CROSSED PRODUCT C*-ALGEBRAS OF MINIMAL DYNAMICAL SYSTEMS ON THE PRODUCT OF THE CANTOR SET AND THE TORUS

by

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Title: CROSSED PRODUCT C*-ALGEBRAS OF MINIMAL DYNAMICAL SYSTEMS ON THE PRODUCT OF THE CANTOR SET AND THE TORUS

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This dissertation is a study of the relationship between minimal dynamical systems on the product of the Cantor set (X) and torus ($\mathbb{T}^2$) and their corresponding crossed product C^*-algebras.

For the case when the cocyles are rotations, we studied the structure of the crossed product C^*-algebra A by looking at a large subalgebra A_x. It is proved that, as long as the cocyles are rotations, the tracial rank of the crossed product C^*-algebra is always no more than one, which then indicates that it falls into the category of classifiable C^*-algebras. In order to determine whether the corresponding crossed product C^*-algebras of two such minimal dynamical systems are isomorphic or not, we just need to look at the Elliott invariants of these C^*-algebras.

If a certain rigidity condition is satisfied, it is shown that the crossed product C^*-algebra has tracial rank zero. Under this assumption, it is proved that for two such dynamical systems, if A and B are the corresponding crossed product C^*-algebras, and we have an isomorphism between $K_i(A)$ and $K_i(B)$ which maps $K_i(C(X \times \mathbb{T}^2))$ to $K_i(C(X \times \mathbb{T}^2))$, then these two dynamical systems are approximately K-conjugate. The proof also indicates that C^*-strongly flip conjugacy implies approximate K-conjugacy in this case.

We also studied the case when the cocyles are Furstenberg transformations, and some results on weakly approximate conjugacy and the K-theory of corresponding crossed product C^*-algebras are obtained.

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CHAPTER I

INTRODUCTION AND NOTATION

I.1 INTRODUCTION

Let X be a compact metric space, and let $\alpha \in \mathrm{Homeo}(X)$ be a minimal homeomorphism of X. We can construct the crossed product C*-algebra from the minimal dynamical system (X, α), denoted by $C^*(\mathbb{Z}, X, \alpha)$.

One interesting question is how properties of the dynamical system (X, α) determine properties of the crossed product C*-algebra, and how properties of the crossed product C*-algebras shed some light on properties of the dynamical system (X, α).

For minimal Cantor dynamical systems, Giodano, Putnam and Skau studied how the relationship between two such dynamical systems and the relationship between the corresponding crossed product C*-algebras interplay with each other. They found (in [GPS]) that for two minimal Cantor dynamical systems, the corresponding crossed product C*-algebras are isomorphic if and only if the minimal Cantor dynamical systems are strongly orbit equivalent.

Lin and Matui studied this problem when the base space is the product of the Cantor set and the circle (see [LM1], [LM2]), and they discovered that in the rigid cases (see Definition 3.1 of [LM1]), for two crossed product C*-algebras to be isomorphic, the dynamical systems must be approximately K-conjugate (a "strengthened" version of weak approximate conjugacy, in the sense that it is compatible with the K-data).

We study minimal dynamical systems on the product of the Cantor set and the torus. For the case that the cocycles take values in the rotation group, similar results are found for the relationship between C*-algebra isomorphisms and approximate K-conjugacy between two dynamical systems. It is also shown that the tracial rank of the crossed product C*-algebra is no more than one.

For the case that the cocycles are Furstenberg transformations, a necessary condition for weak approximate conjugacy between two minimal dynamical systems (via conjugacy maps whose cocycles are Furstenberg transformations) is given.

I.2 NOTATION

Let (X, α) be a minimal dynamical system, by α-invariant probability measure μ, we mean such a probability measure μ on X satisfying $\mu(D) = \mu(\alpha(D))$ for every μ-measurable subset D. Following the Markov-Kakutani fixed point Theorem, it is shown that the set of α-invariant probability measures is not empty (see Lemma 1.9.18 and Theorem 1.9.19 of [Lin2] for details).

Let μ be a measure on X. For $f \in C(X)$, we use $\mu(f)$ to denote $\int_X f(x)\, \mathrm{d}\mu$.

For a minimal dynamical system (X, α) we use $C^*(\mathbb{Z}, X, \alpha)$ to denote $C(X) \times_\alpha \mathbb{Z}$, the crossed product C*-algebra of the dynamical system (X, α).

In a topological space X, we say a subset D is clopen, if D is both closed and open.

In Chapters II, III, IV and V, unless otherwise specified, X denotes the Cantor set, $\mathbb{T}$ denotes the circle, and $\mathbb{T}^2$ denotes the two-dimensional torus.

For a compact Hausdorff space Y, Homeo(Y) is used to denote the set of all the homeomorphisms of Y.

As the Cantor set X is totally disconnected, we can write a homeomorphism of $X \times \mathbb{T}^2$ as $\alpha \times \varphi$ (the skew product form), with $\alpha \in$ Homeo(X) and $\varphi\colon X \to$ Homeo$(\mathbb{T}^2)$ being continuous, and

$$\alpha \times \varphi\colon X \times \mathbb{T}^2 \to X \times \mathbb{T}^2 \text{ defined by } (x, t_1, t_2) \mapsto (\alpha(x), \varphi(x)(t_1, t_2)).$$

For the case that the cocycles take values in rotation groups, we can further express $\alpha \times \varphi$ as $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$, with $\xi, \eta\colon X \to \mathbb{T}$ continuous, and

$$\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta\colon X \times \mathbb{T}^2 \to X \times \mathbb{T}^2 \text{ defined by } (x, t_1, t_2) \mapsto (\varphi(x), t_1 + \xi(x), t_2 + \eta(x)).$$

We use A to denote the corresponding crossed product C*-algebra. For $x \in X$, the subalgebra A_x is defined as below.

Definition I.2.1. *For a minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$, A_x is defined to be the subalgebra of the crossed product C*-algebra generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $u \cdot C_0((X \backslash \{x\}) \times \mathbb{T} \times \mathbb{T})$, with u being the implementing unitary in A satisfying $u^* f u = f \circ (\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)^{-1}$.*

Remark: From the definition, if D is a clopen subset of the Cantor set X, and $1_{D \times \mathbb{T}^2}$ is the characteristic function of $D \times \mathbb{T}^2$, then $u 1_{D \times \mathbb{T}^2} u^* = 1_{D \times \mathbb{T}^2} \circ (\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta) = 1_{\alpha^{-1}(D) \times \mathbb{T}^2}$.

Let $\{\mathcal{P}_n\colon n \in \mathbb{N}\}$ be as in the Bratteli-Vershik model of the minimal Cantor dynamical system (X, α) (see [HPS, Theorem 4.2]), and let Y_n be the roof of $\mathcal{P}_n$ (denoted as $R(\mathcal{P}_n)$). Then $\{Y_n\}$ will be a decreasing sequence of clopen sets such that $\bigcap_{n=1}^\infty Y_n = \{x\}$. Use A_n to denote the subalgebra generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $u \cdot C_0((X \backslash Y_n) \times \mathbb{T} \times \mathbb{T})$.

In a C*-algebra A, for $a, b \in A$, $a \approx_\varepsilon b$ just means $\|a - b\| \leq \varepsilon$. By $a \approx_{\varepsilon_1} b \approx_{\varepsilon_2} c$, we mean $\|a - b\| \leq \varepsilon_1$ and $\|b - c\| \leq \varepsilon_2$. It is clear that $a \approx_{\varepsilon_1} b \approx_{\varepsilon_2} c$ implies $a \approx_{\varepsilon_1 + \varepsilon_2} c$.

In a C*-algebra A, $[a, b]$ (the commutator) is defined to be $ab - ba$.

For a C*-algebra A we use $T(A)$ to denote the convex set of all the tracial states on A, and $\mathrm{Aff}(T(A))$ to denote all the affine linear functions from $T(A)$ to $\mathbb{R}$.

In a C*-algebra A, for $a \in A_+$, we use $\mathrm{Her}(a)$ to denote the smallest hereditary subaglebra that contains a.

For a C*-algebra A we use $\mathrm{TR}(A)$ to denote the tracial rank of A. The detailed definition of tracial rank can be found in [Lin4, Definition 3.6.2]. We use $\mathrm{RR}(A)$ to denote the real rank of A and $\mathrm{tsr}(A)$ to denote the stable rank of A. The detailed definition of real rank and stable rank can be found in [Lin4, Definition 3.1.6] and [Lin4, Definition 3.1.1].

Definition I.2.2. *Let A be a C^*-algebra. Let p be a projection of A and let $a \in A_+$. We say that $p \precsim a$ if p is Murray-von Neumann equivalent to a projection $q \in \mathrm{Her}(a)$.*

Let A be a C*-algebra. We use $U(A)$ to denote the group of all the unitary elements in A. We use $CU(A)$ to denote the norm closure of the group generated by the commutators of $U(A)$. In other words, $CU(A)$ is the norm closure of the group generated by elements in $\{uvu^*v^* : u, v \in U(A)\}$. One can check that $CU(A)$ is a normal subgroup of $U(A)$ and $U(A)/CU(A)$ is an abelian group.

Definition I.2.3. *Let $\varphi : A \longrightarrow B$ be a C^*-algebra homomorphism. We define*

$$\varphi^{\sharp} : U(A)/CU(A) \longrightarrow U(B)/CU(B)$$

to be the map induced by φ which maps $[u] \in U(A)/CU(A)$ to $[\varphi(u)] \in U(B)/CU(B)$.

CHAPTER II

THE STRUCTURE OF THE SUBALGEBRA A_x

In this section, we study properties of a "large" subalgebra of A, namely A_x. The idea of the construction of A_x was first given by Putnam, but the construction here is a bit different from that in the sense that we are removing one fiber $\{x\} \times \mathbb{T} \times \mathbb{T}$ instead of one point. In other words, we define A_x to be the subalgebra generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $u \cdot C_0((X \backslash \{x\}) \times \mathbb{T} \times \mathbb{T})$, with u being the implementing unitary in A (as defined in Section I.2).

II.1 DIRECT LIMIT STRUCTURE OF A_x

The following lemma gives the basic structure of A_x.

Lemma II.1.1. *If $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is minimal, then for any $x \in X$ there are $k_1, k_2, \ldots \in \mathbb{N}$ and $d_{s,n} \in \mathbb{N}$ for $n \in \mathbb{N}$ such that $A_x \cong \varinjlim_n \bigoplus_{s=1}^{k_n} M_{d_{s,n}}(C(\mathbb{T}^2))$.*

Proof. As $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal, it follows that (X, α) is also minimal. For $x \in X$, let $\mathcal{P} = \{X(n, v, k) \colon v \in V_n, k = 1, 2, \ldots, h_n(v)\}$ be as in the Bratteli-Vershik model ([HPS, Theorem 4.2]) for (X, α). Let $R(\mathcal{P}_n)$ be the roof set of $\mathcal{P}_n$, defined by $R(\mathcal{P}_n) = \bigcup_{v \in V_n} X(n, v, h_n(v))$. We can assume that the roof sets satisfy

$$\bigcap_{n \in \mathbb{N}} R(\mathcal{P}_n) = \{x\}.$$

Let A_n be the subalgebra of the crossed product C*-algebra A such that A_n is generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and $u \cdot C_0((X \backslash R(\mathcal{P}_n)) \times \mathbb{T} \times \mathbb{T})$, with u being the implementing unitary element satisfying $ufu^* = f \circ (\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ for all $f \in C(X \times \mathbb{T} \times \mathbb{T})$. Then it is clear that $A_1 \subset A_2 \subset \cdots$.

As we can approximate $f \in C_0((X\backslash\{x\}) \times \mathbb{T} \times \mathbb{T})$ with

$$f_n \in C_0((X\backslash R(\mathcal{P}_n)) \times \mathbb{T} \times \mathbb{T}) = C((X\backslash R(\mathcal{P}_n)) \times \mathbb{T} \times \mathbb{T}),$$

we have $\varinjlim(A_n, \phi_n) = A_x$ with $\phi_n \colon A_n \to A_{n+1}$ being the canonical embedding.

For $C(X\backslash R(\mathcal{P}_n) \times \mathbb{T} \times \mathbb{T})$, it is clear that we have

$$C((X\backslash R(\mathcal{P}_n)) \times \mathbb{T} \times \mathbb{T}) \cong \bigoplus_{v \in V_n} \bigoplus_{1 \le k \le h_n(v)-1} C\left(X(n,v,k) \times \mathbb{T}^2\right).$$

We will show that $A_n \cong \bigoplus_{v \in V_n} M_{h_n(v)}(C(X(n,v,1)) \otimes C(\mathbb{T}^2))$.

Let $e^v_{i,j} = 1_{X(n,v,i)} \cdot u^{i-j}$. Then $e^v_{i,j} \cdot e^{v'}_{i',j'} = 0$ if $v \neq v'$. Note that

$$\begin{aligned} e^v_{i,j} \cdot e^v_{k,s} &= 1_{X(n,v,i)} \cdot u^{i-j} \cdot 1_{X(n,v,k)} \cdot u^{k-s} \\ &= 1_{X(n,v,i)} \cdot 1_{X(n,v,k+i-j)} \cdot u^{i-j+k-s} \\ &= \delta_{k,j} \cdot e^v_{i,s}. \end{aligned}$$

In other words, $\{e^v_{i,j}\}_{i,j=1}^{h(v)}$ is a system of matrix units.

As A_n is generated by

$$\{e^v_{i,j} \otimes C\left(X(n,v,0) \otimes C(T^2)\right) : v \in V_n, 1 \le i,j \le h(v)\},$$

it follows that

$$A_n \cong \bigoplus_{v \in V_n} M_{h_n(v)}\left(C(X(n,v,1)) \otimes C(\mathbb{T}^2)\right).$$

Let $B_n = \bigoplus_{v \in V_n} M_{h_n(v)}(\mathbb{C} \otimes C(\mathbb{T}^2))$. Then it is clear that B_n can be regarded as a subalgebra of A_n.

As for the canonical embedding $\phi_{n,n+1} \colon A_n \to A_{n+1}$, consider

$$a \in A_n \cong \bigoplus_{v \in V_n} M_{h_n(v)}(C(X(n,v,1)) \otimes C(\mathbb{T}^2))$$

such that $a = (f \otimes g) \cdot u^{i-j} \in e^v_{i,j} \otimes C(X(n,v,1) \otimes C(\mathbb{T}^2))$, with $f \in C(X(n,v,i)) \cong C(X(n,v,1))$ and $g \in C(\mathbb{T}^2)$.

Note that the Kakutani-Rokhlin partition of A_{n+1} is finer than that of A_n. We can write

$$f = \sum_{X(n+1,v_s,k)\subset X(n,v,i)} f_{s,k} \quad \text{with } f_{s,k} \in C(X(n+1,v_s,k)).$$

It follows that

$$\phi_{n,n+1}(f \otimes g) = \sum_{X(n+1,v_s,k)\subset X(n,v,i)} f_{s,k} \otimes g.$$

Then we have

$$\begin{aligned}\phi_{n,n+1}(a) &= \left(\sum_{X(n+1,v_s,k)\subset X(n,v,i)} f_{s,k} \otimes g\right) \cdot u^{i-j}\\ &= \sum_{X(n+1,v_s,k)\subset X(n,v,i)} (f_{s,k} \otimes g) \cdot u^{i-j},\end{aligned}$$

with $\sum_{X(n+1,v_s,k)\subset X(n,v,i)} (f_{s,k} \otimes g)\cdot u^{i-j}$ being an element in A_{n+1}. It is then clear that $\phi_{n,n+1}(B_n) \subset B_{n+1}$ if we regard B_n as a subalgebra of A_n and B_{n+1} as a subalgebra of A_{n+1}.

Just abuse notation and use $\phi_{n,n+1}$ to denote the canonical embedding from B_n to B_{n+1}. Then we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & B_n \arrow[r, "\phi_{n,n+1}"] \arrow[d, "j_n"] & B_{n+1} \arrow[r, "\phi_{n+1,n+2}"] \arrow[d, "j_{n+1}"] & B_{n+2} \arrow[r] \arrow[d, "j_{n+2}"] & \cdots \\
\cdots \arrow[r] & A_n \arrow[r, "\phi_{n,n+1}"'] & A_{n+1} \arrow[r, "\phi_{n+1,n+2}"'] & A_{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

For every $a \in A_x = \varinjlim(A_n, \phi_{n,n+1})$ and every $\varepsilon > 0$, there exists $a_n \in A_n$ such that $\|a - a_n\| < \varepsilon/2$ if we identity a_n with $\phi_{n,\infty}(a_n) \in A_x$. Without loss of generality, we can assume that

$$a_n = \sum_{k=1}^{L} \sum_{v\in V_n} \sum_{i,j=1}^{h_n(v)} (f_{k,v,i,j} \otimes g_{k,v,i,j}) \cdot e^v_{i,j},$$

with $f_{k,v,i,j} \in C(X(n,v,0))$ and $g_{k,v,i,j} \in C(\mathbb{T}^2)$.

Let $M = \max_{k,v,i,j}\{\|g_{k,v,i,j}\|\}$. For all k, v, i, j as above, we can find $\delta > 0$ such that for $x, y \in X$, if $\operatorname{dist}(x, y) < \delta$, then

$$\|f_{k,v,i,j}(x) - f_{k,v,i,j}(y)\| < \frac{\varepsilon}{2 \cdot M \cdot L \cdot |V_n| \cdot h_n(v)^2}.$$

According to the Bratteli-Vershik model, $\bigcap_{n\in\mathbb{N}} R(\mathcal{P}_n) = \{x\}$. We may further require that for all $n \in \mathbb{N}$, every block $X(n, v, k)$ in $\mathcal{P}_n$ satisfies $\operatorname{diam}(X(n, v, k)) < 1/n$. Then we can choose $N \in \mathbb{N}$ such that $\operatorname{diam}(R(\mathcal{P}_N)) < \delta$. Without loss of generality, we can assume that $N \geq n$.

In $\mathcal{P}_N$, for every $X(N, v, k)$, choose $w_{N,v,k} \in X(N, v, k)$. For $k = 1, \ldots, L$, $v \in V_n$, $i, j = 1, \ldots, h_n(v)$, define

$$\widetilde{f_{k,v,i,j}} = \sum_{X(N,v',k')\subset X(n,v,k)} f_{k,v,i,j}(w_{N,v',k'}) \cdot 1_{X(N,v',k')}.$$

According to our choice of N, it is clear that $\|f_{k,v,i,j} - \widetilde{f_{k,v,i,j}}\| < \frac{\varepsilon}{2\cdot M\cdot L\cdot|V_n|\cdot h_n(v)^2}$.

For the a_n given above, define $\widetilde{a_n} \in A_n$ by

$$\widetilde{a_n} = \sum_{k=1}^{L} \sum_{v\in V_n} \sum_{i,j=1}^{h_n(v)} \left(\widetilde{f_{k,v,i,j}} \otimes g_{k,v,i,j}\right) \cdot e_{i,j}^{v}.$$

As

$$\|f_{k,v,i,j} - \widetilde{f_{k,v,i,j}}\| < \frac{\varepsilon}{2 \cdot M \cdot L \cdot |V_n| \cdot h_n(v)^2},$$

it follows that $\|a_n - \widetilde{a_n}\| < \varepsilon/2$.

As $\widetilde{f_{k,v,i,j}}$ is constant on $X(N, v', k')$, it follows that $\phi_{n,N}(\widetilde{a_n}) \in B_N$. It is clear that

$$\begin{aligned}
\|\phi_{n,N}(\widetilde{a_n}) - a\| &\leq \|\phi_{n,N}(\widetilde{a_n}) - a_n\| + \|a - a_n\| \\
&= \|\widetilde{a_n} - a_n\| + \|a - a_n\| \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

Note that $a \in A_x$ and $\varepsilon > 0$ are arbitrary. It follows that $\bigcup_{n\in\mathbb{N}} \phi_{n,\infty}(B_n)$ is dense in A_x. In other words, we have $\varinjlim(B_n, \phi_{n,n+1}) \cong A_x$. As $B_n = \bigoplus_{v \in V_n} M_{h_n(v)}(\mathbb{C} \otimes C(\mathbb{T}^2))$, we conclude that $A_x \cong \varinjlim \bigoplus_{s=1}^{k_n} M_{d_{s,n}}(C(\mathbb{T}^2))$.

□

Lemma II.1.2. *Let A_x be defined as above. If $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal, then A_x is simple.*

Proof. This proof is essentially the same as that of Proposition 3.3 (5) in [LM1]. It works like this:

Note that $X \times \mathbb{T} \times \mathbb{T}$ is compact and $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal. It is clear that the positive orbit (under $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$) of (x, t_1, t_2) is dense in $X \times \mathbb{T} \times \mathbb{T}$.

The C*-algebra A corresponds to the groupoid C*-algebra associated with the equivalence relation

$$\mathcal{R} = \{((x, t_1, t_2), (\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)^k(x, t_1, t_2)) \colon (x, t_1, t_2) \in X \times \mathbb{T} \times \mathbb{T}\},$$

and the C*-subalgebra A_x corresponds to the groupoid C*-algebra associated with the equivalence relation

$$\mathcal{R}_x = \mathcal{R} \setminus \{(\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)^k(x, t_1, t_2)), (\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)^l(x, t_1, t_2)) \colon$$

$$(t_1, t_2) \in \mathbb{T} \times \mathbb{T}, k \geq 0, l \leq 0 \text{ or } k \leq 0,\ l \geq 0\}.$$

As the positive orbit of any (x, t_1, t_2) is dense in $X \times \mathbb{T} \times \mathbb{T}$, it follows that each equivalence class of $\mathcal{R}_x$ is dense in $X \times \mathbb{T} \times \mathbb{T}$. According to [Renault, Proposition 4.6], this is equivalent to the simplicity of A_x.

□

II.2 K-THEORY OF A_x

In this section, we study the K-theory of A_x using its direct limit structure.

Lemma II.2.1. *The group $K_0(C(\mathbb{T}^2))$ is order isomorphic to $\mathbb{Z}^2$ with the unit element identified with $(1,0)$ and the positive cone D being $\{(m,n)\colon m>0\}\cup\{(0,0)\}$, and the group $K_1(C(\mathbb{T}^2))$ is isomorphic to $\mathbb{Z}^2$.*

Proof. By the Künneth Theorem, it follows that

$$K_0(C(\mathbb{T}^2)) \cong K_0(C(\mathbb{T}))\otimes K_0(C(\mathbb{T}))\bigoplus K_1(C(\mathbb{T}))\otimes K_1(C(\mathbb{T}))\cong \mathbb{Z}^2,$$

and

$$K_1(C(\mathbb{T}^2)) \cong K_0(C(\mathbb{T}))\otimes K_1(C(\mathbb{T}))\bigoplus K_1(C(\mathbb{T}))\otimes K_0(C(\mathbb{T}))\cong \mathbb{Z}^2.$$

For $C(\mathbb{T}^2)$, it is known that the order on $K_0(C(\mathbb{T}^2))$ is determined by the first copy of $\mathbb{Z}$, which corresponds to the rank of projections. It follows that $K_0(C(\mathbb{T}^2))_+$ can be identified with D.

□

Lemma II.2.2. *There is an isomorphism $\iota\colon K_0(C(X\times\mathbb{T}^2))\longrightarrow C(X,\mathbb{Z}^2)$ which sends $[1]$ to the constant function with value $(1,0)$. Furthermore, ι maps $K_0(C(X\times\mathbb{T}^2))_+$ onto $C(X,D)$, with D as defined in Lemma II.2.1.*

Moreover, for a clopen set U of X and a projection $\eta\in M_k(C(\mathbb{T}^2))$ such that $[\eta]\in K_0(C(\mathbb{T}^2))$ corresponds to (a,b) as in Lemma II.2.1, $\iota([\mathrm{diag}(\underbrace{1_U,\ldots,1_U}_{k})\cdot\eta])=(1_U\cdot a, 1_U\cdot b)$.

Proof. For D as in Lemma II.2.1, define

$$\varphi\colon C(X,D)\to (K_0(C(X\times\mathbb{T}^2)))_+$$

by

$$\varphi(f)=\sum_{(m,n)\in D}\left[\underbrace{(1_{f^{-1}((m,n))},\ldots,1_{f^{-1}((m,n))})}_{d_{m,n}}\cdot\eta_{m,n}\right],$$

where $\eta_{m,n}$ is a projection in $M_{d_{m,n}}(C(\mathbb{T}^2))$ which is identified with (m,n) as in Lemma II.2.1.

If we can show that φ is one-to-one, preserves addition, and maps the constant function with value $(1,0)$ to $[1_{C(X\times\mathbb{T}^2)}]$, then we can extend φ to a group isomorphism from $C(X,\mathbb{Z}^2)$ to $K_0(C(X\times\mathbb{T}^2))$.

It is easy to check that $\varphi((1,0)) = [1_{C(X\times\mathbb{T}^2)}]$. From the definition, it follows that φ preserves addition. We just need to show that φ is one-to-one.

Injectivity of φ:

If $\varphi(f) = 0$ for some $f \in C(X, D)$, then

$$\sum_{(m,n)\in D}\left[\underbrace{(1_{f^{-1}((m,n))},\ldots,1_{f^{-1}((m,n))})}_{d_{m,n}}\cdot\eta_{m,n}\right] = 0$$

in $(K_0(C(X\times\mathbb{T}^2)))_+$. As

$$K_0(C(X\times\mathbb{T}^2)) \cong \bigoplus_{(m,n)\in D} K_0(C(f^{-1}((m,n))\times\mathbb{T}^2)),$$

we get that

$$[\underbrace{(1_{f^{-1}((m,n))},\ldots,1_{f^{-1}((m,n))})}_{d_{m,n}}\cdot\eta_{m,n}] = 0 \text{ in } K_0(C(f^{-1}((m,n))\times\mathbb{T}^2)) \text{ for all } (m,n)\in D.$$

That is, there exists $k\in\mathbb{N}$ such that

$$\underbrace{(1_{f^{-1}((m,n))},\ldots,1_{f^{-1}((m,n))})}_{d_{m,n}}\cdot\eta_{m,n} \bigoplus \operatorname{diag}(\underbrace{1_{C(f^{-1}((m,n))\times\mathbb{T}^2)},\ldots,1_{C(f^{-1}((m,n))\times\mathbb{T}^2)}}_{k})$$

is Murray-von Neumann equivalent to $\operatorname{diag}(\underbrace{1_{C(f^{-1}((m,n))\times\mathbb{T}^2)},\ldots,1_{C(f^{-1}((m,n))\times\mathbb{T}^2)}}_{k})$.

Let $s\in M_{d_{m,n}+k}(f^{-1}((m,n))\times\mathbb{T}^2)$ be the partial isometry corresponding to the Murray-von Neumann equivalence above. Choose $x\in f^{-1}((m,n))$. Then $s(x)$ can be regarded as an element in $M_{d_{m,n}+k}(\mathbb{T}^2)$ that gives a Murray-von Neumann equivalence between

$$\eta_{m,n}\oplus\operatorname{diag}(\underbrace{1_{C(\mathbb{T}^2))},\ldots,1_{C(\mathbb{T}^2)}}_{k}) \text{ and } \operatorname{diag}(\underbrace{1_{C(\mathbb{T}^2))},\ldots,1_{C(\mathbb{T}^2)}}_{k}).$$

It then follows that $\eta_{m,n} = 0$, which proves injectivity.

Surjectivity of φ:

For every projection $p \in M_\infty(C(X \times \mathbb{T}^2))$, we can find a partition $X = \bigsqcup_{i=1}^{M} X_i$ such that $\|p(x) - p(y)\| < 1$ for all $x, y \in X_i$. Choose $x_i \in X_i$ for $i = 1, \ldots, M$, and identify $M_\infty(C(X \times \mathbb{T}^2))$ with $C(X, M_\infty(C(\mathbb{T}^2)))$. Define $p' \in C(X, M_\infty(C(\mathbb{T}^2)))$ by $p'|_{X_i} = p(x_i)$. It is clear that we can regard $p'|_{X_i}$ as an element in $M_\infty(C(\mathbb{T}^2))$.

Use (a_i, b_i) to denote the corresponding element in $K_0(C(\mathbb{T}^2))$ as identified in Lemma II.2.1 and let $f = \sum_{i=1}^{M} 1_{X_i} \cdot (a_i, b_i)$. Then we can check that $\varphi(f) = [p']$ in $(K_0(C(X \times \mathbb{T}^2)))_+$. As $[p] = [p']$, we have proved surjectivity of φ.

As φ is unital, one-to-one and preserves addition, we can extend it to an ordered group isomorphism $\widetilde{\varphi}\colon C(X, \mathbb{Z}^2) \longrightarrow K_0(C(X \times \mathbb{T}^2))$. Let $\iota = \widetilde{\varphi}^{-1}$, and we have finished the proof.

$\square$

Lemma II.2.3. *There is an isomorphism*

$$\gamma_n\colon A_n \longrightarrow \bigoplus_{v \in V_n} M_{h_n(v)}\left(C(X(n, v, 1)) \otimes C(\mathbb{T}^2)\right),$$

such that for every clopen set U in X,

$$\gamma_n(1_{U \times \mathbb{T}^2}) = \bigoplus_{v \in V_n} \operatorname{diag}\left(1_{X(n,v,1) \cap U}, \ldots, 1_{X(n,v,h(v)) \cap U}\right).$$

Proof. The proof is essentially the same as that of [Putnam, Lemma 3.1]. It can also be obtained as a K-theory version of part of the proof of Lemma II.1.1.

$\square$

Lemma II.2.4. *There is a group isomorphism*

$$\phi\colon \bigoplus_{v\in V_n} C\left(X(n,v,1),\mathbb{Z}^2\right) \longrightarrow C\left(X,\mathbb{Z}^2\right)/\{f - f\circ\alpha^{-1}\colon f\,|_{Y_n} = 0\}$$

such that

$$\phi\left((f_1,\ldots,f_{|V_n|})\right) = \sum_{v\in V_n} [1_{X(n,v,1)}\cdot f_v]$$

for $(f_1,\ldots,f_{|V_n|}) \in \bigoplus_{v\in V_n} C\left(X(n,v,1),\mathbb{Z}^2\right)$.

Furthermore, if we define D to be

$$\{(m,n)\in\mathbb{Z}^2\colon m>0\}\cup\{(0,0)\},$$

and if we define the positive cone of $\bigoplus_{v\in V_n} C(X(n,v,1),\mathbb{Z}^2)$ *to be* $\bigoplus_{v\in V_n} C(X(n,v,1),D)$ *and the positive cone of* $C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1}\colon f\,|_{Y_n} = 0\}$ *to be* $C(X,D)/\{f - f\circ\alpha^{-1}\colon f\,|_{Y_n} = 0\}$*, then both ϕ and ϕ^{-1} are order preserving.*

Proof. For $(f_1,\ldots,f_{|V_n|}) \in \bigoplus_{v\in V_n} C\left(X(n,v,1),\mathbb{Z}^2\right)$, define

$$\phi\left(f_1,\ldots,f_{|V_n|}\right) = \sum_{v\in V_n} [1_{X(n,v,1)}\cdot f_v].$$

Injectivity of ϕ:

Suppose

$$(f_1,\ldots,f_{|V_n|}) \in \bigoplus_{v\in V_n} C(X(n,v,1),\mathbb{Z}^2)$$

and that $\phi((f_1,\ldots,f_{|V_n|})) = 0$. That is, there exists $H\in C(X,\mathbb{Z}^2)$ with $H\,|_{Y_n} = 0$ such that

$$\sum_{v=1}^{|V_n|} f_v = H - H\circ\alpha^{-1}.$$

It follows that

$$\left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)}\right)\cdot\left(\sum_{v=1}^{|V_n|} f_v\right) = \left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)}\right)\cdot\left(H - H\circ\alpha^{-1}\right).$$

As $H|_{Y_n} = 0$,

$$\left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)}\right) \cdot (H \circ \alpha^{-1}) = \left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot H\right) \circ \alpha^{-1}.$$

It then follows that

$$\left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)}\right) \cdot \left(H - H \circ \alpha^{-1}\right) = \left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot H\right) - \left(\sum_{k=1}^{h(v)} 1_{X(n,v,k} \cdot H\right) \circ \alpha^{-1}.$$

Use H_v to denote $\left(\sum_{k=1}^{h(v)} 1_{X(n,v,k)}\right) \cdot H$. It is clear that H_v is supported on $\bigsqcup_{k=1}^{h(v)} X(n,v,k)$.

Now we have $f_v = H_v - H_v \circ \alpha^{-1}$. As f_v is supported on $X(n,v,1)$, we get

$$H_v - H_v \circ \alpha^{-1} = 0$$

on $X(n,v,k)$ for $2 \leq k \leq h(v)$, which implies that for all $x \in X(n,v,1)$,

$$H(x) = H_v(\alpha(x)) = \cdots = H_v\left(\alpha^{h(v)-1}(x)\right).$$

As $\alpha^{h(v)-1}(x) \in Y_n$, it follows that $H_v(\alpha^{h(v)-1}(x)) = 0$. Now we can conclude that $H_v = 0$. It is then clear that $f_v = 0$.

Applying the process to all $v = 1, \ldots, h(v)$, we get $H = 0$. It follows that $f_i = 0$ for $i = 1, \ldots, |V_n|$, which proves the injectivity of ϕ.

Surjectivity of ϕ:

For every $g \in C(X, \mathbb{Z}^2)$, we need to find

$$(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C\left(X(n,v,1), \mathbb{Z}^2\right)$$

such that

$$\phi\left((f_1, \ldots, f_{|V_n|})\right) - g = h - h \circ \alpha^{-1}$$

for some $h \in C(X, \mathbb{Z}^2)$ satisfying $h|_{Y_n} = 0$.

Write g as

$$g = 1 \cdot g = \sum_{v \in V_n} \sum_{k=1}^{h(v)} 1_{X(n,v,k)} \cdot g.$$

For every k with $2 \leq k \leq h(v)$, consider $(1_{X(n,v,k)} \cdot g) \circ \alpha$. It is easy to check that $(1_{X(n,v,k)} \cdot g) \circ \alpha \,|_{Y_n} = 0$ and

$$1_{X(n,v,k)} \cdot g + \left((1_{X(n,v,k)} \cdot g) \circ \alpha - (1_{X(n,v,k)} \cdot g) \circ \alpha \circ \alpha^{-1}\right)$$

is supported on $X(n,v,k-1)$.

By repeating this process, we get $s \in C(X, \mathbb{Z}^2)$ such that $1_{X(n,v,k)} \cdot g + (s - s \circ \alpha)$ is supported on $1_{X(n,v,1)}$.

Apply the process for all $1_{X(n,v,k)} \cdot g$ with $v \in V_n$ and $1 < k \leq h(v)$. We can find $H \in C(X, \mathbb{Z}^2)$ such that $g + (H - H \circ \alpha^{-1})$ is supported on $\alpha(R(\mathcal{P}_n)) = \bigoplus_{v \in V_n} X(n,v,1)$. According to the definition, if we set $f_v = 1_{X(n,v,1)} \cdot (g + (H - H \circ \alpha^{-1}))$, then ϕ will map $(f_1, \ldots, f_{|V_n|})$ to g.

