

NON-EXISTENCE OF A
STABLE HOMOTOPY CATEGORY
FOR P-COMPLETE ABELIAN GROUPS

by

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We investigate the existence of a stable homotopy category (SHC) associated to the category of p -complete abelian groups $\mathcal{A}b_p^\wedge$. First we examine $\mathcal{A}b_p^\wedge$ and prove $\mathcal{A}b_p^\wedge$ satisfies all but one of the axioms of an abelian category. The connections between an SHC and homology functors are then exploited to draw conclusions about possible SHC structures for $\mathcal{A}b_p^\wedge$. In particular, let $\mathcal{K}(\mathcal{A}b_p^\wedge)$ denote the category whose objects are chain complexes of $\mathcal{A}b_p^\wedge$ and morphisms are chain homotopy classes of maps. We show that any homology functor from any subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ containing the p -adic integers and satisfying the axioms of an SHC will not agree with standard homology on free, finitely generated (as modules over the p -adic integers) chain complexes. Explicit examples of common functors are included to highlight troubles that arise when working with $\mathcal{A}b_p^\wedge$. We make some first attempts at classifying small objects in $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

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CHAPTER I

INTRODUCTION

A structure in algebraic topology noticed and identified by Puppe [13] and Verdier [16] is that of a triangulated category. Examples include such categories as the derived category of a commutative ring and the category of spectra. These triangulated categories often support additional structures such as monoidal products and function objects. A stable homotopy category (SHC) is an axiomatic generalization of these examples but the axioms have yet to be universally agreed upon [14, 3]. By an SHC we will follow the convention in [4] and mean a triangulated category with a compatible monoidal product and function objects together with a set of weak generators that satisfy Spanier-Whitehead duality. One key characteristic of an SHC is that morphisms sent to isomorphisms by all homology functors are equivalences. Call a morphism f , a quasi-isomorphism if all homology functors send f to an isomorphism. With this language in an SHC, all quasi-isomorphisms are equivalences.

Denote the category of p -complete abelian groups by $\mathcal{A}b_p^\wedge$ and the p -adic integers by Z_p . Let $\mathcal{K}(\mathcal{A}b_p^\wedge)$ be the triangulated category whose objects are chain complexes of p -complete abelian groups and morphisms that are chain homotopy classes of maps. We show any homology functor from any subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ containing the p -adic integers and satisfying the axioms of an SHC will not act like ordinary homology on free, finitely generated (as Z_p -modules) chain complexes. In other words, if there is an SHC for $\mathcal{A}b_p^\wedge$, it does not admit an extension of standard homology.

We begin by explaining our interest in this category and then justify why the standard construction given in [4] does not work for $\mathcal{A}b_p^\wedge$. We give additional properties for $\mathcal{A}b_p^\wedge$ in Chapter II before giving the exact definitions for an SHC in Chapter III. Chapter IV contains the precise statements of the non-existence results for an SHC of $\mathcal{A}b_p^\wedge$.

I.1 Motivation

The motivation for this work comes from searching for an analogue of the derived category for the category known as “Morava modules”. The Morava module of a spectrum X is the input to the Adams-Novikov spectral sequence that computes $\pi_*(L_{K(n)}(X))$ [2]. It is possible that similar techniques used in studying the derived category of modules over the Steenrod Algebra (see [7], [11], [10], and [12]) could be used to study $L_{K(n)}X$ if we had a derived category of Morava modules. Like $\mathcal{A}b_p^\wedge$, the objects in the category of Morava modules are complete with respect to the ideal of a complete local ring. We believe a reasonable SHC structure for $\mathcal{A}b_p^\wedge$ will provide guidance about how to make an SHC for Morava modules and conversely obstructions to a reasonable structure for $\mathcal{A}b_p^\wedge$ will also lead to obstructions to making an SHC for Morava modules.

I.2 Examples and Construction of an SHC

We outline a construction of an SHC that works for the category of R -modules (where R is a unital commutative ring) and highlight how this construction is often used in a more general setting. It is known that the SHC of R -modules is the derived category of R , denoted $\mathcal{D}(R)$. That is, $\mathcal{D}(R)$ is a triangulated category with a compatible monoidal product where all quasi-isomorphisms are equivalences and such that a suitable approximation for R -modules always exists. We present the construction given in [4] that satisfies this definition, though the construction differs significantly from the presentation given in [17].

We begin by creating a triangulated structure for R -modules. Let $Ch(R)$ be a category whose objects are unbounded chain complexes and morphisms are sequences of degree preserving R -module morphisms that commute with the differentials. The quotient category $\mathcal{K}(R)$ has the same objects but morphisms are chain homotopy classes of maps from $Ch(R)$. Define the functor $\Sigma : \mathcal{K}(R) \rightarrow \mathcal{K}(R)$ degree wise by $(\Sigma X)_n = X_{n-1}$ with differentials $(d^{\Sigma X})_n = (-1)d_{n-1}^X$. Let $\Sigma^i X$ be Σ applied i times to X and let $R(i)$ denote the chain complex with the regular module R in the i^{th} degree and 0's in all others. $\mathcal{K}(R)$ is a triangulated category where the cofiber of a morphism is given by the standard cone construction. Furthermore, the tensor product over R induces a canonical monoidal structure on $\mathcal{K}(R)$ that is compatible with the triangulation.

One definition of the derived category is the full subcategory of $\mathcal{K}(R)$ whose objects are all X with a filtration $X^0 \subseteq X^1 \subseteq X^2 \subseteq \dots$ such that $X^n/X^{n-1} \cong \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus R(i) \right)$ for all n (see [8] for details). We call objects that are colimits of sequences of the above form $R(0)$ -cellular, thus $\mathcal{D}(R)$ is the full subcategory of $\mathcal{K}(R)$ whose objects are $R(0)$ -cellular.

May shows concretely in [8] that this construction of $\mathcal{D}(R)$ implies all quasi-isomorphisms are equivalences and a suitable approximation for a given chain complex from $\mathcal{K}(R)$ exists.

The construction of the triangulated category $\mathcal{K}(R)$ generalizes to any additive category \mathcal{A} . We will denote this by $\mathcal{K}(\mathcal{A})$. Furthermore, if \mathcal{A} has a monoidal structure with compatible products and coproducts, a canonical monoidal structure can be built for $\mathcal{K}(\mathcal{A})$. For a suitably nice category \mathcal{A} (explained further in Example III.3.3), the canonical monoidal structure for $\mathcal{K}(\mathcal{A})$ can be compatible with the triangulation.

To continue the parallel construction of $\mathcal{D}(R)$ in the more general case we need a few definitions.

Definition I.2.1. *A set \mathcal{G} of objects in \mathcal{C} is a set of weak generators if $\text{Hom}_{\mathcal{C}}(G, A) = 0$ for all $G \in \mathcal{G}$ implies $A \cong 0$ in \mathcal{C} .*

Note that the definition of a weak generator differs from the categorical definition of a generator. Recall an object Z is a generator of a category \mathcal{C} if given $F, G \in \text{Hom}_{\mathcal{C}}(X, Y)$ with $F \neq G$, there exists an $H \in \text{Hom}_{\mathcal{C}}(Z, X)$ so that $F \circ H \neq G \circ H$. A generator in an additive category in the sense of [17] can be shown to be a weak generator. We do not know what minimal structures a category must have for a weak generator to also qualify as a generator.

Definition I.2.2. *An object A is small in an additive category \mathcal{C} if the natural map*

$$\bigoplus_{i \in I} \text{Hom}_{\mathcal{C}}(A, B^i) \rightarrow \text{Hom}_{\mathcal{C}}(A, \coprod_{i \in I} B^i)$$

is an isomorphism for all coproducts in \mathcal{C} .

Notice in R -modules the regular module is both small and a weak generator.

The following theorem is given in [4] and is the analogue to defining $\mathcal{D}(R)$ as the collection of $R(0)$ -cellular objects in $\mathcal{K}(R)$.

Theorem I.2.3. *Suppose \mathcal{C} is an SHC with a set of small weak generators \mathcal{G} . Then every object X can be written as a sequential colimit of a sequence $0 = X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \dots$ in which the*

cofiber of each map $X^k \rightarrow X^{k+1}$ is equivalent to a coproduct of objects of the form $\Sigma^i G$ for $i \in \mathbb{Z}$ or $G \in \mathcal{G}$.

We call objects that are colimits of sequences of the above form \mathcal{G} -cellular. Notice in $\mathcal{D}(R)$ the collection of $R(i)$, where $i \in \mathbb{Z}$, forms a set of small weak generators. The definition of $\mathcal{D}(R)$ as the full subcategory of $\mathcal{K}(R)$ whose objects are $R(0)$ -cellular is thus consistent with the above theorem.

I.3 Failure of a Standard Construction for an SHC for $\mathcal{A}b_p^\wedge$

Denote the category of abelian groups by $\mathcal{A}b$ and define the category of p -complete abelian groups to be the full subcategory of $\mathcal{A}b$ with objects A such that $A \cong \varprojlim A/p^i A$. A standard example of an object in $\mathcal{A}b_p^\wedge$ is the p -adics integers, denoted by Z_p .

It will be shown in Chapter II that $\mathcal{A}b_p^\wedge$ is an additive category with a monoidal structure. We can thus use the construction from Section I.2 to create the triangulated category with a compatible monoidal structure denoted by $\mathcal{K}(\mathcal{A}b_p^\wedge)$. The remaining steps in the definition of the derived category given in Section I.2 require a choice of small weak generators \mathcal{G} . The full subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ consisting of \mathcal{G} -cellular objects, denoted by $\mathcal{C}(\mathcal{G})$, could be considered and perhaps May's arguments (that showed $\mathcal{D}(R)$ satisfied the conditions of an SHC) could be applied.

If the parallel to $\mathcal{D}(R)$ continued, a reasonable set to consider for \mathcal{G} will be the set of $Z_p(i)$ where $i \in \mathbb{Z}$. Let $\mathcal{C}(Z_p)$ denote the full subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ consisting of $Z_p(0)$ -cellular objects and $H_n(-)$ denote the standard homology functor on chain complexes. For $\mathcal{C}(Z_p)$ to be an SHC, the quasi-isomorphisms in $\mathcal{C}(Z_p)$ must be equivalences and there must exist an object in $\mathcal{C}(Z_p)$ that is a suitable approximation for a given chain complex in $\mathcal{K}(\mathcal{A}b_p^\wedge)$. To show the second of these conditions for $\mathcal{D}(R)$, May constructed complexes $A_i \in \text{ob}(\mathcal{D}(R))$ and used the natural isomorphism $H_n\left(\bigoplus_i A_i\right) \cong \bigoplus_i H_n(A_i)$ to verify $\varinjlim A_i$ was a suitable approximation. An example will be provided in Section IV.1 to show that there exists objects X^i from $\mathcal{K}(\mathcal{A}b_p^\wedge)$ so that

$$H_n\left(\prod_{i=0}^{\infty} X^i\right) \not\cong \bigoplus_{i=0}^{\infty} H_n(X^i).$$

Thus the techniques employed by May to show an object exists in $\mathcal{D}(R)$ that approximates a given chain complex in $\mathcal{K}(R)$ cannot be directly applied to $\mathcal{C}(Z_p)$.

I.4 Non-existence of an SHC for $\mathcal{A}b_p^\wedge$ that Respects $H_n(-)$

We search for a subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ that contains $Z_p(0)$, satisfies the axioms of an SHC, and has equivalences determined by a functor that:

1. acts like ordinary homology on free, finitely generated (as Z_p -modules) chain complexes, and
2. satisfies the axioms for a homology functor on an SHC.

We make the following definition that will be stated in a more general setting in Section III.4.

Definition I.4.1. *Let \mathcal{A} be an additive category with kernels and cokernels. A homology functor on $\mathcal{K}(\mathcal{A}b_p^\wedge)$ is a functor $\mathcal{H} : \mathcal{K}(\mathcal{A}b_p^\wedge) \rightarrow \mathcal{A}$ such that:*

1. *sequences of the form $X \xrightarrow{W} Y \rightarrow \text{Cone}(W)$ induce the following long exact sequence*

$$\cdots \rightarrow \mathcal{H}_{n+1}(\text{Cone}(W)) \rightarrow \mathcal{H}_n(X) \xrightarrow{\mathcal{H}(W)} \mathcal{H}_n(Y) \rightarrow \mathcal{H}_n(\text{Cone}(W)) \rightarrow \mathcal{H}_{n-1}(X) \rightarrow \cdots$$

and

2. *\mathcal{H} preserves coproducts.*

Let $H_n(-)$ denote the standard homology functor on chain complexes to $\mathcal{A}b$. The example referenced in Section I.3 that there exists objects X^i from $\mathcal{K}(\mathcal{A}b_p^\wedge)$ with $H_n\left(\prod_{i=0}^{\infty} X^i\right) \not\cong \bigoplus_{i=0}^{\infty} H_n(X^i)$ implies standard homology does *not* satisfy our conditions to be a homology functor on $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

If we require a homology functor $\mathcal{H}_*(-) : \mathcal{K}(\mathcal{A}b_p^\wedge) \rightarrow \mathcal{A}b$ to return the same groups as $H_n(-)$ on free, finitely generated chain complexes we have the following negative result.

Theorem I.4.2. *There exists no homology functor $\mathcal{H} : \mathcal{K}(\mathcal{A}b_p^\wedge) \rightarrow \mathcal{A}b$ satisfying $\mathcal{H}_n(X) = H_n(X)$ when X is a chain complex that is finitely generated and free as a Z_p -module.*

The above theorem is restated in a stronger form and proved as Theorem IV.2.2. The following main non-existence result follows as a corollary:

Theorem I.4.3. *Let \mathcal{D} be a subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ containing $Z_p(0)$ and satisfying the axioms of an SHC. There is no homology functor $\mathcal{H}_*(-) : \mathcal{D} \rightarrow \mathcal{A}b$ that satisfies $\mathcal{H}_n(X) \cong H_n(X)$ when X is a chain complex that is finitely generated and free as a Z_p -module.*

Results similar to Theorem I.4.2 exist for a homology functor with $\mathcal{A}b_p^\wedge$ as the target category.

Theorem I.4.4. *There exists no homology functor $\mathcal{H} : \mathcal{K}(\mathcal{A}b_p^\wedge) \rightarrow \mathcal{A}b_p^\wedge$ satisfying $\mathcal{H}_n(X) = H_n(X)$ when X is a chain complex that is finitely generated and free as a Z_p -module.*

A stronger form of Theorem I.4.4 is stated and proved in Section IV.2 and implies a second non-existence result.

Theorem I.4.5. *Let \mathcal{D} be a subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ containing $Z_p(0)$ and satisfying the axioms of an SHC. There is no homology functor $\mathcal{H}_*(-) : \mathcal{D} \rightarrow \mathcal{A}b_p^\wedge$ that satisfies $\mathcal{H}_n(X) \cong H_n(X)$ when X is a chain complex that is finitely generated and free as a Z_p -module.*

CHAPTER II

PROPERTIES OF $\mathcal{A}b_p^\wedge$

To familiarize the reader with $\mathcal{A}b_p^\wedge$ a few useful properties and constructions will be introduced. We will see that $\mathcal{A}b_p^\wedge$ has an additive category structure, but is not an abelian category. The failure is that not all monomorphisms are kernels. Before we provide an example of this we need some basic constructions.

Recall $\mathcal{A}b_p^\wedge$ is defined as the full subcategory of $\mathcal{A}b$ with objects A such that $A \cong \varprojlim A/p^i A$. We refer to $\varprojlim A/p^i A$ as the p -completion of A and note that a p -complete group is invariant under p -completion. The construction of the inverse limit lets us write the elements more explicitly as $\{(\dots a_3, a_2, a_1) \in \prod A/(p^i A) \mid \alpha_i(a_i) = a_{i-1}\}$ where $\alpha_i : A/p^i A \rightarrow A/p^{i-1} A$ is induced by the universal property of quotients. The following result is trivial but plays a key role in a number of the proofs that will follow.

Lemma II.0.6. *A morphism $f \in \text{Hom}_{\mathcal{A}b}(A, B)$ induces maps $f_j \in \text{Hom}_{\mathcal{A}b}(A/p^j, B/p^j)$ so that the following diagram commutes.*

$$\begin{array}{ccccc}
 & & & B & \\
 & & & \swarrow & \searrow \pi_j^B \\
 & & A & \xrightarrow{f} & B \\
 & \swarrow \pi_{j+1}^A & & & \\
 & A & \xrightarrow{\beta_{j+1}} & \frac{B}{p^{j+1}B} & \xrightarrow{\beta_{j+1}} & \frac{B}{p^j B} \\
 & \swarrow & & & & \\
 \frac{A}{p^{j+1}A} & \xrightarrow{\alpha_{j+1}} & \frac{A}{p^j A} & \xrightarrow{f_j} & \frac{B}{p^j B}
 \end{array}$$

The morphisms α_j and β_j are induced by the universal property of quotients whereas π_j^A and π_j^B are the natural projections.

We pause to state some immediate corollaries to Lemma II.0.6. The first can be summarized by:

$$\mathrm{Hom}_{\mathcal{A}b_p^\wedge}(A, B) = \mathrm{Hom}_{\mathcal{A}b}(A, B) \cong \mathrm{Hom}_{Z_p\text{-mod}}(A, B)$$

when A and B are p -complete abelian groups. The p -adic numbers \mathbb{Q}_p is an example of a Z_p -module that is not p -complete and $\mathbb{Q}_p \cong \mathrm{Hom}_{Z_p\text{-mod}}(\mathbb{Q}_p, \mathbb{Q}_p) \subsetneq \mathrm{Hom}_{\mathcal{A}b}(\mathbb{Q}_p, \mathbb{Q}_p)$.

A second consequence of Lemma II.0.6 is that p -completion is a functor from $\mathcal{A}b$ to $\mathcal{A}b_p^\wedge$. Define $(-)_p^\wedge : \mathcal{A}b \rightarrow \mathcal{A}b_p^\wedge$ to take objects to their p completions. Given $f \in \mathrm{Hom}_{\mathcal{A}b}(A, B)$, Lemma II.0.6 implies there exists a collection of $f_i \hat{\pi}_i^A : A_p^\wedge \rightarrow B/p^i B$ where $\hat{\pi}_i^A : A_p^\wedge \rightarrow A/p^i A$. Since B_p^\wedge is p -complete, the universal property provides a unique map from A_p^\wedge to B_p^\wedge denoted by f_p^\wedge .

Remark II.0.7. Let $A \in \mathrm{ob}(\mathcal{A}b_p^\wedge)$ and $B \in \mathrm{ob}(\mathcal{A}b)$. Let $\alpha_i : A/p^i A \rightarrow A/p^{i-1} A$ be the natural map and $f_i : B \rightarrow A/p^i A$ be surjections with $\alpha_i \circ f_i = f_{i-1}$ for all i . Then the image of the map $f : B \rightarrow A$ (induced by the universal property of \varprojlim) is dense in A under the subspace topology [18]. Moreover, given an element $a \in A$, the following shows how to collect $b_i \in B$ and $a'_i \in A$ so that $a - f(b_i) = p^i a'_i$.

Since A has no elements divisible by p^k for all k we reduce to the case when $p \nmid a$. Write a as $(\dots a_3, a_2, a_1) \in \prod_i A/p^i A$. Since f_i is onto, choose a lift $b_i \in B$ so that $f_i(b_i) = a_i$.

Let $\pi_k^A : A \rightarrow A/p^k A$ be the natural projection. Then we consider $a - f(b_k)$ in A . Notice

$$\pi_k^A(a - f(b_k)) = 0 \in A/p^k A.$$

So there exists $a'_k \in A$ with $a - f(b_k) = p^k a'_k$.

