FINITE W-ALGEBRAS OF CLASSICAL TYPE

by

JONATHAN BROWN

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2009

University of Oregon Graduate School

Confirmation of Approval and Acceptance of Dissertation prepared by:

Jonathan Brown

Title:

"Finite W-algebras of Classical Type"

This dissertation has been accepted and approved in partial fulfillment of the requirements for the degree in the Department of Mathematics by:

Jonathan Brundan, Co-Chairperson, Mathematics
Victor Ostrik, Co-Chairperson, Mathematics
Arkady Berenstein, Member, Mathematics
Hal Sadofsky, Member, Mathematics
Christopher Wilson, Outside Member, Computer & Information Science

and Richard Linton, Vice President for Research and Graduate Studies/Dean of the Graduate School for the University of Oregon.

June 13, 2009

Original approval signatures are on file with the Graduate School and the University of Oregon Libraries.

An Abstract of the Dissertation of

Jonathan Brown for the degree of Doctor of Philosophy

in the Department of Mathematics to be taken June 2009

Title: FINITE W-ALGEBRAS OF CLASSICAL TYPE

Approved: ______________________________

Dr. Jonathan Brundan

Approved: ______________________________

Dr. Victor Ostrik

In this work we prove that the finite W-algebras associated to nilpotent elements in the symplectic or orthogonal Lie algebras whose Jordan blocks are all the same size are quotients of twisted Yangians. We use this to classify the finite dimensional irreducible representations of these finite W-algebras.

CURRICULUM VITAE

NAME OF AUTHOR: Jonathan Brown

PLACE OF BIRTH: Baltimore, Maryland, USA

DATE OF BIRTH: 10/10/1975

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Virginia Polytechnic Institute and State University, Blacksburg, VA

DEGREES AWARDED:

Doctor of Philosophy, University of Oregon, 2009
Master of Science, University of Oregon, 2006
Bachelor of Science, Virginia Polytechnic Institute and State University, 1997

AREAS OF SPECIAL INTEREST:

Algebraic Lie Theory

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, September 2002 to June 2009

GRANTS, AWARDS AND HONORS:

Borsting Award, University of Oregon Department of Mathematics, 2007

PUBLICATIONS:

J. Brown and J. Brundan, Elementary invariants for centralizers of nilpotent matrices, *J. Austral. Math. Soc.* **86** (2009), 1–15, math.RA/0611024.

J. Brown, Twisted Yangians and finite W-algebras, *Transform. Groups* **14** (2009), 87–114, arXiv:0710.2918.

ACKNOWLEDGMENTS

I thank my advisor, Jonathan Brundan, for suggesting this problem, as well as for his guidance. I greatly appreciate how available and willing he always is to discuss mathematics with me.

I thank the faculty and graduate students at the University of Oregon Departement of Mathematics for providing me with so many stimulating mathematical experiences.

I thank my wife Liz for her love, compassion, empathy, and warmth over the years. I never could have done this without her. I thank Robin for making everything worthwhile.

I thank my parents and sisters for their love, support, and encouragement.

I thank Mark, Meriel, Sammy, Margot, David, Aaron, and all my other friends for making my time here so fulfilling.

To Liz and Robin.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

This work concerns the structure and representation theory of a certain algebra $U(\mathfrak{g}, e)$ associated to a nilpotent element e in a reductive Lie algebra $\mathfrak{g}$. This algebra is called a *finite W-algebra* and is a deformation of the Slodowy slice through the adjoint orbit of the nilpotent element in question. The main focus of this work is the structure and representation theory of the finite W-algebras associated to nilpotent elements in the symplectic or orthogonal Lie algebras whose Jordan blocks are all the same size. We refer to these simply as *rectangular* finite W-algebras.

The general definition of finite W-algebras is due to Premet in [P1], though in some cases they had been introduced much earlier by Lynch in [Ly] following Kostant's celebrated work on Whittaker modules in [K]. The terminology "finite W-algebra" comes from the mathematical physics literature, where finite W-algebras are the finite dimensional analogs of the vertex W-algebras defined and studied for example by Kac, Roan, and Wakimoto in [KRW]. The precise identification between the definitions in [P1] and [KRW] was made only recently by D'Andrea, De Concini, De Sole, Heluani, and Kac in [D^3HK].

There are many remarkable connections between finite W-algebras and other areas of mathematics. The finite W-algebra $U(\mathfrak{g}, e)$ possesses two natural filtrations, the *Kazhdan* and *loop* filtrations. The main structure theorem for finite W-algebras, proved in [P1] and reproved in [GG], is that the associated graded algebra to $U(\mathfrak{g}, e)$ with respect to the Kazhdan filtration is isomorphic to the coordinate algebra of the Slodowy slice, i.e. $U(\mathfrak{g}, e)$ is a *quantization* of the Slodowy slice through the nilpotent orbit containing e. On the other hand, by [P2] the associated graded algebra with respect to the loop filtration is isomorphic to $U(\mathfrak{g}^e)$, the universal enveloping algebra of the centralizer of e in $\mathfrak{g}$. Because of this, the structure of $U(\mathfrak{g}, e)$ is intimately related to the invariant theory of the centralizer $\mathfrak{g}^e$. In [BB] this connection was used to construct a system of algebraically independent generators for the center of the universal enveloping algebra $U(\mathfrak{g}^e)$ in the case $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, giving a constructive proof of the freeness of this center (which had been established earlier by Panyushev, Premet and Yakimova in [PPY] by a different method) and also verifying [PPY, Conjecture 4.1].

The work of Premet in [P2, P3], Losev in [Lo1, Lo2], and Ginzburg in [Gi] has highlighted the importance of the study of finite dimensional representations of $U(\mathfrak{g}, e)$, revealing an intimate relationship with the theory of primitive ideals of the universal enveloping algebra $U(\mathfrak{g})$ itself. At the heart of this connection is an equivalence of categories due to Skryabin in [Sk] between the category of $U(\mathfrak{g}, e)$-modules and a certain category of generalized Whittaker modules for $\mathfrak{g}$.

In [BK2] Brundan and Kleshchev showed that finite W-algebras associated to nilpotent elements in $\mathfrak{gl}_n(\mathbb{C})$ are quotients of a certain subalgebra of Yangians called a *shifted Yangian*, and they exploited this connection to give a complete set of generators and relations for the type A finite W-algebras. This leads to their classification in [BK3] of the finite dimensional irreducible representations of type A finite W-algebras. The results in this thesis extend some of their techniques to the rectangular finite W-algebras of types B, C, and D by replacing the Yangian with the twisted Yangian. The existence of a connection between rectangular finite W-algebras of types B, C and D and twisted Yangians was first observed by Ragoucy in [R] at the "classical" level of associated graded algebras with respect to the Kazhdan filtration (which are commutative Poisson algebras).

I.1 The Main Results of this Thesis

Throughout this thesis we denote the general linear, symplectic, and orthogonal Lie algebras $\mathfrak{gl}_k(\mathbb{C})$, $\mathfrak{sp}_k(\mathbb{C})$, and $\mathfrak{so}_k(\mathbb{C})$ as $\mathfrak{gl}_k$, $\mathfrak{sp}_k$, and $\mathfrak{so}_k$ for short. Let $\mathfrak{g} = \mathfrak{so}_{nl}$ or $\mathfrak{sp}_{nl}$ for some fixed positive integers n and l. We will also need the following index set defined in terms of a positive integer k:

$$\mathcal{I}_k = \{1-k, 3-k, \ldots, k-1\}.$$

Let e be a nilpotent element of Jordan type (l^n) in $\mathfrak{g}$. In order to ensure that such a nilpotent exists one must further assume that if $\mathfrak{g} = \mathfrak{so}_{nl}$ and l is even then n is even, and that if $\mathfrak{g} = \mathfrak{sp}_{nl}$ and l is odd then n is even. Let $U(\mathfrak{g}, e)$ be the finite W-algebra attached to $\mathfrak{g}$ and the nilpotent element e; see §II.1 for the general definition. Let Y_n^+ and Y_n^- denote the twisted Yangians associated to $\mathfrak{so}_n$ and $\mathfrak{sp}_n$, respectively, assuming that n is even in the $\mathfrak{sp}_n$ case. For a sign $\phi \in \{\pm\}$, the twisted Yangian Y_n^ϕ is a certain associative algebra with generators $\{S_{i,j}^{(r)} \mid i, j \in \mathcal{I}_n, r \in \mathbb{Z}_{>0}\}$. See (III.13) and (III.14) below for a full set of relations of the twisted Yangian Y_n^ϕ in terms of these generators.

The main result of Chapter III, the proof of which has also appeared in the author's second published article in [B1], is the following theorem.

Theorem I.1.1. *There exists a surjective algebra homomorphism*

$$Y_n^+ \twoheadrightarrow U(\mathfrak{g}, e) \text{ if } l \text{ is odd and } \mathfrak{g} = \mathfrak{so}_{nl} \text{ or if } l \text{ is even and } \mathfrak{g} = \mathfrak{sp}_{nl};$$

$$Y_n^- \twoheadrightarrow U(\mathfrak{g}, e) \text{ if } l \text{ is odd and } \mathfrak{g} = \mathfrak{sp}_{nl} \text{ or if } l \text{ is even and } \mathfrak{g} = \mathfrak{so}_{nl}.$$

This theorem is proved by constructing explicit elements in $U(\mathfrak{g}, e)$ which are the images of the twisted Yangian elements $S_{i,j}^{(r)}$; see Theorem III.1.2 below.

In Chapter IV we use this theorem and Molev's classification of the finite dimensional irreducible representations of twisted Yangians from [M] to deduce a classification of finite dimensional irreducible representations of the rectangular finite W-algebras. The main combinatorial objects in this classification are skew-symmetric $n \times l$ tableaux. A *skew-symmetric* $n \times l$ *tableaux* is an $n \times l$ matrix of complex numbers, with rows labeled by the set $\mathcal{I}_n$ and columns labeled by the set $\mathcal{I}_l$, and which is skew-symmetric with respect to the center of the matrix, that is, if $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_l}$ is a skew-symmetric $n \times l$ tableaux then $a_{i,j} = -a_{-i,-j}$. Let $\mathrm{Tab}_{n,l}^{\mathfrak{g}}$ denote the set of skew-symmetric $n \times l$ tableaux. We say that two skew-symmetric $n \times l$ tableaux are *row equivalent* if one can be obtained from the other by permuting entries within rows. Let $\mathrm{Row}_{n,l}^{\mathfrak{g}}$ denote the set of row equivalence classes of skew-symmetric $n \times l$ tableaux. A skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_l}$ is *column strict* if

- the entries in every column except for the middle column (which exists only when l is odd) are strictly decreasing from top to bottom, i.e., $a_{1-n,j} > a_{3-n,j} > \cdots > a_{n-1,j}$ for all $0 \neq j \in \mathcal{I}_l$;
- if l is odd and n is even then the entries in the middle column satisfy $a_{1-n,0} > \cdots > a_{-1,0}$, and they also satisfy $a_{-1,0} > 0$ if $\mathfrak{g} = \mathfrak{sp}_{nl}$, and they satisfy

$a_{-3} + a_{-1} > 0$ if $\mathfrak{g} = \mathfrak{so}_{nl}$ and $n \geq 4$;

- if l is odd and n is odd then the entries in the middle column satisfy $a_{1-n,0} > \cdots > a_{-2,0}$, and they also satisfy $2a_{-2,0} > 0$.

In this definition (and from here on) we are using the partial order $\geq$ on $\mathbb{C}$ defined by $a \geq b$ if $a - b \in \mathbb{Z}_{\geq 0}$. Let $\mathrm{Col}^{\mathfrak{g}}_{n,l}$ denote the set of all skew-symmetric $n \times l$ tableaux, and let $\mathrm{Std}^{\mathfrak{g}}_{n,l}$ denote the set of elements of $\mathrm{Row}^{\mathfrak{g}}_{n,l}$ which have a representative in $\mathrm{Col}^{\mathfrak{g}}_{n,l}$.

We relate these sets to certain representations of the twisted Yangian Y^{ϕ}_n. It is convenient to use the power series

$$S_{i,j}(u) = \sum_{r \geq 0} S^{(r)}_{i,j} u^{-r} \in Y^{\phi}_n[[u^{-1}]], \tag{I.1}$$

where $S^{(0)}_{i,j} = \delta_{i,j}$. A Y^{ϕ}_n-module V is called a *highest weight module* if it generated by a vector v such that $S_{i,j}(u)v = 0$ for all $i < j$ and if for all i it happens that $S_{i,i}(u)v = \mu_i(u)v$ for some power series $\mu_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. To a skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in I_n, j \in I_l}$ we associate a (unique up to isomorphism) irreducible highest weight Y^{ϕ}_n-module with highest weight vector v for which

- $(u - \frac{i}{2})^l S_{i,i}(u - \frac{i}{2})v = (u + a_{i,1-l})(u + a_{i,3-l}) \ldots (u + a_{i,l-1})v$ if l is even and $i \geq 0$;

- $(u - \frac{i}{2})^{l-1}(u + \frac{\phi - i}{2})S_{i,i}(u - \frac{i}{2})v = (u + a_{i,1-l})(u + a_{i,3-l}) \ldots (u + a_{i,l-1})v$ if l is odd and $i \geq 0$.

By Corollary III.2.4, this Y^{ϕ}_n-module factors through the surjection $Y^{\phi}_n \twoheadrightarrow U(\mathfrak{g}, e)$ from Theorem I.1.1 to yield an irreducible $U(\mathfrak{g}, e)$-module denoted $L(A)$ for each $A \in \mathrm{Row}^{\mathfrak{g}}_{n,l}$.

Theorem I.1.2. *If l is odd or if l is even and $\mathfrak{g} = \mathfrak{so}_{nl}$ then*

$$\{L(A) \mid A \in \mathrm{Std}_{n,l}^{\mathfrak{g}}\}$$

is a complete set of isomorphism classes of the finite dimensional irreducible representations of $U(\mathfrak{g}, e)$. If l is even and $\mathfrak{g} = \mathfrak{sp}_{nl}$ then

$$\{L(A) \mid A \in \mathrm{Row}_{n,l}^{\mathfrak{sp}_{nl}}, A^+ \in \mathrm{Std}_{n,l+1}^{\mathfrak{so}_{n(l+1)}}\}$$

is a complete set of isomorphism classes of the finite dimensional irreducible representations of $U(\mathfrak{g}, e)$, where A^+ is the skew-symmetric $n \times (l+1)$ tableaux obtained by inserting a middle column into A with entries

$$\frac{n}{2} - 1, \frac{n}{2} - 2, \dots, 1, 0, 0, -1, -2, \dots, 1 - \frac{n}{2}$$

if n is even and

$$\frac{n}{2} - 1, \frac{n}{2} - 2, \dots, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \dots, 1 - \frac{n}{2}$$

if n is odd down the middle column.

This classification meshes well with the general framework of highest weight theory for finite W-algebras developed in [BGK]; see §II.4 and §IV.6 below.

The theorem also helps illuminate the connection between $U(\mathfrak{g})$-modules and $U(\mathfrak{g}, e)$-modules via primitive ideals. As is explained in Chapter II below, in [Lo2] Losev showed that there exists a surjective map

$$\dagger : \mathrm{Prim}_{\mathrm{fin}} U(\mathfrak{g}, e) \to \mathrm{Prim}_{\overline{G.e}} U(\mathfrak{g}).$$

Here $\mathrm{Prim}_{\mathrm{fin}}U(\mathfrak{g},e)$ denotes the primitive ideals of $U(\mathfrak{g},e)$ of finite co-dimension, and $\mathrm{Prim}_{\overline{G.e}}U(\mathfrak{g})$ denotes the primitive ideals of $U(\mathfrak{g})$ whose associated variety is $\overline{G.e}$. Moreover, Losev showed that the fibers of the map $\dagger$ are C-orbits, where C is the component group associated to the nilpotent element e, which acts naturally as automorphisms on $U(\mathfrak{g},e)$ (induced ultimately by its adjoint action of $U(\mathfrak{g})$). As one application of Theorem I.1.2 we can combine it with the results of Losev to obtain a new classification of the primitive ideals in $U(\mathfrak{g})$ whose associated variety is $\overline{G.e}$. This involves computing explicitly the effect of twisting the finite dimensional highest weight modules $L(A)$ from Theorem I.1.2 by the action of C.

I.2 An Example

In this example, which illustrates how Theorem I.1.2 and Losev's results can be played off each other, we assume that $n = l = 2$ and $\mathfrak{g} = \mathfrak{sp}_4$. According to Theorem I.1.2 the following tableaux parameterize the finite dimensional irreducible $U(\mathfrak{g},e)$-modules of the same central character as the trivial $U(\mathfrak{g})$-module:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline -2 & -1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|}\hline -1 & 2 \\ \hline -2 & 1 \\ \hline \end{array}\,, \quad \text{and} \quad \begin{array}{|c|c|}\hline -2 & 1 \\ \hline -1 & 2 \\ \hline \end{array}\,.$$

Of course, the third of these is not row equivalent to any element of $\mathrm{Col}_{2,2}^{\mathfrak{sp}_4}$, but in this case

$$\begin{array}{|c|c|}\hline -2 & 1 \\ \hline -1 & 2 \\ \hline \end{array}^{+} = \begin{array}{|c|c|c|}\hline -2 & 0 & 1 \\ \hline -1 & 0 & 2 \\ \hline \end{array}$$

which is row equivalent to

$$\begin{array}{|c|c|c|}\hline 0 & -2 & 1 \\ \hline -1 & 2 & 0 \\ \hline \end{array}\,,$$

which does belong to $\mathrm{Col}_{2,3}^{\mathfrak{so}_6}$.

To see how this fits with with Losev's results, we note by [C, Chapter 13] that C, the component group, is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In Chapter IV we give an explicit matrix for the generator of C for all rectangular nilpotents (C is only nontrivial in the case that $\mathfrak{g} = \mathfrak{sp}_{nl}$ and l and n are even, in which case $C \cong \mathbb{Z}/2\mathbb{Z}$). We can then use this to directly calculate the action of C on $\mathrm{Prim}_{\mathrm{fin}} U(\mathfrak{g}, e)$. By Theorem IV.5.1 there are exactly two orbits of C on these three tableaux, one containing

1	2
-2	-1

and

-2	1
-1	2

,

the other containing

-1	2
-2	1

.

So we can conclude using Losev's results that there are two primitive ideals of $U(\mathfrak{g})$ of this central character, a result normally obtained via an analysis of the left cells in the Weyl group.

One intriguing thing about this example is that it suggests a connection between finite dimensional irreducible $U(\mathfrak{g}, e)$-modules of certain central characters and the Springer fiber $\mathcal{B}_e$. By definition, the Springer fiber is the subvariety $\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} | [e, \mathfrak{b}] \subseteq \mathfrak{b}\}$ of the flag variety $\mathcal{B}$ (identified with the set of all Borel subalgebras $\mathfrak{b}$ of $\mathfrak{g}$). In the case $\mathfrak{g} = \mathfrak{sp}_4$ and e has Jordan type (2^2), the Springer fiber has three irreducible components, which is equal to the number of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules of trivial central character, as noted above. There is also a C-action on $\mathcal{B}_e$, and in this case two of the irreducible components of $\mathcal{B}_e$ are in the same C-orbit while the third component is invariant under the action of C. In other words, as C-sets, the set of irreducible components of $\mathcal{B}_e$ is isomorphic to the set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules of this central character. At the present time we do not have an explanation for this coincidence.

CHAPTER II

OVERVIEW OF FINITE W-ALGEBRAS

For this chapter $\mathfrak{g}$ denotes an arbitrary reductive Lie algebra over the complex numbers, and e denotes a nilpotent element of $\mathfrak{g}$.

II.1 Definition of the Finite W-algebra $U(\mathfrak{g}, e)$

The finite W-algebra $U(\mathfrak{g}, e)$ is defined in terms of a nilpotent element $e \in \mathfrak{g}$. By the Jacobson-Morozov Theorem, e embeds into an $\mathfrak{sl}_2$-triple (e, h, f). Now the ad h eigenspace decomposition gives a grading on $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i), \tag{II.1}$$

where $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. Define a character $\chi : \mathfrak{g} \to \mathbb{C}$ by $\chi(x) = (x, e)$, where $(.,.)$ is a fixed non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. This defines a non-degenerate symplectic form ω_χ on $\mathfrak{g}(-1)$: $\omega_\chi(x, y) = \chi([x, y])$. Choose a Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}(-1)$ with respect to ω_χ, and let $\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{i \leq -2} \mathfrak{g}(i)$. Let I be the left ideal of $U(\mathfrak{g})$ generated by $\{m - \chi(m) \mid m \in \mathfrak{m}\}$. The adjoint action of $\mathfrak{m}$ on $U(\mathfrak{g})$ leaves the subspace I invariant, so there is an induced adjoint action of $\mathfrak{m}$ on $Q = U(\mathfrak{g})/I$. The space of fixed points $Q^{\mathfrak{m}}$ inherits a well defined multiplication from $U(\mathfrak{g})$, making it an associative algebra, and we define $U(\mathfrak{g}, e) = Q^{\mathfrak{m}}$.

In the case that the grading (II.1) an *even* grading then things are substantially simpler. Let $\mathfrak{p} = \sum_{i \geq 0} \mathfrak{g}(i)$, and by the PBW Theorem,

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I. \tag{II.2}$$

Define $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{p})$ to be the projection along this direct sum decomposition. Then we can identify

$$U(\mathfrak{g}, e) = \{u \in U(\mathfrak{p}) \mid \mathrm{pr}([m, u]) = 0 \text{ for all } m \in \mathfrak{m}\},$$

so $U(\mathfrak{g}, e)$ is a subalgebra of $U(\mathfrak{p})$.

The finite W-algebra $U(\mathfrak{g}, e)$ possesses two natural filtrations. The first of these, the *Kazhdan filtration*, is the filtration on $U(\mathfrak{g}, e)$ induced by the filtration on $U(\mathfrak{g})$ generated by declaring that each element $x \in \mathfrak{g}(i)$ in the grading (II.1) is of degree $\frac{i}{2} + 1$. The fundamental *PBW theorem* for finite W-algebras asserts that the associated graded algebra to $U(\mathfrak{g}, e)$ under the Kazhdan filtration is isomorphic to the coordinate algebra of the Slodowy slice at e; see e.g. [GG, Theorem 4.1].

The second important filtration is called the *good filtration*. Its definition is complicated in the general case, however in the case that the grading (II.1) is even, the good filtration is the filtration induced on $U(\mathfrak{g}, e)$ by the grading (II.1) on $U(\mathfrak{p})$. According to this definition, the associated graded algebra $\mathrm{gr}\, U(\mathfrak{g}, e)$ is identified with a graded subalgebra of $U(\mathfrak{p})$. The fundamental result about the good filtration, which is a consequence of the PBW theorem and [P2, (2.1.2)], is that

$$\mathrm{gr}\, U(\mathfrak{g}, e) = U(\mathfrak{g}^e) \tag{II.3}$$

as graded subalgebras of $U(\mathfrak{p})$, where $\mathfrak{g}^e$ denotes the centralizer of e in $\mathfrak{g}$; see also

[BGK, Theorem 3.5].

Another important result is Skryabin's Theorem. Recall the left $U(\mathfrak{g})$-module Q defined above. Note that Q is also a right $U(\mathfrak{g},e)$-module.

Theorem II.1.1. *(Skryabin, [Sk]) The functor* $Q\otimes_{U(\mathfrak{g},e)}? : U(\mathfrak{g},e)\text{-mod} \to \mathcal{W}(e)$ *is an equivalence of categories.*

Here $\mathcal{W}(e)$ denotes the category of e-Whittaker modules for $U(\mathfrak{g})$; this is the set of $U(\mathfrak{g})$-modules on which $m - \chi(m)$ acts locally nilpotently for all $m \in \mathfrak{m}$.

II.2 Primitive Ideals

One of the main problems in the representation theory of any algebra is to classify its finite dimensional irreducible representations. Losev in [Lo1] has obtained a near classification of the finite dimensional irreducible $U(\mathfrak{g},e)$-modules which uses the theory of primitive ideals in $U(\mathfrak{g})$.

For any ring A, an ideal $I \lhd A$ is *primitive* if $I = \mathrm{Ann}(L)$ for some irreducible A-module L. Primitive ideals in the universal enveloping algebras of reductive Lie algebras were classified by the mid 1980s, see e.g. [Du, Jo1, Jo2, BV]. Primitive ideals in $U(\mathfrak{g})$ have two main invariants, central character and associated variety, which are defined as follows.

If L is an irreducible (possibly infinite dimensional) $\mathfrak{g}$-module, by the Schur-Dixmier Lemma we have that $\mathrm{End}_{U(\mathfrak{g})}(L) \cong \mathbb{C}$. Thus $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$, acts as scalars on L. In other words any irreducible $U(\mathfrak{g})$-module L admits a central character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. This central character depends only on the primitive ideal $\mathrm{Ann}_{U(\mathfrak{g})}L$, so we can talk unambiguously about the central character of a primitive ideal I.

To define $\mathcal{VA}(I)$, the associated variety of I, one uses the standard filtration of $U(\mathfrak{g})$ for which the associated graded algebra $\operatorname{gr} U(\mathfrak{g})$ is naturally identified with $S(\mathfrak{g})$. This induces a filtration on I such that $\operatorname{gr} I \lhd S(\mathfrak{g})$. Identifying $S(\mathfrak{g})$ with $\mathbb{C}[\mathfrak{g}]$, the coordinate algebra of the affine variety $\mathfrak{g}$, via the non-degenerate form $(.,.)$ fixed earlier, it follows that $\operatorname{gr} I$ is an ideal of $\mathbb{C}[\mathfrak{g}]$. The associated variety to I, $\mathcal{VA}(I)$, is defined to be the corresponding Zariski closed subset $Z(\operatorname{gr} I)$ of $\mathfrak{g}$.

Fix a maximal toral subalgebra $\mathfrak{t}$ in $\mathfrak{g}$ and a Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{t}$. Let Φ be the root system of $\mathfrak{g}$ corresponding to $\mathfrak{t}$ and let $\Phi^+ \subset \Phi$ be the set of positive roots corresponding to $\mathfrak{b}$. Let $\rho = 1/2\sum_{\alpha\in\Phi^+}\alpha$, and let $L(\lambda)$ denote the irreducible $U(\mathfrak{g})$-module of highest weight $\lambda - \rho$.

Theorem II.2.1. *(Duflo, [Du]) If $I \lhd U(\mathfrak{g})$ is primitive then $I = \operatorname{Ann}_{U(\mathfrak{g})}(L(\lambda))$ for some $\lambda \in \mathfrak{t}^*$.*

This theorem reduces the problem of classifying primitive ideals in $U(\mathfrak{g})$ to determining when $\operatorname{Ann}_{U(\mathfrak{g})}(L(\lambda)) = \operatorname{Ann}_{U(\mathfrak{g})}(L(\mu))$ for $\lambda, \mu \in \mathfrak{t}^*$. The solution to this is known, but highly non-trivial, being related ultimately by the Kazhdan-Lusztig conjecture to the structure of left cells in the underlying Weyl group.

A useful partial invariant which relates primitive ideals to nilpotent orbits is given by following theorem.

Theorem II.2.2. *(Joseph, [Jo3]) If $I \lhd U(\mathfrak{g})$ is primitive then $\mathcal{VA}(I) = \overline{G.e}$ where G is the adjoint group associated to $\mathfrak{g}$ and e is a nilpotent element in $\mathfrak{g}$.*

II.3 Losev's Near Classification of the Finite Dimensional Irreducible Representations of $U(\mathfrak{g}, e)$

Recall the $(U(\mathfrak{g}), U(\mathfrak{g}, e))$-bimodule Q from §II.1. For a $U(\mathfrak{g}, e)$-module V, let $V^\dagger = Q \otimes_{U(\mathfrak{g},e)} V$. For an algebra A, let Prim A denote the set of primitive ideals of A. In [Lo2] Losev shows that there exists a map

$$\dagger : \text{Prim}\, U(\mathfrak{g}, e) \to \text{Prim}\, U(\mathfrak{g})$$

such that $\text{Ann}_{U(\mathfrak{g})}(L^\dagger) = (\text{Ann}_{U(\mathfrak{g},e)}(L))^\dagger$. Furthermore, he shows that $\dagger$ restricts to a surjective map

$$\dagger : \text{Prim}_{\text{fin}} U(\mathfrak{g}, e) \twoheadrightarrow \text{Prim}_{\overline{G.e}} U(\mathfrak{g}),$$

where

$$\text{Prim}_{\text{fin}} U(\mathfrak{g}, e) = \{I \in \text{Prim}\, U(\mathfrak{g}, e) \mid \text{codim}(I) < \infty\},$$

and

$$\text{Prim}_{\overline{G.e}} U(\mathfrak{g}) = \{I \in \text{Prim}\, U(\mathfrak{g}) \mid \mathcal{VA}(I) = \overline{G.e}\}.$$

Additionally, Losev shows that the fibers of this map are orbits for a natural action of the component group

$$C = C_G(e, h, f)/C_G(e, h, f)^\circ.$$

The action of this group on $\text{Prim}_{\text{fin}} U(\mathfrak{g}, e)$ arises as follows. To start with, the adjoint action of $C_G(e, h, f)$ on $U(\mathfrak{g})$ induces in the obvious fashion an action of $C_G(e, h, f)$ on $U(\mathfrak{g}, e)$. The connected component of the identity, $C_G(e, h, f)^\circ$, acts trivially under this action, because its Lie algebra actually embeds as a subalgebra of $U(\mathfrak{g}, e)$. Hence the adjoint action of $C_G(e, h, f)$ induces an action of C on $U(\mathfrak{g}, e)$ by algebra automorphisms, from which we get the natural action of C on $\text{Prim}_{\text{fin}} U(\mathfrak{g}, e)$.

Since the set of isomorphism classes of finite dimensional irreducible representations of $U(\mathfrak{g}, e)$ is parameterized by $\text{Prim}_{\text{fin}} U(\mathfrak{g}, e)$, this is nearly a classification of the finite dimensional irreducible representations of $U(\mathfrak{g}, e)$. It is a complete classification in the case that $C = \{1\}$, however when $C \neq \{1\}$ other methods are required. In the next section we look at another approach to this problem.

II.4 Highest Weight Theory for $U(\mathfrak{g}, e)$

A different approach to the problem of classifying the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules is given by Brundan, Goodwin, and Kleshchev in [BGK]. In this paper the authors construct Verma modules for $U(\mathfrak{g}, e)$. The key to this is a reductive subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ which contains e. This leads to the "smaller" finite W-algebra $U(\mathfrak{g}_0, e)$ which plays the role of a Cartan subalgebra in defining highest weight modules.

Throughout this section we assume for simplicity that the grading from (II.1) is an even grading, though all the results we mention hold in general. Refer to [BGK] for the general results.

To define $\mathfrak{g}_0$, assume that $\mathfrak{t}$, our fixed maximal toral subalgebra of $\mathfrak{g}$, is chosen so that it contains h and so that $\mathfrak{t}^e$ is a maximal toral subalgebra of $\mathfrak{g}^e \cap \mathfrak{g}(0)$. For $\alpha \in (\mathfrak{t}^e)^*$ let $\mathfrak{g}_\alpha$ denote the α-weight space of $\mathfrak{g}$. So

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi^e} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{t}^e$ in $\mathfrak{g}$ and $\Phi^e \subset (\mathfrak{t}^e)^*$ denotes the set of nonzero weights of $\mathfrak{t}^e$ on $\mathfrak{g}$. Thus we have defined $\mathfrak{g}_0$, which is now a minimal Levi subalgebra of $\mathfrak{g}$ containing e.

