p} + \frac{1}{q} = 1$. The strategy will be to use Theorem XV.2.3, Proposition XIV.3.13, and the fact that every homeomorphism of $S^1$ must map a pair of antipodal points to antipodal points, to reduce this to the case when the group is $\mathbb{Z}_2$, where things can be proved more directly. The fact that the spectrum of $F^p(\mathbb{Z})$ is the circle is crucial in our proof, and we do not know how to generalize these methods to deal with, for example, $\mathbb{Z}^2$.

We begin by looking at the group $L^p$-operator algebra of a finite cyclic group.

**Example XV.3.1.** Let $n$ in $\mathbb{N}$ and let $p \in [1, \infty)$. Consider the group $L^p$-operator algebra $F^p(\mathbb{Z}_n)$ of $\mathbb{Z}_n$. Then $F^p(\mathbb{Z}_n)$ is the Banach subalgebra of $\mathcal{B}(\ell^p_n)$ generated by the cyclic shift of order $n$

$$s_n = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}.$$

(The algebra $\mathcal{B}(\ell^p_n)$ is $M_n$ with the $L^p$-operator norm.) It is easy to check that $F^p(\mathbb{Z}_n)$ is isomorphic, as a complex algebra, to $\mathbb{C}^n$, but the canonical embedding $F^p(\mathbb{Z}_n) \hookrightarrow M_n$ is not as diagonal matrices.

It turns out that computing the norm of a vector in $\mathbb{C}^n \cong F^p(\mathbb{Z}_n)$ is challenging for $p$ different from 1 and 2, essentially because computing $p$-norms of matrices that are not diagonal is difficult. Indeed, set $\omega_n = e^{\frac{2\pi i}{n}}$, and set

$$u_n = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \cdots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{pmatrix}.$$

If $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, then its norm as an element in $F^p(\mathbb{Z}_n)$ is

$$\|\xi\|_{F^p(\mathbb{Z}_n)} = \left\| u_n \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{pmatrix} u_n^{-1} \right\|_p .$$

The matrix $u_n$ is a unitary (in the sense that its conjugate transpose is its inverse), and hence $\|\xi\|_{F^2(\mathbb{Z}_n)} = \|\xi\|_\infty$. The norm on $F^2(\mathbb{Z}_n)$ is therefore well-understood and easy to compute. On the other hand, if $1 \leq p \leq q \leq 2$, then $\|\cdot\|_{F^q(\mathbb{Z}_n)} \leq \|\cdot\|_{F^p(\mathbb{Z}_n)}$ by Corollary XIV.3.20. In particular, the norm $\|\cdot\|_{F^p(\mathbb{Z}_n)}$ always dominates the norm $\|\cdot\|_\infty$.

Computing the automorphism group of $F^p(\mathbb{Z}_n)$ is not easy when $p \neq 2$, since not every permutation of the coordinates of $\mathbb{C}^n \cong F^p(\mathbb{Z}_n)$ induces an isometric isomorphism. Our next result asserts that the cyclic shift on $\mathbb{C}^n$ is isometric.

**Proposition XV.3.2.** Let $n$ in $\mathbb{N}$ and let $p$ in $[1, \infty)$. Denote by $\tau \colon \mathbb{C}^n \to \mathbb{C}^n$ the cyclic forward shift, that is,

$$\tau(x_0, \ldots, x_{n-1}) = (x_{n-1}, x_0, \ldots, x_{n-2})$$

for all $(x_0, \ldots, x_{n-1}) \in \mathbb{C}^n$. Then $\tau \colon F^p(\mathbb{Z}_n) \to F^p(\mathbb{Z}_n)$ is an isometric isomorphism.

*Proof.* We follow the notation from Example XV.3.1, except that we drop the subscript $n$ everywhere, so we write $u$ in place of $u_n$, and we write $s$ in place of $s_n$. (We still denote $\omega_n = e^{\frac{2\pi i}{n}}$.)

For $x$ in $\mathbb{C}^n$, let $\mathrm{d}(x)$ denote the diagonal $n \times n$ matrix with $\mathrm{d}(x)_{j,k} = \delta_{j,k} x_j$ for $0 \leq j, k \leq n-1$. Denote by $\rho \colon \mathbb{C}^n \to M_n$ the algebra homomorphism given by $\rho(x) = u\mathrm{d}(x)u^{-1}$ for $x \in \mathbb{C}^n$.

Then

$$\|x\|_{F^p(\mathbb{Z}_n)} = \|\rho(x)\|_p = \|u\mathrm{d}(x)u^{-1}\|_p$$

for all $x \in \mathbb{C}^n$.

Set $\omega = (1, \omega_n^1, \ldots, \omega_n^{n-1}) \in \mathbb{C}^n$, and denote by $\overline{\omega}$ its (coordinatewise) conjugate. Given $x \in \mathbb{C}^n$, it is easy to check that

$$\mathrm{d}(\tau(x)) = s\mathrm{d}(x)s^{-1}, \quad us = \mathrm{d}(\omega)u, \quad \text{and} \quad s^{-1}u^{-1} = u^{-1}\mathrm{d}(\overline{\omega}).$$

It follows that

$$\|\tau(x)\|_{F^p(\mathbb{Z}_n)} = \|u\mathrm{d}(\tau(x))u^{-1}\|_p = \|us\mathrm{d}(x)s^{-1}u^{-1}\|_p = \|\mathrm{d}(\omega)u\mathrm{d}(x)u^{-1}\mathrm{d}(\overline{\omega})\|_p.$$

Since $\mathrm{d}(\omega)$ and $\mathrm{d}(\overline{\omega})$ are isometries in $\mathcal{B}(\ell_n^p)$, we conclude that

$$\|\tau(x)\|_{F^p(\mathbb{Z}_n)} = \|\mathrm{d}(\omega)u\tau(x)u^{-1}\mathrm{d}(\overline{\omega})\|_p = \|u\mathrm{d}(x)u^{-1}\|_p = \|x\|_{F^p(\mathbb{Z}_n)},$$

as desired. $\square$

Let $k$ and $n$ be positive integers. For each $r$ in $\{0, \ldots, k-1\}$, we define a restriction map

$$\rho_r^{(nk\to n)} \colon \mathbb{C}^{nk} \to \mathbb{C}^n$$

by sending a $nk$-tuple $\beta$ to the $n$-tuple

$$\rho_r^{(nk\to n)}(\beta)_q = \beta_{qk+r}, \quad q = 0, \ldots, n-1.$$

The following lemma asserts that $\rho_r^{(nk\to n)}$ is contractive when regarded as a map $F^p(\mathbb{Z}_{nk}) \to F^p(\mathbb{Z}_n)$.

**Lemma XV.3.3.** Let $k, n \in \mathbb{N}$, and let $p \in [1, \infty)$. For each $r \in \{0, \ldots, k-1\}$, the restriction map $\rho_r^{(nk\to n)}$ is a contractive, unital homomorphism $F^p(\mathbb{Z}_{nk}) \to F^p(\mathbb{Z}_n)$.

*Proof.* Let $\tau \colon F^p(\mathbb{Z}_{nk}) \to F^p(\mathbb{Z}_{nk})$ be the cyclic shift. Then $\tau$ is an isometric isomorphism by Proposition XV.3.2. Note that $\rho_r^{(nk\to n)} = \rho_0^{(nk\to n)} \circ \tau^r$. Thus, it is enough to show that $\rho_0^{(nk\to n)}$

is a contractive, unital homomorphism. This follows immediately from Proposition XV.3.2, so the proof is finished. □

It is a well-known fact that for $p \in [1, \infty) \setminus \{2\}$, the only $n$ by $n$ matrices that are isometries when regarded as linear maps $\ell_n^p \to \ell_n^p$, are precisely the complex permutation matrices. These are the matrices all of whose entries are either zero or a complex number of modulus one, that have exactly one non-zero entry on each column and each row.

Using the above mentioned fact, the proof of the following proposition is straightforward, using the description of the norm on $F^p(\mathbb{Z}_n)$ given in Example XV.3.1.

**Proposition XV.3.4.** Let $n \in \mathbb{N}$, let $p \in [1, \infty) \setminus \{2\}$. Set $\omega_n = e^{\frac{2\pi i}{n}}$. If $x \in F^p(\mathbb{Z}_n)$ is invertible and satisfies $\|x\| = \|x^{-1}\| = 1$, then there exist $\zeta \in S^1$ and $k \in \{0, \ldots, n-1\}$ such that

$$x = \zeta \cdot \left(1, \omega_n^k, \ldots, \omega_n^{(n-1)k}\right).$$

The converse also holds.

Next, we prove an easy fact that will be crucial in our proof of Theorem XVIII.2.12.

**Proposition XV.3.5.** Let $n, d \in \mathbb{N}$ with $d|n$ and $d < n$, and let $p \in [1, \infty) \setminus \{2\}$. There exists $\alpha \in F^p(\mathbb{Z}_n)$ such that

$$\|\alpha\|_{F^p(\mathbb{Z}_n)} > \sup_{b=0,\ldots,\frac{n}{d}-1} \left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)}.$$

*Proof.* Let $\omega = e^{\frac{2\pi i}{n}}$ and set

$$\beta = \left(1, \ldots, 1, \omega^d, \ldots, \omega^d, \ldots, \omega^{nd-d}, \ldots, \omega^{nd-d}\right),$$

regarded as an element in $F^p(\mathbb{Z}_n)$. (There are $\frac{n}{d}$ repetitions of each power of $\omega$.) Then $\beta$ is invertible (the inverse being its coordinate-wise complex conjugate).

We claim that

$$\sup_{b=0,\ldots,\frac{n}{d}-1} \left\|\rho_b^{(n\to d)}(\beta)\right\|_{F^p(\mathbb{Z}_d)} = \sup_{b=0,\ldots,\frac{n}{d}-1} \left\|\rho_b^{(n\to d)}(\beta^{-1})\right\|_{F^p(\mathbb{Z}_d)} = 1.$$

First, note that $\rho_b^{(n\to d)}(\beta) = \rho_a^{(n\to d)}(\beta)$ and $\rho_b^{(n\to d)}(\beta^{-1}) = \rho_a^{(n\to d)}(\beta^{-1})$ for all $a, b = 0, \ldots, \frac{n}{d} - 1$. Since $\rho_0^{(n\to d)}(\beta) = \left(1, \omega^d, \ldots, \omega^{nd-d}\right)$ is the canonical invertible isometry generating

$F^p(\mathbb{Z}_d)$, and $\rho_0^{(n\to d)}(\beta^{-1})$ is its inverse, we conclude that

$$\left\|\rho_0^{(n\to d)}(\beta)\right\|_{F^p(\mathbb{Z}_d)} = \left\|\rho_0^{(n\to d)}(\beta)\right\|_{F^p(\mathbb{Z}_d)} = 1,$$

and the claim follows.

We claim that either $\|\beta\|_{F^p(\mathbb{Z}_n)} > 1$ or $\|\beta^{-1}\|_{F^p(\mathbb{Z}_n)} > 1$.

Based on the description of the invertible isometries of $F^p(\mathbb{Z}_n)$ given in Proposition XV.3.4, it is clear that $\beta$ is not an invertible isometry, so not both $\beta$ and $\beta^{-1}$ have norm one. Since $\|\cdot\|_{F^p(\mathbb{Z}_n)} \geq \|\cdot\|_\infty$ (see Example XV.3.1) and $\|\beta\|_\infty = \|\beta^{-1}\|_\infty = 1$, the claim follows.

The result now follows by setting $\alpha$ equal to either $\beta$ or $\beta^{-1}$, as appropriate. □

The fact that $F^p(\mathbb{Z}_2)$ is isometrically isomorphic to $F^q(\mathbb{Z}_2)$ only in the trivial cases can be shown directly by computing the norm of a specific element. We do not know whether a similar computation can be done for other cyclic groups. However, knowing this for just $\mathbb{Z}_2$ is enough to prove Theorem XV.3.7.

**Proposition XV.3.6.** Let $p$ and $q$ be in $[1,\infty)$. Then $F^p(\mathbb{Z}_2)$ is isometrically isomorphic to $F^q(\mathbb{Z}_2)$ if and only if either $p = q$ of $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* The "if" implication follows from Proposition XIV.2.18. We proceed to show the "only if" implication.

Given $r$ in $[1,\infty)$, we claim that

$$\|(1,i)\|_{F^r(\mathbb{Z}_2)} = 2^{\left|\frac{1}{r}-\frac{1}{2}\right|}.$$

By Proposition XIV.2.18, the quantity on the left-hand side remains unchanged if one replaces $r$ with its conjugate exponent. Since the same holds for the quantity on the right-hand side, it follows that it is enough to prove the claim for $r$ in $[1,2]$.

Define a continuous function $\gamma\colon [1,2] \to \mathbb{R}$ by $\gamma(r) = \|(1,i)\|_{F^r(\mathbb{Z}_2)}$ for $r$ in $[1,2]$. Let $a$ be the matrix

$$a = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}.$$

Then $\gamma(r) = \|a\|_r$ for all $r \in [1,2]$. The values of $\gamma$ at $r = 1$ and $r = 2$ are easy to compute, and we have $\gamma(1) = \|a\|_1 = 2^{\frac{1}{2}}$ and $\gamma(2) = \|a\|_2 = 1$. Fix $r \in [1,2]$ and let $\theta$ in $(0,1)$ satisfy

$$\frac{1}{r} = \frac{1-\theta}{1} + \frac{\theta}{2}.$$

Using the Riesz-Thorin Interpolation Theorem between $r_0 = 1$ and $r_1 = 2$, we conclude that

$$\gamma(r) \leq \gamma(1)^{1-\theta} \cdot \gamma(2)^{\theta} = 2^{\frac{1}{2}(\frac{2}{r}-1)} \cdot 1 = 2^{\frac{1}{r}-\frac{1}{2}}.$$

For the converse inequality, fix $r$ in $[1,2]$ and consider the vector $x = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ in $\ell_2^r$. Then $\|x\|_r = 1$ and $ax = \frac{1}{2}\left(\begin{smallmatrix} 1+i \\ 1-i \end{smallmatrix}\right)$. We compute:

$$\left\| \frac{1}{2} \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \right\|_r = \frac{1}{2}(|1+i|^r + |1-i|^r)^{\frac{1}{r}} = 2^{(\frac{1}{r}-\frac{1}{2})}.$$

We conclude that

$$\gamma(r) = \|a\|_r \geq \frac{\|ax\|_r}{\|x\|_r} = 2^{(\frac{1}{r}-\frac{1}{2})}.$$

This shows that $\gamma(r) = 2^{(\frac{1}{r}-\frac{1}{2})}$ for $r \in [1,2]$, and the claim follows.

Now let $p$ and $q$ be in $[1,\infty)$ and let $\varphi\colon F^p(\mathbb{Z}_2) \to F^q(\mathbb{Z}_2)$ be an isometric isomorphism. Since $\varphi$ is an algebra isomorphism, we must have either $\varphi(x,y) = (x,y)$ or $\varphi(x,y) = (y,x)$ for all $(x,y) \in \mathbb{C}^2$. By Proposition XV.3.2, the flip $(x,y) \mapsto (y,x)$ is an isometric isomorphism of $F^q(\mathbb{Z}_2)$, so we may assume that $\varphi$ is the identity map on $\mathbb{C}^2$. It follows that $\|(1,i)\|_{F^p(\mathbb{Z}_2)} = \|(1,i)\|_{F^q(\mathbb{Z}_2)}$, so $\left|\frac{1}{p} - \frac{1}{2}\right| = \left|\frac{1}{q} - \frac{1}{2}\right|$. We conclude that either $p = q$ or $\frac{1}{p} + \frac{1}{q} = 1$, so the proof is complete. $\square$

We are now ready to show that for $p$ and $q$ in $[1,\infty)$, the algebras $F^p(\mathbb{Z})$ and $F^q(\mathbb{Z})$ are (abstractly) isometrically isomorphic only in the trivial cases $p = q$ and $\frac{1}{p} + \frac{1}{q} = 1$. (Compare this with part (2) of Corollary XIV.3.20, where only the canonical homomorphism is considered.)

**Theorem XV.3.7.** Let $p$ and $q$ be in $[1,\infty)$. Then $F^p(\mathbb{Z})$ is isometrically isomorphic to $F^q(\mathbb{Z})$ if and only if either $p = q$ or $\frac{1}{q} + \frac{1}{q} = 1$.

*Proof.* The "if" implication follows from Proposition XIV.2.18. Let us show the converse.

Recall that the maximal ideal spaces of $F^p(\mathbb{Z})$ and $F^q(\mathbb{Z})$ are canonically homeomorphic to $S^1$ by Proposition XIV.3.13. We let $\Gamma_p\colon F^p(\mathbb{Z}) \to C(S^1)$ denote the Gelfand transform, which sends the generator $u \in F^p(\mathbb{Z})$ to the canonical inclusion $\iota$ of $S^1$ into $\mathbb{C}$.

Let $\varphi\colon F^p(\mathbb{Z}) \to F^q(\mathbb{Z})$ be an isometric isomorphism. Then $\varphi$ induces a homeomorphism $f\colon S^1 \to S^1$ that maps $z$ in $S^1$ to the unique point $f(z)$ in $S^1$ that satisfies

$$\mathrm{ev}_z \circ \varphi = \mathrm{ev}_{f(z)}\colon F^p(\mathbb{Z}) \to \mathbb{C}.$$

It is a classical result in point-set topology that there must exist $\zeta$ in $S^1$ such that $f(-\zeta) = -f(\zeta)$. Denote by $\pi_p\colon F^p(\mathbb{Z}) \to F^p(\mathbb{Z}_2)$ and $\pi_q\colon F^p(\mathbb{Z}) \to F^q(\mathbb{Z}_2)$ the canonical homomorphisms associated with the surjective map $\mathbb{Z} \to \mathbb{Z}_2$. Then $\pi_p$ and $\pi_q$ are quotient maps by Theorem XV.2.3. Let $\omega_\zeta\colon F^p(\mathbb{Z}) \to F^p(\mathbb{Z})$ be the isometric isomorphism induced by multiplying by $\zeta$ the canonical generator in $F^p(\mathbb{Z})$ corresponding to $1 \in \mathbb{Z}$. Analogously, let $\omega_{f(\zeta)}\colon F^q(\mathbb{Z}) \to F^q(\mathbb{Z})$ be the isometric isomorphism induced by multiplying by $f(\zeta)$ the canonical generator in $F^q(\mathbb{Z})$. Then the following diagram is commutative:

$$\begin{tikzcd}
 & C(S^1) \arrow[r, "f^*"] & C(S^1) & \\
F^p(\mathbb{Z}) \arrow[d, "\pi_p"'] & F^p(\mathbb{Z}) \arrow[l, "\omega_\zeta"'] \arrow[u, "\Gamma_p"] \arrow[r, "\varphi"'] & F^q(\mathbb{Z}) \arrow[u, "\Gamma_q"'] \arrow[r, "\omega_{f(\zeta)}"] & F^q(\mathbb{Z}) \arrow[d, "\pi_q"] \\
F^p(\mathbb{Z}_2) \arrow[rrr, dashed, "\widehat{\psi}"'] & & & F^q(\mathbb{Z}_2).
\end{tikzcd}$$

Define a homomorphism $\psi\colon F^p(\mathbb{Z}) \to F^q(\mathbb{Z})$ by

$$\psi = \omega_{f(\zeta)} \circ \varphi \circ \omega_\zeta^{-1}.$$

Then $\psi$ is an isometric isomorphism. One checks that $\psi$ maps the kernel of $\pi_p$ onto the kernel of $\pi_q$. It follows that $\psi$ induces an isometric isomorphism $\widehat{\psi}\colon F^p(\mathbb{Z}_2) \to F^q(\mathbb{Z}_2)$. By Proposition XV.3.6, this implies that $p$ and $q$ are either equal or conjugate, as desired. □

CHAPTER XVI

# THE ISOMORPHISM PROBLEM FOR ALGEBRAS OF CONVOLUTION OPERATORS

This chapter is based on joint work with Hannes Thiel ([99]).

In this short chapter, which is based on [99], we study the structure of contractive, unital homomorphisms between algebras of convolution operators on groups. We focus on discrete groups, but this assumption is not necessary. The results presented here are not the most general possible, but they are interesting enough not to be omitted. We reproduce below the most general form of our results.

**Theorem.** Let $G$ and $H$ be locally compact groups, let $p \in [1,\infty)$ with $p \neq 2$. Let $\varphi\colon F^p_\lambda(G) \to F^p_\lambda(H)$, or $\varphi\colon PM_p(G) \to PM_p(H)$, or $\varphi\colon CV_p(G) \to CV_p(H)$, be a contractive, nondegenerate homomorphism. Then

1. There exist a group homomorphism $\theta\colon G \to H$ with amenable kernel and a group homomorphism $\gamma\colon G \to \mathbb{T}$ such that $\varphi$ (or its extension to $PM_p(G) \to PM_p(H)$) is determined by
$$\varphi(u_s) = \gamma(s)u_{\theta(s)}$$
for all $s \in G$.
2. The range of $\varphi$ is closed.
3. The homomorphism $\varphi$ is isometric if and only if it is injective.

Moreover, $\varphi$ is surjective if and only if $\theta$ is surjective.

Our results generalize Wendel's Theorem from the 60's, that asserts that there is a contractive isomorphism $L^1(G) \cong L^1(H)$ if and only if $G \cong H$. Our techniques differ from those used by Wendel: while he used extreme points of the unit ball, we use invertible isometries. Our main technical device is Lampert'is theorem.

### Contractive Homomorphisms Between Algebras of Convolution

## Operators

We begin by recalling a particular case of Lamperti's theorem. Observe that no $\sigma$-finiteness assumption is needed.

**Theorem XVI.1.1.** (See Theorem XIII.2.4). Let $X$ be a set endowed with the counting measure, let $p \in [1, \infty)$ with $p \neq 2$, and let $u\colon \ell^p(X) \to \ell^p(X)$ be an invertible isometry. Then there exist a function $h\colon X \to \mathbb{T}$ and a bijective map $T\colon X \to X$ such that $u = m_h \cdot u_T$, that is,

$$u(\xi)(x) = h(x)\xi(T^{-1}(x))$$

for $\xi \in \ell^p(X)$ and $x \in X$.

Moreover, this presentation of $u$ is unique in the following sense: Given functions $h_1, h_2\colon X \to \mathbb{T}$ and bijective maps $T_1, T_2\colon X \to X$, we have $m_{h_1} \cdot u_{T_1} = m_{h_2} \cdot u_{T_2}$ if and only if $h_1 = h_2$ and $T_1 = T_2$.

We now specialize to invertible isometries on $\ell^p(G)$ for a discrete group $G$. If one moreover assumes that the invertible isometry in question commutes with the right regular representation, then the conclusion of Lamperti's theorem can be improved significantly, as we show below.

**Proposition XVI.1.2.** Let $G$ be a discrete group, let $p \in [1, \infty)$ with $p \neq 2$, and let $u$ be an invertible isometry in $CV_p(G)$. Then there exist $\alpha \in \mathbb{T}$ and $s \in G$ such that $u = \alpha \cdot u_s$.

*Proof.* By Theorem XVI.1.1, there exist a bijection $T\colon G \to G$ and a function $h\colon G \to \mathbb{T}$ such that $u = m_h \circ u_T$ For $s \in G$, set $u_s = \lambda_p(s)$ and $v_s = \rho_p(s)$, which are invertible isometries on $\ell^p(G)$. Observe that $v_s \in CV_p(G)$. For $s \in G$, we define the left and right translation maps $\mathtt{Lt}_s, \mathtt{Rt}_s\colon G \to G$ on $G$ by

$$\mathtt{Lt}_s(x) = s^{-1}x, \quad \text{and} \quad \mathtt{Rt}_s(x) = xs$$

for $x \in G$. For $\xi \in \ell^p(G)$ and $t \in G$, we have

$$v_s(\xi)(t) = \xi(ts) = \xi(\mathtt{Rt}_s(t)) = u_{\mathtt{Rt}_s^{-1}}(\xi)(t),$$

which implies $v_s = u_{\mathtt{Rt}_s^{-1}}$. Similarly, we have $u_s = u_{\mathtt{Lt}_s^{-1}}$.

Using the assumption that $u$ and $v_s$ commute at the third step, we get

$$m_h \circ u_{T\circ \mathtt{Rt}_s^{-1}} = m_h \circ u_T \circ u_{\mathtt{Rt}_s^{-1}} = u \circ v_s = v_s \circ u = u_{\mathtt{Rt}_s^{-1}} \circ m_h \circ u_T = m_{h\circ \mathtt{Rt}_s} \circ u_{\mathtt{Rt}_s^{-1}\circ T}.$$

This implies that $h = h \circ \mathtt{Rt}_s$ and $T \circ \mathtt{Rt}_s^{-1} = \mathtt{Rt}_s^{-1} \circ T$. Since this holds for all $s \in G$, we immediately deduce that $h$ is constant. Let $\alpha \in \mathbb{T}$ be the constant value of $h$. Denote by $e \in G$ the unit of the group, and set $y = T(e)^{-1}$. For $x \in G$, we compute

$$T(x) = T(ex) = (T \circ \mathtt{Rt}_x)(e) = (\mathtt{Rt}_x \circ T)(e) = T(e)x = \mathtt{Lt}_y^{-1}(x)$$

which implies that $T = \mathtt{Lt}_y^{-1}$. Thus, $u_T = u_y$ and consequently $u = m_h \circ u_T = \alpha \cdot u_y$, as desired. This finishes the proof. $\square$

The next result does not require that $p$ be different from 2. It is surely well known, and we include its proof here for the sake of completeness.

**Lemma XVI.1.3.** Let $G$ be a discrete group, let $p \in [1, \infty)$, let $g, h\colon G \to \mathbb{T}$, and let $S, T\colon G \to G$ be bijective maps.

1. If $S \neq T$, then $\|m_g \circ u_S - m_h \circ u_T\| = 2$.

2. If $S = T$, then $\|m_g \circ u_S - m_h \circ u_T\| = \|g - h\|_\infty$.

*Proof.* (1) Using that $m_g \circ u_S$ is an invertible isometry at the first step, we have

$$\|m_h \circ u_S - m_h \circ u_T\| = \|1 - u_S^{-1} \circ m_g^{-1} \circ m_h \circ u_T\| = \|1 - m_{(\overline{g}h)\circ S^{-1}} u_{S^{-1}T}\|.$$

Thus, it is enough to verify (1) under the additional assumption that $g = 1$ and $S = \mathrm{id}$. Put differently, let $f \in \ell^\infty(G, \mathbb{T})$, and let $Q\colon G \to G$ be a bijective map with $Q \neq \mathrm{id}$. We need to show that

$$\|1 - m_f \circ u_Q\| = 2.$$

Since $Q \neq \mathrm{id}$, there exists $s$ in $G$ such that $Q(s) \neq s$. Set $\xi = \delta_s$, the Dirac function at the point $s$. Then $\xi \in \ell^p(G)$ and $\|\xi\|_p = 1$. By construction, the functions $\xi$ and $(m_f \circ u_Q)(\xi)$ have disjoint support, which implies

$$\|1 - m_f \circ u_Q\| \geq \|(1 - m_f \circ u_Q)(\xi)\|_p = \|\xi\|_p + \|(m_f \circ u_Q)(\xi)\|_p = 2.$$

Conversely, we clearly have $\|1 - m_f \circ u_Q\| \leq \|1\| + \|m_f \circ u_Q\| = 2$.

(2) Using that $u_S$ is an invertible isometry at the last step, we have

$$\|m_h \circ u_S - m_h \circ u_T\| = \|(m_g - m_h) \circ u_S\| = \|m_g - m_h\|.$$

Given $\xi \in \ell^p(G)$, we have

$$\begin{aligned}
\|(m_g - m_h)(\xi)\|_p^p &= \sum_{s \in G} |g(s) - h(s)|^p \cdot |\xi(s)|^p \\
&\leq \sum_{s \in G} \|g - h\|_\infty^p \cdot |\xi(s)|^p \\
&= \|g - h\|_\infty^p \|\xi\|_p^p,
\end{aligned}$$

which implies that $\|m_g - m_h\| \leq \|g - h\|_\infty$.

For the converse inequality, let $\varepsilon > 0$ be given. Choose $s$ in $G$ such that $|g(s) - h(s)| \geq \|g - h\|_\infty - \varepsilon$. Again, we consider the Dirac function $\delta_s$. Then $\|\delta_s\|_p = 1$ and we compute

$$\begin{aligned}
\|m_g - m_h\| &\geq \|(m_g - m_h)(\delta_s)\|_p \\
&= \left( \sum_{x \in G} |g(x) - h(x)|^p \cdot |\delta_s(x)|^p \right)^{1/p} \\
&= (|g(s) - h(s)|^p \cdot)^{1/p} \geq \|g - h\|_\infty - \varepsilon.
\end{aligned}$$

Since this holds for every $\varepsilon > 0$, we conclude that $\|m_g - m_h\| \geq \|g - h\|_\infty$, as desired. □

**Notation XVI.1.4.** Let $A$ be a unital Banach algebra. We let $\mathcal{U}(A)$ denote the group of invertible isometries in $A$. We let $\mathcal{U}(A)_0$ denote the connected component of $\mathcal{U}(A)$ in the norm topology that contains the unit of $A$. Then $\mathcal{U}(A)_0$ is a normal subgroup of $\mathcal{U}(A)$ and we set

$$\pi_0^{\|\cdot\|}(\mathcal{U}(A)) = \mathcal{U}(A)/\mathcal{U}(A)_0,$$

the group of connected components of $\mathcal{U}(A)$ for the norm topology. We denote by $\kappa\colon \mathcal{U}(A) \to \pi_0^{\|\cdot\|}(\mathcal{U}(A))$ the canonical quotient map, which is a group homomorphism.

**Theorem XVI.1.5.** Let $G$ be a discrete group, and let $p \in [1,\infty)$ with $p \neq 2$. Then the map

$$\Delta\colon \mathbb{T} \times G \to \mathcal{U}(CV_p(G)),$$

given by $\Delta(\alpha, s) = \alpha \cdot u_s$ for $\alpha \in \mathbb{T}$ and $s \in G$, is a group isomorphism.

Given $(\alpha_k, s_k) \in \mathbb{T} \times G$ for $k = 1, 2$, the elements $\Delta(\alpha_1, s_1)$ and $\Delta(\alpha_2, s_2)$ lie in the same connected component of $\mathcal{U}(CV_p(G))$ for the norm topology if and only if $s_1 = s_2$. Thus, $\Delta$ induces a group isomorphism $\Delta'$ between $G$ and $\pi_0^{\|\cdot\|}(\mathcal{U}(CV_p(G)))$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{T} \times G \arrow[r, "\Delta"] \arrow[d] & \mathcal{U}(CV_p(G)) \arrow[d, "\kappa"] \\
G \arrow[r, "\Delta'"'] & \pi_0^{\|\cdot\|}(\mathcal{U}(CV_p(G))).
\end{tikzcd}$$

*Proof.* It follows from Proposition XVI.1.2 that the map $\Delta$ is a bijection. Since it is clearly a group homomorphism, the first part of the theorem follows. Finally, Lemma XVI.1.3 implies the rest of the statement. $\square$

**Corollary XVI.1.6.** Let $G$ be a discrete group, and let $p \in [1,\infty)$ with $p \neq 2$. Let $A$ be a closed, unital subalgebra of $\mathcal{B}(\ell^p(G)$ such that $F^p_\lambda(G) \subseteq A \subseteq CV_p(G)$. Then there is a natural group isomorphism:

$$G \cong \pi_0^{\|\cdot\|}(\mathcal{U}(A)).$$

This, in particular, applies when $A$ is one of $F^p_\lambda(G)$, $PM_p(G)$, or $CV_p(G)$.

The following is our main result.

**Theorem XVI.1.7.** Let $G$ and $H$ be discrete groups, and let $p \in [1,\infty)$ with $p \neq 2$. Let $\varphi\colon F^p_\lambda(G) \to F^p_\lambda(H)$ be a contractive, unital homomorphism. Then:

1. There exist an injective group homomorphism $\theta\colon G \to H$ and a group homomorphism $\gamma\colon G \to \mathbb{T}$ such that $\varphi$ is determined by

$$\varphi(u_s) = \gamma(s)u_{\theta(s)}$$

for all $s \in G$.

2. The homomorphism $\varphi$ is isometric.

Moreover, $\varphi$ is surjective if and only if $\theta$ is surjective.

*Proof.* Since $\varphi$ is contractive, we have $\varphi(\mathcal{U}(F^p_\lambda(G))) \subseteq \mathcal{U}(F^p_\lambda(H))$. Using Theorem XVI.1.5, it is easy to see that $\varphi$ induces a group homomorphism $\theta\colon G \to H$, which is clearly injective. There is a group isomorphism

$$\sigma\colon \mathbb{T} \times G \to \mathbb{T} \times H,$$

whose coordinates are denoted by $\sigma_{\mathbb{T}}$ and $\sigma_H$. It is clear that $\sigma_H(\zeta, g) = \zeta u_{\theta(g)}$ for all $(\zeta, g) \in \mathbb{T} \times G$. On the other hand, $\sigma_{\mathbb{T}}(\zeta, g)$ must agree with $\zeta\sigma(1, g)$, and the map $\gamma\colon G \to \mathbb{T}$, given by $\gamma(g) = \sigma(1, g)$ for $g \in G$, is easily seen to be a group homomorphism.

One readily checks that $\gamma$ induces an isometric isomorphism $\Gamma\colon F^p_\lambda(G) \to F^p_\lambda(G)$ given by $\Gamma(u_g) = \gamma(g)u_g$ for $g \in G$. Denote by $\psi\colon F^p_\lambda(G) \to F^p_\lambda(H)$ the isometric homomorphism induced by $\iota$ as in Proposition XV.2.1. Since $\varphi$ and $\psi \circ \Gamma$ agree on $\mathcal{U}(F^p_\lambda(G))$, they must be equal, so the formula given in the statement holds. Finally, since $\psi$ and $\Gamma$ are both isometric, it follows that so is $\varphi$.

If $\varphi$ is surjective, we must have $\varphi(\mathcal{U}(F^p_\lambda(G))) = \mathcal{U}(F^p_\lambda(H))$, so we deduce that $\theta$ is also surjective. The converse implication is immediate. $\square$

Similar conclusions can be obtained for contractive unital homomorphisms between the algebras of pseudomeasures, or the algebras of convolvers.

We close this chapter with an interesting application, which is connected to the Kadison-Kaplansky conjecture for reduced group $C^*$-algebras.

**Theorem XVI.1.8.** Let $G$ be a torsion free discrete group, and let $p \in [1, \infty)$ with $p \neq 2$. Then $F^p_\lambda(G)$ does not contain a nontrivial bicontractive idempotent.

*Proof.* Let $e \in F^p_\lambda(G)$ be a bicontractive idempotent $e$. By Theorem 1 in [11], the element $v = 2e - 1$ is an invertible isometry, which clearly has order two. Since $F^p_\lambda(\mathbb{Z}_2)$ is universal with respect to isometric representations of $\mathbb{Z}_2$, we deduce that there exists a contractive, unital homomorphism $\varphi\colon F^p_\lambda(\mathbb{Z}_2) \to F^p_\lambda(G)$ given by sending the nontrivial generating invertible isometry in $F^p_\lambda(\mathbb{Z}_2)$ to $v$. Observe that $\varphi$ is injective if and only if $e \neq 1, 0$. By Theorem XVI.1.7, if $\varphi$ were injective, then there is an injective group homomorphism $\mathbb{Z}_2 \to G$, which would contradict the fact that $G$ is torsion free. Hence $\varphi$ is not injective, and thus $e$ is either 0 or 1. We conclude that $F^p_\lambda(G)$ has no nontrivial bicontractive idempotents. $\square$

It is a long standing open problem to decide whether for a torsion free group $G$, the $C^*$-algebra $C^*_\lambda(G)$ contains a nontrivial bicontractive idempotent.

# CHAPTER XVII

# CONVOLUTION ALGEBRAS ON $L^P$-SPACES DO NOT ACT ON $L^Q$-SPACES

This chapter is based on joint work with Hannes Thiel ([97]).

Let $G$ be a non-trivial locally compact group, and let $p \in [1, \infty)$. Consider the following Banach algebras: $p$-pseudofunctions $PF_p(G)$, $p$-pseudomeasures $PM_p(G)$, $p$-convolvers $CV_p(G)$, and full group $L^p$-operator algebra $F^p(G)$. We show that none of these Banach algebras are isometrically operator algebras unless $p = 2$. When $G$ is amenable, these Banach algebras are representable on an $L^q$-space if and only if $p$ and $q$ are either equal or conjugate.

## Introduction

Associated to any locally compact group $G$, there are three fundamentally important operator algebras: its reduced group $C^*$-algebra $C^*_\lambda(G)$, its full group $C^*$-algebra $C^*(G)$, and its group von Neumann algebra $L(G)$. These are, respectively, the Banach algebra generated by the left regular representation of $G$ in $\mathcal{B}(L^2(G))$; the universal $C^*$-algebra with respect to unitary representations of $G$ on Hilbert spaces; and the weak-$*$ closure (also called ultraweak closure) of $C^*_\lambda(G)$ in $\mathcal{B}(L^2(G))$. (We identify $\mathcal{B}(L^2(G))$ with the dual of the projective tensor product $L^2(G)\widehat{\otimes}L^2(G)$ canonically.) Equivalently, $L(G)$ is the double commutant of $C^*_\lambda(G)$ in $\mathcal{B}(L^2(G))$.

These operator algebras admit generalizations to representations of $G$ on $L^p$-spaces, for $p \in [1, \infty)$. The analog of $C^*_\lambda(G)$ is the the algebra $PF_p(G)$ of $p$-pseudofunctions on $G$, first introduced by Herz in [116] (Phillips also considered this algebra in [207], where he called it the reduced group $L^p$-operator algebra of $G$). The analog of $C^*(G)$ is the full group $L^p$-operator algebra $F^p(G)$, defined by Phillips in [207]. Finally, the von Neumann algebra $L(G)$ has two analogs: the algebra of $p$-pseudomeasures $PM_p(G)$, which is the weak-$*$ closure of $F^p_\lambda(G)$ in $\mathcal{B}(L^p(G))$ (where we identify $\mathcal{B}(L^p(G))$ with the dual of the projective tensor product $L^p(G)\widehat{\otimes}(L^p(G))^*$ canonically); and the algebra of $p$-convolvers $CV_p(G)$, which is the double commutant of $PF_p(G)$ in $\mathcal{B}(L^p(G))$ (it is also the commutant of the right regular representation). Both these algebras were introduced by Herz in [116].

These objects, and related ones, have been studied by a number of authors in the last three decades. For instance, see [38], [189], [242], [49], and the more recent papers [207], [211], [98], and [94].

Despite the advances in the area, some basic questions remain open. One important open problem is whether $PM_p(G) = CV_p(G)$ for all $p \in [1, \infty)$ and for all locally compact groups $G$. Herz showed in [116] that this is the case for all $p$ if $G$ is amenable, a result that was later generalized by Cowling in [38] to groups with the approximation property.

A less studied problem is the following. By universality of $F^p(G)$, there is a contractive homomorphism $\kappa_p \colon F^p(G) \to PF_p(G)$ with dense range. For $p = 2$, this map is known to be a quotient map, and for $p = 1$ it is an isomorphism regardless of $G$. On the other hand, we do not know if $\kappa_p$ is also a quotient map for all other values of $p$. In fact, we do not even know whether $\kappa_p$ is surjective. If this map is not necessarily surjective, can it be injective without the group being amenable? (By Theorem XIV.3.7, $G$ is amenable if and only if $\kappa_p$ is bijective for some (equivalently, for all) $p \in (1, \infty)$. This result was independently obtained by Phillips in [207] and [211], using different methods.) In this case, it would be interesting to describe precisely for what groups (and Hölder exponents) the map $\kappa^p$ is injective (and not surjective).

Questions of the nature described above would in principle be easier to attack if the objects considered had a more rigid structure (or at least, better understood), as it is the case for operator algebras. Despite the fact that the Banach algebras $PF_p(G), F^p(G), PM_p(G)$ and $CV_p(G)$ have a natural representation as operators on an $L^p$-space, this by itself does not rule out having an isometric representation on a Hilbert space as well; see, for example, [16]. ISince every $C^*$-algebra can be (isometrically) represented on a non-commutative $L^p$-space by Proposition XVII.2.6, it is not a priori clear whether the $L^p$-analogs of group operator algebras can be represented on Hilbert spaces.

In this chapter, which is based on [97], we settle this question negatively. Indeed, we show in Theorem XVII.2.7 that for a non-trivial locally compact group $G$, and for $p \in [1, \infty) \setminus \{2\}$, none of the algebras $PF_p(G), F^p(G), PM_p(G)$, or $CV_p(G)$, can be represented on a Hilbert space. (This result generalizes an unpublished result of Neufang and Runde; Theorem 2.2 in [189], where the authors assume that $G$ is abelian and has an infinite subgroup.) When $G$ is amenable and for $p, q \in [1, \infty)$, we show that one (equivalently, all) of the algebras $PF_p(G), F^p(G), PM_p(G)$, or

$CV_p(G)$, can be represented on an $L^q$-space if and only if either $p = q$ or $\frac{1}{p} + \frac{1}{q} = 1$. Our main tool to prove both these theorems, is that for $1 \leq p \leq q \leq 2$, there exists a canonical, contractive map $\gamma_{p,q}\colon F^p(G) \to F^q(G)$ with dense range; see Theorem XIV.2.30.

As an intermediate result of independent interest, we show that the only $C^*$-algebras that can be isometrically represented on some $L^p$-space, for $p \in [1,\infty) \setminus \{2\}$, are the commutative ones. This result is somewhat surprising at first sight, and it should be compared with the fact that every $C^*$-algebra can be represented on a *noncommutative* $L^p$-space, for any $p \in [1,\infty)$; see Proposition XVII.2.6.

### Convolution Algebras on $L^p$-spaces are not Operator Algebras

Proposition XVII.2.1 is our first preparatory result on representability of full group $L^p$-operator algebras on $L^q$-spaces. We need some notation first. Let $G$ be a locally compact group, and denote by $\Delta\colon G \to \mathbb{R}$ its modular function. For $f \in L^1(G)$, let $f^\sharp\colon G \to \mathbb{C}$ be given by $f^\sharp(s) = \Delta(s^{-1})f(s^{-1})$ for all $s$ in $G$. It is easy to check that the map $\sharp\colon L^1(G) \to L^1(G)$ is an isometric anti-isomorphism of order two.

**Proposition XVII.2.1.** Let $G$ be a locally compact group, and let $p, q \in [1,\infty)$. Suppose that $F^p(G)$ is isometrically representable on an $L^q$-space.

1. If $p, q \in [1,2]$ with $p \leq q$, or $p, q \in [2,\infty)$ with $p \geq q$, then the identity map on $L^1(G)$ extends to an isometric isomorphism $F^p(G) \cong F^q(G)$.

2. If $p \in [1,2]$ and $q \in [2,\infty)$ with $p \leq q'$, or if $q \in [1,2]$ and $p \in [2,\infty)$ with $p' \leq q$, then the map $\sharp$ on $L^1(G)$ extends to an isometric isomorphism $F^p(G) \cong F^q(G)$.

*Proof.* (1). Since the result is trivial when $p = q$, we may suppose, without loss of generality, that $1 \leq p < q \leq 2$. Given $f \in L^1(G)$, we have $\|f\|_{F^q(G)} \leq \|f\|_{F^p(G)}$ by Theorem XIV.2.30. Suppose that there exist an $L^q$-space $E$ and an isometric representation $\varphi\colon F^p(G) \to \mathcal{B}(E)$. Let $\iota_p\colon L^1(G) \to F^p(G)$ be the canonical contractive inclusion with dense range. Then

$$\psi = \varphi \circ \iota_p\colon L^1(G) \to \mathcal{B}(E)$$

is a contractive representation of $L^1(G)$ on an $L^q$-space. Note, however, that this representation is not necessarily non-degenerate, so it does not necessarily induce a representation of $F^q(G)$. In order to circumvent this, we consider the Banach space

$$F = \overline{\{\pi(f)\xi \colon f \in L^1(G), \xi \in E\}}.$$

Then $F$ is a closed subspace of an $L^q$-space, and the compression $\psi\colon L^1(G) \to \mathcal{B}(F)$ is non-degenerate. It is a standard fact that there exists a strongly continuous group representation $\rho\colon G \to \mathrm{Isom}(F)$ of $G$ by invertible isometries of $F$ whose integrated form is $\psi$ (see, for example, Proposition XIV.2.4).

Using the definition of the norm on $F^q_{\mathrm{S}}(G)$ (see Definition XIV.2.1 and Definition XIV.2.5) at the last step, we deduce that

$$\|f\|_{F^p(G)} \leq \|f\|_{F^p(G)} = \|\psi(f)\|_{\mathcal{B}(F)} \leq \|f\|_{F^q_{\mathrm{S}}(G)}.$$

Finally, since $\|f\|_{F^q(G)} = \|f\|_{F^q_{\mathrm{S}}(G)}$ by Theorem XIV.2.26, we deduce that $\|f\|_{F^q(G)} = \|f\|_{F^p(G)}$. Since $f \in L^1(G)$ is arbitrary, it follows that the identity on $L^1(G)$ extends to an isometric isomorphism $F^p(G) \to F^q(G)$.

(2). We can assume, without loss of generality, that $1 \leq p \leq 2 \leq q < \infty$. Denote by $q' \in [1,2]$ Hölder conjugate exponent of $q$. By Proposition XIV.2.18, the map $\sharp\colon L^1(G) \to L^1(G)$ extends to an isometric isomorphism $F^q(G) \to F^{q'}(G)$. (The details are in the proof of Lemma XIV.2.16, the main point being that given a representation of $G$ on an $L^q$-space, its dual representation, is a representation of $G^{op}$ on $L^{q'}$, which induces the same norm on $L^1(G)$. One composes this isometric anti-isomorphism with the map $\sharp$ to get an honest (isometric) isomorphism.)

Since the identity map on $L^1(G)$ extends to an isometric isomorphism $F^p(G) \to F^{q'}(G)$ by part (1), the result follows.

□

Besides $F^p(G)$ and $PF_p(G)$, the other two Banach algebras we will be concerned with are the algebra $PM_p(G)$ of $p$-pseudomeasures, and the algebra $CV_p(G)$ of $p$-convolvers. These are respectively defined as the ultraweak closure, and the bicommutant, of $PF_p(G)$ in $\mathcal{B}(L^p(G))$.

Algebras of pseudomeasures and of convolvers on groups have been thoroughly studied since their inception by Herz in the early 70's; see [116]. It is clear that $PM_p(G) \subseteq CV_p(G)$, and it is conjectured that they are equal for every locally compact group $G$ and every Hölder exponent $p \in [1, \infty)$. The conjecture is known to be true if $p = 2$, or if $G$ is amenable ([116]), or, more generally, if $G$ has the approximation property ([38]).

We recall a result from [99] about multiplier algebras of $L^p$-operator algebras.

**Theorem XVII.2.2.** (Corollary 2.5 in [99].) Let $p \in [1, \infty)$ and let $A$ be an $L^p$-operator algebra with a contractive approximate identity. Assume that $A$ can be non-degenerately represented on an $L^p$-space. Then the multiplier algebra $M(A)$ is an $L^p$-operator algebra. Indeed, if $A$ is non-degenerately representable on an $L^p$-space $E$, then $M(A)$ is unitally representable on $E$.

**Proposition XVII.2.3.** Let $A$ be a $C^*$-algebra and let $p \in [1, \infty) \setminus \{2\}$. Then $A$ can be non-degenerately, isometrically represented on an $L^p$-space if and only if $A$ is commutative.

*Proof.* It is a well-known fact that $C^*$-algebras have contractive approximate identities (one can, for example, take the set of all positive contractions with the canonical order).

Let $E$ be an $L^p$-space and let $A \to \mathcal{B}(E)$ be a non-degenerate isometric homomorphism. Then there is a unital isometric homomorphism $M(A) \to \mathcal{B}(E)$ by Theorem XVII.2.2. Since $A$ is commutative if and only if $M(A)$ is commutative, we can assume that $A$ is unital.

Since any element in a $C^*$-algebra is a linear combination of four positive elements, it is enough to show that such elements commute with each other.

Let $a, b \in A$ with $a, b \geq 0$. We claim that $ab = ba$. For this, it is enough to show that the unitaries $v = e^{ia}$ and $w = e^{ib}$ commute. Since $C^*(1, v, w)$ is a separable unital $C^*$-algebra, we may assume that $A$ itself is separable. In this case, by Proposition 1.25 in [207] (see also Remark 1.18 there), there exist a $\sigma$-finite measure space $(X, \mathcal{A}, \mu)$ and an isometric, unital representation $\pi\colon A \to \mathcal{B}(L^p(X, \mu))$. Without loss of generality, we may assume that $(X, \mathcal{A}, \mu)$ is complete.

The operators $\pi(v)$ and $\pi(w)$ are invertible isometries of $L^p(X, \mu)$. Since $v$ and $w$ are homotopic to the unit of $A$ within the unitaries in $A$, it follows that $\pi(v)$ and $\pi(w)$ can be connected, by norm-continuous paths of invertible isometries in $\mathcal{B}(L^p(X, \mu))$, to the unit of $\mathcal{B}(L^p(X, \mu))$.

By Lamperti's theorem (in the form given in Theorem XIII.2.4; see also [161] for the original statement), there exist measurable functions $h_v, h_w \colon X \to S^1$ and measure class preserving automorphisms $T_v, T_w \colon X \to X$ (see Definition XIII.2.1) such that

$$(v\xi)(x) = h_v(x)\left(\frac{d(\mu\circ T_v^{-1})}{d\mu}(x)\right)^{\frac{1}{p}}\xi(T_v^{-1}(x))$$
$$(w\xi)(x) = h_w(x)\left(\frac{d(\mu\circ T_w^{-1})}{d\mu}(x)\right)^{\frac{1}{p}}\xi(T_w^{-1}(x))$$

for all $\xi \in L^p(X,\mu)$ and $\mu$-almost every $x \in X$. (The transformation $T_v$ is called the *spatial realization* of $v$ in [204].) With the notation from Section XIII.2, we have $v = m_{h_v} \circ u_{T_v}$ and $w = m_{h_w} \circ u_{T_w}$. On the other hand, it is clear that $\mathrm{id}_E = u_{\mathrm{id}_X}$.

By Lemma 6.22 in [204], we must have $T_v(x) = T_w(x) = x$ for $\mu$-almost every $x \in X$. It follows that $v$ and $w$ are, respectively, the operators of multiplication by the functions $h_v$ and $h_w$. We conclude that $v$ and $w$ commute, and the proof of the proposition is complete. □

**Remark XVII.2.4.** We do not know whether the assumption that the representation of $A$ on an $L^p$-space can be chosen to be *non-degenerate* is necessary in Proposition XVII.2.3. On the other hand, we point out that any unital Banach algebra that can be represented on an $L^p$-space, can also be *unitally* represented on an $L^p$-space, by Lemma XIX.2.3.

It is conceivable that a similar result holds for $C^*$-algebras that have an approximate identity consisting of projections.

For the sake of comparison, we will show in Proposition XVII.2.6 that for any $p \in [1,\infty)$, any $C^*$-algebra can be isometrically represented on a noncommutative $L^p$-space (see [107] for the definition of a noncommutative $L^p$-space associated to a semifinite von Neumann algebra). This result is probably known to the experts, but we were not able to find a reference.

We need first a result about $C^*$-algebras which is interesting in its own right. The proof is due to Garth Dales, and it was provided to us by Chris Phillips. We are thankful to both of them for allowing us to include it here. This result will be used in the proofs of Proposition XVII.2.6 and also Theorem XVII.2.7.

**Theorem XVII.2.5.** Let $A$ be a $C^*$-algebra, let $B$ be a Banach algebra, and let $\varphi\colon A \to B$ be a contractive, injective homomorphism. Then $\varphi$ is isometric.

If $\varphi$ is not assumed to be injective, then the conclusion is that it is a quotient map. On the other hand, we must assume that $\varphi$ is contractive, and not merely continuous, for the result to hold (Phillips – private communication). The result also fails for not necessarily self-adjoint operator algebras, even for uniform algebras.

*Proof.* Fix $a \in A$. We want to show that $\|\varphi(a)\| = \|a\|$. Since $\varphi$ is contractive, it is enough to show that $\|\varphi(a)\| \geq \|a\|$. Since this is immediate for $a = 0$, we will assume that $a \neq 0$.

We claim that

$$\|a\|^2 \leq \|\varphi(a)\|\|\varphi(a^*)\|.$$

(This is true even if $\varphi$ is not continuous.) By Gelfand's theorem, the $C^*$-subalgebra $C^*(a^*a)$ of $A$ generated by $a^*a$ is isometrically isomorphic to $C_0(X)$ for $X = \mathrm{sp}(a^*a)$. Since the restriction of $\varphi$ to $C^*(a^*a)$ is obviously an injective homomorphism, it follows from part (2) of Theorem 4.2.3 in [46] that

$$\|\varphi(a^*a)\| \geq \|a^*a\|.$$

Now, $\|a^*a\| = \|a\|^2$ because $A$ is a $C^*$-algebra, and $\|\varphi(a^*a)\| \leq \|\varphi(a^*)\|\|\varphi(a)\|$ because $\varphi$ is multiplicative and $B$ is a Banach algebra. The claim follows.

We use the claim at the third step to deduce that

$$\|a\|\|\varphi(a^*)\| \leq \|a\|\|a^*\| = \|a\|^2 \leq \|\varphi(a)\|\|\varphi(a^*)\|.$$

Now, $\varphi(a^*) \neq 0$ because $\varphi$ is injective. Dividing the inequality above by $\|\varphi(a^*)\|$ gives $\|a\| \leq \|\varphi(a)\|$, and concludes the proof. □

Recall that if $\mathcal{H}$ is a Hilbert space, an operator $T \in \mathcal{B}(\mathcal{H})$ belongs to the Schatten-$p$ class if and only if $\mathrm{Tr}(|T|^p)$ is finite (here, $|T| = (T^*T)^{\frac{1}{2}}$). Moreover, the Schatten-$p$ norm of $T$ is given by

$$\|T\|_p = (\mathrm{Tr}(|T|^p))^{\frac{1}{p}}.$$

Noncommutative $L^p$-spaces were introduced by Haagerup in [107], where the reader will find the definition and basic facts. Here, all we will use is that for a Hilbert space $\mathcal{H}$ and $p \in [1, \infty)$, the Schatten-$p$ class $\mathcal{S}_p(\mathcal{H})$ is a noncommutative $L^p$-space.

