REPRESENTATIONS OF HECKE ALGEBRAS AND THE ALEXANDER POLYNOMIAL

by

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We study a certain quotient of the Iwahori-Hecke algebra of the symmetric group S_d, called the super Temperley-Lieb algebra STL_d. The Alexander polynomial of a braid can be computed via a certain specialization of the Markov trace which descends to STL_d. Combining this point of view with Ocneanu's formula for the Markov trace and Young's seminormal form, we deduce a new state-sum formula for the Alexander polynomial. We also give a direct combinatorial proof of this result.

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This dissertation is dedicated to my parents,

Hank and Mara Black,

who showed me the beauty and power of mathematics,

beginning when I was very young,

and who believed in me unceasingly.

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INTRODUCTION

I.1. Historical Background

The Alexander polynomial has a history dating back to the early days of algebraic topology. The original definition ([1]) came from analyzing the group of deck transformations on the infinite cyclic cover of the complement of a knot. In the same paper, Alexander reduced the calculation of the invariant to a combinatorial procedure using knot diagrams. Decades later, Conway introduced a special normalization of the invariant ([9]) extended to links, denoted $\nabla(L)$, that was recursively computable using a skein relation on the diagrams (see Figure I.1).

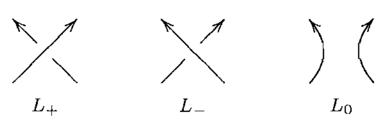


Figure I.1: The Conway skein relation: $\nabla(L_+) - \nabla(L_-) = z\,\nabla(L_0)$

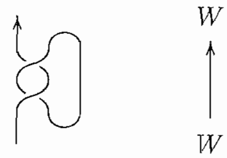
Following the discovery by Jones of a new invariant ([23] and also [10, Chapter 9] for an interesting account of the discovery), there was an explosion of interest in the industry of producing link invariants using skein theories. Within a year, a two-variable polynomial invariant was simultaneously introduced by several independent groups of authors, all of whose names are celebrated in the acronym: HOMFLYPT ([14] and [34]). Using the skein theoretic approach, we first pass to a diagram of the link, which involves the choice of a generic projection onto a plane. The verification that a function on these diagrams is well-defined as a function of the original link reduces to the formal task of checking invariance under the Reidemeister moves. Much of the geometry of three dimensional space is lost in the passage to a diagram, but the trade-off is that many of the tools of representation theory which have no obvious connection to the original link can be brought to bear. The menagerie of quantum invariants that exists today comes about in this manner.

In the seminal work of Reshetikhin and Turaev ([36]), they construct quantum link invariants as tensor functors from the tensor category of colored tangles to the category of modules over the quantum group $\mathcal{U}_v(\mathfrak{g})$, for $\mathfrak{g}$ a simple complex Lie algebra, using the quasi-triangular Hopf algebra structure on $\mathcal{U}_v(\mathfrak{g})$ and choosing representations to label components. Specifically, a (d, d')-tangle connecting d' points on the bottom to d points on the top produces a $\mathcal{U}_v(\mathfrak{g})$-homomorphism $W_{\mu_1} \otimes \cdots \otimes W_{\mu_{d'}} \to W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_d}$, where the modules W_{μ_j} and W_{λ_i} are determined by the colors on the strands. In this context, links appear as $(0,0)$-tangles, as they have no endpoints, and so link invariants are $\mathcal{U}_v(\mathfrak{g})$-module endomorphisms of the ground field $\mathbb{C}(v)$. A family of polynomial invariants P_n $(n \geq 2)$ arises from the choice of $\mathfrak{g} = \mathfrak{sl}(n)$ and the coloring of each component of the link by the standard n-dimensional representation. Each of the invariants P_n are specializations of the HOMFLYPT polynomial at $x = v^n$ (see II.1), and they satisfy the skein relation:

$$v^n P_n(L_+) - v^{-n} P_n(L_-) = (v - v^{-1}) P_n(L_0). \tag{I.1}$$

At $n = 1$ the specialization is trivial, but the $n = 0$ case is evidently the Alexander polynomial ($z = v - v^{-1}$ in Conway's normalization $\nabla(L)$). However, to construct the Alexander polynomial as a quantum invariant, a suitable substitute for the meaningless $\mathfrak{sl}(0)$ is required.

One approach is to use the restricted $\mathcal{U}_\zeta(\mathfrak{sl}(2))$, where $\zeta = \sqrt{-1}$, labeling each component with the entire family of 2-dimensional modules W_λ, $\lambda \in \mathbb{C}$, and recovering the invariant in terms of λ. Another approach is to use the superalgebra $\mathcal{U}_v(\mathfrak{gl}(1|1))$ with a generic v, taking W to be its $(1|1)$-dimensional standard module. In either case, the quantized deformation admits a quasi-triangular Hopf (super-)algebra structure and hence a link invariant (see [40], [28], or [37]). However, there are some subtleties to these two alternative constructions, the main problem being that the quantum invariant evaluates to zero on any link. This is reflection of the fact that the quantum dimension vanishes on W for either algebra. The solution is to cut open the link to form a $(1,1)$-tangle, so that the Reshetikhin-Turaev functor produces an endomorphism of W, which by Schur's Lemma is a scalar.



(I.2)

Figure I.2: A $(1, 1)$-tangle and its associated endomorphism of W.

I.2. Summary of Thesis

In this dissertation, we consider a third alternative, indirectly related to the $\mathfrak{gl}(1|1)$-approach, but we work only with braids, which is Jones' approach via Markov traces ([24]). There is essentially no loss of generality, as any link has a diagram presenting it as the closure of a braid.

In Chapter II, we recall some of the basics regarding links, braids, and invariants. The HOMFLYPT polynomial is constructed from representations of the braid group, factoring through the Iwahori-Hecke algebra $\mathcal{H}_d$ (as in [24], although our notations are substantially different). Also, a converse is presented: an invariant of links defined on braids that satisfies a skein relation necessarily factors through the Hecke algebra. This latter perspective, namely that skein relations determine certain algebras, is a motivation for the results in Chapter III. The remainder of the chapter is devoted to a particular example: the Jones polynomial. We introduce the Temperley-Lieb algebra TL_d, both as an explicit quotient of the Hecke algebra and as a diagrammatic algebra carrying a particularly elegant trace. There are no new results here beyond the considerable task of gathering this wealth of constructions and formulae into one consistent notational framework. The groundwork is laid for the following chapters where certain analogous results are derived for the Alexander polynomial.

In Chapter III, we look at special cases of Schur-Weyl duality between $\mathcal{H}_d$ and $\mathcal{U}_v(\mathfrak{gl}(m|n))$ on tensor space. In particular, the representations of the Hecke algebra from Chapter II are explicitly connected to the Reshetikhin-Turaev intertwiners. The Temperley-Lieb algebra appears here as the centralizer algebra of $\mathcal{U}_v(\mathfrak{g})$ in $\mathrm{End}(V \otimes \cdots \otimes V)$ for $\mathfrak{gl}(2|0) = \mathfrak{gl}(2)$, or equivalently $\mathfrak{sl}(2)$. Hence, using results of Jimbo ([21] and [22]), building on the classical work of Schur ([38]) and Weyl ([41, Chapter III]), TL_d is presented as a product of matrix algebras indexed by partitions of at most two rows. As the Wedderburn components for Young diagrams of more than two rows form a cellular ideal in $\mathcal{H}_d$, the quotient TL_d inherits the cellular structure from $\mathcal{H}_d$. In fact, TL_d has a natural diagram basis that is easy to express in terms of the standard basis for $\mathcal{H}_d$; hence, the Markov trace on $\mathcal{H}_d$ that produces the Jones polynomial can be calculated diagrammatically. The Kauffman state-sum for the Jones polynomial comes directly from this formalism for TL_d. One of the goals of this work, only partially realized, is to find an analogous formalism for the Alexander polynomial.

Alas, the story for the Alexander polynomial is not so simple. The *Super Temperley-Lieb* algebra STL_d is defined as the intertwiner algebra for the quantum group associated to $\mathfrak{gl}(1|1)$. Hence, the Markov trace on $\mathcal{H}_d$ giving the Alexander polynomial on braids factors through STL_d. This algebra has Wedderburn components indexed by hook partitions. However, the ideal that consists of components that are not hooks is not cellular in the dominance order on partitions, and so STL_d does not inherit the cellular structure from $\mathcal{H}_d$ and the map $\mathcal{H}_d \twoheadrightarrow STL_d$ is quite subtle. We can obtain a basis for STL_d, but it has no clear connection to the standard basis for $\mathcal{H}_d$, and so it is not obvious how to calculate the Markov trace directly. In fact, because the transition matrix from the standard basis for $\mathcal{H}_d$ to our basis for STL_d is only defined upon extending scalars to $\mathbb{Q}(v)$, it is no longer clear that the Alexander invariant is a Laurent polynomial in $\mathbb{Z}[v, v^{-1}]$. The main result in this section is a presentation of the relations for STL_d using some of the combinatorics of the symmetric group adapted to $\mathcal{H}_d$.

