MULTINETS IN $\mathbb{P}^2$ AND $\mathbb{P}^3$

by

JEREMIAH DAVID BARTZ

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2013

DISSERTATION APPROVAL PAGE

Student: Jeremiah David Bartz

Title: Multinets in $\mathbb{P}^2$ and $\mathbb{P}^3$

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Sergey Yuzvinsky	Chair
Nicholas Proudfoot	Core Member
Brad Shelton	Core Member
Dev Sinha	Core Member
Andrzej Proskurowski	Institutional Representative

and

Kimberly Andrews Espy	Vice President for Research & Innovation/ Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2013

DISSERTATION ABSTRACT

Jeremiah David Bartz

Doctor of Philosophy

Department of Mathematics

June 2013

Title: Multinets in $\mathbb{P}^2$ and $\mathbb{P}^3$

In this dissertation, a method for producing multinets from a given net in $\mathbb{P}^3$ is presented. Multinets play an important role in the study of resonance varieties of the complement of a complex hyperplane arrangement and very few examples are known. Implementing this method, numerous new and interesting examples of multinets are identified. Each of these examples is the degeneration of a net, supporting the conjecture of Pereira and Yuzvinsky that all multinets are degenerations of nets. Also, a complete description is given of proper weak multinets, a generalization of multinets.

CURRICULUM VITAE

NAME OF AUTHOR: Jeremiah David Bartz

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
University of North Dakota, Grand Forks, ND

DEGREES AWARDED:

Doctor of Philosophy in Mathematics, 2013, University of Oregon
Master of Science in Mathematics, 2006, University of North Dakota
Bachelor of Science in Mathematics, 2004, University of North Dakota
Bachelor of Science in Mechanical Engineering, 2004, University of North Dakota

AREAS OF SPECIAL INTEREST:

Nets and multinets, hyperplane arrangements, combinatorics, discrete geometry

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2006-2013

Graduate Teaching Assistant, University of North Dakota, 2004-2006

GRANTS, AWARDS AND HONORS:

Graduate Teaching Fellowship, University of Oregon, 2006-2013

Graduate Teaching Assistantship, University of North Dakota, 2004-2006

Summa cum Laude, University of North Dakota, 2004

ACKNOWLEDGEMENTS

During my seven years at the University of Oregon, I have received support and encouragement from a great number of sources. First, thank you to my advisor, Sergey Yuzvinsky, for his guidance, endless patience, and introducing me to the area of hyperplane arrangements. Thank you to Nicholas Proudfoot, Brad Shelton, Dev Sinha, and Andrzej Proskurowski for serving as committee members. Thank you to the faculty at the University of Oregon for expanding my mathematical horizons through a wide array of interesting courses. Thank you to Michael Price and Wendy Sullivan for your advice and helpfulness with all aspects of teaching. Thank you to the office staff of the Department of Mathematics, especially Eva Lunnemann and Brandalee Davis, for the friendly emails regarding university deadlines and formal paperwork.

Thank you to my family and friends for always believing that I could do this. Thank you to my fellow classmates in my cohort, especially Robert Fisette, Nicholas Johnson, Aaron Montgomery, Josiah Thornton, Aaron Bennett, and Tristan Brand, for all of the memories in Eugene including homework sessions in Fenton Hall, volleyball, pinochle, tennis, Thanksgiving celebrations, intramural racquetball tournaments, trail running, track meets at Hayward Field, Monday night movies, Thursday morning golf, and Saturday bike rides. Thank you to Eusebio Gardella for the impromptu Spanish lessons at the office and introducing me to empanadas. Thank you to Tyler Kloefkorn for our many conversations on hyperplane arrangements. Thank you to Jim Beyer for the early morning conversations at the pool. Thank you to Curtis Becker, Nathan Hill, Corby Heyne, Kasey Young, Craig Brown, Bjorn

Johnson, and Jon Carpenter for being amazing friends to me throughout the years. To everyone else, thank you for enhancing my time in Eugene.

Lastly, thank you to my parents, David and Darleen, and siblings, Joshua and Jacquelyn, for their unwavering support in my academic endeavors. Thank you for the telephone calls, letters, care packages, homemade cookies, and visits to Eugene. I am very fortunate to have such a wonderful family.

To my family.

TABLE OF CONTENTS

Chapter Page

LIST OF FIGURES

Figure Page

LIST OF TABLES

Table Page

CHAPTER I

INTRODUCTION

The study of resonance varieties of the complement of a complex hyperplane arrangement is an area of current research. The initial allure of these varieties stems from their connections with the jumping loci of the cohomology with local coefficients of the complement. More recently, resonance varieties have played a role in other areas of arrangement theory such as the cohomology of Milnor fibers and roots of b-functions.

Resonance varieties can be defined for general topological spaces.

Definition 1.1. Let X be a connected topological space and $A(X) = \oplus_{i \geq 0} A_i$ denote its graded cohomology algebra over $\mathbb{C}$. Each $a \in A_1$ yields a cochain complex $(A(X), a)$ given by

$$0 \longrightarrow A_0 \xrightarrow{\cdot a} A_1 \xrightarrow{\cdot a} A_2 \xrightarrow{\cdot a} \dots$$

The first-degree resonance variety of X is

$$\begin{aligned} \mathcal{R}^1(X) &= \{a \in A_1 : H^1(A, a) \neq 0\} \\ &= \{a \in A_1 : \exists b \in A_1 \text{ where } b \neq \lambda a, \lambda \in \mathbb{C}, \text{ and } ab = 0\}. \end{aligned}$$

The complement of a complex hyperplane arrangement $\mathcal{A}$ is one type of topological space for which deeper results on its resonance varieties are known. For example, the main result in [6] is that the existence of a nontrivial resonance variety $\mathcal{R}^1(\mathcal{A})$ is equivalent to several different properties. One of these is that $\mathcal{A}$ is a certain special configuration of points and lines in the complex projective plane $\mathbb{P}^2$ called nets and multinets. Another is the existence of a connected pencil of plane curves with irreducible generic fiber and at least three completely reducible fibers. These equivalences provide motivation to study nets and multinets.

The notation introduced here is standard in arrangement theory. A *complex hyperplane arrangement* $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{C}^n$ or $\mathbb{P}^n$. When $n = 2$, the arrangement is referred to as a *line arrangement* and denoted $\mathcal{L}$. For an arrangement $\mathcal{A}$ in $\mathbb{P}^n$, each hyperplane $H \in \mathcal{A}$ can be specified by a homogeneous linear form α_H, up to a multiplicative constant, via $H = \ker \alpha_H$. It is convenient to write $\alpha_H \doteq \alpha'_H$ if $\alpha_H = c\alpha'_H$ for some $c \in \mathbb{C}^\times$. The product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is called a *defining polynomial* of $\mathcal{A}$ and referred to simply as Q when no confusion arises. The *intersection poset* of $\mathcal{A}$, denoted $L = L(\mathcal{A})$, is the set of nonempty intersections of elements of $\mathcal{A}$ with partial ordering given by reverse inclusion. Two arrangements are *lattice equivalent* if there is an order preserving bijection between their intersection posets. An arrangement $\mathcal{A}$ is *central* if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. It is well-known that the intersection poset of a central arrangement is a geometric lattice with rank given by codimension. For this reason, $L(\mathcal{A})$ is referred to as the *intersection lattice* when $\mathcal{A}$ is central. All arrangements considered below are central.

In this dissertation, the main objects of study are nets and multinets, certain line arrangements in $\mathbb{P}^2$. There has been significant developments in the theory of nets as seen in [1], [8], [9], [14], [15], and [16]. On the other hand, less progress has been made regarding multinets, the generalization of nets obtained by allowing multiplicities of points and lines. Notable advances related to multinets occured in [5], [6], [13], and [17]. Nevertheless, there remains very few known general results and examples of multinets.

A generalization of multinets called weak multinets was introduced in [6]. In Chapter II, the relationships between nets, multinets, and a weak mulitnets are explored using definitions, basic properties, and a few other tools. Several examples are given. The main result of this chapter is a complete description of the types of proper weak multinets.

In Chapter III, a method for obtaining multinets from a given net in $\mathbb{P}^3$ is presented. This is done by intersecting the net by a certain choice of hyperplane. The procedure is implemented in one case, and the resulting multinets are classified up to equivalence.

In Chapter IV, an invariant of multinets called graph type is introduced as an aid to distinguish nonisomorphic multinets. The method from the previous chapter for obtaining multinets is applied to additional cases, and the resulting multinets are classified up to graph type. These efforts are rewarded with interesting and unexpected new examples of multinets, several of which are discussed in greater depth at the end of the chapter. Appendix A provides examples of each graph type found in the investigated cases and gives a complete compilation of multinets known at this time. Each of these multinets is a degeneration of a net, supporting the conjecture in [13] that every multinet is a degeneration of a net.

CHAPTER II

WEAK MULTINETS, MULTINETS, AND NETS

Nets have a long association with finite geometries, latin squares, quasigroups, and loops which is discussed in [3]. More recently, it was discovered that nets and their generalization, multinets, play a special part in the study of resonance varieties of complements of complex hyperplane arrangements. Nets initially appeared in this latter context implicitly in [9] and explicity in [16]. Multinets are a fresh notion and were first defined in [6].

In this chapter, the relationships between weak multinets, multinets, and nets are explored using definitions, basic properties, and a few other tools. Several examples are given. The main result is a complete description of proper weak multinets.

2.1. Definitions

Let $\mathcal{L}$ be a line arrangement in $\mathbb{P}^2$ and $m : \mathcal{L} \to \mathbb{Z}_{>0}$ be a function assigning each line $\ell \in \mathcal{L}$ a positive integer $m(\ell)$ called the *multiplicity* of the line. The pair $(\mathcal{L}, m)$ is referred to as a *multi-arrangement*. A *multiple point* is a point which lies on at least three lines of $\mathcal{L}$.

Definition 2.1. A *weak k-multinet* on a multi-arrangement $(\mathcal{L}, m)$ is a pair $(\mathcal{N}, \mathcal{X})$ where $\mathcal{N}$ is a partition of $\mathcal{L}$ into $k \geq 3$ classes $\mathcal{L}_1, \ldots, \mathcal{L}_k$, and $\mathcal{X}$ is a set of multiple points called the *base locus* satisfying the following conditions:

(i) $\sum_{\ell \in \mathcal{L}_i} m(\ell)$ is independent of i;

(ii) for every $\ell \in \mathcal{L}_i$ and $\ell' \in \mathcal{L}_j$ with $i \neq j$, the point $\ell \cap \ell' \in \mathcal{X}$;

(iii) for each $p \in \mathcal{X}$, $\sum_{\ell \in \mathcal{L}_i, p \in \ell} m(\ell)$ is independent of i.

A k-*multinet* is a weak k-multinet satisfying the additional condition:

(iv) for each $1 \leq i \leq k$ and any $\ell, \ell' \in \mathcal{L}_i$, there is a sequence of $\ell = \ell_0, \ell_1, \ldots, \ell_r = \ell'$ such that $\ell_{j-1} \cap \ell_j \notin \mathcal{X}$ for $1 \leq j \leq r$.

It is useful to have a geometrical interpretation of these conditions. The first says that each class $\mathcal{L}_i$ contains the same amount of lines when multiplicities are considered. The second states that any two lines from distinct classes intersect at a point in the base locus $\mathcal{X}$. The third condition establishes that for each point $p \in \mathcal{X}$, the number of lines $\ell \in \mathcal{L}_i$ incident with p is independent of the choice of class when considering multiplicities. The last condition can be viewed as a connectivity condition in the following way. It says that each class of a multinet is connected when the multinet is blown up at its base locus $\mathcal{X}$.

It will be convenient to suppress notation and refer to a weak multinet or multinet simply as $\mathcal{L}$ when no confusion will arise. The common number $\sum_{\ell \in \mathcal{L}_i} m(\ell)$, denoted d, is called the *degree* of the weak k-multinet. A weak k-multinet of degree d is often referred to as a weak (k, d)-multinet. Similar statements are made for multinets.

Remark 2.2. Given any weak (k, d)-multinet $(\mathcal{N}, \mathcal{X})$ on $(\mathcal{L}, m)$, multiplying all $m(\ell)$ by a positive integer c defines a weak (k, cd)-multinet with same $\mathcal{L}$, $\mathcal{N}$, and $\mathcal{X}$. It will be assumed that d is always minimal. In other words, the multiplicities of the lines are assumed to be mutually relatively prime.

Remark 2.3. The base locus $\mathcal{X}$ of a weak multinet $\mathcal{L}$ is determined by its partition $\mathcal{N}$, namely $\mathcal{X} = \{\ell \cap \ell' : \ell \in \mathcal{L}_i, \ell' \in \mathcal{L}_j, i \neq j\}$. If $\mathcal{L}$ is a multinet, $\mathcal{X}$ conversely determines the partition of $\mathcal{L}$. To see this, construct a graph Γ with vertex set $\mathcal{L}$ and an edge from ℓ to ℓ' if $\ell \cap \ell' \notin \mathcal{X}$. Then the classes $\mathcal{L}_i$ are the components of Γ.

For each point $p \in \mathcal{X}$, the *multiplicity* of p is the common number $\sum_{\ell \in \mathcal{L}_i, p \in \ell} m(\ell)$ and labeled as n_p.

Definition 2.4. A *net* is a multinet with $n_p = 1$ for all $p \in \mathcal{X}$.

In particular, nets necessarily have $m(\ell) = 1$ for all $\ell \in \mathcal{L}$ by condition (iii) of Definition 2.1. That is, the multiplicity of each line of a net is one. There are other implications of the condition $n_p = 1$. For instance, conditions (i) and (iv) of Definition 2.1 are direct consequences of the remaining conditions and $n_p = 1$. After reinterpretating condition (iii), the definition for nets can be restated as follows.

Definition 2.5. A *k-net* in $\mathbb{P}^2$ is a pair $(\mathcal{L}, \mathcal{X})$ where $\mathcal{L}$ is a finite collection of lines in $\mathbb{P}^2$ partitioned into $k \geq 3$ classes $\mathcal{L}_1, \ldots, \mathcal{L}_k$, and $\mathcal{X}$ is a finite set of points called the *base locus* satisfying the following conditions:

(i) for every $\ell \in \mathcal{L}_i$ and $\ell' \in \mathcal{L}_j$ with $i \neq j$, the point $\ell \cap \ell' \in \mathcal{X}$;

(ii) for every $p \in \mathcal{X}$ and every i $(i = 1, \ldots, k)$, there exists a unique $\ell \in \mathcal{L}_i$ such that $p \in \ell$.

The next proposition is one of the few general results on weak multinets. Originally appearing in [6], it identifies relationships between the various numerical quantities of a weak multinet. Its proof uses the definition and counting arguments.

Proposition 2.6. Let $\mathcal{L}$ be a weak (k, d)-multinet. Then

(i) $\sum_{\ell \in \mathcal{L}} m(\ell) = dk$;

(ii) $\sum_{p \in \mathcal{X}} n_p^2 = d^2$;

(iii) For each $\ell \in \mathcal{L}$, $\sum_{p \in \mathcal{X} \cap \ell} n_p = d$.

It is immediate from these definitions that all nets are multinets, and all multinets are weak multinets. The next objective is understanding the effects of each additional condition imposed when transitioning from weak multinets to multinets to nets. With this goal in mind, the following terminology is introduced. A *proper weak multinet* is a weak multinet which is not a multinet. A *proper multinet* is a multinet which is not a net. The first issue to address is existence.

Question 2.7. Do proper weak multinets exist?

Question 2.8. Do proper multinets exist?

2.2. Examples of Proper Weak Multinets

Proper weak multinets exist, and two types of examples were exhibited in [6]. Descriptions of these examples are given below after the following definition.

Definition 2.9. A weak multinet is *trivial* if $|\mathcal{X}| = 1$.

Remark 2.10. Alternatively, a weak multinet is *trivial* if it is equivalent to a hyperplane arrangement in $\mathbb{P}^1$.

Example 2.11. A trivial weak multinet is a proper weak multinet if there exists at least one class containing two distinct lines. In this situation, condition (*iv*) of Definition 2.1 fails for the distinct lines lying in the same class. For example,

$$Q = [x^2][y^2][(x-y)(x+y)]$$

defines a proper weak $(3, 2)$-multinet. The three classes are distinguished via brackets, and exponents indicate the multiplicity of each line. The depiction of this arrangement in $\mathbb{RP}^2$ is given in Figure 2.1(a).

To create this proper weak multinet, at least one class contained two distinct lines of the trivial weak multinet. Then multiplicities of lines were appropriately chosen to satisfy the conditions of a weak multinet. This type of construction is always possible when the trivial weak multinet contains at least four distinct lines.

Example 2.12. The Hessian arrangement is a well-known $(4,3)$-net. Denote its classes by C_0, C_1, C_2, and C_3 and let ξ be a primitive third root of unity. The Hessian arrangement is defined by $Q = C_0C_1C_2C_3$ where

$$\begin{aligned}
C_0 &= xyz \\
C_1 &= (x+y+z)(x+\xi y+\xi^2 z)(x+\xi^2 y+\xi z) \\
C_2 &= (x+y+\xi z)(x+\xi y+z)(x+\xi^2 y+\xi^2 z) \\
C_3 &= (x+y+\xi^2 z)(x+\xi y+\xi z)(x+\xi^2 y+z).
\end{aligned}$$

Then $Q' = [C_0C_1][C_2^2][C_3^2]$ defines a proper weak $(3,6)$-multinet. Again, the three classes are distinguished via brackets, and exponents indicate the multiplicities of each line.

In the construction of this proper weak multinet, two classes of the (4,3)-net are combined. Then appropriate multiplicities are assigned to the remaining classes to meet the conditions of a weak multinet. To see this weak multinet is proper, take $\ell \in C_0$ and $\ell' \in C_1$. In Definition 2.1, it follows from condition (ii) that no such sequence in condition (iv) exists.

The Hessian arrangement is a complex arrangement which cannot be realized in $\mathbb{RP}^2$. A diagram of this arrangement appears in Figure 2.1(b). The four classes are indicated by different styled lines. More information about the Hessian arrangement can be found in [2].

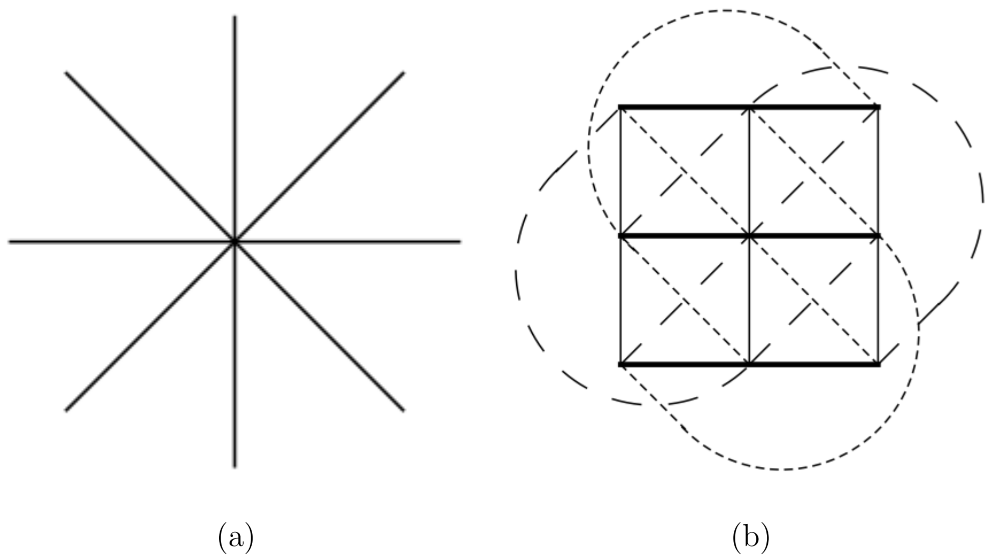


FIGURE 2.1. Examples of proper weak multinets.

Remark 2.13. The curves C_0, C_1, C_2, and C_3 are four completely reducible fibers of the Hesse pencil of plane curves, namely $u(x^3 + y^3 + z^3) + t(xyz)$ where $[u : t] \in \mathbb{P}^1$. These curves are the fibers of the pencil corresponding to $[0, 1]$, $[1 : -3]$, $[1 : -3\xi]$, and $[1 : -3\xi^2]$, respectively, where ξ is a primitive third root of unity. Pencils of plane curves and their connection with multinets are discussed in Chapter IV.

2.3. The Matrix Q and Refinements

A useful tool in studying weak multinets is its associated matrix Q. This matrix first appeared in [9] in the study of nets and reappeared in [6]. The following is a summary of the ideas from these two papers relevelent to the current investigations. These results are used to establish a new theorem regarding the types of examples of proper weak multinets.

Suppose $\mathcal{L}$ is a weak multinet. Let J be its $|\mathcal{X}| \times |\mathcal{L}|$ incidence matrix $(a_{p,\ell})$. That is, $a_{p,\ell} = 1$ if $p \in \ell$ and $a_{p,\ell} = 0$ otherwise. The matrix Q associated to $\mathcal{L}$ is defined to be the $|\mathcal{L}| \times |\mathcal{L}|$ matrix given by $Q = J^T J - E$ where E is the $|\mathcal{L}| \times |\mathcal{L}|$ matrix with every entry 1.

The matrix Q is a generalized Cartan matrix, symmetric with integers on the main diagonal and -1 or 0 off of the main diagonal. Consequently, there is a block direct sum decomposition $Q = Q_1 \oplus \cdots \oplus Q_k$ with each Q_i indecomposable of affine type with regards to Vinberg's classification. Define a graph Γ with vertex set $\mathcal{L}$ and edge connecting ℓ and ℓ' if $\ell \cap \ell' \notin \mathcal{X}$. Then the indecomposable blocks are precisely the restriction of Q to the connected components of Γ. More information on Vinberg's classification can be found in [7].

A *refinement* of a weak multinet $(\mathcal{N}, \mathcal{X})$ on $(\mathcal{L}, m)$ is a weak multinet $(\mathcal{N}', \mathcal{X})$ on $(\mathcal{L}, m')$ where $\mathcal{N}'$ is a refinement of $\mathcal{N}$. Note that m' may be different from m. That is, multiplicities are permitted to be changed in a refinement. Also, there are numerous refinements of a given weak multinet.

The following result appeared in [6].

Proposition 2.14. Any weak k-multinet refines to a k'-multinet where $k' \geq k$.

In the subsequent work on multinets, the following observation will be useful.

Proposition 2.15. Any trivial weak k-multinet refines to a $(k', 1)$-net with $k' \geq k$.

Proof. Let $\mathcal{L}$ be a trivial weak k-multinet. Let $\mathcal{N}'$ be the partition of $\mathcal{L}$ consisting of one line in each equivalence class and m' be the multiplicity function assigning multiplicity one to each line. Then $(\mathcal{N}', \mathcal{X})$ is a $(k', 1)$-net and a refinement of $\mathcal{L}$. •

2.4. Results on Proper Weak Multinets

Examples 2.11 and 2.12 exhibit the only two types of proper weak multinets that occur. This is the main result on proper weak multinets and established below.

Lemma 2.16. A proper weak k-multinet refines to a k'-multinet with $k' \geq k+1$.

Proof. By failure of condition (iv) in Definition 2.1, a proper weak multinet necessarily has a class whose corresponding block in the matrix Q decomposes into at least two indecomposable blocks in the refinement. •

Theorem 2.17. Any proper weak multinet is either trivial or obtained by combining classes of a proper 4-net.

Proof. Suppose $\mathcal{L}$ is a proper weak k-multinet and consider its k'-multinet refinement. Since $k \geq 3$, it follows that $k' \geq 4$ by Lemma 2.16. If $k' > 4$, then the multinet refinement of $\mathcal{L}$ is necessarily a net, and any k'-net with $k' > 4$ is trivial by results in [17]. Since $\mathcal{X}$ is preserved during refinement, $\mathcal{L}$ is a trivial weak multinet. If $k' = 4$, then the refined multinet is a 4-net by [17]. Again noting that $\mathcal{X}$ is preserved during refinement, a proper weak 3-multinet only occurs by combining two classes of the 4-net and assigning appropriate multiplicities to the other two classes. •

Remark 2.18. The Hessian arrangement is the only known example of a 4-net. It is conjectured that no other 4-nets exist. If this conjecture is true, then the only examples of weak proper multinets with $|\mathcal{X}| > 1$ are constructed from the Hessian arrangement.

2.5. Past Examples of Proper Multinets

With a complete understanding of the types of examples of proper weak multinets, the focus shifts to proper multinets. Falk and Yuzvinsky identified several examples of proper multinets in [6]. These were the first and only known examples of proper multinets prior to the subsequent work in this dissertation. A summary of these past examples of proper multinets is given below.

In Chapters III and IV, a method of producing multinets from nets in $\mathbb{P}^3$ is introduced and implemented. This results in the discovery of numerous new examples of proper multinets.

Example 2.19. The first example is a $(3,4)$-multinet with all lines of multiplicity 1, one point of $\mathcal{X}$ of multiplicity 2, and remaining points of $\mathcal{X}$ with multiplicity 1. This arrangement is realizable in $\mathbb{RP}^2$ and depicted in Figure 2.2(a). In Chapter IV, it will be shown that this arrangement has graph type $G_1(2)$ and fits into an infinite family of examples.

Example 2.20. The collection of arrangements defined by

$$Q_n = [(x^n - y^n)z^n][(x^n - z^n)(y^n - 2^n z^n)][(y^n - z^n)(x^n - 2^n z^n)]$$

where $n \geq 2$ gives an infinite family of proper multinets. Each n defines a proper $(3, 2n)$-multinet which has a unique line of multiplicity n. The remaining lines have multiplicity 1. Also, two points of $\mathcal{X}$ have multiplicity n while all other points of $\mathcal{X}$ have multiplicity 1. Figure 2.2(b) gives a depiction in $\mathbb{RP}^2$ of this arrangement when $n = 2$. Later, it will be shown that these arrangements have graph types $G_2(n)$.

