

SUMMABILITY OF FOURIER ORTHOGONAL EXPANSIONS AND A
DISCRETIZED FOURIER ORTHOGONAL EXPANSION
INVOLVING RADON PROJECTIONS FOR
FUNCTIONS ON THE CYLINDER

by

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CHAPTER I

INTRODUCTION

The study of Fourier orthogonal expansions of a function in terms of an orthogonal basis is a classical topic. The most well-known example is the Fourier series of a function on \mathbb{T} , the unit circle in \mathbb{R}^2 , which is an expansion in terms of the orthogonal basis $\{e^{inx} : n \in \mathbb{Z}\}$. An orthogonal polynomial sequence is a basis of polynomials with an added structure, that polynomials of different degrees are orthogonal with respect to an inner product. For orthogonal polynomials in one variable, Fourier orthogonal expansions have been studied extensively; see, for example, [15]. However, for orthogonal polynomials in several variables, there are open questions and active research in the area of expansions and approximations, and orthogonal polynomials in general.

It is frequently the case that Fourier orthogonal expansions will not converge for all functions in a function space. For example, it is well known that the Fourier series of a continuous function converges in $L^p(\mathbb{T})$, for $1 < p < \infty$, but for $p = 1$ and uniform convergence, the Fourier series is not necessarily convergent. Different summability methods must be employed in these function spaces to achieve convergence. In general, Fourier orthogonal expansions may not converge to the original function, and different summability techniques, such as Cesàro means, can be used to achieve convergence. The Cesàro summability of Fourier

orthogonal expansions has been studied on B^d in [17] and I^m , the m -fold Cartesian product of the interval $[-1, 1]$, in [9]. We will generalize these results and investigate the Cesàro summability of the Fourier orthogonal expansion of a function defined on the cylinder, $B^d \times I^m$, where B^d is the closed unit ball in \mathbb{R}^d , and I^m is the m -fold cartesian product of the interval $[-1, 1]$, in terms of the orthogonal polynomials for the space, with respect to the weight function $\prod(1 - z_i)^{\alpha_i}(1 + z_i)^{\beta_i}(1 - |x|^2)^{\lambda-1/2}$, with $z_i \in [-1, 1]$, $\alpha_i, \beta_i > -1$ for $i = 1, 2, \dots, m$, and $x \in B^d$ with $\lambda > -1/2$.

Another interesting field in approximation theory is the study of discretized Fourier orthogonal expansions. For example, if we consider the Fourier series of a function on \mathbb{T} , the Fourier coefficients of a function f are given by the integrals

$$\hat{f}_n = \int_0^{2\pi} f(t)e^{-int} dt,$$

for $n \in \mathbb{Z}$. The data from the function f is continuous data; that is, it relies on the values of f on its entire domain. This integral may be discretized by a quadrature rule, so that the discretized expansion only requires values of f at a finite set of points, rather than values of f on the entire circle. This method of discretizing the coefficients of an expansion by means of a quadrature is called *hyperinterpolation*, and was first suggested by Sloan in [14]. For expansions of functions defined on higher dimensional regions, another approach uses approximation by a finite number of Radon projections, or integrals over hyperplanes intersected with the region. The basis for this approach relies on the connection between orthogonal polynomials and Radon projections of functions on B^2 , which was first studied in [11] and [10]. This connection was generalized to B^d in [13]. Using the original relationship on B^2 , a discretized Fourier orthogonal expansion on B^2 involving

finite Radon projections was found in [18]. We will study the convergence of a discretized Fourier orthogonal expansion on the space $B^2 \times [-1, 1]$, where the discrete data is a finite set of Radon projections of f taken on parallel disks which are perpendicular to the axis of the cylinder. This particular discretization has application in the field of computerized tomography (CT), as Radon projections of f correspond with X-ray data in CT.

The dissertation is organized as follows. In chapter 2, we will present information on orthogonal polynomials, Fourier orthogonal expansions in terms of orthogonal polynomials, and the Cesàro summation technique for orthogonal expansions. In chapter 3, we will present the theorem and proof of the result on the convergence of the Cesàro means of the Fourier orthogonal expansion on $B^d \times I^m$. In chapter 4, we will introduce the Radon transform, and discuss its connection to orthogonal polynomials and Fourier orthogonal expansions. We will then derive the discretized Fourier orthogonal expansion on $B^2 \times [-1, 1]$, and we will prove a result on the Lebesgue constant of this discretized expansion.

CHAPTER II

ORTHOGONAL POLYNOMIALS AND EXPANSIONS

II.1 Definitions and General Theory

II.1.1 Definitions

We first present basic information about orthogonal polynomials. The standard references are [5] for orthogonal polynomials in several variables, and [15] for orthogonal polynomials in a single variable. We let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$, and define $\mathbf{x}^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, \dots, x_d^{\alpha_d})$. We let

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d,$$

and say that a polynomial P is of *total degree* n if

$$P(\mathbf{x}) = \sum_{i=0}^n \sum_{|\alpha|=n} c_\alpha \mathbf{x}^\alpha,$$

where c_α are real coefficients, and at least one of the coefficients c_α with $|\alpha| = n$ is non-zero. For this dissertation, when we say a polynomial is of degree n in d variables, it is meant that P is a polynomial of total degree n . We define Π^d to be the space of polynomials in d variables, and Π_n^d to be the space of polynomials in d variables of total degree less than or equal to n .

Let $\langle \cdot, \cdot \rangle$ be an inner product on Π^d . A polynomial P of degree n is an *orthogonal polynomial* with respect to $\langle \cdot, \cdot \rangle$ if $\langle P, Q \rangle = 0$ whenever Q is a polynomial of degree less than n . A polynomial P is *orthonormal* with respect to $\langle \cdot, \cdot \rangle$ if P is an orthogonal polynomial, and $\langle P, P \rangle = 1$. We let \mathcal{V}_n^d denote the space of orthogonal polynomials in d variables of degree n .

This inner product is often given in the form

$$\langle P, Q \rangle = \int_{\Omega} P(x)Q(x) d\mu(x),$$

where Ω is a subset of \mathbb{R}^d and μ is a Borel measure on Ω . In this case, the space of orthogonal polynomials of degree n is denoted $\mathcal{V}_n(\Omega; \mu)$. In the case that μ is the Lebesgue measure, we will denote the space of orthogonal polynomials by $\mathcal{V}_n(\Omega)$. A natural question is whether a basis of orthogonal polynomials exists for a given space Ω and measure μ . For the situations dealt with in this dissertation, the answer to the question is given by the theorem below; see [5, Cor 3.1.9].

Theorem II.1. *Let μ be a Borel measure on Ω which satisfies*

1. $\int_{\Omega} (P(x))^2 d\mu(x) > 0$ for all $P \in \Pi^d$ with $P \neq 0$, and
2. $\int_{\Omega} \mathbf{x}^{\alpha} d\mu(x) < \infty$ for all $\alpha \in \mathbb{N}_0^d$.

Then there exists an basis of orthogonal polynomials for Π^d .

If the second condition is satisfied, we say that μ has finite moments. If μ is a positive measure, and μ has finite moments, the theorem ensures that a basis of orthogonal polynomials exists. In the specific situations we will study, these conditions will be satisfied. For the remainder of the chapter, we will assume we are working with a space Ω and measure μ which satisfy the conditions of this theorem.

In the case where $d = 1$, the space \mathcal{V}_n^1 has one basis element. We may order the orthogonal polynomials by degree to obtain an *orthogonal polynomial sequence*, which is an ordered basis of orthogonal polynomials. In several dimensions, however, the space \mathcal{V}_n^d is spanned by several polynomials. Specifically, it is well-known that

$$\dim \mathcal{V}_n^d = r_n^d := \binom{n+d-1}{n}.$$

By summing this up, we conclude that

$$\dim \Pi_n^d = \binom{n+d}{n}$$

There are several different bases that may be chosen for \mathcal{V}_n^d . We choose one basis, $P_1^n, P_2^n, \dots, P_{r_n^d}^n$, and define the column vector \mathbb{P}_n by

$$\mathbb{P}_n = \begin{bmatrix} P_1^n \\ P_2^n \\ \vdots \\ P_{r_n^d}^n \end{bmatrix}. \quad (\text{II.1.1})$$

This vector notation, introduced in [7] and [8], and further investigated in [16], allows for the generalization of several properties of orthogonal polynomials of one variable, independently of the choice of basis for \mathcal{V}_n^d , as we will see below.

II.1.2 Properties of Orthogonal Polynomials of One Variable

Assume that μ is a Borel measure on Ω that satisfies the conditions of Theorem (II.1), and let $\{p_0, p_1, \dots\}$, where the degree of p_n is n , be the orthogonal polynomial sequence guaranteed by the theorem. The following properties hold.

Theorem II.2. [15, Thm 3.2.1] For $n > 0$, there exist constants A_n , B_n , and C_n which satisfy the recurrence relation

$$p_n(x) = A_n x p_{n-1}(x) + B_n p_{n-1}(x) - C_n p_{n-2}(x),$$

where p_{-1} and p_{-2} are defined to be the zero polynomial. Furthermore, if k_n denotes the leading coefficient of $p_n(x)$, then A_n and C_n satisfy

$$A_n = \frac{k_n}{k_{n-1}}, \quad C_n = \frac{A_n}{A_{n-1}} = \frac{k_n k_{n-2}}{k_{n-1}^2}.$$

We next define the reproducing kernel. We define the value h_n by

$$h_n := \int_{\Omega} p_n^2(x) d\omega(x). \quad (\text{II.1.2})$$

Definition II.1. The reproducing kernel of degree n , $K_n(x, y)$ of the orthogonal polynomial sequence p_0, p_1, p_2, \dots is a function defined on $\Omega \times \Omega$, given by the formula

$$K_n(x, y) = \sum_{k=0}^n h_k^{-1} p_k(x) p_k(y).$$

The reproducing kernel “reproduces” polynomials of degree less than or equal to n ; that is,

$$\int_{\Omega} K_n(x, y) P(x) d\mu(x) = P(y),$$

if $P \in \Pi_n$. The three-term recurrence relation allows one to write the reproducing kernel in a compact form.

Theorem II.3. [15, Thm. 3.2.2] For $n \geq 0$,

$$K_n(x, y) = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y},$$

for $x \neq y$; if $x = y$, then

$$K_n(x, x) = \frac{k_n}{k_{n+1}} (p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)).$$

One of the most useful properties of orthogonal polynomials is the quadrature rule known as Gaussian quadrature. A quadrature rule is a numerical integration technique. For an m -point quadrature rule, a set of points, x_1, x_2, \dots, x_m , and a set of weights, $\lambda_1, \lambda_2, \dots, \lambda_m$, are fixed, and the rule is given by

$$\int_{\Omega} f(x) d\mu(x) \approx \sum_{j=1}^m \lambda_j f(x_j).$$

In the case where the rule gives equality for a function f , we say the quadrature is exact for f . Often, the effectiveness of a quadrature is gauged by the largest integer n for which the quadrature is exact for all polynomials of degree less than or equal to n . For an m -point quadrature, the largest possible value of n is $2m - 1$; an m -point quadrature is said to be a Gaussian quadrature if the quadrature is exact for all polynomials of degree less than or equal to $2m - 1$. A Gaussian quadrature is fundamentally related to orthogonal polynomials, but before we state this relationship, we present a theorem about the zeroes of an orthogonal polynomial sequence.

Theorem II.4. Let $\{p_0, p_1, \dots\}$ be an orthogonal polynomial sequence on Ω with respect to μ . The polynomial p_n has n real, distinct zeroes, $x_{n,1} < x_{n,2} < \dots < x_{n,n}$. Moreover, for any two adjacent zeroes of p_n , $x_{k,n}$ and $x_{k+1,n}$, there is a zero of

p_{n+1} , $x_{k+1,n+1}$, with $x_{k,n} < x_{k+1,n+1} < x_{k+1,n}$.

The relationship between an orthogonal polynomial sequence and a Gaussian quadrature is given in the following theorem.

Theorem II.5. *Let μ be a weight function on $\Omega \subset \mathbb{R}$ that satisfies the conditions of Theorem (II.1). Let $x_{1,m}, x_{2,m}, \dots, x_{m,m}$ denote the zeroes of the degree m orthogonal polynomial with respect to μ on Ω . The quadrature rule given by*

$$\int_{\Omega} f(x) d\mu(x) \approx \sum_{k=1}^m \lambda_{k,m} f(x_{k,m}) \quad (\text{II.1.3})$$

$$\lambda_{k,m} = \int_{\Omega} \frac{P_n(x)}{(x - x_{k,n})P'_n(x_{k,n})} d\mu(x) \quad (\text{II.1.4})$$

is exact for all polynomials of degree less than or equal than $2m - 1$.

The Gaussian quadrature is considered to be the best quadrature for integration over subsets of \mathbb{R} , because of its high level of exactness.

II.1.3 Properties of Orthogonal Polynomials of Several Variables

Assume that μ is a Borel measure on $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, such that the conditions of Theorem (II.1) are satisfied. Some of the properties of orthogonal polynomials of one variable generalize to orthogonal polynomials of several variables.

Theorem II.6. [5, Thm 3.2.1] *Let $n \geq 0$, and let \mathbb{P}_n denote the column vector of orthogonal polynomials on Ω with respect to μ , where $\Omega \subset \mathbb{R}^d$ with $d \geq 2$. The polynomials satisfy the three-term recurrence relation*

$$x_i \mathbb{P}_n = A_{n,i} \mathbb{P}_{n+1} + B_{n,i} \mathbb{P}_n + C_{n,i} \mathbb{P}_{n-1}, \quad 1 \leq i \leq d,$$

where $\mathbb{P}_{-1} = 0$, and $A_{n,i}$ is an $n \times (n + 1)$ matrix, $B_{n,i}$ is a $n \times n$ matrix, and $C_{n,i}$ is an $n \times (n - 1)$ matrix.

Much more can be said about the matrices in the three-term recurrence relation; see Chapter 3 in [5]. The three term relation also enables one to derive a compact form for the reproducing kernel for orthogonal polynomials in several variables. For an orthogonal polynomial sequence in several variables, the reproducing kernel, $K_n(x, y)$, with $x, y \in \mathbb{R}^d$, is defined by

$$K_n(x, y) = \sum_{k=0}^n \mathbb{P}_k^T(x) H_k^{-1} \mathbb{P}_k(y),$$

with the matrix H_k defined by

$$(H_k)_{i,j} = \int_{\Omega} P_i^n(x) P_j^n(x) d\mu(x).$$

Theorem II.7. [5, Thm 3.5.3] *Let $n \geq 0$ and let \mathbb{P}_n denote the column vector of orthogonal polynomials as described in the previous theorem. For $x, y \in \mathbb{R}^d$, the reproducing kernel may be written in the form*

$$K_n(x, y) = \frac{A_{n,i} [\mathbb{P}_{n+1}(x)]^T H_n^{-1} \mathbb{P}_n(y) - \mathbb{P}_n^T(x) H_n^{-1} [A_{n,i} \mathbb{P}_{n+1}(y)]}{x_i - y_i}, \quad 1 \leq i \leq d,$$

where $x = (x_1, x_2, \dots, x_d)$, $y = (y_1, y_2, \dots, y_d)$.

While the properties above generalize from the theory of orthogonal polynomials of a single variable to orthogonal polynomials of several variables, other properties do not generalize well. Among the properties that do not generalize well are those concerning the zeroes of orthogonal polynomials and Gaussian quadratures. While zeroes of polynomials of a single variable are points,

zeroes of orthogonal polynomials of several variables may be algebraic curves, a fact which does not lend itself to easy generalization. There is much that can be said in this direction, but it is removed from the problems we will be investigating, so we direct the reader to [5].

II.2 Examples of Orthogonal Polynomials

II.2.1 Examples in One Variable

Jacobi Polynomials

On the interval $[-1, 1]$, the Jacobi polynomials are given by the formula

$$P_n^{(\alpha, \beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx} \right)^n \left\{ (1-x)^{n+\alpha} (1+x)^{n+\beta} \right\}.$$

This type of formula is called a Rodriguez type formula. The Jacobi polynomials are orthogonal with respect to the weight function $w^{(\alpha, \beta)}(x) = c_{\alpha, \beta} (1-x)^\alpha (1+x)^\beta$ on $[-1, 1]$, with $\alpha, \beta > -1$, and

$$c_{\alpha, \beta} = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)2^{\alpha + \beta + 1}},$$

where $\Gamma(x)$ is the the gamma function, defined on $(0, \infty)$ by

$$\Gamma(x) = \int_0^\infty x^{t-1} e^{-t} dt.$$

The value h_n defined in (II.1.2) is given by

$$h_n = \frac{2^{2n} n!}{2n + \alpha + \beta + 1} \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(n + \alpha + \beta + 1)} (\alpha + 1)_n (\beta + 1)_n,$$

where the shifted factorial, $(a)_n$, is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (\text{II.2.5})$$

We will be working with the orthonormal Jacobi polynomials, which will be denoted by

$$p_n^{(\alpha,\beta)}(x) = h_n^{-1/2} P_n^{(\alpha,\beta)}(x).$$

Gegenbauer Polynomials

A special subcase of the Jacobi polynomials are the Gegenbauer, or Ultraspherical, polynomials, C_n^λ . These polynomials are defined by the formula

$$C_n^\lambda(x) = (-1)^n \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} 2^n n! P_n^{(\lambda-1/2, \lambda-1/2)}(x), \quad \lambda > -1/2.$$

The Gegenbauer polynomials are orthogonal with respect to the weight $(1-x^2)^{\lambda-1/2}$. The value h_n is given by

$$h_n = \frac{2^{1-2\lambda} \pi}{(\Gamma(\lambda))^2} \frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma(n+1)},$$

and the orthonormal Gegenbauer polynomials, \tilde{C}_n^λ , are defined by

$\tilde{C}_n^\lambda(x) = h_n^{-1/2} C_n^\lambda(x)$. The orthonormal Gegenbauer polynomials and regular

Gegenbauer polynomials are also related by the equation

$$\frac{n+\lambda}{\lambda} C_n^\lambda(x) = \tilde{C}_n^\lambda(1) \tilde{C}_n^\lambda(x). \quad (\text{II.2.6})$$

Another useful equation the Gegenbauer polynomials satisfy is

$$\sum_{k=0}^n \frac{k + \lambda}{\lambda} C_k^\lambda(x) = C_n^{\lambda+1}(x) + C_{n-1}^{\lambda+1}(x). \quad (\text{II.2.7})$$

This relationship plays a key role in deriving a compact formula for the reproducing kernel for orthogonal polynomials on B^d .

Chebyshev Polynomials

The Chebyshev polynomials of the first and second kinds are special cases of the Gegenbauer polynomials. The Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos n\theta, \quad x = \cos \theta,$$

and they form an orthogonal polynomial sequence with respect to the weight function $\omega(x) = \frac{1}{\pi}(1 - x^2)^{-1/2}$. The value of h_n is 1 if $n = 0$ and $1/2$ if $n \geq 1$. We denote by \tilde{T}_n the orthonormal Chebyshev polynomials of the first kind, defined by $\tilde{T}_n(x) = h_n^{-1/2} T_n(x)$. The zeroes of $T_n(x)$ are

$$z_{l,n} := \cos(\gamma_{l,n}) := \cos\left(\frac{2l+1}{2n}\pi\right), \quad l = 0, 1, \dots, n-1 \quad (\text{II.2.8})$$

and the n -point Gaussian quadrature associated with the Chebyshev polynomials of the first kind is given by

$$\frac{1}{\pi} \int_{-1}^1 f(x) \frac{dx}{\sqrt{1-x^2}} \approx \frac{1}{n} \sum_{l=0}^{n-1} f(z_{l,n}). \quad (\text{II.2.9})$$

The Chebyshev polynomials of the second kind are defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin(\theta)}, \quad x = \cos \theta,$$

and are orthogonal with respect to the weight $\frac{1}{\pi}(1-x^2)^{1/2}$ on $[-1, 1]$. The value of h_n is $1/2$ for all n . The zeroes of $U_n(x)$ are given by

$$\cos \theta_{j,n} := \cos \frac{j\pi}{n+1}, \quad j = 1, 2, \dots, n, \quad (\text{II.2.10})$$

and the Gaussian quadrature associated with the Chebyshev polynomials of the second kind is given by

$$\frac{1}{\pi} \int_{-1}^1 f(x) \sqrt{1-x^2} dx \approx \frac{1}{n} \sum_{j=1}^n f(\cos(\theta_{j,n})). \quad (\text{II.2.11})$$

II.2.2 Examples in Several Variables

Product Jacobi Polynomials

On I^m , the Jacobi polynomials are given by the product of the one dimensional Jacobi polynomials; that is,

$$P_{\gamma}^{(\alpha, \beta)}(x) = \prod_{i=1}^m P_{\gamma_i}^{(\alpha_i, \beta_i)}(x_i)$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \dots, \beta_m)$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$ are multi-indices, and $\alpha_i, \beta_i > -1$ for $1 \leq i \leq m$. They are orthogonal with respect to the weight function

$$w^{(\alpha, \beta)}(x) = \prod_{i=1}^m w^{(\alpha_i, \beta_i)}(x_i).$$

Recall that $P_{\gamma}^{(\alpha, \beta)}(x)$ is of degree n if $|\gamma| = n$, where $|\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_m$. The orthonormal Jacobi polynomials are given by

$$p_{\gamma}^{(\alpha, \beta)}(x) = \prod_{i=1}^m p_{\gamma_i}^{(\alpha_i, \beta_i)}(x_i).$$

Given this basis, we denote the column vector of an orthonormal basis of $\mathcal{V}_n(I^m; w^{(\alpha, \beta)})$ by

$$\mathbb{P}_n^{(\alpha, \beta)}(x) = \begin{bmatrix} p_{\gamma_1}^{(\alpha, \beta)}(x) \\ p_{\gamma_2}^{(\alpha, \beta)}(x) \\ \vdots \\ p_{\gamma_l}^{(\alpha, \beta)}(x) \end{bmatrix}. \quad (\text{II.2.12})$$

Orthogonal Polynomials on B^d

On B^d , we denote the orthogonal polynomials with respect to the weight $w_\mu(x) := (1 - \|x\|^2)^{\mu-1/2}$, for $\mu > 0$ and $\|x\|$ denoting the usual Euclidean norm, by $S_\alpha^\mu(x)$, where $\alpha \in \mathbb{N}_0^d$. There are several different bases for $\mathcal{V}_n(B^d, w_\mu)$; we give an explicit basis below.

