KOSZUL AND GENERALIZED KOSZUL PROPERTIES FOR NONCOMMUTATIVE GRADED ALGEBRAS

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Title: KOSZUL AND GENERALIZED KOSZUL PROPERTIES FOR NONCOMMUTATIVE GRADED ALGEBRAS

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We investigate some homological properties of graded algebras. If A is an R-algebra, then $\mathbf{E}(A) := \mathrm{Ext}_A(R, R)$ is an R-algebra under the cup product and is called the Yoneda algebra. (In most cases, we assume R is a field.) A well-known and widely-studied condition on $\mathbf{E}(A)$ is the Koszul property. We study a class of deformations of Koszul algebras that arises from the study of equivariant cohomology and algebraic groups and show that under certain circumstances these deformations are Poincaré–Birkhoff–Witt deformations.

Some of our results involve the $\mathcal{K}_2$ property, recently introduced by Cassidy and Shelton, which is a generalization of the Koszul property. While a Koszul algebra must be quadratic, a $\mathcal{K}_2$ algebra may have its ideal of relations generated in different degrees. We study the structure of the Yoneda algebra corresponding to a monomial $\mathcal{K}_2$ algebra and provide an example of a monomial $\mathcal{K}_2$ algebra whose Yoneda algebra is not also $\mathcal{K}_2$. This example illustrates the difficulty of finding a $\mathcal{K}_2$ analogue of the classical theory of Koszul duality.

It is well-known that Poincaré–Birkhoff–Witt algebras are Koszul. We find a $\mathcal{K}_2$ analogue of this theory. If V is a finite-dimensional vector space with an ordered basis, and $A := \mathbb{T}(V)/I$ is a connected-graded algebra, we can place a filtration F on A as well as $\mathbf{E}(A)$. We show there is a bigraded algebra embedding $\Lambda : \operatorname{gr}^F \mathbf{E}(A) \hookrightarrow \mathbf{E}(\operatorname{gr}^F A)$. If I has a Gröbner basis meeting certain conditions and $\operatorname{gr}^F A$ is $\mathcal{K}_2$, then Λ can be used to show that A is also $\mathcal{K}_2$.

This dissertation contains both previously published and co-authored materials.

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CHAPTER I

INTRODUCTION

This dissertation will explore some homological properties of graded algebras. Homological algebra is a powerful tool used in many areas of mathematics, and noncommutative algebra is no exception. A deep and beautiful homological condition is Koszulity. We also explore a generalization of the Koszul property, known as the $\mathcal{K}_2$ property.

Chapters II and III present previously-known results. In Chapter II, we introduce Yoneda algebras and coalgebras, which are algebras formed from cohomology and homology, respectively. We introduce the Koszul and $\mathcal{K}_2$ properties, as well as describe the theory of Poincaré–Birkhoff–Witt algebras.

Given a connected-graded algebra A, we can study a related algebra $\mathbf{E}(A)$, which is defined cohomologically. If $\mathbf{E}(A)$ satisfies certain conditions, then A is said to be Koszul. The strength of the Koszul condition is seen from the fact that there are any number of equivalent definitions given in very different forms, such as (1) how $\mathbf{E}(A)$ is generated, (2) homological purity, (3) distributive lattices, and (4) canonical resolutions.

While not explicitly stated in the definition, it follows that Koszul algebras must be quadratic. There have been several attempts to formulate a more general condition which admits nonquadratic graded algebras. One such is the $\mathcal{K}_2$ condition, introduced by Cassidy and Shelton [9]. Unlike those in the class of n-Koszul algebras introduced by Berger [2], $\mathcal{K}_2$ algebras can have relations generated in more than one degree. For example, while a commutative algebra with cubic relations has no hope of being 3-Koszul (because the commutativity relations are quadratic), such algebras could be $\mathcal{K}_2$. This suggests we could explore $\mathcal{K}_2$-ness as a geometric property. For example, all complete intersections are $\mathcal{K}_2$.

Suppose V is a finite-dimensional vector space and let $\mathbb{T}(V)$ denote the tensor algebra on V. If $A := \mathbb{T}(V)/I$ is a quadratic algebra and I has a particularly nice Gröbner basis, with respect to some ordered basis for V, then A is said to be a PBW algebra. Priddy proved in [19] that PBW algebras are Koszul.

Chapter III describes the theory of Poincaré–Birkhoff–Witt deformations, which are so named because of the motivating example provided by the PBW theorem from representation theory. We devote most of Chapter III to describing a homological technique, introduced by Cassidy and Shelton [8], for determining when a deformation is PBW. This technique can also be used to determine the regularity of a central element of a noncommutative algebra.

Chapters IV, V, and VI present new results. In Chapter IV, we define a deformation that arose in the study of some algebras related to representation theory and algebraic geometry, and show that it is PBW. In [5], the notion of Goresky–MacPherson duality is introduced, and the main result of Chapter IV is a stepping stone to the development of GM duality. GM duality has been observed in some examples, such as the equivariant cohomology associated to certain algebraic group actions on algebraic varieties.

If A is a Koszul algebra, $\mathbf{E}(A)$ is quadratic and its structure is easily described. Furthermore, $A \simeq \mathbf{E}(\mathbf{E}(A))$. (This relationship is known as Koszul duality.) Finding a $\mathcal{K}_2$-analogue of Koszul duality remains of interest. The results in Chapter V are in pursuit of this goal. In that chaper, we consider (noncommutative) monomial algebras. We provide some more structural results for the Yoneda algebra associated to a monomial algebra. Specifically, we show that if A is a monomial $\mathcal{K}_2$ algebra, we can choose generators of $\mathbf{E}(A)$ so that the defining relations of $\mathbf{E}(A)$ are monomial and binomial. At the of the chapter, we exhibit a monomial $\mathcal{K}_2$ algebra whose Yoneda algebra is not $\mathcal{K}_2$ (in an appropriately generalized sense). This illustrates the difficulty of finding a generalization of Koszul duality to the $\mathcal{K}_2$ world.

In Chapter VI, we provide a $\mathcal{K}_2$ generalization of the theory of PBW algebras. An ordered basis on V induces a filtration F on A. A quadratic algebra $A := \mathbb{T}(V)/I$ is PBW if and only if the associated graded algebra $\mathrm{gr}^F A$ is a quadratic algebra. More generally, if $A := \mathbb{T}(V)/I$ is any connected-graded algebra, there is also a filtration (which we will also denote F) on $\mathbf{E}(A)$, and a bigraded algebra embedding

$$\Lambda : \mathrm{gr}^F \mathbf{E}(A) \hookrightarrow \mathbf{E}(\mathrm{gr}^F A).$$

We prove that if Λ is surjective and $\mathrm{gr}^F A$ is $\mathcal{K}_2$, A is $\mathcal{K}_2$ as well. In fact, the PBW condition on a quadratic algebra is equivalent to requiring that Λ be surjective in the first two cohomological degrees (which is enough to conclude the overall surjectivity of Λ). We find analogous characterizations of the surjectivity of Λ, also involving Gröbner bases, in the $\mathcal{K}_2$ case. This provides another technique to prove that algebras are $\mathcal{K}_2$.

In order to prove the results in Chapter VI, we generalize the definition of $\mathcal{K}_2$ and $\mathbf{E}(A)$ for augmented algebras: that is, $\mathbb{K}$-algebras A with an ideal A_+

with $\dim A/A_+ = 1$. We show the existence of Λ in this context. This more general theory allows one to consider these homological properties under gradings by other monoids than $\mathbb{N}$.

The generalization to the augmented case is interesting in its own right. Since every point in an algebraic variety naturally gives rise to such an augmented algebra, it might be interesting to study these homological properties from a geometric perspective.

Chapters IV and V contain material which was co-authored. Chapters V and VI contain previously published material.

CHAPTER II

THE YONEDA ALGEBRA AND COALGEBRA

II.1 Introduction

We begin our study of homological conditions on a graded algebra with Koszulity. Koszul algebras were first introduced by Priddy in 1970 to study the Steenrod algebra and the universal enveloping algebra of a Lie algebra.[19] Later, Koszul algebras attracted the interest of those studying noncommutative algebraic geometry because some of the Artin–Schelter regular algebras (introduced in 1988 [1]) are Koszul.

Koszul algebras must always be quadratic. There have been some attempts to generalize Koszulity to connected-graded algebras with non-quadratic relations. Hoping to capture more of the Artin–Schelter regular algebras, Berger introduced N-Koszul algberas in 2001 [2]. These algebras are N-homogeneous—that is, their ideal of relations may be generated by degree-N homogeneous elements. (The 2-Koszul algebras are exactly the Koszul algebras. The term N-Koszul as used by Berger is different than the sense of the term seen in [18].) Motivated by problems of deformation theory (see Chapter III), Cassidy and Shelton introduced $\mathcal{K}_2$ algebras in 2007 [8], and explored their properties in depth in [9]. This class of algebras contains all the N-Koszul algebras and the Koszul algebras, but also admits algebras whose ideals of relations are generated by homogeneous elements in different degrees.

II.2 Augmented algebras

Throughout, $\mathbb{K}$ will be a field, and for a $\mathbb{K}$-vector space V, we use $\mathbb{T}(V)$ to denote the tensor algebra of V over $\mathbb{K}$. If R is a semisimple $\mathbb{K}$-algebra and V is an R-bimodule, then $\mathbb{T}_R(V)$ will be the tensor algebra of V over R. We will usually study (R-)augmented algebras, which we define as follows:

Definition II.2.1. An R-algbera is R-**augmented** if there is an ideal $A_+ \subset A$ such that $A = A_+ \oplus R \cdot 1$ as an R-module.

If A is R-augmented, we regard ${}_A R$ as the A-module A/A_+.

We will consider both ungraded and graded versions of some standard functors from homological algebra, but in many of the cases we consider, the two versions will coincide. If a $\mathbb{K}$-algebra A is graded by a monoid $\mathcal{M}$ with identity element e, $\hom_A(M, N)$ is the module of degree-preserving A-linear homomorphisms, $M(\alpha)$ is defined by $M(\alpha)_\beta := M_{\alpha\beta}$, and

$$\mathrm{Hom}_{\mathrm{Gr}}(M, N)_\alpha := \hom_A(M(\alpha), N).$$

Then we have the $\mathcal{M}$-graded Hom functor

$$\mathrm{Hom}_{\mathrm{Gr}}(M, N) := \bigoplus_{\alpha \in \mathcal{M}} \mathrm{Hom}_{\mathrm{Gr}}(M, N)_\alpha.$$

Finally, $\mathrm{Ext}_{\mathrm{Gr}}$ the derived functor of $\mathrm{Hom}_{\mathrm{Gr}}$.

Definition II.2.2. For an R-augmented algebra A, a grading $A = \bigoplus_{n \geq 0} A_n$ is **compatible with the augmentation** if $A = \mathbb{T}_R(V)/I$ for some finite-dimensional

R-bimodule V and finitely-generated homogeneous ideal $I \subset \sum_{i \geq 2} V^{\otimes i}$, $A_n = V^{\otimes n} \mod I$, and $A_+ = \bigoplus_{n>0} A_n$. More specifically, we will say that A is **connected-graded** if $R = \mathbb{K}$.

In either case, A is graded by the monoid $\mathcal{M} = \mathbb{N}$ and R-augmented by setting $A_+ = \sum_{i \geq 1} A_i$. We use notation established by [9], setting

$$A(n_1^{j_1}, n_2^{j_2}, \dots) := \bigoplus_i A(n_i)^{\oplus j_i},$$

where $A(n_j)^{\oplus j_i}$ is j_i direct-sum copies of $A(n_i)$.

Suppose A is an R-augmented algebra. Write $A_+^{\otimes n}$ for the nth tensor power of A_+ over R. We have a canonical resolution of ${}_A R$: the **bar resolution** $\mathrm{Bar}^n(A) := A \otimes_R A_+^{\otimes n}$ with differential $\partial'_n : A \otimes_R A_+^{\otimes n} \to A \otimes_R A_+^{\otimes n-1}$ defined by

$$\partial'_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

Applying the functor $R \otimes_A -$, we get the complex $A_+^{\otimes \bullet}$, with the differential $\partial_n : A_+^{\otimes n} \to A_+^{\otimes n-1}$ defined by

$$\partial_n(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

Now, $\mathrm{Tor}_n^A(R, R)$ is the nth homology module of this complex.

Consider the map

$$\Delta_{n,i} : A_+^{\otimes n} \to A_+^{\otimes i} \otimes A_+^{\otimes n-i}$$

defined by

$$\Delta_{n,i}(a_1 \otimes \cdots \otimes a_n) := (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n).$$

We see that Δ provides a comultiplicative structure for $A_+^{\otimes \bullet}$. Furthermore, we see that Δ respects kernels and images, making $A_+^{\otimes \bullet}$ a differential-graded coalgebra and making $\mathrm{Tor}^A(R, R)$ a graded coalgebra. If A is $\mathcal{M}$-graded, then $\mathrm{Tor}^A(R, R)$ is bigraded, with a homological by $\mathbb{N}$ and an internal grading by $\mathcal{M}$.

The **cobar complex** is the cochain complex $\mathrm{Cob}^\bullet(A)$ defined by

$$\mathrm{Cob}^n(A) := \mathrm{Hom}_A(A_+^{\otimes n}, R),$$

where the differential

$$\partial^* : \mathrm{Cob}^n(A) \to \mathrm{Cob}^{n+1}(A)$$

is the pullback of ∂. Similarly, if A is $\mathcal{M}$-graded, we have a graded version of the cobar complex $\mathrm{Cob}^\bullet_{\mathrm{Gr}}(A)$.

Consider the map

$$\mu_{i,n-i} : \mathrm{Cob}^i(A) \otimes \mathrm{Cob}^{n-i}(A) \to \mathrm{Cob}^n(A)$$

defined by

$$\mu(f \otimes g)(a_1 \otimes a_2) := f(a_1)g(a_2).$$

We see that μ provides a multiplicative structure for Cob $\bullet(A)$ (and similarly for $\mathrm{Cob}^\bullet_{\mathrm{Gr}}(A)$). Furthermore, se see that μ respects kernels and images, making $\mathrm{Cob}^\bullet(A)$ (and $\mathrm{Cob}^\bullet_{\mathrm{Gr}}(A)$) a differential-graded algebra and making $\mathrm{Ext}_A(R, R)$ (and $\mathrm{Ext}_{\mathrm{Gr}}({}_AR, {}_AR)$) a graded algebra. If A is $\mathcal{M}$-graded, then $\mathrm{Ext}_{\mathrm{Gr}}({}_AR, {}_AR)$ is bigraded, with a cohomological grading by $\mathbb{N}$ and an internal grading by $\mathcal{M}$.

Theorem II.2.3. *We have* $\mu_{n,i} = \Delta^\vee$, *where*
$\mu_{n,i} : \mathrm{Cob}^{n-i}(A) \otimes \mathrm{Cob}^i(A) \to \mathrm{Cob}^n(A)$ *and* $\Delta_{n,i} : A_+^{\otimes n} \to A_+^{\otimes n-i} \otimes A_+^{\otimes i}$.

Proof. Let $f_1 \otimes \cdots \otimes f_i \in \mathrm{Cob}^n(A)$, $g_1 \otimes \cdots \otimes g_{n-i} \in \mathrm{Cob}^{n-i}(A)$, and $a_1 \cdots a_n \in A_+^{\otimes n}$. We compute

$$\begin{aligned}
&\Delta^\vee((f_1 \otimes \cdots \otimes f_i) \otimes (g_1 \otimes \cdots \otimes g_{n-i}))(a_1 \otimes \cdots \otimes a_n) \\
&\qquad = ((f_1 \otimes \cdots \otimes f_i) \otimes (g_1 \otimes \cdots \otimes g_{n-i}))\Delta(a_1 \otimes \cdots \otimes a_n) \\
&\qquad = (f_1 \otimes \cdots \otimes f_i)(a_1 \otimes \cdots a_i) \\
&\qquad \cdot (g_1 \otimes \cdots \otimes g_{n-i})(a_{i+1} \otimes \cdots \otimes a_n) \\
&\qquad = \mu((f_1 \otimes \cdots \otimes f_i) \otimes (g_1 \otimes \cdots \otimes g_{n-i}))(a_1 \otimes \cdots \otimes a_n). \square
\end{aligned}$$

The above proof can easily be modified for the case when A is $\mathcal{M}$-graded, yielding the following:

Theorem II.2.4. *The map* $\mu_{n,i} : \mathrm{Cob}_{\mathrm{Gr}}^{n-i}(A) \otimes \mathrm{Cob}_{\mathrm{Gr}}^{i}(A) \to \mathrm{Cob}_{\mathrm{Gr}}^{n}(A)$ *and* $\Delta_{n,i} : A_+^{\otimes n} \to A_+^{\otimes n-i} \otimes A_+^{\otimes i}$ *are dual to one another. Thus, for any subset* $\mathcal{I} \subset \{0, \ldots, n\}$, *the maps* $\sum_{i \in \mathcal{I}} \mu_{n,i}$ *and* $\sum_{i \in \mathcal{I}} \Delta_{n,i}$ *on* $\mathrm{Cob}_{\mathrm{Gr}}^{n}(A)$ *and* $A_+^{\otimes n}$, *respectively, are dual to one another.*

II.3 Homological conditions

Definition II.3.1. For an augmented algebra A, the algebra $\mathbf{E}(A) = \mathrm{Ext}_A(R, R)$ is called the **Yoneda algebra**, and the coalgebra $\mathbf{T}(A) := \mathrm{Tor}^A(R, R)$ is called the **Yoneda coalgebra**. We say A is $\mathcal{K}_m$ if $\mathbf{E}(A)$ is generated as an R-algebra by $\mathbf{E}^1(A), \ldots, \mathbf{E}^m(A)$.

For the most part, we only consider the $\mathcal{K}_1$ and $\mathcal{K}_2$ conditions. The $\mathcal{K}_2$ property was introduced for connected-graded algebras by Cassidy and Shelton in [8] and [9]. The following lemma, which we will prove in Section VI.3, is very useful:

Lemma II.3.2. *If A has a grading compatible with its augmentation, $\mathbf{E}^m(A) = \mathbf{E}^m_{\mathrm{Gr}}(A)$ if an only if $\dim \mathbf{E}^m_{\mathrm{Gr}}(A) < \infty$. Consequently, if A has a grading compatible with its augmentation and is $\mathcal{K}_2$, then $\mathbf{E}(A) = \mathbf{E}_{\mathrm{Gr}}(A)$.*

Definition II.3.3. If A has a grading compatible with its augmentation and A is $\mathcal{K}_1$, then A is called **Koszul**.

In the remainder of the section, we will assume that A has a grading compatible with its augmentation.

The theory of Koszul algebras is very rich: a good reference is the book [18] by Polishchuk and Positselski.

Theorem II.3.4. *The following are equivalent:*

1. *A is Koszul.*
2. *The multiplication $\mu : \mathbf{E}^{n-1}(A) \otimes \mathbf{E}^1(A) \to \mathbf{E}^n(A)$ is surjective for each $n \geq 2$.*
3. *The comultiplication $\Delta : \mathbf{T}^n(A) \to \mathbf{T}^{n-1}(A) \otimes \mathbf{T}^1(A)$ is injective for each $n \geq 2$.*
4. *$\mathbf{E}^n_{\mathrm{Gr}}(A) = \mathbf{E}^{n,n}_{\mathrm{Gr}}(A)$ for all $n \geq 2$.*
5. *$\mathbf{T}^n_{\mathrm{Gr}}(A) = \mathbf{T}^{n,n}_{\mathrm{Gr}}(A)$ for all $n \geq 2$.*
6. *There exists a projective resolution $P^\bullet$ of graded A-modules for ${}_AR$ such that P^n is generated in degree n.*

Conditions 4 and 5 are **homological purity** conditions. The resolution required by Condition 6 is called a **linear projective resolution**.

For a graded algebra $A = \mathbb{T}_R(V)/I$, we may always begin a projective resolution of ${}_AR$ with a sequence

$$A \otimes_R W \xrightarrow{\partial_2} A \otimes_R V \xrightarrow{\partial_1} A \longrightarrow R \longrightarrow 0$$

where W is a minimal subspace of $\mathbb{T}_R(V)$ with the property $I = \langle W \rangle$. Then the duals ∂_1^* and ∂_2^* are both zero. Furthermore, we will have $\partial_3^* = 0$. Therefore, $\mathbf{E}^2(A) = W^*$, and the cohomology $\mathbf{E}^2(A)$ keeps track of the elements that generate the ideal of relations for A. Since a Koszul algebra A has $\mathbf{E}^2(A) = \mathbf{E}^{2,2}(A)$, a Koszul algebra A will always be quadratic.

Suppose $A = \mathbb{T}_R(V)/I$ is a quadratic algebra. We let $\langle -,- \rangle : V^{\otimes 2} \otimes_R (V^*)^{\otimes 2} \longrightarrow R$, where $V^* := \hom_R(V, R)$, be defined by $\langle v_1 \otimes v_2, w_1 \otimes w_2 \rangle := w_1(v_1)w_2(v_2)$. Then, we define the ideal $I^\perp \subset \mathbb{T}_R(V^*)$ by

$$I^\perp := \left\langle w \in (V^*)^{\otimes 2} : \langle v, w \rangle = 0 \text{ for all } v \in I \cap V^{\otimes 2} \right\rangle.$$

Definition II.3.5. The **quadratic dual** to the quadratic algebra $A = \mathbb{T}_R(V)/I$ is the algebra $A^! := \mathbb{T}_R(V^*)/I^\perp$.

Note that for a quadratic algebra A, $(A^!)^! = A$. The following relationship is called **Koszul duality**.

Theorem II.3.6. *A quadratic algebra A is Koszul if and only if $\mathbf{E}(A) = A^!$, so $\mathbf{E}(\mathbf{E}(A)) = A$ for a Koszul algebra.*

Koszul duality is a consequence of a more general result. For a general graded algebra $A = \mathbb{T}_R(V)/I$, we defined the **quadratized version** of A to be the quadratic algebra $qA := \mathbb{T}_R(V)/\left\langle I \cap V^{\otimes 2}\right\rangle$. We think of qA as having relations generated by the quadratic relations of A. Of course, for a quadratic A, $A = qA$. The following is [18, Proposition I.3.1]:

Theorem II.3.7. *For a graded algebra A, $\bigoplus_i \mathbf{E}^{i,i}(A)$ is a subalgebra of $\mathbf{E}(A)$, and in fact is isomorphic to $(qA)^!$. In particular, if A is quadratic, then $A^! \simeq \bigoplus_i \mathbf{E}^{i,i}(A)$.*

The first generalization of Koszulity is the N-Koszul condition introduced by Berger in [2] and is based on the homological purity conditions. Fix $N \geq 2$. Let $\delta : \mathbb{N} \to \mathbb{N}$ be defined by

$$\delta(i) := \begin{cases} \frac{Ni}{2}, & \text{if } i \text{ is even,} \\ \frac{N(i-1)}{2} + 1, & \text{if } i \text{ is odd.} \end{cases}$$

Definition II.3.8. A is N-Koszul if $\mathbf{E}^i(A) = \mathbf{E}^{i,\delta(i)}(A)$ for all i.