Recall that we have fixed a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, and that Φ^+ denotes the corresponding set of positive roots. Let $\mathfrak{q} = \mathfrak{g}_0 + \mathfrak{b}$, which is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. For each simple root $\alpha \in \Phi^+$, the corresponding root space of $\mathfrak{g}$ must lie in $\mathfrak{g}_0$ or the nil-radical of $\mathfrak{q}$. It follows that $\mathfrak{g}_\alpha \subseteq \mathfrak{q}$ or $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{q}$ for each $\alpha \in \Phi^e$. Define $\Phi^e_+ = \{\alpha \in \Phi^e \mid \mathfrak{g}_\alpha \subseteq \mathfrak{q}\}$. This defines the *dominance order* $\geq$ on $(\mathfrak{t}^e)^*$: $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Phi^e_+$, and it is now the case that $\Phi^e = -\Phi^e_+ \sqcup \Phi^e_+$.

Let $\mathfrak{a}$ be $\mathfrak{g}, \mathfrak{g}^e$, or $\mathfrak{p}$, and for $\alpha \in (\mathfrak{t}^e)^*$ let $\mathfrak{a}_\alpha$ denote the α-weight space of $\mathfrak{a}$. Let $\mathfrak{a}_\pm = \bigoplus_{\alpha \in \Phi^e_\pm} \mathfrak{a}_\alpha$, so $\mathfrak{a} = \mathfrak{a}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_+$ and $U(\mathfrak{a}) = \bigoplus_{\alpha \in \mathbb{Z}\Phi^e} U(\mathfrak{a})_\alpha$. In particular, $U(\mathfrak{a})_0$ is a subalgebra. Let $U(\mathfrak{a})_\#$ denote the *left* ideal of $U(\mathfrak{a})$ generated by the roots spaces $\mathfrak{a}_\alpha$ for $\alpha \in \Phi^e_+$. Similarly let $U(\mathfrak{a})_\flat$ denote the *right* ideal of $U(\mathfrak{a})$ generated by the roots spaces $\mathfrak{a}_\alpha$ for $\alpha \in \Phi^e_-$. Let $U(\mathfrak{a})_{0,\#} = U(\mathfrak{a})_0 \cap U(\mathfrak{a})_\#$, and $U(\mathfrak{a})_{\flat,0} = U(\mathfrak{a})_0 \cap U(\mathfrak{a})_\flat$. Now the PBW theorem implies that $U(\mathfrak{a})_{0,\#} = U(\mathfrak{a})_{\flat,0}$, hence $U(\mathfrak{a})_{0,\#}$ is a two-sided ideal of $U(\mathfrak{a})_0$. Moreover, $\mathfrak{a}_0$ is a subalgebra of $\mathfrak{a}$, and we actually have that $U(\mathfrak{a})_0 = U(\mathfrak{a}_0) \oplus U(\mathfrak{a})_{0,\#}$. Let

$$\pi : U(\mathfrak{a})_0 \twoheadrightarrow U(\mathfrak{a}_0)$$

be the algebra homomorphism defined by projection along this decomposition.

It is easy to see that

$$\mathfrak{t}^e \subseteq U(\mathfrak{g}, e) \tag{II.4}$$

since $([m, t], e) = 0$ for all $m \in \mathfrak{m}, t \in \mathfrak{t}^e$. Recall that the good filtration on $U(\mathfrak{g}, e)$ is defined in §II.1, and that $\operatorname{gr} U(\mathfrak{g}, e) = U(\mathfrak{g}^e)$. The following theorem is due to Premet in [P1]:

Theorem II.4.1. *There exists a* $\mathfrak{t}^e$*-equivariant injection* $\Theta : \mathfrak{g}^e \hookrightarrow U(\mathfrak{g}, e)$ *such that* $\operatorname{gr}' \Theta : \mathfrak{g}^e \hookrightarrow U(\mathfrak{g}^e)$ *is the natural embedding.*

It should be noted that Θ is not a Lie algebra homomorphism.

Let $h_1, \dots, h_l$ be a basis of $\mathfrak{g}^e_0$. Let $f_1, \dots, f_m$, and $e_1, \dots, e_m$ be $\mathfrak{t}^e$-weight bases of $\mathfrak{g}^e_-$ and $\mathfrak{g}^e_+$ respectively, such that f_i is of weight $-\gamma_i$, and e_i is of weight

γ_i for $\gamma_1, \ldots, \gamma_m \in \Phi_+^e$. For $i = 1, \ldots, m$, $j = 1, \ldots, l$, let $F_i = \Theta(f_i), E_i = \Theta(e_i)$, and $H_j = \Theta(h_j)$. For $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{\mathbf{m}}$, let $F^{\mathbf{a}} = F_1^{a_1} \ldots F_m^{a_m}$. For $\mathbf{b} \in \mathbb{Z}_{\geq 0}^{\mathbf{l}}, \mathbf{c} \in \mathbb{Z}_{\geq 0}^{\mathbf{m}}$ define $H^{\mathbf{b}}, E^{\mathbf{c}}$ similarly. Theorem II.4.1 implies that the following is a PBW basis of $U(\mathfrak{g}, e)$:

$$\{F^{\mathbf{a}} H^{\mathbf{b}} E^{\mathbf{c}} \mid \mathbf{a}, \mathbf{c} \in \mathbb{Z}_{\geq 0}^{\mathbf{m}}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^{\mathbf{l}}\}.$$

It will be useful to note that for $\mathbf{a}, \mathbf{c} \in \mathbb{Z}_{\geq 0}^{\mathbf{m}}, \mathbf{b} \in \mathbb{Z}_{\geq 0}^{\mathbf{l}}$, the monomial $F^{\mathbf{a}} H^{\mathbf{b}} E^{\mathbf{c}}$ is of weight $\sum (c_i - a_i)\gamma_i$.

Let $U(\mathfrak{g}, e)_{\#}$ be the *left* ideal of $U(\mathfrak{g}, e)$ generated by $\{E_1, \ldots, E_m\}$. Let $U(\mathfrak{g}, e)_{\flat}$ be the *right* ideal of $U(\mathfrak{g}, e)$ generated by $\{F_1, \ldots, F_m\}$. Let $U(\mathfrak{g}, e)_{0,\#} = U(\mathfrak{g}, e)_{\#} \cap U(\mathfrak{g}, e)_0$. Let $U(\mathfrak{g}, e)_{\flat,0} = U(\mathfrak{g}, e)_{\flat} \cap U(\mathfrak{g}, e)_0$. Now from the above PBW basis it is clear that $U(\mathfrak{g}, e)_{0,\#} = U(\mathfrak{g}, e)_{\flat,0}$, and so $U(\mathfrak{g}, e)_{0,\#}$ is a *two-sided* ideal of $U(\mathfrak{g}, e)_0$.

Let $b_1, \ldots, b_r$ be a homogeneous basis for $\mathfrak{m}$ such that b_i is of degree $-d_i$ and $\mathfrak{t}$-weight $\beta_i \in \mathfrak{t}^*$, and let

$$\gamma = \sum_{\substack{1 \leq i \leq r \\ \beta_i|_{\mathfrak{t}^e} \in \Phi_-^e}} \beta_i. \tag{II.5}$$

By [BGK, Lemma 4.1], γ extends uniquely to a character of $\mathfrak{p}_0$. Let $S_{-\gamma} : U(\mathfrak{p}_0) \to U(\mathfrak{p}_0)$ be defined by $S_{-\gamma}(x) = x - \gamma(x)$ for $x \in \mathfrak{p}_0$, so $S_{-\gamma}$ an algebra isomorphism.

Theorem II.4.2. *The restriction of* $S_{-\gamma} \circ \pi : U(\mathfrak{p})_0 \twoheadrightarrow U(\mathfrak{p}_0)$ *to* $U(\mathfrak{g}, e)_0$ *defines a surjective algebra homomorphism*

$$\pi_{-\gamma} : U(\mathfrak{g}, e)_0 \twoheadrightarrow U(\mathfrak{g}_0, e)$$

with $\ker \pi_{-\gamma} = U(\mathfrak{g}, e)_{0,\#}$.

For a $U(\mathfrak{g}, e)$-module V and $\lambda \in (\mathfrak{t}^e)^*$ let

$$V_\lambda = \{v \in V \mid (t + \gamma(t))v = \lambda(t) \text{ for all } t \in \mathfrak{t}^e\}, \tag{II.6}$$

recalling that $\mathfrak{t}^e$ is naturally a subalgebra of $U(\mathfrak{g}, e)$ by (II.4). Now it is the case that $U(\mathfrak{g}, e)_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$, so V_λ is preserved by $U(\mathfrak{g}, e)_0$. We say that V_λ is a *maximal weight space of* V if $U(\mathfrak{g}, e)_\# V_\lambda = 0$. Assuming this is the case, the action of $U(\mathfrak{g}, e)_0$ factors through the homomorphism $\pi_{-\gamma}$ from Theorem II.4.2, thus V_λ is also a $U(\mathfrak{g}_0, e)$-module. Since $\mathfrak{t}^e$ can naturally be considered a subalgebra of $U(\mathfrak{g}_0, e)$ by (II.4) again, restricting the action of $U(\mathfrak{g}_0, e)$ on V_λ to $\mathfrak{t}^e$ gives a new action of $\mathfrak{t}^e$ on V_λ satisfying

$$t.v = \lambda(t)v \quad \text{for all} \quad t \in \mathfrak{t}^e$$

(which is why the shift by γ is included in the definition of the λ-weight space of a $U(\mathfrak{g}, e)$-module from (II.6)).

A $U(\mathfrak{g}, e)$-module is a *highest weight module* if it is generated by a maximal weight space V_λ such that V_λ is finite dimensional and irreducible as a $U(\mathfrak{g}_0, e)$-module. Let

$$\{V_\Lambda \mid \Lambda \in \mathcal{L}\}$$

be a complete set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules for some indexing set $\mathcal{L}$. Since $U(\mathfrak{g}, e)_\#$ is invariant under left multiplication by $U(\mathfrak{g}, e)$ and right multiplication by $U(\mathfrak{g}, e)_0$, we have that $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#$ is a $(U(\mathfrak{g}, e), U(\mathfrak{g}, e)_0)$-bimodule. Moreover the right action of $U(\mathfrak{g}, e)_0$ factors through the homomorphism $\pi_{-\gamma}$ from Theorem II.4.2. Thus we have that $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\#$ is a $(U(\mathfrak{g}, e), U(\mathfrak{g}_0, e))$-bimodule. For $\Lambda \in \mathcal{L}$, define $M(\Lambda, e)$, the *Verma module of* *type* Λ via

$$M(\Lambda, e) = U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_{\#} \otimes_{U(\mathfrak{g}_0, e)} V_\Lambda.$$

It is worth noting since $\mathfrak{t}^e \subseteq U(\mathfrak{g}_0, e)$ by (II.4), that $\mathfrak{t}^e \subseteq Z(U(\mathfrak{g}_0, e))$, so by Schur's Lemma $\mathfrak{t}^e$ acts as scalars on V_Λ, so $M(\Lambda, e)$ is in fact a highest weight module.

By [BGK, Theorem 4.5] $M(\Lambda, e)$ has a unique maximal proper submodule $R(\Lambda, e)$. Let $L(\Lambda, e) = M(\Lambda, e)/R(\Lambda, e)$. Now also by [BGK, Theorem 4.5] we have that $\{L(\Lambda, e) \,|\, \Lambda \in \mathcal{L}\}$ is a complete set of isomorphism classes of irreducible highest weight modules for $U(\mathfrak{g}, e)$. Let

$$\mathcal{L}^+ = \{\Lambda \in \mathcal{L} \,|\, \dim L(\Lambda, e) < \infty\}.$$

By [BGK, Corollary 4.6], $\{L(\Lambda, e) \,|\, \Lambda \in \mathcal{L}^+\}$ is a complete set of isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules.

Unfortunately, an explicit set $\mathcal{L}$ parameterizing the finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules is still unknown in general. In the next section, we focus on a special case in which such a parameterization is available.

II.5 The Case that e Is Regular in $\mathfrak{g}_0$

We assume in this section that e is a regular nilpotent element of $\mathfrak{g}_0$. In this case, by Kostant's Theorem we have that $U(\mathfrak{g}_0, e) \cong Z(\mathfrak{g}_0)$ and in turn by the Harish-Chandra Isomorphism, $Z(\mathfrak{g}_0) \cong S(\mathfrak{t})^{W_0}$ where W_0 is the Weyl group associated to $\mathfrak{g}_0$. We state this more precisely in the following lemma. Let

$$\eta = \frac{1}{2} \sum_{\substack{\alpha \in \Phi \\ \alpha|_{\mathfrak{t}^e} \in \Phi_+^e}} \alpha + \frac{1}{2} \sum_{\substack{1 \le i \le r \\ \beta_i|_{\mathfrak{t}^e} = 0}} \beta_i,$$

where the β_i are defined as in (II.5). The following lemma is essentially [BGK, Lemma 5.1]:

Lemma II.5.1. *Let $\xi : U(\mathfrak{p}_0) \to S(\mathfrak{t})$ be the homomorphism induced by the natural projection $\mathfrak{p}_0 \twoheadrightarrow \mathfrak{t}$. Let $S_{-\eta} : S(\mathfrak{t}) \to S(\mathfrak{t}), x \mapsto x - \eta(x)$ $(x \in \mathfrak{t})$. Then the map $\xi_{-\eta} := S_{-\eta} \circ \xi$ defines an algebra isomorphism $U(\mathfrak{g}_0, e) \xrightarrow{\sim} S(\mathfrak{t})^{W_0}$.*

Since $S(\mathfrak{t})^{W_0}$ is a free polynomial algebra, we have by the isomorphism from Lemma II.5.1 that $\mathcal{L} = \mathfrak{t}^*/W_0 = \text{m-Spec}(S(\mathfrak{t})^{W_0})$. Moreover, in this case there is a conjecture in [BGK], which describes the subset $\mathcal{L}^+$ of $\mathcal{L}$ in combinatorial terms. Recall we have fixed a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{t}$, and Φ^+ is the corresponding set of positive roots. Let $\Phi_0^+ = \{\alpha \in \Phi^+ \mid \mathfrak{g}_\alpha \subseteq \mathfrak{g}_0\}$ denote the resulting system of positive roots for the Levi subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$. Also recall for $\lambda \in \mathfrak{t}^*$ that $L(\lambda)$ denotes the irreducible $U(\mathfrak{g})$-module of highest weight $\lambda - \rho$.

Conjecture II.5.2. *For $\Lambda \in \mathcal{L}$ pick $\lambda \in \Lambda$ such that $(\lambda, \alpha^\vee) \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi_0^+$. Then $L(\Lambda, e)$ is finite dimensional if and only if $\operatorname{Ann}_{U(\mathfrak{g})} L(\lambda) = \overline{G.e}$.*

This conjecture has recently been proved by Losev in [Lo3]. This means that in principle it is possible to determine the subset $\mathcal{L}^+$ of $\mathcal{L}$, applying the existing theory of primitive ideals to determine when $\operatorname{Ann}_{U(\mathfrak{g},e)} L(\lambda) = \overline{G.e}$. In practice this is not very explicit, and there is hope for a much more concrete description of $\mathcal{L}^+$ in many cases. In [BGK] this is been done in type A for finite W-algebras associated to arbitrary nilpotent elements of $\mathfrak{gl}_n$. The rest of this thesis is devoted to this problem in the rectangular cases in types B, C, and D.

CHAPTER III

RECTANGULAR FINITE W-ALGEBRAS AND TWISTED YANGIANS

In this chapter we will prove that rectangular finite W-algebras are quotients of twisted Yangians.

III.1 Statement of the Main Result

We begin by fixing explicit matrix realizations for the classical Lie algebras. For any integer $n \geq 1$, we will label the rows and columns of $n \times n$ matrices by the ordered index set

$$\mathcal{I}_n = \{-n+1, -n+3, \dots, n-1\}.$$

Let $\mathfrak{g}_n = \mathfrak{gl}_n$ with standard basis given by the matrix units $\{e_{i,j} \mid i, j \in \mathcal{I}_n\}$. Let J_n^+ be the $n \times n$ matrix with (i,j) entry equal to $\delta_{i,-j}$, and set

$$\mathfrak{g}_n^+ = \mathfrak{so}_n = \{x \in \mathfrak{g}_n \mid x^T J_n^+ + J_n^+ x = 0\},$$

where x^T denotes the usual transpose of an $n \times n$ matrix. Assuming in addition that n is even, let J_n^- be the $n \times n$ matrix with (i,j) entry equal to $\delta_{i,-j}$ if $j > 0$ and $-\delta_{i,-j}$ if $j < 0$, and set

$$\mathfrak{g}_n^- = \mathfrak{sp}_n = \{x \in \mathfrak{g}_n \mid x^T J_n^- + J_n^- x = 0\}. \tag{III.1}$$

We adopt the following conventions regarding signs. For $i \in \mathcal{I}_n$, define $\hat{\imath} \in \mathbb{Z}/2\mathbb{Z}$ by

$$\hat{\imath} = \begin{cases} 0 & \text{if } i \geq 0; \\ 1 & \text{if } i < 0. \end{cases} \tag{III.2}$$

and define $\tilde{\imath} \in \mathbb{Z}/2\mathbb{Z}$ by

$$\tilde{\imath} = \widehat{-\imath} = \begin{cases} 0 & \text{if } i \leq 0; \\ 1 & \text{if } i > 0. \end{cases} \tag{III.3}$$

We will often identify a sign $\epsilon = \pm$ with the integer ± 1 when writing formulae. For example, $\epsilon^{\hat{\imath}}$ denotes 1 if $\epsilon = +$ or $\hat{\imath} = 0$, and it denotes -1 if $\epsilon = -$ and $\hat{\imath} = 1$. With this notation, $\mathfrak{g}_n^\epsilon$ is spanned by the matrices $\{e_{i,j} - \epsilon^{\hat{\imath}+\hat{\jmath}} e_{-j,-i} \mid i, j \in \mathcal{I}_n\}$.

For the remainder of this work, we fix integers $n, l \geq 1$ and signs $\epsilon, \phi \in \{\pm\}$, assuming that $\phi = \epsilon$ if l is odd, $\phi = -\epsilon$ if l is even, and $\phi = +$ if n is odd. We will show that the finite W-algebra $U(\mathfrak{g}_{nl}^\epsilon, e)$ constructed from a nilpotent matrix e of Jordan type (l^n) in the Lie algebra $\mathfrak{g}_{nl}^\epsilon$ is the level l quotient of the twisted Yangian Y_n^ϕ associated to the Lie algebra $\mathfrak{g}_n^\phi$.

To formulate the main result precisely, first consider the finite W-algebra side. Let $\mathfrak{g} = \mathfrak{g}_{nl}^\epsilon$ and $f_{a,b} = e_{a,b} - \epsilon^{\hat{a}+\hat{b}} e_{-b,-a}$, so $\mathfrak{g}$ is spanned by the matrices $\{f_{a,b} \mid a, b \in \mathcal{I}_{nl}\}$. Up to isomorphism, the finite W-algebra to be defined shortly only depends on $\mathfrak{g}$ and the Jordan type (l^n). However we need to fix an explicit choice of coordinates so that we can be absolutely explicit about the isomorphism in Theorem III.1.1 below. We do this by introducing an $n \times l$ rectangular array of boxes, labeling rows in order from top to bottom by the index set $\mathcal{I}_n$ and columns in order from left to right by the index set $\mathcal{I}_l$. Also label the individual boxes in the array with the elements of the set $\mathcal{I}_{nl}$. For $a \in \mathcal{I}_{nl}$ we let $\mathrm{row}(a)$ and $\mathrm{col}(a)$ denote the row and column numbers of the box in which a appears. We require that the boxes are labeled skew-symmetrically in the sense that $\mathrm{row}(-a) = -\mathrm{row}(a)$ and $\mathrm{col}(-a) = -\mathrm{col}(a)$. If $\epsilon = -$ we require in addition that $a > 0$ either if $\mathrm{col}(a) > 0$ or if $\mathrm{col}(a) = 0$ and $\mathrm{row}(a) > 0$; this additional restriction streamlines some of the signs appearing in formulae below. For example, if $n = 3, l = 2$ and $\epsilon = -, \phi = +$, one could pick the labeling

-5	1
-3	3
-1	5

and get that $\mathrm{row}(1) = -2$ and $\mathrm{col}(1) = 1$. We remark that the above arrays are a special case of the *pyramids* introduced by Elashvili and Kac in [EK]; see also [BG].

Having made these choices, we let $e \in \mathfrak{g}$ denote the following nilpotent matrix of Jordan type (l^n):

$$e = \sum_{\substack{a,b\in\mathcal{I}_{nl} \\ \mathrm{row}(a)=\mathrm{row}(b) \\ \mathrm{col}(a)+2=\mathrm{col}(b)\geq 2}} f_{a,b} + \sum_{\substack{a,b\in\mathcal{I}_{nl} \\ \mathrm{row}(a)=\mathrm{row}(b)>0 \\ \mathrm{col}(a)+2=\mathrm{col}(b)=1}} f_{a,b} + \sum_{\substack{a,b\in\mathcal{I}_{nl} \\ \mathrm{row}(a)=\mathrm{row}(b)=0 \\ \mathrm{col}(a)+2=\mathrm{col}(b)=1}} \tfrac{1}{2} f_{a,b}. \tag{III.4}$$

In the above example, $e = f_{-1,5} + \frac{1}{2} f_{-3,3} = e_{-1,5} + e_{-5,1} + e_{-3,3}$. Also define an even grading

$$\mathfrak{g} = \bigoplus_{r\in\mathbb{Z}} \mathfrak{g}(r) \tag{III.5}$$

with $e \in \mathfrak{g}(2)$ by declaring that $\deg(f_{a,b}) = \mathrm{col}(b) - \mathrm{col}(a)$. Note this grading coincides with the grading obtained by embedding e into the $\mathfrak{sl}_2$-triple (e, h, f) where

$$h = \sum_{a\in\mathcal{I}_{nl}} \mathrm{col}(-a) e_{a,a} \tag{III.6}$$

and considering the $\mathrm{ad}\, h$-eigenspace decomposition of $\mathfrak{g}$. Let $\mathfrak{p} = \bigoplus_{r\geq 0} \mathfrak{g}(r)$ and $\mathfrak{m} = \bigoplus_{r<0} \mathfrak{g}(r)$. For the non-degenerate symmetric invariant bilinear for on $\mathfrak{g}$ from §II.1 we use the form $(x, y) = \frac{1}{2}\mathrm{tr}(xy)$. Define $\chi : \mathfrak{m} \to \mathbb{C}$ by $x \mapsto (e, x)$. An explicit calculation using the formula for the nilpotent matrix e recorded above shows that

$$\chi(f_{a,b}) = -\epsilon^{\hat{a}+\hat{b}}\chi(f_{-b,-a}) = 1 \tag{III.7}$$

if $\mathrm{row}(a) = \mathrm{row}(b), \mathrm{col}(a) = \mathrm{col}(b) + 2$ and either $\mathrm{col}(a) \geq 2$ or $\mathrm{col}(a) = 1, \mathrm{row}(a) \geq 0$; all other $f_{a,b} \in \mathfrak{m}$ satisfy $\chi(f_{a,b}) = 0$. Recall from §II.1 that

$$U(\mathfrak{g}, e) = \{u \in U(\mathfrak{p}) \mid \mathrm{pr}([x, u]) = 0 \text{ for all } x \in \mathfrak{m}\}$$

where $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{p})$ is projection along the the decomposition from (II.2).

To make the connection between $U(\mathfrak{g}, e)$ and the twisted Yangians, we exploit a shifted version of the Miura transform, which we define as follows. Let $\mathfrak{h} = \mathfrak{g}(0)$ be the Levi factor of $\mathfrak{p}$ coming from the grading. It is helpful to bear in mind that there is an isomorphism

$$\mathfrak{h} \cong \begin{cases} \mathfrak{g}_n^{\oplus m} & \text{if } l = 2m; \\ \mathfrak{g}_n^{\epsilon} \oplus \mathfrak{g}_n^{\oplus m} & \text{if } l = 2m+1. \end{cases} \tag{III.8}$$

Although we never need this explicitly, we note for completeness that this isomorphism maps $f_{a,b} \in \mathfrak{h}$ to $f_{\mathrm{row}(a),\mathrm{row}(b)} \in \mathfrak{g}_n^{\epsilon}$ if $\mathrm{col}(a) = \mathrm{col}(b) = 0$ or to $e_{\mathrm{row}(a),\mathrm{row}(b)}$ in the $\lceil \frac{\mathrm{col}(a)}{2} \rceil$th copy of $\mathfrak{g}_n$ if $\mathrm{col}(a) = \mathrm{col}(b) > 0$. For $q \in \mathcal{I}_l$, let

$$\rho_q = \begin{cases} \frac{nq-\epsilon}{2} & \text{if } q > 0; \\ \frac{nq+\epsilon}{2} & \text{if } q < 0; \\ 0 & \text{if } q = 0. \end{cases} \tag{III.9}$$

Let η be the automorphism of $U(\mathfrak{h})$ defined on generators by $\eta(f_{a,b}) = f_{a,b} - \delta_{a,b}\rho_{\mathrm{col}(a)}$.

Let $\xi : U(\mathfrak{p}) \twoheadrightarrow U(\mathfrak{h})$ be the algebra homomorphism induced by the natural projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$. The *Miura transform* $\mu : U(\mathfrak{p}) \to U(\mathfrak{h})$ is the composite map

$$\mu = \eta \circ \xi. \tag{III.10}$$

By [Ly, §2.3] (or Theorem III.3.4 below) the restriction of μ to $U(\mathfrak{g}, e)$ is injective.

Now we turn our attention to the twisted Yangian Y_n^ϕ, recalling that $\phi = -\epsilon$ if l is even and $\phi = \epsilon$ if l is odd. By definition, Y_n^ϕ is a subalgebra of the Yangian Y_n. The Yangian Y_n is a Hopf algebra over $\mathbb{C}$ with countably many generators $\{T_{i,j}^{(r)} \mid i, j \in \mathcal{I}_n, r \in \mathbb{Z}_{>0}\}$. To give the defining relations and other data for the Yangian it is convenient to use the power series

$$T_{i,j}(u) = \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} \in Y_n[[u^{-1}]]$$

where $T_{i,j}^{(0)} = \delta_{i,j}$. Now the defining relations are

$$(u - v)[T_{i,j}(u), T_{k,l}(u)] = T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u).$$

This and subsequent formulae involving generating functions should be interpreted by equating coefficients of the indeterminates u and v on both sides of equations, as discussed in detail in [MNO, §1]. For example, the comultiplication $\Delta : Y_n \to Y_n \otimes Y_n$ making Y_n into a Hopf algebra is defined by the formula

$$\Delta(T_{i,j}(u)) = \sum_{k \in \mathcal{I}_n} T_{i,k}(u) \otimes T_{k,j}(u). \tag{III.11}$$

By [MNO, §3.4], there exists an automorphism $\tau : Y_n \to Y_n$ of order 2 defined by

$$\tau(T_{i,j}(u)) = \phi^{\hat{\imath}+\hat{\jmath}} T_{-j,-i}(-u).$$

We define the twisted Yangian Y_n^ϕ to be the subalgebra of Y_n generated by the elements $\{S_{i,j}^{(r)} \mid i,j \in \mathcal{I}_n, r \in \mathbb{Z}_{>0}\}$ coming from the expansion

$$S_{i,j}(u) = \sum_{r \geq 0} S_{i,j}^{(r)} u^{-r} = \sum_{k \in \mathcal{I}_n} \tau(T_{i,k}(u)) T_{k,j}(u) \in Y_n[[u^{-1}]]. \tag{III.12}$$

This is not the same embedding of Y_n^ϕ into Y_n as used in [MNO, §3]: we have twisted the embedding there by the automorphism τ. The relations for the twisted Yangian are given by

$$\begin{aligned}(u^2 - v^2)[S_{i,j}(u), S_{k,l}(v)] = (u+v)(S_{k,j}(u)S_{i,l}(v) - S_{k,j}(v)S_{i,l}(u)) - \\ (u-v)(\phi^{\hat{k}+\hat{\jmath}} S_{i,-k}(u) S_{-j,l}(v) - \phi^{\hat{k}+\hat{l}} S_{k,-i}(v) S_{-l,j}(u)) + \\ \phi^{\hat{\imath}+\hat{\jmath}} S_{k,-i}(u) S_{-j,l}(v) - \phi^{\hat{\imath}+\hat{\jmath}} S_{k,-i}(v) S_{-j,l}(u)\end{aligned} \tag{III.13}$$

and

$$\phi^{\hat{\imath}+\hat{\jmath}} S_{-j,-i}(-u) = S_{i,j}(u) + \phi \frac{S_{i,j}(u) - S_{i,j}(-u)}{2u}. \tag{III.14}$$

Because of the fact that τ is a coalgebra antiautomorphism of Y_n, we get from [MNO, §4.17] that the restriction of Δ to Y_n^ϕ has image contained in $Y_n^\phi \otimes Y_n$ and

$$\Delta(S_{i,j}(u)) = \sum_{h,k \in \mathcal{I}_n} S_{h,k}(u) \otimes \tau(T_{i,h}(u)) T_{k,j}(u). \tag{III.15}$$

We let $\Delta^{(m)} : Y_n \to Y_n^{\otimes(m+1)}$ denote the mth iterated comultiplication. The preceding formula shows that it maps Y_n^ϕ into $Y_n^\phi \otimes Y_n^{\otimes m}$.

By [MNO, §1.16] there is an evaluation homomorphism $Y_n \to U(\mathfrak{g}_n)$. In view of this and (III.8), we obtain for every $0 < p \in \mathcal{I}_l$ a homomorphism

$$\mathrm{ev}_p : Y_n \to U(\mathfrak{h}), \qquad T_{i,j}(u) \mapsto \delta_{i,j} + u^{-1} f_{a,b}, \tag{III.16}$$

where $a, b \in \mathcal{I}_{nl}$ are defined from $\mathrm{row}(a) = i, \mathrm{row}(b) = j$ and $\mathrm{col}(a) = \mathrm{col}(b) = p$. The image of this map is contained in the subalgebra of $U(\mathfrak{h})$ generated by the $\lceil p/2 \rceil$th copy of $\mathfrak{g}_n$ from the decomposition (III.8). There is also an evaluation homomorphism $Y_n^\phi \to U(\mathfrak{g}_n^\phi)$ defined in [MNO, §3.11]. If we assume that l is odd (so $\epsilon = \phi$), we can therefore define another homomorphism

$$\mathrm{ev}_0 : Y_n^\phi \to U(\mathfrak{h}), \qquad S_{i,j}(u) \mapsto \delta_{i,j} + (u + \tfrac{\phi}{2})^{-1} f_{a,b}, \tag{III.17}$$

where $\mathrm{row}(a) = i, \mathrm{row}(b) = j$ and $\mathrm{col}(a) = \mathrm{col}(b) = 0$; if $\epsilon = -$ this depends on our convention for labeling boxes as specified above. The image of this map is contained in the subalgebra of $U(\mathfrak{h})$ generated by the subalgebra $\mathfrak{g}_n^\epsilon$ in the decomposition (III.8). Putting all these things together, we deduce that there is a homomorphism

$$\kappa_l : Y_n^\phi \to U(\mathfrak{h})$$

defined by

$$\kappa_l = \begin{cases} \mathrm{ev}_1 \bar{\otimes} \mathrm{ev}_3 \bar{\otimes} \cdots \bar{\otimes} \mathrm{ev}_{l-1} \circ \Delta^{(m)} & \text{if } l = 2m + 2; \\ \mathrm{ev}_0 \bar{\otimes} \mathrm{ev}_2 \bar{\otimes} \cdots \bar{\otimes} \mathrm{ev}_{l-1} \circ \Delta^{(m)} & \text{if } l = 2m + 1, \end{cases} \tag{III.18}$$

where $\bar{\otimes}$ indicates composition with the natural multiplication in $U(\mathfrak{h})$. We define the *twisted Yangian of level l* to be the image of this map. Now we are ready to state the main theorem of this chapter.