Theorem II.8. [5, Prop. 2.3.2]. *Let $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = n$. An orthonormal basis of polynomials for $\mathcal{V}_n(B^d, w_\mu)$ is given by*

$$P_\alpha(x) = h_\alpha^{-1} \prod_{j=1}^d \left[(1 - \|\mathbf{x}_{j-1}\|^2)^{\alpha_j/2} \tilde{C}_{\alpha_j}^{\mu + |\alpha^{j+1}| + \frac{d-j}{2}} \left(\frac{x_j}{\sqrt{1 - \|\mathbf{x}_{j-1}\|^2}} \right) \right],$$

where $\mathbf{x}_j = (x_1, x_2, \dots, x_j)$, $\alpha^j = (\alpha_j, \alpha_{j+1}, \dots, \alpha_d)$, and

$$[h_\alpha]^2 = \frac{1}{(\mu + \frac{d+1}{2})_{|\alpha|}} \prod_{j=1}^d \left(\mu + |\alpha^{j+1}| + \frac{d-j+2}{2} \right)_{\alpha_j}.$$

Another explicit basis can be given for $\mathcal{V}_n(B^2)$. This basis is given by

$$U_n(x \cos \theta_{j,n} + y \sin \theta_{j,n}), \quad j = 0, 1, \dots, n \quad (\text{II.2.13})$$

with $\theta_{j,n} := \frac{j\pi}{n+1}$. These polynomials were shown to form an orthonormal basis for $\mathcal{V}_n(B^2, w_{1/2})$ in [10] and play an important role in relating the Radon transform to

the Fourier orthogonal expansion on B^2 .

Given an orthonormal basis for $\mathcal{V}_n(B^d, w_\mu)$, $s_{\alpha_1}^\mu(y), s_{\alpha_2}^\mu(y), \dots, s_{\alpha_{r_n}^\mu}^\mu(y)$, with $y \in B^d$, we write

$$\mathbb{S}_n^\mu(y) = \begin{bmatrix} s_{\alpha_1}^\mu(y) \\ s_{\alpha_2}^\mu(y) \\ \vdots \\ s_{\alpha_{r_n}^\mu}^\mu(y) \end{bmatrix} \quad (\text{II.2.14})$$

One very useful tool when working with orthogonal polynomials on B^d is the product formula. For $\mu > 0$, the formula is given by

$$\begin{aligned} [\mathbb{S}_n^\mu(x)]^T [\mathbb{S}_n^\mu(y)] &= \frac{n + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} \int_{-1}^1 C_n^{(\mu + \frac{d-1}{2})} \left(x \cdot y + \sqrt{1 - |x|^2} \sqrt{1 - |y|^2} t \right) \\ &\quad \times w_{\mu-1/2}(t) dt, \end{aligned} \quad (\text{II.2.15})$$

while if $\mu = 0$, we have

$$\begin{aligned} [\mathbb{S}_n^0(x)]^T [\mathbb{S}_n^0(y)] &= \frac{n + \frac{d-1}{2}}{\frac{d-1}{2}} \left[C_n^{(\frac{d-1}{2})} \left(x \cdot y + \sqrt{1 - |x|^2} \sqrt{1 - |y|^2} \right) \right. \\ &\quad \left. + C_n^{(\frac{d-1}{2})} \left(x \cdot y - \sqrt{1 - |x|^2} \sqrt{1 - |y|^2} \right) \right]. \end{aligned} \quad (\text{II.2.16})$$

Indeed, this formula provides a compact form for the reproducing kernel of degree n for polynomials on B^d , $K_n^\mu(x, y)$, which is a necessary tool when studying the convergence properties of expansions in term of orthogonal polynomials. If $\mu > 0$,

then using (II.2.7) with (II.2.15), we obtain

$$K_n^\mu(x, y) := \sum_{k=0}^n [\mathbb{S}_n^\mu(x)]^T \mathbb{S}_n^\mu(y) = \int_{-1}^1 \left[C_n^{(\mu+\frac{d+1}{2})} \left(x \cdot y + \sqrt{1-|x|^2} \sqrt{1-|y|^2} t \right) \right. \\ \left. + C_{n-1}^{(\mu+\frac{d+1}{2})} \left(x \cdot y + \sqrt{1-|x|^2} \sqrt{1-|y|^2} t \right) \right] w_{\mu-1/2}(t) dt, \quad (\text{II.2.17})$$

$$+ C_{n-1}^{(\mu+\frac{d+1}{2})} \left(x \cdot y + \sqrt{1-|x|^2} \sqrt{1-|y|^2} t \right) \Big] w_{\mu-1/2}(t) dt, \quad (\text{II.2.18})$$

and a similar formula can be obtained for the case of $\mu = 0$.

Orthogonal Polynomials on $B^d \times I^m$

For the weight function $w_\mu(y)w^{(\alpha,\beta)}(x)$, with $y \in B^d$, $x \in I^m$, $\mu > -1/2$ and α, β multi-indices with $\alpha_i, \beta_i > -1$ for $1 \leq i \leq m$, the orthogonal polynomials are given by the product of the orthogonal polynomials with respect to w_μ on B^d and the orthogonal polynomials on I^m with respect to $w^{(\alpha,\beta)}$. Specifically, the polynomials

$$S_{j,k}^\mu(y)P_\gamma^{(\alpha,\beta)}(x), \quad 1 \leq j \leq r_k^d, \quad k + |\gamma| = n, \quad y \in B^d, \quad x \in I^m$$

form a basis for the space $\mathcal{V}_n(B^d \times I^m; w_\mu \times w^{(\alpha,\beta)})$. These polynomials will be studied in Chapter 3.

For $\mu = -1/2$, $d = 2$, $m = 1$ and $\alpha = \beta = -1/2$, a basis for $\mathcal{V}_n(B^2 \times [-1, 1]; w_Z)$, where $w_Z(x, y, z) = \frac{1}{\pi}(1 - z^2)^{-1/2}$ for $(x, y) \in B^2$ and $z \in [-1, 1]$, is given by the polynomials

$$P_{j,k,n}(x, y, z) := U_k(x \cos \theta_{j,k} + y \sin \theta_{j,k})T_{n-k}(z), \\ k = 0, 1, \dots, n, \quad (x, y) \in B^2, \quad z \in [-1, 1]. \quad (\text{II.2.19})$$

This basis will play a role in our discretized expansion in Chapter IV.

II.3 Fourier Orthogonal Expansion and Cesàro Summability

II.3.1 Fourier Orthogonal Expansion

If the conditions in theorem (II.1) are satisfied, then an orthonormal basis of orthogonal polynomials can be obtained for the space Π^d . Moreover, with the inner product defined by

$$\langle f, g \rangle = \int_{\Omega} f(x)g(x) d\mu(x)$$

this orthogonal polynomial sequence also forms a basis for the Hilbert space $L^2(\Omega; \mu)$, and $\mathcal{V}_n(\Omega; \mu)$ form orthogonal subspaces of $L^2(\Omega; \mu)$.

Theorem II.9. *The function space $L^2(\Omega; \mu)$, defined as the set of functions f with*

$$\int_{\Omega} |f(x)|^2 d\mu(x) < \infty$$

can be decomposed as

$$L^2(\Omega; \mu) = \bigoplus_{k=0}^{\infty} \mathcal{V}_k(\Omega, \mu), \quad f = \sum_{k=0}^{\infty} \text{proj}_k f,$$

where proj_k is the projection map from $L^2(\Omega; \mu)$ onto $\mathcal{V}_k(\Omega, \mu)$.

For orthogonal polynomials in one variable, $\text{proj}_k f$ may be written in the form

$$\text{proj}_k f(x) = h_n^{-1} \int_{\Omega} f(t)P_n(t)d\mu(x) P_n(x), \quad (\text{II.3.20})$$

and for orthogonal polynomials of several variables,

$$\text{proj}_k f = \int_{\Omega} [\mathbb{P}_n(t)]^T H_n^{-1} \mathbb{P}_n(x) d\mu(t). \quad (\text{II.3.21})$$

The best approximation to a function f is $L^2(\Omega; \mu)$ by polynomials of degree less than or equal to n is the Fourier partial sum, $S_n f$, defined by

$$S_n f = \sum_{k=1}^n \text{proj}_k f.$$

Using the formulas for proj_k in (II.3.20) and (II.3.21),

$$S_n f(x) = \int_{\Omega} K_n(t, x) f(t) d\mu(t).$$

II.3.2 Cesàro Summability

While $S_n f$ converges to f in $L^2(\Omega; d\mu)$, $S_n f$ may not necessarily converge in other normed spaces, such as $L^p(\Omega; \mu)$ or $C(\Omega)$. For these spaces, different summability techniques may be used. One such technique is Cesàro summability, which we define below.

Definition II.2. Given a sequence $\{a_1, a_2, \dots\}$, the Cesàro means of order δ , s_n^δ , are defined to be

$$s_n^\delta = \sum_{k=0}^n \frac{(-n)_k}{(-n-\delta)_k} a_k,$$

where $(-n)_k$ is defined in (II.2.5). We say that the sequence c_n (or the series $\sum c_n$) is Cesàro summable of order δ , or (C, δ) summable, to a if

$$\lim_{n \rightarrow \infty} s_n^\delta = a.$$

It is known that if a series a_1, a_2, \dots is (C, δ) summable to a for some $\delta > 0$, then the series is $(C, \delta + h)$ summable to a for any $h > 0$ [19, Thm 1.21, Vol. 1].

For a classical application of Cesàro summation, the Fourier series of a continuous function f may not converge to f in the function spaces $L^1(\mathbb{T})$ or $C(\mathbb{T})$. However, the $(C, 1)$ means of the Fourier series will converge to f in these spaces.

For orthogonal polynomials, the Cesàro means of order δ take the form

$$S_n^\delta f := \sum_{k=0}^n \frac{(-n)_k}{(-n-\delta)_k} \text{proj}_k f.$$

Recall that $S_n f$ may be written in the form of an integral of f against the reproducing kernel $K_n(x, y)$. We adopt similar notation for the Cesàro means of order δ . Define

$$K_n^\delta(x, y) = \sum_{k=0}^n \frac{(-n)_k}{(-n-\delta)_k} h_k^{-1} p_k(x) p_k(y)$$

for orthogonal polynomials of one variable, and

$$K_n^\delta(x, y) = \sum_{k=0}^n \frac{(-n)_k}{(-n-\delta)_k} [\mathbb{P}_k(x)]^T H_k^{-1} \mathbb{P}_k(y)$$

for orthogonal polynomials in several variables. We may then express the (C, δ) means as an integral operator,

$$S_n^\delta f(x) = \int_{\Omega} f(y) K_n^\delta(x, y) d\mu(y).$$

Frequently, Cesàro summability is first established for the function spaces $C(\Omega)$ and $L^1(\Omega)$, and then a special case of the Riesz-Thorin theorem [19, Thm 1.11, Vol. 2] is used to extend the result to the spaces $L^p(\Omega; \mu)$, for $1 < p < \infty$.

Theorem II.10. *Suppose that T is a bounded linear operator on $L^1(\Omega; \mu)$ and*

$C(\Omega)$. Then T is also a bounded linear operator on the spaces $L^p(\Omega; \mu)$ for $1 < p < \infty$.

To establish (C, δ) convergence for the spaces $L^1(\Omega; \mu)$ and $C(\Omega)$, the following theorem is used. We only prove this theorem for the case when Ω is a compact set, since we will be dealing with situations of this type. However, generalizations to non-compact domains can be proven.

Theorem II.11. *Let Ω be a compact set, and let μ satisfy the conditions of (II.1), and the additional condition that $\mu(X) > 0$ for every set $X \subset \Omega$ with positive Lebesgue measure. The (C, δ) means $S_n^\delta f$ converge to f for $f \in L^1(\Omega; \mu)$ or $f \in C(\Omega)$ if there exists a constant M so that*

$$\int_{\Omega} |K_n^\delta(x, y)| d\mu(x) \leq M$$

for all $y \in \Omega$ and $n \geq 0$.

Proof. Following the proof of [4, Thm 4.2], we first prove the following claim.

Claim II.1. The norm of the operator S_n^δ as an operator on $C(\Omega)$ and $L^1(\Omega; \mu)$ is given by

$$\sup_{x \in \Omega} \int_{\Omega} |K_n^\delta(x, y)| d\mu(y).$$

Proof. Treating S_n^δ as an operator on $C(\Omega)$, we first note that the norm of S_n^δ , $\|S_n^\delta\|_\infty$, satisfies

$$\begin{aligned} \|S_n^\delta\|_\infty &= \sup_{\substack{f \in C(\Omega) \\ \|f\|_\infty=1}} \sup_{x \in \Omega} \left| \int_{\Omega} K_n^\delta(x, y) f(y) d\mu(y) \right| \\ &\leq \sup_{x \in \Omega} \int_{\Omega} |K_n^\delta(x, y)| d\mu(y) \end{aligned}$$

by moving the absolute value inside the integral and bounding $|f(x)|$ by $\|f\|_\infty$. For the other inequality, we note that since the function

$$\int_{\Omega} |K_n^\delta(x, y)| d\mu(y) \tag{II.3.22}$$

is continuous in x , it achieves its maximum, N , at some point x_0 . Let $h(y) = \text{sign } K_n^\delta(x_0, y)$, and note that h may not be continuous. Let $\varepsilon > 0$, and define a new function h^* which is equal to h on the set

$$A = \{y \in \Omega : |K_n^\delta(x_0, y)| > \varepsilon\},$$

is continuous on Ω , and satisfies $|h^*(y)| \leq 1$. It follows that

$$\begin{aligned} \|K_n^\delta\|_\infty &\geq \left| \int_{\Omega} h^*(y) K_n^\delta(x_0, y) d\mu(y) \right| \\ &= \int_{\Omega} |K_n^\delta(x_0, y)| d\mu(y) + \left| \int_{\Omega \setminus A} (h^*(y) - h(y)) K_n^\delta(x_0, y) d\mu(y) \right| \\ &\geq N - 2\varepsilon\mu(\Omega). \end{aligned}$$

Since ε was arbitrary, this proves the claim for $C(\Omega)$.

We now consider S_n^δ to be an operator on $L^1(\Omega; \mu)$. By moving the absolute value inside of the integral and applying Fubini's theorem, we obtain

$$\begin{aligned} \|S_n^\delta\|_{L^1} &= \sup_{\substack{f \in L^1(\Omega; \mu) \\ \|f\|_1 = 1}} \int_{\Omega} \left| \int_{\Omega} K_n^\delta(x, y) f(x) d\mu(x) \right| d\mu(y) \\ &\leq \int_{\Omega} |K_n^\delta(x, y)| d\mu(y), \end{aligned}$$

where

$$\|f\|_1 = \int_{\Omega} |f(x)| d\mu(x).$$

For the other inequality, we again let x_0 be the point where the function in (II.3.22) achieves its maximum. Let $\varepsilon > 0$. Since $K_n^\delta(x, y)$ is uniformly continuous on Ω , there is a number $\delta > 0$ such that $|K_n^\delta(x_1, y) - K_n^\delta(x_2, y)| < \varepsilon$ for $|x_1 - x_2| < \delta$. Hence, if $f = \chi_{B_\delta(x_0)}(\mu(B_\delta(x_0)))^{-1}$, then f has a norm of 1 in $L^1(\Omega; \mu)$, and we obtain

$$\begin{aligned} \|S_n^\delta\|_{L^1} &\geq \int_{\Omega} \left| \int_{\Omega} K_n^\delta(x, y) f(x) d\mu(x) \right| d\mu(y) \\ &\geq \int_{\Omega} |K_n^\delta(x_0, y)| d\mu(y) - \varepsilon \mu(\Omega). \end{aligned}$$

Since ε was arbitrary, this proves the claim for $L^1(\Omega; \mu)$. This concludes the proof of the claim. \square

We next prove a proposition concerning the Cesàro means of polynomials.

Proposition II.1. *Let $P(x)$ be a polynomial of total degree n and $\delta \geq 0$. Then $S_m^\delta(P)(x) \rightarrow P(x)$ uniformly as $m \rightarrow \infty$.*

Proof. Since $\text{proj}_k P(x) = 0$ if $k > n$,

$$S_m^\delta(P) = \sum_{k=0}^n \frac{(-m)_k}{(-m-\delta)_k} \text{proj}_k(P).$$

The claim follows from the fact

$$\lim_{m \rightarrow \infty} \frac{(-m)_k}{(-m-\delta)_k} = 1.$$

\square

Finally, we may prove the Theorem II.11. We let $f \in C(\Omega)$ and $g \in L^1(\Omega; \mu)$. Since polynomials are dense in the spaces $C(\Omega)$ and $L^1(\Omega; \mu)$, we let $\varepsilon > 0$ and choose polynomials P and Q so that $\|f - P\|_\infty \leq \varepsilon$ and $\|g - Q\|_1 \leq \varepsilon$. We choose n large enough so that $\|S_n^\delta(P) - P\|_\infty < \varepsilon$ and $\|S_n^\delta(Q) - Q\|_1 < \varepsilon$. It then follows that

$$\begin{aligned} \|S_n^\delta(f) - f\|_\infty &\leq \|S_n^\delta(f) - S_n^\delta(P)\|_\infty + \|S_n^\delta(P) - P\|_\infty + \|P - f\|_\infty \\ &\leq M\varepsilon + 2\varepsilon, \end{aligned}$$

and

$$\begin{aligned} \|S_n^\delta(g) - Q\|_1 &\leq \|S_n^\delta(g) - S_n^\delta(Q)\|_1 + \|S_n^\delta(Q) - Q\|_1 + \|Q - g\|_1 \\ &\leq M\varepsilon + 2\varepsilon, \end{aligned}$$

This proves the theorem. □

In the next chapter, we will investigate the Cesàro means of the Fourier orthogonal expansion on $B^d \times I^m$ using these techniques.

CHAPTER III

CONVERGENCE OF THE CESÀRO MEANS ON $B^d \times I^m$

In this section we will prove the result on the convergence of the orthogonal expansion of functions in $L^p(B^d \times I^m; \mu; \alpha, \beta)$, $1 \leq p < \infty$ and $C(B^d \times I^m)$. The proof of this theorem uses Theorem 5.3 in [17]. In addition, the following theorem will play a key role in the proof.

Theorem III.1. [9, Thm 1.1] *Let $\alpha_i > -1$, $\beta_i > -1$, and $\alpha_i + \beta_i \geq -1$ for $1 \leq i \leq m$. The Cesàro means $S_m^\delta f$ of the orthogonal expansion of f in the product Jacobi polynomials on I^m converge to f in $L^p(I^m; \omega^{(\alpha, \beta)})$ for $1 \leq p < \infty$ and $C(I^m)$, if*

$$\delta > \sum_{i=1}^m \max\{\alpha_i, \beta_i\} + \frac{m}{2} + \max \left\{ 0, - \sum_{i=1}^m \min\{\alpha_i, \beta_i\} - \frac{m+2}{2} \right\}.$$

Using the notation from chapter 1, we may write the reproducing kernel on the cylinder as

$$K_n(x, x', y, y') = \sum_{j=0}^n \sum_{k=0}^j \left[\mathbb{P}_{j-k}^{(\alpha, \beta)}(x) \right]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(x') \left[\mathbb{S}_k^\mu(y) \right]^T \mathbb{S}_k^\mu(y'). \quad (\text{III.0.1})$$

For a function $f \in C(B^d \times I^m)$, the Fourier partial sum is given by

$$S_n(f)(x', y') = \int_{I^m} \int_{B^d} K_n(x, x', y, y') f(x, y) w^{(\alpha, \beta)}(x) w_\mu(y) dy dx. \quad (\text{III.0.2})$$

The Cesaro means of order δ , or the (C, δ) means, are given by

$$K_n^\delta(x, x, y, y') = \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(x) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(x') \left[\mathbb{S}_k^\mu(y) \right]^T \mathbb{S}_k^\mu(y'), \quad (\text{III.0.3})$$

where $c_{n,j}^\delta = \frac{(-n)_j}{(-n-\delta)_j}$. Note that the Cesàro means cannot be written as a simple product of two series, one which is written in terms of x and x' , and one which is written in terms of y and y' . This is due to our choice of defining the degree of a polynomial as its total degree, and does not allow the problem to be reduced to a trivial result of separate estimates on B^d and I^m . We now state the theorem concerning the convergence of the Cesàro means on $B^d \times I^m$.

Theorem III.2. *Let f be a continuous function on $B^d \times I^m$, and let $\mu \geq 0$ and $\alpha_i > -1$, $\beta_i > -1$ and $\alpha_i + \beta_i \geq -1$ for $1 \leq i \leq m$. Then (C, δ) means of the orthogonal expansion of f in terms of orthogonal polynomials $S_n^\delta f$ converge to f if*

$$\delta > \sum_{i=1}^m \max\{\alpha_i, \beta_i\} + \mu + \frac{d+m-1}{2} + \max \left\{ 0, - \sum_{i=1}^m \min\{\alpha_i, \beta_i\} - \mu - \frac{d+m+1}{2} \right\}.$$

Proof. The theorem will follow from the fact that the (C, δ) means of the kernel are bounded; that is,

$$\int_{I^m} \int_{B^d} |K_n^\delta(x, x', y, y')| w_\mu(y) dy w^{(\alpha,\beta)}(x) dx \leq M$$

for some constant M which is independent of n , x' , and y' . Throughout the proof, and the rest of the dissertation, c will refer to a constant that may change values from line to line.

We first show that it is enough to consider $x' = e := (1, 1, \dots, 1)$.

Lemma III.1. *In order to prove the convergence of (C, δ) means of the orthogonal expansion, it suffices to prove*

$$\int_{I^m} \int_{B^d} |K_n^\delta(x, e, y, y')| w_\mu(y) dy w^{(\alpha, \beta)}(x) dx \leq M \quad (\text{III.0.4})$$

for M independent of n and y' .

Proof. The proof of the lemma follows from results in [6]. We state these results below.

Theorem III.3. *Let $\alpha, \beta > -1$ and $\alpha \geq \beta$. An integral representation of the form*

$$P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) = \int_{-1}^1 P_n^{(\alpha, \beta)}(1) P_n^{(\alpha, \beta)}(z) K^{(\alpha, \beta)}(x, y, z) w^{(\alpha, \beta)}(z) dz$$

satisfying

$$\int_{-1}^1 |K^{(\alpha, \beta)}(x, y, z)| w^{(\alpha, \beta)}(z) dz \leq M$$

exists for $-1 < x, y < 1$.