Note that the 2-Koszul is synonymous with Koszul. Analogously to the Koszul case, for an N-Koszul algebra A we have $\mathbf{E}^2(A) = \mathbf{E}^{2,N}(A)$, meaning A will have the ideal of relations I generated in degree N—that is, A is N-**homogeneous**.

The following was proved by Green, et. al. in [12] and by Cassidy and Shelton in [9]:

Theorem II.3.9. *A is N-Koszul if and only if A is N-homogeneous and $\mathcal{K}_2$.*

Green, et. al. also proved in [12] a "delayed" version of Koszul duality for N-Koszul algebras:

Theorem II.3.10. *Suppose A is an N-Koszul algebra. Then the Yoneda algebra $\mathbf{E}(A)$ is $\mathcal{K}_1$, and $\mathbf{E}(A)$ can be regraded so that $\mathbf{E}(A)$ is a Koszul algebra. In addition, $\mathbf{E}(\mathbf{E}(\mathbf{E}(A))) = \mathbf{E}(A)$.*

The search for a Koszul duality-like result for $\mathcal{K}_2$ algebras continues, although the results in Section V.4 indicate that this may be difficult.

Another generalization involving purity is the class of bi-Koszul algebras introduced by Lu and Si in [15]. A subclass of bi-Koszul algebras, the strongly bi-Koszul algebras, are $\mathcal{K}_3$.

While N-Koszul and bi-Koszul algebras generalize the purity conditions for Koszul algebras, $\mathcal{K}_2$ algebras generalize the conditions of Ext-generation.

Theorem II.3.11. *The following are equivalent:*

1. *A is $\mathcal{K}_2$.*
2. *The multiplication*

$$\mu : \mathbf{E}^2(A) \otimes \mathbf{E}^{n-2}(A) + \mathbf{E}^1(A) \otimes \mathbf{E}^{n-1}(A) \to \mathbf{E}^n(A)$$

   is surjective for each $n \geq 3$.

3. *The comultiplication*

$$\Delta : \mathbf{T}^n(A) \to \mathbf{T}^2(A) \otimes \mathbf{T}^{n-2}(A) + \mathbf{T}^1(A) \otimes \mathbf{T}^{n-1}(A)$$

   is injective for each $n \geq 3$.

Proof. Suppose A is $\mathcal{K}_2$. Then Condition (2) clearly holds for $n = 3$. But then Condition (2) holds by induction on n. On the other hand, induction shows Condition (2) implies A is $\mathcal{K}_2$.

The equivalence of Conditions (2) and (3) follows from the duality of μ and Δ. □

Recall that I is the ideal of relations for A. Set $I' = I \otimes V + V \otimes I$. Elements of $I \setminus I'$ are called **essential relations**. We may choose a subcomplex $A \otimes_R V^\bullet \subset \mathrm{Bar}^\bullet(A)$, where $V^n \subset A_+^{\otimes n}$, such that the restriction of ∂^* is zero. Such a resolution is called a **minimal projective resolution** of ${}_A R$. We may represent $A \otimes_R V^n$ as a free A-module and the differentials $\partial_n : A \otimes V^n \to A \otimes V^{n-1}$ as matrices with entries in A (such that the differential is right multiplication by the matrix). Lift to a matrix M_n with entries in $\mathbb{T}_R(V)$. Note that $M_n M_{n-1}$ will have entires in I. Set L_n to be the image of M_n after projection of each entry to $A/A_{\geq 2}$ and E_n to be the image of $M_n M_{n-1}$ after projection of each entry to I/I'. The following is due to Cassidy and Shelton [9]:

Theorem II.3.12. *The following are equivalent:*

1. *The comultiplication*

$$\Delta : \mathbf{T}^n(A) \to \mathbf{T}^2(A) \otimes \mathbf{T}^{n-2}(A) + \mathbf{T}^1(A) \otimes \mathbf{T}^{n-1}(A)$$

is injective for each $n \geq 3$ (and hence A is $\mathcal{K}_2$).

2. *The rows of $[L_n : E_n]$ are linearly independent for $3 \leq n \leq \mathrm{gldim}\, A$.*

Example II.3.13. This example will illustrate some of the linear algebra techniques that can be used to find a minimal projective resolution and determine that an algebra is not $\mathcal{K}_2$. Let

$$A := \frac{\mathbb{K}[y,z]}{\langle y^2z^2, y^4 \rangle}.$$

A basis for A is $\{1, y, z, y^2, yz, z^2, y^3, y^2z, yz^2, z^3, yz^3, y^3z, z^4, yz^4, z^5, yz^5, z^6, \dots\}$, and so the Hilbert series for A is

$$H_A(t) := \sum_{n \geq 0} \dim A_n = 1 + 2t + 3t^2 + 4t^3 + 3t^4 + \frac{2t^5}{1-t}.$$

We begin our minimal projective resolution with

$$A(-2,-4,-4) \xrightarrow{M_2} A(-1,-1) \xrightarrow{M_1} A \to \mathbb{K} \to 0$$

where

$$M_2 = \begin{pmatrix} -z & y \\ y^3 & 0 \\ 0 & y^2z \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} y \\ z \end{pmatrix}.$$

From the Hilbert series, we see that

$$\begin{aligned} &\ker M_2 \cap (A_3 \oplus A_1 \oplus A_1) \\ &= \mathbb{K}(y^3, -z, 0) + \mathbb{K}(y^2, 0, -y) + \mathbb{K}(0, y, 0) + \mathbb{K}(0, 0, z). \end{aligned}$$

To find $\ker M_2 \cap (A_4 \oplus A_2 \oplus A_2)$, we calculate

$$\begin{aligned} &(\alpha_1 yz^3 + \alpha_2 y^3 z + \alpha_3 z^4, \beta_1 y^2 + \beta_2 yz + \beta_3 z^3, \gamma_1 y^2 + \gamma_2 yz + \gamma_3 z^2) M_2 \\ &= (-\alpha_1 yz^4 - \alpha_3 z^5, \alpha_3 yz^4). \end{aligned}$$

So,

$$\begin{aligned}\ker M_2 \cap (A_4 \oplus A_2 \oplus A_2) &= \mathbb{K}(y^3z,0,0) + \mathbb{K}(0,y^2,0) + \mathbb{K}(0,yz,0)\\ &\quad + \mathbb{K}(0,z^2,0) + \mathbb{K}(0,0,y^2) + \mathbb{K}(0,0,yz)\\ &\quad + \mathbb{K}(0,0,z^2)\\ &= \mathbb{K}(y^3z,0,0) + A_1(\ker M_2 \cap (A_4 \oplus A_2 \oplus A_2)).\end{aligned}$$

Note we've now shown that for $n \geq 5$,

$$0 \oplus A_{n-3} \oplus A_{n-3} \subset A_{n-5}(\ker M_2 \cap (A_4 \oplus A_2 \oplus A_2)).$$

Now, suppose $n \geq 6$. We calculate

$$(\alpha_1 yz^{n-2} + \alpha_2 z^{n-1}, 0, 0)M_2 = (-\alpha_1 yz^{n-1} - \alpha_2 z^n, \alpha_2 yz^{n-1}),$$

meaning that

$$\ker M_2 \cap (A_{n-1} \oplus 0 \oplus 0) = 0.$$

Thus, we may begin our minimal projective resolution with

$$\begin{aligned}\cdots \to A(-5,-5,-5,-5,-6) \xrightarrow{M_3} A(-2,-4,-4) \xrightarrow{M_2}\\ A(-1,-1) \xrightarrow{M_1} A \to \mathbb{K} \to 0,\end{aligned}$$

where

$$M_3 := \begin{pmatrix} y^3 & z & 0 \\ y^2z & 0 & -y \\ 0 & y & 0 \\ 0 & 0 & z \\ y^3z & 0 & 0 \end{pmatrix}$$

and M_2 and M_1 are defined as above.

So, $\dim \mathbf{E}^{3,6}(A) = 2$ while $\dim(\mathbf{E}^2(A) \otimes \mathbf{E}^1(A) + \mathbf{E}^1(A) \otimes \mathbf{E}^2(A))^6 = 0$. Therefore, A is not $\mathcal{K}_2$.

We can use Theorem II.3.12 to show that A is not $\mathcal{K}_2$. Note that

$$M_3M_2 = \begin{pmatrix} 0 & y^4 \\ -y^2z^2 & 0 \\ y^4 & 0 \\ 0 & y^2z^2 \\ -y^3z^2 & y^4z \end{pmatrix},$$

so

$$[L_3 : E_3] = \begin{pmatrix} 0 & z & 0 & : & 0 & y^4 \\ 0 & 0 & -y & : & -y^2z^2 & 0 \\ 0 & y & 0 & : & y^4 & 0 \\ 0 & 0 & z & : & 0 & y^2z^2 \\ 0 & 0 & 0 & : & 0 & 0 \end{pmatrix}$$

does not have linearly-independent rows.

II.4 The $\mathcal{K}_2$ property for monomial algebras

In this section, we briefly describe a combinatorial perspective for the $\mathcal{K}_2$ condition on monomial algebras, introduced by Cassidy and Shelton in [9]. Because the structure of a monomial algebra is somewhat simple, the structure of its Yoneda algebra is more easily understood. We will go into more detail about this structure in Chapter V.

Definition II.4.1. Fix a basis $\{x_1, \ldots, x_n\}$ for a $\mathbb{K}$-vector space V. A connected-graded algebra $A = \mathbb{T}(V)/I$ is a **monomial algebra** (with respect to the fixed basis) if I is generated by monomials (with respect to the basis).

Suppose $A = \mathbb{T}(V)/I$ is a monomial algebra and let $\mathcal{M}$ be the set of all monomials in $\mathbb{T}(V)$. Let $\mathcal{R}$ be minimal set of monomials generators for I. For $m \in \mathcal{M}$, define the set of **minimal left-annihilators**

$$\mathfrak{A}_m := \{w \in \mathcal{M} \setminus I : wm \in I \text{ but } w'm \notin I \text{ for any } w', w'' \in \mathcal{M} \text{ with } w''w' = w\}.$$

The following is obvious:

Theorem II.4.2. *For $i \geq 1$, set*

$$S^i := \{m_i \otimes m_{i-1} \otimes \cdots \otimes m_0 : m_0 = x_j \text{ for some } j, m_i \in \mathfrak{A}_{m_{i-1}}\} \subset A_+^{\otimes i}.$$

Let $V^i = \operatorname{span} S^i \subset A_+ \otimes V^{i-1}$. Here we identify a monomial $m \in \mathcal{M} \setminus I$ with its image in A. Define a map

$$\partial : A \otimes V^i \to A \otimes V^{i-1}$$

by

$$\partial(a \otimes m_i \otimes \cdots \otimes m_1) = am_i \otimes m_{i-1} \otimes \cdots \otimes m_1.$$

Then $A \otimes V^\bullet$ is a minimal projective resolution for ${}_A\mathbb{K}$ and is in fact a subresolution of the bar resolution $\mathrm{Bar}^n(A)$.

The map $\partial : A \otimes V^i \to A \otimes V^{i+1}$ can be represented by a $|S^{i+1}| \times |S^i|$ matrix in which the columns correspond to elements of S^i, the rows correspond to elements of S^{i+1}, and the row corresponding to $m \otimes u \in S^{i+1}$ (where $m \in \mathcal{M}$ and $u \in S^i$) has an m in the column corresponding to u and a zero in every other column.

Consider the corresponding matrices L_i and E_i from Theorem II.3.12. Each L_i is a $|S^{i+1}| \times |S^i|$ matrix in which the columns correspond to elements of S^i, the rows correspond to elements of S^{i+1}, and the row corresponding to $m \otimes u \in S^{i+1}$ (where $m \in \mathcal{M}$ and $u \in S^i$) has m mod $\mathbb{T}_{\geq 2}(V)$ in the column corresponding to u and a zero in every other column. Each E_i is a $|S^{i+1}| \times |S^{i-1}|$ matrix in which the columns correspond to elements of S^{i-1}, the rows correspond to elements of S^{i+1}, and the row corresponding to $m_2 \otimes m_1 \otimes u \in S^{i+1}$ (where $m_1, m_2 \in \mathcal{M}$ and $u \in S^{i-1}$ has $m_2 m_1$ mod I' in the column corresponding to u and a zero in every other column. Note that the rows of $[L_i : E_i]$ are linearly independent if and only if each row is nonzero. The following consequence is found in [9]:

Theorem II.4.3 (Cassidy–Shelton algorithm for monomial algebras). *For $m \in \mathcal{M}$, set $\mathfrak{A}'_m := \{w \in \mathfrak{A}_m : wm \in I'\}$. Let $\mathfrak{S}_0 = \{x_1, \ldots, x_n\}$, and for $i \geq 1$, $\mathfrak{S}_i := \bigcup_{w \in \mathfrak{S}_{i-1}} \mathfrak{A}_w$ and $\mathfrak{S} := \bigcup \mathfrak{S}_i$. Then a monomial algebra A is $\mathcal{K}_2$ if and only if for every $m \in \mathfrak{S}$, we have $\mathfrak{A}'_m \subset V$.*

We will provide a pictoral view of this condition in Section V.2.

If A is quadratic, then $\mathfrak{A}_m \subset V$ always. So, we have this classic result as a corollary:

Corollary II.4.4. *If A is a quadratic monomial algebra, then A is Koszul.*

II.5 Poincaré–Birkhoff–Witt algebras

Poincaré–Birkhoff–Witt (PBW) algebras are quadratic algebras whose ideals of relations have nice Gröbner bases (with respect to a natural filtration), or equivalently, have the property that $\operatorname{gr} A$ is still quadratic. In the same paper [19] in which he first formulated the concept of a Koszul algebra, Priddy proved the following:

Theorem II.5.1. *PBW algebras are Koszul.*

This result will also be a corollary to our results in Section VI.3. A more complete exposition of the connection between PBW algebras and Koszulity is found in Chapter IV of [18].

Let $\{x_1, \ldots, x_n\}$ be an *ordered* basis for the $\mathbb{K}$-vector space V, and consider a connected-graded algebra $A := \mathbb{T}(V)/I$. Let $\pi : \mathbb{T}(V) \twoheadrightarrow A$ be the canonical surjection. Let $\mathcal{M}$ be the set of monomials (including the empty word e) in $\mathbb{T}(V)$, ordered by degree-lexicographical order. Then $\mathcal{M}$ is an ordered monoid (under concatenation). We filter A by $\mathcal{M}$ via $F_\alpha A = \pi\left(\sum_{\beta \leq \alpha} \mathbb{K}\beta\right)$. The filtration F on A yields an associated graded algebra $\operatorname{gr}^F A$ which is monomial.

Definition II.5.2. 1. We define a (non-linear) function $\tau : \mathbb{T}(V) \to \mathcal{M}$ via

$$\tau\left(c_\alpha \alpha + \sum_{\beta < \alpha} c_\beta \beta\right) := c_\alpha \alpha.$$

The output $\tau(x)$ is called the **leading term** of x.

2. Let $I \subset \mathbb{T}(V)$ be a homogeneous ideal. A generating set $\mathcal{G}$ of I is called a

Gröbner basis if $\langle \tau(\mathcal{G}) \rangle = \langle \tau(I) \rangle$.

The following theorem is standard:

Theorem II.5.3 ([14, Theorem 2.1]). $\operatorname{gr} A \simeq \mathbb{T}(V) / \langle \tau(I) \rangle$.

Thus, gr A is a monomial algebra.

Theorem II.5.4. *Suppose $A = \mathbb{T}(V)/I$ is a quadratic algebra. Then the following are equivalent:*

1. *The monomial algebra* gr A *is a quadratric algebra.*

2. gr A *is a Koszul algebra.*

3. *A has a Gröbner basis $\mathcal{G}$ such that the set $\{x + I' : x \in \mathcal{G}\}$ is linearly independent in I/I'. (Recall that $I' = I \otimes V + V \otimes I$.)*

4. *Every generating set $\mathcal{B}$ such that*

   (a) the set $\{x + I' : x \in \mathcal{B}\}$ is linearly independent in I/I' and

   (b) the no proper subset of $\mathcal{B}$ generates the ideal $\langle \tau(\mathcal{B}) \rangle$

   is a Gröbner basis.

Definition II.5.5. A quadratic algebra A satisfying the equivalent conditions of Theorem II.5.4 is a **Poincaré–Birkhoff-Witt algbera**. We call a generating set $\mathcal{B}$ satisfying Condition (4a) above a **essential generating set**. If an essential generating set also satisfies Condition (4b), we say it has the **leading monomial property**. We call an essential generating set which is also a Gröbner basis an **essential Gröbner basis**.

In Section VI.3, we explore the consequences of having an essential Gröbner basis for I when $A = \mathbb{T}(V)/I$ is a graded algebra with its ideal of relations I generated in arbitrary degrees, generalizing Thoerem II.5.4.

CHAPTER III

PBW DEFORMATION THEORY

III.1 Motivating example and definition

Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Then its universal enveloping algebra

$$U(\mathfrak{g}) = \frac{\mathbb{T}(\mathfrak{g})}{\langle x \otimes y - y \otimes x - [x,y] : x, y \in \mathfrak{g} \rangle}$$

is filtered by degree, and in fact

$$\operatorname{gr} U(\mathfrak{g}) \simeq \mathbb{S}(\mathfrak{g}), \tag{III.1}$$

where $\mathbb{S}(V) := \mathbb{T}(V) / \langle x \otimes y - y \otimes x : x, y \in V \rangle$ denotes the symmetric algebra on the vector space V. The isomorphism in (III.1) is a well-known result from representation theory, known as the Poincaré–Birkhoff–Witt theorem (see, for example, [13, Section 17.3]). Since $U(\mathfrak{g})$ has a very similar structure to $\mathbb{S}(\mathfrak{g})$, we think of $U(\mathfrak{g})$ as a deformed version of $\mathbb{S}(\mathfrak{g})$.

More generally, suppose $R = \bigoplus_\alpha \mathbb{K} e_\alpha$ is a ring where the e_α are orthogonal idempotents, and let $A = \mathbb{T}_R(V)/I$ be a graded algebra, where I has a homogeneous essential generating set $\{r_1, \ldots, r_s\}$. Suppose we have a set of not necessarily homogeneous elements

$$\left\{ l_t \in \sum_{i=0}^{\deg r_t - 1} V^{\otimes i} : t = 1, \ldots, s \right\}.$$

Then let

$$U := \mathbb{T}_R(V) / \langle r_i - l_i : i = 1, \ldots, s \rangle . \tag{III.2}$$

We will say that the ungraded algbera U is a **deformation** of the graded algebra A. Note that U still has a filtration via

$$F_n U = \sum_{i=1}^{n} V^{\otimes i} + \langle r_i - l_i : i = 1, \ldots, s \rangle .$$

Definition III.1.1. U is a **Poincaré–Birkhoff-Witt (PBW) deformation** of A if $\operatorname{gr} U \simeq A$.

In this context, the classic PBW Theorem then implies that $U(\mathfrak{g})$ is a PBW deformation of $\mathbb{S}(\mathfrak{g})$.

PBW deformations were first studied by Braverman and Gaitsgory in 1996 in the quadratic case [6]. Additional results for the quadratic case appeared in the book by Polishchuk and Positselski [18]. Berger and Ginzburg [3] and Fløystad and Vatne [11] extended this study to the N-Koszul algebras in 2006. In 2007, Cassidy and Shelton [8] found a homological technique that works for algebras with mixed-degree relations. Cassidy and Shelton's results are explained in the next section.

III.2 A homological technique for determining if a deformation is PBW

In this section, we will describe a homological technique for determining whether the deformation in (III.2) is a PBW deformation, first described by Cassidy and Shelton in [8]. We begin by introducing another graded algebra which has regularity properties equivalent to the PBW-ness of the deformation:

Definition III.2.1. Suppose U is a deformation of A as in (III.2). Then the **associated central extension** D by z of A is the graded algebra

$$D := \frac{\mathbb{T}_R(V)[z]}{\langle h(r_i - l_i) : i = 1, \ldots, s \rangle},$$

where $\mathbb{T}_R(V)[z]$ is the polynomial ring over $\mathbb{T}_R(V)$ with central indeterminant z and $h : \mathbb{T}_R(V) \to \mathbb{T}_R(V)[z]$ is a (non-linear) function, called the **homogenization**, defined by

$$h\left(\sum_{t=0}^{n} a_t\right) := \sum_{t=0}^{n} a_t z^{n-t},$$

in which each $a_t \in V^{\otimes t}$.

As in Section II.3, fix a minimal projective resolution

$$\cdots \to A \otimes_R V_3 \xrightarrow{\partial_3} A \otimes_R V_2 \xrightarrow{\partial_2} A \otimes V_1 \xrightarrow{\partial_1} A \otimes_R V_0 \to R \to 0$$

of ${}_A R$. Recall, this means each V_i is a graded free R-module and the induced map $\partial_i^* : V_{i-1}^* \to V_i^*$ (where $-^*$ is the *graded* dual) is trivial. Choose the R-modules V_i so that $V_0 = R$, $V_1 = V$, and $V_2 = \bigoplus Rr_i$. We view $A \otimes_R V^\bullet$ as a graded free A-module, and view each ∂_n as a rank $V_{n-1} \times$ rank V_n matrix with entries in A (which acts by left multiplication).

The maximum degree of the entries of V_3 determines how much work is required to verify that U is a PBW deformation.

Definition III.2.2. The **complexity** of the graded algebra A is

$$c(A) := \begin{cases} 0, & \text{if } A \text{ has global dimension less than 2;} \\ \sup\{n : \mathbf{E}^{3,n}(A) \neq 0\} - 1, & \text{otherwise.} \end{cases}$$

Note that V_3 has maximum degree $c(A) + 1$.

Now, choose a lift of the matrix ∂_n to a matrix M_n of elements in $\mathbb{T}_R(V) \subset \mathbb{T}_R(V)[z]$. Choose a rank $V_{n-2} \times$ rank V_n matrix f_n so that

$$\pi_D(M_n M_{n-1} - (-1)^{n-1} z f_n) = 0.$$

Let $\widehat{Q}_i = D \otimes_R (V_n \oplus V_{n-1})$ and define the (rank V_{n-1} + rank V_{n-2}) $\times$ (2 rank V_{n-1}) matrix

$$\widehat{\partial}_n = \pi_D \begin{pmatrix} M_n & f_n \\ (-1)^{n-1} z \cdot \mathrm{id} & M_{n-1} \end{pmatrix}.$$

This gives us a sequence $(\widehat{Q}^\bullet, \widehat{\partial}_\bullet)$, but it may not even be a chain complex, let alone exact.

