PLUMBERS' KNOTS AND UNSTABLE VASSILIEV THEORY

by

CHAD DAVID GIUSTI

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2010

University of Oregon Graduate School

Confirmation of Approval and Acceptance of Dissertation prepared by:

Chad Giusti

Title:

"Plumbers' Knots and Unstable Vassiliev Theory"

This dissertation has been accepted and approved in partial fulfillment of the requirements for the degree in the Department of Mathematics by:

Dev Sinha, Chairperson, Mathematics
Hal Sadofsky, Member, Mathematics
Arkady Berenstein, Member, Mathematics
Daniel Dugger, Member, Mathematics
Andrzej Proskurowski, Outside Member, Computer & Information Science

and Richard Linton, Vice President for Research and Graduate Studies/Dean of the Graduate School for the University of Oregon.

June 14, 2010

Original approval signatures are on file with the Graduate School and the University of Oregon Libraries.

An Abstract of the Dissertation of

Chad David Giusti for the degree of Doctor of Philosophy

in the Department of Mathematics to be taken June 2010

Title: PLUMBERS' KNOTS AND UNSTABLE VASSILIEV THEORY

Approved: ______________________________
Dr. Dev Sinha

We introduce a new finite-complexity knot theory, the theory of plumbers' knots, as a model for classical knot theory. The spaces of plumbers' curves admit a combinatorial cell structure, which we exploit to algorithmically solve the classification problem for plumbers' knots of a fixed complexity. We describe cellular subdivision maps on the spaces of plumbers' curves which consistently make the spaces of plumbers' knots and their discriminants into directed systems.

In this context, we revisit the construction of the Vassiliev spectral sequence. We construct homotopical resolutions of the discriminants of the spaces of plumbers knots and describe how their cell structures lift to these resolutions. Next, we introduce an inverse system of unstable Vassiliev spectral sequences whose limit includes, on its E_∞- page, the classical finite-type invariants. Finally, we extend the definition of the Vassiliev derivative to all singularity types of plumbers' curves and use it to construct canonical chain representatives of the resolution of the Alexander dual for any invariant of plumbers' knots.

CURRICULUM VITAE

NAME OF AUTHOR: Chad David Giusti

PLACE OF BIRTH: Colorado Springs, CO

DATE OF BIRTH: 15 September 1978

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR

DEGREES AWARDED:

Doctor of Philosophy, University of Oregon, 2010
Bachelor of Science, University of Oregon, 2003

AREAS OF SPECIAL INTEREST:

Computational and geometric aspects of algebraic topology, particularly with regards to knot theory and group cohomology

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2003-2010

GRANTS, AWARDS AND HONORS:

Jack and Peggy Borsting for Scholastic Achievement in Graduate Mathematics, University of Oregon Department of Mathematics, 2009
Johnson Fellowship, University of Oregon Department of Mathematics, 2008

ACKNOWLEDGMENTS

Foremost, I want to thank my advisor Dev Sinha, whose knowledge, guidance and patience made this project possible. Similarly, I must thank my office mates Sammy Black and Bill Kronholm, who each contributed invaluable assistance by repeatedly explaining to me things I should already have known. I would also like to extend my thanks to the entire faculty of the Department of Mathematics at the University of Oregon, from whom I have learned a tremendous amount about everything, and in particular to Brad Shelton and Bill Kantor for believing I could do this even when I hadn't done much of anything to deserve their faith.

On a personal note, I am eternally grateful to Elizabeth, my parents Joel and Vicki, my siblings Kim and Nick, and all of the rest of my family and friends for their love, understanding and continual support. I never would have survived this long without all of you.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

In [12], Vassiliev initiated the study of finite-type invariants by constructing the spectral sequence which bears his name and analyzing the combinatorics of its E_1-page. In contrast to classical methods in knot theory which attempt to understand properites of individual knots, the foundation of Vassiliev's approach was to apply the tools of algebraic topology to the study of the space of all knots, $\mathcal{K}$. He discovered that the complement, or *discriminant*, of $\mathcal{K}$ in the space of immersions $S^1 \to \mathbb{R}^3$ carries a rich combinatorial structure. Due to the highly technical nature of its construction, including the use of the weak transversality theorem to perturb polynomial mapping spaces, few other than Vassiliev himself have built upon this approach. The principal starting point in the study of finite-type invariants has instead has been the notion of the Vassiliev derivative introduced by Birman and Lin [4] and made popular by Bar-Natan [2].

We believe that there remains a great deal to be learned through a geometric analysis of the discriminant. Several authors, notably Randell [10, 11], Calvo [5] and Stanford, have approached this area by replacing Vassiliev's choice of polynomial knot spaces by spaces of stick knots. The discriminant in these spaces is constructed from partial cubic hypersurfaces and has hardly been more amenable to comprehensive description.

The starting point of our approach will be to construct a new model for classical knot theory we call the theory of "plumbers' knots". Among the advantages of this model is that its disciminant is the union of partial hyperplanes, so understanding its geometry is a combinatorial problem.

We are able to extend the notion of Vassiliev derivative to any singularity of plumbers' curves. This allows us to produce an inverse system of "unstable" Vassiliev spectral sequences whose limiting sequence's E_∞ page contains that of the classical Vassiliev spectral sequence. In contrast to Vassiliev's stable construction which only carries singularity data from collections of transverse double points and the generic boundaries of such singularities, each such unstable sequence carries information about all singularities arising in the space of plumbers' curves on which it is constructed. In exchange for more intricate combinatorics, this provides us with complete data regarding the evolution of any knot invariant through what we call its Vassiliev system.

In this section, we summarize our main results.

I.1. Plumbers' Curves

The curves we consider, called plumbers' curves, are piecewise linear (hereafter, PL) curves such that the image of each linear map is parallel to one of the coordinate axes and so that these axes alternate in a fixed order. The collection of all such curves decomposes into a directed system of spaces P_m of *plumbers' curves of m moves*, each of which is homeomorphic to a Euclidean cube. Inside of each P_m lies the space of *plumbers' knots of n moves*, K_m, and we determine that

Theorem I.1. $\varinjlim K_m$ *is homotopy equivalent to* $\mathcal{K}$.

As such, these spaces provide us with a model for classical knot theory in addition to presenting an interesting geometric theory in their own right.

We show that each P_m admits a cell decomposition $\textsc{Cell}_\bullet(P_m)$ whose structure can be canonically encoded through triples of permutations and collections of "coincidence data". Within this cell complex we identify a subcomplex for $S_m = P_m \setminus K_m$, the space of singular plumbers' curves. The combinatorial data in the cell complex for S_m allows us to recast the spaces S_m as limits of "coincidence" functors B_m, which provides a concise resolution of its self-intersections using standard tools of homotopy theory.

While is is commonly assumed that classification problems for knots of fixed complexity are as difficult as those for C^1 knots, the structure of these cell complexes provides us with an equivalence relation on top dimensional cells called *elementary geometric isotopy.*

Theorem I.2. *$\pi_0(K_m) \cong \Sigma_{m-1} \times \{x, y, z\}/\sim$, where $\sim$ is the equivalence relation generated by elementary geometric isotopies.*

This characterization leads us to deterministic finite-time algorithms that enumerate the components of K_m and allow us to determine if two knots are isotopic at a fixed complexity. An implementation of the enumeration algorithm has demonstrated that there are seven knot types in K_5, fourty-nine in K_6 and one thousand and eight in K_7. The data, summarized in Table II.1, shows that the seven components of K_5 correspond topologically to the unknot, three right-handed trefoils and three left-handed trefoils. This phenomenon of "stuck knots", knots which are not isotopic at a level of finite complexity but whose topological isotopy classes coincide has been studied in the context of PL knots by a number of authors, including Cantarella and Johnston [6], Calvo [5] and Biedl, Demaine et al. [3].

Plumbers' knots bear strong a resemblance to both lattice knots as considered for example in [8], and the cube diagrams studied in [1]. Any plumbers' knot can be viewed as a lattice knot. We put bounds on number of the classes of lattice knots that arise in each P_m. The similarity to the cube diagrams of Baldridge and Lowrance [1] suggests that the tools we develop for plumbers' knots could be useful in the study of knot Floer homology.

I.2. Vassiliev Theory

Once we have developed the theory of plumbers' knots, we revisit Vassiliev's ideas in this context. We now review the conceptual framework he used for his spectral sequence and summarize our main results.

The discriminant of $\mathcal{K}$ intersects itself in complex ways, as sketched in Figure I.1. Vassiliev's first step was to "resolve" this singular space, replacing it with a union of smooth objects. The most natural such construction replaces points of the discriminant corresponding to curves with n transverse double points by $(n-1)$-simplices, as in Figure I.2. We resolve the discriminant of K_m using an explicit homotopy colimit.

Theorem I.3. *The homotopy colimit of the coincidence functor B_m has the following properties:*

1. $\pi : hocolim B_m \longrightarrow S_m$ *is a homotopy equivalence,*
2. *for all* $x \in S_m$, $\pi^{-1}(x) = \{x\} \times *_{i=1}^{n} \Delta^{k_i}$ *for some* $\{k_1, k_2, \dots k_n\}$,
3. *the cell structure* $\text{CELL}_\bullet(S_m)$ *lifts to such a structure on* $hocolim B_m$, *and*
4. *if* $\mathbf{e}$ *is a codimension one cell in* $\text{CELL}_\bullet(S_m)$, $\pi^{-1}(\mathbf{e}) \cong \mathbf{e}$.

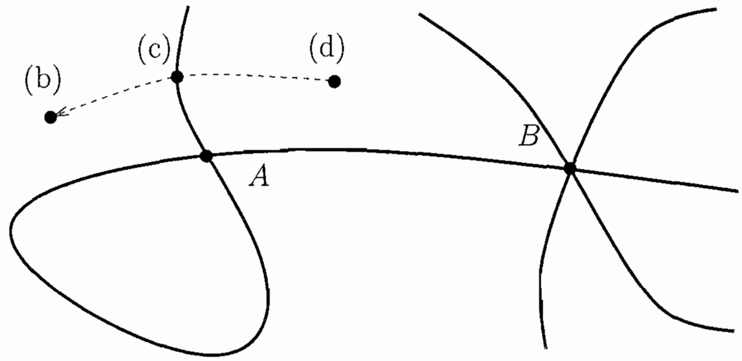
In order to use his resolution, Vassiliev considered only curves with n transverse double points and those singular curves which "generically" occur as boundaries of families of such curves. While this resulted in accurate results in high dimension, the completeness question for finite-type invariants remains one of whether the remaining information is "dense" in the collection of all knot invariants, and is not merely a $\lim^1$ question as is commonly assumed.

Vassiliev introduced a logical ordering on these generic singularities by "complexity", providing a filtration on the resolved discriminant. In the filtration quotient, the boundaries of the simplices introduced in the resolution are identified to a point, leaving a collection of combinatorial codimension one cycles which live on the E^1 page of the spectral sequence of the filtration. Those linear combinations of cycles which survive to the E^∞ page correspond to knot invariants, now called *finite-type* or *Vassiliev* invariants.

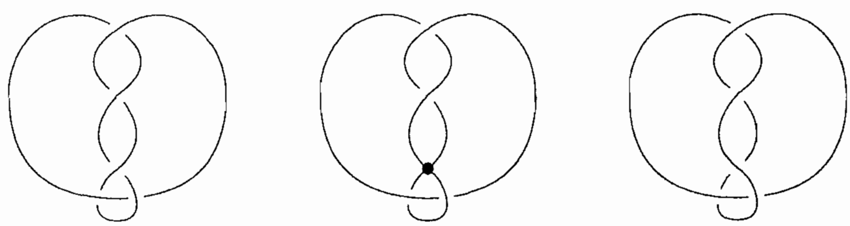
Due to the restrictive nature of the singularity types which can occur in plumbers' curves, we are able to construct a straightforward filtration on the spaces $\tilde{S}_m$ which is compatible with Vassiliev's on "stable" singularities. This allows us to define an inverse system of "unstable" Vassiliev spectral sequences.

Theorem I.4. *There is a directed system of first quadrant homology spectral sequences $E(m)$ whose E^0 pages are given by the filtration quotients of* $\textsc{Cell}_\bullet(\tilde{S}_m)$ *and which converge to $H_*(\tilde{S}_m)$. Further, there is an Alexander dual inverse system of second quadrant cohomology spectral sequences in whose limiting sequence appear the classical finite-type invariants.*

From the standpoint of algebraic topology, knot invariants are classes in $H^0(\mathcal{K})$. Recall that the Alexander dual to a zero dimensional reduced cocycle $[\alpha]$ in a subspace $X \subseteq \mathbb{R}^n$ is a codimension one cycle $[\alpha^\vee]$ in $(\mathbb{R}^n \setminus X)^+$. If X is a CW complex, these cycles have as canonical chain representatives the sum of the codimension one chains of X with coefficients given by the difference in values of $[\alpha]$



(a) A section of the discriminant



(b) A 5_2 knot

(c) A singular curve bounding the 5_2 knot and the trefoil

(d) A trefoil

Fig. I.1: A cross-section of the discriminant of the knot space.

on its cobounding regions. Intuitively, the Alexander dual of a zero cocycle is the collection of its "derivatives" as one changes components along a path like that in Figure I.1.

The new codimension one cycles on the E^1 page of the Vassiliev spectral sequence naturally correspond to higher derivatives. At the chain level, the coefficient of each simplex encodes the change of value of an invariant of "less singular" curves. For example, a path between isotopy classes of curves with a single transverse double point generically passes through a finite number of regions corresponding to curves with two double points, potentially changing the coefficient of the Alexander dual at each crossing, suggesting a "second derivative".

Using the cell structure on the resolution, we are able to define the Vassiliev

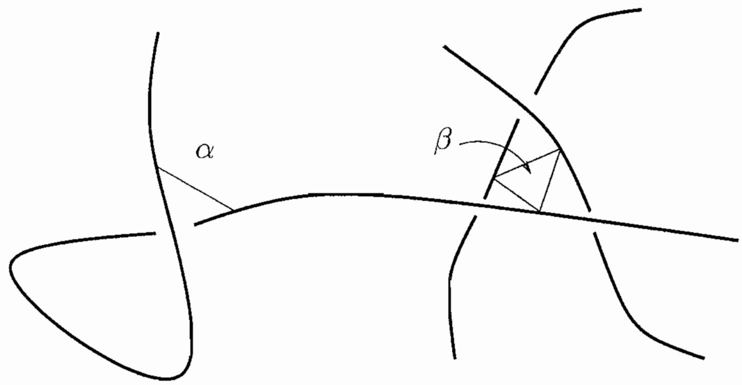


Fig. I.2: Vassiliev's resolution of the discriminant; under the canonical projection, the simplex α maps to the point A in Figure I.1 and β to B.

derivative of a plumbers' knot invariant $[\alpha]$ across all singularity types of plumbers' curves, rather than restricting our study to generic singularities.

Theorem I.5. *For any* $[\alpha] \in \bar{H}^0(K_m)$ *and any* $\mathbf{e} \in \mathrm{CELL}_{3m-4}(hocolim(B_m))$*, there is a canonical Vassiliev derivative* $d_{\mathbf{e}}([\alpha])$ *which is an isotopy invariant of singular plumbers' curves.*

Finally, we use this definition to identify a canonical chain representative for the lift to $\tilde{S}_m$ of the Alexander dual of a plumbers' knot invariant.