Remark II.0.8. A category that will be of some use is the full subcategory of groups whose objects are inverse limits of finite groups. Call a group A *profinite* if $A \cong \varprojlim A_i$ where A_i is a finite group for all i [18]. Notice Z_p is a profinite group but $\prod_{i=0}^\infty Z_p$ is not. In particular, if A is p -complete and finitely generated as a Z_p -module, then A is profinite.

II.1 The p -adic Metric on $\mathrm{ob}(\mathcal{A}b_p^\wedge)$

Let a be a nonzero element of $A \in \mathrm{ob}(\mathcal{A}b_p^\wedge)$. Since A is p -complete, for all nonzero $a \in A$ there exists a well defined $i : A \rightarrow \mathbb{N}$ so that $\pi_{i(a)}^A a = 0$ and $\pi_{i(a)+1}^A a \neq 0$ where $\pi_k^A : A \rightarrow A/p^k A$.

Define $|\cdot|_p : A \rightarrow [0, 1] \subset \mathbb{R}$ by

$$|a|_p = \begin{cases} \frac{1}{p^{i(a)}} & \text{if } a \neq 0 \\ 0 & \text{else.} \end{cases}$$

The definition implies the following for all a and a' in A :

1. $|a|_p = 0$ if and only if $a = 0$.
2. For $r \in Z_p$, $|ra|_p \leq |r|_p |a|_p$.
3. $|a + a'|_p \leq |a|_p + |a'|_p$.

This function will be referred to as the p -adic metric and we adopt the same language associated with a normed space when appropriate. For example, if a series $\sum a_n$ has the property that $\sum |a_n|_p$ converges, we say $\sum a_n$ converges absolutely in the p -adic norm. Standard arguments show if $\sum a_n$ converges absolutely in the p -adic norm to a sum a , then any rearrangement of $\sum a_n$ converges absolutely in the p -adic norm to a as well.

The existence of this function endows objects of $\mathcal{A}b_p^\wedge$ with a topology. We can state one more consequence to Lemma II.0.6:

$$\text{Hom}_{\mathcal{A}b_p^\wedge}(A, B) = \text{Hom}_{\mathcal{A}b}(A, B) \subsetneq \text{Map}_{\mathcal{A}b_p^\wedge}(A, B)$$

where $\text{Map}_{\mathcal{A}b_p^\wedge}(A, B)$ denotes the set of continuous maps from A to B . The containment is strict and we provide an example of a continuous map that is not a group homomorphism. Let $yp^k \in Z_p$,

$$\text{where } p \nmid y, \text{ then define } f : Z_p \rightarrow Z_p \text{ by } f(yp^k) = \begin{cases} yp^{k-1} & \text{if } k > 0 \\ y & \text{if } k = 0 \end{cases}.$$

II.2 Cokernels and Coproducts in $\mathcal{A}b_p^\wedge$

The existence of cokernels and coproducts will follow easily in $\mathcal{A}b_p^\wedge$ once we show that the p -completion functor is left adjoint to the forgetful functor. Let $\mathcal{F} : \mathcal{A}b_p^\wedge \rightarrow \mathcal{A}b$ be the forgetful functor.

Claim II.2.1. *The functor, $(-)_p^\wedge$ is left adjoint to the forgetful functor \mathcal{F} .*

Proof. Let $\phi : A \rightarrow A_p^\wedge$ be the natural map induced by the p -completion and let $\psi : \text{Hom}_{\mathcal{A}b}(A, \mathcal{F}B) \rightarrow \text{Hom}_{\mathcal{A}b_p^\wedge}(A_p^\wedge, B)$ send a morphism to its p -completion. The adjoint isomorphism will be given by ϕ^* and ψ .

We use the same notation that was used in Lemma II.0.6 and write A_p^\wedge as

$$\varprojlim A/p^i A = \{\bar{a} \in \prod A/(p^i A) \mid \alpha_i(a_i) = a_{i-1}\}.$$

Then $\phi(a) = (\dots \pi_3^A a, \pi_2^A a, \pi_1^A a)$. To understand the map ψ let $f \in \text{Hom}_{\mathcal{A}b}(A, \mathcal{F}B)$ and $\hat{\pi}_i^A : A_p^\wedge \rightarrow A/p^i A$. Then $\psi f(\dots a_3, a_2, a_1) = \varprojlim f_i \hat{\pi}_i^A(\dots a_3, a_2, a_1) = (\dots f_3(a_3), f_2(a_2), f_1(a_1))$. The commutativity conditions given in Lemma II.0.6 imply $\phi^* \circ \psi$ is the identity. We calculate $\psi \circ \phi^*$ explicitly.

$$\begin{aligned} \psi(\phi^*(f))(\dots a_3, a_2, a_1) &= (\psi(f \circ \phi))(\dots a_3, a_2, a_1) \\ &= (\varprojlim (f \circ \phi)_i \circ \hat{\pi}_i^A)(\dots a_3, a_2, a_1) \\ &= (\dots (f \circ \phi)_3 a_3, (f \circ \phi)_2 a_2, (f \circ \phi)_1 a_1) \end{aligned}$$

We show that for all i , $(f \circ \phi)_i = f_i$. Let $a_i \in A/p^i A$. Notice A is onto $A/p^i A$ so we can choose $\tilde{a} \in A$ so that $\pi_i^A(\tilde{a}) = a_i$. This implies:

$$\begin{aligned} (f \circ \phi)_i(a_i) &= (f \circ \phi)_i \circ \pi_i^A(\tilde{a}) = \pi_i^B \circ (f \circ \phi)(\tilde{a}) = \pi_i^B \circ f(\dots \pi_2^A(\tilde{a}), \pi_1^A(\tilde{a})) \\ &= \pi_i^B(\dots f_2 \pi_2^A(\tilde{a}), f_1 \pi_1^A(\tilde{a})) = f_i \pi_i^A(\tilde{a}) = f_i(a_i) \end{aligned} \quad \square$$

Since $(-)_p^\wedge$ is left adjoint, the p -completion of cokernels and coproducts taken in $\mathcal{A}b$ will satisfy the universal properties necessary in $\mathcal{A}b_p^\wedge$ [6]. Thus we make the following definitions:

Definition II.2.2. *Let $f \in \text{Hom}_{\mathcal{A}b_p^\wedge}(A, B) = \text{Hom}_{\mathcal{A}b}(A, B)$. The cokernel of f in $\mathcal{A}b_p^\wedge$, is*

$$B \rightarrow (\text{coker}_{\mathcal{A}b} f)_p^\wedge.$$

Definition II.2.3. *Let I be an indexing set and $A_i \in \text{ob}(\mathcal{A}b_p^\wedge)$ for all $i \in I$. The coproduct in $\mathcal{A}b_p^\wedge$ is*

$$\prod_{i \in I} A_i = \left(\bigoplus_{i \in I} A_i \right)_p^\wedge.$$

We reserve the symbol \bigoplus to denote coproducts taken in $\mathcal{A}b$.

Before considering other properties we examine the effect p -completion has when building coproducts in Ab_p^\wedge . For example, consider $\coprod_{\mathbb{N}} Z_p = (\bigoplus_{\mathbb{N}} Z_p)_p^\wedge$. Elements of $\coprod_{\mathbb{N}} Z_p$ can be treated as countable tuples, (z_1, z_2, z_3, \dots) from $\prod_{\mathbb{N}} Z_p$ (where we are taking the product in Ab) with the additional constraint $|z_i|_p \rightarrow 0$ as $i \rightarrow \infty$. Unlike coproducts in Ab , there exist elements in $\coprod_{\mathbb{N}} Z_p$ that have an infinite number of nonzero entries. Some examples of elements of this form are (p, p^2, p^3, \dots) , (p, p, p^2, p^2, \dots) , and $(p, 2p^3, 3p^5, 4p^7, \dots)$.

II.3 Kernels and Products in Ab_p^\wedge

We first consider products in Ab_p^\wedge .

Claim II.3.1. *Let I be an indexing set and $A_i \in \text{ob}(Ab_p^\wedge)$. $\prod_I A_i = \{(a_1, a_2, \dots) \mid a_i \in A_i\}$. is p -complete.*

Proof. Let

$$\alpha_j^{A_i} : \frac{A_i}{p^j A_i} \rightarrow \frac{A_i}{p^{j-1} A_i} \quad \text{and} \quad \alpha_j^{\prod_I A_i} : \frac{\prod_I A_i}{p^j (\prod_I A_i)} \rightarrow \frac{\prod_I A_i}{p^{j-1} (\prod_I A_i)}$$

be the natural projections.

By construction

$$\left(\prod_I A_i \right)_p^\wedge = \left\{ (\dots \bar{a}_3, \bar{a}_2, \bar{a}_1) \mid \bar{a}_j \in \frac{\prod_I A_i}{p^j \prod_I A_i} \text{ and } \alpha_j^{\prod_I A_i}(\bar{a}_j) = \bar{a}_{j-1} \right\}.$$

Since $p^j \prod_I A_i \cong \prod_I p^j A_i$ we can rewrite the above as:

$$\left(\prod_I A_i \right)_p^\wedge = \left\{ (\dots \bar{a}_3, \bar{a}_2, \bar{a}_1) \mid \bar{a}_j \in \prod_I \frac{A_i}{p^j A_i} \text{ and } \prod_I \alpha_j^{A_i}(\bar{a}_j) = \bar{a}_{j-1} \right\}$$

The coherent sequences of vectors in $\prod_I \frac{A_i}{p^j A_i}$ can be reindexed as vectors of coherent sequences from A_i , thus $\left(\prod_I A_i \right)_p^\wedge \cong \prod_I A_i$. \square

The construction of $\prod_I A_i$ for $A_i \in \text{ob}(Ab_p^\wedge)$ can be used to show $\prod_I A_i$ satisfies the universal property of products in Ab_p^\wedge , thus we make the following definition.

Definition II.3.2. *Let I be an indexing set and $A_i \in \text{ob}(Ab_p^\wedge)$ for all i . The product in Ab_p^\wedge is $\prod_{i \in I} A_i$, that is, the set of infinite tuples where the i^{th} factor is an element from A_i .*

Remark II.3.3. The category $\mathcal{A}b_{\hat{p}}$ forms an additive category in the sense of [17]. A consequence is that finite coproducts are naturally isomorphic to finite products and vice versa.

We work towards a definition for kernels in $\mathcal{A}b_{\hat{p}}$ and prove the following claim:

Claim II.3.4. *Given $f \in \text{Hom}_{\mathcal{A}b_{\hat{p}}}(A, B)$, $\ker_{\mathcal{A}b} f = \{a \in A \mid f(a) = 0\} \in \text{ob}(\mathcal{A}b_{\hat{p}})$.*

Proof. Use the same notation set up in Lemma II.0.6 and set $\phi : \ker_{\mathcal{A}b} f \rightarrow (\ker_{\mathcal{A}b} f)_{\hat{p}}$ with natural projections $\phi_k : \ker_{\mathcal{A}b} f \rightarrow \ker_{\mathcal{A}b} f / p^k \ker_{\mathcal{A}b} f$. We will show ϕ is injective and surjective.

The kernel of ϕ is $\bigcap_{k=1}^{\infty} p^k \ker_{\mathcal{A}b} f$. Since A is p -complete, $\bigcap_{k=1}^{\infty} p^k A = 0$, so the containment $p^k \ker_{\mathcal{A}b} f \subset p^k A$ implies no elements in $\ker_{\mathcal{A}b} f$ are sent to zero.

To see surjectivity let $a = (\dots a_3, a_2, a_1) \in (\ker_{\mathcal{A}b} f)_{\hat{p}}$. Lemma II.0.6 implies f is continuous, so $\ker_{\mathcal{A}b} f$ is closed in A and it contains its limit points. We will construct a sequence in $\ker_{\mathcal{A}b} f$ with a limit that maps to a by ϕ .

Choose lifts $\tilde{a}_i \in \ker_{\mathcal{A}b} f$ so that $\phi_i(\tilde{a}_i) = a_i$. Notice $p^i | (\tilde{a}_{i+1} - \tilde{a}_i)$ in $\ker_{\mathcal{A}b} f$ so $p^i | (\tilde{a}_{i+1} - \tilde{a}_i)$ in A . The sequence of \tilde{a}_i is thus Cauchy in A and since A is p -complete it converges to an element \tilde{a} . Since $\ker_{\mathcal{A}b} f$ is closed in A , $\tilde{a} \in \ker_{\mathcal{A}b} f$ and by construction $\phi(\tilde{a}) = a$. \square

Since $\mathcal{A}b_{\hat{p}}$ is a full subcategory of $\mathcal{A}b$ and $\ker_{\mathcal{A}b} f \in \text{ob}(\mathcal{A}b_{\hat{p}})$, $\ker_{\mathcal{A}b} f$ satisfies the universal property of kernels in $\mathcal{A}b_{\hat{p}}$. Thus we can define kernels in $\mathcal{A}b_{\hat{p}}$.

Definition II.3.5. *Let $f \in \text{Hom}_{\mathcal{A}b_{\hat{p}}}(A, B)$. The kernel of f in $\mathcal{A}b_{\hat{p}}$, denoted $\ker_{\mathcal{A}b_{\hat{p}}} f$, is*

$$\ker_{\mathcal{A}b_{\hat{p}}} f = \ker_{\mathcal{A}b} f = \{a \in A \mid f(a) = 0\}.$$

Since $\ker_{\mathcal{A}b} f = \ker_{\mathcal{A}b_{\hat{p}}} f$ when $f \in \text{Hom}_{\mathcal{A}b_{\hat{p}}}(A, B)$, there is no ambiguity if we write $\ker f$. Be aware that the p -adic topology of $\ker f$ may not be the same as the subspace topology induced by A as the following example shows.

Example II.3.6. Consider the kernel of the the following map:

$$\prod_{i \in \mathbb{N}} \mathbb{Z}_p \xrightarrow{\pi} \prod_{i \in \mathbb{N}} \mathbb{Z}/p^i$$

where π is the natural map sending the i th entry to the equivalence class in \mathbb{Z}/p^i .

The kernel of π is $\prod_{i \in \mathbb{N}} p^i Z_p \subset \prod_{i \in \mathbb{N}} Z_p$. Consider the set of elements:

$$\{(p, 0, 0, \dots), (0, p^2, 0, \dots), (0, 0, p^3, \dots), \dots\}.$$

As a subset of $\prod_{i \in \mathbb{N}} Z_p$, these elements form a Cauchy sequence with the p -adic topology and converge to zero. However, when considered as a subset of the kernel $\prod_{i \in \mathbb{N}} p^i Z_p$, the sequence does not converge in the p -adic topology.

II.4 Images in Ab_p^\wedge

We now consider images in Ab_p^\wedge .

Definition II.4.1. Let $g \in \text{Hom}_{Ab_p^\wedge}(A, B)$. Define the image of g in Ab_p^\wedge as done in [15] by $\text{im}_{Ab_p^\wedge}(g) = \ker(B \rightarrow \text{coker}_{Ab_p^\wedge}(g))$.

Claim II.4.2. Let $g \in \text{Hom}_{Ab_p^\wedge}(A, B)$, then $\text{im}_{Ab_p^\wedge}(g) \cong \overline{\text{im}_{Ab}(g)}^B$ where $\overline{\text{im}_{Ab}(g)}^B$ denotes the closure of $\text{im}_{Ab}(g)$ taken as a subspace of B .

Proof. The proof checks double inclusion and follows from the definitions.

Let $b \in \overline{\text{im}_{Ab}(g)}^B \subset B$. There exist elements $b'_i \in \text{im}_{Ab}(g)$ and $b_i \in B$ such that $b = b'_i + p^i b_i$. Since $\text{im}_{Ab_p^\wedge}(g) = \ker \hat{\pi}$, where $\hat{\pi} : B \rightarrow \text{coker}_{Ab_p^\wedge}(g)$, we only have to show $\hat{\pi}(b) = 0$ in $\text{coker}_{Ab_p^\wedge}(g) = (B/\text{im}_{Ab}g)_p^\wedge$. Notice $\hat{\pi}(b) = \hat{\pi}(b'_i + p^i b_i) = p^i \hat{\pi}(b_i)$, thus we have that $p^i | \hat{\pi}(b)$ for all i . However, $\text{coker}_{Ab_p^\wedge}(g)$ is p -complete and has no such nonzero elements implying $\hat{\pi}(b) = 0$.

Let $b \in \text{im}_{Ab_p^\wedge}(g) = \ker \hat{\pi}$, so $\hat{\pi}(b) = 0 \in (B/\text{im}_{Ab}g)_p^\wedge$. Thus there exist elements $b_i \in B/\text{im}_{Ab}(g)$ with $\pi(b) = p^i b_i$ where π is the natural projection. Choose a $\tilde{b}_i \in B$ that projects to b_i by π . Since $\pi(b) = \pi(p^i \tilde{b}_i)$ in $B/\text{im}_{Ab}g$, there exists $b'_i \in \text{im}_{Ab}g$ so that $b = b'_i + p^i \tilde{b}_i$ in B . This means b is a limit point of $\text{im}_{Ab}(g)$ in B . \square

We provide an example in which $\text{im}_{Ab}(g) \not\cong \text{im}_{Ab_p^\wedge}(g)$. This will also provide an example when $\text{coker}_{Ab}g \not\cong \text{coker}_{Ab_p^\wedge}g$, since the definition of kernel implies the following sequences are both exact in Ab :

$$0 \rightarrow \text{im}_{Ab}g \rightarrow B \rightarrow \text{coker}_{Ab}g \rightarrow 0$$

$$0 \rightarrow \text{im}_{Ab_p^\wedge}g \rightarrow B \rightarrow \text{coker}_{Ab_p^\wedge}g \rightarrow 0$$

Example II.4.3. Define $\omega : \prod_{i \in \mathbb{N}} Z_p \rightarrow \prod_{i \in \mathbb{N}} Z_p$ so that $\omega(a_1, a_2, a_3, \dots) = (pa_1, p^2a_2, p^3a_3, \dots)$. Denote the cokernel of ω in $\mathcal{A}b$ by $g : \prod_{i \in \mathbb{N}} Z_p \rightarrow C$, so that the cokernel of ω in $\mathcal{A}b_{\hat{p}}$ would be $g_{\hat{p}} : \prod_{i \in \mathbb{N}} Z_p \rightarrow C_{\hat{p}}$. By Definition II.4.1, $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega = \ker g_{\hat{p}}$. Also by definition of image in $\mathcal{A}b$, $\text{im}_{\mathcal{A}b}\omega = \ker g$.

First examine $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega$. Let $\omega' : \prod_{i \in \mathbb{N}} Z_p \rightarrow \prod_{i \in \mathbb{N}} Z_p$ be defined by multiplying the i^{th} factor by p^i . We will verify ω' is the kernel of $g_{\hat{p}}$ and thus show $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega = \ker g_{\hat{p}} \cong \prod_{i \in \mathbb{N}} Z_p$.