Example 2.21. The collection of arrangements defined by

$$Q_n = [(x^n - y^n)z^n][(x^n - z^n)y^n][(y^n - z^n)x^n]$$

where $n \geq 2$ gives another infinite family of proper multinets. Each n defines a proper $(3, 2n)$-multinet which has three lines of multiplicity n. The remaining lines have multiplicity 1. Also, three points of $\mathcal{X}$ have multiplicity n while all other points of $\mathcal{X}$ have multiplicity 1. A depiction in $\mathbb{RP}^2$ of this arrangement when $n = 2$ is given in Figure 2.2(c). These arrangements will be shown to have graph types $G_3(n)$.

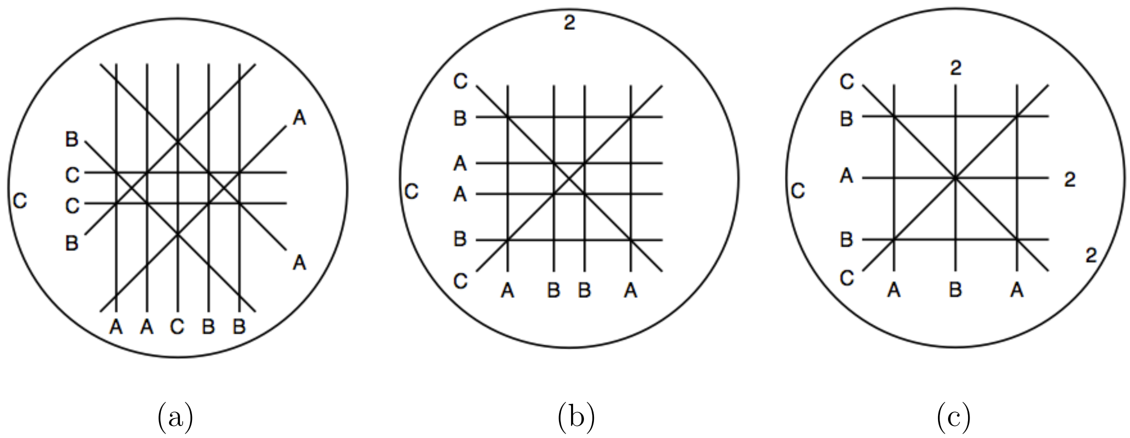


FIGURE 2.2. Past examples of proper (3,4)-multinets.

CHAPTER III

MULTINETS FROM NETS IN $\mathbb{P}^3$

The central idea in this dissertation is a method to produce multinets from nets in $\mathbb{P}^3$. The definition of nets in $\mathbb{P}^2$ given in Definition 2.5 involves conditions on lines and points, objects of rank one and rank two, respectively, in the intersection lattice. It can be naturally extended to define nets in $\mathbb{P}^n$ for $n > 1$ by replacing these conditions on lines and points with analogous ones involving hyperplanes and intersections of hyperplanes. In both situations, the defining conditions of a net depend on rank one and rank two elements of the intersection lattice of the arrangement.

An element of the intersection lattice of an arrangement $\mathcal{A}$ is called *multiple* if it is contained in at least three hyperplanes of $\mathcal{A}$.

Definition 3.1. Let $n > 1$. A *k-net* in $\mathbb{P}^n$ is a pair $(\mathcal{A}, \mathcal{X})$ where $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{P}^n$ partitioned into $k \geq 3$ classes $\mathcal{A}_1, \ldots, \mathcal{A}_k$, and $\mathcal{X}$ is a set of rank two multiple elements of the intersection lattice called the *base locus* satisfying the following conditions:

(i) for every $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ with $i \neq j$, the element $H \cap H' \in \mathcal{X}$

(ii) for every $P \in \mathcal{X}$ and every i $(i = 1, \ldots, k)$, there exists a unique $H \in \mathcal{A}_i$ such that $P \subseteq H$.

Again, it is often convenient to suppress notation and refer to a k-net simply as $\mathcal{A}$ when no confusion will arise and the space $\mathbb{P}^n$ is clear from context. The next proposition generalizes a result for nets in $\mathbb{P}^2$.

Proposition 3.2. Suppose $\mathcal{A}$ is a k-net in $\mathbb{P}^n$. Then $|\mathcal{A}_i|$ is independent of i.

Proof. The definition of k-net in $\mathbb{P}^n$ implies that $|\mathcal{A}_i| = |H \cap \mathcal{X}|$ for every i and every $H \in \mathcal{A}$. $\bullet$

Extending conventions established earlier, the common number $|\mathcal{A}_i|$ is denoted d and called the *degree* of the k-net. Again, a k-net of degree d is often referred to as a (k, d)-net.

The method for producing multinets presented in this dissertation utilizes nets in $\mathbb{P}^3$. It was shown in [13] that there are no nontrivial nets in $\mathbb{P}^n$ for $n > 4$. In addition, there are no nontrivial proper multinets $\mathbb{P}^n$ for $n > 2$. In this context, a net or multinet is considered *nontrivial* if it cannot be realized in a lower dimensional space. Currently, there are no known nontrivial nets in $\mathbb{P}^4$ and only one known family of nontrivial nets in $\mathbb{P}^3$ which appeared in [13].

Let $n \in \mathbb{Z}_{>0}$. Consider the arrangement in $\mathbb{P}^3$ given by

$$Q_n = [(x^n - y^n)(z^n - w^n)][(x^n - z^n)(y^n - w^n)][(x^n - w^n)(y^n - z^n)].$$

It was shown in [13] that this arrangement supports a $(3, 2n)$-net in $\mathbb{P}^3$. In addition, it was observed that intersecting this arrangement by a generic hyperplane produces a $(3, 2n)$-net in $\mathbb{P}^2$. On the other hand, intersecting by the hyperplane defined by $w = 0$ yields the family of multinets

$$Q'_n = [x^n(y^n - z^n)][y^n(x^n - z^n)][z^n(x^n - y^n)],$$

one multinet for each n. These multinets are proper when $n > 1$ and can be viewed as limits of nets through a deformation process. This family was previously mentioned in Example 2.21. A closer examination reveals that other examples of multinets can be obtained by intersecting the arrangement Q_n with different choices of hyperplanes. This is the seminal observation for producing additional examples of proper multinets.

The following terminology will be used throughout the remainder of the dissertation. The process of intersecting a net in $\mathbb{P}^3$ with a hyperplane will be called *slicing*. The intersecting hyperplane will be referred to as the *slicing hyperplane*. The line arrangement in $\mathbb{P}^2$ obtained by slicing will be called the *slice*.

It will be shown that most choices of the slicing hyperplane produce multinets in $\mathbb{P}^2$. However, some choices lead to pathological cases and should be avoided. Consequently, care must be taken in selecting the slicing hyperplane.

If $n = 1$, it will be seen that any choice of slicing hyperplane can be made and yields either the unique $(3, 2)$-net or an arrangement which refines to a trivial multinet. When $n > 1$, a sufficient condition to obtain a multinet is that the classes of hyperplanes in the original multinet structure of Q_n are preserved during slicing. This condition ensures that slicing is done in a manner such that two lines from distinct classes of Q_n do not become identified in the slice. In particular, the slicing hyperplane cannot be one of the hyperplanes of Q_n for $n > 1$. If the hyperplane satisfies these restrictions on the slicing hyperplane, it will be called *allowable*. Otherwise, it will be called *forbidden*. These notions will be discussed further in Chapter IV.

The overall goal of the upcoming analysis is to extract as many examples of multinets as possible from allowable slices of Q_n. The intersection lattice L_n of the arrangement Q_n plays a prominent role in this endeavor. The description of L_n naturally separate into two cases, $n = 1$ and $n > 1$. As a result, the analysis of slicing Q_n occurs in two phases. Slices of Q_1 is the focus for the remainder of this chapter, and slices of Q_n for $n > 1$ are discussed in the next chapter.

3.1. The Intersection Lattice L_1

The arrangement defined by

$$Q_1 = [(x-y)(z-w)][(x-z)(y-w)][(x-w)(y-z)]$$

is the well-known braid arrangement with Coxeter group of type A_3. It is convenient to impose a linear order on the hyperplanes of Q_1 and establish some conventions. Let

$$\begin{array}{lll} H_1 = y - z & H_3 = x - y & H_5 = x - z \\ H_2 = x - w & H_4 = z - w & H_6 = y - w. \end{array}$$

There is a one-to-one correspondence between points and hyperplanes in $\mathbb{P}^3$ given by projective duality. The bijection associates the point $[a : b : c : d] \in \mathbb{P}^3$ to the hyperplane $ax + by + cz + dw = 0$. Consequently, a hyperplane in $\mathbb{P}^3$ can be described by its associated point in $\mathbb{P}^3$ under this correspondence. It will be clear from context whether $[a : b : c : d] \in \mathbb{P}^3$ indicates a point or hyperplane in $\mathbb{P}^3$.

It is common to describe elements of the intersection lattice using set notation and the arbitrary linear order chosen on the hyperplanes of the arrangement. Let the singleton $\{i\}$ denote the hyperplane H_i, and let the subset $\{i_1, \ldots, i_k\}$ denote the intersection of $H_{i_1}, \ldots, H_{i_k}$. Each element of the intersection lattice L_1 of Q_1 are described in Proposition 3.3 in two ways, algebraically and via set notation. Arrows indicate equivalent descriptions.

The symmetric group S_4 acts on $\{x, y, z, w\}$ by permutation. This extends to an action on $\mathbb{C}[x, y, z, w]$ which fixes Q_1 and induces an action of S_4 on the intersection lattice L_1. The groupings used in the description of L_1 correspond to the orbits of this latter action. The choice of names assigned was motivated by the role each orbit plays in slicing.

Proposition 3.3. The intersection lattice L_1 of Q_1 in $\mathbb{P}^3$ has rank 3. Its elements consist of $\mathbb{P}^3$, hyperplanes, lines, and points. More precisely,

• There are 6 hyperplanes, namely

$$\begin{aligned} H_1 &= [0:1:-1:0] &\leftrightarrow& \quad \{1\} \\ H_2 &= [1:0:0:-1] &\leftrightarrow& \quad \{2\} \\ H_3 &= [1:-1:0:0] &\leftrightarrow& \quad \{3\} \\ H_4 &= [0:0:1:-1] &\leftrightarrow& \quad \{4\} \\ H_5 &= [1:0:-1:0] &\leftrightarrow& \quad \{5\} \\ H_6 &= [0:1:0:-1] &\leftrightarrow& \quad \{6\}. \end{aligned}$$

• There are 7 lines which consist of the following two types.

○ There are 4 *locus lines* given by

$$\begin{aligned} [1:0:0:0]u + [0:1:1:1]t &\quad\leftrightarrow\quad \{1,4,6\} \\ [0:1:0:0]u + [1:0:1:1]t &\quad\leftrightarrow\quad \{2,4,5\} \\ [0:0:1:0]u + [1:1:0:1]t &\quad\leftrightarrow\quad \{2,3,6\} \\ [0:0:0:1]u + [1:1:1:0]t &\quad\leftrightarrow\quad \{1,3,5\} \end{aligned}$$

where $[u:t] \in \mathbb{P}^1$.

○ There are 3 *double lines* given by

$$\begin{cases} [1:0:0:1]u + [0:1:1:0]t \quad\leftrightarrow\quad \{1,2\} \\ [1:1:0:0]u + [0:0:1:1]t \quad\leftrightarrow\quad \{3,4\} \\ [1:0:1:0]u + [0:1:0:1]t \quad\leftrightarrow\quad \{5,6\} \end{cases}$$

where $[u:t] \in \mathbb{P}^1$.

• There is a unique point P, namely

$$P = [1:1:1:1] \quad\leftrightarrow\quad \{1,2,3,4,5,6\}.$$

Proof. This is a straightforward computation. Nevertheless, a check is performed. The Poincaré polynomials of braid arrangements are well-known. Using the description of L_1 given in the proposition, the Poincaré polynomial of Q_1 is computed directly from its definition and checked for agreement with the its listing in [12].

Considered as a central arrangement in $\mathbb{C}^4$ in [12], the Poincaré polynomial of Q_1 is

$$\begin{aligned}\pi(Q_1, t) &= 6t^4 + 17t^3 + 17t^2 + 7t + 1 \\ &= (1+t)^2(1+2t)(1+3t).\end{aligned}$$

On the other hand, it follows from the description given in the proposition that L_1 has the properties found in Table 3.1.

TABLE 3.1. Properties of L_1.

Lattice element	Rank r	Number	Value of Möbius function μ
$\mathbb{P}^3$	0	1	$+1$
hyperplanes	1	6	-1
locus lines	2	4	$+2$
double lines	2	3	$+1$
unique point	3	1	-6

The values of the Möbius function μ of the lattice element X are determined from the recurrence relation

$$\sum_{Y \leq X} \mu(Y) = 0$$

with initial condition $\mu(\mathbb{P}^3) = 1$. The definition of the Poincaré polynomial, namely

$$\pi(\mathcal{A}, t) = \sum_{X \in L} \mu(X)(-t)^{r(X)}$$

is used to see

$$\begin{aligned} \pi(Q_1, t) &= 6t^3 + 11t^2 + 6t + 1 \\ &= (1+t)(1+2t)(1+3t). \end{aligned}$$

These two Poincaré polynomials differ by a factor of $1 + t$, reflecting the well-known effect on the Poincaré polynomial of projectivizing a central arrangement. •

Remark 3.4. The arrangement $\mathcal{A}$ defined by Q_1 supports the structure of a (3,2)-net in $\mathbb{P}^3$ with classes $\mathcal{A}_1 = \{H_1, H_2\}$, $\mathcal{A}_2 = \{H_3, H_4\}$, and $\mathcal{A}_3 = \{H_5, H_6\}$. It's base locus $\mathcal{X}$ consists of the four locus lines.

3.2. Isomorphisms of Multinets

The groundwork for the investigation of slices of Q_1 continues by making precise the notion of sameness of a pair of weak multinets or multinets.

Definition 3.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two weak multinets. A *weak multinet isomorphism* $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is a bijection sending $\mathcal{N}_1$ to $\mathcal{N}_2$ that satisfies the following condition: for every $p \in \mathcal{X}_1$, the point $\cap_{i \in S_p} \phi(\ell_i) \in \mathcal{X}_2$ where $S_p = \{i : p \in \ell_i\}$.

It is apparent from this definition that weak multinet isomorphisms preserve all the combinatorial data of weak multinets, namely classes, line multiplicities, and line intersection relations from the base locus. Also, the collection of weak multinets and weak multinet isomorphisms forms a category.

Isomorphisms between multinets are of particular interest for the purposes of this dissertation. As mentioned in Remark 2.3, the partition $\mathcal{N}$ of $\mathcal{L}$ can be recovered from $\mathcal{X}$ when $\mathcal{L}$ is a multinet via components of the graph Γ. Since ϕ preserves the intersection relations of the base locus from $\mathcal{X}_1$ to $\mathcal{X}_2$, the map from Γ_1 to Γ_2 induced by ϕ gives a bijection between components of Γ_1 and Γ_2, hence a bijection between $\mathcal{N}_1$ and $\mathcal{N}_2$. This simplifies the definition of weak mulinet isomorphisms between two multinets.

Definition 3.6. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two multinets in $\mathbb{P}^2$. A *multinet isomorphism* $\phi : \mathcal{L}_1 \to \mathcal{L}_2$ is a bijection that satisfies the following condition; for every $p \in \mathcal{X}_1$, the point $\cap_{i \in S_p} \phi(\ell_i) \in \mathcal{X}_2$ where $S_p = \{i : p \in \ell_i\}$.

It is not difficult to see that the collection of multinets and multinet isomorphisms forms a full subcategory of the category of weak multinets and weak multinets isomorphisms. Now the notion of sameness of a pair of weak multinets or multinets is made precise.

Definition 3.7. Two weak multinets are *isomorphic* if there exists a weak multinet isomorphism between them. Two multinets are *isomorphic* if there exists a multinet isomorphism between them. In particular, two nets are *isomorphic* if there is a multinet isomorphism between them.

Remark 3.8. Two arrangements that differ by a change of coordinates are called *equivalent*. The map induced between two weak multinets or multinets by a change of coordinates is an isomorphism of weak multinets or multinets, respectively.

3.3. Identifications from Slicing

The structure of the slice depends on the choice of slicing hyperplane H. The slice is a line arrangement, consisting of points and lines along with the slicing hyperplane acting as $\mathbb{P}^2$. Its lines are formed from the intersections of H with hyperplanes of Q_1

distinct from H. Its points consist of the intersections of H with lines of L_1 which are not contained in H. There are several ways for identifications to occur during slicing.

If H is one of the six hyperplanes of Q_1, then H will act as $\mathbb{P}^2$ in the slice. The other hyperplanes of Q_1 become lines in the slice with possibly identifications being made. The hyperplanes of Q_1 distinct from H containing a particular locus line or double line are identified as the same line in the slice exactly when H contains that particular locus or double line, respectively.

The point P is contained in every hyperplane of Q_1. When H contains P, any lines formed from slicing pass through the image of P in the slice. In other words, the resulting line arrangement consists of concurrent lines. If H does not contain P, the seven lines of L_1 intersect H in distinct points, resulting in seven distinct points in the slice.

3.4. Classification of Slices of Q_1

A complete description of the slices of Q_1 can now be made up to equivalence. The key observation is that the identifications made during slicing are completely determined by the elements of L_1 contained in the slicing hyperplane. Consequently, it suffices to identify all possible combinations of coplanar lattice elements.

Theorem 3.9. There are five slices of the arrangement defined by Q_1 up to equivalence. A slice of Q_1 supports either the unique $(3,2)$-net or a $(k,1)$-net where $k = 3, 4, 5,$ or 6.

Proof. Assume that the slicing hyperplane H is given by $[a : b : c : d] \in \mathbb{P}^3$. Using the same notation as in Proposition 3.3, Table 3.2 gives algebraic conditions describing when elements of L_1 are contained in H.

TABLE 3.2. Conditions for elements of L_1 to be in slice.

Element of L_1	Type	Condition to be in Slice
$\{1,2\}$	double line	$a+d=0,\ b+c=0$
$\{3,4\}$	double line	$c+d=0,\ a+b=0$
$\{5,6\}$	double line	$b+d=0,\ a+c=0$
$\{1,3,5\}$	locus line	$a+b+c=0,\ d=0$
$\{1,4,6\}$	locus line	$b+c+d=0,\ a=0$
$\{2,3,6\}$	locus line	$a+b+d=0,\ c=0$
$\{2,4,5\}$	locus line	$a+c+d=0,\ b=0$
$\{1,2,3,4,5,6\}$	point	$a+b+c+d=0$

If H contains P, the slice consists of concurrent lines and supports a $(k,1)$-net structure by Proposition 2.15. To determine the possible values of k, it is only necessary to know the possible number of lines that can appear in the slice. This depends on the identifications made in the slicing process.

If H contains a double line or locus line, it also necessarily contains P. There are limitations on the number of locus and double lines contained in H. If H contains three locus lines or three double lines, then $a=b=c=d=0$ and H is not a valid slice. Consequently, H contains at most two locus lines and at most two double lines. Each case is considered separately.

Since $H=[a:b:c:d]\in\mathbb{P}^3$, at least one of a,b,c, and d is nonzero and can be assumed to be 1 by scaling. By permuting the coordinates if necessary, assume $d=1$. Permuting the coordinates is a change of coordinates which respects Q_1, hence any slice is equivalent to a slice with $d=1$.

Suppose H contains P and no locus lines. Then $a, b, c \neq 0$ by Table 3.2. In particular, H is not a hyperplane of Q_1. Under these conditions, H contains no double lines exactly when $a, b, c \neq -1$. To see this last assertion, note that $a = -1$ and $a + b + c + 1 = 0$ (H contains P) implies $b + c = 0$, so H contains a double line. This direction now follows from symmetrical arguments when $b = -1$ or $c = -1$. If H contains a double line, then at least one of a, b, and c is -1 by Table 3.2 since $d = 1$ is assumed. With no double lines and no locus lines contained in the slice, no hyperplanes are identified during slicing. The slice consists of 6 concurrent lines and supports a $(6, 1)$-net.

If H contains no locus lines and precisely one double line, then exactly one of a, b, and c is -1. Without loss of generality assume $c = -1$. Since H contains P, it follows that $a + b = 0$, so $H = [a : -a : -1 : 1]$ where $a \neq 0, \pm 1$. The only identification in the slice occurs from the double line which glues two of the original hyperplanes. Thus, the slice has five concurrent lines and supports a $(5, 1)$-net.

If H contains no locus lines and two double lines, then exactly two of a, b, and c is -1. Without loss of generality, assume that $a = b = -1$. Since H contains P, it follows that $c = 1$ and $H = [-1 : -1 : 1 : 1]$. The two double lines contained in H identify disjoint pairs of hyperplanes of Q_1. As a result, the slice consists of four concurrent lines and supports a $(4, 1)$-net.

Next suppose H contains exactly one locus line. Then exactly one of a, b, and c is zero. In particular, H is not one of the hyperplanes of Q_1. Without loss of generality, suppose $a = 0$. Then $H = [0 : b : c : 1]$ with $b, c \neq 0$ and subject to the condition $b + c + 1 = 0$ as H contains P. No double lines are possible in this case. The only identification occurs from the locus line which identifies three hyperplanes of Q_1. This slice has four lines and supports a $(4, 1)$-net.

Now suppose H contains exactly two locus lines. Then exactly two of the a, b, and c are zero. Without loss of generality, assume $a = b = 0$. Since H contains P, it follows that $c = -1$ so $H = [0 : 0 : -1 : 1]$, one of the hyperplanes of Q_1. There is exactly one double line contained in H. Using the set notation for lattice elements of L_1, $H = \{4\}$ contains the two locus lines given by $\{1, 4, 6\}$ and $\{2, 4, 5\}$ as well as the double line $\{3, 4\}$. In the slice, H_4 will act as $\mathbb{P}^2$ and each disjoint pair of hyperplanes H_1, H_6 and H_2, H_5 are identified, producing two lines in the slice. Lastly, the double line $\{3, 4\}$ will produce one more line in the slice. Thus, this slice has three concurrent lines and supports a $(3, 1)$-net.

Lastly, suppose H does not contain P. Then H does not contain any locus or double lines, and it is not a hyperplane of Q_1. There are no identifications in the slicing process. This slice contains six lines with seven intersections points, namely four locus points and three double points. It supports a $(3, 2)$-net using the classes and base locus obtained by intersection each hyperplane and locus line with H. This establishes the result. •

Depictions of the slices of Q_1 in $\mathbb{RP}^2$ up to equivalence are given in Figure 3.1. The slices of Q_n for $n > 1$ are the focus of the next chapter.

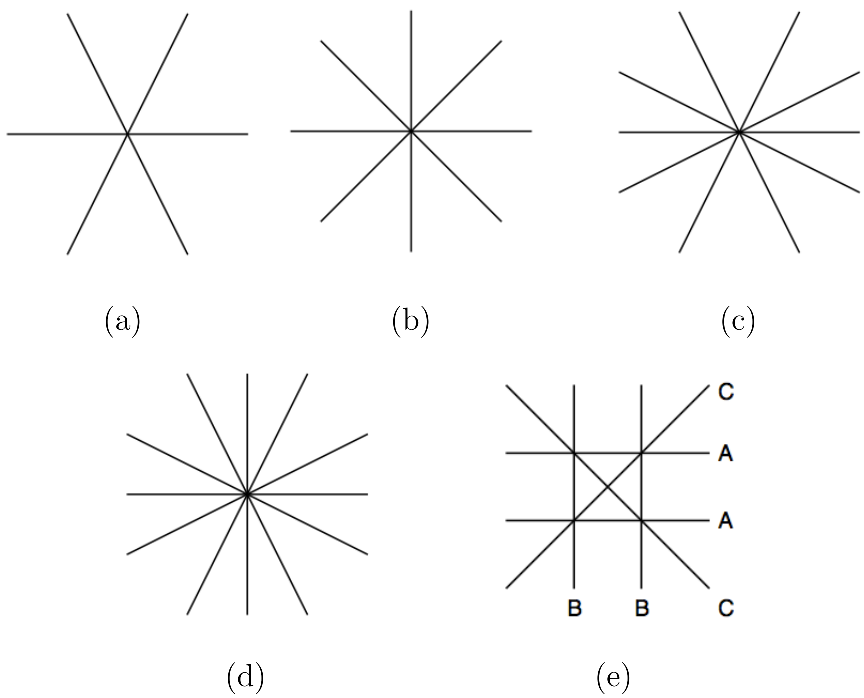


FIGURE 3.1. Slices of Q_1.

CHAPTER IV

SLICES OF Q_N FOR $N > 1$

The main objective of this chapter is to extract as many examples of multinets as possible from slices of Q_n for $n > 1$. With more combinatorial data to navigate, complete analysis up to equivalence is only achieved for certain special slices. The majority of the chapter focuses on obtaining results for small n where the combinatorial data is manageable. These efforts are rewarded with interesting and unexpected new examples of proper multinets.

4.1. The Intersection Lattice L_n

Let $n > 1$. The arrangement defined by

$$Q_n = [(x^n - y^n)](z^n - w^n)][(x^n - z^n)(y^n - w^n)][(x^n - w^n)(y^n - z^n)]$$

is the complex reflection arrangement of the monomial group $G(n, n, 4)$. This group is an irreducible reflection group. It contains the reflections $x_i \mapsto x_j$ for $i \neq j$ and $x_i \mapsto \xi^k x_i$ where ξ is a primitive nth root of unity. In the descriptions of these reflections, the x_i are a relabeling of x, y, z, and w. More information about this arrangement can be found in [12].

The group $G(n, n, 4)$ has a natural action on the intersection lattice L_n of Q_n. The groupings used in the upcoming description of the lattice correspond to the orbits of this action. The choice of names assigned to elements of L_n was motivated by the role each orbit plays in slicing. It is convenient to impose a linear order on the hyperplanes of Q_n.