This is why in Chapter IV we start again using Young's seminormal form for $\mathcal{H}_d$. To begin, STL_d is presented as a diagram algebra in the basis of matrix units of its Wedderburn components. As a result, a lot of complicated information is packaged in the idempotents in $\mathcal{H}_d$ that project onto STL_d. Using character formulas of Geck and Jacon ([16], [17]) for the irreducible representations of $\mathcal{H}_d$ and building upon the work of Jones ([24]), a combinatorial procedure is deduced for calculating the trace, and hence the Alexander polynomial, from a braid diagram directly. This state-sum has the appeal of being computable in terms of certain labeled diagrams (the basis for STL_d) that come from resolving crossings in the original braid. The trade-off for changing basis so drastically is the aforementioned difficulty that the formula gives the Alexander polynomial as a sum of rational functions involving quantum integers, obscuring the fact that the invariant is actually a Laurent polynomial.

In Chapter V, we reprove the main theorem combinatorially. By analyzing possible states in the expansion of a braid diagram, we are able to check invariance under the Markov moves directly. Moreover, agreement with the Alexander polynomial is verified. This "naïve" proof has an unexpected advantage: it may be possible to extend the state-sum formula to the multivariable

Alexander polynomial using colored braids, as in the work of Murakami ([33]), using the combinatorics directly. In a different direction, we hope to recover the Alexander invariant for virtual links ([27]) as a state-sum in an analogous fashion.

CHAPTER II

BRAIDS, LINKS, AND THE JONES POLYNOMIAL

II.1. Links and Braids

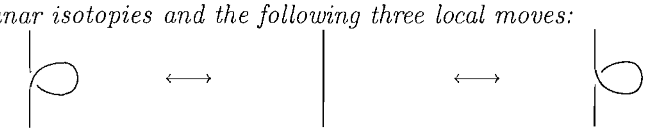
II.1.1. Links

We recall some standard notions and establish some conventions (see [25, chapter 2] or [10, chapter 1] for details). A *link* is a smooth embedding $S^1 \sqcup \cdots \sqcup S^1 \to S^3$. Whenever it is convenient and does not cause confusion, we also refer to the image of the embedding as the link. All links are oriented. A link with one component is called a *knot.* An *isotopy of links* is a smooth path in the space of embeddings, or equivalently a smooth embedding $(S^1 \sqcup \cdots \sqcup S^1) \times I \to S^3$. Isotopy forms an equivalence relation whose equivalence classes are called *link types.* For our purposes, a *link invariant* is a function on link types taking values in a ring.

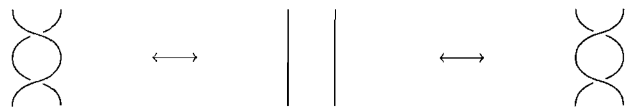
A *link diagram* is a four-valent graph embedded in S^2 along with the choice of over- and under-crossing strands at each vertex. An *isotopy of link diagrams* is an isotopy of the underlying graphs that preserves the crossing data. By compactness, we may assume that the diagram embeds into a disk in $\mathbb{R}^2$ (which is how links are usually drawn), but we allow for isotopies of link diagrams that utilize the connectivity of the sphere S^2. A link diagram specifies a link type in a natural way: compose the planar embedding of the link diagram with the standard embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ and locally perturb the two strands at each crossing in opposite directions normal to the plane. Thus, we have a well-defined surjective map called *realization* $\mathcal{D} \to \mathcal{L}$, where $\mathcal{D}$ is the set of link diagrams up to planar isotopy and $\mathcal{L}$ is the set of link types. Using standard transversality arguments, there exist generic projections $\mathbb{R}^3 \to \Pi$, where Π is a plane, under which the image of a link is a link diagram (there are only finitely many transverse double points). Thus, we have sections $\mathcal{L} \to \mathcal{D}$ of the realization map. The fibers of the realization map are described by Reidemeister's theorem.

Theorem II.1.1. *Two link diagrams realize isotopic links if and only if they are related by a* *sequence of planar isotopies and the following three local moves:*

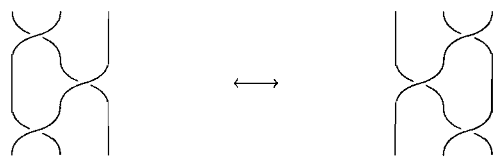
(R I)



(R II)



(R III)



II.1.2. Braids

There are many equivalent ways to define braids (see, e.g. [7] or [25, chapter 1]). A *geometric braid* with d strands is a smooth embedding of of d disjoint arcs in $\mathbb{R}^2 \times [0,1]$ such that the endpoints of each strand have coordinates $(i,0,0)$ and $(\pi(i),0,1)$ for some permutation $\pi \in S_d$ and the height function (projecting onto the third coordinate) has no critical points. By a *geometric braid isotopy*, we mean an isotopy of the ambient Euclidean space keeping the endpoints of the strands fixed. When we speak of a *braid*, we mean an isotopy class of geometric braids. Let $\mathcal{B}_d$ denote collection of braids on d strands.

By analogy with link diagrams, we consider *geometric braid diagrams* consisting of immersed arcs in the strip $\mathbb{R} \times [0,1]$ connecting points $(i,0)$ to $(\pi(i),1)$ for $i = 1,\ldots,d$ satisfying the conditions that projection onto the second coordinate has no critical points and that self-intersections are transverse. Each transverse double point in a geometric braid diagram is called a *crossing* and carries with it the data of which strand passes over and which strand passes under. There is an appropriate notion of *isotopy of geometric braid diagrams* that allows one to speak simply of *braid diagrams*, meaning isotopy classes.

Braids naturally carry the structure of a group whose operation is given by placing diagrams on top of one another. We now define the braid group abstractly, following Artin [3]. Fix $d \geq 1$. Let $\Sigma_d = \{\sigma_1,\ldots,\sigma_{d-1}\}$ be the set of braid generators, and let $\Sigma_d^\bullet$ denote the set of words in the symbols $\Sigma_d \sqcup \Sigma_d^{-1}$. Then, $\mathcal{B}_d$ is the braid group on d strands, that is, the quotient of the free group on Σ_d by the relations

$$\begin{aligned}\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for each } 1 \leq i \leq d-2,\\ \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ when } |i-j| > 1.\end{aligned}$$

For each d, there is a canonical map $\Sigma_d^\bullet \twoheadrightarrow \mathcal{B}_d$. The map $e : \mathcal{B}_d \to \mathbb{Z}$, $\sigma_i \mapsto 1$ induces an isomorphism of groups $\mathcal{B}_d/[\mathcal{B}_d, \mathcal{B}_d] \to \mathbb{Z}$. The image of α under $\mathcal{B}_d \to \mathbb{Z}$ is called the *exponent sum* of α. Also there is a map $\mathcal{B}_d \to \mathcal{L}$, $\alpha \mapsto \widehat{\alpha}$, defined by closing a braid into a link by connecting the top of strand i to the bottom of strand i for each i (see Figure II.1.2).

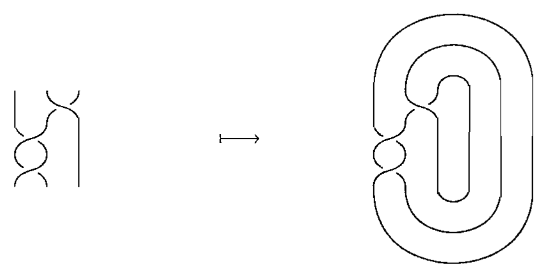


Figure II.1: A diagram of the braid $\alpha = \sigma_2^{-1}\sigma_1^2$ and its closure $\widehat{\alpha}$.

Alexander showed in [2] that this map is surjective by giving an algorithm that transforms a link diagram, using the Reidemeister moves, into an equivalent diagram that is evidently the closure of a braid. We call two braids (not necessarily on the same number of strands) *link equivalent* if they form isotopic links after closure. In [31], Markov made the following characterization of this equivalence relation on $\coprod_{d\geq 1}\mathcal{B}_d$, although the first published proof was due to Birman ([6]).

Theorem II.1.2. *Link of equivalence of braids is generated by the following two local moves.*

$$\begin{array}{llcll} \textit{(M I)} & \alpha\beta & \longleftrightarrow & \beta\alpha, & \\ \textit{(M II)} & \iota(\alpha)\sigma_d^{\pm 1} & \longleftrightarrow & \alpha, & \textit{for } \alpha \in \mathcal{B}_d, \end{array}$$

where $\iota : \mathcal{B}_d \hookrightarrow \mathcal{B}_{d+1}$ *is the natural inclusion defined on generators by* $\sigma_i \mapsto \sigma_i$.

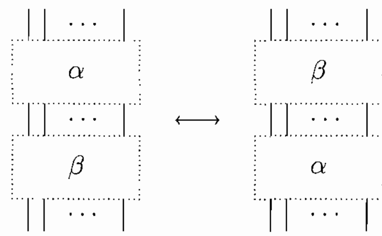


Figure II.2: The cyclic Markov move.

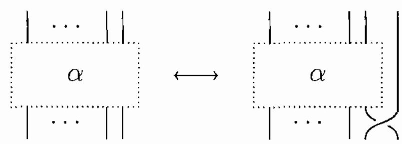


Figure II.3: The stabilization Markov move.

II.2. Representations of the Braid Group and the Iwahori-Hecke Algebra

In this section, we introduce the Iwahori-Hecke algebras of type A, and discuss their central importance in knot theory. Initally, we will take the perspective that a link invariant satisfying a skein relation akin to I.1 is given, and use it to produce a particular quotient of the braid group algebra that the invariant factors through. Afterwards, we take the Hecke algebra as a starting point, and following Jones ([24]), show how to recover those link invariants (satisfying skein relations) from the characters of its representations.