Theorem I.6. *Let* $[\alpha] \in H^0(K_m)$*. There is a canonical choice of chain representative for* $[\tilde{\alpha}^\vee] \in H_{3m-4}(\tilde{S}_m)$ *whose coefficients are given by the Vassiliev derivative.*

CHAPTER II

PLUMBERS' KNOTS

We begin by introducing plumbers' curves and the terminiology we will use to study them. This will immediately lead us to a cell structure on the spaces of plumbers' curves, whose combinatorics we describe. Through analysis of these combinatorics, we construct an algorithm which enumerates the plumbers' knot types appearing in each space.

II.1. Plumbers' Curves

Fix a basis $\{x, y, z\}$ for $\mathbb{R}^3$. Let $\mathbb{I}^3$ be the cube $[0,1]^3 \subseteq \mathbb{R}^3$ and $P_m = (\text{Int } \mathbb{I}^3)^{m-1}$. Given a point $v \in P_m$, we construct a map $\phi_v \colon \mathbb{R} \to \mathbb{R}^3$ with support on the interval $[0,3]$ which we call a *plumbers' curve* with $3m$ "sections of pipe".

Throughout, we write $[\mathbf{m}] = \{1, \ldots, m\}$.

Definition II.1. Let $v = (v_0, v_1, \ldots, v_n) \in P_m$ with $v_0 = (0,0,0)$ and $v_m = (1,1,1)$. For $i \in [\mathbf{m-1}]$, define maps which interpolate between consecutive v_i in three steps, parallel to the coordinate axes. Let the maps parallel to the x-axis, $\mathbf{x}_i(t)$, be of the following form and define $\mathbf{y}_i(t)$ and $\mathbf{z}_i(t)$ analagously.

$$\mathbf{x}_i(t) \colon \left[\frac{3i}{m}, \frac{3i+1}{m}\right] \to \mathbb{R}^3 \quad = \quad \left((3i+1-mt)v_i^x + (mt-3i)v_{i+1}^x, v_i^y, v_i^z\right)$$

Each such map is a *pipe*, and we will sometimes not distinguish between such and its image. Set the *length* of $\mathbf{x}_i$ as $||\mathbf{x}_i|| = |\mathbf{x}_i(\frac{3i}{m}) - \mathbf{x}_i(\frac{3i+1}{m})|$, and its *direction* $s(\mathbf{x}_i)$ to be the sign of $\mathbf{x}_i(\frac{3i+1}{m}) - \mathbf{x}_i(\frac{3i}{m})$ or 0 if they are equal.

Make similar definitions for $\mathbf{y}_i$ and $\mathbf{z}_i$.

Let $\phi_v(t)\colon\ [0,3] \to \mathbb{I}^3$ be the union of these maps. We call ϕ_v an *m-move plumbers' curve.* An example of such a curve is given in Figure II.1

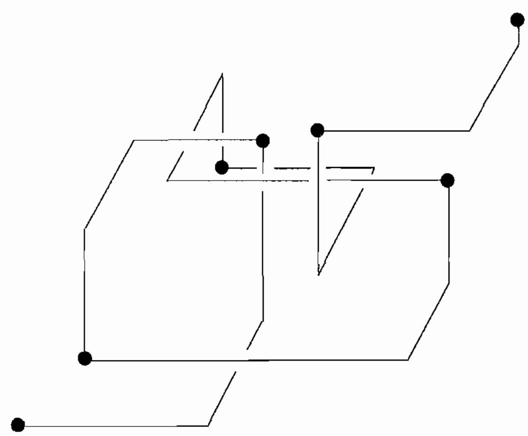


Fig. II.1: A plumbers' curve of 6 moves.

These maps encode piecewise linear motion parallel to the x-, y- and z-axes. A generic curve alternates these three directions of motion in the order $x, y, z, x, \ldots, z$. Up to reparametrization any such map is obtainable via such a construction, so we can identify P_m with the space of such curves, and call P_m the *space of m-move plumber's curves.* Unless we are specifically using the properties of the map ϕ_v, we will make most of our statements about the underlying collection of points v.

There are two types of codimension one singularities in C^1 curves: points at which the derivative vanishes and transverse self-intersections. The rigid nature of plumbers' curves allows for the description of their singularities by a single condition.

Definition II.2. We say two pipes are *distant* if they are separated by at least three intervening pipes. For example, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ are not distant, but $\mathbf{x}_i$ and $\mathbf{y}_{i+1}$ are. A plumbers' curve is *singular* if any pair of its distant pipes intersect. Thus some of the pipes of a plumbers' curve – up to two in a row – can have zero length without introducing a singularity. While this is different from PL knot theory, it is necessary to get an equivalent limiting knot theory.

A non-singular plumbers' curve is a *plumbers' knot*, and the space of all such is $K_m \subsetneq P_m$. The *discriminant* $S_m = P_m \setminus K_m$ is the subspace of singular plumbers' curves.

Definition II.3. Let $\phi_v, \phi_w \in K_m$. We say ϕ_v and ϕ_w are *(geometrically) isotopic* if there is a path $\Phi_{v,w} \colon \mathbb{I} \to K_m$ with $\Phi_{v,w}(0) = v$ and $\Phi_{v,w}(1) = w$.

Later, we will see that this notion of isotopy in any K_m is stronger than the usual topological notion of knot isotopy. However, the two notions converge as we increase the articulation of plumbers' knots.

II.2. A Cell Complex for S_m

The spaces P_m possess an intrinsic combinatorial cell structure given by the threefold product of the standard simplical decomposition of $\mathbb{I}^{m-1}$. This cell structure is compatable with the partition into subspaces S_m and K_m, and we will utilize it throughout our work.

Let Σ_m be the symmetric group on m letters. We sometimes refer to a $\sigma \in \Sigma_m$ using *permutation* notation, $\sigma(1)\sigma(2)\ldots\sigma(m)$. As most of our constructions and results will be symmetric for our x, y and z coordinates, we will often make statements only for x.

Definition II.4. Let $\sigma \in \Sigma_{m-1}$. We say $v \in P_m$ *respects* σ *in* x if

$$0 < v^x_{\sigma_i(1)} < v^x_{\sigma_i(2)} < \cdots < v^x_{\sigma_i(m-1)} < 1 \tag{II.1}$$

Let $\sigma_x, \sigma_y, \sigma_z \in \Sigma_{m-1}$. Define a cell in P_m by

$$\mathbf{e}(\sigma_x, \sigma_y, \sigma_z) \quad = \quad \{v \in P_m \mid v \text{ respects } \sigma_x \text{ in } x, \sigma_y \text{ in } y \text{ and } \sigma_z \text{ in } z\}$$

When we write the permutations in permutation notation, the name of the cell transparently describes the defining inequalities. Such cells are a product of three open $(m-1)$-simplices, and the collection of all such cells $\mathbf{e}$ form the top dimension of the cellular decomposition of P_m given by the threefold product of the standard simplical decomposition of $\mathbb{I}^{m-1}$.

Where possible we will abbreviate the triple $(\sigma_x, \sigma_y, \sigma_z)$ as $\vec{\sigma}$ and write, for example, $\rho_x\vec{\sigma} = (\rho_x\sigma_x, \sigma_y, \sigma_z)$ for the left action of Σ_{m-1} on $\Sigma_{m-1} \times \{x, y, z\}$ in the indicated coordinate.

Each top-dimensional cell described above is non-empty. We distinguish an element as follows.

Definition II.5. Given a cell $\mathbf{e} = \mathbf{e}(\vec{\sigma}) \subseteq P_m$, construct an m-move plumbers' knot by choosing the point $v(\mathbf{e})$ in the cell given by

$$v(\mathbf{e})_i = \left(\frac{\sigma_x^{-1}(i)}{m}, \frac{\sigma_y^{-1}(i)}{m}, \frac{\sigma_z^{-1}(i)}{m}\right).$$

The knot $\phi_{v(\mathbf{e})}$ is called the *representative knot* for $\mathbf{e}$.

Any plumbers' knot ϕ_v in the closure of a cell is geometrically isotopic to the representative knot for that cell via a straight line geometric isotopy. Thus, to study geometric isotopy types of m-move plumbers' knots it suffices to study only representative knots. This discretizes the study of plumbers' knots.

Lemma II.1. *If a plumbers' curve ϕ_v is singular, v is in the closure of a codimension one cell. Further, if some ϕ_v is singular and lies in the interior of a cell $\mathbf{e}$ (of codimension one or greater), all v in the closure of $\mathbf{e}$ are.*

Proof. If ϕ_v is singular, there must be distant pipes a and b which intersect. As the pipes are lines parallel to the coordinate axes, it is easy to characterize plumbers' curves for which given pipes intersect. Here and elsewhere, we write "$(p-r)(q-r) \leq 0$" in place of "either $q \leq r \leq p$ or $p \leq r \leq q$". Then

- $\mathbf{x}_i \cap \mathbf{y}_j$, as illustrated in Figure II.2, if
  1. $(v_i^x - v_{j+1}^x)(v_{i+1}^x - v_{j+1}^x) \leq 0$,
  2. $(v_j^y - v_i^y)(v_{j+1}^y - v_i^y) \leq 0$ and
  3. $v_i^z = v_j^z$,
- $\mathbf{x}_i \cap \mathbf{z}_j$ if
  1. $(v_i^x - v_{j+1}^x)(v_{i+1}^x - v_{j+1}^x) \leq 0$,
  2. $v_i^y = v_{j+1}^y$ and
  3. $(v_j^z - v_i^z)(v_{j+1}^z - v_i^z) \leq 0$, and
- $\mathbf{y}_j \cap \mathbf{z}_i$ if
  1. $v_{i+1}^x = v_{j+1}^x$,
  2. $(v_j^y - v_{i+1}^y)(v_{j+1}^y - v_{i+1}^y) \leq 0$ and
  3. $(v_i^z - v_j^z)(v_{i+1}^z - v_j^z) \leq 0$.

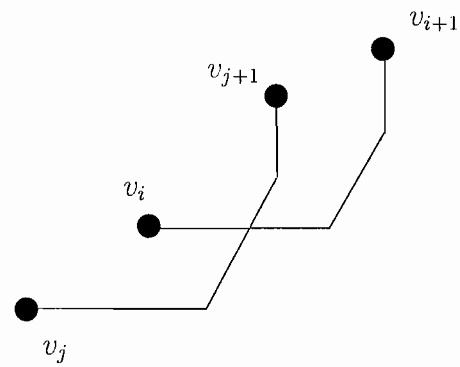


Fig. II.2: An x-y intersection.

Each of these conditions requires an equality of the form $v^x_{i+1} = v^x_{j+1}$, $v^y_i = v^y_{j+1}$ or $v^z_i = v^z_j$. As $\mathbf{e}(\vec{\sigma})$ is a product of simplices, such are satisfied only on the boundary of a cell. If any v in a particular boundary cell of $\mathbf{e}(\vec{\sigma})$ satisfies such a condition, all in that cell do so. □

Lemma II.1 tells us that the discriminant S_m is described by a closed subcomplex of P_m generated by codimension one cells. Thus K_m is an open submanifold of P_m whose combinatorics we now carefully develop.

Boundaries of cells $\mathbf{e}(\vec{\sigma}) \in \textsc{Cell}_{3m-4}(P_m)$ are indexed by collections of coordinate equalities on vertices. We encode such an equality as a transposition decorated with a label indicating which coordinate it involves. For example, $(1\ 2)_x$ means that the first and second vertex share x-coordinates, which is in the boundary of $\mathbf{e}(\vec{\sigma})$ precisely when $\{1\ 2\} = \{\sigma_x(i)\ \sigma_x(i+1)\}$ for some choice of i.

Given a collection τ of transpositions and a cell $\mathbf{e} \in \textsc{Cell}_\bullet(P_m)$ for whose elements all of the equalities indexed by τ hold, we say the cell *respects* τ. For example, $\mathbf{e}(3142_x, 4132_y, 1324_z)$ has several boundary cells which respect the set $\tau = \{(1\,3)_x, (2\,4)_x, (1\,4)_y, (1\,3)_y\}$, all of which also respect the set $\tau' = \{(1\,3)_x, (1\,4)_y\}$.

We use this to establish notation for boundary cells.

Definition II.6. Let $\mathcal{P}(S)$ be the power set of a set S. Fix a permutation $\sigma_x \in \Sigma_{m-1}$. We say that a set $C \subseteq [\mathbf{m-1}]$ is *admissible for* σ_x if C is of the form $\sigma_x(\{i, i+1, \ldots, i+k\})$ for some i and k. Denote such a set $\{\sigma_x(i), \sigma_x(i+1), \ldots, \sigma_x(i+k)\}_x$.

If $\vec{\sigma} \in \Sigma_{m-1} \times \{x, y, z\}$ is a triple of permutations and C is the disjoint union of a collection of sets each of which is admissible for one of the σ_d, we say C is *admissible for* $\vec{\sigma}$.

For example, $C = \{1, 2, 4\}_x$ is admissible for $(3142_x, 4132_y, 1324_z)$. Sets which are admissible for $\vec{\sigma}$ index the collections of coordinate equalities which can occur in the boundary of the cell $\mathbf{e}(\vec{\sigma})$.

Given a set C which is admissible for $\vec{\sigma}$, we produce a collection of transpositions $\tau(C)$ which describe the coordinate equalities in C compatibly with the order of the vertices in $\mathbf{e}(\vec{\sigma})$ by reading off the transpositions in the order they appear in $\vec{\sigma}$.

Definition II.7. Define
$\tau(C, \vec{\sigma}) = \{(\sigma_\alpha(i)\ \sigma_\alpha(i+1)), (\sigma_\alpha(i+1)\ \sigma_\alpha(i+2)), \ldots, (\sigma_\alpha(i+k-1)\ \sigma_\alpha(i+k))\})$.

We say such a collection of transpositions is *sequential for* $\vec{\sigma}$. When $\vec{\sigma}$ is clear from context, we will supress it from notation.

In the example above, $\tau(C, \vec{\sigma}) = \{(1\ 4)_x, (4\ 2)_x\}$.

Definition II.8. Fix a triple of permutations $\vec{\sigma} \in \Sigma_{m-1} \times \{x, y, z\}$. Let $\mathbf{C} = \{C_1, C_2, \ldots, C_k\}$ be a partition of $[\mathbf{m-1}] \times \{x, y, z\}$ into sets which are admissible for $\vec{\sigma}$. Denote by $\mathbf{e}(\vec{\sigma}; \mathbf{C})$ the cell of plumbers' curves obtained by setting equal precisely those coordinates of vertices which appear in the same C_i and otherwise respecting the inequalities defined by $\vec{\sigma}$.

Such a cell is a boundary of $\mathbf{e}(\vec{\sigma})$ of codimension $\sum |C_i| - |\mathbf{C}|$ and is homeomorphic to $\Delta^{a_x} \times \Delta^{a_y} \times \Delta^{a_z}$, where the Δ^{a_d} are the faces of the simplex Δ^{m-1} induced by the coordinate equalities in $\mathbf{C}$. We omit singletons when writing $\mathbf{C}$ as these induce no equalities in the coordinates. We will also write $\mathbf{C}_x$, for example, for the collection of all $C_i \in \mathbf{C}$ which are admissible for σ_x.

To continue our example with $\vec{\sigma} = (3142_x, 4132_y, 1324_z)$, there is a boundary cell of $\mathbf{e}(\vec{\sigma})$ given by $\mathbf{e}(\vec{\sigma}; \{1,3\}_x, \{2,4\}_x, \{1,3,4\}_y)$ whose codimension is $2+2+3-3=4$. Any boundary cell of $\mathbf{e}(\vec{\sigma})$ which respects $\tau = \{(1\,3)_x, (2\,4)_x, (1\,4)_y, (1\,3)_y\}$ is also a boundary of this cell.