Positivity of ϕ:

As

$$\phi\left((f_1, \ldots, f_{|V_n|})\right) = \sum_{v \in V_n} 1_{X(n,v,1)} \cdot f_v,$$

for

$$(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C\left(X(n,v,1), \mathbb{Z}^2\right),$$

if the range of each f_i is in the positive cone D, it is clear that $\sum_{v \in V_n} 1_{X(n,v,1)} \cdot f_v \in C(X, D)$. Thus ϕ is order preserving.

Positivity of ϕ^{-1}:

For $f \in C(X, D)$, we will show that if there is

$$(f_1, \ldots, f_{|V_n|}) \in \bigoplus_{v \in V_n} C\left(X(n,v,1), D\right)$$

such that

$$\phi(f_1, \dots, f_{|V_n|}) = [f],$$

then $f_v \in C(X(n, v, 1)$ for all $1 \leq v \leq |V_n|$.

In fact, such an element $(f_1, \dots, f_{|V_n|})$ can be constructed from f as in the proof of surjectivity of ϕ. The fact that $f \in C(X, D)$ then implies that for all v with $1 \leq v \leq |V_n|$, the image f_k is in D, which finishes the proof.

□

Lemma II.2.5. *There is an order isomorphism*

$$\rho_n \colon K_0(A_n) \longrightarrow C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2), f|_{Y_n} = 0\}$$

with the unit element and positive cone of

$$C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2), f|_{Y_n} = 0\}$$

being $[(1_X, 0)]$ *and*

$$\{[g] \in C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2), f|_{Y_n} = 0\} \colon$$
$$\forall x \in X, g(x) = (0, 0) \text{ or } g(x) = (a, b) \text{ with } a > 0\}.$$

For a clopen subset U *of* X *and* $\eta \in M_k(C(\mathbb{T}^2))$ *such that* $[\eta] \in K_0(C(\mathbb{T}^2))$ *corresponds to* (a, b) *as in Lemma II.2.1,* $\rho_n([\mathrm{diag}(\underbrace{1_U, \dots, 1_U}_{k}) \cdot \eta])$ *is exactly* $[(1_U \cdot a, 1_U \cdot b)]$ *with* 1_U *denoting the continuous function from* X *to* $\mathbb{Z}$ *that is* 1 *on* U *and* 0 *otherwise.*

Proof. Consider the isomorphism

$$\gamma_n \colon A_n \longrightarrow \bigoplus_{v \in V_n} M_{h_n(v)}\left(C(X(n, v, 1)) \otimes C(\mathbb{T}^2)\right)$$

as in Lemma II.2.3. It is clear that

$$(\gamma_n)_{*0}\colon K_0(A_n) \longrightarrow K_0\left(\bigoplus_{v\in V_n} M_{h_n(v)}\left(C(X(n,v,1))\otimes C\left(\mathbb{T}^2\right)\right)\right)$$

is an order isomorphism.

We know that

$$K_0\left(\bigoplus_{v\in V_n} M_{h_n(v)}(C(X(n,v,1))\otimes C(\mathbb{T}^2))\right) \cong \bigoplus_{v\in V_n} K_0\left(M_{h_n(v)}(C(X(n,v,1))\otimes C(\mathbb{T}^2))\right),$$

and use

$$h_n\colon K_0\left(\bigoplus_{v\in V_n} M_{h_n(v)}(C(X(n,v,1))\otimes C(\mathbb{T}^2))\right) \longrightarrow \bigoplus_{v\in V_n} K_0\left(M_{h_n(v)}(C(X(n,v,1))\otimes C(\mathbb{T}^2))\right)$$

to denote the order isomorphism.

There are natural order isomorphisms

$$l_{n,v}\colon K_0(M_{h_n(v)}(C(X(n,v,1))\otimes C(\mathbb{T}^2))) \longrightarrow K_0(C(X(n,v,1))\otimes C(\mathbb{T}^2)).$$

By Lemma II.2.2, we can find order isomorphisms

$$s_{n,v}\colon K_0(C(X(n,v,1))\otimes C(\mathbb{T}^2)) \longrightarrow C(X(n,v,1),\mathbb{Z}^2)$$

such that each $s_{n,v}$ maps $[1_{C(X(n,v,1))\otimes C(\mathbb{T}^2)}]$ to the constant function with value $(1,0)$.

Combining $l_{n,v}$ and $s_{n,v}$ for all v, we get an order isomorphism

$$\varphi\colon \bigoplus_{v\in V_n} K_0(M_{h_n(v)}(C(X(n,v,1))\otimes C(\mathbb{T}^2))) \longrightarrow \bigoplus_{v\in V_n} C(X(n,v,1),\mathbb{Z}^2)$$

with the positive cone of $\bigoplus_{v\in V_n} C(X(n,v,1),\mathbb{Z}^2)$ being $\bigoplus_{v\in V_n} C(X(n,v,1),D)$ (D as defined in Lemma II.2.1). Note that φ is not unital.

According to Lemma II.2.4, there is an order isomorphism

$$\psi\colon \bigoplus_{v\in V_n} C(X(n,v,1),\mathbb{Z}^2) \to C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1}\colon f\,|_{Y_n} = 0\}.$$

Let

$$\rho_n = \psi\circ\varphi\circ h_n\circ(\gamma_n)_{*0}.$$

Then ρ_n is a group isomorphism from $K_0(A_n)$ to

$$C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1}\colon f\in C(X,\mathbb{Z}^2), f\,|_{Y_n} = 0\}$$

because ψ, φ, h_n and $(\gamma_n)_{*0}$ are all group isomorphisms.

According to Lemma II.2.3,

$$\gamma_n(1_{A_n}) = \bigoplus_{v\in V_n} \operatorname{diag}(1_{X(n,v,1)},\ldots,1_{X(n,v,h(v))}).$$

Thus

$$(\gamma_n)_{*0}([1_{A_n}]) = \sum_{v\in V_n}\sum_{1\le k\le h(v)}\left[1_{X(n,v,k)}\right].$$

It is then clear that

$$h_n((\gamma_n)_{*0}([1_{A_n}])) = \left(\sum_{1\le k\le h(1)}\left[1_{X(n,v,k)}\right],\ldots,\sum_{1\le k\le h(|V_n|)}\left[1_{X(n,v,h(k))}\right]\right).$$

Note that $[1_{X(n,v,k)}] = [1_{X(n,v,1)}]$ in $K_0(M_n(X(n,v,1)))$. It follows that

$$\begin{aligned}\varphi\left(h_n((\gamma_n)_{*0}([1_{A_n}]))\right) &= \varphi\left(\sum_{1\le k\le h(1)}[1_{X(n,v,k)}],\ldots,\sum_{1\le k\le h(|V_n|)}[1_{X(n,v,h(k))}]\right)\\ &= \sum_{v\in V_n} h(v)\cdot\left[1_{X(n,v,1)}\right].\end{aligned}$$

According to the definition of ϕ as stated in Lemma II.2.4, we get

$$\psi\left(\varphi(h_n((\gamma_n)_{*0}([1_{A_n}])))\right) = \psi\left(\sum_{v\in V_n} h(v)\cdot [1_{X(n,v,1)}]\right) = \sum_{v\in V_n} [f_v]$$

with $f_v \in C(X,\mathbb{Z}^2)$ satisfying $f_v\,|_{X(n,v,1)} = h(v)$ and $f_v\,|_{X\setminus X(n,v,1)} = 0$.

Let

$$H = \sum_{v\in V_n}\sum_{1\le k\le h(v)-1} 1_{X(n,v,k)}\cdot (h(v)-k).$$

Then it is clear that $H\,|_{Y_n} = 0$ and

$$H_v\circ\alpha^{-1} = \sum_{v\in V_n}\sum_{2\le k\le h(v)} 1_{X(n,v,k)}\cdot (h(v)-k+1).$$

It is easy to check that

$$H - H\circ\alpha^{-1} = \sum_{v\in V_n}\left[\left(\sum_{2\le k\le h(v)} 1_{X(n,v,k)}\cdot(-1)\right) + 1_{X(n,v,1)}\cdot (h(v)-1)\right].$$

In $C(X,\mathbb{Z}^2)$, it is easy to check that $(\sum_{v\in V_n} f_v) - 1_X = H - H\circ\alpha^{-1}$. In other words, we have

$$\psi(\varphi(h_n((\gamma_n)_{*0}([1_{A_n}])))) = \sum_{v\in V_n}[f_v] = [1_X]\ ,$$

which implies that ρ_n is unital.

To show that ρ_n is order preserving, we just need to show that ψ, φ, h_n and $(\gamma_n)_{*0}$ are all order preserving.

It is clear that h_n and $(\gamma_n)_{*0}$ are order preserving. According to Lemma II.2.4, ψ is also order preserving. We just need to show that φ is order preserving.

Note that $\varphi = \bigoplus_{v\in V_n}(s_{n,v}\circ l_{n,v})$. We just need to show that each $s_{n,v}\circ l_{n,v}$ is order preserving. In fact, $l_{n,v}$ is order preserving and $s_{n,v}$ is an order isomorphism. It follows that $s_{n,v}\circ l_{n,v}$ is order preserving. Thus φ is order preserving.

Now we will show that ρ_n is order isomorphism. In fact, we just need to show that for every $(a,b) \in \{(m,n)\colon m > 0, n \in \mathbb{Z}\} \cup \{0,0\}$ and every clopen subset U of X, if we regard $(1_U \cdot a, 1_U \cdot b)$ as a function in $C(X,\mathbb{Z}^2)$ defined by

$$(1_U \cdot a, 1_U \cdot b)(x) = \begin{cases} (a,b) & \text{if } x \in U \\ (0,0) & \text{if } x \notin U \end{cases}$$

and we define

$$\pi\colon C(X,\mathbb{Z}^2) \longrightarrow C(X,\mathbb{Z}^2)/\{f - f \circ \alpha^{-1}\colon f \in C(X,\mathbb{Z}^2), f\,|_{Y_n} = 0\}$$

to be the natural quotient map, then $\pi((1_U \cdot a, 1_U \cdot b))$ is in the image of $\rho_n(K_0(A_n)_+)$.

For a clopen subset U of X and $\eta \in M_k(C(\mathbb{T}^2))$ such that $[\eta] \in K_0(\mathbb{T}^2)$ corresponds to the (a,b) above (see Lemma II.2.1), we have

$$\rho_n([\mathrm{diag}(\underbrace{1_U,\dots,1_U}_{k}) \cdot \eta]) = (\phi \circ \varphi \circ h_n \circ (\gamma_n)_{*0})([\mathrm{diag}(\underbrace{1_U,\dots,1_U}_{k}) \cdot \eta]).$$

According to Lemma II.2.3,

$$\begin{aligned}
&(h_n \circ (\gamma_n)_{*0})([\mathrm{diag}(\underbrace{1_U,\dots,1_U}_{k}) \cdot \eta]) \\
&= (h_n \circ (\gamma_n)_{*0}) \left(\sum_{v \in V_n, 1 \le k \le h(v)} [\mathrm{diag}(\underbrace{1_{X(n,v,k) \cap U},\dots,1_{X(n,v,k) \cap U}}_{k}) \cdot \eta] \right) \\
&= \left(\sum_{1 \le k \le h(v)} \left[1_{X(n,v,k) \cap U} \cdot \eta\right] \right)_{v \in V_n}.
\end{aligned}$$

Then

$$\begin{aligned}
&(\varphi \circ h_n \circ (\gamma_n)_{*0})([\mathrm{diag}(\underbrace{1_U,\dots,1_U}_{k}) \cdot \eta]) \\
&= \left(\sum_{1 \le k \le h(v)} \left(1_{\alpha^{-(k-1)}(X(n,v,k) \cap U)} \cdot a, 1_{\alpha^{-(k-1)}(X(n,v,k) \cap U)} \cdot b\right) \right)_{v \in V_n}
\end{aligned}$$

which is an element of $\bigoplus_{v \in V_n} C(X(n,v,1),\mathbb{Z}^2)$.

According to the definition of ϕ as in Lemma II.2.4, it follows that

$$(\psi \circ \varphi \circ h_n \circ (\gamma_n)_{*0})([\mathrm{diag}(\underbrace{1_U, \ldots, 1_U}_{k}) \cdot \eta]) = (\psi)((\varphi \circ h_n \circ (\gamma_n)_{*0})([\mathrm{diag}(\underbrace{1_U, \ldots, 1_U}_{k}) \cdot \eta]))$$
$$= \sum_{v \in V_n} 1_{X(n,v,1)} \cdot f_v$$

with

$$f_v = \left(\sum_{1 \le k \le h(v)} 1_{\alpha^{-(k-1)}(X(n,v,k) \cap U)} \cdot a, \sum_{1 \le k \le h(v)} 1_{\alpha^{-(k-1)}(X(n,v,k) \cap U)} \cdot b \right).$$

Note that for all k with $1 \le k \le h(v) - 1$, we have $1_{X(n,v,k)} |_{Y_n} = 0$. Also, we can check that

$$1_{X(n,v,k) \cap U} - 1_{X(n,v,k) \cap U} \circ \alpha^{-1} = 1_{X(n,v,k) \cap U} - 1_{\alpha(X(n,v,k) \cap U)}.$$

It follows that

$$[1_{X(n,v,k) \cap U}] = [1_{\alpha(X(n,v,k) \cap U)}] \text{ in } C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}), f |_{Y_n} = 0\}$$

for $k = 1, \ldots, h(v)$. We then get that in $C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}), f |_{Y_n} = 0\}$,

$$\left[\sum_{1 \le k \le h(v)} 1_{\alpha^{-(k-1)}(X(n,v,k) \cap U)} \right] = \left[\sum_{1 \le k \le h(v)} 1_{X(n,v,k) \cap U} \right].$$

It then follows that

$$\left[\sum_{v \in V_n} f_v \right] = \left[\sum_{\substack{v \in V_n \\ 1 \le k \le h(v)}} \left(1_{X(n,v,k) \cap U} \cdot a, 1_{X(n,v,k) \cap U} \cdot b \right) \right]$$
$$= ([1_U] \cdot a, [1_U] \cdot b)$$

in $C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2), f |_{Y_n} = 0\}$.

We have proved that $\rho_n([\mathrm{diag}(\underbrace{1_U, \ldots, 1_U}_{k}) \cdot \eta]) = \pi((1_U \cdot a, 1_U \cdot b))$. It then follows that ρ_n is an order isomorphism, which finishes the proof.

□

Corollary II.2.6. *Let p be a projection in $M_\infty(A_n)$. Then there exists $p' \in M_\infty(C(X \times \mathbb{T}^2)) \subset M_\infty(A_n)$ such that $[p] = [p']$ in $K_0(A_n)$.*

Proof. According to Lemma II.2.5, we have an isomorphism

$$\rho_n \colon K_0(A_n) \to C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2), f\,|_{Y_n} = 0\}.$$

Let $\rho_n([p]) = [g]$ for some $g \in C(X, \mathbb{Z}^2)$. Without loss of generality, we can assume that there is a partition of X as $X = \bigsqcup_{i=1}^N X_i$ such that this partition is finer than $\mathcal{P}_n$ and $g\,|_{X_i}$ is constant for $i = 1, \dots, N$.

As $[p]$ is in $(K_0(A_n))_+$ and ρ_n is an order isomorphism, it follows that $[g]$ is in the positive cone (defined in the statement of Lemma II.2.4). For as g above with $\rho_n([p]) = [g]$, we can assume that on any given X_i, $g\,|_{X_i}$ is either $(0,0)$ or $(a_i, b_i) \in \mathbb{Z}^2$ with $a_i > 0$.

According to Lemma II.2.1, there exist projections $\eta_i \in M_{d(i)}(C(\mathbb{T}^2))$ such that $[\eta_i]$ in $K_0(C(\mathbb{T}^2))$ can be identified with (a_i, b_i).

Let

$$p' = \operatorname{diag}\left(\operatorname{diag}(\underbrace{1_{X_1}, \dots, 1_{X_1}}_{d(1)}) \cdot \eta_1, \dots, \operatorname{diag}(\underbrace{1_{X_N}, \dots, 1_{X_N}}_{d(N)}) \cdot \eta_N \right).$$

Then it is clear that $p' \in M_\infty(C(X \times \mathbb{T}^2))$.

According to Lemma II.2.5, $\rho_n([p']) = [g]$, so that $\rho_n([p']) = \rho_n([p])$. As ρ_n is an isomorphism (by Lemma II.2.5 again), it follows that $[p] = [p']$ in $K_0(A_n)$.

□

Lemma II.2.7. *Let $j_n \colon C(X \times \mathbb{T}^2) \longrightarrow A_n$ be the canonical embedding, and let ι and ρ_n be as in Lemma II.2.2 and Lemma II.2.5. Let $(j_n)_{*0} \colon K_0(C(X \times \mathbb{T}^2)) \to K_0(A_n)$ be the induced map on K_0 and let*

$$\pi_n \colon C(X, \mathbb{Z}^2) \to C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2), f\,|_{Y_n} = 0\}$$

be the canonical quotient map. Then the follow diagram commutes:

$$\begin{tikzcd} K_0(C(X \times \mathbb{T}^2)) \arrow[r, "\iota"] \arrow[d, "(j_n)_{*0}"'] & C(X, \mathbb{Z}^2) \arrow[d, "\pi_n"] \\ K_0(A_n) \arrow[r, "\rho_n"] & C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2), f|_{Y_n} = 0\} \end{tikzcd}.$$

Proof. As $K_0(C(X \times \mathbb{Z}^2))$ is generated by its positive cone $(K_0(C(X \times \mathbb{Z}^2)))_+$, we just need to show that $\pi_n \circ \iota = \rho_n \circ (j_n)_{*0}$ on $(K_0(C(X \times \mathbb{Z}^2)))_+$.

For every projection $p \in M_\infty(C(X \times \mathbb{T}^2))$, according to the proof of surjectivity of φ in Lemma II.2.2, there exist a partition $X = \bigsqcup_{i=1}^M X_i$ and projections $\eta_i \in M_{d_i}(C(\mathbb{T}^2))$ for $i = 1, \ldots, M$ such that

$$[p] = \sum_{i=1}^{M} [(\underbrace{1_{X_i}, \ldots, 1_{X_i}}_{d_i}) \cdot \eta_i].$$

According to Lemma II.2.1, η_i can be identified with $(a_i, b_i) \in D$. By Lemma II.2.2, we get $\iota([p]) = \sum_{i=1}^M (1_{X_i} \cdot a_i, 1_{X_i} \cdot b_i)$.

By Lemma II.2.5,

$$\begin{aligned} \rho_n((j_n)_{*0}([p])) &= \rho_n((j_n)_{*0}(\sum_{i=1}^{M} [(\underbrace{1_{X_i}, \ldots, 1_{X_i}}_{d_i}) \cdot \eta_i])) \\ &= \sum_{i=1}^{M} [(1_{X_i} \cdot a_i, 1_{X_i} \cdot a_i)]. \end{aligned}$$

It is then clear that $(\pi_n \circ \iota)([p]) = (\rho_n \circ (j_n)_{*0})([p])$. Since p is arbitrary, we have finished the proof.

□

Corollary II.2.8. *Let p, q be projections in $M_\infty(C(X \times \mathbb{T}^2)) \subset M_\infty(A_n)$ such that $\iota([p]) - \iota([q]) = h - h \circ \alpha^{-1}$ for some $h \in C(X, \mathbb{Z}^2)$ satisfying $h|_{Y_n} = 0$, with ι as in Lemma II.2.2. Then $(j_n)_{*0}([p]) = (j_n)_{*0}([q])$ in $K_0(A_n)$ with j_n as in Lemma II.2.7.*

Proof. This follows directly from Lemma II.2.7.

□

Lemma II.2.9. *For A_x as defined in the beginning of this chapter,*

$$K_i(A_x) \cong C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1} \colon f \in C(X,\mathbb{Z}^2)\},$$

and

$$K_0(A_x)_+ \cong C(X,D)/\{f - f\circ\alpha^{-1} \colon f \in C(X,\mathbb{Z}^2)\},$$

with D defined to be $\{(a,b)\in\mathbb{Z}^2 \colon a>0, b\in\mathbb{Z}\}\cup\{(0,0)\}$.

Proof. From Lemma II.2.5, we know that

$$K_i(A_n) \cong C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1} \colon f \in C(X,\mathbb{Z}^2) \text{ and } f\,|_{Y_n} = 0\}.$$

As $A_x = \varinjlim A_n$, we get $K_i(A_x) = \varinjlim K_i(A_n)$. Note that the map

$$(j_{n,n+1})_{*i} \colon K_i(A_n) \to K_i(A_{n+1})$$

satisfies $(j_{n,n+1})_{*i}([f]) = [f]$ for all $f\in C(X,\mathbb{Z}^2)$. We can conclude that

$$K_i(A_x) \cong C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1} \colon f \in C(X,\mathbb{Z}^2) \text{ and } f\,|_{Y_n} = 0 \text{ for some } n\in\mathbb{N}\}.$$

As $\bigcap_{n=1}^{\infty} Y_n = \{x\}$, it follows that

$$\{f\in C(X,\mathbb{Z}^2) \colon f\,|_{Y_n} = 0 \text{ for some } n\in\mathbb{N}\} = \{f\in C(X,\mathbb{Z}^2) \colon f(x)=0\}.$$

Then we have

$$K_i(A_x) \cong C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1} \colon f \in C(X,\mathbb{Z}^2) \text{ and } f(x) = 0\}.$$

For every $g \in C(X, \mathbb{Z}^2)$, define $g_0 = g - g(x)$. It is clear that

$$g \in \{f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0\} \text{ and } g(x) = 0.$$

Note that $g_0 - g_0 \circ \alpha^{-1} = g - g \circ \alpha^{-1}$. It follows that

$$K_i(A_x) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\}.$$

Let $j_{n,\infty} \colon A_n \to A_x$ be the embedding of A_n into A_x. Then

$$K_0(A_x)_+ = \bigcup (j_{n,\infty})_{*0}(K_0(A_n)_+).$$

According to Lemma II.2.5,

$$K_0(A_n)_+ \cong C(X, D)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2) \text{ and } f|_{Y_n} = 0\}.$$

Similarly, using the fact that

$$\{f \in C(X, \mathbb{Z}^2) \colon f|_{Y_n} = 0 \text{ for some } n \in \mathbb{N}\} = \{f \in C(X, \mathbb{Z}^2) \colon f(x) = 0\},$$

we can conclude that

$$K_0(A_x)_+ \cong C(X, D)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0\}.$$

As

$$\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2) \text{ and } f(x) = 0\} = \{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\},$$

we get

$$K_0(A_x)_+ \cong C(X, D)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\}.$$

□

Corollary II.2.10. *For A_x as in Definition I.2.1, $K_i(A_x)$ is torsion free for $i = 0, 1$.*

Proof. According to Lemma II.2.9, we just need to show that $C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\}$ is torsion free.

A purely algebraic proof is given like this:

Suppose we have $g \in C(X, \mathbb{Z}^2)$ and $n \in \mathbb{Z} \setminus \{0\}$ such that

$$[ng] = 0 \text{ in } C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\}.$$

If we can show that $[g] = 0$, then we are done. In other words, we need to find $f \in C(X, \mathbb{Z}^2)$ such that $g = f - f \circ \alpha^{-1}$.

As $[ng] = 0$, there exists $F \in C(X, \mathbb{Z}^2)$ such that $ng = F - F \circ \alpha^{-1}$. If $F(x) \in n\mathbb{Z}^2$ for all x, just divide both sides by n. Then we get $g = \left(\frac{F}{n}\right) - \left(\frac{F}{n}\right) \circ \alpha^{-1}$ with $\frac{F}{n} \in C(X, \mathbb{Z}^2)$.

Fix $x_0 \in X$, and define $\widetilde{F} = F - F(x_0)$. It is clear that $\widetilde{F}(x_0) = 0$. As $F - F \circ \alpha^{-1} = ng$, we can easily check that $\widetilde{F} - \widetilde{F} \circ \alpha^{-1} = ng$. It then follows that

$$\widetilde{F}(\alpha(x_0)) = \widetilde{F}(x_0) + ng(\alpha(x_0)) = 0 + ng(\alpha(x_0)) \in n\mathbb{Z}^2,$$

$$\widetilde{F}(\alpha^2(x_0)) = \widetilde{F}(\alpha(x_0)) + ng(\alpha^2(x_0)) \in n\mathbb{Z}^2,$$

$$\cdots$$

So for every $x \in \mathrm{Orbit}_{\mathbb{Z}}(x_0)$, we get $\widetilde{F}(x) \in n\mathbb{Z}^2$. Note that $\widetilde{F}$ is continuous on X and $\mathrm{Orbit}_{\mathbb{Z}}(x_0)$ is dense in X. It follows directly that $\widetilde{F}(x) \in n\mathbb{Z}^2$ for all $x \in X$, thus finishing the proof.

□

Corollary II.2.11. *For A_x as in Definition I.2.1,* $\mathrm{TR}(A_x) \leq 1$.

Proof. From Lemma II.1.1, we know that A_x is a AH algebra with no dimension growth. By Lemma II.1.2, A_x is simple. According to Lemma II.2.9, $K_i(A_x)$ is torsion free.

As A_x is a simple AH algebra with no dimension growth, it follows that $\mathrm{TR}(A_x) \leq 1$. □

CHAPTER III

THE TRACIAL RANK OF THE CROSSED PRODUCT C^*-ALGEBRA A

III.1 THE GENERAL CASE

We start by showing that for the natural embedding $j\colon A_x \to A$, the induced homomorphisms $(j_*)_i\colon K_i(A_x) \to K_i(A)$ are injective for $i = 0, 1$.

Lemma III.1.1. *Let A be $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ and let A_x be as in Definition I.2.1. Let $j\colon A_x \to A$ be the canonical embedding. Then j_{*0} is an injective order homomorphism from $K_0(A_x)$ to $K_0(A)$.*

Proof. It is clear that j_{*0} will induce an order homomorphism from $K_0(A_x)$ to $K_0(A)$ and j_{*0} maps $[1_{A_x}]$ to $[1_A]$.

To show that j_{*0} is injective, we need to show that whenever $p, q \in M_\infty(A_x)$ are projections such that $j_{*0}([p]) = j_{*0}([q])$ in $K_0(A)$, we have $[p] = [q]$ in $K_0(A_x)$. For projections $p, q \in M_\infty(A_x)$, we can find $n \in \mathbb{N}$ and projections $e, f \in M_\infty(A_n)$ such that $[e] = [p]$ and $[f] = [q]$ in $K_0(A_x)$. According to Corollary II.2.6, we can find $e', f' \in M_\infty(C(X \times \mathbb{T}^2))$ such that $[e'] = [e]$ and $[f'] = [f]$ in $K_0(A_n)$. We need to show that if $j_{*0}([p]) = j_{*0}([q])$ in $K_0(A)$, then $[p] = [q]$ in $K_0(A_x)$. In fact, if $j_{*0}([p] - [q]) = 0$, we have $j_{*0}([p]) = j_{*0}([q])$, which implies that $j_{*0}([e']) = j_{*0}([f'])$ in $K_0(A)$.

The Pimsner-Voiculescu six-term exact sequence in our situation reads as follows:

$$\begin{tikzcd}
K_0(C(X \times \mathbb{T}^2)) \arrow[r, "\mathrm{id}_{*0}-\alpha_{*0}"] & K_0(C(X \times \mathbb{T}^2)) \arrow[r, "j_{*0}"] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(C(X \times \mathbb{T}^2)) \arrow[l, "j_{*1}"] & K_1(C(X \times \mathbb{T}^2)) \arrow[l, "\mathrm{id}_{*1}-\alpha_{*1}"]
\end{tikzcd}$$

As $j_{*0}([p_n']) = j_{*0}([q_n'])$, by the exact sequence above, $[p_n'] - [q_n']$ is in the image of $(\mathrm{id}_{*0} - \alpha_{*0})$. That is, there exists x in $K_0(C(X \times \mathbb{T}^2))$ such that $[p_n'] - [q_n'] = x - \alpha_{*0}(x)$. Apply ι as defined in Lemma II.2.2 on both sides. We get

$$\iota([p_n']) - \iota([q_n']) = \iota(x) - \iota(\alpha_{*0}(x)) \text{ in } C(X, \mathbb{Z}^2).$$

Note that $\iota(\alpha_{*0}(x)) = \iota(x) \circ \alpha$. We get $\iota([p_n']) - \iota([q_n']) = (-\iota(x) \circ \alpha) - (-\iota(x) \circ \alpha) \circ \alpha^{-1}$. We can choose $N \in \mathbb{N}$ such that for all $k > N$, $(-\iota(x) \circ \alpha)$ restricted to Y_k will be a constant function, say $c \in \mathbb{Z}^2$. It is clear that

$$\iota([p_n']) - \iota([q_n']) = (-\iota(x) \circ \alpha - c) - (-\iota(x) \circ \alpha - c) \circ \alpha^{-1}.$$

Choose $m \in \mathbb{N}$ such that $m > \max(n, N)$. Then $(-\iota(x) \circ \alpha - c)|_{Y_m} = 0$. According to Corollary II.2.8, we have $(j_m)_{*0}([p_n']) = (j_m)_{*0}([q_n'])$ with j_m as in Lemma II.2.7.

We have shown that $[p_n'] = [q_n']$ in $K_0(A_m)$. Note that $[p_n'] = [p_n]$ and $[q_n'] = [q_n]$ in $K_0(A_n)$ and $m > n$. It follows that $[p_n'] = [p_n]$ and $[q_n'] = [q_n]$ in $K_0(A_m)$. We then have that $[p_n] = [q_n]$ in $K_0(A_m)$, so that $[p_n] = [q_n]$ in $K_0(A_x)$.

Note that $[p_n] = [p]$ and $[q_n] = [q]$ in $K_0(A_x)$. It then follows that $[p] = [q]$ in $K_0(A_x)$, which finishes the proof.

$\square$

Lemma III.1.2. *Let A be $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ and let A_x be as in Definition I.2.1. Let $j\colon A_x \to A$ be the canonical embedding. Then j_{*1} is an injective homomorphism from $K_1(A_x)$ to $K_1(A)$.*

Proof. The proof is similar to the proof of Lemma III.1.1.

For any two unitaries $x, y \in A_x$ such that $j_{*1}([x]) = j_{*1}([y])$ in $K_1(A)$, we need to show that $[x] = [y]$. For x, y as above, we can find $n \in \mathbb{N}$ and $x', y' \in M_\infty(A_n)$ such that $[x] = [x']$ and $[y] = [y']$ in $K_1(A_x)$.

From Lemma II.2.3, we get the structure of A_n, which then implies the fact that

$$K_1(A_n) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2) \text{ and } f\,|_{Y_n} = 0\}.$$

Similar to the analysis of the Pimsner-Voiculescu six-term exact sequence as in the proof of Lemma III.1.1, we get $[x'] = [y']$ in $K_1(A_m)$ for m large enough. It then follows that $[x'] = [y']$ in $K_1(A_x)$, which implies that $[x] = [y]$ in $K_1(A_x)$.

□

The following result is a known fact, and it is used later to show approximate unitary equivalence.

Lemma III.1.3. *Let A be an infinite dimensional simple unital AF algebra and let $CU(A)$ be as in Section I.2. Then $U(A) = CU(A)$.*

Proof. For every unitary $u \in A$ and every $\varepsilon > 0$, we will show that $\operatorname{dist}(u, CU(A)) < \varepsilon$.

As A is unital and infinite dimensional, we can assume that $A \cong \varinjlim A_n$ with each A_n being a finite dimensional C*-algebra and each map $j_{n,n+1} \colon A_n \hookrightarrow A_{n+1}$ being unital. Write

$$A_n \cong \bigoplus_{k=1}^{s_n} M_{d_{n;k}}(\mathbb{C})$$

with $d_{n;1} \leq d_{n;2} \leq \cdots \leq d_{n;s_n}$.

Let $d'_n = \min\{d_{n;s_1}, \ldots, d_{n;s_n}\}$. As A is simple, we have $\lim_{n\to\infty} d'_n = \infty$.

For u and ε as given above, we can choose n large enough such that $d'_n > \frac{2\pi}{\varepsilon}$ and there exists $v \in U(A_n)$ satisfying $\|u - v\| < \varepsilon/2$. Let $\pi_{n;k}$ be the canonical projection from A_n to $M_{d_{n;k}}(\mathbb{C})$.

It is known that for any $w \in U(A)$, we have $w \in CU(A_n)$ if and only if $\det(\pi_{n;k}(w)) = 1$ for $k = 1, \ldots, s_n$. Without loss of generality, we can assume that

$$\pi_{n;k}(u_n) = \mathrm{diag}(\lambda_{k,1}, \ldots, \lambda_{k,d_{n;k}}), \text{ with } |\lambda_{k,d_{n;i}}| = 1.$$

Choose L_k such that $-\pi \leq L_k < \pi$ and $\det(\pi_{n;k}(u_n)) = e^{iL_k}$. For $k = 1, \ldots, s_n$, define

$$v'_k = \mathrm{diag}(\lambda_{k,1} \cdot e^{-iL/d_{n;k}}, \ldots, \lambda_{k,d_{n;k}} \cdot e^{-iL/d_{n;k}}).$$

Let $v' = \mathrm{diag}(v'_1, \ldots, v'_{s_n})$. It is then clear that $\|u_n - u'_n\| \leq \pi/d'_n$. It is easy to check that $\det(\pi_{n;s_k}(v')) = 1$ for all $k = 1, \ldots, s_n$, which then implies that $v' \in CU(A_n) \subset CU(A)$.

Note that $d'_n > \frac{2\pi}{\varepsilon}$. We have

$$\begin{aligned} \mathrm{dist}(u, CU(A)) &\leq \|u - v'\| \\ &\leq \|u - v\| + \|v - v'\| \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

As ε can be chosen to be arbitrarily small, it follows that $u \in CU(A)$.

□

We will need the fact that a cut-down of the crossed product C*-algebra by a projection in $C(X)$ is similar to the original crossed product C*-algebra, and can be regarded as a crossed product C*-algebra of the induced action. Some definitions and facts will be given here.

Let $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ be a minimal topological dynamical system as defined in Section I.2. Let D be a clopen subset of X, and let $x \in D$. For simplicity, we use φ to denote $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$.

Define $\widetilde{\varphi}\colon D \times \mathbb{T} \times \mathbb{T} \to D \times \mathbb{T} \times \mathbb{T}$ by $\widetilde{\varphi}((y, t_1, t_2)) = \varphi^{f(x)}((y, t_1, t_2))$, where $f(x)$ is the first return time function defined by

$$f(x) = \min\{n \in \mathbb{N}\colon n > 0, \varphi^n(x) \in U\}.$$

As φ is minimal on $X \times \mathbb{T} \times \mathbb{T}$, for every $x \in X$, the orbit of x under φ is dense in X. It then follows that the intersection of this orbit with D is dense in D, which implies that $\widetilde{\varphi}$ is also minimal on $D \times \mathbb{T} \times \mathbb{T}$. As the composition of rotations on the circle is still a rotation on the circle, we can find maps $\widetilde{\xi}, \widetilde{\eta}\colon D \to \mathbb{T}$ such that $\widetilde{\varphi} = \widetilde{\alpha} \times \mathrm{R}_{\widetilde{\xi}} \times \mathrm{R}_{\widetilde{\eta}}$ with $\widetilde{\alpha}(x) = \alpha^{f(x)}(x)$ for f as defined above.