**Proposition XVII.2.6.** Let $A$ be a $C^*$-algebra, and let $p \in [1, \infty)$. Then there exist a noncommutative $L^p$-space $E$ and an isometric representation $\varphi\colon A \to \mathcal{B}(E)$.

*Proof.* Since every $C^*$-algebra can be isometrically represented on a Hilbert space, it is enough to show that if $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ can be represented on a noncommutative $L^p$-space.

Consider the Schatten-$p$ class $\mathcal{S}_p(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$; this is the noncommutative $L^p$-space associated to $\mathcal{B}(\mathcal{H})$ with its usual trace Tr. Denote the norm on $\mathcal{S}_p(\mathcal{H})$ by $\|\cdot\|_p$, and by $\|\cdot\|$ the operator norm on $\mathcal{B}(\mathcal{H})$. Define an algebra homomorphism $\varphi_p \colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{S}_p(\mathcal{H}))$ by $\varphi_p(T)(S) = ST$ for $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{S}_p(\mathcal{H})$. We claim that $\varphi_p$ is an isometric representation.

Suppose first that $p \in [1,2]$. Fix $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{S}_p(\mathcal{H})$. It is clear that $S^*T^*TS \leq \|T\|^2 S^*S$. Since $0 < \frac{p}{2} \leq 1$, we use Proposition 6.3 in [258] at the second step to get

$$|TS|^p = (S^*T^*TS)^{\frac{p}{2}} \leq \|T\|^p (S^*S)^{\frac{p}{2}} = \|T\|^p |S|^p.$$

It follows that $\mathrm{Tr}(|TS|^p) \leq \|T\|^p \mathrm{Tr}(|S|^p)$, and thus $\|TS\|_p \leq \|T\|\|S\|_p$. Since $S$ is arbitrary, we conclude that $\varphi_p$ is a contractive representation.

Since $\mathcal{S}_p(\mathcal{H})$ contains all finite rank operators (in particular, all rank one projections), it is clear that $\varphi_p$ is injective. That $\varphi_p$ is isometric now follows from Theorem XVII.2.5. (Alternatively, and in order to conclude that $\varphi_p$ is isometric, one could show directly that $\|\varphi_p\| \geq 1$.)

Suppose now that $p \in (2, \infty)$. Denote by $p' \in [1,2]$ the Hölder conjugate exponent of $p$. It is a standard fact that the dual space $\mathcal{S}_p(\mathcal{H})^*$ of $\mathcal{S}_p(\mathcal{H})$ is isometrically isomorphic to $\mathcal{S}_{p'}(\mathcal{H})$. Moreover, under this canonical identification, we have

$$\varphi_p(T) = \varphi_{p'}(T^*)$$

for all $T \in \mathcal{B}(\mathcal{H})$. Since $\varphi_{p'}$ is isometric by the first part of this proof, it follows that $\varphi_p$ is also isometric. This concludes the proof. $\square$

The following is one of the main results of this chapter. In its proof, for a locally compact group $G$ and $p \in [1, \infty)$, and to emphasize the role played by $G$, we will denote by $\gamma_{p,2}^G \colon F^p(G) \to C^*(G)$ the map constructed in Theorem XIV.2.30.

**Theorem XVII.2.7.** Let $G$ be a locally compact group and let $p \in [1, \infty)$. Then one of $F^p(G)$, $PF_p(G)$, $PM_p(G)$ or $CV_p(G)$ is isometrically an operator algebra if and only if either $p = 2$ or $G$ is the trivial group.

*Proof.* The "if" implication is obvious if $G$ is the trivial group, since the associated Banach algebras are all $\mathbb{C}$, while the statement is true if $p = 2$, essentially by definition.

For the "only if" implication, it is clear that if either $PM_p(G)$ or $CV_p(G)$ is an operator algebra, then so is $PF_p(G)$, since there are isometric inclusions

$$PF_p(G) \subseteq PM_p(G) \subseteq CV_p(G).$$

Assume now that $F^p(G)$ is an operator algebra and that $p \neq 2$. By Proposition XVII.2.1, there is a canonical identification of $F^p(G)$ with $C^*(G) = F^2(G)$. Let $\kappa_p \colon F^p(G) = C^*(G) \to PF_p(G)$ denote the canonical contractive homomorphism with dense range. It is well-known that the quotient $C^*(G)/\ker(\kappa_p)$ is a $C^*$-algebra. The induced map

$$\widehat{\kappa_p} \colon C^*(G)/\ker(\kappa_p) \to PF_p(G)$$

is an injective, contractive homomorphism. Now, Theorem XVII.2.5 shows that $\widehat{\kappa_p}$ is isometric. In particular, $PF_p(G)$ is a $C^*$-algebra, and hence an operator algebra itself.

It is therefore enough to show the statement assuming that $PF_p(G)$ is an operator algebra. Let $\mathcal{H}$ be a Hilbert space and let $\varphi \colon PF_p(G) \to \mathcal{B}(\mathcal{H})$ be an isometric representation.

*Claim: $PF_p(G)$ is a $C^*$-algebra.* Note that $A$ has a contractive approximate identity, since so does $L^1(G)$, and there is a contractive homomorphism $\iota \colon L^1(G) \to PF_p(G)$ with dense range. Moreover, since

$$\overline{\{\varphi(a)\eta \colon a \in PF_p(G), \eta \in \mathcal{H}\}}$$

is itself a Hilbert space, we may assume that the representation $\varphi$ is non-degenerate. It follows from Theorem XVII.2.2 that the algebra $M(PF_p(G)) \subseteq \mathcal{B}(PF_p(G))$ of multipliers on $PF_p(G)$, is unitally representable on $\mathcal{H}$.

By Corollary 2.5 in [99], there is a canonical isometric identification of the multiplier algebra $M(PF_p(G)) \subseteq \mathcal{B}(PF_p(G))$, with the Banach algebra

$$C(PF_p(G)) = \{x \in \mathcal{B}(L^p(G)) \colon xa, ax \in PF_p(G) \text{ for all } a \in PF_p(G)\} \subseteq \mathcal{B}(L^p(G))$$

of centralizers of $PF_p(G)$. Denote by $\psi\colon C(PF_p(G)) \to \mathcal{B}(\mathcal{H})$ the resulting unital, isometric representation.

There is an obvious identification of $G$ with a subgroup of the invertible isometries of $C(PF_p(G))$, given by letting a group element $g \in G$ act on $L^p(G)$ as the convolution operator with respect to the point mass measure $\delta_g$. Now, for $g \in G$, set $u_g = \psi(\delta_g)$. Then $u_g$ is an invertible isometry on $\mathcal{H}$, that is, a unitary operator. Moreover, the map $u\colon G \to \mathcal{U}(\mathcal{H})$ given by $g \mapsto u_g$, is easily seen to be a strongly-continuous unitary representation of $G$ on $\mathcal{H}$. The integrated form $\rho_u\colon L^1(G) \to \mathcal{B}(\mathcal{H})$ of $u$ is therefore a contractive, non-degenerate homomorphism. Whence $\rho_u(L^1(G)) \subseteq \mathcal{B}(\mathcal{H})$ is a $*$-subalgebra. Moreover, it is clear that the following diagram commutes:

$$\begin{tikzcd} L^1(G) \arrow[d, "\iota"'] \arrow[dr, "\rho_u"] & \\ PF_p(G) \arrow[r, "\varphi"'] & \mathcal{B}(\mathcal{H}). \end{tikzcd}$$

We conclude that $\varphi(PF_p(G)) = \overline{\rho_u(L^1(G))}$ is a closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, that is, a $C^*$-algebra. The claim is proved.

It follows from Proposition XVII.2.3 that $PF_p(G)$ is commutative. Thus $G$ is itself commutative, and in particular $PF_q(G) = F^q(G)$ for all $q \in [1, \infty)$, by Theorem XIV.3.7.

The map $\gamma^G_{p,2}\colon PF_p(G) \to C^*_\lambda(G)$ is an isometric isomorphism by Proposition XVII.2.1. The fact that $\gamma^G_{p,2}$ is surjective implies that $G$ is finite, by Corollary XIV.3.20. Using that $\gamma^G_{p,2}$ is isometric, we will show that $G$ must be the trivial group.

Using finiteness of $G$, let $g \in G$ be an element with maximum order. Set $n = \mathrm{ord}(g) \geq 1$, and let $j\colon \mathbb{Z}_n \hookrightarrow G$ be the group homomorphism determined by $j(1) = g$. By Proposition XV.2.1, there are natural isometric embeddings $j_p\colon PF_p(\mathbb{Z}_n) \hookrightarrow PF_p(G)$ and $j_2\colon C^*_\lambda(\mathbb{Z}_n) \hookrightarrow C^*_\lambda(G)$. (Existence of $j_2$ is well known; see [24].) Naturality of the maps involved implies that the following diagram is commutative:

$$\begin{tikzcd} PF_p(\mathbb{Z}_n) \arrow[r, "j_p"] \arrow[d, "\gamma^{\mathbb{Z}_n}_{p,2}"'] & PF_p(G) \arrow[d, "\gamma^G_{p,2}"] \\ C^*_\lambda(\mathbb{Z}_n) \arrow[r, "j_2"'] & C^*_\lambda(G). \end{tikzcd}$$

In particular, $\gamma_{p,2}^{\mathbb{Z}_n}\colon PF_p(\mathbb{Z}_n) \to C^*_\lambda(\mathbb{Z}_n)$ is an isometric isomorphism. (This map is really just the identity on $\mathbb{C}^n$.)

Set $\omega = e^{\frac{2\pi i}{n}} \in S^1$. Using that $p \neq 2$ together with Proposition XV.3.4 (see also the comments above it), we conclude that if $x \in PF_p(\mathbb{Z}_n)$ is an invertible isometry, then there exist $\zeta \in S^1$ and $k \in \{0, \ldots, n-1\}$ such that, under the algebraic identification of $PF_p(\mathbb{Z}_n)$ with $\mathbb{C}^n$, we have

$$x = \left(\zeta, \zeta\omega^k, \ldots, \zeta\omega^{k(n-1)}\right).$$

In particular, if $n > 1$, then not every element in $(S^1)^n \subseteq \mathbb{C}^n$ has norm one in $PF_p(\mathbb{Z}_n)$. Since this certainly is the case in $C^*_\lambda(\mathbb{Z}_n)$, we must have $n = 1$. By the choice of $n$, we conclude that $G$ must be the trivial group, and the proof of the theorem is finished. $\square$

In contrast with Theorem XVII.2.7, we point out that some $L^p$-operator group algebras are *contractively and isomorphically* representable on Hilbert spaces. For example, for any finite group $G$, abelian or not, and for any $p \in [1, \infty)$, the map $\gamma_{p,2}\colon F^p(G) \to C^*(G)$ constructed in Theorem XIV.2.30, is a (non-degenerate) contractive isomorphism.

## $PF_p(G)$, $PM_p(G)$ and $CV_p(G)$ do not Act on $L^q$-spaces

Recall (Theorem 5 in [116]) that if $G$ is an amenable locally compact group, then $PM_p(G) = CV_p(G)$ for all $p \in [1, \infty)$.

**Theorem XVII.3.1.** Let $G$ be a locally compact, amenable group, and let $p, q \in [1, \infty)$. Then one of $PF_p(G)$ or $PM_p(G)$ is isometrically representable on an $L^q$-space if and only if either

$$\left|\frac{1}{2} - \frac{1}{p}\right| = \left|\frac{1}{2} - \frac{1}{q}\right|,$$

or $G$ is the trivial group.

*Proof.* We begin with the "if" implication. When $G$ is the trivial group, we have $PF_p(G) = PM_p(G) = \mathbb{C}$, which is clearly representable on an $L^q$-space. On the other hand, the identity $\left|\frac{1}{2} - \frac{1}{p}\right| = \left|\frac{1}{2} - \frac{1}{q}\right|$ is equivalent to $p$ and $q$ being either equal or conjugate (in the sense that $\frac{1}{p} + \frac{1}{q} = 1$). The case $p = q$ is trivial. If $p$ and $q$ are conjugate, then there the inversion map $G \to G$, given by $g \mapsto g^{-1}$ for $g \in G$, induces an isometric anti-isomorphism $\sharp\colon L^1(G) \to$

$L^1(G)$ (see, for example, Remark 3.4 in [98]), and an isomorphism $\psi\colon PF_p(G) \cong PF_q(G)$ by Proposition 2.18 in [98] and Theorem XIV.3.7. In particular, $PF_p(G)$ acts on $L^q(G)$. For the algebra of $p$-pseudomeasures, recall that $PM_p(G)$ is the closure of $PF_p(G)$ in $\mathcal{B}(L^p(G))$ with respect to the weak*-topology (also called the *ultraweak* topology) induced by the (canonical) identification of $\mathcal{B}(L^p(G))$ with the dual of $L^p(G)\widehat{\otimes}L^q(G)$ given by the pairing

$$\langle a, \xi \otimes \eta\rangle_{\mathcal{B}(L^p(G)),L^p(G)\widehat{\otimes}L^q(G)} = \langle a\xi, \eta\rangle_{L^p(G),L^q(G)}$$

for all $a \in \mathcal{B}(L^p(G))$, all $\xi \in L^p(G)$ and all $\eta \in L^q(G)$. The canonical identification of $L^p(G)\widehat{\otimes}L^q(G)$ with $L^q(G)\widehat{\otimes}L^p(G)$ induces an isomorphism $\kappa\colon \mathcal{B}(L^p(G)) \to \mathcal{B}(L^q(G))$ as Banach spaces, and there is a commutative diagram

$$\begin{tikzcd}
L^1(G) \arrow[r, hook, "\iota_p"'] \arrow[d, "\sharp"'] & PF_p(G) \arrow[r, hook] \arrow[d, "\psi"] & PM_p(G) \arrow[r, hook] & \mathcal{B}(L^p(G)) \arrow[d, "\kappa"] \\
L^1(G) \arrow[r, hook, "\iota_q"'] & PF_q(G) \arrow[r, hook] & PM_q(G) \arrow[r, hook] & \mathcal{B}(L^q(G)).
\end{tikzcd}$$

One checks that $\psi$ extends to an isometric isomorphism $PM_p(G) \cong PM_q(G)$, so $PM_p(G)$ is representable on an $L^q$-space.

We turn to the "only if" implication. It is enough to prove the result for $PF_p(G)$, since we have $PF_p(G) \subseteq PM_p(G)$. Moreover, if either $p = 2$ or $q = 2$, then the result follows from Theorem XVII.2.7. Thus, without loss of generality, we may assume that $p, q \in [1, 2)$ and $p \neq q$. It follows that there is a canonical isometric identification of $PF_p(G)$ with $PF_q(G)$, by Proposition XVII.2.1 and Theorem XIV.3.7.

Consider first the case $p < q$.

We claim that the map $\gamma_{p,2}\colon F^p(G) \to C^*(G)$ constructed in Theorem XIV.2.30 is an isometric isomorphism. In view of Theorem XIV.3.7, this is equivalent to showing that

$$\|\lambda_p(f)\|_{\mathcal{B}(L^p(G))} = \|\lambda_2(f)\|_{\mathcal{B}(L^2(G))}$$

for all $f \in L^1(G)$.

Let $f \in L^1(G)$. Then $\|\lambda_2(f)\|_{\mathcal{B}(L^2(G))} \leq \|\lambda_p(f)\|_{\mathcal{B}(L^p(G))}$ by Corollary XIV.3.20. Let $\theta \in (0,1)$ satisfy $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{2}$. By the Riesz-Thorin interpolation theorem, we have

$$\|\lambda_q(f)\|_{\mathcal{B}(L^q(G))} \leq \|\lambda_2(f)\|^{\theta}_{\mathcal{B}(L^2(G))}\|\lambda_p(f)\|^{1-\theta}_{\mathcal{B}(L^p(G))}.$$

Since $\|\lambda_p(f)\|_{\mathcal{B}(L^p(G))} = \|\lambda_q(f)\|_{\mathcal{B}(L^q(G))}$, we conclude that

$$\|\lambda_p(f)\|^{\theta}_{\mathcal{B}(L^p(G))} \leq \|\lambda_2(f)\|^{\theta}_{\mathcal{B}(L^2(G))},$$

and hence $\|\lambda_p(f)\|_{\mathcal{B}(L^p(G))} \leq \|\lambda_2(f)\|_{\mathcal{B}(L^2(G))}$, as desired. This proves the claim.

Since $\gamma_{p,2}$ is an isometric isomorphism, $F^p(G)$ is an operator algebra. The result now follows from Theorem XVII.2.7.

The case $q < p$ is analogous: the same argument shows that $\gamma_{q,p}$ is an isometric isomorphism, and the Riesz-Thorin interpolation theorem has to be applied to $p \in (q,2)$. We omit the details. □

**Corollary XVII.3.2.** Let $G$ be a locally compact group, and let $p, q \in [1,\infty)$. There is an isometric isomorphism $PF_p(G) \cong PF_q(G)$ or $PM_p(G) \cong PM_q(G)$ if and only if $p$ and $q$ are either conjugate or conjugate, in which case both isomorphisms exist and can be chosen to be natural.

The proof of Theorem XVII.3.1 breaks down both for $F^p(G)$ and $PF_p(G)$ if $G$ is not amenable. Indeed, for the full group algebra, it follows from Proposition XVII.2.3 that if $F^p(G)$ is representable on an $L^q$-space (with $1 \leq p < q \leq 2$), then $\gamma_{p,q} \colon F^p(G) \to F^q(G)$ is an isometric isomorphism. However, with our current knowledge, this by itself does not give us much information, since Theorem 3.18 in [98] requires the *reduced* group algebras to be canonically isomorphic. Moreover, we do not know how to conclude from the isomorphism $F^p(G) \cong F^q(G)$ that $PF_p(G)$ is canonically isometrically isomorphic to $PF_q(G)$, although we suspect that this is the case.

If one tries to show that $PF_p(G)$ is not representable on an $L^q$-space, one runs into similar difficulties, since this by itself does not seem to imply that $PF_p(G) \cong PF_q(G)$ canonically. In this context, the fact that there in general are no maps $\gamma^{\lambda}_{p,q} \colon PF_p(G) \to PF_q(G)$ represents an additional difficulty.

We therefore suggest:

**Problem XVII.3.3.** Generalize Theorem XVII.3.1 to not necessarily amenable groups, including full group $L^p$-operator algebras as well.

One possible strategy is to look more closely at those Banach algebras that are $L^p$- and $L^q$-operator algebras for two different, non conjugate, Hölder exponents $p$ and $q$.

**Question XVII.3.4.** Let $A$ be a unital Banach algebra, and suppose that the set

$$\{v \in A \colon v \text{ is invertible and } \|v\| = \|v^{-1}\| = 1\}$$

has dense linear span in $A$. Let $p, q \in [1, 2]$ with $p \neq q$. Suppose that $A$ can be isometrically represented on an $L^p$-space and on an $L^q$-space. Does it follow that $A$ is commutative? Does it follow that $A \cong C(X)$ for some compact Hausdorff space $X$?

A positive answer to Question XVII.3.4 would likely lead to the desired generalization of Theorem XVII.3.1.

# CHAPTER XVIII

## . BANACH ALGEBRAS GENERATED BY AN INVERTIBLE ISOMETRY OF AN $L^P$-SPACE

This chapter is based on joint work with Hannes Thiel ([96]).

We provide a complete description of those Banach algebras that are generated by an invertible isometry of an $L^p$-space together with its inverse. Examples include the algebra $PF_p(\mathbb{Z})$ of $p$-pseudofunctions on $\mathbb{Z}$, the commutative $C^*$-algebra $C(S^1)$ and all of its quotients, as well as uncountably many 'exotic' Banach algebras.

We associate to each isometry of an $L^p$-space, a spectral invariant called 'spectral configuration', which contains considerably more information than its spectrum as an operator. It is shown that the spectral configuration describes the isometric isomorphism type of the Banach algebra that the isometry generates together with its inverse.

It follows from our analysis that these algebras are semisimple. With the exception of $PF_p(\mathbb{Z})$, they are all closed under continuous functional calculus, and their Gelfand transform is an isomorphism.

As an application of our results, we show that Banach algebras that act on $L^1$-spaces are not closed under quotients. This answers the case $p = 1$ of a question asked by Le Merdy 20 years ago.

### Introduction

Associated to any commutative Banach algebra $A$ is its maximal ideal space $\mathrm{Max}(A)$, which is a locally compact Hausdorff space when endowed with the hull-kernel topology. Gelfand proved in the 1940's that there is a norm-decreasing homomorphism $\Gamma_A \colon A \to C_0(\mathrm{Max}(A))$, now called the Gelfand transform. This representation of $A$ as an algebra of functions on a locally compact Hausdorff space is fundamental to any study of commutative Banach algebras. It is well known that $\Gamma_A$ is injective if and only if $A$ is semisimple, and that it is an isometric isomorphism if and only if $A$ is a $C^*$-algebra. The reader is referred to [142] for a extensive treatment of the theory of commutative Banach algebras.

Despite the usefulness of the Gelfand transform, we are still far from understanding the isometric structure of (unital, semisimple) commutative Banach algebras, since the Gelfand transform is almost never isometric. Commutative Banach algebras for which their Gelfand transform is isometric are called uniform algebras, an example of which is the disk algebra.

In this chapter, which is based on [96], we study those Banach algebras that are generated by an invertible isometry of an $L^p$-space together with its inverse, for $p \in [1, \infty)$. These are basic examples of what Phillips calls $L^p$-operator algebras in [207], which are by definition Banach algebras that can be isometrically represented as operators on some $L^p$-space. $L^p$-operator algebras constitute a large class of Banach algebras, which contains all not-necessarily selfadjoint operator algebras (and in particular, all $C^*$-algebras), as well as many other naturally ocurring examples of Banach algebras. A class worth mentioning is that of the (reduced) $L^p$-operator group algebras, here denoted $F^p_\lambda(G)$, associated to a locally compact group $G$. These are introduced in Section 8 of [116] with the name $p$-pseudofunctions. (We warn the reader that these algebras are most commonly denoted be $PF_p(G)$, for example in [116] and [189].) The notation $F^p_\lambda(G)$ first appeared in [207], following conventions used in [51], and was chosen to match the already established notation in $C^*$-algebra theory.) The Banach algebra $F^p_\lambda(G)$ is the Banach subalgebra of $\mathcal{B}(L^p(G))$ generated by the image of the integrated form $L^1(G) \to \mathcal{B}(L^p(G))$ of the left regular representation of $G$ on $L^p(G)$. It is commutative if and only if the group $G$ is commutative, and together with the universal group $L^p$-operator algebra $F^p(G)$, contains a great deal of information about the group. For example, it is a result in [98] (also proved independently by Phillips), that a locally compact group $G$ is amenable if and only if the canonical map $F^p(G) \to F^p_\lambda(G)$ is an isometric isomorphism, and when $G$ is discrete, this is moreover equivalent to $F^p(G)$ being amenable as a Banach algebra.

Of particular interest in our development are the algebras $F^p(\mathbb{Z})$ for $p \in [1, \infty)$. When $p = 2$, this is a commutative $C^*$-algebra which is canonically identified with $C(S^1)$ under the Gelfand transform. However, $F^p(\mathbb{Z})$ is never a $C^*$-algebra when $p$ is not equal to 2, although it can be identified with a dense subalgebra of $C(S^1)$. As one may expect, it turns out that the norm on $F^p(\mathbb{Z})$ is particularly hard to compute.

Another important example of a commutative $L^p$-operator algebra is the group $L^p$-operator algebra $F^p(\mathbb{Z}_n)$ associated with the finite cyclic group of order $n$. For a fixed $n$, this is the subalgebra of the algebra of $n$ by $n$ matrices with complex entries generated by the cyclic shift of the basis, and hence is (algebraically) isomorphic to $\mathbb{C}^n$. It should come as no surprise that the norm on $\mathbb{C}^n$ inherited via this identification is also very difficult to compute (except in the cases $p = 2$, where the norm is simply the supremum norm, and the case $p = 1$, since the 1-norm of a matrix is easily calculated).

Studying the group of symmetries is critical in the understanding of any given mathematical structure. In the case of classical Banach spaces, this was started by Banach in his 1932 book. There is now a great deal of literature concerning isometries of Banach spaces. See [72], just to mention one example. For $L^p$-spaces, it was Banach who first described the structure of invertible isometries of $L^p([0,1])$ for $p \neq 2$, although a complete proof was not available until Lamperti's 1958 paper [161], where he generalized Banach's Theorem to $L^p(X,\mu)$ for an arbitrary $\sigma$-finite measure space $(X,\mu)$ and $p \neq 2$. The same proof works for invertible isometries between different $L^p$-spaces, yielding a structure theorem for isometric isomorphisms between any two of them. Roughly speaking, an isometric isomorphism from $L^p(X,\mu)$ to $L^p(Y,\nu)$, for $p \neq 2$, is a combination of a multiplication operator by a measurable function $h\colon Y \to S^1$, together with an invertible measurable transformation $T\colon X \to Y$ which preserves null-sets. While this is slighly inaccurate for general $\sigma$-finite spaces, it is true under relatively mild assumptions. (In the general case one has to replace $T\colon X \to Y$ with a Boolean homomorphism between their $\sigma$-algebras, going in the opposite direction.) In the case $p = 2$, the maps described above are also isometries, but they are not the only ones, and we say little about these.

Starting from Lamperti's result, we study the Banach algebra generated by an invertible isometry of an $L^p$-space together with its inverse. It turns out that the multiplication operator and the measurable transformation of the space have rather different contributions to the resulting Banach algebra (more precisely, to its norm). While the multiplication operator gives rise to the supremum norm on its spectrum, the measurable transformation induces a somewhat more exotic norm, which, interestingly enough, is very closely related to the norms on $F^p(\mathbb{Z})$ and $F^p(\mathbb{Z}_n)$ for $n$ in $\mathbb{N}$.

This chapter is organized as follows. In the remainder of this section we introduce the necessary notation and terminology.

In Section XVIII.2, we introduce the notion of spectral configuration; see Definition XVIII.2.3. For $p$ in $[1, \infty)$, we associate to each spectral configuration an $L^p$-operator algebra which is generated by an invertible isometry together with its inverse in Definition XVIII.2.4. We also show that there is a strong dichotomy with respect to the isomorphism type of these algebras: they are isomorphic to either $F^p(\mathbb{Z})$ or to the space of all continuous functions on its maximal ideal space; see Theorem XVIII.2.5. We point out that the isomorphism cannot in general be chosen to be isometric in the second case. The saturation of a spectral configuration (Definition XVIII.2.6) is introduced with the goal of showing that for $p \in [1, \infty) \setminus \{2\}$, two spectral configurations have canonically isometrically isomorphic associated $L^p$-operator algebras if and only if their saturations are equal; see Corollary XVIII.2.13.

Section XVIII.3 contains our main results. Theorem XVIII.3.18 describes the isometric isomorphism type of the Banach algebra $F^p(v, v^{-1})$ generated by an invertible isometry $v$ of an $L^p$-space together with its inverse, for $p \neq 2$. It turns out that this description is very closely related to the dynamic properties of the measurable transformation of the space, and we get very different outcomes depending on whether or not it has arbitrarily long strings (Definition XVIII.3.11). A special feature of algebras of the form $F^p(v, v^{-1})$ is that they are always simisimple, and, except in the case when $F^p(v, v^{-1}) \cong F^p(\mathbb{Z})$, their Gelfand transform is always an isomorphism (although not necessarily isometric); see Corollary XVIII.3.21. Additionally, we show that algebras of the form $F^p(v, v^{-1})$ are closed by functional calculus of a fairly big class of functions, which includes all continuous functions on the spectrum of $v$ except when $F^p(v, v^{-1}) \cong F^p(\mathbb{Z})$, in which case only bounded variation functional calculus is available.

Finally, in Section XVIII.5, we apply our results to answer the case $p = 1$ of a question posed by Le Merdy 20 years ago (Problem 3.8 in [165]). We show that the class of Banach algebras that act on $L^1$-spaces is not closed under quotients. In the following chapter, we use the main results of the present work to give a negative answer to the remaining cases of Le Merdy's question.

Further applications of the results contained in this chapter will appear in [93], where we study Banach algebras generated by two invertible isometries $u$ and $v$ of an $L^p$-space, subject to the relation $uv = e^{2\pi i\theta} vu$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

We mention that, unlike in the case of $C^*$-algebras, the Banach algebra generated by an invertible isometry of an $L^p$-space does not necessarily contain its inverse, even for $p = 2$, as the following example shows. For a function $f$, we denote by $m_f$ the operator of multiplication by $f$.

**Example XVIII.1.1.** Denote by $\mathbb{D}$ the open disk in $\mathbb{C}$, and consider the disk algebra

$$A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) \colon f|_{\mathbb{D}} \text{ is holomorphic}\}.$$

Then $A(\mathbb{D})$ is a Banach algebra when endowed with the supremum norm. Denote by $\mu$ the Lebesgue measure on $S^1$ and define a homomorphism $\rho \colon A(\mathbb{D}) \to \mathcal{B}(L^2(S^1, \mu))$ by $\rho(f) = m_{f|_{S^1}}$ for $f$ in $A(\mathbb{D})$. Then $\rho$ is isometric by the Maximum Modulus Principle.

Denote by $\iota \colon \mathbb{D} \to \mathbb{C}$ the canonical inclusion. Then $\iota$ generates $A(\mathbb{D})$ because every holomorphic function on $\mathbb{D}$ is the uniform limit of polynomials. Moreover, $\rho(\iota)$ is an invertible isometry of $L^2(S^1, \mu)$, but $\iota$ is clearly not invertible in $A(\mathbb{D})$. We conclude that $A(\mathbb{D})$ is an $L^2$-operator algebra generated by an invertible isometry, but it does not contain its inverse.

For $n$ in $\mathbb{N}$, we denote $\omega_n = e^{\frac{2\pi i}{n}}$. If $A$ is a unital Banach algebra and $a \in A$, we denote its spectrum in $A$ by $\mathrm{sp}_A(a)$, or just $\mathrm{sp}(a)$ if no confusion as to where the spectrum is being computed is likely to arise.

If $(X, \mathcal{A}, \mu)$ is a measure space and $Y$ is a measurable subset of $X$, we write $\mathcal{A}_Y$ for the restricted $\sigma$-algebra

$$\mathcal{A}_Y = \{E \cap Y \colon E \in \mathcal{A}\},$$

and we write $\mu|_Y$ for the restriction of $\mu$ to $\mathcal{A}_Y$. Note that if $\{X_n\}_{n\in\mathbb{N}}$ is a partition of $X$ consisting of measurable subsets, then there is a canonical isometric isomorphism

$$L^p(X, \mu) \cong \bigoplus_{n\in\mathbb{N}} L^p(X_n, \mu|_{X_n}).$$

(The direct sum on the right-hand side is the $p$-direct sum.)

We will usually not include the $\sigma$-algebras in our notation for measure spaces, except when they are necessary (particularly in Section XVIII.2). The characteristic function of a measurable set $E$ will be denoted $\mathbb{1}_E$.

## Spectral Configurations

In this section, we study a particular class of commutative Banach algebras. They are naturally associated to certain sequences of subsets of $S^1$ that we call *spectral configurations*; see Definition XVIII.2.3. We show that all such Banach algebras are generated by an invertible isometry of an $L^p$-space together with its inverse. (In fact, the invertible isometry can be chosen to act on $\ell^p$.) We also show that there is a strong dichotomy with respect to the isomorphism type of these algebras: they are isomorphic to either $F^p(\mathbb{Z})$, or to the space of all continuous functions on its maximal ideal space; see Theorem XVIII.2.5. In the last part of the section, we study when two spectral configurations give rise to isometrically isomorphic Banach algebras.

We mention here that one of the main results in Section 5, Theorem XVIII.3.18, states that the Banach algebra generated by an invertible isometry of an $L^p$-space together with its inverse, is isometrically isomorphic to the $L^p$-operator algebra associated to a spectral configuration which is naturally associated to the isometry.

We begin by defining a family of norms on algebras of the form $C(\sigma)$, where $\sigma$ is a certain closed subset of $S^1$. We will later see that these norms are exactly those that arise from spectral configurations consisting of exactly one nonempty set. Recall that if $n$ is a positive integer, we denote $\omega_n = e^{\frac{2\pi i}{n}} \in S^1$.

**Definition XVIII.2.1.** Let $p \in [1, \infty)$ and let $n$ in $\mathbb{N}$. Let $\sigma$ be a nonempty closed subset of $S^1$ which is invariant under rotation by $\omega_n$. For $f$ in $C(\sigma)$, we define

$$\|f\|_{\sigma,n,p} = \sup_{t\in\sigma} \left\| \left(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1} t)\right) \right\|_{F^p(\mathbb{Z}_n)}.$$

We will see in Proposition XVIII.2.2 that, as its notation suggests, the function $\|\cdot\|_{\sigma,n,p}$ is indeed a norm on $C(\sigma)$.

When $p = 2$, we have $\|\cdot\|_{F^p(\mathbb{Z}_n)} = \|\cdot\|_\infty$ and hence the norm $\|\cdot\|_{\sigma,n,2}$ is the supremum norm $\|\cdot\|_\infty$ for all $n$ in $\mathbb{N}$ and every closed subset $\sigma \subseteq S^1$. On the other hand, if $n = 1$ then $F^p(\mathbb{Z}_1) \cong \mathbb{C}$ with the usual norm, so $\|\cdot\|_{\sigma,1,p}$ is the supremum norm for every $p$ in $[1, \infty)$ and every closed subset $\sigma \subseteq S^1$. However, the algebra $(C(\sigma), \|\cdot\|_{\sigma,n,p})$ is never isometrically isomorphic to $(C(\sigma), \|\cdot\|_\infty)$ when $p \neq 2$ and $n > 1$; see part (5) of Theorem XVIII.2.5.

**Proposition XVIII.2.2.** Let $p \in [1,\infty)$, let $n$ in $\mathbb{N}$, and let $\sigma$ be a nonempty closed subset of $S^1$ which is invariant under rotation by $\omega_n$.

1. The function $\|\cdot\|_{\sigma,n,p}$ is a norm on $C(\sigma)$.

2. The norm $\|\cdot\|_{\sigma,n,p}$ is equivalent to $\|\cdot\|_\infty$.

3. The Banach algebra $(C(\sigma), \|\cdot\|_{\sigma,n,p})$ is isometrically representable on $\ell^p$, and hence it is an $L^p$-operator algebra.

*Proof.* (1). This follows immediately from the fact that $\|\cdot\|_{F^p(\mathbb{Z}_n)}$ is a norm.

(2). Since $\|\cdot\|_\infty \leq \|\cdot\|_{F^p(\mathbb{Z}_n)}$, one has $\|\cdot\|_\infty \leq \|\cdot\|_{\sigma,n,p}$. On the other hand, since $F^p(\mathbb{Z}_n)$ is finite dimensional, there exists a (finite) constant $C = C(n,p)$ such that $\|\cdot\|_{F^p(\mathbb{Z}_n)} \leq C\|\cdot\|_\infty$. Thus, for $f \in C(\sigma)$, we have

$$\begin{aligned}
\|f\|_{\sigma,n,p} &= \sup_{t\in\sigma} \left\| \big(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1}t)\big) \right\|_{F^p(\mathbb{Z}_n)} \\
&\leq C \sup_{t\in\sigma} \left\| \big(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1}t)\big) \right\|_\infty \\
&= C\|f\|_\infty.
\end{aligned}$$

We conclude that $\|\cdot\|_\infty \leq \|\cdot\|_{\sigma,n,p} \leq C\|\cdot\|_\infty$, as desired.

(3). Denote by $u_n \in M_n$ the matrix displayed in Example XV.3.1. Let $(t_k)_{k\in\mathbb{N}}$ be a dense sequence in $\sigma$, and consider the linear map

$$\rho\colon C(\sigma) \to \mathcal{B}\left(\bigoplus_{k\in\mathbb{N}} \ell^p_n\right)$$

given by

$$\rho(f) = \bigoplus_{k\in\mathbb{N}} u_n \begin{pmatrix} f(t_k) & & & \\ & f(\omega_n t_k) & & \\ & & \ddots & \\ n & & & f(\omega_n^{n-1}t_k) \end{pmatrix} u_n^{-1}$$

for $f \in C(\sigma)$. It is easy to verify that $\rho$ is a homomorphism. Moreover, given $f$ in $C(\sigma)$, we use the description of the norm $\|\cdot\|_{F^p(\mathbb{Z}_n)}$ from Example XV.3.1 at the second step, and continuity of

$f$ at the last step, to get

$$\begin{aligned}\|\rho(f)\| &= \sup_{k\in\mathbb{N}} \left\| u_n \mathrm{diag}(f(t_k), f(\omega_n t_k), \ldots, f(\omega_n^{n-1} t_k)) u_n^{-1} \right\| \\ &= \sup_{k\in\mathbb{N}} \left\| \big(f(t_k), f(\omega_n t_k), \ldots, f(\omega_n^{n-1} t)\big) \right\|_{F^p(\mathbb{Z}_n)} \\ &= \|f\|_{\sigma,n,p}.\end{aligned}$$

We conclude that $\rho$ is isometric, as desired. □

Now that we have analyzed the basic example of a spectral configuration, we proceed to study the general case.

**Definition XVIII.2.3.** A *spectral configuration* is a sequence $\sigma = (\sigma_n)_{n\in\overline{\mathbb{N}}}$ of closed subsets of $S^1$ such that

1. For every $n \in \mathbb{N}$, the set $\sigma_n$ is invariant under rotation by $\omega_n$;
2. The set $\sigma_\infty$ is either empty or all of $S^1$; and
3. We have $\sigma_n \neq \emptyset$ for at least one $n$ in $\overline{\mathbb{N}}$.

The *order* of the spectral configuration $\sigma$ is defined as

$$\mathrm{ord}(\sigma) = \sup\,\{n \in \overline{\mathbb{N}} \colon \sigma_n \neq \emptyset\}.$$

Note that a spectral configuration may have infinite order and yet consist of only finitely many nonempty sets.

We adopt the convention that for $p \in [1,\infty)$, the function $\|\cdot\|_{\sigma_\infty,\infty,p}$ is the zero function if $\sigma_\infty = \emptyset$, and the norm of $F^p(\mathbb{Z})$ otherwise.

**Definition XVIII.2.4.** Let $\sigma = (\sigma_n)_{n\in\overline{\mathbb{N}}}$ be a spectral configuration and let $p \in [1,\infty)$. Set

$$\overline{\sigma} = \overline{\bigcup_{n\in\overline{\mathbb{N}}} \sigma_n} \subseteq S^1,$$

and for $f$ in $C(\overline{\sigma})$, define

$$\|f\|_{\sigma,p} = \sup_{n\in\overline{\mathbb{N}}} \|f|_{\sigma_n}\|_{\sigma_n,n,p}.$$

The normed algebra associated to $\sigma$ and $p$ is

$$F^p(\sigma) = \{f \in C(\overline{\sigma})\colon \|f\|_{\sigma,p} < \infty\},$$

endowed with the norm $\|\cdot\|_{\sigma,p}$.

Since the Hölder exponent $p \in [1,\infty)$ will be clear from the context (in particular, it is included in the notation for $F^p(\sigma)$), we will most of the times from it from the notation for the norm $\|\cdot\|_{\sigma,p}$, and write $\|\cdot\|_{\sigma}$ instead, except when confusion is likely to arise.

**Theorem XVIII.2.5.** Let $\sigma = (\sigma_n)_{n\in\overline{\mathbb{N}}}$ be a spectral configuration and let $p \in [1,\infty)$.

1. The Banach algebra $F^p(\sigma)$ is an $L^p$-operator algebra that can be represented on $\ell^p$.

2. The Banach algebra $F^p(\sigma)$ is generated by an invertible isometry together with its inverse.

3. If $\mathrm{ord}(\sigma) = \infty$, then there is a canonical isometric isomorphism

$$F^p(\sigma) \cong F^p(\mathbb{Z}).$$

   In particular, if $p \neq 2$, then not every continuous function on $\overline{\sigma}$ has finite $\|\cdot\|_{\sigma,p}$-norm.

4. If $\mathrm{ord}(\sigma) = N < \infty$, then $\|\cdot\|_{\sigma} = \max\limits_{n=1,\ldots,N} \|\cdot\|_{\sigma_n,n}$. Moreover, the identity map on $C(\overline{\sigma})$ is a canonical Banach algebra isomorphism

$$(F^p(\sigma), \|\cdot\|_{\sigma}) \cong (C(\overline{\sigma}), \|\cdot\|_{\infty}),$$

   and thus every continuous function on $\overline{\sigma}$ has finite $\|\cdot\|_{\sigma}$-norm.

5. For $p \in [1,\infty)\setminus\{2\}$, there is a canonical isometric isomorphism

$$(F^p(\sigma), \|\cdot\|_{\sigma}) \cong (C(\overline{\sigma}), \|\cdot\|_{\infty}),$$

   if and only if $\mathrm{ord}(\sigma) = 1$. When $p = 2$, such an isometric isomorphism always exists.

*Proof.* (1). For every $n$ in $\mathbb{N}$, use part (3) in Proposition XVIII.2.2 to find an isometric representation $\rho_n\colon C(\sigma_n) \to \mathcal{B}(\ell^p)$, and let $\rho_\infty\colon F^p(\mathbb{Z}) \to \mathcal{B}(\ell^p)$ be the canonical isometric representation. Define

$$\rho\colon F^p(\sigma) \to \mathcal{B}\left(\bigoplus_{n\in\overline{\mathbb{N}}} \ell^p\right) \quad \text{by} \quad \rho(f) = \bigoplus_{n\in\overline{\mathbb{N}}} \rho_n(f|_{\sigma_n})$$

for all $f$ in $F^p(\sigma)$. It is immediate to check that $\rho$ is isometric. Since $\bigoplus\limits_{n\in\overline{\mathbb{N}}} \ell^p$ is isometrically isomorphic to $\ell^p$, the result follows.

(2). Let $\iota\colon \overline{\sigma} \to \mathbb{C}$ be the inclusion map, and let $\iota^{-1}\colon \overline{\sigma} \to \mathbb{C}$ be its (pointwise) inverse. It is clear that $\|\iota\|_\sigma = \|\iota^{-1}\|_\sigma = 1$. Thus, $\iota$ and $\iota^{-1}$ are invertible isometries, and they clearly generate $F^p(\sigma)$.

(3). It is enough to show that for every $f$ in $\mathbb{C}[\mathbb{Z}]$, one has

$$\|f(\iota)\|_{F^p(\sigma)} = \|\lambda_p(f)\|_{\mathcal{B}(\ell^p)}.$$

Since $\iota$ is an invertible isometry, the universal property of $F^p(\mathbb{Z})$ implies that $\|\lambda_p(f)\| \geq \|f(\iota)\|$.

If $\sigma_\infty \neq \emptyset$, then

$$\|f(\iota)\|_\sigma \geq \|f(\iota)\|_{\sigma_\infty,\infty} = \|f\|_{F^p(\mathbb{Z})},$$

and the result follows. We may therefore assume that $\sigma_\infty = \emptyset$.

In order to show the opposite inequality, let $\varepsilon > 0$ and choose an element $\xi = (\xi_k)_{k\in\mathbb{Z}}$ in $\ell^p$ of finite support with $\|\xi\|_p^p = 1$ and such that

$$\|\lambda_p(f)\xi\|_p > \|\lambda_p(f)\| - \varepsilon.$$

Choose $K$ in $\mathbb{N}$ such that $\xi_k = 0$ whenever $|k| > K$. Find a positive integer $M$ in $\mathbb{N}$ and complex coefficients $a_m$ with $-M \leq m \leq M$ such that $f(x, x^{-1}) = \sum\limits_{m=-M}^{M} a_m x^m$. Since $\mathrm{ord}(\sigma) = \infty$, there exists $n > 2(K + M)$ such that $\sigma_n \neq \emptyset$. Fix $t$ in $\sigma_n$ and define a representation $\rho\colon F^p(\sigma) \to \mathcal{B}(\ell^p_n)$ by

$$\rho(h) = \begin{pmatrix} 0 & & & & h(\omega_n^{n-1}t) \\ h(t) & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ & & & h(\omega_n^{n-2}t) & 0 \end{pmatrix}$$

for all $h$ in $F^p(\sigma)$. It is clear that

$$\|\rho(h)\| = \|(h(t), h(\omega_n t), \ldots, h(\omega_n^{n-1} t))\|_{F^p(\mathbb{Z}_n)} \leq \|h|_{\sigma_n}\|_{\sigma_n,n} \leq \|h\|_\sigma,$$

so $\rho$ is contractive. Denote by $e_0$ the basis vector $(1, 0, \ldots, 0) \in \ell_n^p$. Set $v = \rho(\iota)$, and note that $v^k(e_0) = e_{n-k}\omega_n^{n-k} t$ for all $k = 0, \ldots, n-1$, where indices are taken modulo $n$.

Let $\eta \in \ell_n^p$ be given by $\eta = \sum\limits_{k=-K}^{K} \xi_k e_{n-k}\omega_n^{n-k} t$. Then $\|\eta\|_p = \|\xi\|_p = 1$. Moreover,

$$\begin{aligned}
\rho(f(\iota))\eta &= \sum_{m=-M}^{M} a_m v^m \left( \sum_{k=-K}^{K} \xi_k e_{n-k}\omega_n^{n-k} \right) \\
&= \sum_{m=-M}^{M} \sum_{k=-K}^{K} a_m \xi_k e_{n-m-k} \omega_n^{n-m-k} t \\
&= \sum_{j=-M-K}^{N+K} (\lambda_p(f)\xi)_j v^j(e_0)
\end{aligned}$$

The elements $e_{n-m-k}$ for $-M \leq m \leq M$ and $-K \leq k \leq K$ are pairwise distinct, by the choice of $n$. We use this at the first step to get

$$\|\rho(f(\iota))\eta\|_p^p = \sum_{j=-N-K}^{N+K} \left| [\lambda_p(f)\xi]_j \right|^p \left\| v^j(e_0) \right\| = \|\lambda_p(f)\xi\|_p^p.$$

We deduce that

$$\|f(\iota)\| \geq \|\rho(f(\iota))\| \geq \|\rho(f(\iota))\eta\|_p = \|\lambda_p(f)\xi\|_p > \|\lambda_p(f)\| - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\|f(\iota)\|_{F^p(\sigma)} \geq \|\lambda_p(f)\|_{F^p(\mathbb{Z})}$, as desired.

(4). Let $N = \mathrm{ord}(\sigma)$. It is immediate from the definition that $\|\cdot\|_\sigma = \max\limits_{n=1,\ldots,N} \|\cdot\|_{\sigma_n,n}$. Moreover, for each $n = 1, \ldots, N$, use part (2) in Proposition XVIII.2.2 to find a constant $C(n) > 0$ satisfying

$$\|\cdot\|_\infty \leq \|\cdot\|_{\sigma_n,n} \leq C(n)\|\cdot\|_\infty.$$

Set $C = \max\{C(1), \ldots, C(N)\}$. Then $\|\cdot\|_\infty \leq \|\cdot\|_\sigma \leq C\|\cdot\|_\infty$, and thus $\|\cdot\|_\sigma$ is equivalent to $\|\cdot\|_\infty$, as desired.

(5). It is clear from the comments after Definition XVIII.2.1 that $\|\cdot\|_\sigma = \|\cdot\|_\infty$ if either $p = 2$ or $\mathrm{ord}(\sigma) = 1$. Conversely, suppose that $p \neq 2$ and that $\mathrm{ord}(\sigma) > 1$. Choose $n$ in $\overline{\mathbb{N}}$ with $n > 1$ such that $\sigma_n \neq \emptyset$. If $n = \infty$, then $F^p(\sigma)$ is isometrically isomorphic to $F^p(\mathbb{Z})$ by part (3) of this theorem. The result in this case follows from part (2) of Corollary XIV.3.20 for $G = \mathbb{Z}$ and $p' = 2$. We may therefore assume that $n < \infty$.

Let $t$ in $\sigma_n$ and choose a continuous function $f$ on $\overline{\sigma}$ with $\|f\|_\infty = 1$ such that

$$\big\|\big(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1})\big)\big\|_{F^p(\mathbb{Z}_n)} > 1.$$

(See, for example, the proof of Proposition XV.3.5.) We get

$$\begin{aligned}\|f\|_\sigma &\geq \|f|_{\sigma_n}\|_{\sigma_n, n}\\ &\geq \|(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1} t))\|_{F^p(\mathbb{Z}_n)}\\ &> 1 = \|f\|_\infty.\end{aligned}$$

It follows that $\|\cdot\|_\sigma$ is not the supremum norm, and the claim is proved. □

We now turn to the question of when two spectral configurations determine the same $L^p$-operator algebra.

It turns out that $F^p(\sigma)$ does not in general determine $\sigma$, and thus there are several spectral configurations whose associated $L^p$-operator algebras are pairwise isometrically isomorphic. For example, let $\sigma$ be the spectral configuration given by $\sigma_n = S^1$ for all $n$ in $\overline{\mathbb{N}}$, and let $\tau$ be given by $\tau_n = \mathbb{Z}_n$ for $n$ in $\mathbb{N}$ and $\sigma_\infty = \emptyset$. Then $F^p(\sigma) \cong F^p(\tau) \cong F^p(\mathbb{Z})$ by part (3) of Theorem XVIII.2.5.

However, as we will see later, given $F^p(\sigma)$ one can recover what we call the *saturation* of $\sigma$. See Corollary XVIII.2.13.

**Definition XVIII.2.6.** Let $\sigma$ be a spectral configuration. We define what it means for $\sigma$ to be saturated in two cases, depending on whether its order is infinite or not.

- If $\mathrm{ord}(\sigma) = \infty$, we say that $\sigma$ is *saturated* if $\sigma_n = S^1$ for all $n$ in $\overline{\mathbb{N}}$.

- If $\mathrm{ord}(\sigma) < \infty$, we say that $\sigma$ is *saturated* if $\sigma_m \subseteq \sigma_n$ whenever $n$ divides $m$.

**Remarks XVIII.2.7.** Let $\sigma$ be a saturated spectral configuration.

(1) The set $\sigma_\infty$ is nonempty (in which case it equals $S^1$) if and only if $\mathrm{ord}(\sigma) = \infty$. Since the order of $\sigma$ is determined by the spectral sets $\sigma_n$ for $n$ finite, we conclude that the spectral set $\sigma_\infty$ is redundant.

(2) For every $n$ in $\overline{\mathbb{N}}$, we have $\sigma_n \subseteq \sigma_1$: for $n < \infty$, this is true since 1 divides $n$, and for $n = \infty$ it is true by definition. In particular, since $\sigma_1$ is closed, we must have $\sigma_1 = \overline{\sigma}$.

Denote by $\Sigma$ the family of all saturated spectral configurations. We define a partial order on $\Sigma$ by setting $\sigma \leq \tau$ if $\sigma_n \subseteq \tau_n$ for every $n$ in $\overline{\mathbb{N}}$.

**Lemma XVIII.2.8.** The partial order defined above turns $\Sigma$ into a complete lattice. Moreover, $\Sigma$ has a unique maximal element, and its minimal elements are in bijection with $S^1$.

*Proof.* Let $\Omega$ be a nonempty subset of $\Sigma$. For each $n \in \overline{\mathbb{N}}$, set

$$(\sup \Omega)_n = \overline{\bigcup_{\sigma \in \Omega} \sigma_n} \quad \text{and} \quad (\inf \Omega)_n = \bigcap_{\sigma \in \Omega} \sigma_n.$$

It is readily verified that this defines elements $\sup \Omega$ and $\inf \Omega$ of $\Sigma$ that are the supremum and infimum of $\Omega$, respectively.

The unique maximal element $\sigma^\infty$ is given by $\sigma^\infty_n = S^1$ for all $n \in \overline{\mathbb{N}}$. For each element $\zeta$ in $S^1$, there is a minimal configuration $\sigma(\zeta)$ given by $\sigma(\zeta)_1 = \{\zeta\}$ and $\sigma_n(\zeta) = \emptyset$ for $n \geq 2$. Conversely, if $\sigma$ is a saturated configuration, then $\sigma_1$ is not empty, so we may choose $\zeta \in \sigma_1$. Then $\sigma(\zeta)$ is a minimal configuration and $\sigma(\zeta) \leq \sigma$. ☐

**Definition XVIII.2.9.** Let $\sigma$ be a spectral configuration. The *saturation* of $\sigma$, denoted by $\widetilde{\sigma}$, is the minimum of all saturated spectral configurations that contain $\sigma$.

Note that any spectral configuration (saturated or not) is contained in the maximal configuration $\sigma^\infty$. It follows that the saturation of a spectral configuration is well defined.

The proof of the following lemma is a routine exercise and is left to the reader.

**Lemma XVIII.2.10.** Let $\sigma$ be a spectral configuration and let $\widetilde{\sigma}$ be its saturation. Then:

1. We have $\overline{\sigma} = \overline{\widetilde{\sigma}} = \widetilde{\sigma}_1$.

2. We have $\mathrm{ord}(\sigma) = \mathrm{ord}(\widetilde{\sigma})$.

3. If $\mathrm{ord}(\sigma) < \infty$, then $\widetilde{\sigma}_n = \bigcup\limits_{k=1}^{\infty} \sigma_{kn}$ for every $n$ in $\mathbb{N}$.

If $\sigma$ is a spectral configuration, we denote by $\iota_\sigma \in F^p(\sigma) \subseteq C(\overline{\sigma})$ the canonical inclusion of $\overline{\sigma}$ into $\mathbb{C}$.

**Proposition XVIII.2.11.** Let $\sigma$ be a spectral configuration and let $\widetilde{\sigma}$ be its saturation. Then there is a canonical isometric isomorphism

$$F^p(\sigma) \cong F^p(\widetilde{\sigma}),$$

which sends $\iota_\sigma$ to $\iota_{\widetilde{\sigma}}$.

*Proof.* If $\sigma$ has infinite order, then the result follows from part (3) of Theorem XVIII.2.5.

Assume now that $\sigma$ has finite order. Then the underlying complex algebra of both $F^p(\sigma)$ and $F^p(\widetilde{\sigma})$ is $C(\overline{\sigma})$. We need to show that the norms $\|\cdot\|_\sigma$ and $\|\cdot\|_{\widetilde{\sigma}}$ coincide. Let $f \in C(\overline{\sigma})$. We claim that $\|f\|_\sigma = \|f\|_{\widetilde{\sigma}}$. Since $\sigma_n \subseteq \widetilde{\sigma}_n$, it is immediate that $\|f\|_\sigma \leq \|f\|_{\widetilde{\sigma}}$.