Fix $n \in \mathbb{Z}_{>0}$ and let ξ be a primitive nth root of unity. Put

$$\begin{array}{rclrclrcl}
H_1 &=& y - z & H_{2n+1} &=& x - y & H_{4n+1} &=& x - z \\
H_2 &=& y - \xi z & H_{2n+2} &=& x - \xi y & H_{4n+2} &=& x - \xi z \\
H_3 &=& y - \xi^2 z & H_{2n+3} &=& x - \xi^2 y & H_{4n+3} &=& x - \xi^2 z \\
&\vdots& & &\vdots& & &\vdots& \\
H_n &=& y - \xi^{n-1} z & H_{3n} &=& x - \xi^{n-1} y & H_{5n} &=& x - \xi^{n-1} z \\
H_{n+1} &=& x - w & H_{3n+1} &=& z - w & H_{5n+1} &=& y - w \\
H_{n+2} &=& x - \xi w & H_{3n+2} &=& z - \xi w & H_{5n+2} &=& y - \xi w \\
H_{n+3} &=& x - \xi^2 w & H_{3n+3} &=& z - \xi^2 w & H_{5n+3} &=& y - \xi^2 w \\
&\vdots& & &\vdots& & &\vdots& \\
H_{2n} &=& x - \xi^{n-1} w & H_{4n} &=& z - \xi^{n-1} w & H_{6n} &=& y - \xi^{n-1} w.
\end{array}$$

As in Chapter III, elements of the intersection lattice are described in two ways, algebraically and via set notation. Again, arrows indicated equivalent descriptions.

Proposition 4.1. Let $n > 1$ and ξ be a primitive nth root of unity. The intersection lattice L_n of Q_n in $\mathbb{P}^3$ has rank 3. Its elements consists of $\mathbb{P}^3$, hyperplanes, lines, and points. More precisely,

- There are $6n$ hyperplanes, namely

$$\begin{array}{rclcl rclcl}
H_1 &=& [0:1:-1:0] &\leftrightarrow& \{1\} & H_{3n+1} &=& [0:0:1:-1] &\leftrightarrow& \{3n+1\} \\
H_2 &=& [0:1:-\xi:0] &\leftrightarrow& \{2\} & H_{3n+2} &=& [0:0:1:-\xi] &\leftrightarrow& \{3n+2\} \\
H_3 &=& [0:1:-\xi^2:0] &\leftrightarrow& \{3\} & H_{3n+3} &=& [0:0:1:-\xi^2] &\leftrightarrow& \{3n+3\} \\
&&&\vdots&&&&&\vdots& \\
H_n &=& [0:1:-\xi^{n-1}:0] &\leftrightarrow& \{n\} & H_{4n} &=& [0:0:1:-\xi^{n-1}] &\leftrightarrow& \{4n\}
\end{array}$$

$$
\begin{array}{rclcl}
H_{n+1} & = & [1:0:0:-1] & \leftrightarrow & \{n+1\} \\
H_{n+2} & = & [1:0:0:-\xi] & \leftrightarrow & \{n+2\} \\
H_{n+3} & = & [1:0:0:-\xi^2] & \leftrightarrow & \{n+3\} \\
 & & \vdots & & \\
H_{2n} & = & [1:0:0:-\xi^{n-1}] & \leftrightarrow & \{2n\} \\
H_{2n+1} & = & [1:-1:0:0] & \leftrightarrow & \{2n+1\} \\
H_{2n+2} & = & [1:-\xi:0:0] & \leftrightarrow & \{2n+2\} \\
H_{2n+3} & = & [1:-\xi^2:0:0] & \leftrightarrow & \{2n+3\} \\
 & & \vdots & & \\
H_{3n} & = & [1:-\xi^{n-1}:0:0] & \leftrightarrow & \{3n\}
\end{array}
\qquad
\begin{array}{rclcl}
H_{4n+1} & = & [1:0:-1:0] & \leftrightarrow & \{4n+1\} \\
H_{4n+2} & = & [1:0:-\xi:0] & \leftrightarrow & \{4n+2\} \\
H_{4n+3} & = & [1:0:-\xi^2:0] & \leftrightarrow & \{4n+3\} \\
 & & \vdots & & \\
H_{5n} & = & [1:0:-\xi^{n-1}:0] & \leftrightarrow & \{5n\} \\
H_{5n+1} & = & [0:1:0:-1] & \leftrightarrow & \{5n+1\} \\
H_{5n+2} & = & [0:1:0:-\xi] & \leftrightarrow & \{5n+2\} \\
H_{5n+3} & = & [0:1:0:-\xi^2] & \leftrightarrow & \{5n+3\} \\
 & & \vdots & & \\
H_{6n} & = & [0:1:0:-\xi^{n-1}] & \leftrightarrow & \{6n\}.
\end{array}
$$

• There are $7n^2+6$ lines which consist of the following three types.

∘ There are $4n^2$ *locus lines* given by

$$
\begin{array}{lcl}
[1:0:0:0]u+[0:\xi^{i+j-2}:\xi^{j-1}:1]t & \leftrightarrow & \{i, 3n+j, 5n+k_1\} \\
[0:1:0:0]u+[\xi^{i-1}:0:\xi^{j-1}:1]t & \leftrightarrow & \{n+i, 3n+j, 4n+k_2\} \\
[0:0:1:0]u+[\xi^{i+j-2}:\xi^{i-1}:0:\xi^{j-1}]t & \leftrightarrow & \{n+i, 2n+j, 5n+k_2\} \\
[0:0:0:1]u+[\xi^{i+j-2}:\xi^{i-1}:1:0]t & \leftrightarrow & \{i, 2n+j, 4n+k_1\}
\end{array}
$$

where $[u:t]\in\mathbb{P}^1$, $1\le i,j,k_1,k_2\le n$, and

$$
\begin{aligned}
k_1 &= i+j-1 \pmod{n} \\
k_2 &= i-j+1 \pmod{n}.
\end{aligned}
$$

∘ There are $3n^2$ *double lines* given by

$$
\begin{array}{lcl}
[0:\xi^{i-1}:1:0]u+[\xi^{j-1}:0:0:1]t & \leftrightarrow & \{i, n+j\} \\
[\xi^{i-1}:1:0:0]u+[0:0:\xi^{j-1}:1]t & \leftrightarrow & \{2n+i, 3n+j\} \\
[\xi^{i-1}:0:1:0]u+[0:\xi^{j-1}:0:1]t & \leftrightarrow & \{4n+i, 5n+j\}
\end{array}
$$

where $1\le i,j\le n$ and $[u:t]\in\mathbb{P}^1$.

∘ There are six *n-lines* given by

$$
\begin{aligned}
[1:0:0:0]u+[0:0:0:1]t &\leftrightarrow \{1,2,\ldots,n\}\\
[0:1:0:0]u+[0:0:1:0]t &\leftrightarrow \{n+1,n+2,\ldots,2n\}\\
[0:0:1:0]u+[0:0:0:1]t &\leftrightarrow \{2n+1,2n+2,\ldots,3n\}\\
[1:0:0:0]u+[0:1:0:0]t &\leftrightarrow \{3n+1,3n+2,\ldots,4n\}\\
[0:1:0:0]u+[0:0:0:1]t &\leftrightarrow \{4n+1,4n+2,\ldots,4n\}\\
[1:0:0:0]u+[0:0:1:0]t &\leftrightarrow \{5n+1,5n+2,\ldots,6n\}
\end{aligned}
$$

where $[u:t]\in\mathbb{P}^1$.

• There n^3+6n+4 points of L consist of the following three types.

∘ There are n^3 *double points*

$$[\xi^{i+j+k-3}:\xi^{i+k-2}:\xi^{k-1}:1]\leftrightarrow\{i,n+k_3,2n+j,3n+k,4n+k_4,5n+k_5\}$$

where $1\le i,j,k,k_3,k_4,k_5\le n$ and

$$
\begin{aligned}
k_3 &= i+j+k-2 \pmod n\\
k_4 &= i+j-1 \pmod n\\
k_5 &= i+k-1 \pmod n).
\end{aligned}
$$

∘ There are $6n$ *intraclass points*

$$
\begin{aligned}
[\xi^{i-1}:1:0:0] &\leftrightarrow \{2n+i,3n+1,\ldots,4n\}\\
[\xi^{i-1}:0:1:0] &\leftrightarrow \{4n+i,5n+1,\ldots,6n\}\\
[\xi^{i-1}:0:0:1] &\leftrightarrow \{1,\ldots,n,n+i\}\\
[0:\xi^{i-1}:1:0] &\leftrightarrow \{i,n+1,\ldots,2n\}\\
[0:\xi^{i-1}:0:1] &\leftrightarrow \{4n+1,\ldots,5n,5n+i\}\\
[0:0:\xi^{i-1}:1] &\leftrightarrow \{2n+1,\ldots,3n,3n+i\}
\end{aligned}
$$

where $1\le i\le n$.

∘ There are four *n-points*

$$\begin{aligned}
[1:0:0:0] &\leftrightarrow \{1,\ldots,n,3n+1,\ldots,4n,5n+1,\ldots 6n\}\\
[0:1:0:0] &\leftrightarrow \{n+1,\ldots,2n,3n+1,\ldots,4n,4n+1,\ldots 5n\}\\
[0:0:1:0] &\leftrightarrow \{n+1,\ldots,2n,2n+1,\ldots,4n,5n+1,\ldots 6n\}\\
[0:0:0:1] &\leftrightarrow \{1,\ldots,n,2n+1,\ldots,3n,4n+1,\ldots 5n\}.
\end{aligned}$$

Proof. This is a straightforward computation. As before, a check is conducted. The Poincaré polynomials of reflection groups are well-known. Using the given description of L_n in the proposition, the Poincaré polynomial of Q_n is computed directly from its definition and checked for agreement with the its listing in [12].

It follows from the description given in the proposition that the intersection lattice has the properties presented in Table 4.1.

TABLE 4.1. Properties of L_n for $n > 1$.

Lattice element	Rank r	Number	Value of Möbius function μ
$\mathbb{P}^3$	0	1	$+1$
hyperplane	1	$6n$	-1
locus line	2	$4n^2$	$+2$
double line	2	$3n^2$	$+1$
n-line	2	6	$+(n-1)$
double point	3	n^3	-6
intraclass point	3	$6n$	$-(n-1)$
n-point	3	4	$-(2n^2-2)$

It follows from the definition of the Poincaré polynomial that

$$\begin{aligned}\pi(Q_n, t) &= (6n^3+3n^2-6n-3)t^3+(11n^2-5)t^2+(6n-1)t+1\\ &= (1+(n+1)t)(1+(2n+1)t)(1+(3n-3)t).\end{aligned}$$

On the other hand, the Poincaré polynomial of Q_n, considered as a central arrangement in $\mathbb{C}^4$, is known to be

$$\begin{aligned}\pi(Q_n, t) &= (6n^3+3n^2-6n-3)t^4+(6n^3+14n^2-6n-8)t^3\\ &\quad +(11n^2+6n-6)t^2+(6n)t+1\\ &= (1+t)(1+(n+1)t)(1+(2n+1)t)(1+(3n-3)t).\end{aligned}$$

These two Poincaré polynomials differ by a factor of $1+t$. Again, this reflects the well-known effect on the Poincaré polynomial when projectivizing a central arrangement. •

Remark 4.2. For each $n > 1$, the arrangement defined by Q_n supports a $(3, 2n)$-net in $\mathbb{P}^3$ with classes $\{H_1, \ldots, H_{2n}\}$, $\{H_{2n+1}, \ldots, H_{4n}\}$, and $\{H_{4n+1}, \ldots, H_{6n}\}$. Its base locus $\mathcal{X}$ consists of the $4n^2$ locus lines. Each class breaks naturally into two blocks, giving the six blocks $\{H_1, \ldots, H_n\}$, $\{H_{n+1}, \ldots, H_{2n}\}$, $\{H_{2n+1}, \ldots, H_{3n}\}$, $\{H_{3n+1}, \ldots, H_{4n}\}$, $\{H_{4n+1}, \ldots, H_{5n}\}$, and $\{H_{5n+1}, \ldots, H_{6n}\}$. These blocks will be referred to below.

Remark 4.3. The actions of reflection groups on the intersection lattice of their reflection arrangements have been studied in connection with questions regarding freeness of restriction arrangements. The orbits of these action were computed for irreducible Coxeter groups and unitary reflection groups in [11] and [10], respectively.

4.2. Allowable Slices and Identifications

As mentioned in Chapter III, there are some choices of slicing hyperplane which lead to pathological cases when $n > 1$ and should be avoided. A sufficient condition to ensure a slice yields a multinet is that the classes of the multinet structure of Q_n are preserved during slicing. That is, the slice is made in a manner such that two lines from distinct classes of Q_n do not become identified in the slice. These observations motivate the following definitions.

Definition 4.4. Fix $n > 1$. A hyperplane in $\mathbb{P}^3$ is called *forbidden* if it contains a locus line of Q_n. Otherwise, it is called *allowable.*

In this new language, it will be shown that allowable slicing hyperplanes always yield multinets. Forbidden slicing hyperplanes lead to pathological cases and will not be investigated here. The next observation identifies a restriction on the possible combination of lattice elements contained in an allowable slice.

Proposition 4.5. Let $n > 1$. An allowable slice of Q_n cannot contain an n-point and a double point of L_n.

Proof. Let ξ be a primitive nth root of unity. Suppose the slice contains a double point $P = [\xi^i : \xi^j : \xi^k : 1]$ for some $0 \leq i, j, k < n$ and the n-point $Q = [1 : 0 : 0 : 0]$. It follows that the slice contains the line spanned by P and Q, hence the point $R = [0 : \xi^j : \xi^k : 1]$. The line $Qu + Rt$ where $[u : t] \in \mathbb{P}^3$ is a locus line, so the slice is forbidden. The result follows by making symmetric arguments for the other choices of the n-point Q. $\bullet$

Similar to the situation with Q_1, the structure of the line arrangement in $\mathbb{P}^2$ obtained from slicing Q_n when $n > 1$ depends on the lattice elements of L_n contained in the slicing hyperplane H. With H acting as $\mathbb{P}^2$, the lines and points appearing in the slice are formed from the intersections of H with hyperplanes and lines of L_n not contained in H, respectively. There are several ways for identifications to occur in the slicing process.

Suppose H is allowable. Then hyperplanes of Q_n containing a particular double or n-line are identified as the same line in the slice exactly when H contains that particular double line or n-line, respectively, of L_n. These situations result in lines of multiplicity 2 or n in the slice. These lines are also referred to as double lines and n-lines, respectively. Two lines of L_n which intersect and are not contained in H are identified as same point in the slice when H contains their intersection point. In particular, points of multiplicity 2 and n occur in the slice when H contains double points and n-points, respectively, of L_n. Such points of the slice are also referred to as double points and n-points, respectively.

With the goal of understanding the multinet structure obtained from allowable slices, the focus is primarily placed on the possible identifications of hyperplanes and locus lines in the slicing process. These are the identifications which create lines and points with multiplicity greater than one in the resulting multinet.

4.3. Ceva Pencils of Plane Curves

In this section, the main result is that a slice of Q_n obtained from an allowable hyperplane is a line arrangement which supports a global multinet structure. To establish this assertion, an equivalent property to the existence of a multinet structure is used, namely the existence of a certain pencil of plane curves. The equivalence between these two notions was identified in [6]. A summary of those ideas is given below.

Identify a homogeneous polynomial in three variables with the projective plane curve it defines and refer to either as a curve. A *pencil of plane curves* is a line in the projective space of homogeneous polynomials in three variables of a fixed degree d. Let C_1 and C_2 be any pair of distinct curves in a given pencil. Then the pencil can be described as the set of curves of the form $uC_1 + tC_2$ where $[u : t] \in \mathbb{P}^1$. A pencil has no fixed components if C_1 and C_2 have no common factors. Equivalently, C_1 and C_2 intersect at a finite set of points, $\mathcal{X} = C_1 \cap C_2$, called the *base of the pencil.* Every pair of distinct curves in the pencil intersect precisely at $\mathcal{X}$.

The two curves C_1 and C_2 determine a rational map $\pi : \mathbb{P}^2 \rightarrowtail \mathbb{P}^1$ given by $p \mapsto [C_2(p) : -C_1(p)]$ whose indeterminacy locus is the base of the pencil. The curve $uC_1 + tC_2$ is the closure of the fiber of π over $[u : t]$. Each point outside the base locus lies in a unique such curve. The map π is uniquely determined by the pencil up to a linear change of coordinates in $\mathbb{P}^1$ and referred to as a pencil when no confusion will result.

A curve of the form $\prod_{i=1}^{q} \alpha_i^{m_i}$ where α_i is a linear form and $m_i \in \mathbb{Z}_{>0}$ is called *completely reducible*. Let $\varphi : \mathcal{S} \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the points of $\mathcal{X}$. The rational map $\pi : \mathbb{P}^2 \rightarrowtail \mathbb{P}^1$ lifts to a regular mapping $\tilde{\pi} : \mathcal{S} \to \mathbb{P}^1$. The fibers of $\tilde{\pi}$ are the proper transforms of the fibers of π under the blow-up φ.

A pencil π is called *connected* if every fiber of $\tilde{\pi}$ is connected. Equivalently, π is connected if each completely reducible fiber of π is not the union of finitely many proper subvarieties meeting only in the base locus. A *pencil of Ceva type* or *Ceva pencil* is a connected pencil of plane curves with no fixed components and at least three completely reducible fibers. Lastly, a component R of $\mathcal{R}^1(\mathcal{L})$ is called a *global resonance component* if R is not contained in any coordinate hyperplane in A_1. Here is the main result of [6] which establishes the relationships between resonance varieties, multinets, and pencils of plane curves.

Theorem 4.6. Let $\mathcal{L}$ be a line arrangement in $\mathbb{P}^2$. The following are equivalent:

(i) $\mathcal{L}$ supports a global resonance component of dimension $k-1$.

(ii) $\mathcal{L}$ supports a (k,d) multinet in $\mathbb{P}^2$ for some d.

(iii) $\mathcal{L}$ is the set of components of $k \geq 3$ completely reducible fibers in a Ceva pencil of degree d curves, for some d.

It is useful to illustrate how to obtain the multinet structure on $\mathcal{L}$ from a given Ceva pencil with completely reducible fibers $C_1, \ldots, C_k$. The class $\mathcal{L}_i$ consists of the lines defined by the factors of C_i. Each line $\ell \in \mathcal{L}$ is assigned the multiplicity $m(\ell)$ equal to the multiplicity of its corresponding linear factor in C_i. The base locus $\mathcal{X}$ of the multinet is the base of the pencil.

Here is the main result of the section.

Theorem 4.7. Let $n > 1$. The line arrangement in $\mathbb{P}^2$ obtained from intersecting Q_n with an allowable hyperplane supports a $(3, 2n)$-multinet structure.

Proof. Let $H = [a : b : c : d]$ be an allowable slicing hyperplane. Then at least one of these coefficients is nonzero. By scaling if needed, assume one of the coefficients has value -1, say $d = -1$. Then H is the hyperplane given by $w = ax + by + cz$. Consider the pencil π given by

$$u[(x^n - y^n)(z^n - (ax + by + cz)^n)] + t[(x^n - z^n)(y^n - (ax + by + cz)^n)]$$

where $[u : t] \in \mathbb{P}^1$. There are three singular values: $[1 : 0]$, $[0 : 1]$, and $[-1 : 1]$. The corresponding fibers

$$C_1 = (x^n - y^n)(z^n - (ax + by + cz)^n)$$
$$C_2 = (x^n - z^n)(y^n - (ax + by + cz)^n)$$
$$C_3 = (x^n - (ax + by + cz)^n)(y^n - z^n)$$

are completely reducible and define the arrangement in the slice via $Q = C_1C_2C_3$. Observe that the pencil of space curves in $\mathbb{P}^3$ given by

$$u[(x^n - y^n)(z^n - w^n)] + t[(x^n - z^n)(y^n - w^n)]$$

where $[u : t] \in \mathbb{P}^1$ has no fixed components. Since classes are preserved during an allowable slice, it follows that π also has no fixed components.

Lastly, the pencil π is connected. To see this, suppose ℓ and ℓ' are from the same class $\mathcal{L}_i$ of Q and $p = \ell \cap \ell' \in \mathcal{X}$. From the structure of L_n, p is a double point or an n-point. If p is an n-point, ℓ and ℓ' are from the same block of $\mathcal{L}_i$. (See Remark 4.2.) For any choice of $\ell'' \in \mathcal{L}_i$ in the other block, the sequence of ℓ, ℓ'', ℓ' satisfies condition (iv) in Definition 2.1.

If p is a double point, then ℓ and ℓ' lie in different blocks of $\mathcal{L}_i$. Examining the structure of the intersection lattice, there are restrictions on identifications in the slice made by H containing intraclass points. If H contains a double point, then it can contain at most two intraclass points impacting each class. Assume H contains intraclass points impacting $\mathcal{L}_i$. Then H contains the double line $\ell'' \in \mathcal{L}_i$ passing through these points, and the sequence of ℓ, ℓ'', ℓ' satisfies condition (iv).

If H contains one intraclass point impacting $\mathcal{L}_i$, then all lines of $\mathcal{L}_i$ have multiplicity one. Due to the identifications resulting from the interclass point, all the lines in one block and exactly one line, say $\ell'' \in \mathcal{L}_i$, from the other block are concurrent at a point outside of $\mathcal{X}$. The sequence of ℓ, ℓ'', ℓ' satisfies condition (iv).

Lastly, assume H contains no intraclass points impacting $\mathcal{L}_i$. Then the slice has no double lines and no n-lines. If there are $\ell'', \ell''' \in \mathcal{L}_i$ lying in distinct blocks of ℓ and ℓ', respectively, with $\ell'' \cap \ell''' \notin \mathcal{X}$, the sequence ℓ, ℓ'', ℓ''', ℓ' satisfies condition (iv). If no such ℓ'' and ℓ''' exist, the completely reducible fiber of the pencil corresponding to $\mathcal{L}_i$ is not connected in the blow up at $\mathcal{X}$. As a result, the line arrangement in the slice supports a proper weak multinet structure. This refines to a multinet structure by assigning multiplicity two to the blocks of $\mathcal{L}_i$ and any other classes with disconnected blocks. It follows that the slice supports a k-multinet structure with $k = 4, 5$, or 6. By [13], the slice must be a 4-net. Since classes in a net must contain the same number of lines, this situation is impossible. •

Combining Theorem 4.7, Proposition 4.5, and the observations made about identifications in allowable slices gives the next two results.

Proposition 4.8. Let $n > 1$. An allowable slice of Q_n supports a $(3, 2n)$-multinet with locus points of multiplicity 1, 2, or n. Moreover, none of these multinets can contain both a double point and an n-point simultaneously.

Proposition 4.9. Let $n > 1$. For multinets obtained from allowable slices of Q_n, every point of $\mathcal{X}$ on a line ℓ with $m(\ell) > 1$ has the multiplicity $m(\ell)$.

4.4. Generic Slices of Q_n

It was observed in [16] that every $(3, d)$-net in $\mathbb{P}^2$ can be associated with a $d \times d$ latin square in the following way. Let $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$ denote the three classes of the 3-net. There is a pairing $\mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_3$ given by $(\ell, \ell') \mapsto \ell''$ where ℓ'' is the unique line from $\mathcal{L}_3$ containing the point $\ell \cap \ell' \in \mathcal{X}$. Identify each class with the set $G = \{1, \ldots, d\}$. Then this pairing defines a binary operation on G and gives it the structure of a quasigroup whose Cayley table is a latin square. Identifications can always be made so that G is a loop, an algebraic structure where all group axioms hold except possibly associativity. See [3] for additional information regarding quasigroups, loops, and latin squares.

Since the Cayley table of any finite group is a latin square, there has been interest in which groups can be realized by 3-nets in $\mathbb{P}^2$. Building on the results from [13], [15], and [16], it was shown in [8] that the groups realizable by 3-nets in $\mathbb{P}^2$ are precisely $\mathbb{Z}_n$, $\mathbb{Z}_n \oplus \mathbb{Z}_m$, D_{2n}, and the quaternion group Q_8. On the other hand, it was shown in [14] that there exists a 3-net whose associated latin square is not the Cayley table of a group.

The pairing used in associating a latin square to a 3-net in $\mathbb{P}^2$ utilizes only combinatorial data. It generalizes naturally to 3-nets in higher dimensional projective space using codimension one and two objects in lieu of lines and points. As a result, every 3-net can be associated to a latin square regardless of its ambient projective space $\mathbb{P}^k$ for $k > 1$. In particular, the latin square associated to the $(3, 2n)$-net Q_n in $\mathbb{P}^3$ is computed below. It was mentioned in [1] that the arrangement defined by Q_n appearing in [13] defines a net realizing D_{2n}. This assertion is proven below and used to give explicit equations of a net realizing D_{2n} in Example 4.13.

Proposition 4.10. The arrangement in $\mathbb{P}^3$ defined by Q_1 realizes the group $\mathbb{Z}_2$.

Proof. Let $\mathbb{Z}_2 = \langle g \rangle$. Using the linear ordering imposed on the hyperplanes of Q_1 in Chapter III, the identifications $e \leftrightarrow \{1, 3, 5\}$ and $g \leftrightarrow \{2, 4, 6\}$ give the result. •

Proposition 4.11. Let $n > 1$. The $(3, 2n)$-net in $\mathbb{P}^3$ defined by Q_n realizes the dihedral group of $2n$ elements, namely $D_{2n} = \langle r, s : r^n = s^2 = 1, sr^i s = r^{-i} \text{ for all } i \rangle$.

Proof. Using the linear ordering imposed on Q_n in Proposition 4.1, the classes of the net are $\{1, \ldots, 2n\}$, $\{2n + 1, \ldots, 4n\}$, and $\{4n + 1, \ldots, 6n\}$. Make the following associations between hyperplanes in Q_n and elements of D_{2n}:

$$\begin{aligned} r^{i-1} &\leftrightarrow \{i\}, \{2n+i\}, \{4n+i\} \\ r^{i-1}s &\leftrightarrow \{n+i\}, \{3n+i\}, \{5n+i\} \end{aligned}$$

where $1 \leq i \leq n$. Then the group operations agree with the description given of locus lines which comprises the base locus $\mathcal{X}$ of the net. Explicitly,

$$\begin{aligned} r^{i-1} \times r^{j-1} &= r^{i+j-2} && \leftrightarrow \{i, 2n+j, 4n+k_1\} \\ r^{i-1} \times r^{j-1}s &= r^{i+j-2}s && \leftrightarrow \{i, 3n+j, 5n+k_1\} \\ r^{i-1}s \times r^{j-1} &= r^{i-j} && \leftrightarrow \{n+i, 2n+j, 5n+k_2\} \\ r^{i-1}s \times r^{j-1}s &= r^{i-j} && \leftrightarrow \{n+i, 3n+j, 4n+k_2\} \end{aligned}$$

where

$$\begin{aligned} k_1 &= i+j-1 \pmod{n} \\ k_2 &= i-j+1 \pmod{n} \end{aligned}$$

and $1 \leq i, j, k_1, k_2 \leq n$. •

Theorem 4.12. Slicing Q_n by a generic allowable hyperplane yields a $(3, 2n)$-net in $\mathbb{P}^2$ realizing D_{2n}.