We claim that $\widetilde{\xi}$ and $\widetilde{\eta}$ are both continuous functions. In fact, as D is clopen, we have that f is continuous, which then implies that $\widetilde{\xi}$ and $\widetilde{\eta}$ are continuous.

As $(D \times \mathbb{T} \times \mathbb{T}, \widetilde{\varphi})$ is a minimal dynamical system, the corresponding crossed product C*-algebra $C^*(\mathbb{Z}, D \times \mathbb{T} \times \mathbb{T}, \widetilde{\varphi})$ is simple. Use $\widetilde{u}$ to denote the implementing unitary in $C^*(\mathbb{Z}, D \times \mathbb{T} \times \mathbb{T}, \widetilde{\varphi})$.

Define $\widetilde{A_x}$ to be the subalgebra of $C^*(\mathbb{Z}, D \times \mathbb{T} \times \mathbb{T}, \widetilde{\varphi})$ generated by $C(D \times \mathbb{T} \times \mathbb{T})$ and $\widetilde{u} \cdot C_0((D\backslash\{x\}) \times \mathbb{T} \times \mathbb{T})$.

The lemma below shows that the cut down of the original crossed product C*-algebra is isomorphic to the crossed product C*-algebra of the induced homeomorphism.

Lemma III.1.4. *Let φ and $\widetilde{\varphi}$ be defined as above. There is a C*-algebra isomorphism from $C^*(\mathbb{Z}, D \times \mathbb{T} \times \mathbb{T}, \widetilde{\varphi})$ to $1_{D\times\mathbb{T}\times\mathbb{T}} \cdot A \cdot 1_{D\times\mathbb{T}\times\mathbb{T}}$.*

Proof. Let $f\colon D \to \mathbb{N}$ be the first return time function. As D is clopen, f is continuous. As X is compact and D is closed in X, D is also compact. Continuity of f then implies that $f(D)$ is a compact set, that is, a finite subset of $\mathbb{N}$. Write $f(D) = \{k_1, \ldots, k_N\}$ with $N, k_1, \ldots, k_N \in \mathbb{N}$ and set $D_i = f^{-1}(k_i)$.

In $1_{D\times\mathbb{T}\times\mathbb{T}}\cdot A\cdot 1_{D\times\mathbb{T}\times\mathbb{T}}$, let $w=\sum_{i=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}$. Then we have

$$\begin{aligned}
ww^* &= \left(\sum_{i=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)\cdot\left(\sum_{i=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)^*\\
&= \left(\sum_{i=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)\cdot\left(\sum_{j=1}^{N}u^{-k_j}1_{D_j\times\mathbb{T}\times\mathbb{T}}\right)\\
&= \sum_{i,j=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\cdot u^{-k_j}1_{D_j\times\mathbb{T}\times\mathbb{T}}\\
&= \sum_{i,j=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i-k_j}\cdot 1_{D_j\times\mathbb{T}\times\mathbb{T}}\\
&= \sum_{i,j=1}^{N}1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot(1_{D_j\times\mathbb{T}\times\mathbb{T}}\circ(\alpha\times\mathrm{R}_\xi\times\mathrm{R}_\eta)^{k_i-k_j})\cdot u^{k_i-k_j}.
\end{aligned}$$

We need the following claim to get that $ww^*=1_D$.

Claim: For D_i, k_i as above,

$$1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot(1_{D_j\times\mathbb{T}\times\mathbb{T}}\circ(\alpha\times\mathrm{R}_\xi\times\mathrm{R}_\eta)^{k_i-k_j})=\begin{cases}1_{D_i\times\mathbb{T}\times\mathbb{T}} & i=j\\ 0 & i\neq j\end{cases}.$$

Proof of claim:

If $k_j>k_i$, then $\alpha^{k_j-k_i}(D_j)\subset X\setminus D$. Thus $D_i\cap\alpha^{k_j-k_i}(D_j)=\varnothing$.

If $k_j<k_i$, we claim that $D_i\cap\alpha^{k_j-k_i}(D_j)=\varnothing$. If not, choose $s\in D_i\cap\alpha^{k_j-k_i}(D_j)$. We can assume $s=\alpha^{k_j-k_i}(y)$ for some $y\in D_j$. It is then clear that $\alpha^{k_i-k_j}(s)=y\in D_j\subset D$, contradicting the fact that the first return time of s (in D_i) is k_i.

If $k_j=k_i$, it is clear that $1_{D_i}\cdot(1_{D_j}\circ\alpha^{k_i-k_j})=1_{D_i}$.

This proves the claim.

Using the claim, we get

$$
\begin{aligned}
ww^* &= \sum_{i,j=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}} \cdot \left(1_{D_j\times\mathbb{T}\times\mathbb{T}} \circ (\alpha\times \mathrm{R}_\xi \times \mathrm{R}_\eta)^{k_i-k_j}\right)\cdot u^{k_i-k_j}\\
&= \sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\\
&= 1_{D\times\mathbb{T}\times\mathbb{T}}.
\end{aligned}
$$

Now we calculate w^*w. It is clear that

$$
\begin{aligned}
w^*w &= \left(\sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)^* \cdot \left(\sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)\\
&= \left(\sum_{j=1}^{N} u^{-k_j}\cdot 1_{D_j\times\mathbb{T}\times\mathbb{T}}\right)\cdot \left(\sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)\\
&= \sum_{i,j=1}^{N} u^{-k_j}\cdot 1_{D_j\times\mathbb{T}\times\mathbb{T}}\cdot 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\\
&= \sum_{i=1}^{N} u^{-k_i}\cdot 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\\
&= \sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\circ (\alpha\times\mathrm{R}_\xi\times\mathrm{R}_\eta)^{-k_i}\\
&= \sum_{i=1}^{N} 1_{(\alpha\times\mathrm{R}_\xi\times\mathrm{R}_\eta)^{k_i}(D_i\times\mathbb{T}\times\mathbb{T})}\\
&= \sum_{i=1}^{N} 1_{\widetilde{\varphi}(D_i\times\mathbb{T}\times\mathbb{T})}\\
&= 1_{D\times\mathbb{T}\times\mathbb{T}}.
\end{aligned}
$$

So far, we have shown that w is a unitary in $1_{D\times\mathbb{T}\times\mathbb{T}}\cdot A\cdot 1_{D\times\mathbb{T}\times\mathbb{T}}$.

Define a map

$$\gamma\colon C^*(\mathbb{Z}, D\times\mathbb{T}\times\mathbb{T}, \widetilde{\varphi}) \longrightarrow 1_{D\times\mathbb{T}\times\mathbb{T}}\cdot A\cdot 1_{D\times\mathbb{T}\times\mathbb{T}}$$

by

$$\gamma(f) = f \text{ for all } f\in C(D\times\mathbb{T}\times\mathbb{T}) \text{ and } \gamma(\widetilde{u}) = w.$$

We will check that γ is well-defined and gives the desired isomorphism between $C^*(\mathbb{Z}, D \times \mathbb{T} \times \mathbb{T}, \widetilde{\varphi})$ and $1_{D\times\mathbb{T}\times\mathbb{T}} \cdot A \cdot 1_{D\times\mathbb{T}\times\mathbb{T}}$. In fact, for all $f \in C(D \times \mathbb{T} \times \mathbb{T})$, we have

$$\begin{aligned}\gamma(\widetilde{u}^* f \widetilde{u}) &= \gamma(f \circ \widetilde{\varphi}^{-1}) \\ &= f \circ \widetilde{\varphi}^{-1}.\end{aligned}$$

We also have

$$\begin{aligned}\gamma(\widetilde{u}^* f \widetilde{u}) &= \gamma(\widetilde{u}^*) \cdot \gamma(f) \cdot \gamma(\widetilde{u}) \\ &= w^* \cdot f \cdot w \\ &= \left(\sum_{j=1}^{N} 1_{D_j} \cdot u^{k_j}\right)^* \cdot \left(f \cdot \sum_{i=1}^{N} 1_{D_i}\right) \cdot \left(\sum_{l=1}^{N} 1_{D_l} \cdot u^{k_l}\right) \\ &= \sum_{i,j,k=1}^{N} u^{-k_j} \cdot 1_{D_j} \cdot f \cdot 1_{D_i} \cdot 1_{D_l} \cdot u^{k_l} \\ &= \sum_{i=1}^{N} u^{-k_i} \cdot (f \cdot 1_{D_i}) \cdot u^{k_i} \\ &= f \circ \widetilde{\varphi}^{-1},\end{aligned}$$

which then implies that γ is really a homomorphism.

To show that γ is surjective, we will show that for every $g \in C(X \times \mathbb{T} \times \mathbb{T})$ and $n \in \mathbb{N}$, $1_{D\times\mathbb{T}\times\mathbb{T}} \cdot (gu^n) \cdot 1_{D\times\mathbb{T}\times\mathbb{T}}$ is in the image of γ. Note that

$$\begin{aligned}1_{D\times\mathbb{T}\times\mathbb{T}} \cdot (gu^n) \cdot 1_{D\times\mathbb{T}\times\mathbb{T}} &= (1_{D\times\mathbb{T}\times\mathbb{T}} \cdot g) \cdot (u^n \cdot 1_{D\times\mathbb{T}\times\mathbb{T}}) \\ &= (1_{D\times\mathbb{T}\times\mathbb{T}} \cdot g \cdot 1_{\alpha^{-n}(D)\times\mathbb{T}\times\mathbb{T}}) \cdot u^n.\end{aligned}$$

Without loss of generality, we assume that

$$D \cap \alpha^{-n}(D) \neq \varnothing.$$

Note that there is s with $1 \leq s \leq N$ such that $D \cap \alpha^{-n}(D) = D_s$, $n = k_s$ and D_s is exactly $f^{-1}(n)$.

It follows that

$$\begin{aligned}1_{D\times\mathbb{T}\times\mathbb{T}}\cdot(gu^n)\cdot 1_{D\times\mathbb{T}\times\mathbb{T}} &= (g\cdot 1_{D_n\times\mathbb{T}\times\mathbb{T}})\cdot u^n\\ &= (g\cdot 1_{D_n\times\mathbb{T}\times\mathbb{T}})\cdot(1_{D_n\times\mathbb{T}\times\mathbb{T}}\cdot u^n).\end{aligned}$$

It is clear that we can identify $g\cdot 1_{D_n\times\mathbb{T}\times\mathbb{T}}$ with a function in $C(D\times\mathbb{T}\times\mathbb{T})$. Note that $w = \sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}$. We have

$$\begin{aligned}\gamma\left((g\cdot 1_{D_s\times\mathbb{T}\times\mathbb{T}})\cdot(\widetilde{u})\right) &= \gamma\left((g\cdot 1_{D_n\times\mathbb{T}\times\mathbb{T}})\right)\cdot\gamma(\widetilde{u})\\ &= (g\cdot 1_{D_s\times\mathbb{T}\times\mathbb{T}})\cdot\left(\sum_{i=1}^{N} 1_{D_i\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_i}\right)\\ &= g\cdot 1_{D_s\times\mathbb{T}\times\mathbb{T}}\cdot u^{k_s}\\ &= g\cdot 1_{D_s\times\mathbb{T}\times\mathbb{T}}\cdot u^{n}\\ &= 1_{D\times\mathbb{T}\times\mathbb{T}}\cdot(gu^n)\cdot 1_{D\times\mathbb{T}\times\mathbb{T}}.\end{aligned}$$

Then we have proved that γ is surjective. As $C^*(\mathbb{Z}, D\times\mathbb{T}\times\mathbb{T}, \widetilde{\varphi})$ is a simple C*-algebra, it follows that γ is a C*-algebra isomorphism.

□

The idea of topological full group of the Cantor set is needed in the next lemma, and a definition is given below.

Definition III.1.5. *Let X be the Cantor set and let α be a minimal homeomorphism of X. We say that $\beta\in$ Homeo(X) is in the full group of α if β preserves the orbit of α. That is, for any $x\in X$, $\beta(\{\alpha^n(x)\}_{n\in\mathbb{Z}}) = \{\alpha^n(x)\}_{n\in\mathbb{Z}}$. In this case, there exists a unique function $n\colon X\to\mathbb{Z}$ such that $\beta(x) = \alpha^{n(x)}(x)$ for all $x\in X$.*

We say that $\beta\in$ Homeo(X) is in the topological full group of α if the function n above is continuous.

We use $[\alpha]$ to denote the full group of α, and use $[[\alpha]]$ to denote the topological full group *of* α.

Lemma III.1.6. *Let* X *be the Cantor set and let* α *be a minimal homeomorphism of* X. *Let* Y *and* U *be two clopen subsets of* X *such that* $U \subset Y$. *If there exists* $\beta \in [[\alpha]]$ *such that* $\beta(U) \subset Y$ *and* $U \cap \beta(U) = \varnothing$, *then there exists* $\gamma \in [[\alpha]]$ *such that* $\gamma(Y) = Y$, $\gamma|_U = \beta|_U$ *and* $\gamma|_{X\setminus Y} = \mathrm{id}|_{X\setminus Y}$.

Proof. As $\beta \in [[\alpha]]$, there exists a continuous function $n_1 \colon X \to \mathbb{Z}$ such that $\beta(x) = \alpha^{n_1(x)}(x)$ for all $x \in X$. Let $U_j = U \cap n_1^{-1}(j)$ for $j \in \mathbb{Z}$. As the sets $n_1^{-1}(j)$ are mutually disjoint for $j \in \mathbb{Z}$, so are the sets U_j . Now we have $\beta(U) = \bigsqcup_{j=-\infty}^{\infty} \alpha^j(D_j)$.

Define $\gamma \in \mathrm{Homeo}(X)$ by $\gamma(x) = \alpha^{n_2(x)}(x)$, with

$$n_2(x) = \begin{cases} n_1(x) & x \in U \\ -j & x \in \alpha^j(U_j) \\ 0 & x \notin U \text{ and } x \notin \beta(U) \end{cases}.$$

As $U \cap \beta(U) = \varnothing$, we get $U \cap \alpha^j(U_j) = \varnothing$ for all $j \in \mathbb{Z}$. Thus n_2 is a well-defined function. Then we can check that $\gamma|_U = \beta|_U$ as $n_1|_U = n_2|_U$. It is also obvious that $\gamma(\beta(U)) = U$ and $\gamma|_{Y\setminus(U\sqcup\beta(U))} = \mathrm{id}_{Y\setminus(U\sqcup\beta(U))}$. It follows that $\gamma(Y) = Y$. As $n_2(x) = 0$ when $x \notin Y$, we get $\gamma|_{X\setminus Y} = \mathrm{id}|_{X\setminus Y}$.

□

Lemma III.1.7. *Let* X *be the Cantor set. Let* α *be a minimal homeomorphism of* X, *and let* $x \in X$. *Let* A *be the crossed product* C^**-algebra of the dynamical system* (X, α). *Use* A_x *to denote the subalgebra generated by* $C(X)$ *and* $u \cdot C_0(X\setminus\{x\})$. *Let* D *be a clopen subset of* X *and let* $n \in \mathbb{N}$ *be such that* $x \notin \bigcup_{k=0}^{n-1} \alpha^k(D)$. *In* A_x, *the element* $s = u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D$ *is a partial isometry such that* $s^*s = 1_D$ *and* $ss^* = 1_{\alpha^n(D)}$.

Proof. We just need to check $ss^* = 1_{\alpha^n(D)}$, $s^*s = 1_D$, and $s \in A_x$.

In fact,

$$\begin{aligned} ss^* &= (u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D) \cdot (u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D)^* \\ &= u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D \cdot 1_D \cdot u^* \cdot 1_{\alpha(D)} \cdot u^* \cdots 1_{\alpha^{n-1}(D)} \cdot u^* \\ &= 1_{\alpha^n(D)}, \end{aligned}$$

and

$$\begin{aligned} s^*s &= (u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D)^* \cdot (u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D) \\ &= 1_D \cdot u^* \cdot 1_{\alpha(D)} \cdot u^* \cdots 1_{\alpha^{n-1}(D)} \cdot u^* \cdot u \cdot 1_{\alpha^{n-1}}(D) \cdots u \cdot 1_{\alpha(D)} \cdot u \cdot 1_D \\ &= 1_D. \end{aligned}$$

As $x \notin \bigcup_{k=0}^{n-1} \alpha^k(D)$, it follows that $u \cdot 1_{\alpha^k(D)} \in A_x$ for $k = 0, \ldots, n-1$. Thus $s, s^* \in A_x$.

□

Remark: It is easy to check that $s = u^n \cdot 1_D$ and $s^* = (u^n \cdot 1_D)^* = 1_D \cdot u^{-n}$.

Lemma III.1.8. *Let X be the Cantor set and let α be a minimal homeomorphism of X. Let u be the implementing unitary of the crossed product C^*-algebra $C^*(\mathbb{Z}, X, \alpha)$. For $\gamma \in [[\alpha]]$, there exist mutually disjoint clopen sets $X_1, \ldots, X_N$ and $n_1, \ldots, n_N \in \mathbb{N}$ such that $X = \bigsqcup_{i=1}^N X_i$ and $\gamma(x) = \alpha^{n_i}(x)$ for $x \in X_i$. Furthermore, $w = \sum_{i \in \mathbb{N}} 1_{X_i} \cdot u^{n_i}$ is a unitary element in $C^*(\mathbb{Z}, X, \alpha)$ satisfying $w^* f w = f \circ \gamma^{-1}$ for all $f \in C(X)$.*

Proof. As $\gamma \in [[\alpha]]$, there exists a continuous function $n \colon X \to \mathbb{Z}$ such that $\gamma(x) = \alpha^{n(x)}(x)$ for all $x \in X$. As X is compact and n is continuous, the range $n(X)$ must be finite.

Define

$$w = \sum_{k \in n(X)} 1_{Y_k} \cdot u^k$$

where $Y_k = n^{-1}(k)$. As $n(X)$ is finite, we have finitely many sets Y_k. As γ is a homeomorphism, it follows that $\alpha^k(Y_k) \cap \alpha^j(Y_j) = \varnothing$ if $k \neq j$.

We will check that $ww^* = 1$ and $w^*w = 1$.

Note that

$$\begin{aligned} ww^* &= \Big(\sum 1_{Y_k} \cdot u^k)(\sum 1_{Y_j} \cdot u^j\Big)^* \\ &= \sum_{k,j\in\mathbb{Z}} 1_{Y_k} \cdot u^k \cdot u^{-j} \cdot 1_{Y_j} \\ &= \sum_{k,j\in\mathbb{Z}} 1_{Y_k} \cdot \big(1_{Y_j} \circ \alpha^{k-j}\big) \cdot u^{k-j} \\ &= \sum_{k,j\in\mathbb{Z}} 1_{Y_k} \cdot 1_{\alpha^{j-k}(Y_j)} \cdot u^{k-j}. \end{aligned}$$

As $\alpha^k(Y_k) \cap \alpha^j(Y_j) = \varnothing$ if $k \neq j$, it follows that $\alpha^{j-k}(Y_j) \cap Y_k = \varnothing$ if $k \neq j$. Then we get

$$\begin{aligned} ww^* &= \sum_{k,j\in\mathbb{Z}} 1_{Y_k} \cdot 1_{\alpha^{j-k}(Y_j)} \cdot u^{k-j} \\ &= \sum_k 1_{Y_k} \\ &= 1\ . \end{aligned}$$

As $C^*(\mathbb{Z}, X, \alpha)$ has stable rank one, it is finite. It then follows that $w^*w = 1$. So far, we have shown that w is a unitary element in $C^*(\mathbb{Z}, X, \alpha)$.

To show that $w^*fw = f \circ \gamma^{-1}$, we just need to show that for each i and for every clopen set $D \subset Y_i$, we have $w^*1_D w = 1_D \circ \gamma^{-1}$. As $C(X)$ is generated by

$$\{1_D \colon D \text{ is a clopen set of } Y_i \text{ for some } i \in \mathbb{Z}\},$$

that will imply $w^*fw = f \circ \gamma^{-1}$ for all $f \in C(X)$.

For a clopen set $D \subset Y_i$, it is clear that

$$\begin{aligned} w^* 1_D w &= \left(\sum_{j\in\mathbb{Z}} 1_{Y_j} \cdot u^j\right)^* \cdot 1_D \cdot \left(\sum_{k\in\mathbb{Z}} 1_{Y_k} \cdot u^k\right) \\ &= \sum_{j,k\in\mathbb{Z}} u^{-j} \cdot 1_{Y_j} \cdot 1_D \cdot 1_{Y_k} \cdot u^k \\ &= u^{-i} \cdot 1_D \cdot u^i \\ &= 1_D \circ \alpha^{-i} \\ &= 1_D \circ \gamma^{-1} , \end{aligned}$$

which finishes the proof.

□

Some facts about Cantor dynamical systems that will be needed are given below.

Lemma III.1.9. *Let (X, α) be a minimal Cantor dynamical system and let $x \in X$. Let U and V be two clopen subsets of X. Let A be the crossed product C^*-algebra of (X, α) and let A_x be the subalgebra generated by $C(X)$ and $u \cdot C_0(X\backslash\{x\})$, with u being the implementing unitary element in A satisfying $ufu^* = f \circ \alpha^{-1}$ for all $f \in C(X)$. If there exists an integer $n \geq 1$ such that $\alpha^n(U) = V$ and $x \notin \bigcup_{k=0}^{n-1} \alpha^k(U)$, then there exists $w \in A_x$ such that $w \cdot 1_U \cdot w^* = 1_V$.*

Proof. As $x \notin \bigcup_{k=0}^{n-1} \alpha^k(U)$, we can find a Kakutani-Rokhlin partition $\mathcal{P}$ of X with respect to α such that the roof set $R(\mathcal{P})$ is a clopen set containing x and $R(\mathcal{P}) \cap (\bigcup_{k=0}^{n-1} \alpha^k(U)) = \varnothing$.

Write

$$\mathcal{P} = \bigsqcup_{\substack{1\leq s\leq N \\ 1\leq k\leq h(s)}} X(s,k)$$

with $\alpha(X(s,k)) = X(s,k+1)$ for all $k = 1, \ldots, h(s) - 1$ and $\alpha(R(\mathcal{P})) \subset \bigsqcup_{1\leq s\leq N} X(s,1)$.

Use $A_{\mathcal{P}}$ to denote the subalgebra generated by $C(X)$ and $u \cdot C_0(X\backslash R(\mathcal{P}))$. Then

$$A_{\mathcal{P}} \cong \bigoplus_{s=1}^{N} M_{h(s)}(C(X(s,1))).$$

In other words, there exists a C*-algebra isomorphism

$$\varphi\colon A_{\mathcal{P}} \longrightarrow \bigoplus_{s=1}^{N} M_{h(s)}(C(X(s,1)))$$

satisfying

$$\varphi(1_{X(s,k)}) = \operatorname{diag}(0,\dots,0,1,0,\dots) \in M_{h(s)}(C(X,1))$$

with the k-th diagonal element being $1_{X(s,k)}$.

It is clear that $1_U = \sum_{s,k} 1_{U\cap X(s,k)}$ and $1_V = \sum_{s,k} 1_{V\cap X(s,k)}$. Define U_s to be $\bigsqcup_k (U \cap X(s,k))$ and V_s to be $\bigsqcup_k (V \cap X(s,k))$. It is clear that $1_U = \sum_s 1_{U_s}$ and $1_V = \sum_s 1_{V_s}$. Recall the isomorphism φ above. By abuse of notation, we can regard 1_{U_s} and 1_{V_s} as two diagonal matrices in $M_{h(s)}(C(X_{s,1}))$.

If we can find unitary elements $w_s \in M_{h(s)}(C(X_{s,1}))$ such that $w_s \cdot 1_{U_s} \cdot w_s^* = 1_{V_s}$, by setting $w = w_1 + \dots + w_s$, it is then clear that w is unitary element in $\bigoplus_{s=1}^{N} M_{h(s)}(C(X(s,1)))$ such that $w \cdot 1_U \cdot w^* = 1_V$, which is equivalent to the existence of a unitary in $A_{\mathcal{P}}$ conjugating 1_U to 1_V. As $x \in R(\mathcal{P})$, we can regard $A_{\mathcal{P}}$ as a subalgebra of A_x. Then the unitary w in $A_{\mathcal{P}}$ is also a unitary in A_x.

Let w_s be a unitary element in $M_{h(s)}(C(X_{s,1}))$ satisfying

$$w_s \cdot E_{i,i} \cdot w_s^* = E_{i+1,i+1}$$

for $i = 1, \dots, h(s) - 1$ and

$$w_s E_{h(s),h(s)} w_s^* = E_{1,1},$$

with $(E_{i,j})$ being the standard system of matrix units. It follows that $w_s \cdot 1_{U_s} \cdot w_s^* = 1_{V_s}$, which finishes the proof.

□

Lemma III.1.10. *Let (X, α) be a minimal Cantor dynamical system and let U, V be two clopen subsets of X satisfying $\alpha^n(U) = V$ for some $n \in \mathbb{N}$. Then there exists a partition of U, say*

$U = \bigsqcup_{i=1}^{m} U_i$ with each U_i clopen such that for all $k = 1, \ldots, n$ and $i, j = 1, \ldots, m$ with $i \neq j$, we have $\alpha^k(U_i) \cap \alpha^k(U_j) = \varnothing$.

Proof. We just need to find a partition of U into $U = \bigsqcup_{i=1}^{m} U_i$ such that for every given i with $1 \leq i \leq m$, the clopen sets $\alpha^1(U_i), \ldots, \alpha^n(U_i)$ are mutually disjoint.

For every $y \in U$, as α is a minimal homeomorphism, we can find a clopen set $D_y \subset U$ such that $\alpha^1(D_y), \ldots, \alpha^n(D_y)$ are mutually disjoint. As U is compact, there exists a finite subset of U, say $\{y_1, \ldots, y_N\}$, such that $\bigcup_{s=1}^{N} D_{y_s} = U$.

As the intersection of two clopen sets is still clopen, without loss of generality, we may assume that the sets $D_{y_1}, \ldots, D_{y_N}$ are mutually disjoint. That is, $U = \bigsqcup_{i=1}^{m} D_{y_i}$. It is then clear that for any given s with $1 \leq s \leq N$, $\alpha^k(D_{y_s})$ are mutually disjoint for $k = 1, \ldots, n$, which finishes the proof.

□

The lemma below is the strengthened version of Lemma III.1.9 in the sense that we no longer require $x \notin \bigcup_{k=0}^{n-1} \alpha^k(U)$.

Lemma III.1.11. *Let X be the Cantor set and let $x \in X$. Let α be a minimal homeomorphism of X and let A_x be defined as in Lemma III.1.9. For every $n \in \mathbb{N}$ and clopen subset $U \subset X$, there exists a unitary element $w \in A_x$ such that*

$$w = \sum_{j \in \mathbb{Z}} 1_{D_j} u^j \text{ and } w \cdot 1_U \cdot w^* = 1_{\alpha^n(U)},$$

where D_j for $j \in \mathbb{Z}$ are mutually disjoint clopen subsets of X satisfying $X = \bigsqcup_{j \in \mathbb{Z}} D_j$, and all but finitely many D_j are empty.

Proof. Let d be the metric on X. As (X, α) is a minimal dynamical system, $x, \alpha(x), \ldots, \alpha^n(x)$ are distinct from each other.

Let

$$R = \frac{1}{2} \min_{0 \leq i,j \leq n, i \neq j} d(\alpha^i(x), \alpha^j(x)).$$

It is clear that $R > 0$.

For k with $0 \leq k \leq n$, if $x \in \alpha^k(U)$, as $\alpha^k(U)$ is clopen, there exists $r_k > 0$ such that the open set $\{y \in X\colon d(x,y) < r_k\}$ is a subset of $\alpha^k(U)$. If $x \notin \alpha^k(U)$, as $\alpha^k(U)$ is compact, $\inf_{y\in\alpha^k(U)} d(x,y) = d(x,y')$ for some $y' \in \alpha^k(U)$. In this case, let $r_k = \inf_{y\in\alpha^k(U)} d(x,y)$.

Let

$$r = \min(R, r_0, r_1, \dots, r_n) > 0$$

and define E' to be

$$\{y \in X\colon d(x,y) < r\}.$$

Then E' is an open subset of X. As the topology of the Cantor set X is generated by clopen sets, we can find a clopen subset $E \subset E'$ such that $x \in E$.

According to the definition of r, it follows that for $k = 0, 1, \dots, n$, either $E' \subset \alpha^k(U)$ or $E' \cap \alpha^k(U) = \varnothing$. The fact that $E \subset E'$ implies that for $k = 0, 1, \dots, n$, either $E \subset \alpha^k(U)$ or $E \cap \alpha^k(U) = \varnothing$.

Let $\mathcal{P}$ be a Kakutani-Rokhlin tower such that the roof set is E. As E is the roof set and $E, \alpha(E), \dots, \alpha^n(E)$ are mutually disjoint, it follows that the height of each tower in $\mathcal{P}$ is greater than $n+1$.

Use $X(N, v, s)$ to denote the clopen subset of the partition $\mathcal{P}$ at the v-th tower, with height s. Then

$$X = \bigsqcup_{v\in V, 1\leq k\leq h(v)} X(n, v, s),$$

where $h(v)$ is the height of the v-th tower.

Let $U_{v,k} = U \cap X(N, v, k)$. Then

$$U = \bigsqcup_{v\in V, 1\leq k\leq h(v)} U_{v,k}.$$

For every v, k such that $U_{v,k} \neq \varnothing$, if there exists $m \in \mathbb{N}$ such that $1 \leq m \leq n$ and $\alpha^m(U_{v,k}) \subset \alpha(E)$, then $E \cap \alpha^{m-1}(U) \neq \varnothing$. According to our choice of E, for all s with $1 \leq s \leq n$, either $E \subset \alpha^s(U)$

or $E \cap \alpha^s(U) = \varnothing$. By assumption, we have $\alpha^m(U_{v,k}) \subset \alpha(E)$ and $U_{v,k} \neq \varnothing$. Then

$$E \cap \alpha^{m-1}(U) \supset E \cap \alpha^{m-1}(U_{v,k}) = \alpha^{m-1}(U_{v,k}) \neq \varnothing,$$

which implies that $E \subset \alpha^{m-1}(U)$.

Let A_E be the subalgebra of A generated by $C(X)$ and $u \cdot C_0(X \backslash R(\mathcal{P}))$, with u being the implementing unitary of A. We will show that there exists a unitary element $w \in A_E$ such that

$$w = \sum_{j \in \mathbb{Z}} 1_{D_j} u^j$$

with all the sets D_j for $j \in \mathbb{Z}$ being mutually disjoint and $w \cdot 1_U \cdot w^* = 1_{\alpha^n(U)}$. As A_E can be regarded as a subalgebra of A_x, that is enough to prove the lemma if we can find the unitary w as described above.

If $k+n \leq h(v)$, this is the case that $x \notin \bigsqcup_{j=0}^{n-1} \alpha^j(U_{v,k})$. According to Lemma III.1.7, there exists a partial isometry $s_{v,k} \in A_x$ such that $s_{v,k}^* s_{v,k} = 1_{U_{v,k}}$ and $s_{v,k} s_{v,k}^* = 1_{\alpha^n(U_{v,k})} = 1_{U_{v,k+n}}$. According to the remark after Lemma III.1.7, we have $s_{v,k} = u^n \cdot 1_{U_{v,k}}$.

If there is a nonempty $U_{v,k}$ such that $k + n > h(v)$, then

$$\alpha^{h(v)-k}(U) \cap E \supset \alpha^{h(v)-k}(U_{v,k}) \cap E \neq \varnothing.$$

According to the construction of E, it follows that $E \subset \alpha^{h(v)-k}(U)$, which then implies that $\alpha^{-(h(v)-k)}(E) \subset U$. Intersecting both sets with

$$\alpha^{-(h(v)-k)}(E) = \bigsqcup_{v' \in V} X(n, v', h(v') - (h(v) - k)),$$

we get

$$\begin{aligned}\bigsqcup_{v'\in V} X(n,v',h(v')-(h(v)-k)) &= \alpha^{-(h(v)-k)}(E)\cap \bigsqcup_{v'\in V} X(n,v',h(v')-(h(v)-k))\\ &\subset U\ \cap \bigsqcup_{v'\in V} X(n,v',h(v')-(h(v)-k))\\ &\subset \bigsqcup_{v'\in V} X(n,v',h(v')-(h(v)-k))\ ,\end{aligned}$$

which implies that

$$U\cap \bigsqcup_{v'\in V} X(n,v',h(v')-(h(v)-k)) = \bigsqcup_{v'\in V} X(n,v',h(v')-(h(v)-k)).$$

In other words,

$$U_{v',h(v')-(h(v)-k)} = X(n,v',h(v')-(h(v)-k)) \text{ for all } v\in V'.$$

Now we have

$$\alpha^{-(h(v)-k)}(E) = \bigsqcup_{v'\in V} U_{v',h(v')-(h(v)-k)} = \bigsqcup_{v'\in V} X_{v',h(v')-(h(v)-k)}.$$

It follows that

$$\alpha^n\left(\bigsqcup_{v'\in V} U_{v',h(v')-(h(v)-k)}\right) = \alpha^n\left(\bigsqcup_{v'\in V} X_{v',h(v')-(h(v)-k)}\right) = \bigsqcup_{v'\in V} X_{v',n-(h(v)-k)}.$$

By Lemma III.1.7, there exists a partial isometry $s'_{v,k}$ such that

$$s'_{v,k}s'^{*}_{v,k} = 1_{U(v',h(v')-(h(v)-k))}$$

and

$$\begin{aligned}s'^{*}_{v,k}s'_{v,k} &= 1_{\alpha^n(U(v',h(v')-(h(v)-k))}\\ &= 1_{U(v',h(v')+n-(h(v)-k))-h(v')}.\end{aligned}$$

Furthermore, according to the remark after Lemma III.1.7, $s'_{v,k} \in A_E$.

For every non-empty $U_{v,k}$, either $k + n \leq h(v)$ or $U \supset \alpha^{-(h(v)-k)}(R(\mathcal{P}))$. Thus the above two cases will give a partial isometry $s \in A_E$ such that $ss^* = 1_U$ and $s^*s = 1_{\alpha^n(U)}$.

There exists a partial isometry $\widetilde{s} \in A_E$ such that

$$\widetilde{s}\widetilde{s}^* = 1_{X \setminus U} \text{ and } \widetilde{s}^*\widetilde{s} = 1_{X \setminus \alpha^n(U)}.$$

Let $w = s + \widetilde{s}$. Then w is a unitary element in A_E satisfying $w \cdot 1_u \cdot w^* = 1_{\alpha^n(U)}$, which finishes the proof.