Theorem III.2.3. *The following are equivalent:*

1. *z is regular in D.*

2. *If $zx = 0$ for some nonzero $x \in D_i$, then $i > c(A)$.*

3. *U is a PBW-deformation of A.*

4. *$\widehat{Q}^\bullet$ is a projective resolution of ${}_DR$.*

5. *The sequence $\widehat{Q}^3 \to \widehat{Q}^2 \to \widehat{Q}^1 \to \widehat{Q}^0 \to R \to 0$ is exact.*

6. $\pi_D(M_3 f_2 + f_3 M_1) = 0$.

Cassidy and Shelton proved this theorem in the case where $R = \mathbb{K}$, but their proofs are completely generalizable to the semisimple case.

In the following examples, we will stick to the case where $R = \mathbb{K}$.

Example III.2.4. We recover the classical PBW theorem for the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$, where char $\mathbb{K} \neq 2$. Let

$$U := U(\mathfrak{sl}_2) = \frac{\mathbb{K}\langle x, y, h\rangle}{\langle xy - yx - h, hx - xh - 2x, hy - yh + 2y\rangle}$$

be the universal enveloping algebra for $\mathfrak{sl}_2$ and let and $A := \mathbb{K}[x, y, h]$. So U is a deformation of A, and the associated central extension by z is

$$D = \frac{\mathbb{K}\langle x, y, h\rangle[z]}{\langle xy - yx - hz, hx - xh - 2xz, hy - yh + 2yz\rangle}.$$

A minimal projective resolution for ${}_A\mathbb{K}$ is

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} h & -y & x \end{pmatrix}} A(-2)^3 \xrightarrow{\begin{pmatrix} y & -x & 0 \\ h & 0 & -x \\ 0 & h & -y \end{pmatrix}} A(-1)^3 \xrightarrow{\begin{pmatrix} x \\ y \\ h \end{pmatrix}} A \to \mathbb{K} \to 0.$$

So, we may choose

$$f_3 = \begin{pmatrix} -2y & -2x & -h \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} h \\ -2x \\ 2y \end{pmatrix}.$$

Then it is easy to verify that $M_3 f_2 + f_3 M_1 = 0$, so $\operatorname{gr} U \simeq A$ by Theorem III.2.3.

Example III.2.5. This technique can be used to verify that central elements of a graded algebra are regular. Consider the algebra

$$D := \frac{\mathbb{K}[x,y,z]}{\langle x^3y - y^3x + xyz^2, x^2y^2 \rangle}.$$

We will show that $u := x + y$ is regular in D. Let

$$A := D/uD \simeq \frac{\mathbb{K}[y,z]}{\langle y^2z^4, y^4 \rangle}$$

and let

$$U := D/(u-1)D.$$

In fact, U is a deformation of A and D is the associated central extension by u. In Example II.3.13, we carefully showed that a minimal projective resolution for ${}_A\mathbb{K}$ may begin

$$\begin{aligned} \cdots \to A(-5,-5,-5,-5,-6) \xrightarrow{M_3} A(-2,-4,-4) \xrightarrow{M_2} \\ A(-1,-1) \xrightarrow{M_1} A \to \mathbb{K} \to 0, \end{aligned} \tag{III.3}$$

where

$$M_3 = \begin{pmatrix} y^3 & z & 0 \\ y^2z & 0 & -y \\ 0 & y & 0 \\ 0 & 0 & z \\ y^3z & 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} -z & y \\ y^3 & 0 \\ 0 & y^2z \end{pmatrix}, \text{ and } M_1 = \begin{pmatrix} y \\ z \end{pmatrix}.$$

First, we compute:

$$M_2M_1 = \begin{pmatrix} 0 \\ y^4 \\ y^2z^2 \end{pmatrix} \text{ and } M_3M_2 = \begin{pmatrix} 0 & y^4 \\ -y^2z^2 & 0 \\ y^4 & 0 \\ 0 & y^2z^2 \\ -y^3z^2 & y^4z \end{pmatrix}.$$

Now, to find f_2 and f_3, it helps to know that (remembering that $u = x + y$)

$$\begin{aligned} (y^3 - y^2x)(x+y) &= y^3x - y^2x^2 + y^4 - y^3x \\ &= y^4 - x^2y^2 \\ &= y^4, \end{aligned}$$

and that

$$\begin{aligned} (x+y)(yz^2 - xy^2 + x^2y) &= xyz^2 - x^2y^2 + x^3y + y^2z^2 - xy^3 + x^2y^2 \\ &= xyz^2 + x^3y - xy^3 + y^2z^2 \\ &= y^2z^2. \end{aligned}$$

So, from this, we see that we may choose

$$f_3 = \begin{pmatrix} 0 \\ y^2x - y^3 \\ xy^2 - yz^2 - x^2y \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} 0 & y^3 - y^2x \\ xy^2 - yz^2 - x^2y & 0 \\ y^3 - y^2x & 0 \\ -y^2z^2 + xy^3 - x^2y^2 & y^3z - y^2xz \end{pmatrix}.$$

Now, we calculate

$$M_3f_2 + f_3M_1 = \begin{pmatrix} xy^2z - y^3z \\ -xy^3 + y^2z^2 + x^2y^2 \\ xy^3 - y^4 \\ xy^2z - yz^3 - x^2yz \end{pmatrix} + \begin{pmatrix} y^3z - xy^2z \\ xy^3 - y^2z^2 - x^2y^2 \\ y^4 - y^3x \\ yz^3 - xy^2z - x^2yz \\ xy^4 - x^2y^3 - xy^2z^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ y(y^3x - xyz^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ x^3y^2 \end{pmatrix} = 0.$$

Thus, $u = x + y$ is regular in D by Theorem III.2.3.

Example III.2.6. This example was created by Andrew Conner [10] to show that the Koszulity hypothesis is necessary in Theorem IV.2.1 (see Example IV.2.3). Let $A = \mathbb{K}\langle x, y\rangle \,/\, \langle x^2, y^2 - xy\rangle$. Consider the deformation

$$U = \frac{\mathbb{K}\langle x, y\rangle}{\langle x^2 - 1, y^2 - xy - 1\rangle}$$

and the associated central extension D by t of A is

$$D = \frac{\mathbb{K}\langle x, y\rangle [z]}{\langle x^2 - z^2, y^2 - xy - z^2\rangle}.$$

We may begin a minimal projective resolution for ${}_{\mathbb{K}}A$ with

$$\cdots \to A(-3,-4) \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & yx \end{pmatrix}} A(-2)^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y-x \end{pmatrix}} A(-1)^2 \xrightarrow{\begin{pmatrix} x \\ y \end{pmatrix}} A \to \mathbb{K} \to 0.$$

Note that A is not a Koszul algebra. We can choose our f_3 and f_2 to be represented by the matrices

$$f_3 = \begin{pmatrix} z & 0 \\ 0 & xz \end{pmatrix} \text{ and } f_2 = \begin{pmatrix} -z \\ -z \end{pmatrix}.$$

Then, we compute

$$\pi_D(M_3 f_2 + f_3 M_1) = \begin{pmatrix} 0 \\ xyz - yxz \end{pmatrix},$$

meaning that U is not a PBW deformation of A by Theorem III.2.3.

CHAPTER IV

PBW DEFORMATIONS ARISING FROM KOSZUL ALGEBRAS

IV.1 Introduction

In this section, we study some deformations that arise in the study of a recently-discovered duality. This duality is the subject of a paper [5], currently under review, that the author coauthored with Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster. A preprint of the paper is available on the arXiv preprint server. The results in this chapter are a lemma used to prove the main theorem in this paper, and first appear there.

In this chapter, we consider graded R-algebras of the form $A = \mathbb{T}_R(M)/I$ where R is a semisimple ring over the field $\mathbb{K}$ and M is an R-bimodule. We assume $R = \bigoplus_\alpha \mathbb{K}e_\alpha$, where the e_α are orthogonal idempotents.

First, let us briefly describe the main results of [5].

Definition IV.1.1. Let U be a finite-dimensional complex vector space and $S = \mathbb{S}(U)$ be the symmetric algebra. A **Goresky–MacPherson algebra** is a quadruple $\mathcal{Z} = (U, Z, \mathcal{I}, h)$, where Z is a commutative graded S-algebra, $\mathcal{I}$ is a finite set, and $h : Z \to \bigoplus_{\alpha\in\mathcal{I}} S$ is a map of graded S-algebras. If h is an isomorphism, then $\mathcal{Z}$ is a **strong** GM algebra.

There is a notion of GM duality similar to Koszul duality (Theorem II.3.6). If a graded algebra A is Koszul and meets some additional conditions, then the Yoneda algebra $\mathbf{E}(A)$ will also meet those conditions and we can associate to A a GM algebra $\mathcal{Z}(A)$. Under these stronger conditions, the following holds:

Theorem IV.1.2. *$\mathcal{Z}(A)$ is canonically GM dual to $\mathcal{Z}(\mathbf{E}(A))$.*

This GM duality has been observed in some examples, such as the equivariant cohomology associated to certain algebraic group actions on algebraic varieties.

IV.2 A deformation that arises in the proof of Theorem IV.1.2

In this section, for an algebra A as in Theorem IV.1.2 and $z^! \in Z(A^!)_2$ (the second degree subspace of the center of $A^!$), we define a deformation $\tilde{A}_{z^!}$. (Recall that $A^!$ is the quadratic dual in Definition II.3.5.) In Section IV.3, we will see that this deformation is PBW.

Let $R = \bigoplus_\alpha \mathbb{K} e_\alpha$ be a ring, where the e_α are orthogonal idempotents. Let V and W be R-bimodules, and $\iota : W \hookrightarrow V \otimes_R V$ be an injective R-bimodule homomorphism. We consider the quadratic algebra

$$A := \mathbb{T}_R(V) / \langle \iota(w) : w \in W \rangle .$$

Now, let $z^! \in A^!_2$. Note that by Theorem II.3.7, we can identify $A^!_2 \simeq \mathrm{Ext}^{2,2}_A(R, R)$, and so we have a pairing

$$\langle -, - \rangle : A^!_2 \otimes_R W \to R.$$

We define the deformation

$$\tilde{A}_{z^!} := \frac{\mathbb{T}_R(V)}{\left\langle \iota(w) - e_\alpha \langle z^!, w \rangle e_\beta : w \in e_\alpha W e_\beta \right\rangle}.$$

The proof of Theorem IV.1.2 relies on this theorem as a lemma:

Theorem IV.2.1. *If A is Koszul and $z^! \in Z(A^!)_2 \subset A^!_2$, then $\tilde{A}_{z^!}$ is a PBW deformation.*

Example IV.2.2. The centrality of $z^!$ is a necessary hypothesis for Theorem IV.2.1. Let $A := \mathbb{K}\langle w, x, y \rangle / \langle xy, wx \rangle$, which is a monomial quadratic algebra and hence by Corollary II.4.4 is Koszul. A minimal projective resolution for ${}_A\mathbb{K}$ is

$$0 \to A(-3) \xrightarrow{\begin{pmatrix} 0 & w \end{pmatrix}} A(-2)^2 \xrightarrow{\begin{pmatrix} 0 & w & 0 \\ 0 & 0 & x \end{pmatrix}} A(-1)^3 \xrightarrow{\begin{pmatrix} w \\ x \\ y \end{pmatrix}} A \to \mathbb{K} \to 0.$$

Note

$$\mathbf{E}(A) = A^! = \frac{\mathbb{K}\langle w, x, y \rangle}{\langle w^2, wy, xw, x^2, yw, yx, y^2 \rangle},$$

and the element $xy \in A^!$ is *not* central. Now the deformation is

$$\tilde{A}_{xy} = \frac{\mathbb{K}\langle w, x, y \rangle}{\langle xy - 1, wx \rangle}$$

and the associated central extension is

$$D = \frac{\mathbb{K}\langle w, x, y \rangle [z]}{\langle xy - z^2, wx \rangle}.$$

We may choose $f_3 = 0$ and

$$f_2 = \begin{pmatrix} 0 \\ -z \end{pmatrix},$$

meaning $\pi_D(M3f_2 + f_3M_1) = \pi_D(-wz) \neq 0$, so $\tilde{A}_{xy}$ is not a PBW deformation of A by Theorem III.2.3.

We can also directly observe that in D, we have $wz^2 = wxy = 0$, meaning z is not a regular element in D.

Example IV.2.3. The Koszulity of A is a necessary hypothesis for Theorem IV.2.1. Recall the algebra

$$A = \frac{\mathbb{K}\langle x, y\rangle}{\langle x^2, y^2 - xy\rangle}$$

and the deformation $U = \mathbb{K}\langle x, y\rangle / \langle x^2 - 1, y^2 - xy - 1\rangle$ from Example III.2.6. We showed that U is not a PBW deformation and that A is not Koszul. Consider the element

$$z^! := (x^2) + (y^2 - xy) = x^2 + 2xy \in A^! = \frac{\mathbb{K}\langle x, y\rangle}{\langle yx, y^2 + xy\rangle}.$$

It is easy to show that $z^!$ is central in $A^!$: Noting that $y^3 = -yxy = 0$ in $A^!$, we compute in $A^!$

$$\begin{aligned}(x^2 - 2xy)x &= x^3 - 2xyx = x^3 = x^2 + 2y^3 \\ &= x^3 + 2xy^2 = x^3 - 2x^2y = x(x^2 - 2xy)\end{aligned}$$

and

$$\begin{aligned}(x^2 - 2xy)y &= x^2y - 2xy^2 = x^2y + 2y^3 = x^2y \\ &= -xy^2 = y^3 = 0 = yx^2 - 2yxy = y(x^2 - 2xy).\end{aligned}$$

Furthermore, $\tilde{A}_{z^!} = U$.

IV.3 The deformation $\tilde{A}_{z^!}$ is PBW

The goal of this section is to prove Theorem IV.2.1. We are actually going to prove this slightly more general version:

Theorem IV.3.1. *Suppose $c(A) = 2$ or $c(A) = 0$ (where $c(A)$ is the complexity of A defined in Definition III.2.2) and $z^! \in Z(A^!)_2 = Z(\mathbf{E}(A))_2$. Then $\tilde{A}_{z^!}$ is a PBW deformation of A.*

Proof. Let $\xi \in \mathrm{Hom}_R(W, R)$ be the map

$$\xi : w \mapsto \sum_{\alpha} \left\langle z^!, e_\alpha w e_\alpha \right\rangle e_\alpha .$$

Then we can write

$$D := \frac{\mathbb{T}_R(V)[z]}{\langle \iota(w) - \xi(w) z^2 : w \in W \rangle}$$

and

$$\tilde{A}_{z^!} = D / D(z-1).$$

So $\tilde{A}_{z^!}$ is a deformation of A and D is the associated central extension.

In the following, when we apply the functions π_D or ξ to a matrix, we mean the result of applying the function to each entry in the matrix.

As in Section III.2, choose a minimal projective resolution of ${}_A R$

$$\cdots \to A \otimes_R V_3 \xrightarrow{\partial_3} A \otimes_R V_2 \xrightarrow{\partial_2} A \otimes_R V_1 \xrightarrow{\partial_1} A \otimes_R V_0 \to R \to 0$$

so that $V_0 = R$, $V_1 = V$, and $V_2 = W$. We view each ∂_n as a matrix with entries in

A (which acts by left multiplication), and choose lifts M_n with entries in $\mathbb{T}_R(V) \subset \mathbb{T}_R(V)[z]$.

In general, M_3M_2 has entries in I, which means that we can think of M_3M_2 as a function

$$\mathbb{T}_R(V) \otimes_R V_3 \to I \otimes V_1.$$

However, in this case, $c(A) = 2$ or $c(A) = 0$, and so we actually know that M_3M_2 has entries in W, and we can think of M_3M_2 as a function

$$\mathbb{T}_R(V) \otimes_R V_3 \to (\mathbb{T}_R(V) \otimes_R W) \otimes_R V_1.$$

On the other hand, because $V_2 = W$, $V_1 = V$ and $V_0 = R$, we know that M_2M_1 has entries in W, and so we can think of M_2M_1 as a function

$$\mathbb{T}_R(V) \otimes_R V_2 \to (\mathbb{T}_R(V) \otimes W) \otimes V_0.$$

Indeed, we see that we can view $M_3M_2M_1$ as a function

$$\mathbb{T}_V(R) \otimes_R V_3 \to \mathbb{T}_R(V) \otimes_R (V \otimes_R W \cap W \otimes_R V) \otimes V_0.$$

Set $f_3 := z\xi(M_3M_2)$ and $f_2 := -z\xi(M_2M_1)$. Then

$$\pi_D(M_3M_2 - zf_3) = \pi_D(M_3M_2 - z\xi(M_3M_2)) = 0 \text{ and}$$
$$\pi_D(M_2M_1 + zf_2) = \pi_D(M_2M_1 - z\xi(M_2M_1)) = 0,$$

meaning we may apply Theorem III.2.3.

Now, it suffices to show that

$$M_3 f_2 + f_3 M_1 = 0. \tag{IV.1}$$

We can view the matrix f_3 as the composition

$$\begin{aligned} \mathbb{T}_R(V) \otimes_R V_3 \xrightarrow{\cdot M_3 M_2} (\mathbb{T}_R(V) \otimes_R W) \otimes_R V_1 \xrightarrow{\mathrm{id} \otimes \xi \otimes \mathrm{id}} \mathbb{T}_R(V) \otimes_R V_1 \\ \xrightarrow{\cdot z \otimes \mathrm{id}} \mathbb{T}_R(V) z \otimes_R V_1, \end{aligned}$$

and the matrix f_2 as the composition

$$\begin{aligned} \mathbb{T}_R(V) \otimes_R V_2 \xrightarrow{\cdot M_2 M_1} (\mathbb{T}_R(V) \otimes W) \otimes_R V_0 \xrightarrow{\mathrm{id} \otimes \xi \otimes \mathrm{id}} \mathbb{T}_R(V) \otimes_R V_0 \\ \xrightarrow{\cdot z \otimes \mathrm{id}} \mathbb{T}_R(V) z \otimes_R V_0. \end{aligned}$$

First, note that the diagram

$$\begin{array}{ccccc} \mathbb{T}_R(V) \otimes_R V_3 & \xrightarrow{\cdot M_3 M_2} & \mathbb{T}_R(V) \otimes_R W \otimes_R V_1 & \xrightarrow{\mathrm{id} \otimes \xi \otimes \mathrm{id}} & \mathbb{T}_R(V) \otimes_R V_1 \\ & & \downarrow{\scriptstyle \cdot M_1} & & \downarrow{\scriptstyle \cdot z \otimes \mathrm{id}} \\ & & \mathbb{T}_R(V) \otimes_R (W \otimes_R V \cap V \otimes_R W) \otimes V_0 & & \mathbb{T}_R(V) z \otimes_R V_1 \\ & & \downarrow{\scriptstyle \mathrm{id}_{T_R(V)} \otimes \xi \otimes \mathrm{id}_V \otimes \mathrm{id}_{V_0}} & & \downarrow{\scriptstyle M_1} \\ & & \mathbb{T}_R(V) \otimes_R V \otimes_R V_0 & & \mathbb{T}_R(V) z \otimes_R V_0 \\ & & \downarrow{\scriptstyle \simeq} & \nearrow{\scriptstyle \cdot z \otimes \mathrm{id}} & \\ & & \mathbb{T}_R(V) \otimes_R V_0 & & \end{array}$$

commutes, meaning for $v \in V_3$, $v f_3 M_1 = z(\xi \otimes \mathrm{id})(v M_3 M_2 M_1)$. On the other hand, the commutative diagram

$$
\begin{array}{ccccc}
\mathbb{T}_R(V) \otimes_R V_3 & \xrightarrow{\cdot M_3} & \mathbb{T}_R(V) \otimes_R V_2 & \xrightarrow{\cdot M_2 M_1} & \mathbb{T}_R(V) \otimes_R V_0 \\
& \searrow^{\cdot M_3 M_2 M_1} & & & \big\downarrow \\
& & & & \mathbb{T}_R(V) \otimes_R (V \otimes_R W \cap W \otimes_R V) \otimes_R V_0 \\
& & & & \big\downarrow {\scriptstyle \mathrm{id}_{\mathbb{T}_R(V)} \otimes \mathrm{id}_V \otimes \xi \otimes \mathrm{id}_{V_0}} \\
& & & & \mathbb{T}_R(V) \otimes_R V_0 \\
& & & & \big\downarrow {\scriptstyle \cdot z \otimes \mathrm{id}} \\
& & & & \mathbb{T}_R(V) z \otimes_R V_0
\end{array}
$$

shows us that for $v \in V_3$, $vM_3 f_2 = -z(\mathrm{id} \otimes \xi)(vM_3M_2M_1)$.

Therefore, to show (IV.1), it suffices to show

$$(\xi \otimes \mathrm{id})|_{W \otimes_R V \cap V \otimes_R W} = (\mathrm{id} \otimes \xi)|_{W \otimes_R V \cap V \otimes_R W}. \qquad \text{(IV.2)}$$

The elements $\xi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \xi$ are represented in $A^!_3 = \mathbf{E}^3(A)$ as $z^! \sum x_i$ and $\sum x_i z^!$, respectively, where $\{x_i\}$ is a basis for V^*. (The equality $A^!_3 = \mathbf{E}^3(A)$ holds because $c(A) = 2$ or $c(A) = 0$.) So, (IV.2) follows from the centrality of $z^!$. □

CHAPTER V

YONEDA ALGEBRAS FOR MONOMIAL CONNECTED-GRADED ALGEBRAS

V.1 Introduction

In this chapter, we explore the structure Yoneda algebra and coalgegra for connected-graded monomial algebras. The material in Section V.4 was first described in a paper accepted for publication [7], coauthored with Thomas Cassidy and Brad Shelton. A preprint of the paper is available on the arXiv preprint server.

Recall that a monomial algebra is an algebra which can be written in the form

$$A = \frac{\mathbb{K}\langle x_1, \ldots, x_n \rangle}{\langle r_1, \ldots, r_m \rangle} \tag{V.1}$$

where each relation r_i is a monomial in the generators x_j. A major motivation for studying monomial algebras is convenience: because these algebras are relatively uncomplicated, their Yoneda algebras and coalgebras are easier to understand than in the more general case. This allows us to gain intuition about what may or may not be true in the general case.