Cells of codimension one or greater are not uniquely named; we can rearrange any indices in $\vec{\sigma}$ which appear in the same component of $\mathbf{C}$ and to obtain another permutation $\vec{\sigma}'$ and another label for the same cell, $\mathbf{e}(\vec{\sigma}', \mathbf{C})$. Another name for our example cell is thus $\mathbf{e}(\vec{\sigma}'; \{1,3\}_x, \{2,4\}_x, \{1,3,4\}_y)$ with $\vec{\sigma}' = (1342_x, 1342_y, 1324_z)$. This flexible naming convention will simplify the formula for the Vassiliev derivative in Definition III.9.

Definition II.9. Let $\vec{\sigma}$ and $\mathbf{C}$ be as in Definition II.8. Define $\Sigma_{\mathbf{C}} = \prod_{i=1}^{k} \Sigma_{C_i}$, where Σ_{C_i} is the symmetric group on the elements of C_i.

All possible names for a given cell $\mathbf{e}(\vec{\sigma}; \mathbf{C})$ are given by $\mathbf{e}(\rho\vec{\sigma}; \mathbf{C})$ for $\rho \in \Sigma_{\mathbf{C}}$.

For example, there are two classes of codimension two cells: those for which $\mathbf{C}$ consists of two sets of two indices apeice and those for which $\mathbf{C}$ consists of a single set of three indices. In the former case there are four names for each cell, while in the latter there are six.

We can extend the idea of representative knots for top dimensional cells to produce representative curves for every cell in $\textsc{Cell}_\bullet(P_m)$.

Definition II.10. Fix a cell $\mathbf{e} = \mathbf{e}(\vec{\sigma}; \mathbf{C}) \in \text{CELL}_{\bullet}(P_m)$. A map in such a cell will have vertices in $m - \sum_{C \in \mathbf{C}_x} |C| + |\mathbf{C}_x|$ (y-z)-planes, some of the x vertices now being coplanar. We need to define a function $E[\rho_x](i)$ that counts the number of equalities that occur before v_i in the x-axis equation for e.

Choose an ordering of the elements of $\mathbf{C}_x$ so that the indices $\{1 \dots m\}$ appear in the order $\sigma_x(1), \sigma_x(2), \dots, \sigma_x(m)$, and label the kth set in this order $(\mathbf{C}_x)_k$. For example, if $\sigma_x = 312546$ and $\mathbf{C}_x = \{\{45\}_x, \{123\}_x, \{6\}_x\}$, we would have $(\mathbf{C}_x)_1 = \{3, 1, 2\}_x$, $(\mathbf{C}_x)_2 = \{5, 4\}$ and $(\mathbf{C}_x)_3 = \{6\}$. Now, define $E[\mathbf{C}_x](i) = j$, where $i \in (\mathbf{C}_x)_j$.

Construct an m-move plumbers' curve $\phi_{v(\mathbf{e})} \in \mathbf{e}$ by

$$v(\mathbf{e})_i = \left(\frac{E[\mathbf{C}_x](i)}{m - \sum_{C \in \mathbf{C}_x} |C| + |\mathbf{C}_x|}, \frac{E[\mathbf{C}_y](i)}{m - \sum_{C \in \mathbf{C}_y} |C| + \mathbf{C}_y}, \frac{E[\mathbf{C}_z](i)}{m - \sum_{C \in \mathbf{C}_z} |C + |\mathbf{C}_z|} \right).$$

The knot $\phi_{v(\mathbf{e})}$ is called the *representative curve* (or when appropriate *knot*) for $\mathbf{e}$.

The cell structure leads to a convenient categorical decomposition of the space S_m. Denote by $\binom{I}{2}$ the collection of two element subsets of I.

Definition II.11. Define $\mathcal{S}_m$ be the mth *coincidence category*, whose objects are non-empty elements of $\mathcal{P}\left(\binom{[m-1]}{2} \times \{x, y, z\}\right)$ and whose morphisms are reverse inclusions.

Objects of $\mathcal{S}_m$ are precisely our collections of transpositions, as in Definition II.8.

Definition II.12. Let $B_m : \mathcal{S}_m \to \mathbf{Top}$ be the covariant functor given by $B_m(\tau) = \{\phi \in S_m : \phi \text{ respects } \tau\}$.

Our analysis of the cell complex above and the basic fact that CW complexes are the colimit of their skeleta now immediately gives us that

Proposition II.2. $S_m = \text{colim} B_m$.

II.2.1. An Algorithm for Computing Components of K_m

Our naming convention for cells allows us to resolve several fundamental questions about the spaces of plumbers' knots algorithmically.

By Lemma II.1, there are two types of codimension one cells: those which consist of plumbers' knots and those which consist of singular plumbers' curves. As those which consist of singular plumbers' curves generate a cell complex for S_m, we wish to distinguish them. Fortunately, we can determine combinatorially into which of these classes a given cell falls.

Choose the representative knot for a cell $\mathbf{e}(\vec{\sigma})$ and a transposition $\tau = (a\ b)$ appearing in σ_x. Geometrically, applying τ to σ_x corresponds to exchanging the x-coordinates of the ath and bth vertices, which "pushes" the (y-z)-plane $x = \frac{i}{m}$ past the (y-z)-plane $x = \frac{i+1}{m}$. As they lie in the plane $x = \frac{i}{m}$, this forces the pipes $\mathbf{y}_{a-1}$ and $\mathbf{z}_{a-1}$ to move, possibly intersecting $\mathbf{z}_{b-1}$ or $\mathbf{y}_{b-1}$ respectively, as illustrated in Figure II.3. While an intersection between other pipes than those mentioned is possible, this always occurs in addition to one of the singularities mentioned above.

A similar analysis demonstrates that transpositions appearing in σ_y (respectively, σ_z) can only cause intersections of the form $\mathbf{x}_a \cap \mathbf{z}_{b-1}$ or $\mathbf{x}_b \cap \mathbf{z}_{a-1}$ (respectively, $\mathbf{x}_a \cap \mathbf{y}_b$ or $\mathbf{x}_b \cap \mathbf{y}_a$).

Definition II.13. Let $\mathbf{e}(\vec{\sigma})$ be a cell in P_n, $\tau_x = (a\ b)$ a transposition appearing in σ_x. We say that τ_x *produces an intersection* if one of the following pairs of conditions holds.

Either

1. $\sigma_y^{-1}(b)$ is between $\sigma_y^{-1}(a-1)$ and $\sigma_y^{-1}(a)$ and $\sigma_z^{-1}(a-1)$ is between $\sigma_z^{-1}(b-1)$ and $\sigma_z^{-1}(b)$ or

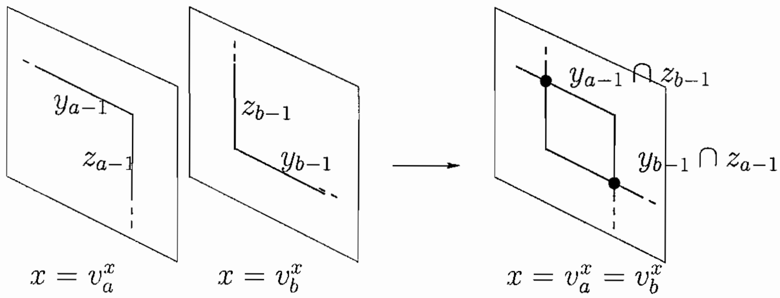


Fig. II.3: Transposing x-coordinates may result in y-z intersections.

2. $\sigma_y^{-1}(a)$ is between $\sigma_y^{-1}(b-1)$ and $\sigma_y^{-1}(b)$ and $\sigma_z^{-1}(b-1)$ is between $\sigma_z^{-1}(a-1)$ and $\sigma_z^{-1}(a)$

This condition is precisely that for $\mathbf{y}_j$ to intersect $\mathbf{z}_i$ in the proof of Lemma II.1. The definitions for y and z are similar.

Theorem II.3. *Let* $\mathrm{e}(\vec{\sigma})$ *and* τ_x *be as in Definition II.13. There is a straight-line geometric isotopy between the representative knots for* $\mathrm{e}(\vec{\sigma})$ *and* $\mathrm{e}(\tau_x\vec{\sigma})$ *if and only if* τ_x *does not produce a* $(y-z)$*-intersection for* $\vec{\sigma}$*.*

If an isotopy such as the one in the theorem exists, it is an *elementary geometric isotopy.* Elementary geometric isotopies play a role for plumbers' knots similar to that played by Reidermeister moves for C^1 knots. Elementary geometric isotopies do not always exist even between neighboring knots which are geometrically isotopic, as shown in Figure II.4.

We notice, however, that unlike Reidermeister moves, elementary geometric isotopies preserve the complexity of the knot, as measured in number of moves necessary to represent it. This affords us greater facility with the equivalence relation generated by such.

Corollary II.4. *Let* $\phi_v, \phi_w \in K_m$*,* $v \in \mathrm{e}(\vec{\sigma})$*,* $w \in e(\vec{\sigma}')$*.* ϕ_v *is geometrically isotopic*

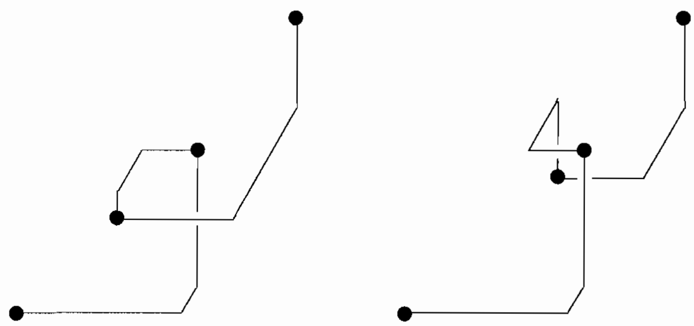


Fig. II.4: Two unknots which are minimally separated by a sequence of three elementary geometric isotopies.

to ϕ_w *if and only if there is a sequence of elementary geometric isotopies connecting the representative knots for* $e(\vec{\sigma})$ *and* $e(\vec{\sigma}')$*.*

Corollary II.5. $H_0(K_m) \cong \mathbb{Z}\langle \mathbf{e}(\vec{\sigma}) \mid \vec{\sigma} \in \Sigma_{m-1} \times \{x, y, z\}\rangle / \sim$, *where* $\sim$ *is the equivalence relation generated by elementary geometric isotopies.*

As an application of Corollary II.5, we can computationally determine if two m-move plumbers' knots ϕ_v and $\phi_{v'}$ are geometrically isotopic. Without loss of generality, v and v' occur in the interior of a $(3m-3)$-cell.

Algorithm II.1.

1. Construct a graph whose vertices V are indexed by $\text{CELL}_{3m-3}(P_m)$ and where there is an edge $(\mathbf{e}, \mathbf{e}') \in E$ if there is an elementary genometric isotopy between the representative knots for $\mathbf{e}$ and $\mathbf{e}'$.

2. Let $\mathbf{e}_0, \mathbf{e}_1 \in V$ with $v \in \mathbf{e}_0$, $v' \in \mathbf{e}_1$.

3. Starting at $\mathbf{e}_0$, perform a graph search for $\mathbf{e}_1$. Such a search recursively traverses edges in the graph until it either locates $\mathbf{e}_1$ or exhausts all vertices in its component. If the search terminates successfully, the knots are isotopic. Otherwise, they are not geometrically isotopic for the given m.

If it is possible to determine to which K_m a pair of plumbers' knots must be lifted to ensure that a lack of geometric isotopy coincides with a lack of isotopy, Algorithm II.1 will determine if given knots are isotopic in $O((m!)^3)$ running time.

A modified version of Algorithm II.1 which instead identifies the components of the graph enumerates the geometric isotopy classes of K_m. The results of this algorithm being run on K_5 and K_6 are included in Table II.1. K_7 has similarly been determined to have one thousand and eight components. The topological isotopy classes of these knots were determined by computation of known knot invariants on representatives of each class.

Such algorithms have not been discovered for other finite-complexity knot theories like stick knots. Indeed, it is usually assumed that enumeration problems are at least as difficult for these theories as they are for that of C^1 knots.

An immediate consequence of Theorem II.3 and Lemma II.1, we have the following characterization of the cell complex for S_m.

Corollary II.6. $\text{CELL}_\bullet(S_m)$ *is generated in dimension* $3m-4$ *as a cell complex by all cells of the form* $e(\vec{\sigma};\mathbf{C})$ *for which some transposition in* $\Sigma_{\mathbf{C}}$ *produces an intersection.*

Thus, a byproduct of the aforementioned modification to Algorithm II.1 is an enumeration of the cells in $\text{CELL}_{3m-4}(S_m)$.

II.3. The Directed System of Spaces of Plumbers' Curves and Knots

For fixed m, elements of P_m are too rigid to properly model classes of C^1 knots. To do so, we require a mechanism by which to increase the articulation of a knot of interest in a fashion which varies continuously across each P_m. As we will also wish to work with duality between the space of knots and the discriminant

Components of K_5				
Type	Cells	Representative	Type	Representative
0_1	13,728	$1234_x, 1234_y, 1234_z$		
3_1 (R)	31	$1342_x, 2413_y, 2413_z$	3_1 (L)	$1342_x, 3142_y, 2413_z$
	16	$1342_x, 2413_y, 3124_z$		$1342_x, 3142_y, 3124_z$
	1	$2431_x, 2413_y, 4213_z$		$2431_x, 3142_y, 4213_z$
Components of K_6				
0_1	1.7m	$12345_x, 12345_y, 12345_z$		
3_1 (R)	19,507	$24135_x, 31245_y, 23145_z$	3_1 (L)	$12453_x, 13524_y, 13524_z$
	5	$42351_x, 24315_y, 24135_z$		$42351_x, 51342_y, 24135_z$
	5	$13524_x, 15324_y, 51342_z$		$13524_x, 42351_y, 51342_z$
4_1	393	$14352_x, 31452_y, 42135_z$	4_1	$31452_x, 31524_y, 32451_z$
	393	$24153_x, 25314_y, 24315_z$		$24513_x, 42135_y, 32415_z$
5_1 (R)	19	$24153_x, 31524_y, 42315_z$	5_1 (L)	$15342_x, 31542_y, 31524_z$
	19	$25134_x, 41253_y, 35241_z$		$52413_x, 24513_y, 25314_z$
	4	$15342_x, 24153_y, 42153_z$		$15342_x, 31542_y, 42513_z$
	4	$31542_x, 31524_y, 42315_z$		$31542_x, 42513_y, 42315_z$
	1	$41523_x, 41352_y, 34152_z$		$41523_x, 25314_y, 34152_z$
5_2 (R)	12	$15342_x, 24513_y, 35124_z$	5_2 (L)	$15342_x, 31542_y, 35124_z$
	12	$25413_x, 35124_y, 25314_z$		$24513_x, 42153_y, 42315_z$
	9	$25134_x, 24153_y, 35241_z$		$25134_x, 35124_y, 35241_z$
	9	$25413_x, 31524_y, 42315_z$		$52413_x, 24513_y, 23514_z$
	4	$15342_x, 24513_y, 42153_z$		$15342_x, 31542_y, 42153_z$
	4	$31542_x, 31524_y, 42351_z$		$31542_x, 42153_y, 42315_z$
	3	$15342_x, 25413_y, 31524_z$		$15342_x, 31452_y, 31524_z$
	3	$24153_x, 35214_y, 42315_z$		$24153_x, 41253_y, 42315_z$
	2	$15342_x, 25413_y, 42513_z$		$15342_x, 31452_y, 42513_z$
	2	$35142_x, 35214_y, 42315_z$		$35142_x, 41253_y, 42315_z$
	1	$15342_x, 24153_y, 41253_z$		$15342_x, 35142_y, 41253_z$
	1	$31452_x, 31524_y, 42315_z$		$31452_x, 42513_y, 42315_z$
	1	$41523_x, 25314_y, 43152_z$		$41523_x, 41352_y, 43152_z$
	1	$41532_x, 41352_y, 34152_z$		$41532_x, 25314_y, 34152_z$

Tab. II.1: Components of K_5 and K_6; the number of cells in a component are the same in the second column inside the directed system, it is desirable to construct the maps $P_m \to P_{m+1}$ so that they take S_m to S_{m+1} and K_m to K_{m+1}. It would also be useful for the map to be cellular with respect to the cell complex structure on P_m, and we will be able to accomplish this using a "barycentric subdivision" for the majority of the maps in each space.