□

Lemma III.1.12. *Let X be the Cantor set and let $x \in X$. Let D be a clopen subset of X satisfying $x \in D$, and use $X \times \mathbb{T}_1 \times \mathbb{T}_2$ to denote the product of the Cantor set and two dimensional torus. Let A be the crossed product C^*-algebra $C^*(\mathbb{Z}, X \times \mathbb{T}_1 \times \mathbb{T}_2, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ and let u be the implementing unitary of A. Let $z_1 \in C(\mathbb{T}_1, \mathbb{C})$ be defined by $z_1(t) = t$ and let $z_2 \in C(\mathbb{T}_2, \mathbb{C})$ be defined by $z_2(t) = t$. By abuse of notation, we identify z_1 with $\mathrm{id}_X \otimes z_1 \otimes \mathrm{id}_{\mathbb{T}_2}$ and z_2 with $\mathrm{id}_X \otimes \mathrm{id}_{\mathbb{T}_1} \otimes z_2$. Suppose that there exists $M \in \mathbb{N}$ such that*

$$\|u^M z_i p u^{-M} - z_i q\| < \varepsilon \text{ for } i = 1, 2, \text{ where } p = 1_D \text{ and } q = u^M p u^{-M}.$$

Then there exists a partial isometry $w \in A_x$ (with A_x as defined in Lemma III.1.9) such that

$$w^*w = p, \; ww^* = q \text{ and } \|wz_i p w^* - z_i q\| < \varepsilon \text{ for } i = 1, 2.$$

Proof. According to Lemma III.1.11, we can find a unitary element $w_1 \in A_x$ such that

$$w_1 = \sum_{k \in \mathbb{Z}} u^k 1_{n^{-1}(k)}$$

for some $n \in C(X, \mathbb{Z})$ and

$$w_1 p w_1^* = q.$$

Let

$$j_0 \colon C(\mathbb{T}_1 \times \mathbb{T}_2) \longrightarrow C(D \times \mathbb{T}_1 \times \mathbb{T}_2)$$

be defined by $j_0(f) = 1_D \otimes f$ for all $f \in C(\mathbb{T}_1 \times \mathbb{T}_2)$. Then it is clear that j is an injective homomorphism.

As $C(D \times \mathbb{T}_1 \times \mathbb{T}_2) \subset pA_x p$ (with $p = 1_D$), we hence get the canonical inclusion map

$$\phi_0 \colon C(D \times \mathbb{T}_1 \times \mathbb{T}_2) \to pA_x p.$$

Define

$$\phi_1 \colon C(D \times \mathbb{T}_1 \times \mathbb{T}_2) \longrightarrow pA_x p$$

by

$$\phi_1(g) = w_1^* \cdot u^M \cdot g \cdot u^{-M} \cdot w_1 \text{ for all } g \in C(D \times \mathbb{T}_1 \times \mathbb{T}_2).$$

As $q = u^M p u^{-M}$ and $p = 1_D$, it follows that $u^M \cdot g \cdot u^{-M} \in qC(X \times \mathbb{T}^2)q \subset qA_x q$.

The fact that $w_1 p w_1^* = q$ implies that $w_1^* q A_x q w_1 = pA_x q$. So far, we have shown that ϕ_1 is really a homomorphism from $C(D \times \mathbb{T}^2)$ to $pA_x p$. As $\|\phi_1(g)\| = \|g\|$, it is clear that ϕ_1 is injective.

Define $\varphi_0 = \phi_0 \circ j_0$ and $\varphi_1 = \phi_1 \circ j_0$. Then φ_0, φ_1 are two injective homomorphisms from $C(\mathbb{T}^2)$ to $pA_x p$.

Let

$$j \colon pA_x p \longrightarrow pAp$$

be the canonical embedding.

By Lemmas III.1.1 and III.1.2,

$$j_{*i} \colon K_i(pA_x p) \longrightarrow K_i(pAp)$$

will induce an injective embedding of $K_i(pA_x p)$ into $K_i(pAp)$ for $i = 0, 1$.

Consider $(\varphi_0)_{*i}$ and $(\varphi_1)_{*i} \colon K_i(C(\mathbb{T}_1 \times \mathbb{T}_2)) \to K_i(pA_x p)$ for $i = 0, 1$. As $\varphi_1(f) =$

$w_1^* u^M f u^{-M} w_1$, it is clear that $(\varphi_0)_{*i}(a) = (\varphi_1)_{*i}(a)$ in $K_i(pAp)$ for all $a \in K_i(\mathbb{T}_1 \times \mathbb{T}_2)$. Since we know that $j_{*i}\colon K_i(pA_xp) \to K_i(pAp)$ is injective, it follows that $(\varphi_0)_{*i}(a) = (\varphi_1)_{*i}(a)$ in $K_i(pA_xp)$ for all $a \in K_i(\mathbb{T}_1 \times \mathbb{T}_2)$.

For a C*-algebra B, recall from Section I.2 that $T(B)$ denotes the convex set of all tracial states on B. For all $\tau \in T(pAp)$ and $g \in C(D \times \mathbb{T}_1 \times \mathbb{T}_2)$, it is clear that

$$\tau(w_1^* u^M g u^{-M} w_1) = \tau(g).$$

As $T(pAp) = T(pA_xp)$, it follows that for every tracial state $\tau' \in T(pA_xp)$, we have

$$\tau'\left(w_1^* u^M g u^{-M} w_1\right) = \tau'(g).$$

It is then clear that for all $\tau' \in T(pA_xp)$ and $f \in C(\mathbb{T}_1 \times \mathbb{T}_2)$,

$$\tau'\left(\varphi_0(f)\right) = \tau'(\varphi_1(f)).$$

Recall from Definition I.2.3 the maps

$$\varphi_0{}^\sharp, \varphi_1{}^\sharp\colon U(C(\mathbb{T}_1 \times \mathbb{T}_2))/CU(C(\mathbb{T}_1 \times \mathbb{T}_2)) \to U(pA_xp)/CU(pA_xp).$$

We will show that $\varphi_0(z_1 \otimes 1_{T_2}) \cdot \varphi_1(z_1 \otimes 1_{T_2})^{-1} \in CU(pA_xp)$. If that is done, then we can show that $\varphi_0(1_{T_1} \otimes z_2) \cdot \varphi_1(1_{T_1} \otimes z_2)^{-1} \in CU(pA_xp)$ in a similar way.

In fact,

$$\begin{aligned}\varphi_1(z_1 \otimes 1_{\mathbb{T}_2}) &= w_1^* \cdot u^M \cdot (1_D \otimes z_1 \otimes 1_{\mathbb{T}_2}) \cdot u^{-M} \cdot w_1 \\ &= w_1^* \cdot \left(1_{\alpha^M(D)} \otimes z_1 \cdot e^{2\pi i s} \otimes 1_{\mathbb{T}_2}\right) \cdot w_1\end{aligned}$$

for some $s \in C(X, \mathbb{R})$. As $w_1 = \sum_{k \in \mathbb{Z}} u^k 1_{n^{-1}(k)}$ and $w_1 1_D w_1^* = u^M p u^{-M}$, we get

$$\begin{aligned}
\varphi_1(z_1 \otimes 1_{\mathbb{T}_2}) &= w_1^* \cdot \left(1_{\alpha^M(D)} \otimes \left(z_1 \cdot e^{2\pi i s}\right) \otimes 1_{\mathbb{T}_2}\right) \cdot w_1 \\
&= \left(\sum_{k \in \mathbb{Z}} u^k 1_{n^{-1}(k) \times \mathbb{T}_1 \times \mathbb{T}_2}\right)^* \cdot \left(1_{\alpha^M(D)} \otimes (z_1 \cdot e^{2\pi i s}) \otimes 1_{\mathbb{T}_2}\right) \cdot \left(\sum_{k \in \mathbb{Z}} u^k 1_{n^{-1}(k) \times \mathbb{T}_1 \times \mathbb{T}_2}\right) \\
&= \sum_{k, j \in \mathbb{Z}} 1_{n^{-1}(k) \times \mathbb{T}_1 \times \mathbb{T}_2} \cdot u^{-k} \cdot \left(1_{\alpha^M(D)} \otimes (z_1 \cdot e^{2\pi i s}) \otimes 1_{\mathbb{T}_2})\right) \cdot u^j \cdot 1_{n^{-1}(j) \times \mathbb{T}_1 \times \mathbb{T}_2} \\
&= \sum_{k \in \mathbb{Z}} 1_{n^{-1}(k) \times \mathbb{T}_1 \times \mathbb{T}_2} \cdot u^{-k} \cdot \left(1_{\alpha^M(D)} \otimes (z_1 \cdot e^{2\pi i s}) \otimes 1_{\mathbb{T}_2})\right) \cdot u^k \cdot 1_{n^{-1}(k) \times \mathbb{T}_1 \times \mathbb{T}_2} \\
&= 1_D \otimes \left(z_1 \cdot e^{2\pi i h}\right) \otimes 1_{\mathbb{T}_2}
\end{aligned}$$

for some $h \in C(X, \mathbb{R})$. Then we have

$$\varphi_0(z_1 \otimes 1_{\mathbb{T}_2}) \cdot \varphi_1(z_1 \otimes 1_{\mathbb{T}_2})^{-1} = 1_D \otimes e^{-2\pi i h} \otimes 1_{\mathbb{T}_2}$$

with $h \in C(X, \mathbb{R})$, and we also have

$$1_D \otimes e^{-2\pi i h} \otimes 1_{\mathbb{T}_2} \in pA_x p \cap pC^*(\mathbb{Z}, X, \alpha)p.$$

Note that $pA_x p \cap pC^*(\mathbb{Z}, X, \alpha)p \cong pC^*(\mathbb{Z}, X, \alpha)_x p$, which is an infinite dimensional simple AF algebra by [HPS]. By Lemma III.1.3, it follows that

$$U(pA_x p \cap pC^*(\mathbb{Z}, X, \alpha)p) = CU(pA_x p \cap pC^*(\mathbb{Z}, X, \alpha)p).$$

Then we get

$$\begin{aligned}
&\varphi_0(z_1 \otimes 1_{\mathbb{T}_2}) \cdot \varphi_1(z_1 \otimes 1_{\mathbb{T}_2})^{-1} \in \\
&\qquad U\left(pA_x p \cap pC^*(\mathbb{Z}, X, \alpha)p\right) = CU\left(pA_x p \cap pC^*(\mathbb{Z}, X, \alpha)p\right) \subset CU(pA_x p).
\end{aligned}$$

So far, we have shown that $\varphi_0^\sharp(z_1 \otimes 1_{\mathbb{T}_2}) = \varphi_1^\sharp(z_1 \otimes 1_{\mathbb{T}_2})$. In the same way, it follows that $\varphi_0^\sharp(1_{\mathbb{T}_1} \otimes z_2) = \varphi_1^\sharp(1_{\mathbb{T}_1} \otimes z_2)$.

According to [Lin1, Theorem 10.10], we conclude that φ_0 and φ_1 are approximately unitarily equivalent. Then there exists a unitary $w_2 \in pA_xp$ such that

$$\|w_1^* u^M z_i u^{-M} w_1 - w_2 z_i p w_2^*\| < \varepsilon - \|u^M z_i p u^{-M} - z_i q\|.$$

Let $w = w_1 w_2$. Then

$$\|u^M z_i p u^{-M} - z_i q\| < \varepsilon \text{ for } i = 1, 2.$$

We can easily check that

$$w^* w = w_2^* w_1^* w_1 w_2 = w_2^* w_2 = p$$

and

$$w w^* = w_1 w_2 w_2^* w_1 = w_1 p w_1^* = q,$$

which finishes the proof.

□

Lemma III.1.13. *We write $X \times \mathbb{T} \times \mathbb{T}$ as $X \times \mathbb{T}_1 \times \mathbb{T}_2$ to distinguish the factors. Let A be the crossed product C^*-algebra $C^*(\mathbb{Z}, X \times \mathbb{T}_1 \times \mathbb{T}_2, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ and let u be the implementing unitary of A. Let $x \in X$. For any $N \in \mathbb{N}$, any $\varepsilon > 0$ and any finite subset $\mathcal{G} \subset C(X \times \mathbb{T} \times \mathbb{T})$, we have a natural number $M > N$, a clopen neighborhood U of x and a partial isometry $w \in A_x$ (with A_x defined as in Lemma III.1.9) satisfying the following:*

(1) $\alpha^{-N+1}(U), \alpha^{-N+2}(U), \ldots, U, \alpha(U), \ldots, \alpha^M(U)$ are mutually disjoint, and $\mu(U) < \varepsilon/M$ for all α-invariant probability measure μ,

(2) $w^* w = 1_U$ and $w w^* = 1_{\alpha^M(U)}$,

(3) $u^{-i} w u^i \in A_x$ for $i = 0, 1, \ldots, M-1$,

(4) $\|wf - fw\| < \varepsilon$ for all $f \in \mathcal{G}$.

Proof. By abuse of notation, we identify $f \in C(X)$ with $f \otimes \mathrm{id}_{\mathbb{T}_1} \otimes \mathrm{id}_{\mathbb{T}_2}$, $g \in C(\mathbb{T}_1)$ with $\mathrm{id}_X \otimes g \otimes \mathrm{id}_{\mathbb{T}_2}$ and $h \in C(\mathbb{T}_2)$ with $\mathrm{id}_X \otimes \mathrm{id}_{\mathbb{T}_2} \otimes h$.

Without loss of generality, we can assume that

$$\mathcal{G} = \{f_1, \ldots, f_k, z_1, z_2\},$$

where $f_i \in C(X) \subset C(X \times \mathbb{T}_1 \times \mathbb{T}_2)$ for $i = 1, \ldots, k$ and $z_i(t_i) = t_i$ for $t_i \in \mathbb{T}_i$, $i = 1, 2$.

There exists a neighborhood E of x such that

$$|f_i(x) - f_i(y)| < \varepsilon/2$$

for all $y \in E$ and $i = 1, \ldots, k$. It then follows that for any $y_1, y_2 \in E$ and i such that $1 \leq i \leq k$, we have

$$|f_i(y_1) - f_i(y_2)| < \varepsilon.$$

As (X, α) is minimal, there exists $M > N$ such that $\alpha^M(x) \in E$. Let

$$K = \max\left\{M, \frac{M}{\varepsilon} + 1\right\}.$$

It is clear that the points $\alpha^{-N+1}(x), \alpha^{-N+2}(x), x, \alpha(x), \ldots, \alpha^K(x)$ are distinct. Then there exists a clopen set U containing x such that $U \subset E$, $\alpha^M(U) \subset E$ and $\alpha^{-N+1}(U)$, $\alpha^{-N+2}(U)$, U, $\alpha(U)$, ..., $\alpha^K(U)$ are disjoint.

As $\alpha^{-N+1}(U), \alpha^{-N+2}(U), U, \alpha(U), \ldots, \alpha^K(U)$ are disjoint, for every α-invariant probability measure μ, we have $\mu(U) < \varepsilon/M$.

By Lemma III.1.12, there exists a partial isometry $w \in A_x$ such that $w^*w = 1_U$ and $ww^* = 1_{\alpha^M(U)}$.

As $U \subset E$ and $\alpha^M(U) \subset E$, it follows that $\|wf_i - f_iw\| < \varepsilon$ for $0 \leq i \leq k$. The fact that $\|u^M z_i p u^{-M} - z_i q\| < \varepsilon$ implies $\|wz_i - z_iw\| < \varepsilon$ for $i = 1, 2$. So far, (4) is checked.

From our construction of U, we have (1). The assertion (2) follows from our construction of w. Note that $U, \alpha(U), \ldots, \alpha^M(U)$ are mutually disjoint. We can check that $u^{-i}wu^i \in A_x$ for $i = 0, \ldots, m-1$, thus finishing the proof.

□

Definition III.1.14. *Let $\mathcal{C}$ be a category of unital separable C^*-algebras. A separable simple C^*-algebra A is called $\mathcal{C}$-Popa if for every finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a nonzero projection $p \in A$ and a unital subalgebra B of pAp (with $1_B = p$) such that $B \in \mathcal{C}$ and*

1) $\|[x,p]\| \leq \varepsilon$ for all $x \in \mathcal{F}$,

2) $p \cdot x \cdot p \in_\varepsilon B$ for all $x \in \mathcal{F}$.

Lemma III.1.15. *Let $\mathcal{C}$ be a category of unital separable C^*-algebras. Let A be a separable simple C^*-algebra. If for every finite set $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a nonzero projection $p \in A$ and a unital subalgebra B of pAp such that B is $\mathcal{C}$-Popa and*

1) $\|[x,p]\| \leq \varepsilon$ for all $x \in \mathcal{F}$,

2) $pxp \in_\varepsilon B$ for all $x \in \mathcal{F}$,

then A is $\mathcal{C}$-Popa.

Proof. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, we can find a subalgebra B such that B is $\mathcal{C}$-Popa and

1) $\|[x, 1_B]\| \leq \varepsilon$ for all $x \in \mathcal{F}$,

2) $1_B \cdot x \cdot 1_B \subset_\varepsilon B$ for all $x \in \mathcal{F}$.

Use $1_B\mathcal{F}1_B$ to denote the set $\{1_Bx1_B \colon x \in \mathcal{F}\}$. As $1_B \cdot x \cdot 1_B \in_\varepsilon B$, for every $x \in \mathcal{F}$, choose an element $y_x \in B$ satisfying $\|y_x - 1_B \cdot x \cdot 1_B\| \leq \varepsilon$. Use $\mathcal{G}$ to denote $\{y_x \colon x \in \mathcal{F}\}$ with y_x as described.

As B is $\mathcal{C}$-Popa, we can find $E \subset B$ such that $E \in \mathcal{C}$ and

a) $\|[1_E, y_x]\| \leq \varepsilon$ for all $y_x \in \mathcal{G}$,

b) $1_E \cdot y_x \cdot 1_E \in_\varepsilon 1_E$ for all $y_x \in \mathcal{G}$.

We then check that

$$\|1_E \cdot y_x - y_x \cdot 1_E\| \approx_{2\varepsilon} \|1_E \cdot 1_B \cdot x \cdot 1_B - 1_B \cdot x \cdot 1_B \cdot 1_E\| \approx_{2\varepsilon} \|1_E \cdot x - x \cdot 1_E\|.$$

It then follows that

$$\|1_E \cdot x - x \cdot 1_E\| \approx_{4\varepsilon} \|1_E \cdot y_x - y_x \cdot 1_E\|.$$

As $\|[1_E, y_x]\| \leq \varepsilon$, we get $\|[x, 1_E]\| \leq 5\varepsilon$.

For any $x \in A$, we have

$$\begin{aligned} \operatorname{dist}(1_E \cdot x \cdot 1_E, E) &= \operatorname{dist}(1_E \cdot (1_B \cdot x \cdot 1_B) \cdot 1_E, E) \\ &\approx_\varepsilon \operatorname{dist}(1_E \cdot y_x \cdot 1_E, E)) \\ &\approx_\varepsilon 0. \end{aligned}$$

Then it is clear that $1_E \cdot x \cdot 1_E \in_{2\varepsilon} E$.

Thus for every finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, we can find the subalgebra E of A as described above such that $E \in \mathcal{C}$ and

1) $\|[x, 1_E]\| \leq 5\varepsilon$ for all $x \in \mathcal{F}$,

2) $1_E \cdot x \cdot 1_E \in_{2\varepsilon} E$ for all $x \in \mathcal{F}$,

which shows that A is $\mathcal{C}$-Popa.

□

This following is a technical result, and the proof is essentially the same as that of [Lin4, Lemma 2.5.5].

Proposition III.1.16. *Let A be a C^*-algebra. For every $a \in A_{sa}$ such that $\|a - a^2\| \leq \delta < \frac{1}{4}$, there exists a projection $p \in C^*(a)$ such that $\|p - a\| \leq \sqrt{\delta}$.*

Proof. According to continuous functional calculus,

$$\|a - a^2\| = \max\{|\lambda - \lambda^2| \colon \lambda \in \operatorname{sp}(a)\}.$$

The fact that $\|a - a^2\| \leq \delta < \frac{1}{4}$ implies that $\operatorname{sp}(a) \subset [-\sqrt{\delta}, \sqrt{\delta}] \sqcup [1 - \sqrt{\delta}, 1 + \sqrt{\delta}]$.

Define $f \in C(\mathrm{sp}(a), \mathbb{R})$ by

$$f(x) = \begin{cases} 0 & x \in \mathrm{sp}(a) \cap [-\sqrt{\delta}, \sqrt{\delta}] \\ 1 & x \in \mathrm{sp} \cap [1-\sqrt{\delta}, 1+\sqrt{\delta}] \end{cases}.$$

Then $f(a)$ is a projection in $C^*(a)$, and it is easy to check that $\|p - a\| \leq \sqrt{\delta}$.

□

Theorem III.1.17. *Let X be the Cantor set and let $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ be a minimal action on $X \times \mathbb{T} \times \mathbb{T}$. Use A to denote the crossed product C^*-algebra of the minimal system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$. Then* $\mathrm{TR}(A) \leq 1$.

Proof. According to [HLX, Lemma 4.3], for simple C*-algebra A, if for every $\varepsilon > 0$, $c \in A_+ \setminus \{0\}$ and finite subset $\mathcal{F} \subset A$, there exists a nonzero projection p and a unital subalgebra B of pAp such that $\mathrm{TR}(B) \leq 1$ and

1) $\|[x, p]\| \leq \varepsilon$ for all $x \in \mathcal{F}$,

2) $\mathrm{dist}(p \cdot x \cdot p, B) \leq \varepsilon$ for all $x \in \mathcal{F}$,

3) $1 - p \preceq c$ as in Definition I.2.2. That is, $1 - 1_B$ is Murray-von Neumann equivalent to a projection in $\mathrm{Her}(c)$,

then it follows that $\mathrm{TR}(A) \leq 1$.

Let A_x be as defined in Lemma I.2.1. According to Lemma II.2.11, $\mathrm{TR}(A_x) = 1$. If we can find a projection $e \in A_x$ such that $B = eA_xe$ satisfies the previous three conditions, then we are done.

As A is generated by $C(X \times \mathbb{T} \times \mathbb{T})$ and the implementing unitary u, we can assume that the finite set is $\mathcal{F} \cup \{u\}$ with $\mathcal{F} \subset C(X \times \mathbb{T} \times \mathbb{T})$.

Choose $N \in \mathbb{N}$ such that $2\pi/N < \varepsilon$ and let

$$\mathcal{G} = \bigcup_{i=0}^{N-1} u^i \mathcal{F} u^{-i}.$$

According to Lemma III.1.13, with respect to $\mathcal{G}$ and ε above, we can find $M > N$, a clopen neighborhood of x and a partial isometry $w \in A_x$ satisfying $w^*w = 1_U$, $ww^* = 1_{\alpha^M(U)}$ and $\|[w, f]\| < \varepsilon$ for all $f \in \mathcal{F}$.

Let $p = 1_U$ and $q = 1_{\alpha^M(U)}$. For $t \in [0, \pi/2]$, define

$$P(t) = p\cos^2 t + \sin t \cos t(w + w^*) + q\sin^2 t.$$

As $pq = 0$ and p, q are Murray-von Neumann equivalent via w, it follows that $t \mapsto P(t)$ is a path of projections with $P(0) = p$ and $P(\pi/2) = q$.

Define

$$e = 1 - \left(\sum_{i=0}^{M-N} u^i p u^{-i} + \sum_{i=1}^{N-1} u^{-i} P(i\pi/2N) u^i \right).$$

According to Lemma III.1.13, $u^{-i}wu^i \in A_x$ for $i = 0, \ldots, m-1$. It is clear that $e \in A_x$. It follows that e is a projection.

We first show that for $e \in A_x$ above, the following hold.

1) $\|[x, e]\| \leq \varepsilon$ for all $x \in \mathcal{F} \cup \{u\}$; (C1)

2) $\mathrm{dist}(exe, eA_xe) \leq \varepsilon$ for all $x \in \mathcal{F} \cup \{u\}$. (C2)

For the part of (C1) involving u, note that

$$\begin{aligned}
ueu^* - e &= 1 - u\left(\sum_{i=0}^{M-N} u^i p u^{-i} + \sum_{i=1}^{N-1} u^{-i} P(i\pi/2N) u^i\right) u^* \\
&\qquad - \left(1 - \left(\sum_{i=0}^{M-N} u^i p u^{-i} + \sum_{i=1}^{N-1} u^{-i} P(i\pi/2N) u^i\right)\right) \\
&= - \sum_{i=1}^{M-N+1} u^i p u^{-i} + \sum_{i=0}^{M-N} u^i p u^{-i} + \sum_{i=1}^{N-1} u^{-i} P(i\pi/2N) u^i \\
&\qquad - \sum_{i=0}^{N-2} u^{-i} P((i+1)\pi/2N) u^i \\
&= p - u^{M-N+1} p (u^*)^{M-N+1} + (u^*)^{N-1} P((N-1)\pi/2N) u^{N-1} - P(\pi/2N) \\
&\qquad + \sum_{i=1}^{N-2} u^{-i} (P(i\pi/2N) - P((i+1)\pi/2N)) u^i \\
&= p - P(\pi/2N) + u^{-(N-1)} P((N-1)\pi/2N) u^{N-1} - u^{M-N+1} p u^{-(M-N+1)} \\
&\qquad + \sum_{i=1}^{N-2} u^{-i} (P(i\pi/2N) - P((i+1)\pi/2N)) u^i.
\end{aligned}$$

As $2\pi/N < \varepsilon$, we get $\|ueu^* - e\| < \varepsilon$. It then follows that $\|ue - eu\| < \varepsilon$. By Lemma III.1.13, $\|fe - ef\| < \varepsilon$ for all $f \in \mathcal{F}$. So far, we have checked (C1).

For $f \in \mathcal{F} \subset C(X \times \mathbb{T} \times \mathbb{T})$, as $f \in A_x$, we get $efe \in eA_xe$. As $eu \in A_x$, it is clear that $eue = e(eu)e \in eA_xe$. Thus we have checked (C2).

Let $\mathcal{C}$ be the set of all the unital separable C*-algebras C such that there exist $N \in \mathbb{N}$ and one dimensional finite CW complexes X_i and $d_i \in \mathbb{N}$ with $1 \leq i \leq N$ and

$$C \cong \bigoplus_{n=1}^{N} M_{d_n}\left(C(X_n)\right).$$

Note that ε can be chosen to be arbitrarily small, and also note that eA_xe has tracial rank no more than one, which implies that eA_xe is $\mathcal{C}$-Popa.

By Lemma III.1.15, A is also $\mathcal{C}$-Popa. According to [Lin4, Lemma 3.6.6], A has property (SP). For the given element $c \in A_+$, there exists a non-zero projection $q \in \mathrm{Her}(c)$. Let $\delta_0 = \inf\{\tau(q) \colon \tau \in T(A)\}$. As A is simple and $q \neq 0$, we get $\tau(q) > 0$ for all $\tau \in T(A)$. As $T(A)$ is a weak* closed subset of the unit ball of A^*, noting that the unit ball of A^* is weak* compact by Alaoglu's Theorem, it follows that $T(A)$ is also compact. Thus $\delta_0 > 0$.

Without loss of generality, we can assume that $\varepsilon < \min\{1, \frac{1}{8}\delta_0, \frac{1}{(40\delta_0)^2}\}$ and $q \in \mathcal{F}$.

It remains to show that $1-e$ is Murray-von Neumann equivalent to a projection in $\mathrm{Her}(c)$.

As $q \in \mathcal{F}$, we have

$$\|[q,e]\| \leq \varepsilon \text{ and } \mathrm{dist}(eqe, eA_xe) \leq \varepsilon.$$

We can find $b \in (eA_xe)_{sa}$ such that $\|eqe - b\| \leq \varepsilon$. Note that $\|[q,e]\| \leq \varepsilon$ implies that $\|(eqe)^2 - eqe\| \leq \varepsilon$. According to Proposition III.1.16, there exists a projection $q' \in A$ such that $\|q' - eqe\| \leq \sqrt{\varepsilon}$ and $q' \preceq eqe$ as in Definition I.2.2.

Note that we have

$$\begin{aligned}\|b^2 - b\| &\leq \|b^2 - (eqe)^2\| + \|(eqe)^2 - eqe\| + \|eqe - b\| \\ &\leq 3\varepsilon + \varepsilon + \varepsilon \\ &= 5\varepsilon.\end{aligned}$$

By Proposition III.1.16 again, there exists a projection $p \in eA_xe$ such that

$$\|p - b\| \leq \sqrt{5\varepsilon} \text{ and } [p] \leq [b].$$

As

$$\|p - q'\| \leq \|p - b\| + \|b - eqe\| + \|eqe - q'\| \leq \sqrt{5\varepsilon} + \varepsilon + \sqrt{\varepsilon},$$

it follows that $[p] = [q']$. As

$$q' \preceq eqe \text{ and } eqe \preceq q,$$

we conclude that $p \lesssim q$ in A.

Note that

$$q = eqe + (1-e)qe + eq(1-e) + (1-e)q(1-e).$$

For every $\tau \in T(A)$, we have

$$\tau(q) = \tau(eqe) + \tau((1-e)q(1-e)) + \tau((1-e)qe + eq(1-e)).$$

According to (C1) and our choice of ε, we have

$$\tau(eqe) + \tau((1-e)q(1-e)) > \tau(q) - \varepsilon > \frac{1}{2}\tau(q).$$

As τ is a tracial state and e is a projection,

$$\tau((1-e)q(1-e)) \leq \tau((1-e)1(1-e)) = \tau(1-e).$$

Note that $\tau(1-e) < \frac{1}{4}\tau(q)$ for all $\tau \in T(A)$ (because $\tau(1-e) < \frac{1}{4}\delta_0$). We can conclude that

$$\tau(eqe) > \frac{1}{2}\tau(q) - \tau((1-e)q(1-e)) \geq \frac{1}{2}\tau(q) - \tau(1-e) > \frac{1}{4}\tau(q) \geq \frac{1}{4}\delta_0 > 0.$$

In our construction, note that

$$\|p - eqe\| \leq \|p - b\| + \|b - eqe\| \leq \sqrt{5\varepsilon} + \varepsilon.$$

It follows that

$$\tau(p) \geq \frac{1}{4}\delta_0 - (\sqrt{5\varepsilon} + \varepsilon) \geq \frac{1}{8}\delta_0 \quad \text{for all } \tau \in T(A).$$

According to our construction, we have

$$\tau(1-e) < M \cdot \frac{\varepsilon}{M} = \varepsilon \leq \frac{1}{8}\delta_0 \leq \tau(p)$$

for all $\tau \in T(A)$, which then implies that $1 - e \lesssim p$. As $[p] \leq [c]$ (as in Definition I.2.2), we get $[1-e] \leq [c]$ (as in Definition I.2.2), which finishes the proof.

$\square$

III.2 THE RIGID CASE

Proposition III.2.1. *Let A be the crossed product C^*-algebra of the minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$. Then*

$$K_0(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2$$

and

$$K_1(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2.$$

Proof. Use $j : C(X \times \mathbb{T}^2) \to A$ to denote the canonical embedding of $C(X \times \mathbb{T}^2)$ into A. We have the Pimsner-Voiculescu six-term exact sequence:

$$\begin{tikzcd}
K_0(C(X \times \mathbb{T}^2)) \arrow[r, "\mathrm{id}_{*0} - \alpha_{*0}"] & K_0(C(X \times \mathbb{T}^2)) \arrow[r, "j_{*0}"] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & K_1(C(X \times \mathbb{T}^2)) \arrow[l, "j_{*1}"] & K_1(C(X \times \mathbb{T}^2)). \arrow[l, "\mathrm{id}_{*1} - \alpha_{*1}"]
\end{tikzcd}$$

We know that

$$K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2, \; K_1(C(\mathbb{T}^2)) \cong \mathbb{Z}^2$$

and

$$K_0(C(X)) \cong C(X, \mathbb{Z}), K_1(C(X))) = 0.$$

According to the Künneth theorem, $K_0(C(X \times \mathbb{T}^2)) \cong C(X, \mathbb{Z}^2)$ and $K_1(C(X \times \mathbb{T}^2)) \cong C(X, \mathbb{Z}^2)$.

For $i = 0, 1$, consider the image of $\mathrm{id}_{*i} - \alpha_{*i}$. They are both isomorphic to

$$\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\}.$$

The kernel of $\mathrm{id}_{*i} - \alpha_{*i}$ for $i = 0, 1$ is

$$\{f \in C(X, \mathbb{Z}^2) \colon f = f \circ \alpha\}.$$

Assume that f is in the kernel of $\mathrm{id}_{*i} - \alpha_{*i}$ for $i = 0, 1$. Fix $x_0 \in X$. We have $f(\alpha^n(x_0)) = f(x_0)$

for all $n \in \mathbb{Z}$. As α is a minimal homeomorphism of the Cantor set X and f is continuous, f must be a constant function from X to $\mathbb{Z}^2$. Now we conclude that

$$\ker(\mathrm{id}_{*i} - \alpha_{*i}) \cong \mathbb{Z}^2.$$

As the six-term sequence above is exact, we have the short exact sequence:

$$0 \longrightarrow \operatorname{coker}(\mathrm{id}_{*0} - \alpha_{*0}) \longrightarrow K_0(A) \longrightarrow \ker(\mathrm{id}_{*1} - \alpha_{*1}) \longrightarrow 0.$$

As $\ker(\mathrm{id}_{*i} - \alpha_{*i}) \cong \mathbb{Z}^2$ and $\mathbb{Z}^2$ is projective, it follows that

$$K_0(A) \cong \operatorname{coker}(\mathrm{id}_{*0} - \alpha_{*0}) \oplus \mathbb{Z}^2.$$

As $\operatorname{coker}(\mathrm{id}_{*0} - \alpha_{*0}) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha \colon f \in C(X, \mathbb{Z}^2)\}$, we get

$$K_0(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha \colon f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2.$$

Similarly, we get that $K_1(A) \cong C(X, \mathbb{Z}^2)/\{f - f \circ \alpha \colon f \in C(X, \mathbb{Z}^2)\} \oplus \mathbb{Z}^2$.

□

If we require a certain "rigidity" condition on the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$, then the tracial rank of the crossed product will be zero.

Definition III.2.2. *Let $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ be a minimal dynamical system. Let μ be an $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$-invariant probability measure on $X \times \mathbb{T} \times \mathbb{T}$. It will induce an α-invariant probability measure on X defined by $\pi(u)(D) = \mu(D \times \mathbb{T} \times \mathbb{T})$ for every Borel set $D \subset X$. We say that $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is rigid if π gives a one-to-one map between the $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$-invariant probability measures and the α-invariant probability measures.*

Remark: For minimal actions on $X \times \mathbb{T} \times \mathbb{T}$ of the type $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$, it is easy to see that π always maps the set of $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$-invariant probability measures over $X \times \mathbb{T} \times \mathbb{T}$ onto the set of

α-invariant measures over X.

According to Theorem 4.6 in [Lin-Phillips], the "rigidity" condition defined above implies that the crossed product C*-algebra has tracial rank zero.

Proposition III.2.3. *Let $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ be a minimal dynamical system. If it is rigid, then the corresponding crossed product C*-algebra $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ has tracial rank zero.*

Proof. Use A to denote $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$. We will show that

$$\rho\colon K_0(A) \longrightarrow \mathrm{Aff}(T(A))$$

has a dense range, which will then imply that $\mathrm{TR}(A) = 0$ according to [Lin-Phillips, Theorem 4.6].