Theorem III.1.1. $\mu(U(\mathfrak{g}, e)) = \kappa_l(Y_n^\phi)$.

We will show moreover that the kernel of κ_l is generated by the elements

$$\begin{aligned} &\left\{ S_{i,j}^{(r)} \,\middle|\, i,j \in \mathcal{I}_n, r > l \right\} \quad \text{if } l \text{ is even;} \\ &\left\{ S_{i,j}^{(r)} + \tfrac{\phi}{2} S_{i,j}^{(r-1)} \,\middle|\, i,j \in \mathcal{I}_n, r > l \right\} \quad \text{if } l \text{ is odd.} \end{aligned} \tag{III.19}$$

Since $U(\mathfrak{g}, e) \cong \mu(U(\mathfrak{g}, e))$ by injectivity of the Miura transform, and we have a full set of relations between the generators $S_{i,j}^{(r)}$ of Y_n^ϕ, this means that we have found a full set of generators and relations for the finite W-algebra $U(\mathfrak{g}, e)$.

The key step in our proof of Theorem III.1.1 is a remarkable explicit formula for the generators of $U(\mathfrak{g}, e)$ corresponding to the elements $S_{i,j}^{(r)} \in Y_n^\phi$. In the remainder of this section we want to explain this formula. Given $i, j \in \mathcal{I}_n$ and $p, q \in \mathcal{I}_l$, let a, b be the elements of $\mathcal{I}_{nl}$ such that $\mathrm{col}(a) = p$, $\mathrm{col}(b) = q$, $\mathrm{row}(a) = i$, and $\mathrm{row}(b) = j$. Define a linear map $s_{i,j} : \mathfrak{g}_l \to \mathfrak{g}$ by setting

$$s_{i,j}(e_{p,q}) = \phi^{\hat{\imath}\hat{p} + \hat{\jmath}\hat{q}} f_{a,b}. \tag{III.20}$$

Let M_n denote the algebra of $n \times n$ matrices over $\mathbb{C}$, with rows and columns labeled by the index set $\mathcal{I}_n$ as usual, and let $T(\mathfrak{g}_l)$ be the tensor algebra on the vector space $\mathfrak{g}_l$. Let

$$s : T(\mathfrak{g}_l) \to M_n \otimes U(\mathfrak{g}) \tag{III.21}$$

be the algebra homomorphism that maps a generator $x \in \mathfrak{g}_l$ to $\sum_{i,j \in \mathcal{I}_n} e_{i,j} \otimes s_{i,j}(x)$. This in turn defines linear maps

$$s_{i,j} : T(\mathfrak{g}_l) \to U(\mathfrak{g}), \tag{III.22}$$

such that

$$s(x) = \sum_{i,j \in \mathcal{I}_n} e_{i,j} \otimes s_{i,j}(x)$$

for every $x \in T(\mathfrak{g}_l)$. Note for any $x, y \in T(\mathfrak{g}_l)$ that

$$s_{i,j}(xy) = \sum_{k \in \mathcal{I}_n} s_{i,k}(x) s_{k,j}(y) \tag{III.23}$$

and also $s_{i,j}(1) = \delta_{i,j}$.

If A is an $l \times l$ matrix with entries in some ring, we define its *row determinant* $\operatorname{rdet} A$ to be the usual Laplace expansion of determinant, but keeping the (not necessarily commuting) monomials that arise in *row order*; see e.g. [BK2, (12.5)]. For $q \in \mathcal{I}_l$ and an indeterminate u, let

$$u_q = u + e_{q,q} + \rho_q \in T(\mathfrak{g}_l)[u],$$

recalling the definition of ρ_q from (III.9). Define $\Omega(u)$ to be the $l \times l$ matrix with entries in $T(\mathfrak{gl}_l)[u]$ whose (p,q) entry for $p, q \in \mathcal{I}_l$ is equal to

$$\Omega(u)_{p,q} = \begin{cases} e_{p,q} & \text{if } p < q; \\ u_q & \text{if } p = q; \\ -1 & \text{if } p = q + 2 < 0; \\ -\phi & \text{if } p = q + 2 = 0; \\ 1 & \text{if } p = q + 2 > 0; \\ 0 & \text{if } p > q + 2. \end{cases} \tag{III.24}$$

For example, if $l = 4$ then

$$\Omega(u) = \begin{pmatrix} u_{-3} & e_{-3,-1} & e_{-3,1} & e_{-3,3} \\ -1 & u_{-1} & e_{-1,1} & e_{-1,3} \\ 0 & 1 & u_1 & e_{1,3} \\ 0 & 0 & 1 & u_3 \end{pmatrix}.$$

If l is odd we also need the $l \times l$ matrix $\bar{\Omega}(u)$ defined by

$$\bar{\Omega}(u)_{p,q} = \begin{cases} \Omega(u)_{p,q} & \text{if } p \neq 0 \text{ or } q \neq 0; \\ e_{0,0} & \text{if } p = q = 0. \end{cases} \tag{III.25}$$

For example, if $l = 5$ then

$$\Omega(u) = \begin{pmatrix} u_{-4} & e_{-4,-2} & e_{-4,0} & e_{-4,2} & e_{-4,4} \\ -1 & u_{-2} & e_{-2,0} & e_{-2,2} & e_{-2,4} \\ 0 & -\phi & u_0 & e_{0,2} & e_{0,4} \\ 0 & 0 & 1 & u_2 & e_{2,4} \\ 0 & 0 & 0 & 1 & u_4 \end{pmatrix},$$

$$\bar{\Omega}(u) = \begin{pmatrix} u_{-4} & e_{-4,-2} & e_{-4,0} & e_{-4,2} & e_{-4,4} \\ -1 & u_{-2} & e_{-2,0} & e_{-2,2} & e_{-2,4} \\ 0 & -\phi & e_{0,0} & e_{0,2} & e_{0,4} \\ 0 & 0 & 1 & u_2 & e_{2,4} \\ 0 & 0 & 0 & 1 & u_4 \end{pmatrix}.$$

Then we let

$$\omega(u) = \sum_{r=-\infty}^{l} \omega_{l-r} u^r = \begin{cases} \operatorname{rdet} \Omega(u) & \text{if } l \text{ is even;} \\ \operatorname{rdet} \Omega(u) + \sum_{r=1}^{\infty} (-2\phi u)^{-r} \operatorname{rdet} \bar{\Omega}(u) & \text{if } l \text{ is odd.} \end{cases} \tag{III.26}$$

This defines elements $\omega_r \in T(\mathfrak{g}_l)$, hence elements $s_{i,j}(\omega_r) \in U(\mathfrak{g})$ for $i, j \in \mathcal{I}_n$ and $r \geq 1$. It is obvious from the definition that each $s_{i,j}(\omega_r)$ actually belongs to $U(\mathfrak{p})$.

Theorem III.1.2. *The elements $\{s_{i,j}(\omega_r) \mid i, j \in \mathcal{I}_n, r \geq 1\}$ generate the subalgebra $U(\mathfrak{g}, e)$. Moreover, $\mu(s_{i,j}(\omega_r)) = \kappa_l(S_{i,j}^{(r)})$.*

The hardest part of the proof is to show that each $s_{i,j}(\omega_r)$ belongs to $U(\mathfrak{g}, e)$. This is established by a lengthy calculation which we postpone until §III.4. In §III.2 we study the twisted Yangian of level l, in particular proving a PBW theorem for this algebra and computing the kernel of κ_l as mentioned above. We also check that $\mu(s_{i,j}(\omega_r)) = \kappa_l(S_{i,j}^{(r)})$. Then in §III.3 we complete the proofs of Theorems III.1.1 and III.1.2. At the same time we obtain a direct proof of the injectivity of the Miura transform in this case.

III.2 Basis Theorem for the Twisted Yangian of Level l

Continuing with notation from the previous section, we begin this section by giving a different description of the map $\kappa_l : Y_n^\phi \to U(\mathfrak{h})$ from (III.18). Let

$$T(u) = \sum_{i,j \in \mathcal{I}_n} e_{i,j} \otimes T_{i,j}(u) \in M_n \otimes Y_n[[u^{-1}]],$$

$$S(u) = \sum_{i,j \in \mathcal{I}_n} e_{i,j} \otimes S_{i,j}(u) \in M_n \otimes Y_n^\phi[[u^{-1}]].$$

For a linear map $f : V \longrightarrow W$, we use the same notation f for the induced map $\mathrm{id} \otimes f : M_n \otimes V \longrightarrow M_n \otimes W$. Thinking of elements of $M_n \otimes V$ (resp. $M_n \otimes W$) as $n \times n$ matrices with entries in V (resp. W), this is just the linear map obtained by applying f simultaneously to all matrix entries. We extend (III.16) by defining a homomorphism $\mathrm{ev}_{-p} : Y_n \longrightarrow U(\mathfrak{h})$ for $0 < p \in \mathcal{I}_l$ by setting

$$\mathrm{ev}_{-p} = \mathrm{ev}_p \circ \tau. \tag{III.27}$$

Since the images of ev_p and ev_q commute for $p \neq \pm q$, it is then the case by (III.18), (III.11), (III.12) and (III.15) that

$$\kappa_l(S(u)) = \begin{cases} \mathrm{ev}_{1-l}(T(u)) \cdots \mathrm{ev}_{-1}(T(u))\, \mathrm{ev}_1(T(u)) \cdots \mathrm{ev}_{l-1}(T(u)) & \text{if } l \text{ is even;} \\ \mathrm{ev}_{1-l}(T(u)) \cdots \mathrm{ev}_{-2}(T(u))\, \mathrm{ev}_0(S(u))\, \mathrm{ev}_2(T(u)) \cdots \mathrm{ev}_{l-1}(T(u)) & \text{if } l \text{ is odd,} \end{cases} \tag{III.28}$$

where the product on the right hand side is in the algebra $M_n \otimes U(\mathfrak{h})[[u^{-1}]]$.

For any $0 \neq p \in \mathcal{I}_l$, (III.27), (III.16), and the labeling convention for boxes implies that

$$\mathrm{ev}_p(T_{i,j}(u)) = \delta_{i,j} + u^{-1} \phi^{\hat{p}(\hat{\imath}+\hat{\jmath})} f_{a,b},$$

where $a, b \in \mathcal{I}_{nl}$ satisfy $\mathrm{row}(a) = i, \mathrm{row}(b) = j$ and $\mathrm{col}(a) = \mathrm{col}(b) = p$. Hence in the notation (III.20) we have that

$$\mathrm{ev}_p(T_{i,j}(u)) = \delta_{i,j} + u^{-1} s_{i,j}(e_{p,p}).$$

Also (III.17) is equivalent to

$$\mathrm{ev}_0(S_{i,j}(u)) = \delta_{i,j} + (u + \tfrac{\phi}{2})^{-1} s_{i,j}(e_{0,0}) = \delta_{i,j} + \sum_{r=0}^{\infty} (-2\phi)^{-r} u^{-1-r} s_{i,j}(e_{0,0}).$$

Using the more sophisticated notation (III.21), we deduce that

$$u\,\mathrm{ev}_p(T(u)) = s(u + e_{p,p}),$$
$$u\,\mathrm{ev}_0(S(u)) = s(u + e_{0,0}) + \sum_{r=1}^{\infty} (-2\phi u)^{-r} s(e_{0,0}).$$

Hence (III.28) is equivalent to the equation

$$u^l \kappa_l(S(u)) = s((u + e_{1-l,1-l}) \cdots (u + e_{-1,-1})(u + e_{1,1}) \cdots (u + e_{l-1,l-1})) \qquad \text{(III.29)}$$

if l is even and

$$u^l \kappa_l(S(u)) = s((u + e_{1-l,1-l}) \cdots (u + e_{-2,-2})(u + e_{0,0})(u + e_{2,2}) \cdots (u + e_{l-1,l-1})) + \sum_{r=1}^{\infty} (-2\phi u)^{-r} s((u + e_{1-l,1-l}) \cdots (u + e_{-2,-2}) e_{0,0} (u + e_{2,2}) \cdots (u + e_{l-1,l-1})) \qquad \text{(III.30)}$$

if l is odd. Equating u^{l-r}-coefficients gives that

$$\kappa_l(S_{i,j}^{(r)}) = \sum_{\substack{p_1,\dots,p_r \in \mathcal{I}_l \\ p_1 < \cdots < p_r}} s_{i,j}(e_{p_1,p_1} \cdots e_{p_r,p_r}) + \sum_{t=1}^{r-1} (-2\phi)^{t-r} \sum_{\substack{p_1,\dots,p_t \in \mathcal{I}_l \\ p_1 < \cdots < p_t \\ 0 \in \{p_1,\dots,p_t\}}} s_{i,j}(e_{p_1,p_1} \cdots e_{p_t,p_t}), \qquad \text{(III.31)}$$

the last term in this formula being zero automatically if l is even. The following theorem verifies the second statement of Theorem III.1.2.

Theorem III.2.1. $u^l \kappa_l(S(u)) = \mu(s(\omega(u)))$.

Proof. The Miura transform (III.10) satisfies $\mu(s(u_p)) = s(u + e_{p,p})$ and $\mu(s(e_{p,q})) = 0$ if $p < q$. So recalling the matrices $\Omega(u)$ and $\bar{\Omega}(u)$ from (III.24) and (III.25) we get that

$$\mu(s(\mathrm{rdet}\,\Omega(u))) = s((u + e_{1-l,1-l}) \cdots (u + e_{l-1,l-1})),$$

and

$$\mu(s(\mathrm{rdet}\,\bar{\Omega}(u))) = s((u + e_{1-l,1-l}) \cdots (u + e_{-2,-2})e_{0,0}(u + e_{2,2}) \cdots (u + e_{l-1,l-1})).$$

The theorem follows on comparing (III.26), (III.29) and (III.30). □

The goal now is to prove a PBW theorem for the twisted Yangian of level l, $\kappa_l(Y_n^\phi)$. We will need the following elementary lemma, which is established in the proof of [BK1, Theorem 3.1].

Lemma III.2.2. *Let X be the variety of tuples $(A_{1-l}, A_{3-l}, \ldots, A_{l-1})$ of $n \times n$ matrices. Let $x_{i,j}^{[r]} \in \mathbb{C}[X]$ be the coordinate function picking out the (i,j) entry of A_r. Let Y be the variety of tuples $(B_1, \ldots, B_l)$ of $n \times n$ matrices. Let $y_{i,j}^{[r]} \in \mathbb{C}[Y]$ be the coordinate function picking out the (i,j) entry of B_r. Define*

$$\theta : X \to Y, \quad (A_{1-l}, \ldots, A_{l-1}) \mapsto (B_1, \ldots, B_l)$$

where

$$B_r = \sum_{\substack{p_1, \ldots, p_r \in \mathcal{I}_l \\ p_1 < \cdots < p_r}} A_{p_1} A_{p_2} \ldots A_{p_r},$$

i.e. B_r *is the* r*th elementary symmetric function in the matrices* $(A_{1-l}, \dots, A_{l-1})$. *Then the comorphism* $\theta^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ *satisfies*

$$\theta^*(y_{i,j}^{[r]}) = \sum_{\substack{i_1,\dots,i_{r-1}\in\mathcal{I}_n \\ p_1,\dots,p_r\in\mathcal{I}_l \\ p_1<\dots<p_r}} x_{i,i_1}^{[p_1]} x_{i_1,i_2}^{[p_2]} \dots x_{i_{r-1},j}^{[p_r]}$$

Moreover the derivative $d\theta_x : T_x(X) \to T_{\theta(x)}(Y)$ *is an isomorphism for any point* $x = (c_{1-l}I_n, \dots, c_{l-1}I_n)$ *such that* $c_{1-l}, \dots, c_{l-1}$ *are pairwise distinct scalars.*

We observe by (III.31) for $i, j \in \mathcal{I}_n$ that

$$\begin{cases} \kappa_l(S_{i,j}^{(r)}) = 0 & \text{if } l \text{ is even and } r > l; \\ \kappa_l(S_{i,j}^{(r)}) = -\frac{\phi}{2}\kappa_l(S_{i,j}^{(r-1)}) & \text{if } l \text{ is odd and } r > l. \end{cases} \tag{III.32}$$

Following [MNO, §3.14], we say (i, j, r) is *admissible* if $i, j \in \mathcal{I}_n$, $1 \le r \le l$, and

$$\begin{cases} i + j \le 0 & \text{if } \phi = + \text{ and } r \text{ is even;} \\ i + j < 0 & \text{if } \phi = + \text{ and } r \text{ is odd;} \\ i + j < 0 & \text{if } \phi = - \text{ and } r \text{ is even;} \\ i + j \le 0 & \text{if } \phi = - \text{ and } r \text{ is odd.} \end{cases}$$

Now consider the standard filtration on $U(\mathfrak{h})$ defined by declaring that each $x \in \mathfrak{h}$ is in degree 1. This induces a filtration on the subalgebra $\kappa_l(Y_n^\phi)$ so that $\operatorname{gr} \kappa_l(Y_n^\phi)$ is a subalgebra of $\operatorname{gr} U(\mathfrak{h})$. Note by (III.31) that each $\kappa_l(S_{i,j}^{(r)})$ belongs to the filtered degree r component of $U(\mathfrak{h})$.

Theorem III.2.3. *The elements*

$$\left\{ \operatorname{gr}_r \kappa_l(S_{i,j}^{(r)}) \,\middle|\, (i,j,r) \textit{ is admissible} \right\}$$

are algebraically independent generators for the commutative algebra $\operatorname{gr} \kappa_l(Y_n^\phi)$*. Hence the monomials in the elements*

$$\left\{ \kappa_l(S_{i,j}^{(r)}) \,\middle|\, (i,j,r) \textit{ is admissible} \right\}$$

taken in some fixed order form a basis for $\kappa_l(Y_n^\phi)$*.*

Proof. By (III.14) and (III.32) monomials in the elements

$$\left\{ \operatorname{gr}_r \kappa_l(S_{i,j}^{(r)}) \,\middle|\, (i,j,r) \text{ is admissible} \right\}$$

taken in some fixed order generate $\operatorname{gr} \kappa_l(Y_n^\phi)$, so it suffices to prove they are algebraically independent. Let notation be as in Lemma III.2.2. Let V be the closed subspace of X defined by the ideal I generated by

$$\left\{ x_{i,j}^{[r]} + \phi^{\hat{\imath}+\hat{\jmath}} x_{-j,-i}^{[-r]} \,\middle|\, i,j \in \mathcal{I}_n, r \in \mathcal{I}_l \right\}.$$

As $\mathfrak{h}$ is the vector space spanned by $\{s_{i,j}(e_{p,p}) \mid i,j \in \mathcal{I}_n, p \in \mathcal{I}_l\}$ subject only to the relations $s_{i,j}(e_{p,p}) = -\phi^{\hat{\imath}+\hat{\jmath}} s_{-j,-i}(e_{-p,-p})$, we can identify $\operatorname{gr} U(\mathfrak{h})$ with $\mathbb{C}[V]$ by declaring that $\operatorname{gr}_1 s_{i,j}(e_{p,p}) = x_{i,j}^{[p]} + I$.

Let W be the closed subspace of Y defined by the ideal J generated by

$$\left\{ y_{i,j}^{[r]} - (-1)^r \phi^{\hat{\imath}+\hat{\jmath}} y_{-j,-i}^{[r]} \,\middle|\, i,j \in \mathcal{I}_n, r = 1,\ldots,l \right\}.$$

We claim that $\theta(V) \subseteq W$, i.e. $\theta^*(J) \subseteq I$. To see this note that

$$\begin{aligned}
\theta^*(y_{i,j}^{[r]}) &= \sum_{\substack{i_1,\dots,i_{r-1}\in\mathcal{I}_n \\ p_1,\dots,p_r\in\mathcal{I}_l \\ p_1<\cdots<p_r}} x_{i,i_1}^{[p_1]} x_{i_1,i_2}^{[p_2]} \cdots x_{i_{r-1},j}^{[p_r]} \\
&\equiv (-1)^r \phi^{\hat{\imath}+\hat{\jmath}} \sum_{\substack{i_1,\dots,i_{r-1}\in\mathcal{I}_n \\ p_1,\dots,p_r\in\mathcal{I}_l \\ p_1<\cdots<p_r}} x_{-j,-i_{r-1}}^{[-p_r]} x_{-i_{r-1},-i_{r-2}}^{[-p_{r-1}]} \cdots x_{-i_1,-i}^{[-p_1]} \pmod{I} \\
&\equiv (-1)^r \phi^{\hat{\imath}+\hat{\jmath}} \theta^*(y_{-j,-i}^{[r]}) \pmod{I}.
\end{aligned}$$

Hence $\theta^*\left(y_{i,j}^{[r]} - (-1)^r \phi^{\hat{\imath}+\hat{\jmath}} y_{-j,-i}^{[r]}\right) \in I$.

Choose $x = (c_{1-l}I_n, \dots, c_{l-1}I_n) \in X$ so that $c_{1-l}, \dots, c_{l-1}$ are pairwise distinct and $c_i + c_{-i} = 0$. Then x belongs to V. Now apply Lemma III.2.2 to deduce that $d\theta_x : T_x(V) \to T_{\theta(x)}(W)$ is injective. An easy calculation shows that $\dim V = \dim W$, hence $d\theta_x : T_x(V) \to T_{\theta(x)}(W)$ is an isomorphism. By [Sp, Theorem 4.3.6(i)] this implies that $\theta : V \to W$ is a dominant morphism, so the comorphism $\theta^* : \mathbb{C}[W] \to \mathbb{C}[V] = \operatorname{gr} U(\mathfrak{h})$ is injective. As $\mathbb{C}[W]$ is freely generated by the elements $\left\{ y_{i,j}^{[r]} \,\middle|\, (i,j,r) \text{ is admissible} \right\}$, we deduce that the elements $\left\{ \theta^*(y_{i,j}^{[r]}) \,\middle|\, (i,j,r) \text{ is admissible} \right\}$ are algebraically independent too. It remains to observe by applying gr_r to (III.31) and using (III.23) that

$$\operatorname{gr}_r \kappa_l(S_{i,j}^{(r)}) = \sum_{\substack{i_1,\dots,i_{r-1}\in\mathcal{I}_n \\ p_1,\dots,p_r\in\mathcal{I}_l \\ p_1<\cdots<p_r}} x_{i,i_1}^{[p_1]} x_{i_1,i_2}^{[p_2]} \cdots x_{i_{r-1},j}^{[p_r]} = \theta^*(y_{i,j}^{[r]}).$$

□

Corollary III.2.4. *The elements*

$$\left\{ S_{i,j}^{(r)} \;\middle|\; i,j \in \mathcal{I}_n, r > l \right\} \qquad \textit{if } l \textit{ is even;}$$

$$\left\{ S_{i,j}^{(r)} + \tfrac{\phi}{2} S_{i,j}^{(r-1)} \;\middle|\; i,j \in \mathcal{I}_n, r > l \right\} \qquad \textit{if } l \textit{ is odd}$$

generate the kernel of κ_l.

Proof. Let I denote the two-sided ideal of Y_n^ϕ generated by the elements listed in Corollary III.2.4. Now κ_l induces a map $\bar{\kappa}_l : Y_n^\phi / I \to \kappa_l(Y_n^\phi)$. Since Y_n^ϕ / I is spanned by the set of all monomials in the elements $\left\{ S_{i,j}^{(r)} + I \;\middle|\; (i,j,r) \text{ is admissible} \right\}$ taken in some fixed order by [MNO, §3.14], and the images of these monomials are linearly independent in $\kappa_l(Y_n^\phi)$ by Theorem III.2.3, we deduce that $\bar{\kappa}_l$ is an isomorphism. □

We also obtain a new proof of the PBW theorem for twisted Yangians, different from the one in [MNO, §3].

Corollary III.2.5. *The set of all monomials in the elements*

$$\left\{ S_{i,j}^{(r)} \;\middle|\; (i,j,r) \textit{ is admissible} \right\}$$

taken in some fixed order forms a basis for Y_n^ϕ.

Proof. It is clear from (III.14) that such monomials span Y_n^ϕ. The fact that they are linearly independent follows from Theorem III.2.3 by taking sufficiently large l. □

III.3 Proof of the Isomorphism Theorem

In §III.4 below we will prove the following theorem:

Theorem III.3.1. *For* $i,j \in \mathcal{I}_n$ *and* $r \geq 1$, *the element* $s_{i,j}(\omega_r)$ *belongs to* $U(\mathfrak{g}, e)$.

In the remainder of this section we explain how to deduce the main results formulated in §III.1 from this theorem.

Recall the definition of the *good filtration* on $U(\mathfrak{g}, e)$ from §II.1 for which the associated graded algebra $\operatorname{gr} U(\mathfrak{g}, e)$ is identified with the graded subalgebra $U(\mathfrak{g}^e)$ of $U(\mathfrak{p})$. The element $s_{i,j}(\omega_{r+1})$ belongs to the subspace of elements of degree r in the good filtration, and we have that $s_{i,j}(\omega_{r+1}) \in U(\mathfrak{g}, e)$ by Theorem III.3.1. So it makes sense to define

$$f_{i,j;r} = \operatorname{gr}_r s_{i,j}(\omega_{r+1}) \in U(\mathfrak{g}^e) \tag{III.33}$$

for $r \geq 0$. Explicitly, we have that

$$f_{i,j;r} = \sum_{\substack{p,q \in \mathcal{I}_l \\ q-p=2r}} \alpha_{p,q} s_{i,j}(e_{p,q}) \tag{III.34}$$

where

$$\alpha_{p,q} = \begin{cases} 1 & \text{if } q < 0; \\ \phi(-1)^{q/2} & \text{if } p < 0 \text{ and } q \geq 0 \text{ and } l \text{ is odd;} \\ (-1)^{(q+1)/2} & \text{if } p < 0 \text{ and } q > 0 \text{ and } l \text{ is even;} \\ (-1)^{(q-p)/2} & \text{if } p \geq 0. \end{cases}$$

This formula comes from the fact that the monomial $e_{p,q}$ where $q - p = 2r$ occurs in $\operatorname{rdet} \Omega(u)$ as a coefficient of $u^{l-(r+1)}$ (and thus in ω_{r+1}) because of the element $\sigma = (p, q, q-2, \ldots, p+2)$ in the symmetric group on $\mathcal{I}_l$. Now $\alpha_{p,q} = \operatorname{sgn}(\sigma) * N$, where N is the number of -1's strictly below and strictly to the left of $e_{p,q}$ in the matrix $\Omega(u)$.

So (III.34) shows that each $f_{i,j;r} \in U(\mathfrak{g}^e)$ is an element of $\mathfrak{g}$, hence belongs to $\mathfrak{g}^e$.

Lemma III.3.2. *The elements* $\{f_{i,j;r} \,|\, (i,j,r+1)$ *is admissible*$\}$ *form a basis for* $\mathfrak{g}^e$.

Proof. We have already observed that each $f_{i,j;r}$ belongs to $\mathfrak{g}^e$. By [Ja, §3.2], the dimension of $\mathfrak{g}^e$ is

$$\begin{cases} n^2l/2 & \text{if } l \text{ is even;} \\ (n^2l - n\epsilon)/2 & \text{if } l \text{ is odd.} \end{cases}$$

An easy calculation shows that this is the same as the number of admissible triples. Now it just remains to show that the elements $f_{i,j;r}$ for all admissible $(i,j,r+1)$ are linearly independent. This is easy to see on noting that all these elements are non-zero, which follows by computing some explicit matrix coefficients. □

Theorem III.3.3. *The elements* $\{s_{i,j}(\omega_r) \mid (i,j,r)$ *is admissible*$\}$ *generate* $U(\mathfrak{g},e)$.

Proof. By (II.3), (III.33) and Lemma III.3.2, the elements

$$\{\mathrm{gr}_r\, s_{i,j}(\omega_{r+1}) \mid (i,j,r+1) \text{ is admissible}\}$$

generate $\mathrm{gr}\, U(\mathfrak{g},e)$, the associated graded algebra in the good filtration. The theorem follows from this statement by induction on the filtration. □

The main theorems of this chapter, Theorems III.1.1 and III.1.2, follow from Theorems III.3.1, III.3.3 and III.2.1. Finally we include a proof of the following theorem, which is originally due to [Ly, Corollary 2.3.2] in a more general setting.

Theorem III.3.4. *The Miura transform* $\mu : U(\mathfrak{g},e) \to U(\mathfrak{h})$ *from (III.10) is injective.*

Proof. Recall the definition of the Kazhdan filtration of $U(\mathfrak{g},e)$ from chapter II. Note that μ is a filtered map with respect to the Kazhdan filtration on $U(\mathfrak{g},e)$ and the standard filtration on $U(\mathfrak{h})$. We actually show that the associated graded map

$\operatorname{gr}\mu : \operatorname{gr} U(\mathfrak{g}, e) \to \operatorname{gr} U(\mathfrak{h})$ is injective, which implies the theorem. Each $s_{i,j}(\omega_r)$ is in degree r under the Kazhdan filtration and $\kappa_l(S_{i,j}^{(r)})$ is in degree r under the standard filtration on $U(\mathfrak{h})$. Moreover Theorem III.2.1 shows that $\mu(s_{i,j}(\omega_r)) = \kappa_l(S_{i,j}^{(r)})$, hence $(\operatorname{gr}\mu)(\operatorname{gr}_r s_{i,j}(\omega_r)) = \operatorname{gr}_r \kappa_l(S_{i,j}^{(r)})$. So by Theorem III.2.3 and the PBW theorem for $U(\mathfrak{g}, e)$ we deduce that $\operatorname{gr}\mu : \operatorname{gr} U(\mathfrak{g}, e) \to \operatorname{gr} U(\mathfrak{h})$ is injective. □

III.4 Proof of Invariance

In this section we prove Theorem III.3.1. We need to show for $i, j \in \mathcal{I}_n$ and $r \geq 1$ that

$$\operatorname{pr}([m, s_{i,j}(\omega_r)]) = 0 \tag{III.35}$$

for all $m \in \mathfrak{m}$. Since $\mathfrak{m}$ is generated by the elements

$$\{s_{i,j}(e_{q+2,q}) \mid i, j \in \mathcal{I}_n, q \in \mathcal{I}_l, -1 \leq q < l-1\}, \tag{III.36}$$

we just need to consider the actions of these elements on each $s_{i,j}(\omega_r)$. Actually we work in terms of the generating series $s_{i,j}(\omega(u))$ from (III.26), and we use the natural extension of pr to a homomorphism $\operatorname{pr} : U(\mathfrak{g})[u] \to U(\mathfrak{p})[u]$. As the calculations are lengthy, we break them up into a series of lemmas.