Since $g_{\hat{p}} \circ \omega' = 0$, the universal property of kernels in $\mathcal{A}b_{\hat{p}}$ induces a map $f : \prod_{i \in \mathbb{N}} Z_p \rightarrow \text{im}_{\mathcal{A}b_{\hat{p}}}\omega$. ω' is an injection so commutativity of the diagram below implies f is an injection.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \curvearrowright & & \\
 \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{\omega'} & \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{g_{\hat{p}}} & C_{\hat{p}} \\
 & \searrow f & \uparrow & \nearrow 0 & \\
 & & \text{im}_{\mathcal{A}b_{\hat{p}}}\omega = \ker g_{\hat{p}} & &
 \end{array}$$

To verify f is onto, let $a \in \text{im}_{\mathcal{A}b_{\hat{p}}}\omega$. Claim II.4.2 lets us treat $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega = \overline{\text{im}_{\mathcal{A}b}\omega}^{\prod_{i \in \mathbb{N}} Z_p} \subset \prod_{i \in \mathbb{N}} Z_p$, so if we write a as (a_1, a_2, a_3, \dots) , there are elements

$$(b_1^i, b_2^i, b_3^i, \dots) \in \prod_{\mathbb{N}} Z_p \quad \text{and} \quad (c_1^i, c_2^i, c_3^i, \dots) \in \prod_{\mathbb{N}} Z_p$$

so that

$$(a_1, a_2, a_3, \dots) - \omega(b_1^i, b_2^i, b_3^i, \dots) = p^i(c_1^i, c_2^i, c_3^i, \dots).$$

The j^{th} entry in the above simplifies to $a_j - p^j b_j^i = p^i c_j^i$. The element $(b_1^i + c_1^i, b_2^i + c_2^i, b_3^i + c_3^i, \dots)$ is in $\prod_{\mathbb{N}} Z_p$ and maps to a under f , thus $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega \cong \prod_{i \in \mathbb{N}} Z_p$.

By contrast, $\text{im}_{\mathcal{A}b}\omega = \ker g \cong \prod_{i \in \mathbb{N}} Z_p$, so $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega \not\cong \text{im}_{\mathcal{A}b}\omega$. We highlight an element in $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega$ that is not in $\text{im}_{\mathcal{A}b}\omega$.

Consider $(p, p^2, p^3, \dots) \in \prod_{i \in \mathbb{N}} Z_p$. Let $[p, p^2, p^3 \dots]_C$ denote the equivalence class of the image of $(p, p^2, p^3 \dots) \in \prod_{i \in \mathbb{N}} Z_p$. By construction $(1, 1, 1, \dots) \notin \prod_{i \in \mathbb{N}} Z_p$ so in C , $[p, p^2, p^3 \dots]_C$ is not zero. By definition of kernels then $(p, p^2, p^3 \dots) \notin \ker g = \text{im}_{\mathcal{A}b}\omega$.

The set $\{(1, 0, 0, 0, \dots), (1, 1, 0, 0, \dots), (1, 1, 1, 0, \dots), \dots\} \in \prod_{i \in \mathbb{N}} Z_p$ when passed through ω gives us a Cauchy sequence that converges to $(p, p^2, p^3 \dots) \in \prod_{i \in \mathbb{N}} Z_p$ using the topology induced by $\prod_{i \in \mathbb{N}} Z_p$. Claim II.4.2 implies that $(p, p^2, p^3 \dots) \in \text{im}_{\mathcal{A}b_{\hat{p}}}\omega = \overline{\text{im}_{\mathcal{A}b}\omega}^{\prod_{i \in \mathbb{N}} Z_p}$. Thus, $(p, p^2, p^3 \dots)$ is an element of $\text{im}_{\mathcal{A}b_{\hat{p}}}\omega$ that is not in $\text{im}_{\mathcal{A}b}\omega$.

Let $g \in \text{Hom}_{\mathcal{A}b_p^\wedge}(A, B)$. We record a few instances when $\text{im}_{\mathcal{A}b}(g) \cong \text{im}_{\mathcal{A}b_p^\wedge}(g)$ and show $\text{im}_{\mathcal{A}b}(g) \in \text{ob}(\mathcal{A}b_p^\wedge)$.

Claim II.4.4. *Let A and B be p -complete and finitely generated as Z_p -modules. Let $g : A \rightarrow B$. Then $\text{im}_{\mathcal{A}b_p^\wedge}(g) = \text{im}_{\mathcal{A}b}(g)$.*

Proof. Note Remark II.0.8 implies A and B are profinite groups. If $\text{im}_{\mathcal{A}b}g$ is closed in B , then $\text{coker}_{\mathcal{A}b}g$ is profinite [18]. Z_p and \mathbb{Z}/p^i are compact with respect to the p -adic topology for all i , thus the finite products A and B are both compact. The image of a compact set is compact and B is Hausdorff so $\text{im}_{\mathcal{A}b}g$ is closed in B and $\text{coker}_{\mathcal{A}b}g$ is profinite.

Since B has finite rank, $\text{coker}_{\mathcal{A}b}g$ will as well. This implies that indexing category \mathcal{N} used to calculate \varprojlim for the p -completion functor is cofinal in the indexing category used to take the profinite completion of an object. Thus, because $\text{coker}_{\mathcal{A}b}g$ is profinite, $\text{coker}_{\mathcal{A}b}g$ is also p -complete. Ergo, $\text{coker}_{\mathcal{A}b}g \cong \text{coker}_{\mathcal{A}b_p^\wedge}g$ and $\text{im}_{\mathcal{A}b_p^\wedge}g \cong \text{im}_{\mathcal{A}b}g$. \square

Claim II.4.5. *Let $g \in \text{Hom}_{\mathcal{A}b_p^\wedge}(A, B)$ and assume $\text{im}_{\mathcal{A}b}g$ is dense in B with the subspace topology. Then $\text{im}_{\mathcal{A}b}g = B$.*

Proof. Let $b \in B$. We will construct a Cauchy sequence in A that is sent to a Cauchy sequence in B which converges to b . The p -completeness of A will then force the limit of our constructed sequence to exist and continuity will force the limit to map to b .

Since $\text{im}_{\mathcal{A}b}g$ is dense in B , we can find $b_1 \in \text{im}_{\mathcal{A}b}(g)$ and $b'_1 \in B$ so that

$$b - b_1 = pb'_1. \tag{II.1}$$

Since $b_1 \in \text{im}_{\mathcal{A}b}(g)$, choose $a_1 \in A$ so that $g(a_1) = b_1$.

Consider the “error”, $b'_1 \in B$. Since $\overline{\text{im}_{\mathcal{A}b}g}^B = B$, we can find $b_2 \in \text{im}_{\mathcal{A}b}(g)$ and $b'_2 \in B$ so that

$$b'_1 - b_2 = pb'_2. \tag{II.2}$$

Since $b_2 \in \text{im}_{\mathcal{A}b}(g)$, choose $a_2 \in A$ so that $g(a_2) = b_2$.

We repeat the process for the “error to the error” b'_2 , and iteratively find $b_i \in \text{im}_{\mathcal{A}b}(g)$ and $b'_i \in B$ so that

$$b'_i - b_{i+1} = pb'_{i+1}. \tag{II.3}$$

Consider the Cauchy sequence

$$\{s_i\} = \{a_1, a_1 + pa_2, a_1 + pa_2 + p^2a_3, a_1 + pa_2 + p^2a_3 + p^3a_4, \dots\} \subset A.$$

Examine the special case $b - g(a_1 + pa_2 + p^2a_3)$ as it is indicative of the general case.

$$\begin{aligned} b - g(a_1 + pa_2 + p^2a_3) &= b - g(a_1) - pg(a_2) - p^2g(a_3) \\ &= (b - b_1) - pb_2 - p^2b_3 && \text{by Equation II.1} \\ &= pb'_1 - pb_2 - p^2b_3 \\ &= p(b'_1 - b_2) - p^2b_3 && \text{by Equation II.2} \\ &= ppb'_2 - p^2b_3 \\ &= p^2(b'_2 - b_3) && \text{by Equation II.3} \\ &= p^3b'_3 \end{aligned}$$

In general $b - g\left(\sum_{i=1}^n p^{i-1}a_i\right) = p^n b'_n$, thus, the sequence $\{g(s_i)\}$ converges to b in B .

Since A is p -complete and $\{s_i\}$ is a Cauchy sequence in A , a limit a exists for s . Continuity of g implies that $g(a) = b$, thus $b \in \text{im}_{\mathcal{A}b}(g)$. \square

Corollary II.4.6. *If $g \in \text{Hom}_{\mathcal{A}b_p}(A, B)$, then $\text{im}_{\mathcal{A}b}(g) = \{b \in B \mid \exists a \in A \text{ such that } g(a) = b\}$ is p -complete.*

Proof. We first show the completion map $\phi : \text{im}_{\mathcal{A}b}(g) \rightarrow (\text{im}_{\mathcal{A}b}(g))_p^\wedge$ is injective. Since,

$$\ker \phi = \bigcap_{i \in \mathbb{N}} (p^i \text{im}_{\mathcal{A}b}(g)) \subset \bigcap_{i \in \mathbb{N}} (p^i B)$$

and B is p -complete we have $\ker \phi \subset \bigcap_{i \in \mathbb{N}} (p^i B) = \{0\}$, thus ϕ is injective.

Let \tilde{g} be the epimorphism between A and $\text{im}_{\mathcal{A}b}g$ induced by the universal property of images in $\mathcal{A}b$. Consider the map $\phi \circ \tilde{g} \in \text{Hom}_{\mathcal{A}b_p}(A, (\text{im}_{\mathcal{A}b}g)_p^\wedge)$. The $\text{im}_{\mathcal{A}b}(\phi \circ \tilde{g})$ is dense in $(\text{im}_{\mathcal{A}b}g)_p^\wedge$ with respect to the p -adic topology of $(\text{im}_{\mathcal{A}b}g)_p^\wedge$, so Claim II.4.5 implies $(\text{im}_{\mathcal{A}b}g)_p^\wedge \cong \text{im}_{\mathcal{A}b}(\phi \circ \tilde{g})$. By definition $\text{im}_{\mathcal{A}b}(\phi \circ \tilde{g}) = \{c \in (\text{im}_{\mathcal{A}b}g)_p^\wedge \mid \exists a \in A \text{ such that } \phi \circ \tilde{g}(a) = c\}$. Since ϕ is one to one in $\mathcal{A}b$, $\text{im}_{\mathcal{A}b}(\phi \circ \tilde{g}) \cong \text{im}_{\mathcal{A}b}\tilde{g}$. Furthermore, \tilde{g} is onto in $\mathcal{A}b$ so $\text{im}_{\mathcal{A}b}\tilde{g} \cong \text{im}_{\mathcal{A}b}g$ implying $\text{im}_{\mathcal{A}b}g \cong (\text{im}_{\mathcal{A}b}g)_p^\wedge$. \square

Note the reference to the subspace topology in Claim II.4.5 is important since in general given $A, B \in \mathcal{A}b_p^\wedge$ with A dense in B under some topology, A may not equal B . Example II.4.3

highlights such a situation where $\text{im}_{\mathcal{A}b}\omega$ is p -complete and dense in $\text{im}_{\mathcal{A}b_p}\omega$ with respect to the topology induced by $\prod_{i \in \mathbb{N}} Z_p$. However, $\text{im}_{\mathcal{A}b}\omega$ is not dense in $\text{im}_{\mathcal{A}b_p}\omega$ with respect to the p -adic topology of $\text{im}_{\mathcal{A}b_p}\omega$. The element $(p, p^2, p^3, \dots) \in \text{im}_{\mathcal{A}b_p}\omega$ is an example of an element not in $\overline{\text{im}_{\mathcal{A}b}\omega}^{\text{im}_{\mathcal{A}b_p}\omega}$, and in fact it was shown that $\text{im}_{\mathcal{A}b}\omega \not\cong \text{im}_{\mathcal{A}b_p}\omega$.

II.5 $\mathcal{A}b_p^\wedge$ Is Not an Abelian Category

Recall that a category \mathcal{A} is *abelian* if it is an additive category satisfying the following [6]:

1. \mathcal{A} has a null object,
2. \mathcal{A} has binary biproducts,
3. every arrow in \mathcal{A} has a kernel and cokernel,
4. every monic arrow is a kernel, and every epi is a cokernel.

The previous sections verified the first three conditions. We will verify the second half of the fourth and show the failure of the remaining condition. Before starting this we recall what it means for a morphism to be an epi or a monic and identify these in $\mathcal{A}b_p^\wedge$.

A morphism m is monic if $m \circ f = m \circ h$ implies that $f = h$. A morphism that is monic is said to have the left cancellation property. Dually, a morphism e is epi if it has the right cancellation property, or $f \circ e = h \circ e$ implies $f = h$.

Claim II.5.1. *A morphism m is monic in $\mathcal{A}b_p^\wedge$ if and only if m is monic in $\mathcal{A}b$.*

Proof. Since $\mathcal{A}b_p^\wedge$ is a full subcategory of $\mathcal{A}b$, we need only verify that if $m \in \text{Hom}_{\mathcal{A}b_p^\wedge}(A, B)$ and is monic, then m is monic in $\mathcal{A}b$.

Assume $C \in \text{ob}(\mathcal{A}b)$, $f, h \in \text{Hom}_{\mathcal{A}b}(C, A)$, and $m \circ f = m \circ h$. We need to show $f = h$, or $f - h = 0$ where we make use of the group structure of $\text{Hom}_{\mathcal{A}b}(C, A)$. We make use of the following diagram in $\mathcal{A}b$ where ι is the natural map.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \curvearrowright & & \\
 C & \xrightarrow{f-h} & A & \xrightarrow{m} & B \\
 & \searrow & \uparrow \iota & \nearrow 0 & \\
 & & \text{ker } m & &
 \end{array}$$

If $\ker m = 0$ in the above diagram the map $f - h$ factors through 0 and thus must be 0. It thus suffices to show $\ker m = 0$.

Recall in Claim II.3.4 that $\ker_{\mathcal{A}b} m = \ker_{\mathcal{A}b_p} m$. To show $\ker m = 0$ it thus suffices to show that 0 satisfies the universal property of kernels in $\mathcal{A}b_p$. This, however, follows from $\ker m \in \text{ob}(\mathcal{A}b_p)$ and m having the left cancellation property in $\mathcal{A}b_p$. Thus, $\ker m = 0$ implying $f - h = 0$ so $f = h$ which is what we had to show. \square

We can import results known in $\mathcal{A}b$ and say such things as morphism in $\mathcal{A}b_p$ are monic if and only if m is one-to-one. A similar result holds for morphisms that are epi.

Claim II.5.2. *A morphism e is epi in $\mathcal{A}b_p$ if and only if e is epi in $\mathcal{A}b$.*

Proof. Since $\mathcal{A}b_p$ is a full subcategory of $\mathcal{A}b$, it suffices to show if $e \in \text{Hom}_{\mathcal{A}b_p}(A, B)$ and is epi then e is an epi in $\mathcal{A}b$. We will show if e is epi in $\mathcal{A}b_p$ then $\text{coker}_{\mathcal{A}b_p} e = 0$ and claims from Section II.4 will finish the proof.

Let $f \in \text{Hom}_{\mathcal{A}b_p}(B, C)$ and $f \circ e = 0$. Since e is epi, $f = 0$ and the following diagram verifies that 0 satisfies the universal property of cokernels in $\mathcal{A}b_p$.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \curvearrowright & & \\
 A & \xrightarrow{e} & B & \xrightarrow{f} & C \\
 & \searrow & \downarrow 0 & \nearrow 0 & \\
 & & 0 & &
 \end{array}$$

Recall by definition of image in $\mathcal{A}b_p$ that

$$\text{coker}_{\mathcal{A}b_p} e = \text{coker}_{\mathcal{A}b}(\text{im}_{\mathcal{A}b_p} e \rightarrow B).$$

Claim II.4.2 implies

$$\text{coker}_{\mathcal{A}b_p} e = \text{coker}_{\mathcal{A}b}(\overline{\text{im}_{\mathcal{A}b} e}^B \rightarrow B).$$

thus $\text{coker}_{\mathcal{A}b_p} e \cong 0$ implies $\text{im}_{\mathcal{A}b} e$ is dense in B under the subspace topology. Since A and B are p -complete, Claim II.4.5 implies $\text{im}_{\mathcal{A}b} e = B$ so e is onto. A morphism in $\mathcal{A}b$ is epi if and only if is onto, thus e is an epi in $\mathcal{A}b$. \square

We return to checking the axioms of an abelian category for $\mathcal{A}b_p$. We verify the second half of the fourth condition but then provide a counterexample to the first half.

Claim II.5.3. *Every epi is a cokernel.*

Proof. Let $f \in \text{Hom}_{\mathcal{A}b_{\hat{p}}}(A, B)$ be an epi in $\mathcal{A}b_{\hat{p}}$, and $\iota : \ker f \rightarrow A$ be the natural map. Note A is in $\mathcal{A}b_{\hat{p}}$ by assumption and $\ker f$ is in $\mathcal{A}b_{\hat{p}}$ by Claim II.3.4, so ι is in $\mathcal{A}b_{\hat{p}}$. It will be shown that f is the cokernel of ι in $\mathcal{A}b_{\hat{p}}$ by showing f satisfies the appropriate universal property.

Assume $g \in \text{Hom}_{\mathcal{A}b_{\hat{p}}}(A, C)$ is such that $g \circ \iota = 0$. The definition of kernel implies $f \circ \iota = 0$, so we concentrate on building the dotted arrow in the following diagram and showing it is unique.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \ker f & \xrightarrow{\iota} & A & \xrightarrow{g} & C \\
 & \searrow & \downarrow f & \nearrow & \\
 & 0 & B & &
 \end{array}$$

In the abelian category $\mathcal{A}b$, there is a factorization of f as follows [6]:

$$A \xrightarrow{e} A/\ker f \xrightarrow{s} \text{im}_{\mathcal{A}b} f \xrightarrow{m} B.$$

The composition $m \circ s \circ e$ equals f , e is an epi, s is an isomorphism, and m is a monic. Expand the above diagram to:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & \curvearrowright & & & \curvearrowleft & & \\
 \ker f & \xrightarrow{\iota} & A & \xrightarrow{id} & A & \xrightarrow{g} & C \\
 & \searrow & \downarrow e & \nearrow \tilde{g} & \downarrow f & \nearrow & \\
 & 0 & A/\ker f & \xrightarrow{m \circ s} & B & &
 \end{array}$$

Since $A/\ker f$ is a cokernel in $\mathcal{A}b$, there exists a unique map $\tilde{g} : A/\ker f \rightarrow C$ that makes the above diagram commute. To define a map from B to C it suffices to show $m \circ s$ is an isomorphism in $\mathcal{A}b$. Since s is an isomorphism, the problem reduces to showing m is onto.

Since f is an epi, $\text{coker}_{\mathcal{A}b_{\hat{p}}} f = 0$. By definition $\text{im}_{\mathcal{A}b_{\hat{p}}} f$ is the kernel of the map $\pi : B \rightarrow \text{coker}_{\mathcal{A}b_{\hat{p}}} f$ thus $\text{im}_{\mathcal{A}b_{\hat{p}}} f \cong B$. Since $\text{im}_{\mathcal{A}b_{\hat{p}}} f = \overline{\text{im}_{\mathcal{A}b} f}^B$ as a subspace of B , $\text{im}_{\mathcal{A}b} f$ is dense in B . Claim II.4.5 finishes the proof. \square

The following is an example of a monic that is not a kernel and thus $\mathcal{A}b_{\hat{p}}$ is not an abelian category.

Example II.5.4. We revisit Example II.4.3 and consider the map $\omega : \prod_{i \in \mathbb{N}} Z_p \rightarrow \prod_{i \in \mathbb{N}} Z_p$ defined by $\omega(a_1, a_2, a_3, \dots) = (pa_1, p^2a_2, p^3a_3, \dots)$. Notice that ω is a monic in $\mathcal{A}b$, so it is also a monic in $\mathcal{A}b_p^\wedge$.