II.2.1. From Link Invariants to Algebras

Fix a commutative ring $\mathbb{F}$ and consider a link invariant Q taking values in $\mathbb{F}$. By composing with the closure map, for each d we get a linear function $\widehat{Q}_d : \mathbb{F}\mathcal{B}_d \to \mathbb{F}$, defined on braids by

$$\widehat{Q}_d(\alpha) = Q(\widehat{\alpha}).$$

where $\mathbb{F}\mathcal{B}_d$ is the group algebra of the braid group. Such a map $\widehat{Q}_d$ is actually a trace on account of the first Markov move, so the linear subspace

$$\ker \widehat{Q}_d = \{\alpha \in \mathbb{F}\mathcal{B}_d \mid \widehat{Q}_d(\alpha\beta) = 0 \text{ for all } \beta \in \mathbb{F}\mathcal{B}_d\}.$$

is a two-sided ideal, maximal among two-sided ideals on which the trace vanishes. Let $A_d(Q) = \mathbb{F}\mathcal{B}_d / \ker \widehat{Q}_d$, and, by abuse of notation, let α denote the image of α under the canonical projection $\mathbb{F}\mathcal{B}_d \twoheadrightarrow A_d(Q)$. Note that $\widehat{Q}_d$ descends to the trace $\overline{Q}_d : A_d(Q) \to \mathbb{F}$ that agrees with the original link invariant in the sense that $\overline{Q}_d(a) = Q(\widehat{\alpha})$ for any braid $\alpha \in \mathcal{B}_d$. Furthermore, $A_d(Q)$ is the smallest quotient of $\mathbb{F}\mathcal{B}_d$ through which the trace factors.

Now, suppose that $\mathbb{F} = \mathbb{Z}[x^{\pm 1}, v^{\pm 1}, (v - v^{-1})^{-1}]$ and that the link invariant Q satisfies the following skein relation

$$xQ(L_+) - x^{-1}Q(L_-) = (v - v^{-1})Q(L_0), \tag{II.1}$$

where the links L_+, L_-, L_0 are identical outside of a small disk and look like the tangles pictured in I.1 within the disk. This relation together with the normalization $Q(\text{unknot}) = 1$ defines a two-variable invariant equivalent to the HOMFLYPT polynomial ([14] and [34]). We investigate the consequence of this skein relation on the maps $\overline{Q}_d$.

For any choice of $\alpha, \beta \in \mathcal{B}_d$ and for any $i = 1, \ldots, d-1$, we have

$$x\widehat{Q}_d(\alpha\sigma_i\beta) - x^{-1}\widehat{Q}_d(\alpha\sigma_i^{-1}\beta) - (v - v^{-1})\widehat{Q}_d(\alpha\beta) = 0 \tag{II.2}$$

Therefore, $x\sigma_i - x^{-1}\sigma_i^{-1} - (v - v^{-1}) \in \ker \widehat{Q}_d$, which is equivalent to the relation

$$(x\sigma_i - v)(x\sigma_i + v^{-1}) = 0 \in A_d(Q). \tag{II.3}$$

The HOMFLYPT invariant factors through this finite dimensional quotient $A_d(Q)$ of $\mathbb{F}\mathcal{B}_d$ by the quadratic relation (II.3).

Lemma II.2.1. *Any $a \in A_d(Q)$ is equal to an $\mathbb{F}$-linear combination of words, each containing exactly one instance of either σ_{d-1} or σ_{d-1}^{-1}.*

Proof. The relation $\sigma_i^{-1} = x^2\sigma_i - x(v - v^{-1})$ shows that it suffices to consider positive braids: words in the generators σ_i but not their inverses. It is a standard result that a is equal to an $\mathbb{F}$-linear combination of words, that contain σ_{d-1} at most once. Any term in the resulting expression that does not contain σ_{d-1} can then be replaced by $(v - v^{-1})^{-1}(x\sigma_i - x^{-1}\sigma_i^{-1})$ to establish the lemma. □

Now we can prove the following result, connecting the algebras $A_d(Q)$ together.

Proposition II.2.2. *The inclusion $\iota : \mathcal{B}_d \hookrightarrow \mathcal{B}_{d+1}$ descends to a well-defined injective algebra homomorphism $\bar{\iota} : A_d(Q) \hookrightarrow A_{d+1}(Q)$.*

Proof. Fix $a \in \mathbb{F}\mathcal{B}_d$. Suppose that $\widehat{Q}_d(ab) = 0$ for all $b \in \mathbb{F}\mathcal{B}_d$ and let $c \in \mathbb{F}\mathcal{B}_{d+1}$. By Lemma II.2.1, we may write

$$c = \sum_i c_i^+ \sigma_d + \sum_j c_j^- \sigma_d^{-1} \qquad \text{for } c_i^+, c_j^- \in \mathbb{F}\mathcal{B}_d. \tag{II.4}$$

Then, by using the second Markov move,

$$\begin{aligned}
\widehat{Q}_{d+1}(\iota(a)c) &= \sum_i \widehat{Q}_{d+1}(\iota(ac_i^+)\sigma_d) + \sum_j \widehat{Q}_{d+1}(\iota(ac_j^-)\sigma_d^{-1}) \\
&= \sum_i \widehat{Q}_d(ac_i^+) + \sum_j \widehat{Q}_d(ac_j^-) \\
&= 0.
\end{aligned}$$

This shows that $\bar{\iota}$ is well-defined. To prove that $\bar{\iota}$ is injective, suppose that $\widehat{Q}_d(\iota(a)c) = 0$ for all $c \in \mathbb{F}\mathcal{B}_{d+1}$ and let $b \in \mathbb{F}\mathcal{B}_d$. We have

$$\widehat{Q}_d(ab) = \widehat{Q}_{d+1}(\iota(ab)\sigma_d) = \widehat{Q}_{d+1}(\iota(a)\,\iota(b)\sigma_d) = 0.$$ □

II.2.2. From the Iwahori-Hecke Algebra to Link Invariants

Definition II.2.3. Let $\mathcal{H}_d$ denote the *Iwahori-Hecke algebra* of type A_{d-1} associated to the symmetric group S_d. This is the algebra over $\mathbb{C}(v)$ generated by $H_1, \ldots, H_{d-1}$, subject to the braid relations

$$H_i H_j H_i = H_j H_i H_j \quad \text{when } |i-j| = 1, \tag{II.5}$$

$$H_i H_j = H_j H_i \quad \text{when } |i-j| > 1, \tag{II.6}$$

and also the quadratic relations

$$(H_i - v)(H_i + v^{-1}) = 0. \tag{II.7}$$

We recognize immediately that $\mathcal{H}_d \twoheadrightarrow \mathbb{C}(v) \otimes_{\mathbb{F}} A_d(Q)$, $H_i \mapsto 1 \otimes x\sigma_i$, where $\mathbb{C}(v)$ is considered as a $(\mathbb{C}(v), \mathbb{F})$ bimodule with $x \in \mathbb{F}$ acting on the right by 1. Let i denote the natural inclusion of algebras $\mathcal{H}_d \hookrightarrow \mathcal{H}_{d+1}$, $H_i \mapsto H_i$.

Lemma II.2.4 (Ocneanu, [14]). *For every $y \in \mathbb{C}(v)$, there is a linear map $\tau : \coprod_{d \geq 1} \mathcal{H}_d \to \mathbb{C}(v)$ uniquely defined by*

(i) $\tau(ab) = \tau(ba)$,

(ii) $\tau\big(i(a)H_d\big) = y\,\tau(a)$ *for* $a \in \mathcal{H}_d$, *and*

(iii) $\tau(1) = 1$.

Such a linear function is often called a *Markov trace* because of the first condition. Although τ does not give a link invariant directly (unless $y = 1$), it is a useful notion for the following two reasons. First of all, it behaves in a controlled manner with respect to the second Markov move, and so it defines an invariant of framed links and can, moreover, be rescaled to make an invariant of ordinary links. The second reason is that there are natural constructions of Markov traces that arise by studying representations of the braid group and their characters for which y is a non-trivial quantity.

Here we how to construct a link invariant from a Markov trace. Following Jones, we choose an invertible x so that $\tau(x^{-1}H_i) = \tau\big((x^{-1}H_i)^{-1}\big)$ for each $i = 1, \ldots, n-1$. Noting that the quadratic relation in $\mathcal{H}_d$ gives $H_i^{-1} = H_i - (v - v^{-1})$, we have $x^{-1}y = x\big(y - (v - v^{-1})\big)$, and finally

$$y = x\left(\frac{v - v^{-1}}{x - x^{-1}}\right). \tag{II.8}$$

Therefore, we have

$$\tau(x^{-1}H_i) = x^{-1}y = \frac{v - v^{-1}}{x - x^{-1}}. \tag{II.9}$$

Now, let $\varphi : \mathcal{B}_d \to \mathcal{H}_d^\times$ be the group homomorphism $\sigma_i \mapsto H_i$. Recall that $e : \mathcal{B}_d \to \mathbb{Z}$ is the exponent sum of a braid (often called the *writhe*).

Theorem II.2.5. *The functions $\widehat{P}_d : \mathcal{B}_d \to \mathbb{F}$ given by*

$$\widehat{P}_d(\alpha) = \left(\frac{x - x^{-1}}{v - v^{-1}}\right)^{d-1} x^{-e(\alpha)}\tau\big(\varphi(\alpha)\big) \tag{II.10}$$

define a link invariant P via the formula $P(\widehat{\alpha}) = \widehat{P}_d(\alpha)$.