Lemma III.4.1. *Let $y_1, \dots, y_m \in \mathfrak{g}_l$. Let $i, j, h, k \in \mathcal{I}_n$. Let $p, q \in \mathcal{I}_l$. Then*

$$\begin{aligned}
&[s_{i,j}(e_{p,q}), s_{h,k}(y_1 \otimes \cdots \otimes y_m)] \\
&\quad = \sum_{t=1}^{m} s_{h,j}(y_1 \otimes \cdots \otimes y_{t-1}) s_{i,k}(e_{p,q} y_t \otimes y_{t+1} \otimes \cdots \otimes y_m) \\
&\quad\quad - \sum_{t=1}^{m} s_{h,j}(y_1 \otimes \cdots \otimes y_{t-1} \otimes y_t e_{p,q}) s_{i,k}(y_{t+1} \otimes \cdots \otimes y_m) \\
&\quad\quad + \gamma \Bigg(- \sum_{t=1}^{m} s_{h,-i}(y_1 \otimes \cdots \otimes y_{t-1}) s_{-j,k}(e_{-q,-p} y_t \otimes y_{t+1} \otimes \cdots \otimes y_m) \\
&\quad\quad\quad + \sum_{t=1}^{m} s_{h,-i}(y_1 \otimes \cdots \otimes y_{t-1} \otimes y_t e_{-q,-p}) s_{-j,k}(y_{t+1} \otimes \cdots \otimes y_m) \Bigg)
\end{aligned}$$

where

$$\gamma = \begin{cases} \phi^{\hat{\imath}\hat{p} + \tilde{\imath}\tilde{p} + \hat{\jmath}\hat{q} + \tilde{\jmath}\tilde{q}} \epsilon^{\hat{p} + \hat{q}} & \text{if } p, q \neq 0; \\ \phi^{\hat{\jmath}\hat{q} + \tilde{\jmath}\tilde{q}} \epsilon^{\hat{\imath} + \hat{q}} & \text{if } p = 0,\ q \neq 0; \\ \phi^{\hat{\imath}\hat{p} + \tilde{\imath}\tilde{p}} \epsilon^{\hat{p} + \hat{\jmath}} & \text{if } p \neq 0,\ q = 0; \\ \epsilon^{\hat{\imath} + \hat{\jmath}} & \text{if } p, q = 0, \end{cases} \tag{III.37}$$

and $e_{p,q} y_t$, $y_t e_{p,q}$, $e_{-q,-p} y_t$, and $y_t e_{-q,-p}$ denote matrix multiplication in M_l.

Proof. First note that for $a, b, c, d \in \mathcal{I}_{nl}$,

$$\begin{aligned}
[f_{a,b}, f_{c,d}] &= [e_{a,b} - \epsilon^{\hat{a}+\hat{b}} e_{-b,-a}, e_{c,d} - \epsilon^{\hat{c}+\hat{d}} e_{-d,-c}] \\
&= \delta_{c,b} e_{a,d} - \delta_{b,-d} \epsilon^{\hat{c}+\hat{d}} e_{a,-c} - \delta_{-a,c} \epsilon^{\hat{a}+\hat{b}} e_{-b,d} + \delta_{a,d} \epsilon^{\hat{b}+\hat{c}} e_{-b,-c} \\
&\quad - \delta_{a,d} e_{c,b} + \delta_{-c,a} \epsilon^{\hat{c}+\hat{d}} e_{-d,b} + \delta_{d,-b} \epsilon^{\hat{a}+\hat{b}} e_{c,-a} - \delta_{c,b} \epsilon^{\hat{a}+\hat{b}} e_{-d,-a} \\
&= \delta_{c,b} f_{a,d} - \delta_{a,d} f_{c,b} + \epsilon^{\hat{a}+\hat{b}} (-\delta_{c,-a} f_{-b,d} + \delta_{-b,d} f_{c,-a})
\end{aligned}$$

Thus for $v, w \in \mathcal{I}_l$ and a, b, c, d such that $\text{row}(a) = i, \text{col}(a) = p, \text{row}(b) = j, \text{col}(b) =$

$q, \mathrm{row}(c) = h, \mathrm{col}(c) = v, \mathrm{row}(d) = k, \mathrm{col}(d) = w$ we have that

$$\begin{aligned}
&[s_{i,j}(e_{p,q}), s_{h,k}(e_{v,w})] \\
&= [\phi^{\hat{\imath}\hat{p}+\hat{\jmath}\hat{q}} f_{a,b}, \phi^{\hat{h}\hat{v}+\hat{k}\hat{w}} f_{c,d}] \\
&= \phi^{\hat{\imath}\hat{p}+\hat{\jmath}\hat{q}+\hat{h}\hat{v}+\hat{k}\hat{w}} (\delta_{c,b} f_{a,d} - \delta_{a,d} f_{c,b} + \epsilon^{\hat{a}+\hat{b}}(-\delta_{c,-a} f_{-b,d} + \delta_{-b,d} f_{c,-a})) \\
&= \phi^{\hat{\imath}\hat{p}+\hat{\jmath}\hat{q}+\hat{h}\hat{v}+\hat{k}\hat{w}} (\phi^{\hat{\imath}\hat{p}+\hat{k}\hat{w}} s_{h,j}(1) s_{i,k}(e_{p,q} e_{v,w}) - \phi^{\hat{h}\hat{v}+\hat{\jmath}\hat{q}} s_{h,j}(e_{v,w} e_{p,q}) s_{i,k}(1) \\
&\quad + \epsilon^{\hat{a}+\hat{b}} (\phi^{\hat{\jmath}\hat{q}+\hat{k}\hat{w}} s_{h,-i}(1) s_{-j,k}(e_{-q,-p} e_{v,w}) + \phi^{\hat{h}\hat{v}+\hat{\imath}\hat{p}} s_{h,-i}(e_{v,w} e_{-q,-p}) s_{-j,k}(1))) \\
&= s_{h,j}(1) s_{i,k}(e_{p,q} e_{v,w}) - s_{h,j}(e_{v,w} e_{p,q}) s_{i,k}(1) \\
&\quad + \gamma(-s_{h,-i}(1) s_{-j,k}(e_{-q,-p} e_{v,w}) + s_{h,-i}(e_{v,w} e_{-q,-p}) s_{-j,k}(1)),
\end{aligned}$$

on noting that the ϵ term in γ equals $\epsilon^{\hat{a}+\hat{b}}$ due to the labeling convention specified in the introduction. Now the linearity of s implies the lemma holds for $m = 1$ and any $y_1 \in \mathfrak{g}_l$, and the lemma follows from induction on m. □

For $p, q \in \mathcal{I}_l$, let $\Omega_{p,q}(u)$ and $\bar{\Omega}_{p,q}(u)$ denote the square submatrices of $\Omega(u)$ and $\bar{\Omega}(u)$, respectively, with rows and columns indexed by $\{p, p+2, \ldots, q\}$.

Lemma III.4.2. *For each $i, j \in \mathcal{I}_n$ and for $q \in \mathcal{I}_l$ such that $q \geq 0$,*

$$\begin{aligned}
&\mathrm{pr}\left(s_{i,j} \left(\mathrm{rdet} \begin{pmatrix} e_{q+2,q} & e_{q+2,q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 1 & u_{q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 0 & 1 & u_{q+4} & \cdots & e_{q+4,l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{l-1} \end{pmatrix} \right) \right) \\
&= (u + \rho_{q+2} - n) s_{i,j}(\mathrm{rdet}\, \Omega_{q+4,l-1}(u)) \\
&= (u + \rho_{q+2} - n) s_{i,j}(\mathrm{rdet}\, \bar{\Omega}_{q+4,l-1}(u)).
\end{aligned}$$

Proof. By (III.7) for any $f, g \in \mathcal{I}_n, \mathrm{pr}(s_{f,g}(e_{q+2,q})) = \delta_{f,g} = s_{f,g}(1)$. So

$$\mathrm{pr}\left(s_{i,j}\left(\mathrm{rdet}\begin{pmatrix} e_{q+2,q} & e_{q+2,q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 1 & u_{q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 0 & 1 & u_{q+4} & \cdots & e_{q+4,l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{l-1} \end{pmatrix}\right)\right)$$

$$= s_{i,j}\left(\mathrm{rdet}\begin{pmatrix} 1 & e_{q+2,q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 1 & u_{q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 0 & 1 & u_{q+4} & \cdots & e_{q+4,l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{l-1} \end{pmatrix}\right)$$

$$+ \sum_{m \in \mathcal{I}_n} \mathrm{pr}([s_{i,m}(e_{q+2,q}), s_{m,j}(\mathrm{rdet}\, \Omega_{q+2,l-1}(u))]). \tag{III.38}$$

Since $u_{q+2} = e_{q+2,q+2} + u + \rho_{q+2}$, doing the obvious row operation gives that

$$\operatorname{rdet}\begin{pmatrix} 1 & e_{q+2,q+2} & e_{q+2,q+4} & \dots & e_{q+2,l-1} \\ 1 & u_{q+2} & e_{q+2,q+4} & \dots & e_{q+2,l-1} \\ 0 & 1 & u_{q+4} & \dots & e_{q+4,l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{l-1} \end{pmatrix}$$

$$= \operatorname{rdet}\begin{pmatrix} 0 & -(u+\rho_{q+2}) & 0 & \dots & 0 \\ 1 & u_{q+2} & e_{q+2,q+4} & \dots & e_{q+2,l-1} \\ 0 & 1 & u_{q+4} & \dots & e_{q+4,l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{l-1} \end{pmatrix}$$

$$= (u + \rho_{q+2}) \operatorname{rdet} \Omega_{q+4,l-1}(u)) \tag{III.39}$$

Next we apply Lemma (III.4.1) to get that

$$[s_{i,m}(e_{q+2,q}), s_{m,j}(\operatorname{rdet} \Omega_{q+2,l-1}(u))] = -s_{m,m}(e_{q+2,q}) s_{i,j}(\operatorname{rdet} \Omega_{q+4,l-1}(u)).$$

By (III.7) $\operatorname{pr}(s_{m,m}(e_{q+2,q})) = 1$, so

$$\operatorname{pr}([s_{i,m}(e_{q+2,q}), s_{m,j}(\operatorname{rdet} \Omega_{q+2,l-1}(u))]) = -s_{i,j}(\operatorname{rdet} \Omega_{q+4,l-1}(u)). \tag{III.40}$$

Combining (III.39) and (III.40) into (III.38) gives that

$$\operatorname{pr}\left(s_{i,j}\left(\operatorname{rdet}\begin{pmatrix} e_{q+2,q} & e_{q+2,q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 1 & u_{q+2} & e_{q+2,q+4} & \cdots & e_{q+2,l-1} \\ 0 & 1 & u_{q+4} & \cdots & e_{q+4,l-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{l-1} \end{pmatrix}\right)\right)$$

$$\begin{aligned} &= (u+\rho_{q+2})s_{i,j}(\operatorname{rdet}\Omega_{q+4,l-1}(u)) - ns_{i,j}(\operatorname{rdet}\Omega_{q+4,l-1}(u)) \\ &= (u+\rho_{q+2}-n)s_{i,j}(\operatorname{rdet}\Omega_{q+4,l-1}(u)) \\ &= (u+\rho_{q+2}-n)s_{i,j}(\operatorname{rdet}\bar{\Omega}_{q+4,l-1}(u)) \end{aligned}$$

since $\Omega_{q+4,l-1}(u) = \bar{\Omega}_{q+4,l-1}(u)$ because $q \geq 0$ by assumption. □

Lemma III.4.3. *For each $i, j, h, k \in \mathcal{I}_n$, for $q \in \mathcal{I}_l$ such that $q > 0$, and for $p \in \mathcal{I}_l$ such that $-q < p < q$,*

$$\operatorname{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\operatorname{rdet}\Omega_{p,l-1}(u))]) = 0,$$

and

$$\operatorname{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\operatorname{rdet}\bar{\Omega}_{p,l-1}(u))]) = 0,$$

Proof. We shall prove the result for $\Omega(u)$, but note that an identical proof holds for $\bar{\Omega}(u)$. We compute using Lemma III.4.1 to get that

$$[s_{i,j}(e_{q+2,q}), s_{h,k}(\operatorname{rdet}\Omega_{p,l-1}(u))] = A - B,$$

where

$$A = s_{h,j}(\operatorname{rdet}\Omega_{p,q-2}(u))s_{i,k}\left(\operatorname{rdet}\begin{pmatrix} e_{q+2,q} & e_{q+2,q+2} & \dots & e_{q+2,l-1} \\ 1 & u_{q+2} & \dots & e_{q+2,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

and

$$B = s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_p & \dots & e_{p,q} & e_{p,q} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_q & e_{q,q} \\ 0 & \dots & 1 & e_{q+2,q} \end{pmatrix}\right)s_{i,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)).$$

We will be more explicit how to calculate B, A is computed in a similar manner. Let $\mathcal{S}(X)$ denote the symmetric group on a set X. Let $M = \Omega_{p,l-1}(u)$. By definition,

$$\operatorname{rdet} M = \sum_{\sigma\in\mathcal{S}(\{p,p+2,\dots,l-1\})} M_{p,\sigma(p)}M_{p+2,\sigma(p+2)}\dots M_{l-1,\sigma(l-1)}.$$

All of the monomials in B come from the second sum in Lemma III.4.1 (all the monomials A come from the first sum in Lemma III.4.1, and the last two sums from that Lemma in the calculation of $[s_{i,j}(e_{q+2,q}), s_{h,k}(\operatorname{rdet}\Omega_{p,l-1}(u))]$ are zero). Furthermore every term in this sum is zero except for those coming from $s_{h,k}$ applied to monomials in rdet M which contain $e_{v,q+2}$ for some $v\in\mathcal{I}_l, v\leq q+2$. Now since the only nonzero terms of M below the diagonal are scalars occurring immediately below the diagonal, if $\sigma\in\mathcal{S}(\{p,p+2,\dots,l-1\})$ contributes a nonzero term to the second sum of Lemma III.4.1, then $\sigma\in\mathcal{S}(\{p,p+l,\dots,q+2\})\times\mathcal{S}(\{q+4,q+6,\dots,l-1\})$. Thus the sum of the terms of $[s_{i,j}(e_{q+2,q}), s_{h,k}(\operatorname{rdet}\Omega_{p,l-1}(u))]$ which come from the second sum in Lemma III.4.1 is precisely B.

Since $\rho_{q+2} - n = \rho_q$, by Lemma III.4.2,

$$\mathrm{pr}(A) = (u + \rho_q) s_{h,j}(\mathrm{rdet}\, \Omega_{p,q-2}(u)) s_{i,k}(\mathrm{rdet}\, \Omega_{q+4,l-1}(u)).$$

By (III.7) for any $f, g \in \mathcal{I}_n$, $\mathrm{pr}(s_{f,g}(e_{q+2,q})) = \delta_{f,g} = s_{f,g}(1)$. So the obvious column operation gives that

$$\mathrm{pr}(B) = s_{h,j}\left(\mathrm{rdet}\begin{pmatrix} u_p & \dots & e_{p,q} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_q & -(u+\rho_q) \\ 0 & \dots & 1 & 0 \end{pmatrix}\right) s_{i,k}(\mathrm{rdet}\, \Omega_{q+4,l-1}(u))$$
$$= (u + \rho_q) s_{h,j}(\mathrm{rdet}\, \Omega_{p,q-2}(u)) s_{i,k}(\mathrm{rdet}\, \Omega_{q+4,l-1}(u)).$$

The lemma now follows. □

Lemma III.4.4. *For each $i, j, h, k \in \mathcal{I}_n$ and for $q \in \mathcal{I}_l$ so that $q > 0$,*

$$\mathrm{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\mathrm{rdet}\, \Omega(u))]) = 0$$

and

$$\mathrm{pr}([s_{i,j}(e_{q+2,q}), s_{h,k}(\mathrm{rdet}\, \bar{\Omega}(u))]) = 0.$$

Proof. We shall prove the result for $\Omega(u)$, but note that an identical proof holds for $\bar{\Omega}(u)$. We compute using Lemma III.4.1 to get that

$$[s_{i,j}(e_{q+2,q}), s_{h,k}(\mathrm{rdet}\, \Omega(u))] = A - B + \phi^{\bar{i}+\bar{j}}(-C + D),$$

where

$$A = s_{h,j}(\operatorname{rdet}\Omega_{1-l,q-2}(u))s_{i,k}\left(\operatorname{rdet}\begin{pmatrix} e_{q+2,q} & e_{q+2,q+2} & \dots & e_{q+2,l-1} \\ 1 & u_{q+2} & \dots & e_{q+2,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

$$B = s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,q} & e_{1-l,q} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_q & e_{q,q} \\ 0 & \dots & 1 & e_{q+2,q} \end{pmatrix}\right)s_{i,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)),$$

$$C = s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-q-4}(u))s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix} e_{-q,-q-2} & e_{-q,-q} & \dots & e_{-q,l-1} \\ -1 & u_{-q} & \dots & e_{-q,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

and

$$D = s_{h,-i}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-q-2} & e_{1-l,-q-2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_{-q-2} & e_{-q-2,-q-2} \\ 0 & \dots & -1 & e_{-q,-q-2} \end{pmatrix}\right) \times s_{-j,k}(\operatorname{rdet}\Omega_{-q+2,l-1}(u)).$$

By Lemma III.4.2,

$$\operatorname{pr}(A) = (u+\rho_q)s_{h,j}(\operatorname{rdet}\Omega_{1-l,q-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)).$$

The obvious column operation gives that

$$\operatorname{pr}(B) = s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,q} & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & u_q & -(u+\rho_q) \\ 0 & \dots & 1 & 0 \end{pmatrix}\right) s_{i,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u))$$

$$= (u+\rho_q)s_{h,j}(\operatorname{rdet}\Omega_{1-l,q-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)).$$

Hence $\operatorname{pr}(A-B)=0$.

Since by (III.7) $\operatorname{pr}(s_{f,g}(e_{-q,-q-2})) = -\delta_{f,g} = s_{f,g}(-1)$ for any $f,g\in\mathcal{I}_n$, we have that

$$\operatorname{pr}(C) = s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-q-4}(u))s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix} -1 & e_{-q,-q} & \dots & e_{-q,l-1} \\ -1 & u_{-q} & \dots & e_{-q,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right)$$

$$+ \sum_{m\in\mathcal{I}_n} s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-q-4}(u))$$

$$\times \operatorname{pr}([s_{-j,m}(e_{-q,-q-2}), s_{m,k}(\operatorname{rdet}\Omega_{-q,l-1}(u))]). \qquad \text{(III.41)}$$

The obvious row operation gives that

$$s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix}-1 & e_{-q,-q} & \dots & e_{-q,l-1}\\ -1 & u_{-q} & \dots & e_{-q,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$= s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix}0 & -(u+\rho_{-q}) & \dots & 0\\ -1 & u_{-q} & \dots & e_{-q,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$= -(u+\rho_{-q})s_{-j,k}(\operatorname{rdet}\Omega_{-q+2,l-1}(u)). \tag{III.42}$$

Next we compute using Lemma III.4.1 to get that

$$[s_{-j,m}(e_{-q,-q-2}), s_{m,k}(\operatorname{rdet}\Omega_{-q,l-1}(u))]$$
$$= -s_{m,m}(e_{-q,-q-2})s_{-j,k}(\operatorname{rdet}\Omega_{-q+2,l-1}(u)) - A' + B',$$

where

$$A' = \phi^{\tilde{j}+\hat{m}}s_{m,j}(\operatorname{rdet}\Omega_{-q,q-2}(u))$$
$$\times s_{-m,k}\left(\operatorname{rdet}\begin{pmatrix}e_{q+2,q} & e_{q+2,q+2} & \dots & e_{q+2,l-1}\\ 1 & u_{q+2} & \dots & e_{q+2,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

and

$$B' = \phi^{\tilde{j}+\hat{m}} s_{m,j}\left(\operatorname{rdet}\begin{pmatrix} u_{-q} & \dots & e_{-q,q} & e_{-q,q} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_q & e_{q,q} \\ 0 & \dots & 1 & e_{q+2,q} \end{pmatrix}\right) s_{-m,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)).$$

By Lemma III.4.2

$$\operatorname{pr}(A') = \phi^{\tilde{j}+\hat{m}}(u+\rho_q)s_{m,j}(\operatorname{rdet}\Omega_{-q,q-2}(u))s_{-m,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)).$$

The usual column operation gives that

$$\begin{aligned}\operatorname{pr}(B') &= \phi^{\tilde{j}+\hat{m}} s_{m,j}\left(\operatorname{rdet}\begin{pmatrix} u_{-q} & \dots & e_{-q,q} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_q & -(u+\rho_q) \\ 0 & \dots & 1 & 0 \end{pmatrix}\right) \\ &\quad \times s_{-m,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)) \\ &= \phi^{\tilde{j}+\hat{m}}(u+\rho_q)s_{m,j}(\operatorname{rdet}\Omega_{-q,q-2}(u))s_{-m,k}(\operatorname{rdet}\Omega_{q+4,l-1}(u)).\end{aligned}$$

Thus $\operatorname{pr}(-A'+B') = 0$.

By Lemma III.4.3, we have that $[s_{m,m}(e_{-q,-q-2}), s_{-j,k}(\operatorname{rdet}\Omega_{-q+2,l-1}(u))] = 0$. Now since $\operatorname{pr}(s_{m,m}(e_{-q,-q-2})) = -1$, we get that

$$\operatorname{pr}(s_{m,m}(e_{-q,-q-2})s_{-j,k}(\operatorname{rdet}\Omega_{-q+2,l-1}(u))) = -s_{-j,k}(\operatorname{rdet}\Omega_{-q+2,l-1}(u)).$$

So

$$\mathrm{pr}([s_{-j,m}(e_{-q,-q-2})s_{m,k}(\mathrm{rdet}\,\Omega_{-q,l-1}(u))] = s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u)). \qquad \text{(III.43)}$$

By combining (III.42) and (III.43) into (III.41) we get that

$$\begin{aligned}\mathrm{pr}(C) &= -(u+\rho_{-q})s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-q-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u)) \\ &\quad + ns_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-q-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u)) \\ &= -(u+\rho_{-q-2})s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-q-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u)).\end{aligned}$$

Finally, we need to apply pr to D. By (III.7) $\mathrm{pr}(s_{f,g}(e_{-q,-q-2})) = -\delta_{f,g} = s_{f,g}(-1)$ for any $f, g \in \mathcal{I}_n$. By Lemma III.4.3 $s_{m,-i}(e_{-q,-q-2})$ commutes with $s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u))$. So the usual column operation gives that

$$\begin{aligned}\mathrm{pr}(D) &= s_{h,-i}\left(\mathrm{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-q-2} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_{-q-2} & -(u+\rho_{-q-2}) \\ 0 & \dots & -1 & -1 \end{pmatrix}\right) \\ &\qquad \times s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u)) \\ &= -(u+\rho_{-q-2})s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-q-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{-q+2,l-1}(u)).\end{aligned}$$

Thus $\mathrm{pr}(-C+D) = 0$. □

Lemma III.4.5. *Suppose that l is even. For each $i, j, h, k \in \mathcal{I}_n$*

$$\mathrm{pr}([s_{i,j}(e_{1,-1}), s_{h,k}(\mathrm{rdet}\,\Omega(u))]) = 0.$$

Proof. Since l is even, $\epsilon = -\phi$, so in all cases by (III.7) we have that for all $f, g \in \mathcal{I}_n$

$$\mathrm{pr}(s_{f,g}(e_{1,-1})) = \delta_{f,g} = s_{f,g}(1). \tag{III.44}$$

We compute using Lemma III.4.1 to get that

$$[s_{i,j}(e_{1,-1}), s_{h,k}(\mathrm{rdet}\,\Omega(u))] = A - B + \phi^{\tilde{\imath}+\tilde{\jmath}}\epsilon(-C + D),$$

where

$$A = s_{h,j}(\mathrm{rdet}\,\Omega_{1-l,-3}(u))s_{i,k}\left(\mathrm{rdet}\begin{pmatrix} e_{1,-1} & e_{1,1} & \dots & e_{1,l-1} \\ 1 & u_1 & \dots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

$$B = s_{h,j}\left(\mathrm{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-1} & e_{1-l,-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_{-1} & e_{-1,-1} \\ 0 & \dots & 1 & e_{1,-1} \end{pmatrix}\right) s_{i,k}(\mathrm{rdet}\,\Omega_{3,l-1}(u)),$$

$$C = s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-3}(u))s_{-j,k}\left(\mathrm{rdet}\begin{pmatrix} e_{1,-1} & e_{1,1} & \dots & e_{1,l-1} \\ 1 & u_1 & \dots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

and

$$D = s_{h,-i}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-1} & e_{1-l,-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_{-1} & e_{-1,-1} \\ 0 & \dots & 1 & e_{1,-1} \end{pmatrix}\right) s_{-j,k}(\operatorname{rdet}\Omega_{3,l-1}(u)).$$

Consider A first. Note that

$$\begin{aligned} &\operatorname{pr}\left(s_{i,k}\left(\operatorname{rdet}\begin{pmatrix} e_{1,-1} & e_{1,1} & \dots & e_{1,l-1} \\ 1 & u_1 & \dots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right)\right) \\ &= s_{i,k}\left(\operatorname{rdet}\begin{pmatrix} 1 & e_{1,1} & \dots & e_{1,l-1} \\ 1 & u_1 & \dots & e_{1,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right) \\ &+ \sum_{m\in\mathcal{I}_n} \operatorname{pr}([s_{i,m}(e_{1,-1}), s_{m,k}(\operatorname{rdet}\Omega_{1,l-1}(u))]). \end{aligned} \tag{III.45}$$

The obvious row operation gives that

$$s_{i,k}\left(\operatorname{rdet}\begin{pmatrix}1 & e_{1,1} & \dots & e_{1,l-1}\\ 1 & u_1 & \dots & e_{1,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$= s_{i,k}\left(\operatorname{rdet}\begin{pmatrix}0 & -(u+\rho_1) & 0 & \dots & 0\\ 1 & u_1 & e_{1,3} & \dots & e_{1,l-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$= (u+\rho_1)s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)). \tag{III.46}$$

Next consider the terms $\operatorname{pr}([s_{i,m}(e_{1,-1}), s_{m,k}(\operatorname{rdet}\Omega_{1,l-1}(u))])$ from (III.45). We calculate using Lemma III.4.1 to get that

$$[s_{i,m}(e_{1,-1}), s_{m,k}(\operatorname{rdet}\Omega_{1,l-1}(u))]$$
$$= -s_{m,m}(e_{1,-1})s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) + \phi^{\tilde{\imath}+\hat{m}}\epsilon s_{m,-i}(e_{1,-1})s_{-m,k}(\operatorname{rdet}\Omega_{3,l-1}(u)).$$

So

$$\operatorname{pr}([s_{i,m}(e_{1,-1}), s_{m,k}(\operatorname{rdet}\Omega_{1,l-1}(u))])$$
$$= -s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) + \phi^{\tilde{\imath}+\hat{m}}\epsilon\delta_{m,-i}s_{-m,k}(\operatorname{rdet}\Omega_{3,l-1}(u)). \tag{III.47}$$

So by combining (III.47) and (III.46) in (III.45) we we get that

$$\begin{aligned}\operatorname{pr}(A) &= (u+\rho_1)s_{h,j}(\operatorname{rdet}\Omega_{1-l,-3}(u))s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) \\ &\quad - ns_{h,j}(\operatorname{rdet}\Omega_{1-l,-3}(u))s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) \\ &\quad + \epsilon s_{h,j}(\operatorname{rdet}\Omega_{1-l,-3}(u))s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) \\ &= (u+\rho_{-1})s_{h,j}(\operatorname{rdet}\Omega_{1-l,-3}(u))s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)), \qquad \text{(III.48)}\end{aligned}$$

since $\rho_1 - n + \epsilon = \rho_{-1}$.

Next we consider B. The usual column operation gives that

$$\begin{aligned}\operatorname{pr}(B) &= s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-1} & e_{1-l,-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & u_{-1} & e_{-1,-1} \\ 0 & \dots & 1 & 1 \end{pmatrix}\right) s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) \\ &= s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & e_{-1,-1} & -(u+\rho_{-1}) \\ 0 & \dots & 1 & 0 \end{pmatrix}\right) s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)) \\ &= (u+\rho_{-1})s_{h,j}(\operatorname{rdet}\Omega_{1-l,-3}(u))s_{i,k}(\operatorname{rdet}\Omega_{3,l-1}(u)). \qquad \text{(III.49)}\end{aligned}$$

So by (III.48) and (III.49), $\operatorname{pr}(A-B) = 0$.