For the reverse inequality, let $n \in \mathbb{N}$. By Lemma XV.3.3, for every $k \in \mathbb{N}$ the restriction map $\rho_0^{(nk\to k)} \colon F^p(\mathbb{Z}_{nk}) \to F^p(\mathbb{Z}_n)$ is contractive. Using this at the fourth step, we obtain

$$\begin{aligned}
\|f\|_{\widetilde{\sigma}_n,n} &= \sup_{t\in\widetilde{\sigma}_n} \|(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1}t))\|_{F^p(\mathbb{Z}_n)} \\
&= \sup_{k\in\mathbb{N}} \sup_{t\in\sigma_{nk}} \|(f(t), f(\omega_n t), \ldots, f(\omega_n^{n-1}t))\|_{F^p(\mathbb{Z}_n)} \\
&= \sup_{k\in\mathbb{N}} \sup_{t\in\sigma_{nk}} \left\|\rho_0^{(nk\to k)}(f(t), f(\omega_{nk} t), \ldots, f(\omega_{nk}^{nk-1}t))\right\|_{F^p(\mathbb{Z}_{nk})} \\
&\leq \sup_{k\in\mathbb{N}} \sup_{t\in\sigma_{nk}} \|(f(t), f(\omega_{nk} t), \ldots, f(\omega_{nk}^{nk-1}t))\|_{F^p(\mathbb{Z}_{nk})} \\
&\leq \sup_{m\in\mathbb{N}} \|f\|_{\sigma_m,m} = \|f\|_\sigma.
\end{aligned}$$

It follows that $\|f\|_{\widetilde{\sigma}} = \sup\limits_{n\in\mathbb{N}} \|f\|_{\widetilde{\sigma}_n,n} \leq \|f\|_\sigma$, as desired. $\square$

**Theorem XVIII.2.12.** Let $\sigma$ and $\tau$ be two saturated spectral configurations and let $p \in [1,\infty) \setminus \{2\}$. Then the following conditions are equivalent:

1. We have $\tau \leq \sigma$, that is, $\tau_n \subseteq \sigma_n$ for every $n$ in $\overline{\mathbb{N}}$.

2. We have $\tau_1 \subseteq \sigma_1$, and

$$\|g_{|\tau_1}\|_\tau \leq \|g\|_\sigma$$

   for every $g \in C(\overline{\sigma})$.

3. There is a contractive, unital homomorphism

$$\varphi\colon F^p(\sigma) \to F^p(\tau)$$

such that $\varphi(\iota_\sigma) = \iota_\tau$.

*Proof.* The implications '(1) $\Rightarrow$ (2) $\Rightarrow$ (3)' are clear. For the implication '(3) $\Rightarrow$ (2)', notice that the sets $\sigma_1$ and $\tau_1$ can be canonically identified with the character spaces of $F^p(\sigma)$ and $F^p(\tau)$, respectively, so that $\sigma_1 \supseteq \tau_1$. Then (2) follows immediately from (3).

Let us show '(2) $\Rightarrow$ (1)'. For $k \in \mathbb{N}$, define a function $\mu^{(k)}\colon \mathbb{R} \to [0,1]$ as

$$\mu^{(k)}(x) = \max(0, 1 - 2kx).$$

These are bump-functions around 0, with support $\left[-\frac{1}{2k}, \frac{1}{2k}\right]$.

Given $n \in \mathbb{N}$, given $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{C}^n$, and given $t \in S^1$, let us define continuous functions $f_{\alpha,t}^{(k)}\colon S^1 \to \mathbb{C}$ for $k \in \mathbb{N}$ as follows:

$$f_{\alpha,t}^{(k)}(x) = \sum_{l=0}^{n-1} \alpha_l \cdot \mu^{(kn)}(\mathrm{dist}(x, \omega_n^l t)).$$

This is a function with $n$ bumps around the points $t, \omega_n t, \ldots, \omega_n^{n-1} t$ taking the values $f_{\alpha,t}^{(k)}(\omega_n^l t) = \alpha_l$ for $l = 0, 1, \ldots, n-1$.

Let $s \in \sigma_m$ and fix $k \in \mathbb{N}$ such that $\frac{1}{km} < \frac{1}{n}$. We compute $\left\| f_{\alpha,t}^{(k)} \right\|_{m,s}$. The support of $f_{\alpha,t}^{(k)}$ is the $\frac{1}{2kn}$-neighborhood of $\{t, t\omega_n, \ldots, t\omega_n^{n-1}\}$. Assume there are $a \in \{0, \ldots, m-1\}$ and $b \in \{0, \ldots, n-1\}$ such that $\omega_m^a s$ belongs to the $\frac{1}{2kn}$-neighborhood of $\omega_n^b t$.

Let $\alpha'$ be the $n$-tuple obtained from $\alpha$ by cyclically rotating by $-b$, that is

$$\alpha' = (\alpha_b, \alpha_{b+1}, \ldots, \alpha_{n-1}, \alpha_0, \ldots, \alpha_{b-1}).$$

Set $t' = \omega_n^b t$. Then $f_{\alpha,t}^{(k)} = f_{\alpha',t'}^{(k)}$. Set $s' = \omega_m^a s$. Since the norm on $F^p(\mathbb{Z}_m)$ is rotation-invariant, we have $\|\cdot\|_{m,s} = \|\cdot\|_{m,s'}$. Thus

$$\left\| f_{\alpha,t}^{(k)} \right\|_{m,s} = \left\| f_{\alpha',t'}^{(k)} \right\|_{m,s'}.$$

We have reduced to the case that $s'$ is in the $\frac{1}{2kn}$-neighborhood of $t'$. Let $d = \gcd(n, m)$. Then $\omega_d^l s'$ is in the $\frac{1}{2kn}$-neighborhood of $\omega_d^l t'$ for each $l = 0, \ldots, d-1$. Since $\frac{1}{kn} < \frac{1}{m}$, the value $f^{(k)}_{\alpha',t'}(\omega_m^r s')$ is zero unless $r$ is a multiple of $\frac{m}{d}$. Let $\delta = \mathrm{dist}(s', t')$. Let $r = i\frac{m}{d} + j$ for $i \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, \frac{m}{d} - 1\}$. Then

$$f^{(k)}_{\alpha',t'}(\omega_m^r s') = \begin{cases} 0, & \text{if } j \neq 0 \\ \mu^{(kn)}(\delta)\alpha'_{i\frac{n}{d}}, & \text{if } j = 0 \end{cases}.$$

We define an inclusion map $\iota^{(d\to m)} \colon \mathbb{C}^d \to \mathbb{C}^m$ as follows. The tuple $\beta \in \mathbb{C}^d$ is sent to the tuple $\iota^{(d\to m)}(\beta)$ which for $r = i\frac{m}{d} + j$ with $i \in \{0, \ldots, d-1\}$ and $j \in \{0, \ldots, \frac{m}{d} - 1\}$ is given by

$$\iota^{(d\to m)}(\beta)_r = \begin{cases} 0, & \text{if } j \neq 0 \\ \beta_i, & \text{if } j = 0 \end{cases}.$$

As shown in Proposition XV.2.1, the map $\iota^{(d\to m)}$ induces an isometric embedding $F^p(\mathbb{Z}_d) \to F^p(\mathbb{Z}_m)$.

We define restriction maps $\rho_j^{(n\to d)} \colon \mathbb{C}^n \to \mathbb{C}^d$ for $j \in \{0, \ldots, \frac{n}{d} - 1\}$ by sending an $n$-tuple $\beta$ to the $d$-tuple $\rho_r^{(n\to d)}(\beta)$ given by

$$\rho^{(n\to d)}(\beta)_i = \beta_{i\frac{n}{d}+j}.$$

It follows that

$$\begin{aligned} \left\| f^{(k)}_{\alpha',t'} \right\|_{s',m} &= \left\| \mu^{(kn)}(\delta) \left( \iota^{(d\to m)} \circ \rho_0^{(n\to d)}(\alpha') \right) \right\|_{F^p(\mathbb{Z}_m)} \\ &= \mu^{(kn)}(\delta) \left\| \rho_0^{(n\to d)}(\alpha') \right\|_{F^p(\mathbb{Z}_d)}. \end{aligned}$$

Let $b(t, s) \in \{0, \ldots, \frac{n}{d} - 1\}$ be the unique number such that $\omega_m^b t$ is in the $\frac{1}{2kn}$-neighborhood of $\{s, \omega_m s, \ldots, \omega_m^{m-1} s\}$. Then

$$\left\| f^{(k)}_{\alpha,t} \right\|_{s,m} = \mu^{(kn)}(\delta) \left\| \rho_{b(t)}^{(n\to d)}(\alpha) \right\|_{F^p(\mathbb{Z}_d)}.$$

Therefore

$$\begin{aligned}
\left\|f_{\alpha,t}^{(k)}\right\|_{m} &= \sup_{s\in\sigma_m}\left\|f_{\alpha,t}^{(k)}\right\|_{m,s} \\
&= \max_{b=0,\ldots,\frac{m}{d}-1}\ \sup_{s\in\sigma_m, b(s,t)=b} \mu^{(kn)}(\mathrm{dist}(s,t))\left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)} \\
&= \max_{b=0,\ldots,\frac{m}{d}-1} \mu^{(kn)}(\mathrm{dist}(\omega_m^b t, \{s, \omega_m s, \ldots, \omega_m^{m-1}s\}))\left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)}
\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{k\to\infty}\left\|f_{\alpha,t}^{(k)}\right\|_{\sigma_m,m} \\
&= \max_{b=0,\ldots,\frac{m}{d}-1}\ \lim_{k\to\infty} \mu^{(kn)}(\mathrm{dist}(t\omega_m^b, \{s, \omega_m s, \ldots, \omega_m^{m-1}s\}))\left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)} \\
&= \max_{b=0,\ldots,\frac{m}{d}-1} \mathbb{1}_{\sigma_m}(\omega_m^b t)\left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)}.
\end{aligned}$$

With this computation at hand, we can 'test' whether some $t \in S^1$ belongs to $\sigma_n$. To that end, let $m \in \mathbb{N}$ and set $d = \gcd(m,n)$.

Assume first that $d < n$. Let $\alpha \in F^p(\mathbb{Z}_n)$ be as in the conclusion of Proposition XV.3.5, and normalize it so that $\max_{b=0,\ldots,\frac{m}{d}-1}\left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)} = 1$. Then

$$\lim_{k\to\infty}\left\|f_{\alpha,t}^{(k)}\right\|_{\sigma_m,m} = \max_{b=0,\ldots,\frac{m}{d}-1} \mathbb{1}_{\sigma_m}(\omega_m^b t)\left\|\rho_b^{(n\to d)}(\alpha)\right\|_{F^p(\mathbb{Z}_d)} \leq 1.$$

On the other hand, if $d = n$ (so that $n$ divides $m$) then

$$\lim_{k\to\infty}\left\|f_{\alpha,t}^{(k)}\right\|_{\sigma_m,m} = \mathbb{1}_{\sigma_m}(t)\|\alpha\|_{F^p(\mathbb{Z}_n)}.$$

Thus, when $n$ divides $m$, then $\lim_{k\to\infty}\left\|f_{\alpha,t}^{(k)}\right\|_{\sigma_m,m} > 1$ if and only if $t \in \sigma_m$. Again, since $n$ divides $m$, this implies $t \in \sigma_m$.

In conclusion, we have

$$\lim_{k\to\infty}\left\|f_{\alpha,t}^{(k)}\right\|_{\sigma} > 1 \ \text{ if and only if } \ t \in \sigma_n.$$

The same computations hold for $\tau$, so that we have

$$\lim_{k\to\infty} \left\| f_{\alpha,t}^{(k)} \right\|_{\tau} > 1 \ \text{ if and only if } \ t \in \tau_n.$$

By assumption, we have $\left\| f_{\beta,t}^{(k)} \right\|_{\tau} \leq \left\| f_{\beta,t}^{(k)} \right\|_{\sigma}$ for every $\beta$, $t$ and $k$. Thus, if $t \in \tau_n$, then

$$1 < \lim_{k\to\infty} \left\| f_{\alpha,t}^{(k)} \right\|_{\tau} \leq \lim_{k\to\infty} \left\| f_{\alpha,t}^{(k)} \right\|_{\sigma},$$

which implies that $t \in \sigma_n$. Hence, $\tau_n \subseteq \sigma_n$. Since this holds for every $n$, we have shown $\tau \leq \sigma$, as desired. $\square$

**Corollary XVIII.2.13.** Let $\sigma$ and $\tau$ be two spectral configurations. The following conditions are equivalent:

1. There is an isometric isomorphism $\varphi\colon F^p(\sigma) \to F^p(\tau)$ such that $\varphi(\iota_\sigma) = \iota_\tau$.
2. We have $\widetilde{\sigma} = \widetilde{\tau}$.

### $L^p$-operator Algebras Generated by an Invertible Isometry

We fix $p \in [1,\infty) \setminus \{2\}$, and a complete $\sigma$-finite standard Borel space $(X, \mathcal{A}, \mu)$. We also fix an invertible isometry $v\colon L^p(X,\mu) \to L^p(X,\mu)$. We will introduce some notation that will be used in most the results of this section. We will recall the standing assumptions in the statements of the main results, but not necessarily in the intermediate lemmas and propositions.

Using Theorem XIII.2.4, we choose and fix a measurable function $h\colon X \to S^1$ and an invertible measure class preserving transformation $T\colon X \to X$ such that $v = m_h \circ u_T$.

Let $n$ in $\mathbb{N}$. Recall that a point $x$ in $X$ is said to have *period* $n$, denoted $\mathcal{P}(x) = n$, if $n$ is the least positive integer for which $T^n(x) = x$. If no such $n$ exists, we say that $x$ has *infinite period*, and denote this by $\mathcal{P}(x) = \infty$.

For each $n$ in $\overline{\mathbb{N}}$, set

$$X_n = \{x \in X\colon \mathcal{P}(x) = n\}.$$

Then $X_n$ is measurable, and $T(X_n) \subseteq X_n$ and $T^{-1}(X_n) \subseteq X_n$ for all $n$ in $\overline{\mathbb{N}}$. For each $n$ in $\overline{\mathbb{N}}$, denote by $h_n \colon X_n \to S^1$, by $T_n \colon X_n \to X_n$, and by $T_n^{-1} \colon X_n \to X_n$, the restrictions of $h$, of $T$, and of $T^{-1}$ to $X_n$. Furthermore, we denote by $\mu_{|X_n}$ the restriction of $\mu$ to the $\sigma$-algebra of $X_n$.

Set $v_n = m_{h_n} \circ u_{T_n}$. Then $v_n$ is an isometric bijection of $L^p(X_n, \mu)$. Since $X$ is the disjoint union of the sets $X_n$ for $n \in \overline{\mathbb{N}}$, there is an isometric isomorphism

$$L^p(X,\mu) \cong \bigoplus_{n\in\overline{\mathbb{N}}} L^p(X_n, \mu_{|X_n})$$

under which $v$ is identified with the invertible isometry

$$\bigoplus_{n\in\overline{\mathbb{N}}} v_n \colon \bigoplus_{n\in\overline{\mathbb{N}}} L^p(X_n, \mu_{|X_n}) \to \bigoplus_{n\in\overline{\mathbb{N}}} L^p(X_n, \mu_{|X_n}).$$

The next lemma shows that each of the transformations $T_n$ acts as essentially a shift of order $n$ on $X_n$, at least when $n < \infty$.

**Lemma XVIII.3.1.** Let $n$ in $\mathbb{N}$. Then there exists a partition $\{X_{n,j}\}_{j=0}^{n-1}$ of $X_n$ consisting of measurable subsets such that

$$T^{-1}(X_{n,j}) = X_{n,j+1}$$

for all $j \in \mathbb{N}$, with indices taken modulo $n$.

*Proof.* Note that every point of $X_n$ is periodic. It follows from Theorem 1.2 in [101] that there is a Borel cross-section, that is, a Borel set $X_{n,0} \subseteq X_n$ such that each orbit of $T$ intersects $X_{n,0}$ exactly once. The result follows by setting $X_{n,j} = T^{-j}(X_{n,0})$ for $j = 1, \ldots, n-1$. $\square$

The following lemma allows us to assume that $h$ is identically equal to 1 on the sets $X_{n,j}$ with $n < \infty$ and $j > 0$. The idea of the proof is to "undo" a certain shift on the space, which is reflected on the construction of the functions $g_n$. One cannot undo the shift completely, and there is a remainder left over which is concentrated on the first set $X_{n,0}$.

**Lemma XVIII.3.2.** There exists a measurable function $g \colon X \to S^1$ such that the function

$$g \cdot (\overline{g} \circ T^{-1}) \cdot h \colon X \to S^1$$

is identically equal to 1 on $X_{n,j}$ whenever $n < \infty$ and $j > 0$.

*Proof.* Let $n < \infty$. Adopt the convention that $h \circ T^0$ is the function identically equal to 1 (this unusual convention is adopted so that the formula below comes out nicer). Using indices modulo $n$, we define $g_n \colon X_n \to S^1$ by

$$g_n = \sum_{j=0}^{n-1} \mathbb{1}_{X_{n,j}} \cdot (h \circ T^0) \cdot (h \circ T) \cdots (h \circ T^{j-1}).$$

We point out that the term corresponding to $j = 0$ is

$$\mathbb{1}_{X_{n,0}} \cdot (h \circ T^0) \cdot (h \circ T) \cdots (h \circ T^{n-1}).$$

Note that $g_n$ is well defined because the sets $X_{n,j}$ are pairwise disjoint for $j = 0, \ldots, n-1$. For $n = \infty$, set $g_\infty = \mathbb{1}_{X_\infty}$.

Finally, we set

$$g = \sum_{n=1}^{\infty} \mathbb{1}_{X_n} \cdot g_n,$$

which is well defined because the sets $X_n$ are pairwise disjoint for $n \in \overline{\mathbb{N}}$. It is a routine exercise to check that $g$ has the desired properties. $\square$

**Remark XVIII.3.3.** Let $g \colon X \to S^1$ be as in Lemma XVIII.3.2. A straightforward computation shows that

$$m_g \circ v \circ m_{\overline{g}} = m_{g \cdot (\overline{g} \circ T^{-1}) \cdot h} \circ u_T.$$

In particular, $v$ is conjugate, via the invertible isometry $m_g$, to another invertible isometry whose multiplication component is identically equal to 1 on $X_{n,j}$ whenever $n < \infty$ and $j > 0$. Since $v$ and $m_g \circ v \circ m_{\overline{g}}$ generate isometrically isomorphic Banach subalgebras of $\mathcal{B}(L^p(X,\mu))$, we may shift our attention to the latter isometry. Upon relabeling its multiplication component, we may and will assume that $h$ itself is identically equal to 1 on $X_{n,j}$ whenever $n < \infty$ and $j > 0$.

Our next reduction refers to the set transformation $T$: we show that we can assume that $T$ preserves the measure of the measurable subsets of $X \setminus X_\infty$. Recall that $\mathcal{A}$ denotes the domain of $\mu$.

**Lemma XVIII.3.4.** There is a measure $\nu$ on $(X, \mathcal{A})$ such that

1. For every measurable set $E \subseteq X \setminus X_\infty$, we have $\nu(T(E)) = \nu(E)$.

2. For every measurable set $E \subseteq X$, we have $\nu(E) = 0$ if and only if $\mu(E) = 0$.

Moreover, $L^p(X,\mu)$ is canonically isometrically isomorphic to $L^p(X,\nu)$.

*Proof.* For every $n$ in $\mathbb{N}$, we define a measure $\nu_n$ on $X_n$ by

$$\nu_n(E) = \sum_{j=0}^{n-1} \mu(T^{-j}(E) \cap X_{n,0})$$

for every $E$ in $\mathcal{A}$. It is clear that $\nu_n(T(E)) = \nu_n(E)$ for every measurable set $E$.

Set $\nu = \sum\limits_{n\in\mathbb{N}} \nu_n + \mu|_{X_\infty}$. It is clear that $\nu \circ T = \nu$ on $X \setminus X_\infty$, so condition (1) is satisfied. In order to check condition (2), assume that $\nu(E) = 0$ for some measurable set $E$. If $n \in \mathbb{N}$, then $\nu_n(E) = 0$, and thus

$$\mu(T^{-j}(E) \cap X_{n,0}) = \mu(T^{-j}(E \cap X_{n,j})) = 0$$

for every $j = 0, \ldots, n-1$. Using that $T$ preserves null-sets, we get $\mu(E \cap X_{n,j}) = 0$ for every $j = 0, \ldots, n-1$. Since the sets $X_{n,j}$ form a partition of $X_n$, we deduce that $\mu(E \cap X_n) = 0$ for $n < \infty$. Since we also have $\mu(E \cap X_\infty) = 0$ and the sets $X_n$ form a partition of $X$, we conclude that $\mu(E) = 0$.

Conversely, if $E$ is measurable and $\mu(E) = 0$, then $\mu(E \cap X_\infty) = 0$ and $\mu(T^{-j}(E)) = 0$ for all $j$ in $\mathbb{Z}$. Thus $\nu_n(E) = 0$ for all $n$ in $\mathbb{N}$, whence $\nu(E) = 0$.

The last claim is a standard fact. Denote by $f$ the Radon-Nikodym derivative $f = \frac{d\mu}{d\nu}$, and define linear maps $\varphi_p \colon L^p(X,\mu) \to L^p(X,\nu)$ and $\psi_p \colon L^p(X,\nu) \to L^p(X,\mu)$ by

$$\varphi_p(\xi) = \xi f^{\frac{1}{p}} \quad \text{and} \quad \psi_p(\eta) = \eta f^{-\frac{1}{p}}$$

for all $\xi$ in $L^p(X,\mu)$ and all $\eta$ in $L^p(X,\nu)$. Then $\varphi_p$ and $\psi_p$ are mutual inverses. Moreover, we have

$$\|\varphi_p(\xi)\|_p^p = \int_X |\xi|^p f \, d\nu = \int_X |\xi|^p \, d\mu = \|\xi\|_p^p$$

for all $\xi$ in $L^p(X,\mu)$. We conclude that $\varphi_p$ is the desired isometric isomorphism. $\square$

**Remark XVIII.3.5.** Adopt the notation of Lemma XVIII.3.4. It is immediate to check that if $\varphi_p\colon L^p(X,\mu)\to L^p(X,\nu)$ is the canonical isometric isomorphism, then

$$\varphi_p(v)(\eta)(x) = h(x)\eta(T^{-1}(x))$$

for all $\eta$ in $L^p(X,\nu)$ and all $x$ in $X\setminus X_\infty$. (Note the absence of the correction term which is present in the statement of Theorem XIII.2.4.) Since the Banach algebras generated by $v$ and $\varphi_p(v)$ are isometrically isomorphic, we have therefore shown that we can always assume that $T$ is measure preserving on $X\setminus X_\infty$.

**Notation XVIII.3.6.** If $g\colon X\to\mathbb{C}$ is a measurable function, we denote by $\mathrm{ran}(g)$ the range of $g$, and by $\mathrm{essran}(g)$ its essential range, which is defined by

$$\mathrm{essran}(g) = \left\{z\in\mathbb{C}\colon \mu\left(g^{-1}(B_\varepsilon(z))\right) > 0 \text{ for all } \varepsilon>0\right\}.$$

It is well known that the spectrum of the multiplication operator $m_g$ is $\mathrm{essran}(g)$, and that

$$\mathrm{essran}(g)\subseteq \overline{\mathrm{ran}(g)}.$$

**Remark XVIII.3.7.** We recall the following fact about spectra of elements in Banach algebras. If $A$ is a unital Banach algebra, $B$ a subalgebra containing the unit of $A$, and $a$ is an element of $B$ such that $\mathrm{sp}_B(a)\subseteq S^1$, then $\mathrm{sp}_A(a)=\mathrm{sp}_B(a)$. In other words, for elements whose spectrum (with respect to a given algebra) is contained in $S^1$, their spectrum does not change when the element is regarded as an element of a larger or smaller algebra.

The following easy lemma will be used a number of times in the proof of Theorem XVIII.3.9, so we state and prove it separately.

**Lemma XVIII.3.8.** Let $n$ in $\mathbb{N}$, let $(Y,\nu)$ be a measure space and let $S\colon Y\to Y$ be a measurable map. Let $E$ be a measurable subset of $Y$ with $0<\nu(E)<\infty$ such that $E, S^{-1}(E),\ldots,S^{n-1}(E)$ are pairwise disjoint. Define a linear map $\psi_E\colon \ell^p_n\to L^p(X,\mu)$ by

$$\psi_E(\eta) = \frac{1}{\nu(E)^{\frac{1}{p}}}\sum_{j=0}^{n-1}\eta_j u_S^j(\mathbb{1}_E)$$

for all $\eta$ in $\ell^p_n$. Then $\psi_E$ is an isometry.

*Proof.* Note that $\psi_E(\eta)$ is a measurable function for all $\eta$ in $\ell^p_n$. For $\eta$ in $\ell^p_n$, we use that the sets $E, S^{-1}(E), \ldots, S^{n-1}(E)$ are pairwise disjoint at the first step to get

$$\|\psi_E(\eta)\|_p^p = \frac{1}{\mu(E)} \sum_{j=0}^{N-1} |\eta_j|^p \|u_T^j(\mathbb{1}_E)\|_p^p = \|\eta\|_p^p,$$

so $\psi_E$ is an isometry and the result follows. □

Denote by $F^p(v, v^{-1})$ the unital Banach subalgebra of $\mathcal{B}(L^p(X,\mu))$ generated by $v$ and $v^{-1}$. Then $F^p(v, v^{-1})$ is an $L^p$-operator algebra and there is a canonical algebra homomorphism $\mathbb{C}[\mathbb{Z}] \to F^p(v, v^{-1})$ given by $x \mapsto v$.

**Theorem XVIII.3.9.** Let $N$ in $\mathbb{N}$. Then $\mathrm{sp}(v_N)$ is a (possibly empty) closed subset of $S^1$, which is invariant under rotation by $\omega_N$. Moreover, the Gelfand transform defines a canonical isometric isomorphism

$$\Gamma\colon F^p(v_N, v_N^{-1}) \to \left(C(\mathrm{sp}(v_N)), \|\cdot\|_{\mathrm{sp}(v_N),N}\right).$$

(See Definition XVIII.2.4 for the definition of the norm $\|\cdot\|_{\mathrm{sp}(v_N),N}$.)

*Proof.* The proposition is trivial if $\mathrm{sp}(v_N)$ is empty (which happens if and only if $\mu(X_N) = 0$), so assume it is not.

We prove the second claim first. Note that $v^N$ is multiplication by the measurable function

$$g = h \cdot (h \circ T^{-1}) \cdots (h \circ T^{-N+1})\colon X_N \to S^1.$$

By Lemma XVIII.3.2 and Remark XVIII.3.3, we may assume that the range and essential of $g$ agree with the range and essential of $h|_{X_{n,0}}$, respectively. By Exercise 6, part (b) in [227], we may assume that the range of $h|_{X_{n,0}}$ is contained in its essential range, and hence that $\mathrm{essran}(h|_{X_{n,0}}) =$

$\overline{\mathrm{ran}(h|_{X_{n,0}})}$. Let $(z_n)_{n\in\mathbb{N}}$ be a dense infinite sequence in $\mathrm{ran}(h|_{X_{n,0}})$, and set

$$w_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ z_n & & & & 0 \end{pmatrix} \in M_N.$$

Then $w = \bigoplus\limits_{n\in\mathbb{N}} w_n$ is an invertible isometry on $\bigoplus\limits_{n\in\mathbb{N}} \ell^p_N \cong \ell^p$.

**Claim:** Let $f$ in $\mathbb{C}[\mathbb{Z}]$. Then $\|f(v)\| = \|f(w)\|$.

It is clear that $\|f(w)\| = \sup\limits_{n\in\mathbb{N}} \|f(w_n)\|$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that whenever $a$ and $b$ are elements in a Banach algebra such that $\|a-b\| < \delta$, then $\|f(a) - f(b)\| < \frac{\varepsilon}{2}$. Choose $n$ in $\mathbb{N}$ and choose $\xi = (\xi_0, \ldots, \xi_{N-1}) \in \ell^p_N$ with $\|\xi\|_p = 1$ such that

$$\|f(w_n)\xi\| > \|f(w)\| - \frac{\varepsilon}{2}.$$

Since $z_n$ is in the essential range of $h|_{X_{n,0}}$, we can find a measurable set $E$ in $X$ with $\mu(E) > 0$ such that

$$|h(x) - z_n| < \delta$$

for all $x$ in $E$. Since $X$ is $\sigma$-finite, we may assume that $\mu(E) < \infty$. We may also assume that the sets $E, T(E), \ldots, T^{N-1}(E)$ are pairwise disjoint. The linear map $\psi_E \colon \ell^p_N \to L^p(X_N, \mu)$ given by

$$\psi_E(\eta) = \frac{1}{\mu(E)^{\frac{1}{p}}} \sum_{j=0}^{N-1} \eta_j u_T^j(\mathbb{1}_E)$$

for all $\eta = (\eta_0, \ldots, \eta_{N-1}) \in \ell^p_N$, is an isometry by Lemma XVIII.3.8.

For notational convenience, we write

$$z_n^{(0)} = z_n \ \text{ and } \ z_n^{(1)} = \cdots = z_n^{(N-1)} = 1,$$

and we take indices modulo $N$. Thus, for $\eta$ in $\ell_N^p$ we have $w_n\eta = \left(\eta_{j-1} z_n^{(j-1)}\right)_{j=0}^{N-1}$. Moreover,

$$\psi_E(w_n\eta) = \frac{1}{\mu(E)^{\frac{1}{p}}} \sum_{j=0}^{N-1} z_n^{(j-1)} \eta_{j-1} u_T^j(\mathbb{1}_E) = \frac{1}{\mu(E)^{\frac{1}{p}}} \sum_{j=0}^{N-1} \eta_j z_n^{(j)} u_T^{j+1}(\mathbb{1}_E)$$

and

$$v\psi_E(\eta) = \frac{1}{\mu(E)^{\frac{1}{p}}} \sum_{j=0}^{N-1} \eta_j (m_h \circ u_T^{j+1})(\mathbb{1}_E)$$

for all $\eta \in \ell_N^p$. Thus,

$$\begin{aligned}
\|\psi_E(w_n\eta) - v\psi_E(\eta)\|_p^p &= \frac{1}{\mu(E)} \sum_{j=0}^{N-1} |\eta_j|^p \left\| z_n^{(j)} u_T^{j+1}(\mathbb{1}_E) - (m_h \circ u_T^{j+1})(\mathbb{1}_E) \right\|_p^p \\
&\leq \frac{1}{\mu(E)} \sum_{j=0}^{N-1} |\eta_j|^p \sup_{x \in T^{-j-1}(E)} \left| z_n^{(j)} - h(x) \right|^p \left\| u_T^{j+1}(\mathbb{1}_E) \right\|_p^p \\
&< \|\eta\|_p^p \, \delta^p,
\end{aligned}$$

for all $\eta \in \ell_N^p$, which shows that $\|\psi_E \circ w_n - v \circ \psi_E\| < \delta$. By the choice of $\delta$, we deduce that

$$\|\psi_E \circ f(w_n) - f(v) \circ \psi_E\| < \frac{\varepsilon}{2}.$$

Using that $\psi_E$ is an isometry at the third step, we get

$$\|f(v)\| \geq \|f(v)\psi_E(\xi)\| \geq \|\psi_E(f(w_n)\xi)\| - \frac{\varepsilon}{2} = \|f(w_n)\xi\| - \frac{\varepsilon}{2} > \|f(w)\| - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\|f(v)\| \geq \|f(w)\|$.

Let us show that $\|f(w)\| \geq \|f(v)\|$. Given $\varepsilon > 0$, choose $g$ in $L^p(X_N, \mu)$ with $\|g\|_p = 1$ such that

$$\|f(v)g\|_p > \|f(v)\| - \varepsilon.$$

Write $X_N$ as a disjoint union $X_N = X_{N,0} \sqcup \ldots \sqcup X_{N,N-1}$, where each of the sets $X_{N,j}$ is measurable and $T^{-1}(X_{N,j}) = X_{N,j+1}$, where the subscripts are taken mod $N$. Given a measurable subset $Y$

of $X_{N,0}$ with $0<\mu(Y)<\infty$, the linear map $\psi_Y\colon \ell^p_N \to L^p(X_N,\mu)$ defined in Lemma XVIII.3.8 is isometric because the sets $Y, T(Y), \ldots, T^{N-1}(Y)$ are pairwise disjoint.

Set

$$\varepsilon_0 = \min\left\{\varepsilon, \frac{\varepsilon}{2\|f(v)g\|_p}\right\}.$$

Choose $\delta_0>0$ such that whenever $a$ and $b$ are elements in a Banach algebra such that $\|a-b\|<\delta_0$, then $\|f(a)-f(b)\|<\varepsilon_0$.

By simultaneously approximating the functions

$$g_{|X_{N,0}}, T^{-1}(g_{|X_{N,1}}), \ldots, T^{-N+1}(g_{|X_{N,N-1}}) \quad \text{and} \quad h_{|X_{N,0}}$$

as functions on $X_{N,0}$, by step-functions, we can find:

- A positive integer $K$ in $\mathbb{N}$ and disjoint, measurable sets $Y^{(k)}$ of positive finite measure with $X_{N,0} = \bigsqcup_{k=1}^K Y^{(k)}$;
- Elements $\eta^{(k)} = (\eta_j^{(k)})_{j=0}^{N-1}$ in $\ell^p_N$;
- Not necessarily distinct positive integers $n_1, \ldots, n_K$,

such that

1. With $\widetilde{h} = \mathbb{1}_{X\setminus X_{N,0}} + \sum_{k=1}^K z_{n_k}\mathbb{1}_{Y^{(k)}}$, we have

$$\|h-\widetilde{h}\|_\infty < \delta_0$$

2. With $\widetilde{g} = \sum_{k=1}^K \psi_{Y^{(k)}}(\eta^{(k)})$, we have

$$\|g-\widetilde{g}\|_p < \frac{\varepsilon_0}{\|f(v)\|+\varepsilon}.$$

Set $\widetilde{v} = m_{\widetilde{h}} \circ u_T$. It follows from condition (1) above that

$$\|v-\widetilde{v}\| \leq \|h-\widetilde{h}\|_\infty \|u_T\| < \delta_0,$$

and by the choice of $\delta_0$, we conclude that $\|f(v)-f(\widetilde{v})\|<\varepsilon_0$.

For $\xi \in \ell^p_N$ and $k = 1, \ldots, K$, we have

$$
\begin{aligned}
\widetilde{v}\psi_{Y^{(k)}}(\xi) &= (m_{\widetilde{h}} \circ u_T)\left(\frac{1}{\mu(Y^{(k)})^{\frac{1}{p}}} \sum_{j=0}^{N-1} \xi_j \mathbb{1}_{T^{-j}(Y^{(k)})}\right) \\
&= \frac{1}{\mu(Y^{(k)})^{\frac{1}{p}}} z_{n_k} \xi_j \mathbb{1}_{Y^{(k)}} + \sum_{j=1}^{N-2} \xi_{j+1} \mathbb{1}_{T^{-j}(Y^{(k)})} + \xi_0 \mathbb{1}_{T^{-N+1}(Y^{(k)})} \\
&= \psi_{Y^{(k)}}\left(w_{n_k}\xi\right).
\end{aligned}
$$

It follows that

$$f(\widetilde{v})\psi_{Y^{(k)}}(\xi) = \psi_{Y^{(k)}}(f(w_{n_k})\xi)$$

for all $\xi$ in $\ell^p_N$. Thus,

$$
\begin{aligned}
\|f(v)g - f(\widetilde{v})\widetilde{g}\|_p &\leq \|f(v)g - f(\widetilde{v})g\|_p + \|f(\widetilde{v})g - f(\widetilde{v})\widetilde{g}\|_p \\
&\leq \|f(v) - f(\widetilde{v})\|\|g\|_p + \|f(\widetilde{v})\|\|g - \widetilde{g}\|_p \\
&\leq \varepsilon_0 + (\|f(v)\| + \varepsilon)\frac{\varepsilon}{2\|f(v)g\|_p} \leq \frac{\varepsilon}{\|f(v)g\|_p}.
\end{aligned}
$$

We therefore conclude that

$$\frac{\|f(\widetilde{v})\widetilde{g}\|_p}{\|\widetilde{g}\|_p} \geq \frac{\|f(v)g\|_p(1+\varepsilon)}{\|g\|_p + \varepsilon} = \frac{\|f(v)g\|_p(1+\varepsilon)}{1+\varepsilon} = \frac{\|f(v)g\|_p}{\|g\|_p} \geq \|f(v)\| - \varepsilon.$$

We have

$$
\begin{aligned}
\|f(\widetilde{v})\widetilde{g}\|_p &= \left\| f(\widetilde{v})\left(\sum_{k=1}^{K} \psi_{Y^{(k)}}(\eta^{(k)})\right)\right\|_p \\
&= \left\|\sum_{k=1}^{K} \psi_{Y^{(k)}}\left(f(w_{n_k})\eta^{(k)}\right)\right\|_p \\
&= \left(\sum_{k=1}^{K} \left\|f(w_{n_k})\eta^{(k)}\right\|_p^p\right)^{\frac{1}{p}},
\end{aligned}
$$

and also

$$\|\widetilde{g}\|_p = \left\| \sum_{k=1}^{K} \psi_{Y^{(k)}}(\eta^{(k)}) \right\|_p = \left( \sum_{k=1}^{K} \left\| \eta^{(k)} \right\|_p^p \right)^{\frac{1}{p}}.$$

Set

$$\widetilde{w} = \bigoplus_{k=1}^{K} w_{n_k} \in \mathcal{B}\left( \bigoplus_{k=1}^{K} \ell_N^p \right),$$

and $\eta = \left(\eta^{(1)}, \ldots, \eta^{(K)}\right) \in \bigoplus\limits_{k=1}^{K} \ell_N^p$. Then $\widetilde{w}$ is an invertible isometry, and the computations above show that

$$\|f(\widetilde{w})\eta\|_p = \|f(\widetilde{v})\widetilde{g}\|_p \quad \text{and} \quad \|\eta\|_p = \|\widetilde{g}\|_p.$$

We clearly have

$$\|f(\widetilde{w})\| = \max_{k=1,\ldots,K} \|f(w_{n_k})\| \leq \sup_{n\in\mathbb{N}} \|f(w_n)\| = \|f(w)\|.$$

We conclude that

$$\|f(w)\| \geq \|f(\widetilde{w})\| \geq \frac{\|f(\widetilde{w})\eta\|_p}{\|\eta\|_p} = \frac{\|f(\widetilde{v})\widetilde{g}\|_p}{\|\widetilde{g}\|_p} \geq \|f(v)\| - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows that $\|f(w)\| \geq \|f(v)\|$, and hence the proof of the claim is complete.

We will now show that $\mathrm{sp}(v_N)$ is invariant under translation by the $N$-th roots of unity in $S^1$. We retain the notation of the first part of this proof. Note that since $\mathrm{sp}(v_N)$ is a subset of $S^1$, it can be computed in any unital Banach algebra that contains $v_N$ by Remark XVIII.3.7; in particular, it can be computed in $F^p(v_N, v_N^{-1})$. Also, the spectrum of $v_N$ in $F^p(v_N, v_N^{-1})$ is equal to the spectrum of $w$ in $\mathcal{B}(\ell^p)$, since we have shown that $v_N \mapsto w$ extends to an isometric isomorphism $F^p(v_N, v_N^{-1}) \cong F^p(w, w^{-1})$. Moreover, it is clear that

$$\mathrm{sp}(w) = \overline{\bigcup_{n\in\mathbb{N}} \mathrm{sp}(w_n)}.$$

It is a straightforward exercise to show that $\mathrm{sp}(w_n) = \{\zeta \in \mathbb{T} \colon \zeta^N = z_n\}$, which is clearly invariant under translation by the $N$-th roots of unity in $S^1$, so the claim follows.

We have shown in the first part that there is an isometric isomorphism

$$F^p(v_N, v_N^{-1}) \cong \left(C(\mathrm{sp}(v_N)), \|\cdot\|_{\mathrm{sp}(v_N),N}\right).$$

Now, the Gelfand transform $\Gamma\colon F^p(v_N, v_N^{-1}) \to C(\mathrm{sp}(v_N))$ maps $v$ to the canonical inclusion of $\mathrm{sp}(v_N)$ into $\mathbb{C}$, so the image of $\Gamma$ is isometrically isomorphic to $\left(C(\mathrm{sp}(v_N)), \|\cdot\|_{\mathrm{sp}(v_N),N}\right)$, as desired. $\square$

The situation for $v_\infty$ is rather different, since the range of the Gelfand transform does not contain all continuous functions on its spectrum (which is either $S^1$ or empty). Indeed, the Banach algebra that $v_\infty$ generates together with its inverse is isometrically isomorphic to $F^p(\mathbb{Z})$ (or the zero algebra if $\mu(X_\infty) = 0$); see Theorem XVIII.3.14 below.

One difficulty of working with with $v_\infty$ is that the analog of Lemma XVIII.3.1 is not in general true, that is, $v_\infty$ need not have an infinite bilateral sub-shift, as the following example shows.

**Example XVIII.3.10.** Let $X = S^1$ with normalized Lebesgue measure $\mu$. Given $\theta$ in $\mathbb{R} \setminus \mathbb{Q}$, consider the invertible transformation $T_\theta\colon S^1 \to S^1$ given by rotation by angle $\theta$. Then $T_\theta$ is measure preserving and every point of $S^1$ has infinite period. We claim that there is no measurable set $E$ with positive measure such that the sets $T_\theta^n(E)$ for $n$ in $\mathbb{Z}$ are pairwise disjoint. If $E$ is any set such that all its images under $T_\theta$ are disjoint, we use translation invariance of $\mu$ to get

$$1 = \mu(S^1) \geq \mu\left(\bigcup_{n\in\mathbb{Z}} T_\theta^n(E)\right) = \sum_{n\in\mathbb{Z}} \mu(E).$$

It follows that $E$ must have measure zero, and the claim is proved.

In order to deal with the absence of infinite sub-shifts, we will show that the set transformation $T_\infty$ has what we call "arbitrarily long strings", which we proceed to define.

**Definition XVIII.3.11.** Let $(Y, \nu)$ be a measure space and let $S\colon Y \to Y$ be an invertible measure class preserving transformation. Given a measurable set $E$ in $Y$ with $\mu(E) > 0$, the finite sequence $E, S^{-1}(E), \ldots, S^{-n+1}(E)$ is called a *string of length* $n$ for $S$ if the sets $E, S^{-1}(E), \ldots, S^{-n+1}(E)$ are pairwise disjoint.

The map $S$ is said to have *arbitrarily long strings* if for all $n$ in $\mathbb{N}$ there exists a string of length $n$.

The following lemma is not in general true without some assumptions on the $\sigma$-algebra. What is needed in our proof is that for every point $x$ in $X$, the intersection of the measurable sets of positive measure that contain $x$ is the singleton $\{x\}$, which is guaranteed in our case since we are working with the (completed) Borel $\sigma$-algebra on a complete metric space.

**Lemma XVIII.3.12.** Let $(Y, \nu)$ be a $\sigma$-finite measure space such that for every $y$ in $Y$, the intersection of the measurable sets of positive measure that contain $y$ is the singleton $\{y\}$. Let $S\colon Y \to Y$ be an invertible measure class preserving transformation such that every point of $Y$ has infinite period. Then $S$ has arbitrarily long strings.

*Proof.* Let $(E_m)_{m\in\mathbb{N}}$ be a decreasing sequence of measurable sets with $\mu(E_m) > 0$ for all $m$ in $\mathbb{N}$ and such that $\bigcap\limits_{m\in\mathbb{N}} E_m = \{x\}$. Without loss of generality, we may assume that $T(x)$ does not belong to $E_m$ for all $m$ in $\mathbb{N}$.

Let $n$ in $\mathbb{N}$. We claim that there exist $m_n$ in $\mathbb{N}$ and a sequence $(F_m^{(n)})_{m\geq m_n}$ of measurable sets such that

1. For all $m \geq m_n$, the set $F_m^{(n)}$ is contained in $E_m$ and $x \in F_m^{(n)}$;

2. $\mu(F_m^{(n)}) > 0$ for all $m \geq m_n$;

3. The sets $F_m^{(n)}, T^{-1}(F_m^{(n)}), \ldots, T^{-n}(F_m^{(n)})$ are pairwise disjoint (up to null-sets).

We proceed by induction on $n$.

Set $n = 1$. If there exists $m_1$ such that $\mu(E_{m_1} \triangle T^{-1}(E_{m_1})) > 0$, take $F_{m_1}^{(1)} = E_{m_1} \cap T^{-1}(E_{m_1})^c$ and $F_m^{(1)} = F_{m_0}^{(1)} \cap E_m$ for all $m \geq m_1$. It is easy to verify that the sequence $(F_m^{(1)})_{m\geq m_1}$ satisfies the desired properties. If no such $m_1$ exists, it follows that $\mu(E_m \triangle T^{-1}(E_m)) = 0$ for all $m$ in $\mathbb{N}$. Upon getting rid of null-sets, we may assume that $E_m = T^{-1}(E_m)$ for all $m$ in $\mathbb{N}$. We have

$$\{x\} = \bigcap_{m\in\mathbb{N}} E_m = \bigcap_{m\in\mathbb{N}} T^{-1}(E_m) = T^{-1}\left(\bigcap_{m\in\mathbb{N}} E_m\right) = \{T^{-1}(x)\},$$

which implies that $x$ is a fixed point for $T$. This is a contradiction, and the case $n = 1$ is proved.

Let $n \geq 2$, and let $m_{n-1}$ and $(F_m^{(n-1)})_{m \geq m_{n-1}}$ be as in the inductive hypothesis for $n-1$. Suppose that there exists $m_n$ such that the sets

$$F_{m_n}^{(n-1)}, T^{-1}(F_{m_n}^{(n-1)}), \ldots, T^{-n}(F_{m_n}^{(n-1)})$$

are not disjoint up to null-sets. Let $j \in \{0, \ldots, n-1\}$ such that

$$\mu(T^{-j}(F_{m_n}^{(n-1)}) \triangle T^{-n}(F_{m_n}^{(n-1)})) > 0.$$

Since $T$ preserves null-sets, it follows that $\mu(T^{-(n-j)}(F_{m_n}^{(n-1)}) \triangle F_{m_n}^{(n-1)}) > 0$. By assumption, the first $n-1$ translates of $F_{m_n}^{(n-1)}$ are pairwise disjoint, so we must have $j = n$ and hence $\mu(T^{-n}(F_{m_n}^{(n-1)}) \triangle F_{m_n}^{(n-1)}) > 0$. Using an argument similar to the one used in the case $n = 1$, one shows that the sequence given by

$$F_{m_n}^{(n)} = F_{m_n}^{(n-1)} \cap T^{-1}(F_{m_n}^{(n-1)})^c \quad \text{and} \quad F_m^{(n)} = F_{m_n}^{(n)} \cap F_m^{(n-1)}$$

for all $m \geq m_n$, satisfies the desired properties.

If no such $m_n$ exists, it follows that $\mu(F_m^{(n-1)} \cap T^{-n}(F_m^{(n-1)})) = 0$ for all $m \geq m_{n-1}$. Again, upon getting rid of null-sets, we may assume that $F_m^{(n-1)} = T^{-n}(F_m^{(n-1)})$ for all $m \geq m_{n-1}$. We have

$$\{x\} = \bigcap_{m \in \mathbb{N}} F_m^{(n-1)} = \bigcap_{m \in \mathbb{N}} T^{-n}(F_m^{(n-1)}) = T^{-n}\left(\bigcap_{m \in \mathbb{N}} F_m^{(n-1)}\right) = \{T^{-n}(x)\},$$

and thus $T^{-n}(x) = x$. This is again a contradiction, which shows that such $m_n$ must exist. This proves the claim, and the proof is finished. $\square$

To see that Lemma XVIII.3.12 is not true in full generality, consider $Y = \mathbb{Z}$ endowed with the $\sigma$-algebra $\{\emptyset, \mathbb{Z}\}$ and measure $\mu(\mathbb{Z}) = 1$. Let $S \colon \mathbb{Z} \to \mathbb{Z}$ be the bilateral shift. Then every point of $Y$ has infinite period, but there are no strings of any positive length, let alone of arbitrarily long length.

**Corollary XVIII.3.13.** The measure class preserving transformation $T \colon X \to X$ has arbitrarily long strings if and only if either $\mu(X_\infty) > 0$ or $\mu(X_n) > 0$ for infinitely many values of $n$ in $\mathbb{N}$.

We are now ready to prove that $F^p(v_\infty, v_\infty^{-1})$ is isometrically isomorphic to $F^p(\mathbb{Z})$. We prove the result in greater generality because the proof is essentially the same and the extra flexibility will be needed later.

**Theorem XVIII.3.14.** Let $p \in [1, \infty)$, let $(Y, \nu)$ be a $\sigma$-finite measure space with $\nu(Y) > 0$, and let $S \colon Y \to Y$ be an invertible measure class preserving transformation with arbitrarily long strings. Let $h \colon Y \to S^1$ be a measurable function and let

$$w = m_h \circ u_S \colon L^p(Y, \nu) \to L^p(Y, \nu)$$

be the resulting invertible isometry. Then $\mathrm{sp}(w) = S^1$ and there is a canonical isometric isomorphism

$$F^p(w, w^{-1}) \cong F^p(\mathbb{Z})$$

determined by sending $w$ to the canonical generator of $F^p(\mathbb{Z})$.

*Proof.* We prove the second claim first. Denote by $\lambda_p \colon \mathbb{C}[\mathbb{Z}] \to \mathcal{B}(\ell^p)$ the left regular representation of $\mathbb{Z}$. It is enough to show that for every $f$ in $\mathbb{C}[\mathbb{Z}]$, one has

$$\|f(w)\|_{\mathcal{B}(L^p(Y,\nu))} = \|\lambda_p(f)\|_{F^p(\mathbb{Z})}.$$

Recall that the norm on $F^p(\mathbb{Z})$ is universal with respect to representations of $\mathbb{Z}$ of $L^p$-spaces. Since $w$ is an invertible isometry, it induces a representation of $\mathbb{Z}$ on $L^p(Y, \nu)$, and universality of the norm $\|\cdot\|_{F^p(\mathbb{Z})}$ implies that $\|\lambda_p(f)\| \geq \|f(w)\|$.

We proceed to show the opposite inequality. Let $\varepsilon > 0$ and choose an element $\xi = (\xi_k)_{k\in\mathbb{Z}}$ in $\ell^p$ of finite support with $\|\xi\|_p^p = 1$, and such that

$$\|\lambda_p(f)\xi\|_p > \|\lambda_p(f)\| - \varepsilon.$$

Choose $K$ in $\mathbb{N}$ such that $\xi_k = 0$ whenever $|k| > K$. Find a positive integer $N$ in $\mathbb{N}$ and complex coefficients $a_n$ with $-N \leq n \leq N$ such that $f(x, x^{-1}) = \sum\limits_{n=-N}^{N} a_n x^n$. By assumption, there exists a measurable subset $E \subseteq Y$ with $\nu(E) > 0$ such that the sets $S^{-N-K}(E), \ldots, S^{N+K}(E)$ are pairwise disjoint. Since $Y$ is $\sigma$-finite, we may assume that $\nu(E) < \infty$.

Define a function $g\colon Y \to \mathbb{C}$ by

$$g = \sum_{k=-K}^{K} \xi_k w^k(\mathbb{1}_E).$$

Clearly $g$ is measurable. Using that the translates of $E$ are pairwise disjoint at the first step, we compute

$$\|g\|_p^p = \sum_{k=-K}^{K} |\xi_k|^p \|w^k(\mathbb{1}_E)\|_p^p = \nu(E)\|\xi\|_p^p = \nu(E) < \infty,$$

so $g \in L^p(Y,\nu)$ and $\|g\|_p = \nu(E)^{\frac{1}{p}}$.

We have

$$\begin{aligned} f(w)g &= \sum_{n=-N}^{N} a_n w^n(g) \\ &= \sum_{n=-N}^{N} a_n w^n \left( \sum_{k=-K}^{K} \xi_k w^k(\mathbb{1}_E) \right) \\ &= \sum_{n=-N}^{N} \sum_{k=-K}^{K} a_n \xi_k w^{n+k}(\mathbb{1}_E) \\ &= \sum_{j=-N-K}^{N+K} \left[\lambda_p(f)\xi\right]_j w^j(\mathbb{1}_E). \end{aligned}$$

We use again that the translates of $E$ are pairwise disjoint at the first step to get

$$\|f(w)g\|_p^p = \sum_{j=-N-K}^{N+K} \left| \left[\lambda_p(f)\xi\right]_j \right|^p \left\| w^j(\mathbb{1}_E) \right\| = \|\lambda_p(f)\xi\|_p^p \, \nu(E).$$

We conclude that

$$\begin{aligned} \|f(w)g\|_p &= \|\lambda_p(f)\xi\|_p \, \nu(E)^{\frac{1}{p}} \\ &= \|\lambda_p(f)\xi\|_p \, \|g\|_p \\ &> (\|\lambda_p(f)\| - \varepsilon) \, \|g\|_p. \end{aligned}$$

The estimate above shows that $\|f(w)\|_{L^p(Y,\nu)} > \|\lambda_p(f)\|_{F^p(\mathbb{Z})} - \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we conclude that $\|f(w)\| \geq \|\lambda_p(f)\|_{F^p(\mathbb{Z})}$, as desired.

We now claim that $\mathrm{sp}(w) = S^1$. We first observe that since $\mathrm{sp}(w)$ is a subset of $S^1$, it can be computed in any unital algebra that contains $w$ by Remark XVIII.3.7. In particular, we can compute the spectrum in $F^p(w, w^{-1})$.

We have shown that there is a canonical isometric isomorphism $\varphi\colon F^p(w, w^{-1}) \to F^p(\mathbb{Z})$ that maps $w$ to the bilateral shift $s$ in $\mathcal{B}(\ell^p)$. We deduce that

$$\mathrm{sp}_{F^p(w,w^{-1})}(w) = \mathrm{sp}_{F^p(\mathbb{Z})}(\varphi(w)) = \mathrm{sp}(s) = S^1,$$

and the proof is complete. □

**Corollary XVIII.3.15.** Assume that $\mu(X_\infty) > 0$. Then $\mathrm{sp}(v_\infty) = S^1$ and there is a canonical isometric isomorphism $F^p(v_\infty, v_\infty^{-1}) \cong F^p(\mathbb{Z})$ determined by sending $v_\infty$ to the canonical generator of $F^p(\mathbb{Z})$.