Recall the notation and results in Section II.4. We have $\mathcal{R} := \{r_1, \ldots, r_m\}$ and $\mathcal{M}$ the set of all monomials in the free algebra $\mathbb{K}\langle x_1, \ldots, x_n \rangle$. For $m \in \mathcal{M}$, we defined a set of minimal left annihilators $\mathfrak{A}_m$, which is used to obtain a bigraded vector space $V^\bullet \subset A_+^{\otimes\bullet}$. This yields a minimal projective resolution $A \otimes V^\bullet$ for ${}_A\mathbb{K}$, which is a subresolution of the bar resolution $\mathrm{Bar}^n(A)$.

V.2 A pictoral method for exhibiting the resolution $A \otimes V^\bullet$

Let A be a monomial algebra presented as in (V.1). In this section, we present a weighted, directed graph exhibiting the resolution $A \otimes V^\bullet$. In constructing such a graph, one will conduct the Cassidy–Shelton algorithm for monomial algebras (Theorem II.4.3). Recall the set $\mathfrak{S} = \bigcup_i \mathfrak{S}_i$ defined in Theorem II.4.3.

Definition V.2.1. The **Cassidy–Shelton algorithm graph** for A is a weighted, directed graph $G(A)$ defined as follows: The set of vertices of $G(A)$ is $\mathfrak{S}$. For each $m_1 \in \mathfrak{S}$ and $m_2 \in \mathfrak{A}_{m_1}$, there is a directed edge $m_2 \longleftarrow m_1$ in $G(A)$ if m_2m_1 is an essential relation and a directed edge $m_2 \leftsquigarrow m_1$ in $G(A)$ if m_2m_1 is a nonessential relation.

When presenting the graph $G(A)$, it is helpful to arrange the vertices so that elements of $\mathfrak{S}_0$ are on the far right, elements of $\mathfrak{S}_1$ are to the immediate left of the elements of $\mathfrak{S}_0$, and so on.

Note that the elements of S_i, which form a basis for V^i, correspond exactly to paths in $G(A)$ which begin at an element of $\mathfrak{S}_0$. (Paths are allowed to have both kinds of edges.) Thus, the global dimension of A will be the length of the longest path in $G(A)$, or infinite if $G(A)$ contains a loop. Also, this is an obvious corollary to Theorem II.4.3:

Theorem V.2.2. *A is $\mathcal{K}_2$ if and only if the only edges $m_2 \rightsquigarrow m_1$ have $m_2 \in \mathfrak{S}_0$.*

Thus the C–S algorithm graph presents a visual way to determine whether a

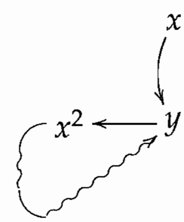

Figure V.1: The C–S algorithm graph for the algebra A in Example V.2.3.

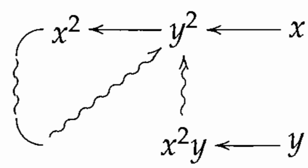

Figure V.2: The C–S algorithm graph for the algebra A in Example V.2.4.

connected-graded monomial algebra has the $\mathcal{K}_2$ property.

Example V.2.3. Let $A := \mathbb{K}\langle x, y\rangle \,/\, \langle x^2y, yx\rangle$. Then the C–S algorithm graph is shown in Figure V.1. We can see that A is $\mathcal{K}_2$ and has infinite global dimension.

Example V.2.4. Let $A := \mathbb{K}\langle x, y\rangle \,/\, \langle x^2y^2, y^2x\rangle$. Then the C–S algorithm graph is shown in Figure V.2. We can see that A has infinite global dimension and is not $\mathcal{K}_2$ (because of the arrow $x^2y \rightsquigarrow y^2$).

Example V.2.5. Let $A := \mathbb{K}\langle x, y, z\rangle \,/\, \langle xy^2, y^2z\rangle$. Then the C-S algorithm graph is shown in Figure V.3. We see that A is $\mathcal{K}_2$ (in fact, 3-Koszul) and has finite global dimension.

V.3 The Yoneda algebra of a monomial $\mathcal{K}_2$ algebra

Suppose A is a $\mathcal{K}_2$ monomial algebra presented as in (V.1). In this section, we will prove:

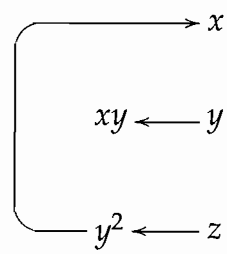

Figure V.3: The C–S algorithm graph for Example V.2.5.

Theorem V.3.1. $\mathbf{E}(A)$ *can be presented with only monomial and binomial relations.*

Recall in Theorem II.4.2, we created a canonical basis S^i for the vector spaces V^i made up of tensors of monomials. Put $S := \bigcup_i S^i$. We write V for V^1 (so, in fact, A is a factor of $\mathbb{T}(V)$). Let $\{\rho_s : s \in S_i\}$ be the dual basis of $(V^i)^*$ to S^i. We identify $(V^i)^* = \mathbf{E}^i(A)$. We may decompose

$$V^2 = (V^2 \cap (V \otimes V)) \oplus W \tag{V.2}$$

where a subset of S^2 is a basis for W.

Example V.3.2. Let $A := \mathbb{K}\langle x, y, z\rangle \,/\, \langle x^2, yx, xy^2, xyz\rangle$. In this case, $S_1 = \{x, y, z\}$ and $S_2 = \{x \otimes x, y \otimes x, xy \otimes y, xy \otimes z\}$. The decomposition (V.2) is

$$V^2 = (\mathbb{K}x \otimes x + \mathbb{K}y \otimes x) \oplus (\mathbb{K}xy \otimes y + \mathbb{K}xy \otimes z).$$

We will identify W^* as the subspace $\bigoplus_{s \in W} \rho_s \subset (V^2)^* = \mathbf{E}^2(A)$. Indeed, we can view W^* as the portion of $\mathbf{E}^2(A)$ *not* generated by $\mathbf{E}^1(A)$. As we have assumed A is $\mathcal{K}_2$, there is a surjective algebra homomorphism

$$\pi_E : \mathbb{T}(V^* \oplus W^*) \twoheadrightarrow \mathbf{E}(A).$$

In other words, $\mathbf{E}(A)$ is an algebra generated by $V^* \oplus W^*$.

Recall that $\mathcal{M}$ is the set of monomials in $\mathbb{K}\langle x_1, \ldots, x_n \rangle$. In the following,

$$\mu : \mathbb{T}(\mathbb{K}\langle x_1, \ldots, x_n \rangle) \to \mathbb{K}\langle x_1, \ldots, x_n \rangle$$

is the multiplication, which is really just removing the tensor product signs (for example, $\mu(x_1 \otimes x_2) = x_1 x_2$). The following is clear from the construction of $S \subseteq \mathbb{T}(\mathbb{K}\langle x_1, \ldots, x_n \rangle)$ (and minimality of $A \otimes V^\bullet$):

Lemma V.3.3. *The restriction $\mu|_S : S \to \mathcal{M}$ is injective.*

Let $\mathcal{M}_E$ be the set of monomials $\rho_{s_1}\rho_{s_2} \cdots \rho_{s_t} \in \mathbb{T}(V^* \oplus W^*)$. For $s \in S$, we set

$$M_s(A) := \{u \in \mathcal{M}_E : \pi_E(u)(s) \neq 0\}.$$

(This is the set of monomials in $\mathcal{M}_E$ which act nontrivially on s.) On the other hand, we define a map $\nu : V^* \oplus W^* \to \mathbb{K}\langle x_1, \ldots, x_n \rangle$ by setting $\nu(\rho_s) := \mu(s)$ for $s \in S^1 \cup S^2$ and extending linearly. Then we can extend ν to an algebra homomorphism

$$\nu : \mathbb{T}(V^* \oplus W^*) \to \mathbb{K}\langle x_1, \ldots, x_n \rangle.$$

We first show that $\mathcal{M}_E$ can be partitioned based on the elements of S upon which $\mathcal{M}_E$ acts nontrivially. This action is easy to describe.

Lemma V.3.4. *$M_s(A) \cap M_{s'}(A) = \emptyset$ for $s, s' \in S$ distinct.*

Proof. For any $s \in S$, $M_s = \nu^{-1}(\mu(s))$. Recall that μ is injective. □

The next lemma follows from our choice of bases.

Lemma V.3.5. *If $u \in M_s(A)$ then $\pi_E(u)(s) = 1$. Therefore, if $u, u' \in M_s(A)$, then $u - u' \in \ker \pi_E$.*

Lemma V.3.6. *If $u \in \mathcal{M}_E$, then $u \notin \bigsqcup_{s \in S} M_s(A)$ if and only if $u \in \ker \pi_E$.*

Proof. Since S is a basis for $V^\bullet$, it is clear that $u \notin \bigsqcup_{s \in S} M_s(A)$ implies $u \in \ker \pi_E$. On the other hand, by definition, $M_s(A) \cap \ker \pi_E = \emptyset$. □

Theorem V.3.1 follows immediately.

For the remainder of the section, we attempt to identify nonessential binomial generators of $\ker \pi_E$. We begin with this cancellation law:

Lemma V.3.7. *Suppose $u_1, u_2, u_2' \in \mathcal{M}_E$. If $u_2 u_1 - u_2' u_1 \in \ker \pi_E$ or $u_1 u_2 - u_1 u_2' \in \ker \pi_E$, then $u_2 = u_2'$.*

Proof. Suppose $u_2 u_1 - u_2' u_1 \in \ker \pi_E$. Note that $u_2 u_1, u_2' u_1 \in \mathcal{M}_s(A)$ for some $s = m_t \otimes \cdots \otimes m_1$. Then there exists $t > t' \geq 1$ so that $u_1 \in M_{m_{t'} \otimes \cdots \otimes m_1}(A)$ while $u_2, u_2' \in M_{m_t \otimes \cdots \otimes m_{t'+1}}(A)$, thus showing that $u_2 - u_2' \in \ker \pi_E$. The proof when $u_1 u_2 - u_1 u_2' \in \ker \pi_E$ is similar. □

By cohomological degree considerations, this is an immediate consequence.

Theorem V.3.8. $\mathbf{E}(A)$ *can be presented so that all binomial relations have the form*

$$\alpha_1 \cdots \alpha_\ell \overline{x_i} = \overline{x_j} \beta_1 \cdots \beta_\ell \text{ or}$$

$$\overline{x_i} \alpha_1 \cdots \alpha_\ell \overline{x_j} = \beta_1 \cdots \beta_{\ell+1},$$

where $\{\overline{x_1}, \ldots, \overline{x_n}\}$ is basis of V^* dual to the basis $\{x_1, \ldots, x_n\}$ of V, and $\alpha_t, \beta_t \in W^*$.

Example V.3.9. Recall the algebra A in Example V.2.5. We can compute $S_1 = \{x, y, z\}$, $S_2 = \{xy \otimes y, y^2 \otimes z\}$, and $S_3 = \{x \otimes y^2 \otimes z\}$. Let $\{\overline{x}, \overline{y}, \overline{z}\}$ be the dual basis elements to x, y, z, respectively. Let α, β be the dual basis elements to

V^{2n+1}:		V^{2n+2}:	
$y \otimes (x^2 \otimes y)^{\otimes n}$	$\overline{y}\alpha^n$	$(x^2 \otimes y)^{\otimes n+1}$	α^{n+1}
$(x^2 \otimes y)^{\otimes n} \otimes x$	$\alpha^n \overline{x}$	$y \otimes (x^2 \otimes y)^{\otimes n} \otimes x$	$\overline{y}\alpha^n \overline{x}$

Figure V.4: The basis for $V^\bullet$ for A as defined in Example V.3.11.

$xy \otimes y$ and $y^2 \otimes z$, respectively. Then we see that $\overline{x}\alpha$ and $\beta\overline{z}$ both act on V^3 via $x \otimes y^2 \otimes z \mapsto 1$. Hence, $\overline{x}\alpha - \beta\overline{z}$ is an essential binomial relation. Indeed, we can easily see

$$\mathbf{E}(A) = \frac{\mathbb{K}\langle \overline{x}, \overline{y}, \overline{z}, \alpha, \beta \rangle}{\left\langle \begin{array}{c} \overline{x}^2, \overline{xy}, \overline{xz}, \overline{y}^2, \overline{yz}, \overline{z}^2, \alpha^2, \alpha\beta, \beta^2\alpha, \beta^2, \\ \overline{y}\alpha, \overline{z}\alpha, \alpha\overline{x}, \alpha\overline{y}\alpha\overline{z}, \overline{x}\beta, \overline{y}\beta, \overline{z}\beta, \beta\overline{x}, \beta\overline{y}, \overline{x}\alpha - \beta\overline{z} \end{array} \right\rangle}.$$

We can decompose elements of S.

Definition V.3.10. Suppose $v = m_t \otimes m_{t-1} \otimes \cdots \otimes m_1 \in S^t$. Then we will call a term m_i in v **perforcedly linear** if $m_i m_{i-1}$ is not an essential relation. (Recall that we have assumed that A is $\mathcal{K}_2$. This property forces m_i to be linear.)

Example V.3.11. Consider the algebra A from Example V.2.4. Figure V.4 shows the bases of V^i and the sets $M_s(A)$. In any element, all but the last y term is perforcedly linear.

Theorem V.3.12. *Suppose that $v = m_t \otimes m_{t-1} \otimes \cdots \otimes m_1 \in S^t$, where m_i is perforcedly linear, and $u_1, u_2 \in \mathcal{M}_E$ where $u_1(v) = u_2(v) = 1$ (so $u_1 - u_2 \in \ker \pi_E$). Let $v' = m_{i-1} \otimes \cdots \otimes m_1$ Then there exists $u_j', u_j'' \in \mathcal{M}_E$ where $u_j = u_j'' u_j'$ and $u_j'(v') = 1$. In other words, essential binomial relations can be obtained by considering only elements in S^t without perforcedly linear terms.*

Proof. Because m_i is perforecedly linear, there is no element $\rho \in V^2$ with

$\rho(m_i \otimes m_{i-1}) \neq 0$. Thus, $u_j = u_j'' u_j'$ where $u_j'' \in M_{m_t \otimes \cdots \otimes m_i}(A)$ and $u_j' \in M_{m_{i-1} \otimes \cdots \otimes m_1}(A)$. The result follows from Lemma V.3.7. □

Example V.3.13. Consider the algebra A from Example V.2.4 and V.3.11. We can see by inspection that

$$\mathbf{E}(A) = \frac{\mathbb{K}\left\langle \overline{x}, \overline{y}, \alpha \right\rangle}{\left\langle \overline{x}^2, \overline{xy}, \overline{y}^2, \overline{x}\alpha, \alpha\overline{y} \right\rangle}.$$

Indeed, the only basis elements without perforcedly linear terms are $x^2 \otimes y \otimes x$ and $x^2 \otimes y$. These yield no binomial relations, and thus $\mathbf{E}(A)$ has no binomial relations.

Note that we can regrade $\mathbf{E}(A)$ so that $\overline{x}, \overline{y}, \alpha$ each have degree one. Under this grading, $\mathbf{E}(A)$ is a monomial quadratic algebra, and hence is Koszul. So, in the original cohomological grading, $\mathbf{E}(A)$ is $\mathcal{K}_1$, by Lemma II.3.2.

V.4 The Yoneda algebra of a $\mathcal{K}_2$ algebra need not be $\mathcal{K}_2$

In this section, we exhibit a monomial $\mathcal{K}_2$ algebra A whose Yoneda algebra $\mathbf{E}(A)$ is not $\mathcal{K}_2$. This example was first described in a paper accepted for publication [7], coauthored with Thomas Cassidy and Brad Shelton. A preprint of the paper is available on the arXiv preprint server. This example illustrates that a delayed version of Koszul duality (motivated by Theorem II.3.10) may be difficult to find for $\mathcal{K}_2$ algebras.

Theorem V.4.1. *Let*

$$A := \frac{\mathbb{K}\left\langle m,n,p,q,r,s,t,u,v,w,x,y,z \right\rangle}{\left\langle mn^2p, n^2pqr, npqrs, pqrst, stu, tuvwx, uvwxy, vwxy^2, xy^2z \right\rangle}. \tag{V.3}$$

A is a $\mathcal{K}_2$ algebra, but $\mathbf{E}(A)$ is not $\mathcal{K}_2$.

Proof. First, we will exhibit the minimal projective resolution $A \otimes V^\bullet$ using Theorem II.4.2. The C–S algorithm graph is shown in Figure V.5.

Figure V.6 is a table showing the basis elements s for $V^\bullet$ as well as the sets $M_s(A)$, which can be read off the C–S algorithm graph. From this table, one can see that A is $\mathcal{K}_2$ as well as all the essential binomial relations. We see that A has a global dimension of 6.

Let $X := \{\overline{m}, \overline{n}, \overline{p}, \overline{q}, \overline{r}, \overline{s}, \overline{t}, \overline{u}, \overline{v}, \overline{w}, \overline{x}, \overline{y}, \overline{z}\}$, and let $R := \{\alpha, \beta, \gamma, \delta, \varepsilon, \eta, \tau, \nu, \rho\}$. Then by careful inspection of Figure V.6, we see that $\mathbf{E}(A)$ is the quotient of $\mathbb{K}\langle X, R\rangle$ by the ideal generated by the following 474 essential relations:

1. All words ab for $a, b \in X$.

2. All words ag and ga for $a \in X, g \in R$ **except** those appearing in the basis for V^3.

3. All words gg' for $g, g' \in R$ **except** those appearing in the basis for V^4.

4. The binomial relations coming from V^3, which are:

$$\overline{n}\gamma - \beta\overline{s}, \overline{n}\delta - \gamma\overline{t}, \overline{t}\nu - \eta\overline{y}, \overline{u}\eta - \tau\overline{y}.$$

5. $\overline{m}\beta\varepsilon$ and $\varepsilon\nu\overline{z}$.

We assign a degree of one to each element of X and R, making $\mathbf{E}(A)$ into a connected-graded algebra with 472 quadratic and 2 cubic relations! Under this grading, using Figure V.6, we see that the Hilbert series of $\mathbf{E}(A)$ is $1 + 22t + 12t^2 + 4t^3$.

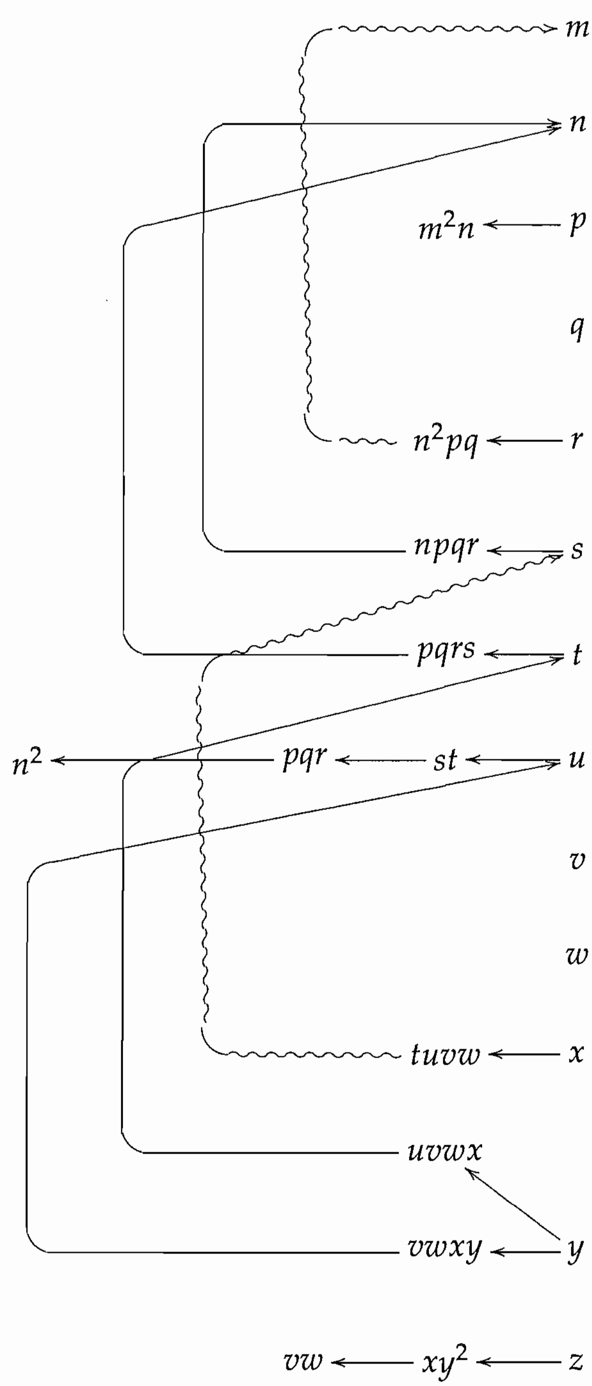

Figure V.5: The C–S algorithm graph for A as defined in (V.3).

V^1:

m	$\overline{m}$
n	$\overline{n}$
p	$\overline{p}$
q	$\overline{q}$
r	$\overline{r}$
s	$\overline{s}$
t	$\overline{t}$
u	$\overline{u}$
v	$\overline{v}$
w	$\overline{w}$
x	$\overline{x}$
y	$\overline{y}$
z	$\overline{z}$

V^2:

$mn^2 \otimes p$	α
$n^2pq \otimes r$	β
$npqr \otimes s$	γ
$pqrs \otimes t$	δ
$st \otimes u$	ε
$tuvw \otimes x$	η
$uvwx \otimes y$	τ
$vwxy \otimes y$	ν
$xy^2 \otimes z$	ρ

V^3:

$m \otimes n^2pq \otimes r$	$\overline{m}\beta$
$n \otimes npqr \otimes s$	$\overline{n}\gamma = \beta\overline{s}$
$n \otimes pqrs \otimes t$	$\overline{n}\delta = \gamma\overline{t}$
$pqr \otimes st \otimes u$	$\delta\overline{u}$
$s \otimes tuvw \otimes x$	$\overline{s}\eta$
$t \otimes uvwx \otimes y$	$\overline{t}\nu = \eta\overline{y}$
$u \otimes vwxy \otimes y$	$\overline{u}\nu = \tau\overline{y}$
$vw \otimes xy^2 \otimes z$	$\nu\overline{z}$

V^4:

$n^2 \otimes pqr \otimes st \otimes u$	$\beta\varepsilon$
$npqr \otimes s \otimes tuvw \otimes x$	$\gamma\eta$
$pqrs \otimes t \otimes uvwx \otimes y$	$\delta\tau$
$st \otimes u \otimes vwxy \otimes y$	$\varepsilon\nu$

V^5:

$n \otimes npqr \otimes s \otimes tuvw \otimes x$	$\overline{n}\gamma\eta = \beta\overline{s}\eta$
$n \otimes pqrs \otimes t \otimes uvwx \otimes y$	$\overline{n}\delta\tau = \gamma\eta\overline{y} = \gamma\overline{t}\tau$
$pqr \otimes st \otimes u \otimes vwxy \otimes y$	$\delta\overline{u}\nu = \delta\tau\overline{y}$

V^6:

$n^2 \otimes pqr \otimes st \otimes u \otimes vwxy \otimes y$	$\beta\varepsilon\nu$

The spaces V^n are zero for $n \geq 7$. The second column lists elements of the sets $M_s(A)$.