Next consider C. Since C is nearly identical to A, an argument nearly identical to that used for A shows that

$$\operatorname{pr}(C) = (u+\rho_{-1})s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-3}(u))s_{-j,k}(\operatorname{rdet}\Omega_{3,l-1}(u)).$$

Since D is nearly identical to B, an argument nearly identical to that used for B shows that

$$\mathrm{pr}(D) = (u+\rho_{-1})s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-3}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{3,l-1}(u)).$$

So $\mathrm{pr}(-C+D)=0$. ☐

Lemma III.4.6. *Suppose that l is odd. For $i,j,h,k \in \mathcal{I}_n$,*

$$\begin{aligned}
&\mathrm{pr}([s_{i,j}(e_{2,0}), s_{h,k}(\mathrm{rdet}\,\Omega(u))]) \\
&\quad = \phi/2 s_{h,j}(\mathrm{rdet}\,\Omega_{1-l,-2}(u))s_{i,k}(\mathrm{rdet}\,\Omega_{4,l-1}(u)) \\
&\qquad + \phi^{\bar{\imath}+\bar{\jmath}+1}/2 s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{4,l-1}(u)) \\
&\qquad - \phi^{\bar{\imath}+\bar{\jmath}}/2 s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{2,l-1}(u))
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{pr}([s_{i,j}(e_{2,0}), s_{h,k}(\mathrm{rdet}\,\bar{\Omega}(u))]) \\
&\quad = (u+\phi/2)s_{h,j}(\mathrm{rdet}\,\Omega_{1-l,-2}(u))s_{i,k}(\mathrm{rdet}\,\Omega_{4,l-1}(u)) \\
&\qquad + \phi^{\bar{\imath}+\bar{\jmath}}(u+\phi/2)s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{4,l-1}(u)) \\
&\qquad - \phi^{\bar{\imath}+\bar{\jmath}+1}(u+\phi/2)s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u))s_{-j,k}(\mathrm{rdet}\,\Omega_{2,l-1}(u)).
\end{aligned}$$

Proof. Since l is odd, $\epsilon=\phi$. We compute using Lemma III.4.1 to get that

$$[s_{i,j}(e_{2,0}), s_{h,k}(\mathrm{rdet}\,\Omega(u))] = A - B + \phi^{\bar{\imath}+\bar{\jmath}}(-C+D),$$

and

$$[s_{i,j}(e_{2,0}), s_{h,k}(\mathrm{rdet}\,\bar{\Omega}(u))] = \bar{A} - \bar{B} + \phi^{\bar{\imath}+\bar{\jmath}}(-\bar{C}+\bar{D}),$$

where

$$A = \bar{A} = s_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}\left(\operatorname{rdet}\begin{pmatrix} e_{2,0} & e_{2,2} & \dots & e_{2,l-1} \\ 1 & u_2 & \dots & e_{2,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

$$B = s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,0} & e_{1-l,0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & e_{0,0}+u & e_{0,0} \\ 0 & \dots & 1 & e_{2,0} \end{pmatrix}\right)s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)),$$

$$\bar{B} = s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,0} & e_{1-l,0} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & e_{0,0} & e_{0,0} \\ 0 & \dots & 1 & e_{2,0} \end{pmatrix}\right)s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)),$$

$$C = s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix} e_{0,-2} & e_{0,0} & \dots & e_{0,l-1} \\ -\phi & e_{0,0}+u & \dots & e_{0,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

$$\bar{C} = s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix} e_{0,-2} & e_{0,0} & \dots & e_{0,l-1} \\ -\phi & e_{0,0} & \dots & e_{0,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right),$$

and

$$D = \bar{D} = s_{h,-i}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-2} & e_{1-l,-2} \\ \vdots & & \vdots & \vdots \\ 0 & \dots & u_{-2} & e_{-2,-2} \\ 0 & \dots & -\phi & e_{0,-2} \end{pmatrix}\right) s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)).$$

By Lemma III.4.2

$$\begin{aligned}\operatorname{pr}(A) = \operatorname{pr}(\bar{A}) &= (u + \rho_2 - n)s_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\ &= (u - \phi/2)s_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)).\end{aligned}$$

By (III.7) for any $f, g \in \mathcal{I}_n$, $\operatorname{pr}(s_{f,g}(e_{2,0})) = \delta_{f,g} = s_{f,g}(1)$. So the obvious column operation gives that

$$\begin{aligned}\operatorname{pr}(B) &= s_{h,j}\left(\operatorname{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,0} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & e_{0,0}+u & -u \\ 0 & \dots & 1 & 0 \end{pmatrix}\right) s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\ &= us_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u))\end{aligned}$$

and

$$\mathrm{pr}(\bar{B}) = 0.$$

So

$$\mathrm{pr}(A - B) = -\phi/2 s_{h,j}(\mathrm{rdet}\,\Omega_{1-l,-2}(u)) s_{i,k}(\mathrm{rdet}\,\Omega_{4,l-1}(u)), \tag{III.50}$$

and

$$\mathrm{pr}(\bar{A} - \bar{B}) = (u - \phi/2) s_{h,j}(\mathrm{rdet}\,\Omega_{1-l,-2}(u)) s_{i,k}(\mathrm{rdet}\,\Omega_{4,l-1}(u)). \tag{III.51}$$

Next we consider $\mathrm{pr}(C)$. Since $\epsilon = \phi$, in all cases we have by (III.7) that for any $f, g \in \mathcal{I}_n$, $\mathrm{pr}(s_{f,g}(e_{0,-2})) = -\phi\delta_{f,g} = s_{f,g}(-\phi)$. So we have that

$$\mathrm{pr}(C) = s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u)) s_{-j,k}\left(\mathrm{rdet}\begin{pmatrix} -\phi & e_{0,0} & \dots & e_{0,l-1} \\ -\phi & e_{0,0}+u & \dots & e_{0,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right)$$
$$+ \sum_{m \in \mathcal{I}_n} s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u))\,\mathrm{pr}([s_{-j,m}(e_{0,-2}), s_{m,k}(\mathrm{rdet}\,\Omega_{0,l-1}(u))]). \tag{III.52}$$

The obvious row operation gives that

$$s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix}-\phi & e_{0,0} & \dots & e_{0,l-1}\\ -\phi & e_{0,0}+u & \dots & e_{0,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$= s_{-j,k}\left(\operatorname{rdet}\begin{pmatrix}0 & -u & \dots & 0\\ -\phi & e_{0,0}+u & \dots & e_{0,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$= -\phi u s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)). \tag{III.53}$$

Next we consider the terms $[s_{-j,m}(e_{0,-2}), s_{m,k}(\operatorname{rdet}\Omega_{0,l-1}(u))]$ from (III.52). By applying Lemma III.4.1, we compute that

$$[s_{-j,m}(e_{0,-2}), s_{m,k}(\operatorname{rdet}\Omega_{0,l-1}(u))]$$

$$= -s_{m,m}(e_{0,-2})s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u))$$

$$- \phi^{\tilde{j}+1+\hat{m}}\delta_{m,j}s_{-m,k}\left(\operatorname{rdet}\begin{pmatrix}e_{2,0} & e_{2,2} & \dots & e_{2,l-1}\\ 1 & u_2 & \dots & e_{2,l-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & u_{l-1}\end{pmatrix}\right)$$

$$+ \phi^{\tilde{j}+1+\hat{m}}s_{m,j}\left(\operatorname{rdet}\begin{pmatrix}e_{0,0}+u & e_{0,0}\\ 1 & e_{2,0}\end{pmatrix}\right)s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \tag{III.54}$$

We need to apply pr to each term of this expression. First we use Lemma III.4.1

again to get that

$$
\begin{aligned}
&\operatorname{pr}(s_{m,m}(e_{0,-2})s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u))) \\
&\quad = -\phi s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) + \operatorname{pr}([s_{m,m}(e_{0,-2}), s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u))]) \\
&\quad = -\phi s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) + \phi^{\hat{m}+1+\hat{m}} \operatorname{pr}(s_{-j,-m}(e_{2,0})s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u))) \\
&\quad = -\phi s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) + \phi\delta_{j,m}s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \qquad \text{(III.55)}
\end{aligned}
$$

Next by applying Lemma III.4.2, we have that

$$
\begin{aligned}
&\operatorname{pr}\left(s_{-m,k}\left(\operatorname{rdet}\begin{pmatrix} e_{2,0} & e_{2,2} & \dots & e_{2,l-1} \\ 1 & u_2 & \dots & e_{2,l-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{l-1} \end{pmatrix}\right)\right) \\
&\quad = (u + \rho_2 - n)s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\quad = (u - \phi/2)s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \qquad \text{(III.56)}
\end{aligned}
$$

Next note that

$$
\begin{aligned}
&\operatorname{pr}\left(s_{m,j}\left(\operatorname{rdet}\begin{pmatrix} e_{0,0}+u & e_{0,0} \\ 1 & e_{2,0} \end{pmatrix}\right) s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u))\right) \\
&\quad = u\delta_{m,j}s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \qquad \text{(III.57)}
\end{aligned}
$$

So by combining (III.55), (III.56), and (III.57) in (III.54) we get that

$$\begin{aligned}
&\operatorname{pr}([s_{-j,m}(e_{0,-2}), s_{m,k}(\operatorname{rdet}\Omega_{0,l-1}(u))]) \\
&\quad = \phi s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) - \phi\delta_{j,m}s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\quad - \phi^{\tilde{j}+1+\hat{m}}\delta_{m,j}(u-\phi/2)s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\quad + \phi^{\tilde{j}+1+\hat{m}}u\delta_{m,j}s_{-m,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \qquad \text{(III.58)}
\end{aligned}$$

So by combining (III.53) and (III.58) in (III.52) we get that

$$\begin{aligned}
\operatorname{pr}(C) &= -\phi u s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) \\
&\quad + \phi n s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) \\
&\quad - \phi s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\quad - \phi^{\tilde{j}+1+\hat{j}}(u-\phi/2)s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\quad + \phi^{\tilde{j}+1+\hat{j}}u s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&= -\phi(u-n)s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) \\
&\quad - \phi/2 s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \qquad \text{(III.59)}
\end{aligned}$$

For the last equality we use that $\phi^{\tilde{j}+\hat{j}} = \phi$, since j cannot be zero if $\phi = -1$.

A very similar calculation shows that

$$\begin{aligned}
\operatorname{pr}(\bar{C}) &= \phi n s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) \\
&\quad - (u+\phi/2)s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)). \qquad \text{(III.60)}
\end{aligned}$$

Finally we must calculate $\mathrm{pr}(D)$. Note that

$$\begin{aligned}\mathrm{pr}(D) &= \mathrm{pr}(\bar{D})\\ &= s_{h,-i}\left(\mathrm{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-2} & e_{1-l,-2}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & u_{-2} & e_{-2,-2}\\ 0 & \dots & -\phi & -\phi\end{pmatrix}\right) s_{-j,k}(\mathrm{rdet}\,\Omega_{2,l-1}(u))\\ &\quad + \sum_{m\in\mathcal{I}_n} s_{h,m}(\mathrm{rdet}\,\Omega_{1-l,-2}(u))\,\mathrm{pr}([s_{m,-i}(e_{0,-2}), s_{-j,k}(\mathrm{rdet}\,\Omega_{2,l-1}(u))]).\end{aligned} \tag{III.61}$$

The obvious column operation gives that

$$\begin{aligned}&s_{h,-i}\left(\mathrm{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-2} & e_{1-l,-2}\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & u_{-2} & e_{-2,-2}\\ 0 & \dots & -\phi & -\phi\end{pmatrix}\right)\\ &= s_{h,-i}\left(\mathrm{rdet}\begin{pmatrix} u_{1-l} & \dots & e_{1-l,-2} & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & u_{-2} & -(u+\rho_{-2})\\ 0 & \dots & -\phi & 0\end{pmatrix}\right)\\ &= -\phi(u+\rho_{-2})s_{h,-i}(\mathrm{rdet}\,\Omega_{1-l,-4}(u)).\end{aligned} \tag{III.62}$$

Next we consider the terms $\mathrm{pr}([s_{m,-i}(e_{0,-2}), s_{-j,k}(\mathrm{rdet}\,\Omega_{2,l-1}(u))])$ from (III.61). We compute using Lemma III.4.1 to get that

$$\begin{aligned}
&\mathrm{pr}([s_{m,-i}(e_{0,-2}), s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u))]) \\
&\quad = \phi^{\hat{m}+1+\tilde{\imath}}\,\mathrm{pr}(s_{-j,-m}(e_{2,0})s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u))) \\
&\quad = \phi^{\hat{m}+1+\tilde{\imath}}\delta_{j,m}s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)).
\end{aligned} \tag{III.63}$$

So by combining (III.62) and (III.63) in (III.61) we have that

$$\begin{aligned}
\mathrm{pr}(D) = \mathrm{pr}(\bar{D}) = &-\phi(u+\rho_{-2})s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)) \\
&+ \phi^{\hat{\jmath}+1+\tilde{\imath}}s_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}((\operatorname{rdet}\Omega_{4,l-1}(u)).
\end{aligned} \tag{III.64}$$

So by (III.50), (III.59), and (III.64) we have that

$$\begin{aligned}
&\mathrm{pr}(A - B + \phi^{\tilde{\imath}+\hat{\jmath}}(-C + D)) \\
&\qquad = \phi/2 s_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\qquad\quad + \phi^{\tilde{\imath}+\hat{\jmath}+1}/2 s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\qquad\quad - \phi^{\tilde{\imath}+\hat{\jmath}}/2 s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)).
\end{aligned}$$

By (III.51), (III.60), and (III.64) we have that

$$\begin{aligned}
&\mathrm{pr}(\bar{A} - \bar{B} + \phi^{\tilde{\imath}+\hat{\jmath}}(-\bar{C} + \bar{D})) \\
&\qquad = (u+\phi/2)s_{h,j}(\operatorname{rdet}\Omega_{1-l,-2}(u))s_{i,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\qquad\quad + \phi^{\tilde{\imath}+\hat{\jmath}}(u+\phi/2)s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{4,l-1}(u)) \\
&\qquad\quad - \phi^{\tilde{\imath}+\hat{\jmath}+1}(u+\phi/2)s_{h,-i}(\operatorname{rdet}\Omega_{1-l,-4}(u))s_{-j,k}(\operatorname{rdet}\Omega_{2,l-1}(u)).
\end{aligned}$$

□

Now we can prove Theorem III.3.1. We need to show that the equation (III.35) holds for all elements x lying in the generating set (III.36) for $\mathfrak{m}$. This follows from Lemmas III.4.4, III.4.5 and III.4.6, using the definition of $\omega(u)$ from (III.26).

CHAPTER IV

REPRESENTATION THEORY OF RECTANGULAR FINITE W-ALGEBRAS

We continue with the notation of the previous chapter. Recall that n, l are positive integers, and that $\epsilon, \phi \in \{\pm\}$, where $\phi = \epsilon$ if l is odd, $\phi = -\epsilon$ if l is even, and $\phi = +$ if n is odd. Also we have

$$\mathfrak{g} = \mathfrak{g}_{nl}^{\epsilon} = \begin{cases} \mathfrak{so}_{nl} & \text{if } \epsilon = +; \\ \mathfrak{sp}_{nl} & \text{if } \epsilon = -. \end{cases}$$

IV.1 Statement of the Main Result

Due to the homomorphism $\kappa_l : Y_n^{\phi} \twoheadrightarrow \mu(U(\mathfrak{g}, e))$ from (III.18), every $U(\mathfrak{g}, e)$-module is a Y_n^{ϕ}-module. The finite dimensional irreducible representations of Y_n^{ϕ} are classified in [M], and in this chapter we use Corollary III.2.4 to determine which of these representations factor through κ_l. This leads to the following classification of the finite dimensional irreducible representations of $U(\mathfrak{g}, e)$.

A Y_n^{ϕ}-module V is called a *highest weight module* if V is generated by a vector v such that $S_{i,j}(u)v = 0$ for all $i < j$, and if for all $i \in \mathcal{I}_n \cap \mathbb{Z}_{\geq 0}$ we have that $S_{i,i}(u)v = \mu_i(u)v$ for some power series $\mu_i \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. In this case we say that V is of highest weight $(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ if n is even or $(\mu_0(u), \mu_2(u), \ldots, \mu_{n-1}(u))$

if n is odd. By Theorem IV.3.1 below (due to Molev) there is a unique (up to isomorphism) irreducible highest weight Y_n^ϕ-module for any given weight. Moreover every irreducible finite dimensional Y_n^ϕ-module is a highest weight module.

By Corollary III.2.4, when l is even the Y_n^ϕ-action on a highest weight module V of highest weight $(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ factors through κ_l precisely when $\mu_i(u)$ is a polynomial in u^{-1} of degree at most l for all $i \in \{1, 3, \ldots, l-1\}$. When l is odd, Corollary III.2.4 implies that the Y_n^ϕ-action on a highest weight module V of highest weight $(\mu_0(u), \mu_2(u), \ldots, \mu_{n-1}(u))$ factors through κ_l precisely when $(1 + \frac{\phi}{2}u^{-1})\mu_i(u)$ is a polynomial of degree at most l for all $i \in \{0, 2, \ldots, l-1\}$. In these cases, for $i \in \mathcal{I}_n \cap \mathbb{Z}_{\geq 0}$ we can write

- $(u - \frac{i}{2})^l S_{i,i}(u - \frac{i}{2})v = (u + a_{i,1-l})(u + a_{i,3-l}) \ldots (u + a_{i,l-1})v$ if l is even;
- $(u - \frac{i}{2})^{l-1}(u + \frac{\phi - i}{2})S_{i,i}(u - \frac{i}{2})v = (u + a_{i,1-l})(u + a_{i,3-l}) \ldots (u + a_{i,l-1})v$ if l is odd.

In the case that V is irreducible, we associate to these polynomials a *skew-symmetric $n \times l$ tableaux*. A skew-symmetric $n \times l$ tableaux is an $n \times l$ matrix of complex numbers, with rows labeled by the set $\mathcal{I}_n$ and columns labeled by the set $\mathcal{I}_l$, and which is skew-symmetric with respect to the center of the matrix, that is, if $B = (b_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_l}$ is a skew-symmetric $n \times l$ tableaux then $b_{i,j} = -b_{-i,-j}$. Let $\mathrm{Tab}^{\mathfrak{g}}_{n,l}$ denote the set of skew-symmetric $n \times l$ tableaux.

Note that in the case $\phi = +$ and n is odd that by (III.14) $S_{0,0}(u) \in Y_n^+[[u^{-2}]]$, so $\mu_0(u) \in \mathbb{C}[[u^{-2}]]$ and we may index so that $a_{0,j} = -a_{0,-j}$ for all $0 \neq j \in \mathcal{I}_l$. Additionally, if n and l are odd then $a_{0,0} = \frac{1}{2}$; we must re-assign $a_{0,0} = 0$ in order to get a skew-symmetric $n \times l$ tableaux. So in all cases by setting $a_{i,j} = -a_{-i,-j}$ for $i, j < 0$, we now have that V determines the skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_l}$, and we let $L(A)$ denote this irreducible highest weight module.

We say that two skew-symmetric $n \times l$ tableaux are *row equivalent* if one can be obtained from the other by permuting entries within rows. Let $\mathrm{Row}_{n,l}^{\mathfrak{g}}$ denote the set of row equivalence classes of skew-symmetric $n \times l$ tableaux. Note that if A, B are in the same row equivalence class in $\mathrm{Row}_{n,l}^{\mathfrak{g}}$ then $L(A) = L(B)$, so $\mathrm{Row}_{n,l}^{\mathfrak{g}}$ parameterizes the irreducible highest weight Y_n^{ϕ}-modules which factor through κ_l.

A skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_l}$ is *column strict* if

- the entries in every column except for the middle column (which exists only when l is odd) are strictly decreasing from top to bottom, i.e., $a_{1-n,j} > a_{3-n,j} > \cdots > a_{n-1,j}$ for all $0 \neq j \in \mathcal{I}_l$;
- if l is odd and n is even then the entries in the middle column satisfy $a_{1-n,0} > \cdots > a_{-1,0}$, and they also satisfy $a_{-1,0} > 0$ if $\mathfrak{g} = \mathfrak{sp}_{nl}$, and they satisfy $a_{-3} + a_{-1} > 0$ if $\mathfrak{g} = \mathfrak{so}_{nl}$ and $n \geq 4$;
- if l is odd and n is odd then the entries in the middle column satisfy $a_{1-n,0} > \cdots > a_{-2,0}$ and they also satisfy $2a_{-2,0} > 0$.

In this definition (and from here on) we are using the partial order $\geq$ on $\mathbb{C}$ defined by $a \geq b$ if $a - b \in \mathbb{Z}_{\geq 0}$. Let $\mathrm{Col}_{n,l}^{\mathfrak{g}}$ denote the set of all skew-symmetric $n \times l$ tableaux, and let $\mathrm{Std}_{n,l}^{\mathfrak{g}}$ denote the set of elements of $\mathrm{Row}_{n,l}^{\mathfrak{g}}$ which have a representative in $\mathrm{Col}_{n,l}^{\mathfrak{g}}$.

Here is our classification of the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules.

Theorem IV.1.1. *If l is odd or if l is even and $\mathfrak{g} = \mathfrak{so}_{nl}$ then*

$$\{L(A) \mid A \in \mathrm{Std}_{n,l}^{\mathfrak{g}}\}$$

is a complete set of isomorphism classes of the finite dimensional irreducible repre-

sentations of $U(\mathfrak{g}, e)$. *If* l *is even and* $\mathfrak{g} = \mathfrak{sp}_{nl}$ *then*

$$\{L(A) \mid A \in \mathrm{Row}_{n,l}^{\mathfrak{sp}_{nl}}, A^+ \in \mathrm{Std}_{n,l+1}^{\mathfrak{so}_{n(l+1)}}\}$$

is a complete set of isomorphism classes of the finite dimensional irreducible representations of $U(\mathfrak{g}, e)$, *where* A^+ *is the skew-symmetric* $n \times (l+1)$ *tableaux obtained by inserting a middle column into* A *with entries*

$$\frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, 1, 0, 0, -1, -2, \ldots, 1 - \frac{n}{2}$$

if n *is even and*

$$\frac{n}{2} - 1, \frac{n}{2} - 2, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{3}{2}, \ldots, 1 - \frac{n}{2}$$

if n *is odd down the middle column.*

In the special case when $l = 1$, so $e = 0$ and $U(\mathfrak{g}, e) = U(\mathfrak{g})$, we can associate to each element of $\mathrm{Tab}_{n,l}^{\mathfrak{g}}$ a weight $\lambda = \sum_{i \in \mathcal{I}_n \cap \mathbb{Z}_{\geq 0}} a_{i,0}\epsilon_i$, where $\{\epsilon_i \mid i \in \mathcal{I}_n\}$ is defined below in §IV.6. Now the requirement that a tableaux be column strict is the usual condition for the corresponding irreducible highest weight $U(\mathfrak{g})$-module $L(\lambda)$ to be finite dimensional; here $L(\lambda)$ is defined in terms of the system of positive roots given in §IV.6.

IV.2 Representation Theory of Yangians

To prove Theorem IV.1.1 we need to review the representation theory of Yangians and twisted Yangians from [M].

We say a Y_n-module V is a *highest weight module* if it is generated by a vector v such that $T_{i,j}(u)v = 0$ for all $i < j$, and if for all i we have that $T_{i,i}(u)v = \lambda_i(u)v$

for some power series $\lambda_i \in 1+u^{-1}\mathbb{C}[[u^{-1}]]$, in which case we say that V is of highest weight $(\lambda_{1-n}(u), \lambda_{3-n}(u), \ldots, \lambda_{n-1}(u))$.

The following theorem is contained in [M, §2].

Theorem IV.2.1. *For each weight*

$$(\lambda_{1-n}(u), \lambda_{3-n}(u), \ldots, \lambda_{n-1}(u)) \in (1+u^{-1}\mathbb{C}[[u^{-1}]])^n$$

there is a unique (up to isomorphism) irreducible highest weight Y_n-module of that highest weight.

For

$$(\lambda_{1-n}(u), \lambda_{3-n}(u), \ldots, \lambda_{n-1}(u)) \in (1+u^{-1}\mathbb{C}[[u^{-1}]])^n$$

let $L(\lambda_{1-n}(u), \lambda_{3-n}(u), \ldots, \lambda_{n-1}(u))$ denote the corresponding irreducible highest weight Y_n-module.

The following is [M, Theorem 2.3].

Theorem IV.2.2. *Every irreducible finite dimensional Y_n-module is a highest weight module.*

To specify which irreducible highest weight modules are finite dimensional, following Molev, we introduce the following notation. Given two power series $\lambda_1(u), \lambda_2(u) \in 1+u^{-1}\mathbb{C}[[u^{-1}]]$ we write $\lambda_1(u) \to \lambda_2(u)$ if there exists a monic polynomial $P(u) \in \mathbb{C}[u]$ such that

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

In fact $P(u)$ must then be unique because if $Q(u)$ is another monic polynomial satisfying $\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{Q(u+1)}{Q(u)}$ then $\frac{Q(u)}{P(u)} = \frac{Q(u+1)}{P(u+1)}$, thus $\frac{Q(u)}{P(u)}$ is periodic, which implies

$P(u) = Q(u)$.

Here is the main classification theorem for finite dimensional irreducible representations of Y_n.

Theorem IV.2.3. *(Drinfeld, [Dr]) The Y_n-module $L(\lambda_{1-n}(u), \lambda_{3-n}(u), \ldots, \lambda_{n-1}(u))$ is finite dimensional if and only if $\lambda_{1-n}(u) \to \lambda_{3-n}(u) \to \cdots \to \lambda_{n-1}(u)$.*

The following lemmas give a more combinatorial description of this notation. Recall that $\geq$ denotes the partial order on $\mathbb{C}$ where $a \geq b$ if $a - b \in \mathbb{Z}_{\geq 0}$.

Lemma IV.2.4. *If $\lambda_1(u), \lambda_2(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ then $\lambda_1(u) \to \lambda_2(u)$ if and only if there exists $\gamma(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that*

$$\gamma(u)\lambda_1(u) = (1 + a_1u^{-1}) \ldots (1 + a_ku^{-1}),$$
$$\gamma(u)\lambda_2(u) = (1 + b_1u^{-1}) \ldots (1 + b_ku^{-1})$$

where $a_i \geq b_i$ for $i = 1, \ldots, k$.

Proof. First assume that $\lambda_1(u) \to \lambda_2(u)$, so there exists a monic polynomial $P(u)$ such that

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

Let k be the degree of $P(u)$, and let $\gamma(u) = \frac{P(u)u^{-k}}{\lambda_2(u)}$. So $\gamma(u)\lambda_1(u) = P(u+1)u^{-k}$ and $\gamma(u)\lambda_2(u) = P(u)u^{-k}$, thus $\gamma(u)$ satisfies the conclusions of the lemma since we can now write $\gamma(u)\lambda_2(u)u^k = P(u) = (u + b_1) \ldots (u + b_k)$ and $\gamma(u)\lambda_1(u)u^k = P(u+1) = (u + b_1 + 1) \ldots (u + b_k + 1)$

Now assume that $\gamma(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ exists such that $\gamma(u)\lambda_1(u) = (1 + a_1u^{-1}) \ldots (1 + a_ku^{-1})$ and $\gamma(u)\lambda_2(u) = (1 + b_1u^{-1})(1 + b_ku^{-1})$ where $a_i \geq b_i$ for

$i = 1, \dots, k$. For $i = 1, \dots, l$ let $P_i(u) = (u + a_i - 1)(u + a_i - 2)\dots(u + b_i + 1)$. Now

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{(u+a_1)P_1(u)\dots(u+a_k)P_k(u)}{P_1(u)(u+b_1)\dots P_k(u)(u+b_k)},$$

so $P(u) = P_1(u)(u+b_1)\dots P_k(u)(u+b_k)$ is the unique polynomial satisfying

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)}.$$

□

Lemma IV.2.5. *Let $\lambda_1(u), \lambda_2(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$, and suppose that $\lambda_1(u) \to \lambda_2(u)$. If $\gamma(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ is such that $\gamma(u)\lambda_1(u), \gamma(u)\lambda_2(u) \in \mathbb{C}[u^{-1}]$ then we can write $\gamma(u)\lambda_1(u) = (1+a_1u^{-1})\dots(1+a_ku^{-1})$ and $\gamma(u)\lambda_2(u) = (1+b_1u^{-1})\dots(1+b_ku^{-1})$ where $a_i \geq b_i$ for $i = 1, \dots, k$.*

Proof. We can write $\gamma(u)\lambda_1(u) = (1 + a_1u^{-1})\dots(1 + a_ku^{-1})$ and $\gamma(u)\lambda_2(u) = (1 + b_1u^{-1})\dots(1+b_ku^{-1})$, and by replacing $\gamma(u)$ we may assume that the sets $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are disjoint. By Lemma IV.2.4 there exists $\gamma'(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that $\gamma'(u)\lambda_1(u) = (1+c_1u^{-1})\dots(1+c_mu^{-1})$ and $\gamma'(u)\lambda_2(u) = (1+d_1u^{-1})\dots(1+d_mu^{-1})$ where $c_i \geq d_i$ for $i = 1, \dots, m$, and by replacing $\gamma'(u)$ we may assume that the sets $\{c_1, \dots, c_m\}$ and $\{d_1, \dots, d_m\}$ are disjoint. So we have that

$$\frac{(1+a_1u^{-1})\dots(1+a_ku^{-1})}{(1+b_1u^{-1})\dots(1+b_ku^{-1})} = \frac{(1+c_1u^{-1})\dots(1+c_mu^{-1})}{(1+d_1u^{-1})\dots(1+d_mu^{-1})}.$$

So $k = m$, $\{a_1, \dots, a_k\} = \{c_1, \dots, c_k\}$, and $\{b_1, \dots, b_k\} = \{d_1, \dots, d_k\}$, and thus the lemma follows by re-indexing $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$. □

Lemma IV.2.6. *Let $\lambda_1(u), \dots \lambda_m(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. If $\lambda_1(u) \to \lambda_2(u) \to \dots \to \lambda_m(u)$ then there exists $\gamma(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that $\gamma(u)\lambda_i(u) \in \mathbb{C}[u^{-1}]$ for*

$i = 1, \ldots, m$.

Proof. Assume that $\lambda_1(u) \to \lambda_2(u) \to \cdots \to \lambda_m(u)$, and for $i = 1, \ldots, m-1$ let $P_i(u)$ be the monic polynomial so that

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)}.$$

Note for $i = 1, \ldots, m-1$ that

$$\lambda_i(u) = \frac{P_i(u+1)P_{i+1}(u+1)\ldots P_{m-1}(u+1)\lambda_m(u)}{P_i(u)P_{i+1}(u)\ldots P_{m-1}(u)}.$$

So

$$\gamma(u) = \frac{u^{-k}P_1(u)\ldots P_{m-1}(u)}{\lambda_m(u)},$$

where $k = \sum_{i=1}^{m-1} \deg(P_i(u))$, satisfies the conclusion of the lemma. □

Now Theorem IV.2.3 and Lemmas IV.2.5 and IV.2.6 imply the following combinatorial description of finite dimensional irreducible Y_n modules.

Corollary IV.2.7. *The Y_n-module $L(\lambda_{1-n}(u), \lambda_{3-n}(u), \ldots, \lambda_{n-1}(u))$ is finite dimensional if and only if there exists $\gamma(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that for all $i \in \mathcal{I}_n$ we have that $\gamma(u)\lambda_i(u) \in C[u^{-1}]$ and can write*

$$\gamma(u)\lambda_i(u) = (1 + a_{i,1}u^{-1})(1 + a_{i,2}u^{-1})\ldots(1 + a_{i,k}u^{-1})$$

such that $a_{i,j} \geq a_{i+2,j}$ for all $j \in \{1, \ldots, k\}$, $i \in \{1-n, 3-n, \ldots, n-3\}$.

IV.3 Representation Theory of Twisted Yangians

Recall that we defined the notion of a highest weight Y_n^ϕ-module in §IV.1. The following theorem is contained in [M, Chapter 3].

Theorem IV.3.1. *For each weight* $(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u)) \in (1+u^{-1}\mathbb{C}[[u^{-1}]])^{n/2}$ *if* n *is even or* $(\mu_0(u), \mu_2(u), \ldots, \mu_{n-1}(u)) \in (1+u^{-1}\mathbb{C}[[u^{-1}]])^{(n+1)/2}$ *if* n *is odd, there is a unique (up to isomorphism) irreducible highest weight* Y_n^ϕ*-module of that highest weight.*

We let $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ or $L(\mu_0(u), \mu_2(u), \ldots, \mu_{n-1}(u))$ denote the corresponding irreducible highest weight Y_n^ϕ-module.

The following is part of [M, Theorem 3.3].

Theorem IV.3.2. *Every irreducible finite dimensional* Y_n^ϕ*-module is a highest weight module.*

Following Molev, to specify which irreducible highest weight modules are finite dimensional, we introduce the following notation. For power series $\mu(u), \nu(u) \in 1+u^{-1}\mathbb{C}[[u^{-1}]]$, we write $\mu(u) \Rightarrow \nu(u)$ if there exists a monic polynomial $P(u) \in \mathbb{C}[u]$ such that $P(u) = P(1-u)$ and

$$\frac{\mu(u)}{\nu(u)} = \frac{P(u+1)}{P(u)}.$$

Note that $P(u) = P(1-u)$ is equivalent to $P(u)$ being of even degree and the roots of $P(u)$ being symmetric about $\frac{1}{2}$.

Here is the classification of the finite dimensional irreducible representations of Y_n^-:

Theorem IV.3.3. *([M, Theorem 4.8]) The Y_n^--module $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is finite dimensional if and only if*

$$\mu_1(-u) \Rightarrow \mu_1(u) \to \mu_3(u) \to \cdots \to \mu_{n-1}(u).$$

To obtain a more combinatorial description of the finite dimensional irreducible representations of Y_n^-, we prove the following lemmas.

Lemma IV.3.4. *If $\mu(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ then $\mu(-u) \Rightarrow \mu(u)$ if and only if there exists $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that $\gamma(u)\mu(u) = (1 - a_1u^{-1})(1 - a_2u^{-1}) \ldots (1 - a_{2k}u^{-1})$ where*

$$a_{2i-1} + a_{2i} \geq 0 \text{ for } i = 1, \ldots, k. \tag{IV.1}$$

Proof. Assume $\mu(-u) \Rightarrow \mu(u)$, so there exists a monic polynomial $P(u)$ of even degree so that $P(u) = P(1-u)$ and

$$\frac{\mu(-u)}{\mu(u)} = \frac{P(u+1)}{P(u)}.$$

Let $2k$ be the degree of $P(u)$, and let

$$\gamma(u) = \frac{P(u)u^{-2k}}{\mu(u)}.$$

So $\gamma(u)\mu(-u) = P(u+1)u^{-2k}$ and $\gamma(u)\mu(u) = P(u)u^{-2k}$. Since the roots of $P(u)$ are symmetric about $\frac{1}{2}$, we can write

$$\gamma(u)\mu(u) = (1 - b_1u^{-1})(1 - (1 - b_1)u^{-1}) \ldots (1 - b_ku^{-1})(1 - (1 - b_k)u^{-1}).$$

Now it is clear that $\gamma(u)\mu(u)$ satisfies (IV.1), so it remains to see that $\gamma(u) \in$

$\mathbb{C}[[u^{-2}]]$. Note that the roots of $P(u+1) = u^{2k}\gamma(u)\mu(-u)$ are $\{b_1 - 1, -b_1, \ldots, b_k - 1, -b_k\}$. Now since these are also the roots of $P(-u) = u^{2k}\gamma(-u)\mu(-u)$, we have that $\gamma(-u)\mu(-u) = \gamma(u)\mu(-u)$, so $\gamma(-u) = \gamma(u)$, and thus $\gamma(u) \in \mathbb{C}[[u^{-2}]]$.