This result is for single variable Jacobi polynomials, but easily extends to the product Jacobi polynomials as

$$P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) = \int_{I^m} P_n^{(\alpha, \beta)}(e) P_n^{(\alpha, \beta)}(z) \mathbf{K}^{(\alpha, \beta)}(x, y, z) w^{(\alpha, \beta)}(z) dz,$$

where $x, y \in I^m$, and

$$\mathbf{K}^{(\alpha, \beta)}(x, y, z) = \prod_{i=1}^m K^{(\alpha_i, \beta_i)}(x_i, y_i, z_i),$$

and $\mathbf{K}^{(\alpha,\beta)}(\cdot, \cdot, \cdot)$ satisfies

$$\int_{I^m} |\mathbf{K}^{(\alpha,\beta)}(x, y, z)| w^{(\alpha,\beta)}(z) dz \leq M,$$

where M is a constant given by the product of the constants in the single variable theorem. By this result,

$$\begin{aligned} A_n^\delta &:= \int_{I^m} \int_{B^d} |K_n^\delta(x, x', y, y')| w_\mu(x) dy w^{(\alpha,\beta)}(x) dx \\ &= \int_{I^m} \int_{B^d} \left| \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(x) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(x') [\mathbb{S}_k^\mu(y)]^T \mathbb{S}_k^\mu(y') \right| \\ &\quad \times w_\mu(y) dy w^{(\alpha,\beta)}(x) dx \\ &\leq \int_{I^m} \int_{B^d} \int_{I^m} \left| \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(e) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(z) [\mathbb{S}_k^\mu(y)]^T \mathbb{S}_k^\mu(y') \right| \\ &\quad |\mathbf{K}(x, x', z)| w^{(\alpha,\beta)}(z) dz w_\mu(y) dy w^{(\alpha,\beta)}(x) dx. \end{aligned}$$

Applying Fubini's theorem gives

$$\begin{aligned} A_n^\delta &\leq \int_{B^d} \int_{I^m} \left| \sum_{j=0}^n c_{n,j}^\delta \sum_{k=0}^j \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(e) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(z) [\mathbb{S}_k^\mu(y)]^T \mathbb{S}_k^\mu(y') \right| \\ &\quad \times \int_{I^m} |\mathbf{K}(x, x', z)| w^{(\alpha,\beta)}(x) dx w^{(\alpha,\beta)}(z) dz w_\mu(y) dy \\ &\leq c \int_{I^m} \int_{B^d} |K_n^\delta(e, z, y, y')| w_\mu(y) dy w^{(\alpha,\beta)}(z) dz, \end{aligned}$$

which proves the lemma. \square

Our next lemma reduces the integral over B^d to an integral over $[-1, 1]$ of a

Gegenbauer polynomial. We define

$$\mathcal{G}_\mu^{(\alpha,\beta)}(y') := \int_{B^d} |K_n^\delta(1, z, y, y')| w_\mu(y) dy$$

and

$$F_{n,\mu}^\delta(\cdot) := \sum_{j=0}^n c_{k,n}^\delta \sum_{k=0}^j \frac{k + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} C_k^{(\mu + \frac{d-1}{2})}(\cdot) \left[\mathbb{P}_{j-k}^{(\alpha,\beta)}(x) \right]^T \mathbb{P}_{j-k}^{(\alpha,\beta)}(e).$$

Lemma III.2. For $\mu \geq 0$,

$$\mathcal{G}_\mu^{(\alpha,\beta)}(y') \leq c \int_{-1}^1 |F_n^\delta(u)| (1-u^2)^{\frac{d-2}{2}+\mu} du, \quad (\text{III.0.5})$$

Proof. We first consider $\mu > 0$ and consider the case of $\mu = 0$ later. Let $x' = e$ and substitute (II.2.15) into (III.0.4) to obtain

$$\mathcal{G}_\mu^{(\alpha,\beta)}(y') = \int_{B^d} \left| \int_{-1}^1 F_n^\delta \left(\langle y, y' \rangle + \sqrt{1-|y|^2} \sqrt{1-|y'|^2} t \right) (1-t^2)^{\mu-1} dt \right| w_\mu(y) dy. \quad (\text{III.0.6})$$

Applying the change of variable $y = r\eta$, where $\eta \in S^{d-1}$, $0 \leq r \leq 1$, gives

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha,\beta)}(y') &= \int_0^1 r^{d-1} \int_{S^{d-1}} \left| \int_{-1}^1 F_n^\delta \left(r \langle \eta, y' \rangle + \sqrt{1-|y'|^2} \sqrt{1-r^2} t \right) (1-t^2)^{\mu-1} dt \right| \\ &\quad (1-r^2)^{\mu-1/2} d\omega(\eta) dr, \end{aligned}$$

where $d\omega$ is the surface measure on S^{d-1} . Now let A be the rotation matrix such that $A(y') = (0, 0, \dots, 0, |y'|)$, and apply the change of basis $\eta \mapsto A^T \eta$ to obtain

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha,\beta)}(y') &= \int_0^1 r^{d-1} \int_{S^{d-1}} \left| \int_{-1}^1 F_n^\delta \left(r\eta_d |y'| + \sqrt{1-|y'|^2} \sqrt{1-r^2} t \right) (1-t^2)^{\mu-1} dt \right| \\ &\quad \times (1-r^2)^{\mu-1/2} d\omega(y') dr \end{aligned}$$

where $\eta = (\eta_1, \dots, \eta_d)$. If we let $\eta_d = s$, then $\eta = (\sqrt{1-s^2}\gamma, s)$ for some $\gamma \in S^{d-2}$, and changing variables gives

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &= \omega_{d-2} \int_0^1 r^{d-1} \int_{-1}^1 \left| \int_{-1}^1 F_n^\delta \left(rs|y'| + \sqrt{1-|y'|^2} \sqrt{1-r^2t} \right) (1-t^2)^{\mu-1} dt \right| \\ &\quad \times (1-r^2)^{\mu-1/2} (1-s^2)^{\frac{d-3}{2}} ds dr \end{aligned} \quad (\text{III.0.7})$$

where ω_{d-2} is the surface area of S^{d-2} . Let $s \mapsto p/r$ so $ds = dp/r$ and move the absolute value inside the inner integral to obtain

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq \omega_{d-2} \int_0^1 \int_{-r}^r \int_{-1}^1 \left| F_n^\delta \left(p|y'| + \sqrt{1-|y'|^2} \sqrt{1-r^2t} \right) (1-t^2)^{\mu-1} \right| dt \\ &\quad \times (1-r^2)^{\mu-1/2} r(r^2-p^2)^{\frac{d-3}{2}} dp dr. \end{aligned}$$

Switching the order of integration of r and p and applying the change of variable $q \mapsto \sqrt{1-r^2}t$, $dq = \sqrt{1-r^2} dt$ gives

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq \omega_{d-2} \int_{-1}^1 \int_{|p|}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \left| F_n^\delta \left(p|y'| + \sqrt{1-|y'|^2}q \right) \right| (1-r^2-q^2)^{\mu-1} dq \\ &\quad \times r(r^2-p^2)^{\frac{d-3}{2}} dr dp. \end{aligned}$$

Switching the order of integration of q and r gives

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq \omega_{d-2} \int_{-1}^1 \int_{-\sqrt{1-|p|^2}}^{\sqrt{1-|p|^2}} \left| F_n^\delta \left(p|y'| + \sqrt{1-|y'|^2}q \right) \right| \\ &\quad \times \int_{|p|}^{\sqrt{1-q^2}} (1-r^2-q^2)^{\mu-1} r(r^2-p^2)^{\frac{d-3}{2}} dr dq dp. \end{aligned}$$

Applying the change of variable $r^2 = u(1-q^2-p^2) + p^2$ shows the inner integral

is $\frac{1}{2}(1 - q^2 - p^2)^{\mu + \frac{d-3}{2}} B(\mu, \frac{d-1}{2})$, where $B(x, y)$ is the beta function, defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0. \quad (\text{III.0.8})$$

Hence, we have the inequality

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq \frac{\omega_{d-2} B(\mu, \frac{d-1}{2})}{2} \int_{-1}^1 \int_{-\sqrt{1-|p|^2}}^{\sqrt{1-|p|^2}} \left| F_n^\delta \left(p|y'| + \sqrt{1-|y'|^2} q \right) \right| \\ &\quad \times (1 - q^2 - p^2)^{\frac{d-3}{2} + \mu} dq dp. \end{aligned}$$

Next, we apply the change of variable $q = \sqrt{1-p^2}s$ to obtain

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq c \int_{-1}^1 \int_{-1}^1 \left| F_n^\delta \left(p|y'| + \sqrt{1-|y'|^2} \sqrt{1-p^2}s \right) \right| \\ &\quad \times (1-p^2)^{\frac{d-2}{2} + \mu} (1-s^2)^{\frac{d-3}{2} + \mu} ds dp. \end{aligned} \quad (\text{III.0.9})$$

$$(\text{III.0.10})$$

$$(\text{III.0.11})$$

Changing variables once again, we let $u = p|y'| + \sqrt{1-|y'|^2} \sqrt{1-p^2}s$ to obtain

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq c \int_{-1}^1 \int_{p|y'| - \sqrt{1-|y'|} \sqrt{1-p^2}}^{p|y'| + \sqrt{1-|y'|} \sqrt{1-p^2}} |F_n^\delta(u)| \frac{(1-|y'|^2 - p^2 - u^2 + 2upv)^{\frac{d-3}{2} + \mu}}{[(1-|y'|^2)(1-p^2)^2(1-u^2)]^{\frac{d-2}{2} + \mu}} \\ &\quad \times (1-p^2)^{\frac{d-2}{2} + \mu} (1-u^2)^{\frac{d-2}{2} + \mu} du dp. \end{aligned}$$

Now we employ the function $D_\lambda(u, v, p)$ introduced in [17], which is defined

as

$$D_\lambda(v, p, u) = \frac{(1 - v^2 - p^2 - u^2 + 2upv)^{\lambda-1/2}}{[(1 - v^2)(1 - u^2)(1 - p^2)]^\lambda}$$

for $1 - v^2 - p^2 - u^2 + 2upv \geq 0$ and 0 otherwise. It is readily verified that

$$\int_{-1}^1 D_\lambda(u, v, p) (1 - p^2)^\lambda dp = 2^{2\lambda} B(\lambda + 1/2, \lambda + 1/2),$$

where $B(x, y)$ is the beta function. Hence, substituting D_λ into the integral and switching the order of integration, we have

$$\begin{aligned} \mathcal{G}_\mu^{(\alpha, \beta)}(y') &\leq c \int_{-1}^1 |F_n^\delta(u)| \int_{-1}^1 D_{\frac{d-2}{2}+\mu}(|y'|, u, p) (1 - p^2)^{\frac{d-2}{2}+\mu} dp (1 - u^2)^{\frac{d-2}{2}+\mu} du \\ &= c \int_{-1}^1 |F_n^\delta(u)| (1 - u^2)^{\frac{d-2}{2}+\mu} du \end{aligned} \quad (\text{III.0.12})$$

This proves the lemma for $\mu > 0$.

Turning our attention now to the case when $\mu = 0$, we substitute (II.2.16) into the left side of (III.0.4) and ignore the integral over I^m as before to obtain

$$\begin{aligned} \mathcal{G}_0^{(\alpha, \beta)}(y') &:= \int_{B^d} |F_n^\delta \left(\langle y, y' \rangle + \sqrt{1 - |y|^2} \sqrt{1 - |y'|^2} \right) \\ &\quad + F_n^\delta \left(\langle y, y' \rangle \sqrt{1 - |y|^2} \sqrt{1 - |y'|^2} \right)| w_0(y) dy. \end{aligned}$$

We perform the same change of variables from the case when $\mu > 0$ to obtain the equivalent of (III.0.7),

$$\begin{aligned} \mathcal{G}_0^{(\alpha, \beta)}(y') &= \omega_{d-2} \int_0^1 r^{d-1} \int_{-1}^1 |F_n^\delta \left(rs|y'| + \sqrt{1 - |y'|^2} \sqrt{1 - r^2} \right) \\ &\quad + F_n^\delta \left(rs|y'| - \sqrt{1 - |y'|^2} \sqrt{1 - r^2} \right)| (1 - r^2)^{-1/2} (1 - s^2)^{\frac{d-3}{2}} ds dr. \end{aligned}$$

Now we substitute $p = \sqrt{1 - r^2}$ and let $v = \sqrt{1 - |y'|^2}$ to obtain

$$\begin{aligned}
\mathcal{G}_0^{(\alpha, \beta)}(y') &= \omega_{d-2} \int_{-1}^1 \int_0^1 |F_n^\delta(\sqrt{1-p^2}\sqrt{1-v^2}s + pv) \\
&\quad + F_n^\delta(\sqrt{1-p^2}\sqrt{1-v^2}s - pv)| (1-p^2)^{\frac{d-2}{2}} (1-s^2)^{\frac{d-3}{2}} dp ds \\
&= \omega_{d-2} \int_{-1}^1 \int_{-1}^1 |F_n^\delta(\sqrt{1-p^2}\sqrt{1-v^2}s + pv)| \\
&\quad \times (1-p^2)^{\frac{d-2}{2}} (1-s^2)^{\frac{d-3}{2}} dp ds. \tag{III.0.13}
\end{aligned}$$

The right side of (III.0.13) is the right side of (III.0.9), with v in place of $|y'|$.

Following the same steps of the proof for $\mu > 0$, we obtain the equivalent of (III.0.12),

$$\mathcal{G}_0^{(\alpha, \beta)}(y') \leq c \int_{-1}^1 |F_n^\delta(u)| (1-u^2)^{\frac{d-2}{2}} du,$$

which proves the case for $\mu = 0$. □

To finish the proof, we substitute (III.0.5) into (III.0.4) to obtain

$$\begin{aligned}
&\int_{I^m} \int_{B^d} |K_n^\delta(x, e, y, y')| (1-|y|^2)^{\mu-\frac{1}{2}} dy d\mu^{(\alpha, \beta)}(x) \\
&\leq c \int_{I^m} \int_{-1}^1 \left| \sum_{j=0}^n c_{k,n}^\delta \sum_{k=0}^j \frac{k + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} C_k^{(\mu + \frac{d-1}{2})}(u) \right. \\
&\quad \left. \times \left[\mathbb{P}_{j-k}^{(\alpha, \beta)}(x) \right]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(e) \right| (1-u^2)^{\frac{d-2}{2} + \mu} du w^{(\alpha, \beta)}(x) dx.
\end{aligned}$$

After substituting in the identity (II.2.6), we arrive at the following inequality.

$$\begin{aligned}
& \int_{I^m} \int_{B^d} |K_n^\delta(x, e, y, y')| w_\mu(y) dy w^{(\alpha, \beta)}(x) \\
& \leq c \int_{I^m} \int_{-1}^1 \left| \sum_{j=0}^n c_{k,n}^\delta \sum_{k=0}^j \tilde{C}_k^{(\mu + \frac{d-1}{2})}(u) \tilde{C}_k^{(\mu + \frac{d-1}{2})}(1) \right. \\
& \quad \left. \times \left[\mathbb{P}_{j-k}^{(\alpha, \beta)}(x) \right]^T \mathbb{P}_{j-k}^{(\alpha, \beta)}(e) \right| (1-u^2)^{\frac{d-2}{2} + \mu} du w^{(\alpha, \beta)}(x) dx.
\end{aligned}
\tag{1}$$

Since the Gegenbauer polynomials are a subset of the Jacobi Polynomials, this expression is equivalent to proving the Cesaro summability of the product Jacobi polynomials on I^{m+1} , with respect to the weight $d\mu^{(\alpha, \beta)}(x)(1-x_{m+1}^2)^{\frac{d-2}{2} + \mu} du$, where $x = (x_1, x_2, \dots, x_m)$. The theorem then follows from Theorem III.1. \square

CHAPTER IV

RADON PROJECTIONS AND DISCRETIZED EXPANSIONS

In this chapter, we will introduce the Radon transform and give an overview of its role in the Fourier orthogonal expansion of a function on B^2 . We will derive a discretized Fourier orthogonal expansion for functions on the domain $B^2 \times [-1, 1]$ in terms of Radon projections of f , which are taken on parallel disks that are perpendicular to the axis of the cylinder $B^2 \times [-1, 1]$. Finally, we will show that the Lebesgue constant of the discretized Fourier expansion is $\approx m(\log(m+1))^2$ where the notation $a \approx b$ means there are positive constants, c_1 and c_2 , such that

$$c_1 a \leq b \leq c_2 a.$$

IV.1 Radon Projections

The Radon transform, named after the mathematician Johann Radon, maps an integrable function on \mathbb{R}^d to the set of integrals of f over all hyperplanes in \mathbb{R}^d . The integrals of f are called the Radon projections of f . More specifically, for f defined on \mathbb{R}^d , we define the Radon projection of f , $\mathcal{R}_\xi(f; t)$, with $\xi \in S^{d-1}$, the unit sphere in \mathbb{R}^d , and $t \in \mathbb{R}$, to be

$$\int_{\langle x, \xi \rangle = t} f(x) dx.$$

Radon's famous result is the following theorem.

Theorem IV.1. *Suppose f is a continuous function on \mathbb{R}^2 satisfying the following properties.*

1. *The integral*

$$\int_{\mathbb{R}^2} \frac{|f(x, y)|}{\sqrt{x^2 + y^2}} dx dy$$

is convergent.

2. *If we define*

$$\bar{f}(x, y; r) := \frac{1}{2\pi} \int_0^{2\pi} f(x + r \cos \phi, y + r \sin \phi) d\phi.$$

for a point $(x, y) \in \mathbb{R}^2$ and $r > 0$, then for any choice of point (x, y) and r ,

$$\lim_{r \rightarrow \infty} \bar{f}_P(r) = 0.$$

Then f can be completely reconstructed from its Radon projections.

Radon gave an explicit inversion formula for the function. This result can be extended to \mathbb{R}^d for $d > 2$; see chapter 2 in [12].

In practice, one often wants to approximate a function f by a finite number of Radon projections. For example, in the two-dimensional setting, Radon projections are closely related to the medical field of computerized tomography. In computerized tomography (CT), the central problem is the reconstruction of images from a finite set of X-ray data. The relative loss of intensity of an X-ray passing through a body is directly related to the line integral of a function $f(x)$, the X-ray attenuation coefficient at a point x in the body. Hence, the problem is

reconstructing f from a finite number of Radon projections. Since f can be completely reconstructed from a complete set of its line integrals for suitably nice functions, we expect that we should be able to approximate a suitably nice function f well using a finite number of Radon projections. In order to do this, we will focus on a relationship between Radon projections and orthogonal polynomials, which was first investigated for functions defined on the domain B^2 in [11] and [10].

Theorem IV.2. [11, Thm. 1] *If $P \in \mathcal{V}_m(B^2)$, then*

$$\mathcal{R}_\theta(P; t) = 2 \frac{\sqrt{1-t^2}}{m+1} U_m(t) P(\cos \theta, \sin \theta),$$

where $U_m(t)$ is the Chebyshev polynomial of the second kind of degree m .

This relationship has been integral in obtaining further results involving Radon projections. In [3], this relationship was used to find a polynomial on B^2 that interpolates the Radon projections of a function taken on sets of parallel lines in directions given by equidistant angles along the unit circle, while in [2], the polynomial interpolating Radon projections on parallel lines in arbitrary directions is considered. We will focus on the result in [18], in which an explicit reconstruction algorithm for a function on B^2 was given in terms of finite Radon projections. This result relies on the relationship (IV.2) and the Fourier orthogonal expansion of a function on B^2 in terms of the orthogonal polynomials given in (II.2.13). We will first introduce some notation, and then give the relationship between Radon projections and the Fourier orthogonal expansion.

Let f be an integrable function defined on \mathbb{R}^2 . We define $L(\theta, t)$, for $\theta \in [0, 2\pi]$ and $t \in \mathbb{R}$, to be the line $\{(s \cos \theta + t \sin \theta, s \sin \theta - t \cos \theta) : s \in \mathbb{R}\}$.

Then $\mathcal{R}_\theta(f; t)$ can be written in the form

$$\mathcal{R}_\theta(f; t) := \int_{L(\theta; t)} f(x, y) \, dx dy.$$

By restricting the domain of f to B^2 , we only need to consider line segments in B^2 in the definition of $\mathcal{R}_\theta(f; t)$. For this reason, we define $I(\theta; t)$ to be the intersection of $L(\theta; t)$ and B^2 ; that is,

$$I(\theta; t) = \{(s \cos \theta + t \sin \theta, s \sin \theta - t \cos \theta) : |s| \leq \sqrt{1 - t^2}\},$$

and so $\mathcal{R}_\theta(f; t)$ can be re-written with the integral taken over $I(\theta; t)$ instead of $L(\theta; t)$,

$$\mathcal{R}_\theta(f; t) = \int_{I(\theta; t)} f(x, y) \, dx dy.$$

Recall the orthogonal polynomials in (II.2.13) form an orthonormal basis for $\mathcal{V}_k(B^2)$. The Fourier coefficients of the Fourier orthogonal expansion of f in terms of this basis of orthogonal polynomials may be written in terms of the Radon projections of f .

Theorem IV.3. [18, Prop. 3.1] *Let $m > 0$ and $f \in L^2(B^2)$. For $0 \leq k \leq 2m$ and $0 \leq j \leq k$,*

$$\begin{aligned} \int_{B^2} f(x, y) U_k(\theta_{j,k}; x, y) \, dx dy \\ = \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^1 \mathcal{R}_{\phi_\nu}(f; t), U_k(t) \, dt U_k(\cos(\theta_{j,k} - \phi_\nu)), \end{aligned}$$

where $\phi_\nu = \frac{2\pi\nu}{2m+1}$.

As a result of this relationship, the orthogonal projection of a function f in

$L^2(B^2)$ onto $\mathcal{V}_k(B^2)$ can be written in terms of the Radon projections of f .