*Proof.* Since $\mathcal{A}$ contains the Borel $\sigma$-algebra on the metric space $X$, Lemma XVIII.3.12 applies and the result follows from Theorem XVIII.3.14. □

It is an immediate consequence of Theorem XVIII.3.9 and Theorem XVIII.3.14 that the sequence $(\mathrm{sp}(v_n))_{n\in\overline{\mathbb{N}}}$ is a spectral configuration, in the sense of Definition XVIII.2.3. We have been working with complete $\sigma$-finite standard Borel spaces in order to use Theorem XIII.2.4, as well as to prove Lemma XVIII.3.12. Our next lemma is the first step towards showing that one can get around this assumption in the general (separable) case.

**Lemma XVIII.3.16.** Let $p \in [1, \infty)$, let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two complete $\sigma$-finite standard Borel spaces such that $L^p(X, \mu)$ and $L^p(Y, \nu)$ are isometrically isomorphic, and let $\varphi\colon L^p(X, \mu) \to L^p(Y, \nu)$ be any isometric isomorphism. Let $v\colon L^p(X, \mu) \to L^p(X, \mu)$ be an invertible isometry and set $w = \varphi^{-1} \circ v \circ \varphi$, which is an invertible isometry of $L^p(Y, \nu)$. If $X_n$ and $Y_n$, for $n$ in $\overline{\mathbb{N}}$, are defined as in the beginning of this section with respect to $v$ and $w$, respectively, then $\varphi$ restricts to an isometric isomorphism $L^p(X_n, \mu|_{X_n}) \to L^p(Y_n, \nu|_{Y_n})$.

*Proof.* Write $v = m_{h_v} \circ u_{T_v}$ and $w = m_{h_w} \circ u_{T_w}$ as in Theorem XIII.2.4. For $n$ in $\overline{\mathbb{N}}$, the set $X_n$ is the set of points of $X$ of period $n$, and similarly with $Y_n$. Recall that there are canonical isometric isomorphisms

$$L^p(X,\mu) \cong \bigoplus_{n\in\overline{\mathbb{N}}} L^p(X_n,\mu|_{X_n}) \quad \text{and} \quad L^p(Y,\nu) \cong \bigoplus_{n\in\overline{\mathbb{N}}} L^p(Y_n,\nu|_{Y_n}).$$

Since $\varphi$ is an isometric isomorphism, Lamperti's theorem also applies to it, so there are a measurable function $g\colon Y \to S^1$ and an invertible measure class preserving transformation $S\colon X \to Y$ such that $\varphi = m_g \circ u_S$. It is therefore enough to show that $S(X_n) = Y_n$ for all $n$ in $\overline{\mathbb{N}}$.

It is an easy exercise to check that the identity $w = \varphi^{-1} \circ v \circ \varphi$ implies $T_w = S^{-1} \circ T_v \circ S$. The period of a point $x$ in $X$ (with respect to $T_v$) equals the period of $S(x)$ (with respect to $T_w$), and the result follows. $\square$

Recall that if $(X,\mu)$ is a measure space for which $L^p(X,\mu)$ is separable, then there is a complete $\sigma$-finite standard Borel space $(Y,\nu)$ for which $L^p(X,\mu)$ and $L^p(Y,\nu)$ are isometrically isomorphic.

**Definition XVIII.3.17.** Let $p \in [1,\infty)$, let $(X,\mathcal{A},\mu)$ be a measure space for which $L^p(X,\mu)$ is separable, and let $v\colon L^p(X,\mu) \to L^p(X,\mu)$ be an invertible isometry. Let $(Y,\mathcal{B},\nu)$ be a complete $\sigma$-finite standard Borel space and let $\psi\colon L^p(X,\mu) \to L^p(Y,\nu)$ be an isometric isomorphism. Set $w = \psi^{-1} \circ v \circ \psi$, which is an invertible isometry of $L^p(Y,\nu)$. Let $\{Y_n\}_{n\in\overline{\mathbb{N}}}$ be the partition of $Y$ into measurable subsets as described in the beginning of this section, and note that $w$ restricts to an invertible isometry $w_n$ of $L^p(Y_n,\nu|_{Y_n})$ for all $n$ in $\overline{\mathbb{N}}$. By the comments before Lemma XVIII.3.16, the sequence $(\mathrm{sp}(w_n))_{n\in\overline{\mathbb{N}}}$ is a spectral configuration.

We define the *spectral configuration* assosited with $v$, denoted $\sigma(v)$, by

$$\sigma(v) = (\mathrm{sp}(w_n))_{n\in\overline{\mathbb{N}}}.$$

We must argue why $\sigma(v)$, as defined above, is independent of the choice of the complete $\sigma$-finite standard Borel space $(Y,\nu)$ and the isometric isomorphism, but this follows immediately from Lemma XVIII.3.16.

The following is the main result of this section, and it asserts that every $L^p$-operator algebra generated by an invertible isometry together with its inverse is as in Theorem XVIII.2.5. The proof will follow rather easily from the results we have already obtained.

**Theorem XVIII.3.18.** Let $p \in [1,\infty)$, let $(X,\mu)$ be a measure space for which $L^p(X,\mu)$ is separable, and let $v\colon L^p(X,\mu) \to L^p(X,\mu)$ be an invertible isometry. Let $\sigma(v)$ be the spectral configuration associated to $v$ as in Definition XVIII.3.17. Then the Gelfand transform defines an isometric isomorphism

$$\Gamma\colon F^p(v,v^{-1}) \to F^p(\sigma(v)).$$

In particular, $F^p(v,v^{-1})$ can be represented on $\ell^p$.

*Proof.* Denote by $\iota\colon \mathrm{sp}(v) \to \mathbb{C}$ the canonical inclusion $\mathrm{sp}(v) \hookrightarrow \mathbb{C}$. For $n$ in $\overline{\mathbb{N}}$, denote by $\iota_n\colon \mathrm{sp}(v_n) \to \mathbb{C}$ the restriction of $\iota$ to $\mathrm{sp}(v_n)$, which is the canonical inclusion $\mathrm{sp}(v_n) \hookrightarrow \mathbb{C}$. Note that $F^p(\sigma(v))$ is generated by $\iota$ and $\iota^{-1}$, and that $\Gamma(v) = \iota$.

Let $f \in \mathbb{C}[\mathbb{Z}]$. We claim that $\|f(v)\| = \|f(\iota)\|$. Once we have proved this, the result will follow immediately.

Using Theorem XVIII.3.9 and Theorem XVIII.3.14 at the second step, and the definition of the norm $\|\cdot\|_{\sigma(v)}$ (Definition XVIII.2.4) at the third step, we have

$$\|f(v)\| = \sup_{n\in\overline{\mathbb{N}}} \|f(v_n)\| = \sup_{n\in\overline{\mathbb{N}}} \|f(\iota_n)\|_{\mathrm{sp}(v_n),n} = \|f(\iota)\|_{\sigma(v)},$$

and the claim is proved.

The last assertion in the statement follows from part (1) of Theorem XVIII.2.5. $\square$

In particular, we have shown that $L^p$-operator algebras generated by an invertible isometry and its inverse can always be isometrically represented on $\ell^p$. We do not know whether this is special to this class of $L^p$-operator algebras, and we believe it is possible that under relatively mild assumptions, any separable $L^p$-operator algebra can be isometrically represented on $\ell^p$. We formally raise this a problem.

**Problem XVIII.3.19.** Let $p \in [1,\infty)$. Find sufficient conditions for a separable $L^p$-operator algebra to be isometrically represented on $\ell^p$.

It is well known that any separable $L^2$-operator algebra can be isometrically represented on $\ell^2$.

We combine Theorem XVIII.3.18 with Theorem XVIII.2.5 to get an explicit description of $F^p(v,v^{-1})$ for an invertible isometry of a not necessarily separable $L^p$-space.

**Corollary XVIII.3.20.** Let $p \in [1, \infty)\backslash\{2\}$, let $(X, \mu)$ be a measure space, and let $v\colon L^p(X, \mu) \to L^p(X, \mu)$ be an invertible isometry. Then one, and only one, of the following holds:

1. There exist a positive integer $N \in \mathbb{N}$ and a (finite) spectral configuration $\sigma = (\sigma_n)_{n=1}^N$ with $\overline{\sigma} = \mathrm{sp}(v)$, and a canonical isometric isomorphism

$$F^p(v, v^{-1}) \cong F^p(\sigma) \cong \left( C(\mathrm{sp}(v)), \max_{n=1,\ldots,N} \|\cdot\|_{\sigma_n, n} \right).$$

   In this case, there is a Banach algebra isomorphism

$$F^p(v, v^{-1}) \cong (C(\mathrm{sp}(v)), \|\cdot\|_\infty),$$

   but this isomorphism cannot in general be chosen to be isometric unless $v$ is a multiplication operator.

2. There is a canonical isometric isomorphism

$$F^p(v, v^{-1}) \cong F^p(\mathbb{Z}).$$

It is obvious that the situations described in (1) and (2) cannot both be true.

*Proof.* It is clear that $F^p(v, v^{-1})$ is separable as a Banach algebra. By Proposition 1.25 in [207], there are a measure space $(Y, \nu)$ for which $L^p(Y, \nu)$ is separable and an isometric representation $\rho\colon F^p(v, v^{-1}) \to \mathcal{B}(L^p(Y, \nu))$. The result now follows from Theorem XVIII.3.18, which assumes that the isometry acts on a separable $L^p$ space, together with Theorem XVIII.2.5. □

We make a few comments about what happens when $p = 2$. Invertible isometries on Hilbert spaces are automatically unitary, so one always has $C^*(v, v^{-1}) = C^*(v) \cong C(\mathrm{sp}(v))$ isometrically, although all known proofs of this fact use completely different methods than the ones that are used here.

Recall that an algebra is said to be *semisimple* if the intersection of all its maximal left (or right) ideals is trivial. For commutative Banach algebras, this is equivalent to the Gelfand transform being injective. It is well-known that all $C^*$-algebras are semisimple.

**Corollary XVIII.3.21.** Let $p \in [1, \infty)$, let $(X, \mu)$ be a measure space, and let

$$v \colon L^p(X, \mu) \to L^p(X, \mu)$$

be an invertible isometry. Then:

1. $F^p(v, v^{-1})$ is semisimple.

2. Except in the case when $F^p(v, v^{-1}) \cong F^p(\mathbb{Z})$ and $p \neq 2$, the Gelfand transform $\Gamma \colon F^p(v, v^{-1}) \to C(\mathrm{sp}(v))$ is an isomorphism, although not necessarily isometric. In particular, if $\mathrm{sp}(v) \neq S^1$, then $\Gamma$ is an isomorphism.

The conclusions in Corollary XVIII.3.21 are somewhat surprising. Indeed, as we will see in the following section, part (1) fails if $v$ is an invertible isometry of a *subspace* of an $L^p$-space. It will follow from Proposition XIX.2.5 that part (2) also fails for isometries of subspaces of $L^p$-spaces.

It is a standard fact that Banach algebras are closed under holomorphic functional calculus, and that $C^*$-algebras are closed under continuous functional calculus. Using the description of the Banach algebra $F^p(v, v^{-1})$ of the corollary above, we can conclude that algebras of this form are closed by functional calculus of a fairly big class of functions, which in some cases includes all continuous functions on the spectrum of $v$.

**Corollary XVIII.3.22.** Let $p \in [1, \infty) \setminus \{2\}$, let $(X, \mu)$ be a measure space, and let $v \colon L^p(X, \mu) \to L^p(X, \mu)$ be an invertible isometry. Then $F^p(v, v^{-1})$ is closed under functional calculus for continuous functions on $\mathrm{sp}(v)$ of bounded variation. Moreover, if $F^p(v, v^{-1})$ is not isomorphic to $F^p(\mathbb{Z})$, then it is closed under continuous functional calculus.

In the context of the corollary above, if $p = 2$ then $F^p(v, v^{-1})$ is always isometrically isomorphic to $C(\mathrm{sp}(v))$, and hence it is closed under continuous functional calculus.

We conclude this work by describing all contractive homomorphisms between algebras of the form $F^p(v, v^{-1})$ that respect the canonical generator.

**Corollary XVIII.3.23.** Let $p \in [1, \infty) \setminus \{2\}$, and let $v$ and $w$ be two invertible isometries on $L^p$-spaces. The following are equivalent:

1. The linear map $\varphi_0 \colon \mathbb{C}[v, v^{-1}] \to \mathbb{C}[w, w^{-1}]$ determined by $v \mapsto w$, extends to a contractive homomorphism
$$\varphi \colon F^p(v, v^{-1}) \to F^p(w, w^{-1}).$$

2. We have $\mathrm{sp}(v) \subseteq \mathrm{sp}(w)$, and for every function $g$ in $F^p(\sigma(w)) \subseteq C(\mathrm{sp}(w))$, the restriction $g|_{\mathrm{sp}(v)}$ belongs to $F^p(\sigma(v))$ and
$$\left\| g|_{\mathrm{sp}(v)} \right\|_{\sigma(v)} \leq \|g\|_{\sigma(w)}.$$

3. We have $\widetilde{\sigma}(v) \subseteq \widetilde{\sigma}(w)$.

*Proof.* This is an immediate consequence of Theorem XVIII.3.18 and Corollary XVIII.2.13. □

**An Application: Quotients of Banach Algebras Acting on $L^1$-spaces**

In this section, we use our description of Banach algebras generated by invertible isometries of $L^p$-spaces to answer the case $p = 1$ of a question of Le Merdy (Problem 3.8 in [165]). In the theorem below, we show that the quotient of a Banach algebra that acts on an $L^1$-space, cannot in general be represented on *any* $L^p$-space for $p \in [1, \infty)$. In Chapter XIX, we give a negative answer to the remaining cases of Le Merdy's question, again using the results of the present work.

We begin with some preparatory notions. Let $A$ be a commutative Banach algebra, and denote by $\Gamma \colon A \to C_0(\mathrm{Max}(A))$ its Gelfand transform. Given a closed subset $E \subseteq \mathrm{Max}(A)$, denote by $k(E)$ the ideal
$$k(E) = \{a \in A \colon \Gamma(a)(x) = 0 \ \text{ for all } x \in E\}$$
in $A$. Similarly, given an ideal $I$ in $A$, set
$$h(I) = \{x \in \mathrm{Max}(A) \colon \Gamma(a)(x) = 0 \ \text{ for all } a \in I\},$$
which is a closed subset of $\mathrm{Max}(A)$. It is clear that $h(k(E)) = E$ for every closed subset $E \subseteq \mathrm{Max}(A)$. It is an easy exercise to check that if $A$ is a semisimple Banach algebra, then the quotient $A/k(E)$ is semisimple as well, essentially because $k(E)$ is the largest ideal $J$ of $A$ satisfying $h(J) = E$.

**Definition XVIII.4.1.** A commutative, semisimple Banach algebra $A$ is said to have (or satisfy) *spectral synthesis* if for every closed subset $E \subseteq \mathrm{Max}(A)$, there is only one ideal $J$ in $A$ satisfying $h(J) = E$ (in which case it must be $J = k(E)$).

It is easy to verify that a semisimple Banach algebra $A$ has spectral synthesis if and only if every quotient of $A$ is semisimple. (We are thankful to Chris Phillips for pointing this out to us.)

It is a classical result due to Malliavin in the late 50's ([179]), that for an abelian locally compact group $G$, its group algebra $L^1(G)$ has spectral synthesis if and only if $G$ is compact.

**Theorem XVIII.4.2.** There is a quotient of $F^1(\mathbb{Z})$ that cannot be isometrically represented on *any* $L^p$-space for $p \in [1, \infty)$. In particular, the class of Banach algebras that act on $L^1$-spaces is not closed under quotients.

*Proof.* The norm of a function $f \in \ell^1(\mathbb{Z})$ as an element of $F^1(\mathbb{Z}) = F^1_\lambda(\mathbb{Z})$ is the norm it gets as a convolution operator on $\ell^1(\mathbb{Z})$. The fact that $\ell^1(\mathbb{Z})$ has a contractive approximate identity is easily seen to imply that in fact $\|f\|_{F^1(\mathbb{Z})} = \|f\|_{\ell^1(\mathbb{Z})}$. It follows that there is a canonical identification $F^1(\mathbb{Z}) = \ell^1(\mathbb{Z})$. (See Proposition 3.14 in [207] for a more general version of this argument.)

Denote by $v \in \ell^1(\mathbb{Z})$ the canonical invertible isometry that, together with its inverse, generates $\ell^1(\mathbb{Z})$. If $I$ is an ideal in $\ell^1(\mathbb{Z})$ and $\pi \colon \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})/I$ is the quotient map, then $\ell^1(\mathbb{Z})/I$ is generated by $\pi(v)$ and $\pi(v^{-1})$. These elements are invertible and have norm one, since we have

$$\|\pi(v)\| \leq 1 \ , \|\pi(v^{-1})\| \leq 1 \ , \quad \text{and} \quad 1 \leq \|\pi(v)\| \|\pi(v^{-1})\|.$$

By Malliavin's result, $\ell^1(\mathbb{Z})$ does not have spectral synthesis. Let $I$ be an ideal in $\ell^1(\mathbb{Z})$ such that $\ell^1(\mathbb{Z})/I$ is not semisimple. Then this quotient cannot be represented on any $L^p$-space, for $p \in [1, \infty)$, by part (1) of Corollary XVIII.3.21. ☐

We close this chapter by showing that our results cannot be extended to invertible isometries of more general Banach spaces, even subspaces of $L^p$-spaces.

**Remark XVIII.4.3.** We keep the notation from the proof of Theorem XVIII.4.2. By Corollary 1.5.2.3 in [140], the quotient $\ell^1(\mathbb{Z})/I$ can be represented on a subspace of an $L^1$-space. In particular, $\ell^1(\mathbb{Z})/I$ is generated by an invertible isometry of a $SL^1$-space, and its inverse. We conclude that part (1) of Corollary XVIII.3.21 fails if we replace $L^p$-spaces with $SL^p$-spaces.

Similarly, it follows from Proposition XIX.2.5 that part (2) also fails for isometries of $SL^p$-spaces, even when the Banach algebra they generate is semisimple, and even if the invertible isometry has non-full spectrum.

# CHAPTER XIX

# QUOTIENTS OF BANACH ALGEBRAS ACTING ON $L^P$-SPACES

This chapter is based on joint work with Hannes Thiel ([95]).

We show that the class of Banach algebras that can be isometrically represented on an $L^p$-space, for $p \neq 2$, is not closed under quotients. This answers a question asked by Le Merdy 20 years ago. Our methods are heavily reliant on our earlier study of Banach algebras generated by invertible isometries of $L^p$-spaces.

## Introduction

An operator algebra is a closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. If $A$ is an operator algebra and $I \subseteq A$ is a closed, two-sided ideal, then the quotient Banach algebra $A/I$ is again an operator algebra, that is, it can be isometrically represented on a Hilbert space. This classical result is due to Lumer and Bernard, although the commutative case (when $A$ is a uniform algebra) was proved earlier by Cole.

Some Banach algebras are naturally given as algebras of operators on certain classes of Banach spaces. If $\mathscr{E}$ is a class of Banach spaces, we say that a Banach algebra $A$ is an *$\mathscr{E}$-operator algebra* if there exist a Banach space $E \in \mathscr{E}$ and an isometric homomorphism $\varphi\colon A \to \mathcal{B}(E)$. If $\mathscr{H}$ is the class of all Hilbert spaces, then an $\mathscr{H}$-operator algebra is just an operator algebra in the usual sense. With this terminology, the Bernard-Cole-Lumer theorem states that $\mathscr{H}$-operator algebras are closed under quotients. A natural question is then:

**Question XIX.1.1.** For what other classes $\mathscr{E}$ of Banach spaces are $\mathscr{E}$-operator algebras closed under quotients?

Given $p \in [1, \infty)$, we say that a Banach space $E$ is a $QSL^p$-space if $E$ is isometrically isomorphic to a quotient of a (closed) subspace of an $L^p$-space. We denote by $\mathscr{Q}\mathscr{S}\mathscr{L}^p$ the class of all $QSL^p$-spaces. In Corollary 3.2 of [165], Christian Le Merdy showed that $\mathscr{Q}\mathscr{S}\mathscr{L}^p$-operator algebras are closed under quotients. This result generalizes the Bernard-Cole-Lumer theorem, which is the case $p = 2$, since $\mathscr{Q}\mathscr{S}\mathscr{L}^2 = \mathscr{L}^2$ is the class of Hilbert spaces.

With $\mathscr{L}^p$ denoting the class of $L^p$-spaces, Problem 3.8 in [165] asks whether $\mathscr{L}^p$-operator algebras are closed under quotients for $p \neq 2$. A partial result in this direction is the work of Marius Junge ([140]) on the class $\mathscr{S}\mathscr{L}^p$ of (closed) subspaces of $L^p$-spaces, which he describes as a first step towards dealing with the class $\mathscr{L}^p$. Indeed, Corollary 1.5.2.3 in [140] asserts that $\mathscr{S}\mathscr{L}^p$-operator algebras are also closed under quotients.

As the authors point out, the arguments used both in [165] and [140] are not suitable to deal with the class $\mathscr{L}^p$, which seems to be the class for which Question XIX.1.1 is more natural.

In this chapter, which is based on [95], we answer Le Merdy's question negatively. In other words, we show that $\mathscr{L}^p$-operator algebras are not closed under quotients when $p \in [1, \infty) \setminus \{2\}$. We do so by exhibiting a concrete example of an $\mathscr{L}^p$-operator algebra $A$ and a closed, two-sided ideal $I$ in $A$ such that $A/I$ cannot be represented on an $L^p$-space. What we show is slightly stronger: in our example, the quotient $A/I$ cannot be represented on *any* $L^q$-space for $q \in [1, \infty)$. The algebra $A$ is a semisimple commutative Banach algebra: the algebra $F^p(\mathbb{Z})$ of $p$-pseudofunctions on $\mathbb{Z}$.

Given the recent attention received by $\mathscr{L}^p$-operator algebras, deciding whether these are closed under quotients becomes more relevant and technically useful. For example, consider the $L^p$-analogs $\mathcal{O}_n^p$ of the Cuntz algebras; see [204]. These are all simple, and any contractive, non-zero representation of any of them on an $L^p$-space is automatically injective (in fact, isometric). For $p = 2$, these two properties are well-known to be equivalent. However, for $p \neq 2$, they are not, since quotients of $\mathscr{L}^p$-operator algebras are not in general representable on $L^p$-spaces. These two properties of $\mathcal{O}_n^p$ therefore require separate and independent proofs. A similar problem arises with the $L^p$-analogs $A_\theta^p$ of irrational rotation algebras; see [93].

If $A$ is a commutative unital Banach algebra, we will denote by $\Gamma_A \colon A \to C(\mathrm{Max}(A))$ its Gelfand transform, which is natural in the following sense. If $\varphi \colon A \to B$ is a unital homomorphism between commutative unital Banach algebras $A$ and $B$, then the assignment $\mathrm{Max}(B) \to \mathrm{Max}(A)$ given by $I \mapsto \varphi^{-1}(I)$ defines a contractive homomorphism $\Gamma(\varphi) \colon C(\mathrm{Max}(A)) \to C(\mathrm{Max}(B))$

making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "\Gamma_A"] \arrow[d, "\varphi"'] & C(\mathrm{Max}(A)) \arrow[d, "\Gamma(\varphi)"] \\ B \arrow[r, "\Gamma_B"'] & C(\mathrm{Max}(B)). \end{tikzcd}$$

**Our Examples**

We begin by introducing some terminology which will be needed later. We adopt the convention that all representations of Banach algebras are contractive, and do not include this in the terminology.

**Definition XIX.2.1.** Let $A$ be a Banach algebra and let $\mathscr{E}$ be a class of Banach spaces.

1. We say that $A$ is *(unitally) representable* on $\mathscr{E}$, if there exist a Banach space $E \in \mathscr{E}$ and a (unital) contractive, injective homomorphism $\varphi\colon A \to \mathcal{B}(E)$.
2. We say that $A$ is *(unitally) isomorphically represented* on $\mathscr{E}$, if there exist a Banach space $E \in \mathscr{E}$ and a (unital) contractive, injective homomorphism $\varphi\colon A \to \mathcal{B}(E)$ with closed range.
3. We say that $A$ is *(unitally) isometrically represented* on $\mathscr{E}$, if there exist a Banach space $E \in \mathscr{E}$ and a (unital) isometric homomorphism $\varphi\colon A \to \mathcal{B}(E)$.

Isometric representability is the notion we are mostly concerned with in this chapter, but we will be able to say things about contractive equivalent representability as well. On the other hand, the notion of (contractive) representability, although natural, is far too weak for our purposes, as is shown below.

**Proposition XIX.2.2.** Let $A$ be a (unital) separable, semisimple, commutative Banach algebra. Let $p$ be an arbitrary Hölder exponent in $[1, \infty)$. Then $A$ is (unitally) representable on an $L^p$-space.

*Proof.* Let $X$ be a separable locally compact Hausdorff space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $X$. For $n \in \mathbb{N}$, denote by $\delta_n$ the atomic measure concentrated on $\{x_n\}$, and set $\mu = \sum\limits_{n\in\mathbb{N}} 2^{-n}\delta_n$. Then $\mu$ is a Borel probability measure on $X$. It is easy to check that the homomorphism

$$\varphi\colon C_0(X) \to \mathcal{B}(L^p(X, \mu))$$

given by $(\varphi(f)\xi)(x) = f(x)\xi(x)$ for every $f \in C_0(X)$, every $\xi \in L^p(X, \mu)$ and almost every $x \in X$, is a isometric representation, which is unital if $X$ is compact.

Now for the Banach algebra $A$ in the statement, its maximal ideal space $\mathrm{Max}(A)$ is locally compact, Hausdorff and separable (and it is compact if $A$ is unital). The Gelfand transform $\Gamma_A \colon A \to C_0(\mathrm{Max}(A))$ is injective and contractive (and unital if $A$ is). By the first paragraph in this proof, there exists a (unital) isometric representation $\varphi_A$ of $C_0(\mathrm{Max}(A))$ on an $L^p$-space. Then $\varphi_A \circ \Gamma_A$ is a (unital) representation of $A$ on an $L^p$-space. $\square$

In the rest of this section, for any $p \in [1, \infty) \setminus \{2\}$, we exhibit an example of a unital $L^p$-operator algebra $A$ and a closed, two-sided ideal $I$ in $A$ such that $A/I$ cannot be isomorphically represented on *any* $L^q$-space for $q \in [1, \infty)$. Our example is a semisimple, commutative Banach algebra: the algebra $F^p(\mathbb{Z})$ of $p$-pseudofunctions on $\mathbb{Z}$, and the quotient $A/I$ is also semisimple. In particular, $A/I$ can be represented on an $L^q$-space for every $q \in [1, \infty)$, by Proposition XIX.2.2.

We begin with some preparatory results. Our first lemma allows us to assume that contractive representations of unital Banach algebras on $L^p$-spaces are unital.

**Lemma XIX.2.3.** Let $A$ be a unital Banach algebra and let $p \in [1, \infty)$. If $A$ can be represented (respectively, isomorphically or isometrically represented) on an $L^p$-space, then it can be unitally represented (respectively, unitally isomorphically or unitally isometrically represented) on an $L^p$-space.

*Proof.* Let $E$ be an $L^p$-space and let $\varphi \colon A \to \mathcal{B}(E)$ be a contractive, injective homomorphism. Then $e = \varphi(1)$ is an idempotent with $\|e\| = 1$ in $\mathcal{B}(E)$. By Theorem 6 in [267], the range $F$ of $e$ is an $L^p$-space. The cut-down homomorphism

$$\psi \colon A \to \mathcal{B}(F) \cong \varphi(1)\mathcal{B}(E)\varphi(1)$$

is the desired unital, contractive, injective representation.

Finally, $\psi$ has closed range (respectively, is isometric) if and only if so does $\varphi$. $\square$

For the rest of this section, we fix $p \in [1, \infty)$. We will abbreviate the Gelfand transform $\Gamma_{F^p(\mathbb{Z})} \colon F^p(\mathbb{Z}) \to C(S^1)$ of $F^p(\mathbb{Z})$ to just $\Gamma$. For an open subset $V$ of $S^1$, we denote

$$I_V = \Gamma^{-1}(C_0(V)),$$

which is a closed, two-sided ideal in $F^p(\mathbb{Z})$. We will abbreviate $F^p(\mathbb{Z})$ to $A$, and the quotient $F^p(\mathbb{Z})/I_V$ to $A_V$. The Gelfand transform $\Gamma_{A_V}\colon A_V \to C(\mathrm{Max}(A_V))$ will be abbreviated to $\Gamma_V$.

**Remark XIX.2.4.** We recall the following fact about spectra of elements in Banach algebras. If $B$ is a unital Banach algebra, $A$ is a subalgebra containing the unit of $B$, and $a$ is an element of $A$ such that $\mathrm{sp}_A(a) \subseteq S^1$, then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$. In other words, if the spectrum of an element of a Banach algebra is a subset of $S^1$, then the spectrum can be computed in any unital algebra containing the element (bigger or smaller than the original algebra).

**Proposition XIX.2.5.** Let $V$ be an open subset of $S^1$. Suppose that there exist $q \in [1, \infty)$ and an $L^q$-space $E$ such that $A_V$ is isomorphically representable on $E$. Then the Gelfand transform $\Gamma_V\colon A_V \to C(S^1 \setminus V)$ is an isomorphism (although not necessarily isometric). In particular, and identifying $F^p(\mathbb{Z})$ with a subalgebra of $C(S^1)$ via $\Gamma$, it follows that every continuous function on $S^1 \setminus V$ is the restriction of a function in $F^p(\mathbb{Z})$.

*Proof.* It is clear that $\mathrm{Max}(A_V)$ is canonically homeomorphic to $S^1 \setminus V$, so the range of $\Gamma_V$ can be canonically identified with a subalgebra of $C(S^1 \setminus V)$. Moreover, it is clear that $A_V$ is semisimple, and hence there are natural identifications

$$A_V \cong \frac{\Gamma(F^p(\mathbb{Z}))}{\Gamma(F^p(\mathbb{Z})) \cap C_0(V)} \cong \frac{\Gamma(F^p(\mathbb{Z})) + C_0(V)}{C_0(V)}.$$

Denote by $\pi\colon A \to A_V$ the canonical quotient map. Observe that $A_V$ is generated by the image $\pi(u)$ of the canonical generator $u$ of $A = F^p(\mathbb{Z})$, which is an invertible isometry. Suppose that there exist $q \in [1, \infty)$, an $L^q$-space $E$, and an isomorphic representation $\varphi\colon A_V \to \mathcal{B}(E)$. By Lemma XIX.2.3, we can assume that $\varphi$ is unital. It is clear that $\varphi(\pi(u))$ generates $\varphi(A_V)$. Since $\varphi$ is unital, $\varphi(\pi(u))$ is an invertible isometry of an $L^q$-space. Moreover, using Remark XIX.2.4 at the first step, we have

$$\mathrm{sp}_{\mathcal{B}(E)}(\varphi(\pi(u))) = \mathrm{sp}_{\varphi(A_V)}(\varphi(\pi(u))) = \mathrm{sp}_{A_V}(\pi(u)) = S^1 \setminus V.$$

We claim that the Gelfand transform $\Gamma_{\varphi(A_V)}\colon \varphi(A_V) \to C(S^1 \setminus V)$ is an isomorphism. Once we show this, it will follow that $\Gamma_V$ is also an isomorphism.

First, $\Gamma_{\varphi(A_V)}$ is clearly injective by semisimplicity of $A_V$. Suppose that $q = 2$. Then $\varphi(A_V)$ is a $C^*$-algebra, because it is generated by an invertible isometry of a Hilbert space (a unitary), and $A_V$ is therefore self-adjoint. The claim is then an immediate consequence of Gelfand's theorem (and in this case $\Gamma_{\varphi(A_V)}$ is isometric). Assume now that $q \in [1, \infty) \setminus \{2\}$. In this case, and since the spectrum of $\varphi(\pi(u))$ in $\mathcal{B}(E)$ is not the whole circle, the result follows from part (1) of Corollary 5.20 in [96]. The claim is proved, and the first part of the proposition follows.

For the second claim, denote by $r\colon C(S^1) \to C(S^1 \setminus V)$ the restriction map. It is clear that $\Gamma(\pi) = r$. Naturality of the Gelfand transform shows that the diagram

$$\begin{tikzcd} A \arrow[r, "\Gamma"] \arrow[d, "\pi"'] & C(S^1) \arrow[d, "r"] \\ A_V \arrow[r, "\Gamma_V"'] & C(S^1 \setminus V) \end{tikzcd}$$

is commutative. It follows that for every $f \in C(S^1 \setminus V)$, there exists $g \in A = F^p(\mathbb{Z})$ such that $\Gamma_V(\pi(g)) = f$. Regarding $g$ as a function on $S^1$, this is equivalent to $g|_{S^1 \setminus V} = f$. $\square$

Let $\theta \in \mathbb{R}$. Then it is easy to show that the homeomorphism $h_\theta\colon S^1 \to S^1$ given by $h_\theta(\zeta) = e^{2\pi i \theta}\zeta$ for $\zeta \in S^1$ induces an isometric automorphism of $F^p(\mathbb{Z})$. (We warn the reader that it is not in general true that any homeomorphism of $S^1$ induces an isometric, or even contractive, automorphism of $F^p(\mathbb{Z})$. In fact, when $p \neq 2$, the only homeomorphisms of $S^1$ that do so are the rotations and compositions of rotations with the homeomorphism $\zeta \mapsto \overline{\zeta}$ of $S^1$.)

The following is the main result of this chapter. Recall our conventions and notations from the comments before Proposition XIX.2.5.

**Theorem XIX.2.6.** Let $p \in [1, \infty) \setminus \{2\}$. Let $V$ be a nontrivial open subset of $S^1$, and assume that $V$ is not dense in $S^1$. Then $A_V$ cannot be isomorphically represented on *any* $L^q$-space for $q \in [1, \infty)$.

*Proof.* We argue by contradiction, so let $V$ be an open subset of $S^1$ as in the statement, and suppose that there exists $q \in [1, \infty)$ such that $A_V$ can be isomorphically representable on an $L^q$-space.

Let $f \in C(S^1)$. We claim that $f$ belongs to $\Gamma(F^p(\mathbb{Z}))$. Once we prove this, it will follow from part (2) of Corollary XIV.3.20 that $p = 2$, and hence the proof will be complete.

Let $W$ be an open subset of $S^1$ such that $V \cap W = \emptyset$. With the notation used in the comments before this theorem, and using compactness of $S^1$, find $n \in \mathbb{N}$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$ such that $\bigcup\limits_{j=1}^{n} h_{\theta_j}(W) = S^1$. For $j \in \{1, \ldots, n\}$, set $V_j = h_{\theta_j}(V)$ and $W_j = h_{\theta_j}(W)$. There is an isometric isomorphism

$$A_{h_{\theta_j}(V)} \cong A_V,$$

so the Banach algebra $A_{h_{\theta_j}(V)}$ can be isomorphically represented on an $L^q$-space. It follows from Proposition XIX.2.5 that every continuous function on $S^1 \setminus V_j$ is in the image of $\Gamma_{V_j}$. In particular, every continuous function on $\overline{W_j}$ is in the image of $\Gamma_{V_j}$.

From now on, we identify the algebras $A, A_{V_1}, \ldots, A_{V_n}$ with their images under their Gelfand transforms. In particular, for $j = 1, \ldots, n$, every continuous function on $\overline{W_j}$ is the restriction of a function in $A$.

Choose continuous functions $k_1, \ldots, k_n \colon S^1 \to \mathbb{R}$ satisfying

1. $0 \leq k_j \leq 1$ for $j = 1, \ldots, n$;
2. $\mathrm{supp}(k_j) \subseteq W_j$ for $j = 1, \ldots, n$;
3. $\sum\limits_{j=1}^{n} k_j(\zeta) = 1$ for all $\zeta \in S^1$;
4. $k_j$ belongs to $F^p(\mathbb{Z})$ for $j = 1, \ldots, n$ (for example, take $k_j \in C^\infty(S^1)$).

For $j = 1, \ldots, n$, choose a function $g_j \in F^p(\mathbb{Z})$ such that $(g_j)|_{W_j} = f|_{W_j}$. Then the product $g_j k_j$ belongs to $F^p(\mathbb{Z})$ because each of the factors does. Since the support of $k_j$ is contained in $W_j$, and $f$ and $g_j$ agree on $W_j$, we have $f k_j = g_j k_j$ for $j = 1, \ldots, n$. Now,

$$f = f \cdot \left( \sum_{j=1}^{n} k_j \right) = \sum_{j=1}^{n} g_j k_j,$$

so $f$ belongs to $F^p(\mathbb{Z})$, and the claim is proved.

We have shown that the Gelfand transform $\Gamma \colon F^p(\mathbb{Z}) \to C(S^1)$ is surjective. Since $F^2(\mathbb{Z})$ is canonically isomorphic to $C(S^1)$, we must have $p = 2$ by part (2) of Corollary XIV.3.20. This is a contradiction, and the result follows. $\square$

In contrast to Theorem XIX.2.6, some (non-trivial) quotients of $F^p(\mathbb{Z})$ are isometrically representable on $L^p$-spaces. For example, if $V$ is the complement of the set of $n$-th roots of unity in $S^1$ for some $n \in \mathbb{N}$, then $F^p(\mathbb{Z})/I_V$ is canonically isometrically isomorphic to $F^p(\mathbb{Z}_n)$. (This identification is induced by the quotient map $\mathbb{Z} \to \mathbb{Z}_n$.) An analogous statement holds for the translates of $V$. We do not know, however, whether these are the only quotients of $F^p(\mathbb{Z})$ that can be represented on $L^p$-spaces. We therefore suggest:

**Problem XIX.2.7.** Characterize those ideals $I$ of $F^p(\mathbb{Z})$ such that $F^p(\mathbb{Z})/I$ can be isometrically represented on an $L^p$-space.

We do not know whether $F^p(\mathbb{Z})$ has spectral synthesis, except for $p = 1$ (in which case it does not) and $p = 2$ (in which case it does). Since Banach algebras generated by an invertible isometry of an $L^p$-space together with its inverse are automatically semisimple by the results in Chapter XVIII, we conclude that for $F^p(\mathbb{Z})/I$ to be isometrically representable on an $L^p$-space, there must exist an open subset $V \subseteq S^1$ such that $I = I_V$ (and $V$ must be dense by Theorem XIX.2.6). This means that Problem XIX.2.7 can be solved without knowing whether $F^p(\mathbb{Z})$ has spectral synthesis, that is, without knowing whether every ideal of $F^p(\mathbb{Z})$ is of the form $I_V$.

We do not know whether density of $V$ is sufficient for $F^p(\mathbb{Z})/I_V$ to be representable on an $L^p$-space.

We conclude this chapter with an observation. If $A$ is a Banach algebra and $a \in A$, we denote by $B(a)$ the smallest Banach subalgebra of $A$ containing $a$.

**Remark XIX.2.8.** Suppose that $p$ is not an even integer, and let $V \subseteq S^1$ be as in Theorem XIX.2.6. It follows from Corollary 1.5.2.3 in [140] that $A_V$ can be isometrically represented on an $SL^p$-space, so there exists an invertible isometry $v$ of an $SL^p$-space $E$ such that $B(u)$ is isometrically isomorphic to $A_V$. By Theorem I in [240], there exist a canonical $L^p$-space $F$ containing $E$ as a closed subspace, and a canonical invertible isometry $w$ of $F$ extending $v$. By naturality of the construction, one may be tempted to guess that $B(v)$ and $B(w)$ are isometrically isomorphic. However, this is not the case since $B(w)$ is trivially representable on an $L^p$-space, while $B(v) \cong A_V$ is not, by Theorem XIX.2.6.

# CHAPTER XX

# REPRESENTATIONS OF ETALE GROUPOIDS ON $L^P$-SPACES

This chapter is based on joint work with Martino Lupini ([89]).

For $p \in (1, \infty)$, we study representations of etale groupoids on $L^p$-spaces. Our main result is a generalization of Renault's disintegration theorem for representations of etale groupoids on Hilbert spaces. We establish a correspondence between $L^p$-representations of an etale groupoid $G$, contractive $L^p$-representations of $C_c(G)$, and tight regular $L^p$-representations of any countable inverse semigroup of open slices of $G$ that is a basis for the topology of $G$. We define analogs $F^p(G)$ and $F^p_\lambda(G)$ of the full and reduced groupoid $C^*$-algebras using representations on $L^p$-spaces. As a consequence of our main result, we deduce that every contractive representation of $F^p(G)$ or $F^p_\lambda(G)$ is automatically completely contractive. Examples of our construction include the following natural families of Banach algebras: discrete group $L^p$-operator algebras, the analogs of Cuntz algebras on $L^p$-spaces, and the analogs of AF-algebras on $L^p$-spaces. Our results yield new information about these objects: their matricially normed structure is uniquely determined. More generally, groupoid $L^p$-operator algebras provide analogs of several families of classical $C^*$-algebras, such as Cuntz-Krieger $C^*$-algebras, tiling $C^*$-algebras, and graph $C^*$-algebras.

## Introduction

Groupoids are a natural generalization of groups, where the operation is no longer everywhere defined. Succinctly, a groupoid can be defined as a small category where every arrow is invertible, with the operations being composition and inversion of arrows. A groupoid is called locally compact when it is endowed with a (not necessarily Hausdorff) locally compact topology compatible with the operations; see [196]. Any locally compact group is in particular a locally compact groupoid. More generally, one can associate to a continuous action of a locally compact group on a locally compact Hausdorff space the corresponding action groupoid as in [176]. This allows one to regard locally compact groupoids as a generalization of topological dynamical systems.

A particularly important class of locally compact groupoids are those where the operations are local homeomorphisms. These are the so-called etale—or $r$-discrete [225]—groupoids, and constitute the groupoid analog of actions of discrete groups on locally compact spaces. In fact, they can be described in terms of partial actions of inverse semigroups on locally compact spaces; see [66]. Alternatively, one can characterize etale groupoids as the locally compact groupoids having an open basis of *slices*, that is, sets where the source and range maps are injective [66, Section 3]. In the étale case, the set of all open slices is an inverse semigroup.

The representation theory of etale groupoids on Hilbert spaces has been intensively studied since the seminal work of Renault [225]; see the monograph [196]. A representation of an etale groupoid $G$ on a Hilbert space is an assignment $\gamma \mapsto T_\gamma$ of an invertible isometry $T_\gamma$ between Hilbert spaces to any element $\gamma$ of $G$. Such an assignment is required to respect the algebraic and measurable structure of the groupoid. The fundamental result of [225] establishes a correspondence between the representations of an etale groupoid $G$ and the nondegenerate $I$-norm contractive representations of $C_c(G)$. (The $I$-norm on $C_c(G)$ is the analogue of the $L^1$-norm for discrete groups. When $G$ is Hausdorff, $C_c(G)$ is just the space of compactly-supported continuous functions on $G$. The non-Hausdorff case is more subtle; see [66, Definition 3.9].) Moreover, such a correspondence is compatible with the natural notions of equivalence for representations of $G$ and $C_c(G)$. In turn, nondegenerate representations of $C_c(G)$ correspond to tight regular representations of any countable inverse semigroup $\Sigma$ of open slices of $G$ that is a basis for the topology. Again, such correspondence preserves the natural notions of equivalence for representations of $C_c(G)$ and $\Sigma$. Tightness is a nondegeneracy condition introduced by Exel in [66, Section 11]. In the case when the set $G^0$ of objects of $G$ is compact and zero dimensional, the one can take $\Sigma$ to be the inverse semigroup of compact open slices of $G$. In this case the semilattice $E(\Sigma)$ of idempotent elements of $\Sigma$ is the Boolean algebra of clopen subsets of $G^0$, and a representation of $G$ is tight if and only if its restriction to $E(\Sigma)$ is a Boolean algebra homomorphism.

In this chapter, which is based on [89], we show how an important chapter in the theory of $C^*$-algebras admits a natural generalization to algebras of operators on $L^p$-spaces, perfectly mirroring the Hilbert space case. We prove that the correspondences described in the paragraph above directly generalize when one replaces representations on Hilbert spaces with representations on $L^p$-spaces for some Hölder exponent $p$ in $(1, \infty)$. For $p = 2$, one recovers Renault's and Exel's results. Interestingly, the proofs for $p = 2$ and $p \neq 2$ differ drastically. The methods when $p \neq 2$ are based on the characterization of invertible isometries of $L^p$-spaces stated by Banach in [6]. (The first available proof is due to Lamperti [161], hence the name Banach-Lamperti theorem.)

Following [216, 164, 48], we say that a representation of a matricially normed algebra $A$ on $L^p(\lambda)$ is *p-completely contractive* if all its amplifications are contractive when the algebra of $n \times n$ matrices of bounded linear operators on $L^p(\lambda)$ is identified with the algebra of bounded linear operators on $\lambda \times c_n$. (Here and in the following, $c_n$ denotes the counting measure on $n$ points.) If $G$ is an etale groupoid, then the identification between $M_n(C_c(G))$ and $C_c(G_n)$ for a suitable amplification $G_n$ of $G$ defines matricial norms on the algebra $C_c(G)$. As a corollary of our analysis a contractive representation of $C_c(G)$ on an $L^p$-space is automatically $p$-completely contractive.

In the case of Hilbert space representations, the universal object associated to $C_c(G)$ is the groupoid $C^*$-algebra $C^*(G)$, as defined in [196, Chapter 3]. One can also define a reduced version $C^*_\lambda(G)$ (see [196, pages 108-109]), that only considers representations of $C_c(G)$ that are induced—in the sense of Rieffel [196, Appendix D]—from a Borel probability measure on the space of objects of $G$. Amenability of the groupoid $G$ implies that the canonical surjection from $C^*(G)$ to $C^*_\lambda(G)$ is an isomorphism. In the case when $G$ is a countable discrete group, these objects are the usual full and reduced group $C^*$-algebras.

A similar construction can be performed for an arbitrary $p$ in $(1, \infty)$, and the resulting universal objects are the full and reduced groupoid $L^p$-operator algebras $F^p(G)$ and $F^p_\lambda(G)$ of $G$. When $G$ is a countable discrete group, these are precisely the full and reduced group $L^p$-operator algebras of $G$ as defined in [207]; see also Chapter XIV. When $G$ is the groupoid associated with a Bratteli diagram as in [226, Section 2.6], one obtains the spatial $L^p$-analog of an AF $C^*$-algebra; see [215]. (The $L^p$-analogs of UHF $C^*$-algebras are considered in [209, 208].) When $G$ is one of the Cuntz groupoids defined in [226, Section 2.5], one obtains the $L^p$-analogs of the corresponding Cuntz algebra from [204, 209, 208].

More generally, this construction provides several new examples of $L^p$-analogs of "classical" $C^*$-algebras, such as Cuntz-Krieger algebras, graph algebras, and tiling $C^*$-algebras (all of which can be realized as groupoid $C^*$-algebras for a suitable etale groupoid; see [160] and [196]). It is worth mentioning here that there seems to be no known example of a nuclear $C^*$-algebra that cannot be described as the enveloping $C^*$-algebra of a locally compact groupoid.

We believe that this point of view is a contribution towards clarifying what are the well-behaved representations of algebraic objects—such as the Leavitt algebras, Bratteli diagrams, or graphs—on $L^p$-spaces. In [204, 209, 208], several characterizations are given for well behaved representations of Leavitt algebras and stationary Bratteli diagrams. The fundamental property considered therein is the uniqueness of the norm that they induce. The groupoid approach shows that these representations are precisely those coming from representations of the associated groupoid or, equivalently, its inverse semigroup of open slices.

Another upshot of the present work is that the groupoid $L^p$-operator algebras $F^p(G)$ and $F^p_\lambda(G)$ satisfy an automatic $p$-complete contractivity property for contractive homomorphisms into other $L^p$-operator algebras. In fact, $F^p(G)$ and $F^p_\lambda(G)$ have canonical matrix norms. Such matrix norm structure satisfies the $L^p$-analog of Ruan's axioms for operator spaces as defined in [48, Subsection 4.1], building on [216, 164]. Using the terminology of [48, Subsection 4.1], this turns the algebras $F^p(G)$ and $F^p_{\mathrm{red}}(G)$ into $p$-operator systems such that the multiplication is $p$-completely contractive. It is a corollary of our main results that any contractive representation of these algebras on an $L^p$-space is automatically $p$-completely contractive. As a consequence the matrix norms on $F^p(G)$ and $F^p_\lambda(G)$ are uniquely determined—as it is the case for $C^*$-algebras.

It is still not clear what are the well-behaved algebras of operators on $L^p$-spaces. Informally speaking, these should be the $L^p$-operator algebras that behave like $C^*$-algebras. The results in this chapter provide strong evidence that $L^p$-operator algebras of the form $F^p(G)$ and $F^p_\lambda(G)$ for some etale groupoid $G$, indeed behave like $C^*$-algebras. Beside having the automatic complete contractiveness property for contractive representations on $L^p$-spaces, another property that $F^p(G)$ and $F^p_\lambda(G)$ share with $C^*$-algebras is being generated by spatial partial isometries as defined in [204]. These are the partial isometries whose support and range idempotents are hermitian operators in the sense of [175]; see also [10]. (In the $C^*$-algebra case, the hermitian idempotents are precisely the orthogonal projections.) In particular, this property forces the algebra to be a $C^*$-algebra in the case $p = 2$. (A stronger property holds for unital $C^*$-algebras, namely being generated by invertible isometries; see [14, Theorem II.3.2.16]. As observed by Chris

Phillips, this property turns out to fail for some important examples of algebras of operators on $L^p$-spaces, such as the $L^p$-analog of the Toeplitz algebra.)

The present work indicates that the properties of being generated by spatial partial isometries, and having automatic complete contractiveness for representations on $L^p$-spaces, are very natural requirements for an $L^p$-operator algebra to behave like a $C^*$-algebra. We believe that the results of this work are a step towards a successful identification of those properties that characterize the class of "well behaved" $L^p$-operator algebras.

*Notation*

We denote by $\omega$ the set of natural numbers including 0. An element $n \in \omega$ will be identified with the set $\{0, 1, \ldots, n-1\}$ of its predecessors. (In particular, 0 is identified with the empty set.) We will therefore write $j \in n$ to mean that $j$ is a natural number and $j < n$.

For $n \in \omega$ or $n = \omega$, we denote by $c_n$ the counting measure on $n$. We denote by $\mathbb{Q}(i)^{\oplus\omega}$ the set of all sequences $(\alpha_n)_{n\in\omega}$ of complex numbers in $\mathbb{Q}(i)$ such that $\alpha_n = 0$ for all but finitely many indices $n \in \omega$.

All Banach spaces will be reflexive, and will be endowed with a (Schauder) basis. Recall that a basis $(b_n)_{n\in\omega}$ of a Banach space $Z$ is said to be *boundedly complete* if $\sum\limits_{n\in\omega} \lambda_n b_n$ converges in $Z$ whenever $\sup\limits_{n\in\omega} \left\| \sum_{j\in n} \lambda_j b_j \right\| < \infty$. By [26, Theorem 7.4], every basis of a reflexive Banach space is boundedly complete.

All Borel spaces will be *standard*. For a standard Borel space $X$, we denote by $B(X)$ the space of complex-valued bounded Borel functions on $X$, and by $\mathcal{B}_X$ the $\sigma$-algebra of Borel subsets of $X$.

For a Borel measure $\mu$ on a standard Borel space $X$, we denote by $\mathcal{B}_\mu$ the *measure algebra* of $\mu$. This is the quotient of the Boolean algebra $\mathcal{B}_X$ of Borel subsets of $X$ by the ideal of $\mu$-null Borel subsets. By [147, Exercise 17.44] $\mathcal{B}_\mu$ is a complete Boolean algebra. The characteristic function of a set $F$ will be denoted by $\chi_F$.

Given a measure space $(X, \mu)$ and a Hölder exponent $p \in (1, \infty)$, we will denote the Lebesgue space $L^p(X, \mu)$ simply by $L^p(\mu)$. Recall that $L^p(\mu)$ is separable precisely if there is a $\sigma$-finite Borel measure $\lambda$ on a standard Borel space such that $L^p(\lambda)$ is isometrically isomorphic to $L^p(\mu)$. Moreover there exists $n \in \omega \cup \{\omega\}$ such that $\lambda$ is Borel-isomorphic to $([0,1] \times n, \nu \times c_n)$, where $\nu$ is the Lebesgue measure on $[0,1]$. The push-forward of a measure $\mu$ under a function $\phi$ will be denote by $\phi_*\mu$ or $\phi_*(\mu)$.

If $X$ and $Z$ are Borel spaces, we say that $Z$ is *fibred* over $X$ if there is a Borel surjection $q\colon Z \to X$. In this case, we call $q$ the *fiber map.* A *section* of $Z$ is a map $\sigma\colon X \to Z$ such that $q \circ \sigma$ is the identity map of $X$. For $x \in X$, we denote the value of $\sigma$ at $x$ by $\sigma_x$, and the fiber $q^{-1}(\{x\})$ over $x$ is denoted by $Z_x$. If $Z^{(0)}$ and $Z^{(1)}$ are Borel spaces fibred over $X$ via fiber maps $q^{(0)}$ and $q^{(1)}$ respectively, then their *fiber product* $Z^{(0)} * Z^{(1)}$ is the Borel space fibred over $X$ defined by

$$Z^{(0)} * Z^{(1)} = \left\{(z^{(0)}, z^{(1)})\colon p^{(0)}(z^{(0)}) = p^{(1)}(z^{(1)})\right\}.$$

We exclude $p = 1$ in our analysis mostly for convenience, because we use duality in many situations. Moreover the theory of $L^1$-operator algebras seems not to be as well-behaved as that for $p \in (1,\infty)$, and is in some sense less interesting. For example:

- The reduced $L^1$-group algebra of the free group $\mathbb{F}_2$ on two generators is not simple, unlike for $p \in (1,\infty)$ (see [217]);
- For a locally compact group $G$, the canonical map $F^1(G) \to F^1_\lambda(G)$ is always an isometric isomorphism (even if $G$ is not amenable), unlike for $p \in (1,\infty)$ (see [207] and Chapter XIV);

We do not know whether the results of this chapter carry over to the case $p = 1$.