Proof. A generic allowable hyperplane does not contain any lattice elements of L_n. Such a slice exists since there are only finitely many lattice elements and infinitely many allowable slicing hyperplanes. By Theorem 4.7, the slice supports a $(3, 2n)$-multinet structure. Each line and point of $\mathcal{X}$ has multiplicity one because no identifications are made, hence the slice is a $(3, 2n)$-net. The pairing used in associating the Latin square to Q_n remains the same in the slice. The result now follows from Proposition 4.11. •

Example 4.13. Let $n > 1$. The slicing hyperplane given by $w = 2x + 4y + 8z$ is allowable and generic since it does not contain any lattice elements of L_n. By Theorem 4.12,

$$Q = [(x^n - y^n)](z^n - (2x+4y+8z)^n)][(x^n - z^n)(y^n - (2x+4y+8z)^n)][(x^n - (2x+4y+8z)^n)(y^n - z^n)]$$

realized the group D_{2n}. These arrangements will be shown to have graph type G_0.

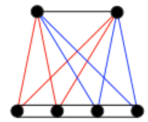
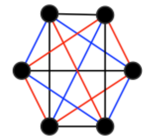
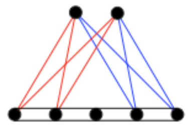
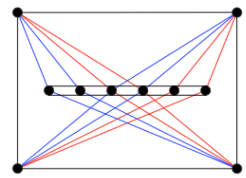
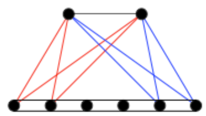
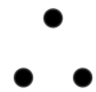
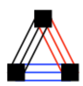
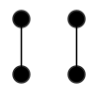
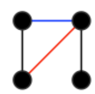
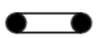
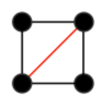
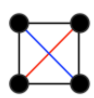
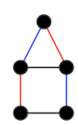
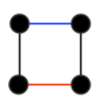
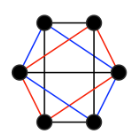
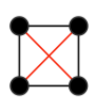
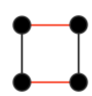
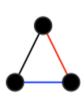
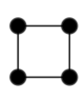
4.5. Graph Types of Multinets

It is convenient to develop a way to distinguish nonisomorphic multinets without wading too deeply through their defining combinatorial data. This provides the motivation to pioneer an invariant of multinets dubbed graph type.

Each multinet can be assigned a graph with weighted vertices and weighted, colored edges. Vertices correspond to points $P \in \mathcal{X}$ of the multinet with $m(P) > 1$ and are assigned the weight $m(P)$. There is an edge between two vertices if the pair of associated points in the multinet lie on a common line ℓ of the arrangement. The edge is colored according to which class contains ℓ and assigned the weight weight $m(\ell)$. By convention, a net is assigned the empty graph, the graph consisting of no vertices and no edges. Also, graphs differing only by the choice of coloring of the edges are considered to be the same.

In Table 4.2, several graphs are presented. It will be shown that each of these is the graph type of certain slices of Q_n. To simplify these graphs, several conventions are employed. Circles and squares indicate vertices of weight 2 and n, respectively. A single edge between circles signifies the multinet contains two double points which lie on a common line of the arrangement. A double edge between circles signals that the double points lie on a double line of the multinet. For the graphs $G_5(n)$ and $G_6(n)$,

TABLE 4.2. Some graph types of Q_n.

G_0		$G_9(4)$		G_{17}	
$G_1(n)$		$G_9(5)$		$G_{18}(6)$	
$G_2(n)$		$G_9(6)$		G_{19}	
$G_3(n)$		G_{10}		G_{20}	
G_4		G_{11}		G_{21}	
$G_5(n)$		G_{12}		G_{22}	
$G_6(n)$ (n even)		G_{13}		G_{23}	
$G_6(n)$ (n odd)		G_{14}		G_{24}	
G_7		G_{15}		G_{25}	
G_8		G_{16}			

only two of the n vertices on each double line are depicted. Edges of weight one are suppressed in $G_6(n)$. Lastly, a triple line appearing between squares indicates the multinet has n-points which lie on an n-line of the arrangement. There are no suppressed vertices in this case. Edges are colored based on the class to which their associated lines belong.

These graphs encode a sufficient amount of combinatorial data to be an effective invariant. Clearly, multinets with different graph types are nonisomorphic multinets. However, nonisomorphic multinets can have the same graph type. For example, the $(3, 2)$-nets realizing $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ both have the empty graph as their graph type, but are nonisomorphic as multinets since their latin squares are not main class isotopic.

4.6. Infinite Families of Multinets

An infinite family of multinets with graph type G_0 was presented in Example 4.13. In this section, infinite families of multinets with other graph types obtained from slices of Q_n for $n > 1$ are identified. There is much interest in examples of multinets which contain at least one line ℓ with $m(\ell) > 1$ due to recent papers such as [4].

Definition 4.14. A multinet $\mathcal{L}$ is called *heavy* if there is a line $\ell \in \mathcal{L}$ with $m(\ell) > 1$. If $m(\ell) = 1$ for all $\ell \in \mathcal{L}$, the multinet is said to be *light*.

Theorem 4.15. Let $n > 1$. Multinets obtained from allowable slices of Q_n with at least one n-point have graph types $G_1(n)$, $G_2(n)$, and $G_3(n)$.

Proof. Let $n > 1$ and consider a multinet obtained from Q_n by an allowable slicing hyperplane H with at least one n-point. Since the four n-points of L_n are not coplanar, H contains at most three n-points.

If H has three n-points, then it is equivalent to $H = [0 : 0 : 0 : 1]$ by a permutation of coordinates. The resulting multinet contains three n-lines, no double lines, and no double points. Its graph type is $G_3(n)$.

If H has two n-points, then H is equivalent by a permutation of coordinates and scaling to $[0 : 0 : 1 : a]$ with $a \neq 0$. There is another condition on a. For H to be allowable, $a \neq -\xi^i$ where ξ is a primitive nth root of unity and $i = 0, \ldots, n-1$. In this case, H contains one n-line, no double lines, and no double points. The graph type of this multinet is $G_2(n)$.

If H has one n-point, then H is equivalent by a permutation of coordinates and scaling to $[0 : 1 : a : b]$ with $a, b \neq 0$. Allowability implies a and b cannot satisfy $1 + \xi^i a + \xi^j b = 0$ for any i, j. It follows that $b \neq (-1 - \xi^i a)/\xi^j$ or equivalently $b \neq -\xi^i a - \xi^j$ for some (different) i, j. In this case, H contains no n-lines, no double lines, and no double points. Its graph type is $G_1(n)$. •

Examples of infinite families of heavy multinets with graph type $G_2(n)$ and $G_3(n)$ are produced by choosing the slicing hyperplanes $w = 2z$ and $w = 0$, respectively. Scaling Q in the former situation yields the two families of multinets exhibited in Example 2.20 and Example 2.21. The following infinite family of light multinets is new and constructed using observations from the proof of Theorem 4.15.

Example 4.16. Let $n > 1$. Slicing Q_n by the hyperplane $w = x + 3y$ produces a $(3, 2n)$-multinet with graph type $G_1(n)$ defined by

$$Q = [(x^n - y^n)(z^n - (x+3y)^n)][(x^n - z^n)(y^n - (x+3y)^n)][(x^n - (x+3y)^n)(y^n - z^n)].$$

This multinet has one n-point, namely $[0:0:1]$, and $m(\ell)=1$ for all lines ℓ. This example was given in [6] for the case $n=2$. The $n=4$ case is examined further in Example 4.33.

One consequence of Theorem 4.15 is that heavy multinets obtained from Q_n with at least one n-line have graph types $G_2(n)$ and $G_3(n)$. The situation is less clear for heavy multinets with at least one double line. However, some general statements can be made. The next sequence of results establishes a maximum on double lines contained in an allowable slice. Examples will follow, showing these maximums are attainable.

Proposition 4.17. Let $n>1$. An allowable slice of Q_n can contain at most two, respectively three, double lines if n is odd, respectively even. Furthermore, if the slice contains two double lines and n is even, it also contains a third double line.

Proof. Suppose H is an allowable slice and contains two double lines. It follows from Proposition 4.1 that the double lines intersect at a point of L_n of the form $[\xi^i:\xi^j:\xi^k:1]$. By a sequence of rotations and reflections fixing L_n, it can be assumed that the two double lines intersect at $P=[1:1:1:1]$. There are three double lines through P, namely

$$\begin{cases} L_1:[1:0:0:1]u+[0:1:1:0]t \\ L_2:[1:1:0:0]u+[0:0:1:1]t \\ L_3:[1:0:1:0]u+[0:1:0:1]t. \end{cases}$$

These lines are not coplanar, so H cannot contain all three. Assume H contains L_1 and L_2. Then $H=[-1:1:-1:1]$. Since H is allowable, any additional double line in H must belong to the third class and have the form:

$$L_{i,j}: \ [\xi^{i-1} : 0 : 1 : 0]u + [0 : \xi^{j-1} : 0 : 1]t$$

where $[u : t] \in \mathbb{P}$ for some $0 < i, j \leq n$. Observe that H contains $L_{i,j}$ exactly when $\xi^{i-1} = -1$ and $\xi^{j-1} = -1$. Both conditions are satisfied for a unique i and j if n is even. If n is odd, -1 is not a root of unity and there is no solution. •

Example 4.18. Let $n > 1$. Slicing Q_n by the hyperplane defined by $w = x + y - z$ produces the heavy $(3, 2n)$-multinet defined by

$$Q = [(x^n-y^n)(z^n-(x+y-z)^n)][(x^n-z^n)(y^n-(x+y-z)^n)][(x^n-(x+y-z)^n)(y^n-z^n)].$$

If n is odd, this slice contains two double lines, $2n-1$ double points, no n-lines, and no n-points. If n is even, it contains three double lines, $3n-3$ double points, no n-lines, and no n-points. These multinets have graph type $G_6(n)$. The case $n = 4$ is discussed in Example 4.36.

There are two additional infinite families of multinets of certain graph types that can be easily described.

Example 4.19. Let $n > 1$. Slicing Q_n by the hyperplane $w = x + \pi y - \pi z$ yields the heavy $(3, 2n)$-multinet specified by

$$Q = [(x^n-y^n)(z^n-(x+\pi y-\pi z)^n)][(x^n-z^n)(y^n-(x+\pi y-\pi z)^n)][(x^n-(x+\pi y-\pi z)^n)(y^n-z^n)].$$

This slice contains one double line, n double points, no n-lines, and no n-points. Its graph type is $G_5(n)$. The case $n = 4$ is examined in Example 4.35.

Example 4.20. Let $n > 1$. Slicing Q_n by the hyperplane $w = -x - y + 3z$ produces a light $(3, 2n)$-multinet given by

$$Q = [(x^n - y^n)(z^n - (-x-y+3z)^n)][(x^n - z^n)(y^n - (-x-y+3z)^n)][(x^n - (-x-y+3z)^n)(y^n - z^n)].$$

This slice contains exactly one double point of multiplicity two, namely $P = [1 : 1 : 1]$, and no other points or lines of multiplicity greater than one. Its graph type is G_4. The case $n = 4$ is treated in Example 4.34.

As will be seen later, there exists heavy multinets which have exactly one double line, say ℓ, and double points not on ℓ. The next two results explore this situation by showing at least two or four double points which are not on ℓ can occur. Examples involving these situations appear as graph types G_9 and G_{18}, respectively.

By a sequence of rotations and reflections fixing L_n, one may take the unique double line contained in the allowable slice H to be $[1 : 0 : 0 : 1]u + [0 : 1 : 1 : 0]t$ where $[u : t] \in \mathbb{P}^1$. Then $H = [-1 : a : -a : 1]$ with $a \in \mathbb{C}^\times$. The proofs of these next two propositions are direct verifications.

Proposition 4.21. Let $n = 2p + 1$ with $p \geq 1$ and ξ be a primitive nth root of unity. Suppose $H = [-1 : a : -a : 1]$ and contains the point $[\xi^i : \xi^j : \xi^k : 1]$ where $0 \leq i, j, k < n$, $i \neq 0$, and $j \neq k$. Then H also contains the point $[\xi^{-i} : \xi^{k-i} : \xi^{j-i} : 1]$.

Proposition 4.22. Let $n = 2p$ with $p \geq 1$ and ξ be a primitive nth root of unity. Suppose $H = [-1 : a : -a : 1]$ and contains the point $[\xi^i : \xi^j : \xi^k : 1]$ where $0 \leq i, j, k < n$, $i \neq 0$, and $j \neq k$. Then H also contains the (not necessarily distinct) points $[\xi^{-i} : \xi^{k-i} : \xi^{j-i} : 1]$, $[\xi^i : \xi^{k+p} : \xi^{j+p} : 1]$, and $[\xi^{-i} : \xi^{j-i+p} : \xi^{k-i+p} : 1]$.

4.7. Classifying Slices of Q_n

The ultimate goal is to classify multinets obtained from Q_n up to isomorphism. This was accomplished for $n = 1$ in Chapter III. When $n > 1$, the first step in this direction is taken by classifying multinets up to graph type, a weaker notion of equivalence than isomorphism.

Many choices of allowable slicing hyperplane produce isomorphic multinets from Q_n. More precisely, the monomial group $G(n, n, 4)$ acts naturally on the intersection lattice L_n and induces an action on the collection $\mathcal{C}$ of linearly closed sets of coplanar elements of L_n. Associate to each set $S \in \mathcal{C}$ the collection of slicing hyperplanes which contain the elements of S and no additional elements of L_n. Two slicing hyperplanes associated to a given $S \in \mathcal{C}$ produce line arrangements which are lattice equivalent. If S does not contain any locus lines, then these line arrangements are isomorphic multinets by Theorem 4.7.

To classify line arrangements obtained from slicing Q_n up to lattice equivalence, it suffices to choose a set of representatives for the orbits of $\mathcal{C}$, say $\{S_i\}$. Then select exactly one slicing hyperplane H_i associated to each S_i and analyze the resulting line arrangements. Unfortunately, the orbits of $\mathcal{C}$ under this action are not completely understood at this time. In particular, it is unknown how to select a set of representatives for the orbits of $\mathcal{C}$ in a pragmatic way.

On the other hand, the interest in this dissertation lies in classifying multinets obtained from allowable slices of Q_n up to isomorphism, a weaker notion than lattice equivalence. This classification can be achieved using a closely related group action.

Let L'_n be the sublattice of L_n formed by excluding the intraclass points. Then $G(n, n, 4)$ acts naturally on L'_n and induces an action on the collection $\mathcal{C}'$ of linearly closed sets of coplanar elements of L'_n which do not contain any locus lines. As before, associate to each set $S \in \mathcal{C}'$ the collection of slicing hyperplanes which contain the elements of S and no additional elements of L_n. Two slicing hyperplanes associated to a given $S \in \mathcal{C}'$ yield line arrangements which are isomorphic multinets.

To classify multinets obtained from Q_n by allowable slicing hyperplanes up to isomorphism, it is sufficient to choose a set of representatives for the orbits of $\mathcal{C}'$, say $\{S_i\}$. Then select exactly one slicing hyperplane H_i associated to each S_i and analyze the resulting multinets. There are issues from implementing this approach. The orbits of $\mathcal{C}'$ under this action are also not well understood at this time. It is unknown how to efficiently select a set of representatives for the orbits of $\mathcal{C}'$. Moreover, determining whether or not two arbitrary multinets are isomorphic directly from the definition is cumbersome.

As a preliminary step, the aforementioned procedure is modified to obtain a classification up to graph type, giving a practical way to investigate multinets from Q_n for small n. This is accomplished by generating a list of elements of $\mathcal{C}'$ which contains representatives from sufficiently many of the orbits of $\mathcal{C}'$ to capture all possible graph types of the associated multinets. The remainder of this section concentrates on the procedural details regarding classification up to graph type. This method will be implemented for small n in the subsequent sections.

As a consequence of Proposition 4.8, a set $S \in \mathcal{C}'$ cannot contain both double points and n-points. Slices of Q_n associated to S containing at least one n-point were considered in Theorem 4.15 and produce multinets with graph types of $G_1(n)$, $G_2(n)$, and $G_3(n)$. Furthermore, slices of Q_n associated to the empty set were examined in Theorem 4.12 and yield multinets with graph type of G_0. It remains to investigate situations where S contains at least one double point.

Since a plane is determined by three non-collinear points, there are three cases to consider for $S \in \mathcal{C}'$ containing at least one double point. Each set S is the linear closure of either one, two, or three double points. The group action of $G(n,n,4)$ on $\mathcal{C}'$ is useful to limit the $S \in \mathcal{C}'$ needed to be considered for determining all possible graph types. Let ξ be a primitive nth root of unity. Two types of linear transformations of $\mathbb{P}^3$ which leave L_n invariant are the rotations $\rho_i^k : x_i \mapsto \xi^k x_i$ and reflections $\sigma_{i,j} : x_i \leftrightarrow x_j$ where x_1, x_2, x_3, x_4 have been identified with x, y, z, w, respectively. All rotations and reflections mentioned below in this section refer to linear transformations of $\mathbb{P}^3$ of the form ρ_i^k and $\sigma_{i,j}$, respectively. Any pair of sets of $\mathcal{C}'$ related by a sequence of rotations and reflections produce multinets with the same graph type.

For convenience, impose an order on $\mathbb{Z}_n$ by $[0] < [1] < \cdots < [n-1]$ and write i for the equivalence class of $[i]$. Using these conventions, the ordering becomes expressed as $0 < 1 < \cdots < n-1$, and statements such as $1 < -1$ are made for $n > 2$.

Suppose S is the linear closure of one double point. That is, S consists of exactly one double point and no other elements of L'_n. Applying a sequence of rotations, this point can be taken to be $P_1 = [1:1:1:1]$. By cardinality, there exists an allowable hyperplane containing P_1 and no other elements of L'_n. This shows $S \in \mathcal{C}'$ and produces a multnet with graph type G_4.

Next suppose S is the linear closure of two double points, P_1 and P_2. Two situations arise. The corresponding double points of the associated multinet either lie on a line formed from a hyperplane from Q_n or do not. If they lie on such a line, S can be taken to be the linear closure of the points $P_1 = [1:1:1:1]$ and $P_2 = [1:\xi^j:\xi^k:1]$ with $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$ and $1 \leq j \leq k$ by a sequence of rotations and reflections. Thus, P_1 and P_2 lie on the line in the slice obtained from $x - w$ of Q_n. Note that $j = 0$ and $k \neq 0$ produce forbidden slices.

There are other ways to reduce the number of sets S to consider in this situation. Applying the rotations ρ_2^{-j} and ρ_3^{-k} followed by the reflection $\sigma_{2,3}$ takes the points P_1 and P_2 to the points $P_1' = [1 : \xi^{-k} : \xi^{-j} : 1]$ and $P_2' = [1 : 1 : 1 : 1]$, respectively, where $-k \leq -j$. This shows that the two corresponding sets lie in the same orbit of $\mathcal{C}'$, hence only one needs to be considered. This is accomplished by taking $j < k \leq -j$. Also, all points with $j = k$ appear on a common double line of L_n'. Consequently, it suffices to consider only $j = k = 1$ in this situation.

If S contains no pair of double points which lie on a common hyperplane of Q_n, then one can take $P_1 = [1 : 1 : 1 : 1]$ and $P_2 = [\xi^i : \xi^j : \xi^k : 1]$ with $0 < i, j, k < n$ by a sequence of rotations and reflections. The condition that P_1 and P_2 do not lie on a common hyperplane of Q_n implies i, j, and k are pairwise distinct. Using a sequence of reflections, it is sufficient to consider $0 < i < j < k < n$.

Lastly, suppose S is the linear closure of three non-collinear double points, namely P_1, P_2, and P_3. These points completely specify the slicing hyperplane. There are two situations to consider in this case. The set S either does or does not contain a pair of double points which lie on common hyperplane of Q_n.

Suppose S contains a pair of double points, say P_1 and P_2, which lie on common hyperplane of Q_n. Let ℓ denote the corresponding line in the associated multinet and refer to this situation as the *collinear case*. The multiplicity of ℓ is one or two.

If $m(\ell) = 2$, then ℓ can be taken to be $[1 : 0 : 0 : 1]u + [0 : 1 : 1 : 0]t$ where $[u : t] \in \mathbb{P}^1$ using a sequence of rotations and reflections. This is the double line spanned by $P_1 = [1 : 1 : 1 : 1]$ and $P_2 = [1 : \xi : \xi : 1]$. Necessary conditions on $P_3 = [\xi^i : \xi^j : \xi^k : 1]$ to ensure the slice is allowable include $i \neq 0$ and $j \neq k$. There are additional reductions. Applying a sequence of rotations which fix ℓ, the double points on ℓ are permuted and P_3 becomes $[\xi^i : 1 : \xi^{k'} : 1]$. The condition

$k' \neq 0$ is required so that the slice is not forbidden. This shows that it is sufficient to consider P_3 with $j = 0$. Also, the reflection $\sigma_{1,4}$ preserves ℓ and sends a double point $P_3 = [\xi^i : 1 : \xi^k : 1]$ off of ℓ to the point $P_3' = [\xi^{-i} : \xi^{-i} : \xi^{k-i} : 1]$. Applying another sequence of rotations which fix ℓ, the point P_3' becomes $P_3'' = [\xi^{-i} : 1 : \xi^{k'} : 1]$ where $k' \neq 0$. As a result, it is sufficient when $m(\ell) = 2$ to only consider P_3 with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, $j = 0$, and $k \neq 0$.

When $m(\ell) = 1$, one can take $P_1 = [1 : 1 : 1 : 1]$ and $P_2 = [1 : \xi^j : \xi^k : 1]$ with $1 \leq j < \lfloor \frac{n+1}{2} \rfloor$ and $j < k \leq -j$ by a sequence of rotations and reflections. Note that $j = k = 1$ implies S contains a double line and was considered previously. There are conditions on $P_3 = [\xi^a : \xi^b : \xi^c : 1]$ necessary to obtain an allowable slice including $a \neq 0$. In addition, P_3 cannot lie on any n-line passing through P_1 and P_2 by Proposition 4.8. This implies P_3 is not one of the points: $[\xi^t : 1 : 1 : 1]$, $[\xi^t : \xi^t : \xi^t : 1]$, $[\xi^{i+t} : \xi^j : \xi^k : 1]$, $[\xi^i : \xi^{j+t} : \xi^k : 1]$, $[\xi^i : \xi^j : \xi^{k+t} : 1]$, or $[\xi^{i+t} : \xi^{j+t} : \xi^{k+t} : 1]$ where $t = 0, 1, \ldots, n-1$. Any remaining choices for $a, b,$ and c produce allowable slices. Observe the reflection $\sigma_{1,4}$ sends the point $P_3 = [\xi^i : \xi^j : \xi^k : 1]$ with $i \neq 0$ to $P_3' = [\xi^{-i} : \xi^{j-i} : \xi^{k-i} : 1]$ and also fixes P_1 and P_2. These define S which lie in the same orbit of $\mathcal{C}'$, so it is sufficient to consider only one of them.

Finally, suppose S contains at least three double points with the property that there is no pair of double points lie on a common hyperplane of Q_n. Refer to this situation as the *noncollinear case*. By a sequence of rotations and reflections, one can take $P_1 = [1 : 1 : 1 : 1]$ and $P_2 = [\xi^i : \xi^j : \xi^k : 1]$ with $0 < i < j < k < n$. The third point and resulting additional points cannot have the property that any two lie on a line in Q_n since those situations were already considered. Thus, it is only necessary to consider S where each hyperplane of Q_n contains at most one of the points: P_1, P_2, and P_3.

Remark 4.23. Using the above approach results in superfluous slices being investigated, however it is the most efficient way known at the present time. Further reductions are possible using the full strength of the group action induced by the monomial group $G(n, n, 4)$ on the intersection lattice L_n.

4.8. Slices of Q_2: (3,4)-Multinets

The method of classifying slices of Q_n up to graph type discussed in the previous section is now implemented for small n. The following conventions are used throughout these investigations. Bold numbers are used to indicated the choices of P_3 needed during analysis of the collinear case with $m(\ell) = 2$. The images of the points in the slice lying off of the line $[1 : 0 : 0 : 1]u + [0 : 1 : 1 : 0]t$ where $[u : t] \in \mathbb{P}^1$ under the reflection $\sigma_{1,4}$ are identified as *reflection points* in the upcoming tables.

Definition 4.24. Write ijk for the point $[\xi^i : \xi^j : \xi^k : 1]$.

Theorem 4.25. Allowable slices of Q_2 yield $(3, 4)$-multinets with the following graph types: G_0, $G_1(2)$, $G_2(2)$, and $G_3(2)$.

Proof. Here $\xi = -1$. Theorem 4.12 shows generic slices produce $(3, 4)$-nets. These give multinets with graph type G_0. From Theorem 4.15, allowable slices containing at least one 2-point yield $(3, 4)$-multinets with graph types $G_1(2)$, $G_2(2)$, and $G_3(2)$. It remains to investigate linearly closed sets $S \in \mathcal{C}'$ with at least one double point.

By Example 4.20 and Example 4.19, there are slices yielding graph types G_4 and $G_5(2)$, respectively. Each choice of P_2 needed in the method lies on a double line of L_n passing through P_1. There is only one case to investigate when S contains at least three double points. The results are summarized in Table 4.3. All slices in the table contain the point $P_1 = [1 : 1 : 1 : 1]$. The second point P_2 is indicated in the table using the short-hand

TABLE 4.3. Collinear case for $n = 2$.