Figure V.6: The basis for $V^\bullet$ for A as defined in (V.3).

Now, consider the bar resolution

$$\mathbf{E}(A) \otimes \mathbf{E}(A)_+^{\otimes \bullet} \to \mathbb{K} \to 0.$$

Truncate and apply the functor $\mathbf{E}(A) \otimes_{\mathbf{E}(A)} -$ to get a complex

$$\mathbf{E}(A)_+^{\otimes \bullet} \to 0$$

whose homology is the Yoneda coalgebra $\mathbf{T}(\mathbf{E}(A))$. We will show that $\mathbf{E}(A)$ is not $\mathcal{K}_2$ by showing that the comultiplication

$$\Delta : \mathbf{T}^3(\mathbf{E}(A)) \to \mathbf{T}^2(\mathbf{E}(A)) \otimes \mathbf{T}^1(A) + \mathbf{T}^1(\mathbf{E}(A)) \otimes \mathbf{T}^2(\mathbf{E}(A))$$

is not injective and appealing to Theorem II.3.11.

Let $\xi := \overline{m}\beta \otimes \varepsilon\nu \otimes \overline{z} \in \mathbf{E}(A)_+ \otimes \mathbf{E}(A)_+ \otimes \mathbf{E}(A)_+$. Note that

$$\partial(\xi) = \overline{m}\beta\varepsilon\nu \otimes \overline{z} - \overline{m}\beta \otimes \varepsilon\nu\overline{z} = 0,$$

so $\xi + \operatorname{im}\partial \in \mathbf{T}^3(\mathbf{E}(A))$. We wish to show that ξ represents a nontrivial homology class and that $\Delta(\xi + \operatorname{im}\partial) = 0$.

First, note that none of the binomial relations of $\mathbf{E}(A)$ involve $\varepsilon\nu$, $\overline{m}\beta$ or $\overline{z}$. We compute

$$\partial(\overline{m} \otimes \beta \otimes \varepsilon\nu \otimes \overline{z}) = \overline{m}\beta \otimes \varepsilon\nu \otimes \overline{z} - \overline{m} \otimes \beta\varepsilon\nu \otimes \overline{z}$$

and

$$\partial(\overline{m}\beta \otimes \varepsilon \otimes \nu \otimes \overline{z}) = -\overline{m}\beta \otimes \varepsilon\nu \otimes \overline{z} + \overline{m}\beta \otimes \varepsilon \otimes \nu\overline{z}.$$

Thus, $\xi \notin \operatorname{im}\partial$.

Now, recall that Δ is induced by

$$\Delta : \mathbf{E}(A)_+^{\otimes 3} \to \mathbf{E}(A)_+^{\otimes 2} \otimes \mathbf{E}(A)_+ + \mathbf{E}(A)_+ \otimes \mathbf{E}(A)_+^{\otimes 2}.$$

We have

$$\Delta(\xi) = (\overline{m}\beta \otimes \varepsilon\nu) \otimes \overline{z} + \overline{m}\beta \otimes (\varepsilon\nu \otimes \overline{z}).$$

However,

$$\partial(-\overline{m}\beta \otimes \varepsilon \otimes \nu) = \overline{m}\beta \otimes \varepsilon\nu,$$

while

$$\partial(\overline{m} \otimes \beta) = \overline{m}\beta.$$

Hence,

$$\begin{aligned}\Delta(\xi + \operatorname{im}\partial) = {}& (\overline{m}\beta \otimes \varepsilon\nu + \operatorname{im}\partial) \otimes (\overline{z} + \operatorname{im}\partial) \\ & + (\overline{m}\beta + \operatorname{im}\partial) \otimes (\varepsilon\nu \otimes \overline{z} + \operatorname{im}\partial) = 0,\end{aligned}$$

as desired. □

CHAPTER VI

A GENERALIZATION OF THE THEORY OF PBW ALGEBRAS

VI.1 Introduction

Our goal in this chapter is to generalize Theorem II.5.4. Recall that a connected-graded $\mathbb{K}$-algebra $A = \mathbb{T}(V)/I$ is a Poincaré–Birkhoff–Witt algebra if there exists a ordered basis for V such that the associated graded algebra gr A is a quadratic algebra. Such algebras A are Koszul. In this chapter, we extend the theory of PBW algebras to algebras that have relations in more than one degree. Some of the material in this chapter appeared in an article published in the *Journal of Algebra* [17], and is reproduced with permission of Elsevier B.V. A preprint of the paper is available on the arXiv preprint server.

At first, we will be considerably more general than just considering the connected-graded case. Let A be a $\mathbb{K}$-augmented algebra. Let $\mathcal{M}$ be an ordered monoid with identity element e. Suppose that there exists a poset isomorphism $p : \mathcal{M} \hookrightarrow \mathbb{N}$—however, we do *not* assume that p is a monoid homomorphism. We do require $p(e) = 0$. We will let $s(\alpha, r) := p^{-1}(p(\alpha) + r)$, the element that comes r places later in the poset. We assume $\mathcal{M}$ filters A so that

1. $\bigcup_{\alpha} F_{\alpha}A = A$;
2. $F_{\alpha}A = \mathbb{K} \oplus F_{\alpha}A_{+}$, where $F_{\alpha}A_{+} := F_{\alpha}A \cap A_{+}$;

3. $F_e A = \mathbb{K}$ and $F_\alpha A_+ \neq 0$ when $\alpha > e$; and

4. $\dim F_\alpha / F_{s(\alpha,-1)} A < \infty$ for all $\alpha > e$.

With these properties, we can put a nice filtration on $A_+^\bullet$ and $\mathrm{Cob}(A)$:

Definition VI.1.1. We filter $A_+^\bullet$ by setting

$$F_\alpha A_+^{\otimes n} := \sum_{\substack{\alpha_1 \cdots \alpha_n < \alpha \\ \alpha_i > e \forall i}} F_{\alpha_1} A_+ \otimes \cdots \otimes F_{\alpha_n} A_+.$$

We put a decreasing filtration on $\mathrm{Cob}(A)$ by setting

$$F_\alpha \mathrm{Cob}^n(A) := \left\{ f : A_+^{\otimes n} \to \mathbb{K} \mid F_{s(\alpha,-1)} A_+^{\otimes n} \subset \ker f \right\}.$$

This induces a filtration $F_\alpha \mathbf{E}^n(A)$ and associated graded algebra $\mathrm{gr}^F \mathbf{E}(A)$. Also, the filtration on A yields the associated graded algebra $\mathrm{gr}^F A$ (graded by $\mathcal{M}$); we set $(\mathrm{gr}^F A)_+ := \bigoplus_{\alpha > e} (\mathrm{gr}^F A)_\alpha$. The algebra $\mathrm{gr}^F A$ is augmented by $\mathrm{gr}^F A = \mathbb{K} \oplus (\mathrm{gr}^F A)_+$. The following is the cornerstone to generalizing the PBW theory, but is also interesting in its own right:

Theorem VI.1.2. *There is a bigraded (with respect to the cohomological and $\mathcal{M}$ gradings) algebra monomorphism*

$$\Lambda : \mathrm{gr}^F \mathbf{E}(A) \hookrightarrow \mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A).$$

An important goal motivating the work in this chapter is a technique for transferring the $\mathcal{K}_2$ property from $\mathrm{gr}^F A$ to A. Recall that $\mathrm{gr}^F A$ is monomial. As seen in Section II.4, it is easy to determine when $\mathrm{gr}^F A$ is $\mathcal{K}_2$.

Theorem VI.1.3. *If* $\mathbf{E}^1_{\mathrm{Gr}}(\mathrm{gr}^F A)$ *and* $\mathbf{E}^2_{\mathrm{Gr}}(\mathrm{gr}^F A)$ *are finite dimensional and generate* $\mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$*, and* Λ^1 *and* Λ^2 *are surjective, then* A *is* $\mathcal{K}_2$*.*

We can then prove our generalization of Theorem II.5.4. Let $A = \mathbb{T}(V)/I$ be a connected-graded algebra and fix an ordered basis for V. The filtration F will be induced by the degree-lexicographical order on monomials $\mathcal{M}$ in $\mathbb{T}(V)$.

Theorem VI.1.4. *If* I *has an essential Gröbner basis and* $\mathrm{gr}^F A$ *is* $\mathcal{K}_2$*, then* A *is* $\mathcal{K}_2$ *as well.*

VI.2 The bigraded embedding of $\mathrm{gr}^F \mathbf{E}(A)$ in $\mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$

In this section, we prove Theorem VI.1.2. We will use A to denote an augmented algebra filtered by the monoid $\mathcal{M}$ as specified above. (Note that $\mathcal{M}$ need not be commutative.)

Throughout, we will denote $\mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) =: V^{\vee}$. We first relate $\mathrm{Cob}^{\bullet}_{\mathrm{Gr}}(\mathrm{gr}^F A)$ to the cobar complex of A.

Proposition VI.2.1. *There is a differential-graded algebra isomorphism*

$$\mathrm{gr}^F \mathrm{Cob}^{\bullet}(A) \simeq \mathrm{Cob}^{\bullet}_{\mathrm{Gr}}(\mathrm{gr}^F A).$$

The proof of Proposition VI.2.1 will follow after two lemmas. Let us fix a $\mathbb{K}$-basis $\mathcal{R} = \coprod_{\alpha \in \mathcal{M}} \mathcal{R}_\alpha$ for A such that:

1. $\bigcup_{\beta \leq \alpha} \mathcal{R}_\beta$ is a basis for $F_\alpha A$.

2. $\mathcal{R}_\alpha \subset F_\alpha A_+$ for $\alpha > e$.

Then $\{r + F_{s(\alpha,-1)} A_+ : r \in \mathcal{R}_\alpha\}$ is a basis for $F_\alpha A_+ / F_{s(\alpha,-1)} A_+$.

For readability, we set $(((\mathrm{gr}^F A)_+)^{\otimes n})_\alpha =: (\mathrm{gr}^F A)^{\otimes n}_{+,\alpha}$.

Lemma VI.2.2. *The map*

$$\varphi : (\mathrm{gr}^F A)^{\otimes n}_{+,\alpha} \to \frac{F_\alpha A_+^{\otimes n}}{F_{s(\alpha,-1)} A_+^{\otimes n}}$$

defined by

$$\varphi((a_1 + F_{s(\alpha_1,-1)}A) \otimes \cdots \otimes (a_n + F_{s(\alpha_n,-1)}A)) := a_1 \otimes \cdots \otimes a_n + F_{s(\alpha,-1)} A_+^{\otimes n}$$

is a chain isomorphism.

Proof. First, if $a_i - a_i' \in F_{s(\alpha_i,-1)}A$ for some $1 \leq i \leq n$ and $\alpha_1 \cdots \alpha_n = \alpha$, then

$$a_1 \otimes \cdots \otimes (a_i - a_i') \otimes \cdots \otimes a_n \in F_{s(\alpha,-1)} A_+^{\otimes n}.$$

Hence, φ is well-defined.

To show that φ is a chain map, suppose $a_i \in F_{\alpha_i} A$ and $\alpha_1 \cdots \alpha_n = \alpha$. We compute

$$\begin{aligned}
(d \circ \varphi) & \left(\left(a_1 + F_{s(\alpha_1,-1)} A \right) \otimes \cdots \left(a_n + F_{s(\alpha_n,-1)} A \right) \right) \\
&= d \left(a_1 \otimes \cdots \otimes a_n + F_{s(\alpha,-1)} A_+^{\otimes n} \right) \\
&= \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + F_{s(\alpha,-1)} A_+^{\otimes n} \\
&= \varphi\Big(\sum_{i=1}^{n-1} (-1)^i \left(a_1 + F_{s(\alpha_1,-1)} A \right) \otimes \cdots \\
&\qquad \otimes \left(a_i a_{i+1} + F_{s(\alpha_i \alpha_{i+1},-1)} A \right) \otimes \cdots \otimes \left(a_n + F_{s(\alpha_n,-1)} A \right) \Big) \\
&= (\varphi \circ d) \left(\left(a_1 + F_{s(\alpha_1,-1)} A \right) \otimes \cdots \otimes \left(a_n + F_{s(\alpha_n,-1)} A \right) \right).
\end{aligned}$$

Now, to show that φ is an isomorphism, note that the set

$$\mathcal{B}_1 := \left\{ \left(a_1 + F_{s(\alpha_1,-1)}A\right) \otimes \cdots \otimes \left(a_n + F_{s(\alpha_n,-1)}A\right) \,\middle|\, a_i \in \mathcal{R}_{\alpha_i}, \alpha_i \neq e \text{ for all } i, \alpha_1 \cdots \alpha_n = \alpha \right\}$$

is a basis for $(\mathrm{gr}^F A)^{\otimes n}_{+,\alpha}$, while

$$\mathcal{B}_2 := \left\{ a_1 \otimes \cdots \otimes a_n + F_{s(\alpha,-1)}A_+^{\otimes n} \,\middle|\, a_i \in \mathcal{R}_{\alpha_i}, \alpha_i \neq e \text{ for all } i, \alpha_1 \cdots \alpha_n = \alpha \right\}$$

is a basis for $F_\alpha A_+^{\otimes n}/F_{s(\alpha,-1)}A_+^{\otimes n}$. Since φ gives a bijection between these bases, φ is an isomorphism. □

Now, because of Condition (4) on the filtration, we have a chain isomorphism

$$\varphi^\vee : \left(\frac{F_\alpha A_+^{\otimes n}}{F_{s(\alpha,-1)}A_+^{\otimes n}} \right)^\vee \xrightarrow{\sim} \mathrm{Cob}^{n,\alpha}_{\mathrm{Gr}}(\mathrm{gr}^F A).$$

The restriction map

$$(A_+^{\otimes n})^\vee \to (F_\alpha A_+^{\otimes n})^\vee$$

induces an injective map

$$\rho : \frac{F_\alpha \mathrm{Cob}^n(A)}{F_{s(\alpha,1)} \mathrm{Cob}^n(A)} \hookrightarrow \left(\frac{F_\alpha A_+^{\otimes n}}{F_{s(\alpha,-1)}A_+^{\otimes n}} \right)^\vee .$$

The following is clear and brings us a long way towards Proposition VI.2.1:

Lemma VI.2.3. *The map ρ is a chain isomorphism.*

We now know that

$$\varphi^\vee \circ \rho : \operatorname{gr}^F \operatorname{Cob}^\bullet(A) \to \operatorname{Cob}^\bullet_{\mathrm{Gr}}(\operatorname{gr}^F A)$$

is a chain isomorphism, graded by $\mathcal{M}$.

Proof of Proposition VI.2.1. It suffices to show that $\varphi^\vee \circ \rho$ is a differential-graded algebra homomorphism. Let $f \in F_\alpha \operatorname{Cob}^n(A)$, $g \in F_\beta \operatorname{Cob}^m(A)$, $a_i \in F_{\alpha_i} A$, $b_i \in F_{\beta_i} A$, $\alpha_1 \cdots \alpha_n = \alpha$, and $\beta_1 \cdots \beta_m = \beta$.

Then

$$\begin{aligned}
&(\varphi^\vee \circ \rho)((f + F_{s(\alpha,1)} \operatorname{Cob}^n(A)) \smile (g + F_{s(\beta,1)} \operatorname{Cob}^m(A))\\
&\qquad \Big((a_1 + F_{s(\alpha_1,-1)} A) \otimes \cdots \otimes (a_n + F_{s(\alpha_n,-1)} A)\\
&\qquad \otimes (b_1 + F_{s(\beta_1,-1)} A) \otimes \cdots \otimes (b_m + F_{s(\beta_m,-1)} A)\Big)\\
&\quad = \rho((f + F_{s(\alpha,1)} \operatorname{Cob}^n(A)) \smile (g + F_{s(\beta,1)} \operatorname{Cob}^m(A)))\\
&\qquad ((a_1 \otimes \cdots \otimes a_n + F_{s(\alpha,-1)} A_+^{\otimes n}) \otimes (b_1 \otimes \cdots \otimes b_m + F_{s(\beta,-1)} A_+^{\otimes m}))\\
&\quad = \rho(f \smile g + F_{s(\alpha\beta,1)} \operatorname{Cob}^{n+m}(A))\\
&\qquad (a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m + F_{s(\alpha\beta,-1)} A_+^{\otimes n+m})\\
&\quad = f(a_1 \otimes \cdots \otimes a_n) g(b_1 \otimes \cdots \otimes b_m).
\end{aligned}$$

Likewise,

$$\begin{aligned}
&\Big((\varphi^\vee \circ \rho)(f + F_{s(\alpha,1)} \operatorname{Cob}^n(A)) \smile (\varphi^\vee \circ \rho)(g + F_{s(\beta,1)} \operatorname{Cob}^m(A))\Big) \\
&\qquad\qquad \Big((a_1 + F_{s(\alpha_1,-1)} A) \otimes \cdots \otimes (a_n + F_{s(\alpha_n,-1)} A) \\
&\qquad\qquad \otimes (b_1 + F_{s(\beta_1,-1)} A) \otimes \cdots \otimes (b_m + F_{s(\beta_m,-1)} A)\Big) \\
&\quad = \rho(f + F_{s(\alpha,1)} \operatorname{Cob}^n(A))(a_1 \otimes \cdots \otimes a_n + F_{s(\alpha,-1)} A_+^{\otimes n}) \\
&\qquad\qquad \cdot \rho(g + F_{s(\beta,1)} \operatorname{Cob}^m(A))(b_1 \otimes \cdots \otimes b_m + F_{s(\beta,-1)} A_+^{\otimes m}) \\
&\quad = f(a_1 \otimes \cdots \otimes a_n) g(b_1 \otimes \cdots \otimes b_m),
\end{aligned}$$

as desired. □

Recall that we give $\mathbf{E}(A)$ a filtration $F_\alpha \mathbf{E}(A)$ induced by the filtration $F_\alpha \operatorname{Cob}^\bullet(A)$.

Definition VI.2.4. Define a surjective map

$$\eta_\infty : F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial \twoheadrightarrow (\operatorname{gr}^F \mathbf{E}(A))^{n,\alpha}$$

to be the composition

$$F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial \twoheadrightarrow F_\alpha \mathbf{E}^n(A) \twoheadrightarrow \frac{F_\alpha \mathbf{E}^n(A)}{F_{s(\alpha,1)} \mathbf{E}^n(A)}.$$

Define a map $\eta_1 : F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial \to \mathbf{E}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr} A)$ to be the composition

$$
\begin{aligned}
F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial &\to \frac{F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial + F_{s(\alpha,1)} \operatorname{Cob}^n(A)}{F_{s(\alpha,1)} \operatorname{Cob}^n(A)} \\
&\to \frac{F_\alpha \operatorname{Cob}^n(A)}{F_{s(\alpha,1)} \operatorname{Cob}^n(A)} \cap \ker(\operatorname{gr}^F \partial) \\
&\xrightarrow{\varphi^\vee \circ \rho} \operatorname{Cob}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr}^F A) \cap \ker \partial \\
&\twoheadrightarrow \mathbf{E}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr}^F A).
\end{aligned}
$$

(Recall that $\varphi^\vee \circ \rho : \operatorname{gr}^F \operatorname{Cob}^\bullet(A) \to \operatorname{Cob}^\bullet_{\mathrm{Gr}}(\operatorname{gr}^F A)$ is a differential-graded algebra isomorphism by Proposition VI.2.1. Here, we restrict $\varphi^\vee \circ \rho$ to $\frac{F_\alpha \operatorname{Cob}^n(A)}{F_{s(\alpha,1)} \operatorname{Cob}^n(A)} \cap \ker(\operatorname{gr}^F \partial)$.)

The maps η_1 and η_∞ appear in the construction of a spectral sequence obtained from the filtration F on $\operatorname{Cob}(A)$. See, for example, [16, Theorem 2.6] and its proof. Even though we will not need this spectral sequence to prove our results, it is nonetheless interesting and is the topic of Section VI.4.

Lemma VI.2.5. $\ker \eta_1 = \ker \eta_\infty$.

Proof. Suppose $f \in \ker \eta_1$, meaning

$$
(\varphi^\vee \circ \rho)(f + F_{s(\alpha,1)} \operatorname{Cob}^n(A)) \in \operatorname{Cob}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr}^F A) \cap \operatorname{im} \partial.
$$

As $\varphi^\vee \circ \rho$ is a differential-graded algebra isomorphism,

$$
f + F_{s(\alpha,1)} \operatorname{Cob}^n(A) \in \frac{F_\alpha \operatorname{Cob}^n(A)}{F_{s(\alpha,1)} \operatorname{Cob}^n(A)} \cap \operatorname{im} \partial;
$$

that is, there exists $g \in F_\alpha \operatorname{Cob}^{n-1}(A)$ such that

$$\partial(g) + F_{s(\alpha,1)} \operatorname{Cob}^n(A) = f + F_{s(\alpha,1)} \operatorname{Cob}^n(A).$$

However, $f - \partial(g) + \operatorname{im}\partial \in F_{s(\alpha,1)}\mathbf{E}^n(A)$. Thus, $\eta_\infty(f) = 0$.

Now, suppose $f \in \ker \eta_\infty$, meaning $f + \operatorname{im}\partial \in F_{s(\alpha,1)}\mathbf{E}^n(A)$. So, $f + \partial(g) \in F_{s(\alpha,1)} \operatorname{Cob}^n(A)$ for some $g \in \operatorname{Cob}^n(A)$. Since $f, f + \partial(g) \in F_\alpha \operatorname{Cob}^n(A)$, $\partial(g) \in F_\alpha \operatorname{Cob}^n(A)$ as well, and

$$f + F_{s(\alpha,1)} \operatorname{Cob}^n(A) = \partial(g) + F_{s(\alpha,1)} \operatorname{Cob}^n(A).$$

Thus,

$$(\varphi^\vee \circ \rho)(f + F_{s(\alpha,1)} \operatorname{Cob}^n(A)) \in \operatorname{Cob}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr}^F A) \cap \operatorname{im}\partial$$

and so $\eta_1(f) = 0$. □

Definition VI.2.6. Since η_∞ is surjective, Lemma VI.2.5 tells us we may define a unique injective map $\Lambda^{n,\alpha}$ such that the diagram

$$\begin{array}{ccc} & F_\alpha \operatorname{Cob}^n(A) \cap \ker\partial & \\ {\scriptstyle \eta_\infty} \swarrow & & \searrow {\scriptstyle \eta_1} \\ (\operatorname{gr}^F \mathbf{E}(A))^{n,\alpha} & \xrightarrow{\quad \Lambda^{n,\alpha} \quad} & \mathbf{E}^{n,\alpha}_{\mathrm{Gr}}(\operatorname{gr}^F A) \end{array}$$

commutes. Set $\Lambda := \bigoplus_{n,\alpha} \Lambda^{n,\alpha}$.