Theorem IV.4. [18, Thm. 3.2] For $m > 0$ and $k \leq 2m$, the projection operator proj_k from $L^2(B^2)$ to $\mathcal{V}_k(B^2)$ can be written in the form

$$\text{proj}_k f(x, y) = \frac{1}{2m+1} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^1 \mathcal{R}_{\phi_\nu}(f; t) U_k(t) dt (k+1) U_k(\phi_\nu; x, y).$$

As a consequence of this theorem, the Fourier partial sum of f , $S_{2m}f$, can also be written in terms of the Radon projections of f ,

$$S_{2m}f(x, y) = \frac{1}{2m+1} \sum_{k=0}^{2m} \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^1 \mathcal{R}_{\phi_\nu}(f; t) U_k(t) dt (k+1) U_k(\phi_\nu; x, y). \quad (\text{IV.1.1})$$

Recalling (IV.2), the expression

$$\frac{\mathcal{R}_{\phi_\nu}(f; t)}{\sqrt{1-t^2}}$$

is a polynomial of degree $2m$ if f is a polynomial of degree $2m$. Hence, by multiplying and dividing by a factor of $\sqrt{1-t^2}$ in (IV.1.1), and then using the $2m$ -point Gaussian quadrature rule given in (II.2.11) to replace the integral with a sum, a discretized Fourier orthogonal expansion which preserves polynomials of degree less than or equal to $2m-1$ is obtained. This discretized expansion, \mathcal{A}_{2m} , is given by

$$\mathcal{A}_{2m}(f)(x, y) = \frac{1}{(2m+1)^2} \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \mathcal{R}_{\phi_\nu}(f; \cos \theta_{j,2m}) \sum_{k=0}^{2m} (k+1) \sin((k+1)\theta_{j,2m}) U_k(\phi_\nu; x, y).$$

The upper limit of the sum in ν is chosen to be $2m$ to eliminate redundancy in the Radon data. With the choice of $2m$, the Radon projections are taken along parallel lines in $(2m+1)$ directions, given by equally spaced points on the unit circle. If the Radon projections were taken along parallel lines in $2m$ directions,

then for $\nu < m$, and $\phi_\nu = \frac{2\pi\nu}{2m}$,

$$\pi + \phi_\nu = \phi_{\nu+m},$$

and the identity

$$\mathcal{R}_{\pi+\phi_\nu}(f; \cos(\theta_{2m+1-j, 2m})) = \mathcal{R}_{\phi_\nu}(f; \cos(\theta_{j, 2m}))$$

shows these two Radon projections are actually the same. In practical settings, more Radon projections are desirable, so we choose to take Radon projections in an odd number of directions.

As an operator on $C(B^2)$, the discretized expansion has a Lebesgue constant of $m \log(m+1)$. In the next section, we will study a version of this algorithm adapted for the cylinder, $B^2 \times [-1, 1]$.

IV.2 Construction of the Discretized Partial Sum Expansion on the Cylinder

We first introduce notation to adapt the Radon projection on B^2 to the cylinder $B^2 \times [-1, 1]$. For an integrable function on $B^2 \times [-1, 1]$, we will take the regular two-dimensional Radon projection of f on disks perpendicular to the axis of the cylinder at position z on the axis of the cylinder. Hence, we use the notation

$$\mathcal{R}_\theta(f(\cdot, \cdot, z); t) := \int_{I(\theta; t)} f(x, y, z) dx dy,$$

where $(x, y) \in B^2$ and $z \in [-1, 1]$, to denote the Radon projection of a function f at position z on the axis of the cylinder, along the line segment $I(\theta; t)$.

With this definition in mind, Theorem (IV.2) can be adapted to the domain $B^2 \times [-1, 1]$.

Lemma IV.1. *If P is a polynomial of degree k on $B^2 \times [-1, 1]$, then for $\theta \in [0, 2\pi]$,*

$$\frac{\mathcal{R}_\theta(P(\cdot, \cdot, s); t)}{\sqrt{1-t^2}}$$

is a polynomial of degree k in t .

Proof. If P is a polynomial of degree k , we may write

$$P(x, y, z) = \sum_{i=0}^k c_i z^i p_{k-i}(x, y),$$

where $p_{k-i}(x, y)$ is a polynomial of degree $k-i$ in x and y . Following the proof of [18, Lem. 2.2], we write

$$\frac{\mathcal{R}_\theta(P(\cdot, \cdot, z); t)}{\sqrt{1-t^2}} = \sum_{i=0}^k c_i z^i \frac{1}{\sqrt{1-t^2}} \int_{I(\theta; t)} p_{k-i}(x, y) dx dy.$$

Rewriting the integral and changing variables,

$$\begin{aligned} & \frac{1}{\sqrt{1-t^2}} \int_{I(\theta; t)} p_{k-i}(x, y) dx dy \\ &= \frac{1}{\sqrt{1-t^2}} \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} p_{k-i}(t \cos \theta + s \sin \theta, t \sin \theta - s \cos \theta) ds \\ &= \int_{-1}^1 p_{k-i}(t \cos \theta + \sqrt{1-t^2} s \sin \theta, t \sin \theta - \sqrt{1-t^2} s \cos \theta) ds. \end{aligned}$$

After expanding $p_{k-i}(x, y)$ in the integrand, we note that each odd power of $\sqrt{1-t^2}$ is accompanied by an odd power of s , which becomes 0 after integrating.

Hence,

$$\frac{1}{\sqrt{1-t^2}} \int_{I(\theta,t)} p_{k-i}(x,y) dx dy = \sum_{j=0}^{\lfloor \frac{k-i}{2} \rfloor} b_j t^{k-i-2j} (1-t^2)^j$$

for some coefficients b_j . The lemma follows. \square

Recall the definition of $P_{j,k,n}$ from (II.2.19). We let proj_k be the projection operator from the space $L^2(B^2 \times [-1, 1]; \omega_Z)$ onto $\mathcal{V}_k(B^2 \times [-1, 1]; \omega_Z)$, and prove the equivalent of theorem (IV.4).

Theorem IV.5. *Let $m \geq 0$ and let $n \leq 2m$. Define $\phi_\nu := \frac{2\nu\pi}{2m+1}$, and $\sigma_\nu(x, y) := \arccos(x \cos(\phi_\nu) + y \sin(\phi_\nu))$. The operator proj_n can be written as*

$$\text{proj}_n f(x, y, z) = \frac{1}{\pi} \sum_{\nu=0}^{2m} \int_{-1}^1 \int_{-1}^1 \mathcal{R}_{\phi_\nu}(f(\cdot, \cdot, s); t) \Psi_\nu(x, y, z; s, t) dt \frac{ds}{\sqrt{1-s^2}},$$

(IV.2.2)

$$\Psi_{\nu,n}(x, y, z; s, t) = \frac{1}{2m+1} \sum_{k=0}^n (k+1) U_k(t) U_k(\cos(\sigma_\nu(x, y))) \tilde{T}_{n-k}(s) \tilde{T}_{n-k}(z)$$

Proof. Since the polynomials $P_{j,k,n}$ in (II.2.19) form an orthonormal basis for $\mathcal{V}_n(B^2 \times [-1, 1]; \omega_Z)$,

$$\text{proj}_n(f)(x, y, z) = \sum_{k=0}^n \sum_{j=0}^k \frac{1}{\pi} \int_{-1}^1 \int_{B^2} f(x, y, z) P_{j,k,n}(x, y, z) \omega_Z(x, y, z).$$

After expanding $P_{j,k,n}(x, y, z)$, Theorem IV.4 gives the result. \square

This relationship between the projection operator and the Radon projection again yields a connection between the partial sum operator S_{2m} and the Radon projection.

Corollary IV.6. *Let $m \geq 0$. The partial sum operator S_{2m} may be written as*

$$S_{2m}(f)(x, y, z) = \sum_{\nu=0}^{2m} \frac{1}{\pi} \int_{-1}^1 \int_{-1}^1 \mathcal{R}_{\phi_\nu}(f(\cdot, \cdot, s); t) \Phi_\nu(x, y, z; s, t) \frac{ds}{\sqrt{1-s^2}} dt, \quad (\text{IV.2.3})$$

where

$$\Phi_\nu(x, y, z; s, t) = \sum_{n=0}^{2m} \Psi_{\nu,n}(x, y, z; s, t).$$

To discretize this Fourier orthogonal expansion, we discretize the two integrals by $2m$ -point Gaussian quadratures. Using Lemma (IV.1), we divide and multiply the integrand in (IV.2.3) by a factor of $\sqrt{1-t^2}$. For the integral in t , we use the quadrature formula (II.2.11), while for the integral in s , the quadrature formula (II.2.9) is used. The discretized partial sum operator, \mathcal{B}_{2m} , is given below.

Definition IV.1. For $m \geq 0$, $(x, y) \in B^2$ and $z \in [-1, 1]$, we define

$$\mathcal{B}_{2m}(f)(x, y, z) := \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \sum_{l=0}^{2m-1} \mathcal{R}_{\phi_\nu}(f(\cdot, \cdot, z_{l,2m}; \cos(\theta_{j,2m}))) T_{\nu,j,l}(x, y, z), \quad (\text{IV.2.4})$$

where

$$T_{\nu,j,l}(x, y, z) = \frac{1}{(2m+1)^3} \sum_{n=0}^{2m} \sum_{k=0}^n (k+1) \sin((k+1)\theta_{j,2m}) U_k(\cos(\sigma_\nu(x, y))) \\ \times \tilde{T}_{n-k}(z_l) \tilde{T}_{n-k}(z). \quad (\text{IV.2.5})$$

As a result of Lemma IV.1 and the fact that $2m$ -point Gaussian quadratures are exact for polynomials of degrees up to $4m-1$, we obtain the following theorem.

Theorem IV.7. *The algorithm \mathcal{B}_{2m} preserves polynomials of degree less than or*

equal to $2m - 1$; that is, for $(x, y) \in B^2$ and $z \in [-1, 1]$,

$$\mathcal{B}_{2m}(f)(x, y, z) = f(x, y, z)$$

for $f \in \Pi_{2m-1}$.

The proof of the following theorem is contained in the next section.

Comparing this with the result in [18], we see that the extension of the algorithm to the cylinder introduces a factor of $\log(m + 1)$.

Theorem IV.8. *For $m \geq 0$, the norm of the operator \mathcal{B}_{2m} on $C(B^2 \times [-1, 1])$ is given by*

$$\|\mathcal{B}_{2m}\|_\infty \approx m (\log(m + 1))^2.$$

The proof of this theorem is not trivial. Since we have defined the degree of a polynomial to be its total degree, the series in the definition of $T_{\nu,j,l}(x, y, z)$,

$$\sum_{n=0}^{2m} \sum_{k=0}^n (k + 1) \sin((k + 1)\theta_{j,2m}) U_k(\cos(\sigma_\nu(x, y))) \times \tilde{T}_{n-k}(z_l) \tilde{T}_{n-k}(z),$$

cannot be written as the product of a two series, one in terms of z and z_l , and one in terms of θ_j and $\sigma_\nu(x, y)$. As a result, the estimate of the Lebesgue constant cannot be trivially reduced to an estimate on B^2 and an estimate on $[-1, 1]$. In particular, for the upper bound of the estimate, a different approach from that in [18] is used to obtain this result.

As a result of Theorems (IV.8) and (IV.7), we obtain the following corollary.

Corollary IV.9. *For $f \in C^2(B^2 \times [-1, 1])$, $\mathcal{B}_{2m}(f)$ converges to f in the uniform norm.*

Proof. If $f \in C^2(B^2 \times [-1, 1])$, then by Theorem 1 in [1], there exists a polynomial p_n of degree n on $B^2 \times [-1, 1]$, and a constant $C > 0$, so that

$$\|f - p_n\|_\infty \leq \frac{C}{n^2} \omega_{f,2} \left(\frac{1}{n} \right),$$

where

$$\omega_{f,2} \left(\frac{1}{n} \right) = \sup_{|\gamma|=2} \left(\sup_{\substack{x,y \in B^2 \times [-1,1] \\ |x-y| \leq 1/n}} |D^\gamma f(x) - D^\gamma f(y)| \right).$$

We let $n = 2m - 1$ to obtain

$$\begin{aligned} \|\mathcal{B}_{2m}(f) - f\|_\infty &\leq \|\mathcal{B}_{2m}(f - p_{2m-1})\|_\infty + \|f - p_{2m-1}\|_\infty \\ &\leq \|f - p_{2m-1}\|_\infty (1 + \|\mathcal{B}_{2m}\|_\infty) \\ &\leq c \frac{1}{(2m-1)^2} (m(\log(m+1))^2 + 1), \end{aligned}$$

which converges to zero as m approaches infinity. \square

Before proceeding to the proof of Theorem IV.8, we make one comment.

We believe that the Lebesgue constant of the Fourier partial sum of the orthogonal expansion $\|S_{2m}\|$ is $m \log(m+1)$, although we have yet to prove it. If this is true, the discretization of the expansion adds a factor of $\log(m+1)$.

IV.3 Proof of Theorem IV.8

We first derive an expression with which we may estimate $\|\mathcal{B}_{2m}\|_\infty$.

Proposition IV.1. *The norm of \mathcal{B}_{2m} as an operator on $C(B^2 \times [-1, 1])$ is given by*

$$\|\mathcal{B}_{2m}\|_\infty = 2 \max_{\nu=0} \sum_{j=1}^{2m} \sum_{l=0}^{2m-1} \sin \theta_{j,2m} |T_{\nu,j,l}(x, y, z)| \quad (\text{IV.3.1})$$

where the maximum is taken over all points (x, y, z) in $B^2 \times [-1, 1]$.

Proof. By definition,

$$\begin{aligned} \mathcal{R}_{\phi_\nu}(f(\cdot, \cdot, z_l), \cos \theta_{j,2m}) &= \int_{I(\cos \theta_{j,2m}, \phi_\nu)} f(\tilde{x}, \tilde{y}, z_l) d\tilde{x}d\tilde{y} \\ &= \int_{-\sin \theta_{j,2m}}^{\sin \theta_{j,2m}} f(\cos \theta_{j,2m} \cos \phi_\nu - s \cos \phi_\nu, \cos \theta_{j,2m} \sin \phi_\nu + s \cos \phi_\nu, z_l) ds. \end{aligned} \quad (\text{IV.3.2})$$

Taking absolute value of both sides and using the triangle inequality, we immediately have

$$\|\mathcal{B}_{2m}\|_\infty \leq 2 \max_{\nu=0}^{2m} \sum_{j=1}^{2m} \sum_{l=0}^{2m-1} \sin \theta_{j,2m} |T_{\nu,j,l}(x, y, z)|$$

On the other hand, if we define

$$\mathcal{T}(x, y, z) := 2 \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \sum_{l=0}^{2m-1} \sin \theta_{j,2m} |T_{\nu,j,l}(x, y, z)|,$$

then $\mathcal{T}(x, y, z)$ is a continuous function on $B^2 \times [-1, 1]$, and hence achieves its maximum at some point (x_0, y_0, z_0) on the cylinder. We would like to choose a function f so that $f(x, y, z) = \text{sign}(T_{\nu,j,l}(x_0, y_0, z_0))$ on the set of lines $\{(I(\cos \theta_{j,2m}, \phi_\nu), z_l)\}$, for $1 \leq j \leq 2m$, $0 \leq \nu \leq 2m$, and $0 \leq l \leq 2m - 1$, since this would immediately give us the result. However, such a function may not be continuous at the points of intersection of these lines. To allow for continuity, we instead take neighborhoods of volume ε around each point of intersection of the lines, and define a function f^* which is equal to $\text{sign}(T_{\nu,j,l}(x_0, y_0, z_0))$ on the lines $\{(I(\cos \theta_{j,2m}, \phi_\nu), z_l)\}_{j,\nu,l}$ except on the ε -neighborhoods at the points of intersection; on the rest of the cylinder, f^* is chosen so that it takes values

between 1 and -1 and is continuous. It then follows that

$$\|\mathcal{B}_{2m}\|_\infty \geq |\mathcal{B}_{2m}(f^*(x_0, y_0, z_0))| \geq 2 \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \sum_{l=0}^{2m-1} \sin \theta_{j,2m} |T_{\nu,j,l}(x_0, y_0, z_0)| - c\varepsilon,$$

where c denotes the number of points of intersection of the lines

$\{(I(\cos \theta_{j,2m}, \phi_\nu), z_l)\}_{j,\nu,l}$. Since ε is arbitrary, this proves the proposition. \square

For the remainder of the proof, the number n in (II.2.8) and (II.2.10) will be fixed as $2m$. For this reason, we define

$$\begin{aligned} \theta_j = \theta_{j,2m} &= \frac{j\pi}{2m+1}, \quad \gamma_l = \gamma_{l,2m} = \frac{2l+1}{4m}\pi, \quad z_l = z_{l,2m} = \cos \gamma_l, \\ \phi_\nu &= \frac{2\pi\nu}{2m+1}, \quad \sigma_\nu(x, y) = \arccos(x \cos \phi_\nu + y \sin \phi_\nu). \end{aligned} \quad (\text{IV.3.3})$$

The proof will be separated into two parts: a lower bound, to show

$\|\mathcal{B}_{2m}\| \geq c_1 m (\log(m+1))^2$ for some constant c_1 ; and an upper bound, to show $\mathcal{B}_{2m} \leq c_2 m (\log(m+1))^2$ for some constant c_2 .

IV.3.1 Lower Bound

We will establish there exists a constant $c > 0$ so that

$\|\mathcal{B}_{2m}\|_\infty \geq cm (\log(m+1))^2$ for all $m > 0$. By (IV.3.1), it suffices to show

$$\sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \sum_{l=0}^{2m-1} \sin \theta_j |T_{\nu,j,l}(x_1, y_1, z_1)| \geq cm (\log(m+1))^2$$

for the point $(x_1, y_1, z_1) = (\cos \frac{\pi}{4m+2}, \sin \frac{\pi}{4m+2}, 1)$ for some $c > 0$. We begin by deriving a compact formula for $T_{\nu,j,l}(x_1, y_1, 1)$. Using the Christoffel-Darboux

formula for \tilde{T}_n , letting $\cos \gamma_z = z$,

$$|T_{\nu,j,l}(x, y, z)| = \frac{1}{(2m+1)^3} \left| \frac{1}{\sin \sigma_\nu(x, y)} \sum_{k=0}^{2m} \sin((k+1)\sigma_\nu(x, y)) \sin((k+1)\theta_j) \right. \\ \left. \times \frac{\cos((2m-k+1)\gamma_z) \cos((2m-k)\gamma_l) - \cos((2m-k+1)\gamma_l) \cos((2m-k)\gamma_z)}{\cos(\gamma_z) - \cos(\gamma_l)} \right|.$$

Substituting in $z = 1$ and applying the identity for the difference of cosines,

$$|T_{\nu,j,l}(x, y, 1)| = \frac{1}{(2m+1)^3} \frac{1}{\sin \sigma_\nu(x, y)} \frac{1}{\sin \frac{\gamma_l}{2}} \\ \times \left| \sum_{k=0}^{2m} (k+1) \sin((k+1)\sigma_\nu(x, y)) \sin((k+1)\theta_j) \sin((2m-k+1/2)\gamma_l) \right|.$$

Applying the product formula for sine and the product formula for sine and cosine,

$$|T_{\nu,j,l}(x, y, 1)| = \frac{1}{4} \frac{1}{(2m+1)^3} \frac{1}{\sin \frac{\gamma_l}{2}} \frac{1}{\sin \sigma_\nu(x, y)} \quad (\text{IV.3.4}) \\ \times \left| \sum_{k=0}^{2m} (k+1) \left[\sin((k+1)(\theta_j - \sigma_\nu(x, y) + \gamma_l) - \frac{3}{2}\gamma_l - \frac{\pi}{2}) \right. \right. \\ - \sin((k+1)(\theta_j - \sigma_\nu(x, y) - \gamma_l) + \frac{3}{2}\gamma_l + \frac{\pi}{2}) \\ - \sin((k+1)(\theta_j + \sigma_\nu(x, y) + \gamma_l) - \frac{3}{2}\gamma_l - \frac{\pi}{2}) \\ \left. \left. + \sin((k+1)(\theta_j + \sigma_\nu(x, y) - \gamma_l) + \frac{3}{2}\gamma_l + \frac{\pi}{2}) \right] \right|.$$

Next, apply the formula

$$\sum_{k=0}^{2m} (k+1) \sin((k+1)\theta + \phi) \\ = \frac{1}{2} \frac{(2m+2) \sin((2m+1)\theta + \phi) - (2m+1) \sin((2m+2)\theta + \phi) + \sin(\phi)}{\sin^2\left(\frac{\theta}{2}\right)} \\ = \frac{1}{2} \frac{\sin((2m+1)\theta + \phi) - (4m+2) \cos((2m+3/2)\theta + \phi) \sin(\theta/2) + \sin(\phi)}{\sin^2\left(\frac{\theta}{2}\right)},$$

to (IV.3.4). Under our choice of x_1 and y_1 , $\cos \sigma_\nu(x_1, y_1) = \cos \frac{2\nu-1/2}{2m+1}\pi$, so $\sigma_\nu(x_1, y_1) = \frac{2\nu-1/2}{2m+1}\pi$ if $1 \leq \nu \leq m$. We will only be considering ν within this range, so we define $\sigma_\nu := \frac{2\nu-1/2}{2m+1}\pi$. Define

$$F_{j,l}^\pm(\theta, \phi, \gamma) := \frac{(-1)^{j+l+1} \cos\left(\frac{\gamma}{2}\right) + (-1)^{j+l+1} (4m+2) \sin\left(\frac{\theta+\phi}{2}\right) \sin\left(\frac{\theta+\phi+\gamma}{2}\right) \pm \cos\left(\frac{3\gamma}{2}\right)}{\sin^2\left(\frac{\theta+\phi+\gamma}{2}\right)}.$$

Taking into account

$$(2m+1)(\theta_{j,2m} \pm_1 \sigma_\nu \pm_2 \gamma_l) = (j \pm_1 (2\nu - 1/2) \pm_2 (l + 1/2))\pi \pm_2 \gamma_l,$$

where the subscripts indicate the signs of \pm_1 and \pm_2 are not related (a convention we will adopt for the remainder of the dissertation), we are able to write

$$\begin{aligned} |T_{\nu,j,l}(\cos \frac{\pi}{4m+2}, \sin \frac{\pi}{4m+2}, 1)| &= \frac{1}{(2m+1)^3} \frac{1}{8 \sin \sigma_\nu \sin \frac{\gamma_l}{2}} \quad (\text{IV.3.5}) \\ &\times |F_{j,l}^+(\theta_j, -\sigma_\nu, \gamma_l) - F_{j,l}^-(\theta_j, \sigma_\nu, \gamma_l) - F_{j,l}^-(\theta_j, -\sigma_\nu, -\gamma_l) + F_{j,l}^+(\theta_j, \sigma_\nu, -\gamma_l)|. \end{aligned}$$

We will show the lower bound is attained if we restrict the summation in (IV.3.1) to the set of indices where $\pi/4 \leq \theta_j \leq 3\pi/8$, $\pi/4 \leq \gamma_l < \theta_j$, and $0 \leq \sigma_\nu < \theta_j - \gamma_l$, so we only take the sums over the following range of indices:

- $\lfloor \frac{m}{2} \rfloor + 5 \leq j \leq 3 \lfloor \frac{m}{4} \rfloor$
- $\lfloor \frac{m}{2} \rfloor + 1 \leq l \leq j - 4$
- $1 \leq \nu \leq \lfloor \frac{j-l}{2} \rfloor - 1$

We assume $m \geq 24$, so that these inequalities make sense. With this restriction of summation, $\sin \theta_j$ and $\sin \frac{\gamma_l}{2}$ are bounded away from zero by a positive constant.