P_2	Additional Points	Slice	Graph Type
011	**101**	$[-1 : -1 : 1 : 1]$	$G_6(2)$

notation introduced in Definition 4.24. The third point and any additional points are indicated in the column labeled *Additional Points*. The result now follows from observing $G_1(2) = G_4$, $G_2(2) = G_5(2)$, and $G_3(2) = G_6(2)$. •

Each multinets obtained from Q_2 is realizable in $\mathbb{RP}^2$. Depictions of the proper $(3, 4)$-multinets obtained from Q_2 are given in Table 4.4. Different intraclass structures occur for slices of graph type $G_1(2)$ by choosing slicing hyperplanes containing different number of intraclass points of L_2. This gives three arrangements which support a multinet structure with graph type $G_1(2)$, but have non-isomorphic intersection lattices. These three examples are equivalent to slices formed using the slicing hyperplanes $w = -x - 2y + 4z$, $w = -2x - 3y + 6z$, and $w = -x - y + 3z$.

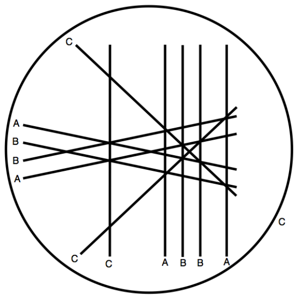
4.9. Slices of Q_3: (3,6)-Multinets

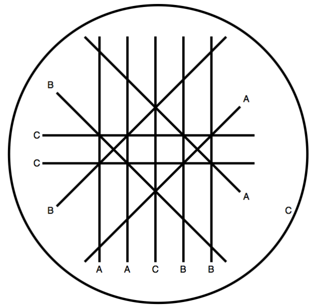
Theorem 4.26. Allowable slices of Q_3 yield $(3, 6)$-multinets with the following graph types: G_0, $G_1(3)$, $G_2(3)$, $G_3(3)$, G_4, $G_5(3)$, $G_6(3)$, G_7, and G_8.

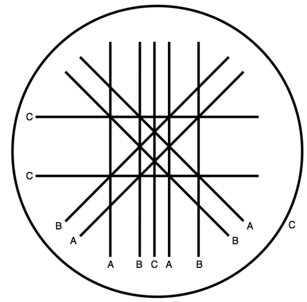
Proof. Let ξ be a primitive third root of unity. Theorem 4.12 shows generic slices produce $(3, 6)$-nets. These give multinets with graph type G_0. From Theorem 4.15, allowable slices containing at least one 3-point yield $(3, 6)$-multinets with graph types $G_1(3)$, $G_2(3)$, and $G_3(3)$. It remains to investigate linearly closed sets $S \in \mathcal{C}'$ with at least one double point.

TABLE 4.4. Types of $(3,4)$-multinets.

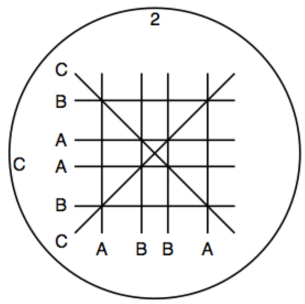
$G_1(2)$



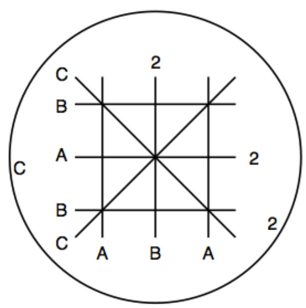




$G_2(2)$



$G_3(2)$



By Example 4.20 and Example 4.19, there are slices yielding graph type G_4 and $G_5(3)$, respectively. For additional graph types of slices involving P_1 and P_2, only the collinear situation via a line of multiplicity 1 is possible. Appendix A gives an example of such a slice with graph type G_7. Next consider S with at least three double points. Only the collinear case is possible and needs to be considered for P_3. A summary of the findings in the collinear case is given in Table 4.5. This completes the analysis and gives the result. $\bullet$

TABLE 4.5. Collinear case for $n = 3$.

P_2	Additional Points	Reflection Points	Slice	Graph Type
011	022, **101**, 202	-	$[-1 : -1 : 1 : 1]$	$G_6(3)$
011	022, **102**, 212	-	$[-1 : \xi : -\xi : 1]$	$G_6(3)$
012	102	221	$[-1 : -1 : -\xi : \xi + 2]$	G_8
012	121	210	$[\xi - 1 : -\xi - 1 : 1 : 1]$	G_8

4.10. Slices of Q_4: (3,8)-Multinets

Theorem 4.27. Allowable slices of Q_4 yield $(3, 8)$-multinets with the following graph types: G_0, $G_1(4)$, $G_2(4)$, $G_3(4)$, G_4, $G_5(4)$, $G_6(4)$, G_7, G_8, $G_9(4)$, G_{10}, G_{11}, G_{12}, and G_{13}.

Proof. Let ξ be a primitive fourth root of unity. Theorem 4.12 shows generic slices produce $(3, 8)$-nets. These give multinets with graph type G_0. From Theorem 4.15, allowable slices containing at least one 4-point yield $(3, 8)$-multinets with graph types $G_1(4)$, $G_2(4)$, and $G_3(4)$. It remains to investigate linearly closed sets $S \in \mathcal{C}'$ with at least one double point.

By Example 4.20 and Example 4.19, there are slices yielding graph type G_4 and $G_5(4)$, respectively. For additional graph types of slices involving P_1 and P_2, the collinear via a line of multiplicity 1 and noncollinear situations are both possible. Examples of such slices are given in Appendix A and have graph types G_7 and G_{10}, respectively.

Lastly, consider S with at least three double points. Table 4.6 gives a summary of the analysis for the collinear case. It is necessary to check if three double points

TABLE 4.6. Collinear case for $n = 4$.

P_2	Additional Points	Reflection Points	Slice	Graph Type
011	022, 033, **101**, 132, **202**, 303, 312	-	$[-1 : -1 : 1 : 1]$	$G_6(4)$
011	022, 033, **102**, 313	-	$[-\xi - 1 : -1 : 1 : \xi + 1]$	$G_9(4)$
011	022, 033, **103**,112, 213, 310, 323	-	$[-1 : \xi : -\xi : 1]$	$G_6(4)$
011	022, 033, **201**, 232	-	$[-\xi + 1 : -1 : 1 : \xi - 1]$	$G_9(4)$
011	022, 033, **203**, 212	-	$[-1 : \xi - 1 : -\xi + 1 : 1]$	$G_9(4)$
012	101, 133, 202, 303	110, 220, 322, 330	$[-1 : -\xi - 1 : 1 : \xi + 1]$	$G_9(4)$
012	102	331	$[2 : 2 : \xi - 1 : -\xi - 3]$	G_8
012	103, 313	120, 332	$[1 : -\xi + 1 : \xi : -2]$	G_{13}
012	113, 210, 311, 323	122, 130, 232, 302	$[1 : -\xi - 1 : 1 : \xi - 1]$	$G_9(4)$
012	121	310	$[2\xi - 1 : -\xi - 1 : 1 : -\xi + 1]$	G_8
012	131	320	$[\xi - 2 : -\xi - 1 : 1 : 2]$	G_{12}
012	132	321	$[-2 : -\xi - 1 : 1 : \xi + 2]$	G_{12}
012	201, 233	211, 223	$[\xi : -2 : -\xi + 1 : 1]$	G_{13}
012	203	221	$[-2 : -\xi - 1 : 1 : \xi + 2]$	G_{12}
012	213	231	$[1 : -2\xi : \xi + 1 : \xi - 2]$	G_{12}
013	101, 112, 123, 130, 202, 233, 303	110, 211, 220, 301, 312, 323, 330	$[\xi : -1 : -\xi : 1]$	$G_6(4)$
013	102, 133	322, 331	$[-\xi + 1 : 1 : \xi : -2]$	G_{13}
013	103	332	$[1 : 1 : \xi : -\xi - 2]$	G_8
013	121	310	$[\xi - 2 : -\xi : 1 : 1]$	G_8
013	122, 131	311, 320	$[-2 : -\xi : 1 : \xi + 1]$	G_{13}
013	132	321	$[-\xi - 2 : -\xi : 1 : 2\xi + 1]$	G_{11}
013	201, 212, 223, 230	201, 212, 223, 230	$[1 : \xi - 1 : -\xi - 1 : 1]$	$G_9(4)$
013	203	221	$[1 : \xi + 1 : \xi - 1 : -2\xi - 1]$	G_8
013	210	232	$[1 : -\xi - 1 : -\xi + 1 : 2\xi - 1]$	G_8

of L_4 exist with the property of being pairwise noncollinear in Q_4. Table 4.7 identifies candidates for this situation.

TABLE 4.7. Candidates for P_3 in noncollinear case for $n = 4$.

P	Noncollinear Points with P
000	123, 132, 213, 231, 312, 321
123	000, 031, 202, 211, 310, 332

Inspecting Table 4.7, there is no point in L_4 which is noncollinear with both points $[1 : 1 : 1 : 1]$ and $[\xi : \xi^2 : \xi^3 : 1]$ simultaneously. Thus, there does not exist a set of three double points with the property that no pair lies on a line of Q_4. This completes the analysis and gives the result. •

4.11. Slices of Q_5: (3,10)-Multinets

Theorem 4.28. Allowable slices of Q_5 yield $(3, 10)$-multinets with the following graph types: G_0, $G_1(5)$, $G_2(5)$, $G_3(5)$, G_4, $G_5(5)$, $G_6(5)$, G_7, G_8, $G_9(5)$, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, and G_{17}.

Proof. Let ξ be a primitive fifth root of unity. Theorem 4.12 shows generic slices produce $(3, 10)$-nets. These give multinets with graph type G_0. From Theorem 4.15, allowable slices containing at least one 5-point yield $(3, 10)$-multinets with graph types $G_1(5)$, $G_2(5)$, and $G_3(5)$. It remains to investigate linearly closed sets $S \in \mathcal{C}'$ with at least one double point.

By Example 4.20 and Example 4.19, there are slices yielding graph type G_4 and $G_5(5)$, respectively. For additional graph types of slices involving P_1 and P_2, the collinear via a line of multiplicity 1 and noncollinear situations are both possible. Examples of such slices are given in Appendix A and have graph types G_7 and G_{10}, respectively.

Next consider S with at least three double points. A summary of the analysis for the collinear case is given in Table 4.8.

TABLE 4.8. Collinear case for $n = 5$.

P_2	Additional Points	Reflection Points	Slice	Type
011	022, 033, 044, **101**, **202**, 303, 404	-	$[-1 : -1 : 1 : 1]$	$G_6(5)$
011	022, 033, 044, **102**, 414	-	$[-\xi + 1 : 1 : -1 : -\xi - 1]$	$G_9(5)$
011	022, 033, 044, **103**, 424	-	$[\xi^2 + \xi + 1 : 1 : -1 : -\xi^2 - \xi - 1]$	$G_9(5)$
011	022, 033, 044, **104**, 214, 324, 434	-	$[-\xi^4 : 1 : -1 : \xi^4]$	$G_6(5)$
011	022, 033, 044, **201**, 343	-	$[1 : \xi + 1 : -\xi - 1 : -1]$	$G_9(5)$
011	022, 033, 044, 143, **203**, 313, 423	-	$[\xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^2 - \xi - 1]$	$G_6(5)$
011	022, 033, 044, **204**, 323	-	$[\xi^2 + 1 : 1 : -1 : -\xi^2 - 1]$	$G_9(5)$
012	101, 144, 202, 303, 404	110, 220, 330, 433, 440	$[1 : \xi + 1 : -1 : -\xi - 1]$	$G_9(5)$
012	102	441	$[1 : 1 : \xi^3 + \xi : -\xi^3 - \xi - 2]$	G_8
012	103, 414	120, 442	$[-\xi^2 - x - 1 : -\xi - 1 : 1 : \xi^2 + 2x + 1]$	G_{13}
012	104, 314	231, 443	$[\xi^4 : -\xi - 1 : 1 : -\xi^4 + \xi]$	G_{14}
012	113, 214, 310, 411, 423	122, 134, 232, 342, 402	$[\xi^2 : \xi + 1 : -1 : -\xi^2 - \xi]$	$G_9(5)$
012	114, 320, 422, 434	133, 140, 242, 403	$[-\xi^3 - \xi^2 : -\xi - 1 : 1 : \xi^3 + \xi^2 + \xi]$	G_{17}
012	121	410	$[\xi^2 + 2\xi : -\xi - 1 : 1 : -\xi^2 - \xi]$	G_8
012	124, 331	203, 413	$[-\xi^3 + \xi : -\xi - 1 : 1 : \xi^3]$	G_{14}
012	130	424	$[\xi^3 + 2\xi^2 + 2\xi + 1 : -\xi - 1 : 1 : -\xi^3 - 2\xi^2 - \xi - 1]$	G_{12}
012	131	420	$[\xi^3 + 2\xi^2 + 2\xi : -\xi - 1 : 1 : -\xi^3 - 2\xi^2 - \xi]$	G_{12}
012	132	421	$[\xi^3 + 2\xi^2 + \xi : -\xi - 1 : 1 : -\xi^3 - 2\xi^2]$	G_{12}
012	141	430	$[\xi^3 + \xi^2 + \xi - 1 : -\xi - 1 : 1 : -\xi^3 - \xi^2 + 1]$	G_{12}
012	142	431	$[\xi^3 + \xi^2 - 1 : -\xi - 1 : 1 : -\xi^3 - \xi^2 + \xi + 1]$	G_{12}
012	143	432	$[\xi^3 - 1 : -\xi - 1 : 1 : -\xi^3 + \xi + 1]$	G_{12}
012	201, 244	322, 334	$[1 : \xi^2 + 2\xi + 1 : -\xi - 1 : -\xi^2 - \xi - 1]$	G_{15}
012	204, 313	230, 332	$[-\xi^2 - 1 : -\xi - 1 : 1 : \xi^2 + \xi + 1]$	G_{13}
012	210	343	$[1 : -\xi - 1 : 1 : \xi - 1]$	G_8
012	211, 223	301, 344	$[\xi : -\xi^2 - 2\xi - 1 : \xi + 1 : \xi^2]$	G_{13}
012	213	341	$[\xi^2 : \xi^2 + 2\xi + 1 : -\xi - 1 : -2\xi^2 - \xi]$	G_{12}
012	221	304	$[\xi^2 + 2\xi : -\xi^2 - 2\xi - 1 : \xi + 1 : -\xi]$	G_{12}
012	224	302	$[-\xi^2 + \xi : -x - 1 : 1 : \xi^2]$	G_8
012	233, 240	311, 323	$[\xi^3 + \xi^2 + \xi : -\xi^2 - 2\xi - 1 : \xi + 1 : -\xi^3]$	G_{13}
012	241	324	$[-\xi^4 + \xi^3 + \xi : -\xi - 1 : 1 : \xi^4 - \xi^3]$	G_{11}
012	243	321	$[\xi^3 - 1 : -\xi^2 - 2\xi - 1 : \xi + 1 : -\xi^3 + \xi^2 + \xi + 1]$	G_{11}
013	101, 202, 244, 303, 404	110, 220, 322, 330, 440	$[1 : \xi^2 + \xi + 1 : -1 : -\xi^2 - \xi - 1]$	$G_9(5)$
013	102, 144	433, 441	$[\xi + 1 : \xi^2 + \xi + 1 : -1 : -\xi^2 - 2\xi - 1]$	G_{13}

TABLE 4.8. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
013	103	442	$[\xi^4+\xi^3 : \xi^4+\xi^3 : 1 : -2\xi^4-2\xi^3-1]$	G_8
013	104, 414	120, 443	$[\xi^4 : \xi^4+\xi^3 : 1 : -2\xi^4-\xi^3-1]$	G_{13}
013	112, 240, 302, 344	211, 224, 323, 401	$[\xi^2 : -\xi^2-\xi-1 : 1 : \xi]$	G_{17}
013	114, 210, 311, 324, 412	123, 233, 241, 343, 403	$[\xi^3 : \xi^2+\xi+1 : -1 : -\xi^3-\xi^2-\xi]$	$G_9(5)$
013	121	410	$[\xi^3+2\xi^2+2\xi : -\xi^2-\xi-1 : 1 : -\xi^3-\xi^2-\xi]$	G_8
013	122, 130	411, 424	$[\xi^3+2\xi^2+\xi : -\xi^2-\xi-1 : 1 : -\xi^3-\xi^2]$	G_{15}
013	131	420	$[\xi^3+2\xi^2+\xi-1 : -\xi^2-\xi-1 : 1 : -\xi^3-\xi^2+1]$	G_{12}
013	132	421	$[\xi^3+2\xi^2-1 : -\xi^2-\xi-1 : 1 : -\xi^3-\xi^2+\xi+1]$	G_{11}
013	133, 141	422, 430	$[\xi^3+\xi^2-1 : -\xi^2-\xi-1 : 1 : -\xi^3+\xi+1]$	G_{13}
013	134, 242	320, 423	$[\xi^2-1 : -\xi^2-\xi-1 : 1 : \xi+1]$	G_{14}
013	140, 243	321, 434	$[\xi^3+\xi^2 : -\xi^2-\xi-1 : 1 : -\xi^3+\xi]$	G_{14}
013	142	431	$[\xi^3+\xi^2-\xi-1 : -\xi^2-\xi-1 : 1 : -\xi^3+2\xi+1]$	G_{11}
013	143	432	$[\xi^3-\xi-1 : -\xi^2-\xi-1 : 1 : -\xi^3+\xi^2+2\xi+1]$	G_{12}
013	201	334	$[1 : -\xi^4+\xi^2+\xi : -\xi-1 : \xi^4-\xi^2]$	G_{12}
013	203	331	$[\xi^2+\xi+1 : -\xi^4+\xi^2+\xi : -\xi-1 : \xi^4-2\xi^2-\xi]$	G_8
013	204	332	$[\xi^2+1 : \xi^2+\xi+1 : -1 : -2\xi^2-\xi-1]$	G_{12}
013	212, 301	223, 340	$[\xi^2 : -\xi^3-2\xi^2-2\xi-1 : \xi+1 : \xi^3+\xi^2+\xi]$	G_{15}
013	214	342	$[\xi^3 : \xi^3+2\xi^2+2\xi+1 : -\xi-1 : -2\xi^3-2\xi^2-\xi]$	G_{12}
013	221	304	$[\xi^3+2\xi^2+2\xi : -\xi^3-2\xi^2-2\xi-1 : \xi+1 : -\xi]$	G_{12}
013	231	314	$[\xi^3+\xi^2+2\xi : -\xi^2-\xi-1 : 1 : -\xi^3-\xi]$	G_{12}
013	232	310	$[\xi^3+2\xi^2-1 : -\xi^3-2\xi^2-2\xi-1 : \xi+1 : \xi+1]$	G_8
013	234	312	$[\xi^2-1 : -\xi^3-2\xi^2-2\xi-1 : \xi+1 : \xi^3+\xi^2+\xi+1]$	G_{12}
014	101, 124, 202, 234, 303 344, 404	110, 211, 220, 312, 330 413, 440	$[1 : -\xi^4 : -1 : \xi^4]$	$G_6(5)$
014	102, 134	423, 441	$[\xi+1 : -\xi^4 : -1 : \xi^4-\xi]$	G_{14}
014	103, 144	433, 442	$[\xi^2+\xi+1 : -\xi^4 : -1 : \xi^4-\xi^2-\xi]$	G_{13}
014	104	443	$[\xi^4 : \xi^4 : 1 : -2\xi^4-1]$	G_8
014	112, 130, 233, 242	311, 320, 401, 424	$[\xi^3+\xi^2 : \xi^4 : 1 : \xi]$	G_{17}
014	113, 140, 203, 244	322, 331, 402, 434	$[\xi^3 : \xi^4 : 1 : \xi^2+\xi]$	G_{17}
014	121	410	$[-\xi^4-2 : \xi^4 : 1 : 1]$	G_8
014	122, 131	411, 420	$[\xi^3+\xi^2-1 : \xi^4 : 1 : \xi+1]$	G_{13}
014	123, 141	412, 430	$[\xi^3-1 : \xi^4 : 1 : -\xi^4-\xi^3]$	G_{14}
014	132	421	$[\xi^3+\xi^2-\xi-1 : \xi^4 : 1 : 2\xi+1]$	G_{11}
014	133, 142	422, 431	$[\xi^3-\xi-1 : \xi^4 : 1 : \xi^2+2\xi+1]$	G_{16}
014	143	432	$[5\xi^3+3\xi^2+4\xi+3 : -\xi^3-\xi^2-3 : \xi^2-2\xi+1 : -4\xi^3-3\xi^2-2\xi-1]$	G_{11}
014	201, 224, 313, 340	212, 230, 302, 334	$[1 : \xi^3+\xi^2+\xi : -\xi-1 : -\xi^3-\xi^2]$	G_{17}
014	204	332	$[\xi^2+1 : -\xi^4 : -1 : \xi^4-\xi^2]$	G_8
014	210	343	$[\xi^2+1 : \xi^4 : 1 : -\xi^4-\xi^2-2]$	G_8
014	213, 240	323, 341	$[\xi^3 : -\xi^3-\xi^2-\xi : \xi+1 : \xi^2-1]$	G_{14}
014	221	304	$[-\xi^4-2 : \xi^4+1 : \xi+1 : -\xi]$	G_8
014	223, 241	301, 324	$[\xi^3-1 : \xi^4+1 : \xi+1 : \xi^2]$	G_{14}
014	232	310	$[\xi^2-1 : \xi^4 : 1 : -\xi^4-\xi^2]$	G_8
014	243	321	$[5\xi^3+3\xi^2+4\xi+3 : -\xi^3-2\xi-2 : \xi^3-\xi^2-\xi+1 : -5\xi^3-2\xi^2-\xi-2]$	G_{11}
023	101, 133, 202, 243, 303, 404, 413	110, 124, 220, 321, 330 422, 440	$[\xi+1 : \xi^2+\xi+1 : -\xi-1 : -\xi^2-\xi-1]$	$G_6(5)$
023	102, 143	432, 441	$[\xi^2+2\xi+1 : \xi^2+\xi+1 : -\xi-1 : -2\xi^2-2\xi-1]$	G_{14}
023	103	442	$[\xi^3+2\xi^2+2\xi+1 : \xi^2+\xi+1 : -\xi-1 : -\xi^3-3\xi^2-2\xi-1]$	G_8
023	104, 113, 410, 424	121, 130, 402, 443	$[-\xi^2-\xi-1 : \xi^3+\xi^2 : -\xi^3-\xi^2-\xi : \xi^2+2\xi+1]$	G_{17}

TABLE 4.8. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
023	112, 144, 304, 313	221, 230, 401, 433	$[\xi : \xi^2+\xi+1 : -\xi-1 : -\xi^2-\xi]$	G_{17}
023	114	403	$[\xi^3-1 : \xi^2+\xi+1 : -\xi-1 : -\xi^3-\xi^2+1]$	G_8
023	120	414	$[\xi^3+2\xi^2+2\xi+1 : -\xi^2-\xi-1 : \xi+1 : -\xi^3-\xi^2-2\xi-1]$	G_8
023	122, 140, 302, 343	210, 224, 411, 434	$[\xi^3+\xi^2 : -\xi^2-\xi-1 : \xi+1 : -\xi^3]$	G_{17}
023	131	420	$[\xi^3+2\xi^2-1 : -\xi^2-\xi-1 : \xi+1 : -\xi^3-\xi^2-1]$	G_8
023	132, 141	421, 430	$[\xi^3+\xi^2-\xi-1 : -\xi^2-\xi-1 : \xi+1 : -\xi^3+\xi+1]$	G_{14}
023	142	431	$[\xi^3-2\xi-1 : -\xi^2-\xi-1 : \xi+1 : -\xi^3+\xi^2+2\xi+1]$	G_{11}
023	201, 233	311, 334	$[1 : \xi^2+\xi+1 : -\xi-1 : -\xi^2-1]$	G_{13}
023	203	331	$[\xi^2+\xi+1 : \xi^2+\xi+1 : -\xi-1 : -2\xi^2-\xi-1]$	G_8
023	204, 213	332, 341	$[\xi^3+\xi^2+1 : \xi^2 : -\xi^3-\xi^2-\xi-1 : -\xi^2+\xi]$	G_{14}
023	211, 234	312, 344	$[\xi^2 : -\xi^3-2\xi^2-2\xi-1 : \xi^2+2\xi+1 : \xi^3]$	G_{16}
023	212, 244	322, 340	$[\xi : \xi^3+2\xi^2+2\xi+1 : -\xi^2-2\xi-1 : -\xi^3-\xi^2-\xi]$	G_{13}
023	214	342	$[\xi^3-1 : \xi^3+2\xi^2+2\xi+1 : -\xi^2-2\xi-1 : -2\xi^3-\xi^2+1]$	G_{11}
023	231	314	$[\xi^3+2\xi^2-1 : -\xi^3-2\xi^2-2\xi-1 : \xi^2+2\xi+1 : -\xi^2+1]$	G_{11}
023	232, 241	310, 324	$[\xi^3+\xi^2-\xi-1 : -\xi^3-2\xi^2-2\xi-1 : \xi^2+2\xi+1 : \xi+1]$	G_{14}
023	242	320	$[\xi^2-\xi-1 : -\xi^2-\xi-1 : \xi+1 : \xi+1]$	G_8

It is necessary to check if three double points of L_5 exist with the property of being pairwise noncollinear in Q_5. Table 4.9 identifies candidates for this situation.

TABLE 4.9. Candidates for P_3 in noncollinear case for $n = 5$.

P	Reflection Points	Noncollinear Points with P
000	-	123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432
123	124, 134, 234	000, 004, 030, 031, 041, 044, 200, 202, 210, 211, **241**, 242, 302, 314, 311, **314**, 331, 332, 410, 414, 430, **432**, 442, 444

Only combinations of three points which are pairwise noncollinear in Q_5 need to be analyzed. These result from finding common entries in the last column in the two rows of Table 4.9. For example, 241 is common in the third column to the rows corresponding to 000 and 123. This indicates 000, 123, and 241 are three double points which are pairwise noncollinear in Q_5. Thus, the slice specified by the three points $[1 : 1 : 1 : 1]$, $[\xi : \xi^2 : \xi^3 : 1]$, and $[\xi^2 : \xi^4 : \xi : 1]$ is analyzed. A summary of such slices are given below in Table 4.10.

TABLE 4.10. Noncollinear case for $n = 5$.