Proof. Regarding the first Markov move, invariance of P follows from invariance of τ and the fact that the exponent sums match: $e(\alpha\beta) = e(\beta\alpha)$. It suffices to check invariance under the second

Markov move. Let $\alpha \in \mathcal{B}_d$ and calculate

$$\begin{aligned}\widehat{P}_{d+1}(\alpha\sigma_d) &= \left(\frac{x-x^{-1}}{v-v^{-1}}\right)^d x^{-e(\alpha)-1}\tau\big(\varphi(\alpha)H_d\big)\\ &= \left(\frac{x-x^{-1}}{v-v^{-1}}\right)^d x^{-e(\alpha)-1}\left(x\frac{v-v^{-1}}{x-x^{-1}}\right)\tau\big(\varphi(\alpha)\big)\\ &= \left(\frac{x-x^{-1}}{v-v^{-1}}\right)^{d-1} x^{-e(\alpha)}\tau\big(\varphi(\alpha)\big)\\ &= \widehat{P}_d(\alpha)\end{aligned}$$

The equality $\widehat{P}_{d+1}(\alpha\sigma_d^{-1}) = \widehat{P}_d(\alpha)$ follows from a similar calculation, noting that

$$\tau\big(\varphi(\alpha)H_d^{-1}\big) = \tau\big(\varphi(\alpha)\big(H_d-(v-v^{-1})\big)\big) = \left(x^{-1}\frac{v-v^{-1}}{x-x^{-1}}\right)\tau\big(\varphi(\alpha)\big)$$

□

The invariants $\{\widehat{P}_d\}_{d\geq 1}$ are normalized so that $\widehat{P}_1(1) = 1$, where 1 is the trivial braid on one strand, and they satisfy the following skein relation for any $\alpha \in \mathcal{B}_k$ and any $i = 1, \ldots, d-1$,

$$x\widehat{P}_{d+1}(\alpha\sigma_i) - x^{-1}\widehat{P}_{d+1}(\alpha\sigma_i^{-1}) = (v-v^{-1})\widehat{P}_d(\alpha). \tag{II.11}$$

Therefore, the link invariant P satisfies the relation II.1 and we have the isomorphism

$$\mathcal{H}_d \cong \mathbb{C}(v) \otimes_{\mathbb{F}} A_d(P). \tag{II.12}$$

Making the substitutions $l = -\sqrt{-1}x$ and $m = \sqrt{-1}(v-v^{-1})$ we recover the original normalization of the HOMFLYPT polynomial $P' : \mathcal{L} \to \mathbb{Z}[l^{\pm}, m^{\pm}]$ ([14]) satisfying $P'(\text{unknot}) = 1$ and

$$lP'(L_+) + l^{-1}P'(L_-) + mP'(L_0) = 0. \tag{II.13}$$

II.2.3. Specializing to the Jones Polynomial

In the sequel, we will be concerned with two different specializations of Q, which will be shown to be equivalent to the Jones polynomial and the Alexander polynomial, respectively. For now, we consider the Jones polynomial.

Definition II.2.6. The *Jones polynomial* is the function $J : \mathcal{L} \to \mathbb{Z}[v, v^{-1}]$, defined by setting $x = v^2$ in $P(L)$.

Thus, J is the unique isotopy invariant of links normalized so that $J(\text{unknot}) = 1$ and satisfying the skein relation

$$v^2 J(L_+) - v^{-2} J(L_-) = (v - v^{-1}) J(L_0). \tag{II.14}$$

Extending the Jones polynomial to braids, we have $\widehat{J}_d : \mathcal{B}_d \to \mathbb{Z}[v, v^{-1}]$ for each $d \geq 1$. Since $\widehat{J}_d$ is a specialization of the HOMFLYPT polynomial, it factors through the Hecke algebra. In fact, it factors through a smaller quotient, $A_d(J)$ which we now describe.

II.3. The Temperley-Lieb Algebra

Definition II.3.1. The *Temperley-Lieb algebra* $TL_d(\delta)$ is the unital, associative algebra over $\mathbb{C}(v)$ generated by $U_1, \ldots, U_{d-1}$ subject to the relations

$$U_i U_j U_i = U_i \qquad \text{when } |i - j| = 1, \tag{II.15}$$

$$U_i U_j = U_j U_i \qquad \text{when } |i - j| > 1, \text{and} \tag{II.16}$$

$$U_i^2 = \delta U_i \tag{II.17}$$

The quantity $\delta \in \mathbb{C}(v)^\times$ is called the *loop value* (the name will be explained in II.4). Let $C_d = \frac{1}{d+1}\binom{2d}{d}$ denote the dth *Catalan number*, which is the solution to the recurrence

$$C_0 = 1 \tag{II.18}$$

$$C_{d+1} = \sum_{j=0}^{d} C_j C_{d-j}, \qquad d \geq 0. \tag{II.19}$$

Lemma II.3.2 ([25, Lemma 5.26]). *The Temperley-Lieb algebra is spanned by the set of words of the form*

$$(U_{i_1} U_{i_1 - 1} \cdots U_{j_1})(U_{i_2} U_{i_2 - 1} \cdots U_{j_2}) \cdots (U_{i_m} U_{i_m - 1} \cdots U_{j_m}), \tag{II.20}$$

where $0 \leq m < d$,

$$0 < i_1 < \cdots < i_m < d, \quad 0 < j_1 < \cdots < j_m < d,$$

and

$$i_1 \geq j_1, \quad i_2 \geq j_2, \quad \ldots, \quad i_m \geq j_m.$$

There are C_d *such words; hence,* $\dim TL_d(\delta) \leq C_d$.

A word in the generators $U_1, \ldots, U_{d-1}$ corresponding to the indices $(i_1, \ldots, i_m, j_1, \ldots, j_m)$ as in the Lemma will be referred to as a *normal form*. For our purposes, we choose $\delta = [2] = (v + v^{-1}) \in \mathbb{C}(v)$ and let $TL_d = TL_d([2])$. We now describe TL_d as a quotient of $\mathcal{H}_d$.

Theorem II.3.3 (Compare [25, Thm. 5.29]). *The map* $\psi : \mathcal{H}_d \to TL_d$*, defined on generators by*

$$\psi(H_i) = v - U_i \tag{II.21}$$

is a surjective $\mathbb{C}(v)$*-algebra homomorphism.*

Proof. Since $U_i = \psi(v - H_i)$ and $U_1, \ldots, U_{d-1}$ generate TL_d, ψ is surjective. We verify directly that the map is well-defined. For relation II.5, suppose that $|i - j| = 1$ and calculate in TL_d:

$$\begin{aligned}\psi(H_i H_j H_i) &= (v - U_i)(v - U_j)(v - U_i) \\ &= v^3 - 2v^2 U_i - v^2 U_j + vU_iU_j + vU_jU_i + vU_i^2 - U_iU_jU_i \\ &= v^3 - v^2U_i - v^2U_j + vU_iU_j + vU_jU_i.\end{aligned}$$

This last expression is symmetric in i and j, and so $\psi(H_iH_jH_i) = \psi(H_jH_iH_j)$. The verification that $\psi(H_iH_j) = \psi(H_jH_i$ for $|i - j| > 1$ is similar and easier. Finally, for the quadratic relations II.7, we check:

$$\psi\big((H_i - v)(H_i + v^{-1})\big) = -U_i(v + v^{-1} - U_i) = -[2]U_i + U_i^2 = 0. \qquad \square$$

For $n = 2$, $\dim \mathcal{H}_2 = 2 = \dim TL_2$, so ψ is an isomorphism. For $d \geq 3$, we can describe $\ker \psi$ explicitly, as follows.

Theorem II.3.4. *The kernel of* ψ *is the two-sided ideal of* $\mathcal{H}_d$ *generated by*

$$\begin{aligned}&(v - H_1)(v - H_2)(v - H_1) - (v - H_1) \\ &\qquad = v^3 - v^2H_1 - v^2H_2 + vH_1H_2 + vH_2H_1 - H_1H_2H_1\end{aligned} \tag{II.22}$$

Proof. Clearly, $\ker\psi$ is the two sided ideal generated by $v^3-v^2H_i-v^2H_j+vH_iH_j+vH_jH_i-H_iH_jH_i$ for all i,j such that $|i-j|=1$. Using the braid relation (II.5), it suffices to consider the case $j=i+1$, so the ideal is generated by

$$v^3-v^2H_i-v^2H_{i+1}+vH_iH_{i+1}+vH_{i+1}H_i-H_iH_{i+1}H_i.$$

In $\mathcal{H}_d$, conjugation by the invertible element $(H_1\cdots H_{d-1})^{i-1}$ defines an automorphism sending H_1 to H_i. Therefore, the kernel is generated by

$$v^3-v^2H_1-v^2H_2+vH_1H_2+vH_2H_1-H_1H_2H_1. \qquad \square$$

Although it's not obvious now, we will see in Chapter III that $\mathcal{H}_d$ is a semisimple algebra over $\mathbb{C}(v)$, and a suitable multiple of ψ is a projection onto certain Wedderburn matrix factors.

II.4. Diagram Algebras

II.4.1. Temperley-Lieb Arc Diagrams

Definition II.4.1. A (d,d')-*arc diagram* D is a topological embedding of the disjoint union of $(d+d')/2$ intervals in the strip $\mathbb{R}\times[0,1]$ such that the boundary of the embedded 1-manifold consists of the points $\{(1,0),\ldots,(d',0)\}\sqcup\{(1,1),\ldots,(d,1)\}$. A (d,d)-diagram is called a *Temperley-Lieb diagram on d strands.* Let $[D]$ denote the isotopy class of an arc diagram D, keeping its endpoints fixed.