We may now prove Theorem VI.1.2, which we restate:

Theorem VI.2.7. *The map*

$$\Lambda : \operatorname{gr}^F \mathbf{E}(A) \hookrightarrow \mathbf{E}_{\mathrm{Gr}}(\operatorname{gr}^F A)$$

is an algebra monomorphism.

Proof. It remains only to prove Λ is an algebra homomorphism. Let $f \in (\operatorname{gr}^F \mathbf{E}(A))^{n,\alpha}$ and $g \in (\operatorname{gr}^F \mathbf{E}(A))^{m,\beta}$. Choose preimages (under η_1)

$$\tilde{f} \in F_\alpha \operatorname{Cob}^n(A) \cap \ker \partial$$

and

$$\tilde{g} \in F_\beta \operatorname{Cob}^m(A) \cap \ker \partial$$

for f and g, respectively. We have

$$\tilde{f} \otimes \tilde{g} \in F_{\alpha\beta} \operatorname{Cob}^{n+m}(A) \cap \ker \partial.$$

Now, we compute

$$\begin{aligned}
\eta_\infty(\tilde{f} \otimes \tilde{g}) &= ((\tilde{f} \otimes \tilde{g}) + \operatorname{im} \partial) + F_{s(\alpha\beta,1)} \mathbf{E}^{n+m}(A) \\
&= ((\tilde{f} + \operatorname{im} \partial) \smile (\tilde{g} + \operatorname{im} \partial)) + F_{s(\alpha\beta,1)} \mathbf{E}^{n+m}(A) \\
&= ((\tilde{f} + \operatorname{im} \partial) + F_{s(\alpha,1)} \mathbf{E}^{n}(A)) \smile ((\tilde{g} + \operatorname{im} \partial) + F_{s(\beta,1)} \mathbf{E}^{m}(A)) \\
&= \eta_\infty(\tilde{f}) \smile \eta_\infty(\tilde{g}) \\
&= f \smile g.
\end{aligned}$$

□

Before proving Theorem VI.1.3, we prove a general fact about filtered algebras:

Lemma VI.2.8. *Let $R = \bigoplus_i R_i$ be a graded algebra with a decreasing filtration F by a totally-ordered monoid $\mathcal{M}$. Put $F_\alpha R_i = F_\alpha R \cap R_i$ and assume $F_\alpha R = \bigoplus_i F_\alpha R_i$ for all i. Let R' be the subalgebra of R generated by $R_1, \ldots, R_m$. Suppose, for each i, $F_\alpha R_i \subset R'$ for α sufficiently large. If $(\operatorname{gr}^F R)_1, \ldots, (\operatorname{gr}^F R)_m$ generate $\operatorname{gr}^F R$, then $R_1, \ldots, R_m$ generate R.*

Proof. Suppose that $F_{s(\alpha,1)} R_i \subset R'$. Let $a \in F_\alpha R_i \setminus F_{s(\alpha,1)} R_i$. As $\operatorname{gr}^F R$ is generated by $(\operatorname{gr}^F R)_1, \ldots, (\operatorname{gr}^F R)_n$, there exists $a' \in R' \cap F_\alpha R_i$ such that

$$a - a' \in F_{s(\alpha,1)} R_i \subset R'.$$

As $a \in R'$, we know $a \in R'$. Thus, $F_\alpha R \subset R'$. By (decreasing) induction on α, $R = R'$. □

Lemma VI.2.9. *If* $\dim \operatorname{gr}^F \mathbf{E}^n(A) < \infty$ *then* $F_\alpha \mathbf{E}^n(A) = 0$ *for some* α*, and consequently,* $\dim \operatorname{gr}^F \mathbf{E}^n(A) = \dim \mathbf{E}^n(A)$.

Proof. Let $\{\xi + F_{s(\alpha_i,1)} \mathbf{E}^n(A) : 1 \le i \le m\}$ be a basis for $\operatorname{gr}^F \mathbf{E}^n(A)$, and choose $\alpha > \alpha_i$ for $1 \le i \le m$. For $\beta \ge \alpha$, $F_\beta \mathbf{E}^n(A) / F_{s(\beta,1)} \mathbf{E}^n(A) = 0$, meaning $F_\beta \mathbf{E}^n(A) = F_\alpha \mathbf{E}^n(A)$.

Now, choose any $\xi \in F_\alpha \mathbf{E}^n(A)$. For $\beta \ge \alpha$, there exists $f_\beta \in F_\beta \operatorname{Cob}^n(A)$ and $f'_\beta \in \operatorname{Cob}^{n-1}(A)$ such that $f_\beta + \operatorname{im} d = \xi$ and $f_\beta = f_{s(\beta,1)} + d(f'_\beta)$.

Then, for $\beta \geq \alpha$,

$$\begin{aligned} f_\alpha &= f_{s(\alpha,1)} + d(f'_\alpha) \\ &= f_{s(\alpha,2)} + d(f'_{s(\alpha,1)}) + d(f'_\alpha) \\ &\vdots \\ &= f_\beta + \sum_{\alpha \leq \gamma < \beta} d(f'_\gamma). \end{aligned}$$

So, for $x \in F_\beta A_+^{\otimes n}$ and $\gamma > \beta$,

$$\begin{aligned} f_\alpha(x) &= f_\gamma(x) + \sum_{\alpha \leq \delta < \gamma} d(f'_\gamma)(x) \\ &= \sum_{\alpha \leq \delta < \gamma} (f'_\delta \circ \partial)(x). \end{aligned}$$

Thus, there exists $f' : A_+^{\otimes n-1} \to \mathbb{K}$ such that $f_\alpha = f' \circ \partial$. Therefore, $\xi = 0$. □

We may now prove Theorem VI.1.3, which we restate:

Theorem VI.2.10. *If $\mathbf{E}^1_{\mathrm{Gr}}(\mathrm{gr}^F A)$ and $\mathbf{E}^2_{\mathrm{Gr}}(\mathrm{gr}^F A)$ are finite dimensional and generate $\mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$, and Λ^1 and Λ^2 are surjective, then A is $\mathcal{K}_2$.*

Proof. The map Λ is an algebra isomorphism. Apply Lemma VI.2.8 when $m = 2$ and $R = \mathbf{E}(A)$. □

Although the main purpose of these results is to study connected-graded algebras, it is nevertheless interesting to consider ungraded applications:

Example VI.2.11. Let $p \in \mathbb{K}[x, y]_2$. Let

$$A = \frac{\mathbb{K}[x,y]}{\langle x^3 - p \rangle}.$$

Define $\varepsilon : A \twoheadrightarrow \mathbb{K}$ via $\varepsilon(x) := 0$ and $\varepsilon(y) := 0$.

The standard $\mathbb{N}$-grading on $\mathbb{K}\langle x, y\rangle$ induces a filtration F on A which satisfies the conditions in the introduction. Then

$$\operatorname{gr}^F A \simeq \frac{\mathbb{K}[x,y]}{\langle x^3\rangle}.$$

Note that $\operatorname{gr}^F A$ is a complete intersection, and therefore is $\mathcal{K}_2$ by [9, Corollary 9.2].

One can easily compute $\dim E^1(\operatorname{gr}^F A) = \dim E^2(\operatorname{gr}^F A) = 2$. Furthermore, using $\operatorname{Cob}^\bullet(A)$, one can find the necessary linearly-independent cohomology classes to show $\dim \mathbf{E}^1(A) = \dim \mathbf{E}^2(A) = 2$, implying that Λ^1 and Λ^2 are surjective. Hence A is $\mathcal{K}_2$.

VI.3 Connected-graded algebras with monomial filtrations

In this section, we assume that $A = \mathbb{T}(V)/I$ is a connected-graded algebra. We consider A as a filtered algebra as in (and adopting the notation from) Section II.5. Our goal is to prove Theorem VI.1.4. We begin by proving Lemma II.3.2, which we restate:

Lemma VI.3.1. *For a connected-graded algebra A, $\mathbf{E}^m(A) = \mathbf{E}^m_{\mathrm{Gr}}(A)$ if and only if $\dim \mathbf{E}^m_{\mathrm{Gr}}(A) < \infty$.*

Proof. Projective modules in the category Gr-A of graded A-modules are graded-free [2, Proposition 2.1]. So, there exists a projective resolution (in both the category of graded A-modules and of all A-modules)

$$\cdots \to A\otimes V^m \xrightarrow{\partial^m} \cdots \to A\otimes V^1 \to A\otimes V^0 \to A \to_A \mathbb{K} \to 0$$

such that each V^i is a graded vector space and $\partial^i(A \otimes V^i) \subseteq A_+ \otimes V^{i-1}$. So, for *any* A-module homomorphism $f : A \otimes V^{i-1} \to \mathbb{K}$, $\partial^i \circ f = 0$. Thus, all the differentials in both $\mathrm{Hom}(A \otimes V^\bullet, {}_A\mathbb{K})$ and $\mathrm{Hom}_{\mathrm{Gr}}(A \otimes V^\bullet, {}_A\mathbb{K})$ are zero. So,

$$E^m(A) = \mathrm{Hom}(A \otimes V^m, {}_A\mathbb{K})$$

while

$$\mathbf{E}^m_{\mathrm{Gr}}(A) = \mathrm{Hom}_{\mathrm{Gr}}(A \otimes V^m, {}_A\mathbb{K}). \qquad \square$$

Since, in our case, $\mathbf{E}^1(A)$ and $\mathbf{E}^2(A)$ are finite dimensional, any $\mathcal{K}_2$ algebra will have $\mathbf{E}_{\mathrm{Gr}}(A) = \mathbf{E}(A)$, and the $\mathcal{K}_2$ condition is equivalent to $\mathbf{E}_{\mathrm{Gr}}(A)$ being generated by $\mathbf{E}^1_{\mathrm{Gr}}(A)$ and $\mathbf{E}^2_{\mathrm{Gr}}(A)$. However, it is possible for $\mathbf{E}(A) \neq \mathbf{E}_{\mathrm{Gr}}(A)$ for a connected-graded algebra:

Example VI.3.2. Consider the algebra

$$A = \frac{\mathbb{K}\langle w, x, y, z\rangle}{\langle yz, zx - xz, zw\rangle}.$$

introduced in [8, Example 5.2]. A minimal projective resolution for ${}_A\mathbb{K}$ is

$$0 \to A(-3, -4, -5, \dots) \to A(-2)^{\oplus 2} \to A(-1)^{\oplus 4} \to A \to \mathbb{K} \to 0.$$

Thus, the dimension of $\mathbf{E}^3_{\mathrm{Gr}}(A)$ is countably infinite, while the dimension of $\mathbf{E}^3(A)$ is uncountable.

Recall that $\mathcal{M}$ denotes the monomials of $\mathbb{T}(V)$, which (with respect the ordered basis for V) form a monoid which is totally ordered by degree lexicographical order. For $\alpha \in \mathcal{M}$, we set $F_\alpha A := \mathrm{span}\{\pi(\beta) : \beta \leq \alpha\}$. As $\mathcal{M}$ is itself $\mathbb{N}$-graded, the we may put an $\mathbb{N}$-grading on $\mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$ by setting

$$\mathbf{E}^{i,j}_{\mathrm{Gr}}(\mathrm{gr}^F A) := \bigoplus_{|\alpha|=j} \mathbf{E}^{i,\alpha}_{\mathrm{Gr}}(\mathrm{gr}^F A).$$

(Here $|\alpha|$ denotes the length of α.) The algebra $\mathbf{E}(A)$ inherits the grading on A, and so does $\mathrm{gr}^F E(A)$. Indeed, it is clear that

$$(\mathrm{gr}^F \mathbf{E}(A))^{i,j} = \bigoplus_{|\alpha|=j} (\mathrm{gr}^F \mathbf{E}^j(A))^{\alpha}$$

Furthermore, the monomorphism

$$\Lambda : \mathrm{gr}^F \mathbf{E}(A) \hookrightarrow \mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$$

defined in Theorem VI.1.2 is homogeneous with respect to this internal $\mathbb{N}$-grading.

The goal of this section is to apply Theorem VI.1.3 to connected-graded algebras, using this monomial filtration. Note that Λ^1 is always surjective, so to apply Theorem VI.1.3, we need only check:

1. $\mathrm{gr}^F A$ is $\mathcal{K}_2$, and
2. $\Lambda^2 : \mathrm{gr}^F \mathbf{E}^2(A) \hookrightarrow \mathbf{E}^2(\mathrm{gr}^F A)$ is surjective.

As discussed in Sections II.4 and II.5, the algebra $\mathrm{gr}^F A$ is a monomial algebra and it is easy to determine if $\mathrm{gr}^F A$ is $\mathcal{K}_2$. Theorem VI.1.4 is then a consequence of the following:

Lemma VI.3.3. *The map $\Lambda^2 : \mathrm{gr}^F \mathbf{E}^2(A) \hookrightarrow \mathbf{E}^2(\mathrm{gr}^F A)$ is surjective if and only if I has an essential Gröbner basis.*

Note that a generating set $\mathcal{B}^e$ for I is essential if and only if $|\mathcal{B}^e| = \dim I/I' = \dim E^2(A)$. We will show later that the existence of an essential Gröbner basis is equivalent to the surjectivity of Λ^2. At the same time, it is desirable to know when an essential generating set is a Gröbner basis.

Example VI.3.4. Consider the ideal $I := \langle x^3, y^2 \rangle$ in $\mathbb{K}\langle x, y \rangle$. Under the order $x < y$, the set $\mathcal{B}^e := \{y^2, x^3 - y^2 x\}$ is an essential generating set for I. However $\mathcal{B}^e$ is not a Gröbner basis. On the other hand, the slightly modified set $\mathcal{G} := \{y^2, x^3\}$ is an essential Gröbner basis. The failure of I to be a Gröbner basis was due to the needless redundancy of leading monomials.

The following lemma is easy.

Lemma VI.3.5. *Let $\mathcal{B}^e$ be an essential generating set for I. Then the following are equivalent:*

1. *$\tau(\mathcal{B}^e)$ is an essential generating set for $\langle \tau(\mathcal{B}^e) \rangle$.*

2. *For every $r, r' \in \mathcal{B}^e$ and $\alpha', \alpha'' \in \mathcal{M}$, $\tau(r) \notin \mathbb{K}\alpha'\tau(r')\alpha''$.*

3. *For every $r, r' \in \mathcal{B}^e$ and $\alpha', \alpha'' \in \mathcal{M}$, $\tau(r) \notin \mathbb{K}\tau(\alpha' r' \alpha'')$.*

Definition VI.3.6. If an essential generating set $\mathcal{B}^e$ meets the equivalent conditions of Lemma VI.3.5, we say $\mathcal{B}^e$ has the **leading monomial property**.

In Example VI.3.4, the set $\mathcal{B}^e$ failed to be a Gröbner basis because it failed to have the leading monomial property.

Lemma VI.3.7. *Essential Gröbner bases have the leading monomial property.*

Proof. Suppose that $\mathcal{G}$ is an essential Gröbner basis. As we have an injective map $\Lambda^2 : \operatorname{gr}^F \mathbf{E}^2(A) \hookrightarrow \mathbf{E}^2_{\mathrm{Gr}}(\operatorname{gr}^F A)$,

$$|\mathcal{G}| = \dim \mathbf{E}^2(A) \leq \dim \mathbf{E}^2_{\mathrm{Gr}}(\operatorname{gr}^F A).$$

On the other hand, if $\tau(r) \in \mathbb{K}\tau(\alpha' r' \alpha'')$ for some $\alpha', \alpha'' \in \mathcal{M}$ and $r, r' \in \mathcal{B}^e$, then

$$\langle \tau(\mathcal{G}) \rangle = \langle \tau(\mathcal{G}) \setminus \{\tau(r)\} \rangle$$

and so $\dim \mathbf{E}^2_{\mathrm{Gr}}(\mathrm{gr}^F A) < \dim \mathbf{E}^2(A)$, which is absurd. □

Theorem VI.3.8. *There exist homogeneous bases $\mathcal{B}$ for I and $\mathcal{B}'$ for I' such that $\mathcal{B}' \subset \mathcal{B}$, and the essential generating set $\mathcal{B}^e := \mathcal{B} \setminus \mathcal{B}'$ has the leading monomial property.*

The proof of this theorem will follow after two technical lemmas.

Lemma VI.3.9. *For $W \subset I$ and $\alpha \in \mathcal{M}$, define*

$$\mathcal{A}^{\alpha}_m(W) := \{r \in I_m : r \notin \operatorname{span} W \text{ and } \tau(r) \notin \mathbb{K}\tau(s) \text{ for any } s \in W \text{ with } \tau(s) \geq \alpha\}.$$

If $\mathcal{A}^{x_1^{m+1}}_m(W) \neq \emptyset$, then $\mathcal{A}^{x_1^m}_m(W) \neq \emptyset$; that is, there exists $r \in I_m$ such that $\tau(r) \notin \mathbb{K}\alpha'\tau(s)\alpha''$ for any $\alpha', \alpha'' \in \mathcal{M}$ and $s \in W$.

Proof. We need only show that $\mathcal{A}^{\alpha}_m(W) \neq \emptyset$ implies that $\mathcal{A}^{s(\alpha,-1)}_m(W) \neq \emptyset$. Let $r \in \mathcal{A}^{\alpha}_m(W)$. Suppose $\tau(r) = \tau(s)$ for some $s \in W$. Then $r - s \in \mathcal{I}_m$ but $r - s \notin \operatorname{span} W$. Also, $\tau(r-s) < \tau(s) < \alpha$, so $r - s \in \mathcal{A}^{s(\alpha,-1)}_m(W)$. □

We will use the following lemma to build our basis degree-by-degree:

Lemma VI.3.10. *Suppose $\mathcal{B}$ is a homogeneous basis for $\bigoplus_{i=0}^{m-1} I_i$ and $\mathcal{B}' \subset \mathcal{B}$ is a basis for $\bigoplus_{i=0}^{m-1} I'_i$. Then there exists $\mathcal{B}'' \subset I_m$ and $r_1, \ldots, r_\ell \in I_m$ such that:*

1. *$\mathcal{B}''$ is a basis for I'_m.*

2. *$r_i \notin \mathbb{K}\alpha'\tau(r)\alpha''$ for any $i = 1, \ldots, \ell$, $\alpha', \alpha'' \in \mathcal{M}$, and $r \in \mathcal{B}$.*

3. *$\mathcal{B}'' \cup \{r_1, \ldots, r_\ell\}$ is a basis of I_m.*

Proof. Set

$$\mathcal{B}^{(0)} = \{\alpha' r' \alpha'' \in I_m : \alpha', \alpha'' \in \mathcal{M}, r' \in \mathcal{B}\}.$$

Let $\mathcal{B}'' \subset \mathcal{B}^{(0)}$ such that $\mathcal{B}'_m$ is linearly independent. Since $\mathcal{B}^{(0)}$ spans I'_m, $\mathcal{B}''$ is a basis for I'_m.

Now, suppose we have constructed $\mathcal{B}^{(j)} = \mathcal{B}^{(j-1)} \cup \{r_j\}$ for $1 \leq j \leq i$ such that $(\mathcal{B}^{(i)} \setminus \mathcal{B}^{(0)}) \cup \mathcal{B}''$ is linearly independent and $\tau(r_j) \notin \mathbb{K}\tau(s)$ for any $s \in \mathcal{B}^{(i-1)}$.

If $\mathcal{B}^{(i)}$ spans I_m, then $\mathcal{B}'' \cup \{r_1, \ldots, r_i\}$ also spans I_m, and the claim is proved. Otherwise, $\mathcal{A}_m^{x_1^{m+1}}(\mathcal{B}^{(j)}) \neq \emptyset$, and so by Lemma VI.3.9, there exists $r_{i+1} \in I_m$ such that $\tau(r_{i+1}) \notin \mathbb{K}\tau(s)$ for any $s \in \mathcal{B}^{(i)}$. Set $\mathcal{B}^{(i+1)} = \mathcal{B}^{(i)} \cup \{r_{i+1}\}$. □

Proof of Theorem VI.3.8. Set $\mathcal{B}_m = \mathcal{B}'_m = \mathcal{B}^e_m = \emptyset$ for $m \leq 1$. Apply Lemma VI.3.10 and induction on m. □

We are now ready to prove Lemma VI.3.3 and Theorem VI.1.4, both of which are incorporated in the following:

Theorem VI.3.11. *The following are equivalent:*

1. *Every essential generating set for I with the leading monomial property is a Gröbner basis.*

2. *There is an essential Gröbner basis for I.*

3. $\dim E^2(A) = \dim E^2(\mathrm{gr}^F A)$.

4. *The injective map $\Lambda^2 : \mathrm{gr}^F \mathbf{E}^2(A) \hookrightarrow \mathbf{E}^2(\mathrm{gr}^F A)$ defined in Theorem VI.1.2 is surjective.*

Therefore, if I has an essential Gröbner basis and $\operatorname{gr}^F A$ is $\mathcal{K}_2$, then A is $\mathcal{K}_2$ as well.

Proof. We set $J = \ker(\widehat{\pi} : \mathbb{T}(V) \twoheadrightarrow \operatorname{gr}^F A)$ and $J' = J \otimes V + V \otimes J$.

In light of Theorem VI.3.8, is clear that Condition (1) implies Condition (2).

Suppose $\mathcal{G}$ is an essential Gröbner basis for I. Then $|\mathcal{G}| = \dim I/I'$. Also, since $\mathcal{G}$ has the leading monomial property, $|\mathcal{G}| = |\tau(\mathcal{G})| = \dim J/J'$. So, Condition (2) implies Condition (3).

Clearly, Condition (3) and Condition (4) are equivalent.

Finally, assume (4). Suppose $\mathcal{B}^e$ is an essential generating set for I with the leading monomial property. Let $\mathcal{B}^e_J$ be an essential generating set of J such that

$$\{\tau(x) : x \in \mathcal{B}^e\} \subset \mathcal{B}^e_J.$$

Then, $|\mathcal{B}^e| = \dim I/I' = \dim J/J' = |\mathcal{B}^e_J|$. So, $\mathcal{B}^e$ is a Gröbner basis. Thus, Condition (4) implies Condition (1). □

Example VI.3.12. Consider

$$A := \frac{\mathbb{K}\langle x, y\rangle}{\langle xy - x^2, yx, y^3\rangle}$$

with a monomial order induced by $x < y$. We know from [9, Example 4.5] that A is not a $\mathcal{K}_2$ algebra. The Hilbert series of A is $H_A(t) = 1 + 2t + 2t^2$. Since $\pi(x^3) = 0$, we see that $\widehat{\pi}(x^3) = 0$, and $\operatorname{gr}^F A \simeq \mathbb{K}\langle x, y\rangle / \langle xy, yx, x^3, y^3\rangle$. We may apply [9, Theorem 5.3] to see that $\operatorname{gr}^F A$ is $\mathcal{K}_2$. The essential generating set $\{xy - x^2, yx, y^3\}$ is not a Gröbner basis for $\ker \pi$. The behavior is similar under $y < x$ (although $\operatorname{gr}^F A$ is a different $\mathcal{K}_2$ algebra).