Before showing that ω is not a kernel we show why ω is not the kernel to the most obvious candidate, mainly the kernel of $g_p^\wedge : \prod_{i \in \mathbb{N}} Z_p \rightarrow \text{coker}_{\mathcal{A}b_p^\wedge} \omega$. Notice $\ker g_p^\wedge = \text{im}_{\mathcal{A}b_p^\wedge} \omega$ which was shown to be isomorphic to $\prod_{i \in \mathbb{N}} Z_p$ in Example II.4.3. Thus ω is not the kernel of g_p^\wedge .

To show that ω is not the kernel of any map, we assume ω is the kernel of a map h and derive a contradiction. Since $h \circ \omega = 0$, the universal property of $\text{coker}_{\mathcal{A}b_p^\wedge} \omega$ induces s in the diagram below. Commutativity implies $h \circ f' = s \circ g_p^\wedge \circ f' = 0$, so the universal property of kernels from ω induces the map r . Let ι be the map induced by the universal property of $\ker g_p^\wedge$.

$$\begin{array}{ccccccc}
 & & \prod_{i \in \mathbb{N}} Z_p & & & & \\
 & \swarrow \iota_2 & \downarrow \iota & \searrow \omega & & & \\
 \prod_{i \in \mathbb{N}} Z_p \oplus A \cong \ker g_p^\wedge & \xrightarrow{f'} & \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{g_p^\wedge} & \text{coker}_{\mathcal{A}b_p^\wedge} \omega & & \\
 \downarrow \text{proj}_1 & & \downarrow id & \circ & \downarrow s & & \\
 \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{\omega} & \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{h} & D & & \\
 & & \downarrow r & & & & \\
 & & \prod_{i \in \mathbb{N}} Z_p & & & &
 \end{array}$$

Since $r \circ \iota = id$ we can write $\ker g_p^\wedge \cong \prod_{i \in \mathbb{N}} Z_p \oplus A$ for some $A \in \text{ob}(\mathcal{A}b)$. Example II.4.3 shows that $\ker g_p^\wedge \not\cong \prod_{i \in \mathbb{N}} Z_p$, thus $A \neq 0$. Tracing a nonzero element $(0, a) \in \prod_{i \in \mathbb{N}} Z_p \oplus A$, we find that $\omega \circ \text{proj}_1(0, a) = 0$. However, $\prod_{i \in \mathbb{N}} Z_p \oplus A$ also can be sent to $\prod_{i \in \mathbb{N}} Z_p$ by the injective map f' , so by commutativity of the diagram $\omega \circ \text{proj}_1(0, a) \neq 0$. Thus we have a contradiction and h must not exist.

II.6 Failure of Five Lemma in $\mathcal{A}b_p^\wedge$

Before leaving the chapter we stress that $\mathcal{A}b_p^\wedge$ is not abelian so theorems relying on Property 4 in the definition of an abelian category may not hold. In particular the Five Lemma no longer holds as the following example illustrates.

Example II.6.1. Consider again the map from Example II.4.3, $\omega : \prod_{i \in \mathbb{N}} Z_p \rightarrow \prod_{i \in \mathbb{N}} Z_p$ defined by $\omega(a_1, a_2, a_3, \dots) = (pa_1, p^2a_2, p^3a_3, \dots)$. Recall $\text{im}_{\mathcal{A}b_p^\wedge} \omega = \overline{\text{im}_{\mathcal{A}b} \omega}^{\prod_{i \in \mathbb{N}} Z_p}$, thus the sequences below

are both short exact in $\mathcal{A}b_p^{\hat{}}$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{\omega} & \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{g} & \text{coker}_{\mathcal{A}b_p^{\hat{}}} \omega \longrightarrow 0 \\
 \text{id} \downarrow & & \cong \downarrow & & \text{id} \downarrow & & \text{id} \downarrow \\
 0 & \longrightarrow & \ker g_p & \longrightarrow & \prod_{i \in \mathbb{N}} Z_p & \xrightarrow{g_p^{\hat{}}} & \text{coker}_{\mathcal{A}b_p^{\hat{}}} \omega \longrightarrow 0
 \end{array}$$

CHAPTER III

DEFINITION AND EXAMPLES OF STABLE HOMOTOPY CATEGORIES

The definition of an SHC given in [4] requires some preliminary definitions which we review before giving the definition. The definition of an SHC is from [4]; other definitions that are provided below can be found in [4] or [9].

III.1 Closed Symmetric Monoidal Categories

Definition III.1.1. *A category \mathcal{C} has a closed symmetric monoidal structure if there exists a monoidal functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ that is associative up to coherent natural isomorphism and the following hold [6]:*

1. *There exists a unit S and coherent natural isomorphisms in \mathcal{C} so that $S \otimes A \cong A \cong A \otimes S$.*
2. *(symmetric) There is a coherent natural isomorphism between $A \otimes B$ and $B \otimes A$.*
3. *(closed) There is a function object $F_{\mathcal{C}}(A, B)$ that is covariant in the second variable and contravariant in the first. There also exists an isomorphism*

$$\text{Hom}_{\mathcal{C}}(A, F_{\mathcal{C}}(B, C)) \cong \text{Hom}_{\mathcal{C}}(A \otimes B, C)$$

that is natural with respect to all three variables.

An example is the category $\mathcal{A}b$ where \otimes is the tensor product over \mathbb{Z} and the function object $F_{\mathcal{C}}(A, B)$ is $\text{Hom}_{\mathcal{A}b}(A, B)$. More generally, the category of R -modules (where R is a commutative unital ring) is a closed symmetric monoidal category with the monoidal structure given by the tensor product over R and the function object is $\text{Hom}_R(A, B)$. The unit is R and the closed condition is satisfied since \otimes_R and Hom_R are an adjoint pair.

A topological example is the category of compactly generated Hausdorff spaces, denoted $CGHaus$. The monoidal structure arises from the cartesian product and the unit is the space consisting of a single point. The function object $F_{CGHaus}(A, B)$ is the set of continuous maps from A to B with the compact open topology. More examples of closed symmetric monoidal categories can be found in Section VII of [6].

III.2 Triangulated Categories

Definition III.2.1. A triangulation on an additive category \mathcal{C} is an additive (suspension) functor Σ that gives an automorphism of \mathcal{C} , together with a collection Δ of diagrams, called exact triangles, of the form $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ so that the following hold:

1. (T1)

- Any diagram equivalent to an exact triangle is also in Δ .
- Any diagram of the form $0 \rightarrow X \xrightarrow{1} X \rightarrow 0$ is in Δ .
- For every $F \in \text{Hom}_{\mathcal{C}}(X, Y)$ there exists an exact triangle of the form

$$X \xrightarrow{F} Y \rightarrow Z \rightarrow \Sigma X.$$

2. (T2) If $X \xrightarrow{F} Y \rightarrow Z \rightarrow \Sigma X$ is in Δ , then $Y \rightarrow Z \rightarrow \Sigma X \xrightarrow{-\Sigma F} \Sigma Y$ is in Δ .

3. (T3) Let

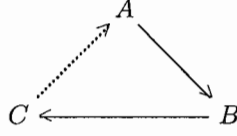
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ F \downarrow & & \circ & & \downarrow & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

where the rows are in Δ . Then there exists a map $G \in \text{Hom}_{\mathcal{C}}(Z, Z')$, not necessarily unique, so that the following diagram commutes.

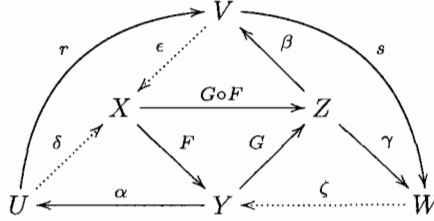
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ F \downarrow & & \circ & & \downarrow & \circ & G \downarrow & \circ & \downarrow \Sigma F \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

4. (T4) Verdier's octahedral axiom holds. To state this more precisely, denote the exact triangle

$A \rightarrow B \rightarrow C \rightarrow \Sigma A$ in \mathcal{C} by:



Assume there are maps $F \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $G \in \text{Hom}_{\mathcal{C}}(Y, Z)$. Suppose that there are $X \xrightarrow{F} Y \xrightarrow{\alpha} U$, $X \xrightarrow{G \circ F} Z \xrightarrow{\beta} V$, and $Y \xrightarrow{G} Z \xrightarrow{\gamma} W$ in Δ as shown below.



Then there exists maps $r : U \rightarrow V$ and $s : V \rightarrow W$ so that $U \xrightarrow{r} V \xrightarrow{s} W$ is in Δ and $\beta \circ G = r \circ \alpha$, $\zeta \circ s = (\Sigma F) \circ \epsilon$, $s \circ \beta = \gamma$, and $\epsilon \circ r = \delta$.

We refer to a category with a triangulation as a triangulated category.

Example III.2.2. An example of a triangulated category is $\mathcal{K}(R)$ referred to in Section I.2. Recall the objects of $\mathcal{K}(R)$ are unbounded chain complexes of R -modules where R is a unital commutative ring. The morphisms of $\mathcal{K}(R)$ are degree preserving chain homotopy classes of maps. The suspension functor is defined degree-wise by $(\Sigma X)_n = X_{n-1}$ with differentials $(d^{\Sigma X})_n = (-1)d_{n-1}^X$. The exact triangles are the collection of all sequences in $\mathcal{K}(R)$ equivalent to the sequence

$$X \xrightarrow{F} Y \rightarrow \text{Cone}(F)$$

for some $F \in \text{Hom}_{\mathcal{K}(R)}(X, Y)$ where $\text{Cone}(F)_n = X_{n-1} \oplus Y_n$ with the differential

$$d_n^{\text{Cone}(F)}(x, y) = (-d_{n-1}^X(x), d_n^Y(y) - F_{n-1}(x)).$$

Other examples of triangulated categories can be found by considering subcategories of $\mathcal{K}(R)$ with the triangulated structure just defined. Given an R -module A , let $A(i)$ denote the chain complex with A in the i^{th} degree and the zero object in all others.

Example III.2.3. We revisit the derived category $\mathcal{D}(R)$, first introduced in Chapter I, and provide a more precise definition:

Definition III.2.4. We say an object Z is a coproduct of free modules if Z is chain homotopy equivalent to $\bigoplus_{j \in \mathbb{Z}} \left(\bigoplus_{i \in I_j} R(j) \right)$. An object X is $R(0)$ -cellular if X is a colimit of a sequence $\{X^0, X^1, X^2, \dots\}$ where $X^0 \cong 0$ and $X^i \cong \text{Cone}(F^{i-1})$ for some $F^{i-1} : Z^{i-1} \rightarrow X^{i-1}$ where Z^{i-1} is a coproduct of free modules. Define $\mathcal{D}(R)$ to be the full subcategory of $\mathcal{K}(R)$ with objects that are $R(0)$ -cellular.

The Σ and Δ structures for $\mathcal{K}(R)$ restrict to make $\mathcal{D}(R)$ a triangulated category in its own right.

Example III.2.5. We can generalize Example III.2.2 to any additive category \mathcal{A} with arrows that are degree preserving chain homotopy classes of maps. Denote this category $\mathcal{K}(\mathcal{A})$ and note that the same definitions for Σ and Δ in $\mathcal{K}(R)$ makes $\mathcal{K}(\mathcal{A})$ a triangulated category.

We mention a non-example. Recall $Ch(\mathcal{A})$ is the category with the same objects as $\mathcal{K}(\mathcal{A})$ but with arrows that are degree preserving chain maps. The Σ functor and Cone construction both make sense in $Ch(\mathcal{A})$, but the set Δ defined as the set of sequences of the form

$$X \xrightarrow{F} Y \rightarrow \text{Cone}(F)$$

for some $F \in \text{Hom}_{Ch(\mathcal{A})}(X, Y)$, does not satisfy the (T1) condition since $\text{Cone}(\text{Id}) \not\cong 0$ in $Ch(\mathcal{A})$.

Definition III.2.6. A thick subcategory is a full subcategory \mathcal{C} , of a triangulated category such that the following two conditions are satisfied:

1. When any two objects in the exact triangle $X \rightarrow Y \rightarrow Z$ are in \mathcal{C} , the third is also in \mathcal{C} .
2. If Y is in \mathcal{C} , and $X \xrightarrow{i} Y \xrightarrow{p} X$ is such that $p \circ i = \text{id}_X$, then X is in \mathcal{C} .

A localizing subcategory of a triangulated category is a thick subcategory \mathcal{C} with the property that any coproduct of objects from \mathcal{C} is also in \mathcal{C} .

Given a triangulated category \mathcal{C} , the trivial examples \mathcal{C} and the zero subcategory are both localizing. The derived category $\mathcal{D}(R)$ is a nontrivial example of a localizing subcategory in $\mathcal{K}(R)$.

III.3 Triangulation Compatible with the Product in Monoidal Categories

Definition III.3.1. *If \mathcal{C} is both a triangulated and a closed symmetric monoidal category, the two structures are compatible if the following hold:*

- *The monoidal product preserves suspensions in the following way. There is a natural equivalence $e_{XY} : (\Sigma X) \otimes Y \rightarrow \Sigma(X \otimes Y)$. Let $r_X : X \otimes S \rightarrow X$ be the unital equivalence, then $\Sigma r_X \circ e_{XS} = r_{\Sigma X}$. Let $a_{XYZ} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ be the associative equivalence, then the following diagram commutes.*

$$\begin{array}{ccc}
 & \Sigma(A \otimes B) \otimes C & \\
 e_{AB} \otimes 1_C \nearrow & & \searrow e_{(A \otimes B)C} \\
 (\Sigma A \otimes B) \otimes C & \circ & \Sigma((A \otimes B) \otimes C) \\
 a_{(\Sigma A)BC} \downarrow & & \downarrow \Sigma a_{ABC} \\
 \Sigma A \otimes (B \otimes C) & \xrightarrow{e_{A(B \otimes C)}} & \Sigma(A \otimes (B \otimes C))
 \end{array}$$

- *Given an object X of \mathcal{C} , the functor $- \otimes X$ takes exact triangles to exact triangles.*
- *Given an object W of \mathcal{C} , the functor $F_{\mathcal{C}}(W, -)$ takes exact triangles to exact triangles. (The natural equivalence of e can be used to show $F(W, \Sigma X) \cong \Sigma F(W, X)$ and $F(\Sigma X, W) \cong \Sigma^{-1} F(X, W)$ as in [4].) The functor $F_{\mathcal{C}}(-, W)$ takes an exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, to the following triangle in Δ .*

$$\Sigma^{-1} F_{\mathcal{C}}(X, W) \xrightarrow{-F_{\mathcal{C}}(h, 1)} F_{\mathcal{C}}(Z, W) \xrightarrow{F_{\mathcal{C}}(g, 1)} F_{\mathcal{C}}(Y, W) \xrightarrow{F_{\mathcal{C}}(f, 1)} F_{\mathcal{C}}(X \otimes W)$$

- *The monoidal product interacts with the suspension functor in a graded-commutative manner. In more detail, the following diagram must be commutative for all integers r and s , where t is the natural equivalence responsible for the symmetric structure.*

$$\begin{array}{ccc}
 \Sigma^r S \otimes \Sigma^s S & \rightarrow & \Sigma^{r+s} S \\
 t \downarrow & \circ & \downarrow (-1)^{rs} \\
 \Sigma^s S \otimes \Sigma^r S & \rightarrow & \Sigma^{r+s} S
 \end{array}$$

Example III.3.2. The category $\mathcal{K}(R)$ has a closed symmetric monoidal structure that is compatible with the triangulated structure already introduced. The monoidal functor is the standard

tensor product of chain complexes over R . Given chain complexes X and Y , the function object $F_{\mathcal{K}(R)}(X, Y)$ is the chain complex with $(F_{\mathcal{K}(R)}(X, Y))_n = \prod_j \text{Hom}_R(X_j, Y_{j+n})$. The component of the differential $d_n^{F_{\mathcal{K}(R)}(X, Y)}$, landing in $\text{Hom}_R(X_j, Y_{j+n-1})$ comes from

$$d^{F_{\mathcal{K}(R)}(X, Y)} : \text{Hom}_R(X_{j-1}, Y_{j+n-1}) \oplus \text{Hom}_R(X_j, Y_{j+n}) \rightarrow \text{Hom}_R(X_j, Y_{j+n-1})$$

where $d^{F_{\mathcal{K}(R)}(X, Y)}(f_{j-1}, f_j) = (-1)^{n+1} f_{j-1} d_j^X + d_{j+n}^Y f_j$. Notice that $F_{\mathcal{K}(R)}(X, Y)_n$ only depends on the graded group structures of X and Y . The differentials in X and Y are taken into consideration by $d^{F_{\mathcal{K}(R)}(X, Y)}$.

Example III.3.3. We can generalize the previous example to any additive category \mathcal{A} with products, coproducts, and a closed symmetric monoidal structure provided coproducts and the monoidal product satisfy some compatibility conditions. Let $\mathcal{K}(\mathcal{A})$ be the triangulated category described in Example III.2.5 and $A(i)$ be the chain complex with the object A in the i^{th} degree and the zero object in all others. If $\otimes_{\mathcal{A}} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is the associative functor and $S_{\mathcal{A}}$ the unit in \mathcal{A} define $\otimes : \mathcal{K}(\mathcal{A}) \times \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ degree-wise by:

$$(X \otimes Y)_n = \prod_{i+j=n} (X_i \otimes_{\mathcal{A}} Y_j)$$

with the differential

$$d_n^{X \otimes Y} = \sum_{i+j=n} (d_i^X \otimes_{\mathcal{A}} id_{Y_j} + (-1)^i id_{X_i} \otimes_{\mathcal{A}} d_j^Y).$$

Notice that we use the summation sign in the definition of $d^{X \otimes Y}$ but this may not be a finite sum. In order for \otimes to be an associative functor for $\mathcal{K}(\mathcal{A})$, \mathcal{A} must have a natural equivalence between $(\prod_I A_i) \otimes_{\mathcal{A}} B$ and $\prod_I (A_i \otimes_{\mathcal{A}} B)$. If $0 \otimes_{\mathcal{A}} A$ is equivalent to 0 in \mathcal{A} , the unit object in $\mathcal{K}(\mathcal{A})$ will be $S_{\mathcal{A}}(0)$.

Define the function object degree-wise by $(F_{\mathcal{K}(\mathcal{A})}(X, Y))_n = \prod_j F_{\mathcal{A}}(X_j, Y_{j+n})$ where $F_{\mathcal{A}}$ denotes the function object in \mathcal{A} . The component of the differential landing in $F_{\mathcal{A}}(X_j, Y_{j+n-1})$ is defined as

$$d^{F_{\mathcal{K}(\mathcal{A})}(X, Y)} : F_{\mathcal{A}}(X_{j-1}, Y_{j+n-1}) \oplus F_{\mathcal{A}}(X_j, Y_{j+n}) \rightarrow F_{\mathcal{A}}(X_j, Y_{j+n-1})$$

where $d^{F_{\mathcal{K}(\mathcal{A})}(X, Y)}(f_{j-1}, f_j) = (-1)^{n+1} f_{j-1} d_j^X + d_{j+n}^Y f_j$.

Recall the collection of Δ defined for $\mathcal{K}(\mathcal{A})$ are all sequences chain homotopy equivalent to a sequence of the form:

$$X \xrightarrow{F} Y \rightarrow \text{Cone}(F)$$

for some $F \in \text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$. The restriction of our exact triangles to the above form allows one to verify each of the compatibility conditions stated in Definition III.3.1.

III.4 Stable Homotopy Categories

We introduce two more definitions before stating the axioms of an SHC.