Hence, we are left with proving the estimate

$$\begin{aligned}
& \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{(2m+1)^3} \frac{1}{\sin \sigma_\nu} \\
& \times \left| F_{j,l}^+(\theta_j, -\sigma_\nu, \gamma_l) - F_{j,l}^-(\theta_j, \sigma_\nu, \gamma_l) - F_{j,l}^-(\theta_j, -\sigma_\nu, -\gamma_l) + F_{j,l}^+(\theta_j, \sigma_\nu, -\gamma_l) \right| \\
& \geq cm(\log(m+1))^2
\end{aligned} \tag{IV.3.6}$$

Also note that, under this restriction of summation,

$$0 < \frac{\theta_j - \sigma_\nu - \gamma_l}{2} \leq \frac{\theta_j + \sigma_\nu + \gamma_l}{2} \leq \frac{3}{8}\pi,$$

so

$$\sin\left(\frac{\theta_j \pm_1 \sigma_\nu \pm_2 \gamma_l}{2}\right) \approx \theta_j \pm_1 \sigma_\nu \pm_2 \gamma_l,$$

where we have used the fact

$$\sin \theta \approx \theta$$

if $-15\pi/16 \leq \theta \leq 15\pi/16$, a fact we will use repeatedly throughout the proof.

The dominating terms in the summation will be the terms

$$\frac{(4m+2) \sin\left(\frac{\theta_j \pm \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j \pm \sigma_\nu - \gamma_l}{2}\right)},$$

the middle term in the numerator of $F_{j,l}^\pm(\theta_j, \pm\sigma_\nu, -\gamma_l)$. We first prove two lemmas to eliminate the non-dominating terms. The first lemma eliminates the first and third terms in the numerators of $F_{j,l}^{\pm 1}(\theta_j, \pm_2\sigma_\nu, \pm_3\gamma_l)$.

Lemma IV.2. *Recalling (IV.3.3),*

$$J_1(\pm_1, \pm_2) := \frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin \sigma_\nu \sin^2 \left(\frac{\theta_j \pm_1 \sigma_\nu \pm_2 \gamma_l}{2} \right)} \leq cm \log(m).$$

Proof. First, considering $\theta_j \pm \sigma_\nu + \gamma_l$, apply the inequalities

$$\theta_j + \sigma_\nu + \gamma_l > \theta_j + \gamma_l, \text{ and } \theta_j - \sigma_\nu + \gamma_l > \theta_j + \gamma_l - \pi/8,$$

to obtain

$$\begin{aligned} J_1(\pm_1, +) &\leq \frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{\sin^2 \left(\frac{\theta_j + \gamma_l - \pi/8}{2} \right)} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin \sigma_\nu} \\ &\leq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \frac{1}{j^2} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{2\nu - 1/2} \\ &\leq cm \log(m+1). \end{aligned}$$

For $J(+, -)$, using the inequality $\theta_j + \sigma_\nu - \gamma_l > \theta_j - \gamma_l$,

$$\begin{aligned} J_1(+, -) &\leq \frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{\sin^2 \left(\frac{\theta_j - \gamma_l}{2} \right)} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin(\sigma_\nu)} \\ &\leq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{(j-l-2)^2} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{2\nu - 1/2} \\ &\leq cm \log(m+1). \end{aligned}$$

For the remaining case of $J(-, -)$, we split the sum in ν ,

$$\left(\sum_{\nu=1}^{\lfloor \frac{i-l}{4} \rfloor - 1} + \sum_{\nu=\lfloor \frac{i-l}{4} \rfloor}^{\lfloor \frac{i-l}{2} \rfloor - 1} \right) \frac{1}{\sin(\sigma_\nu) \sin^2\left(\frac{\theta_j - \sigma_\nu - \gamma_l}{2}\right)}.$$

We are only considering values of $\nu \geq 1$, so we ignore any instances of $\nu = 0, -1$ in the sums. For the first sum, $\theta_j - \sigma_\nu - \gamma_l > (\theta_j - \gamma_l)/2$, so a bound of $cm \log(m+1)$ is found as in the case of $J(+, -)$. For the second sum,

$$\frac{2\nu - 1/2}{2m+1} \geq \frac{(j-l-3)/2}{2m+1} \geq \frac{1}{4m+2},$$

so it readily follows that

$$\begin{aligned} J_1(-, -) &\leq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{j-l-3} \sum_{\nu=1}^{\lfloor \frac{i-l}{2} \rfloor - 1} \frac{1}{\left(j - 2\nu - l - 1 - \frac{1}{4m}\right)^2} \\ &\quad + cm \log(m+1) \\ &\leq cm \log(m+1). \end{aligned}$$

□

The next lemma eliminates the parts of $F_{j,l}^\pm(\theta, \mp\sigma_\nu, \gamma_l)$ with $4m+2$ in the numerator.

Lemma IV.3. *Recalling (IV.3.3),*

$$J_2(\pm) := \frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{i-l}{2} \rfloor - 1} \frac{(4m+2) \sin\left(\frac{\theta_j \pm \sigma_\nu}{2}\right)}{\sin \sigma_\nu \sin\left(\frac{\theta_j \pm \sigma_\nu + \gamma_l}{2}\right)} \leq cm \log(m+1).$$

Proof. Since $3\pi/4 \geq \theta_j + \sigma_\nu + \gamma_l \geq \theta_j - \sigma_\nu \geq \pi/4$, both $\sin\left(\frac{\theta_j \pm \sigma_\nu}{2}\right)$ and

$\sin\left(\frac{\theta_j \pm \sigma_\nu + \gamma_l}{2}\right)$ are bounded away from zero by a positive constant. The lemma then follows from

$$\begin{aligned} J_2(\pm) &\leq \frac{c}{2m+1} \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{2\nu - 1/2} \\ &\leq cm \log(m+1). \end{aligned}$$

□

By applying the triangle inequality to (IV.3.6), we obtain

$$\begin{aligned} \|\mathcal{B}_{2m}\|_\infty &\geq \frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-2} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin(\sigma_\nu)} \\ &\quad \times \left(|F_{j,l}^+(\theta_j, \sigma_\nu, -\gamma_l) - F_{j,l}^-(\theta_j, -\sigma_\nu, -\gamma_l)| - |F_{j,l}^+(\theta_j, -\sigma_\nu, \gamma_l)| - |F_{j,l}^-(\theta_j, \sigma_\nu, \gamma_l)| \right). \end{aligned}$$

The two lemmas show that

$$\begin{aligned} &\frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-2} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin(\sigma_\nu)} \left(|F_{j,l}^+(\theta_j, -\sigma_\nu, \gamma_l)| + |F_{j,l}^-(\theta_j, \sigma_\nu, \gamma_l)| \right) \\ &\leq cm \log(m+1). \end{aligned}$$

We also have

$$\begin{aligned} |F_{j,l}^+(\theta_j, \sigma_\nu, -\gamma_l) - F_{j,l}^-(\theta_j, -\sigma_\nu, -\gamma_l)| &\geq (4m+2) \left| \frac{\sin\left(\frac{\theta_j + \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l + \sigma_\nu}{2}\right)} - \frac{\sin\left(\frac{\theta_j - \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l - \sigma_\nu}{2}\right)} \right| \\ &\quad - c \left(\frac{1}{\sin^2\left(\frac{\theta_j - \sigma_\nu - \gamma_l}{2}\right)} + \frac{1}{\sin^2\left(\frac{\theta_j + \sigma_\nu - \gamma_l}{2}\right)} \right). \end{aligned}$$

Lemma (IV.2) shows that

$$\begin{aligned} & \frac{1}{(2m+1)^3} \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-2} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin(\sigma_\nu)} \left(\frac{1}{\sin^2\left(\frac{\theta_j - \sigma_\nu - \gamma_l}{2}\right)} + \frac{1}{\sin^2\left(\frac{\theta_j + \sigma_\nu - \gamma_l}{2}\right)} \right) \\ & \leq cm \log(m+1). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \|\mathcal{B}_{2m}\|_\infty & \geq \frac{1}{(2m+1)^2} \sum_{j=\lfloor \frac{m}{2} \rfloor + 5}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-2} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin(\sigma_\nu)} \left| \frac{\sin\left(\frac{\theta_j + \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l + \sigma_\nu}{2}\right)} - \frac{\sin\left(\frac{\theta_j - \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l - \sigma_\nu}{2}\right)} \right| \\ & \tag{IV.3.7} \end{aligned}$$

$$- cm \log(m+1).$$

We now show the dominant part achieves the bound of $cm(\log(m+1))^2$.

Using the formula for the product of sines,

$$\left| \frac{\sin\left(\frac{\theta_j - \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l - \sigma_\nu}{2}\right)} - \frac{\sin\left(\frac{\theta_j + \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l + \sigma_\nu}{2}\right)} \right| = \frac{\sin(\sigma_\nu) \sin\left(\frac{\gamma_l}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l - \sigma_\nu}{2}\right) \sin\left(\frac{\theta_j - \gamma_l + \sigma_\nu}{2}\right)},$$

and it follows that

$$\begin{aligned}
& \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\sin(\sigma_\nu)} \left| \frac{\sin\left(\frac{\theta_j - \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l - \sigma_\nu}{2}\right)} - \frac{\sin\left(\frac{\theta_j + \sigma_\nu}{2}\right)}{\sin\left(\frac{\theta_j - \gamma_l + \sigma_\nu}{2}\right)} \right| \\
& \geq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{(\theta_j - \gamma_l - \sigma_\nu)(\theta_j - \gamma_l + \sigma_\nu)} \\
& = c \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{\theta_j - \gamma_l} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \left(\frac{1}{\theta_j - \gamma_l - \sigma_\nu} + \frac{1}{\theta_j - \gamma_l + \sigma_\nu} \right) \\
& \geq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{\theta_j - \gamma_l} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\theta_j - \gamma_l - \sigma_\nu}.
\end{aligned}$$

Dividing by $(2m+1)^2$, we have

$$\begin{aligned}
& \frac{1}{(2m+1)^2} \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{\theta_j - \gamma_l} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{\theta_j - \sigma_\nu - \gamma_l} \\
& \geq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{j - (2l+1)\left(\frac{2m+1}{4m}\right)} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{j - 2\nu - (2l+1)\left(\frac{2m+1}{4m}\right)} \\
& \geq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{1}{j - l - 1/2} \sum_{\nu=1}^{\lfloor \frac{j-l}{2} \rfloor - 1} \frac{1}{j - 2\nu - l - 1/2} \\
& \geq c \sum_{j=\lfloor \frac{m}{2} \rfloor + 1}^{3\lfloor \frac{m}{4} \rfloor} \sum_{l=\lfloor \frac{m}{2} \rfloor + 1}^{j-4} \frac{\log(j-l)}{j - l - 1/2} \\
& \geq cm(\log(m+1))^2,
\end{aligned}$$

which completes the proof of the lower bound.

IV.3.2 Upper Bound

As mentioned previously, the estimate of the Lebesgue constant on $B^2 \times [-1, 1]$ does not trivially reduce to separate estimates on B^2 and $[-1, 1]$. Moreover, a straightforward estimate of the upper bound of the Lebesgue constant, as done in [18] for the case on B^2 , would be extremely difficult, if not impossible. We instead use a different approach, by deriving generating functions for the series in the definition of $T_{\nu,j,l}(x, y, z)$, and then writing $T_{\nu,j,l}(x, y, z)$ as the Fourier coefficient of the product of the generating functions. The idea for this approach comes from [9], and provides an alternative proof for the upper bound of [18, Theorem 5.2].

Lemma IV.4. *For $0 < |r| < 1$ and $m \geq 0$,*

$$T_{\nu,j,l}(x, y, z) = \frac{1}{(2m+1)^3 \sin \sigma_\nu(x, y)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - re^{i\theta}} G_1(re^{i\theta}, z, z_l) G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y)) e^{-2mi\theta} d\theta r^{-2m},$$

where

$$G_1(r, z, z_l) := \sum_{k=0}^{\infty} \tilde{T}_k(z) \tilde{T}_k(z_l) r^k,$$

and

$$G_2(r, \theta_j, \sigma_\nu(x, y)) := \sum_{k=0}^{\infty} (k+1) \sin((k+1)(\theta_j)) \sin((k+1)(\sigma_\nu(x, y))) r^k.$$

Proof. Our first step is to derive the generating function of the function

$$\begin{aligned} R_N(\theta_j, \sigma_\nu(x, y), z, z_l) \\ := \sum_{n=0}^N \sum_{k=0}^n (k+1) \sin((k+1)\theta_j) \sin((k+1)\sigma_\nu(x, y)) \tilde{T}_{n-k}(z_l) \tilde{T}_{n-k}(z) \end{aligned}$$

in $T_{\nu,j,l}(x, y, z)$. Since the coefficient of r^N in

$$\sum_{n=0}^{\infty} \sum_{k=0}^n \sin((k+1)\theta_j) \sin((k+1)\sigma_\nu(x, y)) r^n \sum_{j=0}^{\infty} \tilde{T}_j(z) \tilde{T}_j(z_l) r^j$$

is precisely $R_N(\theta_j, \sigma_\nu(x, y), z, z_l)$, and

$$\begin{aligned} \frac{1}{1-r} \sum_{k=0}^{\infty} \sin((k+1)\theta_j) \sin((k+1)\sigma_\nu(x, y)) r^k \\ = \sum_{n=0}^{\infty} \sum_{k=0}^n \sin((k+1)\theta_j) \sin((k+1)\sigma_\nu(x, y)) r^n, \end{aligned}$$

it follows that

$$\sum_{N=0}^{\infty} R_N(\theta_j, \sigma_\nu(x, y), z, z_l) r^N = \frac{1}{1-r} G_1(r, z, z_l) G_2(r, \theta_j, \sigma_\nu(x, y)).$$

Since both sides of the above equation are analytic functions of r for $|r| < 1$, we may replace r with $re^{i\theta}$ to obtain analytic, complex-valued functions of r . Since

$$R_{2m}(\theta_j, \sigma_\nu(x, y), z, z_l) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{N=0}^{\infty} R_N(\theta_j, \sigma_\nu(x, y), z, z_l) r^{2m} e^{2mi\theta} e^{-2mi\theta} d\theta r^{-2m},$$

it follows that

$$\begin{aligned} R_{2m}(\theta_j, \sigma_\nu(x, y), z, z_l) \\ = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1 - re^{i\theta}} G_1(re^{i\theta}, z, z_l) G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y)) e^{-2mi\theta} d\theta r^{-2m}. \end{aligned}$$

The lemma follows from the fact that

$$(2m + 1)^3 \sin \sigma_\nu(x, y) R_{2m}(\theta_j, \sigma_\nu(x, y), z, z_l) = T_{\nu, j, l}(x, y, z).$$

□

We next obtain compact formulas for $G_1(re^{i\theta}, z, z_l)$ and $G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y))$, and obtain estimates for these functions.

Lemma IV.5. For $m \geq 0$, and $r = 1 - \frac{1}{m}$,

$$\begin{aligned} |T_{\nu, j, l}(x, y, z)| \leq & \frac{1}{(2m + 1)^3 \sin \sigma_\nu(x, y)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|} \\ & \times (|A_1^+(x, y) - A_1^-(x, y)| + |A_2^+(x, y) - A_2^-(x, y)|) \\ & \times (|P(re^{i\theta}, \gamma_z + \gamma_l)| + |P(re^{i\theta}, \gamma_z - \gamma_l)|) d\theta. \end{aligned} \quad (\text{IV.3.8})$$

where

$$A_1^\pm(x, y) = \frac{1}{1 - 2re^{i\theta} \cos(\theta_j \pm \sigma_\nu(x, y)) + r^2 e^{2i\theta}}, \quad (\text{IV.3.9})$$

and

$$A_2^\pm(x, y) = \frac{(1 - r^2 e^{2i\theta})(re^{i\theta} - \cos(\theta_j \pm \sigma_\nu(x, y)))}{(1 - 2re^{i\theta} \cos(\theta_j \pm \sigma_\nu(x, y)) + r^2 e^{2i\theta})^2}. \quad (\text{IV.3.10})$$

Proof. First, it follows from Lemma (IV.4) that

$$|T_{\nu,j,l}(x, y, z)| \leq \frac{1}{(2m+1)^3 \sin \sigma_\nu(x, y)} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1 - re^{i\theta}|} |G_1(re^{i\theta}, z, z_l)| |G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y))| d\theta r^{-2m}.$$

We next derive estimates for $|G_1(re^{i\theta}, z, z_l)|$ and $|G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y))|$. The compact formula for the generating function $G_1(re^{i\theta}, z, z_l)$ is well-known,

$$G_1(re^{i\theta}, z, z_l) = \frac{1}{4} [P(re^{i\theta}, \gamma_z + \gamma_l) + P(re^{i\theta}, \gamma_z - \gamma_l)], \quad (\text{IV.3.11})$$

where $P(r, \phi)$ is the Poisson kernel, defined by

$$P(r, \phi) := 1 + 2 \sum_{n=1}^{\infty} \cos(n\phi) r^n = \frac{1 - r^2}{1 - 2r \cos \phi + r^2}, \quad 0 \leq r \leq 1. \quad (\text{IV.3.12})$$

For $G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y))$, we use the identity for the product of sines to obtain

$$\begin{aligned} G_2(re^{i\theta}, \theta_j, \sigma_\nu) &= \frac{1}{2} \frac{d}{dr} \sum_{k=1}^{\infty} [\cos((k)(\theta_j - \sigma_\nu(x, y))) - \cos((k+1)(\theta_j + \sigma_\nu(x, y)))] r^k e^{ik\theta} \\ &= \frac{1}{8} \frac{d}{dr} [P(re^{i\theta}, \theta_j - \sigma_\nu(x, y)) - P(re^{i\theta}, \theta_j + \sigma_\nu(x, y))]. \end{aligned}$$

Using the formula

$$\frac{d}{dr} P(r, \phi) = -2 \left(\frac{r}{1 - 2r \cos \phi + r^2} + \frac{(r - \cos \phi)(1 - r^2)}{(1 - 2r \cos \phi + r^2)^2} \right),$$

we obtain

$$G_2(re^{i\theta}, \theta_j, \sigma_\nu) = -\frac{1}{4} \times \left[\frac{re^{i\theta}}{1 - 2re^{i\theta} \cos(\theta_j - \sigma_\nu(x, y)) + r^2e^{2i\theta}} + \frac{[re^{i\theta} - \cos(\theta_j - \sigma_\nu(x, y))](1 - r^2e^{2i\theta})}{[1 - 2re^{i\theta} \cos(\theta_j - \sigma_\nu(x, y)) + r^2e^{2i\theta}]^2} \right. \\ \left. - \frac{re^{i\theta}}{1 - 2re^{i\theta} \cos(\theta_j + \sigma_\nu(x, y)) + r^2e^{2i\theta}} + \frac{[re^{i\theta} - \cos(\theta_j + \sigma_\nu(x, y))](1 - r^2e^{2i\theta})}{[1 - 2re^{i\theta} \cos(\theta_j + \sigma_\nu(x, y)) + r^2e^{2i\theta}]^2} \right].$$

It follows that

$$|G_2(re^{i\theta}, \theta_j, \sigma_\nu(x, y))| \leq (|A_1^+(x, y) - A_1^-(x, y)| + |A_2^+(x, y) - A_2^-(x, y)|).$$

Finally, since the inequality in (IV.4) holds for all values of r with $0 < r < 1$, we set $r = 1 - \frac{1}{m}$. With this choice of r , r^{-2m} converges to e^{-2} as m approaches infinity, and so r^{-2m} is bounded by a constant for all m . \square

Before beginning the estimate, we make several reductions in the range of the sums and values of x, y, z that need to be considered.