Occasionally it will be useful to refer to our basis vectors (x, y, z) as (e_0, e_1, e_2), although this often decreases readability and we will avoid it where possible. When we write $[m]_3$ we are referring to the reduction of m modulo 3.

We require two pieces of notation in order to understand how the maps in the directed system interact with the cell structure. We first need to build new permutations that reflect the insertion of vertices into the plumbers' maps.

Definition II.14. Let $k \in [\mathbf{m-1}]$ and $\sigma \in \Sigma_{m-1}$. Define $j_{k,k+1}(\sigma) \in \Sigma_m$ in two steps. First, increase by one the image of the elements of the permutation whose image is already greater than k. For example, if $k = 2$, 124563_x becomes 125674_x. Write the new permutation as $\hat{\sigma}[k]$, defined by

$$\hat{\sigma}[k](i) = \begin{cases} \sigma(i) & \sigma(i) \leq k \\ \sigma(i) + 1 & \sigma(i) > k. \end{cases}$$

Now, insert $k+1$ in the "middle", lexicographically, of k and $k+2$, so in the above example we end up with 1253674_x. Let $a = \min\{\sigma^{-1}(k), \sigma^{-1}(k+1)\}$ and define

$$j_{k,k+1}(\sigma)(i) = \begin{cases} \hat{\sigma}[k](i) & i < \sigma^{-1}(a) + \lfloor \frac{1}{2}(\sigma^{-1}(k) + \sigma^{-1}(k+1)\rfloor) \\ k+1 & i = \sigma^{-1}(a) + \lfloor \frac{1}{2}(\sigma^{-1}(k) + \sigma^{-1}(k+1)\rfloor) \\ \hat{\sigma}[k](i-1) & i > \sigma^{-1}(a) + \lfloor \frac{1}{2}(\sigma^{-1}(k) + \sigma^{-1}(k+1)\rfloor). \end{cases}$$

/

Alternately, define an element $j_k(\sigma)$ by inserting $k+1$ immediately following k. In the current example, the result would be 123564_x.

$$j_k(\sigma)(i) = \begin{cases} \hat{\sigma}[k](i) & i < \sigma^{-1}(k)+1 \\ k+1 & i = \sigma^{-1}(k)+1 \\ \hat{\sigma}[k](i-1) & i > \sigma^{-1}(k)+1 \end{cases}$$

We abuse terminology by calling the product of the barycentric subdivisions of each of the simplicies in a cell again barycentric subdivision. In order to name the subdivided cells in the image, we use the following notation.

$$\begin{aligned} \mathbf{e}(\sigma_x, \sigma_y, \sigma_z; \langle j \rangle_x) &= \{v \in \mathbf{e}(\sigma_x, \sigma_y, \sigma_z) | \\ & \qquad v_j^x = \frac{1}{2}(v^x_{\sigma_x(\sigma_x^{-1}(j)-1)} + v^x_{\sigma_x(\sigma_x^{-1}(j)+1)})\} \end{aligned}$$

The codimension one subset of the simplex represented by σ_x in which v_j^x is the average of its two neighboring coordinates is a union of faces in the barycentric subdivision, as in Figure II.5. Thus, the cell $\mathbf{e}(\sigma_x, \sigma_y, \sigma_z; \langle j \rangle_x)$ is a product of the simplicies represented by σ_y and σ_z with these faces. Since this set is a subset of an existing cell, we can use our transposition notation for faces in the usual fashion.

Schematically, we want our sequence of maps to insert new vertices in a "sufficiently distributed" fashion across the pipes in a plumbers' curve. To choose the pipe into which to insert a vertex at each stage, we will fix a function $\alpha\colon \mathbb{N} \to \mathbb{N}$ and insert a vertex into the middle of the a pipe travelling in the $e_{[m]_3}$ direction in the $\alpha(m)$th move, per Figure II.6. We choose α so that each move in a map in P_M, for some fixed M, will be chosen in lexicographic order, skipping any newly created moves until vertices have been inserted in each of the existing M pipes, then begin

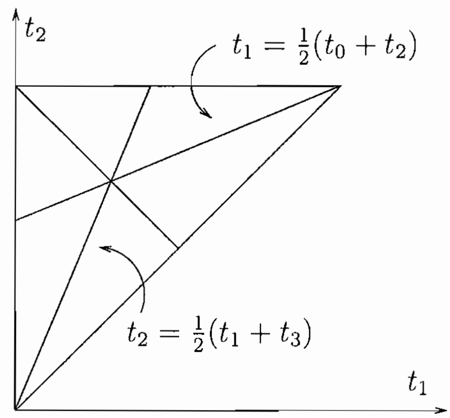


Fig. II.5: Averages of coordinates in barycentric subdivision; here, $t_0 = 0$ and $t_3 = 1$

again at P_{2M}.

We now define the maps on both the pointwise and cellular level. Let $v \in P_m$ so that there is at least one vertex between v_α and $v_{\alpha+1}$ along the $e_{[m]_3}$ axis. Such points live in cells for which $(\alpha\ \alpha{+}1)$ does not appear in $\sigma_{e_{[m]_3}}$. Any cell in with this property will be called *good*.

Definition II.15. For $m > 2$ and $v \in \mathbf{e} \in \textsc{Cell}_\bullet(P_m)$ with $\mathbf{e}$ a good cell, $\alpha = \alpha(m) + 1$ and $\delta_{i,j}$ the Kronecker delta. Define $\iota_m^{good}\colon P_m \to P_{m+1}$ by

$$(\iota_m^{good}(v))_i \quad = \quad \begin{cases} v_i & i \leq \alpha \\ v_{i-1} & i > \alpha + 1 \end{cases}$$

and

$$\begin{aligned} (\iota_m^{good}(v))_{\alpha+1}^x &= v_{\alpha+1}^x - \tfrac{1}{2}(v_{\alpha+1}^x - v_\alpha^x)\delta_{[m]_3,0} \\ (\iota_m^{good}(v))_{\alpha+1}^y &= v_\alpha^y + \tfrac{1}{2}(v_{\alpha+1}^y - v_\alpha^y)[m-1]_3 \\ (\iota_m^{good}(v))_{\alpha+1}^z &= v_\alpha^z + \tfrac{1}{2}(v_{\alpha+1}^z - v_\alpha^z)\delta_{[m]_3,2}. \end{aligned}$$

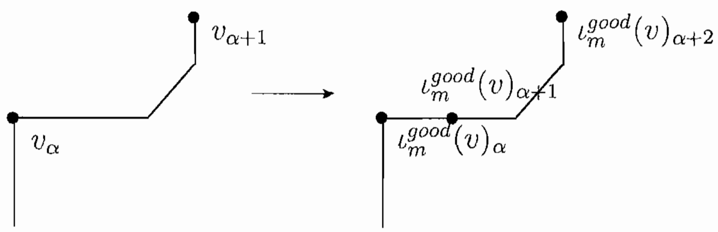


Fig. II.6: $\iota_m^{good}(v)$

The image of $\phi_{\iota_m^{good}(v)}$ is the same as that of ϕ_v, but the curve contains a new vertex. This is homotopic to a curve in which the third coordinate is inserted instead at the average of those of the two vertices closest to the middle of the pipe.

Observe that with this definition ι_m^{good} is not cellular, but is cellular when we barycentrically subdivide the codomain. This allows us to use the j maps of Definition II.14 to determine the image of the cell.

Lemma II.7. *Let* $\mathbf{e} \in \mathrm{CELL}_\bullet(P_m)$ *be a good cell. Then the image of* $\mathbf{e}$ *under the induced chain map* $(\iota_m^{good})_\#$ *is*

$$\begin{aligned}(\iota_m^{good})_\#(\mathbf{e}(\sigma_x, \sigma_y, \sigma_z)) &= \mathbf{e}(j_{\alpha,\alpha+1}\sigma_x, j_\alpha\sigma_y, j_\alpha\sigma_z; \\ &\qquad \langle\alpha{+}1\rangle_x, (\alpha\ \alpha{+}1)_y, (\alpha\ \alpha{+}1)_z).\end{aligned}$$

If the pipe into which we wish to insert a vertex has zero length, such a map may insert a vertex coincident to an existing one. This poses no problem in the discriminant, but would result in a knot mapping to a singular curve. For example, if $v_i = (v_i^x, v_i^y, v_i^z)$, and $v_{i+1} = (v_i^x, v_i^y, v_{i+1}^z)$, inserting a vertex in either of $\mathbf{x}_i$ or $\mathbf{y}_i$ results in $\iota_m^{good}(v)_i = \iota_m^{PL}(v)_{i+1} = v_i$, so $\iota_m^{good}(v)\left(\frac{3i+1}{m}\right) = \iota_m^{good}(v)\left(\frac{3i+4}{m}\right)$, an intersection of distant pipes.

We resolve this issue by borrowing length from the two surrounding pipes, at least one of which is guaranteed to have non-zero length if the curve is non-singular.

In some cases, this requires changing the coordinates of the two existing neighboring vertices in a manner which does not change the knot type of the image, as in Figure II.7. To ensure that we don't change the knot type, we limit our deformation of the knot in either borrowing direction to half the distance to the closest pipe, which must run perpendicular to the plane in which the two vertices occur.

This issue only arises if two specific consecutive vertices in the curve lie in the same coordinate plane. These are precisely the curves produced by points lying in the closure of codimension one cells of the form $\mathbf{e}(\sigma_x, \sigma_y, \sigma_z; (\alpha\ \alpha{+}1)_{e_{[m]_3}})$. Call any such cell *$\alpha$-planar*. All cells which are neither good nor α-planar, namely those for which $(\alpha\ \alpha{+}1)$ appears in $\sigma_{[m]_3}$ but which are not in the closure of an α-planar cell, will be called *interpolating*.

We make the following sequence of definitions for $e_{[m]_3} = e_0$ (or x). Similar definitions can be made for y and z.

Definition II.16. Let $\alpha(m)$ and α be as in Definition II.15 and $v \in \mathbf{e} \in \textsc{Cell}_\bullet(P_m)$ with $\mathbf{e}$ an α-planar cell

Define $\iota_m^P(v)$ by

$$(\iota_m^P(v))_i \quad = \quad \begin{cases} v_i & i \leq \alpha - 1 \\ (v_\alpha^x, v_\alpha^y, \frac{1}{2}(v_\alpha^z + v^z_{\sigma_z(\sigma_z^{-1}(\alpha)+s(\mathbf{z}_{\alpha-1}))})) & i = \alpha \\ (v_\alpha^x, \frac{1}{2}(v_\alpha^y + v^y_{\sigma_y(\sigma_y^{-1}(\alpha)+s(\mathbf{y}_\alpha))}), v_\alpha^z) & i = \alpha + 1 \\ v_{i-1} & i \geq \alpha + 2. \end{cases}$$

ι_m^P is a cellular map from α-planar cells to the the barycentric subdivision of the codomain.

Lemma II.8. *Let* $\mathbf{e} \in \textsc{Cell}_\bullet(P_m)$ *be an α-planar cell. Then the image of* $\mathbf{e}$ *under*

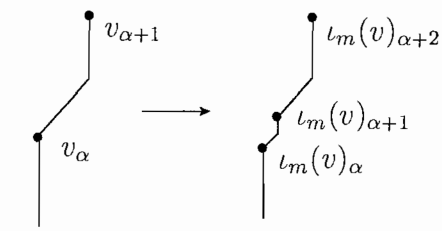


Fig. II.7: $\iota_m^P(v)$

the induced chain map $(\iota_m^P)_\#$ *is*

$$\begin{aligned}(\iota_m^P)_\#(\mathbf{e}(\sigma_x,\sigma_y,\sigma_x;(\alpha\ \alpha{+}1)_x)) &= \mathbf{e}(j_{\alpha,\alpha+1}\sigma_x, j_\alpha\sigma_y, j_\alpha\sigma_z;\\ &\qquad (\alpha\ \alpha{+}1)_x\langle\alpha{+}1\rangle_x, \langle\alpha{+}1\rangle_y, \langle\alpha{+}1\rangle_z).\end{aligned}$$

It now remains to continuously extend the maps ι_m^{good} and ι_m^P across the interpolating cells. When within such a cell, rather than just inserting a vertex we will also slightly perturb the vertices using the borrowing construction from Definition II.16. See Figure II.8 for an example of how the map functions on a point which requires such an interpolation.

Definition II.17. Let $\alpha(m)$ and α be as in Definition II.15 and $v \in \mathbf{e} \in \mathrm{CELL}_\bullet(P_m)$ with $\mathbf{e}$ an interpolating cell.

We define the interpolating parameter $p(v)$ to be the ratio of the distance between v_α^x and $v_{\alpha+1}^x$ to the maximal distance between them within the cell if we fix the other vertices, namely

$$p(v) = \frac{||\mathbf{x}_\alpha||}{|v^x_{\sigma_x(\sigma_x^{-1}(\alpha)-s(\mathbf{x}_\alpha)))} - v^x_{\sigma_x(\sigma_x^{-1}(\alpha+1)+s(\mathbf{x}_\alpha)))}|}.$$

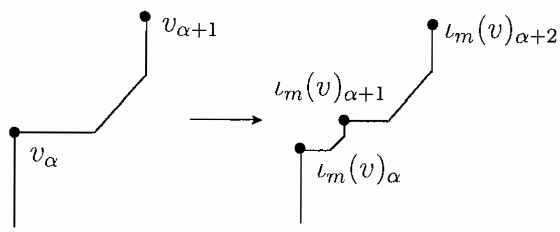


Fig. II.8: $\iota^I_m(v)$

Define $\iota^I_m(v)$ by

$$(\iota^0_m(v))_i \quad = \begin{cases} v_i & i \leq \alpha - 1 \\ (v^x_\alpha, v^y_\alpha, v^z_\alpha - \frac{p(v)}{2}(v^z_\alpha - v^z_{\sigma_z(\sigma_z^{-1}(\alpha)+s(\mathbf{z}_{\alpha-1}))})) & i = \alpha \\ (v^x_\alpha, v^y_\alpha - \frac{p(v)}{2}(v^y_\alpha - v^y_{\sigma_y(\sigma_y^{-1}(\alpha)+s(\mathbf{y}_\alpha))}), v^z_\alpha) & i = \alpha + 1 \\ v_{i-1} & i \geq \alpha + 2. \end{cases}$$

Now, we define ι_m as the union of these three maps. For $M > m$, write $\iota_{m,M} = \iota_{M-1} \circ \iota_{M-2} \circ \cdots \circ \iota_m$ and let $\iota_{m,m}$ be the identity.