P_2	Additional Points	Slice	Type
123	013, 233, **241**, 343, 403	$[\xi^3 + 2\xi^2 + 2\xi : -\xi^2 - 2\xi - 2 : \xi^2 + 1 : -\xi^3 + 1]$	$G_9(5)$
123	034, 144, 204, **314**, 424	$[\xi^3 + 2\xi^2 + \xi + 1 : \xi^3 + \xi^2 + 2\xi + 1 : -\xi^2 - 2\xi - 2 : -2\xi^3 - 2\xi^2 - \xi]$	$G_9(5)$
123	110, 220, 330, **432**, 440	$[\xi^3 + 2\xi^2 + 2\xi : -\xi^3 - 2\xi^2 - 2\xi : \xi^3 + 2\xi^2 + \xi + 1 : -\xi^3 - 2\xi^2 - \xi - 1]$	$G_9(5)$

This completes the analysis and gives the result. •

4.12. Slices of Q_6: (3,12)-Multinets

Theorem 4.29. Allowable slices of Q_6 yield $(3, 12)$-multinets with the following graph types: G_0, $G_1(6)$, $G_2(6)$, $G_3(6)$, G_4, $G_5(6)$, $G_6(6)$, G_7, G_8, $G_9(6)$, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, $G_{18}(6)$, G_{19}, G_{20}, G_{21}, G_{22}, G_{23}, G_{24}, and G_{25}.

Proof. Let ξ be a primitive sixth root of unity. Theorem 4.12 shows generic slices produce $(3, 12)$-nets. These give multinets with graph type G_0. From Theorem 4.15, allowable slices containing at least one 6-point yield $(3, 12)$-multinets with graph types $G_1(6)$, $G_2(6)$, and $G_3(6)$. It remains to investigate linearly closed sets $S \in \mathcal{C}'$ with at least one double point.

By Example 4.20 and Example 4.19, there are slices yielding graph type G_4 and $G_5(6)$, respectively. For additional graph types of slices involving P_1 and P_2, the collinear via a line of multiplicity 1 and noncollinear situations are both possible. Examples of such slices are given in Appendix A and have graph types G_7 and G_{10}, respectively.

Next consider S with at least three double points. A summary of the analysis for the collinear case is given in Table 4.11.

TABLE 4.11. Collinear case for $n = 6$.

P_2	Additional Points	Reflection Points	Slice	Type
011	022, 033, 044, 055, **101**, 143, **202**, 253, **303**, 404, 413, 505, 523	-	$[-1 : -1 : 1 : 1]$	$G_6(6)$
011	022, 033, 044, 055, **102**, 153, 515, 524	-	$[\xi + 1 : 1 : -1 : -\xi - 1]$	G_{18}
011	022, 033, 044, 055, **103**, 525	-	$[2\xi : 1 : -1 : -2\xi]$	$G_9(6)$
011	022, 033, 044, 055, **104**, 113, 520, 535	-	$[2\xi - 1 : 1 : -1 : -2\xi + 1]$	G_{18}
011	022, 033, 044, 055, **105**, 123, 215, 224, 325, 420, 435, 521, 545	-	$[\xi - 1 : 1 : -1 : -\xi + 1]$	$G_6(6)$
011	022, 033, 044, 055, **201**, 243, 412, 454	-	$[-1 : -\xi - 1 : \xi + 1 : 1]$	G_{18}
011	022, 033, 044, 055, **203**, 414	-	$[-2\xi : -\xi - 1 : \xi + 1 : 2\xi]$	$G_9(6)$
011	022, 033, 044, 055, 112, 154, **204**, 213, 314, 415, 424, 510, 534	-	$[-\xi : -1 : 1 : \xi]$	$G_6(6)$
011	022, 033, 044, 055, **205**, 223, 410, 434	-	$[-\xi : \xi - 2 : -\xi + 2 : \xi]$	G_{18}
011	022, 033, 044, 055, **301**, 343	-	$[-1 : -2\xi : 2\xi : 1]$	$G_9(6)$
011	022, 033, 044, 055, **302**, 353	-	$[-\xi - 1 : -2\xi : 2\xi : \xi + 1]$	$G_9(6)$
011	022, 033, 044, 055, **304**, 313	-	$[-\xi - 1 : -2 : 2 : \xi + 1]$	$G_9(6)$
011	022, 033, 044, 055, **305**, 323	-	$[-\xi : -2 : 2 : \xi]$	$G_9(6)$
012	101, 155, 202, 254, 303, 404, 505, 513	110, 124, 220, 330, 432 440, 544, 550	$[1 : \xi + 1 : -1 : -\xi - 1]$	G_{18}
012	102, 154, 514	125, 543, 551	$[\xi + 1 : \xi + 1 : -1 : -2\xi - 1]$	G_{23}
012	103, 515	120, 552	$[\xi^2 + x + 1 : \xi + 1 : -1 : -\xi^2 - 2\xi - 1]$	G_{22}
012	104, 510, 524	121, 135, 553	$[\xi - 2 : 2\xi - 1 : -\xi : -2\xi + 3]$	G_{23}
012	105, 113, 214, 315, 410,	122, 134, 232, 240, 342,	$[\xi - 1 : \xi + 1 : -1 : -2\xi + 1]$	G_{18}

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
	424, 511, 523	452, 502, 554		
012	114, 420, 522, 534	133, 145, 242, 503	$[\xi-2:\xi+1:-1:-2\xi+2]$	G_{24}
012	115, 430, 533, 545	144, 150, 252, 504	$[-2:\xi+1:-1:-\xi+2]$	G_{24}
012	130	525	$[4\xi-2:-\xi-1:1:-3\xi+2]$	G_{12}
012	131	520	$[4\xi-3:-\xi-1:1:-3\xi+3]$	G_{12}
012	132, 140	521, 535	$[3\xi-3:-\xi-1:1:-2\xi+3]$	G_{25}
012	141	530	$[3\xi-4:-\xi-1:1:-2\xi+4]$	G_{12}
012	142	531	$[2\xi-4:-\xi-1:1:-\xi+4]$	G_{12}
012	143, 151	532, 540	$[\xi-3:-\xi-1:1:3]$	G_{25}
012	152	541	$[-3:-\xi-1:1:\xi+3]$	G_{12}
012	153	542	$[-\xi-2:-\xi-1:1:2\xi+2]$	G_{12}
012	201, 255	433, 445	$[1:3\xi:-\xi-1:-2\xi]$	G_{22}
012	203	441	$[2\xi:3\xi:-\xi-1:-4\xi+1]$	G_{12}
012	204, 414	230, 442	$[\xi:\xi+1:-1:2\xi]$	G_{13}
012	205, 213	443, 451	$[-\xi:-3:-\xi+2:2\xi+1]$	G_{25}
012	210, 224, 422, 434	244, 250, 402, 454	$[1:-\xi-1:1:\xi-1]$	G_{17}
012	211, 223	401, 455	$[\xi:-3\xi:\xi+1:\xi-1]$	G_{13}
012	215	453	$[-2:3\xi:-\xi-1:-2\xi+3]$	G_{12}
012	221, 235	405, 413	$[3\xi-1:-3\xi:\xi+1:-\xi]$	G_{25}
012	225	403	$[-2\xi-1:3\xi:-\xi-1:2]$	G_{12}
012	231	415	$[4\xi-3:-3\xi:\xi+1:-2\xi+2]$	G_{12}
012	233, 245	411, 423	$[2\xi-2:-3\xi:\xi+1:1]$	G_{22}
012	241	425	$[3\xi-4:-3\xi:\xi+1:-\xi+3]$	G_{11}
012	243, 251	421, 435	$[\xi-3:-3\xi:\xi+1:\xi+2]$	G_{20}
012	253	431	$[-\xi-2:-3\xi:\xi+1:3\xi+1]$	G_{11}
012	301, 355	322, 334	$[1:4\xi-2:-2\xi:-2\xi+1]$	G_{15}
012	302, 354	321, 335	$[\xi+1:4\xi-2:-2\xi:-3\xi+1]$	G_{21}
012	304	331	$[-\xi-1:-2\xi-2:2:3\xi+1]$	G_{12}
012	305, 313	332, 340	$[-\xi:-2\xi-2:2:3\xi]$	G_{15}
012	310, 324	343, 351	$[\xi+1:-4\xi+2:2\xi:\xi-3]$	G_{21}
012	311, 323	344, 350	$[\xi:-4\xi+2:2\xi:\xi-2]$	G_{15}
012	314	341	$[\xi-2:4\xi-2:-2\xi:-3\xi+4]$	G_{12}
012	320	353	$[3\xi:-4\xi+2:2\xi:-\xi-2]$	G_{12}
012	325	352	$[-2\xi-1:4\xi-2:-2\xi:3]$	G_{12}
013	101, 202, 255, 303, 404	110, 220, 330, 433, 440,	$[1:2\xi:-1:-2\xi]$	$G_9(6)$

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
	505	550		
013	102, 155	544, 551	$[\xi+1 : 2\xi : -1 : -3\xi]$	G_{22}
013	103	552	$[2\xi : 2\xi : -1 : -4\xi+1]$	G_8
013	104, 515	120, 553	$[2\xi-1 : 2\xi : -1 : -4\xi+2]$	G_{15}
013	105, 415	231, 554	$[\xi-1 : 2\xi : -1 : -3\xi+2]$	G_{14}
013	112, 350, 402, 455	211, 224, 323, 501	$[\xi-1 : -2\xi : 1 : \xi]$	G_{17}
013	114, 215, 310, 411, 424, 512	123, 233, 240, 343, 453, 503	$[-1 : 2\xi : -1 : -2\xi+2]$	$G_9(6)$
013	115, 320, 422, 435	244, 251, 353, 504	$[-\xi-1 : 2\xi : -1 : -\xi+2]$	G_{24}
013	121	510	$[4\xi-3 : -2\xi : 1 : -2\xi+2]$	G_8
013	122, 135	511, 524	$[3\xi-3 : -2\xi : 1 : -\xi+2]$	G_{22}
013	125, 442	204, 514	$[-3\xi+1 : 2\xi : -1 : \xi]$	G_{14}
013	130	525	$[4\xi-4 : -2\xi : 1 : -2\xi+3]$	G_{12}
013	131	520	$[4\xi-5 : -2\xi : 1 : -2\xi+4]$	G_{12}
013	132	521	$[3\xi-5 : -2\xi : 1 : -\xi+4]$	G_{11}
013	133, 140	522, 535	$[2\xi-4 : -2\xi : 1 : 3]$	G_{15}
013	134, 242	420, 523	$[2\xi-3 : -2\xi : 1 : 2]$	G_{14}
013	141	530	$[2\xi-5 : -2\xi : 1 : 4]$	G_{12}
013	142	531	$[\xi-5 : -2\xi : 1 : \xi+4]$	G_{12}
013	143	532	$[-4 : -2\xi : 1 : 2\xi+3]$	G_{12}
013	144, 151	533, 540	$[-3 : -2\xi : 1 : 2\xi+2]$	G_{15}
013	145	534	$[\xi-3 : -2\xi : 1 : \xi+2]$	G_{11}
013	150, 253	431, 545	$[-2 : -2\xi : 1 : 2\xi+1]$	G_{14}
013	152	541	$[-\xi-3 : -2\xi : 1 : 3\xi+2]$	G_{11}
013	153	542	$[-2\xi-2 : -2\xi : 1 : 4\xi+1]$	G_{12}
013	154	543	$[-2\xi-1 : -2\xi : 1 : 4\xi]$	G_{12}
013	201	445	$[1 : 4\xi-2 : -\xi-1 : -3\xi+2]$	G_{12}
013	203	441	$[2\xi : 4\xi-2 : -\xi-1 : -5\xi+3]$	G_8
013	205, 414	230, 443	$[-\xi : -2\xi-2 : -\xi+2 : 4\xi]$	G_{22}
013	210, 423	245, 454	$[2\xi : -4\xi+2 : \xi+1 : \xi-3]$	G_{21}
013	212, 401	223, 450	$[\xi-1 : -4\xi+2 : \xi+1 : 2\xi-2]$	G_{15}
013	214	452	$[-1 : 4\xi-2 : -\xi-1 : -3\xi+4]$	G_{12}
013	221	405	$[4\xi-3 : -4\xi+2 : \xi+1 : -\xi]$	G_{12}
013	225, 434	250, 403	$[-3\xi+1 : 4\xi-2 : -\xi-1 : 2]$	G_{21}
013	232	410	$[3\xi-5 : -4\xi+2 : \xi+1 : 2]$	G_8

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
013	234	412	$[2\xi-3:-4\xi+2:\xi+1:\xi]$	G_{12}
013	241	425	$[2\xi-5:-4\xi+2:\xi+1:\xi+2]$	G_{11}
013	243	421	$[-4:-4\xi+2:\xi+1:3\xi+1]$	G_{12}
013	252	430	$[4\xi-1:2\xi-4:-2\xi+1:-4\xi+4]$	G_{12}
013	254	432	$[3\xi-2:2\xi-4:-2\xi+1:-3\xi+5]$	G_{11}
013	301	334	$[1:4\xi-4:-2\xi:-2\xi+3]$	G_{12}
013	302, 355	322, 335	$[\xi+1:4\xi-4:-2\xi:-3\xi+3]$	G_{22}
013	304	331	$[-\xi+2:4:2\xi-2:-\xi-4]$	G_{12}
013	305	332	$[1:4:2\xi-2:-2\xi-3]$	G_{12}
013	311, 324	344, 351	$[2\xi-1:-4\xi+4:2\xi:-3]$	G_{22}
013	312	345	$[\xi-1:-4\xi+4:2\xi:\xi-3]$	G_{12}
013	314	341	$[-1:4\xi-4:-2\xi:-2\xi+5]$	G_{12}
013	315	342	$[-\xi-1:4\xi-4:-2\xi:-\xi+5]$	G_{12}
013	321	354	$[4\xi-3:-4\xi+4:2\xi:-2\xi-1]$	G_{11}
013	325	352	$[-3\xi+1:4\xi-4:-2\xi:\xi+3]$	G_{11}
014	101, 113, 202, 250, 303, 355, 404, 505	110, 220, 322, 330, 434, 440, 502, 550	$[1:2\xi-1:-1:-2\xi+1]$	G_{18}
014	102, 150, 254	432, 545, 551	$[\xi+1:2\xi-1:-1:-3\xi+1]$	G_{23}
014	103, 155	544, 552	$[2\xi:2\xi-1:-1:-4\xi+2]$	G_{15}
014	104	553	$[2\xi-1:2\xi-1:-1:-4\xi+3]$	G_8
014	105, 515	120, 554	$[\xi-1:2\xi-1:-1:-3\xi+3]$	G_{13}
014	112, 124, 234, 240, 344, 352, 454, 504	115, 210, 311, 325, 412, 424, 501, 513	$[\xi-2:-2\xi+1:1:\xi]$	G_{18}
014	121	510	$[3\xi-4:-2\xi+1:1:-\xi+2]$	G_8
014	122, 130	511, 525	$[2\xi-4:-2\xi+1:1:2]$	G_{15}
014	123, 135, 242	420, 512, 524	$[\xi-3:2\xi+1:1:\xi+1]$	G_{23}
014	131	520	$[2\xi-5:-2\xi+1:1:3]$	G_{12}
014	132	521	$[\xi-5:-2\xi+1:1:\xi+3]$	G_{11}
014	133, 141	522, 530	$[-4:-2\xi+1:1:2\xi+2]$	G_{22}
014	134, 140	523, 535	$[-3:-2\xi+1:1:2\xi+1]$	G_{25}
014	142	531	$[-\xi-4:-2\xi+1:1:3\xi+2]$	G_{11}
014	143	532	$[5\xi-2:\xi-2:-\xi:-5\xi+4]$	G_{11}
014	144, 152	533, 541	$[-2\xi-2:-2\xi+1:1:4\xi]$	G_{22}
014	145, 151	534, 540	$[3\xi-1:\xi-2:-\xi:-3\xi+3]$	G_{25}
014	153	542	$[4\xi-3:\xi-2:-\xi:-4\xi+5]$	G_{11}

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
014	154	543	$[-3\xi : -2\xi+1 : 1 : 5\xi-2]$	G_{12}
014	201, 213	445, 451	$[1 : 3\xi-3 : -\xi-1 : -2\xi+3]$	G_{25}
014	203, 255	433, 441	$[2\xi : 3\xi-3 : -\xi-1 : -4\xi+4]$	G_{22}
014	204	442	$[\xi : 2\xi-1 : -1 : -3\xi+2]$	G_8
014	205	443	$[1 : 3 : 2\xi-1 : -2\xi-3]$	G_{12}
014	211, 225	403, 455	$[2\xi-2 : -3\xi+3 : \xi+1 : -2]$	G_{13}
014	212, 224, 402, 450	212, 224, 402, 450	$[\xi-1 : -2\xi+1 : 1 : \xi-1]$	G_{17}
014	215	453	$[-\xi : 3\xi-3 : -\xi-1 : -\xi+4]$	G_{12}
014	221	405	$[3\xi-4 : -3\xi+3 : \xi+1 : -\xi]$	G_{12}
014	223, 235	401, 413	$[\xi-3 : -3\xi+3 : \xi+1 : \xi+1]$	G_{25}
014	231	415	$[2\xi-5 : -3\xi+3 : \xi+1 : 1]$	G_{12}
014	232	410	$[2\xi-3 : -2\xi+1 : 1 : 1]$	G_8
014	233, 241	411, 425	$[-4 : -3\xi+3 : \xi+1 : 2\xi]$	G_{22}
014	243	421	$[5\xi-2 : -3 : -2\xi+1 : -3\xi+4]$	G_{11}
014	244, 252	422, 430	$[-2 : -2\xi+1 : 1 : 2\xi]$	G_{13}
014	245, 251	423, 435	$[3\xi-1 : -3 : -2\xi+1 : -\xi+3]$	G_{20}
014	253	431	$[4\xi-3 : -3 : -2\xi+1 : -2\xi+5]$	G_{11}
014	301, 313	334, 340	$[1 : 2\xi-4 : -2\xi : 3]$	G_{15}
014	302, 350	323, 335	$[\xi+1 : 2\xi-4 : -2\xi : -\xi+3]$	G_{21}
014	304	331	$[-\xi-1 : -4\xi+2 : 2 : 5\xi-3]$	G_8
014	305	332	$[1 : 2\xi+2 : 2\xi-2 : -4\xi-1]$	G_{12}
014	310	343	$[-\xi-1 : 4\xi-2 : -2 : -3\xi+5]$	G_8
014	312, 324	345, 351	$[\xi-2 : -2\xi+4 : 2\xi : -\xi-2]$	G_{21}
014	315	342	$[-\xi : 2\xi-4 : -2\xi : \xi+4]$	G_{12}
014	320	353	$[3\xi-3 : -2\xi+4 : 2\xi : -3\xi-1]$	G_{12}
014	321	354	$[\xi+3 : -2\xi-2 : -2\xi+2 : 3\xi-3]$	G_{12}
015	101, 125, 202, 213, 224, 235, 240, 251, 303, 345, 404, 455, 505	110, 211, 220, 312, 330, 402, 413, 424, 435, 440, 451, 514, 550	$[1 : \xi-1 : -1 : -\xi+1]$	$G_6(6)$
015	102, 113, 124, 135, 140, 151, 244, 253	422, 431, 502, 513, 524, 535, 540, 551	$[\xi+1 : \xi-1 : -1 : -2\xi+1]$	G_{18}
015	103, 145	534, 552	$[2\xi : \xi-1 : -1 : -3\xi+2]$	G_{14}
015	104, 155	544, 553	$[-\xi+2 : 1 : \xi : -3]$	G_{13}
015	105	554	$[\xi-1 : \xi-1 : -1 : -2\xi+3]$	G_8
015	112, 130, 233, 242	411, 420, 501, 525	$[-2 : -\xi+1 : 1 : \xi]$	G_{17}

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
015	114, 150, 204, 255	433, 442, 503, 545	$[-\xi : -\xi+1 : 1 : 2\xi-2]$	G_{17}
015	121	510	$[\xi-3 : -\xi+1 : 1 : 1]$	G_8
015	122, 131	511, 520	$[-3 : -\xi+1 : 1 : \xi+1]$	G_{13}
015	123, 141	512, 530	$[-\xi-2 : -\xi+1 : 1 : 2\xi]$	G_{14}
015	132	521	$[4\xi-1 : -1 : -\xi : -3\xi+2]$	G_{11}
015	133, 142	522, 531	$[4\xi-2 : -1 : -\xi : -3\xi+3]$	G_{16}
015	134, 152	523, 541	$[3\xi-2 : -1 : -\xi : -2\xi+3]$	G_{20}
015	143	532	$[4\xi-3 : -1 : -\xi : -3\xi+4]$	G_{11}
015	144, 153	533, 542	$[-3\xi : -\xi+1 : 1 : 4\xi-2]$	G_{16}
015	154	543	$[2\xi-3 : -1 : -\xi : -\xi+4]$	G_{11}
015	201, 225, 414, 450	212, 230, 403, 445	$[1 : \xi-2 : -\xi-1 : 2]$	G_{24}
015	203, 245	423, 441	$[-2\xi+2 : \xi+1 : 2\xi-1 : -\xi-2]$	G_{14}
015	205	443	$[-\xi : -2\xi+1 : -\xi+2 : 4\xi-3]$	G_8
015	210, 401, 425	223, 241, 454	$[-\xi : 2\xi-1 : \xi-2 : -2\xi+3]$	G_{23}
015	214, 250, 405	221, 434, 452	$[-\xi : -\xi+2 : \xi+1 : \xi-3]$	G_{23}
015	232	410	$[4\xi-1 : -\xi-1 : -2\xi+1 : -\xi+1]$	G_8
015	234, 252	412, 430	$[3\xi-2 : -\xi-1 : -2\xi+1 : 2]$	G_{14}
015	243	421	$[4\xi-3 : -\xi-1 : -2\xi+1 : -\xi+3]$	G_{11}
015	254	432	$[2\xi-3 : -\xi-1 : -2\xi+1 : \xi+3]$	G_{11}
015	301, 325	334, 352	$[-\xi : 2\xi : 2\xi-2 : -3\xi+2]$	G_{14}
015	302, 313, 324, 335, 340, 351	302, 313, 324, 335, 340, 351	$[-2\xi+1 : 2\xi : 2\xi-2 : -2\xi+1]$	$G_9(6)$
015	304, 355	322, 331	$[-\xi+2 : 2\xi : 2\xi-2 : -3\xi]$	G_{13}
015	305	332	$[-\xi : -2\xi+2 : 2 : 3\xi-4]$	G_8
015	310	343	$[-\xi : 2\xi-2 : -2 : -\xi+4]$	G_8
015	311, 320	344, 353	$[\xi+1 : -2\xi : -2\xi+2 : 3\xi-3]$	G_{13}
015	314, 350	323, 341	$[-\xi : 2 : 2\xi : -\xi-2]$	G_{14}
015	321	354	$[2\xi+1 : -2\xi : -2\xi+2 : 2\xi-3]$	G_{11}
023	101, 144, 202, 303, 404, 505	110, 220, 330, 440, 533, 550	$[\xi+1 : 2\xi : -\xi-1 : -2\xi]$	$G_9(6)$
023	102	551	$[3\xi : 2\xi : -\xi-1 : -4\xi+1]$	G_{12}
023	103	552	$[-4\xi+2 : -2\xi : \xi+1 : 5\xi-3]$	G_8
023	104, 525	130, 553	$[3\xi-3 : 2\xi : -\xi-1 : -4\xi+4]$	G_{22}
023	105, 425	241, 554	$[\xi+1 : -2\xi+2 : 2\xi-1 : -\xi-2]$	G_{14}
023	112, 155, 313, 405	221, 340, 501, 544	$[\xi : 2\xi : -\xi-1 : -2\xi+1]$	G_{24}

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
023	113, 510	121, 502	$[2\xi-2 : 2\xi : -\xi-1 : -3\xi+3]$	G_{13}
023	114, 535	140, 503	$[\xi-3 : 2\xi : -\xi-1 : -2\xi+4]$	G_{21}
023	115	504	$[-\xi-2 : 2\xi : -\xi-1 : 3]$	G_{12}
023	120, 543	154, 515	$[4\xi-2 : -2\xi : \xi+1 : -3\xi+1]$	G_{21}
023	122, 145, 343, 402	224, 310, 511, 534	$[\xi-2 : -2\xi : \xi+1 : 1]$	G_{24}
023	124, 225, 320, 421, 522, 545	133, 150, 243, 353, 403, 513	$[-\xi-1 : 2\xi : -\xi-1 : 2]$	$G_9(6)$
023	125	514	$[-3\xi : 2\xi : -\xi-1 : 2\xi+1]$	G_{12}
023	131	520	$[3\xi-5 : -2\xi : \xi+1 : -2\xi+4]$	G_8
023	132	521	$[\xi-4 : -2\xi : \xi+1 : 3]$	G_{12}
023	135, 442	204, 524	$[-3\xi+2 : 2\xi : -\xi-1 : 2\xi-1]$	G_{14}
023	141	530	$[\xi-5 : -2\xi : \xi+1 : 4]$	G_{12}
023	142	531	$[-\xi-4 : -2\xi : \xi+1 : 2\xi+3]$	G_{11}
023	143	532	$[-2\xi-2 : -2\xi : \xi+1 : 3\xi+1]$	G_{12}
023	151	540	$[-\xi-3 : -2\xi : \xi+1 : 2\xi+2]$	G_{12}
023	152	541	$[-3\xi-2 : -2\xi : \xi+1 : 4\xi+1]$	G_{11}
023	153	542	$[-4\xi : -2\xi : \xi+1 : 5\xi-1]$	G_{12}
023	201, 244	422, 445	$[1 : 2\xi : -\xi-1 : -\xi]$	G_{13}
023	203	441	$[2\xi : 2\xi : -\xi-1 : -3\xi+1]$	G_8
023	205, 424	240, 443	$[-\xi : -2 : -\xi+2 : 2\xi]$	G_{13}
023	210	454	$[2\xi : -4\xi+2 : 3\xi : -\xi-2]$	G_{12}
023	211, 234	412, 455	$[\xi-1 : -4\xi+2 : 3\xi : -1]$	G_{16}
023	212, 255	433, 450	$[\xi : 4\xi-2 : -3\xi : -2\xi+2]$	G_{15}
023	213	451	$[2\xi-2 : 4\xi-2 : -3\xi : -3\xi+4]$	G_{12}
023	214	452	$[\xi-3 : 4\xi-2 : -3\xi : -2\xi+5]$	G_{11}
023	215	453	$[-\xi-2 : 4\xi-2 : -3\xi : 4]$	G_{12}
023	230	414	$[4\xi-4 : -4\xi+2 : 3\xi : -3\xi+2]$	G_{12}
023	231	415	$[3\xi-5 : -4\xi+2 : 3\xi : -2\xi+3]$	G_{11}
023	232	410	$[\xi-4 : -4\xi+2 : 3\xi : 2]$	G_{12}
023	233, 250	411, 434	$[-2 : -4\xi+2 : 3\xi : \xi]$	G_{15}
023	235	413	$[-3\xi+2 : 4\xi-2 : -3\xi : 2\xi]$	G_{12}
023	242	420	$[\xi-3 : -2\xi : \xi+1 : 2]$	G_8
023	251	435	$[4\xi-1 : 2\xi-4 : -3\xi+3 : -3\xi+2]$	G_{11}
023	252	430	$[-3\xi-2 : -4\xi+2 : 3\xi : 4\xi]$	G_{12}
023	253	431	$[-4\xi : -4\xi+2 : 3\xi : 5\xi-2]$	G_{12}