### Borel Bundles of Banach Spaces

**Definition XX.2.1.** Let $X$ be a Borel space. A (standard) *Borel Banach bundle* over $X$ is a Borel space $\mathcal{Z}$ fibred over $X$ together with

1. Borel maps $+\colon \mathcal{Z} * \mathcal{Z} \to \mathcal{Z}$, $\cdot\colon \mathbb{C} \times \mathcal{Z} \to \mathcal{Z}$, and $\|\cdot\|\colon \mathcal{Z} \to \mathbb{C}$,
2. a Borel section $\mathbf{0}\colon X \to \mathcal{Z}$, and
3. a sequence $(\sigma_n)_{n\in\omega}$ of Borel sections $\sigma_n\colon X \to \mathcal{Z}$

   such that the following holds:

- $\mathcal{Z}_x$ is a reflexive Banach space with zero element $\mathbf{0}_x$ for every $x \in X$;

– there is $K > 0$ such that, for every $x \in X$, the sequence $(\sigma_{n,x})_{n\in\omega}$ is a basis of $\mathcal{Z}_x$ with basis constant $K$, and the sequence $(\sigma'_{n,x})_{n\in\omega}$ is a basis of $\mathcal{Z}'_x$ with basis constant $K$.

The sequence $(\sigma_n)_{n\in\omega}$ is called a *basic sequence* for $\mathcal{Z}$, and $K$ is called *basis constant* for $(\sigma_n)_{n\in\omega}$. We say that $(\sigma_n)_{n\in\omega}$ is an *unconditional basic sequence* if there exists $K > 0$ such that for every $x \in X$, the sequences $(\sigma_{n,x})_{n\in\omega}$ and $\left(\sigma'_{n,x}\right)_{n\in\omega}$ are unconditional bases of $\mathcal{Z}_x$ and $\mathcal{Z}'_x$ with unconditional basis constant $K$. Finally, we say that $(\sigma_n)_{n\in\omega}$ is a *normal basic sequence* if $\|\sigma_{n,x}\| = \left\|\sigma'_{n,x}\right\| = 1$ for every $n \in \omega$ and $x \in X$.

**Example XX.2.2** (Constant bundles)**.** Let $X$ be a Borel space, let $Z$ be a reflexive Banach space, and set $\mathcal{Z} = X \times Z$. Then $\mathcal{Z}$ with the product Borel structure is naturally a Borel Banach bundle, where each fiber $\mathcal{Z}_x$ is isomorphic to $Z$. In the particular case when $Z$ is the field of complex numbers, this is called the *trivial bundle* over $X$.

Let $q\colon \mathcal{Z} \to X$ be a Borel Banach bundle. Then the space of Borel sections of $\mathcal{Z}$ has a natural structure of $B(X)$-module. Accordingly, if $\xi_1$ and $\xi_2$ are Borel sections of $\mathcal{Z}$ and $f \in B(X)$, we denote by $\xi_1 + \xi_2$ and $f\xi$ the Borel sections given by

$$(\xi_1 + \xi_2)_x = (\xi_1)_x + (\xi_2)_x \quad \text{and} \quad (f\xi)_x = f(x)\xi_x$$

for every $x$ in $X$.

If $E$ is a Borel subset of $X$, then $q^{-1}(E)$ is canonically a Borel Banach bundle over $E$, called the *restriction* of $\mathcal{Z}$ to $E$, and denoted by $\mathcal{Z}|_E$.

**Remark XX.2.3.** A Borel Banach bundle where each fiber is a Hilbert space is called a *Borel Hilbert bundle*. Such bundles (usually called just Hilbert bundles) are the key notion in the study of representation of groupoids on Hilbert spaces; see [268, Appendix F], [196, Section 3.1], and [222, Section 2]. The Gram-Schmidt process shows that a Borel Hilbert bundle $\mathcal{H}$ over $X$ always has a basic sequence $(\sigma_n)_{n\in\omega}$ such that for all $x$ in $X$, the sequence $(\sigma_{n,x})_{n\in\omega}$ is an *orthonormal basis* of $\mathcal{H}_x$.

Let $X$ be a Borel space, and let $\mathcal{Z}$ be a set (with no Borel structure) fibred over $X$. Assume there are operations

$$+\colon \mathcal{Z} * \mathcal{Z} \to \mathcal{Z} \quad , \quad \cdot\colon \mathbb{C} \times \mathcal{Z} \to \mathcal{Z} \quad \text{and} \quad \|\cdot\|\colon \mathcal{Z} \to \mathbb{C},$$

making each fiber a Banach space. In this situation, we will say that $\mathcal{Z}$ is a *bundle of Banach spaces* over $X$, and will denote it by $\bigsqcup\limits_{x\in X} \mathcal{Z}_x$. Let $\mathcal{Z}'$ be the set

$$\mathcal{Z}' = \left\{(x, v) : x \in X, \, v \in \mathcal{Z}_x'\right\}.$$

Then $\mathcal{Z}'$ is also a bundle of Banach spaces over $X$.

Suppose further that there is a sequence $(\sigma_n)_{n\in\omega}$ of Borel sections $\sigma_n\colon X \to \mathcal{Z}$ such that, for every $x \in X$, the sequence $(\sigma_{n,x})_{n\in\omega}$ is a basis of $\mathcal{Z}_x$. For every $x \in X$, denote by $(\sigma'_{n,x})_{n\in\omega}$ the dual basis of $\mathcal{Z}'_x$. Assume that for every $m \in \omega$ and every sequence $(\alpha_j)_{j\in m}$ in $\mathbb{Q}(i)^{\oplus m}$, the map $X \to \mathbb{R}$ given by $x \mapsto \left\|\sum\limits_{j\in m} \alpha_j \sigma_{j,x}\right\|$ is Borel. Set

$$Z = \left\{(x, (\alpha_n)_{n\in\omega}) \in X \times \mathbb{C}^\omega \colon \sup_{n\in\omega} \left\|\sum_{j\in n} \alpha_j \sigma_{j,x}\right\| < \infty\right\}.$$

We claim that $Z$ is a Borel subset of $X \times \mathbb{C}^\omega$. To see this, note that a pair $(x, (\alpha_n)_{n\in\omega})$ in $X \times \mathbb{C}^\omega$ belongs to $Z$ if and only if there is $N \in \omega$ such that for every $m, k \in \omega$ there is $(\beta_j)_{j\in m}$ in $\mathbb{Q}(i)^{\oplus m}$ such that

$$\max_{j\in n} |\alpha_j - \beta_j| \leq \frac{1}{2^k} \quad \text{and} \quad \left\|\sum_{j\in m} \beta_j \sigma_{j,x}\right\| < \infty.$$

Since the map $x \mapsto \left\|\sum\limits_{j\in m} \beta_j \sigma_{j,x}\right\|$ is Borel, this proves the claim.

The assignment $(x, (\alpha_n)_{n\in\omega}) \mapsto \sum\limits_{n\in\omega} \alpha_n \sigma_{n,x}$ induces a bijection $Z \to \mathcal{Z}$ since, for every $x \in X$, the sequence $(\sigma_{n,x})_{n\in\omega}$ is a boundedly complete basis of $\mathcal{Z}_x$. This bijection induces a standard Borel structure on $\mathcal{Z}$, and it is not difficult to verify that such Borel structure turns $\mathcal{Z}$ into a Borel Banach bundle.

A similar argument shows that the set

$$Z' = \left\{ (x, (\alpha_n)_{n\in\omega}) \in X \times \mathbb{C}^\omega \colon \sup_{n\in\omega} \left\| \sum_{j\in n} \alpha_j \sigma'_{j,x} \right\| < \infty \right\}$$

is Borel, and that the map from $Z'$ to $\mathcal{Z}'$ given by $(x, (\alpha_n)_{n\in\omega}) \mapsto \sum_{n\in\omega} \alpha_n \sigma'_{n,x}$ is a bijection. This induces a standard Borel structure on $\mathcal{Z}'$ that makes $\mathcal{Z}'$ a Borel Banach bundle.

It follows from the definition of the Borel structures on $\mathcal{Z}$ and $\mathcal{Z}'$, that the canonical pairing $\mathcal{Z} * \mathcal{Z}' \to \mathbb{C}$ is Borel. In fact, for $(x, (\alpha_n)_{n\in\omega}) \in Z$ and $(x, (\beta_n)_{n\in\omega}) \in Z'$, we have

$$\left\langle \sum_{n\in\omega} \alpha_n \sigma_{n,x}, \sum_{m\in\omega} \beta_m \sigma'_{m,x} \right\rangle = \sum_{n\in\omega} \alpha_n \overline{\beta}_n.$$

The standard Borel structures on $\mathcal{Z}$ and $\mathcal{Z}'$ here described will be referred to as the *canonical Borel structures* associated with the sequence $(\sigma_n)_{n\in\omega}$ of Borel sections $X \to \mathcal{Z}$. By [147, Theorem 14.12], these can be equivalently described as the Borel structures generated by the sequence of functionals on $\mathcal{Z}$ and $\mathcal{Z}'$ given by

$$z \mapsto \left\langle z, \sigma'_{n,q(z)} \right\rangle \quad \text{and} \quad w \mapsto \left\langle \sigma_{n,q(w)}, w \right\rangle$$

for $n \in \omega$.

As a consequence of the previous discussion, we conclude that if $\mathcal{Z}$ is a Borel Banach bundle, then the Borel structure on $\mathcal{Z}$ is generated by the sequence of maps $\mathcal{Z} \to \mathbb{C}$ given by $z \mapsto \left\langle z, \sigma'_{n,q(z)} \right\rangle$ for $n$ in $\omega$. Moreover, the dual bundle $\mathcal{Z}'$ has a *unique* Borel Banach bundle structure making the canonical pairing Borel. In the following, whenever $\mathcal{Z}$ is a Borel Banach bundle, we will always consider $\mathcal{Z}'$ as a Borel Banach bundle endowed with such canonical Borel structure.

The following criterion to endow a Banach bundle with a Borel structure is an immediate consequence of the observations contained in this subsection.

**Lemma XX.2.4.** Let $(Z_k)_{k\in\omega}$ be a sequence of reflexive Banach spaces. For every $k \in \omega$, let $(b_{n,k})_{n\in\omega}$ be a basis of $Z_k$ with dual basis $(b'_{n,k})_{n\in\omega}$, and suppose that both $(b_{n,k})_{n\in\omega}$ and $(b'_{n,k})_{n\in\omega}$ have basis constant $K$ independent of $k$. Let $\mathcal{Z}$ be a bundle of Banach spaces over $X$, and assume there exist a Borel partition $(X_k)_{k\in\omega}$ of $X$, and isometric isomorphisms $\psi_x \colon Z_k \to \mathcal{Z}_x$

and $\psi'_x\colon Z'_k \to \mathcal{Z}'_x$ for $k \in \omega$ and $x \in X_k$. For $k, n \in \omega$ and $x \in X_k$, set $\sigma_{n,x} = \psi_x(b_{n,k})$ and $\sigma'_{n,x} = \psi'_x(b'_{n,k})$. There are unique Borel Banach bundle structures on $\mathcal{Z}$ and $\mathcal{Z}'$ such that $(\sigma_n)_{n\in\omega}$ and $(\sigma'_n)_{n\in\omega}$ are basic sequences, and such that the canonical pairing between $\mathcal{Z}$ and $\mathcal{Z}'$ is Borel.

*Banach space valued $L^p$-spaces*

For the remainder of this section, we fix a Borel Banach bundle $q\colon \mathcal{Z} \to X$ over the standard Borel space $X$, a basic sequence $(\sigma_n)_{n\in\omega}$ of $\mathcal{Z}$ with basis constant $K$, a $\sigma$-finite Borel measure $\mu$ on $X$, and a Hölder exponent $p \in (1, \infty)$.

**Definition XX.2.5.** Denote by $\mathcal{L}^p(X, \mu, \mathcal{Z})$ the space of Borel sections $\xi\colon X \to \mathcal{Z}$ such that

$$N_p(\xi)^p = \int \|\xi_x\|^p \; d\mu(x) < \infty.$$

It follows from the Minkowski inequality that $\mathcal{L}^p(X, \mu, \mathcal{Z})$ is a seminormed complex vector space. We denote by $L^p(X, \mu, \mathcal{Z})$ the normed space obtained as a quotient of the seminormed space $(\mathcal{L}^p(X, \mu, \mathcal{Z}), N_p)$.

When $\mathcal{Z}$ is the trivial bundle over $X$, then $L^p(X, \mu, \mathcal{Z})$ coincides with the Banach space $L^p(X, \mu)$. Consistently, we will abbreviate $\mathcal{L}^p(X, \mu, \mathcal{Z})$ and $L^p(X, \mu, \mathcal{Z})$ to $\mathcal{L}^p(\mu, \mathcal{Z})$ and $L^p(\mu, \mathcal{Z})$, respectively.

**Theorem XX.2.6.** The normed vector space $L^p(\mu, \mathcal{Z})$ is a Banach space.

*Proof.* We need to show that the norm on $L^p(\mu, \mathcal{Z})$ is complete. In order to show this, it is enough to prove that if $(\xi_n)_{n\in\omega}$ is a sequence in $\mathcal{L}^p(\mu, \mathcal{Z})$ such that $\sum\limits_{n\in\omega} N_p(\xi_n) < \infty$, then there is $\xi \in \mathcal{L}^p(\mu, \mathcal{Z})$ such that

$$\lim_{m\to\infty} N_p\left(\xi - \sum_{n\in m} \xi_n\right) = 0.$$

Let $(\xi_n)_{n\in\omega}$ be such a sequence. We use Fatou's Lemma at the second step and Jensen's inequality at the fourth to obtain

$$\begin{aligned}\int\left(\sum_{n\in\omega}\|\xi_{n,x}\|\right)^p d\mu(x) &= \int \lim_{n\to\infty}\left(\sum_{j\in n}\|\xi_{j,x}\|\right)^p d\mu(x)\\ &\leq \liminf_{n\to\infty}\int\left(\sum_{j\in n}\|\xi_{j,x}\|\right)^p d\mu(x)\\ &= \liminf_{n\to\infty} N_p\left(\sum_{j\in n}\xi_j\right)^p\\ &\leq \liminf_{n\to\infty}\left(\sum_{j\in n}N_p(\xi_j)\right)^p\\ &= \left(\sum_{n\in\omega}N_p(\xi_n)\right)^p < \infty.\end{aligned}$$

Therefore, the Borel set

$$F = \left\{x\in X\colon \sum_{n\in\omega}\|\xi_{n,x}\| < \infty\right\}$$

is $\mu$-conull. Using that $\mathcal{Z}_x$ is a Banach space for all $x \in X$, we conclude that the sequence $\left(\sum_{j\in n}\xi_{j,x}\right)_{n\in\omega}$ converges to an element $\xi_x$ of $\mathcal{Z}_x$ for all $x\in F$. Set $\xi_x = \mathbf{0}_x$ for $x \in X\backslash F$. The resulting map $\xi\colon X\to\mathcal{Z}$ is a section, and we claim that it is Borel. To see this, it is enough to observe that the identity

$$\langle \xi_x, \sigma'_{k,x}\rangle = \begin{cases}\sum_{n\in\omega}\langle \xi_{n,x}, \sigma'_k(x)\rangle & \text{if } x\in F\\ 0 & \text{otherwise}\end{cases}$$

implies that the assignment $x\mapsto \langle \xi_x, \sigma'_{k,x}\rangle$ is Borel. The claim now follows.

Finally,

$$N_p\left(\xi - \sum_{j\in n}\xi_n\right) = \int\left\|\sum_{j\geq n}\xi_{j,x}\right\|^p d\mu(x) \leq \left(\sum_{j\geq n}N_p(\xi_{j,x})\right)^p,$$

and since $\lim_{n\to\infty}\left(\sum_{j\geq n}N_p(\xi_{j,x})\right)^p = 0$, the proof is complete. □

As it is customary, we will identify an element of $\mathcal{L}^p(\mu, \mathcal{Z})$ with its image in the quotient $L^p(\mu, \mathcal{Z})$. We will also write $\|\cdot\|_p$, or just $\|\cdot\|$ if no confusion is likely to arise, for the norm on $L^p(\mu, \mathcal{Z})$ induced by $N_p$.

**Lemma XX.2.7.** Suppose that $(\xi_n)_{n\in\omega}$ is a sequence in $L^p(\mu, \mathcal{Z})$ converging in norm to an element $\xi$ in $L^p(\mu, \mathcal{Z})$. Then there are a $\mu$-conull Borel subset $X_0$ of $X$, and a subsequence $(\xi_{n_k})_{k\in\omega}$ such that $\lim\limits_{k\to\infty} \|\xi_{n_k,x} - \xi_x\| = 0$ for every $x \in X_0$.

*Proof.* Given $\varepsilon > 0$ and $n \in \omega$, set

$$F_{n,\varepsilon} = \{x \in X\colon\ \|\xi_{n,x} - \xi_x\| \geq \varepsilon\}.$$

Then $\lim\limits_{n\to\infty} \mu(F_{n,\varepsilon}) = 0$ by Chebyshev's inequality. Find an increasing sequence $(n_k)_{k\in\omega}$ in $\omega$ such that $\mu\left(F_{m,2^{-k}}\right) \leq 2^{-k}$ for every $m \geq n_k$, and set

$$F = \bigcap_{k\in\omega} \bigcup_{m\geq n_k} F_{m,2^{-k}}.$$

Then $\mu(F) = 0$ and moreover $\lim\limits_{k\to\infty} \|\xi_{n_k,x} - \xi_x\| = 0$ for all $x \in X\backslash F$. This concludes the proof. $\square$

**Proposition XX.2.8.** Let $\xi \in L^p(\mu, \mathcal{Z})$. Then:

1. The function $\langle \xi, \sigma'_n\rangle : X \to \mathbb{R}$ defined by $x \mapsto \left\langle \xi_x, \sigma'_{n,x}\right\rangle$ belongs to $\mathcal{L}^p(\mu)$;
2. The sequence $\left(\sum\limits_{k\in n} \langle \xi, \sigma'_k\rangle \sigma_k\right)_{n\in\omega}$ converges to $\xi$.

*Proof.* (1). The function $\langle \xi, \sigma'_n\rangle$ is Borel because the canonical pairing map is Borel. Moreover, the estimate

$$\int \left|\left\langle \xi_x, \sigma'_{n,x}\right\rangle\right|^p \ d\mu(x) \leq (2K)^p \int \|\xi_x\|^p \ d\mu(x) = (2K\|\xi\|)^p$$

shows that $\langle \xi, \sigma'_n\rangle$ belongs to $\mathcal{L}^p(\mu)$.

(2). For every $x \in X$, and using that $K$ is a basis constant for $(\sigma_{n,x})_{n\in\omega}$, we have

$$\left\| \sum_{k\in n} \left\langle \xi_x, \sigma'_{k,x}\right\rangle \sigma_{k,x} \right\| \leq K \|\xi_x\|.$$

Given $\varepsilon > 0$ and $n \in \omega$, define the Borel set

$$F_{n,\varepsilon} = \left\{ x \in X \colon \left\| \sum_{k \in n} \left\langle \xi_x, \sigma'_{k,x} \right\rangle \sigma_{k,x} - \xi_x \right\| \leq \varepsilon \right\}.$$

Then $\bigcup\limits_{n \in \omega} F_{n,\varepsilon} = X$. By the dominated convergence theorem, there is $n_0 \in \omega$ such that

$$\int_{X \setminus F_{n_0,\varepsilon}} \|\xi_x\|^p \, d\mu(x) < \varepsilon.$$

Thus, for $n \geq n_0$, we have

$$\begin{aligned}
\left\| \sum_{k \in n} \langle \xi, \sigma'_k \rangle \sigma_k - \xi \right\|_p^p &= \int \left\| \sum_{k \in n} \left\langle \xi_x, \sigma'_{k,x} \right\rangle \sigma_{k,x} - \xi_x \right\|^p \, d\mu(x) \\
&\leq \mu \left( F_{n,\varepsilon} \right) \varepsilon + (K+1)^p \int_{X \setminus F_{n,\varepsilon}} \|\xi_x\|^p \, d\mu(x) \\
&\leq \left( (K+1)^p + 1 \right) \varepsilon.
\end{aligned}$$

This shows that the sequence $\left( \sum\limits_{k \in n} \langle \xi, \sigma'_k \rangle \sigma_k \right)_{n \in \omega}$ converges to $\xi$. □

In view of Proposition XX.2.8, the sequence $(\sigma_n)_{n \in \omega}$ can be thought as a basis of $L^p(\mu, \mathcal{Z})$ over $L^p(\mu)$. In particular, Proposition XX.2.8 implies that $L^p(\mu, \mathcal{Z})$ is a *separable* Banach space. It is not difficult to verify that, if $(\sigma_n)_{n \in \omega}$ is an *unconditional* basic sequence for $\mathcal{Z}$, then the series $\sum\limits_{k \in \omega} \langle \xi, \sigma'_k \rangle \sigma_k$ converges *unconditionally* to $\xi$ for every $\xi \in L^p(\mu, \mathcal{Z})$.

**Proposition XX.2.9.** Let $(f_n)_{n \in \omega}$ be a sequence in $L^p(\mu)$ such that

$$\sup_{n \in \omega} \left\| \sum_{k \in n} f_k \sigma_k \right\|_p$$

is finite. Then the sequence

$$\left( \sum_{k \in n} f_k \sigma_k \right)_{n \in \omega}$$

of partial sums converges in $L^p(\mu, \mathcal{Z})$.

*Proof.* Set $M = \sup_{n\in\omega} \left\| \sum_{k\in n} f_k \sigma_k \right\|^p$ and fix $N \in \omega$. Given $n \in \omega$, define

$$F_n^N = \left\{ x \in X \colon \left\| \sum_{k\in n} f_k(x)\sigma_{k,x} \right\| \leq 2NM \right\}.$$

Then $F_n^N$ is Borel and $\mu(F_n^N) \geq 1 - \frac{1}{N}$. Set

$$F^N = \bigcap_{n\in\omega} \bigcup_{k\geq n} F_k^N.$$

Then $\mu(F^N) \geq 1 - \frac{1}{N}$. Since $(\sigma_{n,x})_{n\in\omega}$ is a basis for $\mathcal{Z}_x$ with basis constant $K$, we have

$$\sup_{m\in\omega} \left\| \sum_{k\in m} f_k(x)\sigma_{k,x} \right\| \leq 2NMK < \infty$$

for every $x \in F^N$. We conclude that the Borel set

$$F = \left\{ x \in X \colon \sup_{m\in\omega} \left\| \sum_{k\in m} f_k(x)\sigma_{k,x} \right\| < \infty \right\}$$

is $\mu$-conull. Given $x \in F$, and since $(\sigma_{n,x})_{n\in\omega}$ is a boundedly complete basis of $\mathcal{Z}_x$, the series $\sum_{n\in\omega} f_n(x)\sigma_{n,x}$ converges to an element $\xi_x$ of $\mathcal{Z}_x$. Defining $\xi_x = \mathbf{0}_x$ for $x \in X\backslash F$, one obtains a Borel section $\xi \colon X \to \mathcal{Z}$. Moreover,

$$\int \|\xi_x\|^p \, d\mu(x) \leq \sup_{n\in\omega} \int \left\| \sum_{k\in n} f_k(x)\sigma_{k,x} \right\|^p \, d\mu(x) \leq M,$$

and hence $\xi$ belongs to $L^p(\mu, \mathcal{Z})$. It follows from Proposition XX.2.8 that $\xi$ is the limit in $L^p(\mu, \mathcal{Z})$ of $\left( \sum_{k\in n} f_k \sigma_k \right)_{n\in\omega}$. □

*Pairing*

In this subsection, we show that there is a natural pairing between $L^p(\mu, \mathcal{Z})$ and $L^{p'}(\mu, \mathcal{Z}')$, under which we may identify $L^p(\mu, \mathcal{Z})'$ with $L^{p'}(\mu, \mathcal{Z}')$. We describe this pairing first.

Define a map

$$\langle \cdot, \cdot \rangle \colon L^p(\mu, \mathcal{Z}) \times L^{p'}(\mu, \mathcal{Z}') \to \mathbb{C} \quad \text{by} \quad \langle \xi, \eta \rangle = \int \langle \xi_x, \eta_x \rangle \, d\mu(x)$$

for all $\xi \in L^p(\mu, \mathcal{Z})$ and all $\eta \in L^{p'}(\mu, \mathcal{Z}')$. To show that this map is well-defined, we must check that the assignment $x \mapsto \langle \xi_x, \eta_x \rangle$ is integrable. For this, assuming without loss of generality that $\|\xi\|_p = \|\eta\|_{p'} = 1$, we use Young's inequality at the second step to get

$$\begin{aligned} \int |\langle \xi_x, \eta_x \rangle| d\mu(x) &\leq \int \|\xi_x\| \|\eta_x\| d\mu(x) \\ &\leq \frac{1}{p} \int \|\xi_x\|^p d\mu(x) + \frac{1}{p'} \int \|\eta_x\|^{p'} d\mu(x) = 1. \end{aligned}$$

**Theorem XX.2.10.** The function from $L^{p'}(\mu, \mathcal{Z}')$ to $L^p(\mu, \mathcal{Z})'$ given by

$$\eta \mapsto \langle \cdot, \eta \rangle = \int \langle \cdot_x, \eta_x \rangle \, d\mu(x)$$

is an isometric isomorphism.

*Proof.* We first show that such a function is isometric. Fix $\varepsilon > 0$ and fix $\eta \in L^{p'}(\mu, \mathcal{Z}')$ with $\|\eta\|_{p'} = 1$. Set $\mathcal{Z}_0 = \left\{ z \in \mathcal{Z} \colon \left( \langle z, \sigma_{n,q(z)} \rangle \right)_{n \in \omega} \in \mathbb{Q}(i)^{\oplus \omega} \right\}$ and

$$F = \left\{ z \in \mathcal{Z}_0 \colon \|z\| \leq 1 \text{ and } (1-\varepsilon) \left\| \eta_{q(z)} \right\| \leq \left| \left\langle z, \eta_{q(z)} \right\rangle \right| \right\}.$$

Then $q(F) = X$, and the fiber map $q$ is countable-to-one on $F$. By [147, Theorem 18.10], there is a Borel section $\tau \colon X \to \mathcal{Z}$ such that $\tau_x \in F$ for every $x \in X$. Define a Borel section $\xi \colon X \to \mathcal{Z}$ by $\xi_x = \|\eta_x\|^{p'-1} \tau_x$ for $x \in X$. Then

$$\|\xi\|^p = \int \|\xi_x\|^p \, d\mu(x) \leq \int \|\eta_x\|^{p'} \, d\mu(x) = 1,$$

and thus $\xi$ belongs to $L^p(\mu, \mathcal{Z})$. Finally,

$$\int |\langle \xi_x, \eta_x \rangle| \, d\mu(x) \geq (1-\varepsilon) \int \|\eta_x\|^{p'} \, d\mu(x) = 1 - \varepsilon$$

and thus $\|\langle \cdot, \eta \rangle\| \geq \|\eta\|$. Since the opposite inequality is immediate, we conclude that the function $\eta \mapsto \langle \cdot, \eta \rangle$ is isometric, as desired.

We will now show that such a function is surjective. Let $\Phi \in L^p(\mu, \mathcal{Z})'$ be given. For every $n \in \omega$, define the Borel measure $\lambda_n$ on $X$ by $\lambda_n(E) = \Phi(\chi_E \sigma_n)$ for $E \subseteq X$. Then $\lambda_n$ is absolutely continuous with respect to $\mu$. Denote by $g_n = \frac{d\lambda_n}{d\mu}$ the corresponding Radon-Nikodym derivative, which belongs to $L^1(\mu)$. Then

$$\Phi(\chi_E\sigma_n) = \int \chi_E g_n \; d\mu$$

for all Borel subsets $E \subseteq X$ and all $n \in \omega$. By continuity, we have $\Phi(f\sigma_n) = \int f g_n \; d\mu$ for every bounded Borel function $f$ on $X$.

Fix $n \in \omega$. We claim that $g_n$ belongs to $L^{p'}(\mu)$. Let $h$ be a Borel function on $X$ of modulus one such that $hg_n = |g_n|$. Given $k$ in $\omega$, set $E_k = \{x\colon |g_n(x)| \leq k\}$ and define a bounded Borel function $h_k\colon X \to \mathbb{C}$ by

$$h_k = \chi_{E_k} h \, |g_n|^{p'-1}.$$

It is readily checked that $|h_k|^p$ coincides with $|g_n|^{p'}$ on $E_k$. We use this at the last step to get

$$\begin{aligned}\|\Phi\| \left(\int_{E_k} |g_n|^{p'} \, d\mu\right)^{\frac{1}{p}} &= \|\Phi\| \left(\int |h_k|^p \, d\mu\right)^{\frac{1}{p}} \geq \Phi(h_k\sigma_n)\\ &= \int h_k g_n \; d\mu = \int_{E_k} |g_n|^{p'} \, d\mu(x).\end{aligned}$$

It follows that $\left(\int_{E_k} |g_n|^{p'} \, d\mu(x)\right)^{\frac{1}{p'}} \leq \|\Phi\|$. Since $k$ is arbitrary, an application of the monotone convergence theorem yields $\|g_n\|_{p'} \leq \|\Phi\|$, and hence $g_n \in L^{p'}(\mu)$. The claim is proved. Let $K$ be a basis constant for $(\sigma_n)_{n\in\omega}$. We claim that

$$\sup_{n\in\omega} \left\| \sum_{j\in n} g_j \sigma'_j \right\|_{p'} \leq K \, \|\Phi\| \, .$$

Fix $\xi \in L^p\,(\mu, \mathcal{Z})$, and write $\xi = \sum\limits_{n\in\omega} f_n\sigma_n$ as in Proposition XX.2.8. Then

$$\begin{aligned}\left| \int \left\langle \xi, \sum_{j\in n} g_j\sigma'_j \right\rangle \; d\mu \right| &= \left| \sum_{j\in n} \int f_j g_j \; d\mu \right| = \left| \sum_{j\in n} \Phi(f_j\sigma_j) \right|\\ &= \left| \Phi\left(\sum_{j\in n} f_j\sigma_j\right) \right| \leq \|\Phi\| \left\| \sum_{j\in n} f_j\sigma_j \right\|\\ &\leq K \, \|\Phi\| \, \|\xi\|.\end{aligned}$$

This being true for every $\xi \in L^p(\mu, \mathcal{Z})$ implies that $\left\| \sum_{j\in n} g_j\sigma'_j \right\| \leq K \, \|\Phi\|$, and the claim has been proved. We can now conclude from Proposition XX.2.9 that the sequence $\left(\sum_{j\in n} g_j\sigma'_j\right)_{n\in\omega}$

of partial sums converges to an element $\eta$ in $L^{p'}(\mu, \mathcal{Z}')$. Using Proposition XX.2.8 it is immediate to verify that

$$\Phi(\xi) = \int \langle \xi, \eta \rangle \, d\mu$$

for every $\xi \in L^p(\mu, \mathcal{Z})$. Thus $\Phi = \langle \cdot, \eta \rangle$, and this finishes the proof. $\square$

It follows that the Banach space $L^p(\mu, \mathcal{Z})$ is reflexive. (Recall that the Banach bundle is assumed to have a basic sequence $(\sigma_n)_{n \in \omega}$, and, in particular, all its fibers are reflexive.)

*Bundles of $L^p$-spaces*

Consider a Borel probability measure $\mu$ on a standard Borel space $X$. Let $\lambda$ be a Borel probability measure on a standard Borel space $Z$ fibred over $X$ via a fiber map $q$ such that $q_*(\lambda) = \mu$. By [147, Exercise 17.35], the measure $\lambda$ admits a disintegration $(\lambda_x)_{x \in X}$ with respect to $\mu$, which is also written as $\lambda = \int \lambda_x \, d\mu(x)$. In other words,

- there is a Borel assignment $x \mapsto \lambda_x$, where $\lambda_x$ is a probability measure on $\mathcal{Z}_x$, and
- for every bounded Borel function $f \colon Z \to \mathbb{C}$, we have

$$\int f \, d\lambda = \int \left( \int f \, d\lambda_x \right) \, d\mu(x).$$

Consider the Banach bundle $\mathcal{Z} = \bigsqcup_{x \in X} L^p(\lambda_x)$ over $X$, where the fiber $\mathcal{Z}_x$ over $x$ is $L^p(\lambda_x)$.

**Theorem XX.2.11.** There is a canonical Borel Banach bundle structure on $\mathcal{Z}$ such that $L^p(\mu, \mathcal{Z})$ is isometrically isomorphic to $L^p(\lambda)$.

*Proof.* Let us assume for simplicity that $\mu$ and $\lambda_x$ are atomless for every $x \in X$. In this case, by [106, Theorem 2.2], we can assume without loss of generality that []

- $X$ is the unit interval $[0, 1]$ and $\mu$ is its Lebesgue measure;
- $Z$ is the unit square $[0, 1]^2$ and $\lambda$ is its Lebesgue measure;
- $q \colon Z \to X$ is the projection onto the first coordinate; and
- $\lambda_x$ is the Lebesgue measure on $\{x\} \times [0, 1]$ for every $x \in X$.

Let $(h_n)_{n\in\omega}$ be the Haar system on $[0,1]$ defined as in [26, Chapter 3]. For $n \in \omega$ and $x \in [0,1]$, define $h_{n,x}^{(p)} \colon [0,1] \to \mathbb{R}$ by

$$h_{n,x}^{(p)}(t) = \frac{h_n(t)}{\|h_n\|_p}$$

for every $t \in [0,1]$. Then $\left(h_{n,x}^{(p)}\right)_{n\in\omega}$ is a normalized basis of $L^p(\lambda_x)$ for every $x \in [0,1]$. It follows from the discussion in Subsection XX.2 that there are unique Borel Banach bundle structures on $\mathcal{Z}$ and $\mathcal{Z}' = \bigsqcup_{x\in X} L^{p'}(\lambda_x)$ such that $(h_n^{(p)})_{n\in\omega}$ and $(h_n^{(p')})_{n\in\omega}$ are normal basic sequences for $\mathcal{Z}$ and $\mathcal{Z}'$, and that the canonical pairing between $\mathcal{Z}$ and $\mathcal{Z}'$ is Borel.

We claim that $L^p(\mu, \mathcal{Z})$ can be canonically identified with $L^p(\lambda)$. Given $f \in L^p(\lambda)$, consider the Borel section $s_f \colon X \to \mathcal{Z}$ defined by $s_{f,x}(t) = f(x,t)$ for $x, t \in [0,1]$. It is clear that $s_{f,x}$ belongs to $L^p(\mu, \mathcal{Z})$ and that

$$\left(\int \|s_{f,x}\|_p^p \; d\mu(x)\right)^{\frac{1}{p}} = \|f\|_p .$$

It follows that the map $f \mapsto s_{f,x}$ induces an isometric linear map $s \colon L^p(\lambda) \to L^p(\mu, \mathcal{Z})$. The fact that $s$ is surjective is a consequence of Proposition XX.2.8, since the range of $s$ is a closed linear subspace of $L^p(\mu, \mathcal{Z})$ that contains $h_n^{(p)}$ for every $n \in \omega$.

The case when $\lambda$ and $\mu$ are arbitrary Borel probability measures can be treated similarly, using the classification of disintegration of Borel probability measures given in [106, Theorem 3.2], together with Lemma XX.2.4. In fact, the results of [106] show that the same conclusions hold if $\lambda$ is a Borel $\sigma$-*finite* measure. ☐

**Definition XX.2.12.** Let $X$ be a Borel space, and let $\mu$ be a Borel probability measure on $X$. An $L^p$-*bundle* over $(X, \mu)$ is a Borel Banach bundle $\mathcal{Z} = \bigsqcup_{x\in X} L^p(\lambda_x)$ obtained from the disintegration of a $\sigma$-finite Borel measure $\lambda$ on a Borel space $Z$ fibred over $X$, as described in Theorem XX.2.11.

*Decomposable operators*

Let $q_X \colon \mathcal{Z} \to X$ and $q_Y \colon \mathcal{W} \to Y$ be standard Borel Banach bundles with basic sequences $(\sigma_n)_{n\in\omega}$ and $(\tau_n)_{n\in\omega}$, respectively, and let $\phi \colon X \to Y$ be a Borel isomorphism.

**Definition XX.2.13.** Let $B(\mathcal{Z}, \mathcal{W}, \phi)$ be the space of contractive linear maps of the form $T \colon \mathcal{Z}_x \to \mathcal{W}_{\phi(x)}$ for some $x \in X$. For such a map $T$, we denote the corresponding point $x$ in $X$ by $x_T$.

Consider the Borel structure on $B(\mathcal{Z}, \mathcal{W}, \phi)$ generated by the maps $T \mapsto x_T$ and $T \mapsto \left\langle T\sigma_{n,x_T}, \tau'_{m,\phi(x_T)} \right\rangle$ for $n, m \in \omega$. It is not difficult to check that the operator norm and composition of operators are Borel functions on $B(\mathcal{Z}, \mathcal{W}, \phi)$, which make $B(\mathcal{Z}, \mathcal{W}, \phi)$ into a Borel space fibred over $X$.

**Lemma XX.2.14.** The Borel space $B(\mathcal{Z}, \mathcal{W}, \phi)$ is standard.

*Proof.* Let $V$ be set of elements $(x, (c_{n,m})_{n,m\in\omega})$ in $X \times \mathbb{C}^{\omega\times\omega}$ such that, for some $M \in \omega$ and every $(\alpha_n)_{n\in\omega} \in \mathbb{Q}(i)^{\oplus\omega}$, we have

$$\sup_{m\in\omega} \left\| \sum_{k\in m} \left( \sum_{n\in\omega} a_n c_{n,m} \right) \tau_{\phi(x),m} \right\| \leq M \sup_{n\in\omega} \left\| \sum_{k\in n} \alpha_k \sigma_{x,k} \right\|.$$

Then $V$ is a Borel subset of $X \times \mathbb{C}^{\omega\times\omega}$, and it is therefore a standard Borel space by [147, Corollary 13.4]. The result follows since the function $B(\mathcal{Z}, \mathcal{W}, \phi) \to X \times \mathbb{C}^{\omega\times\omega}$ given by

$$T \mapsto \left( x_T, \left( \left\langle T\sigma_{n,x_T}, \tau'_{m,\phi(x_T)} \right\rangle \right)_{(n,m)\in\omega\times\omega} \right)$$

is a Borel isomorphism between $B(\mathcal{Z}, \mathcal{W}, \phi)$ and $V$. □

Fix Borel $\sigma$-finite measures $\mu$ on $X$ and $\nu$ on $Y$ with $\phi_*(\mu) \sim \nu$.

**Proposition XX.2.15.** If $x \mapsto T_x$ is a Borel section of $B(\mathcal{Z}, \mathcal{W}, \phi)$ such that, for some $M \geq 0$ and $\mu$-almost every $x \in X$, we have

$$\|T_x\|^p \leq M^p \frac{d\phi_*(\mu)}{d\nu}(\phi(x)), \tag{XX.1}$$

then the linear operator $T\colon L^p(\mu, \mathcal{Z}) \to L^p(\nu, \mathcal{W})$ defined by

$$(T\xi)_y = T_{\phi^{-1}(y)}\xi_{\phi^{-1}(y)}.$$

for all $y \in Y$, is bounded. Moreover, the norm of $T$ is the minimum $M \geq 0$ such that the inequality in (XX.1) holds for $\mu$-almost every $x \in X$.

*Proof.* For $\xi$ in $L^p(\mu, \mathcal{Z})$, we have

$$\begin{aligned}
\|T\xi\|^p &= \int \|(T\xi)_y\|^p \ d\nu(y) = \int \left\|T_{\phi^{-1}(y)}\xi_{\phi^{-1}(y)}\right\|^p \ d\nu(y) \\
&= \int \left\|T_{\phi^{-1}(y)}\xi_{\phi^{-1}(y)}\right\|^p \ d\nu(y) \leq \int \left\|T_{\phi^{-1}(y)}\right\|^p \left\|\xi_{\phi^{-1}(y)}\right\|^p \ d\nu(y) \\
&\leq \int M^p \frac{d\phi_*(\mu)}{d\nu}(y) \left\|\xi_{\phi^{-1}(y)}\right\|^p \ d\nu(y) \leq M^p \|\xi\|^p.
\end{aligned}$$

This shows that $T$ is bounded with norm at most $M$. It remains to show that

$$\|T_x\|^p \leq \|T\|^p \frac{d\phi_*(\mu)}{d\nu}(\phi(x))$$

for $\mu$-almost every $x \in X$. For $\alpha \in \mathbb{Q}(i)^{\oplus\omega}$, set

$$\sigma_\alpha = \sum_{n\in\omega} \alpha_n \sigma_n \in L^p(\mu, \mathcal{Z})$$

and observe that the set $\{\sigma_{\alpha,x} \colon \alpha \in \mathbb{Q}(i)^{\oplus\omega}\}$ is dense in $\mathcal{Z}_x$ for every $x \in X$. It is therefore enough to show that

$$\left\|T_{\phi^{-1}(y)}\sigma_{n,\phi^{-1}(y)}\right\|^p \leq \|T\|^p \frac{d\phi_*(\mu)}{d\nu}(y) \left\|\sigma_{n,\phi^{-1}(y)}\right\|^p$$

for every $\alpha \in \mathbb{Q}(i)^{\oplus\omega}$, and for $\nu$-almost every $y \in Y$. In order to show this, let $g \colon Y \to \mathbb{C}$ be a bounded Borel function. Then

$$\begin{aligned}
&\int |g(y)|^p \left\|T_{\phi^{-1}(y)}\sigma_{\alpha,\phi^{-1}(y)}\right\|^p d\nu \\
&= \|T(g\circ\phi)\sigma_\alpha\|^p \\
&\leq \|T\|^p \int \|g(\phi(x))\sigma_{\alpha,x}\|^p \, d\mu(x) \\
&\leq \|T\|^p \int |g(y)|^p \left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p d\phi_*(\mu)(x) \\
&= \|T\|^p \int |g(y)|^p \left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p \frac{d\phi_*(\mu)}{d\nu}(y) \ d\nu(y).
\end{aligned}$$

Since $g$ is arbitrary, this concludes the proof. □

**Definition XX.2.16.** An operator $T \colon L^p(\mu, \mathcal{Z}) \to L^p(\nu, \mathcal{W})$ obtained from a Borel section $x \mapsto T_x$ of $B(\mathcal{Z}, \mathcal{W}, \phi)$ as in Proposition XX.2.15, is called *decomposable* with respect to the Borel isomorphism $\phi\colon X \to Y$. The Borel section $x \mapsto T_x$ corresponding to the decomposable operator $T$ is called the *disintegration* of $T$ with respect to the Borel isomorphism $\phi\colon X \to Y$.

**Remark XX.2.17.** It is not difficult to verify that the disintegration of a decomposable operator $T$ is essentially unique, in the sense that if $x \mapsto T_x$ and $x \mapsto \widetilde{T}_x$ are two Borel sections defining the same decomposable operator, then $T_x = \widetilde{T}_x$ for $\mu$-almost every $x$ in $X$.

Given a bounded Borel function $g\colon Y \to \mathbb{C}$, denote by $\Delta_g \in B(L^p(\nu, \mathcal{W}))$ the corresponding multiplication operator.

We have the following characterization of decomposable operators.

**Proposition XX.2.18.** For a bounded map $T\colon L^p(\mu, \mathcal{Z}) \to L^p(\nu, \mathcal{W})$, the following are equivalent:

1. $T$ is decomposable with respect to $\phi$;
2. $\Delta_g T = T\Delta_{g\circ\phi}$ for every bounded Borel function $g\colon Y \to \mathbb{C}$;
3. There is a countable collection $\mathcal{F}$ of Borel subsets of $Y$ that separates the points of $Y$, such that $\Delta_{\chi_F} T = T\Delta_{\chi_{\phi^{-1}[F]}}$ for every $F \in \mathcal{F}$.

*Proof.* (1) implies (2). Let $x \mapsto T_x$ be a Borel section of $B(\mathcal{Z}, \mathcal{W}, \phi)$ such that

$$(T\xi)_y = T_{\phi^{-1}(y)}\xi_{\phi^{-1}(y)}$$

for every $\xi \in L^p(\nu, \mathcal{W})$ and every $y \in Y$. Then

$$(\Delta_g T)_y = g(y)T_{\phi^{-1}(y)}\xi_{\phi^{-1}(y)} = (T\Delta_{g\circ\phi}\xi)_y$$

for all $y \in Y$.

(2) implies (3). Obvious.

(3) implies (1). For $(\alpha_n)_{n\in\omega} \in \mathbb{Q}(i)^{\oplus\omega}$, set

$$\sigma_\alpha = \sum_{n\in\omega} \alpha_n \sigma_n \in L^p(\mu, \mathcal{Z}) \quad \text{and} \quad \widehat{\sigma}_\alpha = T\sigma_\alpha \in L^p(\nu, \mathcal{W}).$$

Using the assumption (3) at the second step, we get

$$\begin{aligned}
\int_F \|\widehat{\sigma}_{\alpha,y}\|^p \, d\nu(y) = \|\Delta_{\chi_F}\widehat{\sigma}_\alpha\|^p &= \left\|T\Delta_{\phi^{-1}[F]}\sigma_\alpha\right\|^p \\
&\leq \|T\|^p \left\|\Delta_{\phi^{-1}[F]}\sigma_\alpha\right\|^p \\
&= \|T\|^p \int_{\phi^{-1}[F]} \|\sigma_{\alpha,x}\|^p \ d\mu(x) \\
&= \|T\|^p \int_F \left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p d\phi_*(\mu)(y) \\
&= \|T\|^p \int_F \left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p \frac{d\phi_*(\mu)}{d\nu}(y) \ d\nu(y)
\end{aligned}$$

for every $F \in \mathcal{F}$. We conclude that

$$\|\widehat{\sigma}_{\alpha,y}\|^p \leq \|T\|^p \left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p \frac{d\phi_*(\mu)}{d\nu}(y)$$

for $\nu$-almost every $y \in Y$ and every $\alpha \in \mathbb{Q}(i)^{\oplus\omega}$. It follows that for $\mu$-almost every $x \in X$, the linear map $\sigma_{\alpha,x} \mapsto \widehat{\sigma}_{\alpha,\phi(x)}$ extends to a bounded linear map $T_x \colon \mathcal{Z}_x \to \mathcal{W}_{\phi(x)}$ that satisfies

$$\|T_x\|^p \leq \|T\|^p \|\sigma_{\alpha,x}\|^p \frac{d\phi_*(\mu)}{d\nu}(\phi(x)).$$

Since the assignment $x \mapsto \left\langle T_x\sigma_{n,x}, \tau_{m,\phi(x)}\right\rangle$ is Borel, it follows that the map $x \mapsto T_x$ defines a Borel section of $B(\mathcal{Z},\mathcal{W},\phi)$ satisfying

$$(T\xi)_y = T_{\phi^{-1}(y)}\xi_{\phi^{-1}(y)}$$

for $\xi \in L^p(\mu,\mathcal{Z})$ and $\nu$-almost every $y \in Y$. This concludes the proof. □

**Definition XX.2.19.** A $\phi$-*isomorphism* from $\mathcal{Z}$ to $\mathcal{W}$ is a Borel section $x \mapsto T_x$ of the bundle $B(\mathcal{Z},\mathcal{W},\phi)$ such that $T_x$ is a surjective isometry for every $x \in X$.

If $T = (T_x)_{x\in X}$ is a $\phi$-isomorphism from $\mathcal{Z}$ to $\mathcal{W}$, we denote by $T^{-1}$ the $\phi^{-1}$-isomorphism $\left(T^{-1}_{\phi^{-1}(y)}\right)_{y\in Y}$ from $\mathcal{W}$ to $\mathcal{Z}$.

**Definition XX.2.20.** If $X = Y$, then $\mathcal{Z}$ and $\mathcal{W}$ are said to be *isomorphic* if there is an $\mathrm{id}_X$-isomorphism from $\mathcal{Z}$ to $\mathcal{W}$. In this case an $\mathrm{id}_X$-isomorphism is simply called an *isomorphism.*

**Theorem XX.2.21.** Let $\mathcal{F}$ be a countable collection of Borel subsets of $Y$ that separates the points of $Y$, and let $T\colon L^p(\mu,\mathcal{Z})\to L^p(\nu,\mathcal{W})$ be an invertible isometry such that

$$\Delta_{\chi_F}T = T\Delta_{\chi_{\phi^{-1}(F)}}$$

for every $F\in\mathcal{F}$. Then there are a $\mu$-conull subset $X_0$ of $X$, a $\nu$-conull subset $Y_0$ of $Y$, and a $\phi$-isomorphism $\mathcal{Z}|_{X_0}\to\mathcal{W}|_{Y_0}$ such that $T$ is the decomposable operator associated with the Borel section

$$x\mapsto\left(\frac{d\phi_*(\mu)}{d\nu}\phi(x)\right)^{\frac{1}{p}}T_x.$$

Moreover, if $\widetilde{T}$ is another decomposable operator associated with said Borel section, then $\widetilde{T}_x = T_x$ for $\mu$-almost every $x\in X$.

*Proof.* Given $(\alpha_n)_{n\in\omega}$ in $\mathbb{Q}(i)^{\oplus\omega}$, set $\sigma_\alpha=\sum\limits_{n\in\omega}\alpha_n\sigma_n\in L^p(\mu,\mathcal{Z})$, and set

$$\widehat{\sigma}_\alpha=\Delta_{\left(\frac{d\phi_*(\mu)}{d\nu}\right)^{-\frac{1}{p}}}(T\sigma_\alpha).$$

Let $F\in\mathcal{F}$. Then

$$\Delta_{\chi_F}\widehat{\sigma}_\alpha=\Delta_{\chi_F}T\Delta_{\left(\frac{d\phi_*(\mu)}{d\nu}\circ\phi\right)^{-\frac{1}{p}}}\sigma_\alpha=T\Delta_{\chi_{\phi^{-1}(F)}\left(\frac{d\phi_*\mu}{d\nu}\circ\phi\right)^{-\frac{1}{p}}}\sigma_\alpha.$$

Thus,

$$\begin{aligned}\int_F\|\widehat{\sigma}_{\alpha,y}\|^p\ d\nu(y)&=\|\Delta_{\chi_F}\widehat{\sigma}_\alpha\|^p\\&=\left\|T\Delta_{\chi_{\phi^{-1}(F)}\left(\frac{d\phi_*(\mu)}{d\nu}\circ\phi\right)^{-\frac{1}{p}}}\sigma_\alpha\right\|^p\\&=\left\|\Delta_{\chi_{\phi^{-1}(F)}\left(\frac{d\phi_*(\mu)}{d\nu}\circ\phi\right)^{-\frac{1}{p}}}\sigma_\alpha\right\|^p\\&=\int_{\phi^{-1}[F]}\left(\frac{d\phi_*(\mu)}{d\nu}\circ\phi\right)\|\sigma_{\alpha,x}\|^p\ d\mu(x)\\&=\int_F\left(\frac{d\nu}{d\phi_*(\mu)}\right)\left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p d\phi_*(\mu)(y)\\&=\int_F\left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|^p\ d\nu(y).\end{aligned}$$

We conclude that $\|\widehat{\sigma}_{\alpha,y}\| = \left\|\sigma_{\alpha,\phi^{-1}(y)}\right\|$ for $\nu$-almost every $y \in Y$. Therefore, for $\mu$-almost every $x \in X$, the linear map $\sigma_{\alpha,x} \mapsto \widehat{\sigma}_{\alpha,\phi(x)}$ extends to a linear isometry $T_x \colon \mathcal{Z}_x \to \mathcal{W}_{\phi(x)}$. It can be verified, as in the proof of Proposition XX.2.18, that $x \mapsto T_x$ is a Borel section of $B(\mathcal{Z}, \mathcal{W}, \phi)$. Moreover, it is clear that $T$ is the decomposable operator associated with the section $x \mapsto \left(\left(\frac{d\phi_*(\mu)}{d\nu}\right)(\phi(x))\right)^{\frac{1}{p}} T_x$.

We claim that $T_x$ is surjective for $\mu$-almost every $x \in X$. We will do so by constructing a left inverse.

Reasoning as before on $T^{-1}$, one obtains a Borel section $y \mapsto S_y$ of $B(\mathcal{W}, \mathcal{Z}, \phi)$ such that $T^{-1}$ is the decomposable operator associated with the Borel section given by $y \mapsto \left(\frac{d\phi^{-1}\nu}{d\mu}\left(\phi^{-1}(y)\right)\right)^{\frac{1}{p}} S_y$. Since the disintegration of a decomposable operator is unique, we have

$$\mathrm{id}_{\mathcal{Z}_x} = \left(\frac{d\phi^{-1}\nu}{d\mu}\left(\phi^{-1}(\phi(x))\right)\right)^{\frac{1}{p}} S_{\phi(x)} \left(\left(\frac{d\phi_*(\mu)}{d\nu}\right)(\phi(x))\right)^{\frac{1}{p}} T_x = S_{\phi(x)} T_x$$

for $\mu$-almost every $x \in X$. This shows that $S_{\phi(x)}$ is the inverse of $T_x$ for $\mu$-almost every $x \in X$, and the claim is proved.

The last assertion follows again from uniqueness of the disintegration of a decomposable operator. $\square$

## Banach Representations of Etale Groupoids

### *Some background notions on groupoids*

A *groupoid* can be defined as a (nonempty) small category where every arrow is invertible. The set of objects of a groupoid $G$ is denoted by $G^0$. Identifying an object with its identity arrow, one can regard $G^0$ as a subset of $G$. We will denote the source and range maps on $G$ by $s, r \colon G \to G^0$, respectively. The set of pairs of composable arrows

$$\{(\gamma, \rho) \in G \times G \colon s(\gamma) = r(\rho)\}$$

will be denoted, as customary, by $G^2$. If $(\gamma, \rho)$ is a pair of composable arrows of $G$, we denote their composition by $\gamma\rho$. If $A$ and $B$ are subsets of $G$, we denote by $AB$ the set

$$\left\{\gamma\rho : (\gamma, \rho) \in (A \times B) \cap G^2\right\}.$$

Similarly, if $A$ is a subset of $G$ and $\gamma \in G$, then we write $A\gamma$ for $A\{\gamma\}$ and $\gamma A$ for $\{\gamma\}A$. In particular, when $x$ is an object of $G$, then $Ax$ denotes the set of elements of $A$ with source $x$, while $xA$ denotes the set of elements of $A$ with range $x$.

A *slice* of a groupoid $G$ is a subset $A$ of $G$ such that source and range maps are injective on $A$. (Slices are called $G$-sets in [225, 196].) If $U \subseteq G^0$, then the set of elements of $G$ with source and range in $U$ is again a groupoid, called the *restriction* of $G$ to $U$, and will be denoted by $G_{|U}$.

A *locally compact groupoid* is a groupoid endowed with a topology having a countable basis of Hausdorff open sets with compact closures, such that

1. composition and inversion of arrows are continuous maps, and
2. the set of objects $G^0$, as well as $Gx$ and $xG$ for every $x \in G^0$, are locally compact Haudorff spaces.