Only ι^I_m fails to be cellular, though the image of an interpolating cell is contained in a cell we can name.

Lemma II.9. *Let $\alpha = \alpha(m) + 1$ and $\mathbf{e}$ an interpolating cell. The image of $\mathbf{e}$ under ι^I_m satisfies*

$$\iota^I_m(\mathbf{e}(\sigma_x, \sigma_y, \sigma_x)) \quad \subseteq \quad \mathbf{e}(j_{\alpha,\alpha+1}\sigma_x, j_\alpha \sigma_y, j_\alpha \sigma_z; \langle \alpha + 1 \rangle_x).$$

By construction, the following lemma is immediate.

Lemma II.10. $\iota_m \colon K_m \hookrightarrow K_{m+1}$ *and* $\iota_m \colon S_m \hookrightarrow S_{m+1}$.

That is, the $\{K_m\}$ and the $\{S_m\}$ form compatible directed systems under the ι_m. In Definition III.12, we will use the compatibility of these systems to define an inverse system of cohomology spectral sequences as the Alexander dual of a directed system of homology spectral sequences.

Let $\iota_{m,\infty}\colon\ K_m \to \varinjlim K_m$ be the induced map. Similarly for S_m.

Definition II.18. Let $v \in K_m$ and $w \in K_M$. We say ϕ_v and ϕ_w are *isotopic* if there is a path $\bar{\Phi}_{v,w}\colon\ \mathbb{I} \to \varinjlim K_m$ with $\bar{\Phi}(0) = \iota_{m,\infty}(v)$ and $\bar{\Phi}(1) = \iota_{M,\infty}(w)$.

It follows from Lemma II.10 that if ϕ_v and $\phi_w \in K_m$ are geometrically isotopic, so are $\phi_{\iota_m(v)}$ and $\phi_{\iota_m(w)}$.

We wish to compare these new spaces K_m to more familiar spaces of (long) knots. We will use PL maps of m segments whose segments are each parameterized by intervals of fixed length $\frac{3}{m}$, so the ith segment has domain $[\frac{3i}{m}, \frac{3(i+1)}{m}]$. Given a point $v \in (\text{Int}\ \mathbb{I}^3)^{m-1}$, a *PL map* of "m segments" is a map $\psi_v\colon\ \mathbb{R} \to \mathbb{R}^3$ with support on $[0, 3]$, built in a manner similar to that for a plumbers' curve.

Definition II.19. Let $v \in (\text{Int}\ \mathbb{I}^3)^{m-1}$, and for $i \in [\mathbf{m-1}]$ construct linear maps between consecutive v_i as follows.

$$\ell_i(t)\colon\ \left[\frac{3i}{m}, \frac{3(i+1)}{m}\right] \to \mathbb{R}^3 \quad = \quad (3i + 1 - mt)v_i + (mt - 3i)v_{i+1}$$

Each such map is a *segment*, and as with pipes we will sometimes fail to distinguish between the map and its image. Let $\psi_v\colon\ [0,3] \to \mathbb{R}^3$ be the union of the ℓ_i, so for t in the domain of ℓ_i, $\psi_v(t) = \ell_i(t)$. Such a map is a *PL map with m segments.*

Definition II.20. Let $v \in (\text{Int}\ \mathbb{I}^3)^{m-1}$ and ψ_v a PL map with m segments. ψ_v is *non-singular* if $\psi_v(s) = \psi_v(t)$ only if $s = t$. A non-singular PL map with n segments is a *PL knot with m segments.* Denote space of all such maps by L_m.

The maps which make L_m into a directed system are constructed in a manner similar to the maps between the P_m, but are much more straightforward, defined simply to be "inserting a vertex in the middle of the $\alpha(m)$th move". This requires a reparameterization of the map in order to make it a valid element of L_{m+1}, which we produce by reconstructing the knot from the new list of vertices.

Definition II.21. Let $\alpha(m), \alpha$ be defined as before. For $v \in (\text{Int } \mathbb{I}^3)^{m-1}$, define

$$(I_m(v))_i = \begin{cases} v_i & i < \alpha \\ \frac{1}{2}(v_{i-1} + v_i) & i = \alpha \\ v_{i-1} & i > \alpha. \end{cases}$$

It is apparent that I_m is injective for all m. For $M > m$, write $I_{m,M} = I_{M-1} \circ I_{M-2} \circ \cdots \circ I_m$, $I_{m,m}$ is the identity. The spaces $\{L_m\}$ form a directed system under I_m, so let $I_{m,\infty} \colon L_m \to \varinjlim L_m$ be the induced map.

Geometric isotopy and isotopy of PL knots are defined in precisely the same manner as for plumbers' knots.

Let $\mathcal{K}$ be the space of C^1 long knots in $\mathbb{R}^3$. With the construction we have exhibited, it is known that $\varinjlim L_m$ is homotopy equivalent to $\mathcal{K}$. We wish to establish that $\pi_*(\varinjlim K_m) \cong \pi_*(\varinjlim L_m)$ in order to use the space of plumbers' knots as a model for studying $\mathcal{K}$. We begin by proving Theorem II.13, which shows that the components of $\varinjlim K_m$ are the same as those of $\varinjlim L_m$, using a standard but technically involved method: we produce maps from each directed system to the other which are, up to isotopy, inverses in the limit.

To simplify the proof of this theorem, we will want to be able to say that a map from one space of knots to another "respects the isotopy type" of a knot. To make this rigorous, we approximate both PL and plumbers' knots with C^1 knots and compare these approximations, thus allowing us to compare C^1-isotopy classes across knot spaces. It is known that we can approximate a PL knot $\psi_w \in L_m$ to arbitrary precision with a C^1 knot $\psi_{\tilde{w}}$ via a careful choice of corner smoothing, so it remains to do the same with plumbers' knots. This approxmiation will not be canonical, but it is well defined on components of the space.

We proceed by producing an appropriate map $K_m \to L_{3m}$. We would like to use the naive map that sends a plumbers' knot to the PL knot defined by the same set of vertices. However not all plumbers' knots consist of vertices which produce valid PL knots, as plumbers' knots can have up to two consecutive pipes of zero length. In order to produce continuous maps, we rely on techniques similar to those used to produce maps between the K_m, "buckling" knots which have segments whose length is below a particular threshold.

Definition II.22. Let $v \in K_m$. Define the *global perturbation distance* $\epsilon(v)$ for the knot ϕ_v to be a small fraction of the minimum of the distances between distant pipes and between pipes (other than the first two and the last two) and the boundary of the unit cube. By perturbing the knot no more than this distance, we do not to change the isotopy type or make it singular.

Define the *buckling* function for a pipe p in ϕ_v by

$$\beta(p) = \begin{cases} \sqrt{\frac{\epsilon(v)^2 - ||p||^2}{2}} & \epsilon > ||p|| \\ 0 & \epsilon \leq ||p||. \end{cases}$$

The buckling function acts as the borrowing function did before, providing an interpolation which allows us to make our function continuous. As the length of a particular pipe shrinks, we deform the the image of the knot in L_{3m} by moving the vertices at its endpoints into the pipes which neighbor it, as in Figure II.9.

Produce a map $f_m\colon K_m \to L_{3m}$. For each $i \in \{0, \dots, m-1\}$ define

$$\begin{aligned}
(f_m(v))_{3i} &= (v_i^x + s(\mathbf{x}_i)||\mathbf{x}_i||\beta(\mathbf{z}_{i-1}), v_i^y, v_i^z - s(\mathbf{z}_{i-1})||\mathbf{z}_{i-1}||\beta(\mathbf{x}_i)),\\
(f_m(v))_{3i+1} &= (v_i^x - s(\mathbf{x}_i)||\mathbf{x}_i||\beta(\mathbf{y}_i), v_i^y + s(\mathbf{y}_i)||\mathbf{y}_i||\beta(\mathbf{x}_i), v_i^z),\\
(f_m(v))_{3i+2} &= (v_i^x, v_i^y - s(\mathbf{y}_i)||\mathbf{y}_i||\beta(\mathbf{z}_i), v_i^z + s(\mathbf{z}_i)||\mathbf{z}_i||\beta(\mathbf{y}_i)).
\end{aligned}$$

Here, for convenience, define $\beta(\mathbf{z}_{-1}) = ||\mathbf{z}_{-1}|| = 0$.

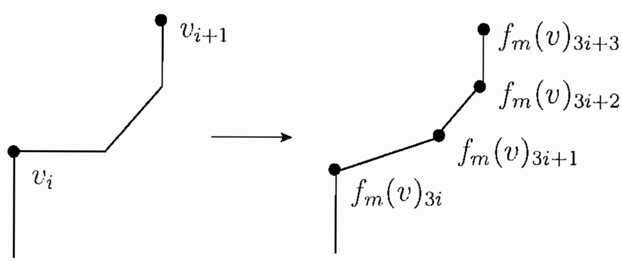


Fig. II.9: Buckling a plumbers' knot to create a PL knot

The image of f_m has the property that $|f_m(v)(t) - v(t)| < \epsilon(v)$.

Lemma II.11. *There is a map of sets* SM : $K_m \to \mathcal{K}$ *which passes to a well-defined map on* π_0.

Sketch of proof. Consider a "tubular neighborhood" of $\phi \in K_m$. SM(ϕ) is a C^1 curve which "intersects each fiber" in precisely one point. Such a choice of curves can be chosen to be continuously parameterized by elements of K_m and the tubular neighborhood guarantees that the knot type is preserved. □

There is an analagous map which smooths PL curves which by abuse we also call SM.

Definition II.23. Let $g\colon K_m \to L_m$ (respectively $h\colon L_m \to K_m$). We say g *respects* *the knot type of* ϕ_v if $\text{SM}(g(v))$ is isotopic to $\text{SM}(v)$ (respectively $\text{SM}(h(v))$ is isotopic to $\text{SM}(v)$). If g respects the knot type of all ϕ_v, we say g *respects knot types.*

We will make use of the maps f_m in our proof of Theorem II.13. The inverse maps are defined as follows:

Definition II.24. Let $w \in L_m$. Define $A_m\colon L_m \to K_m$ to be the map which takes the PL knot ψ_w to the plumbers' knot ϕ_w. In general, A_m does not preserve knot types or even non-singularity. However, in order to ensure that A_m respects knot types it is sufficient to force the lengths of segments in ψ_w to be sufficiently small.

Define the *maximal segment length* function δ for $w \in L_m$ by

$$\begin{aligned} m_1 &= \min\{d(\ell_i, \ell_j) \mid |i-j| > 1\} \\ m_2 &= \min\{d(w_i, \partial(\mathbb{I}^3) \mid i \in \{1, \ldots, n-1\}\} \\ \delta(w) &= \frac{1}{10}\min(m_1, m_2). \end{aligned}$$

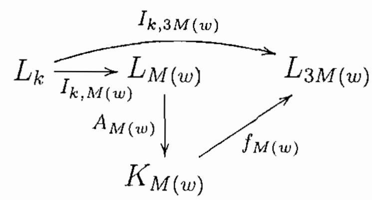
Let $M(w)$ be the minimum number so that $\psi_{I_{m,M(w)}(w)}$ is a PL knot with the property that no segment has length greater than $\delta(w)$.

Lemma II.12. *$A_{M(w)} \circ I_{m,M(w)}$ respects the knot type of ψ_w.*

The "isotopy" between the approximated C^1 knots is itself approximated by the triangulated surfaces bounded by each segment of the PL knot and the corresponding pipes of the plumbers' knot.

Theorem II.13. *The induced map $f_*\colon \pi_0(\varinjlim K_m) \to \pi_0(\varinjlim L_m)$ is an isomorphism of sets.*

Proof. Let $\bar{w} \in \varinjlim L_m$ and k an integer and $w \in L_k$ so that $I_{k,\infty}(w) = \bar{w}$. We require that in following family of diagrams indexed by $\bar{w}$, the images of ψ_w under both composites of maps are geometrically isotopic.



However, as A_m, f_m and I_m respect the knot type of ψ_w, there is a geometric isotopy between the images. Since the ι_m are injective, this says that for $\bar{w} \in \varinjlim L_m$ there exists $\bar{v} \in \varinjlim K_m$ which maps to the isotopy class of $\psi_{\bar{w}}$ under f_*.

Now, let $\psi_{\bar{w}}$ and $\psi_{\bar{w}'}$ be isotopic elements of $\varinjlim L_m$. We can lift an isotopy between them to a geometric isotopy at some finite stage, $\Psi_{w,w'}: \mathbb{I} \to L_m$. Let $M = max\{M(\Psi(t)) \mid t \in \mathbb{I}\}$, where $M(w)$ is as in Definition II.24. Precompose $\Psi_{w,w'}$ by $I_{m,M}$ to produce a geometric isotopy $\hat{\Psi}$ between $\psi_{I_M(w)}$ and $\psi_{I_M(w')}$. Now we can apply A_M to get a geometric isotopy between $\phi_{(A_M \circ I_M)(w)}$ and $\phi_{(A_M \circ I_M)(w')}$. Per the proof of surjectivity, under f_M these map to elements geometrically isotopic to $\psi_{I_{3M}(w)}$ and $\psi_{I_{3M}(w')}$ respectively. That is, if $\psi_{\bar{w}}, \psi_{\bar{w}'} \in \varinjlim L_m$ are isotopic, we can construct an isotopy between elements of $\varinjlim K_m$ which map to knots isotopic to $\psi_{\bar{w}}$ and $\psi_{\bar{w}'}$, so f_* is injective.

□

Using compactness to choose the maximum number of vertices necessary for any higher isotopy, it is now straightforward to conclude that

Theorem II.14. $\varinjlim K_m$ *is homotopy equivalent to* $\varinjlim L_m$.

II.4. Relationships with Lattice Knots and Cube Diagrams

We observe that plumbers' knots bear strong resemblance to a number of other finite-complexity knot theories. Two of particular interest are lattice knots and cube diagrams. Lattice knots are studied because they can be used to model physical data like length and thickness of the material from which a knot is constructed. Cube diagrams are used in [1] to construct chain complexes of knots with which one can study knot Floer homology.

II.4.1. Lattice knots

A lattice knot is a PL knot whose segments lie parallel to the coordinate axes and meet one another on points of the integer lattice $\mathbb{Z}^3 \subseteq \mathbb{R}^3$. Clearly, such knots are very closely related to the representative knots of Definition II.10. In fact, a suitable representative of a cell of non-singular plumbers' knots can be rescaled and "closed" to produce a lattice knot, as in Figure II.10.

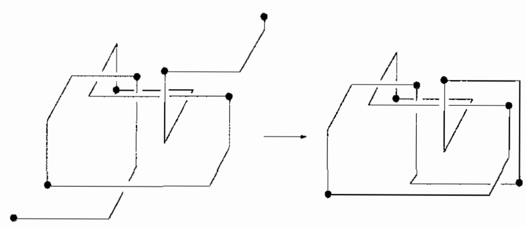


Fig. II.10: Closing a plumbers' knot to obtain a lattice knot.

Definition II.25. Recall that L_m is the space of m-segment piecewise linear knots and suitably modify the definitions given so that maps in L_m have their images in $[0, N]^3$ for some large N. Let $Lat_m \subseteq L_m$ be the subspace of lattice knots.