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
023	254	432	$[-3\xi+1 : -4\xi+2 : 3\xi : 4\xi-3]$	G_{11}
023	301, 344	311, 334	$[\xi+1 : 4\xi-4 : -4\xi+2 : -\xi+1]$	G_{22}
023	302	335	$[3\xi : 4\xi-4 : -4\xi+2 : -3\xi+2]$	G_{12}
023	304	331	$[-3\xi : -4\xi : 2\xi+2 : 5\xi-2]$	G_{12}
023	305	332	$[\xi+1 : 4 : 2\xi-4 : -3\xi-1]$	G_{12}
023	312, 355	322, 345	$[\xi : 4\xi-4 : -4\xi+2 : -\xi+2]$	G_{22}
023	314	341	$[\xi-3 : 4\xi-4 : -4\xi+2 : -\xi+5]$	G_{11}
023	315	342	$[-3\xi+1 : -4 : -2\xi+4 : 5\xi-1]$	G_{11}
023	321	354	$[3 : -4 : -2\xi+4 : 2\xi-3]$	G_{11}
023	324	351	$[-\xi-1 : 4\xi-4 : -4\xi+2 : \xi+3]$	G_{12}
023	325	352	$[-3\xi : 4\xi-4 : -4\xi+2 : 3\xi+2]$	G_{12}
024	101, 112, 123, 134, 145, 150, 202, 244, 303, 354, 404, 505, 514	110, 125, 220, 321, 330, 422, 440, 501, 512, 523, 534, 545, 550	$[1 : \xi : -1 : -\xi]$	$G_6(6)$
024	102, 144	533, 551	$[\xi+1 : \xi : -1 : -2\xi]$	G_{13}
024	103, 154	543, 552	$[2\xi : \xi : -1 : -3\xi+1]$	G_{14}
024	104	553	$[2\xi-1 : \xi : -1 : -3\xi+2]$	G_8
024	105, 114, 510, 525	121, 130, 503, 554	$[\xi-1 : \xi : -1 : -2\xi+2]$	G_{17}
024	113, 155, 204, 515	120, 442, 502, 544	$[\xi : \xi : -1 : -2\xi+1]$	G_{17}
024	115, 420, 511, 535	122, 140, 242, 504	$[-1 : \xi : -1 : -\xi+2]$	G_{17}
024	131	520	$[2\xi-3 : -\xi : 1 : -\xi+2]$	G_8
024	132, 141	521, 530	$[\xi-3 : -\xi : 1 : 2]$	G_{14}
024	133, 151	522, 540	$[-2 : -\xi : 1 : \xi+1]$	G_{13}
024	142	531	$[-3 : -\xi : 1 : \xi+2]$	G_{11}
024	143, 152	532, 541	$[-\xi-2 : -\xi : 1 : 2\xi+1]$	G_{20}
024	153	542	$[-2\xi-1 : -\xi : 1 : 3\xi]$	G_{11}
024	201, 212, 223, 234, 245, 250, 413, 455	211, 235, 401, 412, 423, 434, 445, 450	$[1 : 2\xi-1 : -\xi-1 : -\xi+1]$	G_{18}
024	203, 254	432, 441	$[2\xi : 2\xi-1 : -\xi-1 : -3\xi+2]$	G_{14}
024	205, 214, 415	231, 443, 452	$[\xi-1 : 2\xi-1 : -\xi-1 : -2\xi+3]$	G_{23}
024	210, 225, 411, 435	233, 251, 403, 454	$[\xi : -2\xi+1 : \xi+1 : -2]$	G_{24}
024	213, 255, 405, 414	221, 230, 433, 451	$[\xi : 2\xi-1 : -\xi-1 : -2\xi+2]$	G_{24}
024	215, 410, 425	232, 241, 453	$[-1 : 2\xi-1 : -\xi-1 : -\xi+3]$	G_{23}
024	243, 252	421, 430	$[-\xi-2 : -2\xi+1 : \xi+1 : 2\xi]$	G_{14}
024	253	431	$[-2\xi-1 : -2\xi+1 : \xi+1 : 3\xi-1]$	G_{11}

TABLE 4.11. Continued from previous page.

P_2	Additional Points	Reflection Points	Slice	Type
024	301, 312, 323, 334, 345, 350	301, 312, 323, 334, 345, 350	$[1 : 2\xi - 2 : -2\xi : 1]$	$G_9(6)$
024	302, 344	311, 335	$[\xi + 1 : 2\xi - 2 : -2\xi : -\xi + 1]$	G_{13}
024	304	331	$[-\xi - 1 : -2\xi : 2 : 3\xi - 1]$	G_8
024	305, 314	332, 341	$[\xi + 1 : 2\xi + 2 : 2\xi - 4 : -5\xi + 1]$	G_{14}
024	310, 325	343, 352	$[\xi : -2\xi + 2 : 2\xi : -\xi - 2]$	G_{14}
024	313, 355	322, 340	$[\xi : 2\xi - 2 : -2\xi : -\xi + 2]$	G_{13}
024	315	342	$[-1 : 2\xi - 2 : -2\xi : 3]$	G_{11}
024	320	353	$[2\xi - 1 : -2\xi + 2 : 2\xi : -2\xi - 1]$	G_8

It is necessary to check if three double points of L_6 exists with the property of being pairwise noncollinear in Q_6. Table 4.12 identifies candidates for this situation. Reflection points indicate other double points obtained from P_2 by reflections fixing $P_1 = [1 : 1 : 1 : 1]$. Consequently, slices involving these double points are equivalent to the one involving P_3.

Only combinations of three points which are pairwise non-collinear in Q_6 need to be analyzed. A summary of such slices is given below in Table 4.13. This completes the analysis and gives the result. •

4.13. Conjectures on Heavy Multinets from Q_n

The current method used to classify multinets obtained from Q_n becomes more cumbersome as n increases. Due to the current interest in heavy multinets, these situations are investigated for $n = 7, 8, 9$, and 10. A summary of analysis of such slices is given in the tables found in Appendix B.

The infinite families of heavy multinets identified in this dissertation have graph types of $G_2(n)$, $G_3(n)$, $G_5(n)$, and $G_6(n)$. There are two other types of heavy nets which appear, namely G_9 and G_{18}. It is suspected that these examples fit into another

TABLE 4.12. Candidates for P_3 in noncollinear case for $n = 6$.

P_2	Reflection Points	Noncollinear Points with P
000	-	123, 124, 125, 132, 134, 135, 142, 143, 145, 152, 153, 154, 213, 214, 215, 231, 234, 235, 241, 243, 245, 251, 253, 254, 312, 314, 315, 321, 324, 325, 341, 342, 345, 351, 352, 354 412, 413, 415, 421, 423, 425, 431, 432, 435, 451, 452, 453 512, 513, 514, 521, 523, 524, 531, 532, 534, 541, 542, 543
123	125, 145, 345	000, 004, 005, 030, 031, 035, 040,041, 044, 051, 054, 055, 200,202, 205, 210, 211, **215**, 240, **241**, 242, **251**, 252, 255 300, 302, 304, 310, 311, **314**, 330, 331, 332, **351**, **352**, **354**, 402, 414, 415, 411, 414, **415**, **431**, **432**, **435**, 441,442, 444, 510, **514**, 515, 530, **532**, 535, 540, **542**, 544, 552, 554, 555
124	135, 234, 245	000, 001, 005, 030, 031, 032, 041, 042, 045, 050, 052, 055, 200, 201, 203, 210, 211, 212, **241**, 242, **243**, 250, 252, **253**, 301, 303, 305, 311, **312**, **315**, 331, 332, 333, **352**, 353, 355, 400, 403, 405, 410, **412**, **415**, 430, **432**, 433, 442, 443, 445, 510, 511, 515, 530, **531**, 533, **541**, **543**, 545, 550, 553, 555
134	235	000, 002, 005, 010, 011, 015, 040, 041, 042, 051, 052, 055, 200, 202, 203, 210, 211, **213**, 220, 221, 222, **251**, 252, **253**, 302, 303, 305, 311, 313, **315**, **321**, 322, **325**, **341**, **342**, 343, 410, **413**, **415**, 420, 422, **425**, 440, 442, 443, **452**, **453**, 455, 500, 503, 505, 520, **521**, 525, 540, **541**, **543**, 551, 553, 555

TABLE 4.13. Noncollinear case for $n = 6$.

P_2	Additional Points	Slice	Graph Type
123	011, 022, 033, 044, 055, 105, **215**, 224, 325, 420, **435**, 521, 545	$[2\xi - 3 : \xi + 2 : -\xi - 2 : -2\xi + 3]$	$G_6(6)$
123	**241**	$[5\xi - 4 : -3\xi - 1 : -\xi + 2 : -\xi + 3]$	G_{19}
123	**251**	$[3\xi - 4 : -3\xi - 1 : -\xi + 3 : \xi + 2]$	G_{19}
123	**314**	$[3\xi - 2 : 3\xi - 1 : -3\xi - 1 : -3\xi + 4]$	G_{19}
123	**351**	$[3\xi - 4 : -5\xi + 1 : \xi + 2 : \xi + 1]$	G_{19}
123	**352**	$[2\xi - 5 : -5\xi + 2 : \xi + 2 : 2\xi + 1]$	G_{19}
123	024, 101, 112, 134, 145, 150, 202, 244, 303, **354**, 404, 505, **514**	$[-\xi - 2 : -3\xi + 1 : \xi + 2 : 3\xi - 1]$	$G_6(6)$
123	**415**	$[2\xi - 3 : 3\xi - 2 : -4\xi + 1 : -\xi + 4]$	G_{19}
123	**431**	$[5\xi - 2 : -5\xi + 3 : 3\xi - 2 : -3\xi + 1]$	G_{19}
123	**432**	$[4\xi - 3 : -5\xi + 4 : 3\xi - 2 : -2\xi + 1]$	G_{19}
123	**532**	$[4\xi - 3 : -3\xi + 4 : 2\xi - 3 : -3\xi + 2]$	G_{19}
123	**542**	$[4\xi - 5 : -3\xi + 4 : \xi - 2 : -2\xi + 3]$	G_{19}
124	**241**	$[4\xi - 5 : -4\xi + 1 : -\xi + 2 : \xi + 2]$	G_{19}
124	**243**	$[\xi - 5 : -3\xi + 2 : -\xi + 2 : 3\xi + 1]$	G_{19}
124	015, 102, 113, 124, 135, 140, 151, 244, **253**	$[-\xi - 4 : -3\xi + 2 : -\xi + 3 : 5\xi - 1]$	G_{18}
124	**312**	$[\xi - 2 : -5\xi + 4 : 3\xi + 1 : \xi - 3]$	G_{19}
124	**315**	$[\xi - 2 : 3\xi - 2 : -3\xi - 1 : -\xi + 5]$	G_{19}

TABLE 4.13. Continued from previous page.

P_2	Additional Points	Slice	Graph Type
124	014, 112, 234, 240, 344, **352**, 454, 504	$[\xi - 5 : -5\xi + 4 : \xi + 2 : 3\xi - 1]$	G_{18}
124	**412**	$[\xi - 2 : -4\xi + 5 : 4\xi - 1 : -\xi - 2]$	G_{19}
124	**415**	$[\xi - 2 : 2\xi - 3 : -4\xi + 1 : \xi + 4]$	G_{19}
124	012, 110, 220, 330, **432**, 440, 544, 550	$[4\xi - 5 : -4\xi + 5 : 3\xi - 2 : -3\xi + 2]$	G_{18}
124	011, 022, 033, 044, 055, 115, **531**, 540	$[4\xi + 1 : -3\xi + 1 : 3\xi - 1 : -4\xi - 1]$	G_{18}
124	**541**	$[5\xi - 1 : -3\xi + 1 : 2\xi - 1 : -4\xi + 1]$	G_{19}
124	**543**	$[5\xi - 4 : -4\xi + 3 : 2\xi - 1 : -3\xi + 2]$	G_{19}
134	**213**	$[3\xi - 1 : 3\xi - 2 : -3\xi - 1 : -3\xi + 4]$	G_{19}
134	**251**	$[4\xi - 3 : -\xi - 3 : -2\xi + 3 : -\xi + 3]$	G_{19}
134	**253**	$[-3\xi - 2 : -3\xi + 2 : \xi + 2 : 5\xi - 2]$	G_{19}
134	045, 155, 205, 224, **315**, **425**, 510, 535	$[\xi - 3 : 3\xi - 2 : -5\xi + 1 : \xi + 4]$	G_{18}
134	**321**	$[3\xi - 2 : -5\xi + 3 : 5\xi - 2 : -3\xi + 1]$	G_{19}
134	**325**	$[-\xi - 2 : 3\xi - 2 : -5\xi + 2 : 3\xi + 2]$	G_{19}
134	**341**	$[5\xi - 2 : -3\xi - 2 : \xi + 2 : -3\xi + 2]$	G_{19}
134	012, 122, 232, 240, **342**, **452**, 502, 554	$[2\xi - 1 : -\xi - 1 : 1 : -\xi + 1]$	G_{18}
134	**413**	$[3\xi - 1 : 3\xi - 5 : -5\xi + 3 : -\xi + 3]$	G_{19}
134	**415**	$[3\xi - 2 : 3\xi - 1 : -3\xi - 2 : -3\xi + 5]$	G_{19}

TABLE 4.13. Continued from previous page.

P_2	Additional Points	Slice	Graph Type
134	**453**	$[2\xi - 5 : -5\xi + 2 : 2\xi + 1 : \xi + 2]$	G_{19}
134	**521**	$[3\xi - 2 : -2\xi + 3 : 3\xi - 4 : -4\xi + 3]$	G_{19}
134	**541**	$[5\xi - 2 : -3\xi + 1 : 3\xi - 2 : -5\xi + 3]$	G_{19}
134	**543**	$[3\xi - 4 : -4\xi + 3 : 3\xi - 2 : -2\xi + 3]$	G_{19}

two infinite families. Table 4.14 identifies the corresponding graphs associated to examples found of this type. This gives support for the following conjectures.

Conjecture 4.30. There is an infinite family of heavy multinets obtained from slices of Q_n with graph type $G_9(n)$ for $n \geq 4$.

Conjecture 4.31. There is an infinite family of heavy multinets obtained from slices of Q_n with graph type $G_{18}(n)$ for even $n \geq 6$.

Conjecture 4.32. For $n > 1$, any heavy multinet obtained from slices of Q_n has one of the following graph types: $G_2(n), G_3(n), G_5(n), G_6(n), G_9(n),$ or $G_{18}(n)$.

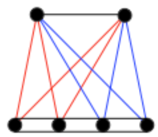
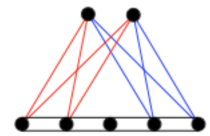
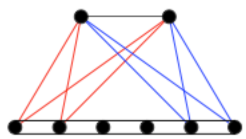
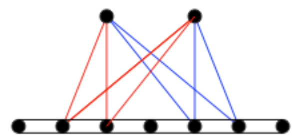
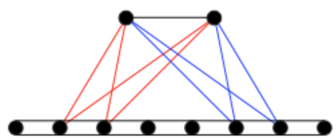
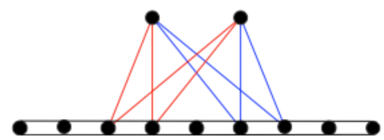
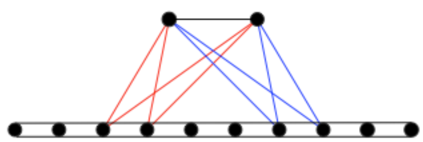
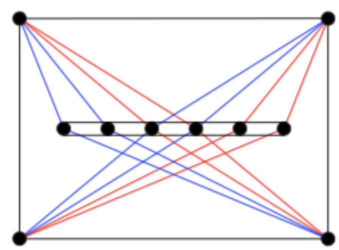
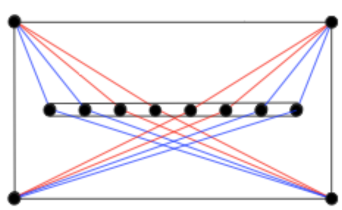
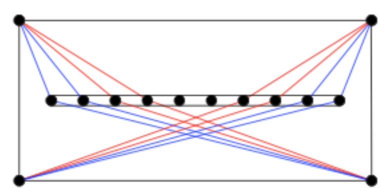
4.14. Selected Examples of Multinets

Examples of multinets with the graph type found from the investigated slices of

$$Q_n = [(x^n - y^n)](z^n - w^n)][(x^n - z^n)(y^n - w^n)][(x^n - w^n)(y^n - z^n)]$$

can be expressed via equations using the tables found in Appendix A. In this section, seven of the new examples are given explicitly.

TABLE 4.14. Additional graph types of Q_n.

$G_9(4)$		$G_9(5)$	
$G_9(6)$		$G_9(7)$	
$G_9(8)$		$G_9(9)$	
$G_9(10)$		$G_{18}(6)$	
$G_{18}(8)$		$G_{18}(10)$	

Example 4.33. Slicing Q_4 by $w = x + 3y$ produces a light $(3, 8)$-multinet of graph type $G_1(4)$ with defining polynomial

$$\begin{aligned} Q &= [(x^4 - y^4)(z^4 - (x+3y)^4)][(x^4 - z^4)(y^4 - (x+3y)^4)][(x^4 - (x+3y)^4)(y^4 - z^4)] \\ &\doteq C_1 C_2 C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)(x-iy)(x+iy)(x+3y-z)(x+3y+z)(x+3y-iz)(x+3y+iz) \\ C_2 &= (x+2y)(x+4y)(x+(3-i)y)(x+(3+i)y)(x-z)(x+z)(x-iz)(x+iz) \\ C_3 &= y(2x+3y)((1-i)x+3y)((1+i)x+3y)(y-z)(y+z)(y-iz)(y+iz). \end{aligned}$$

Each class consists of eight lines of multiplicity 1. Its base locus $\mathcal{X}$ has 48 points of multiplicity 1 and a unique point of multiplicity 4, namely $P = [0 : 0 : 1]$. There are exactly four lines from each class passing through P.

Example 4.34. Slicing Q_4 by $w = -x - y + 3z$ produces a light $(3, 8)$-multinet of graph type G_4 with defining polynomial

$$\begin{aligned} Q &= [(x^4 - y^4)(z^4 - (-x - y + 3z)^4)][(x^4 - z^4)(y^4 - (-x - y + 3z)^4)][(x^4 - (-x - y + 3z)^4)(y^4 - z^4)] \\ &\doteq C_1C_2C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)(x-iy)(x+iy)(x+y-2z)(x+y-4z)(x+y+(-3-i)z)(x+y+(-3+i)z) \\ C_2 &= (x-z)(x+z)(x-iz)(x+iz)(x+2y-3z)(x-3z)(x+(1-i)y-3z)(x+(1+i)y-3z) \\ C_3 &= (y-z)(y+z)(y-iz)(y+iz)(2x+y-3z)(y-3z)((1+i)x+y-3z)((1-i)x+y-3z). \end{aligned}$$

Each class contains eight lines of multiplicity 1. Its base locus $\mathcal{X}$ has 60 points of multiplicity 1 and a unique double point, namely $P = [1 : 1 : 1]$. There are exactly two lines from each class passing through P.

Example 4.35. Slicing Q_4 by $w = x + 3y$ produces a heavy $(3, 8)$-multinet of graph type $G_5(4)$ with defining polynomial

$$\begin{aligned} Q &= [(x^4 - y^4)(z^4 - (x + \pi y - \pi z)^4)][(x^4 - z^4)(y^4 - (x + \pi y - \pi z)^4)][(x^4 - (x + \pi y - \pi z)^4)(y^4 - z^4)] \\ &\doteq C_1C_2C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)(x-iy)(x+iy)(x+\pi y-(\pi-1)z)(x+\pi y-(\pi+1)z)(x+\pi y-(\pi-i)z)(x+\pi y-(\pi+i)z) \\ C_2 &= (x-z)(x+z)(x-iz)(x+iz)(x+(\pi+1)y-\pi z)(x+(\pi-1)y-\pi z)(x+(\pi-i)y-\pi z)(x+(\pi+i)y-\pi z) \\ C_3 &= (y-z)^2(y+z)(y-iz)(y+iz)(2x+\pi y-\pi z)((1-i)x+\pi y-\pi z)((1+i)x+\pi y-\pi z). \end{aligned}$$

One class has one double line and six lines of multiplicity 1. The other two classes consist of eight lines of multiplicity 1. Its base locus $\mathcal{X}$ has 48 points of multiplicity 1 and four double points. The latter points are: $[1 : 1 : 1]$, $[-1 : 1 : 1], [i : 1 : 1]$, and $[-i : 1 : 1]$.

Example 4.36. Slicing Q_4 by $w = x + y - z$ produces a heavy $(3,8)$-multinet of graph type $G_6(4)$ with defining polynomial

$$\begin{aligned} Q &= [(x^4 - y^4)(z^4 - (x+y-z)^4)][(x^4 - z^4)(y^4 - (x+y-z)^4)][(x^4 - (x+y-z)^4)(y^4 - z^4)] \\ &\doteq C_1C_2C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)^2(x-iy)(x+iy)(x+y-2z)(x+y+(-1+i)z)(x+y+(-1-i)z) \\ C_2 &= (x-z)^2(x+z)(x-iz)(x+iz)(x+2y-z)(x+(1+i)y-z)(x+(1-i)y-z) \\ C_3 &= (y-z)^2(y+z)(y-iz)(y+iz)(2x+y-z)((1-i)x+y-z)((1+i)x+y-z). \end{aligned}$$

Each class is composed of one double line and six lines of multiplicity 1. Its base locus $\mathcal{X}$ has 28 points of multiplicity 1 and nine double points. The latter points are:

$$[1:1:1], [1:-1:1], [1:i:1], [1:-i:1], [-1:1:1], [i:1:1], [i:-i:1], [-i:1:1], \text{ and } [-i:i:1].$$

Example 4.37. Slicing Q_4 by $w = x + y + (2+i)z$ produces a light $(3,8)$-multinet of graph type G_8 with defining polynomial

$$\begin{aligned} Q &= [(x^4 - y^4)(z^4 - (x+y+(2+i)z)^4)][(x^4 - z^4)(y^4 - (x+y+(2+i)z)^4)][(x^4 - (x+y+(2+i)z)^4)(y^4 - z^4)] \\ &\doteq C_1C_2C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)(x-iy)(x+iy)(x+y+2z)(x+y+(2+2i)z)(x+y+(1+i)z)(x+y+(3+i)z) \\ C_2 &= (x-z)(x+z)(x-iz)(x+iz)(x+(2+i)z)(x+2y+(2+i)z)(x+(1-i)y+(2+i)z)(x+(1+i)y+(2+i)z) \\ C_3 &= (y-z)(y+z)(y-iz)(y+iz)(y+(2+i)z)(2x+y+(2+i)z)(2x+(1+i)y+(1+3i)z)(2x+(1-i)y+(3-i)z). \end{aligned}$$

Each class has eight lines of multiplicity 1. Its base locus $\mathcal{X}$ has 52 points of multiplicity 1 and three double points. The latter points are:

$$[-1:-1:1], [-1:-i:1], \text{ and } [-i:-1:1].$$

There are exactly two lines from each class passing through these double points.