1. First, we can reduce the interval of integration to $[0, \pi]$. To see this, replace θ with $2\pi - \theta$. This change of variable amounts to conjugation of the complex number $re^{i\theta}$, and hence the norms of the expression are unchanged.
2. We may also restrict γ_z to the interval $[0, \pi/2m]$. To see this, replace γ_z with $\gamma_z + \frac{\pi}{2m}$ in $P(re^{i\theta}, \gamma_z + \theta_j)$ and $P(re^{i\theta}, \gamma_z - \gamma_l)$. We see that, upon changing the summation index from l to $2m - 1 - l$,

$$\sum_{l=0}^{2m-1} \frac{1 - r^2e^{2i\theta}}{1 - 2re^{i\theta} \cos(\gamma_z + \frac{\pi}{2m} + \gamma_l) + r^2e^{2i\theta}} \\ = \sum_{l=1}^{2m-1} \frac{1 - r^2e^{2i\theta}}{1 - 2re^{i\theta} \cos(\gamma_z + \gamma_l) + r^2e^{2i\theta}} + \frac{1 - r^2e^{2i\theta}}{1 - 2re^{i\theta} \cos(\gamma_z - \frac{4m-1}{4m}\pi) + r^2e^{2i\theta}},$$

and

$$\begin{aligned} & \sum_{l=0}^{2m-1} \frac{1 - r^2 e^{2i\theta}}{1 - 2re^{i\theta} \cos\left(\gamma_z + \frac{\pi}{2m} - \gamma_l\right) + r^2 e^{2i\theta}} \\ &= \sum_{l=0}^{2m-2} \frac{1 - r^2 e^{2i\theta}}{1 - 2re^{i\theta} \cos(\gamma_z - \gamma_l) + r^2 e^{2i\theta}} + \frac{1 - r^2 e^{2i\theta}}{1 - 2re^{i\theta} \cos\left(\gamma_z + \frac{\pi}{4m}\right) + r^2 e^{2i\theta}}. \end{aligned}$$

It follows that the expression

$$\sum_{l=0}^{2m-1} |P(re^{i\theta}, \gamma_z + \gamma_l) + P(re^{i\theta}, \gamma_z - \gamma_l)|$$

is invariant under translations of γ_z by $\pi/2m$, so we only need to consider

$$\gamma_z \in \left[0, \frac{\pi}{2m}\right].$$

3. The sum in j may be reduced to $1 \leq j \leq m$. Replacing j with $2m + 1 - j$, $\sin \theta_{2m+1-j} = \sin \theta_j$, and $\cos(\theta_{2m+1-j} \pm \sigma_\nu(x, y)) = \cos(\theta_j \mp (\pi - \sigma_\nu(x, y)))$. It follows from the definition of $\sigma_\nu(x, y)$ that $\pi - \sigma_\nu(x, y) = \sigma_\nu(-x, -y)$, which implies that $\sin \sigma_\nu(-x, -y) = \sin \sigma_\nu(x, y)$. Hence,

$$\begin{aligned} & \sum_{j=m+1}^{2m} \frac{\sin \theta_j}{\sin \sigma_\nu(x, y)} (|A_1^+(x, y) - A_1^-(x, y)| + |A_2^+(x, y) - A_2^-(x, y)|) \\ &= \sum_{j=1}^m \frac{\sin \theta_j}{\sin \sigma_\nu(-x, -y)} \\ & \quad \times (|A_1^+(-x, -y) - A_1^-(-x, -y)| + |A_2^+(-x, -y) - A_2^-(-x, -y)|), \end{aligned}$$

which shows we only need to consider $1 \leq j \leq m$.

4. We also only need to consider (x, y) in the region

$$\Gamma_m = \left\{ (\rho \cos \eta, \rho \sin \eta) : -\frac{\pi}{4m+2} \leq \eta \leq \frac{\pi}{4m+2}, 0 \leq \rho \leq 1 \right\}.$$

To see this, let $x = \rho \cos \eta$ and $y = \rho \sin \theta$, so that $\cos \sigma_\nu(x, y) = \rho \cos(\eta - \phi_\nu)$. Note that the collection of points $\rho \cos(\eta - \phi_\nu)$, for $\nu = 0, 1, \dots, 2m$, is unchanged by a rotation of η by ϕ_ν . Moreover, every expression involving $\sigma_\nu(x, y)$ in the right side of (IV.3.8) can be written in terms of $\cos \sigma_\nu(x, y)$. Since $0 \leq \sigma_\nu(x, y) \leq \pi$, $\sin \sigma_\nu(x, y) = \sqrt{1 - \cos^2 \sigma_\nu(x, y)}$, and the expressions $\cos(\theta_j \pm \sigma_\nu(x, y))$ can be expanded using the cosine addition identity. Hence, every expression involving $\sigma_\nu(x, y)$ in

$$\sum_{\nu=0}^{2m} \frac{1}{\sin \sigma_\nu(x, y)} (|A_1^+(x, y) - A_1^-(x, y)| + |A_2^+(x, y) - A_2^-(x, y)|)$$

is the same at the points $(\rho \cos \eta, \rho \sin \eta)$ and $(\rho \cos(\eta + \phi_\nu), \rho \sin(\eta + \phi_\nu))$.

5. Finally, we may also reduce the sum in ν to $0 \leq \nu \leq m$. First note that

$$\cos \sigma_{2m+1-\nu}(x, y) = \cos \sigma_\nu(x, -y), \text{ and } \sin \sigma_{2m+1-\nu}(x, y) = \sin \sigma_\nu(x, -y).$$

Hence, we obtain

$$\begin{aligned} & \sum_{\nu=m+1}^{2m} \frac{1}{\sin \sigma_\nu(x, y)} (|A_1^+(x, y) - A_1^-(x, y)| + |A_2^+(x, y) - A_2^-(x, y)|) \\ &= \sum_{\nu=1}^m \frac{1}{\sin \sigma_\nu(x, -y)} (|A_1^+(x, -y) - A_1^-(x, -y)| + |A_2^+(x, -y) - A_2^-(x, -y)|), \end{aligned}$$

and since Γ_m is symmetric with respect to y , we only need to consider

$$0 \leq \nu \leq m.$$

From these reductions, it follows that

$$\begin{aligned} \|\mathcal{B}_{2m}\|_\infty &\leq c \max_{\substack{(x,y) \in \Gamma_m \\ \gamma_z \in [0, \frac{\pi}{2m}]}} \int_0^{2\pi} \frac{1}{2m+1} \sum_{l=0}^{2m-1} |P(re^{i\theta}, \gamma_z + \gamma_l) + P(re^{i\theta}, \gamma_z - \gamma_l)| \\ &\quad \times \frac{1}{(2m+1)^2} \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \frac{1}{|1 - re^{i\theta}|} \frac{\sin \theta_j}{\sin \sigma_\nu(x, y)} \\ &\quad \times (|A_1^+(x, y) - A_1^-(x, y)| + |A_2^+(x, y) - A_2^-(x, y)|) d\theta. \end{aligned}$$

The proof of Theorem IV.8 will follow from the following three lemmas.

The proofs are contained in the next chapter.

Lemma IV.6. For $z \in [0, \pi/2m]$,

$$\frac{1}{2m+1} \sum_{l=0}^{2m-1} |P(re^{i\theta}, \gamma_z + \gamma_l)| + |P(re^{i\theta}, \gamma_z - \gamma_l)| \leq c \log(m+1)$$

for some c which is independent of θ , z , and m .

Lemma IV.7. For $(x, y) \in \Gamma_m$,

$$\begin{aligned} \frac{1}{(2m+1)^2} \int_0^\pi \frac{1}{|1 - re^{i\theta}|} \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \frac{\sin(\theta_{j,2m})}{\sin(\sigma_\nu(x, y))} |A_1^+(x, y) - A_1^-(x, y)| d\theta \\ \leq cm \log(m+1), \end{aligned}$$

for some c which is independent of x, y , and m .

Lemma IV.8. For $(x, y) \in \Gamma_m$,

$$\begin{aligned} \frac{1}{(2m+1)^2} \int_0^\pi \frac{1}{|1 - re^{i\theta}|} \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \frac{\sin(\theta_{j,2m})}{\sin(\sigma_\nu(x, y))} |A_2^+(x, y) - A_2^-(x, y)| d\theta \\ \leq cm(\log(m+1)), \end{aligned}$$

for some constant c which is independent of x, y , and m .

CHAPTER V

PROOFS OF LEMMAS IV.6, IV.7, AND IV.8

Proof of Lemma IV.6. Fix $z \in [0, \frac{\pi}{2m}]$. We first note that we may factor

$$1 - 2re^{i\theta} \cos(\phi) + r^2 e^{2i\theta} = (1 - re^{i(\theta+\phi)})(1 - re^{i(\theta-\phi)}), \quad (\text{V.0.1})$$

and, recalling $r = 1 - \frac{1}{m}$, we may estimate

$$|1 - re^{i\theta}| \approx |\sin \frac{\theta}{2}| + m^{-1}. \quad (\text{V.0.2})$$

to obtain

$$\begin{aligned} & |P(re^{i\theta}, \gamma_z + \gamma_l)| + |P(re^{i\theta}, \gamma_z - \gamma_l)| \\ & \leq c \left(\frac{(|\sin(\frac{\pi-\theta}{2})| + m^{-1})(|\sin(\frac{\theta}{2})| + m^{-1})}{(|\sin(\frac{\theta+\gamma_z+\gamma_l}{2})| + m^{-1})(|\sin(\frac{\theta-\gamma_z-\gamma_l}{2})| + m^{-1})} \right. \\ & \quad \left. + \frac{(|\sin(\frac{\pi-\theta}{2})| + m^{-1})(|\sin(\frac{\theta}{2})| + m^{-1})}{(|\sin(\frac{\theta+\gamma_z-\gamma_l}{2})| + m^{-1})(|\sin(\frac{\theta-\gamma_z+\gamma_l}{2})| + m^{-1})} \right) \\ & := c(\Psi_1(\theta, \gamma_z, \gamma_l) + \Psi_2(\theta, \gamma_z, \gamma_l)) \end{aligned}$$

We will consider two separate cases, $0 \leq \theta \leq \pi/2$ and $\pi/2 \leq \theta \leq \pi$.

Case 1: $0 \leq \theta \leq \pi/2$. Since $\sin(\frac{\pi-\theta}{2}) \geq \sqrt{2}/2$, we may ignore this factor in

the numerator. We also have

$$\left| \frac{\theta + \theta_z + \gamma_l}{2} \right| \leq \frac{3\pi}{4} + \frac{\pi}{4m},$$

so that $\sin((\theta \pm_1 \gamma_z \pm_2 \gamma_l)/2) \approx \theta \pm_1 \gamma_z \pm_2 \gamma_l$. Fixing θ , we have

$$\begin{aligned} \frac{1}{2m+1} \sum_{l=0}^{2m-1} \Psi_1(\theta, \gamma_z, \gamma_l) &\leq \frac{c}{2m+1} \sum_{l=0}^{2m-1} \frac{\theta + m^{-1}}{(\theta + \gamma_z + \gamma_l + m^{-1})(|\theta - \gamma_z - \gamma_l| + m^{-1})} \\ &\leq \frac{c}{2m+1} \sum_{l=0}^{2m-1} \frac{1}{|\theta - \gamma_z - \gamma_l| + m^{-1}} \\ &\leq c \sum_{l=1}^{2m} \frac{1}{|m(\theta - \gamma_z) - \frac{2l+1}{4}\pi| + 1}. \end{aligned}$$

Since

$$\frac{1}{|m(\theta - \gamma_z) - \frac{2l+1}{4}\pi| + 1} \leq \frac{1}{|l + 2 - \frac{2m}{\pi}(\theta - \gamma_z)| + 1},$$

we let $d_{m,z,\theta} = 2 - \frac{2m}{\pi}(\theta - \gamma_z)$ and split the sum in l in two pieces, so

$$\begin{aligned} \frac{1}{2m+1} \sum_{l=0}^{2m-1} \Psi_1(\theta, \gamma_z, \gamma_l) &\leq c \left(\sum_{l \geq d_{m,z,\theta}} + \sum_{l \leq d_{m,z,\theta}} \right) \frac{1}{|l + 2 - \frac{2m}{\pi}(\theta - \gamma_z)| + 1} \\ &\leq c \left(2 \sum_{l=1}^{2m} \frac{1}{l} \right) \\ &\leq c \log(m+1). \end{aligned}$$

This type of estimate will be very common throughout the proof, and we will omit the repetitive details.

For $\Psi_2(\theta, \gamma_z, \gamma_l)$, we first consider the sum starting from $l = 1$, so that

$\theta + \gamma_l - \gamma_z > \theta$, and consider the case when $l = 0$ later. We obtain

$$\begin{aligned} \frac{1}{2m+1} \sum_{l=1}^{2m-1} \Psi_2(\theta, \gamma_z, \gamma_l) &= \frac{1}{2m+1} \sum_{l=1}^{2m-1} \frac{|\theta| + m^{-1}}{(|\theta + \gamma_z - \gamma_l| + m^{-1})(|\theta - \gamma_z + \gamma_l| + m^{-1})} \\ &\leq \frac{1}{2m+1} \sum_{l=1}^{2m-1} \frac{1}{|\theta + \gamma_z - \gamma_l| + m^{-1}} \\ &\leq c \log(m+1). \end{aligned}$$

For the term corresponding to $l = 0$, either $\gamma_z \geq \pi/4m$ or $\gamma_z \leq \pi/4m$; assuming, without loss of generality, the former, it then follows that

$$\begin{aligned} &\frac{1}{2m+1} \frac{\theta + m^{-1}}{(|\theta + \gamma_z - \frac{\pi}{4m}| + m^{-1})(|\theta - \gamma_z + \frac{\pi}{4m}| + m^{-1})} \\ &\leq \frac{1}{2m+1} \frac{1}{|\theta - \gamma_z + \frac{\pi}{4m}| + m^{-1}} \\ &\leq 1, \end{aligned}$$

which shows that

$$\frac{1}{2m+1} \sum_{l=0}^{2m-1} \Psi_2(\theta, \gamma_z, \gamma_l) \leq c \log(m+1)$$

for the case when $\theta \in [0, \pi/2]$.

Case 2: $\pi/2 \leq \theta \leq \pi$. For this case, $\sin(\theta/2) \geq \sqrt{2}/2$, and so this factor may be ignored. For $\Psi_1(\theta, \gamma_z, \gamma_l)$, note that

$$-\frac{\pi}{4m} \leq \frac{2\pi - \theta - \gamma_z - \gamma_l}{2} \leq \frac{3\pi}{4}, \quad -\frac{\pi}{4} - \frac{\pi}{8m} \leq \frac{\theta - \gamma_z - \gamma_l}{2} \leq \frac{\pi}{2},$$

so we may approximate the sine functions accordingly and obtain

$$\begin{aligned} & \frac{1}{2m+1} \sum_{l=0}^{2m-1} \Psi_1(\theta, \gamma_z, \gamma_l) \\ & \leq \frac{c}{2m+1} \sum_{l=0}^{2m-1} \frac{|\pi - \theta| + m^{-1}}{(|2\pi - \theta - \gamma_z - \gamma_l| + m^{-1})(|\theta - \gamma_z - \gamma_l| + m^{-1})} \\ & = \frac{c}{2m+1} \sum_{l=0}^{2m-1} \frac{|\pi - \theta| + m^{-1}}{(|\pi - \theta - \gamma_z + \gamma_l| + m^{-1})(|\pi - \theta + \gamma_z - \gamma_l| + m^{-1})}, \end{aligned}$$

where we have substituted $2m - 1 - l$ for l in the last equality. This estimate is very similar to the estimate of $\Psi_2(\theta, \gamma_z, \gamma_l)$ in Case 1, and hence for $\pi/2 \leq \theta \leq \pi$,

$$\frac{1}{2m+1} \sum_{l=1}^{2m} \Psi_1(\theta, \gamma_z, \gamma_l) \leq c \log(m+1).$$

For Ψ_2 ,

$$0 \leq \frac{2\pi - \theta + \gamma_z - \gamma_l}{2} \leq \frac{3\pi}{4} + \frac{\pi}{4m}, \quad -\frac{\pi}{2} \leq \frac{\theta + \gamma_z - \gamma_l}{2} \leq \frac{\pi}{2} + \frac{\pi}{4m},$$

so we may approximate the sine functions, to obtain

$$\begin{aligned} & \frac{1}{2m+1} \sum_{l=0}^{2m-1} \Psi_2(\theta, \gamma_z, \gamma_l) \\ & \leq \frac{c}{2m+1} \sum_{l=0}^{2m-1} \frac{|\pi - \theta| + m^{-1}}{(|2\pi - \theta + \gamma_z - \gamma_l| + m^{-1})(|\theta + \gamma_z - \gamma_l| + m^{-1})} \\ & = \frac{c}{2m+1} \sum_{l=0}^{2m-1} \frac{|\pi - \theta| + m^{-1}}{(|\pi - \theta + \gamma_z + \gamma_l| + m^{-1})(|\pi - \theta - \gamma_z - \gamma_l| + m^{-1})}, \end{aligned}$$

which is very similar to our estimate for $\Psi_1(\theta, \gamma_z, \gamma_l)$ in Case 1, and we again get a estimate of $c \log(m+1)$. \square

Proof of Lemma IV.7. The proof of the remaining two lemmas will proceed by

separating the integral into three different regions, then dividing the sums in j and ν into several sections, and performing estimates on each resulting section.

Frequently, obtaining an estimate consists of bounding quotients by a constant, and then estimating similar types of sums and integrals. For the sake of brevity, we list these types here, and then direct the reader to the type of estimate that arises in the each piece. The symbols ϕ_1, ϕ_2 and ξ refer to values that are specific to the section under investigation. For two different expressions f_1 and f_2 , the notation $\{f_1, f_2\}$ indicates that either expression satisfies that type. Finally, we note that these estimates also hold for sums whose range of indices are a subset of those listed below.

Type 1:

$$\begin{aligned} & \frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \frac{1}{\{|\theta - \phi_1|, |\phi_2 - \theta|\} + m^{-1}} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{|\theta_j + \xi| + m^{-1}} d\theta \\ & \leq c(\log(m+1))^2. \end{aligned}$$

Type 2:

$$\frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{(|\theta_j + \xi| + m^{-1})^2} d\theta \leq cm.$$

Type 3:

$$\begin{aligned} & \frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \frac{1}{(\{|\theta - \phi_1|, |\phi_2 - \theta|\} + m^{-1})^2} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{|\theta_j + \xi| + m^{-1}} d\theta \\ & \leq cm \log(m+1). \end{aligned}$$

Type 4:

$$\begin{aligned} & \frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \frac{1}{\{|\phi_2 - \theta|, |\theta - \phi_1|\} + m^{-1}} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{(|\theta_j + \xi| + m^{-1})^2} d\theta \\ & \leq cm \log(m+1). \end{aligned}$$

Type 5:

$$\begin{aligned} & \frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \frac{1}{\{|\phi_2 - \theta|, |\theta - \phi_1|\} + m^{-1}} \sum_{\nu=0}^m \frac{1}{\{|\sigma_\nu(x, y)|, |\pi - \sigma_\nu(x, y)|\} + m^{-1}} \\ & \quad \times \sum_{j=1}^m \frac{1}{|\theta_j + \xi| + m^{-1}} d\theta \\ & \leq c(\log(m+1))^3. \end{aligned}$$

Type 6:

$$\begin{aligned} & \frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \sum_{\nu=0}^m \frac{1}{\{|\sigma_\nu(x, y)|, |\pi - \sigma_\nu(x, y)|\} + m^{-1}} \sum_{j=1}^m \frac{1}{|\theta_j + \xi| + m^{-1}} d\theta \\ & \leq c(\log(m+1))^2. \end{aligned}$$

Type 7:

$$\begin{aligned} & \frac{1}{(2m+1)^2} \int_{\phi_1}^{\phi_2} \sum_{\nu=0}^m \frac{1}{\{|\sigma_\nu(x, y)|, |\pi - \sigma_\nu(x, y)|\} + m^{-1}} \sum_{j=1}^m \frac{1}{(|\theta_j + \xi| + m^{-1})^2} d\theta \\ & \leq cm \log(m+1). \end{aligned}$$

The above estimates are easily obtained with the following lemma and propositions.

Proposition V.1. For any real number ξ ,

$$\sum_{j=1}^m \frac{1}{(|\theta_j + \xi| + m^{-1})^k} \leq \begin{cases} c(2m+1) \log(m+1) & k=1 \\ c(2m+1)^2 & k=2 \end{cases}$$

Proof. By the definition of θ_j ,

$$\begin{aligned} \sum_{j=1}^m \frac{1}{(|\theta_j + \xi| + m^{-1})^k} &= (2m+1)^k \sum_{j=1}^m \frac{1}{|j\pi + (2m+1)\xi| + 2 + \frac{1}{m}} \\ &\leq (2m+1)^k 2 \sum_{j=1}^m \frac{1}{(j+1)^k}, \end{aligned}$$

from which the proposition easily follows. \square

Lemma V.1. For $(x, y) \in \Gamma_m$,

$$\sum_{\nu=0}^m \frac{1}{|\sigma_\nu(x, y)| + m^{-1}} \leq cm \log(m+1) \quad (\text{V.0.3})$$

$$\sum_{\nu=0}^m \frac{1}{|\pi - \sigma_\nu(x, y)| + m^{-1}} \leq cm \log(m+1). \quad (\text{V.0.4})$$

Proof. Recall that $(x, y) = (r \cos(\phi), r \sin(\phi))$. If we restrict ν to $0 \leq \nu \leq m/2$, then $\phi_\nu - \phi \leq \pi/2$, and $|\sigma_\nu(x, y)| \geq |\phi_\nu - \phi|$, since $\cos(\sigma_\nu(x, y)) \leq \cos(\phi_\nu - \phi)$. On the other hand, if $\phi_\nu - \phi > \pi/2$, then $\sigma_\nu(x, y) > \pi/2$. The first inequality follows, since

$$\sum_{\nu=0}^m \frac{1}{|\sigma_\nu(x, y)| + m^{-1}} \leq \sum_{\nu=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{|\phi_\nu - \phi| + m^{-1}} + \frac{m}{2} \frac{2}{\pi} \leq cm \log(m+1).$$

The proof of the second inequality is similar to the first. Recall that

$\pi - \sigma_\nu(x, y) = \sigma_\nu(-x, -y)$, and write $(-x, -y) = (r \cos(\phi), r \sin(\phi))$, where

$\phi \in (\pi - \frac{\pi}{4m+2}, \pi + \frac{\pi}{4m+2})$. A similar argument shows $\sigma_\nu(-x, -y) > \pi/2$ for $\nu \leq m/2$, while $\sigma_\nu(-x, -y) > |\phi - \phi_\nu|$ for $m/2 < \nu \leq m$, and the remainder of the proof is identical to the proof of the first inequality. \square

Proposition V.2. For $0 \leq \phi_1 < \phi_2 \leq \pi$,

$$\int_{\phi_1}^{\phi_2} \frac{1}{(\{|\theta - \phi_1|, |\phi_2 - \theta|\} + m^{-1})^k} d\theta \leq \begin{cases} c \log(m+1) & k = 1 \\ cm & k = 2 \end{cases}$$

Proof. The proposition follows from a change of variables in the integral. \square

We introduce new notation to simplify the proof of the remaining estimates. The notation $\mathcal{I}_\nu(\phi_1, \phi_2)$ denotes the set of indices ν such that $\phi_1 \leq \sigma_\nu(x, y) \leq \phi_2$, and the symbol $\mathcal{I}_j(\phi_1, \phi_2)$ denotes the equivalent set of indices such that $\phi_1 \leq \theta_j \leq \phi_2$.