Definition III.4.1. *Let \mathcal{C} be a triangulated category and \mathcal{A} an additive category. A cohomology functor $\mathcal{H} : \mathcal{C} \rightarrow \mathcal{A}$, is a contravariant functor that:*

1. *takes exact triangles in \mathcal{C} to long exact sequences in \mathcal{A} , and*
2. *takes coproducts in \mathcal{C} to products in \mathcal{A} .*

Given a closed symmetric monoidal category \mathcal{C} there exists an evaluation map $\eta_{A \rightarrow B} : F_{\mathcal{C}}(A, B) \otimes A \rightarrow B$ that corresponds to the identity morphism under the adjoint

$$\text{Hom}_{\mathcal{C}}(F_{\mathcal{C}}(A, B), F_{\mathcal{C}}(A, B)) \cong \text{Hom}_{\mathcal{C}}(F_{\mathcal{C}}(A, B) \otimes A, B).$$

The identity morphism in $\text{Hom}_{\mathcal{C}}(B, B)$ corresponds to a map $F_{\mathcal{C}}(A, S) \otimes B \rightarrow F_{\mathcal{C}}(A, B)$ by the following natural morphisms:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(B, B) & & \text{Hom}_{\mathcal{C}}(F_{\mathcal{C}}(A, S) \otimes B, F_{\mathcal{C}}(A, B)) \\ \cong \downarrow & & \cong \uparrow \\ \text{Hom}_{\mathcal{C}}(S \otimes B, B) & \xrightarrow{\eta_{A \rightarrow S}^*} & \text{Hom}_{\mathcal{C}}((F_{\mathcal{C}}(A, S) \otimes A) \otimes B, B) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}((F_{\mathcal{C}}(A, S) \otimes B) \otimes A, B) \end{array}$$

Definition III.4.2. *Let \mathcal{C} be a closed symmetric monoidal category with a unit object S . An object A is strongly dualizable if the natural map $F_{\mathcal{C}}(A, S) \otimes B \rightarrow F_{\mathcal{C}}(A, B)$ described above, is an isomorphism for all B .*

In $\mathcal{A}b$ the finitely generated strongly dualizable objects are exactly the finitely generated free abelian groups. More generally, finitely generated free R -modules in $R\text{-mod}$ are strongly dualizable. In $CGHaus$ the space consisting of only one point is an example of a strongly dualizable object.

We at last state the axioms for a stable homotopy category given in [4].

Definition III.4.3. *A stable homotopy category (SHC) is a category \mathcal{D} with the following five properties:*

1. \mathcal{D} has a triangulated structure.
2. \mathcal{D} has a closed, symmetric monoidal structure denoted by \otimes and that is compatible with the triangulated structure.
3. There exists a set \mathcal{G} of strongly dualizable objects of \mathcal{D} , such that the only localizing subcategory of \mathcal{D} containing \mathcal{G} is \mathcal{D} .
4. Arbitrary coproducts of objects exist in \mathcal{D} .
5. Every cohomology functor on \mathcal{D} is representable.

It is shown in [4] that the derived category $\mathcal{D}(R)$ satisfies the above axioms. The closed symmetric monoidal structure that is compatible with a triangulation is inherited from $\mathcal{K}(Ab)$ and the set \mathcal{G} consists of only $R(0)$.

Not all conditions in Definition III.4.3 are assumed by all authors who study SHC's. For example, the closed symmetric monoidal structure was not assumed in [9] and [3] suggests eliminating the strongly dualizable condition. However, most alternative definitions provided in the literature include conditions 1 and 4 which we focus on in this paper. A more complete discussion of axioms assumed for an SHC is in [14].

Recall the close connection drawn between an SHC and homology functors in Chapter I. For example, in $\mathcal{D}(R)$ morphisms sent to isomorphisms by the functor $H_n : \mathcal{K}(R) \rightarrow Ab$ defined by $\text{Hom}_{\mathcal{K}(R)}(\Sigma^n R(0), -)$ are equivalences. This property is shown explicitly in [8] and [17], but a general property holds for an arbitrary SHC [4]. Let $\Sigma^* \mathcal{G}$ denote $\{\Sigma^i Z \mid Z \in \mathcal{G} \text{ and } i \in \mathbb{Z}\}$.

Lemma III.4.4. *Let \mathcal{C} be an SHC and $\mathcal{H}^W : \mathcal{C} \rightarrow Ab$ be the functor defined by $\text{Hom}_{\mathcal{C}}(W, -)$. If $F \in \text{Hom}_{\mathcal{C}}(X, Y)$ is such that $\mathcal{H}^W(F)$ is an isomorphism for all $W \in \Sigma^* \mathcal{G}$, then F is an isomorphism.*

The proof makes use of the uniqueness assumption with respect to localizing subcategories containing the set of weak generators in the definition of an SHC.

We return to the example $\mathcal{D}(R)$, where quasi isomorphisms with respect to the functor $\text{Hom}_{\mathcal{D}(R)}(\Sigma^n R(0), -)$ are equivalences in $\mathcal{D}(R)$. Instead of generalizing to functors of the form \mathcal{H}^W as done in Lemma III.4.4 we might have instead considered a generic homology functor defined below.

Definition III.4.5. *Let \mathcal{C} be a triangulated category and \mathcal{A} an additive category with kernels and cokernels. A homology functor $\mathcal{H} : \mathcal{C} \rightarrow \mathcal{A}$ is a covariant functor that:*

1. *takes exact triangles in \mathcal{C} to long exact sequences in \mathcal{A} , and*
2. *preserve coproducts. That is, for any indexing set I , the object $\mathcal{H}(\coprod_I X^i)$ and morphisms $\mathcal{H}(\iota_i) : \mathcal{H}(X^i) \rightarrow \mathcal{H}(\coprod_I X^i)$ satisfy the universal properties of coproducts in \mathcal{A} .*

In the case of \mathcal{H}^W as defined in Lemma III.4.4, \mathcal{H}^W is a homology functor if and only if W is small. Recall W is small if the natural map

$$\bigoplus_{i \in I} \text{Hom}_{\mathcal{C}}(W, Y^i) \rightarrow \text{Hom}_{\mathcal{C}}(W, \coprod_{i \in I} Y^i)$$

is an isomorphism for all coproducts. Notice in $\mathcal{D}(R)$, the object $R(0)$ is small, so the functor $\text{Hom}_{\mathcal{D}(R)}(\Sigma^n R(0), -)$ is a homology functor. An SHC whose objects from \mathcal{G} are small is called *algebraic*.

Theorem 2.3.2 from [4] provides an effective way to check if a category is an algebraic SHC.

Theorem III.4.6. *Let \mathcal{C} be a triangulated category with a compatible closed symmetric monoidal structure. Suppose \mathcal{C} has arbitrary products and coproducts. Suppose \mathcal{G} is a set of small strongly dualizable objects of \mathcal{C} that are weak generators of \mathcal{C} . Then \mathcal{C} is an algebraic SHC.*

A second example of an SHC is the category of CW spectra, \mathcal{S} . The triangulated structure comes from the usual suspension functor and cone construction of spectra. The smash product of spectra and the function spectra makes \mathcal{S} into a closed symmetric monoidal category. The sphere spectrum S^0 is the only small strongly dualizable object in \mathcal{G} . J.F. Adams gives a proof in Part III of [1] that an $X \in \mathcal{S}$ with the property $\text{Hom}_{\mathcal{S}}(\Sigma^i S^0, X) = 0$ for all i , implies that $X = 0$. Theorem III.4.6 thus implies \mathcal{S} is an algebraic SHC.

Additional examples including some non-algebraic SHC's are discussed in detail in Section 9 of [4].

CHAPTER IV

STRUCTURES ON $\mathcal{A}b_p^\wedge$

We show that $\mathcal{A}b_p^\wedge$ is a closed symmetric monoidal category and apply Section III.3 to build a compatible triangulated structure. We will exhibit a family of homology functors that do not exist on $\mathcal{K}(\mathcal{A}b_p^\wedge)$ and use this to state the main result. Some results about homology functors of the form $\text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(W, -)$ are then given by way of considering the small objects of $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

IV.1 A Triangulated and Monoidal Structure for $\mathcal{A}b_p^\wedge$

The closed symmetric monoidal structure on the category of R -modules (when $R = Z_p$) introduced in Section III.1 does not provide the monoidal structure for $\mathcal{A}b_p^\wedge$. The tensor product over Z_p of two objects in $\mathcal{A}b_p^\wedge$ is not, in general, p -complete.

Define the product $A \otimes_{\mathcal{A}b_p^\wedge} B := (A \otimes_{Z_p} B)_p^\wedge$. Notice Z_p acts as a unit and the symmetry of \otimes_{Z_p} descends to make $\otimes_{\mathcal{A}b_p^\wedge}$ symmetric. The product $\otimes_{\mathcal{A}b_p^\wedge}$ will endow $\mathcal{A}b_p^\wedge$ with a symmetric monoidal structure.

Claim IV.1.1. *If $B \in \text{ob}(\mathcal{A}b_p^\wedge)$ then $\text{Hom}_{\mathcal{A}b}(A, B)$ is p -complete.*

Proof. Let $\phi : \text{Hom}_{\mathcal{A}b}(A, B) \rightarrow (\text{Hom}_{\mathcal{A}b}(A, B))_p^\wedge$ with natural projections $\phi_k : \text{Hom}_{\mathcal{A}b}(A, B) \rightarrow \text{Hom}_{\mathcal{A}b}(A, B)/p^k \text{Hom}_{\mathcal{A}b}(A, B)$.

If $f \in \ker \phi$ then for all $a \in A$, $p^n | f(a)$ for all n . B is p -complete so if p^n divides $f(a)$ for all n , $f(a) = 0$. Since this is true for all $a \in A$, f is the zero morphism and ϕ is injective.

To show ϕ is surjective we construct a sequence of morphisms much as we did in Claim II.3.4 for a given $f = (\dots, f_3, f_2, f_1) \in (\text{Hom}_{\mathcal{A}b}(A, B))_p^\wedge$. Since B is p -complete, we can point-wise define a map from A to B , but we take more care in the construction to guarantee this map is a group morphism.

Choose a lift $\tilde{f}_1 \in \text{Hom}_{\mathcal{A}b}(A, B)$ so that $\phi_1(\tilde{f}_1) = f_1$. Let α'_k be the natural projection from $\text{Hom}_{\mathcal{A}b}(A, B)/p^k \text{Hom}_{\mathcal{A}b}(A, B)$ to $\text{Hom}_{\mathcal{A}b}(A, B)/p^{k-1} \text{Hom}_{\mathcal{A}b}(A, B)$.

Notice $\alpha'_2(\phi_2(\tilde{f}_1) - f_2) = 0 \in \text{Hom}_{\mathcal{A}b}(A, B)/p^1 \text{Hom}_{\mathcal{A}b}(A, B)$, so $\phi_2(\tilde{f}_1) - f_2 \in \ker \alpha'_2$.

Observe

$$\ker \alpha'_2 \cong p (\text{Hom}_{\mathcal{A}b}(A, B)/p^2 \text{Hom}_{\mathcal{A}b}(A, B)),$$

so there exists a $g_2 \in (\text{Hom}_{\mathcal{A}b}(A, B)/p^2 \text{Hom}_{\mathcal{A}b}(A, B))$ with

$$\phi_2(\tilde{f}_1) - f_2 = pg_2, \text{ and } pg_2 \in \ker \alpha'_2.$$

Choose a lift $\tilde{g}_2 \in \text{Hom}_{\mathcal{A}b}(A, B)$ so that $\phi_2(\tilde{g}_2) = g_2$.

Notice the element $\tilde{f}_2 := \tilde{f}_1 - p\tilde{g}_2$ has been designed so that

$$\phi_2(\tilde{f}_2) = \phi_2(\tilde{f}_1 - p\tilde{g}_2) = f_2, \text{ and } \alpha'_2 \phi_2(\tilde{f}_1 - p\tilde{g}_2) = f_1.$$

We can inductively construct a sequence of $\tilde{f}_n = \tilde{f}_1 - \sum_{i=1}^n p^i \tilde{g}_{i+1}$ so that

$$\phi_n \left(\tilde{f}_1 - \sum_{i=1}^n p^i \tilde{g}_{i+1} \right) = f_n, \text{ and } \alpha'_n \phi_n \left(\tilde{f}_1 - \sum_{i=1}^n p^i \tilde{g}_{i+1} \right) = f_{n-1}.$$

Fix $a \in A$. Define $\tilde{f}(a)$ to be the limit of the Cauchy sequence $\tilde{f}_n(a)$ in B . By construction \tilde{f} maps to f under ϕ , so it only remains to show that the point-wise defined f is a group morphism.

Let a and a' be in A . We are comparing

$$\tilde{f}(a + a') = \lim_{n \rightarrow \infty} \left(\tilde{f}_1(a + a') + \sum_{k=1}^n p^k \tilde{g}_{k+1}(a + a') \right)$$

$$\text{to } \tilde{f}(a) + \tilde{f}(a') = \lim_{n \rightarrow \infty} \left(\tilde{f}_1(a) + \sum_{k=1}^n p^k \tilde{g}_{k+1}(a) \right) + \lim_{n \rightarrow \infty} \left(\tilde{f}_1(a') + \sum_{k=1}^n p^k \tilde{g}_{k+1}(a') \right).$$

Since \tilde{f}_1 and \tilde{g}_k are group morphisms for all k it suffices to show the series $\sum_{k=1}^{\infty} \sum_{i=1}^2 p^k \tilde{g}_{k+1}(x_i)$ where $x_1 = a$ and $x_2 = a'$, is invariant under reordering. By results stated in Section II.1 it suffices to show this series converges absolutely in the p -adic norm.

Property 2 from Section II.1 and the observation that $0 \leq |\tilde{g}_{k+1}(x_i)|_p \leq 1$ for all $x_i \in A$ implies

$$|p^k \tilde{g}_{k+1}(x_i)|_p \leq \frac{1}{p^k} |\tilde{g}_{k+1}(x_i)|_p \leq \frac{1}{p^k}.$$

So

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{k=1}^n \sum_{i=1}^2 |p^k \tilde{g}_{k+1}(x_i)|_p &= \lim_{n \rightarrow \infty} \sum_{k=1}^n (|p^k \tilde{g}_{k+1}(a)|_p + |p^k \tilde{g}_{k+1}(a')|_p) \\ &\leq \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{1}{p^k} + \frac{1}{p^k} \right) = \frac{2}{p-1}. \end{aligned}$$

The series defining $\tilde{f}(a+a')$ thus absolutely converges in the p norm implying $\tilde{f}(a+a') = \tilde{f}(a) + \tilde{f}(a')$ so \tilde{f} is a group morphism. \square

Definition IV.1.2. Define the function object for $\mathcal{A}b_p^\wedge$, $F_{\mathcal{A}b_p^\wedge}(A, B)$ as

$$\text{Hom}_{\mathcal{A}b_p^\wedge}(A, B) = \text{Hom}_{\mathcal{A}b}(A, B).$$

The closed condition is justified below:

$$\begin{aligned} \text{Hom}_{\mathcal{A}b_p^\wedge}(A \otimes_{\mathcal{A}b_p^\wedge} B, C) &\cong \text{Hom}_{\mathcal{A}b}(A \otimes_{Z_p} B, \mathcal{F}(C)) && ((-)_p^\wedge, \mathcal{F}) \text{ is an adjoint pair} \\ &\cong \text{Hom}_{\mathcal{A}b}(A, \text{Hom}_{Z_p}(B, \mathcal{F}(C))) && (\otimes_{Z_p}, \text{Hom}_{Z_p}) \text{ is an adjoint pair} \\ &\cong \text{Hom}_{\mathcal{A}b}(A, \text{Hom}_{\mathcal{A}b}(B, \mathcal{F}(C))) && \text{consequence of Lemma II.0.6} \\ &\cong \text{Hom}_{\mathcal{A}b}(A, \text{Hom}_{\mathcal{A}b_p^\wedge}(B, C)) && ((-)_p^\wedge, \mathcal{F}) \text{ is an adjoint pair} \\ &\cong \text{Hom}_{\mathcal{A}b}(A, F_{\mathcal{A}b_p^\wedge}(B, C)) && \text{by definition of } F_{\mathcal{A}b_p^\wedge}(B, C). \end{aligned}$$

This establishes a closed symmetric monoidal structure on $\mathcal{A}b_p^\wedge$.

Section III.3 outlines a method to extend the closed symmetric monoidal category $\mathcal{A}b_p^\wedge$ into a compatible triangulated category $\mathcal{K}(\mathcal{A}b_p^\wedge)$. $\mathcal{K}(\mathcal{A}b_p^\wedge)$ may be thought of as a triangulated subcategory of $\mathcal{K}(Z_p)$, however, the different monoidal functor and coproduct structure on $\mathcal{A}b_p^\wedge$ gives $\mathcal{K}(\mathcal{A}b_p^\wedge)$ a closed symmetric monoidal structure distinct from that of $\mathcal{K}(Z_p)$. For example, consider the chain complex X with zero differentials and $X_i = Z_p$ for all i . In $\mathcal{K}(Z_p)$, $(X \otimes_{\mathcal{K}(Z_p)} X)_0 \cong \bigoplus_{\mathbb{Z}} Z_p$, where as in $\mathcal{K}(\mathcal{A}b_p^\wedge)$, $(X \otimes_{\mathcal{K}(\mathcal{A}b_p^\wedge)} X)_0 \cong \coprod_{\mathbb{Z}} Z_p$.

We record one immediate consequence for any subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ containing $Z_p(0)$ that satisfies the axioms of an SHC.

Claim IV.1.3. *Let \mathcal{D} be a subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ containing the chain complex $Z_p(0)$ that satisfies the axioms of an SHC. Then \mathcal{D} is not algebraic.*

Proof. The construction of \otimes in $\mathcal{K}(\mathcal{A}b_p^\wedge)$ implies

$$Z_p(0) \otimes X \cong X \cong X \otimes Z_p(0)$$

for all $X \in \text{ob}(\mathcal{K}(\mathcal{A}b_p^\wedge))$. Since \mathcal{D} is subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ and $Z_p(0) \in \text{ob}(\mathcal{D})$, $Z_p(0)$ also acts as a unit with respect to the monoidal structure in \mathcal{D} .

We now show that $Z_p(0)$ is not small in \mathcal{D} . Recall there exists elements in $\coprod_{\mathbb{N}} Z_p$ with an infinite number of nonzero entries such as (p, p^2, p^3, \dots) . Since \mathcal{D} contains its coproducts, $\coprod_{\mathbb{N}} Z_p(0) \in \text{ob}(\mathcal{D})$. The morphism F , defined by sending the generator of $Z_p(0)_0$ to $(p, p^2, p^3, \dots) \in (\coprod_{\mathbb{N}} Z_p(0))_0$ is a well defined morphism in $\text{Hom}_{\mathcal{D}}(Z_p(0), \coprod_{\mathbb{N}} Z_p(0))$ but has no corresponding morphism in $\bigoplus_{\mathbb{N}} \text{Hom}_{\mathcal{D}}(Z_p(0), Z_p(0))$. \square

IV.2 Non-existence of Homology Functors on $\mathcal{K}(\mathcal{A}b_p^\wedge)$

We focus on homology functors that return the same groups as standard homology when given a chain complex that is free and finitely generated as an ungraded Z_p -module. Terminology will be introduced before we can state the non-existence results explicitly. The proofs will rely on a lemma and a carefully chosen morphism in $\mathcal{A}b_p^\wedge$ to derive contradictions.