Example VI.3.13. Consider

$$A := \frac{\mathbb{K}\langle x,y,z\rangle}{\langle x^2y - x^3, yz^2 - yx^2, x^3z - x^4\rangle}$$

with the monomial order induced by $x < y < z$. We may use the diamond lemma [4, Theorem 1.2] to show that

$$\mathrm{gr}^F A \simeq B := \frac{\mathbb{K}\langle x,y,z\rangle}{\langle x^2y, yz^2, x^3z\rangle}.$$

Thus, $\{x^2y - x^3, yz^2 - yx^2, x^3z - x^4\}$ is an essential Gröbner basis for $\ker \pi$. However, application of [9, Theorem 5.3] shows that B is not $\mathcal{K}_2$. By inspection,

$$0 \to B(-5) \xrightarrow{\begin{pmatrix} 0 & x^2 & 0 \end{pmatrix}} B(-3^2, -4) \xrightarrow{\begin{pmatrix} 0 & x^2 & 0 \\ 0 & 0 & yz \\ 0 & 0 & x^3 \end{pmatrix}} B(-1^3) \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} B \to \mathbb{K} \to 0$$

is a minimal projective resolution for ${}_B\mathbb{K}$. By Theorem VI.1.2, $\dim \mathbf{E}^{i,j}(A) \leq \dim \mathbf{E}^{i,j}(B)$. So, the chain complex of projective A-modules

$$0 \to A(-5) \xrightarrow{\begin{pmatrix} 0 & x^2 & -x \end{pmatrix}} A(-3^2, -4) \xrightarrow{\begin{pmatrix} x^2 & -x^2 & 0 \\ y^2 & 0 & -yx \\ x^3 & 0 & -x^3 \end{pmatrix}} A(-1^3) \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} A \to \mathbb{K} \to 0$$

is a minimal projective resolution for ${}_A\mathbb{K}$. Applying [9, Theorem 4.4], we see that A is $\mathcal{K}_2$. Hence, the converse to the last sentence of Theorem VI.3.11 is false.

Example VI.3.14. Let

$$A := \frac{\mathbb{K}\langle x, y\rangle}{\langle yx - xy, y^3 + x^2y\rangle}.$$

Then under the order $x < y$, the essential generating set $\{yx - xy, y^3 + x^2y\}$ is a Gröbner basis for $\ker \pi$, and

$$\mathrm{gr}^F A = \frac{\mathbb{K}\langle x, y\rangle}{\langle yx, y^3\rangle}.$$

We may use [9, Theorem 5.3] to show that $\mathrm{gr}^F A$ is $\mathcal{K}_2$. Thus, by Theorem VI.3.11, A is $\mathcal{K}_2$. (This can also be verified directly using [9, Corollary 9.2].)

It turns out that the classical PBW algebra theory is a corollary:

Theorem VI.3.15 ([18, Theorem IV.3.1]). *If A is a quadratic algebra, and* $\mathrm{gr}^F A$ *is also quadratic, then A is Koszul.*

Proof. Quadratic monomial algebras are Koszul [18, Corollary II.4.3]. The theorem follows directly from Theorem VI.1.3. □

VI.4 Spectral sequence approach to connected-graded algebras with monomial filtrations

In this section, we will present an alternate approach to the material in the previous section. This approach uses the spectral sequence E in which $\mathbf{E}_{\mathrm{Gr}}(\mathrm{gr}^F A)$ is the E_1 page and $\mathrm{gr}^F \mathbf{E}(A)$ is the E_∞ page. We then show that the multiplication on the E_1 page can be pushed to the E_∞ page. This yields an alternate proof to Theorem VI.1.4. The material in this section was not included in the article [17].

Recall we had the poset (but not monoid) embedding

$$p : \mathcal{M} \to \mathbb{N}$$

We introduce some notation from the proof of Theorem 3.8 in [16]:

$$\begin{aligned} Z_r^{\alpha,q} &:= F_\alpha \operatorname{Cob}^{p(\alpha)+q} \cap \partial^{-1}(F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+q+1}(A)), \\ B_r^{\alpha,q} &:= F_\alpha \operatorname{Cob}^{p(\alpha)+q} \cap \partial(F_{s(\alpha,-r)} \operatorname{Cob}^{p(\alpha)+q-1}, \\ Z_\infty^{\alpha,q} &:= \bigcap_r Z_r^{\alpha,q} = F_\alpha \operatorname{Cob}^{p(\alpha)+q}(A) \cap \ker \partial, \end{aligned}$$

and

$$B_\infty^{\alpha,q} := \bigcup_r Z_r^{\alpha,q} = F_\alpha \operatorname{Cob}^{p(\alpha)+q}(A) \cap \operatorname{im} \partial.$$

With this notation, we view

$$E_r^{\alpha,q} := Z_r^{\alpha,q} / (Z_{r-1}^{s(\alpha,1),q-1} + B_{r-1}^{\alpha,q})$$

for $r < \infty$ and

$$E_\infty^{\alpha,q} := Z_\infty^{\alpha,q} / (Z_\infty^{s(\alpha,1),q-1} + B_\infty^{\alpha,q}),$$

and denote by $\eta_r^{\alpha,q} : Z_r^{\alpha,q} \twoheadrightarrow E_r^{\alpha,q}$ the projection. Because

$$E_1^{\alpha,q} \simeq \mathbf{E}^{p(\alpha)+q,\alpha}(\operatorname{gr}^F A)$$

and

$$E_\infty^{\alpha,q} \simeq \frac{F_\alpha \mathbf{E}^{p(\alpha)+q}(A)}{F_{s(\alpha,1)} \mathbf{E}^{p(\alpha)+q}(A)},$$

the use of η is consistent with that in Section VI.2. Set $\widehat{E}_r^{\alpha,q} = \eta_r(Z_\infty^{\alpha,q})$. Note

$\Lambda\left(E_\infty^{\alpha,q}\right) = \widehat{E}_1^{\alpha,q}$. We have a string of inequalities

$$\dim E_\infty^{\alpha,q} \leq \dim \widehat{E}_1^{\alpha,q} \leq \dim E_1^{\alpha,q}.$$

It would be nice if E_r was a spectral sequence of algebras (see [16, Definition 2.13]). However, the natural tensor product of $E_r^{\alpha,i}$ and $E_r^{\beta,j}$ does not land in $E_r^{\alpha\beta,i+j}$, as required by the definition, since $E_r^{\alpha\beta,i+j}$ is a factor of $\mathrm{Cob}^{p(\alpha\beta)+i+j}(A)$, and $p(\alpha\beta)+i+j$ is not necessarily the same as $p(\alpha)+i+p(\beta)+j$. However, we can define a similar multiplicative structure

$$E_r^{\alpha,i} \otimes E_r^{\beta,j} \to E_r^{\alpha\beta,i+jR(\alpha,\beta)},$$

where $R(\alpha,\beta) = p(\alpha)+p(\beta)-p(\alpha\beta)$.

Lemma VI.4.1 (Odometer formulas). *The following two formulas hold:*

$$p(x_{i_m}x_{i_{m-1}}\cdots x_{i_1}) = \sum_{t=1}^{m} i_t n^{t-1}$$

and

$$p(\alpha\beta) = p(\alpha)n^{|\beta|} + p(\beta).$$

Proof. To prove the first formula, we induct on $\alpha = x_{i_m}\cdots x_{i_1}$. The formula obviously holds when $\alpha = x_1$. Suppose the formula holds when $\alpha = x_{i_m}\cdots x_{i_1}$.

If $\alpha = x_n^m$, then $s(\alpha, 1) = x_1^{m+1}$ and

$$\begin{aligned} p(s(\alpha,1)) &= p(\alpha) + 1 \\ &= \sum_{t=1}^{m} n \cdot n^{t-1} + 1 \\ &= \sum_{t=1}^{m+1} n^{t-1}. \end{aligned}$$

Now, suppose $\alpha \neq x_n^m$. Let $j \in \{1, \ldots, m\}$ be the smallest such that $i_j \neq n$. Then $\alpha = x_{i_m} \cdots x_{i_j} x_n^{j-1}$ and $s(\alpha, 1) = x_{i_m} \cdots x_{i_j+1} x_1^{j-1}$.

Also,

$$\begin{aligned} &\sum_{t=j+1}^{m} i_t n^{t-1} + (i_j + 1)n^{j-1} + \sum_{t=1}^{j-1} n^{t-1} \\ &= \sum_{t=j}^{m} i_t n^{t-1} + n^{j-1} + \sum_{t=1}^{j-1} n^{t-1} \\ &= \sum_{t=j}^{m} i_t n^{t-1} + (n^{j-1} - 1) + \sum_{t=1}^{j-1} n^{t-1} + 1 \\ &= \sum_{t=j}^{m} i_t n^{t-1} + (n-1) \sum_{t=1}^{j-1} n^{t-1} + \sum_{t=1}^{j-1} n^{t-1} + 1 \\ &= \sum_{t=j}^{m} i_t n^{t-1} + \sum_{t=1}^{j-1} n \cdot n^{t-1} + 1 \\ &= p(\alpha) + 1 = p(s(\alpha, 1)), \end{aligned}$$

as desired.

To prove the second formula, write $\beta = x_{i_{|\beta|}} \cdots x_{i_1}$ and $\alpha = x_{i_{|\alpha\beta|}} \cdots x_{i_{|\beta|+1}}$.

Then

$$
\begin{aligned}
p(\alpha\beta) &= \sum_{t=1}^{|\alpha\beta|} i_t n^{t-1} \\
&= \sum_{t=1}^{|\alpha\beta|} i_t n^{t-1} + \sum_{t=1}^{|\beta|} i_t n^{t-1} \\
&= \sum_{t=|\beta|+1}^{|\alpha\beta|} i_t n^{t-|\beta|-1} n^{|\beta|} + p(\beta) \\
&= p(\alpha) n^{|\beta|} + p(\beta).
\end{aligned}
$$

□

Lemma VI.4.2. *We have* $\alpha s(\beta, r) \geq s(\alpha\beta, r)$.

Proof. This follows from the estimate

$$
\begin{aligned}
p(\alpha, s(\beta, r)) &= p(\alpha) n^{|s(\beta,r)|} + p(s(\beta, r)) \\
&\geq p(\alpha) n^{|\beta|} + p(\beta) + r \\
&= p(s(\alpha\beta, r)).
\end{aligned}
$$

□

Lemma VI.4.3. *We have* $s(\alpha, r)\beta \geq s(\alpha\beta, r)$.

Proof. This follows from the estimate

$$
\begin{aligned}
p(s(\alpha, r)\beta) &= p(s(\alpha, r)) n^{|\beta|} + p(\beta) \\
&= (p(\alpha) + r) n^{|\beta|} + p(\beta) \\
&\geq p(\alpha) n^{|\beta|} + r + p(\beta) \\
&= p(\alpha\beta) + r = p(s(\alpha\beta, r)).
\end{aligned}
$$

□

Lemma VI.4.4. *The multiplication*

$$F_\alpha \operatorname{Cob}^{p(\alpha)+i}(A) \otimes F_\beta \operatorname{Cob}^{p(\beta)+j}(A) \to F_{\alpha\beta} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j}(A)$$

restricts to a map

$$Z_r^{\alpha,i} \otimes Z_r^{\beta,j} \to Z_r^{\alpha\beta,i+j+R(\alpha,\beta)}.$$

Proof. Let $f \in Z_r^{\alpha,i}$ and $g \in Z_r^{\beta,j}$. Then

$$\partial(f) \in F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+i+1}(A)$$

and

$$\partial(g) \in F_{s(\beta,r)} \operatorname{Cob}^{p(\beta)+j+1}(A).$$

So,

$$\begin{aligned}
\partial(f \otimes g) &= \partial(f) \otimes g + (-1)^i f \otimes \partial(g) \\
&\in F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+i+1}(A) F_\beta \operatorname{Cob}^{p(\beta)+j}(A) \\
&\qquad + F_\alpha \operatorname{Cob}^{p(\alpha)+i}(A) F_{s(\beta,r)} \operatorname{Cob}^{p(\beta)+j+1}(A) \\
&= F_{s(\alpha,r)\beta} \operatorname{Cob}^{p(\alpha)+i+p(\beta)+j+1}(A) + F_{\alpha s(\beta,r)} \operatorname{Cob}^{p(\alpha)+i+p(\beta)+j+1}(A) \\
&\subseteq F_{s(\alpha\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A).
\end{aligned}$$

Hence, $f \otimes g \in Z_r^{\alpha\beta,i+j-R(\alpha,\beta)}$. □

Theorem VI.4.5. *The map*

$$Z_r^{\alpha,i} \otimes Z_r^{\beta,j} \to Z_r^{\alpha\beta,i+j+R(\alpha,\beta)}$$

induces a map

$$\mu_r : E_r^{\alpha,i} \otimes E_r^{\beta,j} \longrightarrow E_r^{\alpha\beta,i+j+R(\alpha,\beta)}.$$

Proof. First suppose $f \in Z_{r-1}^{s(\alpha,1),i-1} + B_{r-1}^{\alpha,i}$ and $g \in Z_r^{\beta,j}$. So, we may write $f = f_1 + f_2$ where $f_1 \in Z_{r-1}^{s(\alpha,1),i-1}$ and $f_2 \in B_{r-1}^{\alpha,i}$. Thus,

$$\partial(f_1) \in F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+r+i}(A),$$

and hence

$$\begin{aligned}
\partial(f_1 \otimes g) &= \partial(f_1) \otimes g + (-1)^{p(\alpha)+i} f_1 \otimes \partial(g) \\
&\in F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+i+1}(A) \otimes F_\beta \operatorname{Cob}^{p(\beta)+j}(A) \\
&\qquad + F_{s(\alpha,1)} \operatorname{Cob}^{p(\alpha)+i}(A) \otimes F_{s(\beta,r)} \operatorname{Cob}^{p(\beta)+j+1}(A) \\
&\subseteq F_{s(\alpha,r)\beta} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A) + F_{s(\alpha,1)s(\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A) \\
&\subseteq F_{s(\alpha\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A).
\end{aligned}$$

Hence, $f_1 \otimes g \in Z_{r-1}^{s(\alpha\beta,1),i+j+R(\alpha,\beta)}$.

Also,

$$\begin{aligned}
\partial(f_2 \otimes g) &= \partial(f_2) \otimes g + (-1)^{p(\alpha)+i} f_2 \otimes \partial(g) \\
&= (-1)^{p(\alpha)+i} f_2 \otimes \partial(g) \\
&\in F_{\alpha s(\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A) \\
&\subseteq F_{s(\alpha\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A).
\end{aligned}$$

So, $f \otimes g \in Z_{r-1}^{s(\alpha\beta,1),i+j+R(\alpha,\beta)} \subseteq Z_{r-1}^{s(\alpha\beta,1),i+j+R(\alpha,\beta)} + B_{r-1}^{\alpha\beta,i+j+R(\alpha,\beta)}$.

On the other hand, suppose that $g \in Z_{r-1}^{s(\beta,1),j-1} + B_{r-1}^{\beta,j}$ and $f \in Z_r^{\alpha,i}$. Then $g = g_1 + g_2$ where $g_1 \in Z_{r-1}^{s(\beta,1),j-1}$ and $g_2 \in B_{r-1}^{\beta,j}$.

We compute

$$\begin{aligned}
\partial(f \otimes g) &= \partial(f \otimes (g_1 + g_2)) \\
&= \partial(f \otimes g_1) + \partial(f \otimes g_2) \\
&= \partial(f) \otimes (g_1 + g_2) + (-1)^{p(\alpha)+i} f \otimes \partial(g_1) + (-1)^{p(\alpha)+i} f \otimes \partial(g_2) \\
&= \partial(f) \otimes (g_1 + g_2) + (-1)^{p(\alpha)+i} f \otimes \partial(g_1).
\end{aligned}$$

Note that

$$\begin{aligned}
\partial(f) \otimes g_1 &\in F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+i+1}(A) \otimes F_{s(\beta,1)} \operatorname{Cob}^{p(\beta)+j}(A) \\
&\subseteq F_{s(\alpha,r)s(\beta,1)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A) \\
&\subseteq F_{s(\alpha\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A).
\end{aligned}$$

Also,

$$\begin{aligned}
\partial(f) \otimes g_2 &\in F_{s(\alpha,r)} \operatorname{Cob}^{p(\alpha)+i+1}(A) \otimes F_{\beta} \operatorname{Cob}^{p(\beta)+j}(A) \\
&\subseteq F_{s(\alpha,r)\beta} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A) \\
&\subseteq F_{s(\alpha\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A).
\end{aligned}$$

Finally,

$$\begin{aligned}
f \otimes \partial(g_1) &\in F_{\alpha} \operatorname{Cob}^{p(\alpha)+i}(A) \otimes F_{s(\beta,r)} \operatorname{Cob}^{p(\beta)+j+1}(A) \\
&\subseteq F_{s(\alpha\beta,r)} \operatorname{Cob}^{p(\alpha)+p(\beta)+i+j+1}(A).
\end{aligned}$$

So, in general, we have

$$\begin{aligned} d(f \otimes g) &\in Z_{r-1}^{s(\alpha\beta,1),i+j+R(\alpha,\beta)} \\ &\subseteq Z_{r-1}^{s(\alpha\beta,1),i+j+R(\alpha,\beta)} + B_{r-1}^{\alpha\beta,i+j+R(\alpha,\beta)}, \end{aligned}$$

as desired. □

As E is a spectral sequence, there is a differential

$$d_r : E_r^{\alpha,q} \to E_r^{s(\alpha,r),q-r+1}.$$

This differential is defined so that the diagram

$$\begin{array}{ccc} Z_r^{\alpha,q} & \xrightarrow{\partial} & Z_r^{s(\alpha,r),q-r+1} \\ \downarrow{\scriptstyle \eta_r} & & \downarrow{\scriptstyle \eta_r} \\ E_r^{\alpha,q} & \xrightarrow{d_r} & E_r^{s(\alpha,r),q-r+1} \end{array}$$

commutes. Furthermore, the map η_r sends $Z_{r+1}^{\alpha,q} \subseteq Z_r^{\alpha,q}$ to $\ker d_r$, and the composition

$$\gamma : Z_{r+1}^{\alpha,q} \xrightarrow{\eta_r} \ker d_r \to H^{\alpha,q}(E_r, d_r)$$

induces an isomorphism $\gamma : E_{r+1}^{\alpha,q} \simeq H^{\alpha,q}(E_r, d_r)$. (For more details, see [16, pp. 34–37].) We will show that the map $\mu_{r+1} : E_{r+1} \otimes E_{r+1} \to E_{r+1}$ is the same as an induced map $H(\mu_r) : H(E_r) \otimes H(E_r) \to H(E_r)$ in cohomology.

First, we need to show that μ_r induces a map on cohomology.

Lemma VI.4.6. *The map*

$$\mu_r : E_r^{\alpha,i} \otimes E_r^{\beta,j} \to E_r^{\alpha\beta,i+j+R(\alpha,\beta)}$$

restricts to a map

$$\mu_r : \ker d_r^{\alpha,i} \otimes \ker d_r^{\beta,j} \to \ker d_r^{\alpha\beta, i+j+R(\alpha,\beta)}.$$

This restriction induces a well-defined map

$$H(\mu_r) : H^{\alpha,i}(E_r, d_r) \otimes H^{\beta,j}(E_r, d_r) \to H^{\alpha\beta,i+j+R(\alpha,\beta)}(E_r, d_r).$$

Furthermore, this map is compatible with μ_{r+1}; that is, the diagram

$$\begin{array}{ccc} E_{r+1}^{\alpha,i} \otimes E_{r+1}^{\beta,j} & \xrightarrow{\simeq} & H^{\alpha,i}(E_r, d_r) \otimes H^{\beta,j}(E_r, d_r) \\ \downarrow \mu_r & & \downarrow H(\mu_r) \\ E_{r+1}^{\alpha\beta,i+j+R(\alpha,\beta)} & \xrightarrow{\simeq} & H^{\alpha\beta,i+j+R(\alpha,\beta)}(E_r, d_r) \end{array}$$

commutes.

Proof. Recall that for $f \in Z_r^{\alpha,i}$, $d_r(\eta_r(f)) := \eta_r(\partial(f))$.

First suppose that $f \in Z_r^{\alpha,i}$ and $g \in Z_r^{\beta,j}$.

$$\begin{aligned} d_r(\mu_r(\eta_r f \otimes \eta_r g)) &= d_r(\eta_r(f \otimes g)) \\ &= \eta_r(\partial(f \otimes g)) \\ &= \eta_r(\partial(f) \otimes g) + (-1)^i \eta_r(f \otimes \partial(g)) \\ &= \mu_r(\eta_r(\partial(f)) \otimes \eta_r(g)) + (-1)^i \mu_r(\eta_r(f) \otimes \eta_r(\partial(g))) \\ &= \mu_r(d_r(\eta_r(f)) \otimes \eta_r(g)) + (-1)^i \mu_r(\eta_r(f) \otimes d_r(\eta_r(g))) \end{aligned}$$

So, if $\eta_r(f) \in \ker d_r$ and $\eta_r(g) \in \ker d_r$, then $\mu_r(\eta_r(f) \otimes \eta_r(g)) \in \ker d_r$ as well.

Now, suppose that $\eta_r(f) \in \operatorname{im} d_r$ and $\eta_r(g) \in \ker d_r$. Then there exists

$\widehat{f} \in Z_r^{s(\alpha,-r),i+r-1}$ such that $d_r(\eta_r(\widehat{f})) = \eta_r(f)$. Thus, $\eta_r(\partial(\widehat{f})) = \eta_r(f)$. So,

$$\begin{aligned}
d_r(\eta_r(\widehat{f} \otimes g)) &= \eta_r(\partial(\widehat{f} \otimes g)) \\
&= \eta_r(\partial(\widehat{f}) \otimes g + (-1)^{i+r-1}\widehat{f} \otimes \partial(g)) \\
&= \mu_r(\eta_r(f) \otimes \eta_r(g)) + (-1)^{i+r-1}\mu_r(\eta_r(\widehat{f}) \otimes d_r(\eta_r(g))) \\
&= \mu_r(\eta_r(f) \otimes \eta_r(g)).
\end{aligned}$$

So, $\mu_r(\eta_r(f) \otimes \eta_r(g)) \in \operatorname{im} d_r$ as well. A similar argument holds when $\eta_r(g) \in \operatorname{im} d_r$ and $\eta_r(f) \in \ker(d_r)$.