For the crossed product C*-algebra $B = C^*(\mathbb{Z}, X, \alpha)$, we know that B has tracial rank zero and $\rho_B\colon K_0(B) \to T(B)$ has the dense range. According to [Putnam, Theorem 1.1], $K_0(A) \cong C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1}\}$. For every $x \in K_0(A)$, we can find $f \in C(X, \mathbb{Z})$ such that $\hat{x}(\tau) := \tau(x)$ equals $\tau(f) = \int_X f \, \mathrm{d}\mu_\tau$.

As $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is rigid, there is a one-to-one correspondence between $(\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$-invariant measures and α-invariant measures. In other words, $T(A)$ is homeomorphic to $T(B)$ (as two convex compact sets). Let $h \in C(X)$ be a projection. Then $h \otimes 1_{C(\mathbb{T} \times \mathbb{T})}$ is a projection in A.

As ρ_B has a dense range in $\mathrm{Aff}(T(B))$, we have that ρ has dense range in $\mathrm{Aff}(T(A))$. As $X \times \mathbb{T} \times \mathbb{T}$ is an infinite finite dimensional metric space and $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal, according to [Lin-Phillips, Theorem 4.6], $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ has tracial rank zero.

□

III.3 EXAMPLES

We start with a criterion for determining whether a dynamical system of $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is minimal or not. This result is a special case of the remark of page 582 in [Furstenberg]. The proof here essentially follows that of Lemma 4.2 of [LM1].

Lemma III.3.1. *Let Y be a compact metric space, and let $\beta\times \mathrm{R}_\eta$ be a skew product homeomorphism of $Y \times \mathbb{T}$ with $\beta \in Homeo(Y)$, $\eta\colon Y \to \mathbb{T}$ and*

$$(\beta \times \mathrm{R}_\eta)(y,t) = (\beta(y), t+\eta(y)) \text{ with } \mathbb{T} \text{ identified with } \mathbb{R}/\mathbb{Z}.$$

Then $\beta \times \mathrm{R}_\eta$ is minimal if and only if (Y,β) is minimal and there exist no $f \in C(Y,\mathbb{T})$ and non-zero integer n such that

$$n\eta = f \circ \beta - f.$$

Proof. Proof of the "if" part:

If (Y,β) is minimal and there exist no $f \in C(Y,\mathbb{T})$ and non-zero integer n such that $n\eta = f\circ\beta - f$, we will prove that $\beta \times \mathrm{R}_\eta$ is minimal.

If $\beta \times \mathrm{R}_\eta$ is not minimal, then there exists a proper minimal subset E of $Y \times \mathbb{T}$. Let $\pi_Y : Y \times \mathbb{T} \to Y$ be the canonical projection onto Y. Note that $\pi_Y \circ (\beta \times \mathrm{R}_\eta) = \beta \circ \pi_Y$. It follows that $\pi_Y(E)$ is an invariant subset of Y. As Y is compact, so is $\pi_Y(E)$. Since (Y,β) is minimal, the closed invariant set $\pi_Y(E)$ must be Y.

Let's consider

$$D := \{t \in \mathbb{T} : (\mathrm{id}_Y \times \mathrm{R}_t)(E) = E\}.$$

As $(\mathrm{id}_Y \times \mathrm{id}_\mathbb{T})(E) = E$, the set D is not empty. Note that D is a subgroup of $\mathbb{T}$. It follows that D is a non-empty subgroup of $\mathbb{T}$ (with $\mathbb{T}$ identified with the quotient group $\mathbb{R}/\mathbb{Z}$).

If we have $\{t_n\}_{n\in\mathbb{N}} \subset D$ such that $t_n \to t$, then for any $\omega \in E$, we have $(\mathrm{id}\times \mathrm{R}_{t_n})\omega \in E$. Then $t_n \to t$ implies that $(\mathrm{id}\times\mathrm{R}_{t_n})w \to (\mathrm{id}\times\mathrm{R}_t)w$. As E is closed, $(\mathrm{id}\times\mathrm{R}_t)w \in E$.

So far, we have shown that if $t_n \in D$ for $n \in \mathbb{N}$ and $t_n \to t$, then $t \in D$. Note that "$\{t_n\}_{n\in\mathbb{N}} \subset D$ and $t_n \to t$" is equivalent to "$\{-t_n\}_{n\in\mathbb{N}} \subset D$ and $-t_n \to -t$". It follows that $-t \in D$. In other words, we have

$$(\mathrm{id}\times\mathrm{R}_t)(E) \subset E \text{ and } (\mathrm{id}\times\mathrm{R}_{-t})(E) \subset E.$$

Then we get

$$E = (\mathrm{id}\times\mathrm{R}_t)((\mathrm{id}\times\mathrm{R}_{-t})(E)) \subset (\mathrm{id}\times\mathrm{R}_t)(E) \subset E,$$

which implies that $(\mathrm{id} \times \mathrm{R}_t)E = E$. In other words, D is closed.

As E is a proper subset and $\pi_Y(E) = Y$, D must be a proper subgroup of $\mathbb{T}$. Otherwise, for any $(y,t) \in Y \times \mathbb{T}$, as $\pi_Y(E) = Y$, there exists $t' \in \mathbb{T}$ such that $(y,t') \in E$. Since $t - t' \in D = \mathbb{T}$, $(y,t) = (\mathrm{id} \times \mathrm{R}_{t-t'})(y,t') \in E$, which indicates that $E = Y \times T$, contradicting the fact that E is a proper subset.

As a proper closed subgroup of $\mathbb{T}$, D must be

$$\left\{\frac{k}{n}\right\}_{0 \le k \le n-1} \quad \text{with } n = |D|.$$

Let $\pi_{\mathbb{T}}$ be the canonical projection from $Y \times \mathbb{T}$ onto $\mathbb{T}$. For $y \in Y$, use E_y to denote $\pi_{\mathbb{T}}(E \cap \pi_Y^{-1}(\{y\}))$.

Using the fact that E is a minimal subset of (β, R_η), we will show that E_y must be n points distributed evenly on the circle for all $y \in Y$.

We claim that if $t, t' \in E_y$, then for any $m \in \mathbb{Z}$, $t + m(t'-t)$ must be in E_y. To prove this claim, if $t, t' \in E_y$, then there exists $\{k_n\}_{n\in\mathbb{N}}$ such that $k_n \to \infty$ and $\mathrm{dist}((\beta \times \mathrm{R}_\eta)^{k_n}(y,t),(y,t')) \to 0$. Note that

$$\mathrm{dist}((\beta \times \mathrm{R}_\eta)^{k_n}(y,t),(y,t')) = \mathrm{dist}((\beta \times \mathrm{R}_\eta)^{k_n}(y,t'),(y,t+2(t'-t))).$$

It follows that $(y, t+2(t'-t)) \in \overline{\mathrm{Orbit}_{\beta\times\mathrm{R}_\eta}((y,t))}$. By induction, we conclude that if $t, t' \in E_y$, then for any $m \in \mathbb{Z}$, $t + m(t'-t)$ is also in E_y, proving the claim.

For any $y \in Y$, consider E_y, which is a non-empty closed subset of $\mathbb{T}$. Let

$$l_y = \inf_{t_1, t_2 \in E_y} \mathrm{dist}(t_1, t_2).$$

Note that if $t, t' \in E_y$, then $t + m(t'-t) \in E_y$. The fact that $E_y \subsetneq \mathbb{T}$ implies that $l_y > 0$. It is then clear that E_y is made up of $1/l_y$ points distributed evenly on $\mathbb{T}$.

Claim: For every $y \in Y$, $1/l_y = |D|$.

For given $y \in Y$, as $(\mathrm{id} \times \mathrm{R}_t)(E) = E$ for all $t \in D$, we get that E_y is invariant under R_t

for all $t \in D$. It then follows that $1/l_y = kn$ with $k \in \mathbb{N}$ and $n = |D|$.

If $k > 1$, write

$$E_y = \{(y, t_1), \ldots, (y, t_{kn})\}.$$

Use $\mathrm{Orbit}_{\beta \times \mathrm{R}_\eta}(E_y)$ to denote $\bigcup_{m=1}^{\infty} (\beta \times \mathrm{R}_\eta)^m(E_y)$.

As β is minimal, for every $y' \in Y$, there is a sequence $(m_k)_{k \in \mathbb{N}}$ such that

$$\beta^{m_k}(y) \to y'.$$

The fact that $\mathrm{Orbit}_{\beta \times \mathrm{R}_\eta}(E_y)$ is dense implies that there exists $t' \in \mathbb{T}$ such that (y', t') is in the closure of $\mathrm{Orbit}_{\beta \times \mathrm{R}_\eta}(E_y)$. Note that for every $m \in \mathbb{N}$, $(\beta \times \mathrm{R}_\eta)^m(E_y)$ consists of kn points distributed evenly on the circle. It follows that $E_{y'}$ contains at least nk points distributed evenly on the circle.

Now we have shown that for every $a \in Y$, E_a is made up of at least nk evenly distributed points on the circle, which then implies that D contain at least nk elements. The assumption that $k > 1$ gives a contradiction.

We then conclude that $k = 1$, which proves the claim.

By the claim above, for all $y \in Y$, the set E_y is made up of n points distributed evenly on $\mathbb{T}$. If we define

$$nE = \{(x, nt) \colon (x, t) \in E\},$$

then nE is the graph of some continuous map $g \colon Y \to \mathbb{T}$. As E is closed, so is nE, which implies that g is continuous. As E is $(\beta \times \mathrm{R}_\eta)$-invariant, for every $(x, t) \in E$, it follows that

$$(\beta \times \mathrm{R}_\eta)(x, t) = (\beta(x), t + \eta(x)) \in E.$$

In other words, we have $n(t + \eta(x)) = g(\beta(x))$. As $nt = g(x)$, it follows that $n\eta = g \circ \beta - g$, which finishes the proof of "if" part.

Proof of the "only if" part:

Suppose $\beta \times \mathrm{R}_\eta$ is minimal. Then it is clear that (Y, β) is a minimal system.

Suppose that there exists nonzero $n \in \mathbb{Z}$ such that $n\eta = g \circ \beta - g$ for some $g \in C(X, T)$. Let

$$E = \{(y,t) \in Y \times \mathbb{T} \colon nt = g(y)\}.$$

For $(y,t) \in E$, we have $(\beta \times \mathrm{R}_\eta)(y,t) = (\beta(y), t + \eta(y))$. As

$$n(t + \eta(y)) = nt + n\eta(y) = g(y) + n\eta(y) = g(\beta(y)),$$

it follows that E is $(\beta \times \mathrm{R}_\eta)$-invariant.

As g is continuous, E is closed. And it is clear that E is a proper subset of $Y \times \mathbb{T}$. Now we have a proper closed $(\beta \times \mathrm{R}_\eta)$-invariant set in $Y \times \mathbb{T}$, contradicting the minimality of $\beta \times \mathrm{R}_\eta$.

$\square$

Lemma III.3.1 provides an inductive approach to determine the minimality of some dynamical systems. Following this lemma, we get the proposition below.

Proposition III.3.2. *Let $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ be a homeomorphism of $X \times \mathbb{T} \times \mathbb{T}$. Then $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal if and only if*

i) (X, α) is minimal,

ii) ξ is not a torsion element in $C(X, \mathbb{T})/\{f \circ \alpha - f\}$,

iii) For $\tilde{\eta} \in C(X \times \mathbb{T}, \mathbb{T})$ defined by $\tilde{\eta}(x,t) = \eta(x)$, the map $\tilde{\eta}$ is not a torsion element in

$$C(X \times \mathbb{T}, \mathbb{T})/\{f \circ (\alpha \times \mathrm{R}_\xi) - f \colon f \in C(X \times \mathbb{T}, \mathbb{T})\}.$$

Proof. Proof of the "if" part:

If i), ii) and iii) are true, we need to show that $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal.

Note that $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is a skew product of $\alpha \times \mathrm{R}_\xi$ and $\widetilde{\mathrm{R}_\eta}$, where $\widetilde{\mathrm{R}_\eta}$ is defined by

$$\widetilde{\mathrm{R}_\eta} \colon X \times \mathbb{T} \to \mathrm{Homeo}(\mathbb{T}), \text{ with } (\widetilde{\mathrm{R}_\eta}(x,t))(t') = t' + \eta(x).$$

From i) and ii), using Lemma 4.2 of [LM1], $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$ is minimal.

According to Lemma III.3.1, and by iii), we conclude that $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is minimal.

Proof of the "only if" part:

As $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is the skew product of $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$ and $\widetilde{\mathrm{R}_\eta} \colon X \times \mathbb{T} \to \mathrm{Homeo}(\mathbb{T})$, with $\widetilde{\mathrm{R}_\eta}$ defined as above, the minimality of $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ implies the minimality of $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$. By Lemma 4.2 of [LM1], that implies (i) and (ii).

For (iii), suppose that $\tilde{\eta}$ is a torsion element, that is, there is non-zero $n \in \mathbb{Z}$ and $f \in C(X \times \mathbb{T}, \mathbb{T})$ such that $n\tilde{\eta} = f \circ (\alpha \times \mathrm{R}_\xi) - f$. By Lemma III.3.1, it follows that $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ is not minimal, a contradiction.

□

Proposition III.3.2 enables us to construct minimal dynamical systems on $X \times \mathbb{T} \times \mathbb{T}$ inductively. In fact, we have the following lemma.

Lemma III.3.3. *Given any minimal dynamical system $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$, there exist uncountably many $\theta \in [0, 1]$ such that if we use θ to denote the constant function in $C(X, \mathbb{T})$ defined by $\theta(x) = \theta$ for all $x \in X$ (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$), then the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is still minimal.*

Proof. Note that the dynamical system $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$ is minimal. According to Lemma III.3.1, (X, α) must be a minimal dynamical system, and ξ is not a torsion element in

$$C(X, \mathbb{T})/\{f - f \circ \alpha \colon f \in C(X, \mathbb{T})\}.$$

This implies that conditions i) and ii) in Proposition III.3.2 are already satisfied.

According to Proposition III.3.2, for $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ to be minimal, we just need to find $\theta \in \mathbb{R}$ such that for every $n \in \mathbb{Z} \setminus \{0\}$ and $f \in C(X \times \mathbb{T}, \mathbb{T})$, we have

$$n\theta \neq f - f \circ (\alpha \times \mathrm{R}_\xi).$$

If this is not true, then we have

$$n\theta = f - f \circ (\alpha \times \mathrm{R}_\xi).$$

Let $F\colon X \times \mathbb{T} \to \mathbb{R}$ be a lifting of f. That is, $F \in C(X \times \mathbb{T}, \mathbb{R})$ and the following diagram commutes:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ X \times \mathbb{T} \arrow[ur, "F"] \arrow[r, "f"'] & \mathbb{T}, \end{tikzcd}$$

with $\pi(t) = t$ for all $t \in \mathbb{R}$ (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$).

We use $[F]$ to denote $\pi \circ F$.

It follows that

$$\begin{aligned} n\theta &= [F] - [F \circ (\alpha \times \mathrm{R}_\xi)] \\ &= [F - F \circ (\alpha \times \mathrm{R}_\xi)]. \end{aligned}$$

In other words, there exists $g \in C(X \times \mathbb{T}, \mathbb{Z})$ such that

$$n\theta - (F - F \circ (\alpha \times \mathrm{R}_\xi)) = g.$$

For every $(\alpha \times \mathrm{R}_\xi)$-invariant probability measure μ, we have

$$\mu(n\theta) = \mu(g),$$

with $\mu(n\theta) = \int_{X \times \mathbb{T}} n\theta \, \mathrm{d}\mu$ and $\mu(g) = \int_{X \times \mathbb{T}} g \, \mathrm{d}\mu$

Since $\mu(n\theta) = n\mu(\theta)$, it follows that

$$\mu(\theta) = \frac{\mu(g)}{n} = \mu\left(\frac{g}{n}\right).$$

Let A be the crossed product C*-algebra of $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$. Define

$$\rho : A_{sa} \longrightarrow \mathrm{Aff}(T(A))$$

by $\rho(a)(\tau) = \tau(a)$ for all $a \in A_{sa}$ and $\tau \in T(A)$. Then we have

$$\rho(\theta) = \rho\left(\frac{g}{n}\right)$$

in $\mathrm{Aff}(T(A))$.

Now we have show that if θ (as a constant function) is a torsion element in

$$C(X \times \mathbb{T}, \mathbb{T})/\{f - f \circ \alpha \colon f \in C(X \times \mathbb{T}, \mathbb{T})\}$$

with order n, then there exists $g \in C(X \times \mathbb{T}, \mathbb{Z})$ such that $\rho(\theta) = \rho\left(\frac{g}{n}\right)$.

As $\mathbb{T}$ is connected, we have $C(X \times \mathbb{T}, \mathbb{Z}) \cong C(X, \mathbb{Z})$. Note that the set

$$\left\{\frac{g}{n} \colon g \in C(X \times \mathbb{T}, \mathbb{Z}) \cong C(X, \mathbb{Z}), n \in \mathbb{Z} \setminus \{0\}\right\}$$

contains countably many elements. It follows that its image under ρ contains at most countably many elements. The fact that $[0, 1]$ contains uncountably many elements and $\rho(\theta) = 0$ if and only if $\theta = 0$ implies that there exists (uncountably many, in fact) $\theta \in \mathbb{R}$ such that θ (as a constant function) is not a torsion element in

$$C(X \times \mathbb{T}, \mathbb{T})/\{f - f \circ \alpha \colon f \in C(X \times \mathbb{T}, \mathbb{T})\},$$

which then implies that $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is still minimal.

$\square$

We now give examples of rigid and non-rigid minimal actions of on $X \times \mathbb{T} \times \mathbb{T}$.

Let $\varphi_0 \colon \mathbb{T} \to \mathbb{T}$ be a Denjoy homeomorphism (see [PSS, Definition 3.3] or [KatokHasselblatt, Prop 12.2.1]) with rotation number $r(\gamma) = \theta$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. It is known that φ_0 has a unique proper invariant closed subset of $\mathbb{T}$, which is a Cantor set, and that φ_0 restricted on this Cantor set is minimal.

Let X be the Cantor set and use $\varphi\colon X \to X$ to denote the restriction of φ_0 to X.

According to the Poincare Classification Theorem (see [KatokHasselblatt, Theorem 11.2.7]), there is a non-invertible continuous monotonic map $h\colon \mathbb{T} \to \mathbb{T}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\varphi_0"] \arrow[d, "h"'] & \mathbb{T} \arrow[d, "h"] \\ \mathbb{T} \arrow[r, "\mathrm{R}_\theta"'] & \mathbb{T} \end{tikzcd} .$$

Using the restriction of φ to the invariant subset (which is the Cantor set X), we get a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "h|_X"'] & X \arrow[d, "h|_X"] \\ \mathbb{T} \arrow[r, "\mathrm{R}_\theta"'] & \mathbb{T} \end{tikzcd} .$$

It is known that for a Denjoy homeomorphism, $h|_X$ maps X onto $\mathbb{T}$.

Recall that for $\xi, \eta\colon \mathbb{T} \to \mathbb{T}$, the action

$$\gamma\colon (s, t_1, t_2) \mapsto (s + \theta, t_1 + \xi(s), t_2 + \eta(s))$$

is called a Furstenberg transformation. Consider the action

$$\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}\colon X \times \mathbb{T} \times \mathbb{T} \to X \times \mathbb{T} \times \mathbb{T}.$$

It is clear that we have the commutative diagram below :

$$\begin{tikzcd} X \times \mathbb{T} \times \mathbb{T} \arrow[r, "\alpha\times \mathrm{R}_{\xi o h}\times \mathrm{R}_{\eta o h}"] \arrow[d, "h|_X \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}}"'] & X \times \mathbb{T} \times \mathbb{T} \arrow[d, "h|_X \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}}"] \\ \mathbb{T} \times \mathbb{T} \times \mathbb{T} \arrow[r, "\gamma"] & \mathbb{T} \times \mathbb{T} \times \mathbb{T} \ . \end{tikzcd} \tag{III.1}$$

In this case, if γ is minimal, then $\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h}$ is also minimal, as will be shown in the next proposition.

Proposition III.3.4. *For the minimal dynamical systems as in diagram (III.1), if* $(\mathbb{T} \times \mathbb{T} \times \mathbb{T}, \gamma)$ *is a minimal dynamical system, then* $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})$ *is also a minimal dynamical system.*

Proof. Assume that $(\mathbb{T} \times \mathbb{T} \times \mathbb{T}, \gamma)$ is minimal and $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})$ is not minimal. It then follows that there exist $(x, t_1, t_2) \in X \times \mathbb{T} \times \mathbb{T}$, nonempty open subset $D \subset X$ and open subsets $U, V \subset \mathbb{T}$ such that

$$\{(\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^n(x, t_1, t_2)\}_{n \in \mathbb{N}} \cap (D \times U \times V) = \varnothing. \tag{III.2}$$

Define

$$\pi_1, \pi_2 \colon X \times \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{T}$$

by

$$\pi_1(x, t_1, t_2) = t_1 \text{ and } \pi_2(x, t_1, t_2) = t_2.$$

As α is a minimal action on the Cantor set X, the statement III.2 implies that for every $k \in \mathbb{N}$ such that $\alpha^k(x) \in D$, we have

$$\pi_1\left((\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^k(x)\right) \notin U \text{ and } \pi_2\left((\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^k(x)\right) \notin V. \tag{III.3}$$

Note that if we regard the Cantor set X as a subset of $\mathbb{T}$, then $h|_X : X \to \mathbb{T}$ is a noninvertible continuous monotone function. For the open set $D \subset X$, without loss of generality, we can assume that (by identifying X as a subset of $\mathbb{T}$ and identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$)

$$D = (a, b) \cap X \text{ with } a, b \in (0,1) \text{ and } a < b.$$

It then follows that there exists $c, d \in (0,1)$ with $c < d$ (without loss of generality, we can assume that $0 \notin h|_X(D)$ such that $h|_X(D)$ is one of the following:

$$(c, d),\ (c, d],\ [c, d) \text{ or } [c, d].$$

In either case, there exists $c', d' \in (0,1)$ with $c' < d'$ such that

$$(c', d') \subset h|_X(D).$$

Let $t_x = h|_X(x)$. It is then clear that

$$h|_X((\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^n(x, t_1, t_2)) = \gamma^n(t_x, t_1, t_2)$$

for all $n \in \mathbb{N}$. As $h|_X(D)$ is monotone, for every $k \in \mathbb{N}$, if $\mathrm{R}_\theta^k(t_x) \in (c', d')$, then we have $\alpha^k(x) \in D$, which implies (see (III.3)) that

$$\pi_1\left((\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^k(x, t_1, t_2)\right) \notin U \text{ and } \pi_2\left((\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^k(x, t_1, t_2)\right) \notin V.$$

Define

$$\rho_1, \rho_2 \colon \mathbb{T} \times \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T} \times \mathbb{T}$$

by $\rho_1(t_0, t_1, t_2) = t_1$ and $\rho_2(t_0, t_1, t_2) = t_2$. It is easy to check that for all $n \in \mathbb{N}$, we have

$$\pi_i\left((\alpha \times \mathrm{R}_{\xi o h} \times \mathrm{R}_{\eta o h})^k(x, t_1, t_2)\right) = \rho_i\left(\gamma^k(t_x, t_1, t_2)\right).$$

Then we have that for every $k \in \mathbb{N}$ such that $\mathrm{R}_\theta^k(t_x) \in (c', d')$,

$$\rho_1\left(\gamma^k(td_x, t_1, t_2)\right) \notin U \text{ and } \rho_2\left(\gamma^k(t_x, t_1, t_2)\right) \notin V.$$

According to the definition of the Furstenberg transformation γ, it follows that

$$\{\gamma^n(t_x, t_1, t_2)\}_{n\in\mathbb{N}} \cap ((c', d') \times U \times V) = \varnothing,$$

contradicting the minimality of γ, which finishes the proof.

$\square$

The proposition below shows that if the two dynamical systems in Prop III.3.4 are minimal, then there is a one-to-one correspondence between the invariant measures on them.

Proposition III.3.5. *If the dynamical systems* $(\mathbb{T}\times\mathbb{T}\times\mathbb{T}, \gamma)$ *and* $(X\times\mathbb{T}\times\mathbb{T}, \alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h})$ *(as in diagram (III.1)) are minimal, then there is a one-to-one correspondence between the* $\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}$*-invariant probability measures and the* γ*-invariant probability measures.*

Proof. First of all, we will define the correspondence between the $\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}$-invariant probability measures and the γ-invariant probability measures.

For simplicity, we use H to denote the function $h|_X$ in diagram (III.1). We use $M_{\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}}$ to denote the set of $\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}$-invariant probability measures on $X\times\mathbb{T}\times\mathbb{T}$ and M_γ to denote the set of γ-invariant probability measures on $\mathbb{T}\times\mathbb{T}\times\mathbb{T}$.

Define

$$\varphi : M_{\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}} \longrightarrow M_\gamma \text{ and } \psi : M_\gamma \longrightarrow M_{\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}}$$

by

$$\varphi(\mu)(D) = \mu\left((H\times \mathrm{id}_{\mathbb{T}}\times \mathrm{id}_{\mathbb{T}})^{-1}(D)\right) \text{ and } \psi(\nu)(E) = \nu\left((H\times \mathrm{id}_{\mathbb{T}}\times \mathrm{id}_{\mathbb{T}})(E)\right)$$

for all Borel subsets D of $\mathbb{T}\times\mathbb{T}\times\mathbb{T}$, Borel subsets E of $X\times\mathbb{T}\times\mathbb{T}$, $\mu\in M_{\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}}$ and $\nu\in M_\gamma$.

We need to show that the φ and ψ above are well-defined.

As every $\mu\in M_{\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}}$ is a probability measure, it follows that $\varphi(\mu)(\mathbb{T}\times\mathbb{T}\times\mathbb{T}) = 1$.

For every Borel subset $D\subset \mathbb{T}\times\mathbb{T}\times\mathbb{T}$, as both $\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h}$ and γ are homeomorphisms, it follows that

$$(H\times \mathrm{id}_{\mathbb{T}}\times \mathrm{id}_{\mathbb{T}})^{-1}(\gamma(D)) = (\alpha\times \mathrm{R}_{\xi\circ h}\times \mathrm{R}_{\eta\circ h})\left((H\times \mathrm{id}_{\mathbb{T}}\times \mathrm{id}_{\mathbb{T}})^{-1}(D)\right),$$

which implies that $\varphi(\mu)$ is γ-invariant.

For a sequence of Borel subsets $D_1, D_2, \dots$ of $\mathbb{T} \times \mathbb{T} \times \mathbb{T}$ such that $D_i \cap D_j = \varnothing$ if $i \neq j$, it is clear that $(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}(D_1), (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}(D_2), \dots$ are Borel subsets of $X \times \mathbb{T} \times \mathbb{T}$ (as $H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}}$ is continuous) satisfying $(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}(D_i) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}(D_j) = \varnothing$ if $i \neq j$. Then we have that

$$\varphi(\mu)\left(\bigsqcup_{n=1}^{\infty} D_n\right) = \sum_{n=1}^{\infty} \varphi(\mu)(D_n).$$

So far, we have shown that φ is a well-defined map from $M_{\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}}$ to M_γ.

Now we will check the map ψ.

As every $\nu \in M_\gamma$ is a probability measure, it follows that

$$\psi(\nu)(X \times \mathbb{T} \times \mathbb{T}) = \nu(\mathbb{T} \times \mathbb{T} \times \mathbb{T}) = 1.$$

For every Borel subset $E \subset X \times \mathbb{T} \times \mathbb{T}$, we will show that $\psi(\nu)(E)$ is well-defined. According to the definition of $\psi(\nu)$, we just need to show that $(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(E)$ is ν-measurable.

For any two open subsets S_1 and S_2 of $X \times \mathbb{T} \times \mathbb{T}$, we have

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1 \cup S_2) = (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1) \cup (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_2),$$

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_i^c) = ((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_i))^c \quad \text{for } i = 1, 2.$$

As H is not one-to-one, we cannot get

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1 \cap S_2) = (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_2),$$

but we still have

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1 \cap S_2) \subset (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_2).$$

We will consider $((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_2)) \setminus (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1 \cap S_2)$.

Note that H is just the restriction of h to X, where h is a noninvertible continuous monotone map from $\mathbb{T}$ to $\mathbb{T}$ (see [KatokHasselblatt, Theorem 11.2.7]). It follows that $H : X \to \mathbb{T}$ is one-to-one except at countablely many points of X. Use X_0 to denote this subset consists of countably many points. Then we have that

$$((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_2)) \setminus (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1 \cap S_2) \subset H(X_0) \times \mathbb{T} \times \mathbb{T}.$$

As $\nu(\mathbb{T} \times \mathbb{T} \times \mathbb{T}) = 1$ and the minimal action γ has the skew product structure, it follows that for every $t \in \mathbb{T}$, $\nu(\{t\} \times \mathbb{T} \times \mathbb{T}) = 0$, which then implies that $\nu(H(X_0) \times \mathbb{T} \times \mathbb{T}) = 0$. Then we get that

$$((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_2)) \setminus (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(S_1 \cap S_2)$$

is of measure zero for all γ-invariant measure ν.

For two sets A and B, we use $A \triangle B$ to denote $(A \cap B^c) \cup (A^c \cap B)$.

For every Borel subset F of $X \times \mathbb{T} \times \mathbb{T}$, as F is generated by open sets via taking complements, countably many unions and intersections, it follows that there exists a Borel set F', such that

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(F) \triangle F'$$

is of measure zero for all γ-invariant measure ν. Note that F' is a Borel set. For every γ-invariant measure ν, F' is both ν-measurable. It then follows that $(H \times \mathrm{id}_{\mathbb{T} \times \mathbb{T}})(F)$ is measurable. Recall that

$$\psi(\nu)(F) = \nu\left((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(F)\right).$$

It follows that for $\psi(v)$ is well-defined on all the Borel subsets of $X \times \mathbb{T} \times \mathbb{T}$.

For a sequence of Borel subsets $E_1, E_2, \ldots$ of $X \times \mathbb{T} \times \mathbb{T}$ such that $D_i \cap D_j = \varnothing$ if $i \neq j$, and for every γ-invariant probability measure ν, we will show that

$$\psi(\nu)\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \psi(\nu)(E_n).$$

According to the definition, we have

$$\psi(\nu)\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \nu\left((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})\left(\bigsqcup_{n=1}^{\infty} E_n\right)\right)$$

Note that

$$\left((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})\left(\bigsqcup_{n=1}^{\infty} E_n\right)\right) = \left(\bigcup_{i=1}^{\infty}(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(E_n)\right)$$

and

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(E_i) \cap (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(E_j) \subset H(X_0) \times \mathbb{T} \times \mathbb{T} \text{ for } i \neq j.$$

Recall that $H(X_0) \times \mathbb{T} \times \mathbb{T}$ is a set of measure zero for every γ-invariant probability measure. It follows that

$$\psi(\nu)\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \psi(\nu)(E_n).$$

For every Borel subset $E \subset X \times \mathbb{T} \times \mathbb{T}$, according to the commutative diagram (III.1), we have

$$(\gamma \circ (H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}}))E = ((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}}) \circ (\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}))\,(E).$$

It then follows that

$$\begin{aligned}
\psi(\nu)(E) &= \nu((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})E) \\
&= \nu(\gamma\,((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})E)) \\
&= \nu\,((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})((\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h})E)) \\
&= \psi(\nu)\,((\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h})E)\,,
\end{aligned}$$

which implies that $\psi(\nu)$ is $\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}$-invariant.

So far, we have shown that ψ is a well-defined map from M_γ to $M_{\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}}$.

Now we will show that for every $\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}$-invariant measure μ and γ-invariant measure ν, we have

$$(\varphi \circ \psi)(\nu) = \nu \text{ and } (\psi \circ \varphi)(\mu) = \mu.$$

In fact, we just need to show that for every Borel subset D of $\mathbb{T} \times \mathbb{T} \times \mathbb{T}$ and every Borel subset E of $X \times \mathbb{T} \times \mathbb{T}$,

$$\nu\left((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}(D)) \triangle D\right) = 0 \tag{III.4}$$

and

$$\mu\left((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(E)) \triangle E\right) = 0. \tag{III.5}$$

As

$$(H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}(D)) = D,$$

the equation (III.4) holds.

Note that

$$\left((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})^{-1}((H \times \mathrm{id}_{\mathbb{T}} \times \mathrm{id}_{\mathbb{T}})(E)) \triangle E\right) \subset X_0 \times \mathbb{T} \times \mathbb{T}.$$

The fact that X_0 consists of countably many points and the minimal action $\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}$ has skew product structure implies that

$$\mu(X_0 \times \mathbb{T} \times \mathbb{T}) = 0.$$

It then follows that the equation (III.5) holds, which finishes the proof.

□

By Proposition III.3.5 above, there is a one-to-one correspondence between the $\alpha \times \mathrm{R}_{\xi \circ h} \times \mathrm{R}_{\eta \circ h}$-invariant probability measures and the γ-invariant probability measures (because if two measures coincide on all the Borel sets, they must be the same measure).

It follows that a minimal Furstenberg transformation on $\mathbb{T}^3$ that is uniquely ergodic will yield an example of a rigid minimal action on $X \times \mathbb{T} \times \mathbb{T}$, and a minimal transformation on $\mathbb{T}^3$ that is not uniquely ergodic will yield an example of a non-rigid minimal action on $X \times \mathbb{T} \times \mathbb{T}$.

Example III.3.6. *This is an example of rigid minimal dynamical system* $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$.

Let (X, α) be a Denjoy homeomorphism with rotation number $\theta_1 \in \mathbb{R} \setminus \mathbb{Q}$.

Choose θ_2, θ_3 such that $1, \theta_1, \theta_2, \theta_3 \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$. That is, if $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q}$ and satisfy

$$\lambda_0 + \lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3 = 0,$$

then $\lambda_i = 0$ for $i = 0, \dots, 3$.

The dynamical system $(\mathbb{T} \times \mathbb{T} \times \mathbb{T}, \mathrm{R}_{\theta_1} \times \mathrm{R}_{\theta_2} \times \mathrm{R}_{\theta_3})$ is minimal and uniquely ergodic.

Define $\varphi \colon X \to \mathrm{Homeo}(\mathbb{T}^2)$ by

$$\varphi(x)(z_1, z_2) = (z_1 e^{2\pi i\theta_2}, z_2 e^{2\pi i \theta_3}).$$

As $(\mathbb{T} \times \mathbb{T} \times \mathbb{T}, \mathrm{R}_{\theta_1} \times \mathrm{R}_{\theta_2} \times \mathrm{R}_{\theta_3})$ is uniquely ergodic, so is $(X \times \mathbb{T}^2, \alpha \times \varphi)$. This gives an example of a rigid minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$.

Example III.3.7. *We will give an example of minimal dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ such that it is not rigid.*

According to [Furstenberg] (see page 585), there exists a minimal a Furstenberg transformation

$$\gamma_0 \colon \mathbb{T}^2 \longrightarrow \mathbb{T}^2$$

such that

$$\gamma_0(z_1, z_2) = (z_1 e^{2\pi i \theta}, f(z_1) z_2) \text{ for some } \theta \in \mathbb{R} \setminus \mathbb{Q} \text{ and contractible } f \in C(\mathbb{T}, \mathbb{T}),$$

and γ_0 is not uniquely ergodic.