Definition IV.2.1. *Let $\mathcal{B}(\mathcal{A}b_p^\wedge)$ be the full subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ consisting of chain complexes that are free and finitely generated as graded Z_p -modules. Let \mathcal{K} be a triangulated category with $\mathcal{B}(\mathcal{A}b_p^\wedge) \subset \mathcal{K} \subset \mathcal{K}(\mathcal{A}b_p^\wedge)$. Let $\mathcal{E}_i : \mathcal{B}(\mathcal{A}b_p^\wedge) \rightarrow \mathcal{A}$ be a functor to an additive category \mathcal{A} . A functor $\mathcal{H}_i : \mathcal{K} \rightarrow \mathcal{A}$ extends \mathcal{E}_i to \mathcal{K} , if \mathcal{H}_i restricted to $\mathcal{B}(\mathcal{A}b_p^\wedge)$ is naturally equivalent to \mathcal{E}_i . That is for any $F \in \text{Hom}_{\mathcal{B}(\mathcal{A}b_p^\wedge)}(X, Y)$, there exists isomorphisms $\eta_X : \mathcal{E}_i(X) \rightarrow \mathcal{H}_i(X)$ so that*

$$\begin{array}{ccc} \mathcal{E}_i(X) & \xrightarrow{\eta_X} & \mathcal{H}_i(X) \\ H_i(F) \downarrow & \circ & \downarrow \mathcal{H}_i(F) \\ \mathcal{E}_i(Y) & \xrightarrow{\eta_Y} & \mathcal{H}_i(Y) \end{array}$$

The rest of the section is dedicated to proving the following two non-existence results:

Theorem IV.2.2. *Let $\mathcal{B}(\mathcal{A}b_p^\wedge) \subset \mathcal{K} \subset \mathcal{K}(\mathcal{A}b_p^\wedge)$ as triangulated categories. Assume \mathcal{K} is a localizing subcategory and contains arbitrary coproducts, then there exists no homology functor $\mathcal{H}_* : \mathcal{K} \rightarrow \mathcal{A}b$ that extends the standard homology functor H_* to \mathcal{K} .*

Theorem IV.2.3. *Let $\mathcal{B}(\mathcal{A}b_p^\wedge) \subset \mathcal{K} \subset \mathcal{K}(\mathcal{A}b_p^\wedge)$ as triangulated categories. Assume \mathcal{K} is a localizing subcategory and contains arbitrary coproducts. Let $H_i : \mathcal{B}(\mathcal{A}b_p^\wedge) \rightarrow \mathcal{A}b_p^\wedge$ take X to $\ker d_i^X / \text{im } \mathcal{A}b d_{i+1}^X$. There exists no homology functor \mathcal{H}_* that extends H_* to \mathcal{K} .*

The statement of Theorem IV.2.3 requires some justification as an arbitrary $X \in \mathcal{K}(\mathcal{A}b_p^\wedge)$ may not return a p -complete group under standard homology (see Example II.4.3). However, when restricted to $\mathcal{B}(\mathcal{A}b_p^\wedge)$, Claim II.4.4 implies standard homology will return a p -complete group.

Given a functor that extends standard homology $H_*(-)$, there is an analog to the fact $f = H_0(f)$ where $f \in \text{Hom}_{\mathcal{A}b}(A, B)$ and A and B are treated as chain complexes concentrated in degree zero. Recall \mathcal{M}_n is the functor from $\mathcal{A}b_p^\wedge$ to $\mathcal{K}(\mathcal{A}b_p^\wedge)$ that sends an object to the chain complex concentrated in degree n .

Lemma IV.2.4. *Let \mathcal{K} be a subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$ and \mathcal{H} be a homology functor that extends standard homology H to \mathcal{K} . Let A^i and B^k be objects from $\mathcal{A}b_p^\wedge$ that are finitely generated as Z_p -modules. Let*

$$f \in \text{Hom}_{\mathcal{A}b_p^\wedge} \left(\coprod_I A^i, \coprod_K B^k \right) \quad g \in \text{Hom}_{\mathcal{A}b_p^\wedge} \left(\coprod_I A^i, \coprod_K B^k \right) \quad h \in \text{Hom}_{\mathcal{A}b_p^\wedge} \left(\coprod_I A^i, \coprod_K B^k \right)$$

and set

$$\mathcal{M}_0 f = F, \mathcal{M}_0 g = G, \text{ and } \mathcal{M}_0 h = H.$$

There exist isomorphisms in $\mathcal{A}b$ for all n that make the following diagrams commute.

$$\begin{array}{ccc} \coprod_I A^i & \xrightarrow{f} & \coprod_K B^k \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{H}_n \mathcal{M}_n \coprod_I A^i & \xrightarrow{\mathcal{H}_n(F)} & \mathcal{H}_n \mathcal{M}_n \coprod_K B^k \end{array}$$

$$\begin{array}{ccc} \coprod_I A^i & \xrightarrow{g} & \coprod_K B^k \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{H}_n \mathcal{M}_n \coprod_I A^i & \xrightarrow{\mathcal{H}_n(G)} & \mathcal{H}_n \mathcal{M}_n \coprod_K B^k \end{array} \quad \begin{array}{ccc} \coprod_I A^i & \xrightarrow{h} & \coprod_K B^k \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{H}_n \mathcal{M}_n \coprod_I A^i & \xrightarrow{\mathcal{H}_n(H)} & \mathcal{H}_n \mathcal{M}_n \coprod_K B^k \end{array}$$

Proof. The top diagram commutes since \mathcal{H} is a homology functor that extends H . The remaining diagrams will follow from Theorem 22 of [5].

Consider the diagram with the morphisms g and $\mathcal{H}_n(G)$. Theorem 22 in [5] implies there exists an indexing set J and an isomorphism $\psi : \coprod_J C^j \rightarrow \prod_{i \in I} A^i$, where C^j is isomorphic to Z_p or Z/p^l for some l . Denote $\mathcal{M}_n(\psi)$ as Ψ . The result follows since \mathcal{H} is a functor and the following diagram commutes.

$$\begin{array}{ccccc} \prod_I A^i & \xrightarrow[\cong]{\psi^{-1}} & \prod_J C^j & \xrightarrow{g\psi} & \prod_K B^k \\ & & \cong \downarrow & & \downarrow \cong \\ \mathcal{H}_n \mathcal{M}_n \prod_I A^i & \xrightarrow[\cong]{\mathcal{H}_n(\Psi^{-1})} & \mathcal{H}_n \mathcal{M}_n \prod_J C^j & \xrightarrow{\mathcal{H}_n(G\Psi)} & \mathcal{H}_n \mathcal{M}_n \prod_K B^k \end{array}$$

A similar application of Theorem 22 in [5] gives the last diagram involving the morphisms h and $\mathcal{H}_n(H)$. \square

The proofs for Theorem IV.2.2 and IV.2.3 will make use of the following morphisms in $\mathcal{A}b_p^\wedge$. Set the following notation for the remainder of the section.

$$\begin{array}{ccccc} \prod_{\mathbb{N}} Z_p & \xrightarrow{\iota} & \prod_{\mathbb{N}} Z_p & \xrightarrow{\pi_j} & Z_p \\ & \searrow \bar{\omega} & \downarrow \omega' & & \downarrow \times p^j \\ \prod_{\mathbb{N}} Z_p & \xrightarrow{\iota} & \prod_{\mathbb{N}} Z_p & \xrightarrow{\pi_j} & Z_p \end{array}$$

The morphism ω' is induced by the universal property of products in $\mathcal{A}b_p^\wedge$. If we identify $\prod_{\mathbb{N}} Z_p$ with its image in $\prod_{\mathbb{N}} Z_p$, $\bar{\omega}$ is well defined. These maps are related to ω from Example II.4.3 by $\omega = \omega' \circ \iota$.

Let $\mathcal{M}_0\omega$, $\mathcal{M}_0\bar{\omega}$, and $\mathcal{M}_0\omega'$ be denoted by Ω , $\bar{\Omega}$, and Ω' respectively.

Proof of Theorem IV.2.2. Let \mathcal{K} be a localizing subcategory containing arbitrary coproducts and $\mathcal{B}(\mathcal{A}b_p^\wedge) \subset \mathcal{K} \subset \mathcal{K}(\mathcal{A}b_p^\wedge)$ as triangulated categories. Assume \mathcal{H}_* is a homology functor that extends H_* . Let $\psi : \coprod_J Z_p \rightarrow \prod_{\mathbb{N}} Z_p$ be the isomorphism guaranteed to exist from Theorem 22 in [5]. Note that the cardinality of J is greater than that of \mathbb{N} .

We will consider the exact triangle of the form

$$\mathcal{M}_0(\coprod_J Z_p) \xrightarrow{\bar{\Omega} \circ \Psi} \mathcal{M}_0(\prod_{\mathbb{N}} Z_p) \rightarrow \text{Cone}(\bar{\Omega} \circ \Psi).$$

This triangle is in \mathcal{K} since $\mathcal{M}_0(\coprod_J Z_p) \cong \coprod_J(\mathcal{M}_0(Z_p))$ and \mathcal{K} is a localizing subcategory that contains its coproducts. The assumed existence of \mathcal{H} will provide a long exact sequence in $\mathcal{A}b$ with which we can arrive at a contradiction.

Consider the long exact sequence obtained by applying $\mathcal{H}_*(-)$ to the above exact triangle.

$$\begin{aligned} \cdots \rightarrow \mathcal{H}_1(\text{Cone}(\bar{\Omega} \circ \Psi)) \rightarrow \\ \mathcal{H}_0 \mathcal{M}_0 \left(\coprod_J Z_p \right) \xrightarrow{\mathcal{H}(\bar{\Omega} \circ \Psi)} \mathcal{H}_0 \mathcal{M}_0 \left(\coprod_{\mathbb{N}} Z_p \right) \rightarrow \mathcal{H}_0 \text{Cone}(\bar{\Omega} \circ \Psi) \rightarrow \\ \mathcal{H}_{-1} \mathcal{M}_0 \left(\coprod_J Z_p \right) \rightarrow \cdots \end{aligned}$$

For any n not equal to 0, $\mathcal{H}_n(\mathcal{M}_0(\coprod_J Z_p)) \cong \bigoplus_J \mathcal{H}_n(\mathcal{M}_0(Z_p))$ because \mathcal{H}_n preserves coproducts. Since \mathcal{H}_n extends H_n , $\bigoplus_J \mathcal{H}_n(\mathcal{M}_0(Z_p)) \cong \bigoplus H_n(\mathcal{M}_0(Z_p)) \cong 0$ for $n \neq 0$. Similarly $\mathcal{H}_n(\mathcal{M}_0(\coprod_{\mathbb{N}} Z_p)) \cong 0$ when $n \neq 0$. The above long exact sequence thus becomes:

$$0 \rightarrow \mathcal{H}_1(\text{Cone}(\bar{\Omega} \circ \Psi)) \rightarrow \mathcal{H}_0(\mathcal{M}_0(\coprod_J Z_p)) \xrightarrow{\mathcal{H}(\bar{\Omega} \circ \Psi)} \mathcal{H}_0(\mathcal{M}_0(\coprod_{\mathbb{N}} Z_p)) \rightarrow \mathcal{H}_0(\text{Cone}(\bar{\Omega} \circ \Psi)) \rightarrow 0$$

We can further simplify the above long exact sequence by applying Lemma IV.2.4 to $\mathcal{H}(\bar{\Omega} \circ \Psi)$ and reduce to the following long exact sequence.

$$\cdots \rightarrow 0 \rightarrow \bigoplus_J Z_p \xrightarrow{\mathcal{H}(\bar{\Omega} \circ \Psi)} \bigoplus_{\mathbb{N}} Z_p \rightarrow \mathcal{H}_0(\text{Cone}(\bar{\Omega} \circ \Psi)) \rightarrow 0 \rightarrow \cdots$$

The above exact sequence is in $\mathcal{A}b$, but the objects also have a Z_p -modules structure. Note \mathbb{Q}_p is flat so when passed through the functor $- \otimes \mathbb{Q}_p$ the above becomes:

$$0 \rightarrow \bigoplus_J \mathbb{Q}_p \rightarrow \bigoplus_{\mathbb{N}} \mathbb{Q}_p \rightarrow \mathcal{H}_0(\text{Cone}(\bar{\Omega} \circ \Psi)) \otimes \mathbb{Q}_p \rightarrow 0$$

We have thus found an injective map of vector spaces mapping $\bigoplus_J \mathbb{Q}_p$ into a space with strictly smaller cardinality. \square

Proof of Theorem IV.2.3. Let \mathcal{K} be a localizing subcategory containing arbitrary coproducts and $\mathcal{B}(\mathcal{A}b_p) \subset \mathcal{K} \subset \mathcal{K}(\mathcal{A}b_p)$ as triangulated categories. Theorem IV.2.3 results from assuming the existence of a homology functor \mathcal{H}_* that extends H_* to \mathcal{K} and calculating $\mathcal{H}_*(-)$ of the chain complex below. Note that the products written below do exist in \mathcal{K} since Theorem 22 from [5] implies the products can be written as coproducts for some indexing set and \mathcal{K} contains arbitrary coproducts.

$$\begin{array}{ccc}
\text{degree} & & \\
2 & & \prod_{\mathbb{N}} Z_p \\
& & \downarrow [\iota^{-\omega}] \\
1 & & \prod_{\mathbb{N}} Z_p \oplus \prod_{\mathbb{N}} Z_p \\
& & \downarrow [\text{id } \omega'] \\
0 & & \prod_{\mathbb{N}} Z_p
\end{array}$$

Denote the above chain complex by Z and note that Z is isomorphic to the cone of

$$\text{Cone}(-\Omega) \xrightarrow{I} \text{Cone}(-\Omega') \quad \text{and} \quad \text{Cone}(\iota) \xrightarrow{\Phi} \text{Cone}(-\text{Id})$$

where I and Φ are defined respectively by:

$$\begin{array}{ccc}
\prod_{\mathbb{N}} Z_p & \xrightarrow{-\iota} & \prod_{\mathbb{N}} Z_p & & \prod_{\mathbb{N}} Z_p & \xrightarrow{\omega} & \prod_{\mathbb{N}} Z_p \\
\omega \downarrow & & \omega' \downarrow & & -\iota \downarrow & & \text{id} \downarrow \\
\prod_{\mathbb{N}} Z_p & \xrightarrow{-\text{id}} & \prod_{\mathbb{N}} Z_p & & \prod_{\mathbb{N}} Z_p & \xrightarrow{-\omega'} & \prod_{\mathbb{N}} Z_p
\end{array}$$

We will calculate $\mathcal{H}_*(Z)$ by first using $\text{Cone}(I)$. We then repeat the calculation of $\mathcal{H}_*(Z)$ with $\text{Cone}(\Phi)$ and conclude $\mathcal{H}_*(Z)$ is not well defined.

To begin calculating $\mathcal{H}_*(\text{Cone}(I))$ we first consider the two triangles

$$\Delta_{-\Omega} := \mathcal{M}_0(\prod_{\mathbb{N}} Z_p) \xrightarrow{-\Omega} \mathcal{M}_0(\prod_{\mathbb{N}} Z_p) \rightarrow \text{Cone}(-\Omega)$$

and

$$\Delta_{-\Omega'} := \mathcal{M}_0(\prod_{\mathbb{N}} Z_p) \xrightarrow{-\Omega'} \mathcal{M}_0(\prod_{\mathbb{N}} Z_p) \rightarrow \text{Cone}(-\Omega').$$

There is a map of triangles that induces a map between their respective long exact sequences. The same techniques in the proof of Theorem IV.2.2 reduce the long exact sequences to the following.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_1(\text{Cone}(-\Omega)) & \longrightarrow & \mathcal{H}_0(\prod_{\mathbb{N}} Z_p) & \xrightarrow{\mathcal{H}_0(-\Omega)} & \mathcal{H}_0(\prod_{\mathbb{N}} Z_p) & \longrightarrow & \mathcal{H}_0(\text{Cone}(-\Omega)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \mathcal{H}_0(\mathcal{M}_0(\iota)) \downarrow & & \mathcal{H}_0(\text{id}) \downarrow & & \mathcal{H}_0(I) \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{H}_1(\text{Cone}(-\Omega')) & \longrightarrow & \mathcal{H}_0(\prod_{\mathbb{N}} Z_p) & \xrightarrow{\mathcal{H}_0(-\Omega')} & \mathcal{H}_0(\prod_{\mathbb{N}} Z_p) & \longrightarrow & \mathcal{H}_0(\text{Cone}(-\Omega')) & \longrightarrow & 0
\end{array}$$

Lemma IV.2.4 implies $\mathcal{H}_0(-\Omega)$ and $\mathcal{H}_0(-\Omega')$ are injective. The definition of kernel in $\mathcal{A}b_p^\wedge$ thus implies that $\mathcal{H}_1(\text{Cone}(-\Omega)) \cong 0 \cong \mathcal{H}_1(\text{Cone}(-\Omega'))$. This and Lemma IV.2.4 implies that we can simplify the above to:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \prod_{\mathbb{N}} Z_p & \xrightarrow{-\omega} & \prod_{\mathbb{N}} Z_p & \longrightarrow & \mathcal{H}_0(\text{Cone}(-\Omega)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \iota & & \downarrow \text{id} & & \mathcal{H}_0(I) \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & \prod_{\mathbb{N}} Z_p & \xrightarrow{-\omega'} & \prod_{\mathbb{N}} Z_p & \longrightarrow & \mathcal{H}_0(\text{Cone}(-\Omega')) & \longrightarrow & 0
\end{array}$$

The rows are exact in $\mathcal{A}b_p^\wedge$, so

$$\mathcal{H}_0(\text{Cone}(-\Omega)) \cong \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega) \quad \text{and} \quad \mathcal{H}_0(\text{Cone}(-\Omega')) \cong \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega').$$

Let $\pi : \prod_{\mathbb{N}} Z_p \rightarrow \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega)$ and $\pi' : \prod_{\mathbb{N}} Z_p \rightarrow \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega')$ be the natural projections. Remark II.0.7 implies that the image of $\prod_{\mathbb{N}} Z_p$ is dense in $\text{coker}_{\mathcal{A}b_p^\wedge}(-\omega)$ and $\text{coker}_{\mathcal{A}b_p^\wedge}(-\omega')$. Claim II.4.5 then implies π and π' is onto. The following sequence is thus exact in $\mathcal{A}b$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \pi & \longrightarrow & \prod_{\mathbb{N}} Z_p & \xrightarrow{\pi} & \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{id} & & \mathcal{H}_0(I) \downarrow & & \downarrow \\
0 & \longrightarrow & \ker \pi' & \longrightarrow & \prod_{\mathbb{N}} Z_p & \xrightarrow{\pi'} & \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega') & \longrightarrow & 0
\end{array}$$

It was shown in Example II.4.3 that $\ker \pi \cong \prod_{i \in \mathbb{N}} p^i Z_p$. It can also be shown that $\ker \pi' \cong \prod_{i \in \mathbb{N}} p^i Z_p$. Since the above diagram is in $\mathcal{A}b$, the Five Lemma implies $\text{coker}_{\mathcal{A}b_p^\wedge}(-\omega) \cong \text{coker}_{\mathcal{A}b_p^\wedge}(-\omega')$ and $\mathcal{H}_0(I)$ is an isomorphism.

We return to calculating $\mathcal{H}_n(Z)$ by with the triangle

$$\Delta_Z := \text{Cone}(-\Omega) \xrightarrow{I} \text{Cone}(-\Omega') \rightarrow Z.$$

The long exact sequence in $\mathcal{A}b_p^\wedge$ induced by \mathcal{H} simplifies to

$$0 \rightarrow \mathcal{H}_1(Z) \rightarrow \mathcal{H}_0(\text{Cone}(-\Omega)) \xrightarrow{\mathcal{H}_0(I)} \mathcal{H}_0(\text{Cone}(-\Omega')) \rightarrow \mathcal{H}_0(Z) \rightarrow 0.$$

Since $\mathcal{H}_0(I)$ is an isomorphism, $\mathcal{H}_n(Z) = 0$ for all n .