Let $\phi_v \in P_m$ be a representative knot for the cell $e(\sigma_x, \sigma_y, \sigma_z)$ and define a lattice knot $LK(v) = LK(\phi_v) \in Lat_{3m-1}$ by

$$\begin{aligned} LK(v)_0 &= \big(m+1, m \cdot v^y_{m-1}, 0\big) \\ LK(v)_1 &= \big(m \cdot v^x_1, m \cdot v^y_{m-1}, 0\big) \\ LK(v)_2 &= (m \cdot v^x_1, m \cdot v^y_1, 0) \end{aligned}$$

$$\begin{aligned} LK(v)_{3k} &= (m \cdot v^x_k, m \cdot v^y_k, n \cdot v^z_k) \\ LK(v)_{3k+1} &= \big(m \cdot v^x_{k+1}, m \cdot v^y_k, m \cdot v^z_k\big) \\ LK(v)_{3k+2} &= \big(m \cdot v^x_{k+1}, m \cdot v^y_{k+1}, m \cdot v^z_k\big) \end{aligned}$$

$$\begin{aligned} LK(v)_{3n-3} &= \big(m \cdot v^x_{m-1}, m \cdot v^y_{m-1}, m \cdot v^z_{m-1}\big) \\ LK(v)_{3n-2} &= \big(m+1, m \cdot v^y_{m-1}, m \cdot v^z_{m-1}\big) \end{aligned}$$

where k ranges from 1 to $m-2$.

Each representative knot for a codimension 0 cell of P_m maps to a lattice knot with $3m-1$ segments. However, these knots tend to use more segments than necessary and there is some interest in discovering the minimal number of segments required to create a lattice knot of a given topological knot type. Note that, as in Definition II.10, the idea of a representative knot is extensible to cells of any dimension in $\text{CELL}_\bullet(K_m)$. As some of the plumbers' knots that appear in these cells contain zero-length pipes which must be omitted from their image in Lat_m, we can study them as a means to find lattice knots with fewer segments. A pipe has zero length precisely when the two vertices which define its move are in the same appropriate coordinate plane.

Proposition II.15. *Let $v \in e(\vec{\sigma}; \mathbf{C})$. Define $\mu(\mathbf{C})$ to be the number of pairs of consecutive indices which appear in the some component of $\mathbf{C}$. The number of* *zero-length pipes in* ϕ_v *is* $\mu(\mathbf{C})$.

Also, notice that when we close an m-move plumbers' knot for which $v_1^y = v_{m-1}^y$, there is a zero length segment produced in the closure. This occurs when 1 and $m-1$ appear in the same set in $\mathbf{C}_y$.

Further, since adjacent pipes can move along the same coordinate axis, it is permissible to omit a vertex in the middle of the segment when we construct the lattice knot.

Proposition II.16. *Let* $\nu(x)$ *be the number of pairs of consecutive integers which appear in the same set in* $\mathbf{C}_y$ *and* $\mathbf{C}_z$. *Let* $v \in e(\vec{\sigma};\mathbf{C})$. *The number of consecutive moves which travel only along single axes in* ϕ_v *is* $\nu(v) = \nu(x) + \nu(y) + \nu(z)$.

Notice that if ϕ_v is a plumbers' knot, the same pair of consecutive integers can never appear in the same set in all three of $\mathbf{C}_x$, $\mathbf{C}_y$ and $\mathbf{C}_z$, as this would produce three consecutive zero-length pipes.

Lemma II.17. *For* $v \in K_m$, *let* $\mu(v) = \mu(\mathbf{C})$ *for the cell* $\mathbf{e}(\vec{\sigma};\mathbf{C})$ *containing* v. $\mu(v) + \nu(v) + \delta_{v_1^y, v_{m-1}^y} \leq 3(m-2) + 1$.

Definition II.26. Fix a cell $e(\vec{\sigma};\mathbf{C})$ and let $\phi_v \in K_m$ be the representative knot for e. Let $\mu(v)$ and $\nu(v)$ be as above. Define a lattice knot $LK(v) \in Lat_{3m-1-\mu(v)-\nu(v)}$ in the same manner as in Definition II.25, omitting vertices which would coincide or which bound two segments which move along the same coordinate axis.

Using this definition, Lemma II.17 says that the smallest number of segments that can occur in a lattice knot arising as the closure of a plumbers' knot of m moves is 4. Clearly, such a lattice knot is an unknot.

The lattice knots which are produced by plumbers' knots are characterized by one or two segments lying in the $z = 0$ plane and one in the plane with the highest x coordinate. Every lattice knot can be deformed to such, under the appropriate notion of isotopy. It seems likely, therefore, that our classification of plumbers' knots will illuminate the study of lattice knots.

II.4.2. Remarks on Cube Diagrams

The cube diagrams of Baldridge and Lowrance [1] bear strong resemblance to lattice knots and can be considered as plumbers' knot representatives of particular cells. A plumbers' knot which *satisfies the $x - y$ crossing condition* (as described in [1]) is one whose projection as a knot diagram onto the $x - y$ plane results in the y-parallel pipe crossing over the x-parallel pipe for each crossing. Such a diagram is called a grid diagram, and can be used to compute the knot Floer homology of a knot [9].

Proposition II.18. *Let* $\mathbf{e} = \mathbf{e}(\vec{\sigma}) \in \text{CELL}_{3m-3}(K_m)$ *and* $v \in \mathbf{e}$. *The plumbers' knot* ϕ_v *satisfies the $x - y$ crossing condition if whenever* $\sigma_x^{-1}(b+1)$ *is between* $\sigma_x^{-1}(a)$ *and* $\sigma_x^{-1}(a+1)$ *and* $\sigma_y^{-1}(a)$ *is between* $\sigma_y^{-1}(b)$ *and* $\sigma_y^{-1}(b+1)$, *then* $\sigma_z^{-1}(b) > \sigma_z^{-1}(a)$.

This follows immediately from the definitions, and symmetric statements exist for the y-z and z-x crossing conditions. We can consider the subspace of cube knots of m moves, $C_m \subseteq K_m$, generated by such cells, which are precisely the cube knots of grid number m. If we do not allow stabilization moves, isotopy of cube knots in each finite space is the same as geometric isotopy through cube knots. Application of the algorithm for classification of plumbers' knots yields that C_5 has precisely one cell, a right-handed trefoil, while C_6 has 11 components and C_7 has 108. While these spaces are significantly smaller than the general spaces of plumbers' knots, their geometry is significantly less straightforward and are therefore unsuitable for our current purposes.

Allowing for the stabilization moves described in [1], the authors prove the following analog to Theorem II.13.

Theorem II.19 ([1]). $\pi_0(\varinjlim C_m) \cong \pi_0(\varinjlim K_m)$

We expect that our development of the combinatorics of plumbers' knots will illuminate computations in the cube diagram chain complex for knot Floer homology. Further, our extension of the theory of finite-type invariants may provide a method of understanding connections between the two theories.

CHAPTER III

UNSTABLE VASSILIEV THEORY

We describe an unstable Vassiliev spectral sequence for the homology of the spaces of plumbers' knots. Using the categorical description of the discriminant S_m given in Proposition II.2 and standard tools of homotopy theory, we first construct a resolution $\tilde{S}_m$ of this space. This resolution is compatible with the existing cell structure on S_m, allowing us to lift it to a cell structure on $\tilde{S}_m$ which encodes both the geometry of the discriminant and singularity data similar to that in Vassiliev's resolution of Σ.

By understanding the possible singularity types that can arise in plumbers' curves, we introduce a filtration on the discriminant and lift it to the resolution. Due to the rigid geometry of plumbers' maps, very few configurations of transverse intersections of pipes are possible. Isolated triple points only occur when three pipes, one parallel to each of the coordinate axes, intersect in a single point. It is impossible to produce an isolated quadruple (or higher) point. Since double and triple points are the only singularities which contribute to cycles in classical Vassiliev theory, plumbers' knots are well suited to this analysis. The fact that plumbers' knots serve as a model for classical knot theory coupled with this observation gives circumstantial evidence that Vassiliev's invariants should be a complete system of knot invariants.

We extend the Vassiliev derivative to these spaces, using the cell structure to provide a canonical choice of chain representative on all of $\tilde{S}_m$ for the lift of the Alexander dual to an invariant of plumbers' knots. Finally, we introduce the inverse system of unstable Vassiliev spectral sequences induced by our filtration and see that the E^∞-page of its limit contains the finite type invariants.

III.1. The Homotopical Blowup of the Discriminant

The problem of understanding the geometry of the discriminant is that of understanding an arrangement of partial real hyperplanes. It is natural to record which pairs planes intersect, and a standard technique in singularity theory is to "blow up" the points of the intersection by replacing them with simplices whose vertices are labelled by these pairs. The combinatorial description of the discriminant as a colimit gives us the information we require to perform this blowup using the homotopy colimit.

Definition III.1. Let B_m be as in Definition II.12. The *homotopical blowup of the discriminant* is $\tilde{S}_m = \text{hocolim} B_m$.

In order to use these spaces $\tilde{S}_m$ to study S_m, we need the following.

Proposition III.1. *The projection map* $\pi : \tilde{S}_m \to S_m$ *is a homotopy equivalence.*

The coincidence categories $\mathcal{S}_m$ of Definition II.11 are directed Reedy categories, so Proposition III.1 is an instance of the general construction considered, for example, as Application 13.6 in Dugger's expository paper on homotopy colimits [7]. While blowing up the discriminant in this manner is a standard technique, both Vassiliev's filtration and the one we define in Definition III.4 produce spectral sequences which are not equivalent to the one recorded in [7] which arises from the usual simplicial filtration.

Vassiliev's analysis of the spectral sequence in [12] relies on a cell structure that arises in the "stable" filtration quotients of his resolution of the discriminant Σ. While the blowup in Proposition III.1 is similar in spirit to Vassiliev's resolution, illustrated in Figure I.2, ours carries a canonical cell structure $\text{CELL}_\bullet(\tilde{S}_m)$ lying over $\text{CELL}_\bullet(S_m)$ whose combinatorics we can understand before stabilization or filtration.

Definition III.2. Let $\mathbf{e} = \mathbf{e}(\vec{\sigma}; \mathbf{C}) \in \text{CELL}_\bullet(S_m)$ and let ρ be a nonempty collection of transpositions in $\Sigma_{\mathbf{C}}$. Denote by $*$ the topological join and by $\rho(C_i)$ the transpositions in ρ with support on C_i.

Define $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho) = \mathbf{e} \times *_{i=1}^{\ell} \Delta^{\binom{\rho(C_i)}{2}-1} \in \text{CELL}_\bullet(\tilde{S}_m)$ to be the face of the simplex "lying over" $\mathbf{e}(\vec{\sigma}; \mathbf{C})$ indexed by the elements of ρ.

This is schematically illustrated in Figure III.1 for the blowup of of the cell $\mathbf{e}((25134_x, 41253_y, 35241_z); \{1, 2, 5\}_y)$, which arises as the non-transverse intersection of three codimension-one cells. In this case, $\mathbf{C}$ consists of a single component resulting in a simplex whose vertices are indexed by $\{(1\ 2)_y\}, \{(1\ 5)_y\}$ and $\{(2\ 5)_y\}$ respectively. Also illustrated are the "blowups" of the codimension one cells cobounding $\mathbf{e}$, which are homeomorphic copies.

By construction, the collection of all such cells $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho)$ gives rise to a cell structure for $\tilde{S}_m$. Write $\pi_\#$ for the induced map $\text{CELL}_\bullet(\tilde{S}_m) \to \text{CELL}_\bullet(S_m)$ which "forgets ρ".

It will be useful to abuse notation and extend our naming conventions to plumbers' knots, which by necessity have empty singularity data, denoting by $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \emptyset)$ the cell $\mathbf{e}(\vec{\sigma}) \in \text{CELL}_{3m-3}(K_m)$.

By the Leibniz rule, the boundaries of cells in $\text{CELL}_\bullet(\tilde{S}_m)$ decompose into an *external* component inherited from the boundary maps in $\text{CELL}_\bullet(S_m)$ and an *internal* component induced by the combinatorics of the join of simplices. The

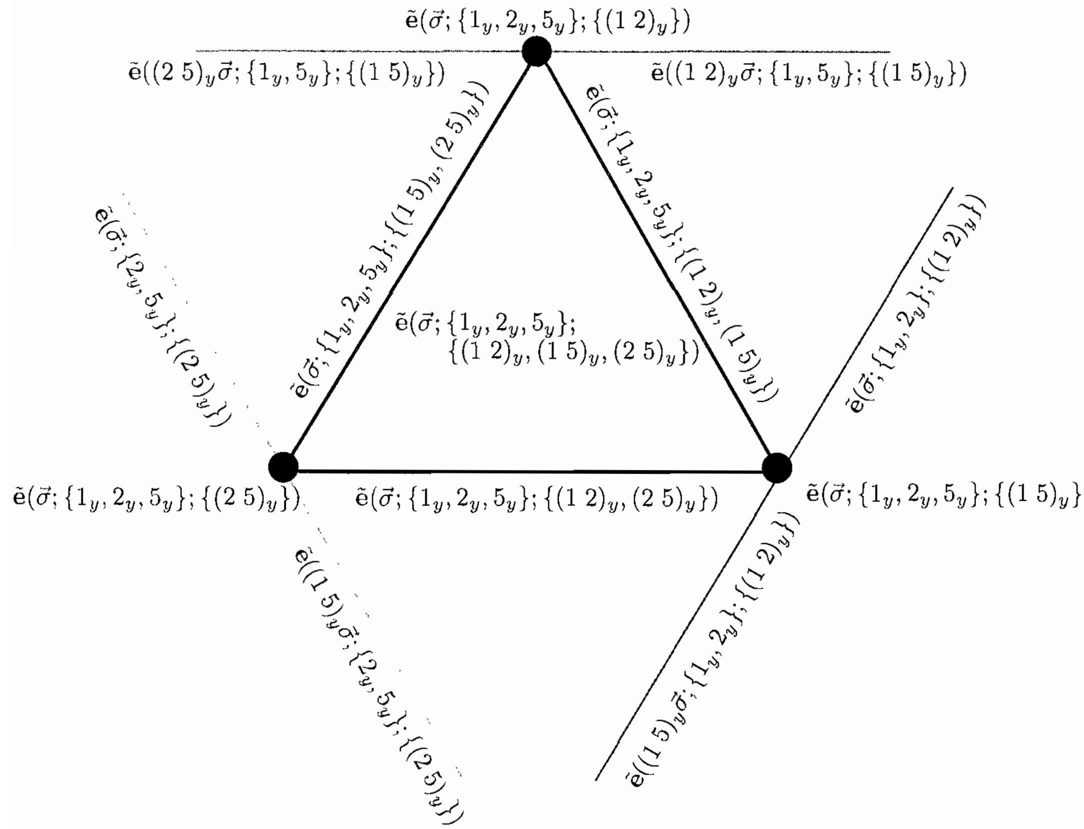


Fig. III.1: The cell stucture over a non-transverse triple intersection in the discriminant at $\mathrm{e}((25134_x, 41253_y, 35241_z); \{1, 2, 5\}_y)$.

external boundary component introduces new equalities of coordinates in $\mathbf{C}$, whereas the internal component deletes transpositions from ρ.

That is, the boundary of $\mathbf{e}(\vec{\sigma}; \mathbf{C}; \rho) \in C_*(\tilde{S}_m)$ is given by

$$d(\mathbf{e}(\vec{\sigma}; \mathbf{C}; \rho)) = \sum_{\mathbf{C}'} \pm \mathbf{e}(\vec{\sigma}; \mathbf{C}'; \rho) + \sum_{\rho_i \in \rho} \pm \mathbf{e}(\vec{\sigma}; \mathbf{C}; \rho \setminus \{\rho_i\}),$$

where $\mathbf{C}'$ range over coarsenings of the partition $\mathbf{C}$ produced by combining precisely two elements of $\mathbf{C}$ so that the resulting sets are admissible for $\vec{\sigma}$, and the signs alternate with respect to the lexicographic ordering in both sums.