Conversely, we now assume that there exists $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that $\gamma(u)\mu(u) = (1 - a_1u^{-1})(1 - a_2u^{-1})\ldots(1 - a_{2k}u^{-1})$, where $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$. Let $P_i(u) = (u + a_{2i-1} - 1)(u + a_{2i-1} - 2)\ldots(u - a_{2i} + 1)$, and let $Q_i(u) = (u + a_{2i} - 1)(u + a_{2i} - 2)\ldots(u - a_{2i-1} + 1)$. Now it is the case that

$$\frac{\mu(-u)}{\mu(u)} = \frac{(u + a_1)P_1(u)(u + a_2)Q_1(u)\ldots(u + a_{2k-1})P_k(u)(u + a_{2k})Q_k(u)}{P_1(u)(u - a_2)Q_1(u)(u - a_1)\ldots P_k(u)(u - a_{2k})Q_k(u)(u - a_{2k-1})},$$

so $P(u) = P_1(u)(u-a_2)Q_1(u)(u-a_1)\ldots P_k(u)(u-a_{2k})Q_k(u)(u-a_{2k-1})$ is the unique monic polynomial of even degree such that $P(u) = P(1-u)$ and

$$\frac{\mu(-u)}{\mu(u)} = \frac{P(u+1)}{P(u)}.$$

□

Lemma IV.3.5. *Suppose $\mu(u) \in 1 + u^{-1}\mathbb{C}[u^{-1}]$, $\mu(-u) \Rightarrow \mu(u)$, and there exists $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that $\gamma(u)\mu(u) = (1 - a_1u^{-1})(1 - a_2u^{-1})\ldots(1 - a_{2k}u^{-1})$ where $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$. Then there exists $\gamma'(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that after re-indexing $\{a_1, \ldots, a_{2k}\}$ we can write $\gamma'(u)\mu(u) = (1 - a_1u^{-1})(1 - a_2u^{-1})\ldots(1 - a_{2m}u^{-1})$ where $m \leq k$, for each $i \neq j \in \{1, \ldots, 2m\}$ we have that $a_i \neq -a_j$, and $\{a_1, \ldots, a_{2m}\}$ satisfies $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, m$.*

Proof. We proceed by induction on k, and assume that $a_i = -a_j$ for some $i \neq j \in \{1, \ldots, 2k\}$. After re-indexing we may assume that $a_1 = -a_2$ or $a_1 = -a_3$. If $a_1 = -a_2$ then

$$\frac{\gamma(u)}{1 - a_1^2u^{-2}} = (1 - a_3u^{-1})\ldots(1 - a_{2k}u^{-1})$$

satisfies the hypotheses of the lemma, so the lemma follows by induction. If $a_1 = -a_3$ then we have that $a_2 + a_4 = a_1 + a_2 + a_3 + a_4 \geq 0$, so

$$\frac{\gamma(u)}{1 - a_1^2 u^{-2}} = (1 - a_2 u^{-1})(1 - a_4 u^{-1}) \dots (1 - a_{2k} u^{-1})$$

satisfies the hypotheses of the lemma, so the lemma follows by induction. □

Lemma IV.3.6. *Let $\mu(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. If $\mu(-u) \Rightarrow \mu(u)$ and $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ is such that $\gamma(u)\mu(u) \in \mathbb{C}[u^{-1}]$ then we can write $\gamma(u)\mu(u) = (1 - a_1 u^{-1})(1 - a_2 u^{-1}) \dots (1 - a_{2k} u^{-1})$ so that $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \dots, k$.*

Proof. By Lemmas IV.3.4 and IV.3.5 there exists $\gamma'(u) \in 1 + u^{-1}\mathbb{C}[[u^{-2}]]$ such that $\gamma'(u)\mu(u) = (1 - b_1 u^{-1}) \dots (1 - b_{2m} u^{-1})$ so that $b_i \neq -b_j$ for all $i \neq j \in \{1, \dots, 2m\}$ and so that $b_{2i-1} + b_{2i} \geq 0$ for $i = 1, \dots, m$. Write

$$\gamma(u)\mu(u) = (1 - a_1 u^{-1}) \dots (1 - a_p u^{-1})(1 - a_{p+1}^2 u^{-2}) \dots (1 - a_q^2 u^{-2})$$

such that $a_i \neq -a_j$ for all $i \neq j \in \{1, \dots, p\}$. Thus

$$\frac{\gamma(u)}{\gamma'(u)} = \frac{(1 - a_1 u^{-1})(1 - a_p u^{-1})(1 - a_{p+1}^2 u^{-2}) \dots (1 - a_q^2 u^{-2}) \dots}{(1 - b_1 u^{-1})(1 - b_{2m} u^{-1})}$$

which implies

$$\begin{aligned} &(1 - a_1 u^{-1}) \dots (1 - a_p u^{-1})(1 + b_1 u^{-1}) \dots (1 + b_{2m} u^{-1}) \\ &= \frac{\gamma(u)(1 - b_1^2 u^{-2})(1 - b_{2m}^2 u^{-2})}{\gamma'(u)(1 - a_{p+1}^2 u^{-2}) \dots (1 - a_q^2 u^{-2})} \in \mathbb{C}[[u^{-2}]], \end{aligned}$$

and thus $p = 2m$ and after re-indexing we must have that $a_i = b_i$ for all $i \in \{1, \dots, 2m\}$. □

Lemma IV.3.7. *Let $\mu_1(u), \mu_2(u), \dots, \mu_m(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. Suppose $\mu_1(-u) \Rightarrow \mu_1(u) \to \mu_2(u) \to \cdots \to \mu_m(u)$. Then there exists $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that $\gamma(u)\mu_i(u) \in \mathbb{C}[u^{-1}]$ for $i = 1, \dots, m$.*

Proof. By Lemma IV.2.6 there exists $v(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ such that $v(u)\mu_i(u) \in \mathbb{C}[u^{-1}]$. So we can write $v(u)\mu_1(u) = (1 + b_1u^{-1}) \dots (1 + b_su^{-1})$. Let $v'(u) = v(u)(1 - b_1u^{-1}) \dots (1 - b_su^{-1})$, so $v'(u)\mu_1(u) \in \mathbb{C}[u^{-2}]$, and $v'(u)\mu_i(u) \in \mathbb{C}[u^{-1}]$ for $i = 1, \dots, m$. By Lemma IV.3.4 there exists $\eta(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that $\eta(u)\mu_1(u) \in \mathbb{C}[u^{-1}]$. Let $\gamma(u) = \eta(u)v'(u)\mu_1(u)$. Now $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ and $\gamma(u)\mu_i(u) \in \mathbb{C}[u^{-1}]$ for $i = 1, \dots, m$. □

Next we turn our attention to the classification of finite dimensional irreducible representations of Y_n^+ when n is even. The $n = 2$ case needs to be treated separately from the $n > 2$ cases.

Theorem IV.3.8. *([M, Proposition 5.3]) The Y_2^+ representation $L(\mu(u))$ is finite dimensional if and only if there exists $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that*

$$(1 + \frac{1}{2}u^{-1})\gamma(u)\mu(u) = (1 - a_1u^{-1})(1 - a_2u^{-1}) \dots (1 - a_{2k+1}u^{-1})$$

where $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \dots, k$.

We need a slight generalization of this theorem.

Lemma IV.3.9. *Let $\mu(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. If the Y_2^+-module $L(\mu(u))$ is finite dimensional and $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ is such that $(1 + \frac{1}{2}u^{-1})\gamma(u)\mu(u) \in \mathbb{C}[u^{-1}]$ then we can write*

$$(1 + \frac{1}{2}u^{-1})\gamma(u)\mu(u) = (1 - a_1u^{-1})(1 - a_2u^{-1}) \dots (1 - a_{2k+1}u^{-1}),$$

where $a_{2i-1} + a_{2i} \geq 0$ *for* $i = 1, \ldots, k$

Proof. Suppose that such a $\gamma(u)$ exists. By [M, Theorem 5.4] $L(\mu(u))$ is finite dimensional if and only if there exists a monic polynomial $P(u) \in \mathbb{C}[u]$ with $P(u) = P(-u+1)$ and $c \in \mathbb{C}$ such that $P(-c) \neq 0$ and

$$\frac{\mu(-u)}{\mu(u)} = \frac{P(u+1)(u+c)(2u+1)}{P(u)(u-c)(2u-1)}.$$

Let $\lambda(u) = \mu(u)(1+cu^{-1})(1+\frac{1}{2}u^{-1})$. Thus we have that $\lambda(-u) \Rightarrow \lambda(u)$, and since $\gamma(u)\lambda(u) = (1-a_1u^{-1})(1-a_2u^{-1})\ldots(1-a_{2k+1}u^{-1})(1+cu^{-1})$, by Lemma IV.3.6 after re-indexing we have that $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$. □

Next we will give the classification of finite dimensional Y_n^+ representations for n even, $n > 2$. This depends on a certain Y_n^+ automorphism ψ:

$$\psi : Y_n^+ \to Y_n^+, \quad S_{i,j}(u) \mapsto S_{i',j'}(u), \tag{IV.2}$$

where $i' = i$ if $i \neq \pm 1$, and $i' = -i$ if $i = \pm 1$.

If L is a Y_n^+-module, we let $L^\#$ denote the module created by twisting with ψ, that is, if $v \in L$, $y \in Y_n^+$, then $L^\#$ is the module created by the action $y.v = \psi(y)v$, where $\psi(y)v$ denotes the action given by L. Of course, if $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is a finite dimensional Y_n^ϕ-module, then so is $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))^\#$, and by Theorem IV.3.2 $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))^\#$ is another highest weight module. To determine which highest weight module, we need to define the notation of a *#-special* element of a list of complex numbers. A list $\{a_1, a_2, \ldots, a_{2k+1}\}$ of complex numbers can be indexed so that the following condition is satisfied:

$$\begin{aligned}&\text{for every } i = 1, \ldots, k \text{ we have:}\\&\text{if the set } \{a_p + a_q \mid 2i-1 \leq p < q \leq 2k+1\} \cap \mathbb{Z}_{\geq 0}\} \text{ is non-empty}\\&\text{then } a_{2i-1} + a_{2i} \text{ is its minimal element.}\end{aligned} \tag{IV.3}$$

For an element a in a list $\{a_1, a_2, \ldots, a_{2k+1}\}$ of complex numbers, we say that a is a *#-special* element of $\{a_1, a_2, \ldots, a_{2k+1}\}$ if $a = a_{2k+1}$ when $\{a_1, \ldots, a_{2k+1}\}$ is indexed so that (IV.3) holds.

Lemma IV.3.10. *If a_{2k+1} is a #-special element of $\{a_1, \ldots, a_{2k+1}\}$ where $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$ then a_{2k+1} is the maximal element in $\{b_{2k+1}\}$ where $\{b_1, \ldots, b_{2k+1}\}$ is a re-indexing of $\{a_1, \ldots, a_{2k+1}\}$ satisfying $b_{2i-1} + b_{2i} \geq 0$ for $i = 1, \ldots, k$. In particular, the #-special element is unique in these circumstances.*

Proof. We proceed by induction on k, the case $k = 0$ being clear. Let $\{a_1, \ldots, a_{2k+1}\}$ be a list for which $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$, and for which (IV.3) holds. Let $\{b_1, \ldots, b_{2k+1}\}$ be a re-indexing of $\{a_1, \ldots, a_{2k+1}\}$ such that $b_{2i-1} + b_{2i} \geq 0$ for $i = 1, \ldots, k$. Assume that $b_{2k+1} \neq a_{2k+1}$. Then after re-indexing we may assume that $b_1 = a_{2k+1}$. Let i be such that $a_i = b_2$. Without loss of generality we may assume that i is odd. Since $\{a_1, \ldots, a_{2k+1}\}$ satisfies (IV.3), we have that $a_i + a_{i+1} \leq a_i + a_{2k+1}$, so $a_{i+1} \leq a_{2k+1}$. Now after re-indexing we may assume that $a_{i+1} = b_3$, so $a_{2k+1} + b_4 \geq 0$. Now we have that the lists $\{a_1, \ldots, a_{i-1}, a_{i+2}, \ldots, a_{2k+1}\}$ and $\{a_{2k+1}, b_4, \ldots, b_{2k+1}\}$ also satisfy the hypotheses of the lemma, so by induction $b_{2k+1} \leq a_{2k+1}$. □

Suppose $\mu(u) \in 1+u^{-1}\mathbb{C}[[u^{-1}]]$ is such that there exists $\gamma(u) \in 1+u^{-2}\mathbb{C}[[u^{-2}]]$ such that $(1 + \frac{1}{2}u^{-1})\gamma(u)\mu(u) = (1 - a_1u^{-1}) \ldots (1 - a_{2k+1}u^{-1})$, where $a_{2i-i} + a_{2i} \geq 0$

for $i = 1, \ldots, k$ and $\{a_1, \ldots, a_{2k+1}\}$ satisfies (IV.3). If these conditions are met then we say that $\mu(u)^{\#}$ is *well-defined.* Now we define

$$\mu^{\#}(u) = \gamma(u)^{-1}(1 + \frac{1}{2})(1 - a_1 u^{-1}) \ldots (1 - a_{2k}u^{-1})(1 + (1 + a_{2k+1})u^{-1}) \quad \text{(IV.4)}$$

Lemma IV.3.11. *The definition of $\mu^{\#}(u)$ is well-defined, that is, it does not depend on $\gamma(u)$.*

Proof. First we make the following observation. If $\{a_1, \ldots, a_{2k+1}\}$ satisfies (IV.3) and $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$ then for an arbitrary $a \in \mathbb{C}$ the ordered list $\{a, -a, a_1, \ldots, a_{2k+1}\}$ also satisfies (IV.3). Note this also implies that if a and $-a$ both occur in $\{a_1, \ldots, a_{2k+1}\}$ then the #-special element of the list $\{a_1, \ldots, a_{2k+1}\}$ with one occurrence of a and $-a$ removed is also a_{2k+1}.

Now suppose that for $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ that $(1 + \frac{1}{2}u^{-1})\gamma(u)\mu(u) = (1 - a_1 u^{-1}) \ldots (1 - a_{2k+1}u^{-1})$, where $a_{2i-i} + a_{2i} \geq 0$ for $i = 1, \ldots, k$ and $\{a_1, \ldots, a_{2k+1}\}$ satisfies (IV.3). Also suppose for some $\gamma'(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ that $(1 + \frac{1}{2}u^{-1})\gamma'(u)\mu(u) = (1 - b_1 u^{-1}) \ldots (1 - b_{2k'+1}u^{-1})$, where $b_{2i-i} + b_{2i} \geq 0$ for $i = 1, \ldots, k'$ and $\{b_1, \ldots, b_{2k'+1}\}$ satisfies (IV.3).

By re-indexing we may write

$$(1 - a_1 u^{-1}) \ldots (1 - a_{2k+1}u^{-1}) = (1 - a_1 u^{-1}) \ldots (1 - a_p u^{-1})(1 - a_{p+1}^2 u^{-2}) \ldots (1 - a_q^2 u^{-2})$$

and

$$(1 - b_1 u^{-1}) \ldots (1 - b_{2k'+1}u^{-1}) = (1 - b_1 u^{-1}) \ldots (1 - b_{p'} u^{-1})(1 - b_{p'+1}^2 u^{-2}) \ldots (1 - b_{q'}^2 u^{-2}),$$

where $a_i \neq a_j$ for all $i \neq j \in \{1, \ldots, p\}$ and $b_i \neq b_j$ for all $i \neq j \in \{1, \ldots, p'\}$. So

$$\frac{(1-a_1u^{-1})\ldots(1-a_pu^{-1})(1-a_{p+1}^2u^{-2})\ldots(1-a_q^2u^{-2})}{(1-b_1u^{-1})\ldots(1-b_{p'}u^{-1})(1-b_{p'+1}^2u^{-2})\ldots(1-b_{q'}^2u^{-2})} = \frac{\gamma(u)}{\gamma'(u)},$$

so

$$\begin{aligned}(1-a_1u^{-1})&\ldots(1-a_pu^{-1})(1+b_1u^{-1})\ldots(1+b_{p'}u^{-1})\\ &= \frac{\gamma(u)(1-b_1^2u^{-2})\ldots(1-b_{q'}^2u^{-2})}{\gamma'(u)(1-a_1^2u^{-2})\ldots(1-a_p^2u^{-2})} \in \mathbb{C}[[u^{-2}]].\end{aligned}$$

Thus $p = p'$, and after re-indexing, $a_i = b_i$ for $i = 1, \ldots, p$. Now the lemma follows from the above observation. □

The following theorem is contained in the proof of [M, Theorem 5.9].

Theorem IV.3.12. *Let $\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$ where $\mu_1^{\#}(u)$ is well-defined. Then $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))^{\#} = L(\mu_1^{\#}(u), \mu_3(u), \ldots, \mu_{n-1}(u))$.*

We will also need to following lemma.

Lemma IV.3.13. *If $\{a_1, \ldots, a_{2k+1}\}$ satisfies $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \ldots, k$ and a_{2k+1} is the #-special element of $\{a_1, \ldots, a_{2k+1}\}$ then $-1 - a_{2k+1}$ is the #-special element of $\{a_1, \ldots, a_{2k}, -1 - a_{2k+1}\}$.*

Proof. By Theorem IV.3.8 and Lemma IV.3.9 the Y_2^+-module $L((1-a_1u^{-1})\ldots(1-a_{2k+1}u^{-1})(1+\frac{1}{2}u^{-1})^{-1})$ is finite dimensional, and by (IV.4) and Theorem IV.3.12

$$\begin{aligned}L((1-a_1u^{-1})&\ldots(1-a_{2k+1}u^{-1})(1+\frac{1}{2}u^{-1})^{-1})^{\#}\\ &= L((1-a_1u^{-1})\ldots(1-a_{2k}u^{-1})(1+(1+a_{2k+1})u^{-1})(1+\frac{1}{2}u^{-1})^{-1}).\end{aligned}$$

Since ψ from (IV.2) is an involution, we must have that

$$L((1-a_1u^{-1})\dots(1-a_{2k}u^{-1})(1+(1+a_{2k+1})u^{-1})(1+\frac{1}{2}u^{-1})^{-1})^{\#}$$
$$= L((1-a_1u^{-1})\dots(1-a_{2k+1}u^{-1})(1+\frac{1}{2}u^{-1})^{-1}).$$

Now suppose #-special element of $\{a_1,\dots,a_{2k},-1-a_{2k+1}\}$ is $a_j \neq -1-a_{2k+1}$ for some $j \in \{1,\dots,2k\}$. Then by Theorem IV.3.12 we have that $\{a_1,\dots,a_{2k+1}\} = \{a_1,\dots,a_{j-1},-1-a_j,a_{j+1},\dots,a_{2k},-1-a_{2k+1}\}$, so we must have that $\{a_j,a_{2k+1}\} = \{-1-a_j,-1-a_{2k+1}\}$. Since $a_j \neq -1-a_{2k+1}$, we must have that $a_{2k+1} = -1-a_{2k+1}$, which implies that $a_{2k+1} = -\frac{1}{2} = -1-a_{2k+1}$, so by Lemma IV.3.10 the #-special element of $\{1,\dots,a_{2k},-1-a_{2k+1}\}$ is in fact $-1-a_{2k+1}$. □

Here is the classification of the finite dimensional representations of Y_n^+ for even $n > 2$.

Theorem IV.3.14. *([M, Theorem 5.9]) Let $n > 2$ be even. Then the Y_n^+-module $L(\mu_1(u),\mu_3(u),\dots,\mu_{n-1}(u))$ is finite dimensional if and only if $\mu_1^{\#}(u)$ is well defined and any of the following four conditions holds:*

(i) $\mu_1(-u) \Rightarrow \mu_1(u) \to \mu_3(u) \to \cdots \to \mu_{n-1}(u)$,

(ii) $\frac{2u-1}{2u+1}\mu_1(-u) \Rightarrow \mu_1(u) \to \mu_3(u) \to \cdots \to \mu_{n-1}(u)$,

(iii) $\mu_1^{\#}(-u) \Rightarrow \mu_1^{\#}(u) \to \mu_3(u) \to \cdots \to \mu_{n-1}(u)$,

(iv) $\frac{2u-1}{2u+1}\mu_1^{\#}(-u) \Rightarrow \mu_1^{\#}(u) \to \mu_3(u) \to \cdots \to \mu_{n-1}(u)$.

In order to give a more combinatorial description of this classification we need the following for lemmas. In each of the lemmas we assume for some $\gamma(u) \in$

$1 + u^{-2}\mathbb{C}[[u^{-2}]]$ that

$$\mu(u) = \gamma(u)^{-1}(1 + \frac{1}{2}u^{-1})^{-1}(1 - a_1u^{-1})\dots(1 - a_{2k+1}u^{-1})$$

where a_{2k+1} is the #-special element of $\{a_1, \dots, a_{2k+1}\}$ and $a_{2i-1} + a_{2i} \geq 0$ for $i = 1, \dots, k$.

Lemma IV.3.15. *Let $\mu(u)$ be as above. Then $\mu(-u) \Rightarrow \mu(u)$ if and only if $a_{2k+1} \geq -\frac{1}{2}$.*

Proof. Suppose $\mu(-u) \Rightarrow \mu(u)$. Since $(1 - \frac{1}{4}u^{-2})\gamma(u)\mu(u) \in \mathbb{C}[u^{-1}]$, by Lemma IV.3.6 the set $\{a_1, \dots, a_{2k+1}, \frac{1}{2}\}$ can be re-indexed as $\{b_1, \dots, b_{2k+2}\}$ where $b_{2i-1} + b_{2i} \geq 0$ for $i = 1, \dots, k+1$. Now if $b_j = \frac{1}{2}$, with out loss of generality we may assume that j is odd. So by Lemma IV.3.10 $b_{j+1} \leq a_{2k+1}$, so $0 \leq b_{j+1} + \frac{1}{2} \leq a_{2k+1} + \frac{1}{2}$.

To prove the converse note that $\gamma(u)(1 - \frac{1}{4}u^{-2})\mu(u) = (1 - a_1u^{-1})\dots(1 - a_{2k+1}u^{-1})(1 - \frac{1}{2}u^{-1})$, then apply Lemma IV.3.4.

□

Lemma IV.3.16. *Let $\mu(u)$ be as above. Then $\frac{2u-1}{2u+1}\mu(-u) \Rightarrow \mu(u)$ if and only if $a_{2k+1} \geq 0$.*

Proof. Note that

$$\frac{2u-1}{2u+1}\mu(-u) \Rightarrow \mu(u) \text{ if and only if } (1 - \frac{1}{2}u^{-1})\mu(-u) \Rightarrow (1 + \frac{1}{2}u^{-1})\mu(u). \quad \text{(IV.5)}$$

Suppose that $\frac{2u-1}{2u+1}\mu(-u) \Rightarrow \mu(u)$. So by Lemma IV.3.6 and (IV.5), the set $\{a_1, \dots, a_{2k+1}, 0\}$ can be re-indexed as $\{b_1, \dots, b_{2k+2}\}$ where $b_{2i-1} + b_{2i} \geq 0$ for $i = 1, \dots, k+1$. Now if $b_j = 0$, with out loss of generality we may assume that j is odd. So by Lemma IV.3.10 $b_{j+1} \leq a_{2k+1}$, so $0 \leq b_{j+1} \leq a_{j+1}$.

The converse follows immediately from Lemma IV.3.4 and (IV.5) since $\gamma(u)(1+\frac{1}{2}u^{-1})\mu(u) = (1-a_1u^{-1})\dots(1-a_{2k+1}u^{-1})(1-0u^{-1})$. □

Lemma IV.3.17. *Let $\mu(u)$ be as above. Then $\mu^{\#}(-u) \Rightarrow \mu^{\#}(u)$ if and only if $a_{2k+1} \leq -\frac{1}{2}$.*

Proof. This follows from (IV.4), Lemma IV.3.13, and Lemma IV.3.15. □

Lemma IV.3.18. *Let $\mu(u)$ be as above. Then $\frac{2u-1}{2u+1}\mu^{\#}(-u) \Rightarrow \mu^{\#}(u)$ if and only if $a_{2k+1} \leq -1$.*

Proof. This follows from (IV.4), Lemma IV.3.13, and Lemma IV.3.16. □

Next we give the classification of the finite dimensional irreducible Y_n^+ representations for n odd. Note that for a highest weight representation of highest weight $(\mu_0(u), \mu_2(u), \dots, \mu_{n-1}(u)) \in (1+u^{-1}\mathbb{C}[[u^{-1}]])^{(n+1)/2}$, by the relation (III.14), we must have that $\mu_0(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$.

Theorem IV.3.19. *([M, Theorem 6.7]) Assume that $n \in \mathbb{Z}_{>0}$ is odd. Then the Y_n^+-module $L(\mu_0(u), \dots, \mu_{n-1}(u))$ is finite dimensional if and only if either one of the the following two relations holds:*

(i) $\mu_0(u) \to \mu_2(u) \to \cdots \to \mu_{n-1}(u)$,

(ii) $\frac{2u}{2u+1}\mu_0(u) \to \mu_2(u) \to \cdots \to \mu_{n-1}(u)$.

We now wish to establish that every finite dimensional irreducible Y_n^{ϕ}-module is a twist of a finite W-algebra module. By [MNO, Proposition 3.10] for every $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ there exists a Y_n^{ϕ} automorphism ν_γ defined via $\nu_\gamma(S_{i,j}(u)) = \gamma(u)S_{i,j}(u)$. For a Y_n^{ϕ}-module L, let $\gamma(u)L$ denote the Y_n^{ϕ}-module defined by the action $y.v = \nu_\gamma(y)v$ for $y \in Y_n^{\phi}$, $v \in L$.

Corollary IV.3.20. *Every finite dimensional irreducible Y_n^ϕ-module can be realized as $\gamma(u)L$ where $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ and L is a finite dimensional irreducible $U(\mathfrak{g}_{nk}^\phi, e')$-module where e' is a nilpotent element in $\mathfrak{g}_{nk}^\phi$ of Jordan type (k^n). Every finite dimensional irreducible Y_n^ϕ-module can also be realized as $\gamma(u)L'$ where $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ and L' is a finite dimensional irreducible $U(\mathfrak{g}_{nk'}^{-\phi}, e'')$-module where e'' is a nilpotent element in $\mathfrak{g}_{nk'}^{-\phi}$ of Jordan type $((k')^n)$.*

Proof. By Corollary III.2.4 it suffices to prove that for every finite dimensional Y_n^ϕ-module $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is n if even or $L(\mu_0(u), \mu_2(u), \dots, \mu_{n-1}(u))$ is n if odd that there exists $\gamma(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$ such that $\gamma(u)\mu_i(u) \in \mathbb{C}[u^{-1}]$ for all $i \in \mathcal{I}_n \cap \mathbb{Z}_{\geq 0}$. By Theorems IV.3.3, IV.3.8, IV.3.14, and IV.3.19 and Lemma IV.3.7 this is true in all cases (in the case that n and l are odd and $\phi = +$ note that $\mu_0(-u) \Rightarrow \mu_0(u)$ since $\mu_0(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$). ☐

IV.4 Proof of the Classification Theorem

In this section we prove Theorem IV.1.1 on a case by case basis.

Lemma IV.4.1. *Theorem IV.1.1 holds in the case that $\phi = -, \epsilon = +$, n is even, and l is even.*

Proof. In this case, by Corollary III.2.4, an irreducible highest weight Y_n^--module $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ factors through κ_l if $\mu_i(u)$ is a polynomial of degree l or less for all $i \in \{1, 3, \dots, n-1\}$. Furthermore, if $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is finite dimensional then by Theorem IV.3.3, Lemma IV.3.6, and Lemma IV.2.5 we can write $\mu_i(u) = (1 + c_{i,1-1}u^{-1})(1 + c_{i,l-3}u^{-1}) \dots (1 + c_{i,l-1}u^{-1})$ for $i \in \{1, 3, \dots, n-1\}$ such that $c_{1,j} + c_{1,-j} \leq 0$ for all $j \in \mathcal{I}_l$ and $c_{i,j} \geq c_{i+2,j}$ for all $j \in \mathcal{I}_l$, $i \in \{1, \dots, n-3\}$. Now associate to this data the $n \times l$ skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j}$

where $a_{i,j} = c_{i,j} - \frac{i}{2}$ for $i \in \{1, 3, \ldots, n-1\}$, $j \in \mathcal{I}_l$, and $a_{i,j} = -a_{-i,-j}$ for $i \in \{1-n, 3-n, \ldots, -1\}$, $j \in \mathcal{I}_l$. Now it is clear that $A \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$.

This process is easily reversed: starting with a skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j} \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$, for $i \in \{1, 3, \ldots, n-1\}$, $j \in \mathcal{I}_l$ set $c_{i,j} = a_{i,j} + \frac{i}{2}$, then set $\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,3-l}u^{-1}) \ldots (1 + c_{i,l-1}u^{-1})$. Now we have that $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is a finite dimensional Y_n^--module by Theorem IV.3.3, Lemma IV.3.4, and Lemma IV.2.4. □

Lemma IV.4.2. *Theorem IV.1.1 holds in the case that $\phi = -, \epsilon = -$, n is even, and l is odd.*

Proof. In this case, by Corollary III.2.4, an irreducible highest weight Y_n^--module $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ factors through κ_l if $(1 - \frac{1}{2}u^{-1})\mu_i(u)$ is a polynomial of degree at most l for all $i \in \{1, 3, \ldots, n-1\}$. If $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is finite dimensional then by Theorem IV.3.3, Lemma IV.3.6, and Lemma IV.2.5 for $i \in \{1, 3, \ldots, n-1\}$ we can write

$$(1 - \frac{1}{4}u^{-2})\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,3-l}u^{-1}) \ldots (1 + c_{i,,l-1}u^{-1})(1 + \frac{1}{2}u^{-1})$$

such that $c_{1,0} \leq -\frac{1}{2}$, $c_{1,j} + c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$ and $c_{i,j} \geq c_{i+2,j}$ for $j \in \mathcal{I}_l$, $i \in \{1, \ldots, n-3\}$. Now associate to this data the skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j}$ where $a_{i,j} = c_{i,j} - \frac{i}{2}$ for $i \in \{1, 3, \ldots, n-1\}$, $j \in \mathcal{I}_l$, and $a_{i,j} = -a_{-i,-j}$ for $i \in \{1-n, 3-n, \ldots, -1\}$, $j \in \mathcal{I}_l$. Now it is clear that $A \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$.