Consider calculating $\mathcal{H}_n(Z)$ with the following triangle instead.

$$\Delta_{Z'} := \text{Cone}(\iota) \xrightarrow{\Omega} \text{Cone}(-\text{id}) \rightarrow Z$$

Note that $\mathcal{H}_n(\text{Cone}(-\text{id})) = 0$ for all n . The long exact sequence will imply that $\mathcal{H}_{n+1}(Z) \cong \mathcal{H}_n(\text{Cone}(\iota))$ for all n . Lemma IV.2.4 implies that $\mathcal{H}_0(\text{Cone}(\iota)) \cong \text{coker}_{\mathcal{A}b_p^\wedge} \iota \neq 0$ thus $\mathcal{H}_1(Z) \neq 0$ which is a contradiction. \square

IV.3 Non-existence of a SHC for $\mathcal{A}b_p^\wedge$ with a Homology Extending $H_*(-)$

Recall an SHC is a localizing subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$. If \mathcal{D} is an SHC containing the p -adic integers in the form of $\mathcal{M}_n(Z_p)$, \mathcal{D} will also contain the chain complexes used in the proof of Theorem IV.2.2. We can thus state the following non-existence result as a corollary to Theorem IV.2.2.

Corollary IV.3.1. *Let \mathcal{D} be an SHC containing a chain complex of the form $\mathcal{M}_n(Z_p)$ that is a subcategory of $\mathcal{K}(\mathcal{A}b_p^\wedge)$. If $\mathcal{H}_* : \mathcal{D} \rightarrow \mathcal{A}b$ is a homology functor, then it does not extend $H_*(-)$.*

IV.4 Small Objects in $\mathcal{K}(\mathcal{A}b_p^\wedge)$

If W is a small object in $\mathcal{K}(\mathcal{A}b_p^\wedge)$, the functor \mathcal{H}^W defined by $\text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(W, -)$ is a homology functor [9]. We thus investigate the small objects of $\mathcal{K}(\mathcal{A}b_p^\wedge)$. To begin, we consider $\mathcal{A}b_p^\wedge$ which naturally sits in $\mathcal{K}(\mathcal{A}b_p^\wedge)$. We will show $\mathcal{A}b_p^\wedge$ has no small objects and state some observations about the small objects in $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

Remark IV.4.1. Let \mathcal{A} be an additive category and let $A, B \in \text{ob}(\mathcal{A})$. If B is not small, the group structure on the Hom sets imply $A \oplus B$ is also not small.

Lemma IV.4.2. \mathbb{Z}/p^r is not small in $\mathcal{A}b_p^\wedge$ for all r .

Proof. Fix r . The morphism f defined by $f(1) = (1, p, p^2, p^3, \dots)$ is a well defined morphism in $\text{Hom}_{\mathcal{A}b_p^\wedge}(\mathbb{Z}/p^r, \coprod_{i=r}^\infty \mathbb{Z}/p^i)$ that has no corresponding morphism in $\bigoplus_{i=r}^\infty \text{Hom}_{\mathcal{A}b_p^\wedge}(\mathbb{Z}/p^r, \mathbb{Z}/p^i)$. Thus, the group \mathbb{Z}/p^r is not small. \square

Lemma IV.4.3. There are no nonzero small objects in $\mathcal{A}b_p^\wedge$.

Proof. Assume A is a nonzero object in $\mathcal{A}b_p^\wedge$. Theorem 22 from [5] states there are indexing sets I_j for $j = 0, 1, 2, \dots, n$, not all empty, so that

$$A \cong \left(\bigoplus_{I_0} Z_p \oplus \bigoplus_{I_1} \mathbb{Z}/p \oplus \bigoplus_{I_2} \mathbb{Z}/p^2 \oplus \dots \right)_p^\wedge.$$

We will show A is not small by considering the two cases $I_0 \neq \emptyset$ and $I_0 = \emptyset$.

If $I_0 \neq \emptyset$, let $B \cong Z_p$ and C be the remaining factors so that $A \cong (B \oplus C)_p^\wedge$. Remark II.3.3 lets us write $B \oplus C$ as a product, which commutes with p -completion. Thus $A \cong B \oplus C_p^\wedge$. The proof of Claim IV.1.3 showed Z_p is not small so Remark IV.4.1 implies A is not small.

If $I_0 = \emptyset$ we can find an r so that $I_r \neq \emptyset$ and let $B \cong \mathbb{Z}/p^r$. Let C be the remaining factors and repeat the arguments above so we can write $A \cong B \oplus C_p^\wedge$. Lemma IV.4.2 states B is not small, so Remark IV.4.1 again implies A is not small. \square

Recall the functor $\mathcal{M}_j : \mathcal{A}b_p^\wedge \rightarrow \mathcal{K}(\mathcal{A}b_p^\wedge)$ takes objects $A \in \mathcal{A}b_p^\wedge$ to the chain complex with A in the j^{th} degree and zeros in all others. Chain homotopies between objects of the form $\mathcal{M}_j(A)$ and $\mathcal{M}_j(B)$ for $A, B \in \text{ob}(\mathcal{A}b_p^\wedge)$ are thus not possible and $\text{Hom}_{\mathcal{A}b_p^\wedge}(A, B) \cong \text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(\mathcal{M}_j(A), \mathcal{M}_j(B))$. The above lemma implies the following.

Corollary IV.4.4. There are no small objects in $\mathcal{K}(\mathcal{A}b_p^\wedge)$ of the form $\mathcal{M}_j(A)$ for some $A \in \text{ob}(\mathcal{A}b_p^\wedge)$.

Claim IV.4.5. Let X^k be the chain complex whose only nonzero degrees are the n^{th} and the $n+1^{\text{th}}$. Let the nontrivial differential be $d_{n+1}^{X^k} : Z_p \rightarrow Z_p$ where $d_{n+1}^{X^k}(1) = p^k$. The object X^k is not small in $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

Proof. Without loss of generality assume $n = 0$, so X^k is the chain complex with Z_p in the first and zeroth degrees and the nontrivial differential is $d_1^X(1) = p^k$. Fix k , we will produce Y^i and show $\text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(X^k, \coprod_{\mathbb{N}} Y^i)$ is strictly larger than $\bigoplus_{\mathbb{N}} \text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(X^k, Y^i)$.

Define $F^i : X^k \rightarrow Y^i$ to be the following chain map where all other degrees are zero.

$$\begin{array}{ccc} \text{degree} & X^k & \xrightarrow{F^i} Y^i \\ \\ 1 & Z_p & \xrightarrow{0} 0 \\ & d_1^X \downarrow & \downarrow 0 \\ 0 & Z_p & \xrightarrow{\times p^{\max(k, i-k)}} Z/p^i \end{array}$$

Notice that the collection of F^i induce a map in $\text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(X^k, \coprod_{\mathbb{N}} Y^i)$ that we denote by $\coprod F^i$. In particular, $(\coprod F^i)_0(1) \in \coprod_{\mathbb{N}} Z/p^i$ will be of the form:

$$(0, 0, \dots, p^k, p^k, \dots, p^k, p^{k+1}, p^{k+2}, p^{k+3}, \dots)$$

where the first nonzero entry is in the Z/p^{k+1} factor and p^{k+1} is in the Z/p^{2k+1} factor.

The placement of the zero differentials in X^k and $\coprod_{\mathbb{N}} Y^i$ will force any maps homotopic to $\coprod F^i$ to be equal to $\coprod F^i$ in degree 0. Notice $(\coprod F^i)_0 \in \text{Hom}_{\mathcal{A}b_{\hat{p}}}(Z_p, \coprod_{\mathbb{N}} Z/p^i)$ but there is no corresponding morphism in $\bigoplus_{\mathbb{N}} \text{Hom}_{\mathcal{A}b_{\hat{p}}}(Z_p, Z/p^i)$. We thus have a morphism $\coprod F^i$ in $\text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(X^k, \coprod_{\mathbb{N}} Y^i)$ with no corresponding morphism in $\bigoplus_{\mathbb{N}} \text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(X^k, Y^i)$ implying X^k is not small. \square

Claim IV.4.6. *If X is small in $\mathcal{K}(\mathcal{A}b_{\hat{p}})$, then $H_n(X) = 0$ for all but finitely many n .*

Proof. Assume $H_n(X) \neq 0$ infinitely often. Let I be an index set so that $i \in I$ when $H_i(X) \neq 0$. For each $i \in I$, a chain complex Y^i and chain map $F^i : X \rightarrow Y^i$ will be constructed so that $Y^i \not\cong 0$ and $F^i \not\cong 0$ in $\mathcal{K}(\mathcal{A}b_{\hat{p}})$. A map is constructed that exists in $\text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(X, \coprod_{i \in I} Y^i)$ that does not correspond to a morphism in $\bigoplus_{i \in I} \text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(X, Y^i)$, implying that X is not small.

Define the nonzero degrees of Y^i and F^i by the following.

$$\begin{array}{ccc}
 X_{i+2} & \xrightarrow{\overline{d_{i+2}}} & \ker d_{i+1}^X \\
 d_{i+2}^X \downarrow & \circ & \downarrow \iota \\
 X_{i+1} & \xrightarrow{id} & X_{i+1} \\
 d_{i+1}^X \downarrow & \circ & d_{i+1}^X \downarrow \\
 X_i & \xrightarrow{id} & X_i \\
 d_i^X \downarrow & \circ & d_i^X \downarrow \\
 X_{i-1} & \xrightarrow{id} & X_{i-1}
 \end{array}$$

where ι and $\overline{d_{i+2}}$ are induced by the universal property of $\ker d_{i+1}^X$. By construction $H_i(F^i)$ is an isomorphism so $F^i \not\cong 0$ and $Y^i \not\cong 0$ in $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

Consider the map $\prod_{i \in I} F^i : X \rightarrow \prod_{i \in I} Y^i$, that has a nontrivial image in each factor in some degree. Recall products in $\mathcal{K}(\mathcal{A}b_p^\wedge)$ are taken degree wise. For each $j \in \mathbb{N}$,

$$\left(\prod_I Y^i \right)_j \cong Y_j^{j+1} \oplus Y_j^j \oplus Y_j^{i-1} \oplus Y_j^{j-2}$$

so Remark II.3.3 implies $(\prod_{i \in I} Y^i)_j \cong (\coprod_{i \in I} Y^i)_j$ for all j . The map $\prod_{i \in I} F^i$ is thus a well defined morphism in $\text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(X, \prod_{i \in I} Y^i)$ even though there is no corresponding map in $\bigoplus_{i \in I} \text{Hom}_{\mathcal{K}(\mathcal{A}b_p^\wedge)}(X, Y^i)$. \square

We make the following conjecture:

Conjecture IV.4.7. There are no small objects in $\mathcal{K}(\mathcal{A}b_p^\wedge)$.

CHAPTER V

HOMOLOGY FUNCTOR CANDIDATES THAT EXTEND $\text{Hom}_{\mathcal{K}(\mathcal{A}b)}(\mathcal{M}_n(\mathbb{Z}), -)$

Theorems IV.2.2 and IV.2.3 state that there are no examples of homology functors from $\mathcal{K}(\mathcal{A}b\hat{p})$ to either $\mathcal{A}b$ or $\mathcal{A}b\hat{p}$ that extend standard homology. We record here the particular problems that arise for a number of homology functor candidates.

V.1 Properties of the Functor $\text{Hom}_{\mathcal{K}(\mathcal{A}b)}(\mathcal{M}_n(\mathbb{Z}), -)$

It is worth returning to the definition of $H_n(-)$. The description “kernel mod image” introduces ambiguity since the image in $\mathcal{A}b\hat{p}$ differs from the image in $\mathcal{A}b$. We can define $H_n(-) : \mathcal{K}(\mathcal{A}b) \rightarrow \mathcal{A}b$ for each n by $H_n(X) = \text{Hom}_{\mathcal{K}(\mathcal{A}b)}(\mathcal{M}_n(\mathbb{Z}), X)$. Given our interest in $\mathcal{K}(\mathcal{A}b\hat{p})$ we instead restrict to the subcategory $\mathcal{K}(\mathcal{A}b\hat{p})$ and make the following definition.

Definition V.1.1. *Define standard homology as the functor $H_n(-) : \mathcal{K}(\mathcal{A}b\hat{p}) \rightarrow \mathcal{A}b$ where $H_n(X) = \text{Hom}_{\mathcal{K}(\mathcal{A}b)}(\mathcal{M}_n(\mathbb{Z}), X)$.*

This definition lets us write $H_n(-) \cong \ker d_n / \text{im}_{\mathcal{A}b} d_{n+1}$ where both the image and cokernel is taken with respect to $\mathcal{A}b$ and not $\mathcal{A}b\hat{p}$. We note $H_n(-)$ cannot be treated as a functor to $\mathcal{A}b\hat{p}$ since $H_n(X)$ may not, in general, be p -complete. An example of $H_n(X) \notin \text{ob}(\mathcal{A}b\hat{p})$ can be constructed using ω from Example II.4.3.

Claim V.1.2. *$H_n(-)$ is naturally equivalent to the functor $\text{Hom}_{\mathcal{K}(\mathcal{A}b\hat{p})}(\mathcal{M}_n(\mathbb{Z}_p), -)$*

Proof. Let $X \in \mathcal{K}(\mathcal{A}b\hat{p})$. Claim II.3.4 states that $\ker d_n$ is p -complete. The adjoint condition implies

$$\ker d_n \cong \text{Hom}_{\mathcal{A}b}(\mathbb{Z}, \ker d_n) \cong \text{Hom}_{\mathcal{A}b\hat{p}}(\mathbb{Z}_p, \ker d_n)$$

and

$$\text{Hom}_{\mathcal{A}b}(\mathbb{Z}, X_{n+1}) \cong \text{Hom}_{\mathcal{A}b\hat{p}}(\mathbb{Z}_p, X_{n+1}).$$

The definition of chain maps and chain homotopy then imply $\text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(\mathcal{M}_n Z_p, -)$ is naturally equivalent to $H_n(-) = \text{Hom}_{\mathcal{K}(\mathcal{A}b)}(\mathcal{M}_n(\mathbb{Z}), -)$. \square

Given an exact triangle $X \rightarrow Y \rightarrow Z$ in $\mathcal{K}(\mathcal{A}b_{\hat{p}})$, the functor $H_*(-)$ induces a long exact sequence in $\mathcal{A}b$. The proof is straightforward and makes heavy use of the cone structure used when defining exact triangles [6].

The functor $H_*(-)$ fails to preserve coproducts in general and thus does not qualify as a homology functor. The following provides an example of when $H_*(\coprod_I X^j) \not\cong \bigoplus_I H_*(X^j)$.

Example V.1.3. Let X^j be $\mathcal{M}_0(Z_p)$ for all j . Since the differentials are all zero we can compute $H_*(\coprod_{\mathbb{N}} X^j)$ and $\bigoplus_{\mathbb{N}} H_*(X^j)$ directly.

$$H_0(\coprod_{\mathbb{N}} X^j) \cong \prod_{\mathbb{N}} Z_p \not\cong \bigoplus_{\mathbb{N}} Z_p \cong \bigoplus_{\mathbb{N}} H_0(X^j)$$

V.2 Properties of the Functor $(\text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(\mathcal{M}_n(Z_p), -))_{\hat{p}}$

The previous section allows us to write $(\text{Hom}_{\mathcal{K}(\mathcal{A}b_{\hat{p}})}(\mathcal{M}_n(Z_p), -))_{\hat{p}}$ compactly as $(H_n(-))_{\hat{p}}$. Both $\mathcal{A}b$ and $\mathcal{A}b_{\hat{p}}$ may be target categories for the functor $(H_n(-))_{\hat{p}}$. Since the target categories involved have different coproducts we distinguish the different functors explicitly.

Definition V.2.1. Define $(H_n(-))_{\hat{p}} : \mathcal{K}(\mathcal{A}b_{\hat{p}}) \rightarrow \mathcal{A}b_{\hat{p}}$ by taking X to $(H_n(X))_{\hat{p}}$.

Define $(\mathcal{H}_n(-))_{\hat{p}} : \mathcal{K}(\mathcal{A}b_{\hat{p}}) \rightarrow \mathcal{A}b$ by taking X to $(H_n(X))_{\hat{p}}$.

The chain complexes given in Example V.1.3 provides a situation in which

$$(\mathcal{H}_0(\coprod_{\mathbb{N}} X^j))_{\hat{p}} \not\cong \bigoplus_{\mathbb{N}} (\mathcal{H}_0(X^j))_{\hat{p}}.$$

The functor $(\mathcal{H}_n(-))_{\hat{p}}$ is thus not a homology functor.

Before examining the second functor $(H_n(-))_{\hat{p}}$, we consider the object $(H_n(X))_{\hat{p}}$ where $X \in \text{ob}(\mathcal{K}(\mathcal{A}b_{\hat{p}}))$. Let π be the natural map in $\mathcal{A}b$ from $\ker d_n^X$ to $H_n(X)$. Notice Claim II.4.5 implies $\pi_{\hat{p}} : \ker d_n^X \rightarrow (H_n(X))_{\hat{p}}$ is also onto so $(H_n(X))_{\hat{p}}$ may also be treated as a quotient of $\ker d_n^X$. We will show

$$0 \rightarrow \overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{\ker d_n^X} \rightarrow \ker d_n^X \xrightarrow{\pi_{\hat{p}}} (H_n(X))_{\hat{p}} \rightarrow 0 \quad (\text{V.1})$$

is exact in $\mathcal{A}b$, where $\overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{\ker d_n^X}$ denotes the closure of $\text{im}_{\mathcal{A}b} d_{n+1}^X$ with respect to $\ker d_n^X$. Since $\mathcal{A}b$ is an abelian category we will be able to compare $H_n(X)$ and $(H_n(X))_p^\wedge$ more directly with this fact.

To show V.1 is exact, we work in $\mathcal{A}b$ and consider $\ker \pi_p^\wedge$. By definition of kernel:

$$\begin{aligned} \ker \pi_p^\wedge &= \{x \in \ker d_n^X \mid \pi_p^\wedge x = 0 \in (H_n(X))_p^\wedge\} \\ &= \{x \in \ker d_n^X \mid \text{for all } k \text{ there exists } y_k \in \ker d_n^X \text{ with } \pi_p^\wedge x = p^k \pi_p^\wedge y_k\} \\ &= \{x \in \ker d_n^X \mid \forall k \exists y_k \in \ker d_n^X \text{ and } z_k \in \text{im}_{\mathcal{A}b} d_{n+1}^X \text{ so that } x = p^k y_k + z_k\} \\ &= \overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{\ker d_n^X} \end{aligned}$$

The following example shows the functor $(H_*(-))_p^\wedge$ does not take exact triangles to long exact sequences.