This cell structure contains both the singularity data from the original discriminant and combinatorial data analogous to that in Vassiliev's auxillary spectral sequences from [12]. This wealth of data allows us to perform detailed analysis at the chain level. Indeed, we will see that there is a canonical choice of chain representative for a plumbers' knot invariant in our blowup.

III.2. The Complexity Filtration

In order to construct the unstable Vassiliev spectral sequence on the spaces of plumbers' knots, we require an increasing, cellular *complexity* filtration on the space S_m. Because all maps in a cell share the same singularity data, we define the filtration on $\text{C}\small{\text{ELL}}_\bullet(S_m)$, which then lifts to $\text{C}\small{\text{ELL}}_\bullet(\tilde{S}_m)$.

As we wish to compare our spectral sequence to the classical Vassiliev spectral sequence, we construct our filtration so that "stable" singularity data appears in the expected filtration. The filtration on other plumbers' curves is then determined by choosing the greatest complexity amongst cells which such a cell bounds.

Definition III.3. Call a plumbers' curve whose only singularities are transverse double points *simple.* If $\mathbf{e} \in \text{CELL}_\bullet(S_m)$ is a cell whose points are simple plumbers' curves, call e *simple* as well.

Simple curves have generic singularity data after subdivision. However, some pipes may intersect two or more other pipes, which is an unstable condition.

Definition III.4. Let $\mathbf{e} \in \text{CELL}_\bullet(S_m)$ be simple. Define the *complexity* of $\mathbf{e}$, $\text{CX}(\mathbf{e})$, to be the number of double points of a curve in e.

For $\mathbf{e} \in \text{CELL}_k(S_m)$ which contains a singularity other than isolated double points, define

$$\text{CX}(\mathbf{e}) = \max\{\text{CX}(\mathbf{f}) | \mathbf{f} \in \text{CELL}_{k+1}(S_m) \text{ and } \mathbf{e} \in \partial(\mathbf{f})\}.$$

As triple points only occur in the boundary of cells with a pair of double points, we recover that isolated triple points have complexity 2. Similarly, pauses of three consecutive zero-length pipes are in the boundary of the transverse double point where the curve "turns back through itself" in the span of four pipes.

Definition III.5. Let $F_p\text{CELL}_\bullet(S_m) = \{\mathbf{e} \in \text{CELL}_\bullet(S_m) | \text{CX}(\mathbf{e}) \leq p\}$.

We observe that the maximal number of transverse self intersections of a plumbers' curve occurs when all of its defining vertices lie in a single plane. By computing maximal number of transverse intersections rectilinear motion can produce in a fixed number of pipes we have $F_{\binom{m-1}{2}}\text{CELL}_\bullet(S_m) = \text{CELL}_\bullet(S_m)$.

Using this filtration on the space S_m, we may generalize the notion of isotopy to all plumbers' curves.

Definition III.6. Two singular plumbers' curves $\phi, \phi' \in S_m$ are *isotopic* if there exists a path $\Phi : I \to S_m$ with $\Phi(0) = \phi, \Phi(1) = \phi'$ and $\Phi(I) \subseteq (F_p \setminus F_{p-1})S_m$ for some p.

Finally, we lift this filtration by $F_p\text{CELL}_\bullet(\tilde{S}_m) = \pi_{\#}^{-1}(F_p\text{CELL}_\bullet(S_m))$. By construction, the boundary maps in $\text{CELL}_\bullet(\tilde{S}_m)$ can never decrease complexity. As the suspension maps $S_m \hookrightarrow S_{m+1}$ do not change the image of a curve, they also respect this filtration.

III.3. The Vassiliev Derivative of a Plumbers' Knot Invariant

The standard approach to the study of Vassiliev invariants has been through the Vassiliev derivative, introduced by Birman and Lin [4] and popularized by Bar-Natan [2]. The classical Vassiliev derivative of a knot invariant is defined for curves with n transverse double points and can be extended through the 4-term relation to triple points, but fails to see more degenerate singularities of smooth knots. We define an analogue of the Vassiliev derivative for invariants of plumbers' curves across any choice of singular cell.

Definition III.7. Fix $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho) \in C_*(\tilde{S}_m)$, $C_i \in \mathbf{C}$. Let $\rho(C_i)$ denote those transpositions of ρ supported on C_i. If $\rho(C_i)$ is not sequential for any $\vec{\sigma}$, define the C_i coboundary of $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho)$ to be zero.

If $\rho(C_i)$ is sequential for some $\vec{\sigma}'$, it is sequential for exactly two such, both with the property that $\tilde{\mathbf{e}}(\vec{\sigma}'; \mathbf{C}; \rho) = \tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho)$. These two choices of $\vec{\sigma}'$ will differ by reversal of the order in which the elements of C_i appear in the permutation. Let $\vec{\sigma}[\rho(C_i)]^+$ be the one which occurs first in the underlying lexicographic ordering and $\vec{\sigma}[\rho(C_i)]^-$ the other. Define the C_i-coboundary of $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho)$ to be

$$\delta_{C_i}(\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho)) = \tilde{\mathbf{e}}(\vec{\sigma}[\rho(C_i)]^+; \mathbf{C}; \rho \setminus \rho(C_i)) - (-1)^{|C_i|}\tilde{\mathbf{e}}(\vec{\sigma}[\rho(C_i)]^-; \mathbf{C}; \rho \setminus \rho(C_i))$$

Note that the coboundary formula involves a "signed difference" of cells determined by the geometry of the blowup.

The fact that the C_i are disjoint immediately implies

Lemma III.2. *Let* $\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho) \in \mathrm{CELL}_\bullet(\tilde{S}_m)$. *For any pair* $C_1, C_2 \in \mathbf{C}$, $\delta_{C_1}\delta_{C_2}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho) = \delta_{C_2}\delta_{C_1}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho)$.

Definition III.8. Define the *total coboundary* of $\tilde{\mathbf{e}} = \tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho)$, written $\delta_{\mathbf{C}}(\tilde{\mathbf{e}})$, to be the element of $\mathrm{CELL}_{3n-3}(K_m)$ resulting from composing, in any order, all of the δ_{C_i} for $C_i \in \mathbf{C}$.

Lemma III.2 then says that the total coboundary is well defined.

Note that when $\mathbf{C}$ has a single component $\delta_{\mathbf{C}}(\tilde{\mathbf{e}})$ is the "signed difference" of two cells in $\mathrm{CELL}_{3n-3}(K_m)$, as in Figure III.1. In this sense, certain codimension one cells in the blowup "separate" pairs of cells from P_m containing plumbers' curves. Recalling Section I.2 we make the following definition.

Definition III.9. Let $[\alpha] \in \bar{H}^0(K_m)$ and $\tilde{\mathbf{e}} = \tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho) \in \mathrm{CELL}_{3m-4}(\tilde{S}_m)$. The *Vassiliev derivative of* $[\alpha]$ *at* $\tilde{\mathbf{e}}$, $d_{\tilde{\mathbf{e}}}([\alpha])$, is $[\alpha](\delta_{\mathbf{C}}(\tilde{\mathbf{e}}))$.

In contrast to the classical Vassiliev derivative, this definition works for any singularity of plumbers' curve.

Proposition III.3. *The Vassiliev derivative is an isotopy invariant for singular plumbers' curves.*

Sketch of proof. Analogously to our isotopy classification of plumbers' knots, we can consider representative curves in cells of singular plumbers' curves and restrict our attention to straight-line "elementary" isotopies between them. Given a plumbers' knot invariant $[\alpha]$ and two cells of singular plumbers' curves $\mathbf{e}$ and $\mathbf{e}'$ which share a boundary $\mathbf{f}$, it is straightforward to compute $d_{\mathbf{f}}([\alpha])$ and that $d_{\mathbf{e}}([\alpha]) = d_{\mathbf{e}}([\alpha])$ if there is an elementary isotopy between their representative curves. □

It is vital that our definition agree with the classical definition when the plumbers' curves in question are sufficiently articulated and the singularities in the "stable" class studied by Vassiliev.

Definition III.10. Call an isolated singularity of a plumbers' curve *stable* if it is separated from any other singular point by at least one vertex. A cell consisting of singular plumbers' curves whose singularities consist only of stable double and triple points is a *stable cell.*

Our notion of stability differs from Vassiliev's. In particular, our definition is at the cellular level, while his was at the spectral sequence level.

Recall that a singular curve is said to respect a chord diagram if the endpoints of each chord are identified in the image of the map (c.f. [2]). Chord diagrams are usually considered up to diffeomorphisms of the spine which do not change the order of the endpoints of the chords, and we abuse terminology and call such a class of chord diagrams a chord diagram. When we say a map respects a chord diagram, we mean that it respects some member of its equivalence class.

Each stable cell $\mathbf{e}$ has associated to it some maximal chord diagram which its elements respect, so to evaluate the Vassiliev derivative of $[\alpha] \in H^0(\mathcal{K})$ on its lift $\tilde{\mathbf{e}}$, we evaluate a representative weight system for $[\alpha]$ on this chord diagram.

The following lemma justifies the term "stable" and follows immediately from Definition III.9. It says that on stable cells our notion of Vassiliev derivative agrees with that of Birman and Lin [4].

Let $\mathbf{e} \in \textsc{Cell}_\bullet(S_m)$ be a stable cell of complexity n and $[\alpha] \in H^0(K_m)$. We compute that the codimension one lift of such a cell, $\tilde{\mathbf{e}}$, is $\pi_\#^{-1}(\mathbf{e}) = \mathbf{e} \times \Delta^{n-1}$. In particular, it consists of a single cell which is analagous to the one which arises in the classical Vassiliev spectral sequence for the same singularity data, and so will have as its derivative a single coefficient.

Lemma III.4. *Let* $\mathbf{e}, \tilde{\mathbf{e}}$ *and* $[\alpha]$ *be as above, then* $d_{\tilde{\mathbf{e}}}([\alpha])$ *is given by evaluation of a representative weight system for* $[\alpha]$ *on the maximal chord diagram respected by* $\mathbf{e}$ *or, equivalently, by evaluation of* $[\alpha]$ *on an alternating sum of plumbers' knots produced by resolving in all possible ways the singularities of some map in* $\mathbf{e}$.

The case most referred to is where $\mathbf{e} \in \mathrm{CELL}_{3m-3-n}(S_m)$ be a stable cell whose points are singular curves with precisely n double points. The cell $\mathbf{e}$ has singularity data $\mathbf{C} = \{\{a_i, b_i\}_{d_i}\}$, $i \in \{1 \ldots n\}$ and lifts to a codimension one cell $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho) = \mathbf{e} \times \Delta^{n-1}$ whose second factor has vertices which are labeled by transpositions $\rho_i = \rho(\{a_i, b_i\}_{d_i}) = (a_i\ b_i)_{d_i}$.

One computes that $\delta_{\mathbf{C}}(\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho)) = \sum_i (-1)^i \tilde{\mathbf{e}}(\vec{\sigma}[\rho_1]^{\epsilon_{i,1}}[\rho_2]^{\epsilon_{i,2}} \cdots [\rho_n]^{\epsilon_{i,n}}; \mathbf{C}; \emptyset)$, where $\epsilon_{i,j}$ is the jth digit of the binary representation of i using digits from $\{+, -\}$. Therefore, for $[\alpha] \in H^0(K_m)$, $d_{\tilde{\mathbf{e}}}([\alpha])$ is the alternating sum of the value of $[\alpha]$ on the 2^n cells of K_m cobounding $\mathbf{e}$.

All other stable cells contain triple points. Vassiliev studies such singularities in [13], but we do not single out such cells for further analysis here.

The ability to define the Vassiliev derivative for any singularity of plumbers' curves along with the cell structure on $\tilde{S}_m$ provides a canonical chain representative for the dual to a plumber's knot invariant as given by the following "Taylor's theorem", showing how the derivatives of a plumbers' knot invariant determine that invariant.

Theorem III.5. *Let* $[\alpha] \in \bar{H}^0(K_m)$. *The lift of its Alexander dual cycle* $[\alpha^\vee]$ *to* $H_{3m-4}(\tilde{S}_m)$ *has a chain representative given by*

$$\tilde{\alpha}^\vee = \sum_{\tilde{\mathbf{e}} \in \mathrm{CELL}_{3m-4}(\tilde{S}_m)} (-1)^{o(\tilde{\mathbf{e}})} d_{\tilde{\mathbf{e}}}([\alpha]) \tilde{\mathbf{e}}.$$

To prove Theorem III.5, we first need to understand which cells in $\textsc{Cell}_\bullet(\tilde{S}_m)$ can carry non-zero coefficients as chain representatives and how these cells fit together. The condition in Definition III.7 that each $\rho(C_i)$ be sequential for $\vec{\sigma}$ implies that non-zero Vassiliev derivatives only occur for cells of dimension $3m-4$. The internal faces of such a cell are indexed by forgetting one transposition τ in some ρ_i. In order to have internal faces, $|\rho|$ must be greater than one.

Given a face $\tilde{\mathbf{f}} = \tilde{\mathbf{f}}(\vec{\sigma}; \mathbf{C}; \rho \setminus \{\tau\})$, the collection of cells which are incident to the face are of two types: internal cofaces which also lie in $\pi_{\#}^{-1}(\mathbf{e}(\vec{\sigma}; \mathbf{C}))$ and external cofaces which appear in some $\pi_{\#}^{-1}(\mathbf{e}(\vec{\sigma}'; \mathbf{C}; \rho \setminus \{\tau\}))$. The internal cofaces which are incident to $\tilde{\mathbf{f}}$ are of the form $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; (\rho \setminus \{\tau\}) \cup \{\tau'\})$, for some $\tau' \in \Sigma_{C_i}$. Precisely those τ' whose addition to $\rho \setminus \{\tau\}$ results in a new collection sequential for $\vec{\sigma}$ correspond to cells with non-zero Vassiliev derivative.

Write $\rho_i = \{(\rho_i(1)\ \rho_i(2)), (\rho_i(2)\ \rho_i(3)), \ldots, (\rho_i(k-1)\ \rho_i(k))\}$. There are two possibilities: τ is an "endpoint", either $(\rho_i(1)\ \rho_i(2))$ or $(\rho_i(k-1)\ \rho_i(k))$, or removing $\tau = (\rho_i(\ell)\ \rho_i(\ell+1))$ splits ρ_i into two disjoint collections sequential for $\vec{\sigma}$.

In the first case, there are two choices of transposition $\tau' \in \Sigma_{C_i}$ whose addition will result in a collection sequential for $\vec{\sigma}$: τ and $(\rho_i(1)\ \rho_i(k))$. In the second case, any of the transpositions $(\rho_i(1)\ \rho_i(\ell+1))$, $(\rho_i(\ell)\ \rho_i(\ell+1))$, $(\rho_i(1)\ \rho_i(k))$ or $(\rho_i(\ell)\ \rho_i(k))$ "reattach" them into a single collection sequential for $\vec{\sigma}$, while the rest result in non-sequential collections. Thus, there are always either two or four internal cofaces which can contribute a non-zero coefficient to $\tilde{\mathbf{f}}$.