See Figure II.4.1 for an illustration of Temperley-Lieb diagrams on three stands.

Figure II.4: The five Temperley-Lieb diagrams on 3 strands.

There is a natural concatenation operation given by placing one arc diagram above another, identifying boundary points, and scaling down the result to fit in the strip. However, this process may produce extra disjoint components, each homeomorphic to a circle. For diagrams D_1 and D_2, let $D_1\circ D_2$ denote the arc diagram obtained by placing D_1 on top of D_2 followed by removal of the resulting circles, and let $c(D_1,D_2)\in\mathbb{Z}_{\geq 0}$ denote the number of such circles. We now make an algebra out of the linear span of isotopy classes of Temperley-Lieb diagrams.

Definition II.4.2. Let $TL'_d(\delta)$ consist of the $\mathbb{C}(v)$-vector space on the set of isotopy classes of Temperley-Lieb diagrams $[D]$ on d strands, endowed with a multiplication defined by

$$[D_1][D_2] = \delta^{c(D_1,D_2)}[D_1 \circ D_2] \tag{II.23}$$

The quantity δ is called the *loop value*, which we assume is non-zero.

Lemma II.4.3. *Up to isotopy, there are C_d distinct arc diagrams on d strands. Hence,*

$$\dim TL'_d(\delta) = C_d \tag{II.24}$$

Proof. There is a bijection between isotopy classes of Temperley-Lieb diagrams on d strands and isotopy classes of $(0, 2d)$-arc diagrams, as illustrated in (Figure II.5). It is well known that the latter set has size C_d □

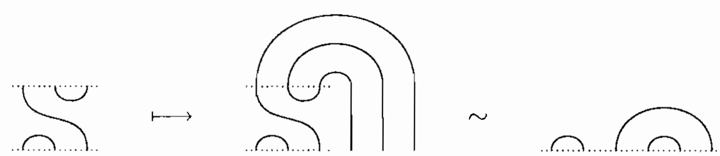


Figure II.5: The correspondence between (d, d)-diagrams and $(0, 2d)$-diagrams.

Let I_d denote the TL diagram in which each strand is a vertical segment. It's clear that $[I_d]$ is the unit for $TL'_d(\delta)$. For each $i = 1, \ldots, d-1$, let U'_i be the TL diagram with a single cap-cup pair occurring at positions $(i, i+1)$ and vertical line segments elsewhere (see Figure II.6).

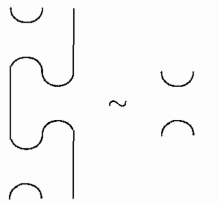
Figure II.6: The generating diagrams I_4 and U'_1, U'_2, U'_3 in $TL'_4(\delta)$.

Lemma II.4.4. *Every TL diagram D on d strands is isotopic to the composition $U'_{i_1} \circ \cdots \circ U'_{i_\ell}$ for some sequence $(i_1, \ldots, i_\ell)$ of integers with $0 < i_j < d$. Therefore, $[D] = [U'_{i_1}] \cdots [U'_{i_\ell}] \in TL'_d(\delta)$.*

Proof. The proof is by induction on d and sub-induction on the location of the leftmost cap on the bottom of the diagram D (see [25, Theorem 5.34]). □

Theorem II.4.5. *The map $\theta : TL_d(\delta) \to TL'_d(\delta)$ defined on generators by $\theta(U_i) = [U'_i]$ is an isomorphism of $\mathbb{C}(v)$-algebras.*

Proof. Showing that θ is well-defined amounts to finding planar isotopies between the two diagrams corresponding to the relations (II.15)-(II.17). The most interesting relation is illustrated below. Note that these pictures are meant to represent TL diagrams on d strands, where the remaining vertical strands are supressed.



(II.25)

By Lemma II.4.4, every diagram D is isotopic to $[U'_{i_1}]\cdots[U'_{i_\ell}] = \theta(U_{i_1}\cdots U_{i_\ell})$ for some sequence $(i_1,\ldots,i_\ell)$ of indices. Therefore, θ is surjective. Since $\dim TL_d(\delta) \le C_d = \dim TL'_d(\delta)$, we conclude that θ is an isomorphism. □

Corollary II.4.6. *The set of words in Lemma II.3.2 forms a basis for $TL_d(\delta)$. Hence,*

$$\dim TL_d(\delta) = C_d. \tag{II.26}$$

From now on, we use the isomorphism θ to identify TL_d and TL'_d.

II.4.2. Evaluation of Jones via Temperley-Lieb

Here we set $\delta = [2] = v + v^{-1}$. Let $[\widehat{D}]$ denote the *closure* of an element of TL'_d by connecting the ith vertex at the top of the diagram D to the ith vertex at the bottom of D by a simple curve for each $i = 1,\ldots,d$. The result is a disjoint union of circles, each of which can be replaced by $[2] = v + v^{-1} \in \mathbb{C}(v)$, as in the definition of multiplication in TL'_d. See Figure II.7.

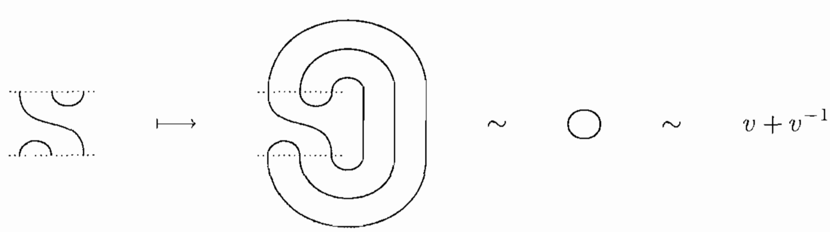


Figure II.7: An example of the closure operation on arc diagrams.

Let $\overline{\tau} : \mathcal{B}_d \to \mathbb{C}(v)$ denote the composition

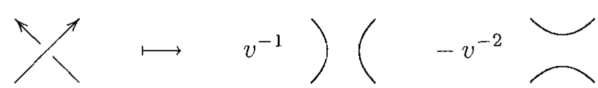
$$\mathcal{B}_d \xrightarrow{\varphi} \mathcal{H}_d \xrightarrow{\psi} TL_d \xrightarrow{\widehat{\ }} \mathbb{C}(v) \tag{II.27}$$

Theorem II.4.7. *The function* $\overline{J}_d : \mathcal{B}_d \to \mathbb{C}(v)$ *defined by*

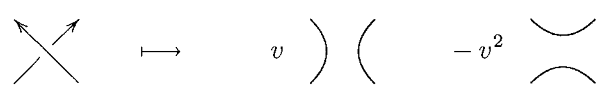
$$\overline{J}_d(\alpha) = v^{-2e(\alpha)}\overline{\tau}(\alpha) \tag{II.28}$$

is an invariant of links, and $\overline{J}_d(\alpha) = (v + v^{-1})\ \widehat{J}_d(\alpha)$.

Before proving the theorem, we introduce the following diagrammatic way to calculate $\overline{J}_d$. Resolve each crossing of a braid diagram in a manner representing the map $\alpha \mapsto v^{-2e(\alpha)}\ \psi \circ \varphi(\alpha)$. This is suggested by the pictures below, where the remaining vertical strands are suppressed.



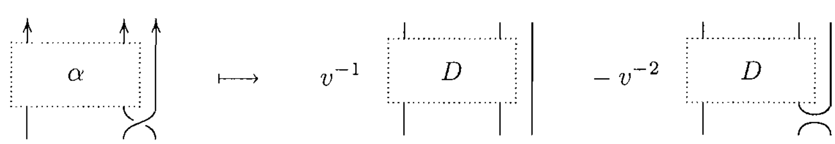
(II.29)



(II.30)

Now, close the corresponding arc diagrams to form a collection of circles, each of which evaluates to the scalar $v + v^{-1}$. As an immediate consequence, the trivial braid on d strands representing $1 \in \mathcal{B}_d$, closes to form d unlinked, unknotted circles, and takes the value $\overline{J}_d(1) = (v + v^{-1})^d$.

Proof of Theorem. We check invariance under the second Markov move. Let $\alpha \in \mathcal{B}_d$ and let D be an arc diagram representing a term in the expansion $\psi \circ \varphi(\alpha) \in TL'_d$. Calculate the resolution of the last crossing in the braid $\alpha\sigma_d \in \mathcal{B}_{d+1}$.