Let $(\mathbb{T}, \varphi)$ be a Denjoy homeomorphism with rotation number θ. Let (X, α) be the minimal Cantor dynamical system derived from $(\mathbb{T}, \varphi)$ which factors through $(\mathbb{T}, \mathrm{R}_\theta)$. In other words,

$\alpha = \varphi|_X$ and we have the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{T} \arrow[r, "\mathrm{R}_\theta"'] & \mathbb{T} \end{tikzcd} \quad , \tag{III.6}$$

with $\pi\colon X \to \mathbb{T}$ being a surjective map.

Define $\xi\colon X \to \mathrm{Homeo}(\mathbb{T})$ by $\xi(x)(z) = f(\pi(x))z$. We can then check that the following diagram commutes:

$$\begin{tikzcd} X \times \mathbb{T} \arrow[r, "\alpha\times \mathrm{R}_\xi"] \arrow[d, "\pi\times \mathrm{id}_{\mathbb{T}}"'] & X \times \mathbb{T} \arrow[d, "\pi\times \mathrm{id}_{\mathbb{T}}"] \\ \mathbb{T}^2 \arrow[r, "\gamma_0"'] & \mathbb{T}^2 \end{tikzcd} \quad .$$

As π is surjective, so is $\pi \times \mathrm{id}_{\mathbb{T}}$. Minimality of γ_0 then implies minimality of $\alpha \times \mathrm{R}_\xi$. As γ_0 is not uniquely ergodic, similarly to the proof of Proposition III.3.5, it follows that $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$ is not uniquely ergodic.

In the commutative diagram (III.6), note that π is onto, and $(\mathbb{T}, \mathrm{R}_\theta)$ is uniquely ergodic. It follows that (X, α) is also uniquely ergodic.

As $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$ is not uniquely ergodic, there exist more than one $(\alpha \times \mathrm{R}_\xi)$-invariant probability measure. Let μ and ν to be two such measures on $X \times \mathbb{T}$ that are different from each other.

According to Lemma III.3.3, there exists $\theta \in \mathbb{R}$ such that if we use R_θ to denote the function in $C(X, \mathrm{Homeo}(\mathbb{T}))$ defined by

$$\mathrm{R}_\theta(x)(z) = ze^{2\pi i\theta} \text{ for all } x \in X \text{ and } z \in \mathbb{T},$$

then the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is still minimal.

Use m to denote the Lebesgue measure on $\mathbb{T}$. For the $(\alpha \times \mathrm{R}_\xi)$-invariant probability measures μ and ν, as R_θ is a rotation of the circle, we can check that both $\mu \times m$ and $\nu \times m$ are $(\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$-invariant probability measures on $X \times \mathbb{T} \times \mathbb{T}$.

As μ and ν are different measures, it is clear that $\mu \times m$ is different from $\nu \times m$.

Now we have at least two $(\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$-invariant measures. Note that (X, α) is uniquely ergodic. We have that the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is not uniquely ergodic.

Remark: For this example, the corresponding crossed product C*-algebra has tracial rank one and the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is not rigid. The reason is as follows.

Consider the dynamical system $(X \times \mathbb{T}_1, \alpha \times \mathrm{R}_\xi)$. It is not uniquely ergodic. As (X, α) is uniquely ergodic, it follows that $(X \times \mathbb{T}_1, \alpha \times \mathrm{R}_\xi)$ is not rigid.

Use A to denote the crossed product C*-algebra $C^*(\mathbb{Z}, X \times \mathbb{T}_1, \alpha \times \mathrm{R}_\xi)$. According to Theorem 4.3 of [LM2], the algebra A has tracial rank one. By Proposition 1.10 (1) of [Ph2], $\rho_A(K_0(A))$ is not dense in $\mathrm{Aff}(T(A))$.

Note that A is an A$\mathbb{T}$-algebra. According to Theorem 2.1 of [EGL], A is approximately divisible. By Theorem 1.4 (e) of [BKR], and noting that real rank of A is not zero (as tracial rank of A is one and A is A$\mathbb{T}$-algebra), we have that the projections in A does not separate traces of A. In other words, there exist two $(\alpha \times \mathrm{R}_\xi)$-invariant measures μ and ν such that

$$\mu \neq \nu, \text{ and } \mu(x) = \nu(x) \text{ for all } x \in K_0(A).$$

Define measures μ_X, ν_X by

$$\mu_X(D) = \mu(D \times \mathbb{T}) \quad \text{and} \quad \nu_X(D) = \nu(D \times \mathbb{T})$$

for all Borel sets $D \subset X$. It is clear that both μ_X and ν_X are α-invariant probability measures on X.

Note that $C(X, \mathbb{Z})$ is generated by the projections in $C(X)$. Also note that the $\mathbb{C}$-linear span of $C(X, \mathbb{Z})$ is dense in $C(X, \mathbb{R})$. The fact that the projections in A do not separate μ and ν implies that $C(X, \mathbb{Z})$ do not separate μ_X and ν_X, which then implies that $\mu_X = \nu_X$.

Use B to denote $C^*(\mathbb{Z}, X \times \mathbb{T}_1 \times \mathbb{T}_2, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$. Let m be the Lebesgue measure on $\mathbb{T}$. It is clear that $\mu \times m$ and $\nu \times m$ are two $(\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$-invariant probability measures.

We will show that the projections in B do not separate $\mu \times m$ and $\nu \times m$.

From Proposition III.2.1,

$$K_0(B) \cong C(X, \mathbb{Z}^2)/\{(f,g) - (f,g) \circ \alpha^{-1} \colon f, g \in C(X, \mathbb{Z})\} \oplus \mathbb{Z} \oplus \mathbb{Z}. \tag{III.7}$$

The two copies of $\mathbb{Z}$ correspond to the two generalized Rieffel projections e_1 and e_2, given by $e_1 = g_1 u^* + f_1 + u g_1$, and $e_2 = g_2 u^* + f_2 + u g_2$, where e_i, f_i, g_i are defined similarly to the functions defined in Section 6 of [LM1], $f_1(x, z_1, z_2) = f_1(x, z_1, z_2')$ and $f_2(x, z_1, z_2) = f_1(x, z_1', z_2)$ for all $z_1, z_1' \in \mathbb{T}_1, z_2, z_2' \in \mathbb{T}_2$.

As the projections in A do not distinguish μ and ν, it follows that the elements in $K_0(B)$ that correspond to the first two summands of III.7 do not separate $\mu \times m$ and $\nu \times m$.

For the generalized Rieffel projection e_2, as $f_2(x, z_1, z_2)$ is independent of z_1, we have $f(x, z_1, z_2) = F_2(x, z_2)$ for some $F \in C(X \times \mathbb{T}_2, \mathbb{R})$.

Recall that for a measure σ on X and $f \in C(X)$, we use $\sigma(f)$ to denote $\int_X f(x)\, \mathrm{d}\mu$ (see Section I.2). We check that

$$\begin{aligned}
(\mu \times m)(e_2) &= (\mu \times m)(f_2) \\
&= \int_{(X \times \mathbb{T}_1) \times \mathbb{T}_2} f_2(x, z_1, z_2)\ \mathrm{d}(\mu \times m) \\
&= \int_{X \times \mathbb{T}_2} F_2(x, z_2)\ \mathrm{d}(\mu_X \times m) \\
&= \int_{X \times \mathbb{T}_2} F_2(x, z_2)\ \mathrm{d}(\nu_X \times m) \\
&= \int_{(X \times \mathbb{T}_1) \times \mathbb{T}_2} f_2(x, z_1, z_2)\ \mathrm{d}(\nu \times m) \\
&= (\nu \times m)(f_2) \\
&= (\nu \times m)(e_2).
\end{aligned}$$

Then we have shown that e_2 does not separate $\mu \times m$ and $\nu \times m$ either, which then implies that the projections in B cannot separate traces of B.

According to Theorem 1.4 of [BKR], the real rank of B is not zero. Then it follows that the tracial rank of B is not zero.

By Theorem III.1.17, the tracial rank of B must be one.

According to Proposition III.2.3, the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\theta)$ is not rigid.

CHAPTER IV

APPROXIMATE K-CONJUGACY

In this chapter, we start with a sufficient condition for approximate K-conjugacy between two minimal dynamical systems $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$. Then we give an if and only if condition for weak approximate conjugacy of these two dynamical systems, showing that weak approximate conjugacy just depends on α and β. In Section IV.3, an if and only if condition for approximate K-conjugacy between these two dynamical systems is given.

In [LM3], several notions of approximate conjugacy between dynamical systems are introduced. In [LM1], it is shown that for rigid minimal systems on $X \times \mathbb{T}$ (with X being the Cantor set and $\mathbb{T}$ being the circle; see Definition 3.1 of [LM1]), the corresponding crossed product C*-algebras are isomorphic if and only if the dynamical systems are approximately K-conjugate.

For two minimal rigid dynamical systems $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi} \times \mathrm{R}_{\eta})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$, we study the relationship between approximate K-conjugacy and the isomorphism of crossed product C*-algebras.

We start with basic definitions and facts about conjugacy and approximate conjugacy.

Definition IV.0.1. *Let X, Y be two compact metric spaces, and let $\alpha \in \mathrm{Homeo}(X)$ and $\beta \in \mathrm{Homeo}(Y)$ be two minimal actions. We say that (X, α) and (Y, β) are conjugate if there exists $\sigma \in \mathrm{Homeo}(X, Y)$ such that $\sigma \circ \alpha = \beta \circ \sigma$. We say that (X, α) and (Y, β) are flip conjugate if (X, α) is conjugate to (Y, β) or (Y, β^{-1}).*

Definition IV.0.2. *Let X, Y be two compact metric spaces, and let $\alpha \in \mathrm{Homeo}(X)$ and $\beta \in$*

Homeo(Y) *be two minimal actions. We say that* (X,α) *and* (Y,β) *are weakly approximately conjugate if there exist* $\sigma_n \in \mathrm{Homeo}(X,Y)$ *and* $\gamma_n \in \mathrm{Homeo}(Y,X)$ *for* $n \in \mathbb{N}$ *such that*

$$\mathrm{dist}(f \circ \sigma_n \circ \alpha, f \circ \beta \circ \sigma_n) \to 0 \quad \text{and} \quad \mathrm{dist}(g \circ \alpha \circ \gamma_n, g \circ \gamma_n \circ \beta) \to 0 \quad \text{as } n \to \infty$$

for all $f \in C(X)$ *and* $g \in C(Y)$*, where* $\mathrm{dist}(f_1, f_2)$ *is defined to be* $\sup_{x \in D} \mathrm{dist}(f_1(x), f_2(x))$ *for all continuous functions* f_1, f_2 *on the metric space* D.

It is clear that if two minimal dynamical systems are conjugate, they are weakly approximately conjugate. Generally speaking, the inverse implication does not hold.

IV.1 C^*-STRONG APPROXIMATE CONJUGACY

Given minimal dynamical systems (X,α) and (Y,β), if they are flip conjugate, then it is easy to check that the corresponding crossed product C*-algebras $C^*(\mathbb{Z}, X, \alpha)$ and $C^*(\mathbb{Z}, Y, \beta)$ are isomorphic.

According to [Tomiyama] (Corollary of Theorem 2), for two minimal dynamical systems (X,α) and (Y,β), there exists an isomorphism

$$\varphi\colon C^*(\mathbb{Z}, X, \alpha) \longrightarrow C^*(\mathbb{Z}, Y, \beta)$$

satisfying $\varphi(C(X)) = C(Y)$ if and only if these two dynamical systems are flip conjugate.

In view of Tomiyama's result above, C^*-strong approximate flip conjugacy is defined as below.

Definition IV.1.1. *Let* (X,α) *and* (X,β) *be two minimal dynamical systems such that* $\mathrm{TR}(C^*(\mathbb{Z}, X, \alpha)) = \mathrm{TR}(C^*(\mathbb{Z}, X, \beta)) = 0$*, we say that* (X,α) *and* (X,β) *are* C^**-strongly approximately flip conjugate if there exists a sequence of isomorphisms*

$$\varphi_n\colon C^*(\mathbb{Z}, X, \alpha) \to C^*(\mathbb{Z}, X, \beta),\ \psi_n\colon C^*(\mathbb{Z}, X, \beta) \to C^*(\mathbb{Z}, X, \alpha)$$

and a sequence of isomorphisms $\chi_n, \lambda_n \colon C(X) \to C(X)$ *such that*

1) $[\varphi_n] = [\varphi_m] = [\psi_n^{-1}]$ *in* $KL(C^*(\mathbb{Z}, X, \alpha), C^*(\mathbb{Z}, X, \alpha))$ *for all* $m, n \in \mathbb{N}$,

2) $\lim_{n\to\infty} \|\varphi_n \circ j_\alpha(f) - j_\beta \circ \chi_n(f)\| = 0$ *and* $\lim_{n\to\infty} \|\psi_n \circ j_\beta(f) - j_\alpha \circ \lambda_n(f)\| = 0$ *for all* $f \in C(X)$, *with* j_α, j_β *being the injections from* $C(X)$ *into* $C^*(\mathbb{Z}, X, \alpha)$ *and* $C^*(\mathbb{Z}, X, \beta)$.

Some notation will be introduced before the next result about C^*-strong approximate conjugacy.

Let A be a separable amenable C*-algebra that satisfies UCT. For $\theta \in KL(A, B)$, there are induced homomorphisms $\Gamma(\theta)_i \colon K_i(A) \to K_i(B)$ for $i = 0, 1$. Define $\rho_A \colon A_{sa} \longrightarrow \mathrm{Aff}(T(A))$ by $\rho_A(a)(\tau) = \tau(a)$ for all $a \in A_{sa}$ and $\tau \in T(A)$. Suppose A and B are two unital simple C*-algebras with tracial rank zero and $\gamma \colon K_0(A) \to K_0(B)$ is an order preserving homomorphism. As A has real rank zero, γ will induce a positive homomorphism $\gamma_\rho \colon \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$.

The theorem below ([Lin4, Theorem 2.5]) gives one necessary condition for C^*-strong approximate flip conjugacy between two crossed product C*-algebras.

Theorem IV.1.2. *Let* (X, α) *and* (X, β) *be two minimal dynamical systems such that the corresponding crossed product C*-algebras* A_α *and* A_β *both have tracial rank zero. Then* α *and* β *are* C^**-strongly approximately flip conjugate if the following holds: There is an isomorphism* $\chi \colon C(X) \to C(X)$ *and there is* $\theta \in KL(A_\alpha, A_\beta)$ *such that* $\Gamma(\theta)$ *gives an isomorphism*

$$\Gamma(\theta) \colon (K_0(A_\alpha), K_0(A_\alpha)_+, [1], K_1(A_\alpha)) \to (K_0(A_\beta), K_0(A_\beta)_+, [1], K_1(A_\beta)),$$

and such that

$$[j_\alpha] \times \theta = [j_\beta \circ \chi] \text{ in } KL(C(X), A_\beta)$$

and

$$\rho_{A_\beta} \circ j_\beta \circ \chi(f) = ((\Gamma(\theta)_0)_\rho) \circ \rho_{A_\alpha} \circ j_\alpha(f)$$

for all $f \in C(X)_{sa}$.

If $K_i(C(X))$ is torsion free, then a simplified version of this result holds ([Lin4, Corollary

2.6]).

Corollary IV.1.3. *Let X be a compact metric space with torsion free K-theory. Let (X,α) and (X,β) be two minimal dynamical systems such that $\mathrm{TR}(A_\alpha) = \mathrm{TR}(A_\beta) = 0$. Suppose that there is an order isomorphism that maps $[1_{A_\alpha}]$ to $[1_{A_\beta}]$:*

$$\gamma\colon (K_0(A_\alpha), K_0(A_\alpha)_+, [1_{A_\alpha}], K_1(A_\alpha)) \to (K_0(A_\beta), K_0(A_\beta)_+, [1_{A_\beta}], K_1(A_\beta)),$$

such that there exists an isomorphism $\chi\colon C(X) \to C(X)$ satisfying

$$\gamma \circ (j_\alpha)_{*i} = (j_\beta \circ \chi)_{*i} \text{ for } i = 0,1 \text{ and } \gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi \text{ on } C(X)_{sa}.$$

Then (X,α) and (X,β) are C^*-strongly approximately flip conjugate.

In the rest of this chapter, for a minimal homeomorphism α on the Cantor set X, we will use $K^0(X,\alpha)$ to denote the ordered group

$$C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1}\colon f \in C(X,\mathbb{Z}^2)\}$$

with the positive cone being (denoted by $K^0(X,\alpha)_+$)

$$C(X,D)/\{f - f\circ\alpha^{-1}\colon f \in C(X,\mathbb{Z}^2)\}$$

where D is as defined in Lemma II.2.9. In $K^0(X,\alpha)$, we define the unit element to be

$$[(1,0)_{C(X,\mathbb{Z}^2)}] \in C(X,\mathbb{Z}^2)/\{f - f\circ\alpha^{-1}\colon f \in C(X,\mathbb{Z}^2)\},$$

with $(1,0)_{C(X,\mathbb{Z}^2)}$ being the constant function in $C(X,\mathbb{Z}^2)$ that maps every $x \in X$ to $(1,0) \in \mathbb{Z}^2$. We use $1_{K^0(X,\alpha)}$ to denote this unit element.

Lemma IV.1.4. *Let X be the Cantor set. For every minimal action $\alpha \in Homeo(X)$, if there is* *an order isomorphism*

$$\varphi\colon (K^0(X,\alpha), K^0(X,\alpha)_+, 1_{K^0(X,\alpha)}) \longrightarrow (K^0(X,\beta), K^0(X,\beta)_+, 1_{K^0(X,\beta)}),$$

then there is an order isomorphism

$$\widetilde{\varphi}\colon (C(X,\mathbb{Z}^2), C(X,D), (1,0)_{C(X,\mathbb{Z}^2)}) \longrightarrow (C(X,\mathbb{Z}^2), C(X,D), (1,0)_{C(X,\mathbb{Z}^2)})$$

such that the following diagram commutes:

$$\begin{array}{ccc} (C(X,\mathbb{Z}^2), C(X,D)) & \xrightarrow{\quad\widetilde{\varphi}\quad} & (C(X,\mathbb{Z}^2), C(X,D)) \\ \pi_\alpha \downarrow & & \downarrow \pi_\beta \\ (K^0(X,\alpha), K^0(X,\alpha)_+) & \xrightarrow{\quad\varphi\quad} & (K^0(X,\beta), K^0(X,\beta)_+) \end{array} \quad , \tag{IV.1}$$

where π_α, π_β are the canonical projections from $C(X,\mathbb{Z}^2)$ to $K^0(X,\alpha)$ and $K^0(X,\beta)$. In fact, there exists $\sigma \in Homeo(X)$ such that $\varphi(F) = F \circ \sigma^{-1}$ for all $F \in C(X,\mathbb{Z}^2)$.

Proof. The proof is based on [LM3, Theorem 2.6].

Define $\overline{K^0(X,\alpha)}$ to be

$$C(X,\mathbb{Z})/\{g - g\circ\alpha^{-1} : g \in C(X,\mathbb{Z})\}$$

and $\overline{K^0(X,\alpha)}_+$ to be

$$C(X,\mathbb{Z}^+ \cup \{0\})/\{g - g\circ\alpha^{-1} : g \in C(X,\mathbb{Z})\}.$$

We can check that $(\overline{K^0(X,\alpha)}, \overline{K^0(X,\alpha)}_+)$ gives an ordered group with order unit.

Define

$$h\colon K^0(X,\alpha) \to \overline{K^0(X,\alpha)} \text{ by } h([f]) = [f_1]$$

for every $f = (f_1, f_2) \in C(X,\mathbb{Z}^2)$, with $f_1, f_2 \in C(X,\mathbb{Z})$.

From the definition, we can check that h is surjective and $h(K^0(X,\alpha)_+) = \overline{K^0(X,\alpha)}_+$.

For the isomorphism

$$\varphi\colon (K^0(X,\alpha), K^0(X,\alpha)_+) \to (K^0(X,\beta), K^0(X,\beta)_+),$$

define

$$\varphi_0\colon \overline{K^0(X,\alpha)} \to \overline{K^0(X,\beta)} \text{ by } \varphi_0([f]) = h(\varphi([(f,0)]))$$

for all $f \in C(X,\mathbb{Z})$.

Suppose that there exist $f_1, f_2, g \in C(X,\mathbb{Z})$ such that $f_1 - f_2 = g - g \circ \alpha^{-1}$. Then it follows that $(f_1, 0) - (f_2, 0) = (g, 0) - (g, 0) \circ \alpha^{-1}$, which implies that $\varphi([(f_1, 0)]) = \varphi([(f_1, 0)])$. It is now clear that φ_0 is well-defined.

Note that $\varphi_0([1_{C(X,\mathbb{Z})}]) = h(\varphi([(1,0)_{C(X,\mathbb{Z}^2)}]))$. As φ is unital, $\varphi(1_{K^0(X,\alpha)}) = 1_{K^0(X,\beta)}$, which then implies that $\varphi_0([1_{C(X,\mathbb{Z})}]) = h([(1,0)_{C(X,\mathbb{Z}^2)}]) = [1_{C(X,\mathbb{Z})}]$. We can now claim that φ_0 is unital.

For any $f \in C(X, \mathbb{Z}^+ \cup \{0\})$, $\varphi_0([f]) = h(\varphi([(f,0)]))$. As both φ and h are order preserving, φ_0 is also order preserving.

So far, we have that $\varphi_0\colon \overline{K^0(X,\alpha)} \to \overline{K^0(X,\beta)}$ is untial and order preserving. According to [LM3, Theorem 2.6], there exists a continuous order preserving map

$$\widetilde{\varphi_0}\colon (C(X,\mathbb{Z}), C(X,\mathbb{Z})_+, 1_{C(X,\mathbb{Z})}) \to (C(X,\mathbb{Z}),, C(X,\mathbb{Z})_+, 1_{C(X,\mathbb{Z})}),$$

such that the following diagram commutes:

$$\begin{tikzcd}
(C(X,\mathbb{Z}), C(X,Z)_+) \arrow[r, "\widetilde{\varphi_0}"] \arrow[d, "\pi'_\alpha"'] & (C(X,\mathbb{Z}), C(X,Z)_+) \arrow[d, "\pi'_\beta"] \\
(\overline{K^0(X,\alpha)}, \overline{K^0(X,\alpha)_+}) \arrow[r, "\varphi_0"] & (\overline{K^0(X,\beta)}, \overline{K^0(X,\beta)_+})
\end{tikzcd} \quad . \tag{IV.2}$$

Now we need to construct the unital positive linear map

$$\widetilde{\varphi}\colon (C(X,\mathbb{Z}^2), C(X,D)) \to (C(X,\mathbb{Z}^2), C(X,D)),$$

such that diagram (IV.1) commutes.

For the $\widetilde{\varphi_0}$ we get, note that $\widetilde{\varphi_0}$ is a unital positive isomorphism from $K_0(C(X))$ to $K_0(C(X))$. As $C(X)$ is a unital AF-algebra, by the existence theorem of classification of unital AF-algebras, there exists an isomorphism $\psi\colon C(X) \to C(X)$ such that (identifying $K_0(C(X))$ with $C(X,\mathbb{Z})$ and $K_0(C(X))_+$ with $C(X,\mathbb{Z})_+$)

$$\psi_{*0}\colon (C(X,\mathbb{Z}), C(X,\mathbb{Z})_+, [1]) \to (C(X,\mathbb{Z}), C(X,\mathbb{Z})_+, [1])$$

coincides with $\widetilde{\varphi_0}$.

As ψ is an isomorphism, there exists $\sigma\colon X \to X$ such that $\psi(f) = f \circ \sigma^{-1}$ for all $f \in C(X)$.

Define $\widetilde{\varphi}\colon C(X,\mathbb{Z}^2) \to C(X,\mathbb{Z}^2)$ by $\widetilde{\varphi}((f,g)) = (\psi(f), \psi(g))$ for all $f, g \in C(X,\mathbb{Z})$. In other words, $\widetilde{\varphi}((f,g)) = (f,g) \circ \sigma^{-1}$ for all $(f,g) \in C(X,\mathbb{Z}^2)$.

For the $\widetilde{\varphi}$ above-defined, it is easy to check that it is unital and linear. It remains to show that $\widetilde{\varphi}$ maps positive cone to positive cone, and makes the diagram commute.

For every $(f,g) \in C(X,D)$, we get $\widetilde{\varphi}((f,g)) = (f,g) \circ \sigma^{-1}$. As $(f,g) \in C(X,D)$, it is clear that $(f,g) \circ \sigma^{-1} \in C(X,D)$. So far, we proved that $\widetilde{\varphi}$ is a positive map.

We can check that

$$\begin{aligned}
\pi_\beta \circ \widetilde{\varphi}((f,g)) &= \pi_\beta(h(f), h(g)) \\
&= \pi_\beta(\widetilde{\varphi_0}(f), \widetilde{\varphi_0}(g)) \\
&= \pi_\beta(\widetilde{\varphi_0}(f), 0) + \pi_\beta(0, \widetilde{\varphi_0}(g)) \\
&= (\pi'_\beta \circ \widetilde{\varphi_0}(f), 0) + (0, \pi'_\beta \circ \widetilde{\varphi_0}(g)) \\
&= (\varphi_0 \circ \pi'_\alpha(f), 0) + (0, \varphi_0 \circ \pi'_\alpha(g)) \\
&= \varphi \circ \pi_\alpha((f,0)) + \varphi \circ \pi_\alpha((0,g)) \\
&= \varphi \circ \pi_\alpha((f,g)),
\end{aligned}$$

which implies the commutativity of diagram (IV.1).

As $\widetilde{\varphi}((f,g)) = (f,g) \circ \sigma^{-1}$ for all $f, g \in C(X,\mathbb{Z})$, we get that $\widetilde{\varphi}$ is an isomorphism, which finishes the proof.

□

Theorem IV.1.5. *Let $(X\times\mathbb{T}\times\mathbb{T},\alpha\times\mathrm{R}_{\xi_1}\times\mathrm{R}_{\eta_1})$ and $(X\times\mathbb{T}\times\mathbb{T},\beta\times\mathrm{R}_{\xi_2}\times\mathrm{R}_{\eta_2})$ be two minimal rigid Cantor dynamical systems. Use A, B to denote the two corresponding crossed product C^*-algebras. According to Proposition III.2.1, $K^0(X,\alpha)$ is a direct summand of $K_0(A)$ and $K^0(X,\beta)$ is a direct summand of $K_0(B)$. Let*

$$j_A : K^0(X,\alpha) \to K_0(A) \cong K^0(X,\alpha)\oplus\mathbb{Z}^2 \quad \textit{and} \quad j_B : K^0(X,\beta) \to K_0(B) \cong K^0(X,\alpha)\oplus\mathbb{Z}^2$$

be defined by

$$j_A(x) = (x,0) \text{ and } j_B(x) = (x,0).$$

If there is an order preserving isomorphism ρ from $K_0(A)$ to $K_0(B)$ that maps $K^0(X,\alpha)$ onto $K^0(X,\beta)$, then these two dynamical systems are C^*-strongly approximately conjugate.

Proof. We have the following commutative diagram:

$$\begin{tikzcd} K_0(A) \arrow[r, "\rho"] & K_0(B) \\ K^0(X,\alpha) \arrow[u, "j_A"] \arrow[r, "\rho|_{K^0(X,\alpha)}"] & K^0(X,\beta) \arrow[u, "j_B"'] \end{tikzcd} .$$

According to Lemma IV.1.4, we can lift

$$\rho\,\big|_{K^0(X,\alpha)} : K^0(X,\alpha) \longrightarrow K^0(X,\beta)$$

to

$$\widetilde{\rho}\colon C(X,\mathbb{Z}^2) \longrightarrow C(X,\mathbb{Z}^2),$$

which will yield the commutative diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "\rho"] & K_0(B) \\
K^0(X,\alpha) \arrow[u, "j_A"] \arrow[r, "\rho|_{K^0(X,\alpha)}"] & K^0(X,\beta) \arrow[u, "j_B"'] \\
C(X,\mathbb{Z}^2) \arrow[u, "\pi_\alpha"] \arrow[r, "\widetilde{\rho}"] & C(X,\mathbb{Z}^2) \arrow[u, "\pi_\beta"'] .
\end{tikzcd}$$

In fact, according to Lemma IV.1.4, there exists $\sigma \in \mathrm{Homeo}(X)$ such that $\widetilde{\rho}(F) = F \circ \sigma^{-1}$. Define

$$\chi \colon C(X \times \mathbb{T}^2) \to C(X \times \mathbb{T}^2)$$

by $\chi(f) = f \circ (\sigma \times \mathrm{id}_{\mathbb{T}^2})$ for all $f \in C(X \times \mathbb{T}^2)$.

According to the Künneth Theorem, we get that $K_0(C(X \times \mathbb{T}^2)) \cong C(X, \mathbb{Z}^2)$. By Lemma II.2.1, if we identify $K_0(C(X \times \mathbb{T}^2))$ with $C(X, \mathbb{Z}^2)$, the positive cone will be identified with $C(X, D)$, with D as defined in Lemma II.2.1. Choose $x \in X$. According to Lemma II.2.9, we know that $K_0(A_x) \cong K^0(X, \alpha)$ and $K_0(B_x) \cong K^0(X, \beta)$, with A_x, B_x being the subalgebras of A and B, as in Definition I.2.1.

Now we have the commutative diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "\rho"] & K_0(B) \\
K_0(C(X \times \mathbb{T}^2)) \arrow[u, "(j_\alpha)_{*0}"] \arrow[r, "\widetilde{\rho}"] & K_0(C(X \times \mathbb{T}^2)) \arrow[u, "(j_\beta)_{*0}"'] .
\end{tikzcd}$$

Note that $\widetilde{\rho}$ is induced by the $\chi \colon C(X \times \mathbb{T}^2) \to C(X \times \mathbb{T}^2)$ defined above. We have shown that $\rho \circ (j_\alpha)_{*i} = (j_\beta \circ \chi)_{*i}$, i $= 0, 1$.

We will show that $\gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi$ on $C(X)_{sa}$.

For every tracial state $\tau \in T(C^*(\mathbb{Z}, X, \beta))$, we know that it corresponds to a β-invariant probability meausure μ_B (in such sense that $\tau(a) = \mu(E(a))$, with E being the conditional expectation from $C^*(\mathbb{Z}, X, \beta)$ to $C(X)$).

For every β-invariant probability measure μ_B on X, if we use v to denote standard Lebesgue measure on $\mathbb{T}$, it is then clear that $\mu_B \times v \times v$ is $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$-invariant. As the dynamical system $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$ is rigid, for every $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$-invariant probability measure, it must be $\mu \times v \times v$, with μ being an β-invariant probability measure and v being the Lebesgue probability measure.

Note that A denotes $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$ and B denotes $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$. According to Proposition III.2.1, the fact that $K_0(A)$ is isomorphic to $K_0(B)$ implies that $K_1(A)$ is also isomorphic to $K_1(B)$. According to Proposition III.2.3, the tracial rank of A and B are both zero, thus classifiable via the K-data.

Let $\varphi\colon A \to B$ be the C*-algebra isomorphism such that

$$\varphi_{*0}\colon K_0(A) \longrightarrow K_0(B)$$

coincides with the ρ in the statement. Define

$$\varphi^*\colon T(B) \longrightarrow T(A)$$

as $\varphi^*(\tau_B)(a) = \tau_B(\varphi(a))$ for all $a \in A$ and $\tau_B \in T(B)$.

Note that a C*-algebra with tracial rank zero must have real rank zero. We can now claim that for every $a \in C^*(\mathbb{Z}, X, \alpha)_{sa}$ and $\tau_B \in T(B)$ given by $\mu_B \times v \times v$,

$$(\gamma_\rho \circ j_\alpha(a))(\tau_B) = \varphi^*(\tau_B)(a).$$

Consider

$$a = f \otimes g \otimes h \in C(X \times \mathbb{T} \times \mathbb{T})_{sa} \subset A_{sa}$$

with $f \in C(X)_{sa}, g \in C(\mathbb{T})_{sa}$ and $h \in C(\mathbb{T})_{sa}$, and use τ_A to denote $\varphi^*(\tau_B)$. As $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ is rigid, there exists an α-invariant measure μ_A such that $\tau_A(a) = (\mu_A \times v \times v)(E(a))$, with E being the conditional expectation from A to $C(X \times \mathbb{T} \times \mathbb{T})$ and v being the Lebesgue measure on the circle. It follows that $(\gamma_\rho \circ j_\alpha(a))(\tau_B) = \tau_A(a) = \mu_A(f) \cdot v(g) \cdot v(h)$.

As for $((\rho_{A_\beta} \circ j_\beta \circ \chi)(a))(\tau_B)$, we know from the definition that

$$((\rho_{A_\beta} \circ j_\beta \circ \chi)(a))(\tau_B) = \tau_B(\chi(f \otimes g \otimes h)) = (\mu_B \times v \times v)(\chi(f \otimes g \otimes h)).$$

Recall the definition of χ. We have

$$(\mu_B \times v \times v)(\chi(f \otimes g \otimes h)) = \mu_B(f \circ \sigma^{-1}) \cdot v(g) \cdot v(h).$$

If we can show that $\mu_B(f \circ \sigma^{-1}) = \mu_A(f)$, then it follows that

$$(\mu_B \times v \times v)(\chi(f \otimes g \otimes h)) = \mu_A(f) \cdot v(g) \cdot v(h) = (\mu_A \times v \times v)(f \otimes g \otimes h),$$

and we can then get

$$\gamma_\rho \circ j_\alpha = \rho_{A_\beta} \circ j_\beta \circ \chi \text{ on } C(X \times \mathbb{T}^2)_{sa}.$$

We will show that for all $f \in C(X, \mathbb{Z})$ and μ_A, μ_B as given above, we have $\mu_B(f \circ \sigma^{-1}) = \mu_A(f)$. If that is done, noting that the $\mathbb{C}$-linear span of $C(X, \mathbb{Z})$ is dense in $C(X)_{sa}$, we get $\mu_B(f \circ \sigma^{-1}) = \mu_A(f)$ for all $f \in C(X)$.

According to our notation, for $g \in C(X)$, we have

$$\begin{aligned} \mu_A(g) &= (\mu_A \times v \times v)(g \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\ &= \tau_A(g \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\ &= \varphi^*(\tau_B)(g \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\ &= \tau_B(\varphi(g \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}})). \end{aligned}$$

According to digram (IV.2) in the proof of Lemma IV.1.4, we have the commutative diagram

$$\begin{tikzcd} K_0(C(X)) \arrow[r, "\widetilde{\varphi_0}"] \arrow[d, "\pi'_\alpha"'] & K_0(C(X)) \arrow[d, "\pi'_\beta"] \\ K_0(C^*(\mathbb{Z}, X, \alpha)) \arrow[r, "\varphi_0"] & K_0(C^*(\mathbb{Z}, X, \beta)) \end{tikzcd} \quad , \tag{IV.3}$$

where $C^*(\mathbb{Z}, X, \alpha)$ and $C^*(\mathbb{Z}, X, \beta)$ are the crossed product C*-algebras of dynamical systems

(X, α) and (X, β) respectively, $\widetilde{\varphi_0}$, φ_0 are order preserving isomorphisms, and $\widetilde{\varphi_0}$ agrees with χ as a map from $C(X, \mathbb{Z})$ to $C(X, \mathbb{Z})$.