In constrast, there are always two external cofaces incident to a given $\tilde{\mathbf{f}}$ which can contribute non-zero coefficients. As mentioned above, these are the cells $\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}'; \rho \setminus \{\tau\}) \in \pi_{\#}^{-1}(\mathbf{e}(\vec{\sigma}; \mathbf{C}'))$ and $\tilde{\mathbf{e}}(\tau\vec{\sigma}; \mathbf{C}'; \rho \setminus \{\tau\}) \in \pi_{\#}^{-1}(\mathbf{e}(\tau\vec{\sigma}; \mathbf{C}'))$, where $\mathbf{C}'$ is the refinement obtained from $\mathbf{C}$ by "splitting the appropriate C_i along τ" and τ acts on $\vec{\sigma}$ by block permutation of the elements in these two new partition elements in $\mathbf{C}$.

Proof of Theorem III.5. We must check two things: that $\tilde{\alpha}^\vee$ agrees with a chain representative of $[\alpha^\vee]$ on cells which lift to homeomorphic copies of themselves, and that it is a cycle.

The cells in $\text{CELL}_{3m-4}(S_m)$ are all of the form $\mathbf{e}(\vec{\sigma}; \{a_i, b_i\})$. These cells lift to cells of the form $\tilde{\mathbf{e}} = \tilde{\mathbf{e}}(\vec{\sigma}; \{a_i, b_i\}; \{(a\ b)_i\})$. We see that $d_{\tilde{\mathbf{e}}}([\alpha]) = (-1)^{o(\tilde{\mathbf{e}})}[\alpha](\mathbf{e}(\vec{\sigma}) - \mathbf{e}((a\ b)_i\vec{\sigma}))$, the difference of the value of the invariant on the cobounding cells of $\mathbf{e}$, which is precisely the coefficient assigned to the cell by Alexander duality. Thus if $\tilde{\alpha}^\vee$ is a cycle, it is a chain representative of the lift.

It remains to show that for each cell $\tilde{\mathbf{f}} \in \text{CELL}_{3m-5}(\tilde{S}_m)$, the total contribution of cells incident to $\mathbf{f}$ under the boundary map d is zero. To do so, we consider an arbitrary cell $\tilde{\mathbf{e}} \in \text{CELL}_{3m-4}(\tilde{S}_m)$ for which Vassiliev derivatve of $[\alpha]$ is non-zero, select one of its internal faces and compute the sum of the incidence coefficients of each of the face's cobounding cells. It suffices to consider internal faces of such cells: every cell with a coface whose Vassiliev derivative is non-zero arises as such an internal face, since if internal cells do not contribute the two external cells' contribution must be equal with opposite sign by Proposition III.3.

Write $C_i(\tau) = C_i' \sqcup C_i''$ for the refinement of C_i by splitting at τ, $\mathbf{C}(\tau)$ for the corresponding refinement to $\mathbf{C}$ and $\rho[\tau, i, j] = (\rho \setminus \tau) \cup (\rho_i\ \rho_j)$. Let ∂ be the standard coboundary of a cell in $\text{CELL}_\bullet(\tilde{S}_m)$ and use our analysis of cofaces to compute that (up to a sign depending on choice of $\vec{\sigma}$),

$$\begin{aligned}\partial(\tilde{\mathbf{f}}(\vec{\sigma}; \mathbf{C}; \rho \setminus \{\tau\})) = -\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho) &+ (-1)^\ell \tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho[\tau, 1, \ell+1]) \\ &+ (-1)^{k-\ell}\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho[\tau, \ell, k]) - (-1)^{k-1}\tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}; \rho[\tau, 1, k]) \\ &+ \tilde{\mathbf{e}}(\vec{\sigma}; \mathbf{C}(\tau); \rho \setminus \{\tau\}) - \tilde{\mathbf{e}}(\tau\vec{\sigma}; \mathbf{C}(\tau); \rho \setminus \{\tau\}) \\ &+ \text{cells with zero Vassiliev derivative.}\end{aligned}$$

Using Lemma III.2, we can rewrite $\delta_{\mathbf{C}}$ as $\delta_{\mathbf{C}\setminus C_i}\delta_{C_i}$ and $\delta_{\mathbf{C}\setminus C_i}\delta_{C_i'}\delta_{C_i''}$ for these two different types of cells. We now expand the coboundaries.

$$\begin{aligned}
d_{\delta(\tilde{\mathbf{f}})}([\alpha]) &= [\alpha]\big(\delta_{\mathbf{C}}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho) + \delta_{\mathbf{C}}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho[\tau,1,k]) + \delta_{\mathbf{C}}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho[\tau,1,\ell+1]) \\
&\qquad + \delta_{\mathbf{C}}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho[\tau,\ell,k]) + \delta_{\mathbf{C}(\tau)}\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C}(\tau);\rho\setminus\{\tau\}) \\
&\qquad + \delta_{\mathbf{C}(\tau)}\tilde{\mathbf{e}}(\tau\vec{\sigma};\mathbf{C}(\tau);\rho\setminus\{\tau\})\big) \\
&= [\alpha]\big(\delta_{\hat{\mathbf{C}}_i}(\delta_{C_i}(\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho) + \tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho[\tau,1,k]) \\
&\qquad + \tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho[\tau,1,\ell+1]) + \tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho[\tau,\ell,k])) \\
&\qquad + \delta_{C_i'}\delta_{C_i''}(\tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C}(\tau);\rho\setminus\{\tau\}) + \tilde{\mathbf{e}}(\tau\vec{\sigma};\mathbf{C}(\tau);\rho\setminus\{\tau\})))\big) \\
&= [\alpha]\big(\delta_{\hat{\mathbf{C}}_i}(\tilde{\mathbf{e}}(\vec{\sigma}[\rho(C_i)]^-;\mathbf{C};\rho\setminus\rho(C_i)) - (-1)^{|C_i|}\tilde{\mathbf{e}}(\vec{\sigma}[\rho(C_i)]^+;\mathbf{C};\rho\setminus\rho(C_i)) \\
&\qquad + \tilde{\mathbf{e}}(\vec{\sigma};\mathbf{C};\rho\setminus\rho(C_i)) - (-1)^{|C_i|}\tilde{\mathbf{e}}(w_0(C_i)\vec{\sigma};\mathbf{C}\rho\setminus\rho(C_i)\big) = [\alpha](0)
\end{aligned}$$

So $d_{\delta\tilde{\mathbf{f}}}([\alpha]) = [\alpha](0) = 0$, as required. □

This chain representative $\tilde{\alpha}^\vee$ is canonical in the sense that the coefficient on a given cell is the "signed difference" of the coefficients on the cells it "separates". We call this canonical representative the *Vassiliev-Taylor series for* $[\alpha]$. For purposes of computation, one can identify representatives of $[\alpha^\vee]$ in $\text{CELL}_\bullet(\tilde{S}_m)$ with fewer non-zero coefficients by choosing certain boundary contributions also to be non-zero.

Definition III.11. Let $[\alpha] \in \bar{H}^0(\mathcal{K})$. Define the *Vassiliev system* of $[\alpha]$ to be the collection $\{\tilde{\alpha}_5^\vee, \tilde{\alpha}_6^\vee, \ldots\}$ of the Vassiliev-Taylor series of its restrictions $[\alpha_m] \in H^0(K_m)$.

We note that the Vassiliev system is defined for any knot invariant, even those not of finite type.

Theorem III.6. *Any* $[\alpha] \in \bar{H}^0(\mathcal{K})$ *is completely determined by its Vassiliev system.*

Proof. Any knot invariant is a locally constant function on isotopy classes of knots $k \in H_0(\mathcal{K})$. By Theorem II.13, any $k \in H_0(\mathcal{K})$ has a representative k_m in K_m for some m. The restriction $[\alpha_m] \in H^0(K_m)$ has Vassiliev-Taylor series $\tilde{\alpha}_m^{\vee}$, which in particular contains sufficient information to deduce the value of $[\alpha_m](k_m) = [\alpha](k)$. □

Conceptually, the Vassiliev system decomposes Vassiliev's filtration of knot invariants by degree into a sequence of finite objects which each carry such a filtration. These filtrations are finite at each stage and provide us with a mix of stable and unstable data with which to analyze invariants. However, we have not yet identified the associated graded for these filtered complexes, nor have we investigated how the filtration behaves with respect to the inverse limit.

III.4. The Unstable Vassiliev Spectral Sequence

We now construct analogues of the Vassiliev spectral sequence for the spaces of plumbers' knots. The cell structure on each plumbers' knot space allows us to analyze these sequences explicitly and to identify how the inverse limit of these spectral sequences contains the collection of finite-type invariants.

Definition III.12. Let $E^r_{p',q'}(m)$ be the homology spectral sequence of the complexity filtration on $\tilde{S}_m$ with E^0 page given by $E^0_{p',q'}(m) = (F_{p'}/F_{p'-1})\, C_{q'-p'}(\tilde{S}_m)$ and converging to $H_*(\tilde{S}_m) \cong H_*(S_m)$. The corresponding spectral sequence in cohomology $E_r^{p,q}(m)$ is obtained by reindexing $p = -p', q = (3m-4) - q' + 2p'$, We call this the mth *unstable Vassiliev spectral sequence.*

By Alexander duality, $E_r^{p,q}(m) \implies \bar{H}^*(K_m)$.

A stable cell $\tilde{\mathbf{e}}$ cannot lie in the boundary of any $\tilde{f} \in \textsc{Cell}_{3m-3}(\tilde{S}_m)$ because such cobounding cells only arise internally to $\pi_{\#}^{-1}(\mathbf{e})$, and $\tilde{\mathbf{e}}$ is the cell of highest dimension lying over $\mathbf{e}$. Thus, any cycle in which a stable cell occurs with a non-zero coefficient will represent non-zero homology class.

Lemma III.7. *$(F_p/F_{p-1})\tilde{S}_m$ is homotopy equivalent to a disjoint union of wedges of spheres*

Sketch of proof. Let $\mathbf{e} \in \textsc{Cell}(S_m)$ with be a cell of complexity p. The lift $\pi^{-1}(\mathbf{e})$ is the product of $\mathbf{e}$ with a join of simplices, and applying the filtration quotient identifies some collection of faces of this join to the basepoint. The image of $\tilde{\mathbf{e}}$ under this identification is homotopy equivalent to a wedge of spheres.

If two cells $\mathbf{e}$ and $\mathbf{e}'$ consisting of complexity p curves share a boundary $\mathbf{f}$ whose elements are also curves of complexity p, then $\pi^{-1}(\mathbf{f})$ retracts onto intersection of $\tilde{\mathbf{e}}$ and $\tilde{\mathbf{e}}'$, and their union is, up to homotopy, $\tilde{\mathbf{e}}$. On the other hand, if $\mathbf{f}$ has complexity greater than p, it is not included in $F_p\tilde{S}_m$. □

This data allows us to identify a collection of non-zero cycles in $E_1^{-n,n}(m)$.

Definition III.13. Fix $\tilde{\mathbf{e}} \in \textsc{Cell}_{3m-4}(\tilde{S}_m)$ a stable cell of complexity n. Let $[N(\tilde{\mathbf{e}})] \in E_1^{-n,n}(m)$ be the unique minimal cycle containing $\tilde{\mathbf{e}}$, as guaranteed by Lemma III.7.

Using such cycles we can see that finite type invariants arise in the proper complexity in limit of the unstable spectral sequences.

Theorem III.8. *Let $\mathcal{FT}_n \subseteq H^0(\mathcal{K})$ be the collection of finite-type invariants of type n, then $\mathcal{FT}_n \hookrightarrow \varprojlim E_\infty^{-n,n}(m)$.*

Proof. Let $[\alpha] \in \bar{H}^0(\mathcal{K})$ be an invariant of type n. Fix a representative linear combination of weight systems for $[\alpha]$, a linear combination of chord diagrams $\sum c_i$ with which this representative pairs non-trivially and a collection of singular curves $\Gamma = \{\gamma_i | \gamma_i \text{ respects } c_i\}$.

Choose an integer $m(\Gamma)$ large enough that all of the $\gamma_i \in \Gamma$ are represented in $S_{m(\Gamma)}$ by stable curves lying in cells $\mathbf{e}_i$. This ensures that we can resolve the singularities individually, so all of the topological knot types necessary to apply Lemma III.4 are represented in $K_{m(\Gamma)}$. Write $[\alpha_{m(\Gamma)}] \in H^0(K_{m(\Gamma)})$ for the restriction of $[\alpha]$ to $K_{m(\Gamma)}$. Then $d_{\sum_i \tilde{\mathbf{e}}(\gamma_i)}([\alpha_{m(\Gamma)}]) = \sum_i \langle [\alpha], c_i \rangle \neq 0$ and so the class $\sum_i \langle \alpha, c_i \rangle [N(\tilde{\mathbf{e}}_i)] \in E_\infty^{-n,n}(m)$ is nontrivial.

Applying the universal property of the inverse limit, we see that $[\alpha]$ maps non-trivially to $\lim E_\infty^{-n,n}$. □

REFERENCES

[1] S. Baldridge and A. Lowrance, *Cube diagrams and a homology theory for knots*, arXiv:0811.0225v1 [math.GT], 2008.

[2] Dror Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995), no. 2, 423–472. MR MR1318886 (97d:57004)

[3] T. Biedl, E. Demaine, M. Demaine, and et al., *Locked and unlocked polygonal chains in three dimensions*, Discrete Comput. Geom. **26** (2001), no. 3, 269–281. MR MR1854101 (2002h:52018)

[4] Joan S. Birman and Xiao-Song Lin, *Knot polynomials and Vassiliev's invariants*, Invent. Math. **111** (1993), no. 2, 225–270. MR MR1198809 (94d:57010)

[5] Jorge Alberto Calvo, *Geometric knot spaces and polygonal isotopy*, J. Knot Theory Ramifications **10** (2001), no. 2, 245–267, Knots in Hellas '98, Vol. 2 (Delphi). MR MR1822491 (2002b:57002)

[6] Jason Cantarella and Heather Johnston, *Nontrivial embeddings of polygonal intervals and unknots in 3-space*, J. Knot Theory Ramifications **7** (1998), no. 8, 1027–1039. MR MR1671500 (99m:57002)

[7] D. Dugger, *A primer on homotopy colimits*, http://math.uoregon.edu/ ddugger/hocolim.pdf, 2008.

[8] E. J. Janse van Rensburg and S. D. Promislow, *The curvature of lattice knots*, J. Knot Theory Ramifications **8** (1999), no. 4, 463–490. MR MR1697384 (2000i:57009)

[9] Ciprian Manolescu, Peter Ozsváth, Zoltán Szabó, and Dylan Thurston, *On combinatorial link Floer homology*, Geom. Topol. **11** (2007), 2339–2412. MR MR2372850

[10] Richard Randell, *Invariants of piecewise-linear knots*, Knot theory (Warsaw, 1995), Banach Center Publ., vol. 42, Polish Acad. Sci., Warsaw, 1998, pp. 307–319. MR MR1634464 (99c:57031)

[11] ______, *The space of piecewise-linear knots*, Physical knots: knotting, linking, and folding geometric objects in $\mathbb{R}^3$ (Las Vegas, NV, 2001), Contemp. Math., vol. 304, Amer. Math. Soc., Providence, RI, 2002, pp. 31–36. MR MR1953009 (2003m:55026)

[12] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of singularities and its applications, Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 23–69. MR MR1089670 (92a:57016)

[13] ______, *Topology of two-connected graphs and homology of spaces of knots*, Differential and symplectic topology of knots and curves, Amer. Math. Soc. Transl. Ser. 2, vol. 190, Amer. Math. Soc., Providence, RI, 1999, pp. 253–286. MR MR1738399 (2001b:55030)