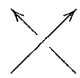
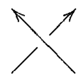


(II.31)

Upon closing the strands for both of the diagrams on the right, the first diagram has an extra disjoint circle. Therefore, we obtain:

$$\overline{J}_{d+1}(\alpha\sigma_d) = \big(v^{-1}(v + v^{-1}) - v^{-2}\big)\overline{J}_d(\alpha) = \overline{J}_d(\alpha). \tag{II.32}$$

The calcuation that $\overline{J}_{d+1}(\alpha\sigma_d^{-1}) = \overline{J}_d(\alpha)$ is completely analogous, replacing v by v^{-1}. It remains to check that $\overline{J}_d = (v+v^{-1})\widehat{J}_d(\alpha)$. Working in $\mathbb{C}(v)\mathcal{B}_d$, where we may take $\mathbb{C}(v)$-linear combinations of braids,

v^2  $-\, v^{-2}$  $\longmapsto$ $(v - v^{-1})$  (II.33)

Upon closing the resulting arc diagrams, we recover the skein relation for the Jones polynomial

$$v^2\, \overline{J}_d(\alpha\sigma_i\beta) - v^{-2}\, \overline{J}_d(\alpha\sigma_i^{-1}\beta) = (v - v^{-1})\, \overline{J}_d(\alpha\beta) \tag{II.34}$$

This shows that $\overline{J}_d$ is a scalar multiple of $\widehat{J}_d$, and so we compare the value of each function on the unknot to complete the proof: $\overline{J}_d(1) = v + v^{-1} = (v+v^{-1})\widehat{J}_d(1)$. □

Perhaps the most important consequence of Theorem II.4.7 is that the Jones polynomial can be calculated as a state-sum. This observation, as well as the diagrammatic expansion of crossings (as in formulas II.29 and II.30) are due to Kauffman ([26]). By opening parentheses, a braid diagram with m crossings yields 2^m terms in TL_d. Starting from these states, Khovanov was able to construct a link homology theory, categorifying the Jones polynomial ([30]).

Fix a braid α and an explicit diagram for it with exponent sum e. A *state* is a diagram of unlinked circles that results from smoothing each crossing in the closure of the braid. For each state s, let $c = c(\mathbf{s})$ be the number of circles in the state, and let $r = r(\mathbf{s})$ be the number of crossings that are smoothed horizontally to form a cap-cup pair in the diagram $\mathbf{s}$. Then,

Theorem II.4.8 (Kauffman).

$$\overline{J}_d(\alpha) = \sum_{\mathbf{s}} (-1)^r v^{-e(1+r)}(v+v^{-1})^c. \tag{II.35}$$

CHAPTER III

SCHUR-WEYL DUALITY AND QUOTIENTS OF THE HECKE ALGEBRA

III.1. Schur-Weyl Duality

We review the classical notion of Schur-Weyl duality between the general linear groups and the symmetric group. Then, we generalize this in two directions, giving a duality theorem between the quantum group associated to the Lie superalgebra $\mathfrak{gl}(m|n)$ and the Iwahori-Hecke algebra $\mathcal{H}_d$.

III.1.1. Classical Schur-Weyl Duality

Fix the ground field $\mathbb{C}$. Let $\mathrm{Par}(d)$ denote the set of all integer partitions of d. To the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, associate its *Young diagram*, that is, the left-justified diagram with λ_1 boxes on the first row, λ_2 boxes on the second row, etc. Let $S(\lambda)$ denote the irreducible $\mathbb{C}S_d$-module, called the *Specht module* associated to $\lambda \in \mathrm{Par}(d)$.

Let $\Lambda^+(n,d)$ denote the set of all $\lambda \in Par(d)$ whose Young diagram has d boxes in at most n rows. The irreducible (polynomial) highest weight $GL(n)$-module of highest weight λ will be denoted $V(\lambda)$, where we fix the standard choice of maximal torus and Borel subalgebra consisting of diagonal matrices and upper triangular matrices, respectively.

Let $\mathbb{C}^n = V(1, 0, \ldots, 0)$ denote the natural n-dimensional vector representation of the group $GL(n)$, and consider the tensor space

$$\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n \tag{III.1}$$

with d tensor factors. This is a left $GL(n)$-module via the diagonal action:

$$g \cdot (x_1 \otimes \cdots \otimes x_d) = (g \cdot x_1) \otimes \cdots \otimes (g \cdot x_d). \tag{III.2}$$

Also, the symmetric group acts on the right by permuting factors:

$$(x_1 \otimes \cdots \otimes x_d) \cdot s = x_{s(1)} \otimes \cdots \otimes x_{s(d)}. \tag{III.3}$$

Theorem III.1.1. *Consider the representations of $GL(n)$ and S_d on tensor space:*

$$GL(n) \xrightarrow{\rho_d} \mathrm{End}\big(\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n\big) \xleftarrow{\pi_d} S_d \tag{III.4}$$

1. *The actions of $GL(n)$ and S_d on $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ commute.*
2. *As a $(GL(n), S_d)$-bimodule, tensor space decomposes into simples*

$$\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n \cong \bigoplus_{\lambda \in \Lambda^+(n,d)} V(\lambda) \otimes S(\lambda). \tag{III.5}$$

3. *The image of each group in* $\mathrm{End}(\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ *generates the full centralizer algebra of the other group.*

For $d \leq n$, the homomorphism $\pi_d : \mathbb{C}S_d \to \mathrm{End}(\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ is injective. In the sequel, we will be concerned with the other extreme, where π_d is far from injective, so $\pi_d(\mathbb{C}S_d)$ is a proper quotient of the group algebra. But, first we must generalize Schur-Weyl duality is several ways. In ([5]), Berele and Regev use the Lie superalgebra $\mathfrak{gl}(m|n)$ and its standard $(m|n)$-dimensional module to establish an analogous duality theorem for $(\mathcal{U}(\mathfrak{gl}(m|n)), \mathbb{C}S_d)$, where the symmetric group acts by signed permutations. The indexing set for the decomposition generalizing (III.5) is the set $\Lambda^+(m|n, d)$ consisting of all Young diagrams that are $(m|n)$-hooks: $\lambda_i \leq n$ for all $i > m$. This is further generalized by Mitsuhashi ([32]) to a duality statement about the quantum deformation $\mathcal{U}_v(\mathfrak{gl}(m|n))$ of $\mathcal{U}(\mathfrak{gl}(m|n))$ and the Iwahori-Hecke algebra $\mathcal{H}_d$. However, we shall only consider the cases $(m|n) = (2|0)$ in connection to the Jones polynomial and $(1|1)$ in connection to the Alexander polynomial.

III.2. Representations of the Iwahori-Hecke Algebra

We summarize some important facts about the Iwahori-Hecke algebra $\mathcal{H}_d$ and its representations, most of which are modified versions of well know facts about the symmetric group S_d and its representations (see, e.g. [12]). Recall the standard generators $H_1, \ldots, H_{d-1}$ for $\mathcal{H}_d$, analogous to the generators $s_1, \ldots, s_{d-1}$ of S_d, where s_i is the elementary transposition $(i\ i+1)$.

1. The *standard basis* for $\mathcal{H}_d$ consists of the elements H_w, where $w \in S_d$. In terms of the standard generators,

$$H_w = H_{i_1} \cdots H_{i_m}, \tag{III.6}$$

where $w = s_{i_1} \cdots s_{i_m}$ is any reduced expression in S_d. The non-negative integer m is called the *length* of w and denoted $\ell(w)$.

2. Multiplication in this basis is

$$H_w H_i = \begin{cases} H_{ws_i}, & \text{if } \ell(ws_i) > \ell(w) \\ H_{ws_i} + (v - v^{-1})H_w, & \text{if } \ell(ws_i) < \ell(w). \end{cases} \tag{III.7}$$

3. There is an irreducible module $S(\lambda)$ of H_d corresponding to each integer partition λ. These are pairwise non-isomorphic, and any finite-dimensional representation decomposes as a direct sum of thse simple modules. An explicit construction of these modules using Young's semi-normal form is given in Chapter IV.

4. $\mathcal{H}_d$ is semi-simple. Hence, we have the isomorphisms of algebras

$$\mathcal{H}_d \cong \prod_\lambda \operatorname{End} S(\lambda) \cong \prod_\lambda M_{d_\lambda}, \tag{III.8}$$

where M_{d_λ} is the full matrix algebra of degree $d_\lambda = \dim S(\lambda)$. In fact, generically, we have the isomorphism $\mathcal{H}_d \cong \mathbb{C}(v)S_d$.

5. The branching rules for $\mathcal{H}_d$ are same as for S_d. Specifically, restriction of a simple $\mathcal{H}_d$ module $S(\lambda)$ to $\mathcal{H}_{d-1}$ decomposes into a multiplicity-free sum of simple modules $S(\mu)$, where μ is obtained from λ by removing a corner box.

Corresponding to each indecomposable block in the Wedderburn decomposition (III.8), there is a primitive, central idempotent $e(\lambda) \in \mathcal{H}_d$, which can be constructed explicitly, as follows. This is analogous to the Young symmetrizer in S_d and is originally due to Gyoja ([18]).

Corresponding to the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, let $S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_r}$ be the row stabilizer of the Young diagram. Let λ^T denote the transpose partition. Define the elements

$a_\lambda, b_\lambda \in \mathcal{H}_d$ (the symmetrizer and the skew-symmetrizer, respectively) by

$$a_\lambda = \sum_{w \in S_\lambda} v^{\ell(w)} H_w \qquad \text{and} \qquad b_\lambda = \sum_{w \in S_{\lambda^\mathrm{T}}} (-v)^{-\ell(w)} H_w \tag{III.9}$$

Define the permutation w_λ as follows. Fill the boxes of the Young diagram for λ with the numbers $1, \ldots, d$ across rows from left to right and top to bottom. Now, form the sequence $(w(1), \ldots, w(d))$ by reading the numbers down the columns from left to right. This sequence is the one-line notation for w_λ.

Theorem III.2.1 (Compare [13]). *The element $c_\lambda \in \mathcal{H}_d$ defined by*

$$c_\lambda = a_\lambda H_{w_\lambda} b_\lambda H_{w_\lambda}^{-1} \tag{III.10}$$

satisfies $c_\lambda c_\lambda = h_\lambda c_\lambda$ for some $h_\lambda \in \mathbb{C}(v)^\times$. Moreover, for any two partitions $\lambda \neq \mu$, $c_\lambda c_\mu = 0$.