Example 4.38. Slicing Q_4 by the hyperplane $w = x + (1+i)y + (1+i)z$ produces a heavy $(3,8)$-multinet of graph type $G_9(4)$ with defining polynomial

$$\begin{aligned} Q &= [(x^4-y^4)(z^4-(x+(1+i)y+(1+i)z)^4)][(x^4-z^4)(y^4-(x+(1+i)y+(1+i)z)^4)][(x^4-(x+(1+i)y+(1+i)z)^4)(y^4-z^4)] \\ &\doteq C_1C_2C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)(x-iy)(x+iy)(x+(1+i)y+z)(x+(1+i)y+iz)(x+(1+i)y+(1+2i)z)(x+(1+i)y+(2+i)z) \\ C_2 &= (x-z)(x+z)(x-iz)(x+iz)(x+y+(1+i)z)(x+iy+(1+i)z)(x+(1+2i)y+(1+i)z)(x+(2+i)y+(1+i)z) \\ C_3 &= (y-z)(y+z)^2(y-iz)(y+iz)(x+y+z)(x+iy+iz)(2x+(1+i)y+(1+i)z). \end{aligned}$$

One class has one double line and six lines of multiplicity 1. The other two classes are composed of eight lines of multiplicity 1. Its base locus $\mathcal{X}$ has 40 points of multiplicity 1 and six double points. The latter points are:

$$[1:-1:1], [-1:-1:1], [i:-1:1], [-i:-1:1], [-1:-i:1], \text{ and } [-i:i:1].$$

Example 4.39. Let ξ be a primitive sixth root of unity. Slicing Q_6 by the hyperplane $w = x + (1+\xi)y + (1+\xi)z$ produces a heavy $(3,12)$-multinet of graph type $G_{18}(6)$ with defining polynomial

$$\begin{aligned} Q &= [(x^6-y^6)(z^6-(x+(1+\xi)y+(1+\xi)z)^6)][(x^6-z^6)(y^6-(x+(1+\xi)y+(1+\xi)z)^6)][(x^6-(x+(1+\xi)y+(1+\xi)z)^6)(y^6-z^6)] \\ &\doteq C_1C_2C_3 \end{aligned}$$

where

$$\begin{aligned} C_1 &= (x-y)(x+y)(x-\xi y)(x+\xi y)(x-\xi^2 y)(x+\xi^2 y)(x+(1+\xi)y+\xi z)(x+(1+\xi)y+(2+\xi)z)\cdots \\ &\quad (x+(1+\xi)y+z)(x+(1+\xi)y+(1+2\xi)z)(x+(1+\xi)y+2z)(x+(1+\xi)y+2\xi z) \\ C_2 &= (x-z)(x+z)(x-\xi z)(x+\xi z)(x-\xi^2 z)(x+\xi^2 z)(x+\xi y+(1+\xi)z)(x+(2+\xi)y+(1+\xi)z)\cdots \\ &\quad (x+y+(1+\xi)z)(x+(1+2\xi)y+(1+\xi)z)(x+2y+(1+\xi)z)(x+2\xi y+(1+\xi)z) \\ C_3 &= (y-z)(y+z)^2(y-\xi z)(y+\xi z)(y-\xi^2 z)(y+\xi^2 z)(x+y+z)(x+\xi y+\xi z)\cdots \\ &\quad (2x+(1+\xi)y+(1+\xi)z)(x+(-1+2\xi)y+(-1+2\xi)z)(x+(2-\xi)y+(2-\xi)z). \end{aligned}$$

One class has one double line and ten lines of multiplicity 1. The other two classes have twelve lines of multiplicity 1. Its base locus $\mathcal{X}$ has 104 points of multiplicity 1 and ten double points. Noting that $\xi^2 - \xi + 1 = 0$, the latter points are: $[1:-1:1]$, $[-1:-1:1], [\xi:-1:1], [-\xi:-1:1], [\xi^2:-1:1], [-\xi^2:-1:1], [-1:-\xi:1]$, $[-\xi:\xi^2:1], [\xi^2:-\xi:1]$, and $[-\xi^2:\xi^2:1]$.

APPENDIX A

SUMMARY OF EXAMPLES OF MULTINETS FROM Q_N

This appendix summarizes the various examples of multinets found from the investigated slices of

$$Q_n = [(x^n - y^n)](z^n - w^n)][(x^n - z^n)(y^n - w^n)][(x^n - w^n)(y^n - z^n)].$$

The infinite families of multinets appear in Table A.1. The remaining sporadic examples of multinets are listed in Table A.2. These tables include examples of all of the proper multinets known at this time. In Table A.2, ξ denotes a primitive nth root of unity where n is listed for each example.

TABLE A.1. Examples of multinets from infinite families.

n	Graph Type	Slice
$n \geq 2$	G_0	$w = 2x + 4y + 8z$
	$G_1(n)$	$w = x + 3y$
	$G_2(n)$	$w = 2z$
	$G_3(n)$	$w = 0$
$n \geq 3$	G_4	$w = -x - y + 3z$
	$G_5(n)$	$w = x + \pi y - \pi z$
	$G_6(n)$	$w = x + y - z$

TABLE A.2. Examples of sporadic multinets.

n	Graph Type	Slice
3	G_7	$w = -x + 2y - (\xi + 2)z$
3	G_8	$w = -x - y - (\xi - 1)z$
4	G_7	$w = 3x + (\xi + 1)y - (\xi + 3)z$
4	G_8	$w = x + y + (\xi + 2)z$
4	$G_9(4)$	$w = x + (\xi + 1)y + (\xi + 1)z$
4	G_{10}	$w = 2x + (\xi - 3)y + (2\xi + 3)z$
4	G_{11}	$w = x + (\xi + 2)y + (2\xi + 1)z$
4	G_{12}	$w = 2x + (\xi + 1)y + (\xi + 2)z$
4	G_{13}	$w = x + 2y + (\xi + 1)z$
5	G_7	$w = 2x + (\xi + 1)y - (\xi - 2)z$
5	G_8	$w = -x + (\xi + 1)y + (\xi - 1)z$
5	$G_9(5)$	$w = x + (\xi + 1)y - (\xi + 1)z$
5	G_{10}	$w = -2x + (\xi^3 + \xi - 2)y - (2\xi^3 + 2\xi + 3)z$
5	G_{11}	$w = (\xi + 1)x - (\xi - 1)y + (\xi^3 - \xi^2 - 1)z$
5	G_{12}	$w = (\xi + 1)x + (\xi + 2)y - (x^2 + 2\xi + 1)z$
5	G_{13}	$w = -x - (\xi + 1)y + (\xi^2 + 2\xi + 1)z$
5	G_{14}	$w = -x + (\xi + 1)y + (\xi^2 - 1)z$
5	G_{15}	$w = (\xi + 1)x + (\xi^2 + \xi + 1)y - (\xi^2 + 2\xi + 1)z$
5	G_{16}	$w = -x - (\xi^2 + 2\xi + 1)y - (\xi^3 - \xi - 1)z$
5	G_{17}	$w = (\xi + 1)x + (\xi + 1)y - (\xi^2 + \xi + 1)z$
6	G_7	$w = 4x + (\xi + 1)y - (\xi + 4)z$
6	G_8	$w = 2x + 2y + (\xi + 3)z$

TABLE A.2. Continued from previous page.

n	Graph Type	Slice
6	$G_9(6)$	$w = x + 2y - 2z$
6	G_{10}	$w = -2x + \frac{1}{2}(2\xi + 3)y + \frac{1}{2}(3\xi + 1)z$
6	G_{11}	$w = x + 3y + (\xi + 2)z$
6	G_{12}	$w = 3x + (\xi + 1)y + (\xi + 3)z$
6	G_{13}	$w = x + 2y + (\xi + 1)z$
6	G_{14}	$w = x + 2y + (\xi - 3)z$
6	G_{15}	$w = 2x + (\xi + 1)y + 2(\xi - 2)z$
6	G_{16}	$w = x + 3y + 2(2\xi - 1)z$
6	G_{17}	$w = x + y + (\xi - 2)z$
6	$G_{18}(6)$	$w = x + (\xi + 1)y + (\xi + 1)z$
6	G_{19}	$w = \frac{1}{7}(10\xi - 9)x + \frac{1}{7}(5\xi + 6)y + \frac{1}{7}(5\xi - 8)z$
6	G_{20}	$w = x + (\xi + 2)y + (2\xi + 1)z$
6	G_{21}	$w = \frac{1}{2}(\xi + 1)x + \frac{1}{2}(\xi + 2)y + (\xi + 1)z$
6	G_{22}	$w = 2x + 3y + (\xi + 1)z$
6	G_{23}	$w = (\xi + 1)x + (\xi - 3)y + (2\xi - 1)z$
6	G_{24}	$w = 2x + (\xi + 1)y + (\xi - 2)z$
6	G_{25}	$w = 3x + (\xi + 1)y + (2\xi + 1)z$
7	$G_9(7)$	$w = x + (\xi + 1)y - (\xi + 1)z$
8	$G_9(8)$	$w = w = x + (\xi^2 + 1)y + (\xi^2 + 1)z$
8	$G_{18}(8)$	$w = x + (\xi + 1)y - (\xi + 1)z$
9	$G_9(9)$	$w = x + (\xi + 1)y - (\xi + 1)z$
10	$G_9(10)$	$w = 2x + (\xi + 1)y + (\xi - 2)z$

TABLE A.2. Continued from previous page.

n	Graph Type	Slice
10	$G_{18}(10)$	$w = x + (2\xi^3 + 2\xi)y - (2\xi^3 + 2\xi)z$

APPENDIX B

ADDITIONAL COMPUTATIONS

This appendix contains the computations from investigating heavy multinets obtained from Q_n for $n = 7, 8, 9,$ and 10.

TABLE B.1. Heavy multinet case for $n = 7$.

P_2	Additional Points	Slice	Graph Type
011	022, 033, 044, 055, 066, **101**, 202 303, 404, 505, 606	$[1 : 1 : -1 : -1]$	$G_6(7)$
011	022, 033, 044, 055, 066, **102**, 616	$[\xi + 1 : 1 : -1 : -\xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **103**, 626	$[\xi^2 + \xi + 1 : 1 : -1 : -\xi^2 - \xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **104**, 636	$[\xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **105**, 646	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **106**, 216, 326, 436, 546, 656	$[\xi^5 + \xi^4 + \xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^5 - \xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_6(7)$
011	022, 033, 044, 055, 066, **201**, 565	$[1 : \xi + 1 : -\xi - 1 : -1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **203**, 515	$[\xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^2 - \xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **204**, 525	$[\xi^2 + 1 : 1 : -1 : -\xi^2 - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, 165, **205**, 315, 425, 535, 645	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_6(7)$
011	022, 033, 044, 055, 066, **206**, 545	$[\xi^4 + \xi^2 + 1 : 1 : -1 : -\xi^4 - \xi^2 - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **301**, 454	$[1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **302**, 464	$[\xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, 154, 264, **304**, 414, 524, 634	$[\xi^3 + \xi^2 + \xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_6(7)$
011	022, 033, 044, 055, 066, **305**, 424	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(7)$
011	022, 033, 044, 055, 066, **306**, 434	$[\xi^3 + 1 : 1 : -1 : -\xi^3 - 1]$	$G_9(7)$

TABLE B.2. Heavy multinet case for $n = 8$.

P_2	Additional Points	Slice	Graph Type
011	022, 033, 044, 055, 066, 077, **101**, 154, **202**, 264, **303**, 374, **404**, 505, 514, 606, 624, 707, 734	$[1 : 1 : -1 : -1]$	$G_6(8)$
011	022, 033, 044, 055, 066, 077, **102**, 164, 717, 735	$[\xi + 1 : 1 : -1 : -\xi - 1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **103**, 174, 727, 736	$[\xi^2 + \xi + 1 : 1 : -1 : -\xi^2 - \xi - 1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **104**, 737	$[\xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **105**, 114, 730, 747	$[\xi^3 + \xi^2 + \xi : 1 : -1 : -\xi^3 - \xi^2 - \xi]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **106**, 124, 731, 757	$[\xi^3 + \xi^2 : 1 : -1 : -\xi^3 - \xi^2]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **107**, 134, 217, 235, 327, 336, 437, 530, 547, 631, 657, 732, 767	$[\xi^3 + \xi^2 : 1 : -1 : -\xi^3 - \xi^2]$	$G_6(8)$
011	022, 033, 044, 055, 066, 077, **201**, 254, 623, 676	$[1 : \xi + 1 : -\xi - 1 : -1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **203**, 274, 616, 625	$[\xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^2 - \xi - 1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **204**, 626	$[\xi^2 + 1 : 1 : -1 : -\xi^2 - 1]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **205**, 214, 627, 636	$[\xi^3 + \xi^2 + \xi : \xi + 1 : -\xi - 1 : -\xi^3 - \xi^2 - \xi]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, 123, 176, **206**, 224, 316, 325, 426, 527, 536, 620, 646, 721, 756	$[\xi^2 : 1 : -1 : -\xi^2]$	$G_6(8)$
011	022, 033, 044, 055, 066, 077, **207**, 234, 621, 656	$[\xi^3 : \xi + 1 : -\xi - 1 : -\xi^3]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **301**, 354, 512, 565	$[1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **302**, 364, 513, 575	$[\xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi - 1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **304**, 515	$[\xi^3 + \xi^2 + \xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, 112, 165, 213, 275, **305**, 314, 415, 516, 525, 617, 635, 710, 745	$[\xi^3 + \xi^2 + \xi : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^3 - \xi^2 - \xi]$	$G_6(8)$
011	022, 033, 044, 055, 066, 077, **306**, 324, 517, 535	$[\xi^3 + 1 : 1 : -1 : -\xi^3 - 1]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **307**, 334, 510, 545	$[\xi^3 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^3]$	$G_{18}(8)$
011	022, 033, 044, 055, 066, 077, **401**, 454	$[1 : \xi^3 + \xi^2 + \xi + 1 : -\xi^3 - \xi^2 - \xi - 1 : -1]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **402**, 464	$[1 : \xi^2 + 1 : -\xi^2 - 1 : -1]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **403**, 474	$[\xi^2 + \xi + 1 : \xi^3 + \xi^2 + \xi + 1 : -\xi^3 - \xi^2 - \xi - 1 : -\xi^2 - \xi - 1]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **405**, 414	$[\xi^3 + \xi^2 + \xi : \xi^3 + \xi^2 + \xi + 1 : -\xi^3 - \xi^2 - \xi - 1 : -\xi^3 - \xi^2 - \xi]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **406**, 424	$[\xi^2 : \xi^2 + 1 : -\xi^2 - 1 : -\xi^2]$	$G_9(8)$
011	022, 033, 044, 055, 066, 077, **407**, 434	$[\xi^3 : \xi^3 + \xi^2 + \xi + 1 : -\xi^3 - \xi^2 - \xi - 1 : -\xi^3]$	$G_9(8)$

TABLE B.3. Heavy multinet case for $n = 9$.

P_2	Additional Points	Slice	Graph Type
011	022, 033, 044, 055, 066, 077, 088, **101**, **202**, **303**, **404**, 505, 606, 707, 808	$[1 : 1 : -1 : -1]$	$G_6(9)$
011	022, 033, 044, 055, 066, 077, 088, **102**, 818	$[\xi + 1 : 1 : -1 : -\xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **103**, 828	$[\xi^2 + \xi + 1 : 1 : -1 : -\xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **104**, 838	$[\xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **105**, 848	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **106**, 858	$[\xi^5 + \xi^4 + \xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^5 - \xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **107**, 868	$[\xi^5 + \xi^4 + \xi^2 + \xi : 1 : -1 : -\xi^5 - \xi^4 - \xi^2 - \xi]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **108**, 218, 328, 438, 548, 658, 768, 878	$[\xi^5 + \xi^2 : 1 : -1 : -\xi^5 - \xi^2]$	$G_6(9)$
011	022, 033, 044, 055, 066, 077, 088, **201**, 787	$[1 : \xi + 1 : -\xi - 1 : -1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **203**, 717	$[\xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **204**, 727	$[\xi^2 + 1 : 1 : -1 : -\xi^2 - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **205**, 737	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **206**, 747	$[\xi^4 + \xi^2 + 1 : 1 : -1 : -\xi^4 - \xi^2 - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, 187, **207**, 317, 427, 537, 647, 757, 867	$[\xi^5 + \xi^4 + \xi^2 + \xi : \xi + 1 : -\xi - 1 : -\xi^5 - \xi^4 - \xi^2 - \xi]$	$G_6(9)$
011	022, 033, 044, 055, 066, 077, 088, **208**, 767	$[\xi^4 - \xi^3 + \xi^2 : 1 : -1 : -\xi^4 + \xi^3 - \xi^2]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **301**, 676	$[1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **302**, 686	$[\xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **304**, 616	$[\xi^3 + \xi^2 + \xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **305**, 626	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, 176, 286, **306**, 416, 526, 636, 746, 856	$[\xi^3 + 1 : 1 : -1 : -\xi^3 - 1]$	$G_6(9)$
011	022, 033, 044, 055, 066, 077, 088, **307**, 646	$[\xi^5 + \xi^4 + \xi^2 + \xi : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^5 - \xi^4 - \xi^2 - \xi]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **308**, 656	$[\xi^5 + \xi^2 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^5 - \xi^2]$	$G_9(9)$

TABLE B.3. Continued from previous page.

P_2	Additional Points	Slice	Graph Type
011	022, 033, 044, 055, 066, 077, 088, **401**, 565	$[1 : \xi^3+\xi^2+\xi+1 : -\xi^3-\xi^2-\xi-1 : -1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **402**, 575	$[1 : \xi^2+1 : -\xi^2-1 : -1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **403**, 585	$[\xi^2+\xi+1 : \xi^3+\xi^2+\xi+1 : -\xi^3-\xi^2-\xi-1 : -\xi^2-\xi-1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, 165, 275, 385, **405**, 515, 625, 735, 845	$[\xi^4+\xi^3+\xi^2+\xi+1 : \xi^3+\xi^2+\xi+1 : -\xi^3-\xi^2-\xi-1 : -\xi^4-\xi^3-\xi^2-\xi-1]$	$G_6(9)$
011	022, 033, 044, 055, 066, 077, 088, **406**, 525	$[\xi^4+\xi^2+1 : \xi^2+1 : -\xi^2-1 : -\xi^4-\xi^2-1]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **407**, 535	$[\xi^5+\xi^4+\xi^2+\xi : \xi^3+\xi^2+\xi+1 : -\xi^3-\xi^2-\xi-1 : -\xi^5-\xi^4-\xi^2-\xi]$	$G_9(9)$
011	022, 033, 044, 055, 066, 077, 088, **408**, 545	$[\xi^4+1 : 1 : -1 : -\xi^4-1]$	$G_9(9)$

TABLE B.4. Heavy multinet case for $n = 10$.

P_2	Additional Points	Slice	Graph Type
011	022, 033, 044, 055, 066, 077, 088, 099, **101**, 165, **202**, 275, **303**, 385, **404**, 495, **505**, 606, 615, 707, 725, 808, 835, 909, 945	$[1 : 1 : -1 : -1]$	$G_6(10)$
011	022, 033, 044, 055, 066, 077, 088, 088, 099, **102**, 175, 919, 946	$[\xi + 1 : 1 : -1 : -\xi - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **103**, 185, 929, 947	$[\xi^2 + \xi + 1 : 1 : -1 : -\xi^2 - \xi - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **104**, 195, 939, 948	$[\xi^3 + \xi^2 + \xi + 1 : 1 : -1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **105**, 949	$[2\xi^3 + 2\xi : 1 : -1 : -2\xi^3 - 2\xi]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **106**, 115, 940, 959	$[2\xi^3 + 2\xi - 1 : 1 : -1 : -2\xi^3 - 2\xi + 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **107**, 125, 941, 969	$[2\xi^3 + \xi - 1 : 1 : -1 : -2\xi^3 - \xi + 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **108**, 135, 942, 979	$[2\xi^3 - \xi^2 + \xi - 1 : 1 : -1 : -2\xi^3 + \xi^2 - \xi + 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **109**, 145, 219, 246, 329, 347, 439, 448, 549, 640, 659, 741, 769, 842, 879, 943, 989	$[\xi^3 - \xi^2 + \xi - 1 : 1 : -1 : -\xi^3 + \xi^2 - \xi + 1]$	$G_6(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **201**, 265, 834, 898	$[1 : \xi + 1 : -\xi - 1 : -1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **203**, 285, 818, 836	$[\xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^2 - \xi - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **204**, 295, 828, 837	$[\xi^2 + 1 : 1 : -1 : -\xi^2 - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **205**, 838	$[\xi^4 + \xi^3 + \xi^2 + \xi + 1 : \xi + 1 : -\xi - 1 : -\xi^4 - \xi^3 - \xi^2 - \xi - 1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **206**, 215, 839, 848	$[\xi^4 + \xi^2 + 1 : 1 : -1 : -\xi^4 - \xi^2 - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **207**, 225, 830, 858	$[2\xi^3 + \xi - 1 : \xi + 1 : -\xi - 1 : -2\xi^3 - \xi + 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, 134, 198, **208**, 235, 318, 336, 428, 437, 538, 639, 648, 730, 758, 831, 868, 932, 978	$[\xi^3 : 1 : -1 : -\xi^3]$	$G_6(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **209**, 245, 832, 878	$[\xi^3 - \xi^2 + \xi - 1 : \xi + 1 : -\xi - 1 : -\xi^3 + \xi^2 - \xi + 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **301**, 365, 723, 787	$[1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **302**, 375, 724, 797	$[\xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **304**, 395, 717, 726	$[\xi^3 + \xi^2 + \xi + 1 : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -\xi^3 - \xi^2 - \xi - 1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **305**, 727	$[2\xi^3 + 2\xi : \xi^2 + \xi + 1 : -\xi^2 - \xi - 1 : -2\xi^3 - 2\xi]$	$G_9(10)$

TABLE B.4. Continued from previous page.

P_2	Additional Points	Slice	Graph Type
011	022, 033, 044, 055, 066, 077, 088, 099, **306**, 315, 728, 737	$[\xi^3+1:1:-1:-\xi^3-1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, 123, 187, 224, 297, **307**, 325, 417, 426, 527, 628, 637, 729, 747, 820, 857, 921, 967	$[2\xi^3+\xi-1:\xi^2+\xi+1:-\xi^2-\xi-1:-2\xi^3-\xi+1]$	$G_6(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **308**, 335, 720, 757	$[2\xi^3-\xi^2+\xi-1:\xi^2+\xi+1:-\xi^2-\xi-1:-2\xi^3+\xi^2-\xi+1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **309**, 345, 721, 767	$[\xi^3-\xi+1:1:-1:-\xi^3+\xi-1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **401**, 465, 612, 676	$[1:\xi^3+\xi^2+\xi+1:-\xi^3-\xi^2-\xi-1:-1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **402**, 475, 613, 686	$[1:\xi^2+1:-\xi^2-1:-1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **403**, 485, 614, 696	$[\xi^2+\xi+1:\xi^3+\xi^2+\xi+1:-\xi^3-\xi^2-\xi-1:-\xi^2-\xi-1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **405**, 616	$[2\xi^3+2\xi:\xi^3+\xi^2+\xi+1:-\xi^3-\xi^2-\xi-1:-2\xi^3-2\xi]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, 112, 176, 213, 286, 314, 396, **406**, 415, 516, 617, 626, 718, 736, 819, 846, 910, 956	$[\xi^3+\xi:\xi^2+1:-\xi^2-1:-\xi^3-\xi]$	$G_6(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **407**, 425, 614, 636	$[2\xi^3+\xi-1:\xi^3+\xi^2+\xi+1:-\xi^3-\xi^2-\xi-1:-2\xi^3-\xi+1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **408**, 435, 619, 646	$[\xi^3-\xi^2+\xi:1:-1:-\xi^3+\xi^2-\xi]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **409**, 445, 610, 656	$[\xi^3-\xi^2+\xi-1:\xi^3+\xi^2+\xi+1:-\xi^3-\xi^2-\xi-1:-\xi^3+\xi^2-\xi+1]$	$G_{18}(10)$
011	022, 033, 044, 055, 066, 077, 088,	$[1:2\xi^3+2\xi:-2\xi^3-2\xi:-1]$	$G_9(10)$
	099, **501**, 565		
011	022, 033, 044, 055, 066, 077, 088, 099, **502**, 575	$[\xi+1:2\xi^3+2\xi:-2\xi^3-2\xi:-\xi-1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **503**, 585	$[\xi^2+\xi+1:2\xi^3+2\xi:-2\xi^3-2\xi:-\xi^2-\xi-1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **504**, 595	$[\xi^3+\xi^2+\xi+1:2\xi^3+2\xi:-2\xi^3-2\xi:-\xi^3-\xi^2-\xi-1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **506**, 515	$[2\xi^3+2\xi-1:2\xi^3+2\xi:-2\xi^3-2\xi:-2\xi^3-2\xi+1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **507**, 525	$[2\xi^3+\xi-1:2\xi^3+2\xi:-2\xi^3-2\xi:-2\xi^3-\xi+1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **508**, 535	$[2\xi^3-\xi^2+\xi-1:2\xi^3+2\xi:-2\xi^3-2\xi:-2\xi^3+\xi^2-\xi+1]$	$G_9(10)$
011	022, 033, 044, 055, 066, 077, 088, 099, **509**, 545	$[\xi^3-\xi^2+\xi-1:2\xi^3+2\xi:-2\xi^3-2\xi:-\xi^3+\xi^2-\xi+1]$	$G_9(10)$

REFERENCES CITED

[1] A. Blokhuis, G. Korchmáros, and F. Mazzocca. On the structure of 3-nets embedded in a projective plane. *ArXiv e-prints*, November 2009.

[2] E. Brieskorn and H. Knörrer. *Plane Algebraic Curves.* Birkhäuser Verlag, Basel, 1986.

[3] J. Dénes and A. Keedwell. *Latin Squares and their Applications.* Academic Press, 1974.

[4] G. Denham and A. Suciu. Multinets, parallel connections and milnor vibrations of arrangements. *ArXiv e-prints*, September 2012.

[5] A. Dimca and S. Papadima. Finite galois covers, cohomology jump loci, formality properties, and multinets. *Ann. Sci. Norm. Sup. Pisa*, 10:253–268, 2011.

[6] M. Falk and S. Yuzvinsky. Multinets, resonance varieties, and pencils of plane curves. *Compositio Math.*, 143:1069–1088, 2007.

[7] V. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, 1990.

[8] G. Korchmáros, G. Nagy, and N. Pace. 3-nets realizing a group in a projective plane. *ArXiv e-prints*, June 2011.

[9] A. Libgober and S. Yuzvinsky. Cohomology of the orlik-solomon algebras and local systems. *Compositio Math.*, 121:337–361, 2000.

[10] P. Orlik and L. Solomon. Arrangements defined by unitary reflection groups. *Math. Ann.*, 261:339–357, 1982.

[11] P. Orlik and L. Solomon. Coxeter arrangements. In *Singularities*, volume 40 Part 2 of *Proc. Symp. Pure Math.*, pages 269–292. Amer. Math. Soc., 1983.

[12] P. Orlik and H. Terao. *Arrangements of Hyperplanes*, volume 54. Springer-Verlag, Berlin, Heidelberg, New York, 1992.

[13] J. Pereira and S. Yuzvinsky. Completely reducible hyper surfaces in a pencil. *Adv. Math.*, 219:672–688, 2008.

[14] J. Stipins. Old and new examples of k-nets in $\mathbb{P}^2$. *ArXiv e-prints*, January 2007.

[15] G. Urzúa. On line arrangements with applications to 3-nets. *ArXiv e-prints*, October 2009.

[16] S. Yuzvinsky. Realization of finite abelian groups by nets in $\mathbb{P}^2$. *Compositio Math.*, 140:1614–1624, 2004.

[17] S. Yuzvinsky. A new bound on the number of special fibers in a pencil of curves. *Proc. Amer. Math. Soc.*, 137:1641–1648, 2009.