It follows that also source and range maps are continuous, since $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$ for all $\gamma \in G$. It should be noted that the topology of a locally compact groupoid might not be (globally) Hausdorff. Examples of non-Hausdorff locally compact groupoids often arise in the applications, such as the holonomy groupoid of a foliation; see [196, Section 2.3]. Locally compact groups are the locally compact groupoids with only one object.

**Definition XX.3.1.** An *etale groupoid* is a locally compact groupoid such that composition of arrows—or, equivalently, the source and range maps—are local homeomorphisms. This in particular implies that $G\gamma$ and $\gamma G$ are countable discrete sets.

etale groupoids can be regarded as the analog of countable discrete groups. In fact, countable discrete groups are precisely the etale groupoids with only one object.

**Definition XX.3.2.** Let $G$ be an etale groupoid. If $U$ is an open Hausdorff subset of $G$, then $C_c(U)$ is the space of compactly supported continuous functions on $U$. Recall that $B(G)$ denotes the space of complex-valued Borel functions on $G$. We define $C_c(G)$ to be the linear span inside $B(G)$ of the union of all $C_c(U)$, where $U$ ranges over the open Hausdorff subsets of $G$. (Equivalently, $U$ ranges over a covering of $G$ consisting of open slices [66, Proposition 3.10].) One can define the convolution product and inversion on $C_c(G)$ by

$$(f * g)(\gamma) = \sum_{\rho_0\rho_1=\gamma} f(\rho_0)g(\rho_1) \quad \text{and} \quad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

for $f, g \in C_c(G)$. For $f \in C_c(G)$, its $I$-norm is given by

$$\|f\|_I = \max\left\{\sup_{x\in G}\sum_{\gamma\in xG}|f(\gamma)|, \sup_{x\in G}\sum_{\gamma\in Gx}|f(\gamma)|\right\}.$$

These operations turn $C_c(G)$ into a normed *-algebra; see [196, Section 2.2].

Similarly, one can define the space $B_c(G)$ as the linear span inside $B(G)$ of the space of complex-valued bounded functions on $G$ vanishing outside a compact Hausdorff subset of $G$. Convolution product, inversion, and the $I$-norm can be defined exactly in the same way on $B_c(G)$ as on $C_c(G)$, making $B_c(G)$ a normed *-algebra; see [196, Section 2.2]. Both $C_c(G)$ and $B_c(G)$ have a contractive approximate identity.

**Remark XX.3.3.** When $G$ is a *Hausdorff* etale groupoid, then $C_c(G)$ as defined above coincides with the space of compactly supported continuous functions on $G$.

**Definition XX.3.4.** A *representation* of $C_c(G)$ on a Banach space $Z$ is a homomorphism $\pi\colon C_c(G) \to B(Z)$. We say that $\pi$ is *contractive* if it is contractive with respect to the $I$-norm on $C_c(G)$.

Let $G$ be an etale groupoid, and let $\mu$ is a Borel probability measure on $G^0$. Then $\mu$ induces $\sigma$-finite Borel measures $\nu$ and $\nu^{-1}$ on $G$, which are given by

$$\nu(A) = \int_{G^0} |xA| \ d\mu(x)$$

and

$$\nu^{-1}(A) = \nu(A^{-1}) = \int_{G^0} |Ax| \ d\mu(x)$$

for every Borel subset $A$ of $G$.

Observe that $\nu$ is the measure obtained integrating the Borel family $(c_{xG})_{x\in X}$—where $c_{xG}$ denotes the counting measure on $xG$—with respect to $\mu$. Similarly, $\nu$ is the measure obtained integrating $(c_{Gx})_{x\in X}$ with respect to $\mu$.

The measure $\mu$ is said to be *quasi-invariant* if $\nu$ and $\nu^{-1}$ are equivalent, in symbols $\nu \sim \nu^{-1}$. In such case, the Radon-Nikodym derivative $\frac{d\nu}{d\nu^{-1}}$ will be denoted by $D$. Results of Hahn [108] and Ramsay [223, Theorem 3.20] show that one can always choose—as we will do in the following—$D$ to be a Borel homomorphism from $G$ to the multiplicative group of strictly positive real numbers.

**Definition XX.3.5.** An *open slice* of a groupoid $G$ is an open subset $A$ of $G$ such that source and range maps are injective on $A$. If $A$ is an open slice, then there is a local homeomorphism $\theta_A\colon A^{-1}A \to AA^{-1}$ given by $\theta_A(x) = r(Ax)$ for $x \in A^{-1}A$.

Proposition 3.2.2 of [196] shows that, if $A$ is an open slice, then

$$D(yA)^{-1} = \frac{d(\theta_A)_*\mu_{|A^{-1}A}}{d\mu_{|AA^{-1}}}(y)$$

for every $y \in G$. Moreover, $\mu$ is quasi-invariant if and only if $d(\theta_A)_*\mu_{|A^{-1}A} \sim d\mu_{|AA^{-1}}$ for every open slice $A$.

It is easy to verify that if $\mu$ and $\widetilde{\mu}$ are equivalent quasi-invariant measures on $G^0$, and $\nu$ and $\widetilde{\nu}$ are the corresponding measures on $G$, then $\nu \sim \widetilde{\nu}$ and

$$\frac{d\nu}{d\widetilde{\nu}}(\gamma) = \frac{d\mu}{d\widetilde{\mu}}(r(\gamma)) \quad \text{and} \quad \frac{d\nu^{-1}}{d(\widetilde{\nu})^{-1}}(\gamma) = \frac{d\mu}{d\widetilde{\mu}}(s(\gamma))$$

for all $\gamma \in G$. The chain rule then shows that

$$\frac{d\widetilde{\nu}}{d\widetilde{\nu}^{-1}}(\gamma) = \frac{d\widetilde{\mu}}{d\mu}(r(\gamma))\frac{d\nu}{d\nu^{-1}}(\gamma)\frac{d\mu}{d\widetilde{\mu}}(s(\gamma))$$

for all $\gamma \in G$.

**Remark XX.3.6.** etale groupoids can be characterized as those locally compact groupoids whose topology admits a countable basis of *open slices*.

Closely related to the notion of an etale groupoid is that of an inverse semigroup.

**Definition XX.3.7.** An *inverse semigroup* is a semigroup $S$ such that for every element $s$ of $S$, there exists a unique element $s^*$ of $S$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

Let $G$ be an etale groupoid, and denote by $\Sigma(G)$ the set of open slices of $G$. The operations

$$AB = \{\gamma\rho \colon (\gamma, \rho) \in (A \times B) \cap G^2\} \quad \text{and} \quad A^{-1} = \{\gamma^{-1} \colon \gamma \in A\}$$

turn $\Sigma(G)$ into an inverse semigroup. The set $\Sigma_c(G)$ of *precompact* open slices of $G$ is a subsemigroup of $\Sigma(G)$. Similarly, the set $\Sigma_{\mathcal{K}}(G)$ of *compact* open slices of $G$ is also a subsemigroup of $\Sigma(G)$.

**Definition XX.3.8.** An etale groupoid $G$ is called *ample* if $\Sigma_{\mathcal{K}}(G)$ is a basis for the topology of $G$. This is equivalent to the assertion that $G^0$ has a countable basis of compact open sets.

*Representations of etale groupoids on Banach bundles*

Throughout the rest of this section, we fix an etale groupoid $G$, and a Borel Banach bundle $q \colon \mathcal{Z} \to G^0$.

**Definition XX.3.9.** We define the *groupoid of fiber-isometries* of $\mathcal{Z}$ by

$$\mathrm{Iso}(\mathcal{Z}) = \left\{(T, x, y) \colon T \colon \mathcal{Z}_x \to \mathcal{Z}_y \text{ is an invertible isometry, and } x, y \in G^0\right\}.$$

We denote the elements of $\mathrm{Iso}(\mathcal{Z})$ simply by $T \colon \mathcal{Z}_x \to \mathcal{Z}_y$.

The set $\mathrm{Iso}(\mathcal{Z})$ has naturally the structure of groupoid with set of objects $G^0$, where the source and range of the fiber-isometry $T \colon \mathcal{Z}_x \to \mathcal{Z}_y$ are $s(T) = x$ and $r(T) = y$, respectively. If $(\sigma_n)_{n\in\omega}$ is a basic sequence for $\mathcal{Z}$, then the Borel structure generated by the maps

$$T \mapsto \left\langle T\sigma_{n,s(T)}, \sigma_{m,r(T)}\right\rangle$$

for $n, m \in \omega$, is standard, and makes $\mathrm{Iso}(\mathcal{Z})$ a standard Borel groupoid. This means that $\mathrm{Iso}\,(\mathcal{Z})$ is a groupoid endowed with a standard Borel structure that makes composition and inversion of arrows Borel.

**Definition XX.3.10.** Let $\mu$ be a quasi-invariant Borel probability measure on $G^0$. A Borel map $T \colon G \to \mathrm{Iso}(\mathcal{Z})$ is said to be a *$\mu$-almost everywhere homomorphism*, if there exists a $\mu$-conull subset $U$ of $G^0$ such that the restriction of $T$ to $G|_U$ is a groupoid homomorphism which is the identity on $U$.

**Definition XX.3.11.** A *representation* of $G$ on $\mathcal{Z}$, is a pair $(\mu, T)$ consisting of an invariant Borel probability measure $\mu$ on $G^0$, and a $\mu$-almost everywhere homomorphism $T\colon G \to \mathrm{Iso}(\mathcal{Z})$.

If $G$ is a discrete group, then a Borel Banach bundle over $G^0$ is just a Banach space $Z$, and a representation of $G$ on $Z$ is a Borel group homomorphism from $G$ to the Polish group $\mathrm{Iso}(Z)$ of invertible isometries of $Z$ endowed with the strong operator topology. (It should be noted here that a Borel group homomorphism from $G$ to $\mathrm{Iso}(Z)$ is automatically continuous by [147, Theorem 9.10].)

**Example XX.3.12** (Dual representation)**.** Let $(\mu, T)$ be a representation of $G$ on $\mathcal{Z}$. The *dual representation* of $(\mu, T)$ is the representation $(\mu, T')$ of $G$ on $\mathcal{Z}'$ defined by

$$(T')_\gamma = \left(T_{\gamma^{-1}}\right)' : \mathcal{Z}'_{s(\gamma)} \to \mathcal{Z}'_{r(\gamma)}$$

for all $\gamma \in G$.

There is a natural notion of equivalence for representations of $G$ on Banach bundles.

**Definition XX.3.13.** Two representations $(\mu, T)$ and $(\widetilde{\mu}, \widetilde{T})$ of $G$ on Borel Banach bundles $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$ over $G^0$, are said to be *equivalent*, if $\mu \sim \widetilde{\mu}$ and there are a $\mu$-conull (and hence $\widetilde{\mu}$-conull) Borel subset $U$ of $G^0$, and an isomorphism $v\colon \mathcal{Z}|_U \to \widetilde{\mathcal{Z}}|_U$ (see Definition XX.2.20) such that

$$\widetilde{T}_\gamma v_{s(\gamma)} = v_{r(\gamma)} T_\gamma$$

for every $\gamma \in G|_U$.

It is clear that two representations are equivalent if and only if their dual representations are equivalent. (Recall our standing assumption that all Borel Banach bundles are endowed with a basic sequence and, in particular, all the fibers are reflexive Banach spaces.)

We now show how to integrate groupoid representations to $L^p$-bundles. From now on, we fix a Hölder exponent $p \in (1, \infty)$.

**Theorem XX.3.14.** Let $(\mu, T)$ be a representation of $G$ on $\mathcal{Z}$. Then the equation

$$(\pi_T(f)\xi)_x = \sum_{\gamma \in xG} f(\gamma) D(\gamma)^{-\frac{1}{p}} T_\gamma \xi_{s(\gamma)} \tag{XX.2}$$

for $f \in C_c(G)$, $\xi \in L^p(\mu, \mathcal{Z})$, and $x \in G^0$, defines an $I$-norm contractive, nondegenerate representation $\pi_T \colon C_c(G) \to B(L^p(\mu, \mathcal{Z}))$.

*Proof.* Fix $f \in C_c(G)$, $\xi \in L^p(\mu, \mathcal{Z})$, and $\eta \in L^{p'}(\mu, \mathcal{Z}')$. We claim that the complex-valued function on $G$ defined by

$$\gamma \mapsto D^{-\frac{1}{p}}(\gamma) f(\gamma) \left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)} \right\rangle$$

is $\nu$-integrable. This follows from the following estimate where we use Hölder's inequality at the third step:

$$\begin{aligned}
&\int \left| f(\gamma) \left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)} \right\rangle \right| D^{-\frac{1}{p}}(\gamma)\, d\nu(\gamma) \\
&= \int \left| f(\gamma) \left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)} \right\rangle \right| D^{-\frac{1}{p}}(\gamma)\, d\nu(\gamma) \\
&\leq \int |f(\gamma)|^{\frac{1}{p}} \left\| \xi_{s(\gamma)} \right\| D^{-\frac{1}{p}}(\gamma)\, |f(\gamma)|^{\frac{1}{p'}} \left\| \eta_{r(\gamma)} \right\| \, d\nu(\gamma) \\
&\leq \left( \int |f(\gamma)| \left\| \xi_{s(\gamma)} \right\|^p D^{-1}(\gamma)\, d\nu(\gamma) \right)^{\frac{1}{p}} \left( \int |f(\gamma)| \left\| \eta_{r(\gamma)} \right\|^{p'} \, d\nu(\gamma) \right)^{\frac{1}{p'}} \\
&= \left( \int |f(\gamma)| \left\| \xi_{s(\gamma)} \right\|^p \, d\nu^{-1}(\gamma) \right)^{\frac{1}{p}} \left( \int |f(\gamma)| \left\| \eta_{r(\gamma)} \right\|^{p'} \, d\nu(\gamma) \right)^{\frac{1}{p'}} \\
&= \left( \int \sum_{\gamma \in Gx} |f(\gamma)| \, \|\xi_x\|^p \, d\mu(x) \right)^{\frac{1}{p}} \left( \int \sum_{\gamma \in yG} f(y) \, \|\eta_y\|^{p'} \, d\mu(y) \right)^{\frac{1}{p'}} \\
&\leq \|f\|_I \|\xi\| \, \|\eta\| \, .
\end{aligned}$$

Therefore, the linear functional $\phi_{T,\xi}(f) \colon L^{p'}(\mu, \mathcal{Z}') \to \mathbb{C}$ given by

$$\phi_{T,\xi}(f)(\eta) = \int_G f(\gamma) D^{-\frac{1}{p}}(\gamma) \left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)} \right\rangle \, d\nu(\gamma)$$

for $\eta \in L^{p'}(\mu, \mathcal{Z}')$, satisfies $\|\phi_{T,\xi}(f)\| \leq \|f\|_I \|\xi\|$. By Theorem XX.2.10, there is a unique element $\pi_T(f)\xi$ of $L^p(\mu, \mathcal{Z})$ of norm at most $\|f\|_I\|\xi\|$, such that

$$\begin{aligned} \int \langle (\pi_T(f)\xi)_x, \eta_x \rangle \ d\mu(x) &= \phi_{T,\xi}(f)(\eta) \\ &= \int \left\langle \sum_{\gamma \in xG} f(\gamma) D(\gamma)^{-\frac{1}{p}} T_\gamma \xi_{s(\gamma)}, \eta_x \right\rangle \ d\mu(x) \end{aligned}$$

for all $\eta \in L^{p'}(\mu, \mathcal{Z}')$. In particular, Equation (XX.2) defines a bounded linear operator $\pi_T(f) \in B(L^p(\mu, \mathcal{Z}))$ of norm at most $\|f\|_I$.

We claim that $\pi_T \colon C_c(G) \to B(L^p(\mu, \mathcal{Z}))$ is a nondegenerate homomorphism. Given $f$ and $g$ in $C_c(G)$, we have

$$\begin{aligned} &(\pi_T(f * g)\xi)_x \\ &= \sum_{\gamma \in xG} (f * g)(\gamma) D(\gamma)^{-\frac{1}{p}} T_\gamma \xi_{s(\gamma)} \\ &= \sum_{\gamma \in xG} \sum_{\substack{(\rho_0, \rho_1) \in G^2 \\ \rho_0\rho_1 = \gamma}} f(\rho_0) g(\rho_1) D(\rho_0)^{-\frac{1}{p}} D(\rho_1)^{-\frac{1}{p}} T_{\rho_0} T_{\rho_1} \xi_{s(\rho_1)} \\ &= \sum_{\rho_0 \in xG} f(\rho_0) D(\rho_0)^{-\frac{1}{p}} T_{\rho_0} \left( \sum_{\rho_1 \in s(\rho_0)} g(\rho_1) D(\rho_1)^{-\frac{1}{p}} T_{\rho_1} \xi_{s(\rho_1)} \right) \\ &= \sum_{\rho_0 \in xG} f(\rho_0) D(\rho_0)^{-\frac{1}{p}} T_{\rho_0} \left(\pi_T(g)\xi\right)_{s(\rho_0)} \\ &= (\pi_T(f)\pi_T(g)\xi)_x \end{aligned}$$

for $\mu$-almost every $x \in G^0$. We conclude that $\pi_T$ is multiplicative.

Let us now show that $\pi_T$ is nondegenerate. Assume that $\eta \in L^{p'}(\mu, \mathcal{Z}')$ satisfies $\langle \pi(f)\xi, \eta \rangle = 0$ for every $f \in C_c(G)$ and every $\xi \in L^p(\mu, \mathcal{Z})$. We claim that $\eta = 0$. Let $(\sigma_n)_{n\in\omega}$ be a basic sequence for $\mathcal{Z}$. For $\alpha \in \mathbb{Q}(i)^{\oplus\omega}$, set $\sigma_\alpha = \sum\limits_{n\in\omega} \alpha_n \sigma_n \in L^p(\mu, \mathcal{Z})$. Then $\{\sigma_{\alpha,x} \colon \alpha \in \mathbb{Q}(i)^{\oplus\omega}\}$ is a dense subspace of $\mathcal{Z}_x$ for every $x \in X$, and therefore $\{T_\gamma \sigma_{\alpha,x} \colon \alpha \in \mathbb{Q}(i)^{\oplus\omega}\}$ is a dense subspace of $\mathcal{Z}_{r(\gamma)}$ for every $\gamma \in Gx$. By assumption,

$$0 = \langle \pi(f)\sigma_\alpha, \eta \rangle = \int D^{-\frac{1}{p}}(\gamma) f(\gamma) \left\langle T_\gamma \sigma_{\alpha, s(\gamma)}, \eta_{r(\gamma)} \right\rangle \ d\nu(\gamma)$$

for every $f \in C_c(G)$. Therefore, $\langle T_\gamma \sigma_{\alpha,s(\gamma)}, \eta_{r(\gamma)} \rangle = 0$ for $\nu$-almost every $\gamma \in G$ and for every $\alpha \in \mathbb{Q}(i)^{\oplus\omega}$. This implies that $\eta_x = 0$ for $\mu$-almost every $x \in G^0$, and thus $\eta = 0$. This finishes the proof. $\square$

**Definition XX.3.15.** Let $(\mu, T)$ be a representation of $G$ on $\mathcal{Z}$. We call the representation $\pi_T \colon C_c(G) \to B(L^p(\mu, \mathcal{Z}))$ constructed in the theorem above, the *integrated form* of $(\mu, T)$.

**Remark XX.3.16.** Given a representation $(\mu, T)$ of $G$ on $\mathcal{Z}$, one can show that there is an $I$-norm contractive nondegenerate representation $\pi_T \colon B_c(G) \to B(L^p(\mu, \mathcal{Z}))$ defined by the same expression as in the statement of Theorem XX.3.14.

**Definition XX.3.17.** Let $\mu$ be a Borel $\sigma$-finite measure on $G^0$ and let $\pi \colon C_c(G) \to B(L^p(\mu, \mathcal{Z}))$ be an $I$-norm contractive nondegenerate representation. The *dual representation* of $\pi$ is the $I$-norm contractive nondegenerate representation $\pi' \colon C_c(G) \to B(L^{p'}(\mu, \mathcal{Z}'))$ given by $\pi'(f) = \pi(f^*)'$ for all $f \in C_c(G)$.

**Lemma XX.3.18.** Let $(\mu, T)$ be a representation of $G$ on $\mathcal{Z}$. Then $\pi_T(f)' = \pi_{T'}(f^*)$ for all $f \in C_c(G)$.

*Proof.* The result follows from the following computation, valid for all $\xi \in L^p(\mu, \mathcal{Z})$ and all $\eta \in L^{p'}(\mu, \mathcal{Z}')$:

$$\begin{aligned}
\langle \pi_{T'}(f^*)\eta, \xi \rangle &= \int f'(\gamma) D(\gamma)^{-\frac{1}{p}} \left\langle T_{\gamma^{-1}} \eta_{s(\gamma)}, \xi_{r(\gamma)} \right\rangle \, d\nu(\gamma) \\
&= \int \overline{f(\gamma^{-1})} \left\langle T_{\gamma^{-1}} \eta_{s(\gamma)}, \xi_{r(\gamma)} \right\rangle D(\gamma)^{\frac{1}{p'}} \, d\nu^{-1}(\gamma) \\
&= \int \overline{f(\gamma)} \left\langle T_\gamma \eta_{r(\gamma)}, \xi_{s(\gamma)} \right\rangle D(\gamma)^{-\frac{1}{p'}} \, d\nu(\gamma) \\
&= \overline{\int f(\gamma) \left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)} \right\rangle \, d\nu(\gamma)} \\
&= \overline{\langle \pi_T(f)\xi, \eta \rangle}.
\end{aligned}$$

$\square$

Our next goal is to show that two representations of a groupoid on Borel Banach bundles are equivalent if and only if their integrated forms are equivalent.

**Theorem XX.3.19.** Let $(\mu, \mathcal{Z})$ and $(\lambda, \mathcal{W})$ be Borel Banach bundles over $G^0$, and let $T$ and $S$ be groupoid representations of $G$ on $\mathcal{Z}$ and $\mathcal{W}$, respectively. Then $T$ and $S$ are equivalent if and only if $\pi_T$ and $\pi_S$ are equivalent.

*Proof.* Suppose that $T$ and $S$ are equivalent. Thus $\mu \sim \lambda$ and there are a $\mu$-conull Borel subsets $U$ of $G^0$ and an isomorphism $v\colon \mathcal{Z}|_U \to \mathcal{W}_U$ (see Definition XX.2.20) such that $v_{r(\gamma)}T_\gamma = S_\gamma v_{s(\gamma)}$ for every $\gamma \in G|_U$. Define a linear map $u\colon L^p(\mu, \mathcal{Z}) \to L^p(\lambda, \mathcal{W})$ by

$$(u\xi)_x = \left(\frac{d\mu}{d\lambda}(x)\right)^{\frac{1}{p}} v_x \xi_x$$

for $\xi$ in $L^p(\mu, \mathcal{Z})$ and $x \in U$. It is easy to check that $u$ is isometric. We claim that $u$ is bijective. For this, it suffices to check that its inverse is given by

$$(u^{-1}\xi)_y = \left(\frac{d\lambda}{d\mu}(y)\right)^{\frac{1}{p}} v_y^{-1} \xi_y$$

for all $\xi \in L^p(\lambda, \mathcal{W})$ and all $y \in G^0$. We omit the details.

We claim that $u$ intertwines $\pi_T$ and $\pi_S$. Once we show this, the "only if" implication will be proved. To prove the claim, fix $\xi$ in $L^p(\lambda, \mathcal{W})$ and $x \in G^0$. We have

$$\begin{aligned}
&(u\pi_T(f)u^{-1}\xi)_x \\
&= \left(\frac{d\mu}{d\lambda}(x)\right)^{\frac{1}{p}} v_x (\pi_T(f)u^{-1}\xi)_x \\
&= \left(\frac{d\mu}{d\lambda}(x)\right)^{\frac{1}{p}} v_x \left(\sum_{\gamma \in xG} f(\gamma)\left(\frac{d\nu_\mu}{d\nu_\mu^{-1}}(\gamma)\right)^{-\frac{1}{p}} T_\gamma (u^{-1}\xi)_{s(\gamma)}\right) \\
&= \sum_{\gamma \in xG} f(\gamma)\left(\frac{d\mu}{d\lambda}(x)\right)^{\frac{1}{p}} \left(\frac{d\nu_\mu}{d\nu_\mu^{-1}}(\gamma)\right)^{-\frac{1}{p}} \left(\frac{d\lambda}{d\mu}(s(\gamma))\right)^{\frac{1}{p}} v_x T_\gamma v_{s(\gamma)}^{-1} \xi_{s(\gamma)} \\
&= \sum_{\gamma \in xG} f(\gamma)\left(\frac{d\nu_\lambda}{d(\nu_\lambda)^{-1}}(\gamma)\right)^{-\frac{1}{p}} S_\gamma \xi_{s(\gamma)} \\
&= (\pi_S(f)\xi)_x
\end{aligned}$$

for all $x \in G$, and the claim is proved.

Conversely, assume that $\pi_T$ and $\pi_S$ are equivalent, and let $u\colon L^p(\mu, \mathcal{Z}) \to L^p(\lambda, \mathcal{W})$ be a surjective isometry such that

$$u\pi_T(f) = \pi_S(f)u \tag{XX.3}$$

for every $f \in C_c(G)$. Denote by $\mathcal{I}$ the set of those functions $f$ in $B_c(G)$ that satisfy Equation (XX.3). Fix an open subset $U$ of $G$ contained in some compact Hausdorff set. It follows from the dominated convergence theorem that if $(f_n)_{n\in\omega}$ is a uniformly bounded sequence in $\mathcal{I}$ converging to a function $f \in B(U)$, then $f \in \mathcal{I}$. By [196, Lemma 2.2.1], we have $B(U) \subseteq \mathcal{I}$. In particular, if $A$ is an open slice of $G$ contained in some compact Hausdorff set, then $\chi_A$ belongs to $\mathcal{I}$.

Let $\mathcal{F}$ be a countable basis for the topology of $G$ consisting of open slices each one of which is contained in some compact Hausdorff set. Apply Theorem XX.2.21 to find a Borel conull set $X_0$ of $G^0$ and an isomorphism $v\colon \mathcal{Z}|_{X_0} \to \mathcal{W}|_{X_0}$ such that

$$(u\xi)_x = \left(\frac{d\mu}{d\lambda}(x)\right)^{\frac{1}{p}} v_x\xi_x$$

for all $x \in X_0$ and all $\xi \in L^p(\mu, \mathcal{Z})$. It is not difficult to verify that $S_\gamma v_{s(\gamma)} = v_{r(\gamma)}T_\gamma$ for all $\gamma$ in $G$. This finishes the proof. $\square$

*Amplification of representations*

Given a natural number $n \geq 1$, regard $M_n(C_c(G))$ as a normed *-algebra with respect to the usual matrix product and involution, and the $I$-norm

$$\begin{aligned} &\|[f_{ij}]_{i,j\in n}\|_I \\ = \;&\max\left\{\max_{x\in G^0}\max_{j\in n}\sum_{j\in n}\sum_{\gamma\in xG}|f_{ij}(\gamma)|\;,\;\max_{x\in G^0}\max_{j\in n}\sum_{j\in n}\sum_{\gamma\in Gx}|f_{ij}(\gamma)|\right\}.\end{aligned}$$

**Definition XX.3.20.** Let $\mu$ be a $\sigma$-finite Borel measure on $G^0$, and let $\pi\colon C_c(G) \to B(L^p(\mu, \mathcal{Z}))$ be an $I$-norm contractive representation. We define its *amplification* $\pi^{(n)}\colon M_n(C_c(G)) \to B(\ell^p(n, L^p(\mu, \mathcal{Z})))$ by

$$\pi^{(n)}([f_{ij}]_{i,j\in n})[\xi_j]_{j\in n} = \left[\sum_{j\in n}\pi(f_{ij})\xi_j\right]_{i\in n}.$$

If one starts with a representation $T$ of a groupoid on a Borel Banach bundle, one may take its integrated form, and then its amplification to matrices over $C_c(G)$, as in the definition above. The resulting representation $\pi_T^{(n)}$ is the integrated form of a representation of an amplified groupoid, which we proceed to describe.

**Definition XX.3.21.** Given $n \geq 1$, denote by $G_n$ the groupoid $n \times G \times n$ endowed with the product topology, with set of objects $G^0 \times n$, and operations defined by

$$s(i,\gamma,j) = (s(\gamma),j)\ ,\quad r(i,\gamma,j) = (r(\gamma),i) \quad \text{and} \quad (i,\gamma,j)(j,\rho,k) = (i,\gamma\rho,k).$$

Denote by $\mathcal{Z}^{(n)}$ the Borel Banach bundle over $G^0 \times n$ such that $\mathcal{Z}^{(n)}_{(x,j)} = \mathcal{Z}_x$, with basic sequence $(\sigma_k^{(n)})_{k\in\omega}$ defined by

$$\sigma^{(n)}_{k,(x,j)} = \sigma_{k,x}$$

for $(x,j) \in G^0 \times n$. Endow $G^0 \times n$ with the measure $\mu^{(n)} = \mu \times c_n$, and define the *amplification* $T^{(n)} \colon G_n \to \mathrm{Iso}(\mathcal{Z}^{(n)})$ of $T$ by $T^{(n)}_{(i,\gamma,j)} = T_\gamma$ for $(i,\gamma,j) \in G_n$.

**Proposition XX.3.22.** Let $(T,\mu)$ be a representation of $G$ on $\mathcal{Z}$. Given $n \geq 1$, the representations $\pi_T^{(n)}$ and $\pi_{T^{(n)}}$ are equivalent.

*Proof.* Under the canonical identifications

$$M_n(C_c(G)) \cong C_c(G_n)$$

and

$$\ell^p(n, L^p(\mu,\mathcal{Z})) \cong L^p(\mu^{(n)}, \mathcal{Z}^{(n)}),$$

it is easy to verify that $\pi_T^{(n)}$ is (equivalent to) the integrated form of the representation $T^{(n)}$. We omit the details. □

*Representations of etale groupoids on $L^p$-bundles*

In this section, we want to isolate a particularly important and natural class of representations of an etale groupoid on Banach spaces.

We fix a quasi-invariant measure $\mu$ on $G^0$. Let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space $Z$ fibred over $G^0$ via $q$, and assume that $\mu = q_*(\lambda)$. Denote by $\mathcal{Z}$ the $L^p$-bundle $\bigsqcup_{x\in G^0} L^p(\lambda_x)$ over $\mu$ obtained from the disintegration $\lambda = \int \lambda_x \, d\mu(x)$ as in Theorem XX.2.11.

**Definition XX.3.23.** Adopt the notation from the comments above. A representation $T\colon G \to \mathrm{Iso}(\mathcal{Z})$ is called an *$L^p$-representation* of $G$ on $\mathcal{Z}$. Under the identification $L^p(\mu,\mathcal{Z}) \cong L^p(\mu)$ given by Theorem XX.2.11, the integrated form $\pi_T\colon C_c(G) \to B(L^p(\lambda))$ of $T$, is an $I$-norm contractive nondegenerate representation.

It will be shown in Theorem XX.6.9 that every $I$-norm contractive nondegenerate representation of $C_c(G)$ on an $L^p$-space is the integrated form of some $L^p$-representation of $G$.

**Remark XX.3.24.** It is clear that an $L^2$-representation of $G$ in the sense of Definition XX.3.23, is a representation of $G$ on a Borel Hilbert bundle. Conversely, any representation of $G$ on a Borel Hilbert bundle is equivalent—as in Definition XX.3.13—to an $L^2$-representation. In fact, if $\mathcal{H}$ is a Borel Hilbert bundle over $G^0$, then for every $0 \leq \alpha \leq \omega$ the set $X_\alpha = \{x \in G^0 \colon \dim(\mathcal{H}_x) = \alpha\}$ is Borel. Thus, $\mathcal{H}$ is isomorphic to the Hilbert bundle

$$\mathcal{Z}_0 = \bigsqcup_{0 \leq \alpha \leq \omega} X_\alpha \times \ell^2(\alpha).$$

Set $Z = \bigsqcup_{0 \leq \alpha \leq \omega} (Z_\alpha \times \alpha)$, and define a $\sigma$-finite Borel measure $\lambda$ on $Z$ by $\lambda = \bigsqcup_{0 \leq \alpha \leq \omega} (\mu \times c_\alpha)$. It is immediate that $\mathcal{Z}_0$ is (isomorphic to) the Borel Hilbert bundle $\bigsqcup_{x \in G^0} L^2(\lambda_x)$ obtained from the disintegration of $\lambda$ with respect to $\mu$.

In view of the above remark, there is no difference, up to equivalence, between $L^2$-representations and representations on Borel Hilbert bundles. The theory of $L^p$-representations of $G$ for $p \in (1, \infty)$ can therefore be thought of as a generalization of the theory of representations of $G$ on Borel Hilbert bundles.

**Example XX.3.25** (Left regular representation)**.** Take $Z = G$ and $\lambda = \nu$, in which case the disintegration of $\lambda$ with respect to $\mu$ is $(c_{xG})_{x \in X}$. For $\gamma \in G$, define the surjective linear isometry

$$T^{\mu,p}_\gamma \colon \ell^p(s(\gamma)G) \to \ell^p(r(\gamma)G)$$

by

$$(T^{\mu,p}_\gamma \xi)(\rho) = \xi(\gamma^{-1}\rho).$$

The assignment $\gamma \mapsto T^{\mu,p}_\gamma$ defines a representation

$$T^{\mu,p} \colon G \to \mathrm{Iso}\left(\bigsqcup_{x \in G^0} \ell^p(xG)\right),$$

which we shall call the *left regular $L^p$-representation* of $G$ associated with $\mu$. When the Hölder exponent $p$ is clear from the context, we will write $T^\mu$ in place of $T^{\mu,p}$.

**Lemma XX.3.26.** The dual $(T^{\mu,p})'$ of the left regular $L^p$-representation associated with $\mu$ is the left regular $L^{p'}$-representation $T^{\mu,p'}$ associated with $\mu$.

*Proof.* The result follows from the following computation, valid for all $\eta \in \ell^{p'}(r(\gamma)G)$ and all $\xi \in \ell^p(s(\gamma)G)$:

$$\begin{aligned}\left\langle T_\gamma^{\mu,p'}\eta, \xi\right\rangle &= \sum_{\rho\in r(\gamma)G} (T_\gamma^{\mu,p'}\eta)(\rho)\overline{\xi(\rho)} = \sum_{\rho\in r(\gamma)G} \eta(\gamma^{-1}\rho)\overline{\xi(\rho)} \\ &= \sum_{\theta\in s(\gamma)G} \eta(\theta)\overline{\xi(\gamma\theta)} = \overline{\sum_{\rho\in r(\gamma^{-1})G} \xi(\gamma\rho)\overline{\eta(\rho)}} \\ &= \overline{\left\langle T^{\mu,p}_{\gamma^{-1}}\xi, \eta\right\rangle} = \left\langle \left(T^{\mu,p}_{\gamma^{-1}}\right)'\eta, \xi\right\rangle.\end{aligned}$$

□

We will now compute the integrated form of the left regular $L^p$-representation $T^\mu$ of $G$ associated with a quasi-invariant Borel probability measure $\mu$ on $G^0$. Following Rieffel's induction theory and for consistency with [196, Section 3.1 and Appendix D] we will denote such representation by $\mathrm{Ind}(\mu)$.

**Proposition XX.3.27.** The integrated form $\mathrm{Ind}(\mu)$ of the left regular representation $T^\mu$ associated with $\mu$, is the left action of $C_c(G)$ on $L^p(\nu^{-1})$ by convolution.

*Proof.* It is easy to check that multiplication by $D^{\frac{1}{p}}$ and $D^{\frac{1}{p'}}$ define isometric isomorphisms $L^p(\nu) \cong L^p(\nu^{-1})$ and $L^{p'}(\nu) \cong L^{p'}(\nu^{-1})$, respectively.

Given $\xi \in L^p(\nu)$ and $\eta \in L^{p'}(\nu)$, set $\widehat{\xi} = D^{\frac{1}{p}}\xi \in L^p(\nu^{-1})$ and $\widehat{\eta} = D^{\frac{1}{p'}}\eta \in L^{p'}(\nu^{-1})$. Then

$$\begin{aligned}
&\langle \mathrm{Ind}(\mu)(f)\xi, \eta\rangle_{L^p(\nu)} \\
&= \int f(\gamma)\left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)}\right\rangle \, d\nu(\gamma) \\
&= \int \sum_{\gamma\in xG} f(\gamma)\left\langle T_\gamma \xi_{s(\gamma)}, \eta_x\right\rangle D^{-\frac{1}{p}}(\gamma)\, d\mu(x) \\
&= \int \sum_{\gamma\in xG} f(\gamma)\left(\sum_{\rho\in xG} \xi(\gamma^{-1}\rho)\overline{\eta(\rho)}\right) D^{-\frac{1}{p}}(\gamma)\, d\mu(x) \\
&= \int \sum_{x\in xG}\sum_{\rho\in xG}\left(\sum_{\gamma\in xG} f(\gamma)\widehat{\xi}(\gamma^{-1}\rho)\right)\overline{\widehat{\eta}(\rho)}D^{-1}(\rho)\, d\mu(x) \\
&= \int \sum_{\gamma\in xG}\sum_{\rho\in xG} (f*\widehat{\xi})(\rho)\overline{\widehat{\eta}(\rho)}D^{-1}(\rho)\, d\mu(x) \\
&= \left\langle f*\widehat{\xi}, \widehat{\eta}\right\rangle_{L^p(\nu^{-1})}
\end{aligned}$$

This finishes the proof. $\square$

**Lemma XX.3.28.** A function $f$ in $C_c(G)$ belongs to $\mathrm{Ker}(\mathrm{Ind}(\mu))$ if and only if it vanishes on the support of $\nu$.

*Proof.* Suppose that $f$ vanishes on the support of $\nu$. Then

$$\langle \mathrm{Ind}(\mu)(f)\xi, \eta\rangle_{L^p(\nu)} = \int f(\gamma)\left\langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)}\right\rangle D^{-\frac{1}{p}}(\gamma)\, d\nu(\gamma) = 0.$$

for every $\xi \in L^p(\nu)$ and $\eta \in L^{p'}(\nu)$, so $\mathrm{Ind}(\mu)(f) = 0$.

Conversely, if $\mathrm{Ind}(\mu)(f) = 0$ then $f * \xi = 0$ for every $\xi \in L^p(\nu^{-1})$. In particular, $f = f * \chi_{G^0} = 0$ in $L^p(\nu^{-1})$. Thus $f(\gamma) = 0$ for $\nu^{-1}$-almost every $\gamma \in G$ and hence also for $\nu$-almost every $\gamma \in G$. By continuity of $f$, this implies that $f$ vanishes on the support of $\nu$. $\square$

**Definition XX.3.29.** Let us say that a family $\mathcal{M}$ of quasi-invariant probability measures on $G^0$ *separates points*, if for every nonzero function $f \in C_c(G)$, there is a measure $\mu \in \mathcal{M}$ such that $f$ does not vanish on the support of the integrated measure $\nu = \int c_{xG}\, d\mu(x)$. Similarly, a collection $\mathcal{R}$ of representations of $C_c(G)$ on Banach algebras is said to *separate points* if for every nonzero function $f \in C_c(G)$, there is a representation $\pi \in \mathcal{R}$ such that $\pi(f)$ is nonzero.

By Lemma XX.3.28, a family $\mathcal{M}$ of Borel probability measures on $G^0$ separates points if and only if the collection of left regular representations associated with elements of $\mathcal{M}$ separates points.

**Proposition XX.3.30.** The family of left regular representations associated with quasi-invariant Borel probability measures on $G^0$ separates points.

*Proof.* A quasi-invariant Borel probability measure is said to be transitive if it is supported by an orbit. Every orbit carries a transitive measure, which is unique up to equivalence; see [225, Definition 3.9 of Chapter 1 ]. It is well known that the transitive measures constitute a collection of quasi-invariant Borel probability measures on $G^0$ that separates points; see [225, Proposition 1.11 of Chapter 2], so the proof is complete. $\square$

## Representations of Inverse Semigroups on $L^p$-spaces

### *The Banach-Lamperti theorem*

Let $\mu$ and $\nu$ be Borel probability measures on standard Borel spaces $X$ and $Y$, and let $p \in [1, \infty)$. For $f \in L^p(\mu)$, the *support* of $f$, denoted $\mathrm{supp}(f)$, is the largest element $F$ of $\mathcal{B}_\mu$ such that $f\chi_F = f$. (See Subsection XX.1 for the definition of the Boolean algebra $\mathcal{B}_\mu$.) The completeness of $\mathcal{B}_\mu$ implies that such a largest element exists.

**Lemma XX.4.1** (Lamperti-Clarkson; see [72, Proposition 3.2.2])**.** Adopt the notation of the comments above, and suppose that $p \neq 2$. If $f, g \in L^p(\mu)$ satisfy

$$\|f + g\|^p + \|f - g\|^p = 2\|f\|^p + 2\,\|g\|^p\,,$$

then the supports of $f$ and $g$ are disjoint elements of $\mathcal{B}_\mu$.

**Theorem XX.4.2** (Banach-Lamperti)**.** Let $p \in [1, \infty)\backslash\{2\}$. If $u\colon L^p(\mu) \to L^p(\nu)$ is a surjective linear isometry, then there are conull Borel subsets $X_0$ and $Y_0$ of $X$ and $Y$, a Borel isomorphism $\phi\colon X_0 \to Y_0$ such that $\phi_*(\mu)|_{X_0} \sim \nu|_{Y_0}$, and a Borel function $g\colon Y \to \mathbb{C}$ with $|g(y)|^p = \frac{d\phi_*(\mu)}{d\nu}(y)$ for $\nu$-almost every $y \in Y$, such that

$$u\xi = g \cdot \left(\xi \circ \phi^{-1}\right)$$

for every $\xi \in L^p(\nu)$.

*Proof.* Define maps $\Psi\colon \mathcal{B}_\mu \to \mathcal{B}_\nu$ and $\Phi\colon \mathcal{B}_\nu \to \mathcal{B}_\mu$ by

$$\Psi(F) = \mathrm{supp}(u^{-1}\chi_F) \quad \text{and} \quad \Phi(E) = \mathrm{supp}(u\chi_E)$$

for all $F \in \mathcal{B}_\mu$ and all $E \in \mathcal{B}_\nu$. It follows from Lemma XX.4.1 that $\Phi$ and $\Psi$ are mutually inverse $\sigma$-complete Boolean algebra homomorphisms. By [147, Theorem 15.10] there are conull subsets $X_0$ and $Y_0$ of $X$ and $Y$ respectively, and a Borel isomorphism $\phi\colon X_0 \to Y_0$ such that $\Phi(E) = \phi^{-1}(E \cap Y_0)$ for every $E \in \mathcal{B}_\nu$. It follows that $\phi_*(\mu|_{X_0}) \sim \nu|_{Y_0}$. Set $g = u1_X$, and observe that

$$u\chi_F = g \cdot \chi_{\phi(F)} = g \cdot \left(\chi_F \circ \phi^{-1}\right)$$

for every $F \in \mathcal{B}_\mu$. Thus $u\xi = g \cdot (\xi \circ \phi^{-1})$ for every $\xi \in L^p(\mu)$. We conclude that $|g(y)|^p = \frac{d\phi_*(\mu)}{d\nu}(y)$ for $\nu$-almost every $y \in Y$, and this finishes the proof. $\square$

*Hermitian idempotents and spatial partial isometries*

Let $X$ be a complex vector space. The following definition is taken from [175].

**Definition XX.4.3.** A *semi-inner product* on $X$ is a function $[\cdot,\cdot]\colon X \times X \to \mathbb{C}$ satisfying:

1. $[\cdot,\cdot]$ is linear in the first variable;
2. $[x, \lambda y] = \overline{\lambda}\,[x, y]$ for every $\lambda \in \mathbb{C}$ and $x, y \in X$;
3. $[x, x] \geq 0$ for every $x \in X$, and equality holds if and only if $x = 0$;
4. $|[x, y]| \leq [x, x]\,[y, y]$ for every $x, y \in X$.

The *norm* on $X$ associated with the semi-inner product $[\cdot,\cdot]$ is defined by $\|x\| = [\cdot,\cdot]^{\frac{1}{2}}$ for $x \in X$.

In general, there might be different semi-inner products on $X$ that induce the same norm. Nonetheless, it is not difficult to see that on a smooth Banach space—and in particular on $L^p$-spaces—there is at most one semi-inner product compatible with its norm; see the remark after the proof of Theorem 3 in [175].

**Definition XX.4.4.** A semi-inner product on a Banach space that induces its norm is called *compatible*. A Banach space $X$ endowed with a compatible semi-inner product is called a *semi-inner product space*.

By the above discussion, if $X$ is a smooth Banach space, then a compatible semi-inner product—when it exists—is unique.

**Remark XX.4.5.** It is easy to verify that the norm of $L^p(\lambda)$ is induced by the semi-inner product

$$[f,g] = \|g\|_p^{2-p} \int f(x)\overline{g(x)}\,|g(x)|^{p-2}\, d\lambda(x)$$

for $f, g \in L^p(\lambda)$ with $g \neq 0$.

An *inner product* on $X$ is precisely a semi-inner product such that moreover $[x,y] = \overline{[y,x]}$ for every $x, y \in X$. Semi-inner products allow one to generalize notions for operators on Hilbert spaces to more general Banach spaces.

**Definition XX.4.6.** Let $X$ be a semi-inner product space, and let $T \in B(X)$. The *numerical range* $W(T)$ of $T$, is the set

$$\{[Tx,x] \colon x \in X, [x,x] = 1\} \subseteq \mathbb{C}.$$

The operator $T$ is called *hermitian* if $W(T) \subseteq \mathbb{R}$.

Adopt the notation and terminology from the definition above. In view of [175] the following statements are equivalent:

1. $T$ is hermitian;
2. $\|1 + irT\| \leq 1 + o\,(r)$ for $r \to 0$;
3. $\|\exp\,(irT)\| = 1$ for all $r \in \mathbb{R}$.

It is clear that when $X$ is a Hilbert space, an operator is hermitian if and only if it is self-adjoint. In particular, the hermitian idempotents on a Hilbert space are exactly the orthogonal projections.

Let $\lambda$ be a Borel measure on a standard Borel space $Z$. Hermitian idempotents on $L^p(\lambda)$, for $p \neq 2$, have been characterized by Banach in [6]: these are precisely the multiplication operators associated with characteristic functions on Borel subsets of $Z$.

Recall that a bounded linear operator $s$ on a Hilbert space is a *partial isometry* if there is another bounded linear operator $t$ such that $st$ and $ts$ are orthogonal projections. The following is a generalization of partial isometries on Hilbert spaces to $L^p$-spaces. We use the term 'spatial' in accordance to the terminology in [204, 209, 208, 207].

**Definition XX.4.7.** Let $X$ be a Banach space and $s \in B(X)$. We say that $s$ is a *partial isometry* if $\|s\| \leq 1$ and there exists $t \in B(X)$ such that $\|t\| \leq 1$ and $st$ and $ts$ are idempotent.

**Definition XX.4.8.** Let $X$ be a semi-inner product space and $s \in B(X)$. We say that $s$ is a *spatial partial isometry* if $\|s\| \leq 1$ and there exists $t \in B(X)$ such that $\|t\| \leq 1$ and $st$ and $ts$ are hermitian idempotents.

Following [204], we call an element $t$ as in Definition XX.4.8 a *reverse* of $s$. (It is in general not unique.) We call $ts$ and $st$ the *source* and *range* idempotents of $s$, respectively. We denote by $\mathcal{S}(B(X))$ the set of all spatial partial isometries in $R$, and by $\mathcal{E}(B(X))$ the set of hermitian idempotents in $R$.

It is a standard fact in Hilbert space theory that all partial isometries on a Hilbert space are spatial. Moreover, the reverse of a partial isometry on a Hilbert space is unique, and it is given by its adjoint. The situation for partial isometries on $L^p$-spaces, for $p \neq 2$, is rather different. The following proposition can be taken as a justification for the term "spatial".

**Proposition XX.4.9.** Let $p \in (1, \infty) \setminus \{2\}$ and let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space $Z$. If $s$ is a spatial partial isometry on $L^p(\lambda)$, then there are Borel subsets $E$ and $F$ of $Z$, a Borel isomorphism $\phi\colon E \to F$, and a Borel function $g\colon F \to \mathbb{C}$ such that

$$(s\xi)(y) = \begin{cases} g(y) \cdot (\xi \circ \phi^{-1})(y) & \text{if } y \in F, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

for all $\xi$ in $L^p(\lambda)$ and for $\lambda$-almost every $y \in Z$.

Moreover, if $e$ is a hermitian idempotent in $L^p(\lambda)$, then there is a Borel subset $E$ of $Z$ such that $e = \Delta_{\chi_E}$.

*Proof.* The result follows from the characterization of hermitian idempotents mentioned above, together with Theorem XX.4.2. □

**Remark XX.4.10.** Adopt the notation of the above proposition. It is easy to check that the reverse of $s$ is also spatial, and that it is given by

$$(t\xi)(y) = \begin{cases} \overline{(g\circ\phi)(y)}\cdot(\xi\circ\phi)(y) & \text{if } y \in E, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

for all $\xi$ in $L^p(\lambda)$ and for $\lambda$-almost every $y \in Z$. In particular, the reverse of a spatial partial isometry of an $L^p$-space is unique. We will consequently write $s^*$ for the reverse of a spatial partial isometry $s$.

The set $\mathcal{S}(L^p(\lambda))$ of spatial partial isometries on $L^p(\lambda)$ is an inverse semigroup, and the set $\mathcal{E}(L^p(\lambda))$ of hermitian idempotents on $L^p(\lambda)$ is precisely the semilattice of idempotent elements of $\mathcal{S}(L^p(\lambda))$. Moreover, the map $\mathcal{B}_\lambda \to \mathcal{E}(L^p(\lambda))$ given by $F \mapsto \Delta_{\chi_F}$ is an isomorphism of semilattices. Thus, $\mathcal{E}(L^p(\lambda))$ is a complete Boolean algebra.

**Remark XX.4.11.** If $(e_j)_{j\in I}$ is an increasing net of hermitian idempotents, then $\sup\limits_{j\in I} e_j$ is the limit of the sequence $(e_j)_{j\in I}$ in the strong operator topology.

*Representations of inverse semigroups*

We now turn to inverse semigroup representations on $L^p$-spaces by spatial partial isometries. Fix an inverse semigroup $\Sigma$, and recall that $M_n(\Sigma)$ has a natural structure of inverse semigroup for every $n \geq 1$ by [196, Proposition 2.1.4].

**Definition XX.4.12.** Let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space. A *representation* of $\Sigma$ on $L^p(\lambda)$ is a semigroup homomorphism $\rho\colon \Sigma \to \mathcal{S}(L^p(\lambda))$.

For $n \geq 1$ denote by $\lambda^{(n)}$ the measure $\lambda \times c_n$, where $c_n$ is the counting measure on $n$. We define the *amplification* $\rho^{(n)}\colon M_n(\Sigma) \to \mathcal{S}(L^p(\lambda^{(n)}))$ of $\rho$, by $\rho^n([\sigma_{ij}]_{i,j\in n}) = [\rho(\sigma_{ij})]_{i,j\in n}$, where we identify $B(L^p(\lambda^{(n)}))$ with $M_n(B(L^p(\lambda)))$ in the usual way.

The *dual* of $\rho$ is the representation $\rho'\colon \Sigma \to \mathcal{S}(L^{p'}(\lambda))$ given by $\rho'(\sigma) = \rho(\sigma^*)'$ for $\sigma \in \Sigma$.

**Definition XX.4.13.** Denote by $\mathbb{C}\Sigma$ complex *-algebra of formal linear combinations of elements of $\Sigma$, with operations determined by $\delta_\sigma\delta_\tau = \delta_{\sigma\tau}$ and $\delta_\sigma^* = \delta_{\sigma^*}$ for all $\sigma, \tau \in \Sigma$, and endowed with the $\ell^1$-norm. The canonical identification of $\mathbb{C}M_n(\Sigma)$ with $M_n(\mathbb{C}\Sigma)$ for $n \geq 1$, defines matrix norms on $\mathbb{C}\Sigma$.

**Remark XX.4.14.** Every representation $\rho\colon \Sigma \to \mathcal{S}(L^p(\lambda))$ induces a contractive representation $\pi_\rho\colon \mathbb{C}\Sigma \to B(L^p(\lambda))$ such that $\pi_\rho(\delta_\sigma) = \rho(\sigma)$ for $\sigma \in \Sigma$. It is not difficult to verify the following facts:

1. Since, for $n \geq 1$, the amplification $\pi_\rho^{(n)}$ of $\pi_\rho$ to $M_n(\mathbb{C}\Sigma)$ is the representation associated with the amplification $\rho^{(n)}$ of $\rho$, it follows that $\pi_\rho$ is $p$-completely contractive.

2. The representation $\pi_{\rho'}$ associated with the dual $\rho'$ of $\rho$ is the dual of the representation $\pi_\rho$ associated with $\rho$.

**Definition XX.4.15.** Let $\lambda$ and $\mu$ be $\sigma$-finite Borel measure on standard Borel spaces, and let $\rho$ and $\kappa$ be representations of $\Sigma$ on $L^p(\lambda)$ and $L^p(\mu)$ respectively. We say that $\rho$ and $\kappa$ are *equivalent* if there is a surjective linear isometry $u\colon L^p(\lambda) \to L^p(\mu)$ such that $u\rho(\sigma) = \kappa(\sigma)u$ for every $\sigma \in \Sigma$.

Adopt the notation of the definition above. If $\rho$ and $\kappa$ are equivalent, then their dual representations $\rho'$ and $\kappa'$ are also equivalent. Similarly, if $\rho$ and $\kappa$ are equivalent, then the corresponding representations $\pi_\rho$ and $\pi_\kappa$ of $\mathbb{C}\Sigma$ are equivalent.

*Tight representations of semilattices*

In the following, all semilattices will be assumed to have a minimum element 0. Consistently, all inverse semigroups will be assumed to have a neutral element 0, which is the minimum of the associated idempotent semilattice. In the rest of this subsection we recall some definitions from Section 11 of [66].

**Definition XX.4.16.** Let $E$ be a semilattice and let $\mathcal{B} = (\mathcal{B}, 0, 1, \wedge, \vee, \neg)$ be a Boolean algebra. A *representation* of $E$ on $\mathcal{B}$ is a semilattice morphism $E \to (\mathcal{B}, \wedge)$ satisfying $\beta(0) = 0$.

Two elements $x, y$ of $E$ are said to be *orthogonal*, written $x \perp y$, if $x \wedge y = 0$. Furthermore, we say that $x$ and $y$ *intersect (each other)* if they are not orthogonal.