Hence, the collection of $e(\lambda) = h_\lambda^{-1} c_\lambda$ for all partitions of d are mutually orthogonal, primitive, central idempotents that sum to one, which explicitly realize the decomposition in (III.8).

III.3. The Quantum Group $\mathcal{U}_v(\mathfrak{sl}(2))$

III.3.1. Definitions and Conventions

Let $\mathcal{U}$ be the quantum group $\mathcal{U}_v(\mathfrak{sl}(2))$ (see [21] or [20]). This is the unital, associative $\mathbb{C}(v)$-algebra generated by E, F, K, subject to the relations

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KE &= v^2 EK, \\ KF &= v^{-2} FK, \\ EF - FE &= \frac{K - K^{-1}}{v - v^{-1}}. \end{aligned}$$

The representation theory of $\mathcal{U}$ is analogous to that of $\mathfrak{sl}(2)$. Specifically, certain finite-dimensional simple modules for $\mathcal{U}$ are highest weight modules (the ones so-called type **1**), in the following multiplicative sense. In a $\mathcal{U}$-module W, a vector w has weight λ if $K.w = v^\lambda w$. For each $\lambda \in \mathbb{Z}_{\geq 0}$, there is a unique finite-dimensional simple $\mathcal{U}$-module $L(\lambda)$ of highest weight λ.

It is a standard fact that $\mathcal{U}$ is a Hopf algebra with comultiplication Δ, counit ε, and antipode S given by

$$\begin{aligned}
\Delta(K^{\pm 1}) &= K^{\pm 1} \otimes K^{\pm 1}, \\
\Delta(E) &= E \otimes K^{-1} + 1 \otimes E, \\
\Delta(F) &= F \otimes 1 + K \otimes F, \\
\varepsilon(K) &= 1, \quad \varepsilon(E) = \varepsilon(F) = 0. \\
S(K) &= K^{-1}, \\
S(E) &= -EK, \quad S(F) = -K^{-1}F.
\end{aligned}$$

Let W denote the natural module for $\mathcal{U}$, represented in the basis $\{w_1, w_2\}$ by the matrices

$$K \mapsto \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}, \qquad E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{III.11}$$

Let $\rho : \mathcal{U} \to \mathrm{End}(W)$ be the corresponding representation. Using the comultiplication in the Hopf algebra, it extends to a representation $\rho_d : \mathcal{U} \to \mathrm{End}(W^{\otimes d})$. Set $\Delta^{(1)} = \Delta$ and for $i > 1$, $\Delta^{(i)} = (\Delta \otimes \mathrm{Id}^{\otimes i-1}) \circ \Delta^{(i-1)}$. Then, $\rho_d = \rho^{\otimes d} \circ \Delta^{(d-1)}$, producing the formulae:

$$\begin{aligned}
\rho_d(K) &= \rho(K)^{\otimes d} \\
\rho_d(E) &= \sum_{i=1}^{d} \mathrm{Id}^{\otimes i-1} \otimes \rho(E) \otimes \rho(K^{-1})^{\otimes d-i} \\
\rho_d(F) &= \sum_{i=1}^{d} \rho(K)^{\otimes i-1} \otimes \rho(F) \otimes \mathrm{Id}^{\otimes d-i}.
\end{aligned}$$

We make $W^{\otimes d}$ into a right $\mathcal{H}_d$-module by defining a homomorphism $\pi_d : \mathcal{H}_d \to \mathrm{End}(W^{\otimes d})^{\mathrm{op}}$, as follows. First, define the right operator H on $W \otimes W$ by:

$$w_i \otimes w_j H = \begin{cases} v w_i \otimes w_j & \text{if } i = j, \\ w_j \otimes w_i + (v - v^{-1}) w_i \otimes w_j & \text{if } i < j, \\ w_j \otimes w_i & \text{if } i > j. \end{cases} \tag{III.12}$$

Now, set

$$\pi_d(H_i) = \mathrm{id}^{\otimes(i-1)} \otimes H \otimes \mathrm{id}^{\otimes(d-i-1)}. \tag{III.13}$$

III.3.2. Schur-Weyl Duality for $\mathcal{U}_v(\mathfrak{sl}(2))$ and $\mathcal{H}_d$

The following theorem is well-known.

Theorem III.3.1. *Consider the maps*

$$\mathcal{U} \xrightarrow{\rho_d} \mathrm{End}(W \otimes \cdots \otimes W) \xleftarrow{\pi_d} \mathcal{H}_d. \tag{III.14}$$

1. *The actions of $\mathcal{U}$ and $\mathcal{H}_d$ on $W \otimes \cdots \otimes W$ commute.*

2. *As a $(\mathcal{U}, \mathcal{H}_d)$-bimodule, tensor space decomposes into simples*

$$W \otimes \cdots \otimes W \cong \bigoplus_{\lambda \in \Lambda^+(2,k)} L(\lambda) \otimes S(\lambda). \tag{III.15}$$

3. *The map π_d factors through the Temperley Lieb algebra TL_d, via ψ which was defined in (II.21).*

$$\begin{array}{ccc} \mathrm{End}_{\mathcal{U}}(W \otimes \cdots \otimes W) & \xleftarrow{\pi_d} & \mathcal{H}_d \\ & \nwarrow^{\overline{\pi}_d} & \downarrow \psi \\ & & TL_d \end{array} \tag{III.16}$$

Proof. Statements 1 and 2 follow from Mitsuhashi, [32] [Theorem 5.1]. Part 3 is a consequence of Mitsuhashi's result, combined with Lemma III.2.1. We will explain a similar result in detail later on, so we omit a full explanation here (See Remark III.5.5 below). □

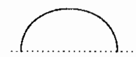
III.3.3. Jones Polynomial as a Quantum Invariant

The tensor square of the standard $\mathcal{U}$-module $W = L(1)$ decomposes as

$$W \otimes W \cong S^2W \oplus \Lambda^2 W \tag{III.17}$$

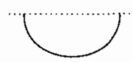
where $S^2W \cong L(2)$ is the v-symmetrized submodule on which $\mathcal{H}_2$ acts diagonally as the scalar v and $\Lambda^2 W \cong L(0) \cong \mathcal{C}(v)$ is the v-skew-symmetrized submodule on which $\mathcal{H}_2$ acts as $-v^{-1}$. This is the $k = 2$ example of Theorem III.29, which we now interpret with arc diagrams.

We interpret the $(0,2)$-tangle (a cap, reading up the page) as the linear map $b : W \otimes W \to \mathbb{C}(v)$, given by the formulas



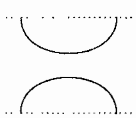
$$W \otimes W \to \mathbb{C}(v): \quad w_1 \otimes w_1 \mapsto 0, \quad w_1 \otimes w_2 \mapsto v^{-1}, \quad w_2 \otimes w_1 \mapsto -1, \quad w_2 \otimes w_2 \mapsto 0 \tag{III.18}$$

Dually, we interpret the $(2,0)$-tangle (a cup) as the map $c : \mathbb{C}(v) \to W \otimes W$, given by the formula $c(1) = w_1 \otimes w_2 - v w_2 \otimes w_1$.



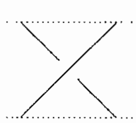
$$\mathbb{C}(v) \to W \otimes W: \quad 1 \mapsto w_1 \otimes w_2 - v w_2 \otimes w_1 \tag{III.19}$$

The $(2,2)$-tangle (cup over cap) gives the composition $c \circ b : W \otimes W \to W \otimes W$, which is a non-zero multiple of the canonical map $W \otimes W \twoheadrightarrow \Lambda^2 W \hookrightarrow W \otimes W$, and is given by the formulas

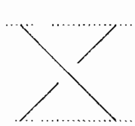


$$W \otimes W \to W \otimes W: \quad w_1 \otimes w_1 \mapsto 0, \quad w_1 \otimes w_2 \mapsto v^{-1} w_1 \otimes w_2 - w_2 \otimes w_1, \quad w_2 \otimes w_1 \mapsto -w_1 \otimes w_2 + v w_2 \otimes w_1, \quad w_2 \otimes w_2 \mapsto 0 \tag{III.20}$$

The map $\mathrm{Id}^{\otimes i-1} \otimes (c \circ b) \otimes \mathrm{Id}^{\otimes d-i-1} : W^{\otimes d} \to W^{\otimes d}$ is precisely the image of the standard generator U_i of TL_d in the representation π_d. Hence, we recover a matrix representation for the generator H_i of $\mathcal{H}_d$, using the relations $\pi_d(H_i) = \overline{\pi}_d(v - U_i)$ and $\pi_d(H_i^{-1}) = \overline{\pi}_d(v^{-1} - U_i)$, we recover the action (III.12) of $\mathcal{H}_d$.



$$W \otimes W \to W \otimes W: \quad w_1 \otimes w_1 \mapsto v, \quad w_1 \otimes w_2 \mapsto (v - v^{-1}) w_1 \otimes w_2 + w_2 \otimes w_1, \quad w_2 \otimes w_1 \mapsto w_1 \otimes w_2, \quad w_2 \otimes w_2 \mapsto v \tag{III.21}$$



$$W \otimes W \to W \otimes W: \quad w_1 \otimes w_1 \mapsto v^{-1}, \quad w_1 \otimes w_2 \mapsto w_2 \otimes w_1, \quad w_2 \otimes w_1 \mapsto w_1 \otimes w_2 - (v - v^{-1}) w_2 \otimes w_1, \quad w_2 \otimes w_2 \mapsto v^{-1} \tag{III.22}$$

III.4. The Quantum Supergroup $\mathcal{U}_v(\mathfrak{gl}(1|1))$

We introduce the *general linear Lie superalgebra* $\mathfrak{g} \doteq \mathfrak{gl}(1|1)$, as well as its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and its quantization $\mathcal{U}_v(\mathfrak{g})$. (See [40]) and [4]).