This process is easily reversed: starting with a skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j} \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$, for $i \in \{1, 3, \ldots, n-1\}$, $j \in \mathcal{I}_l$ set $c_{i,j} = a_{i,j} + \frac{i}{2}$, then set $\mu_i(u) = (1 + c_{i,1-1}u^{-1})(1 + c_{i,l-3}u^{-1}) \ldots (1 - c_{i,l-1}u^{-1})(1 - \frac{1}{2}u^{-1})^{-1}$. Now $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is a finite dimensional Y_n^--module by Theorem IV.3.3, Lemma IV.3.4, and Lemma IV.2.4. □

Lemma IV.4.3. *Theorem IV.1.1 holds in the case that $\phi = +, \epsilon = +$, $n = 2$, and l is odd.*

Proof. By Corollary III.2.4, an irreducible highest weight Y_2^+-module $L(\mu(u))$ factors through κ_l if $\mu(u)(1 + \frac{1}{2}u^{-1})$ is a polynomial of degree l or less. Now if $L(\mu(u))$ is finite dimensional then by Theorem IV.3.8 and Lemma IV.3.9 $\mu(u) = (1 + c_{1-l}u^{-1})(1 + c_{3-l}u^{-1}) \dots (1 + c_{l-1}u^{-1})$ where $c_j + c_{-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$. Now associate to this data the skew-symmetric $2 \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_2, j \in \mathcal{I}_j}$ where $a_{1,j} = c_{1,j} - \frac{1}{2}$ for $j \in \mathcal{I}_l$, and $a_{-1,j} = -a_{1,-j}$ for $j \in \mathcal{I}_l$. Now it is clear that $A \in \mathrm{Col}^{\mathfrak{g}}_{2,l}$.

This process is easily reversed: starting with a skew-symmetric $2 \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_2, j \in \mathcal{I}_j} \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$, for $j \in \mathcal{I}_l$ set $c_j = a_{1,j} + \frac{1}{2}$, then set $\mu(u) = (1 + c_{1-1}u^{-1})(1 + c_{l-3}u^{-1}) \dots (1 - c_{l-1}u^{-1})(1 - \frac{1}{2}u^{-1})^{-1}$. Now $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is a finite dimensional Y_n^--module by Theorem IV.3.8. □

Lemma IV.4.4. *If $\phi = \epsilon = +$, $n > 2$ is even, and l is odd then the Y_n^+-module $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ factors though κ_l and is finite dimensional if and only if for all $i \in \{1, 3, \dots, n-1\}$ we can write $\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,3-l}u^{-1}) \dots (1 + c_{i,l-1}u^{-1})(1 + \frac{1}{2}u^{-1})^{-1}$ where $c_{1,j} + c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, $c_{1,0} + c_{3,0} \leq 1$, and $c_{i,j} \geq c_{i+2,j}$ for $i \in \{1, 3, \dots, n-3\}, j \in \mathcal{I}_l$.*

Proof. By Corollary III.2.4 we have that an irreducible highest weight Y_n^+-module $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ factors through κ_l if $\mu_i(u)(1 + \frac{1}{2}u^{-1})$ is a polynomial of degree l or less for all $i \in \{1, 3, \dots, n-1\}$. So we can write $\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,3-l}u^{-1}) \dots (1 + c_{i,l-1}u^{-1})(1 + \frac{1}{2}u^{-1})^{-1}$ for all $i \in \{1, 3, \dots, n-1\}$.

If $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is finite dimensional then we need to examine the implications from the four conditions in Theorem IV.3.14 separately.

If condition (i) or condition (ii) holds from Theorem IV.3.14 then by Lemma IV.2.5 we can re-index each row of the matrix $(c_{i,j})_{i\in\{1,3,\dots,n-1\},j\in\mathcal{I}_l}$ so that

$$-c_{1,0} \quad \text{is the \#-special element of} \quad \{-c_{1,1-l}, -c_{1,3-l}, \dots, -c_{1,l-1}\} \tag{IV.6}$$

and

$$c_{i,j} \geq c_{i+2,j} \quad \text{for} \quad j \in \mathcal{I}_l, i \in \{1,\dots,n-3\}. \tag{IV.7}$$

If condition (i) from Theorem IV.3.14 holds then since $(1-\frac{1}{4}u^{-2})\mu_i(u) = (1+c_{i,1-l}u^{-1})(1+c_{i,3-l}u^{-1})\dots(1+c_{i,l-1}u^{-1})(1-\frac{1}{2}u^{-1})$ for all $i \in \{1,3,\dots,n-1\}$, we have by Lemmas IV.3.6 and IV.3.15 that can further re-index so that (IV.6) and (IV.7) still hold, $c_{1,j}+c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, and $c_{1,0} \leq \frac{1}{2}$. Since $c_{3,0} \leq c_{1,0}$ we now have that $c_{1,0}+c_{3,0} \leq 1$

If condition (ii) holds from Theorem IV.3.14 then since $(1-\frac{1}{2}u^{-1})\mu_1(-u) \Rightarrow (1+\frac{1}{2}u^{-1})\mu_1(u) = (1+c_{i,1-l}u^{-1})(1+c_{i,3-l}u^{-1})\dots(1+c_{i,l-1}u^{-1})$, by Lemmas IV.3.6 and IV.3.16 we can further re-index so that (IV.6) and (IV.7) still hold, $c_{1,j}+c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, and $c_{1,0} \leq 0$. Since $c_{3,0} \leq c_{1,0}$ we now have that $c_{1,0}+c_{3,0} \leq 0$.

If condition (iii) or condition (iv) holds from Theorem IV.3.14 then by Lemma IV.2.5 and (IV.4) we can re-index each row of the matrix $(c_{i,j})_{i\in\{1,3,\dots,n-1\},j\in\mathcal{I}_l}$ so that

$$-c_{1,0} \quad \text{is the \#-special element of} \quad \{-c_{1,1-l}, -c_{1,3-l}, \dots, -c_{1,l-1}\}, \tag{IV.8}$$

$$c_{i,j} \geq c_{i+2,j} \quad \text{for} \quad 0 \neq j \in \mathcal{I}_l, i \in \{1,\dots,n-3\}, \tag{IV.9}$$

$$c_{i,0} \geq c_{i+2,0} \quad \text{for} \quad i \in \{3,\dots,n-3\}, \tag{IV.10}$$

and

$$c_{1,0}+c_{3,0} \leq 1. \tag{IV.11}$$

If condition (iii) holds from Theorem IV.3.14 then since $(1-\frac{1}{4}u^{-1})\mu_1^{\#}(u) = (1+c_{i,1-l}u^{-1})\dots(1+(1-c_{1,0})u^{-1})\dots(1+c_{i,l-1}u^{-1})(1-\frac{1}{2}u^{-1})$, by Lemmas IV.3.6 and IV.3.17 we can further re-index so that (IV.8), (IV.9), (IV.10), and (IV.11) still hold, $c_{1,j}+c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, and $c_{1,0} \geq \frac{1}{2}$. Since $-c_{3,0} \geq c_{1,0}-1$ we now have that $c_{1,0}-c_{3,0} \geq c_{1,0}+c_{1,0}-1 \geq 0$.

If condition (iv) holds from Theorem IV.3.14 then since $(1-\frac{1}{2}u^{-1})\mu_1^{\#}(-u) \Rightarrow (1+\frac{1}{2}u^{-1})\mu_1^{\#}(u) = (1+c_{i,1-l}u^{-1})\dots(1+(1-c_{1,0})u^{-1}))\dots(1+c_{i,l-1}u^{-1})$, by Lemmas IV.3.6 and IV.3.18 we can further re-index so that (IV.8), (IV.9), (IV.10), and (IV.11) still hold, $c_{1,j}+c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, and $c_{1,0} \geq 1$. Since $-c_{3,0} \geq c_{1,0}-1$ we now have that $c_{1,0}-c_{3,0} \geq c_{1,0}+c_{1,0}-1 \geq 1$.

To prove the converse suppose that we are given a matrix $(c_{i,j})_{i\in\{1,3,\dots,n-1\},j\in\mathcal{I}_l}$ where $c_{1,j}+c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, $c_{1,0}+c_{3,0} \leq 1$, and $c_{i,j} \geq c_{i+2,j}$ for $i \in \{1,3,\dots,n-3\}, j \in \mathcal{I}_l$. For $i \in \{1,3,\dots,n-1\}$ let $\mu_i(u) = (1+c_{i,1-l}u^{-1})(1+c_{i,3-l}u^{-1})\dots(1+c_{i,l-1}u^{-1})(1+\frac{1}{2}u^{-1})^{-1}$. Now it is clear by Lemma IV.2.4 that $\mu_1(u) \to \mu_3(u) \to \dots \to \mu_{n-1}(u)$. Since $c_{1,0}+c_{3,0} \leq 1$ and $c_{1,0}-c_{3,0} \geq 0$, we have that $2c_{1,0} \in \mathbb{Z}$. If $c_{1,0} \leq \frac{1}{2}$ then by Lemma IV.3.15 $\mu_1(-u) \Rightarrow \mu_1(u)$. If $c_{1,0} \leq 0$ then by Lemma IV.3.16 $\frac{2u-1}{2u+1}\mu_1(-u) \Rightarrow \mu_1(u)$. If $c_{1,0} > \frac{1}{2}$ then by Lemma IV.3.17 $\mu_1^{\#}(-u) \Rightarrow \mu_1^{\#}(u)$. If $c_{1,0} > 0$ then by Lemma IV.3.18 $\frac{2u-1}{2u+1}\mu_1^{\#}(-u) \Rightarrow \mu_1^{\#}(u)$. So in all cases by Theorem IV.3.14 $L(\mu_1(u),\mu_3(u),\dots,\mu_{n-1}(u))$ is finite dimensional. □

Lemma IV.4.5. *Theorem IV.1.1 holds in the case that $\phi = +, \epsilon = +$, $n > 2$ is even, and l is odd.*

Proof. If the Y_n^+-module $L(\mu_1(u),\mu_3(u),\dots,\mu_{n-1}(u))$ is finite dimensional then by Lemma IV.4.4 we can write $\mu_i(u) = (1+c_{i,1-l}u^{-1})(1+c_{i,3-l}u^{-1})\dots(1+c_{i,l-1}u^{-1})(1+\frac{1}{2}u^{-1})^{-1}$ for all $i \in \{1,3,\dots,n-1\}$ where $c_{1,j}+c_{1,-j} \leq 0$ for $0 \neq j \in \mathcal{I}_l$, $c_{1,0}+c_{3,0} \leq 1$, and $c_{i,j} \geq c_{i+2,j}$ for $i \in \{1,3,\dots,n-3\}, j \in \mathcal{I}_l$. Associate to this data the skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j}$ where $a_{i,j} = c_{i,j} - \frac{i}{2}$ for $i \in \{1, 3, \ldots, n-1\}$, $j \in \mathcal{I}_l$, and $a_{i,j} = -a_{-i,-j}$ for $i \in \{1-n, 3-n, \ldots, -1\}$, $j \in \mathcal{I}_l$, and now it is clear that $A \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$.

We still need to show that every element of $\mathrm{Col}^{\mathfrak{g}}_{n,l}$ can be obtained from these representations. Let $A \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$, and for $i \in \{1, 3, \ldots, n-1\}$, $j \in \mathcal{I}_l$ set $c_{i,j} = a_{i,j} + \frac{\mathrm{row}(i)}{2}$. For $i \in \{1, 3, \ldots, n-1\}$ let

$$\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,3-l}u^{-1}) \ldots (1 + c_{i,l-1}u^{-1})(1 + \frac{1}{2}u^{-1})^{-1}.$$

Let $j \in \mathcal{I}_l$ be such that $-c_{1,j}$ is the #-special element of $\{c_{1,l-1}, c_{1,3-l}, \ldots, c_{1,l-1}\}$.

If $j = 0$ then by Lemma IV.4.4 $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is finite dimensional, so suppose that $j \neq 0$. By Lemma IV.3.10 we can choose a matrix $(d_{i,j})_{i \in \{1,3,\ldots,n-1\}, j \in \mathcal{I}_l}$ such that each row is a permutation of the numbers in the corresponding row in $(c_{i,j})$ and so that $d_{1,0} = c_{1,j}$, $d_{1,2} = c_{1,0}$, $d_{3,2} = c_{3,0}$, $d_{1,k} + d_{1,-k} \leq 0$ for $0 \neq k \in \mathcal{I}_l$, and so that $d_{i,k} \geq d_{i+2,k}$ for all $i \in \{1, 3, \ldots, n-10\}, k \in \mathcal{I}_l$. Now we have the following: $d_{1,2} \geq d_{3,2}, d_{1,0} \geq d_{3,0}$, and $d_{1,2} + d_{3,2} \leq 1$. Furthermore by Lemma IV.3.10 we have that $d_{1,0} \leq d_{1,2}$. If $d_{1,0} + d_{3,0} \leq 1$ then by Lemma IV.4.4 $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is finite dimensional. So suppose that $d_{1,0} + d_{3,0} \geq 2$. Since $d_{1,0} \geq d_{3,0}$ this implies that $d_{1,0} \geq 1$, which in turn implies that $d_{1,2} \geq 1$. Since $d_{1,2} + d_{3,2} \leq 1$, this implies that $d_{3,2} \leq 0$, so $d_{1,0} - d_{3,2} \geq 0$. We also have that $d_{1,0} + d_{3,2} \leq d_{1,2} + d_{3,2} \leq 1$. Finally we have that $d_{1,2} - d_{3,0} \geq 0$ since $d_{1,2} \geq d_{1,0} \geq d_{3,0}$. So if we swap $d_{3,0}$ with $d_{3,2}$ then $(d_{i,j})_{i \in \{1,3,\ldots,n-1\}, j \in \mathcal{I}_l}$ satisfies all the properties from Lemma IV.4.4, so this lemma implies that $L(\mu_1(u), \mu_3(u), \ldots, \mu_{n-1}(u))$ is finite dimensional. ☐

Lemma IV.4.6. *Theorem IV.1.1 holds in the case that $\phi = +, \epsilon = -$, n is even, and l is even.*

Proof. In this case, by Corollary III.2.4, an irreducible highest weight Y_n^+-module $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ factors through κ_l if $\mu_i(u)$ is a polynomial of degree l or less for all $i \in \{1, 3, \dots, n-1\}$. Furthermore, if $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is finite dimensional then by Theorem IV.3.14, Lemma IV.3.6 and Lemma IV.2.5 we can write $\mu_i(u) = (1 + c_{i,1-1}u^{-1})(1 + c_{i,l-3}u^{-1}) \dots (1 + c_{i,l-1}u^{-1})$ for $i \in \{1, 3, \dots, n-1\}$. Since $(1 + \frac{1}{2}u^{-1})\mu_i(u)$ is now a polynomial of odd degree we can apply the conditions from the case where $\phi = +, \epsilon = +, n$ is even, and l is odd to determine precisely when $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is finite dimensional. This exactly translates to the condition that $A^+ \in \mathrm{Col}_{n,l+1}^{\mathfrak{so}_n(l+1)}$, and the proof of the $\phi = +, \epsilon = +, n$ is even, and l is odd case shows that every element of $\{A \in \mathrm{Col}_{n,l}^{\mathfrak{g}} \mid A^+ \in \mathrm{Col}_{n,l+1}^{\mathfrak{so}_{n(l+1)}}\}$ corresponds to a finite dimensional irreducible $U(\mathfrak{g}, e)$-module. ☐

Lemma IV.4.7. *Theorem IV.1.1 holds in the case that $\phi = +, \epsilon = +$, n is odd, and l is odd.*

Proof. By Corollary III.2.4 we have that an irreducible highest weight Y_n^+-module $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ factors through κ_l if $\mu_i(u)(1 + \frac{1}{2}u^{-1})$ is a polynomial of degree l or less for all $i \in \{1, 3, \dots, n-1\}$. So for all $i \in \{1, 3, \dots, n-1\}$ we can write $\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,3-l}u^{-1}) \dots (1 + c_{i,l-1}u^{-1})(1 + \frac{1}{2}u^{-1})^{-1}$ for all $i \in \{1, 3, \dots, n-1\}$. Additionally, since $\mu_0(u) \in 1 + u^{-2}\mathbb{C}[[u^{-2}]]$, we must have that $\mu_0(u)$ is a polynomial of degree $l-1$ or less, and we can re-index so that

$$c_{0,j} = -c_{0,-j} \quad \text{for} \quad 0 \neq j \in \mathcal{I}_l \quad \text{and} \quad c_{0,0} = \frac{1}{2}. \tag{IV.12}$$

If condition (i) holds from Theorem IV.3.19 then by Lemma IV.2.5 we can re-index so that (IV.12) holds and $c_{i,j} \geq c_{i+2,j}$ for $i \in \{0, 2, \dots, n-3\}$, $j \in \mathcal{I}_l$. In particular, we now have that $\frac{1}{2} \geq c_{2,0} \geq c_{4,0} \geq \cdots \geq c_{n-1,0}$. Associate to this data the skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j}$ where $a_{0,0} = 0$,

$a_{i,j} = c_{i,j} - \frac{i}{2}$ for $i \in \{0, 2, \dots, n-1\}$, $j \in \mathcal{I}_l$, $(i,j) \neq (0,0)$, and $a_{i,j} = -a_{-i,-j}$ for $i \in \{1-n, 3-n, \dots, -1\}$, $j \in \mathcal{I}_l$. Now it is clear that $A \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$.

If condition (ii) holds from Theorem IV.3.19 then $\mu_0(u) \to (1+\frac{1}{2}u^{-1})\mu_2(u) \to (1+\frac{1}{2}u^{-1})\mu_4(u) \to \cdots \to (1+\frac{1}{2}u^{-1})\mu_{n-1}(u)$. So by Lemma IV.2.5 we can re-index so that (IV.12) holds, $0 \geq c_{2,0} \geq c_{4,0} \geq \cdots \geq c_{n-1,0}$, and $c_{i,j} \geq c_{i+2,j}$ for $i \in \{0, 2, \dots, n-3\}$, $0 \neq j \in \mathcal{I}_l$. Associate to this data the skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j}$ where $a_{0,0} = 0$, $a_{i,j} = c_{i,j} - \frac{i}{2}$ for $i \in \{0, 2, \dots, n-1\}$, $j \in \mathcal{I}_l$, $(i,j) \neq (0,0)$, and $a_{i,j} = -a_{-i,-j}$ for $i \in \{1-n, 3-n, \dots, -1\}$, $j \in \mathcal{I}_l$. Now it is clear that $A \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$.

This process is easily reversed: starting with a skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_j} \in \mathrm{Col}^{\mathfrak{g}}_{n,l}$, set $c_{0,0} = \frac{1}{2}$, for $i \in \{1, 3, \dots, n-1\}$, $j \in \mathcal{I}_l$, $(i,j) \neq (0,0)$ set $c_{i,j} = a_{i,j} + \frac{i}{2}$, then set $\mu_i(u) = (1 + c_{i,1-l}u^{-1})(1 + c_{i,l-3}u^{-1}) \dots (1 - c_{i,l-1}u^{-1})(1 + \frac{1}{2}u^{-1})^{-1}$. Now if $c_{2,0} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ then by Lemma IV.2.4 $\mu_0(u) \to \mu_2(u) \to \cdots \to \mu_{n-1}(u)$, and if $c_{2,0} \in \mathbb{Z}$ then by Lemma IV.2.4 $\frac{2u}{2u+1}\mu_0(u) \to \mu_2(u) \to \mu_4(u) \to \cdots \to \mu_{n-1}(u)$, so $L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ is a finite dimensional Y_n^--module by Theorem IV.3.19. $\square$

Lemma IV.4.8. *Theorem IV.1.1 holds in the case that $\phi = +, \epsilon = -$, n is odd, and l is even.*

Proof. In this case, by Corollary III.2.4, an irreducible highest weight Y_n^+-module $L(\mu_0(u), \mu_0(u), \dots, \mu_{n-1}(u))$ factors through κ_l if $\mu_i(u)$ is a polynomial of degree l or less for all $i \in \{0, 2, \dots, n-1\}$. Since $(1 + \frac{1}{2}u^{-1})\mu_i(u)$ is now a polynomial of odd degree, we can apply the conditions from the case where $\phi = +, \epsilon = +, n$ is odd, and l is odd to determine precisely when $L(\mu_0(u), \mu_2(u), \dots, \mu_{n-1}(u))$ is finite dimensional. This exactly translates to the condition that $A^+ \in \mathrm{Col}^{\mathfrak{so}_{n(l+1)}}_{n,l+1}$, and the proof of the $\phi = +, \epsilon = +, n$ is odd, and l is odd case shows that every element of $\{A \in \mathrm{Col}^{\mathfrak{g}}_{n,l} \mid A^+ \in \mathrm{Col}^{\mathfrak{so}_{n(l+1)}}_{n,l+1}\}$ corresponds to a finite dimensional irreducible $U(\mathfrak{g}, e)$-module. □

IV.5 Action of the Component Group C

In this section we show how to explicitly calculate the action of the component group $C = C_G(e,h,f)/C_G(e,h,f)^\circ$ on the set of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules. Here $C_G(e,h,f)$ denotes the centralizer of the $\mathfrak{sl}_2$-triple (e,h,f) in the adjoint group G of $\mathfrak{g}$. Recall Losev's near classification of finite dimensional irreducible representations of $U(\mathfrak{g}, e)$: there exists a map

$$\dagger : \mathrm{Prim}\, U(\mathfrak{g}, e) \to \mathrm{Prim}\, U(\mathfrak{g})$$

such that $\mathrm{Ann}(L^\dagger) = \mathrm{Ann}(L)^\dagger$. He also shows that on restricting $\dagger$ to $\mathrm{Prim}_{\mathsf{fin}} U(\mathfrak{g}, e)$, the set of primitive ideals of $U(\mathfrak{g}, e)$ of finite co-dimension, that

$$\dagger : \mathrm{Prim}_{\mathsf{fin}} U(\mathfrak{g}, e) \twoheadrightarrow \mathrm{Prim}_{\overline{G.e}} U(\mathfrak{g}),$$

where

$$\mathrm{Prim}_{\overline{G.e}} U(\mathfrak{g}) = \{I \in \mathrm{Prim} U(\mathfrak{g}) \mid \mathcal{VA}(I) = \overline{G.e}\}.$$

Furthermore Losev shows that the fibers of this map are C orbits.

In our special cases we can calculate explicitly the action of C on the set of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules, and therefore on $\mathrm{Prim}_{\mathsf{fin}} U(\mathfrak{g}, e)$. By [C, Chapter 13] the only cases where C is not trivial are the cases when $\epsilon = -$, and n and l are both even; *so we assume this for the rest of this section.*

Now we claim that

$$c = \sum_{\substack{a \in \mathcal{I}_{nl} \\ \mathrm{row}(a) \notin \{\pm 1\}}} e_{a,a} + \sum_{\substack{a,b \in \mathcal{I}_{nl} \\ \mathrm{col}(a)=\mathrm{col}(b) \\ \mathrm{row}(a)=1 \\ \mathrm{row}(b)=-1}} e_{a,b} + e_{b,a}$$

generates C. To see this note that conjugating with c simply transposes each pair of indices $a, b \in \mathcal{I}_{nl}$ where $\mathrm{col}(a) = \mathrm{col}(b), \mathrm{row}(a) = 1, \mathrm{row}(b) = -1$. Since this is an even number of transpositions, we have that $\det c = 1$. It is also clear that $c.J^{-}.c = J^{-}$ (recall that $\mathfrak{g}$ is defined with J^{-} in (III.1)) since for $a \in \mathcal{I}_{nl} \cap \mathbb{Z}_{>0}$ we have that $c.e_{-a,a}.c = e_{-b,b}$ and $c.e_{a,-a}.c = e_{b,-b}$ for some $b \in \mathcal{I}_{nl} \cap \mathbb{Z}_{>0}$. Thus we have that $c \in G$. Furthermore, $c.h.c = h$ (see (III.6) for the definition of h) since for $a \in \mathcal{I}_{nl}$ $c.e_{a,a}.c = e_{b,b}$ for some b such that $\mathrm{col}(b) = \mathrm{col}(a)$. Next note $c.e.c = e$ (see (III.4) for the definition of e) since for $a, b \in \mathcal{I}_{nl}$ such that $\mathrm{row}(a) = \mathrm{row}(b), \mathrm{col}(a)+2 = \mathrm{col}(b)$, $c.f_{a,b}.c = f_{a,b}$ if $\mathrm{row}(a) \notin \{\pm 1\}$, and if $\mathrm{row}(a) = 1$ and $\mathrm{col}(b) \geq 1$ then $c.f_{a,b}.c = f_{a',b'}$ where $\mathrm{col}(a') = \mathrm{col}(a), \mathrm{row}(a') = -1$ and $\mathrm{col}(b') = \mathrm{col}(b), \mathrm{row}(b') = -1$. So we have that $c \in C_G(e, h, f)$. To see that $c \notin C_G(e, h, f)^{\circ}$, we calculate explicitly $\mathfrak{c}_{\mathfrak{g}}(e, h, f)$. By Lemma III.3.2 $\{f_{i,j;r} \mid (i, j, r+1) \text{ is admissible}\}$ is a basis for $\mathfrak{g}^e$. We see from (III.34) that $f_{i,j;r} \in \mathfrak{g}^h = \mathfrak{h}$ if and only if $r = 0$, so $\{f_{i,j;0} \mid (i, j, 1) \text{ is admissible}\}$ is a basis of $\mathfrak{c}_{\mathfrak{g}}(e, h, f)$. Since each of these is a diagonal matrix, it is clear that $c \notin C_G(e, f, h)^{\circ}$. Therefore $C =< c >$.

To understand the action of C on $U(\mathfrak{g}, e)$-mod, we calculate the action of C on $\{\mathrm{pr}\, s_{i,j}(\omega(u)) \mid i, j \in \mathcal{I}_n\}$. Recall the definition of $s_{i,j}$ from (III.20). Note that $c.s_{i,j}(e_{p,p}) = s_{i',j'}(e_{p,p})$ where $i' = i$ if $i \notin \{\pm 1\}$, $i' = -i$ otherwise. Thus $c.\,\mathrm{pr}\, s_{1,1}(\omega(u)) = \mathrm{pr}\, s_{-1,-1}(\omega(u))$, and $c.\,\mathrm{pr}\, s_{i,i}(\omega(u)) = \mathrm{pr}\, s_{i,i}(\omega(u))$ for $i \notin \{\pm 1\}$. Since by Theorem III.1.2 $\kappa_l(S_{i,j}(u) = \mu(s_{i,j}(\omega(u)))$, we see that c acts by the automorphism ψ from (IV.2), factored though κ_l.

Theorem IV.5.1. *Let $A \in \mathrm{Row}^{\mathfrak{g}}_{n,l}$ be such that $A^+ \in \mathrm{Std}^{\mathfrak{so}_{n(l+1)}}_{n,l}$ and let $L(A)$ denote the corresponding finite dimensional irreducible representation of $U(\mathfrak{g}, e)$. Let $(a^+_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_{l+1}} \in \mathrm{Tab}^{\mathfrak{so}_{n(l+1)}}_{n,l+1}$ be a representative of A^+ so that $a^+_{-1,0} - \frac{1}{2}$ is the $\#$-special element of $\{a^+_{-1,1-l} - \frac{1}{2}, a^+_{-1,3-l} - \frac{1}{2}, \dots, a^+_{-1,l-1} - \frac{1}{2}\}$. Let $B = (b_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_{l+1}}$ be the skew-symmetric $n \times (l+1)$ tableaux such that for $i \in \mathcal{I}_n, j \in \mathcal{I}_l$,*

$$b_{i,j} = \begin{cases} a^+_{i,j} & \text{if } (i,j) \neq (1,0),\ (i,j) \neq (-1,0); \\ a^+_{-1,0} & \text{if } i = 1, j = 0; \\ a^+_{1,0} & \text{if } i = -1, j = 0. \end{cases}$$

Then $c.L(A) = L(D)$ where $D \in \mathrm{Tab}^{\mathfrak{g}}_{n,l}$ satisfies $D^+ = B$.

Proof. Observe that the Y^+_n-module $L(A^+) = L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))$ where for $i \in \{1, 3, \dots, n-1\}$ $\mu_i(u) = (1 + c_{i,1-1}u^{-1})(1 + c_{i,l-3}u^{-1}) \dots (1 + c_{i,l-1}u^{-1})$, where $c_{i,j} = a^+_{i,j} + \frac{i}{2}$ for $i \in \{1, 3, \dots, n-1\}, j \in \mathcal{I}_l$.

Now by Theorem IV.3.12,

$$\psi.L(A) = L(\mu_1(u), \mu_3(u), \dots, \mu_{n-1}(u))^{\#} = L(\mu_1^{\#}(u), \mu_3(u), \dots, \mu_{n-1}(u)),$$

and by (IV.4), $\mu_1^{\#}(u) = (1 + c_{1,1-1}u^{-1}) \dots (1 + (1 - c_{1,0})u^{-1}) \dots (1 + c_{1,l-1}u^{-1})$. Since $b_{1,0} = -a^+_{1,0} = -(c_{1,0} - \frac{1}{2}) = 1 - c_{1,0} - \frac{1}{2}$, we have that $L(\mu_1^{\#}(u), \mu_3(u), \dots, \mu_{n-1}(u)) = L(B)$. □

IV.6 BGK Highest Weight Theory for Rectangular Finite W-algebras

In this section we show how the highest weight theory from §II.4 is related to the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules given in Theorem

IV.1.1. Recall that the BGK highest weight theory is defined in terms of a the finite dimensional irreducible representations of a "smaller" finite W-algebra $U(\mathfrak{g}_0, e)$. In our cases it turns out that e is regular in $\mathfrak{g}_0$, so by Kostant's Theorem and the Harish-Chandra isomorphism $U(\mathfrak{g}_0, e) \cong S(\mathfrak{t})^{W_0}$, a free polynomial algebra. By determining which finite dimensional irreducible representations of $S(\mathfrak{t})^{W_0}$ correspond to finite dimensional irreducible representations of $U(\mathfrak{g}, e)$ we obtain a potentially different classification of the finite dimensional irreducible representations of $U(\mathfrak{g}, e)$. In this section we will show that this "other" classification is in fact the same as the one given in Theorem IV.1.1.

First we need to fix choices of $\mathfrak{t}$, a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{b}$, a Borel subalgebra of $\mathfrak{g}$ as in §II.5. We let $\mathfrak{t}$ be the span of diagonal matrices in $\mathfrak{g}$. We choose our Borel subalgebra $\mathfrak{b}$ by specifying a system of positive roots. For $a \in \mathcal{I}_{nl}$ let $\epsilon_a \in \mathfrak{t}^*$ be the restriction to $\mathfrak{t}$ of the diagonal coordinate function of $\mathfrak{g}_{nl}$ given by $\epsilon_a(e_{b,b}) = \delta_{a,b}$. If $\epsilon = -$ (so nl is even) our positive root system is

$$\begin{aligned}\Phi^+ = \{\epsilon_a - \epsilon_b \mid a, b \in \mathcal{I}_{nl}, \mathrm{row}(a) < \mathrm{row}(b)\} \\ \cup \{\epsilon_a - \epsilon_b \mid a, b \in \mathcal{I}_{nl}, \mathrm{row}(a) = \mathrm{row}(b), \mathrm{col}(a) < \mathrm{col}(b)\}.\end{aligned}$$

If $\epsilon = +$ then

$$\begin{aligned}\Phi^+ = \{\epsilon_a - \epsilon_b \mid a, b \in \mathcal{I}_{nl}, \mathrm{row}(a) < \mathrm{row}(b), a \neq -b\} \\ \cup \{\epsilon_a - \epsilon_b \mid a, b \in \mathcal{I}_{nl}, \mathrm{row}(a) = \mathrm{row}(b), \mathrm{col}(a) < \mathrm{col}(b), a \neq -b\}.\end{aligned}$$

Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}$ corresponding to this choice of positive roots.