If $X \subseteq Y \subseteq E$, then $X$ is a *cover* for $Y$ if every nonzero element of $Y$ intersects an element of $X$.

It is easy to verify that a representation of a semilattice $E$ on a Boolean algebra sends orthogonal elements to orthogonal elements. It is also immediate to check that a cover for the set of predecessors of some $x \in E$ is also a cover for $\{x\}$.

**Notation XX.4.17.** If $X$ and $Y$ are (possibly empty) subsets of $E$, we denote by $E^{X,Y}$ the set

$$E^{X,Y} = \{z \in E \colon z \leq x \ \text{ for all } x \in X, \text{ and } z \perp y \ \text{ for all } y \in Y\}.$$

We are now ready to state the definition of tight representation of a semilattice.

**Definition XX.4.18.** Let $E$ be a semilattice and let $\mathcal{B}$ be a Boolean algebra. A representation $\beta \colon E \to \mathcal{B}$ is said to be *tight* if for every pair $X, Y$ of (possibly empty) finite subsets of $E$ and every finite cover $Z$ of $E^{X,Y}$, we have

$$\bigvee_{z \in Z} \beta(z) = \bigwedge_{x \in X} \beta(x) \wedge \bigwedge_{y \in Y} \neg\beta(y). \tag{XX.4}$$

**Lemma XX.4.19.** Let $E$ be a semilattice, let $\mathcal{B}$ be a Boolean algebra and let $\beta \colon E \to \mathcal{B}$ be a tight representation. If $z_0, \ldots, z_{n-1}$ are elements of $E$ such that for every $w \in E$, there exists $j \in n$ such that $z_j \wedge w \neq 0$, then $\bigvee_{j \in n} \beta(z_j) = 1$. In particular, if $E$ has a largest element 1, then $\beta(1) = 1$.

*Proof.* The result is immediate since the assumptions imply that $\{z_0, \ldots, z_{n-1}\}$ is a cover of $E^{\varnothing,\varnothing}$. $\square$

**Lemma XX.4.20** ([66, Proposition 11.9])**.** Suppose that $E$ is a Boolean algebra. If $\beta$ is a representation of $E$ on $\mathcal{B}$, then $\beta$ is tight if and only if $\beta$ is a Boolean algebra homomorphism.

**Definition XX.4.21.** Suppose that $E$ is a semilattice. A subsemilattice $F$ of $E$ is *dense* if for every $x \in E$ nonzero there is $y \in F$ nonzero such that $y \leq x$.

**Lemma XX.4.22.** Suppose that $E$ is a semilattice, and $F$ is a dense subsemilattice of $E$. If $\beta$ is a tight representation of $E$ on $\mathcal{B}$, then the restriction of $\beta$ to $F$ is a tight representation of $F$.

*Proof.* Suppose that $X, Y \subset F$ and that $Z$ is a cover for $F^{X,Y}$. We claim that $Z$ is a cover for $E^{X,Y}$. Pick $x \in E^{X,Y}$ nonzero. Then there is $y \in F$ nonzero such that $y \leq x$. Since $y \in F^{X,Y}$ and $Z$ is a cover for $F^{X,Y}$, there is $z \in Z$ such that $z$ and $y$ intersect. Therefore also $z$ and $x$ intersect. This shows that $Z$ is a cover for $E^{X,Y}$. Therefore Equation (XX.4) holds. This concludes the proof that the restriction of $\beta$ to $F$ is tight. $\square$

*Tight representations of inverse semigroups on $L^p$-spaces*

As in the case of representation of inverse semigroups on Hilbert spaces (see [66, Section 13]), we will isolate a class of "well behaved" representations of inverse semigroups on $L^p$-spaces. The following definition is a natural generalization of [66, Definition 13].

**Definition XX.4.23.** Let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space. A representation $\rho\colon \Sigma \to \mathcal{S}(L^p(\lambda))$ is said to be *tight* if its restriction to the idempotent semilattice $E(\Sigma)$ of $\Sigma$ is a tight representation on the Boolean algebra $\mathcal{E}(L^p(\lambda))$ of hermitian idempotents.

**Remark XX.4.24.** If $\rho\colon \Sigma \to \mathcal{S}(L^p(\lambda))$ is a tight representation as above, then the net $(\rho(\sigma))_{\sigma\in E(\Sigma)}$ converges to the identity in the strong operator topology. Thus, tightness should be thought of as a *nondegeneracy* condition for representations of inverse semigroups

**Definition XX.4.25.** A tight representation $\rho$ of $\Sigma$ on $L^p(\lambda)$ is said to be *regular* if, for every idempotent open slice $U$ of $G$, $\rho\left(U\right)$ is the limit of the net $\left(\rho\left(V\right)\right)_V$ where $V$ ranges among all idempotent open slices of $G$ with compact closure contained in $U$, ordered by inclusion. In formulas

$$\rho(U) = \lim_{V\in E(\Sigma_c(G)),\overline{V}\subseteq U} \rho(V). \tag{XX.5}$$

*Representations of semigroups of slices*

Let $G$ be an etale groupoid, let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space, and let $\pi$ be a contractive nondegenerate representation of $C_c(G)$ on $L^p(\lambda)$. Denote by $\Sigma_c(G)$ the inverse semigroup of precompact open slices of $G$. In this subsection, we show how to associate to $\pi$ a tight, regular representation of $\rho_\pi$ of $\Sigma_c(G)$ on $L^p(\lambda)$.

Given a precompact open slice $A$ of $G$, $\xi \in L^p(\lambda)$, and $\eta \in L^{p'}(\lambda)$, the assignment $f \mapsto \langle\pi(f)\xi,\eta\rangle$ is a $\|\cdot\|_\infty$-continuous linear functional on $C_c(A)$ of norm at most $\|\xi\|\|\eta\|$. By the Riesz-Markov-Kakutani representation theorem, there is a Borel measure $\mu_{A,\xi,\eta}$ supported on $A$, of total mass at most $\|\xi\|\|\eta\|$, such that

$$\langle\pi(f)\xi,\eta\rangle = \int f \; d\mu_{A,\xi,\eta} \tag{XX.6}$$

for every $f \in C_c(G)$. If $A, B \in \Sigma_c(G)$, then $\mu_{A,\xi,\eta}$ and $\mu_{B,\xi,\eta}$ coincide on $A\cap B$. Arguing as in [196, page 87, and pages 98-99], we conclude that there is a Borel measure $\mu_{\xi,\eta}$ defined on all of

$G$, such that $\mu_{A,\xi,\eta}$ is the restriction of $\mu_{\xi,\eta}$ to $A$, for every $A \in \Sigma_c(G)$, and moreover $\langle \pi(f)\xi, \eta\rangle = \int f \, d\mu_{\xi,\eta}$ for every $f \in C_c(G)$.

**Lemma XX.4.26.** The linear span of $\{\pi(\chi_A)\xi \colon A \in \Sigma_c(G), \xi \in L^p(\lambda)\}$ is dense in $L^p(\lambda)$.

*Proof.* Let $\eta \in L^{p'}(\lambda)$ satisfy $\langle \rho_\pi(A)\xi, \eta\rangle = 0$ for every $\xi \in L^p(\lambda)$ and every $A \in \Sigma_c(G)$. Since $\langle \rho_\pi(A)\xi, \eta\rangle = \int \chi_A \, d\mu_{\xi,\eta}$, we conclude that $\mu_{\xi,\eta}(A) = 0$ for every $\xi \in L^p(\lambda)$ and every $A \in \Sigma_c(G)$. Thus $\langle \pi(f)\xi, \eta\rangle = 0$ for every $f \in C_c(G)$ and every $\xi \in L^p(\lambda)$. Since $\pi$ is nondegenerate, we conclude that $\eta = 0$, which finishes the proof. ☐

Equation (XX.6) allows one to extend $\pi$ to $B_c(G)$ by defining

$$\langle \pi(f)\xi, \eta\rangle = \int f d\mu_{\xi,\eta}$$

for $f \in B_c(G)$, $\xi \in L^p(\lambda)$, and $\eta \in L^p(\eta)$. Lemma 2.2.1 of [196] shows that $\pi$ is indeed a nondegenerate representation of $B_c(G)$ on $L^p(\lambda)$. In particular the function $\rho_\pi : A \mapsto \pi(\chi_A)$ is a semigroup homomorphism from $\Sigma_c(G)$ to $B\left(L^p(\lambda)\right)$. We will show below that such a function is a tight, regular representation of $\Sigma_c(G)$ on $L^p(\lambda)$

Suppose that $f \in B(G^0)$. Define $\pi(f) \in B\left(L^p(\lambda)\right)$ by

$$\langle \pi(f)\xi, \eta\rangle = \int f d\mu_{\xi,\eta} \tag{XX.7}$$

for $\xi \in L^p(\lambda)$, and $\eta \in L^{p'}(\lambda)$. Since

$$\pi\left(fg\right) = \pi(f)\pi(g) \tag{XX.8}$$

for $f, g \in B_c(G^0)$, it follows via a monotone classes argument that Equation (XX.8) holds for any $f, g \in B(G^0)$. In particular, $\pi(\chi_A)$ is an idempotent for every $A \in \mathcal{B}(G)$. It follows from Lemma XX.4.26 that $\pi\left(\chi_{G^0}\right)$ is the identity operator on $L^p(\lambda)$. Fix now $A \in \mathcal{B}_{G^0}$ and $r \in \mathbb{R}$. For any $\xi \in L^p(\lambda)$ and $\eta \in L^{p'}(\lambda)$ we have that

$$\begin{aligned} |\langle (1 + ir\pi(\chi_A))\,\xi, \eta \rangle| &= |\langle \pi\left(\chi_{G^0} + ir\chi_A\right)\xi, \eta \rangle| \\ &= \left| \int \left(\chi_{G^0} + ir\chi_A\right) d\mu_{\xi,\eta} \right| \\ &\leq \|\chi_{G^0} + ir\chi_A\|_\infty \|\xi\| \, \|\eta\| \\ &\leq \left(1 + \frac{1}{2}r^2\right) \|\xi\| \, \|\eta\| \,. \end{aligned}$$

Therefore

$$\|1 + ir\pi(\chi_A)\| \leq 1 + \frac{1}{2}r^2.$$

This shows that $\pi(\chi_A)$ is an hermitian idempotent of $L^p(\lambda)$.

It follows from Equation (XX.7) and Equation (XX.8) that the function $A \mapsto \pi(\chi_A)$ is a $\sigma$-complete homomorphism of Boolean algebras from $\mathcal{B}(G^0)$ to $\mathcal{E}\left(L^p(\lambda)\right)$. Therefore by Lemma XX.4.20 and Lemma XX.4.22 the function $\rho_\pi : A \mapsto \pi(\chi_A)$ for $A \in \Sigma_c(G)$ is a tight, regular representation of $\Sigma_c(G)$ on $L^p(\lambda)$.

The same argument shows that if $\Sigma$ is an inverse subsemigroup of $\Sigma_c(G)$ which is a basis for the topology of $G$, then the restriction of $\rho_\pi$ to $\Sigma$ is a tight, regular representation of $\Sigma$ on $L^p(\lambda)$.

**Remark XX.4.27.** It is clear that if $\pi$ and $\widetilde{\pi}$ are $I$-norm contractive nondegenerate representations of $C_c(G)$ on $L^p$-spaces, then $\pi$ and $\widetilde{\pi}$ are equivalent if and only if $\rho_\pi$ and $\rho_{\widetilde{\pi}}$ are equivalent. The easy details are left to the reader.

## Disintegration of Representations

Throughout this section, we let $G$ be a locally compact groupoid and $\Sigma$ be an inverse subsemigroup of $\Sigma_c(G)$ that generates the topology of $G$. Denote by $\Sigma_c$ the inverse semigroup of precompact elements of $\Sigma$. Let $\lambda$ be a $\sigma$-finite measure on a standard Borel space $Z$.

### *The disintegration theorem*

**Theorem XX.5.1.** If $\rho\colon G \to \mathcal{S}(L^p(\lambda))$ is a tight, regular representation, then there are a quasi-invariant measure $\mu$ on $G^0$, and, with $\lambda = \int \lambda_x \, d\mu(x)$ denoting the disintegration of $\lambda$ with respect to $\mu$, a representation $T$ of $G$ on the Borel Banach bundle $\bigsqcup_{x\in G^0} L^p(\lambda_x)$, such that

$$\langle \rho(A)\xi, \eta\rangle = \int_A D(\gamma)^{-\frac{1}{p}} \langle T_\gamma \xi_{s(\gamma)}, \eta_{r(\gamma)}\rangle \, d\nu(\gamma)$$

for $A \in \Sigma$, for $\xi \in L^p(\lambda)$, and for $\eta \in L^{p'}(\lambda)$.

The rest of this section is dedicated to the proof of the theorem above. For simplicity and without loss of generality, we will focus on the case where $\lambda$ is a probability measure. In the following, we fix a representation $\rho$ as in the statement of Theorem XX.5.1.

*Fibration*

Define $\Phi\colon E(\Sigma) \to \mathcal{B}_\lambda$ by $\Delta_{\chi_{\Phi(U)}} = \rho(U)$ for $U \in E(\Sigma)$. Denote by $\mathcal{U}$ the semilattice of open subsets of $G^0$. Extend $\Phi$ to a function $\mathcal{U} \to \mathcal{B}_\lambda$ by setting

$$\Phi(V) = \bigcup_{W\in E(\Sigma_c), \overline{W}\subseteq U} \Phi(W).$$

Then $\Delta_{\chi_{\Phi(V)}}$ is the limit in the strong operator topology of the increasing net $\left(\Delta_{\chi_{\rho(W)}}\right)_{W\in E(\Sigma_c),\overline{W}\subseteq V}$. By Equation (XX.5), the expression above indeed defines an extension of $\Phi$. Moreover, a monotone classes argument shows that $\Phi$ is a representation. Tightness of $\rho$ together with Equation (XX.5) further imply that $\Phi(U \cup V) = \Phi(U) \cup \Phi(V)$ whenever $U$ and $V$ are disjoint, and that

$$\Phi\left(\bigcup_{n\in\omega} U_n\right) \subseteq \bigcup_{n\in\omega} \Phi\left(U_n\right) \tag{XX.9}$$

for any sequence $(U_n)_{n\in\omega}$ in $\mathcal{U}$. For $U \in \mathcal{U}$, set $m(U) = \lambda(\Phi(U))$. Using [196, Proposition 3.2.7], one can extend $m$ to a Borel measure on $G^0$ by setting

$$m(E) = \inf\left\{m(U) \colon U \in \mathcal{U},\, E \subseteq U\right\}$$

for $E \in \mathcal{B}_{G^0}$. Extend $\Phi$ to a homomorphism from $\mathcal{B}_{G^0}$ to $\mathcal{B}_\lambda$, by setting

$$\Phi(E) = \bigwedge \left\{\Phi(U) \colon U \in \mathcal{U},\, U \supseteq E\right\}.$$

(The infimum exists by completeness of $\mathcal{B}_\lambda$.)

**Lemma XX.5.2.** The map $\Phi\colon \mathcal{B}_{G^0} \to \mathcal{B}_\lambda$ is a $\sigma$-complete Boolean algebra homomorphism.

*Proof.* We claim that given $E_0$ and $E_1$ in $\mathcal{B}_{G^0}$, we have $\Phi(E_0 \cap E_1) = \Phi(E_0) \cap \Phi(E_1)$.

To prove the claim, observe that if $U_j$ is an open set containing $E_j$ for $j \in \{0,1\}$, then $\Phi(U_0 \cap U_1) = \Phi(U_0) \cap \Phi(U_1)$, and thus $\Phi(E_0 \cap E_1) \subseteq \Phi(E_0) \cap \Phi(E_1)$. In order to prove that equality holds, it is enough to show that given $\varepsilon > 0$, we have

$$\lambda\left(\Phi(E_0 \cap E_1)\right) \geq \lambda\left(\Phi\left(E_0\right) \cap \Phi\left(E_1\right)\right) - \varepsilon.$$

Fix an open set $U$ containing $E_0 \cap E_1$ such that $m(U) \leq m(E_0 \cap E_1) + \varepsilon$. Let $V_0$ and $V_1$ be open sets satisfying $E_j \backslash (E_0 \cap E_1) \subseteq V_j$ for $j = 0, 1$, and

$$\mu(V_j) \leq \mu(E_j \backslash (E_0 \cap E_1)) + \varepsilon.$$

For $j = 0, 1$, set $U_j = U \cup V_j$. Then $U_j \supseteq E_j$ and

$$\begin{aligned}
\lambda(\Phi(E_0 \cap E_1)) &= m(E_0 \cap E_1) \geq m(U) - \varepsilon \geq m(U_0 \cap U_1) - 3\varepsilon \\
&= \lambda(\Phi(U_0 \cap U_1)) - 3\varepsilon = \lambda(\Phi(U_0) \cap \Phi(U_1)) - 3\varepsilon \\
&\geq \lambda(\Phi(E_0) \cap \Phi(E_1)) - 3\varepsilon.
\end{aligned}$$

We have therefore shown that $\Phi(E_0 \cap E_1) = \Phi(E_0) \cap \Phi(E_1)$, so the claim is proved.

It remains to show that if $(E_n)_{n\in\omega}$ is a sequence of pairwise disjoint Borel subsets of $G^0$, then

$$\Phi\left(\bigcup_{n\in\omega} E_n\right) = \bigcup_{n\in\omega} \Phi\left(E_n\right).$$

By Equation (XX.9), the left-hand side is contained in the right-hand side. On the other hand, we have

$$\begin{aligned}
\lambda\left(\Phi\left(\bigcup_{n\in\omega} E_n\right)\right) &= m\left(\bigcup_{n\in\omega} E_n\right) = \sum_{n\in\omega} m(E_n) \\
&= \sum_{n\in\omega} \lambda(\Phi(E_n)) = \lambda\left(\bigcup_{n\in\omega} \Phi(E_n)\right),
\end{aligned}$$

so we conclude that equality holds, and the proof is complete. $\square$

By [147, Theorem 15.9], there is a Borel function $q\colon Z \to G^0$ such that $\Phi(E) = q^{-1}(E)$ for every $E \in \mathcal{B}_{G^0}$. Moreover, the map $q$ is unique up to $\lambda$-almost everywhere equality.

*Measure*

Define a Borel probability measure $\mu$ on $G^0$ by $\mu = q_*(\lambda)$. Consider the disintegration $\lambda = \int \lambda_x \, d\mu(x)$ of $\lambda$ with respect to $\mu$, the Borel Banach bundle $\mathcal{Z} = \bigsqcup\limits_{x \in G^0} L^p(\lambda_x)$, and identify $L^p(\lambda)$ with $L^p(\mu, \mathcal{Z})$ as in Theorem XX.2.11.

For $A \in \Sigma$, denote by $\theta_A\colon A^{-1}A \to AA^{-1}$ the homomorphism defined by $\theta_A(x) = r(Ax)$ for $x \in A^{-1}A$. Since $\rho(A)$ is a spatial partial isometry with domain $\Phi(s(A))$ and range $\Phi(r(A))$, there are a Borel function $g_A\colon \Phi(r(A)) \to \mathbb{C}$ and a Borel isomorphism $\phi_A\colon \Phi(s(A)) \to \Phi(r(A))$ such that

$$(\rho(A)\xi)_z = g_A(z)\xi(\phi_A^{-1}(z)) \tag{XX.10}$$

for $z \in \Phi(r(A))$. We claim that $(q \circ \phi_A)(z) = (\theta_A \circ q)(z)$ for $\lambda$-almost every $z \in \Phi(r(A))$.

By the uniqueness assertion in [147, Theorem 15.9], it is enough to show that $(\theta_A \circ q \circ \phi_A^{-1})^{-1}(U) = \Phi(U)$ for every $U \in E(\Sigma)$ with $U \subseteq r(A)$. We have

$$(\theta_A \circ q \circ \phi_A^{-1})^{-1}(U) = (\phi_A \circ q^{-1} \circ \theta_A^{-1})(U) = \phi_A(\Phi(\theta_A^{-1}(U))) = \phi_A(\Phi(A^{-1}UA)).$$

Given $\xi \in L^p(\lambda|_{\Phi(r(A))})$, set $\eta = \xi \circ \phi_A$. Then

$$\begin{aligned}
\Delta_{\chi_{\phi_A(\Phi(A^{-1}UA))}}\xi &= \left(\Delta_{\chi_{\Phi(A^{-1}UA)}}\eta\right) \circ \phi_A^{-1} = \left(\rho\left(A^{-1}UA\right)\eta\right) \circ \phi_A^{-1} \\
&= \left(\rho(A)^{-1}\rho(U)\rho(A)\eta\right) \circ \phi_A^{-1} = \left(\rho(A)^{-1}\rho(U)g_A\xi\right) \circ \phi_A^{-1} \\
&= \left(\rho(A)^{-1}\chi_{\Phi(U)}g_A\xi\right) \circ \phi_A^{-1} \\
&= \left((g_A \circ \phi_A)^{-1}\left(\chi_{\Phi(U)} \circ \phi_A\right)(g_A \circ \phi_A)\eta\right) \circ \phi_A^{-1} \\
&= \chi_{\Phi(U)}\xi = \Delta_{\chi_{\Phi(U)}}\xi.
\end{aligned}$$

Thus $\Phi(U) = \phi_A(\Phi(A^{-1}UA)) = (\theta_A \circ q \circ \phi_A^{-1})^{-1}(U)$, and hence $(\theta_A \circ q \circ \phi_A^{-1})(z) = q(z)$ for $\lambda$-almost every $z \in \Phi(r(A))$, as desired. The claim is proved.

It is shown in [196, Proposition 3.2.2] that $\mu$ is quasi-invariant whenever $(\theta_A)_*\mu|_{s(A)} \sim \mu|_{r(A)}$ for every open slice $A$ of $G$. The same proof in fact shows that it is sufficient to check this condition for $A \in \Sigma$. Given $A \in \Sigma$, we have

$$\begin{aligned}\mu|_{r(A)} &= q_*\lambda|_{\Phi(r(A))} \sim q_*((\phi_A)_*\lambda|_{\Phi(s(A))}) = (q \circ \phi_A)_*\lambda|_{\Phi(s(A))} = (\theta_A \circ q)_*\lambda|_{\Phi(s(A))} \\ &= (\theta_A)_*(q_*\lambda|_{\Phi(s(A))}) = (\theta_A)_*\mu|_{s(A)},\end{aligned}$$

so $\mu$ is quasi-invariant.

*Disintegration*

For $x \in G^0$, set $Z_x = q^{-1}(\{x\})$, and note that $Z_x = \Phi(\{x\})$. Given $A \in \Sigma$, regard $\rho(A)$ as a surjective linear isometry

$$\rho(A)\colon L^p(\lambda|_{\Phi(s(A))}) \to L^p(\lambda|_{\Phi(r(A))}).$$

Let $\mathcal{Z}$ denote the Borel Banach bundle

$$\bigsqcup_{x\in G^0} L^p(\lambda_x),$$

and identify $L^p(\lambda|_{\Phi(s(A))})$ and $L^p(\lambda|_{\Phi(r(A))})$ with $L^p(\mu|_{s(A)}, \mathcal{Z}|_{s(A)})$ and $L^p(\mu|_{r(A)}, \mathcal{Z}|_{r(A)})$, respectively.

If $U \in E(\Sigma)$ satisfies $U \subseteq r(A)$, one uses $\rho(A^{-1}UA) = \rho(A)^{-1}\rho(U)\rho(A)$ to show that

$$\Delta_U \circ \rho(A) = \rho(A) \circ \Delta_{\theta_A(U)}.$$

By Theorem XX.2.21, there is a Borel section $x \mapsto T_x^A$ of $B(\mathcal{Z}|_{s(A)}, \mathcal{Z}|_{r(A)}, \theta_A)$ consisting of invertible isometries, such that

$$(\rho(A)\xi)|_{Z_y} = \left(\frac{d(\theta_A)_*\mu}{d\mu}(y)\right)^{\frac{1}{p}} T^A_{\theta_A^{-1}(y)}\xi|_{Z_{\phi^{-1}(y)}}$$

for $\mu$-almost every $y \in r(A)$. Since

$$(\rho(A)\xi)\,|_{Z_y} = (g_A)\,|_{Z_y} \cdot \left(\xi|_{Z_y} \circ \left((\phi_A)|^{|Z_y}_{Z_{\theta_A^{-1}(y)}}\right)^{-1}\right)$$

for $\mu$-almost every $y \in r(A)$, we have

$$T_x^A \xi = \left( \frac{d(\theta_A)_* \mu}{d\mu} (\theta_A(x)) \right)^{\frac{1}{p}} (g_A)\,|_{Z_{\theta_A(x)}} \left( \xi \circ \left( (\phi_A)|_{Z_x}^{|Z_{\theta_A(x)}} \right)^{-1} \right)$$

for $\mu$-almost every $x \in s(A)$. Arguing as in the proof of [196, Theorem 3.2.1], one can see that if $A$ and $B$ are in $\Sigma$ and $U \in E(\Sigma)$, then

- $T_x^A = T_x^B$ for $\mu$-almost every $x \in s(A \cap B)$,
- $(T_x^A)^{-1} = T_{\theta_A(x)}^{A^{-1}}$ for $\mu$-almost every $x \in s(A)$, and
- $T_x^U$ is the identity operator of $L^p(\lambda_x)$ for $\mu$-almost every $x \in U$.

Moreover, up to discarding a $\nu$-null set, the assignment $T\colon G \to \mathrm{Iso}(\mathcal{Z})$ given by $T_\gamma = T_\gamma^A$ for some $A \in \Sigma$ containing $\gamma$, is well defined and determines a representation of $G$ on $\mathcal{Z}$. It is a consequence of Equation (XX.10) that

$$\langle \rho(A)\xi, \eta \rangle = \int D\,(xA)^{-\frac{1}{p}} \left\langle T_{xA} \xi_{\theta_A^{-1}(x)}, \eta_x \right\rangle \, d\mu(x),$$

for every $\xi \in L^p(\mathcal{Z}, \mu)$ and every $\eta \in L^{p'}(\mu, \mathcal{Z}')$. This concludes the proof of Theorem XX.5.1.

## $L^p$-operator Algebras of Etale Groupoids

Throughout this section, we fix a Hölder exponent $p \in (1, \infty)$.

### *$L^p$-operator algebras*

**Definition XX.6.1.** A *concrete $L^p$-operator algebra* is a subalgebra $A$ of $B(L^p(\lambda))$ for some $\sigma$-finite Borel measure $\lambda$ on a standard Borel space. The identification of $M_n(A)$ with a subalgebra of $B(L^p(\lambda^{(n)}))$ induces a norm on $M_n(A)$. The collection of such norms defines a $p$-operator space structure on $A$ as in [48, Section 4.1]. Moreover the multiplication on $A$ is a $p$-completely contractive bilinear map. (Equivalently $M_n(A)$ is a Banach algebra for every $n \in \mathbb{N}$.)

An *abstract $L^p$-operator algebra* is a Banach algebra $A$ endowed with a $p$-operator space structure, which is $p$-completely isometrically isomorphic to a concrete $L^p$-operator algebra.

Let $A$ be a separable matricially normed algebra and let $\mathcal{R}$ be a collection of $p$-completely contractive nondegenerate representations of $A$ on $L^p$-spaces. Set $I_{\mathcal{R}} = \bigcap\limits_{\pi\in\mathcal{R}} \mathrm{Ker}(\pi)$. Then $I_{\mathcal{R}}$ is an ideal in $A$. Arguing as in [17, Section 1.2.16], the completion $F^{\mathcal{R}}(A)$ of $A/I_{\mathcal{R}}$ with respect to the norm

$$\|a + I\| = \sup\{\|\pi(a)\| \colon \pi \in \mathcal{R}\}$$

for $a \in A$, is a Banach algebra. Moreover, $F^{\mathcal{R}}(A)$ has a natural $p$-operator space structure that makes it into an (abstract) $L^p$-operator algebra.

**Remark XX.6.2.** If $\mathcal{R}$ separates the points of $A$, then the ideal $I_{\mathcal{R}}$ is trivial, and hence the canonical map $A \to F^{\mathcal{R}}(A)$ is an injective $p$-completely contractive homomorphism.

**Definition XX.6.3.** Let $\mathcal{R}^p$ denote be the collection of *all* $p$-completely contractive nondegenerate representations of $A$ on $L^p$-spaces associated with $\sigma$-finite Borel measures on standard Borel spaces. Then $F^{\mathcal{R}^p}(A)$ is abbreviated to $F^p(A)$, and called the *enveloping $L^p$-operator algebra* of $A$.

Suppose further that $A$ is a matricially normed *-algebra with a completely isometric involution $a \mapsto a^*$. If $\pi \colon A \to B(L^p(\lambda))$ is a $p$-completely contractive nondegenerate representation as before, then the dual representation of $\pi$ is the $p'$-completely contractive nondegenerate representation $\pi'$ given by $\pi'(a) = \pi(a^*)'$ for all $a \in A$.

Let $\mathcal{R}$ be a collection of $p$-completely contractive nondegenerate representations of $A$ on $L^p$-spaces, and denote by $\mathcal{R}'$ the collection of duals of elements of $\mathcal{R}$. It is immediate that the involution of $A$ extends to a $p$-completely isometric anti-isomorphism $F^{\mathcal{R}}(A) \to F^{\mathcal{R}'}(A)$. Finally, since $(\mathcal{R}^p)' = \mathcal{R}^{p'}$, the discussion above shows that the involution of $A$ extends to a $p$-completely isometric anti-isomorphism $F^p(A) \to F^{p'}(A)$.

*The full $L^p$-operator algebra of an etale groupoid*

Let $G$ be an etale groupoid.

**Definition XX.6.4.** We define the *full $L^p$-operator algebra* $F^p(G)$ of $G$ to be the enveloping $L^p$-operator algebra of the matricially normed *-algebra $C_c(G)$.

**Remark XX.6.5.** By Proposition XX.3.30, the family of $p$-completely contractive nondegenerate representations of $C_c(G)$ on $L^p$-spaces separates the points of $C_c(G)$, and hence the canonical map $C_c(G) \to F^p(G)$ is injective.

The proof of the following is straightforward, and is left to the reader.

**Proposition XX.6.6.** The correspondence sending a $p$-completely contractive representation of $F^p(G)$ on an $L^p$-space to its restriction to $C_c(G)$, is a bijective correspondence between $p$-completely contractive representations of $F^p(G)$ on $L^p$-spaces and $p$-completely contractive representations of $C_c(G)$ on $L^p$-spaces.

**Definition XX.6.7.** Let $\Sigma$ be an inverse semigroup, and consider the matricially normed *-algebra $\mathbb{C}\Sigma$. Denote by $\mathcal{R}^p_{\mathrm{tight}}$ the collection of tight representations of $\Sigma$ on $L^p$-spaces. We define the *tight enveloping $L^p$-operator algebra* of $\Sigma$, denoted $F^p_{\mathrm{tight}}(\Sigma)$, to be $F^{\mathcal{R}^p_{\mathrm{tight}}}(\mathbb{C}\Sigma)$.

**Remark XX.6.8.** Since the dual of a tight representation is also tight, it follows that the involution on $\mathbb{C}\Sigma$ extends to a $p$-completely isometric anti-isomorphism $F^p_{\mathrm{tight}}(\Sigma) \to F^{p'}_{\mathrm{tight}}(\Sigma)$.

Let $\Sigma$ be an inverse semigroup of open slices of $G$ that is a basis for its topology. Let $\mathcal{Z}$ be a Borel Banach bundle over $G^0$, and let $(T, \mu)$ be a representation of $G$ on $\mathcal{Z}$. Then $T$ induces a tight representation $\rho_T \colon \Sigma \to \mathcal{S}(L^p(\mathcal{Z}))$ determined by

$$\langle \rho_T(A)\xi, \eta \rangle = \int_{r(A)} D^{-\frac{1}{p}}(xA) \left\langle T_{xA}\xi_{\theta_A^{-1}(x)}, \eta_x \right\rangle \, d\mu(x)$$

for all $A \in \Sigma$, for all $\xi \in L^p(\mathcal{Z})$ and all $\eta \in L^{p'}(\mathcal{Z}')$. We also denote by $\pi_T$ the integrated form of $T$ as in Theorem XX.3.14.

It is shown in Subsection XX.4 that a contractive representation $\pi$ of $C_c(G)$ on $L^p(\lambda)$ induces a tight regular representation $\rho_\pi$ of $\Sigma$ on $L^p(\lambda)$.

**Theorem XX.6.9.** Adopt the notation of the comments above.

1. The assignment $T \mapsto \rho_T$ determines a bijective correspondence between representations of $G$ on $L^p$-bundles and tight regular representations of $\Sigma$ on $L^p$-spaces.

2. The assignment $\pi \mapsto \rho_\pi$ determines a bijective correspondence between contractive representations of $C_c(G)$ on $L^p$-spaces and tight regular representations of $\Sigma$ on $L^p$ spaces.

3. The assignment $T \mapsto \pi_T$ assigning is a bijective correspondence between representations of $G$ on $L^p$-bundles and contractive representations of $C_c(G)$ on $L^p$-spaces.

Moreover, the correspondences in (1), (2), and (3) preserve the natural relations of equivalence of representations.

The difference between (1) and (3) above is that in (3) the representations of $\Sigma$ are not necessarily assumed to be regular. In fact this condition is trivially satisfied by any representation in the case of the inverse semigroup of compact open slices.

*Proof.* (1). This is an immediate consequence of the Disintegration Theorem XX.5.1.

(2). Suppose that $\rho$ is a tight representation of $\Sigma$ on $L^p(\lambda)$. Applying the Disintegration Theorem XX.5.1 one obtains a representation $(\mu, T)$ of $G$ on the bundle $\bigsqcup_{x \in G^0} L^p(\lambda)$ for a disintegration $\lambda = \int \lambda_x d\mu(x)$. One can then assign to $\rho$ the integrated form $\pi_\rho$ of $(\mu, T)$. It is easy to verify that the maps $\rho \mapsto \pi_\rho$ and $\pi \mapsto \rho_\pi$ are mutually inverse.

Finally (3) follows from combining (1) and (2). □

Observe that when $G$ is ample, and $\Sigma$ is the inverse semigroup of compact open slices, any tight representation of $\Sigma$ on an $L^p$-space is automatically regular.

**Corollary XX.6.10.** If $A$ is an $L^p$-operator algebra, then any contractive homomorphism from $C_c(G)$ or $F^p(G)$ to $A$ is automatically $p$-completely contractive.

*Proof.* It is enough to show that any contractive nondegenerate representation of $C_c(G)$ on an $L^p$-space is $p$-completely contractive. This follows from part (3) of Theorem XX.6.9, together with the fact that the integrated form of a representation of $G$ on an $L^p$-bundle is $p$-completely contractive, as observed in Subsection XX.3. □

**Corollary XX.6.11.** Adopt the assumptions of Theorem XX.6.9, and suppose moreover that $G$ is ample. Then the $L^p$-operator algebras $F^p(G)$ and $F^p_{\mathrm{tight}}(\Sigma)$ are $p$-completely isometrically isomorphic. In particular, $F^p(G)$ is generated by its spatial partial isometries.

*Proof.* This follows from part (2) of Theorem XX.6.9. □

Let $\mu$ be a (not necessarily quasi-invariant) Borel probability measure on $G^0$, and let $\nu$ be the measure on $G$ associated with $\mu$ as in Subsection XX.3. Denote by $\mathrm{Ind}(\mu)\colon C_c(G) \to B(L^p(\nu^{-1}))$ the left action by convolution. Then $\mathrm{Ind}(\mu)$ is contractive and nondegenerate.

**Remark XX.6.12.** When $\mu$ is quasi-invariant, the representation $\mathrm{Ind}(\mu)$ is the integrated form of the left regular representation $T^\mu$ of $G$ on $\bigsqcup_{x\in G^0} \ell^p(xG)$ as defined in Subsection XX.3.27. The same argument as in Lemma XX.3.28 shows that a function $f$ in $C_c(G)$ belongs to $\mathrm{Ker}(\mathrm{Ind}(\mu))$ if and only if $f$ vanishes on the support of $\nu$.

**Definition XX.6.13.** Define $\mathcal{R}^p_\lambda$ red to be the collection of representations $\mathrm{Ind}(\mu)$ where $\mu$ varies among the Borel probability measures on $G^0$. The *reduced $L^p$-operator algebra* $F^p_\lambda(G)$ of $G$ is the enveloping $L^p$-operator algebra $F^{\mathcal{R}^p_\lambda}(C_c(G))$. The norm on $F^p_\lambda(G)$ is denoted by $\|\cdot\|_\lambda$.

By Proposition XX.3.30, the family $\mathcal{R}^p_\lambda$ separates points, and hence the canonical map $C_c(G) \to F^p_\lambda(G)$ is injective. It follows that the identity map on $C_c(G)$ extends to a canonical $p$-completely contractive homomorphism $F^p(G) \to F^p_\lambda(G)$ with dense range.

**Remark XX.6.14.** The dual of $\mathrm{Ind}(\mu)\colon C_c(G) \to B(L^p(\nu^{-1}))$ is the representation $\mathrm{Ind}(\mu)\colon C_c(G) \to B(L^{p'}(\nu))$, and thus the involution on $C_c(G)$ extends to a $p$-completely isometric anti-isomorphism $F^p_{\mathrm{red}}(G) \to F^{p'}_\lambda(G)$.

For $x \in G^0$, we write $\delta_x$ for its associated point mass measure, and write $\mathrm{Ind}(x)$ in place of $\mathrm{Ind}(\delta_x)$. In this case, $\nu$ is the counting measure $c_{xG}$ on $xG$, and $\nu^{-1}$ is the counting measure $c_{Gx}$ on $Gx$. Moreover, $\mathrm{Ind}(x)$ is given by

$$(\mathrm{Ind}(x)f(\xi))(\rho) = \sum_{\gamma\in r(\rho)G} f(\gamma)\xi(\gamma^{-1}\rho)$$

for $f \in C_c(G)$, $\xi \in L^p(\nu^{-1})$, and $\rho \in Gx$.

**Proposition XX.6.15.** Let $\mu$ be a probability measure on $G^0$. If $f \in C_c(G)$, then

$$\|\mathrm{Ind}(\mu)f\| = \sup_{x\in\mathrm{supp}(\mu)} \|\mathrm{Ind}(x)(f)\|.$$

*Proof.* Denote by $C$ the support of $\mu$ and fix $f \in C_c(G)$. Set

$$M = \sup_{x \in \mathrm{supp}(\mu)} \|\mathrm{Ind}(x)(f)\|.$$

We will first show that $\|\mathrm{Ind}(\mu)f\| \leq M$. Given $\xi \in L^p(\nu^{-1})$ and $\eta \in L^{p'}(\nu^{-1})$ with $\|\xi\|, \|\eta\| \leq 1$, we use Hölder's inequality at the second to last step to get

$$\begin{aligned}
|\langle \mathrm{Ind}(\mu)(f)\xi, \eta\rangle| &= \left| \int \left(\mathrm{Ind}(\mu)(f)\xi\right)(\rho)\overline{\eta(\rho)}\ d\nu^{-1}(\rho) \right| \\
&= \left| \int \sum_{\gamma \in r(\rho)G} f(\gamma)\xi(\gamma^{-1}\rho)\overline{\eta(\rho)}\ d\nu^{-1}(\rho) \right| \\
&= \left| \int_C \sum_{\rho \in Gx} \sum_{\gamma \in r(\rho)G} f(\gamma)\xi(\gamma^{-1}\rho)\overline{\eta(\rho)}\ d\mu(x) \right| \\
&= \left| \int_C \langle \mathrm{Ind}(x)(f)\xi|_{Gx}, \eta|_{Gx}\rangle\ \ d\mu(x) \right| \\
&\leq \int_C |\langle \mathrm{Ind}(x)(f)\xi|_{Gx}, \eta|_{Gx}\rangle|\ \ d\mu(x) \\
&\leq M \int_C \|\xi|_{Gx}\|\ \|\eta|_{Gx}\|\ \ d\mu(x) \\
&= M \int_C \left( \sum_{\gamma \in Gx} |\xi(\gamma)|^p \right)^{\frac{1}{p}} \left( \sum_{\gamma \in Gx} |\eta(\gamma)|^{p'} \right)^{\frac{1}{p'}} \ d\mu(x) \\
&\leq M \left( \int_C \sum_{\gamma \in Gx} |\xi(\gamma)|^p\ \ d\mu(x) \right)^{\frac{1}{p}} \left( \int_C \sum_{g \in Gx} |\eta(\gamma)|^{p'} \right)^{\frac{1}{p'}} \\
&\leq M,
\end{aligned}$$

which implies that $\|\mathrm{Ind}(\mu)f\| \leq M$, as desired.

Conversely, fix $x \in C$ and let $(V_n)_{n\in\omega}$ be a decreasing sequence of open sets containing $x$ such that $\{V_n\}_{n\in\omega}$ is a basis for the neighborhoods of $x$. Then $\mu(V_n) > 0$ for all $n$ in $\omega$, since $x$ is in the support of $\mu$. For $n \in \omega$, choose a positive function $f_n \in C_c(V_n) \subseteq C_c(G)$ satisfying $f_n(x) = 1$ and $\int f_n\ d\mu = 1$.

Given $\xi \in L^p(\nu^{-1})$ and $\eta \in L^{p'}(\nu^{-1})$, set

$$\xi_n = \left( f_n^{\frac{1}{p}} \circ s \right) \xi \quad \text{and} \quad \eta_n = \left( f_n^{\frac{1}{p'}} \circ s \right) \eta.$$

Then

$$\langle \mathrm{Ind}(y) f(\xi_n), \eta_n\rangle = f_n(y) \sum_{\rho\in Gy} \sum_{\gamma\in r(\rho)G} f(\gamma)\xi(\gamma^{-1}\rho)\overline{\eta(\rho)}$$

for all $y \in G^0$ and all $n \in \omega$. Fix $\varepsilon > 0$. Since the map $y \mapsto \langle \mathrm{Ind}(y)f(\xi), \eta\rangle$ is continuous on $G^0$, there is $N \in \omega$ such that if $n \geq N$, then

$$|\langle \mathrm{Ind}(y)(f)\xi, \eta\rangle - \langle \mathrm{Ind}(x)(f)\xi, \eta\rangle| < \varepsilon$$

for every $y \in V_n$. For $n \geq N$, we have

$$\begin{aligned} & |\langle \mathrm{Ind}(\mu)(f)\xi_n, \eta_n\rangle - \langle \mathrm{Ind}(x)(f)\xi, \eta\rangle| \\ = \; & \int_{V_n} f_n(y) |\langle (\mathrm{Ind}(y)(f) - \mathrm{Ind}(x)(f))\xi, \eta\rangle| d\mu(y) < \varepsilon. \end{aligned}$$

Therefore

$$\begin{aligned} & |\langle \mathrm{Ind}(x) f(\xi), \eta\rangle| \\ & = \lim_{n\to\infty} |\langle \mathrm{Ind}(\mu) f(\xi_n), \eta_n\rangle| \leq \|\mathrm{Ind}(\mu) f\| \lim_{n\to\infty} \|\xi_n\|\|\eta_n\| \\ & = \|\mathrm{Ind}(\mu) f\| \lim_{n\to\infty} \left(\sum_{\gamma\in Gx} |\xi_n(\gamma)|^p\right)^{\frac{1}{p}} \lim_{n\to\infty} \left(\sum_{\gamma\in Gx} |\eta_n(\gamma)|^{p'}\right)^{\frac{1}{p'}} \\ & = \|\mathrm{Ind}(\mu) f\|\|\xi\|\|\eta\|. \end{aligned}$$

This concludes the proof. □

**Corollary XX.6.16.** The algebra $F^p_\lambda(G)$ of $G$ is $p$-completely isometrically isomorphic to the enveloping $L^p$-operator algebra $F^{\mathcal{R}}(C_c(G))$ with respect to the family of representations $\mathcal{R} = \{\mathrm{Ind}(x) \colon x \in G^0\}$.

*Amenable groupoids and their $L^p$-operator algebras*

There are several equivalent characterizations of amenability for etale groupoids. By [2, Theorem 2.2.13], an etale groupoid is amenable if and only if has an approximate invariant mean.

**Definition XX.6.17.** An *approximate invariant mean* on $G$ is a sequence $(f_n)_{n\in\omega}$ of positive continuous compactly supported functions on $G$ such that

1. $\sum\limits_{\gamma\in xG} f_n(\gamma) \leq 1$ for every $n \in \omega$ and every $x \in G^0$,
2. the sequence of functions $x \mapsto \sum\limits_{\gamma\in xG} f_n(\gamma)$ converges to 1 uniformly on compact subsets of $G^0$, and
3. the sequence of functions
$$\gamma \to \sum_{\rho\in r(\gamma)G} |f_n(\rho^{-1}\gamma) - f_n(\rho)|$$
converges to 0 uniformly on compact subsets of $G$.

**Lemma XX.6.18.** If $G$ is amenable and $m \geq 1$, then its amplification $G_m$ is amenable.

*Proof.* Let $(f_n)_{n\in\omega}$ be an approximate invariant mean for $G$. For $n \in \omega$, define $f_n^{(m)} \colon C_c(G_m) \to \mathbb{C}$ by $f_n^{(m)}(i,\gamma,j) = \frac{1}{m} f_n(\gamma)$ for $(i,\gamma,j) \in G_m$. It is not difficult to verify that $(f_n^{(m)})_{n\in\omega}$ is an approximate invariant mean for $G_m$. We omit the details. $\square$

**Definition XX.6.19.** A pair of sequences $(g_n)_{n\in\omega}$ and $(h_n)_{n\in\omega}$ of positive functions in $C_c(G)$ is said to be an *approximate invariant* $p$-*mean* for $G$, if they satisfy the following:

1. $\sum\limits_{\gamma\in xG} g_n(\gamma)^p \leq 1$ and $\sum\limits_{\gamma\in xG} h_n(\gamma)^{p'} \leq 1$ for every $n \in \omega$ and every $x \in G^0$,
2. the sequence of functions $x \mapsto \sum\limits_{\rho\in xG} g_n(\rho)h_n(\rho)$ converges to 1 uniformly on compact subsets of $G^0$, and
3. the sequences of functions
$$\gamma \mapsto \sum_{\rho\in r(\gamma)G} |g_n(\gamma^{-1}\rho) - g_n(\rho)|^p$$
and
$$\gamma \mapsto \sum_{\rho\in r(\gamma)G} |h_n(\gamma^{-1}\rho) - h_n(\rho)|^{p'}$$
converges to 0 uniformly on compact subsets of $G$.

It is not difficult to see that any amenable groupoid has an approximate invariant $p$-mean. Indeed, if $(f_n)_{n\in\omega}$ is any approximate invariant mean on $G$, then the sequences $(f_n^{1/p})_{n\in\omega}$ and $(f_n^{1/p'})_{n\in\omega}$ define an approximate invariant $p$-mean on $G$.

**Remark XX.6.20.** It is easy to check that if $(g_n, h_n)_{n\in\omega}$ is an approximately invariant $p$-mean on $G$, then $(h_n * g_n)_{n\in\omega}$ converges to 1 uniformly on compact subsets of $G$.

The following theorem asserts that full and reduced $L^p$-operator algebras of amenable etale groupoids are canonically isometrically isomorphic.

**Theorem XX.6.21.** Suppose that $G$ is amenable. Then the canonical homomorphism $F^p(G) \to F^p_\lambda(G)$ is a $p$-completely isometric isomorphism.

*Proof.* In view of Corollary XX.6.10 and Lemma XX.6.18, it is enough to show that the canonical $p$-completely contractive homomorphism from $F^p(G)$ to $F^p_\lambda(G)$ is isometric. Let $\mu$ be a quasi-invariant measure on $G^0$, let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space, and let $\lambda = \int \lambda_x \, d\mu(x)$ be the disintegration of $\lambda$ with respect to $\mu$. Let $T$ be a representation of $G$ on $\mathcal{Z} = \bigsqcup\limits_{x\in G^0} L^p(\lambda_x)$, and let $\pi_T \colon C_c(G) \to B(L^p(\mu, \mathcal{Z}))$ be its integrated form. We want to show that $\|\pi_T(f)\| \leq \|f\|_\lambda$.

Set $\mathcal{W} = \bigsqcup\limits_{x\in G^0} \ell^p(Gx, L^p(\lambda_x))$ and let $(g_n, h_n)_{n\in\omega}$ be an approximate invariant $p$-mean for $G$. For $\xi \in L^p(\mu, \mathcal{Z})$ and $\eta \in L^{p'}(\mu, \mathcal{Z}')$, define $\widehat{\xi}_n \in L^p(\mu, \mathcal{W})$ and $\widehat{\eta}_n \in L^{p'}(\mu, \mathcal{W}')$ by

$$\widehat{\xi}_{n,x}(\gamma) = D^{\frac{1}{p}}(\gamma) g_n(\gamma) T_{\gamma^{-1}} \xi_{r(\gamma)} \quad \text{and} \quad \widehat{\eta}_{n,x}(\gamma) = D^{\frac{1}{p'}}(\gamma) h_n(\gamma) T_{\gamma^{-1}} \eta_{r(\gamma)}.$$

Then

$$\begin{aligned}
\int \left\|\widehat{\xi}_{n,x}\right\|^p \, d\mu(x) &= \int \sum_{\gamma\in Gx} D(\gamma) \left|g_n(\gamma)\right|^p \left\|\xi_{r(\gamma)}\right\|^p \, d\mu(x) \\
&= \int D(\gamma) \left|g_n(\gamma)\right|^p \left\|\xi_{r(\gamma)}\right\|^p \, d\nu^{-1}(\gamma) \\
&= \int \left|g_n(\gamma)\right|^p \left\|\xi_{r(\gamma)}\right\|^p \, d\nu(\gamma) \\
&= \int \sum_{\gamma\in xG} \left|g_n(\gamma)\right|^p \left\|\xi_x\right\|^p \, d\mu(x) \leq \int \left\|\xi_x\right\|^p \, d\mu(x).
\end{aligned}$$

This shows that $\widehat{\xi}_n$ belongs to $L^p(\mu, \mathcal{W})$ and that $\|\widehat{\xi}_n\| \leq \|\xi\|$. Similarly, $\widehat{\eta}_n$ belongs to $L^{p'}(\mu, \mathcal{W}')$ and $\|\widehat{\eta}_n\| \leq \|\eta\|$.

Given $x \in G^0$, identify $\ell^p(Gx, L^p(\lambda_x))$ with $\ell^p(Gx) \otimes L^p(\lambda_x)$ and consider the representation $\mathrm{Ind}(x) \otimes 1 \colon C_c(G) \to B(\ell^p(Gx, L^p(\lambda_x)))$, which for $f \in C_c(G)$ is given by is given byQTOallowdisplaybreaks

$$\langle (\mathrm{Ind}(x)\otimes 1)(f)v,w\rangle = \sum_{\gamma\in Gx}\sum_{\rho\in r(\gamma)G} f(\rho)\left\langle v(\rho^{-1}\gamma),w(\gamma)\right\rangle$$

for $v\in \ell^p(Gx,L^p(\lambda_x))$ and $w\in \ell^{p'}(Gx,L^{p'}(\lambda_x))$.

Set $\pi=\int(\mathrm{Ind}(x)\otimes 1)\ d\mu(x)\colon C_c(G)\to B(L^p(\mu,\mathcal{W}))$. Fix $v\in L^p(\mu,\mathcal{W})$ and $w\in L^{p'}(\nu,\mathcal{W}')$ with $\|v\|,\|w\|\leq 1$. Then

$$\begin{aligned}\langle \pi(f)v,w\rangle &= \int \langle (\mathrm{Ind}(x)\otimes 1)(f)v_x,w_x\rangle \ d\mu(x)\\ &= \int \sum_{\gamma\in Gx}\sum_{\rho\in r(\gamma)G} f(\rho)\left\langle v_x(\rho^{-1}\gamma),w_x(\gamma)\right\rangle \ d\mu(x)\end{aligned}$$

and hence

$$\begin{aligned}|\langle \pi(f)v,w\rangle| &\leq \int |\langle (\mathrm{Ind}(x)\otimes 1)(f)v_x,w_x\rangle| d\mu\\ &\leq \int \|(\mathrm{Ind}(x)\otimes 1)(f)\|\|v_x\|\|w_x\|d\mu\\ &\leq \int \|\mathrm{Ind}(x)(f)\|\|v_x\|\|w_x\|d\mu\\ &\leq \|f\|_\lambda \int \|v_x\|\|w_x\|d\mu(x)\leq \|f\|_\lambda .\end{aligned}$$

We conclude that $\|\pi(f)\|\leq \|f\|_\lambda$ for all $f\in C_c(G)$. In particular, for $v=\widehat{\xi}_n$ and $w=\widehat{\eta}_n$, one gets

$$\begin{aligned}&\left\langle \pi(f)\widehat{\xi}_n,\widehat{\eta}_n\right\rangle\\ &=\int \sum_{\gamma\in Gx}\sum_{\rho\in s(\gamma)G} f(\rho)\langle \widehat{\xi}_{n,x}(\rho^{-1}\gamma),\widehat{\eta}_{n,x}(\gamma)\rangle \ d\mu\\ &=\int \sum_{\gamma\in xG}\sum_{\rho\in xG} f(\rho)D^{-\frac{1}{p}}(\rho)g_n(\rho^{-1}\gamma)h_n(\gamma)\langle T_\rho \xi_{s(\rho)},\eta_x\rangle \ d\mu(x)\\ &=\int \sum_{\rho\in xG}\left(\sum_{\gamma\in xG} g_n\left(\rho^{-1}\gamma\right)h_n(\gamma)\right) f(\rho)D^{-\frac{1}{p}}(\rho)\langle T_\rho\xi_{s(\rho)},\eta_x\rangle \ d\mu(x)\end{aligned}$$

$$= \int \sum_{\rho \in xG} \left( \sum_{\gamma \in xG} h_n(\gamma) g_n^*(\gamma^{-1}\rho) \right) f(\rho) D^{-\frac{1}{p}}(\rho) \langle T_\rho \xi_{s(\rho)}, \eta_x \rangle \; d\mu(x)$$
$$= \int \sum_{\rho \in xG} (h_n * g_n)(\rho) f(\rho) D^{-\frac{1}{p}}(\rho) \langle T_\rho \xi_{s(\rho)}, \eta_x \rangle \; d\mu(x)$$

and thus

$$\begin{aligned} &\lim_{n\to\infty} \langle \pi(f)\widehat{\xi}_n, \widehat{\eta}_n \rangle \\ &= \lim_{n\to\infty} \int \sum_{\rho \in xG} (h_n * g_n)(\rho) f(\rho) D^{-\frac{1}{p}}(\rho) \langle T_\rho \xi_{s(\rho)}, \eta_x \rangle \; d\mu(x) \\ &= \int \sum_{\rho \in xG} f(\rho) D^{-\frac{1}{p}}(\rho) \langle T_\rho \xi_{s(\rho)}, \eta_x \rangle \; d\mu(x) = \langle \pi_T(f)\xi, \eta \rangle \, . \end{aligned}$$

We conclude that

$$\|\pi_T(f)\| \le \lim_{n\to\infty} \left| \left\langle \pi(f)\widehat{\xi}_n, \widehat{\eta}_n \right\rangle \right| \le \|\pi(f)\| \le \|f\|_{\mathrm{red}},$$

as desired. □

**Examples: Analogs of Cuntz Algebras and AF-algebras**

Throughout this section, we let $p \in (1, \infty)$.