Finally, let $f \in Z_{r+1}^{\alpha,i}$ and $g \in Z_{r+1}^{\beta,j}$. Then,

$$\begin{aligned}
H(\mu_r)(\gamma(\eta_{r+1}(f)) \otimes \gamma(\eta_{r+1}(g))) &= H(\mu_r)([\eta_r(f)] \otimes [\eta_r(g)]) \\
&= [\mu_r(\eta_r(f) \otimes \eta_r(g))] \\
&= [\eta_r(f \otimes g)].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\gamma(\mu_{r+1}(\eta_{r+1}(f) \otimes \eta_{r+1}(g))) &= \gamma(\eta_{r+1}(f \otimes g)) \\
&= [\eta_r(f \otimes g)],
\end{aligned}$$

as desired. □

In particular, the map μ_1 is the cup product on $\mathbf{E}(\operatorname{gr}^F A)$.

Corollary VI.4.7. *The map*

$$\mu_1 : E_1^{\alpha,i} \otimes E_1^{\beta,j} \to E_1^{\alpha\beta,i+j+R(\alpha,\beta)}$$

is the same as the cup product

$$\mathbf{E}^{\alpha,p(\alpha)+i}(\mathrm{gr}^F A)\otimes \mathbf{E}^{\beta,p(\beta)+j}(\mathrm{gr}^F A)\to \mathbf{E}^{\alpha\beta,p(\alpha)+p(\beta)+i+j}(\mathrm{gr}^F A).$$

Proof. The differential d_0 is the differential induced by ∂ on $\mathrm{gr}^F \mathrm{Cob}(A)$. Lemma VI.2.1 establishes an isomorphism

$$\mathrm{gr}^F \mathrm{Cob}(A)\simeq \mathrm{Cob}(\mathrm{gr}^F A)$$

as differential-graded algebras. □

Theorem VI.4.8. *The diagram*

$$\begin{array}{ccc}
E_\infty^{\alpha,i}\otimes E_\infty^{\beta,j} & \xrightarrow{\mu_\infty} & E_\infty^{\alpha\beta,i+j+R(\alpha,\beta)} \\
\downarrow\simeq & & \downarrow\simeq \\
\dfrac{F_\alpha \mathbf{E}^{p(\alpha)+i}(A)}{F_{s(\alpha,1)}\mathbf{E}^{p(\alpha)+i}(A)}\otimes\dfrac{F_\beta \mathbf{E}^{p(\beta)+j}(A)}{F_{s(\beta,1)}\mathbf{E}^{p(\beta)+j}(A)} & \longrightarrow & \dfrac{F_{\alpha\beta} \mathbf{E}^{p(\alpha)+p(\beta)+i+j}(A)}{F_{s(\alpha\beta,1)}\mathbf{E}^{p(\alpha)+p(\beta)+i+j}(A)}
\end{array}$$

commutes. In other words, the multiplication on E converges to the cup product structure on $\mathrm{gr}^F \mathrm{Ext}_A(\mathbb{K},\mathbb{K})$.

Proof. Let

$$\pi_h : \ker(\partial : \mathrm{Cob}(A)\to\mathrm{Cob}(A)) \twoheadrightarrow \mathrm{Ext}_A(\mathbb{K},\mathbb{K}).$$

Then the isomorphism

$$E_\infty^{\alpha,q}\simeq \frac{F_\alpha \mathbf{E}_{(}^{p(\alpha)+q} A)}{F_{s(\alpha,1)}\mathbf{E}^{p(\alpha)+q}(A)}$$

is provided by the induced map

$$\pi_h : E_\infty^{\alpha,q}\to \frac{F_\alpha \mathbf{E}^{p(\alpha)+q}(A)}{F_{s(\alpha,1)}\mathbf{E}^{p(\alpha)+q}(A)}$$

yielded by the inclusion

$$\pi_h(\ker \eta_\infty^{\alpha,q}) \subseteq F_{s(\alpha,1)}\mathbf{E}^{p(\alpha)+q}(A).$$

Now, let $f \in Z_\infty^{\alpha,i}$ and $g \in Z_\infty^{\beta,j}$. Then

$$\begin{aligned}
&\pi_h(\mu_\infty(\eta_\infty(f) \otimes \eta_\infty(g))) \\
&\quad = \pi_h(\eta_\infty(f \otimes g)) \\
&\quad = \pi_h(f \otimes g) + F_{s(\alpha\beta,1)}\mathbf{E}^{p(\alpha)+p(\beta)+i+j}(A) \\
&\quad = \pi_h(f) \otimes \pi_h(g) + F_{s(\alpha\beta,1)}\mathbf{E}^{p(\alpha)+p(\beta)+i+j}(A) \\
&\quad = (\pi_h(f) + F_{s(\alpha,1)}\mathbf{E}^{p(\alpha)+p(\beta)+i+j}(A))(\pi_h(g) + F_{s(\beta,1)}\mathbf{E}^{p(\alpha)+p(\beta)+i+j}(A)) \\
&\quad = \pi_h(\eta_\infty(f)) \otimes \pi_h(\eta_\infty(g)),
\end{aligned}$$

as desired.

□

Write

$$D_r^{\alpha,q} := \sum_{\substack{\alpha_1\cdots\alpha_z=\alpha \\ i_1+\cdots+i_z=q-p(\alpha) \\ i_t=1 \text{ or } i_t=2}} E_r^{\alpha_1,i_1-p(\alpha_1)} \otimes \cdots \otimes E_r^{\alpha_z,i_z-p(\alpha_z)}.$$

These lemmas are exercises in unraveling notation:

Lemma VI.4.9. *The algebra* $\operatorname{gr}^F A$ *is* $\mathcal{K}_2$ *if and only if the map* $D_1^{\alpha,q} \to E_1^{\alpha,q}$ *induced by* μ_1 *is surjective.*

Lemma VI.4.10. *The algebra* A *is* $\mathcal{K}_2$ *if the map* $D_\infty^{\alpha,q} \to E_\infty^{\alpha,q}$ *induced by* μ_∞ *is surjective.*

We can now prove Theorem VI.3.11, which we restate:

Theorem VI.4.11. *Suppose there exists an essential Gröbner basis for $I = \ker \pi$. If $\mathrm{gr}^F A$ is $\mathcal{K}_2$, then A is also $\mathcal{K}_2$.*

Proof. Set

$$G_r^{\alpha,q} = \sum_{\substack{\alpha_1 \cdots \alpha_z = \alpha \\ i_1 + \cdots + i_z = q - p(\alpha) \\ i_t = 1 \text{ or } i_t = 2}} Z_r^{\alpha_1, i_1 - p(\alpha)} \otimes \cdots \otimes Z_r^{\alpha_z, i_z - p(\alpha_z)}.$$

(So, $\eta_r(G_r^{\alpha,q}) = D_r^{\alpha,q}$.) By Lemma VI.4.4, we can view $G_r^{\alpha,q} \subset Z_r^{\alpha,q}$; the diagram

$$\begin{array}{ccc} G_r^{\alpha,q} & \hookrightarrow & Z_r^{\alpha,q} \\ \downarrow \eta_r & & \downarrow \eta_r \\ D_r^{\alpha,q} & \xrightarrow{\mu_r} & E_r^{\alpha,q} \end{array}$$

commutes.

Let $x \in E_\infty^{\alpha,q}$. Then $x = \eta_\infty(\tilde{x})$ for some $\tilde{x} \in Z_\infty^{\alpha,q} \subset Z_1^{\alpha,q}$.

Note that there exists $y \in G_1^{\alpha,q}$ such that $y - \tilde{x} \in \ker \eta_1$, because $\mathrm{gr}^F A$ is $\mathcal{K}_2$. Now since $\widehat{E}_1^{\alpha_i, 1-p(\alpha_i)} = E_1^{\alpha_i, 1-p(\alpha_i)}$ and $\widehat{E}_1^{\alpha_i, 2-p(\alpha_i)} = E_1^{\alpha_i, 2-p(\alpha_i)}$, we have $\eta_1(G_r^{\alpha,q}) = \eta_1(G_\infty^{\alpha,q})$ and there exists $\tilde{y} \in G_\infty^{\alpha,q}$ such that $\tilde{y} - \tilde{x} \in \ker \eta_1$. But then $\mu_\infty(\eta_\infty(\tilde{y})) = \eta_\infty(\tilde{x}) = x$, since $\tilde{y} - \tilde{x} \in \ker \eta_1|_{Z_\infty} \subseteq \ker \eta_\infty$.

Thus,

$$D_\infty^{\alpha,q} \to E_\infty^{\alpha,q}$$

is surjective. □

VI.5 Upper-triangularity condition for having an essential Gröbner basis

An alternate approach to proving that PBW algebras are Koszul is to show they meet a special distributivity condition. Our goal is to generalize this distributivity condition to algebras whose ideals of relations have essential Gröbner bases.

A triple (W_1, W_2, W_3) of subspaces of a vector space W is called **distributive** if $(W_i + W_j) \cap W_j = W_i \cap W_k + W_j \cap W_k$ for any i, j, k. Given $W_1, \ldots, W_z$ subspaces of a vector space W, we may consider the sublattice of subspaces of W generated by $W_1, \ldots, W_z$ by the operations of intersection and summation. If every triple in that sublattice is distributive, we say that $\{W_1, \ldots, W_z\}$ is distributive.

We will deploy the following useful lemma:

Lemma VI.5.1 ([18, Proposition 1.7.1]). *Let $W_1, \ldots, W_z$ be subspaces of some vector space W. Then $\{W_1, \ldots, W_z\}$ is distributive if and only if there exists a basis $\mathcal{B}$ for W such that $\mathcal{B} \cap W_i$ is a basis of W_i for each i.*

Let $\mathcal{R}$ be the set of $\alpha \in \mathcal{M}$ which have this property:

$$\text{If } |\alpha| = n \text{ and } \pi(\alpha) + \sum_{\alpha \neq \beta \in \mathcal{M}_n} c_\beta \pi(\beta) = 0 \text{ then } c_\beta \neq 0 \text{ for some } \beta > \alpha. \quad \text{(VI.1)}$$

These are exactly the monomials that cannot be reduced modulo I to a sum of lower monomials. For an essential generating set $\mathcal{B}^e$ with the leading monomial property, let
$\mathcal{L}(\mathcal{B}^e) := \{\alpha \in \mathcal{M} : \beta \neq \tau(r) \text{ for any } r \in \mathcal{B}^e, \text{for any subword } \beta \text{ of } \alpha\}$.
Equivalently, $\mathcal{L}(\mathcal{B}^e)$ is the set of monomials not in $\mathbb{T}(V) \otimes \operatorname{span} \tau(\mathcal{B}^e) \otimes \mathbb{T}(V)$.
We will write $\mathcal{L}(\mathcal{B}^e)_m := \mathcal{L}(\mathcal{B}^e) \cap \mathcal{M}_m$.

This is a special case of [14, Proposition 2.2] and the proof is omitted:

Lemma VI.5.2. *Let $\mathcal{B}^e$ be an essential generating set for I with the leading monomial property. Then $\mathcal{L}(\mathcal{B}^e) = \mathcal{R}$ if and only if $\mathcal{B}^e$ is an essential Gröbner basis.*

Theorem VI.5.3. *Let $\mathcal{B}^e$ be an essential basis for $I := \ker(\pi : \mathbb{T}(V) \twoheadrightarrow A)$ with the leading monomial property. The following are equivalent:*

1. *There exists a basis $\mathcal{B}_m = \{y_\alpha : \alpha \in \mathcal{M}_m\}$ for $V^{\otimes m}$ such that*

   (a) $V^{\otimes i} \otimes \operatorname{span} \mathcal{B}^e_k \otimes V^{\otimes j} \cap \mathcal{B}_m$ is a basis of $V^{\otimes i} \otimes \operatorname{span} \mathcal{B}^e_k \otimes V^{\otimes j}$ for any i, j, k such that $i + j + k = m$ and

   (b) $y_\alpha = \alpha - \sum_{\beta<\alpha} q_{\beta,\alpha}\beta$.

2. *$\mathcal{R} = \mathcal{L}(\mathcal{B}^e)$ (and hence $\mathcal{B}^e$ is a Gröbner basis).*

Remark VI.5.4. The equivalence in the quadratic case Condition (2) and Condition (1) is [18, Proposition 4.5.1]. By Lemma VI.5.1, Condition (1) implies that for each $m \geq 2$, the set $\{V^{\otimes i} \otimes \operatorname{span} \mathcal{B}^e_k \otimes V^{\otimes j} : i + j + k = m\}$ of $V^{\otimes m}$ is distributive. If A is a *quadratic* algebra and $\{V^{\otimes i} \otimes \operatorname{span} \mathcal{B}^e_2 \otimes V^{\otimes j} : i + j + 2 = m\}$ is distributive for each $m \geq 2$, then A is Koszul [18, Theorem 2.4.1]; this yields another proof that PBW algebras are Koszul.

Proof of Theorem VI.5.3. Note that

$$I_m = \sum_{i+j+k=m} V^{\otimes i} \otimes \operatorname{span} \mathcal{B}^e_k \otimes V^{\otimes j}.$$

Suppose Condition (2) holds. Set for $\alpha \in \mathcal{M}_n$

$$y_\alpha := \begin{cases} \alpha & \text{if } \alpha \in \mathcal{L}_m, \\ \alpha' \otimes r \otimes \alpha'' & \text{if } \alpha \notin \mathcal{L}_m \text{ and } \alpha' \in \mathcal{L}_{|\alpha'|} \text{ is the largest element} \\ & \text{such that } \alpha = \alpha' \otimes \tau(r) \otimes \alpha'' \text{ for some } r \in \mathcal{B}^e. \end{cases}$$

Set $D_m := I \cap \{y_\alpha : \alpha \in \mathcal{M}\}$. Note that

$$V^{\otimes i} \otimes \text{span}\, \mathcal{B}_k^e \otimes V^{\otimes j} = \text{span}\{\alpha' \otimes r \otimes \alpha'' : \alpha' \in \mathcal{M}_i, \alpha'' \in \mathcal{M}_j, r \in \mathcal{B}_k^e\}.$$

Thus, $I = \text{span}\, D_m$. Also,

$$\begin{aligned}\alpha' \otimes r \otimes \alpha'' &= \alpha' \otimes \left(\tau(r) - \sum_{\beta<\tau(r)} c_{\tau(r),\beta}\beta\right) \otimes \alpha'' \\ &= \alpha' \otimes \tau(r) \otimes \alpha'' - \sum_{\beta<\tau(r)} c_{\tau(r),\beta}\alpha' \otimes \beta \otimes \alpha''.\end{aligned}$$

We have $|\mathcal{L}_m(\mathcal{B}^e)| = \dim A_m = \dim V^{\otimes m} - \dim I_m$, and so $|D_m| = \dim I_m$ Hence, the $\{y_\alpha\}$ are linearly independent, and Condition (1) holds.

Now, suppose Condition (1) holds. Set

$$Y_m := \{y_\alpha : \alpha \in \mathcal{M}_m\} \setminus I_m$$

and

$$D_m := \{y_\alpha : \alpha \in \mathcal{M}_m\} \cap I_m.$$

Note that $A_m = \text{span}\, \pi(Y_m)$. Also, D_m is a basis of I_m, implying that $Y_m = \dim A_m$ and hence $\pi(Y_m)$ is a basis of A_m. Furthermore, if $y_\alpha \in I_m$ then $\pi(\alpha) = \sum_{\beta<\alpha} c_{\beta,\alpha}\pi(\beta)$, and hence $y_\alpha \notin \mathcal{R}_m$. Thus,

$$\mathcal{R}_m \subseteq \{\alpha : y_\alpha \in Y_m\} = \tau(Y_m).$$

As $\pi(\mathcal{R}_m)$ is a basis of A_m as well, we have $\mathcal{R}_m = \tau(Y_m)$.

Now, suppose $y_\alpha \in I$. Then

$$\begin{aligned} y_\alpha &= \sum_{r\in\mathcal{B}^e,i} \alpha'_{r,i} \otimes q_{r,i} r \otimes \alpha''_{r,i} \\ &= \sum_{r\in\mathcal{B}^e,i} \alpha'_{r,i} \otimes q_{r,i} \left(\tau(r) - \sum_{\beta<\tau(r)} z_{\beta,i}\beta \right) \otimes \alpha''_{r,i}, \end{aligned}$$

and so $\tau(y_\alpha) = \alpha'_{r,i} \otimes \tau(r) \otimes \alpha''_{r,i}$ for some $r \in \mathcal{B}^e$ and i.

So, $\alpha = \tau(y_\alpha) = \alpha'_{r,i} \otimes \tau(r) \otimes \alpha''_{r,i} \in \mathbb{T}(V) \otimes \operatorname{span}\{\tau(r) : r \in \mathcal{B}^e\} \otimes \mathbb{T}(V)$, and thus $\alpha \notin \mathcal{L}_m(\mathcal{B}^e)$.

Hence, $\mathcal{L}_m(\mathcal{B}^e) \subseteq \tau(Y_\alpha) = \mathcal{R}_m$. So Condition (2) holds. □

VI.6 Anticommutative analogues to face rings

In this section, use the results from Section VI.3 to show some anticommutative rings analogous to face rings are $\mathcal{K}_2$. In particular, we prove:

Theorem VI.6.1. *The algebra*

$$\frac{\bigwedge_{\mathbb{K}}(x_1, \ldots, x_n)}{(x_1 \cdots x_n)}$$

is $\mathcal{K}_2$.

Suppose $X := \{x_1, \ldots, x_n\}$ is a finite set and Δ is a simplicial complex on X—that is, Δ is a subset of the power set 2^X such that $\{x_i\} \in \Delta$ for $1 \leq i \leq n$ and if $Y \in \Delta$, then $2^Y \subset \Delta$. We define an algebra

$$A[\Delta] := \bigwedge_{\mathbb{K}}(x_1, \ldots, x_n) / \left\langle x_{i_1} \cdots x_{i_r} | i_1 < i_2 < \cdots < i_r, \{x_{i_1}, \ldots x_{i_r}\} \notin \Delta \right\rangle,$$

the anticommutative analogue of the face ring of Δ. (Face rings are studied in detail in [20].)

Definition VI.6.2. If $Y \subset X$, $Y \notin \Delta$, but $2^Y \setminus \{Y\} \subset \Delta$, then we say Y is a **minimally missing face** of Δ.

Theorem VI.6.3. *Suppose Δ is a simplicial complex on $X := \{x_1, \ldots, x_n\}$. Under the order $x_1 < \cdots < x_n$, $\ker \pi_{A[\Delta]}$ has an essential Gröbner basis if and only if every minimally missing face $Y := \{x_{i_1}, \ldots, x_{i_m}\} \subset X$ (where $i_1 < i_2 < \cdots < i_m$) satisfies the following property:*

$$\textit{If } u \notin Y \textit{ and } i_1 < u < i_m, \textit{ then } (Y \setminus \{x_{i_1}\}) \cup \{x_u\} \notin \Delta \textit{ or } (Y \setminus \{x_{i_m}\}) \cup \{x_u\} \notin \Delta. \tag{VI.2}$$

Proof. An essential generating set with the leading monomial property for

$$I := \ker(\pi : \mathbb{K}\langle x_1, \ldots, x_n\rangle \to A[\Delta])$$

is $\mathcal{B}^e = \{x_j x_i + x_i x_j | i < j\} \cup \{x_i^2 | i = 1 \ldots n\} \cup \{x_{i_1} \cdots x_{i_m} | i_1 < \cdots < i_m \text{ and } \{x_{i_1}, \ldots, x_{i_m}\} \text{ is a minimally missing face}\}$.

If Y is a minimally missing face which fails (VI.2) for some $u \notin Y$, then

$$x_{i_1} \cdots x_{i_t} x_u x_{i_{t+1}} \cdots x_m$$

is an essential relation of $\mathrm{gr}^F A$ for some t, meaning that $\mathcal{B}^e$ is not a Gröbner basis.

On the other hand, suppose $\mathcal{B}^e$ is not a Gröbner basis. Then $\mathrm{gr}^F A$ has some new essential relation r such that $r \neq \tau(x)$ for $x \in \mathcal{B}^e$. Pick such r minimally. Then

$$r = x_{i_1} \cdots x_{i_m} x_u \bmod \langle x_i x_j + x_j x_i \rangle$$

for some minimally missing face $Y = \{x_{i_1}, \ldots, x_{i_m}\}$. So Y fails (VI.2). $\square$

Proof of Theorem VI.6.1. Let $X := \{x_1, \ldots, x_n\}$ and $\Delta = 2^X \setminus \{X\}$. Then by Theorem VI.6.3, $\ker(\pi : \mathbb{K}\langle x_1, \ldots, x_n\rangle \to A[\Delta])$ has an essential Gröbner basis.

So, applying [9, Theorem 5.3] to

$$\operatorname{gr}^F A = \mathbb{K}\langle x_1, \ldots, x_n\rangle \,/\, \langle x_1 \cdots x_n, x_j x_i : 1 \leq i \leq j \leq n\rangle,$$

we see that $\operatorname{gr}^F A$ is $\mathcal{K}_2$, and hence A is $\mathcal{K}_2$. □

Not every simplicial complex Δ on a set X has an ordering of X which yields an essential Gröbner basis for $\ker \pi_{A[\Delta]}$.

Example VI.6.4. Set $X := \{t, u, w, x, y, z\}$ and

$$\Delta := \left(2^{\{u,x,y,z\}} \cup 2^{\{t,u,x,z\}} \cup 2^{\{u,w,x,z\}}\right) \setminus \{\{u,x,y,z\},$$
$$\{t,u,x,z\}, \{u,w,x,z\}, \{x,y,z\}, \{t,u,z\}, \{u,w,x\}\}.$$

Suppose we have an order $<$ of X under which $\ker \pi_{A[\Delta]}$ has an essential Gröbner basis. Note that $\{x,y,z\}$ is a minimally missing face, but $\{u,x,y\}, \{u,y,z\}, \{u,x,z\} \in \Delta$. So either $u < x,y,z$ or $u > x,y,z$. Without loss of generality, $u < x,y,z$. Also, $\{t,u,z\}$ is a minimally missing face, but $\{u,x,z\}, \{t,x,z\}, \{t,u,x\} \in \Delta$. So as $u < x$ we have $x > t,u,z$. Finally, $\{u,w,x\}$ is a minimally missing face, but $\{u,x,z\}, \{u,w,z\}, \{w,x,z\} \in \Delta$. However, as $x > z$, we cannot have $z > x,u,w$. However, as $u < z$, we cannot have $z < x,u,w$ either.

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