Example V.2.2. Consider,

$$\begin{array}{ccccccc} \text{(degree)} & 0 & & 0 & & \prod_{\mathbb{N}} Z_p \oplus 0 & \\ & \downarrow & & \downarrow & & \downarrow d_2^{\text{Cone } F} & \\ (1) & \prod_{\mathbb{N}} Z_p & \xrightarrow{F_1} & \prod_{\mathbb{N}} Z_p & \xrightarrow{\iota_1} & \prod_{\mathbb{N}} Z_p \oplus \prod_{\mathbb{N}} Z_p & \\ & \downarrow d_1^X & & \downarrow d_1^Y & & \downarrow d_1^{\text{Cone } F} & \\ (0) & \prod_{\mathbb{N}} Z_p & \xrightarrow{F_0} & \prod_{\mathbb{N}} Z_p & \xrightarrow{\iota_0} & 0 \oplus \prod_{\mathbb{N}} Z_p & \\ & \downarrow d_0^X & & \downarrow d_0^Y & & \downarrow & \\ (-1) & 0 & \longrightarrow & 0 & \xrightarrow{\iota_{-1}} & 0 \oplus 0 & \\ & \vdots & & \vdots & & \vdots & \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

We refer to the chain complexes on the left, center, and right as X , Y , and Z respectively. If we use the same notation used in Section IV.2, the nontrivial differentials d_1^X and d_1^Y will be $\bar{\omega} \circ \iota$ and $\bar{\omega}$ respectively. Explicitly, if we treat elements of X_1 and Y_1 as column vectors

$$d_1^X = \begin{bmatrix} p & 0 & 0 & \dots \\ 0 & p^2 & 0 & \\ 0 & 0 & p^3 & \\ \vdots & & & \ddots \end{bmatrix} \quad \text{and} \quad d_1^Y = \begin{bmatrix} p & 0 & 0 & \dots \\ 0 & p^2 & 0 & \\ 0 & 0 & p^3 & \\ \vdots & & & \ddots \end{bmatrix}.$$

Set F_1 to be the natural inclusion and F_0 to be the identity map. Notice since Z is the cone of the map between X and Y , this is an exact triangle in $\mathcal{K}(\mathcal{A}b_{\hat{p}})$.

For any $W \in \text{ob}(\mathcal{K}(\mathcal{A}b_{\hat{p}}))$,

$$(H_n(W))_{\hat{p}} \cong (\ker d_n^W / \text{im}_{\mathcal{A}b} d_{n+1}^W)_{\hat{p}} \cong \ker d_n^W / \overline{\text{im}_{\mathcal{A}b} d_{n+1}^W}^{\ker d_n^W}.$$

Let π_p^W be the surjection from $\ker d_n^W$ to $(H_n(W))_{\hat{p}}$. The universal property of quotients in $\mathcal{A}b$ gives the following.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \searrow & & \searrow & \\
 \text{im}_{\mathcal{A}b} d_{n+1}^W & \xrightarrow{\quad} & \ker d_n^W & \xrightarrow{\pi_p^W} & \ker d_n^W / \overline{\text{im}_{\mathcal{A}b} d_{n+1}^W} \cong (H_n(W))_{\hat{p}} \\
 & \searrow & \downarrow & \nearrow & \\
 & & \ker d_n^W / \text{im}_{\mathcal{A}b} d_{n+1}^W \cong H_n(W) & &
 \end{array}$$

Commutativity of the above diagram implies the induced map $H_n(W) \rightarrow (H_n(W))_{\hat{p}}$ is a surjection. Recall $H_*(-)$ takes exact triangles to long exact sequences in $\mathcal{A}b$ and consider the following commutative diagram.

$$\begin{array}{ccccccccccc}
 \cdots & \rightarrow & H_1(X) & \xrightarrow{F_*} & H_1(Y) & \xrightarrow{\iota_*} & H_1(Z) & \xrightarrow{\delta_*} & H_0(X) & \cdots & \rightarrow \\
 & & \downarrow & & \beta \downarrow & & \gamma \downarrow & & \alpha \downarrow & & \\
 \cdots & \rightarrow & (H_1(X))_{\hat{p}} & \xrightarrow{\hat{F}_*} & (H_1(Y))_{\hat{p}} & \xrightarrow{\hat{\iota}_*} & (H_1(Z))_{\hat{p}} & \xrightarrow{\hat{\delta}_*} & (H_0(X))_{\hat{p}} & \cdots & \rightarrow
 \end{array}$$

The element $[(p, p^2, p^3, \dots)]$ is nonzero in $H_0(X)$ but divisible by p^n for all n and so is in the kernel of α . Note that $F_*[(p, p^2, p^3, \dots)] = [(p, p^2, p^3, \dots)] = [d_1^Y(1, 1, 1, \dots)] = [0] \in H_0(Y)$. Since the top row in the above diagram is exact, there is a nonzero element in $H_1(Z)$ that maps to $[p, p^2, p^3, \dots]$. We will show $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)]$ is an element of $H_1(Z)$, that it maps to $[(p, p^2, p^3, \dots)]$ under δ_* , and it is not in the kernel of γ . A diagram chase will then imply that $(H_*(-))_{\hat{p}}$ does not take exact triangles to long exact sequences.

Consider $((p, p^2, p^3, \dots), (1, 1, 1, \dots)) \in Z_1$. Note that

$$d_1^Z((p, p^2, p^3, \dots), (1, 1, 1, \dots)) = (-d_0^X(p, p^2, p^3, \dots), d_1^Y(1, 1, 1, \dots) - F_0(p, p^2, p^3, \dots)) = (0, 0)$$

thus $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \in H_1(Z)$. We see also that

$$\delta_*[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] = [p, p^2, p^3, \dots].$$

To verify $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \notin \ker \gamma$ we will show that $p \nmid [(p, p^2, p^3, \dots), (1, 1, 1, \dots)]$ in $H_1(Z)$. Since $\ker d_1^Z \subset Z_1$, it suffices to show $p \nmid [(p, p^2, p^3, \dots), (1, 1, 1, \dots)]$ in $Z_1/\text{im}_{\mathcal{A}b} d_2^Z$. Consider the lift $((p, p^2, p^3, \dots), (1, 1, 1, \dots))$ in Z_1 . The second coordinate $(1, 1, 1, \dots) \in \prod_{\mathbb{N}} Z_p \subset Z_1$, has an infinite number of nonzero entries whose respective p -norms do not tend to zero. The image of d_2^Z restricted to $\prod_{\mathbb{N}} Z_p \subset Z_1$, however is the natural embedding of the coproduct in $\prod_{\mathbb{N}} Z_p$ that cannot alter an infinite number of entries if the p -norm does not tend to zero. Thus $p \nmid [(p, p^2, p^3, \dots), (1, 1, 1, \dots)]$ in $Z_1/\text{im}_{\mathcal{A}b} d_2^Z$, so $p \nmid [(p, p^2, p^3, \dots), (1, 1, 1, \dots)]$ in $H_1(Z)$.

Assume towards contradiction that $(H_*(-))_p^\wedge$ takes exact triangles to long exact sequences in $\mathcal{A}b$. Commutativity of the above diagram implies $\hat{\delta}_* \gamma[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] = [0]$, thus there exists a $[0] \neq [y] \in (H_1(Y))_p^\wedge$ such that $\hat{\iota}_*([y]) = \gamma[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \in (H_1(Z))_p^\wedge$. However $H_1(Y) = 0$, thus $(H_1(Y))_p^\wedge = 0$, implying that $[y] = [0]$ so we have our contradiction.

This example can be generalized by letting d_1^X be any injective map that is not a kernel. Define Y_1 to be the kernel of d_1^X . The failure of injectives being kernels thus provides a family of counterexamples.

V.3 Properties of the Categorically Defined “Kernel Mod Image”

Let X be an object in $\mathcal{K}(\mathcal{A}b_p^\wedge)$. A categorical definition of homology from $\mathcal{K}(\mathcal{A}b_p^\wedge)$ to $\mathcal{A}b_p^\wedge$ is provided by the following diagram.

$$\begin{array}{ccccc}
 & & \text{coker}_{\mathcal{A}b_p^\wedge} d_n^X & & \\
 & & \uparrow & & \\
 \text{im}_{\mathcal{A}b_p^\wedge} d_{n+1}^X & \xrightarrow{\quad 0 \quad} & X_n & \xrightarrow{d_n^X} & X_{n-1} \\
 & \searrow \text{dotted} & \uparrow & \nearrow 0 & \\
 & & \ker d_n^X & \xrightarrow{\hat{\pi}} & \left(\frac{\ker d_n^X}{\text{im}_{\mathcal{A}b_p^\wedge} d_{n+1}^X} \right)_p^\wedge
 \end{array}$$

Denote $\left(\frac{\ker d_n^X}{\text{im}_{\mathcal{A}b_p} d_{n+1}^X}\right)_p$ by $H_n^C(X)$. We perform an analysis similar to that done in Section V.2 to write the following exact sequence in $\mathcal{A}b$:

$$0 \rightarrow \overline{\text{im}_{\mathcal{A}b_p} d_{n+1}^X}^{\ker d_n^X} \rightarrow \ker d_n^X \xrightarrow{\hat{\pi}} H_n^C(X) \rightarrow 0. \quad (\text{V.2})$$

To justify V.2, we work in $\mathcal{A}b$ and consider the kernel of $\hat{\pi}$. By definition:

$$\begin{aligned} \ker \hat{\pi} &= \{x \in \ker d_n^X \mid \hat{\pi}x = 0 \in H_n^C(X)\} \\ &= \{x \in \ker d_n^X \mid \text{for all } k \text{ there exists } y_k \in \ker d_n^X \text{ so that } \hat{\pi}x = p^k \hat{\pi}y_k\} \\ &= \{x \in \ker d_n^X \mid \forall k \exists y_k \in \ker d_n^X \text{ and } z_k \in \text{im}_{\mathcal{A}b_p} d_{n+1}^X \text{ so that } x = p^k y_k + z_k\} \\ &= \overline{\text{im}_{\mathcal{A}b_p} d_{n+1}^X}^{\ker d_n^X} \end{aligned}$$

We can further refine the sequence in V.2 to the short exact sequence in $\mathcal{A}b$:

$$0 \rightarrow \overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{X_n} \rightarrow \ker d_n^X \xrightarrow{\hat{\pi}} H_n^C(X) \rightarrow 0. \quad (\text{V.3})$$

To see this recall for an arbitrary map $f \in \text{Hom}_{\mathcal{A}b_p}(A, B)$, that $\text{im}_{\mathcal{A}b_p} f \cong \overline{\text{im}_{\mathcal{A}b} f}^B$ where $\overline{\text{im}_{\mathcal{A}b} f}^B$ denotes the closure of $\text{im}_{\mathcal{A}b} f$ with respect to the ambient group B . Then, $\text{im}_{\mathcal{A}b_p} d_{n+1}^X \cong \overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{X_n}$. The above calculation of $\ker \hat{\pi}$ thus shows $\ker \hat{\pi} = \overline{\text{im}_{\mathcal{A}b_p} d_{n+1}^X}^{\ker d_n^X} = \overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{X_n}$ where the first closure is taken with respect to X_n and the second with respect to $\ker d_n^X$. Since $\ker d_n^X$ a subset of X_n , we can simply write $\ker \hat{\pi} = \overline{\text{im}_{\mathcal{A}b_p} d_{n+1}^X}^{X_n}$. Thus the above sequence is exact in $\mathcal{A}b$ and we can write $H_n^C(X) \cong \ker d_n^X / \overline{\text{im}_{\mathcal{A}b} d_{n+1}^X}^{X_n}$ where the cokernel is taken in $\mathcal{A}b$.

Once we have $H_n^C(W) \cong \ker d_n^W / \overline{\text{im}_{\mathcal{A}b} d_{n+1}^W}^W$ we can use the universal property of quotients in $\mathcal{A}b$. The following diagrams guarantee surjections from $H_n(-)$ and $(H_n(-))_p^\wedge$ onto $H_n^C(-)$.

$$\begin{array}{ccc} & 0 & \\ & \curvearrowright & \\ \text{im}_{\mathcal{A}b} d_{n+1}^X & \longrightarrow & \ker d_n^W \xrightarrow{\hat{\pi}} H_n^C(W) \\ & \searrow & \downarrow \\ & & H_n(W) \end{array} \quad \begin{array}{ccc} & 0 & \\ & \curvearrowright & \\ \overline{\text{im}_{\mathcal{A}b} d_{n+1}^W}^{\ker d_n^W} & \longrightarrow & \ker d_n^W \xrightarrow{\hat{\pi}} H_n^C(W) \\ & \searrow & \downarrow \\ & & (H_n(W))_p^\wedge \end{array}$$

$H_*^C(-)$ does not send exact triangles to long exact sequences and the counterexample used for $(H_n(-))_p^\wedge$ will again work here. Assume the same notation used in Example V.2.2. Notice that $X_0 = \ker d_0^X$, so $\overline{\text{im}_{\mathcal{A}b} d_1^X}^{X_0} = \overline{\text{im}_{\mathcal{A}b} d_1^X}^{\ker d_0^X}$. Thus $H_0^C(X) \cong (H_0(X))_p^\wedge$. In particular we still have an element $[p, p^2, p^3, \dots] \in H_0(X)$ that is sent to zero under the completion morphism α .

We record this information in the following commutative diagram.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_1(X) & \xrightarrow{F_*} & H_1(Y) & \xrightarrow{\iota_*} & H_1(Z) & \xrightarrow{\delta_*} & H_0(X) & \cdots \\
& & \downarrow & & \downarrow \beta & & \downarrow \gamma & & \downarrow \alpha & \\
\cdots & \longrightarrow & (H_1(X))_p^\wedge & \xrightarrow{\hat{F}_*} & (H_1(Y))_p^\wedge & \xrightarrow{\hat{\iota}_*} & (H_1(Z))_p^\wedge & \xrightarrow{\hat{\delta}_*} & (H_0(X))_p^\wedge & \cdots \\
& & \downarrow & & \downarrow \beta^C & & \downarrow \gamma^C & & \downarrow \cong & \\
\cdots & \longrightarrow & H_1^C(X) & \xrightarrow{F_*^C} & H_1^C(Y) & \xrightarrow{\iota_*^C} & H_1^C(Z) & \xrightarrow{\delta_*^C} & H_0^C(X) & \cdots
\end{array}$$

Consider the element:

$$[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \in H_1^C(Z).$$

Example V.2.2 showed $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \in (H_1(Z))_p^\wedge$ is in the kernel of $\hat{\delta}_*$ so commutivity in the above diagram implies $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \in H_1^C(Z)$ is in the kernel of δ_*^C . We will show $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \neq 0$ in $H_1^C(Z)$, but $H_1^C(Y) = 0$ which will imply $H_*^C(-)$ does not take exact triangles to long exact sequences.

Recall $H_1^C(Z) \cong \ker d_1^Z / \overline{\text{im}_{\mathcal{A}b} d_2^Z}^{Z_1}$, so to show $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \neq 0$ in $H_1^C(Z)$ it suffices to show $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \notin \overline{\text{im}_{\mathcal{A}b} d_2^Z}^{Z_1}$. If $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \in \overline{\text{im}_{\mathcal{A}b} d_2^Z}^{Z_1}$ there would exist elements $x_k \in \text{im}_{\mathcal{A}b} d_2^Z$ and $y_k \in Z_1$ with

$$((p, p^2, p^3, \dots), (1, 1, 1, \dots)) = x_k + p^k y_k$$

which implies that $p^k [(p, p^2, p^3, \dots), (1, 1, 1, \dots)]$ in $Z_1 / \overline{\text{im}_{\mathcal{A}b} d_2^Z}$. Example V.2.2 showed this is not possible, thus $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] \neq 0$ in $H_1^C(Z)$ which is what we wanted.

The difference between $H_*^C(-)$ and $(H(-))_p^\wedge$ is subtle but they are distinct as the following example shows.

Example V.3.1. $(H_n X)_p \not\cong H_n^C(X)$

Define X to be the chain complex with nonzero degrees shown below.

$$\begin{array}{ccc}
 \text{degree} & & X \\
 & & \prod_{\mathbb{N}} Z_p \\
 & & \downarrow d_3^X \\
 & & \prod_{\mathbb{N}} Z_p \\
 & & \downarrow d_2^X \\
 & & \prod_{\mathbb{N}} Z/p^k
 \end{array}$$

Let d_2^X project the k th factor of $\prod Z_p$ onto Z/p^k and d_3^X send the generator in the k^{th} entry to $(p, p^2, \dots, p^k, 0, \dots)$.

Consider the element $(p, p^2, p^3, \dots) \in \prod Z_p = X_2$. Note this element is not in $\text{im}_{\mathcal{A}b} d_3^X$. We can approximate this element in X_2 with the sequence $\{d_3^X(1, 0, 0, \dots), d_3^X(0, 1, 0, \dots), d_3^X(0, 0, 1, \dots), \dots\} \subset \text{im}_{\mathcal{A}b} d_3^X$. The difference $(p, p^2, p^3, \dots) - d_3^X(0, 0, \dots, 1, 0, \dots)$ is divisible by p^k in X_2 , thus (p, p^2, p^3, \dots) is contained within $\overline{\text{im}_{\mathcal{A}b} d_3^X}^{X_2}$ and $[(p, p^2, p^3, \dots), (1, 1, 1, \dots)] = [0]$ in $H_2^C(X)$. However, there exists no such sequences that approximate (p, p^2, p^3, \dots) with respect to $\ker d_2^X$ so (p, p^2, p^3, \dots) is a nonzero element in $(H_2 X)_p$.

V.4 Conditions so that $H_n^C(X) \cong (H_n(X))_p$

Though $(H_n(-))_p \not\cong H_n^C(-)$ on $\mathcal{K}(\mathcal{A}b_p)$, there exists subcategories \mathcal{K} of $\mathcal{K}(\mathcal{A}b_p)$ in which $H_*^C(-)$ is naturally equivalent to $(H_*(-))_p$. We provide a condition on \mathcal{K} that will imply $H_*^C(-) \cong (H_*(-))_p$ when treated as functors from \mathcal{K} .

Claim V.4.1. $(H_n(Y))_p$ is naturally equivalent to $H_n^C(Y)$ if and only if there exists an N so that there is no $y \in Y$ with p^i torsion when $i > N$.

Proof. The necessity of the condition is provided in Example V.3.1.

To show the condition is sufficient, recall $(H_n(Y))_p \cong \ker d_n^Y / \overline{\text{im}_{\mathcal{A}b} d_{n+1}^Y}^{\ker d_n^Y}$ and $H_n^C(Y) \cong \ker d_n^Y / \overline{\text{im}_{\mathcal{A}b} d_{n+1}^Y}^{Y_n}$ where we are considering $\text{im}_{\mathcal{A}b} d_{n+1}^Y$ under two different closures. Notice that we have an inclusion from $\overline{\text{im}_{\mathcal{A}b} d_{n+1}^Y}^{\ker d_n^Y}$ into $\overline{\text{im}_{\mathcal{A}b} d_{n+1}^Y}^{Y_n}$. We will show the conditions given above imply $\overline{\text{im}_{\mathcal{A}b} d_{n+1}^Y}^{\ker d_n^Y} = \overline{\text{im}_{\mathcal{A}b} d_{n+1}^Y}^{Y_n}$ and thus $(H_n(Y))_p \cong H_n^C(Y)$.

Let $y \in \overline{\text{im}_{\mathcal{A}b}d_{n+1}}^{Y_n}$. By definition there exists elements $x_k \in \text{im}_{\mathcal{A}b}d_{n+1}$ and $y_k \in Y_n$ so that

$$y = x_k + p^k y_k \tag{V.4}$$

for all k . Since Y_n is p -complete equation V.4 implies

$$0 = p^k d_n(y_k) \tag{V.5}$$

so $d_n(y_k)$ has p^k torsion. The assumption in the claim guarantees there exists an N so that $i > N$ will imply $d_n y_i = 0$ so $y_i \in \ker d_n$. The elements $x_i \in \text{im}_{\mathcal{A}b}d_{n+1}$ and $y_i \in \ker d_n$ for $i > N$ imply $y \in \overline{\text{im}_{\mathcal{A}b}d_{n+1}}^{\ker d_n}$. \square

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