*The Cuntz $L^p$-operator algebras*

Fix $d \in \omega$ with $d \geq 2$. The following is [204, Definition 1.1] and [204, Definition 7.4 (2)]. Algebra representations of complex unital algebras are always assumed to be unital.

**Definition XX.7.1.** Define the *Leavitt algebra* $L_d$ to be the universal (complex) algebra with generators $s_0, \ldots, s_{d-1}, s_0^*, \ldots, s_{d-1}^*$, subject to the relations

1. $s_j^* s_k = \delta_{j,k}$ for $j, k \in d$; and
2. $\sum_{j \in d} s_j s_j^* = 1$.

If $\lambda$ is a $\sigma$-finite Borel measure on a standard Borel space, a *spatial representation* of $L_d$ on $L^p(\lambda)$ is an algebra homomorphism $\rho\colon L_d \to \mathcal{S}(L^p(\lambda))$ such that for $j \in d$, the operators $\rho(s_j)$ and $\rho(s_j^*)$ are mutually inverse spatial partial isometries, that is, $\rho(s_j^*) = \rho(s_j)^*$.

It is a consequence of a fundamental result of J. Cuntz from [39] that any two *-representations of $L_d$ on a Hilbert space induce the same norm on $L_d$. The corresponding completion is the Cuntz $C^*$-algebra $\mathcal{O}_d$.

Cuntz's result was later generalized by N.C. Phillps in [204] to spatial representations of $L_d$ on $L^p$-spaces. Theorem 8.7 of [204] asserts that any two spatial $L^p$-representations of $L_d$ induce the same norm on it. The corresponding completion is the Cuntz $L^p$-operator algebra $\mathcal{O}_d^p$; see [204, Definition 8.8]. We now to explain how one can realize $\mathcal{O}_d^p$ as a groupoid $L^p$-operator algebra.

Denote by $d^\omega$ the space of infinite sequences of elements of $d$, endowed with the product topology. (Recall that $d$ is identified with the set $\{0, 1, \dots, d-1\}$ of its predecessors.) Denote by $d^{<\omega}$ the space of (possibly empty) finite sequences of elements of $d$. The length of an element $a$ of $d^{<\omega}$ is denoted by $\mathrm{lh}(a)$. For $a \in d^{<\omega}$ and $x \in d^\omega$, define $a^\frown x \in d^\omega$ to be the concatenation of $a$ and $x$. For $a \in d^{<\omega}$, denote by $[a]$ the set of elements of $d^\omega$ having $a$ as initial segment, that is,

$$[a] = \left\{ a^\frown x \colon x \in d^\omega \right\}.$$

Clearly $\{[a] \colon a \in d^{<\omega}\}$ is a clopen basis for $d^\omega$.

**Definition XX.7.2.** The *Cuntz inverse semigroup* $\Sigma_d$ is the inverse semigroup generated by a zero 0, a unit 1, and elements $s_j$ for $j \in d$, satisfying $s_j^* s_k = 0$ whenever $j \neq k$.

Set $s_\varnothing = 1$ and $s_a = s_{a_0} \cdots s_{a_{lh(a)-1}} \in \Sigma_d$ for $a \in d^{<\omega} \setminus \{\varnothing\}$. Every element of $\Sigma_d$ can be written uniquely as $s_a s_b^*$ for some $a, b \in d^{<\omega}$.

**Remark XX.7.3.** The nonzero idempotents $E(\Sigma_d)$ of $\Sigma_d$ are precisely the elements of the form $s_a s_a^*$ for $a \in d^{<\omega}$. Moreover, the function $d^{<\omega} \cup \{0\} \to E(\Sigma)$ given by $a \mapsto s_a s_a^*$ and $0 \mapsto 0$, is a semilattice map, where $d^{<\omega}$ has its (downward) tree ordering defined by $a \leq b$ if and only if $b$ is an initial segment of $a$, and 0 is a least element of $d^{<\omega} \cup \{0\}$.

Observe that if $a, b \in d^{<\omega}$, then $ab = 0$ if and only if $a(j) \neq b(j)$ for some $j \in \min\{\mathrm{lh}(a), \mathrm{lh}(b)\}$

**Lemma XX.7.4.** Let $\mathcal{B}$ be a Boolean algebra and let $\beta\colon d^{<\omega} \to \mathcal{B}$ be a representation. Then $\beta$ is tight if and only if $\beta(\varnothing) = 1$ and

$$\beta(a) \leq \bigvee_{j\in d} \beta(a^\frown j)$$

for every $a \in d^{<\omega}$.

*Proof.* Suppose that $\beta$ is tight. Since 1 is a cover of $E^{\varnothing,\varnothing}$, we have $\beta(\varnothing) = 1$. Similarly, $\{a^\frown j\colon j \in d\}$ is a cover of $E^{\{a\},\varnothing}$ and thus $\beta(a) \leq \bigvee\limits_{j\in d} \beta(a^\frown j)$. Let us now show the "if" implication. By [66, Proposition 11.8], it is enough to show that for every $a \in d^{<\omega}$ and every finite cover $Z$ of $\{a\}$, one has $\beta(a) \leq \bigvee\limits_{z\in Z} \beta(z)$. That this is true follows from the hypotheses, using induction on the maximum length of elements of $Z$. □

**Lemma XX.7.5.** Let $\lambda$ be a $\sigma$-finite Borel measure on a standard Borel space, and $\rho$ be a representation of $\Sigma_d$ on $L^p(\lambda)$. Then $\rho$ is tight if and only if

$$\sum_{j\in d} \rho(s_j s_j^*) = \rho(1) = 1.$$

*Proof.* Suppose that $\rho$ is tight. Then $\rho|_{E(\Sigma)}$ is tight and therefore

$$1 = \rho(1) = \bigvee_{j\in d} \rho(s_j s_j^*) = \sum_{j\in d} \rho(s_j s_j^*)$$

by Lemma XX.7.4. Conversely, given $a \in d^{<\omega}$, we have

$$\begin{aligned}
\sum_{j\in d} \rho(s_{a^\frown j} s_{a^\frown j}^*) &= \sum_{j\in d} \rho(s_a s_j s_j^* s_a^*) = \sum_{j\in d} \rho(s_a)\rho(s_j s_j^*)\rho(s_a^*) \\
&= \rho(s_a)\left(\sum_{j\in d} \rho(s_j s_j^*)\right)\rho(s_a^*) = \rho(s_a)\rho(1)\rho(s_a^*) \\
&= \rho(s_a s_a^*),
\end{aligned}$$

which shows that $\rho$ is tight, concluding the proof. □

**Proposition XX.7.6.** The algebra $F^p_{\mathrm{tight}}(\Sigma_d)$ is $p$-completely isometric isomorphic to $\mathcal{O}^p_d$.

*Proof.* Observe that the Leavitt algebra $L_d$ (see Definition XX.7.1) is isomorphic to the quotient of $\mathbb{C}\Sigma_d$ by the ideal generated by the elements $\delta_1 - \sum\limits_{j\in d} \delta_{s_j s_j^*}$ and $\delta_0$. (Here, $\delta_s$ denotes the canonical element in $\mathbb{C}\Sigma_d$ corresponding to $s \in \Sigma_d$.) By Lemma XX.7.5, tight representations of $\Sigma_d$ correspond precisely to spatial representations of the Leavitt algebra $L_d$ as defined in [204, Definition 7.4]. The result then follows. $\square$

It is well known that $\Sigma_d$ an inverse semigroup of compact open slices of an ample groupoid $\mathcal{G}_d$. We now proceed to define $\mathcal{G}_d$. Let $T\colon d^\omega \to d^\omega$ denote the unilateral shift on $d^\omega$, and observe that $T$ is one-to-one on $[a]$ whenever $\mathrm{lh}(a) \geq 1$. Denote by $\mathcal{G}_d$ the groupoid

$$\mathcal{G}_d = \{(x, m-n, y)\colon x, y \in d^\omega, m, n \in \mathbb{N}, T^m x = T^n y\},$$

with operations defined by

$$s(x, m-n, y) = x \quad , \quad r(x, m-n, y) = y$$

$$(x, m-n, y)(y, k-r, z) = (x, m-n+k-r, z)$$

$$(x, k, y)^{-1} = (y, -k, x).$$

For $a$ and $b$ in $d^{<\omega}$, set

$$[a, b] = \left\{\left(a^\frown x, \mathrm{lh}(a) - \mathrm{lh}(b), b^\frown x\right) : x \in d^\omega\right\} \subseteq \mathcal{G}_d.$$

The collection $\{[a, b]\colon a, b \in d^{<\omega}\}$ is a basis of clopen slices for $\mathcal{G}_d$, and $\mathcal{G}_d$ is therefore ample.

**Theorem XX.7.7.** Let $d \geq 2$ be a positive integer, and let $\mathcal{G}_d$ denote the corresponding Cuntz groupoid. Then $F^p(\mathcal{G}_d)$ is canonically $p$-completely isometrically isomorphic to $\mathcal{O}_d^p$.

*Proof.* It is easy to check that the function $s_a s_b^* \mapsto [a, b]$ defines an injective homomorphism from $\Sigma_d$ to the inverse semigroup of compact open slices of $\mathcal{G}_d$. It is well known that $\mathcal{G}_d$ is amenable; see [226, Exercise 4.1.7]. It follows from Theorem XX.6.21, Corollary XX.6.11, and Proposition XX.7.6, that there are canonical $p$-completely isometric isomorphisms

$$F^p_\lambda(\mathcal{G}_d) \cong F^p(\mathcal{G}_d) \cong F^p_{\mathrm{tight}}(\Sigma_d) \cong \mathcal{O}_d^p. \qquad \square$$

In this subsection, we show how one can use the machinery developed in the previous sections to construct those $L^p$-analogs of AF-algebras that look like $C^*$-algebras, and which are called "spatial" in [215].

Fix $n \in \mathbb{N}$. The algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex coefficients can be (algebraically) identified with $B(\ell^p(n))$. This identification turns $M_n(\mathbb{C})$ into an $L^p$-operator algebra that we will denote—consistently with [204]—by $M_n^p$. It is not difficult to verify that $M_n^p$ can be realized as a groupoid $L^p$-operator algebra, and we proceed to outline the argument.

Denote by $T_n$ the principal groupoid determined by the trivial equivalence relation on $n$. It is well-known (see [225, page 121]) that $T_n$ is amenable. Moreover, the inverse semigroup $\Sigma_{\mathcal{K}}(T_n)$ of compact open slices of $T_n$, is the inverse semigroup generated by a zero element 0, a unit 1, and elements $e_{jk}$ for $j,k \in n$, subject to the relations $e_{jk}^* e_{\ell m} = \delta_{k\ell} e_{jm}$ for $j,k,\ell,m \in n$. It is not difficult to verify, using Lemma XX.4.19, that a tight $L^p$-representation $\rho$ of $\Sigma_{\mathcal{K}}(T)$ satisfies

$$1 = \rho(1) = \sum_{j \in n} \rho(e_{jj}).$$

It thus follows from [204, Theorem 7.2] that the map from $M_n^p$ to the range of $\rho$, defined by assigning $\rho(e_{jk})$ to the $jk$-th matrix unit in $M_n^p$, is isometric. We conclude that $F^p(T_n)$ is isometrically isomorphic to $M_n^p$. Reasoning in the same way at the level of amplifications shows that $F^p(T_n)$ and $M_n^p$ are in fact $p$-completely isometrically isomorphic.

If $k \in \mathbb{N}$ and $\mathbf{n} = (n_0, \ldots, n_{k-1})$ is a $k$-tuple of natural numbers, then the Banach algebra $M_{n_0}^p \oplus \cdots \oplus M_{n_{k-1}}^p$ acts naturally on the $L^p$-direct sum $\ell^p(n_0) \underset{p}{\oplus} \cdots \underset{p}{\oplus} \ell^p(n_{k-1}) \cong \ell^p(n_0 + \cdots + n_{k-1})$. The Banach algebra $M_{n_0}^p \oplus \cdots \oplus M_{n_{k-1}}^p$ can also be realized as groupoid $L^p$-operator algebra by considering the disjoint union of the groupoids $T_{n_0}, T_{n_1}, \ldots, T_{n_{k-1}}$.

Here is the definition of spatial $L^p$-operator AF-algebras

**Definition XX.7.8.** A separable Banach algebra $A$ is said to be a *spatial $L^p$-operator AF-algebra* if there exists a direct system $(A_n, \varphi_n)_{n \in \omega}$ of $L^p$-operator algebras $A_n$ which are isometrically isomorphic to algebras of the form $M_{n_0}^p \oplus \cdots \oplus M_{n_k}^p$, with isometric connecting maps $\varphi_n \colon A_n \to A_{n+1}$, and such that $A$ is isometrically isomorphic to the direct limit $\varinjlim (A_n, \varphi_n)_{n \in \omega}$.

Banach algebras as in the definition above, as well as more general direct limits of semisimple finite dimensional $L^p$-operator algebras, will be studied in [215].

In the rest of this subsection, we will show that spatial $L^p$-operator AF-algebras can be realized as groupoid $L^p$-operator algebras.

For simplicity, we will start by observing that spatial $L^p$-operator UHF-algebras are groupoid $L^p$-operator algebras. Spatial $L^p$-operator UHF-algebras are the spatial $L^p$-operator AF-algebras where the building blocks $A_n$ appearing in the definition are all full matrix algebras $M^p_{d_n}$ for some $d_n \in \omega$. These have been defined and studied in [208].

Let $d = (d_n)_{n\in\omega}$ be a sequence of positive integers. Denote by $A^p_d$ the corresponding $L^p$-operator UHF-algebra defined as above; see also [209, Definition 3.9]. In the following we will show that $A^p_d$ is the enveloping algebra of a natural groupoid associated with the sequence $d$. Define $Z_d = \prod_{j\in n} d_j$, and consider the groupoid

$$G_d = \left\{ \left(\alpha^\frown x, \beta^\frown x\right) : \alpha, \beta \in \prod_{j\in n} d_j, x \in \prod_{j\geq n} d_j, n \in \omega \right\}$$

having $Z_d$ as set of objects. (Here we identify $x \in Z_d$ with the pair $(x,x) \in G_d$.) The operations are defined by

$$\begin{aligned} s(\alpha^\frown x, \beta^\frown x) &= \beta^\frown x, \\ (\alpha^\frown x, \beta^\frown x)^{-1} &= (\beta^\frown x, \alpha^\frown x), \text{ and} \\ (\alpha^\frown x, \beta^\frown x)(\gamma^\frown y, \delta^\frown y) &= (\alpha^\frown x, \delta^\frown y) \ \text{ whenever } \ \beta^\frown x = \gamma^\frown y. \end{aligned}$$

It is well-known that $G_d$ is amenable; see [226, Chapter III, Remark 1.2].

Given $k \in \omega$ and given $\alpha$ and $\beta$ in $\prod_{j\in k} d_j$, define

$$U_{\alpha\beta} = \left\{ \left(\alpha^\frown x, \beta^\frown x\right) \in G_d \colon x \in \prod_{j\geq k} d_j \right\}.$$

Then

$$\left\{ U_{\alpha\beta} \colon \alpha, \beta \in \prod_{j\in k} d_j, k \in \omega \right\}$$

is a basis of compact open slices for an ample groupoid topology on $G_d$.

Fix $k \in \omega$ and consider the compact groupoid

$$G_d^k = \bigcup \left\{ U_{\alpha,\beta} \colon \alpha, \beta \in \prod_{j \in k} d_j \right\}.$$

The groupoid $G_d$ can be seen as the topological direct limit of the system $(G_d^k)_{k\in\omega}$. It is clear that, if $n = d_0 \cdots d_{k-1}$, then $G_d^k$ is isomorphic to the groupoid $T_n$ defined previously. Therefore $F^p(G_d^k)$ is isometrically isomorphic to $M^p_{d_0 \cdots d_{k-1}}$.

For $k \in \mathbb{N}$, identify $C(G_d^k)$ with a *-subalgebra of $C_c(G_d)$, by setting $f \in C(G_d^k)$ to be 0 outside $G_d^k$. For $k < n$, we claim that the inclusion map from $C(G_d^k)$ to $C(G_d^n)$ induces an isometric embedding

$$\varphi_n \colon F^p(G_d^k) \to F^p(G_d^n).$$

This can be easily verified by direct computation, after noticing that $G_d^k$ and $G_d^n$ are amenable, and hence the full and reduced norms on $C(G_d^k)$ and $C(G_d^n)$ coincide. One then obtains a direct system $\left(F^p(G_d^k), \varphi_n\right)_{n\in\mathbb{N}}$ with isometric connecting maps whose limit is $F^p(G)$. Since $F^p(G_d^k) \cong M^p_{d_0 \cdots d_{k-1}}$ as observed above, we conclude that $F^p(G_d) \cong A_d^p$.

We now turn to spatial AF-algebras. As in the $C^*$-algebra case, there is a natural correspondence between $L^p$-operator AF-algebras and Bratteli diagrams. Let $(E, V)$ be a Bratteli diagram, and $A^{(E,V)}$ be the associated $L^p$-operator AF-algebra. In the following, we will explain how to realize $A^{(E,V)}$ as a groupoid $L^p$-operator algebra.

Denote by $X$ the set of all infinite paths in $(E, V)$. Then $X$ is a compact zero dimensional space. Denote by $G^{(E,V)}$ the tail equivalence relation on $X$, regarded as a principal groupoid having $X$ as set of objects. It is well known that $G^{(E,V)}$ is amenable; see [226, Chapter III, Remark 1.2]. If $\alpha, \beta$ are *finite* paths of the same length and with the same endpoints, define $U_{\alpha\beta}$ to be the set of elements of $G^{(E,V)}$ of the form $\left(\alpha^\frown x, \beta^\frown x\right)$. The collection of all the sets $U_{\alpha\beta}$ is a basis for an ample groupoid topology on $G^{(E,V)}$. For $k \in \omega$, let $G_k^{(E,V)}$ be the union of $U_{\alpha\beta}$ over all finite paths $\alpha, \beta$ as before that moreover have length at most $k$. Then $G_n^{(E,V)}$ is a compact groupoid and $G$ is the topological direct limit of $(G_k^{(E,V)})_{k\in\omega}$.

Fix $k \in \omega$. Denote by $l$ the cardinality of the $k$-th vertex set $V_k$. Denote by $n_0, \ldots, n_{l-1}$ the *multiplicities* of the vertices in $V_k$. (The multiplicity of a vertex in a Bratteli diagram is defined in the usual way by recursion.) Set $\mathbf{n} = (n_0, \ldots, n_{l-1})$, and observe that $G_k^{(E,V)}$ is isomorphic to the groupoid $T_{\mathbf{n}}$ as defined above. In particular

$$F^p\left(G_n^{(E,V)}\right) \cong M_{n_0}^p \oplus \cdots \oplus M_{n_{l-1}}^p.$$

As before, one can show that the direct system $(F^p(G_n^{(E,V)}))_{n\in\omega}$ has isometric connecting maps, and that the inductive limit is $F^p(G^{(E,V)})$. This concludes the proof that $A^{(E,V)}$ is $p$-completely isometrically isomorphic to $F^p\left(G^{(E,V)}\right)$. In particular, this shows that $A^{(E,V)}$ is indeed an $L^p$-operator algebra.

## Concluding Remarks and Outlook

It is not difficult to see that the class of $L^p$-operator algebras is closed—within the class of all matricially normed Banach algebras—under taking subalgebras and ultraproducts. As observed by Ilijas Farah and Chris Phillips, this observation, together with a general result from logic for metric structures, implies that the class of $L^p$-operator algebras is—in model-theoretic jargon—*universally axiomatizable*. This means that $L^p$-operator algebras can be characterized as those matricially normed Banach algebras satisfying certain expressions only involving

- the algebra operations,
- the matrix norms,
- continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$, and
- suprema over balls of matrix amplifications.

Determining what these expressions are seems to be, in our opinion, an important problem in the theory of algebras of operators on $L^p$-spaces.

**Problem XX.8.1.** Find an explicit intrinsic characterization of $L^p$-operator algebras within the class of matricially normed Banach algebras.

An explicit characterization of algebras acting on *subspaces of quotients* of $L^p$-spaces was provided by Le Merdy in [165]. These are precisely the matricially normed Banach algebras that are moreover $p$-operator spaces in the terminology of [48], and such that multiplication is $p$-completely contractive. Similar results have been obtained by Junge for algebras of operators on subspaces of $L^p$-spaces; see [140, Corollary 1.5.2.2].

It is shown in [20, Theorem 5.1] that the reduced $C^*$-algebra of an etale groupoid is simple if and only the groupoid is minimal and topologically principal. (A groupoid is called *minimal* if it has no nontrivial invariant open set of objects, and *topologically principal* if the set of objects with trivial isotropy group is dense.) We believe that the same should be true for the reduced $L^p$-operator algebras of etale groupoids. This has been shown for $L^p$-analogs of UHF-algebras and Cuntz algebras in [209] by seemingly *ad hoc* methods.

**Problem XX.8.2.** Is $F^p_\lambda(G)$ simple whenever $G$ is a minimal and topologically principal etale groupoid?

A potential application of groupoids to the theory of $L^p$-operator algebras comes from the technique of Putnam subalgebras. Let $X$ be a compact metric space and let $h\colon X \to X$ be a homeomorphism. Denote by $u$ the canonical unitary in the C*-crossed product $C(X) \rtimes_h \mathbb{Z}$ implementing $h$. If $Y$ is a closed subset of $X$, then the corresponding *Putnam subalgebra* $(C(X) \rtimes_h \mathbb{Z})_Y$ is the $C^*$-subalgebra of $C(X) \rtimes_h \mathbb{Z}$ generated by $C(X)$ and $uC_0(X\backslash Y)$. It is known that $(C(X) \rtimes_h \mathbb{Z})_Y$ can be described as the enveloping $C^*$-algebra of a suitable etale groupoid.

In the context of $C^*$-algebras, Putnam subalgebras are fundamental in the study of transformation group $C^*$-algebras of minimal homeomorphisms. For example, Putnam showed in [218, Theorem 3.13] that if $h$ is a minimal homeomorphism of the Cantor space $X$, and $Y$ is a nonempty clopen subset of $X$, then $(C(X) \rtimes_h \mathbb{Z})_Y$ is an AF-algebra. This is then used in [218] to prove that the crossed product $(C(X) \rtimes_h \mathbb{Z})_Y$ is a simple A$\mathbb{T}$-algebra of real rank zero. Similarly, Putnam subalgebras were used by Huaxin Lin and Chris Phillips in [172] to show that, under a suitable assumption on $K$-theory, the crossed product of a finite dimensional compact metric space by a minimal homeomorphism is a simple unital $C^*$-algebra with tracial rank zero.

Considering the groupoid description of Putnam subalgebras provides a natural application of our constructions to the theory of $L^p$-crossed products introduced in [207]. It is conceivable that with the aid of groupoid $L^p$-operator algebras, Putnam subalgebras could be used to obtain generalizations of the above mentioned results to $L^p$-crossed products.

CHAPTER XXI

# NONCLASSIFIABILITY OF UHF $L^P$-OPERATOR ALGEBRAS

This chapter is based on joint work with Martino Lupini ([90]).

We prove that simple, separable, monotracial UHF $L^p$-operator algebras are not classifiable up to (complete) isomorphism using countable structures, such as K-theoretic data, as invariants. The same assertion holds even if one only considers UHF $L^p$-operator algebras of tensor product type obtained from a diagonal system of similarities. For $p = 2$, it follows that separable nonselfadjoint UHF operator algebras are not classifiable by countable structures up to (complete) isomorphism. Our results, which answer a question of N. Christopher Phillips, rely on Borel complexity theory, and particularly Hjorth's theory of turbulence.

## Introduction

Suppose that $X$ is a standard Borel space and $\lambda$ is a Borel probability measure on $X$. For $p \in [1, \infty)$, we denote by $L^p(\lambda)$ the Banach space of Borel-measurable complex-valued functions on $X$ (modulo null sets), endowed with the $L^p$-norm. Let $B(L^p(\lambda))$ denote the Banach algebra of bounded linear operators on $L^p(\lambda)$ endowed with the operator norm. We will identify the Banach algebra $M_n(B(L^p(\lambda)))$ of $n \times n$ matrices with entries in $B(L^p(\lambda))$, with the algebra $B(L^p(\lambda)^{\oplus n})$ of bounded linear operators on the $p$-direct sum $L^p(\lambda)^{\oplus n}$ of $n$ copies of $L^p(\lambda)$.

A (concrete) separable, unital *$L^p$-operator algebra*, is a separable, closed subalgebra of $B(L^p(\lambda))$ containing the identity operator. (Such a definition is consistent with [207, Definition 1.1], in view of [207, Proposition 1.25].) In the following, all $L^p$-operator algebras will be assumed to be separable and unital. Every unital $L^p$-operator algebra $A \subseteq B(L^p(\lambda))$ is in particular a $p$-operator space in the sense of [48, 4], with matrix norms obtained by identifying $M_n(A)$ with a subalgebra of $M_n(B(L^p(\lambda)))$.

If $A$ is a unital complex algebra, then an *$L^p$-representation* of $A$ on a standard Borel probability space $(X, \lambda)$ is a unital algebra homomorphism $\rho\colon A \to B(L^p(\lambda))$. The closure inside $B(L^p(\lambda))$ of $\rho(A)$ is an $L^p$-operator algebra, called the *$L^p$-operator algebra associated with $\rho$*. It can be identified with the completion of $A$ with respect to the operator seminorm structure

$\|[a_{ij}]\|_\rho = \|[\rho(a_{ij})]\|_{M_n(B(L^p(\lambda)))}$ for $[a_{ij}] \in M_n(A)$; see [17, 1.2.16]. A unital homomorphism $\varphi$ is *completely bounded* if every amplification $\varphi^{(n)}$ is bounded and

$$\|\varphi\|_{cb} = \sup_{n\in\mathbb{N}} \left\|\varphi^{(n)}\right\|$$

is finite.

**Definition XXI.1.1.** Let $A$ and $B$ be unital $L^p$-operator algebras.

1. $A$ and $B$ are said to be *(completely) isomorphic*, if there is a (completely) bounded unital isomorphism $\varphi\colon A \to B$ with (completely) bounded inverse $\varphi^{-1}\colon B \to A$.

2. $A$ and $B$ are said to be *(completely) commensurable* if there are (completely) bounded unital homomorphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$.

For $d \in \mathbb{N}$, we denote by $M_d$ the unital algebra of $d \times d$ complex matrices, with matrix units $\{e_{i,j}\}_{1\leq i,j\leq d}$. Let $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$, and let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ be a sequence of representations $\rho_n\colon M_{d_n} \to B(L^p(X_n, \lambda_n))$. Define $M_{\boldsymbol{d}}$ to be the algebraic infinite tensor product $\bigotimes_{n\in\mathbb{N}} M_{d_n}$. Let $X = \prod_{n\in\mathbb{N}} X_n$ be the product Borel space and $\lambda = \bigotimes_{n\in\mathbb{N}} \lambda_n$ be the product measure. We naturally regard the algebraic tensor product $\bigotimes_{n\in\mathbb{N}} B(L^p(\lambda_n))$ as a subalgebra of $B(L^p(\lambda))$. The correspondence

$$M_{\boldsymbol{d}} \to \bigotimes_{n\in\mathbb{N}} B(L^p(\lambda_n)) \subseteq B(L^p(\lambda))$$

$$a_1 \otimes \cdots \otimes a_k \mapsto \rho_1(a_1) \otimes \cdots \otimes \rho_k(a_k),$$

extends to a unital homomorphism $M_{\boldsymbol{d}} \to B(L^p(\lambda))$.

**Definition XXI.1.2.** The algebra $A(\boldsymbol{d}, \boldsymbol{\rho})$ as defined in [204, Example 3.8], is the $L^p$-operator algebra associated with $\boldsymbol{\rho}$. A *UHF $L^p$-operator algebra of tensor product type $\boldsymbol{d}$* is an algebra of the form $A(\boldsymbol{d}, \boldsymbol{\rho})$ for some sequence $\boldsymbol{\rho}$ as above; see [204, Definition 3.9] and [208, Definition 1.7].

A special class of UHF $L^p$-operator algebras of tensor product type $\boldsymbol{d}$ has been introduced in [208, Section 5]. For $d \in \mathbb{N}$, denote by $c_d$ the normalized counting measure on $d = \{0, 1, 2, \ldots, d-1\}$, and set $\ell^p(d) = L^p(\{0, \ldots, d-1\}, c_d)$. The *(canonical) spatial representation* $\sigma^d$

of $M_d$ on $\ell^p(d)$ is defined by setting

$$\left(\sigma^d(a)\xi\right)(j) = \sum_{i=0,\ldots,d-1} a_{ij}\xi(i)$$

for $a \in M_d$, for $\xi \in \ell^p(d)$ and $j = 0, \ldots, d-1$; see [204, Definition 7.1]. Observe that the corresponding matrix norms on $M_d$ are obtained by identifying $M_d$ with the algebra of bounded linear operators on $\ell^p(d)$.

Fix a real number $\gamma$ in $[1, +\infty)$, and an enumeration $(w_{d,\gamma,k})_{k\in\mathbb{N}}$ of all diagonal $d \times d$ matrices with entries in $[1, \gamma] \cap \mathbb{Q}$. Let $X$ be the disjoint union of countably many copies of $\{0, 1, \ldots, d-1\}$, and let $\lambda_d$ be the Borel probability measure on $X$ that agrees with $2^{-k}c_d$ on the $k$-th copy of $\{0, 1, \ldots, d-1\}$. We naturally identify the algebraic direct sum $\bigoplus_{n\in\mathbb{N}} B(\ell^p(d))$ with a subalgebra of $B(L^p(\lambda_d))$. The map

$$\begin{aligned} M_d &\to \bigoplus_{n\in\mathbb{N}} B(\ell^p(d)) \subseteq B(L^p(\lambda_d)) \\ x &\mapsto \left(\sigma^d\left(w_{d,\gamma,k}\,x\,w_{d,\gamma,k}^{-1}\right)\right)_{k\in\mathbb{N}} \end{aligned}$$

defines a representation $\rho^\gamma \colon M_d \to B(L^p(\lambda_d))$.

For a sequence $\boldsymbol{\gamma}$ in $[1, +\infty)$, we will denote by $\boldsymbol{\rho}^{\boldsymbol{\gamma}}$ the sequence of representations $\rho^{\gamma_n} \colon M_{d_n} \to B(L^p(\lambda_{d_n}))$ described in the paragraph above. Following the terminology in [208, Section 3 and Section 5], we say that the corresponding UHF $L^p$-operator algebras $A(\boldsymbol{d}, \boldsymbol{\rho}^{\boldsymbol{\gamma}})$ are *obtained from a diagonal system of similarities* .

**Definition XXI.1.3.** If $A$ is a unital Banach algebra, a *normalized trace* on $A$ is a continuous linear functional $\tau \colon A \to \mathbb{C}$ with $\tau(1) = 1$, satisfying $\tau(ab) = \tau(ba)$ for all $a, b \in A$. The algebra $A$ is said to be *monotracial* if $A$ has a unique normalized trace.

Recall that a Banach algebra is said to be *simple* if it has no nontrivial closed two-sided ideals.

**Remark XXI.1.4.** It was shown in [209, Theorem 3.19(3)] that UHF $L^p$-operator algebras obtained from a diagonal system of similarities are always simple and monotracial.

Problem 5.15 of [208] asks to provide invariants which classify, up to isomorphism, some reasonable class of UHF $L^p$-operator algebras, such as those constructed using diagonal similarities. The following is the main result of the present chapter, which is based on [90].

**Theorem XXI.1.5.** The simple, separable, monotracial UHF $L^p$-operator algebras are not classifiable by countable structures up to any of the following equivalence relations:

1. complete isomorphism;
2. isomorphism;
3. complete commensurability;
4. commensurability.

The same conclusions hold even if one only considers UHF $L^p$-operator algebras of tensor product type $\boldsymbol{d}$ obtained from a diagonal system of similarities for a fixed sequence $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ of positive integers such that, for every distinct $n, m \in \mathbb{N}$, neither $d_n$ divides $d_m$ nor $d_m$ divides $d_n$.

It follows from Theorem XXI.1.5 that simple, separable, UHF $L^p$-operator algebras with a unique tracial state are not classifiable by K-theoretic data, even after adding to the K-theory a countable collection of invariants consisting of countable structures. When $p = 2$, Theorem XXI.1.5 asserts that separable nonselfadjoint UHF operator algebras are not classifiable by countable structures up to isomorphism. This conclusion is in stark constrast with Glimm's classification of UHF $C^*$-algebras by their corresponding supernatural number [102]. (Observe that, in view of Glimm's classification, Banach-algebraic isomorphism and $*$-isomorphism coincide for UHF $C^*$-algebras.)

### Borel Complexity Theory

In order to obtain our main result, we will work in the framework of Borel complexity theory. In such a framework, a classification problem is regarded as an equivalence relation $E$ on a standard Borel space $X$. If $F$ is another equivalence relation on another standard Borel space $Y$, a *Borel reduction* from $E$ to $F$ is a Borel function $g\colon X \to Y$ with the property that

$$xEx' \quad \text{if and only if} \quad g(x)Fg(x').$$

The map $g$ can be seen as a classifying map for the objects of $X$ up to $E$. The requirement that $g$ is Borel captures the fact that $g$ is *explicit* and *constructible* (and not, for example, obtained by using the Axiom of Choice). The relation $E$ is *Borel reducible* to $F$ if there is a Borel reduction from $E$ to $F$. This can be interpreted as asserting that is it possible to explicitly classify the elements of $X$ up to $E$ using $F$-classes as invariants.

The notion of Borel reducibility provides a way to compare the complexity of classification problems in mathematics. Some distinguished equivalence relations are then used as benchmarks of complexity. The first such a benchmark is the relation $=_{\mathbb{R}}$ of equality of real numbers. (One can replace $\mathbb{R}$ with any other Polish space.) An equivalence relation is called *smooth* if it is Borel reducible to $=_{\mathbb{R}}$. Equivalently, an equivalence relation is smooth if its classes can be explicitly parametrized by the points of a Polish space. For instance, the above mentioned classification of UHF $C^*$-algebras due to Glimm [102] shows that the classification problem of UHF $C^*$-algebras is smooth. Smoothness is a very restrictive notion, and many natural classification problems transcend such a benchmark. For instance, the relation of isomorphism of rank 1 torsion-free abelian groups is not smooth; see [127].

A more generous notion of classifiability is being *classifiable by countable structures.* Informally speaking, an equivalence relation $E$ on a standard Borel space $X$ is classifiable by countable structures if it is possible to explicitly assign to the elements of $X$ complete invariants up to $E$ that are countable structures, such as as countable (ordered) groups, countable (ordered) rings, etcetera. To formulate precisely this definition, let $\mathcal{L}$ be a countable first order language [180, Definition 1.1.1]. The class $\mathrm{Mod}(\mathcal{L})$ of $\mathcal{L}$-structures supported by the set $\mathbb{N}$ of natural numbers can be regarded as a Borel subset of $\prod_{n\in\mathbb{N}} 2^{\mathbb{N}^n}$. As such, $\mathrm{Mod}(\mathcal{L})$ inherits a Borel structure making it a standard Borel space. Let $\cong_{\mathcal{L}}$ be the relation of isomorphism of elements of $\mathrm{Mod}(\mathcal{L})$.

**Definition XXI.2.1.** An equivalence relation $E$ on a standard Borel space is said to be *classifiable by countable structures*, if there exists a countable first order language $\mathcal{L}$ such that $E$ is Borel reducible to $\cong_{\mathcal{L}}$.

The Elliott-Bratteli classification of AF $C^*$-algebras ([57] and [19]) shows, in particular, that AF $C^*$-algebras are classifiable by countable structures up to $*$-isomorphism. Any smooth equivalence relation is in particular classifiable by countable structures.

Many naturally occurring classification problems in mathematics, and particularly in functional analysis and operator algebras, have recently been shown to transcend countable structures. This has been obtained, for examample, for the relation of unitary conjugacy of irreducible representations and automorphisms of non type I $C^*$-algebras ([124], [148], [69], [177]); for the relation of conjugacy for automorphisms of $\mathcal{Z}$-stable $C^*$-algebras and McDuff $\mathrm{II}_1$ factors ([149]); and for the relation of isomorphism for von Neumann factors ([244], [245]). The main tool involved in these results is the theory of turbulence developed by Hjorth in [125].

Suppose that $G \curvearrowright X$ is a continuous action of a Polish group $G$ on a Polish space $X$. The corresponding orbit equivalence relation $E_G^X$ is the relation on $X$ obtained by setting $xE_G^X x'$ if and only if $x$ and $x'$ belong to the same orbit. Hjorth's theory of turbulence provides a dynamical condition, called *(generic) turbulence*, that ensures that a Polish group action $G \curvearrowright X$ yields an orbit equivalence relation $E_G^X$ that is not classifiable by countable structures. This provides, directly or indirectly, useful criteria to prove that a given equivalence relation is not classifiable by countable structures. A prototypical example of turbulent group action is the action of $\ell^1$ on $\mathbb{R}^{\mathbb{N}}$ by translation. A standard argument allows one to deduce the following nonclassification criterion from turbulence of the action $\ell^1 \curvearrowright \mathbb{R}^{\mathbb{N}}$ and Hjorth's turbulence theorem [125, Theorem 3.18]; see for example [177, Lemma 3.2 and Criterion 3.3].

Recall that a subspace of a topological space is *meager* if it is contained in the union of countably many closed nowhere dense sets.

**Proposition XXI.2.2.** Suppose that $E$ is an equivalence relation on a standard Borel space $X$. If there is a Borel map $f\colon [0,+\infty)^{\mathbb{N}} \to X$ such that

1. $f(\boldsymbol{t})Ef(\boldsymbol{t}')$ whenever $\boldsymbol{t}, \boldsymbol{t}' \in [0,+\infty)^{\mathbb{N}}$ satisfy $\boldsymbol{t} - \boldsymbol{t}' \in \ell^1$, and

2. the preimage under $f$ of any $E$-class is meager,

then $E$ is not classifiable by countable structures.

We will apply such a criterion to establish our main result.

### Nonclassification

Fix a sequence $\boldsymbol{d} = (d_n)_{n\in\mathbb{N}}$ of integers such that for every distinct $n, m \in \mathbb{N}$, neither $d_n$ divides $d_m$ nor $d_m$ divides $d_n$. In particular, this holds if the numbers $d_n$ are pairwise coprime.

The same argument works if one only assumes that all but finitely many values of $\boldsymbol{d}$ satisfy such an assumption. We endow $[1,+\infty)^{\mathbb{N}}$ with the product topology, and regard it as the parametrizing space for UHF $L^p$-operator algebras of type $\boldsymbol{d}$ obtained from a diagonal system of similarities, as described in the previous section; see also [208, Section 3 and Section 5]. We therefore regard (complete) isomorphism and (complete) commensurability of UHF $L^p$-operator algebras of type $\boldsymbol{d}$, obtained from a diagonal system of similarities, as equivalence relations on $[1,+\infty)^{\mathbb{N}}$.

For $\boldsymbol{\gamma} \in [1,+\infty)^{\mathbb{N}}$, we denote by $A^{\boldsymbol{\gamma}}$ the corresponding UHF $L^p$-operator algebra. In the following, we will denote by $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}'$ sequences $(\gamma_n)_{n\in\mathbb{N}}$ and $(\gamma'_n)_{n\in\mathbb{N}}$ in $[1,+\infty)^{\mathbb{N}}$. For $\gamma \in [1,+\infty)$, we denote by $M_d^{\gamma}$ the $L^p$-operator algebra structure on $M_d$ induced by the representation $\rho^{\gamma}$ defined in Section XXI.1. The corresponding matrix norms on $M_d^{\gamma}$ are denoted by $\|\cdot\|_{\gamma}$. In particular, when $\gamma = 1$ one obtains the matrix norms induced by the spatial representation $\sigma^d$ of $M_d$. The algebra $A^{\boldsymbol{\gamma}}$ can be seen as the $L^p$-operator tensor product $\bigotimes_{n\in\mathbb{N}}^{p} M_{d_n}^{\gamma_n}$, as defined in [209, Definition 1.9]. (Note that, unlike in [209], we write the Hlder exponent $p$ as a superscript in the notation for tensor products.)

**Lemma XXI.3.1.** Let $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in [1,+\infty)^{\mathbb{N}}$ satisfy

$$L := \prod_{n\in\mathbb{N}} \frac{\gamma_n}{\gamma'_n} < +\infty.$$

Then the identity map on the algebraic tensor product $M_{\boldsymbol{d}} = \bigotimes_{n\in\mathbb{N}} M_{d_n}$ extends to a completely bounded unital homomorphism $A^{\boldsymbol{\gamma}} \to A^{\boldsymbol{\gamma}'}$, with $\|\varphi\|_{cb} \leq L$. In other words, the matrix norms $\|\cdot\|_{\boldsymbol{\gamma}}$ and $\|\cdot\|_{\boldsymbol{\gamma}'}$ on the algebraic tensor product $\bigotimes_{n\in\mathbb{N}} M_{d_n}$ satisfy

$$\|\cdot\|_{\boldsymbol{\gamma}'} \leq L\|\cdot\|_{\boldsymbol{\gamma}}.$$

*Proof.* For $j \in \mathbb{N}$, let $L_j = \frac{\gamma_j}{\gamma'_j}$. Fix $\varepsilon > 0$. In order to prove our assertion, it is enough to show that if $k \in \mathbb{N}$ and $x$ is an element of $M_k\left(\bigotimes_{j\in\mathbb{N}} M_{d_j}\right)$, then $\|x\|_{\boldsymbol{\gamma}'} \leq (1+\varepsilon)L\|x\|_{\boldsymbol{\gamma}}$. Let $x \in M_k\left(\bigotimes_{j\in\mathbb{N}} M_{d_j}\right)$, and choose $n, m \in \mathbb{N}$ and $X_{i,j} \in M_k(M_{d_i})$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, satisfying

$$x = \sum_{1\leq j\leq m} X_{1,j} \otimes \cdots \otimes X_{n,j}$$

By definition of the matrix norms on $A_{\boldsymbol{\gamma}}$, for $1 \leq i \leq n$ there exists a diagonal matrix $w_i \in M_{d_i}$ with entries in $[1, \gamma_i]$ such that, if $W_i \in M_k(M_{d_i})$ is the diagonal matrix with entries in $M_{d_i}$, and nonzero entries equal to $w_i$ (in other words, $W_i = 1_{M_k} \otimes w_i$), then

$$\|x\|_{\boldsymbol{\gamma}} \leq (1+\varepsilon) \left\| \sum_{1 \leq j \leq m} W_1 X_{1,j} W_1^{-1} \otimes \cdots \otimes W_n X_{n,j} X_n^{-1} \right\|.$$

For $1 \leq i \leq n$, we denote the diagonal entries of $w_i \in M_{d_j}$ by $a_{i,\ell}$, for $\ell = 1, \ldots, d_i$. We will define two other diagonal matrices

$$w_i' = \operatorname{diag}(a_{i,1}', \ldots, a_{i,d_i}') \quad \text{and} \quad r_i = \operatorname{diag}(r_{i,1}, \ldots, r_{i,d_i})$$

in $M_{d_i}$, with entries in $[1, \gamma_i']$ and $[1, L_i]$, respectively, as follows. For $1 \leq \ell \leq d_i$, we set

$$a_{i,\ell}' = \begin{cases} a_{i,\ell}, & \text{if } a_{i,\ell} < \gamma_i'; \\ \gamma_i', & \text{if } a_{j,\ell} \geq \gamma_i'. \end{cases}$$

and

$$r_{i,\ell} = \begin{cases} 1, & \text{if } a_{i,\ell} < \gamma_i'; \\ \frac{1}{\gamma_i'} a_{i,\ell}, & \text{if } a_{i,\ell} \geq \gamma_i'. \end{cases}$$

Observe that $r_{i,\ell}$ belongs to $[1, L_i]$ (since $a_{i,\ell} \leq \gamma_i \leq L_i \gamma_i'$), and that $a_{i,\ell}'$ belongs to $[1, \gamma_i']$ for all $1 \leq i \leq n$ and $1 \leq \ell \leq d_i$.

Define $w_i'$ and $r_i$ to be the diagonal $d_i \times d_i$ matrices with diagonal entries $a_{i,\ell}'$ and $r_{i,\ell}$ for $1 \leq \ell \leq d_i$. Let $W_i', R_i \in M_k(M_{d_i})$ be the diagonal $k \times k$ matrices with entries in $M_{d_i}$ having diagonal entries equal to, respectively, $w_i'$ and $r_i$. (In other words, $W_i' = 1_{M_k} \otimes w_i'$ and $R_i = 1_{M_k} \otimes r_i$.)

Then $W_i = R_i W_i'$ for all $1 \leq i \leq n$. Additionally,

$$\|R_i\| \leq L_i \quad \text{and} \quad \left\|R_i^{-1}\right\| \leq 1.$$

Therefore,

$$\begin{aligned}
\|x\|_{\gamma} &\leq (1+\varepsilon)\left\| \sum_{1\leq j\leq m} W_1 X_{1,j} W_1^{-1} \otimes \cdots \otimes W_n X_{n,j} W_n^{-1} \right\| \\
&= (1+\varepsilon)\left\| \sum_{1\leq j\leq m} R_1 W_1' X_{1,j} W_1^{'-1} R_1^{-1} \otimes \cdots \otimes R_n W_n' X_{n,j} W_n^{'-1} R_n^{-1} \right\| \\
&\leq (1+\varepsilon)\,\|R_1\|\,\|R_2\|\cdots\|R_n\| \left\| \sum_{1\leq j\leq m} W_1' X_{1,j} W_1'^{-1} \otimes \cdots \otimes W_n' X_{n,j} W_n'^{-1} \right\| \\
&\leq (1+\varepsilon) L_1 \cdots L_n \left\| \sum_{1\leq j\leq m} W_1' X_{1,j} W_1'^{-1} \otimes \cdots \otimes W_n' X_{n,j} W_n'^{-1} \right\| \\
&\leq (1+\varepsilon) L \, \|x\|_{\gamma'} .
\end{aligned}$$

This concludes the proof. $\square$

**Corollary XXI.3.2.** If $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in [1,+\infty)^{\mathbb{N}}$ satisfy

$$\prod_{n\in\mathbb{N}} \max\left\{ \frac{\gamma_n}{\gamma_n'}, \frac{\gamma_n'}{\gamma_n} \right\} < +\infty,$$

then $A^{\gamma}$ and $A^{\gamma'}$ are completely isomorphic.

The following lemma can be proved in the same way as [208, Lemma 5.11] with the extra ingredient of [208, Lemma 5.8]. As before, we denote by $\otimes^p$ the $L^p$-operator tensor product; see [209, Definition 1.9].

**Lemma XXI.3.3.** Let $L > 0$ and let $d \in \mathbb{N}$. Then there is a constant $R(L,d) > 0$ such that the following holds. Whenever $A$ is a unital $L^p$-operator algebra, whenever $\gamma, \gamma' \in [1,+\infty)$ satisfy

$$\gamma' \geq R(L,d)\gamma,$$

and $\varphi\colon M_d^{\gamma} \to M_d^{\gamma'} \otimes^p A$ is a unital homomorphism with $\|\varphi\| \leq L$, there exists a unital homomorphism $\psi\colon M_d^{\gamma} \to A$ with $\|\psi\| \leq L+1$.

Our assumption on the values of $\boldsymbol{d}$ will be used for the first time in the next lemma, where it is shown that sufficiently different sequences yield noncommensurable UHF $L^p$-operator algebras.

The $K_0$-group of a Banach algebra $A$ is defined using idempotents in matrices over $A$, and a suitable equivalence relation involving similarities of such idempotents. We refer the reader to [13, Chapters 5,8,9] for the precise definition and some basic properties. What we will need here is the following:

**Remark XXI.3.4.** For $n \in \mathbb{N}$ and a unital Banach algebra $A$, if there exists a unital, continuous homomorphism $M_n \to A$, then the class of unit of $A$ in $K_0(A)$ must be divisible by $n$.

**Lemma XXI.3.5.** Suppose that $\boldsymbol{\gamma}, \boldsymbol{\gamma}' \in [1, +\infty)^{\mathbb{N}}$ satisfy $\gamma_n' \geq R(n, d_n)\gamma_n$ for infinitely many $n \in \mathbb{N}$. Then there is no continuous unital homomorphism $\varphi\colon A^{\boldsymbol{\gamma}} \to A^{\boldsymbol{\gamma}'}$.

*Proof.* Assume by contradiction that $\varphi\colon A^{\boldsymbol{\gamma}} \to A^{\boldsymbol{\gamma}'}$ is a continuous unital homomorphism and set $L = \|\varphi\|$. Pick $n \in \mathbb{N}$ such that $n \geq L$ and $\gamma_n' \geq R(n, d_n)\gamma_n$. Set

$$A = \bigotimes_{m \in \mathbb{N}, m \neq n}^{p} M_{d_m}^{\gamma_m}.$$

Apply Lemma XXI.3.3 to the unital homomorphism $\varphi\colon M_{d_n}^{\gamma_n} \to M_{d_n}^{\gamma_n} \otimes^p A$, to get a unital homomorphism $\psi\colon M_{d_n}^{\gamma_n} \to A$ with $\|\psi\| \leq L + 1$.

Using Remark XXI.3.4, we conclude that the class of the unit of $A$ in $K_0(A)$ is divisible by $d_n$. On the other hand, the $K$-theory of $A$ is easy to compute using that $K$-theory for Banach algebras commutes with direct limits (with contractive maps). We get

$$K_0(A) = \mathbb{Z}\left[\frac{1}{b} : b \neq 0 \text{ divides } d_m \text{ for some } m \neq n\right]$$

with the unit of $A$ corresponding to $1 \in K_0(A) \subseteq \mathbb{Q}$.

Since there is a prime appearing in the factorization of $d_n$ that does not divide any $d_m$, for $m \neq n$, we deduce that the class of the unit of $A$ in $K_0(A)$ cannot be divisible by $d_n$. This contradiction shows that there is no continuous unital homomorphism $\varphi\colon A^{\boldsymbol{\gamma}} \to A^{\boldsymbol{\gamma}'}$ $\square$

We say that a set is *comeager* if its complement is meager. Observe that, by definition, a nonmeager set interescts every comeager set. Recall that we regard $[1, +\infty)^{\mathbb{N}}$ as the parametrizing space of the UHF $L^p$-operator algebras of tensor product type $\boldsymbol{d}$ obtained from a diagonal system of similarities. Consistently, we regard (complete) isomorphism and commensurability of such algebras as equivalence relations on $[1, +\infty)^{\mathbb{N}}$.

*Proof of Theorem XXI.1.5.* By [209, Theorem 3.19(3)], every UHF $L^p$-operator algebra of tensor product type $\boldsymbol{d}$ obtained from a diagonal system of similarities is simple and monotracial. Therefore, it is enough to prove the second assertion of Theorem XXI.1.5. For $\boldsymbol{t} \in [0,+\infty)^{\mathbb{N}}$, define $\exp(\boldsymbol{t})$ to be the sequence $(\exp(t_n))_{n\in\mathbb{N}}$ of real numbers in $[1,\infty)$. By Corollary XXI.3.2, if $\boldsymbol{t},\boldsymbol{t'} \in [0,+\infty)^{\mathbb{N}}$ satisfy $\boldsymbol{t}-\boldsymbol{t'} \in \ell^1$, then $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t'})}$ are completely isomorphic. We claim that for any nonmeager subset $C$ of $[0,+\infty)^{\mathbb{N}}$ one can find $\boldsymbol{t},\boldsymbol{t'} \in C$ such that $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t'})}$ are not commensurable. This fact together with Corollary XXI.3.2 will show that the Borel function

$$\begin{aligned}[0,+\infty)^{\mathbb{N}} &\to [1,+\infty)^{\mathbb{N}}\\ \boldsymbol{t} &\mapsto \exp(\boldsymbol{t})\end{aligned}$$

satisfies the hypotheses of Proposition XXI.2.2 for any of the equivalence relations $E$ in the statement of Theorem XXI.1.5, yielding the desired conclusion.

Let then $C$ be a nonmeager subset of $[0,+\infty)^{\mathbb{N}}$, and fix $\boldsymbol{t} \in C$. We want to find $\boldsymbol{t'} \in C$ such that $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t'})}$ are not commensurable. The set

$$\begin{aligned}&\left\{\boldsymbol{t'} \in [0,+\infty)^{\mathbb{N}} \colon \text{ for all but finitely many } n\in\mathbb{N},\ \exp(t'_n) \leq R(n,d_n)\exp(t_n)\right\}\\ &= \bigcup_{k\in\mathbb{N}} \left\{\boldsymbol{t'} \in [0,+\infty)^{\mathbb{N}} \colon \forall n \geq k,\ \exp(t'_n) \leq R(n,d_n)\exp(t_n)\right\}\end{aligned}$$

is a countable union of closed nowhere dense sets, hence meager. Therefore, its complement

$$\left\{\boldsymbol{t'} \in [0,+\infty)^{\mathbb{N}} \colon \text{ for infinitely many } n\in\mathbb{N},\ \exp(t'_n) > R(n,d_n)\exp(t_n)\right\},$$

is comeager. In particular, since $C$ is nonmeager, there is $\boldsymbol{t'} \in C$ such that $\exp(t'_n) \geq R(n,d_n)\exp(t_n)$ for infinitely many $n \in \mathbb{N}$. By Lemma XXI.3.3, there is no continuous unital homomorphism from $A^{\exp(\boldsymbol{t})}$ to $A^{\exp(\boldsymbol{t'})}$. Therefore $A^{\exp(\boldsymbol{t})}$ and $A^{\exp(\boldsymbol{t'})}$ are not commensurable. This concludes the proof of the above claim. ☐

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