Combining $A_1^+(x, y)$ and $A_1^-(x, y)$, we obtain

$$\begin{aligned} & |A_1^+(x, y) - A_1^-(x, y)| \\ &= \frac{4re^{i\theta} \sin \theta_j \sin \sigma_\nu(x, y)}{(1 - 2re^{i\theta} \cos(\theta_j + \sigma_\nu(x, y)) + r^2e^{2i\theta})(1 - 2re^{i\theta} \cos(\theta_j - \sigma_\nu(x, y)) + r^2e^{2i\theta})}. \end{aligned}$$

Upon substituting this into (IV.3.8) and using (V.0.1) and (V.0.2), we are left with

estimating

$$\frac{1}{(2m+1)^2} \int_0^\pi \frac{1}{1-re^{i\theta}} \sum_{\nu=0}^{2m} \sum_{j=1}^{2m} \frac{\sin \theta_j}{\sin \sigma_\nu(x,y)} |A_1^+(x,y) - A_1^-(x,y)| \quad (\text{V.0.5})$$

$$\leq c \frac{1}{(2m+1)^2} \int_0^\pi \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{\sin \theta + m^{-1}} \sin^2 \theta_j \quad (\text{V.0.6})$$

$$\begin{aligned} & \times \frac{1}{\left(\left| \sin \left(\frac{\theta + \theta_j + \sigma_\nu(x,y)}{2} \right) \right| + m^{-1} \right) \left(\left| \sin \left(\frac{\theta - \theta_j - \sigma_\nu(x,y)}{2} \right) \right| + m^{-1} \right)} \\ & \times \frac{1}{\left(\left| \sin \left(\frac{\theta + \theta_j - \sigma_\nu(x,y)}{2} \right) \right| + m^{-1} \right) \left(\left| \sin \left(\frac{\theta - \theta_j + \sigma_\nu(x,y)}{2} \right) \right| + m^{-1} \right)} d\theta. \end{aligned}$$

In order to estimate the sine functions, the integral over $[0, \pi]$ is divided into integrals over three subintervals: $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{3\pi}{4}]$, and $[\frac{3\pi}{4}, \pi]$. We will use the notation \mathcal{H}_1^1 , \mathcal{H}_1^2 , and \mathcal{H}_1^3 to denote the left side of the inequality (V.0.5) restricted over these respective sub-intervals.

Case 1: $0 \leq \theta \leq \pi/4$. With this restriction on θ , the sine functions in (V.0.5) are estimated by

$$\left| \sin \left(\frac{\theta + \theta_j + \sigma_\nu(x,y)}{2} \right) \right| \approx |\theta + \theta_j + \sigma_\nu(x,y)| \quad (\text{V.0.7})$$

$$\left| \sin \left(\frac{\theta - \theta_j + \sigma_\nu(x,y)}{2} \right) \right| \approx |\theta - \theta_j + \sigma_\nu(x,y)| \quad (\text{V.0.8})$$

$$\left| \sin \left(\frac{\theta + \theta_j - \sigma_\nu(x,y)}{2} \right) \right| \approx |\theta + \theta_j - \sigma_\nu(x,y)| \quad (\text{V.0.9})$$

$$\left| \sin \left(\frac{\theta + \theta_j + \sigma_\nu(x,y)}{2} \right) \right| \approx |\theta + \theta_j + \sigma_\nu(x,y)| \quad (\text{V.0.10})$$

to obtain

$$\begin{aligned} \mathcal{H}_1^1 &\leq c \frac{1}{(2m+1)^2} \int_0^{\frac{\pi}{4}} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{\theta + m^{-1}} \theta_j^2 \\ &\quad \times \frac{1}{(|\theta + \theta_j + \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})} \\ &\quad \times \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})}. \end{aligned} \quad (\text{V.0.11})$$

First, use the inequality

$$\frac{\theta_j}{(\theta + \theta_j + \sigma_\nu(x, y) + m^{-1})} < 1.$$

If $\nu \in \mathcal{I}_\nu(0, \theta)$, we use the inequality

$$\frac{\theta_j}{\theta_j + \theta - \sigma_\nu(x, y) + m^{-1}} < 1,$$

and then note that if $j \in \mathcal{I}_j(0, \theta)$,

$$\begin{aligned} &\frac{1}{\theta + m^{-1}} \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} \\ &\leq \frac{1}{\theta + m^{-1}} \frac{1}{\sigma_\nu(x, y) + m^{-1}} \frac{1}{|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}} \end{aligned}$$

while if $j \in \mathcal{I}_j(\theta, \frac{\pi}{2})$,

$$\begin{aligned} &\frac{1}{\theta + m^{-1}} \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} \\ &\leq \frac{1}{\theta + m^{-1}} \frac{1}{\sigma_\nu(x, y) + m^{-1}} \frac{1}{|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1}} \end{aligned}$$

so by splitting the sum in j in this way, we obtain two estimates of Type 5. For

$\nu \in \mathcal{I}_\nu(\theta, \pi)$, use the inequality

$$\frac{\theta_j}{\theta_j + \sigma_\nu(x, y) - \theta + m^{-1}} < 1,$$

and then note that if $j \in \mathcal{I}_j(0, \sigma_\nu(x, y))$,

$$\begin{aligned} \frac{1}{\theta + m^{-1}} \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} \\ \leq \frac{1}{(\theta + m^{-1})^2} \frac{1}{|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\sigma_\nu(x, y), \frac{\pi}{2})$,

$$\begin{aligned} \frac{1}{\theta + m^{-1}} \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} \\ \leq \frac{1}{(\theta + m^{-1})^2} \frac{1}{|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

so splitting the sum in j in this way yields two estimates of Type 3.

Case 2: $\pi/4 \leq \theta \leq 3\pi/4$. In this case, the factor of $\sin(\theta) + m^{-1}$ in the denominator of (V.0.5) is greater than $\sqrt{2}/2$ and may be ignored. The sine functions in (V.0.5) are approximated by (V.0.7), (V.0.9), (V.0.8), and

$$\left| \sin\left(\frac{\theta + \theta_j + \sigma_\nu(x, y)}{2}\right) \right| \approx |2\pi - \theta - \theta_j - \sigma_\nu(x, y)| \quad (\text{V.0.12})$$

to obtain

$$\begin{aligned}
\mathcal{H}_1^2 &\leq c \frac{1}{(2m+1)^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sum_{\nu=0}^m \sum_{j=1}^m \theta_{j,2m}^2 \\
&\quad \times \frac{1}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}) (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})} \\
&\quad \times \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}) (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} d\theta.
\end{aligned} \tag{V.0.13}$$

First consider $\nu \in \mathcal{I}_\nu(0, \theta)$. Under this restriction,

$$2\pi - \theta - \theta_j - \sigma_\nu(x, y) > 2 \left(\frac{3\pi}{4} - \theta \right)$$

and

$$\frac{\theta_j}{\theta_j + \theta - \sigma_\nu(x, y) + m^{-1}} < 1.$$

If $j \in \mathcal{I}_j(0, \theta)$, then

$$\begin{aligned}
&\frac{1}{\frac{3}{4}\pi - \theta + m^{-1}} \frac{\theta_j}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}) (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} \\
&\leq \frac{1}{\frac{3}{4}\pi - \theta + m^{-1}} \frac{1}{\sigma_\nu(x, y) + m^{-1}} \frac{1}{|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}},
\end{aligned}$$

while if $j \in \mathcal{I}_j(\theta, \frac{\pi}{2})$,

$$\begin{aligned}
&\frac{1}{\frac{3}{4}\pi - \theta + m^{-1}} \frac{\theta_j}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}) (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})} \\
&\leq \frac{1}{\frac{3}{4}\pi - \theta + m^{-1}} \frac{1}{\sigma_\nu(x, y) + m^{-1}} \frac{1}{|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1}},
\end{aligned}$$

so we obtain two estimates of Type 5. If $\nu \in \mathcal{I}_\nu(\theta, \pi)$, we use the inequalities

$$\frac{\theta_j}{\theta_j + \sigma_\nu(x, y) - \theta + m^{-1}} \leq 1,$$

and

$$\theta - \theta_j + \sigma_\nu(x, y) \geq 2(\theta - \pi/4).$$

Substituting $2m + 1 - j$ for j , note that if $j \in \mathcal{I}_j(\frac{\pi}{2}, \theta)$,

$$\begin{aligned} & \frac{1}{\theta - \frac{\pi}{4} + m^{-1}} \frac{\pi - \theta_j}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}) (|\pi + \theta - \theta_j - \sigma_\nu(x, y)| = m^{-1})} \\ & \leq \frac{1}{\theta - \frac{\pi}{4} + m^{-1}} \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

and if $j \in \mathcal{I}_j(\theta, \pi)$,

$$\begin{aligned} & \frac{1}{\theta - \frac{\pi}{4} + m^{-1}} \frac{\pi - \theta_j}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}) (|\pi + \theta - \theta_j - \sigma_\nu(x, y)| = m^{-1})} \\ & \leq \frac{1}{\theta - \frac{\pi}{4} + m^{-1}} \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

so we obtain two estimates of Type 5.

Case 3: $3\pi/4 \leq \theta \leq \pi$. We may again ignore the factor of $\sin(\theta) + m^{-1}$ in the denominator of (V.0.5). The sine functions are approximated by (V.0.12), (V.0.7), (V.0.9) and

$$\sin\left(\frac{\theta - \theta_j + \sigma_\nu(x, y)}{2}\right) \approx 2\pi - \theta + \theta_j - \sigma_\nu(x, y) \quad (\text{V.0.14})$$

to obtain

$$\begin{aligned} \mathcal{H}_1^3 &\leq c \frac{1}{(2m+1)^2} \int_{\frac{3\pi}{4}}^{\pi} \sum_{\nu=0}^m \sum_{j=1}^m \theta_j^2 \\ &\quad \times \frac{1}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})} \\ &\quad \times \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|2\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})} d\theta. \end{aligned} \tag{V.0.15}$$

First observe that $\theta + \sigma_\nu(x, y) \leq 2\pi$, so that

$$\frac{\theta_j}{2\pi - \theta + \theta_j - \sigma_\nu(x, y) + m^{-1}} < 1.$$

For $\nu \in \mathcal{I}_\nu(0, \theta)$, we use the inequality

$$\frac{\theta_j}{\theta + \theta_j - \sigma_\nu(x, y) + m^{-1}} < 1,$$

and substitute $2m+1-j$ for j . For j into $\mathcal{I}_j(\frac{\pi}{2}, \sigma_\nu(x, y))$,

$$\begin{aligned} &\frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\pi - \theta - \theta_j + \sigma_\nu| + m^{-1})} \\ &\leq \frac{1}{\pi - \theta + m^{-1}} \frac{1}{|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\sigma_\nu(x, y), \pi)$,

$$\begin{aligned} &\frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\pi - \theta - \theta_j + \sigma_\nu| + m^{-1})} \\ &\leq \frac{1}{\pi - \theta + m^{-1}} \frac{1}{|\pi - \theta - \theta_j + \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

so two estimates of Type 1 are obtained. For $\nu \in \mathcal{I}_\nu(\theta, \pi)$, use the inequality

$$\frac{\theta_j}{\theta_j + \sigma_\nu(x, y) - \theta + m^{-1}} < 1,$$

substitute $2m + 1 - j$ for j , and note that if $j \in \mathcal{I}_j(\frac{\pi}{2}, \theta)$,

$$\begin{aligned} & \frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\pi + \theta - \theta_j - \sigma_\nu| + m^{-1})} \\ & \leq \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\theta, \pi)$,

$$\begin{aligned} & \frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})(|\pi + \theta - \theta_j - \sigma_\nu| + m^{-1})} \\ & \leq \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

so we obtain two estimates of Type 6. This concludes the proof of Lemma

IV.7. □

Proof of Lemma IV.8. First, we let $\zeta = re^{i\theta}$. As in the proof of Lemma IV.7, we combine terms to obtain

$$\begin{aligned} & |A_2^+(x, y) - A_2^-(x, y)| \\ & = 2 \sin \theta_j \sin \sigma_\nu(x, y) |1 - r^2 e^{2i\theta}| \\ & \quad \times \frac{|P(\zeta, \theta_j, \sigma_\nu(x, y))|}{(1 - re^{i\theta} \cos(\theta_j + \sigma_\nu(x, y)) + (re^{i\theta})^2)^2 (1 - re^{i\theta} \cos(\theta_j - \sigma_\nu(x, y)) + (re^{i\theta})^2)^2}, \end{aligned}$$

where

$$\begin{aligned}
P(\zeta, \theta_j, \sigma_\nu(x, y)) & \tag{V.0.16} \\
&= 1 - 3\zeta^4 - 2\zeta^2(3 - 2\sin^2(\theta_j) - 2\sin^2(\sigma_\nu(x, y))) \\
&+ 8\zeta^3 \left(1 - 2\sin^2(\sigma_\nu(x, y)/2) - 2\sin^2(\theta_j/2) + 4\sin^2(\sigma_\nu(x, y)/2)\sin^2(\theta_j/2) \right).
\end{aligned}$$

Substituting this into (IV.3.8) and using (V.0.1) and (V.0.2), it remains to estimate

$$\begin{aligned}
& \frac{1}{(2m+1)^2} \int_0^\pi \sum_{\nu=0}^m \sum_{j=1}^m (\sin(\pi - \theta) + m^{-1}) \sin^2(\theta_j) \tag{V.0.17} \\
& \times \frac{|P(\zeta, \theta_j, \sigma_\nu(x, y))|}{\left(\left| \sin \left(\frac{\theta + \theta_j + \sigma_\nu(x, y)}{2} \right) \right| + m^{-1} \right)^2 \left(\left| \sin \left(\frac{\theta - \theta_j - \sigma_\nu(x, y)}{2} \right) \right| + m^{-1} \right)^2} \\
& \times \frac{1}{\left(\left| \sin \left(\frac{\theta + \theta_j - \sigma_\nu(x, y)}{2} \right) \right| + m^{-1} \right)^2 \left(\left| \sin \left(\frac{\theta - \theta_j + \sigma_\nu(x, y)}{2} \right) \right| + m^{-1} \right)^2} d\theta.
\end{aligned}$$

We will again split this integral into three sub-integrals over $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{3\pi}{4}]$, and $[\frac{3\pi}{4}, \pi]$, and denote the part of (V.0.17) associated with these subintervals by \mathcal{H}_2^1 , \mathcal{H}_2^2 , and \mathcal{H}_2^3 , respectively. The crucial part of the estimate is suitably approximating $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$. Several different approximations will be used. These approximations are given in the following lemma, and are referenced as needed.

Lemma V.2. *The function $P(\zeta, \theta_j, \sigma_\nu(x, y))$ satisfies the following inequalities.*

(E1) For $0 \leq \theta \leq \pi/2$,

$$\begin{aligned}
& |P(\zeta, \theta_j, \sigma_\nu(x, y))| && \text{(V.0.18)} \\
& \leq c \left((\theta + \theta_j + m^{-1})^2 (|\theta - \theta_j| + m^{-1}) \right. \\
& \quad \left. + (\sigma_\nu(x, y))^2 (\theta + (\sigma_\nu(x, y))^2 + \theta_j^2 + m^{-1}) \right).
\end{aligned}$$

(E2) For $0 \leq \theta \leq \pi/2$,

$$\begin{aligned}
& |P(\zeta, \theta_j, \sigma_\nu(x, y))| && \text{(V.0.19)} \\
& \leq c \left((\theta + m^{-1})^3 + (\theta_j + \sigma_\nu(x, y))^2 (|\theta_j - \sigma_\nu(x, y)|)^2 \right. \\
& \quad \left. + (\theta + m^{-1})(\theta_j^2 + (\sigma_\nu(x, y))^2) \right).
\end{aligned}$$

(E3) For $\pi/4 \leq \theta \leq 3\pi/4$,

$$\begin{aligned}
& |P(\zeta, \theta_j, \sigma_\nu(x, y))| \leq c \left((|\theta + \theta_j| + m^{-1}) (|\theta - \theta_j| + m^{-1}) + (\sigma_\nu(x, y))^2 \right). \\
& && \text{(V.0.20)}
\end{aligned}$$

(E4) For $\pi/4 \leq \theta \leq \pi$,

$$\begin{aligned}
& |P(\zeta, \theta_j, \sigma_\nu(x, y))| && \text{(V.0.21)} \\
& \leq c \left((|\pi - \theta + \theta_j| + m^{-1})^2 (|\pi - \theta - \theta_j| + m^{-1}) + (\pi - \sigma_\nu(x, y))^2 \right).
\end{aligned}$$

Proof. We first prove the estimate (V.0.18). We define

$$P_1(\zeta, \theta_{j,2m}) := 1 - 2\zeta^2(3 - 2\sin^2 \theta_j) + 8\zeta^3 \cos \theta_j - 3\zeta^4, \quad \text{(V.0.22)}$$

$$P_2^\pm(\zeta, \theta_j, \sigma_\nu(x, y)) := 4\zeta^2 \sin^2 \sigma_\nu(x, y) \pm 16\zeta^3 \sin^2 \frac{\sigma_\nu(x, y)}{2} \cos \theta_j,$$

so we may write $P(\zeta, \theta_j, \sigma_\nu(x, y)) = P_1(\zeta, \theta_j) + P_2^-(\zeta, \theta_j, \sigma_\nu(x, y))$. It is possible to factor $P_1(\zeta, \theta_j)$ as

$$P_1(\zeta, \theta_j) = (1 + 2\zeta \cos \theta_j - 3\zeta^2)(1 - 2\zeta \cos \theta_j + \zeta^2). \quad (\text{V.0.23})$$

The second factor of (V.0.23) is approximated by (V.0.1) and (V.0.2) as before, and the first factor may be further factored as

$$\begin{aligned} (1 + 2\zeta \cos \theta_j - 3\zeta^2) &= -3 \left(\zeta + \frac{1}{3} \left(\sqrt{4 - \sin^2 \theta_j} - \cos \theta_j \right) \right) \\ &\times \left(\zeta - \frac{1}{3} \left(\sqrt{4 - \sin^2 \theta_j} + \cos \theta_j \right) \right). \end{aligned} \quad (\text{V.0.24})$$

The first factor of (V.0.24) will not be used. Using the double angle identity for cosine, the second factor of (V.0.24) is approximated by

$$\begin{aligned} &\left| \zeta - \frac{1}{3} \left(\sqrt{4 - \sin^2 \theta_j} + \cos \theta_j \right) \right| \\ &\leq c \left(3 \sin \theta + \left| 3 \cos \theta - \cos \theta_j - \sqrt{4 - \sin^2 \theta_j} \right| + m^{-1} \right) \\ &\leq c \left(\theta + \theta_j^2 + \left| 2 - \sqrt{4 - \sin^2 \theta_j} \right| + m^{-1} \right). \end{aligned}$$

Since $2 - \sqrt{4 - \sin^2 \theta_j} \leq \sin \theta_j$, we obtain

$$P_1(\zeta, \theta_j) \leq c \left(\left(\left| \frac{\theta + \theta_j}{2} \right| + m^{-1} \right) \left(\left| \frac{\theta - \theta_j}{2} \right| + m^{-1} \right) (\theta + \theta_j + m^{-1}) \right). \quad (\text{V.0.25})$$

Now considering $P_2^-(\zeta, \theta_j)$, the double angle identities for sine and cosine are

used to obtain

$$\begin{aligned}
& |P_2^-(\zeta, \theta_j, \sigma_\nu(x, y))| && \text{(V.0.26)} \\
& \leq c(\sigma_\nu(x, y))^2 \left| \cos^2 \frac{\sigma_\nu(x, y)}{2} - z \cos \theta_j \right| \\
& \leq c(\sigma_\nu(x, y))^2 (\theta + (\sigma_\nu(x, y))^2 + \theta_j^2 + m^{-1}).
\end{aligned}$$

Adding the estimates (V.0.25) and (V.0.26), we arrive at the estimate (V.0.18).

The proof of the estimate (V.0.20) follows from replacing θ with a constant in (V.0.18).

We next prove the estimate (V.0.19). We first re-write $P(\zeta, \theta_j, \sigma_\nu(x, y))$ as

$$\begin{aligned}
P(\zeta, \theta_j, \sigma_\nu(x, y)) &= 1 - 3\zeta^4 - 6\zeta^2 + 8\zeta^3 \\
&+ 4\zeta^2 \left[\sin^2 \theta_j + \sin^2 \sigma_\nu(x, y) \right. \\
&+ 4 \left(2 \sin^2 \frac{\theta_j}{2} \sin^2 \frac{\sigma_\nu(x, y)}{2} - \sin^2 \frac{\theta_j}{2} - \sin^2 \frac{\sigma_\nu(x, y)}{2} \right) \\
&\left. + 4(\zeta - 1) \left(2 \sin^2 \frac{\theta_j}{2} \sin^2 \frac{\sigma_\nu(x, y)}{2} - \sin^2 \frac{\theta_j}{2} - \sin^2 \frac{\sigma_\nu(x, y)}{2} \right) \right].
\end{aligned}$$

It is easily checked that $1 - 3\zeta^4 - 6\zeta^2 + 8\zeta^3 = -(\zeta - 1)^3(3\zeta + 1)$, and applying the double angle identity for sines,

$$\begin{aligned}
& \sin^2 \theta_j + \sin^2 \sigma_\nu(x, y) + 4 \left(2 \sin^2 \frac{\theta_j}{2} \sin^2 \frac{\sigma_\nu(x, y)}{2} - \sin^2 \frac{\theta_j}{2} - \sin^2 \frac{\sigma_\nu(x, y)}{2} \right) \\
&= -4 \left(\sin^2 \frac{\theta_j}{2} - \sin^2 \frac{\sigma_\nu(x, y)}{2} \right)^2.
\end{aligned}$$

Finally, we may approximate the sine functions to obtain (V.0.19).