Next we give an explicit basis for $\mathfrak{t}^e$, the centralizer of e in $\mathfrak{t}$. In Lemma III.3.2 a basis for $\mathfrak{g}^e$ is given in terms of certain elements $f_{i,j;r}$, and it is easy to see that each of these is nilpotent except for when $r = 0$ and $i = j$. So by Lemma III.3.2 a basis for $\mathfrak{t}^e$ is given by

$$\{f_{i,i;0} \mid i \in \{1-n, 3-n, \ldots, -1\}\}.$$

More explicitly, for $i \in \mathcal{I}_n$ we have that

$$f_{i,i;0} = \sum_{\substack{a \in \mathcal{I}_{nl} \\ \mathrm{row}(a)=i}} f_{a,a}.$$

Next we give basis for $(\mathfrak{t}^e)^*$. Let $\delta_i \in (\mathfrak{t}^e)^*$ be defined via $\delta_i(f_{j,j,;0}) = \delta_{i,j}$ for $i, j \in \mathcal{I}_n$, $i, j < 0$.

Now $\mathfrak{g}_0$ is the span of $\{f_{a,b} \mid a, b \in \mathcal{I}_{nl}, \mathrm{row}(a) = \mathrm{row}(b)\}$, so

$$\mathfrak{g}_0 \cong \begin{cases} \mathfrak{g}_l^{\oplus n/2} & \text{if } n \text{ is even;} \\ \mathfrak{g}_l^{\epsilon} \oplus \mathfrak{g}_l^{\oplus (n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

We also have that the parabolic $\mathfrak{q} = \mathfrak{b} + \mathfrak{g}_0$ is the span of

$$\{f_{a,b} \mid a, b \in \mathcal{I}_{nl}, \mathrm{row}(a) \leq \mathrm{row}(b)\}.$$

Note that for $a, b \in \mathcal{I}_{nl}$, we have that $f_{a,b} \in \mathfrak{g}_{\delta_{\mathrm{row}(b)} - \delta_{\mathrm{row}(a)}}$. Thus

$$\Phi_+^e = \{\delta_i - \delta_j \mid i, j \in \mathcal{I}_n, i < j\}.$$

Recall for $i, j \in \mathcal{I}_n$ that there is a map $s_{i,j} : T(\mathfrak{g}_l) \to U(\mathfrak{g})$ defined in (III.22). This definition makes it clear that for any $v \in T(\mathfrak{g}_l)$, $s_{i,j}(v) \in U(\mathfrak{g})_{\delta_j - \delta_i}$. Thus we can explicitly state a choice for the $\mathfrak{t}^e$-equivariant map $\Theta : \mathfrak{g}^e \to U(\mathfrak{g}, e)$ from (II.4): For

$i, j \in \mathcal{I}_n$, $r \geq 0$ we set $\Theta(f_{i,j;r}) = s_{i,j}(\omega_{r+1})$. Thus $\{s_{i,j}(\omega_{r+1}) \,|\, r \geq 0, i, j \in \mathcal{I}_n, i < j\}$ generate the left $U(\mathfrak{g}, e)$ ideal $U(\mathfrak{g}, e)_{\#}$.

Recall the maps $\pi_{-\gamma} : U(\mathfrak{g}, e)_0 \twoheadrightarrow U(\mathfrak{g}_0, e)$ from Theorem II.4.2 and $\xi_{-\eta} : U(\mathfrak{g}_0, e) \tilde{\to} S(\mathfrak{t})^{W_0}$ from Lemma II.5.1. These maps make every $S(\mathfrak{t})^{W_0}$-module a $U(\mathfrak{g}, e)_0$-module. We need to calculate the action of $s_{i,i}(\omega_{r+1})$ on a $S(\mathfrak{t})^{W_0}$-module, so we need to calculate $\xi_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega_r))$. We do this with a series of lemmas.

This lemma is a special case of Lemma III.4.1:

Lemma IV.6.1. *For $i, j \in \mathcal{I}_n, p, q \in \mathcal{I}_l$*

$$\begin{aligned}
&[s_{i,j}(e_{p,q}), s_{h,k}(e_{v,w})] \\
&\quad = \delta_{h,j}\delta_{q,v}s_{i,k}(e_{p,w}) - \delta_{i,k}\delta_{p,w}s_{h,j}(e_{v,q}) \\
&\qquad + \gamma(-\delta_{h,-i}\delta_{v,-p}s_{-j,k}(e_{-q,w}) + \delta_{-j,k}\delta_{w,-q}s_{h,-i}(e_{v,-p})),
\end{aligned}$$

Note that $s_{i,i}(\omega_r)$ is a linear combination of monomials of the form

$$s_{i,i_1}(e_{p_1,q_1})s_{i_1,i_2}(e_{p_2,q_2}) \dots s_{i_{m-1},i}(e_{p_m,q_m}), \tag{IV.13}$$

where $i_j \in \mathcal{I}_n$ for $j = 1, \dots, m-1$, $p_i \leq q_i$ for $i = 1, \dots, m$, and $q_i < q_{i+1}$ for $i = 1, \dots, m-1$. So to calculate $\xi_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega_r))$ we first prove a lemma about applying $\pi : U(\mathfrak{p})_0 \to U(\mathfrak{p}_0)$ to such monomials.

Lemma IV.6.2. *Let*

$$v = s_{i,i_1}(e_{p_1,q_1})s_{i_1,i_2}(e_{p_2,q_2}) \dots s_{i_{m-1},i}(e_{p_m,q_m})$$

be as in (IV.13). *If $i \geq 0$ then $\pi(v) = 0$ unless $i_1 = i_2 = \dots = i_{m-1} = i$.*

Proof. For uniformity, let $i_0, i_m = i$. The key fact used repeatedly in this proof is that if $w = s_{j_1,j_2}(e_{r_1,r_2}) \ldots s_{j_k,j_{k+1}}(e_{r_k,r_{k+1}}) \in U(\mathfrak{p})_0$ satisfies $j_1 > j_2$ or $j_k < j_{k+1}$ then $w \in U(\mathfrak{p})_{0,\#} = U(\mathfrak{p})_{\flat,0}$, so $\pi(w) = 0$.

By Lemma IV.6.1 we see that each term $s_{i_{j-1},i_j}(e_{p_j,q_j})$ of v commutes with all terms $s_{i_{k-1},i_k}(e_{p_k,q_k})$ in v unless $p_k = q_j$, $q_k = p_j$, $p_k = -p_j$, or $q_k = -q_j$.

Suppose that there exists j such that $i_{j-1} < i_j$ and $p_j, q_j > 0$. Then $s_{i_{j-1},i_j}(e_{p_j,q_j})$ commutes with every term to its right, so $\pi(v) = 0$. Next suppose that there exists a j such that $i_{j-1} > i_j$ and $p_j, q_j < 0$. Then $s_{i_{j-1},i_j}(e_{p_j,q_j})$ commutes with every term to its left, so π of such a monomial is 0. So $\pi(v) = 0$ unless v satisfies $i_{j-1} \leq i_j$ if $q_j \leq 0$ and $i_{j-1} \geq i_j$ if $p_j \geq 0$, so for the rest of this proof we assume that this is the case.

Now suppose that there exists a j such that $p_j < 0$, $q_j > 0$ and $i_{j-1} < i_j$. Then for $k \neq j$ we must have that $i_k \geq 0$. Note that $s_{i_{j-1},i_j}(e_{p_j,q_j})$ must commute with every term to its right unless there exists $k > j$ such that $p_k = -p_j$. In this case, $[s_{i_{j-1},i_j}(e_{p_j,q_j}), s_{i_{k-1},i_k}(e_{p_k,q_k})]$ is a multiple of $s_{-i_j,i_k}(e_{-q_j,q_k})$, which commutes with every term to the right of $s_{i_{k-1},i_k}(e_{p_k,q_k})$. Furthermore since $i_j > 0$ and $i_k \geq 0$ we have that $-i_j < i_k$. Thus $\pi(v) = 0$.

Next suppose that there exists a j such that $p_j < 0$, $q_j > 0$ and $i_{j-1} > i_j$. Then for $k \neq j-1$ we must have that $i_k \geq 0$. Note that $s_{i_{j-1},i_j}(e_{p_j,q_j})$ must commute with every term to its left unless there exists $k < j$ such that $q_k = -q_j$. In this case, $[s_{i_{k-1},i_k}(e_{p_k,q_k}), s_{i_{j-1},i_j}(e_{p_j,q_j})]$ is a multiple of $s_{i_{j-1},-i_{k-1}}(e_{p_j,-p_k})$, which commutes with every term to the left of $s_{i_{k-1},i_k}(e_{p_k,q_k})$, and it also satisfies $i_{j-1} > -i_{k-1}$. Thus $\pi(v) = 0$.

So for the rest of the proof we will assume that if there exists a j such that $p_j < 0$ and $q_j > 0$ then $i_{j-1} = i_j$.

Let j be such that $i_{j-1} < i_j$ and i_j is maximal in $\{i_1, \ldots, i_m\}$. Now it must be the case that $q_j < 0$. Since $i_k \geq 0$ for all k, by Lemma IV.6.1 $s_{i_{j-1},i_j}(e_{p_j,q_j})$

must commute with every term to its right unless $i = 0$. So if $i \neq 0$, then π of this monomial is 0, so assume that $i = 0$. Even in the case that $i = 0$, since $i_j > 0$ we still have that $s_{i_{j-1},i_j}(e_{p_j,q_j})$ commutes with every term to its right unless there exists a $k > j$ such that $p_k = -p_j$. In this case, $[s_{i_{j-1},i_j}(e_{p_j,q_j}), s_{i_{k-1},i_k}(e_{p_k,q_k})]$ is a multiple of $s_{-i_j,i_k}(e_{-q_j,q_k})$ which commutes with every term to the right of $s_{i_{k-1},i_k}(e_{p_k,q_k})$. Furthermore, $-i_j < i_k$ since $i_j > 0$, so $\pi(v) = 0$.

Thus we have proven that $\pi(v) \neq 0$ if and only if $i_1 = i_2 = \cdots = i_m = i$. $\square$

Suppose that

$$v = s_{i,i_1}(e_{p_1,q_1})s_{i_1,i_2}(e_{p_2,q_2}) \ldots s_{i_{m-1},i}(e_{p_m,q_m}) \in U(\mathfrak{p}_0)$$

Now it is clear that

$$\xi(v) = 0 \text{ unless } p_j = q_j \text{ for } j = 1, \ldots, m. \tag{IV.14}$$

Lemma IV.6.3. *Let $i \in \mathcal{I}_n \cap \mathbb{Z}_{\geq 0}, p \in \mathcal{I}_l$. Then*

$$S_{-\eta}(S_{-\gamma}(s_{i,i}(e_{p,p} + \rho_p))) = \begin{cases} s_{i,i}(e_{p,p}) + \frac{i}{2} & \text{if } p \neq 0; \\ s_{i,i}(e_{p,p}) + \frac{i}{2} - \frac{\epsilon}{2} & \text{if } p = 0,\ i \neq 0; \\ s_{i,i}(e_{p,p}) & \text{if } p, i = 0; \end{cases}$$

Proof. Recall that the weight γ is defined by choosing a weight basis $\{b_1, \ldots, b_r\}$ for $\mathfrak{m}$, where each b_i is of weight $\beta_i \in \mathfrak{t}^*$. A natural basis to choose is

$$\{f_{a,b} \mid a, b \in \mathcal{I}_{nl}, (a, b, 1) \text{ is admissible, } \operatorname{col}(a) > \operatorname{col}(b)\}.$$

Here $f_{a,b}$ is of weight $\epsilon_a - \epsilon_b$. Recall that γ is now defined by

$$\gamma = \sum_{\substack{1 \le i \le r \\ \beta_i|_{\mathfrak{t}^e} \in \Phi_-^e}} \beta_i.$$

Thus in these cases

$$\gamma = \sum_{\substack{a,b \in \mathcal{I}_{nl} \\ \operatorname{col}(a) > \operatorname{col}(b) \\ \operatorname{row}(a) > \operatorname{row}(b) \\ (a,b,1) \text{ is admissible}}} \epsilon_a - \epsilon_b.$$

To calculate $\gamma(f_{a,a})$, first we include a few useful calculations. Let $\mathrm{RA}(a)$ denote the number of rows occurring above the number a in the $n \times l$ array used to define $U(\mathfrak{g}, e)$ in §III.1. Similarly $\mathrm{RB}(a)$ denotes the number of rows below a, $\mathrm{CL}(a)$ denotes the number of columns left of a, and $\mathrm{CR}(a)$ denotes the number of columns to the right of a. Then

$$\mathrm{RA}(a) = \frac{n - 1 + \operatorname{row}(a)}{2},$$
$$\mathrm{RB}(a) = \frac{n - 1 - \operatorname{row}(a)}{2},$$
$$\mathrm{CL}(a) = \frac{l - 1 + \operatorname{col}(a)}{2},$$

and

$$\mathrm{CR}(a) = \frac{l - 1 - \operatorname{col}(a)}{2}.$$

Naturally $\gamma(f_{a,a})$ is the number of times ϵ_a appears in the above equation. Every root involving ϵ_a is of the form $\epsilon_a - \epsilon_b$ or $\epsilon_b - \epsilon_a$. Roots of the first type occur in γ when b is north west of a, that is b is above and to the left of a. Roots of the second type occur when b is south east of a. Finally we need make a special adjustment when $b = -a$. This root only occurs in γ when $\mathfrak{g} = \mathfrak{sp}_{nl}$ and a is in the upper left or lower right quadrant. So we need to add or subtract 1 when a is in the upper left or lower right quadrant depending on whether $\mathfrak{g} = \mathfrak{sp}_{nl}$ or $\mathfrak{so}_{nl}$. Note that the number of boxes north west of a is $\mathrm{RA}(a)\mathrm{CL}(a)$. The number of boxes south east of a is $\mathrm{RB}(a)\mathrm{CR}(a)$. We divide things into three cases: Case 1 is when a is not in the lower left or upper right quadrant, Case 2 is when a is in the lower right quadrant, and Case 3 is when a is in the upper left quadrant. Then

$$\gamma(f_{a,a}) = \begin{cases} \mathrm{RA}(a)\mathrm{CL}(a) - \mathrm{RB}(a)\mathrm{CR}(a) & \text{in Case 1;} \\ \mathrm{RA}(a)\mathrm{CL}(a) - \mathrm{RB}(a)\mathrm{CR}(a) - \epsilon & \text{in Case 2;} \\ \mathrm{RA}(a)\mathrm{CL}(a) - \mathrm{RB}(a)\mathrm{CR}(a) + \epsilon & \text{in Case 3.} \end{cases}$$

In calculations below we use the following simplification:

$$\mathrm{RA}(a)\mathrm{CL}(a) - \mathrm{RB}(a)\mathrm{CR}(a) = \frac{1}{2}((n-1)\,\mathrm{col}(a) + (l-1)\,\mathrm{row}(a)).$$

Now we turn our attention to the shift S_η. Recall that

$$\eta = \frac{1}{2} \sum_{\substack{\alpha \in \Phi \\ \alpha|_{\mathfrak{t}^e} \in\ Phi^e_+}} \alpha + \frac{1}{2} \sum_{\substack{1 \le i \le r \\ \beta_i|_{\mathfrak{t}^e} = 0}} \beta_i.$$

Let η_1 denote the left sum, η_2 denote the right sum. Now $\eta_1(f_{a,a})$ is the number of of boxes below a minus the number of boxes above a, with an adjustment for when a multiple of ϵ_a is root. Thus

$$\eta_1 = l(\mathrm{RB}(a) - \mathrm{RA}(a)) + z = -l\,\mathrm{row}(a) + z$$

where

$$z = \begin{cases} 0 & \text{if } \operatorname{row}(a) = 0; \\ \epsilon & \text{if } \operatorname{row}(a) > 0; \\ -\epsilon & \text{if } \operatorname{row}(a) < 0. \end{cases}$$

Also we calculate using the fact that $\mathrm{CL}(a) - \mathrm{CR}(a) = \operatorname{col}(a)$ to get that

$$\eta_2(f_{a,a}) = \begin{cases} \operatorname{col}(a) & \text{if } \operatorname{row}(a) \neq 0; \\ \operatorname{col}(a) - \epsilon & \text{if } \operatorname{row}(a) = 0, \text{ and } \operatorname{col}(a) > 0; \\ \operatorname{col}(a) + \epsilon & \text{uf } \operatorname{row}(a) = 0, \text{ and } \operatorname{col}(a) < 0. \end{cases}$$

Now we are ready to calculate. We do this in a case by case basis.

First we consider the case when a is in the lower right section. We calculate using the above results to get that

$$\begin{aligned} \gamma(f_{a,a}) + \eta(f_{a,a}) &= \frac{1}{2}((n-1)\operatorname{col}(a) + (l-1)\operatorname{row}(a)) - \epsilon \\ &\quad + \frac{1}{2}(-l\operatorname{row}(a) + \epsilon) + \frac{1}{2}\operatorname{col}(a) \\ &= \frac{1}{2}(-\epsilon + n\operatorname{col}(a) - \operatorname{row}(a)). \end{aligned}$$

Note that

$$\rho_{\operatorname{col}(a)} - \gamma(f_{a,a}) - \eta(f_{a,a}) = \frac{\operatorname{row}(a)}{2}.$$

Next we consider the case when a is in the bottom half of the middle column.

We calculate using the above results to get that

$$\begin{aligned}\gamma(f_{a,a})+\eta(f_{a,a}) &= \frac{1}{2}((n-1)\operatorname{col}(a)+(l-1)\operatorname{row}(a)) \\ &\quad+\frac{1}{2}(-l\operatorname{row}(a)+\epsilon)+\frac{1}{2}\operatorname{col}(a) \\ &= \frac{1}{2}(\epsilon+n\operatorname{col}(a)-\operatorname{row}(a)).\end{aligned}$$

Note that

$$\rho_{\operatorname{col}(a)}-\gamma(f_{a,a})-\eta(f_{a,a})=\frac{\operatorname{row}(a)}{2}-\frac{\epsilon}{2}.$$

Next we consider he case when a is in the right half of the middle row. We calculate using the above results to get that

$$\begin{aligned}\gamma(f_{a,a})+\eta(f_{a,a}) &= \frac{1}{2}((n-1)\operatorname{col}(a)+(l-1)\operatorname{row}(a)) \\ &\quad+\frac{1}{2}(-l\operatorname{row}(a))+\frac{1}{2}(\operatorname{col}(a)-\epsilon) \\ &= \frac{1}{2}(-\epsilon+n\operatorname{col}(a)-\operatorname{row}(a)).\end{aligned}$$

Note that

$$\rho_{\operatorname{col}(a)}-\gamma(f_{a,a})-\eta(f_{a,a})=\frac{\operatorname{row}(a)}{2}.$$

Next we consider he case when a is in the upper right. We calculate using the above results to get that

$$\begin{aligned}\gamma(f_{a,a})+\eta(f_{a,a}) &= \frac{1}{2}((n-1)\operatorname{col}(a)+(l-1)\operatorname{row}(a)) \\ &\quad+\frac{1}{2}(-l\operatorname{row}(a)-\epsilon)+\frac{1}{2}\operatorname{col}(a) \\ &= \frac{1}{2}(-\epsilon+n\operatorname{col}(a)-\operatorname{row}(a)).\end{aligned}$$

Note that

$$\rho_{\mathrm{col}(a)} - \gamma(f_{a,a}) - \eta(f_{a,a}) = \frac{\mathrm{row}(a)}{2}.$$

So in all cases we have that

$$S_{-\eta}(S_{-\gamma}(s_{i,i}(e_{p,p} + \rho_p))) = \begin{cases} \frac{i}{2} & \text{if } p \neq 0; \\ \frac{i}{2} - \frac{\epsilon}{2} & \text{if } p = 0,\ i \neq 0; \\ 0 & \text{if } p, i = 0. \end{cases}$$

□

Let $E_i^{(r)}$ denote the rth elementary symmetric function in

$$\{f_{a,a} + \frac{\mathrm{row}(a)}{2} \mid a \in \mathcal{I}_{nl}, \mathrm{col}(a) \in \mathcal{I}_l\}.$$

Lemma IV.6.4. *Let $i \in \mathcal{I}_n$ If $i > 0$, and l is even then*

$$S_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega_r)) = E_i^{(r)}.$$

If $i \geq 0$ and l is odd then

$$S_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega_r)) = \sum_{i=0}^{r-1} (-2\epsilon)^i E_i^{(r-i)}.$$

Proof. Now if l is even then by Lemma IV.6.2, (IV.14), and Lemma IV.6.3 we have that

$$S_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega(u)) = s_{i,i}(u + e_{1-l,1-l} + i/2) \ldots s_{i,i}(u + e_{l-1,l-1} + i/2)$$

so the lemma holds in this case.

Now we consider the l odd case. Let

$$P_i(u) = s_{i,i}(u + e_{1-l,1-l} + i/2) \dots s_{i,i}(u + e_{-2,-2} + i/2) \\ \times s_{i,i}(u + e_{0.0} + i/2 - \epsilon/2) s_{i,i}(u + e_{2,2} + i/2) + \dots s_{i,i}(u + e_{l-1,l-1} + i/2),$$

and

$$Q_i(u) = s_{i,i}(u + e_{1-l,1-l} + i/2) \dots s_{i,i}(u + e_{-2,-2} + i/2) \\ \times s_{i,i}(e_{0,0} + i/2 - \epsilon/2) s_{i,i}(u + e_{2,2} + i/2) + \dots s_{i,i}(u + e_{l-1,l-1} + i/2).$$

So

$$S_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega(u))) = P_i(u) + \sum_{r=1}^{\infty} (-2\epsilon u)^{-r} Q_i(u).$$

Observe that $P_i(u) = P_i'(u) - \frac{\epsilon}{2} P_i''(u)$ and $Q_i(u) = Q_i'(u) - \frac{\epsilon}{2} P_i''(u)$ where

$$P_i'(u) = s_{i,i}(u + e_{1-l,1-l} + i/2) \dots s_{i,i}(u + e_{-2,-2} + i/2) \\ \times s_{i,i}(u + e_{0,0} + i/2) s_{i,i}(u + e_{2,2} + i/2) + \dots s_{i,i}(u + e_{l-1,l-1} + i/2),$$

$$P_i''(u) = s_{i,i}(u + e_{1-l,1-l} + i/2) \dots s_{i,i}(u + e_{-2,-2} + i/2) \\ \times s_{i,i}(u + e_{2,2} + i/2) + \dots s_{i,i}(u + e_{l-1,l-1} + i/2),$$

and

$$Q_i'(u) = s_{i,i}(u + e_{1-l,1-l} + i/2) \dots s_{i,i}(u + e_{-2,-2} + i/2) \\ \times s_{i,i}(e_{0,0} + i/2) s_{i,i}(u + e_{2,2} + i/2) + \dots s_{i,i}(u + e_{l-1,l-1} + i/2).$$

Also observe that

$$P_i''(u) + \frac{1}{u}Q_i'(u) = \frac{1}{u}P'(u).$$

So

$$\begin{aligned} P(u) - \frac{\epsilon}{2u}Q(u) &= P_i'(u) - \frac{\epsilon}{2}P_i''(u) - \frac{\epsilon}{2u}Q_i'(u) + \frac{1}{4u}P_i''(u) \\ &= P_i'(u) - \frac{\epsilon}{2u}P_i'(u) + \frac{1}{4u}P_i''(u). \end{aligned}$$

Thus

$$\begin{aligned} S_{-\eta} \circ \pi_{-\gamma}(s_{i,i}(\omega(u))) &= P_i(u) + \sum_{r=1}^{\infty}(-2\epsilon u)^{-r}Q_i(u) \\ &= P_i'(u) - \frac{\epsilon}{2}P_i''(u) + \sum_{r=1}^{\infty}(-2\epsilon u)^{-r}\left(Q_i'(u) - \frac{\epsilon}{2u}P_i''(u)\right) \\ &= P_i'(u) + \sum_{r=1}^{\infty}(-2\epsilon u)^{-r}P_i'(u), \end{aligned}$$

which implies the lemma. □

Now we explain how irreducible highest weight $U(\mathfrak{g}, e)$-modules under the BGK highest weight theory are related to the irreducible highest weight $U(\mathfrak{g}, e)$-modules from Theorem IV.1.1. To each element of $\mathfrak{t}^*$ we associate a skew-symmetric $n \times l$ tableaux in the following way. For a weight

$$\lambda = \sum_{a \in \mathcal{I}_{nl} \cap \mathbb{Z}_{>0}} \alpha_a \epsilon_a \in \mathfrak{t}^*$$

we associate the skew-symmetric $n \times l$ tableaux $A = (a_{i,j})_{i \in \mathcal{I}_n, j \in \mathcal{I}_l}$ where $a_{i,j} = \alpha_a$ if $\mathrm{row}(a) = i, \mathrm{col}(a) = j$, $a_{i,j} = -\alpha_a$ if $\mathrm{row}(-a) = i, \mathrm{col}(-a) = j$, and $a_{0,0} = 0$ if n and l are odd. Under this association, $\mathfrak{t}^* = \mathrm{Tab}^{\mathfrak{g}}_{n,l}$, and $\mathfrak{t}^*/W_0 = \mathrm{Row}^{\mathfrak{g}}_{n,l}$. Let Λ denote the W_0-orbit containing λ, and let V_Λ denote the one-dimensional $U(\mathfrak{g}_0, e)$-module obtained by lifting the one-dimensional $S(\mathfrak{t})^{W_0}$-module corresponding to Λ through $\xi_{-\eta}$.

Now by Lemma IV.6.4 and the Miura transform from (III.10) we see that for $i \in \mathcal{I}_n, i \geq 0$ $s_{i,i}(\omega(u))$ acts in the same way on the highest weight spaces of the irreducible $U(\mathfrak{g}, e)$-module $L(A)$ as it does on $L(\Lambda, e)$. So the BGK highest weight theory implies that $L(\Lambda, e) \cong L(A)$, since $L(A)$ is itself a highest weight module by the definition of highest weight module from §II.4. Thus the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules coming from BGK highest weight theory is in fact the same as the classification in Theorem IV.1.1.

Remark IV.6.5. In the course of this work we also understood how to apply the results of this chapter and the algorithms for calculating $\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})} L(\lambda))$ from [BV] to verify that Conjecture II.5.2 holds in these cases. This conjecture has recently been proved in the general case by Losev in [Lo3], so we have not included this material here. In fact it is possible to use Losev's proof of Conjecture II.5.2 and Corollary IV.3.20 to recover Molev's classification of the finite dimensional irreducible representations of Y_n^Φ from the classification of the finite dimensional irreucible representations of $U(\mathfrak{g}, e)$ obtained via BGK highest weight theory.

REFERENCES

[B1] J.Brown, Twisted Yangians and finite W-algebras, *Transform. Groups* **14** (2009), 87–114, `arXiv:0710.2918`.

[BB] J. Brown and J. Brundan, Elementary invariants for centralizers of nilpotent matrices, *J. Austral. Math. Soc.* **86** (2009), 1–15, `math.RA/0611024`.

[BG] J. Brundan and S. Goodwin, Good grading polytopes, *Proc. London Math. Soc.* **94** (2007), 155–180, `math.QA/0510205`.

[BGK] J. Brundan, S. Goodwin and A. Kleshchev, Highest weight theory for finite W-algebras, *Int. Math. Res. Notices* (2008), `arXiv:0801.1337v1`.

[BK1] J. Brundan and A. Kleshchev, Parabolic presentations of the Yangian $Y(\mathfrak{gl}_n)$, *Commun. Math. Phys.* **254** (2005), 191–220, `math.QA/0407011`.

[BK2] J. Brundan and A. Kleshchev, Shifted Yangians and finite W-algebras, *Adv. Math.* **200** (2006), 136–195, `math.QA/0407012`.

[BK3] J. Brundan and A. Kleshchev, Representations of shifted Yangians and finite W-algebras, *Mem. Amer. Math. Soc.* **196** no. 918 (2008), `arXiv:math.RT/0508003`.

[BV] D. Barbash and D.Vogan, Primitive ideals and orbital integrals in complex classical groups, *Math. Ann.* **259** (1982), 153–199.

[C] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, John Wiley & Sons Ltd., 1985.

[DK] A. De Sole and V. Kac, Finite vs affine W-algebras, *Jpn. J. Math.* **1** (2006), 137–261, `math-ph/0511055`.

[D^{3}HK] A. D'Andrea, C. De Concini, A. De Sole, R. Heluani and V. Kac, Three equivalent definitions of finite W-algebras, appendix to [DK].

[Dr] V. G. Drinfeld, A new realization of Yangians and quantized affine algebras, *Soviet Math. Dokl.* **36** (1988), 212–216.

[Du] M. Duflo, Sur la classification des ideaux primitifs dans l'algebra envelop-pante d'une algebra de Lie semi-simple, *Ann. of Math.*, **105** (1977) 107–120.

[EK] P. Elashvili and V. Kac, Classification of good gradings of simple Lie alge-bras, *Amer. Math. Soc. Transl.* **213** (2005), 85–104, `math-ph/0312030`.

[GG] W. L. Gan and V. Ginzburg, Quantization of Slodowy slices, *Int. Math. Res. Notices* **5** (2002) 243–255, `math.RT/0105225`.

[Gi] V. Ginzburg, Harish-Chandra bimodules for quantized Slodowy slices, `arXiv:0807.0339`.

[Ja] J. C. Jantzen, *Nilpotent orbits in representation theory*, Prog. Math **228** (2004).

[Jo1] A. Joseph, Towards the Janzten conjecture *Comp. Math.*, **40** (1980) 35–67.

[Jo2] A. Joseph, Goldie rank in the enveloping algebra of a semisimple Lie alge-bra, *J. Algebra*, **65** (1980) 284–306.

[Jo3] A. Joseph, On the associated variety of a primitive ideal, *J. Algebra*, **93** (1983) 509–523.

[K] B. Kostant, On Whittaker vectors and representation theory, *Inventiones Math.* **48** (1978), 101–184.

[KRW] V. Kac, S. Roan, and M. Wakimoto, Quantum Reduction for Affine Super-algebras, *Commun. Math. Phys.* **241** (2003), 307–342, `math-ph/0302015`.

[Lo1] I. Losev, Quantized symplectic actions and W-algebras, `math.RT/0707.3108`.

[Lo2] I. Losev, Finite dimensional representations of W-algebras, `arXiv:0807.1023v1`.

[Lo3] I. Losev, On the structure of the category $\mathcal{O}$ for W-algebras, `arXiv:0812.1584`.

[Ly] T. E. Lynch, *Generalized Whittaker vectors and representation theory*, Ph.D. Thesis, MIT, Cambridge, MA, 1979.

[M] A. Molev, Finite-dimensional irreducible representations of twisted Yan-gians, *J. Math. Phys.* **39** (1998), 5559-5600, `arXiv:q-alg/9711022`.

[MNO] A. Molev, M. Nazarov and G. Olshanskii, *Yangians and classical Lie alge-bras*, Russian Math. Surveys **51** (1996), 205–282.

[P1] A. Premet, Special transverse slices and their enveloping algebras, *Adv. Math.* **170** (2002), 1–55.

[P2] A. Premet, Enveloping algebras of Slodowy slices and the Joseph ideal, *J. Eur. Math. Soc.* **9** (2007), 487–543, `math.RT/0504343`.

[P3] A. Premet, Commutative quotients of finite W-algebras, `arXiv:0809.0663`.

[PPY] D. Panyushev, A. Premet, and O. Yakimova, On symmetric invariants of centralisers in reductive Lie algebras, *J. Algebra* **313** (2007), 343–391, `math.RT/0610049`.

[R] E. Ragoucy, Twisted Yangians and folded W-algebras, *Int. J. Mod. Phys. A* **16** 13 (2001), 2411–2433, `math.QA/0012182`.

[Sk] S. Skryabin, A category equivalence, Appendix to [P1].

[Sp] T. A. Springer, *Linear algebraic groups*, Birkhäuser, second edition, 1998.