By the proof of Lemma IV.1.4, for all $f \in C(X, \mathbb{Z})$, if we identify $C(X, \mathbb{Z})$ with $K_0(C(X))$, we get

$$\widetilde{\varphi_0}(f) = f \circ \sigma^{-1}.$$

From the commutative diagram (IV.3), we can conclude that (although we cannot claim that $\varphi(f \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) = \chi(f) \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}$)

$$\tau_B(\varphi(f \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}})) = \tau_B(\chi(f) \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}).$$

As $\chi(f) = f \circ \sigma^{-1}$, it follows that

$$\begin{aligned}
\mu_A(f) &= (\mu_A \times v \times v)(f \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\
&= \tau_A(f \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\
&= \varphi^*(\tau_B)(f \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\
&= \tau_B(\varphi(f \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}})) \\
&= \tau_B(\chi(f) \otimes \mathrm{id}_{\mathbb{T}} \otimes \mathrm{id}_{\mathbb{T}}) \\
&= \mu_B(\chi(f)) \\
&= \mu_B(f \circ \sigma^{-1}).
\end{aligned}$$

Now we have that $\mu_A(f) = \mu_B(f \circ \sigma^{-1})$ for all $f \in C(X, \mathbb{Z})$. Note that the $\mathbb{C}$-linear span of $C(X, \mathbb{Z})$ is dense in $C(X)$, we get

$$\mu_A(f) = \mu_B(f \circ \sigma^{-1}) \text{ for all f } \in C(X)_{sa}.$$

As both dynamical systems $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$ are rigid, by Proposition III.2.3, we have $\mathrm{TR}(A) = \mathrm{TR}(B) = 0$. According to Corollary IV.1.3, these two dynamical systems $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$ are C^*-strongly approximately conjugate.

$\square$

IV.2 WEAK APPROXIMATE CONJUGACY

For minimal homeomorphisms $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$, the following lemma shows that whether they are weakly approximately conjugate or not is determined by α and β only, and has nothing to do with R_{ξ_i} and R_{η_i} for $i = 1, 2$.

Lemma IV.2.1. *Let (X, α) and (X, β) be two minimal Cantor dynamical systems. For continuous maps $\xi_1, \xi_2, \eta_1, \eta_2 \colon X \to \mathbb{T}$, $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$ are weakly approximately conjugate if and only if (X, α) and (X, β) are weakly approximately conjugate.*

Proof. The "if" part:

For every $\varepsilon > 0$, we will show that there exists $\sigma_n \in \mathrm{Homeo}(X \times \mathbb{T} \times \mathbb{T})$ such that

$$\mathrm{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}, \beta) < \varepsilon.$$

As (X, β) is a minimal Cantor dynamical system, there exists a Kakutani-Rokhlin partition

$$\{X_{s,k} \colon 1 \leq s \leq n, 0 \leq k < h(s)\}$$

such that $h(s) > 5/\varepsilon$, and $\mathrm{diam}(X_{s,j}) < \varepsilon/5$, where $\mathrm{diam}(X_{s,j})$ is defined to be $\sup_{x,y \in X_{s,j}} \mathrm{dist}(x, y)$.

For any two clopen sets X_{s_1,j_1} and X_{s_2,j_2} in the Kakutani-Rokhlin partition, there exists $\delta_{s_1,j_1;s_2,j_2} > 0$ such that if $x, y \in X_{s_1,j_1} \bigsqcup X_{s_2,j_2}$ and $\mathrm{dist}(x, y) < \delta_{s_1,j_1;s_2,j_2}$, then either $x, y \in X_{s_1,j_1}$ or $x, y \in X_{s_2,j_2}$.

Let $\delta = \min \delta_{s,j;s',j'}$, where $X_{s,j}$ and $X_{s',j'}$ traverse through all pairs of distinct clopen sets in the Kakutani-Rokhlin partition above.

As (X, α) and (X, β) are weakly approximately conjugate, there exists $\gamma_n \in \mathrm{Homeo}(X)$ such that

$$\mathrm{dist}(\gamma \circ \alpha \circ \gamma^{-1}(x), \beta(x)) < \delta.$$

According to the definition of δ, it follows that for every $X_{s,j}$ in the Kakutani-Rokhlin partition above, we have

$$\gamma \circ \alpha \circ \gamma^{-1}(X_{s,j}) = \beta(X_{s,j}).$$

Without loss of generality (replacing α with $\gamma \circ \alpha \circ \gamma^{-1}$), we can assume that α and β satisfies

$$\alpha(X_{s,j}) = \beta(X_{s,j}).$$

Identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, and define π by $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}, t \mapsto t + \mathbb{Z}$. For all $x \in X_{s,0}$, define $h(x) = 0$. For $x \in X_{s,k}$ with $0 < k < h(s)$, define

$$f_1(x) = \sum_{j=1}^{k} (\xi_2 - \xi_1)(\alpha^{-j}(x)).$$

As ξ_1 and ξ_2 are both in $C(X, \mathbb{T})$, it follows that the above defined f_1 is a continuous function from X to $\mathbb{T}$.

For $x \in X_{s.k}$, define

$$g_1(x) = \sum_{j=1}^{h(s)} (\xi_2 - \xi_1)(\alpha^{-j}(\alpha^{h(s)-k}(x))).$$

It is also clear that $g_1 \in C(X, \mathbb{T})$.

As X is totally disconnected, we can divide X into $\bigsqcup_{k=1}^{N} X_k$, with every X_k being a clopen subset of X satisfying $\mathrm{dist}(h(x), h(y)) < \frac{1}{4}$ for x, y in the same X_k. For $g_1|_{X_k}$, we can lift it to continuous function $G_{1,k}\colon X_k \to [0 - \frac{1}{4}, 1 + \frac{1}{4}]$ satisfying $g_1|_{X_k} = \pi \circ G_{1,k}$.

Define $G_1\colon X \to \mathbb{R}$ by setting $G_1(x)$ to be $G_{1,k}(x)$ if $x \in X_k$. It is then easy to check that G_1 is a lifting of g_1 satisfying

$$g_1 = \pi \circ G_1 \text{ and } G_1(x) \in [0 - \frac{1}{4}, 1 + \frac{1}{4}] \text{ for all x } \in X.$$

For $x \in X_{s,k}$, define

$$s_1(x) = f_1(x) - \frac{G_1(x) \cdot k}{h(s)} + \mathbb{Z}.$$

Similarly, define $f_2(x) = 0$ if $x \in X_{s,0}$ and

$$f_2(x) = \sum_{j=1}^{k} (\eta_2 - \eta_1)(\alpha^{-j}(x))$$

for $x \in X_{s,k}$ with $0 < k < h(s)$. Define

$$g_2(x) = \sum_{j=1}^{h(s)} (\eta_2 - \eta_1)\left(\alpha^{-j}\left(\alpha^{h(s)-k}(x)\right)\right).$$

As X is totally disconnected, we can find a lifting $G_2 \in C(X, \mathbb{R})$ such that

$$g_2 = \pi \circ G_2 \text{ and } G_2(x) \in \left[0 - \frac{1}{4}, 1 + \frac{1}{4}\right]$$

for all $x \in X$.

For $x \in X_{s,k}$, define

$$s_2(x) = f_2(x) - \frac{G_2(x) \cdot k}{h(s)} + \mathbb{Z}.$$

For the s_1 and s_2 we have defined, it is easy to check that they are continuous function from X to $\mathbb{R}/\mathbb{Z}$. According to our identification, we can regard s_1 and s_2 as functions in $C(X, \mathbb{T})$.

We will show that $(\mathrm{id}_X \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2})$ will approximately conjugate $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$.

For every $(x, t_1, t_2) \in X \times \mathbb{T} \times \mathbb{T}$, we have

$$\begin{aligned}
&(\mathrm{id}_x \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2}) \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ (\mathrm{id}_x \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2})^{-1}(x, t_1, t_2) \\
&= (\mathrm{id}_x \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2}) \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})(x, t_1 - s_1(x), t_2 - s_2(x)) \\
&= (\mathrm{id}_x \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2})(\alpha(x), t_1 - s_1(x) + \xi_1(x), t_2 - s_2(x) + \eta_1(x)) \\
&= (\alpha(x), t_1 + \xi_1(x) - s_1(x) + s_1(\alpha(x)), t_2 + \eta_1(x) - s_2(x) + s_2(\alpha(x))),
\end{aligned}$$

and it is clear that

$$(\beta \times \xi_2 \times \eta_2)(x, t_1, t_2) = (\beta(x), t_1 + \xi_2(x), t_2 + \eta_2(x)).$$

As $\alpha(X_{s,j}) = \beta(X_{s,j})$ and $\operatorname{diam}(X_{s,j}) < \varepsilon/5$, we have $\operatorname{dist}(\alpha(x), \beta(x)) < \varepsilon/5$ for all $x \in X$. Consider the distance between $t_1 + \xi_1(x) - s_1(x) + s_1(\alpha(x))$ and $t_1 + \xi_2(x)$. We get

$$|t_1 + \xi_1(x) - s_1(x) + s_1(\alpha(x)) - (t_1 + \xi_2(x))| = |s_1(\alpha(x)) - s_1(x) + \xi_1(x) - \xi_2(x)|.$$

According to the definition of s_1, if $x \in X_{s,h(s)}$ (that is, x is on the roof), then

$$\begin{aligned} s_1(x) &= \sum_{j=1}^{h(s)} (\xi_2 - \xi_1)\left(\alpha^{-j}(x)\right) - G_1(x) \\ &= \sum_{j=1}^{h(s)} (\xi_2 - \xi_1)\left(\alpha^{-j}(x)\right) - \sum_{j=0}^{h(s)} (\xi_2 - \xi_1)(\alpha^{-j}(x)) \\ &= -(\xi_2 - \xi_1)(x) \\ &= 0. \end{aligned}$$

We know that $s_1(\alpha(x)) = 0$ as $(\alpha^{-h(s)})(x) \in X_{s,0}$. It is then clear that

$$|s_1(\alpha(x)) - s_1(x) + \xi_1(x) - \xi_2(x)| = 0$$

if x is in the roof set.

If x is not in the roof, in other words, for $x \in X_{s,k}$ with $0 \leq k < h(s) - 1$, we have

$$s_1(\alpha(x)) - s_1(x) = (\xi_2 - \xi_1)(x) - \frac{G_1(x)}{h(s)}.$$

As $G_1(x) \in [0 - \frac{1}{4}, 1 + \frac{1}{4}]$ for all x, and we have $h(s) > 5/\varepsilon$ for all s, it then follows that

$$|s_1(\alpha(x)) - s_1(x) + \xi_1(x) - \xi_2(x)| < 2\varepsilon/5 \text{ for all } x \in X.$$

Similarly, we have

$$|t_2 + \eta_1(x) - s_2(x) + s_2(\alpha(x)) - (t_2 + \eta_2(x))| = |s_2(\alpha(x)) - s_2(x) + \eta_1(x) - \eta_2(x)|$$

and

$$|s_2(\alpha(x)) - s_2(x) + \eta_1(x) - \eta_2(x)| < 2\varepsilon/5 \text{ for all } x \in X.$$

So far, we have proved that

$$\begin{aligned}\operatorname{dist}\big((\mathrm{id}_x \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2}) \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ (\mathrm{id}_x \times \mathrm{R}_{s_1} \times \mathrm{R}_{s_2})^{-1}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}\big) \\ &< \varepsilon/5 + 2\varepsilon/5 + 2\varepsilon/5 \\ &= \varepsilon.\end{aligned}$$

As we can construct such conjugacy maps for all $\varepsilon > 0$, it follows that $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ is weakly approximately conjugate to $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$ if α is weakly approximately conjugate to β.

The "only if" part.

If a sequence of σ_n in $\mathrm{Homeo}(X \times \mathbb{T}^2)$ approximately conjugates $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ to $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$, as X is totally disconnected, we can write σ_n as $\gamma_n \times \varphi$, with $\gamma_n \in \mathrm{Homeo}(X)$ and $\varphi\colon X \to \mathrm{Homeo}(\mathbb{T}^2)$ being a continuous map.

Let $P\colon X \times \mathbb{T}^2 \to X$ be defined by $P(x, (t_1, t_2)) = x$ (the canonical projection onto X). We can easily check that

$$P((\sigma_n \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ \sigma_n^{-1})(x, (t_1, t_2))) = (\gamma_n \circ \alpha \circ \gamma_n^{-1})(x).$$

As $(\sigma_n \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ \sigma_n^{-1}) \longrightarrow \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$, we have

$$P((\sigma_n \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ \sigma_n^{-1})(x, (t_1, t_2))) \longrightarrow P((\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})(x, (t_1, t_2))),$$

which then implies that

$$(\gamma_n \circ \alpha \circ \gamma_n^{-1})(x) \longrightarrow \beta(x) \text{ for all } x \in X.$$

We have finished the proof of the "only if" part.

$\square$

IV.3 APPROXIMATE K-CONJUGACY

From Lemma IV.2.1, we know that the if and only if condition for $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$ to be weakly approximately conjugate is that α and β are weakly approximately conjugate.

One might be wondering whether we have weak approximate conjugacy between $\alpha \times R_{\xi_1} \times R_{\eta_1}$ and $\beta \times R_{\xi_2} \times R_{\eta_2}$, can we expect to have the isomorphism between C*-algebras $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \alpha \times R_{\xi_1} \times R_{\eta_1})$ and $C^*(\mathbb{Z}, X \times \mathbb{T} \times \mathbb{T}, \beta \times R_{\xi_2} \times R_{\eta_2})$?

Generally speaking, weak approximate conjugacy is not enough to imply that the corresponding crossed product C*-algebras are isomorphic. Examples can be found in [M1], [LM1] and [LM3].

As guessed by Lin in [LM1], if we strengthen the definition of weak approximate conjugacy (in the sense that those conjugacies will induce an isomorphism of K-data of these two crossed product C*-algebras), this might be equivalent to the isomorphism of two crossed product C*-algebras.

That "strengthened" version of weak approximate conjugacy is called approximate K-conjugacy. Before the definition of approximate K-conjugacy is given, the definition of asymptotic morphism will be given and a technical result needs to be mentioned.

Definition IV.3.1. *A sequence of contractive completely positive linear maps $\{\varphi_n\}$ from C*-algebra A to C*-algebra B is said to be an asymptotic morphism, if*

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0 \textit{ for all } a, b \in A.$$

Proposition IV.3.2. *[Lin4]*

Let (X, α) and (X, β) be two dynamical systems. If there exists a sequence of homeomorphisms $\sigma_n \colon X \to X$ such that $\lim_{n\to\infty} \operatorname{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}, \beta) = 0$, then for a sequence of unitaries $\{z_n\}$ in A_α with

$$\lim_{n\to\infty} \|z_n j_\alpha(f) - j_\alpha(f) z_n\| = 0 \textit{ for all } f \in C(X),$$

there exists a unital asymptotic morphism $\{\varphi_n^\sigma\}$ from A_β to A_α such that

$$\lim_{n\to\infty} \|\psi_n^\sigma(u_\beta) - u_\alpha z_n\| = 0 \text{ and}$$

$$\lim_{n\to\infty} \|\psi_n^\sigma(j_\beta(f)) - j_\alpha(f \circ \sigma_n)\| = 0$$

for all $f \in C(X)$.

Proof. This is Proposition 3.1 in [Lin4]. The main ingredient in the proof is to use weakly approximate conjugacies to construct a C*-algebra homomorphism from A_β to $\prod_1^\infty A_\alpha / \bigoplus_1^\infty A_\alpha$, and apply the lifting property of completely contractive positive linear maps.

It works like this:

Let $\pi\colon \prod_1^\infty A_\alpha \to \bigoplus_1^\infty A_\alpha$ be the quotient map. Define

$$\Psi\colon A_\beta \to \prod_1^\infty A_\alpha / \bigoplus_1^\infty A_\alpha$$

by setting

$$\Psi(j_\beta(f)) = \pi(\{j_\alpha(f \circ \sigma_n) : n \in \mathbb{N}\}) \text{ and } \Psi(u_\beta) = \pi(\{u_\alpha z_n : n \in \mathbb{N}\}).$$

To show that Ψ is a well-defined homomorphism, we just need to check that

$$\|(u_\alpha z_n)^* \cdot j_\alpha(f \circ \sigma_n) \cdot (u_\alpha z_n) - j_\alpha(f \circ \beta \circ \sigma_n)\| \longrightarrow 0.$$

As $\operatorname{dist}(\sigma_n \circ \alpha \circ \sigma_n^{-1}, \beta) \to 0$, we have

$$\lim_{n\to\infty} \|(u_\alpha z_n)^* \cdot j_\alpha(f \circ \sigma_n) \cdot (u_\alpha z_n) - j_\alpha(f \circ \beta \circ \sigma_n)\| = \lim_{n\to\infty} \|f \circ \sigma_n \circ \alpha - f \circ \beta \circ \sigma_n\| = 0.$$

Thus $\Psi\colon A_\beta \to \prod_1^\infty A_\alpha / \bigoplus_1^\infty A_\alpha$ is a C*-algebra homomorphism.

Consider

$$
\begin{tikzcd}
 & \prod_1^\infty A_\alpha \arrow[d, "\pi"] \\
A_\beta \arrow[r, "\Psi"'] & \prod_1^\infty A_\alpha / \bigoplus_1^\infty A_\alpha
\end{tikzcd}
$$

As A_β is amenable, according to [CE, Theorem 3.10], there exists a sequence of contractive completely positive linear maps $\varphi_n^\sigma \colon A_\beta \to A_\alpha$ such that

$$\pi(\{\varphi_n^\sigma(b) \colon n \in \mathbb{N}\}) = \Psi(b) \text{ for all } b \in A_\beta.$$

As Ψ is a homomorphism, it follows that

$$\lim_{n\to\infty} \|\varphi_n^\sigma(ab) - \varphi_n^\sigma(a)\varphi_n^\sigma(b)\| = 0 \text{ for all } a, b \in A_\beta,$$

which indicates that $\{\varphi_n^\sigma \colon A_\beta \to A_\alpha : n \in \mathbb{N}\}$ gives a unital discrete asymptotic morphism.

□

Now we can give the definition of approximate K-conjugacy between two dynamical systems (X, α) and (X, β).

Definition IV.3.3. *For two minimal dynamical systems (X, α) and (Y, β), with X and Y being compact metrizable spaces, we say that (X, α) and (Y, β) are approximately K-conjugate if there exist homeomorphisms $\sigma_n \colon X \to Y$, $\tau_n \colon Y \to X$, and an isomorphism*

$$\rho \colon K_*(C^*(\mathbb{Z}, Y, \beta)) \to K_*(C^*(\mathbb{Z}, X, \alpha))$$

between K-groups such that

$$\sigma_n \circ \alpha \circ \sigma_n^{-1} \to \beta, \quad \tau_n \circ \beta \circ \tau_n^{-1} \to \alpha,$$

and the associated discrete asymptotic morphisms $\psi_n \colon B \to A$ and $\varphi_n \colon A \to B$ induce the isomorphisms ρ and ρ^{-1} respectively.

Remark: According to Proposition IV.3.2, the weak approximate conjugacy maps will induce asymptotic morphisms. But it is not generally true that the asymptotic morphisms will induce a homomorphism of K_0 and K_1 data. In Definition IV.3.3, those approximate conjugacies must not only induce a pair of homomorphisms between $K_i(A)$ and $K_i(B)$, in addition, these homomorphisms must be a pair of isomorphisms that are inverses of each other.

For the classical case of minimal Cantor dynamical systems, it is shown in [LM3] that two Cantor minimal dynamical systems are approximately K-conjugate if and only if the corresponding crossed product C*-algebras are isomorphic.

For the case of $(X \times \mathbb{T}, \alpha \times \mathrm{R}_\xi)$, with $\alpha \in \mathrm{Homeo}(X)$ being minimal homeomorphism and $\xi\colon X \to \mathbb{T}$ being a continuous map, similar results are obtained in Theorem 7.8 of [LM1].

Based on Theorem IV.1.5 and Lemma IV.2.1, we will give an if and only if condition for approximate K-conjugacy between $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$.

Theorem IV.3.4. *Let X be the Cantor set. Let $\alpha, \beta \in Homeo(X)$ be minimal homeomorphisms, and let $\xi_1, \xi_2, \eta_1, \eta_2\colon X \to \mathbb{T}$ be continuous map such that both $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$ are minimal rigid homeomorphism of $X \times \mathbb{T} \times \mathbb{T}$ (as in Definition III.2.2). Use A to denote the crossed product C*-algebra corresponding to the minimal system $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$, and B to denote the one corresponding to $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$. Use $K^0(X, \alpha)$ to denote $C(X, \mathbb{Z})/\{f - f \circ \alpha^{-1} \colon f \in C(X, \mathbb{Z}^2)\}$ and $K^0(X, \beta)$ to denote $C(X, \mathbb{Z})/\{f - f \circ \beta^{-1} \colon f \in C(X, \mathbb{Z}^2)\}$.*

The following are equivalent:

1) $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$ are approximately K-conjugate,

2) There is an order isomorphism $\rho\colon K_0(B) \to K_0(A)$ that maps $K^0(X, \beta)$ to $K^0(X, \alpha)$.

Proof. $1) \Rightarrow 2)$:

If $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})$ and $(X \times \mathbb{T} \times \mathbb{T}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2})$ are approximately K-conjugate, according to the definition of approximate K-conjugacy (Definition IV.3.3), there exists $\sigma_n \in \mathrm{Homeo}(X \times \mathbb{T} \times \mathbb{T})$ such that

$$\mathrm{dist}(\sigma_n \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ \sigma_n^{-1}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}) \longrightarrow 0,$$

and the discrete asymptotic morphism induced by $\{\sigma_n : n \in \mathbb{N}\}$ will yield an isomorphism from $K_*(B)$ to $K_*(A)$.

That is, there exists an isomorphism

$$\phi_0 \colon (K_0(B), K_0(B)_+, [1_B], K_1(B)) \to (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Define ϕ to be the restriction of ϕ_0 on $K_0(A)$. We just need to show that ϕ maps $K^0(X, \beta)$ to $K^0(X, \alpha)$.

According to the Pimsner-Voiculescu six-term exact sequence (as in the proof of Proposition III.2.1), we have

$$(j_\beta)_0(C(X \times \mathbb{T} \times \mathbb{T})) \cong K^0(X, \beta) = C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1} : f \in C(X, \mathbb{Z}^2)\}.$$

As $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$ are approximately K-conjugate, for given projection $p \in M_\infty(B)$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have $[p \circ \sigma_n] = [p \circ \sigma_m]$ in $K_0(A)$.

It is obvious that $[p \circ \sigma_n] \in (j_\alpha)_*(C(X \times \mathbb{T} \times \mathbb{T}))$. Then we can conclude that the isomorphism ρ induced by the conjugacy maps will map $K^0(X, \beta)$ to $K^0(X, \alpha)$.

2) $\Rightarrow$ 1) :

It is easy to check that 2) implies the following commutative diagram:

$$\begin{tikzcd} K_0(B) \arrow[r, "\rho"] & K_0(A) \\ K^0(X,\beta) \arrow[u, "(j_\beta)_{*0}"] \arrow[r, "\rho|_{K^0(X,\beta)}"'] & K^0(X,\alpha) \arrow[u, "(j_\alpha)_{*0}"'] \end{tikzcd} .$$

According to Theorem IV.1.5, the two minimal homeomophisms $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$ are C^*-strongly flip conjugate.

The map ρ above induces an order preserving isomorphism between $K^0(X, \beta)$ (which is isomorphic to $C(X, \mathbb{Z}^2)/\{f - f \circ \beta^{-1}\}$, with order described as in Lemma II.2.9) and $K^0(X, \alpha)$ (which is isomorphic to $C(X, \mathbb{Z}^2)/\{f - f \circ \alpha^{-1}\}$, with order described as in Lemma II.2.9). Note that

$$K_0(C^*(\mathbb{Z},X,\alpha)) \cong C(X,\mathbb{Z})/\{g - g\circ\alpha^{-1}\colon g \in C(X,\mathbb{Z})\},$$

with

$$K_0(C^*(\mathbb{Z},X,\alpha))_+ \cong C(X,\mathbb{Z})/\{g - g\circ\alpha^{-1}\colon g \in C(X,\mathbb{Z}), g \geq 0\}.$$

It follows that there is an order isomorphism

$$\begin{aligned}\widetilde{\rho}\colon &(K_0(C^*(\mathbb{Z},X,\beta)), K_0(C^*(\mathbb{Z},X,\beta))_+, [1_{C^*(\mathbb{Z},X,\beta)}])\\ &\longrightarrow (K_0(C^*(\mathbb{Z},X,\alpha)), K_0(C^*(\mathbb{Z},X,\alpha))_+, [1_{C^*(\mathbb{Z},X,\alpha)}]).\end{aligned}$$

According to Theorem 5.4 of [LM3], (X,α) and (X,β) are approximately K-conjugate. Thus they are weakly approximately conjugate.

For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset C(X\times\mathbb{T}\times\mathbb{T})$, as β is minimal, we can find Kakutani-Rokhlin partition

$$\mathcal{P} = \{X(s,k)\colon s \in S, 1 \leq k \leq H(s)\}$$

such that $H(s) > \dfrac{32\pi}{\varepsilon}$ for all $s \in S$ and $\operatorname{diam}(X(s,k)) < \dfrac{\varepsilon}{16}$.

As $C(X\times\mathbb{T}_1\times\mathbb{T}_2)$ is generated by

$$\{1_D, z_1, z_2\colon D \text{ is a clopen subset of } X, z_i \text{ is the identity function on } T_i\},$$

without loss of generality, we can assume that

$$\mathcal{F} = \{1_{X(s,k)}, z_1 1_{X(s,k)}, z_2 1_{X(s,k)}\colon s \in S, 1 \leq k \leq H(s)\}.$$

The fact that (X,α) and (X,β) are approximately K-conjugate implies that there exist $\{\sigma_n \in \mathrm{Homeo}(X) : n \in \mathbb{N}\}$ such that

$$\sigma_n\circ\alpha\circ\sigma_n^{-1} \longrightarrow \beta.$$

By choosing n large enough, just as in the proof of the "if" part of Theorem IV.2.1, we get

$$(\sigma_n \circ \alpha \circ \sigma_n^{-1})(X(s,k)) = \beta(X(s,k)) \text{ for } s \in S, 1 \leq k \leq H(s).$$

Without loss of generality, we can assume that

$$\alpha(X(s,k)) = \beta(X(s,k)) \text{ for } s \in S, 1 \leq k \leq H(s).$$

As in the proof of "if" part of Theorem IV.2.1, there exist maps $\{\mathrm{id}_X \times \mathrm{R}_{g_n} \times \mathrm{R}_{h_n}\}_{n \in \mathbb{N}}$ such that

$$(\mathrm{id}_X \times \mathrm{R}_{g_n} \times \mathrm{R}_{h_n}) \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}) \circ (\mathrm{id}_X \times \mathrm{R}_{g_n} \times \mathrm{R}_{h_n})^{-1} \longrightarrow (\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}),$$

with all the $g_n, h_n \colon X \to \mathbb{T}$ being continuous functions as defined in the proof of Theorem IV.2.1.

We will show that the conjugacy maps $\{\mathrm{id}_X \times \mathrm{R}_{g_n} \times \mathrm{R}_{h_n} : n \in \mathbb{N}\}$ will induce an isomorphism between $K_*(B)$ and $K_*(A)$.

The idea is like this:

We know that these two dynamical systems $\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1}$ and $\beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}$ are C^*-strongly flip conjugate. Thus there exists $\psi_n \colon B \to A$ such that the following diagram approximately commutes:

$$\begin{tikzcd} B \arrow[r, "\psi_n"] & A \\ C(X \times \mathbb{T} \times \mathbb{T}) \arrow[u, "j_B"] \arrow[r, "\chi_n"] & C(X \times \mathbb{T} \times \mathbb{T}). \arrow[u, "j_A"'] \end{tikzcd}$$

As we had assumed that (without loss of generality) $\alpha(X(s,k)) = \beta(X(s,k))$ for $s \in S, k = 1, \ldots, H(s)$, the χ_n in the diagram above satisfies

$$\mathrm{dist}(\chi_n(x), x) < \mathrm{diam}(X(s,k)) < \varepsilon/M$$

for $x \in X(s,k)$. In other words, restricted on $C(X \times \mathbb{T} \times \mathbb{T})$, χ_n is close to the identity map.

Note that $\{\psi_n\}$ are isomorphisms and $[\psi_n] = [\psi_m]$ in $KL(B, A)$ for m, n large enough. If we can find $W_n \in U(A)$ such that $f \circ \sigma_n$ is close to $W_n^* \psi_n(f) W_n$ in A, and $W_n^* \psi_n(u_B) W_n$ is close to $u_A z_n$ in A, where z_n is a unitary element that "almost" commutes with $C(X \times \mathbb{T} \times \mathbb{T})$, then it follows that the conjugacy maps $\{\mathrm{id}_X \times \mathrm{R}_{g_n} \times \mathrm{R}_{h_n} : n \in \mathbb{N}\}$ will induce an isomorphism between $K_*(B)$ and $K_*(A)$.

The complete proof is as below:

Let g_1, g_2, f_1, f_2 be as defined in the proof of Lemma IV.2.1, and let

$$\mathcal{F}_1 = \{g_i \cdot 1_{X(s,k)}, f_i \cdot 1_{X(s,k)} : s \in S, 1 \leq k \leq H(s)\}.$$

We can further divide $\alpha^{-1}(X(s,1))$ into the disjoint union of clopen sets $Y(s,1)$, $Y(s,2)$, $\ldots$, $Y(s, N(s))$, and choose $x_{s,j} \in Y(s,j)$ such that

$$|f(x) - f(x_{s,j})| < \varepsilon/16 \text{ for all } f \in \mathcal{F}_1, 1 \leq j \leq N(s), s \in S.$$

Let G_1, G_2 be the same as the one defined in the proof of Theorem IV.2.1. That is, G_1 is the lifting of $g_1(x) = \sum_{j=1}^{h(s)} (\xi_2 - \xi_1)(\alpha^{-j}(\alpha^{h(s)-k}(x)))$, G_2 is the lifting of $g_2(x) = \sum_{j=1}^{h(s)} (\eta_2 - \eta_1)(\alpha^{-j}(\alpha^{h(s)-k}(x)))$, and $G_i(x) \in [0 - \frac{1}{4}, 1 + \frac{1}{4}]$. As both G_1, G_2 are path connected to the zero function, it is clear that

$$[z_i \cdot 1_{Y(s,j)}] = [z_i \cdot e^{-i2\pi G_k / H(s)} \cdot 1_{Y(s,j)}]$$

in $K_1(A)$ for $i = 1, 2$ and $k = 1, 2$.

Let

$$\iota_{s,j} : C(1_{Y_{s,j}} \times \mathbb{T} \times \mathbb{T}) \longrightarrow 1_{Y_{s,j}} \cdot A \cdot 1_{Y_{s,j}}$$

be the inclusion map. Let two homomorphisms

$$\Delta_{s,j},\ \delta_{s,j}\ : C(\mathbb{T}^2) \longrightarrow C(1_{Y_{s,j}} \times \mathbb{T} \times \mathbb{T})$$

be defined by

$$\Delta_{s,j}(f) = \mathrm{id}_{Y(s,j)} \otimes f$$

and

$$\delta_{s,j}(f)(x,z_1,z_2) = \mathrm{id}_{Y_{s,j}}(x)\cdot f(z_1\cdot e^{i2\pi G_1(x_{s,j})/H(s)}, z_2\cdot e^{i2\pi G_2(x_{s,j})/H(s)}).$$

Consider the maps

$$\iota_{s,j}\circ\Delta_{s,j},\ \iota_{s,j}\circ\delta_{s,j}\ : C(\mathbb{T}^2)\longrightarrow 1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}}.$$

It is clear that these two maps are monomorphisms.

By Proposition III.2.3, $\mathrm{TR}(A)=0$, and it follows that $\mathrm{TR}(1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}})=0$.

As G_1, G_2 are contractible, we can claim that

$$[\iota_{s,j}\circ\Delta_{s,j}] = [\iota_{s,j}\circ\delta_{s,j}] \text{ in } KL(C(\mathbb{T}^2), 1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}}).$$

For every $f\in 1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}}$, and for every tracial state τ on $1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}}$, consider $\tau((\iota_{s,j}\circ\Delta_{s,j})(f))$ and $\tau((\iota_{s,j}\circ\delta_{s,j})(f))$. By Lemma III.1.4, we can regard $1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}}$ as the crossed product C*-algebra of the induced minimal homeomorphism of $Y_{s,j}\times\mathbb{T}\times\mathbb{T}$. As $\alpha\times\mathrm{R}_\xi\times\mathrm{R}_\eta$ is rigid, it follows that the traces on $1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}}$ also corresponds to such measures like $\mu\times v$, with v being the Lebesgue measure on the torus.

Now we have

$$\begin{aligned}
\tau\left((\iota_{s,j}\circ\Delta_{s,j})(f)\right) &= \tau\left(\mathrm{id}_{Y(s,j)}\otimes f\right)\\
&= \mu(Y(s,j))\cdot\int_{\mathbb{T}^2} f\left((z_1,z_2)\right)\,\mathrm{d}v\\
&= \mu(Y(s,j))\cdot\int_{\mathbb{T}^2} f\left(z_1\cdot e^{i2\pi G_1(x_{s,j})/H(s)}, z_2\cdot e^{i2\pi G_2(x_{s,j})/H(s)}\right)\,\mathrm{d}v\\
&= \tau\left((\iota_{s,j}\circ\delta_{s,j})(f)\right).
\end{aligned}$$

As $\mathrm{TR}(1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}})=0$, $[\iota_{s,j}\circ\Delta_{s,j}]=[\iota_{s,j}\circ\delta_{s,j}]$ and

$$\tau((\iota_{s,j}\circ\Delta_{s,j})(f)) = \tau((\iota_{s,j}\circ\delta_{s,j})(f))$$

for all $\tau\in T(1_{Y_{s,j}}\cdot A\cdot 1_{Y_{s,j}})$. According to Theorem 3.4 of [Lin3], the two monomorphisms $\iota_{s,j}\circ\Delta_{s,j}$ and $\iota_{s,j}\circ\delta_{s,j}$ are approximately unitarily equivalent. Thus there exists a unitary element

$v_{s,j} \in 1_{Y_{s,j}} \cdot A \cdot 1_{Y_{s,j}}$ such that

$$\|v_{s,j}^* z_i q_{s,j} v_{s,j} - z_i e^{-i2\pi G_i(x_{s,j})/H(s) \cdot 1_{Y_{s,j}}}\| < \varepsilon/(16K) \quad \text{for all } s \in S, 1 \leq k \leq H(s), 1 \leq j \leq N(s).$$

Let $v_s = \sum_{j=1}^{N(s)} v_{s,j}$. As $Y_{s,1}, Y_{s,2}, \ldots, Y_{s,N(s)}$ are mutually disjoint, we have

$$\begin{aligned}\|(v_s^k)^* z_i f(x) 1_{\alpha^{-1}(X(s,1))} v_s^k - z e^{-2\pi k G_i(x)/H(s)} f(x) 1_{\alpha^{-1}(X(s,1))}\| &< \varepsilon/16 + K\varepsilon/(16K) + \varepsilon/16 \\ &< \varepsilon/4.\end{aligned}$$

for all $f \in \mathcal{F}_1, s \in S$.