Finally, we prove the estimate (V.0.21). First, replace $\sin^2 \sigma_\nu(x, y)$ with

$\sin^2(\pi - \sigma_\nu(x, y))$ and $\sin^2 \frac{\sigma_\nu(x, y)}{2}$ with $1 - \sin^2 \frac{\pi - \sigma_\nu(x, y)}{2}$ in (V.0.16) to obtain

$$\begin{aligned} & |P(\zeta, \theta_j, \sigma_\nu(x, y))| \\ & \leq \left| 1 - 3\zeta^4 - 4\zeta^2(3 - 2\sin^2 \theta_j) + 8\zeta^3(2\sin^2 \frac{\theta_j}{2} - 1) \right| \\ & + c |P_2^+(\zeta, \theta_j, \pi - \sigma_\nu(x, y))|. \end{aligned} \tag{V.0.27}$$

The inequality $|P_2^+(\zeta, \theta_j, \pi - \sigma_\nu(x, y))| < c(\pi - \sigma_\nu(x, y))^2$ follows easily from the definition of P_2^+ . The first term in (V.0.27) becomes $|P_1(\zeta, \pi - \theta_j)|$, after replacing $3 - 2\sin^2 \theta_j$ with $3 - 2\sin^2(\pi - \theta_j)$ and $2\sin^2 \frac{\theta_j}{2} - 1$ with $1 - 2\sin^2 \left(\frac{\pi - \theta_j}{2}\right)$. $P_1(\zeta, \pi - \theta_j)$ factors as in (V.0.23). The factor of $1 + 2\cos(\theta_j)\zeta - 3\zeta^2$ can be factored further, as

$$\begin{aligned} & 1 + 2\zeta \cos(\pi - \theta_j) - 3\zeta^2 \\ & = 1 - 2\zeta \cos \theta_j - 3\zeta^2 \\ & = -3 \left(\zeta + \frac{1}{3} \left(\cos \theta_j - \sqrt{4 - \sin^2 \theta_j} \right) \right) \left(\zeta + \frac{1}{3} \left(\cos \theta_j + \sqrt{4 - \sin^2 \theta_j} \right) \right). \end{aligned}$$

The first factor will not be used, but the absolute value of the second factor may be approximated by

$$\begin{aligned} & \left| \zeta + \frac{1}{3} \left(\cos \theta_j + \sqrt{4 - \sin^2 \theta_j} \right) \right| \\ & \leq c \left(|\sin \theta| + \left| \cos \theta + \frac{1}{3} \left(\cos \theta_j + \sqrt{4 - \sin^2 \theta_j} \right) \right| + m^{-1} \right) \\ & \leq c \left(\pi - \theta + \left| 6\sin^2 \frac{\pi - \theta}{2} - 2\sin^2 \frac{\theta_j}{2} + \sqrt{4 - \sin^2 \theta_j} - 2 \right| + m^{-1} \right) \\ & \leq c(\pi - \theta + \theta_j + m^{-1}), \end{aligned}$$

and we are able to obtain the estimate (V.0.21). □

Case 1: $0 \leq \theta \leq \pi/4$. Approximating the sine functions in (V.0.17) with (V.0.10), (V.0.7), (V.0.8), and (V.0.9), we obtain

$$\begin{aligned} \mathcal{H}_2^1 &\leq c \frac{1}{(2m+1)^2} \int_0^{\pi/4} \sum_{\nu=0}^m \sum_{j=1}^m \theta_{j,2m}^2 \\ &\quad \times \frac{|P(\zeta, \theta_j, \sigma_\nu(x, y))|}{(|\theta + \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ &\quad \times \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

We first consider $\nu \in \mathcal{I}_\nu(0, \theta)$ and let $\mathcal{H}_2^1(\sigma_\nu(x, y) \leq \theta)$ denote \mathcal{H}_2^1 with this restriction on ν , a notation we will adopt for the rest of this proof. Note that $\theta_j^2/(\theta + \theta_j - \sigma_\nu(x, y) + m^{-1})^2 < 1$, and approximate $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$ using (V.0.18), to obtain

$$\begin{aligned} \mathcal{H}_2^1(\sigma_\nu(x, y) \leq \theta) &\leq c \frac{1}{(2m+1)^2} \int_0^{\pi/4} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \\ &\quad \times \frac{(\theta + \theta_j + m^{-1})^2 (|\theta - \theta_j| + m^{-1}) + (\sigma_\nu(x, y))^2 (\theta + (\sigma_\nu(x, y))^2 + \theta_j^2 + m^{-1})}{(|\theta + \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

For the first term in the sum in (V.0.18), first use the inequality

$$(\theta + \theta_j + m^{-1})^2 / (\theta + \theta_j + \sigma_\nu(x, y) + m^{-1})^2 < 1.$$

and then note that for j in $\mathcal{I}_j(0, \theta)$,

$$\begin{aligned} &\frac{|\theta - \theta_j| + m^{-1}}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \\ &\leq \frac{1}{|\sigma_\nu(x, y)| + m^{-1}} \frac{1}{|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

while, if $j \in \mathcal{I}_j(\theta, \frac{\pi}{2})$,

$$\begin{aligned} & \frac{|\theta - \theta_j| + m^{-1}}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{|\sigma_\nu(x, y)| + m^{-1}} \frac{1}{|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

so by splitting the sum in j in this way, two estimates of Type 7 are obtained. For the second term in (V.0.18), first use the inequality

$$\frac{\theta + (\sigma_\nu(x, y))^2 + \theta_j^2 + m^{-1}}{(\theta + \theta_j + \sigma_\nu(x, y) + m^{-1})^2} < c \frac{1}{\theta + m^{-1}}.$$

Note that if $j \in \mathcal{I}_j(0, \theta)$, then

$$\begin{aligned} & \frac{(\sigma_\nu(x, y))^2}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\theta, \frac{\pi}{2})$,

$$\begin{aligned} & \frac{(\sigma_\nu(x, y))^2}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1}}, \end{aligned}$$

so by splitting the sum in j as above, we obtain two estimates of Type 4.

For $\nu \in \mathcal{I}_\nu(\theta, \pi)$, first use the inequality

$$\frac{\theta_j^2}{(\theta_j + \sigma_\nu(x, y) - \theta + m^{-1})^2} < 1,$$

and then approximate $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$ using (V.0.19) to obtain

$$\begin{aligned} \mathcal{H}_2^1(\theta \leq \sigma_\nu(x, y)) &\leq c \frac{1}{(2m+1)^2} \int_0^{\frac{\pi}{4}} \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ &\times \frac{(\theta + m^{-1})^3 + (\theta_j + \sigma_\nu(x, y))^2 (|\theta_j - \sigma_\nu(x, y)|)^2 + (\theta + m^{-1})(\theta_j^2 + (\sigma_\nu(x, y))^2)}{(|\theta + \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

For the first term of the sum in (V.0.19), use the inequality

$$\frac{(\theta + m^{-1})^2}{(\theta + \theta_j + \sigma_\nu(x, y) + m^{-1})^2} < 1$$

and then note that if $j \in \mathcal{I}_j(0, \sigma_\nu(x, y))$,

$$\begin{aligned} &\frac{\theta + m^{-1}}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ &\leq \frac{1}{\theta + m^{-1}} \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\sigma_\nu(x, y), \frac{\pi}{2})$,

$$\begin{aligned} &\frac{\theta + m^{-1}}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ &\leq \frac{1}{\theta + m^{-1}} \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so we obtain two estimates of type 4. For the second term of the sum in (V.0.19), first use the inequality

$$\frac{(\theta_j + \sigma_\nu(x, y))^2}{(\theta + \theta_j + \sigma_\nu(x, y) + m^{-1})^2} < 1,$$

and then split the sum in j into $\mathcal{I}_j(0, \sigma_\nu(x, y))$ and $\mathcal{I}_j(\sigma_\nu(x, y), \frac{\pi}{2})$. For the first

sum, use the inequality

$$\begin{aligned} & \frac{|\theta_j - \sigma_\nu(x, y)|^2}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\theta + m^{-1}} \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

and for the second sum, use

$$\begin{aligned} & \frac{|\theta_j - \sigma_\nu(x, y)|^2}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\theta + m^{-1}} \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

and we obtain two estimates of Type 2. Finally, for the third term in (V.0.19), use the inequality

$$\frac{(\sigma_\nu(x, y))^2 + \theta_j^2}{\theta + \theta_j + \sigma_\nu(x, y) + m^{-1}} < 1,$$

and then use the fact that if $j \in \mathcal{I}_j(0, \sigma_\nu(x, y))$,

$$\begin{aligned} & \frac{\theta + m^{-1}}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\theta + m^{-1}} (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2, \end{aligned}$$

while if $j \in \mathcal{I}_j(\sigma_\nu(x, y), \frac{\pi}{2})$,

$$\begin{aligned} & \frac{\theta + m^{-1}}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\theta + m^{-1}} (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2, \end{aligned}$$

so we obtain two estimates of Type 4.

Case 2: $\pi/4 \leq \theta \leq 3\pi/4$: If $\sigma_\nu(x, y) \leq \pi/2$, the sine functions in (V.0.17)

may be approximated as in Case 1, while if $\sigma_\nu(x, y) \geq \pi/2$, the sine functions may be approximated by (V.0.12), (V.0.7), (V.0.8), and (V.0.9). Hence, we obtain

$$\begin{aligned} \mathcal{H}_2^2 &\leq c \frac{1}{(2m+1)^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &\left[\sum_{\nu \in \mathcal{I}_\nu(0, \frac{\pi}{2})} \sum_{j=1}^m \theta_j^2 \frac{|P(\zeta, \theta_j, \sigma_\nu(x, y))|}{(|\theta + \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \right. \\ &\times \left. \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \right] \\ &+ \left[\sum_{\nu \in \mathcal{I}_\nu(\frac{\pi}{2}, \pi)} \sum_{j=1}^m \theta_j^2 \frac{|P(\zeta, \theta_j, \sigma_\nu(x, y))|}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \right. \\ &\times \left. \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \right] d\theta. \end{aligned}$$

We consider first $\nu \in \mathcal{I}_\nu(0, \min(\frac{\pi}{2}, \theta))$. With this restriction on ν , we may use the inequality $\theta + \theta_j + \sigma_\nu(x, y) \geq \pi/4$, so this factor may be ignored. We may also use the inequality

$$\frac{\theta_j^2}{(\theta + \theta_j - \sigma_\nu(x, y) + m^{-1})^2} < 1,$$

and approximate $P(\zeta, \theta_j, \sigma_\nu(x, y))$ by (V.0.20) to obtain

$$\begin{aligned} &\mathcal{H}_2^2(\sigma_\nu(x, y) \leq \min(\frac{\pi}{2}, \theta)) \\ &\leq c \frac{1}{(2m+1)^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sum_{\nu \in \mathcal{I}_\nu(0, \min(\frac{\pi}{2}, \theta))} \sum_{j=1}^m \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \\ &\times \frac{(\theta + \theta_j + m^{-1})(|\theta - \theta_j| + m^{-1}) + [\sigma_\nu(x, y)]^2}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

If $j \in \mathcal{I}_j(0, \theta)$, then

$$\begin{aligned} & \frac{(\theta + \theta_j + m^{-1})(|\theta - \theta_j| + m^{-1})}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\sigma_\nu(x, y) + m^{-1}} \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

which yields an estimate of Type 7, and

$$\begin{aligned} & \frac{[\sigma_\nu(x, y)]^2}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

which yields an estimate of Type 2.

Next, we consider $\nu \in \mathcal{I}_\nu(\theta, \frac{\pi}{2})$. We again have

$\theta + \theta_j + \sigma_\nu(x, y) + m^{-1} > \pi/4$, so this factor may be ignored, and the inequality

$$\frac{\theta_j^2}{(\theta_j^2 + \sigma_\nu(x, y) - \theta + m^{-1})^2} < 1$$

is used. We only need to approximate $P(\zeta, \theta_j, \sigma_\nu(x, y))$ by a constant, and we obtain

$$\begin{aligned} & \mathcal{H}_2^2(\theta \leq \sigma_\nu(x, y) \leq \frac{\pi}{2}) \\ & \leq c \frac{1}{(2m+1)^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sum_{\nu \in \mathcal{I}_\nu(\theta, \frac{\pi}{2})} \sum_{j=1}^m \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \\ & \times \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

If $j \in \mathcal{I}_j(0, \sigma_\nu(x, y))$, then

$$\begin{aligned} & \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{4}{\pi} \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\sigma_\nu(x, y), \frac{\pi}{2})$,

$$\begin{aligned} & \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{4}{\pi} \frac{1}{(|\theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so in both cases, estimates of Type 2 are obtained.

Now we consider $\nu \in \mathcal{I}_\nu(\frac{\pi}{2}, \theta)$. For this range of ν , we may use the inequalities $\theta - \theta_j + \sigma_\nu(x, y) \geq \pi/4$ and

$$\theta_j^2 / (\theta + \theta_j - \sigma_\nu(x, y) + m^{-1})^2 < 1.$$

We again approximate $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$ by a constant, and then substitute $2m + 1 - j$ for j to obtain

$$\begin{aligned} & \mathcal{H}_2^2 \left(\frac{\pi}{2} \leq \sigma_\nu(x, y) \leq \theta \right) \\ & \leq c \frac{1}{(2m+1)^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sum_{\nu \in \mathcal{I}_\nu(\theta, \frac{\pi}{2})} \sum_{j=m+1}^{2m} \frac{1}{(|\pi - \theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2} \\ & \times \frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

If $j \in \mathcal{I}_j(\frac{\pi}{2}, \sigma_\nu(x, y))$,

$$\begin{aligned} & \frac{1}{(|\pi - \theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{4}{\pi} \frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\sigma_\nu(x, y), \pi)$,

$$\begin{aligned} & \frac{1}{(|\pi - \theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2 (|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{4}{\pi} \frac{1}{(|\pi - \theta - \theta_j + \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

both of which produce estimates of Type 2.

To complete this case, we consider $\nu \in \mathcal{I}_\nu(\max(\theta, \frac{\pi}{2}), \pi)$. Under these restrictions, $\theta - \theta_j + \sigma_\nu(x, y) + m^{-1} \geq \theta + m^{-1}$ and hence may be ignored, and

$$\frac{\theta_j^2}{(\theta_j + \sigma_\nu(x, y) - \theta + m^{-1})^2} < 1.$$

Use the estimate for $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$ in (V.0.21), and substitute $2m + 1 - j$ for j , to obtain

$$\begin{aligned} & \mathcal{H}_2^2(\max(\theta, \frac{\pi}{2}) \leq \sigma_\nu(x, y) \leq \pi) \\ & \leq c \frac{1}{(2m+1)^2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sum_{\nu \in \mathcal{I}_\nu(\theta, \frac{\pi}{2})} \sum_{j=m+1}^{2m} \frac{1}{(|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \times \frac{(\pi - \theta + \pi - \theta_j + m^{-1})(|\theta - \theta_j| + m^{-1}) + |\pi - \sigma_\nu(x, y)|^2}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

If $j \in \mathcal{I}_j(\frac{\pi}{2}, \theta)$, then

$$\begin{aligned} & \frac{(|\pi - \theta + \pi - \theta_j| + m^{-1})(|\pi - \theta - \theta_j| + m^{-1})}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq c \frac{1}{\pi - \sigma_\nu(x, y)} \frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so we obtain an estimate of Type 4, while

$$\begin{aligned} & \frac{|\pi - \sigma_\nu(x, y)|^2}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

which yields an estimate of Type 2. If $j \in \mathcal{I}_j(\theta, \pi)$, then

$$\begin{aligned} & \frac{(|\pi - \theta + \pi - \theta_j| + m^{-1})(|\pi - \theta - \theta_j| + m^{-1})}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq c \frac{1}{\pi - \sigma_\nu(x, y)} \frac{1}{(|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

which gives an estimate of Type 4, and the inequality

$$\begin{aligned} & \frac{|\pi - \sigma_\nu(x, y)|^2}{(|\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{(|\pi + \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

yields an estimate of Type 2.

Case 3: $3\pi/4 \leq \theta \leq \pi$. Recall from (IV.3.10) that a factor of $\pi - \theta + m^{-1}$ is present in the numerator. After approximating the sine functions in (V.0.17) by

(V.0.12), (V.0.7), (V.0.14) and (V.0.9), we obtain

$$\begin{aligned} \mathcal{H}_2^3 &\leq c \frac{1}{(2m+1)^2} \int_{\frac{3\pi}{4}}^{\pi} (\pi - \theta + m^{-1}) \sum_{\nu=0}^m \sum_{j=1}^m \theta_j^2 \\ &\quad \times \frac{|P(\zeta, \theta_j, \sigma_\nu(x, y))|}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ &\quad \times \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|2\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

We first consider $\nu \in \mathcal{I}_\nu(0, \frac{\pi}{2})$. Note that $|2\pi - \theta - \sigma_\nu(x, y) + \theta_j| \geq \pi/2$, and $|\theta - \sigma_\nu(x, y) + \theta_j| > \pi/2$, so we may ignore those factors. Also, note that $|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| > |\pi - \theta|$. We approximate $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$ by a constant to obtain

$$\begin{aligned} \mathcal{H}_2^3(\sigma_\nu(x, y) \leq \frac{\pi}{2}) &\leq c \frac{1}{(2m+1)^2} \int_{\frac{3\pi}{4}}^{\pi} \frac{1}{\pi - \theta + m^{-1}} \\ &\quad \times \sum_{\nu \in \mathcal{I}_\nu(0, \frac{\pi}{2})} \sum_{j=1}^m \theta_j^2 \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta, \end{aligned}$$

which immediately gives an estimate of type 4.

We next consider $\nu \in \mathcal{I}_\nu(\frac{\pi}{2}, \theta)$. First use the inequality

$$\frac{\theta_j^2}{(\theta + \theta_j - \sigma_\nu(x, y) + m^{-1})^2} < 1$$

and approximate $|P(\zeta, \theta, \theta_j, \sigma_\nu(x, y))|$ with (V.0.21) to obtain

$$\begin{aligned} & \mathcal{H}_2^3\left(\frac{\pi}{2} \leq \sigma_\nu(x, y) \leq \theta\right) \\ & \leq c \frac{1}{(2m+1)^2} \int_{\frac{3\pi}{4}}^{\pi} (\pi - \theta + m^{-1}) \sum_{\nu=0}^m \sum_{j=1}^m \frac{1}{(|2\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \quad \times \frac{(\pi - \theta + \theta_j + m^{-1})^2 (|\pi - \theta - \theta_j| + m^{-1}) + |\pi - \sigma_\nu(x, y)|^2}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

For the first term in the sum in the numerator, we use the inequality

$$\frac{(\pi - \theta + \theta_j + m^{-1})^2}{2\pi - \theta + \theta_j - \sigma_\nu(x, y) + m^{-1}} < 1.$$

If $j \in \mathcal{I}_j(0, \pi - \sigma_\nu(x, y))$,

$$\begin{aligned} & \frac{(\pi - \theta + m^{-1}) (|\pi - \theta - \theta_j| + m^{-1})}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so we obtain an estimate of type 2. If $j \in \mathcal{I}_j(\pi - \sigma_\nu(x, y), \frac{\pi}{2})$,

$$\begin{aligned} & \frac{(\pi - \theta + m^{-1}) (|\pi - \theta - \theta_j| + m^{-1})}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq c \frac{1}{\pi - \theta + m^{-1}} \frac{1}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so we obtain an estimate of type 4. For the second term of the sum in the numerator, use the inequality

$$\frac{(\pi - \sigma_\nu(x, y))^2}{(2\pi - \theta + \theta_j - \sigma_\nu(x, y) + m^{-1})^2} < 1.$$

If $j \in \mathcal{I}_j(0, \pi - \sigma_\nu(x, y))$,

$$\begin{aligned} & \frac{\pi - \theta + m^{-1}}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\pi - \theta + m^{-1}} \frac{1}{(|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\pi - \sigma_\nu(x, y), \frac{\pi}{2})$,

$$\begin{aligned} & \frac{\pi - \theta + m^{-1}}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq c \frac{1}{\pi - \theta + m^{-1}} \frac{1}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so two estimates of type 4 are obtained.

Finally, we consider $\nu \in \mathcal{I}_\nu(\theta, \pi)$. Use the inequality

$$\frac{\theta_j^2}{(\theta_j + \sigma_\nu(x, y) - \theta + m^{-1})^2} < 1$$

and approximate $|P(\zeta, \theta_j, \sigma_\nu(x, y))|$ using (V.0.21) to obtain

$$\begin{aligned} & \mathcal{H}_2^3(\theta \leq \sigma_\nu(x, y) \leq \pi) \\ & \leq c \frac{1}{(2m+1)^2} \int_{\frac{3\pi}{4}}^{\pi} (\pi - \theta + m^{-1}) \sum_{\nu \in \mathcal{I}_\nu(\theta, \pi)}^m \sum_{j=1}^m \frac{1}{(|2\pi - \theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \quad \times \frac{(\pi - \theta + \theta_j + m^{-1})^2 (|\pi - \theta - \theta_j| + m^{-1}) + |\pi - \sigma_\nu(x, y)|^2}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} d\theta. \end{aligned}$$

For the first term in the sum in (V.0.21), use the inequality

$$\frac{(\pi + \theta_j - \theta + m^{-1})^2}{(2\pi - \theta - \sigma_\nu(x, y) + \theta_j + m^{-1})^2} < 1,$$

and then note that if $j \in \mathcal{I}_j(0, \pi - \theta)$,

$$\begin{aligned} & \frac{(\pi - \theta + m^{-1})(|\pi - \theta - \theta_j| + m^{-1})}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so we obtain an estimate of type 2. If $j \in \mathcal{I}_j(\pi - \theta, \frac{\pi}{2})$,

$$\begin{aligned} & \frac{(\pi - \theta + m^{-1})(|\pi - \theta - \theta_j| + m^{-1})}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so we obtain an estimate of type 7. For the second term in the sum in (V.0.21),

use the inequality

$$\frac{(\pi - \sigma_\nu(x, y))(\pi - \theta + m^{-1})^2}{(2\pi - \theta - \sigma_\nu(x, y) + \theta_j + m^{-1})} < 1$$

and then note that if $j \in \mathcal{I}_j(0, \pi - \theta)$,

$$\begin{aligned} & \frac{\pi - \sigma_\nu(x, y) + m^{-1}}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{(|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

while if $j \in \mathcal{I}_j(\pi - \theta, \frac{\pi}{2})$,

$$\begin{aligned} & \frac{\pi - \sigma_\nu(x, y) + m^{-1}}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2 (|\theta + \theta_j - \sigma_\nu(x, y)| + m^{-1})^2} \\ & \leq \frac{1}{\pi - \sigma_\nu(x, y) + m^{-1}} \frac{1}{(|2\pi - \theta - \theta_j - \sigma_\nu(x, y)| + m^{-1})^2}, \end{aligned}$$

so two estimates of type 7 are obtained. This completes the proof of Lemma (IV.8)

and Theorem (IV.8). \square

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