Let

$$\mathcal{F}_2 = \mathcal{F} \cup \{1_{Y_{s,j}}, z_i 1_{Y_{s,j}}, z f 1_{\alpha^{-1}(X(s,1))} \colon f \in \mathcal{F}_1, s \in S, 1 \leq k \leq H(s)\}.$$

As $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$ is C^*-strongly flip conjugate to $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$, for any $\delta > 0$, and for the $\mathcal{F}_2 \subset C(X \times \mathbb{T} \times \mathbb{T})$, there exists a C*-algebra isomorphism $\psi \colon B \to A$ such that

$$\|\psi(j_\beta(f)) - j_\alpha(f)\| < \delta \text{ and } \|\psi(u_B)^* j_\alpha(f) \psi(u_B) - j_\alpha(f \circ \beta)\| < \delta \text{ for all } f \in \mathcal{F}_2.$$

Note that $1_{X(s,k)}$, for $s \in S$ and $1 \leq k \leq H(s)$, are mutually orthogonal projections and add up to 1_B, and $\{1_{X(s,k)} \colon s \in S, 1 \leq k \leq H(s)\} \subset \mathcal{F}_2$. According to the perturbation lemma [Lin2, Lemma 2.5.7], by taking δ to be small enough, the fact that $\|\psi(j_\beta(f)) - j_\alpha(f)\| < \delta$ will imply that there exists $v \in U(A)$ such that

$$v \approx_{\varepsilon/(16K^2)} \psi(u_B)$$

and

$$v^* 1_{X(s,k)} v = 1_{X(s,k)} \circ \beta \text{ and } \|v^* f v - f \circ \beta\| < \varepsilon/(4K) \text{ for all} f \in \mathcal{F}_2.$$

Define $W = \sum_{s\in S} \sum_{k=1}^{H(s)} 1_{X(s,k)} v^{-k} v_s^k u^k$. Then we can check that

$$\begin{aligned}
W^*W &= \left(\sum_{s\in S} \sum_{k=1}^{H(s)} 1_{X(s,k)} v^{-k} v_s^k u^k \right)^* \cdot \sum_{s'\in S} \sum_{k'=1}^{H(s)} 1_{X(s',k')} v^{-k'} v_{s'}^{k'} u^{k'} \\
&= \sum_{s\in S} \sum_{k=1}^{H(s)} \left(u^{-k} v_s^{-k} v^k 1_{X(s,k)} 1_{X(s,k)} v^{-k} v_s^k u^k \right) \\
&= \sum_{s\in S} \sum_{k=1}^{H(s)} u^{-k} v_s^{-k} 1_{\alpha^{-1}(X(s,1))} v_s^k u^k \\
&= \sum_{s\in S} \sum_{k=1}^{H(s)} u^{-k} 1_{\alpha^{-1}(X(s,1))} u^k \\
&= \sum_{s\in S} \sum_{k=1}^{H(s)} 1_{\alpha^k(\alpha^{-1}(X(s,1)))} \\
&= \sum_{s\in S} \sum_{k=1}^{H(s)} 1_{X(s,k)} \\
&= 1_A.
\end{aligned}$$

As $\mathrm{TR}(A) = 0$, we have $\mathrm{tsr}(A) = 1$. Thus $W^*W = 1_A$ implies that $WW^* = 1_A$. So far, it is checked that W is a unitary element in A.

As

$$\|(v_s^k)^* z_i f(x) 1_{\alpha^{-1}(X(s,1))} v_s^k - z e^{-2\pi k G_i(x)/H(s)} f(x) 1_{\alpha^{-1}(X(s,1))}\| < \varepsilon/4$$

and

$$\|v^* f v - f \circ \beta\| < \varepsilon/(4K) \text{ for all } f \in \mathcal{F}_2 \text{ and for all } f \in \mathcal{F}_2,$$

we have

$$
\begin{aligned}
W^* z_i 1_{X(s,k)} W &= \left(\sum_{s_1\in S}\sum_{k_1=1}^{H(s_1)} 1_{X(s_1,k_1)} v^{-k_1} v_{s_1}^{k_1} u^{k_1}\right)^* z_i 1_{X(s,k)} \left(\sum_{s_2\in S}\sum_{k=1}^{H(s_2)} 1_{X(s_2,k_2)} v^{-k_2} v_{s_2}^{k_2} u^{k_2}\right)\\
&= \left(\sum_{s_1\in S}\sum_{k_1=1}^{H(s_1)} u^{-k_1} v_{s_1}^{-k_1} v^{k_1} 1_{X(s_1,k_1)}\right) z_i 1_{X(s,k)} \left(\sum_{s_2\in S}\sum_{k_2=1}^{H(s_2)} 1_{X(s_2,k_2)} v^{-k_2} v_{s_2}^{k_2} u^{k_2}\right)\\
&= u^{-k} v_s^{-k} v^k 1_{X(s,k)} z_i 1_{X(s,k)} 1_{X_{s,k}} v^{-k} v_s^k u^k\\
&= u^{-k} v_s^{-k} v^k (z_i 1_{X(s,k)}) v^{-k} v_s^k u^k\\
&\approx_{\varepsilon/(4K)} u^{-k} v_s^{-k} \left((z_i 1_{X(s,k)}) \circ \beta^k\right) v_s^k u^k\\
&\approx_{\varepsilon/(4K)+\varepsilon/4} (z 1_{X(s,k)}) \circ \sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma(x,t_1,t_2) \quad &= \\
&\left(x, t_1 + \left(\sum_{j=1}^{k} \xi_2\left(\alpha^{j-1}(\beta^{-k}(x))\right) - \xi_1\left(\beta^{-j}(x)\right)\right) - kG_1(x)/H(s),\right.\\
&\left. t_2 + \left(\sum_{j=1}^{k} \eta_2\left(\alpha^{j-1}(\beta^{-k}(x))\right) - \eta_1\left(\beta^{-j}(x)\right)\right) - kG_1(x)/H(s)\right),
\end{aligned}
$$

for $x \in X(s,k)$ with $s \in S$ and $1 \le k \le H(s)$.

Then it follows that

$$\|W^* z_i 1_{X(s,k)} W - (z_i 1_{X(s,k)}) \circ \sigma\| < K(\varepsilon/4K) + \varepsilon/4 < \varepsilon.$$

Similar to the proof of Theorem IV.2.1, we have

$$\mathrm{dist}(\sigma \circ (\alpha \times \mathrm{R}_{\xi_1} \times \mathrm{R}_{\eta_1})\sigma^{-1}, \beta \times \mathrm{R}_{\xi_2} \times \mathrm{R}_{\eta_2}) < \varepsilon.$$

Consider the map $\mathrm{ad}W \circ \psi$, we have that

$$\|(\mathrm{ad}W \circ \psi)(j_\beta(f)) - j_\alpha(f \circ \sigma)\| < \varepsilon + \delta.$$

If $(\mathrm{ad}W \circ \psi)$ maps u_B to u_A or $u_A \cdot y$ such that $\|yf - fy\| < \varepsilon$ for all $f \in \mathcal{F}$, then it follows that the K-map induced by approximate conjugacy map σ (restricted to $\mathcal{F}$) will coincide with $[\mathrm{ad}W \circ \psi] \in KL(B, A)$.

In fact, we can check that

$$W^*v^*Wz_i1_{X(s,k)}W^*vW \approx_d \varepsilon u_A^* z_i 1_{X(s,k)} u_A,$$

which then implies that $\|yf - fy\| < \varepsilon$ if we define $y = u_A^*(W^*vW) \in U(A)$.

As

$$(\mathrm{ad}W \circ \psi)(u_B) = W\psi(u_B)W \approx_{\varepsilon/(16K^2)} W^*vW = u_A y,$$

we may claim that the K-map induced by approximate conjugacy map σ (restricted to $\mathcal{F}$) will coincide with $[\mathrm{ad}W \circ \psi] \in KL(B, A)$.

As $C(X \times \mathbb{T} \times \mathbb{T})$ is separable, by taking $\mathcal{F}$ to be large enough and $\varepsilon \to 0$, it follows that the weak approximate conjugacy map σ will induce an isomorphism from $K_i(B)$ to $K_i(A)$, which finishes the proof.

□

CHAPTER V

THE CASES WITH COCYCLES BEING FURSTENBERG TRANSFORMATIONS

We had studied properties of dynamical systems and the corresponding crossed product C*-algebras if the action on $X \times \mathbb{T} \times \mathbb{T}$ is $\alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta$. That is, in the skew product, the actions on the torus are just rotations.

If the actions on torus are Furstenberg transformations, do we have similar results? This chapter studies weak approximate conjugacy between two such systems and the K_i of such crossed product C*-algebras (which might be different from the case in the previous chapter), and shows that there are two types of such minimal dynamical systems that will yield different K-theory for the crossed product C*-algebras.

A definition of Furstenberg transformation on $\mathbb{T}^2$ is given below.

Definition V.0.1. *A map $F\colon \mathbb{T}^2 \to \mathbb{T}^2$ is called a Furstenberg transformation of degree d if there exist $\theta \in \mathbb{T}$ and continuous function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x+1) - f(x) = d$ for all $x \in \mathbb{R}$ such that (identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$)*

$$F(t_1, t_2) = (t_1 + \theta, t_2 + f(t_1)).$$

For the F above, d is called the degree of Furstenberg transform F, and is denoted by $\deg(F)$. The number d is also called the degree of f, and denoted by $\deg(f)$.

V.1 WEAK APPROXIMATE CONJUGACY BETWEEN TWO FURSTENBERG TRANSFORMATIONS

Use $\mathrm{FT}(\mathbb{T}^2)$ to denote the set of all Furstenberg transformations on $\mathbb{T}^2$. We will consider the relationship between $\alpha \times \varphi$ and $\beta \times \psi$, with $\alpha, \beta \in \mathrm{Homeo}(X)$, and $\varphi, \psi \colon X \to \mathrm{FT}(\mathbb{T}^2)$.

Proposition V.1.1. *Let F, G be two Furstenberg transformations on $\mathbb{T}^2$ (as defined above). If the degree of F is m, and the degree of G is n, then $F \circ G$ is still a Furstenberg transformation, and the degree of $F \circ G$ is $m + n$.*

Proof. Let $F(t_1, t_2) = (t_1 + \theta, t_2 + f(t_1))$ and $G(t_1, t_2) = (t_1 + \delta, t_2 + g(t_1))$. It follows that

$$\begin{aligned} F \circ G(t_1, t_2) &= F(t_1 + \delta, t_2 + g(t_1)) \\ &= (t_1 + \delta + \theta, t_2 + g(t_1) + f(t_1 + \delta)). \end{aligned}$$

According to definition V.0.1, $F \circ G$ is a Furstenberg transformation.

As $\deg F = m$ and $\deg G = n$, it follows that

$$\begin{aligned} g(t_1 + 1) + f(t_1 + 1 + \delta) - (g(t_1) + f(t_1)) &= g(t_1 + 1) - g(t_1) + f(t_1 + 1 + \delta) - f(t_1) \\ &= m + n. \end{aligned}$$

Thus the degree of $F \circ G$ is $m + n$.

$\square$

In this chapter, we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$. For $t_1, t_2 \in \mathbb{R}/\mathbb{Z}$, we define the distance between them by

$$\mathrm{dist}(t_1, t_2) = \min\{|t_1 - t_2 + n| \colon n \in \mathbb{Z}\}.$$

The following observation will be used.

Proposition V.1.2. *Let $f, g \in C(\mathbb{T}, \mathbb{T})$, and define* $\mathrm{dist}(f, g) = \sup_{t \in \mathbb{T}} \mathrm{dist}(f(t), g(t))$. *If* $\mathrm{dist}(f, g) < 1/2$, *then* $\deg(f) = \deg(g)$.

Proof. Suppose that $\mathrm{dist}(f, g) < 1/2$ and $\deg(f) \neq \deg g$.

Note that $f - g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ is of degree $\deg(f) - \deg(g)$, which is not zero. According to the Intermediate Value Theorem, there exists $t \in \mathbb{R}$ and $n \in \mathbb{N}$ such that

$$|f(t) - g(t) + n| = 1/2.$$

It then follows that $\mathrm{dist}(f - g) = 1/2$, contradicting with our assumption. So far, we have finished the proof.

$\square$

For two minimal homeomorphisms $\alpha \times \varphi$ and $\alpha \times \psi$ (with $\varphi, \psi \colon X \to \mathrm{FT}(\mathbb{T}^2)$), a necessary condition for weak approximate conjugacy between them (with conjugacy maps having cocycles in Furstenberg transformations) is given:

Proposition V.1.3. *Let $\alpha \times \varphi$ and $\beta \times \psi$ be two minimal homeomorphisms on $X \times \mathbb{T}^2$ with $\varphi, \psi \colon X \to FT(\mathbb{T}^2)$. If there exists $\gamma_n \times \phi_n \in Homeo(X \times \mathbb{T}^2)$ with $\phi_n \colon X \to FT(\mathbb{T}^2)$ continuous such that $(\gamma_n \times \phi_n) \circ (\alpha \times \varphi) \circ (\gamma_n \times \phi_n)^{-1} \to \beta \times \psi$, then*

1) $\{\gamma_n \colon n \in \mathbb{N}\}$ approximately conjugates α to β,

2) there exists $N \in \mathbb{N}$ such that

$$\deg(\psi(\gamma_n(x))) + \deg(\phi_n(x)) = \deg(\varphi(x)) + \deg(\phi_n(\alpha(x)))$$

for all $n > N$.

Proof. As $(\gamma_n \times \phi_n) \circ (\alpha \times \varphi) \circ (\gamma_n \times \phi_n)^{-1} \to \beta \times \psi$, we have

$$(\gamma_n \times \phi_n) \circ (\alpha \times \varphi) \circ (\gamma_n \times \phi_n)^{-1}(x, (t_1, t_2)) \to (\beta \times \psi)(x, (t_1, t_2)),$$

which is equivalent to

$$\mathrm{dist}((\gamma_n \times \phi_n) \circ (\alpha \times \varphi)(x, (t_1, t_2)), (\beta \times \psi) \circ (\gamma_n \times \phi_n)(x, (t_1, t_2))) \to 0.$$

Assume that $\varphi, \psi, \phi_n \colon X \to \mathrm{FT}(\mathbb{T}^2)$ are defined by

$$\varphi(x)((t_1,t_2)) = (t_1 + \theta_1(x), t_2 + f_x(t_1)),$$

$$\psi(x)((t_1,t_2)) = (t_1 + \theta_2(x), t_2 + g_x(t_1)),$$

$$\phi_n(x)((t_1,t_2)) = (t_1 + \xi_n(x), t_2 + h_{n,x}(t_1)),$$

with f_x, g_x, h_x just like the function f in definition V.0.1.

Note that

$$\begin{aligned}(\gamma_n \times \phi_n) \circ (\alpha \times \varphi)(x,(t_1,t_2)) &= (\gamma_n \times \phi_n) \circ (\alpha(x), (t_1 + \theta_1(x), t_2 + f_x(t_1))) \\ &= (\gamma_n(\alpha(x)), (t_1 + \theta_1(x) + \xi_n(\alpha(x)), t_2 + f_x(t_1) + h_{n,\alpha(x)}(t_1))),\end{aligned}$$

and

$$\begin{aligned}(\beta \times \psi) \circ (\gamma_n \times \phi_n)(x,(t_1,t_2))) &= (\beta \times \psi)(\gamma_n(x), (t_1 + \xi(x), t_2 + h_x(t_1))) \\ &= (\beta(\gamma_n(x)), (t_1 + \xi_n(x) + \theta_2(\gamma_n(x)), t_2 + h_{n,x}(t_1) + g_{\gamma_n(x)}(t_1))).\end{aligned}$$

It follows that $\mathrm{dist}(\gamma_n(\alpha(x)), \beta(\gamma_n(x))) \to 0$ and $\mathrm{dist}(H_{n,x}(t_1), G_{n,x}(t_1)) \to 0$, where $H_{n,x}(t_1) = f_x(t_1) + h_{n,\alpha(x)}(t_1)$ and $G_{n,x}(t_1) = h_{n,x}(t_1) + g_{\gamma_n(x)}(t_1)$.

Choose $N \in \mathbb{N}$ such that if $n > N$. Then $\mathrm{dist}(H_{n,x}(t_1), G_{n,x}(t_1)) < 1/2$.

As $f_x, h_{n,\alpha(x)}, h_{n,x}$ and $g_{\gamma_n(x)}$ can be regarded as maps from $\mathbb{T}$ to $\mathbb{T}$, we can identify $H_{n,x}$ and $G_{n,x}$ as functions in $C(\mathbb{T},\mathbb{T})$. According to Proposition V.1.2, it follows that for all $n > N$, we have

$$\deg(H_{n,x}) = \deg(G_{n,x}).$$

Note that $\deg(f_x) = \deg(\varphi(x))$, $\deg(g_x) = \deg(\psi(x))$, and $\deg(h_{n,x}) = \deg(\phi_n(x))$. We then have

$$\deg(\varphi(x)) + \deg(\phi_n(\alpha(x))) = \deg(\phi_n(x)) + \deg(\psi(\gamma_n(x))),$$

which finishes the proof.

□

V.2 K-THEORY OF THE CROSSED PRODUCT C^*-ALGEBRA

For the minimal dynamical system $(X\times\mathbb{T}^2, \alpha\times\varphi)$, let A be the crossed product C*-algebra. We will use the Pimsner-Voiculescu six-term exact sequence to get the K-data of A.

We use $K^0(X,\alpha)$ to denote $C(X,\mathbb{Z})/\{f - f\circ\alpha\colon f\in C(X,\mathbb{Z})\}$. Note that $\deg(\varphi(x))\in C(X,\mathbb{Z})$. Let $\pi\colon C(X,\mathbb{Z})\to K^0(X,\alpha)$ be the canonical projection, and use $[\deg(\varphi(x))]$ to denote $\pi(\deg(\varphi(x)))$.

Proposition V.2.1. *For the minimal dynamical system $(X\times\mathbb{T}^2, \alpha\times\varphi)$ with cocycles being Furstenberg transformations, use A to denote the crossed product C^*-algebra of this dynamical system.*

1) If $[\deg(\varphi(x))]\neq 0$ in $K^0(X,\alpha)$, then

$$K_0(A)\cong C(X,\mathbb{Z}^2)/\{f - f\circ\alpha\colon f\in C(X,\mathbb{Z}^2)\}\oplus\mathbb{Z}$$

and

$$K_1(A)\cong C(X,\mathbb{Z}^2)/\{(f,g)-(f,g)\circ\alpha-(\deg(\varphi)\cdot(g\circ\alpha),0)\colon f,g\in C(X,\mathbb{Z})\}\oplus\mathbb{Z}^2.$$

2) If $[\deg(\varphi(x))]=0$ in $K^0(X,\alpha)$, then

$$K_0(A)\cong C(X,\mathbb{Z}^2)/\{f - f\circ\alpha\colon f\in C(X,\mathbb{Z}^2)\}\oplus\mathbb{Z}^2$$

and

$$K_1(A)\cong C(X,\mathbb{Z}^2)/\{(f,g)-(f,g)\circ\alpha-(\deg(\varphi)\cdot(g\circ\alpha),0)\colon f,g\in C(X,\mathbb{Z})\}\oplus\mathbb{Z}^2.$$

Proof. According to the Pimsner-Voiculescu six-term exact sequence, we have

$$\begin{array}{ccccc} K_0(C(X\times\mathbb{T}^2)) & \xrightarrow{\mathrm{id}-(\alpha\times\varphi)_{*0}} & K_0(C(X\times\mathbb{T}^2)) & \xrightarrow{j_{A*0}} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \xleftarrow[j_{A*1}]{} & K_1(C(X\times\mathbb{T}^2)) & \xleftarrow[\mathrm{id}-(\alpha\times\varphi)_{*1}]{} & K_1(C(X\times\mathbb{T}^2)) \end{array} .$$

It then follows that we have the exact sequences

$$0 \longrightarrow \mathrm{coker}(\mathrm{id}-(\alpha\times\varphi)_{*0}) \longrightarrow K_0(A) \longrightarrow \ker(\mathrm{id}-(\alpha\times\varphi)_{*1}) \longrightarrow 0 \tag{V.1}$$

and

$$0 \longrightarrow \mathrm{coker}(\mathrm{id}-(\alpha\times\varphi)_{*1}) \longrightarrow K_1(A) \longrightarrow \ker(\mathrm{id}-(\alpha\times\varphi)_{*0}) \longrightarrow 0\ . \tag{V.2}$$

We will study $K_i(A)$ by looking at the kernel and co-kernel of $\mathrm{id}-(\alpha\times\varphi)_{*i}$ (for $i=0,1$).

From Lemma II.2.1, we know that $K_i(C(\mathbb{T}^2))$ is isomorphic to $\mathbb{Z}^2$ for $i=0,1$. Note that $K_0(C(X))\cong C(X,\mathbb{Z})$ and $K_1(C(X))=0$. According to the Künneth Theorem,

$$K_0(C(X\times\mathbb{T}^2)\cong K_0(C(X))\otimes K_0(C(\mathbb{T}^2))\bigoplus K_1(C(X))\otimes K_1(C(\mathbb{T}^2))\cong C(X,\mathbb{Z}^2)$$

and

$$K_1(C(X\times\mathbb{T}^2)\cong K_0(C(X))\otimes K_1(C(\mathbb{T}^2))\bigoplus K_1(C(X))\otimes K_0(C(\mathbb{T}^2))\cong C(X,\mathbb{Z}^2).$$

We will identify both $K_0(C(X\times\mathbb{T}^2)$ and $K_1(C(X\times\mathbb{T}^2)$ with $C(X,\mathbb{Z}^2)$.

According to Example 4.9 of [Ph1], for every $x\in X$,

$$\varphi(x)_{*0}\colon K_0(C(\mathbb{T}^2))\to K_0(C(\mathbb{T}^2))$$

is the identify map, and $\varphi(x)_{*1}\colon K_1(C(\mathbb{T}^2)) \to K_1(C(\mathbb{T}^2))$ is given by

$$\varphi(x)_{*1}\colon \mathbb{Z}^2 \to \mathbb{Z}^2, \begin{pmatrix} m \\ n \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ \deg(\varphi(x)) & 1 \end{pmatrix} \cdot \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m \\ \deg(\varphi(x)) \cdot m + n \end{pmatrix}.$$

For $(f,g) \in C(X, \mathbb{Z}^2) \cong K_1(C(X \times \mathbb{T}^2))$, we can consider $H \in U(C(X \times \mathbb{T}^2))$ defined by

$$H(x, z_1, z_2) = z_1^{f(x)} \cdot z_2^{g(x)},$$

with $z_i \in \mathbb{T}_i$, and each $\mathbb{T}_i$ is identified with the unit circle in the complex plane $\mathbb{C}$. It is then clear that this H corresponds to (f,g) in $K_1(C(X \times \mathbb{T}^2))$.

Let $\varphi(x)((z_1, z_2)) = (z_1 \cdot e^{i2\pi\theta(x)}, z_2 \cdot z_1^{w_x} \cdot s(z_1))$ such that $\theta \in C(X, \mathbb{R})$ is continuous, and $s_x \in U(C(\mathbb{T}_1))$ is path connected to $1_{C(\mathbb{T}_1)}$ for all $x \in X$. We can check that

$$\begin{aligned} H \circ (\alpha \times \varphi)(x, z_1, z_2) &= H(\alpha(x), z_1 \cdot e^{i2\pi\theta(x)}, z_2 \cdot z_1^{w(x)} \cdot s_x(z_1)) \\ &= (z_1 \cdot e^{i2\pi\theta(x)})^{f(\alpha(x))} \cdot (z_2 \cdot z_1^{w(x)} \cdot s_x(z_1))^{g(\alpha(x))}. \end{aligned}$$

In $U(C(X \times \mathbb{T}^2))$, it is clear that $H \circ (\alpha \times \varphi)$ is path connected to G, with G defined to be

$$G(x, z_1, z_2) = z_1^{f(\alpha(x))} \cdot (z_2 \cdot z_1^{w(x)})^{g(\alpha(x))} = z_1^{f(\alpha(x)) + w(x) g(\alpha(x))} \cdot z_2^{g(\alpha(x))}.$$

Noting that $w(x) = \deg(\varphi(x))$, it then follows that

$$\varphi_{*1}((f,g))(x) = (f(\alpha(x)) + \deg(\varphi(x)) \cdot g(\alpha(x)), g(\alpha(x))).$$

Now we will study $\ker(\mathrm{id} - (\alpha \times \varphi)_{*0})$. For $f, g \in C(X, \mathbb{Z})$, we use (f,g) to denote a function in $C(X, \mathbb{Z}^2)$. If (f,g) satisfies $(\mathrm{id} - (\alpha \times \varphi_{*0}))((f,g)) = 0$, as $\varphi(x)_{*0}\colon K_0(C(\mathbb{T}^2)) \to K_0(C(\mathbb{T}^2))$ is the identify map, we get

$$f \circ \alpha = f \quad \text{and} \quad g \circ \alpha = g.$$

The minimality of α then implies that both f and g are constant functions in $C(X, \mathbb{Z})$. So far, we have shown that $\ker(\mathrm{id} - (\alpha \times \varphi)_{*0}) \cong \mathbb{Z}^2$.

As for $\ker(\mathrm{id}-(\alpha\times\varphi)_{*1})$, if there exists $(f,g) \in C(X,\mathbb{Z}^2)$ such that $(\mathrm{id}-(\alpha\times\varphi)_{*1})((f,g)) = 0$, it follows that

$$f(x) = f(\alpha(x)) + \deg(\varphi(x)) \cdot g(\alpha(x)) \quad \text{and} \quad g(x) = g(\alpha(x)).$$

As α is minimal, we conclude that $g \in C(X,\mathbb{Z})$ must be a constant function, say, $g(x) \equiv C$ for all $x \in X$.

To further study the kernel of $\mathrm{id} - (\alpha \times \varphi)_{*1}$, we will consider two cases.

Case One: $[\deg(\varphi(x))] \neq 0$ in $K^0(X,\alpha)$.

In this case, if $g(x) \equiv C \neq 0$, we will show that there is no solution for

$$f(x) = f(\alpha(x)) + C \deg(\varphi(x)).$$

In fact, if such $f \in C(X,\mathbb{Z})$ exists, it follows that $C[\deg(\varphi(x))] = 0$ in $K^0(X,\alpha)$. Similar to the proof of Corollary II.2.10, we can show that $K^0(X,\alpha)$ is torsion free, which then implies that $[\deg(\varphi(x))] = 0$, a contradiction.

If $g(x) \equiv 0$, note that α is a minimal action on X. It is then clear that $f(x) = f(\alpha(x)) + \deg(\varphi(x)) \cdot g(\alpha(x))$ implies $f(x)$ is a constant function.

So far, we have proved that if $[\deg(\varphi(x))] \neq 0$ in $K^0(X,\alpha)$, then

$$\ker(\mathrm{id} - (\alpha \times \varphi)_{*1}) \cong \{(f,0)\colon f \equiv C \text{ for } C \in \mathbb{Z}\} \cong \mathbb{Z}.$$

Case Two: $[\deg(\varphi(x))] = 0$ in $K^0(X,\alpha)$

In this case, there exists $h \in C(X,\mathbb{Z})$ such that $h(x) - h \circ \alpha(x) = \deg(\varphi(x))$.

For $(f,g) \in \ker(\mathrm{id} - (\alpha \times \varphi)_{*1})$, if $g \equiv 0$, similar to Case One, we can still get that $f \equiv C$ (with $C \in \mathbb{Z}$).

If $g \equiv M \neq 0$, then f need to satisfy

$$f(x) = f(\alpha(x)) + M \deg(\varphi(x)).$$

If there are two functions $f_1, f_2 \in C(X, \mathbb{Z})$ satisfying

$$f_i(x) = f_i(\alpha(x)) + M \deg(\varphi(x)) \text{ for } i = 1, 2,$$

then it follows that

$$(f_1 - f_2)(x) = (f_1 - f_2)(\alpha(x)),$$

which implies that $f_1 - f_2$ is a constant function.

According to our assumption, there exists $h \in C(X, \mathbb{Z})$ such that $h(x) - h \circ \alpha(x) = \deg(\varphi(x))$, it is clear that $Mh(x) - M \cdot h \circ \alpha(x) = M \deg(\varphi(x))$.

It then follows that any $f \in C(X, \mathbb{Z})$ satisfying $f(x) - f \circ \alpha(x) = M \deg(\varphi(x))$ must be in $\{M \cdot h + N : N \in \mathbb{Z}\}$.

So far, we conclude that

$$\ker(\mathrm{id} - (\alpha \times \varphi)_{*1}) \cong \{(C, 0) : C \in \mathbb{Z}\} \bigcup \{(M \cdot h + N, M) : M \neq 0, N \in \mathbb{Z}\},$$

which is isomorphic to

$$\{(M \cdot h + N, M) : M, N \in \mathbb{Z}\}.$$

So far, we showed that in this case,

$$\ker(\mathrm{id} - (\alpha \times \varphi)_{*1}) \cong \mathbb{Z}^2.$$

For either of the cases, as $\varphi(x)_{*0} : K_0(C(\mathbb{T}^2)) \to K_0(C(\mathbb{T}^2))$ is the identify map for all $x \in X$, we have

$$\operatorname{coker}(\mathrm{id} - (\alpha \times \varphi)_{*0}) \cong C(X, \mathbb{Z}^2) / \{f - f \circ \alpha : f \in C(X, \mathbb{Z}^2)\}.$$

For $(f,g) \in C(X,\mathbb{Z}^2)$, note that $(\alpha \times \varphi)_{*1}(f,g)(x) = (f(\alpha(x)) + \deg(\varphi(x)) \cdot g(\alpha(x)), g(\alpha(x)))$. It follows that

$$\operatorname{coker}(\mathrm{id} - (\alpha \times \varphi)_{*1}) \cong C(X,\mathbb{Z}^2)/\{(f,g) - (f,g)\circ\alpha - (\deg(\varphi)\cdot(g\circ\alpha), 0)\colon f,g \in C(X,\mathbb{Z})\}.$$

For either case, note that $\ker(\mathrm{id} - (\alpha \times \varphi)_{*1})$ is a free $\mathbb{Z}$-module. It follows from short exact sequences V.1 and V.2 that

$$K_0(A) \cong \operatorname{coker}(\mathrm{id} - (\alpha \times \varphi)_{*0}) \oplus \ker(\mathrm{id} - (\alpha \times \varphi)_{*1})$$

and

$$K_1(A) \cong \operatorname{coker}(\mathrm{id} - (\alpha \times \varphi)_{*1}) \oplus \ker(\mathrm{id} - (\alpha \times \varphi)_{*0}).$$

For both cases, as we know the kernel and co-kernel of $\mathrm{id} - (\alpha \times \varphi)_{*i}$ (for $i = 0, 1$), the K-data of A follows easily, which finishes the proof.

□

V.3 RIGIDITY

Similar to the idea of rigidity as in Definition III.2.2, we can define the rigidity condition for the case that cocycles are Furstenberg transformations.

Definition V.3.1. *Let $(X \times \mathbb{T}^2, \alpha \times \varphi)$ be a minimal dynamical system with each $\varphi(x)$ being a Furstenberg transformation. Let μ be an $\alpha \times \varphi$-invariant probability measure on $X \times \mathbb{T}^2$. It will induce an α-invariant probability measure on X defined by $\pi(u)(D) = \mu(D \times \mathbb{T}^2)$. We say that $(X \times \mathbb{T}^2, \alpha \times \varphi)$ is rigid if π gives a one-to-one map between the $\alpha \times \varphi$-invariant probability measures and the α-invariant probability measures.*

V.4 EXAMPLES

Several examples of rigid minimal dynamical systems $(X \times \mathbb{T}^2, \alpha \times \varphi)$ are given, with $\varphi(x)$ being a Furstenberg transformation for all $x \in X$.

a) The examples of rigid (or non-rigid) minimal dynamical systems $(X \times \mathbb{T} \times \mathbb{T}, \alpha \times \mathrm{R}_\xi \times \mathrm{R}_\eta)$ are definitely the examples of rigid (or non-rigid) minimal dynamical systems of type $(X \times \mathbb{T}^2, \alpha \times \varphi)$. For example, Example III.3.6 and Example III.3.7 in Section III.3.

b) The example of a rigid minimal dynamical system $(X \times \mathbb{T}^2, \alpha \times \varphi)$, with $\varphi(x)$ being a Furstenberg transformation for all $x \in X$, and $[\deg(\varphi(x))] \neq 0$ in $K^0(X, \alpha)$.

Let $(\mathbb{T}^3, \gamma)$ be a topological dynamical system on $\mathbb{T}^3$, with γ defined by

$$\gamma(z_1, z_2, z_3) = (z_1 e^{i2\pi\theta}, z_1 z_2, z_2 z_3)$$

for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.

According to Theorem 2.1 of [Furstenberg], the dynamical system $(\mathbb{T}^3, \gamma)$ is uniquely ergodic. Then there is only one γ-invariant probability measure on $\mathbb{T}^3$ (in fact, this measure is the standard Lebesgue measure on $\mathbb{T}^3$).

Let $(\mathbb{T}, \varphi)$ be a Denjoy homeomorphism of rotation number θ. Let (X, α) be the minimal Cantor dynamical system derived from $(\mathbb{T}, \varphi)$ such that it factors through $(\mathbb{T}, \mathrm{R}_\theta)$. In other words, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha := \varphi|_X"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{T} \arrow[r, "\mathrm{R}_\theta"'] & \mathbb{T} \end{tikzcd}$$

with $\pi \colon X \to \mathbb{T}$ being a surjective map.

Regard $\pi(x)$ as a unitary element in $\mathbb{C}$ (as $\pi(x) \subset \mathbb{T}$), and define $\varphi \colon X \to \mathrm{Homeo}(\mathbb{T}^2)$ by

$$\varphi(x)(z_1, z_2) = (\pi(x) z_1, z_1 z_2).$$

It is then clear that the following diagram commutes:

$$\begin{tikzcd} X \times \mathbb{T}^2 \arrow[r, "\alpha\times\varphi"] \arrow[d, "\pi\times \mathrm{id}_{\mathbb{T}^2}"'] & X \times \mathbb{T}^2 \arrow[d, "\pi\times \mathrm{id}_{\mathbb{T}^2}"] \\ \mathbb{T}^3 \arrow[r, "\gamma"'] & \mathbb{T}^3 \end{tikzcd} .$$

According to Proposition III.3.5, there exists a one-to-one correspondence between the invariant probability measure of $(\mathbb{T}^3, \gamma)$ and that of $(X \times \mathbb{T}^2, \alpha \times \varphi)$. Thus $(X \times \mathbb{T}^2, \alpha \times \varphi)$ is an example of rigid dynamical system with cocycles being Furstenberg transformations, and $[\deg(\varphi(x))] \neq 0$ in $K^0(X, \alpha)$.

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