III.4.1. Lie Superalgebra $\mathfrak{gl}(1|1)$

We work over the base ring $\mathbb{C}$. A vector superspace of superdimension $(m|n)$ is a $\mathbb{Z}_2$-graded vector space $W = W_0 \oplus W_1$ whose even part W_0 is of dimension m and whose odd part W_1 is of dimension n. For homogeneous elements $w \in W_i$, write $|w| = i \in \mathbb{Z}_2$. Let $\mathbb{C}^{m|n}$ denote the superspace $\mathbb{C}^m \oplus \mathbb{C}^n$. The endomorphisms of $\mathbb{C}^{m|m}$ form the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$, consisting of block matrices of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an $m \times m$ matrix, B an $m \times n$ matrix, C an $n \times m$ matrix, and D an $n \times n$ matrix. There is a $\mathbb{Z}_2$-grading: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ consists of block diagonal matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ and $\mathfrak{g}_1$ consists of matrices of the form $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$. The Lie superbracket is defined on homogeneous elements to be the supercommutator

$$[X,Y] = XY - (-1)^{ij}YX, \qquad \text{where } |X| = i, |Y| = j. \tag{III.23}$$

and extended bilinearly. This multiplication satisfies the axioms of a superbracket, namely for homogeneous elements X, Y, Z with $|X| = i, |Y| = j, |Z| = k$,

$$\begin{aligned} &[X,Y] + (-1)^{ij}[Y,X] = 0 && \text{(super-skew commutativity)} \\ &(-1)^{ki}[X,[Y,Z]] + (-1)^{ij}[Y,[Z,X]] + (-1)^{jk}[Z,[X,Y]] = 0 && \text{(super-Jacobi).} \end{aligned} \tag{III.24}$$

III.4.2. The Quantum Supergroup $\mathcal{U}_v(\mathfrak{gl}(1|1))$

In the sequel, we shall only be interested in the quantum supergroup associated to the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1|1)$. Following [4], let $\mathcal{U}$ be the quantum supergroup $\mathcal{U}_v(\mathfrak{gl}(1|1))$. This is the unital, associative $\mathbb{C}(v)$-algebra generated by E, F, K_a and K_a^{-1} ($a \in \{1, 2\}$) subject to the relations

$$K_a K_a^{-1} = K_a^{-1} K_a = 1,$$
$$K_a K_b = K_b K_a,$$
$$E^2 = F^2 = 0,$$
$$K_1 E = vEK_1, \qquad K_2 E = v^{-1} E K_2$$
$$K_1 F = v^{-1} F K_1, \qquad K_2 F = vFK_2$$
$$EF + FE = \frac{K - K^{-1}}{v - v^{-1}},$$

where $K = K_1 K_2^{-1}$. Note that K_1 and K_2 are even while E and F are odd.

The weight of a vector w in a $\mathcal{U}$-module refers to $\lambda \in \mathbb{Z}$ such that $K.w = v^\lambda w$. Let $L(\lambda)$ denote the indecomposable weight module of highest weight λ. Because E and F are nilpotent in $\mathcal{U}$, any indecomposable cyclic module has dimension at most four. Note that there are indecomposable modules that are not simple, such as the the adjoint module. However, see Theorem III.4.1 below.

It is a standard fact that $\mathcal{U}$ is a Hopf superalgebra with comultiplication Δ, counit ε, and antipode S given by

$$\Delta(K_a) = K_a \otimes K_a,$$
$$\Delta(E) = E \otimes K^{-1} + 1 \otimes E,$$
$$\Delta(F) = F \otimes 1 + K \otimes F,$$
$$\varepsilon(K_a^{\pm 1}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.$$
$$S(K_a) = K_a^{-1},$$
$$S(E) = -EK, \quad S(F) = -K^{-1}F.$$

Let W denote the natural module for $\mathcal{U}$, represented in the basis $\{w_1, w_2\}$ (let w_1 be even) by the matrices

$$K_1 \mapsto \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}, \qquad K_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}, \qquad E \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{III.25}$$

Let $\rho : \mathcal{U} \to \mathrm{End}(W)$ be the representation. Using the comultiplication in the Hopf algebra, it extends to a representation of $\rho_d : \mathcal{U} \to \mathrm{End}(W^{\otimes d})$. Set $\Delta^{(1)} = \Delta$ and for $i > 1$, $\Delta^{(i)} = (\Delta \otimes \mathrm{Id}^{\otimes i-1}) \circ \Delta^{(i-1)}$. Then, $\rho_d = \rho^{\otimes d} \circ \Delta^{(d-1)}$.

Theorem III.4.1 ([4], Prop. 3.1). *The representations ρ_d are completely reducible.*

Make $W^{\otimes d}$ into a right $\mathcal{H}_d$-module by defining the homomorphism $\pi_d : \mathcal{H}_d \to \mathrm{End}(W^{\otimes d})^{\mathrm{op}}$, as follows. First, define the right operator H on $W \otimes W$ by:

$$w_i \otimes w_j H = \begin{cases} v w_i \otimes w_j & \text{if } i = j \text{ and } |w_i| = 0\,, \\ -v^{-1} w_i \otimes w_j & \text{if } i = j \text{ and } |w_i| = 1\,, \\ w_j \otimes w_i + (v - v^{-1}) w_i \otimes w_j & \text{if } i < j\,, \\ w_j \otimes w_i & \text{if } i > j\,. \end{cases} \tag{III.26}$$

Now, set

$$\pi_d(H_i) = \mathrm{Id}^{\otimes(i-1)} \otimes H \otimes \mathrm{Id}^{\otimes(d-i-1)} \tag{III.27}$$

and extend by linearity.

III.4.3. Schur-Weyl Duality for $\mathcal{U}_v(\mathfrak{gl}(1|1))$ and $\mathcal{H}_d$

Recall that $\Lambda^+(1|1, d)$ is the set of Young diagrams of hook shape with d boxes.

Theorem III.4.2 ([32], Theorem 5.1). *Consider the maps*

$$\mathcal{U} \xrightarrow{\rho_d} \mathrm{End}(W \otimes \cdots \otimes W) \xleftarrow{\pi_d} \mathcal{H}_d. \tag{III.28}$$

1. *The actions of $\mathcal{U}$ and $\mathcal{H}_d$ on $W \otimes \cdots \otimes W$ commute.*

2. *As a $(\mathcal{U}, \mathcal{H}_d)$-bimodule, tensor space decomposes into simples*

$$W \otimes \cdots \otimes W \cong \bigoplus_{\lambda \in \Lambda^+(1|1,d)} L(\lambda) \otimes S(\lambda). \tag{III.29}$$

III.5. The Super Temperley-Lieb Algebra

III.5.1. Definition of STL_d.

Definition III.5.1. The *super Temperley-Lieb* algebra STL_d is the centralizer algebra

$$\mathrm{End}_{\mathcal{U}_v\left(\mathfrak{gl}(1|1)\right)}(W^{\otimes d})^{\mathrm{op}}. \tag{III.30}$$

By Theorem III.4.2, this is $\pi_d(\mathcal{H}_d)$.

Proposition III.5.2.

$$\dim STL_d = \sum_{j=0}^{d-1} \binom{d-1}{j}^2 = \binom{2d-2}{d-1} \tag{III.31}$$

Proof. As abstract $\mathbb{C}(v)$-algebras, $STL_d \cong \prod_\lambda \mathrm{End}\, S(\lambda)$, where the product is over λ whose Young diagrams are hooks. Let $\lambda^{(j)} = (d-j, 1^j)$ be the hook partition of leg length j for $0 \le j < d$. The irreducible representation $S(\lambda^{(1)})$ is of degree $d-1$, and for each $j = 0, \dots, d-1$, $S(\lambda^{(j)}) \cong \Lambda^j S(\lambda^{(1)})$, hence has degree $\binom{d-1}{j}$.

The second equality follows from looking at the coefficient of t^{d-1} in the expansion of $\left((1+t)^{d-1}\right)^2 = (1+t)^{2d-2}$. □

We have the presentation $STL_d = \mathcal{H}_d/\mathcal{I}_d$, where $\mathcal{I}_d$ is the two sided ideal generated by the idempotents $e(\lambda)$, where λ is not a hook partition. For $1 \le d \le 3$, $\mathcal{I}_d$ is trivial, hence the representation π_d is injective. Now, define $\mathcal{J}$ to be the two-sided ideal of $\mathcal{H}_4$ generated by

$$(H_1 - v)(H_3 - v)H_2(H_1 + v^{-1})(H_3 + v^{-1}). \tag{III.32}$$

Lemma III.5.3.

$$\mathcal{J} = \mathcal{I}_4 \tag{III.33}$$

Proof. The only partition of 4 that is not a hook is $\lambda = (2^2)$. The symmetrizer is

$$c_\lambda = (H_1 - v)(H_3 - v)H_2(H_1 + v^{-1})(H_3 + v^{-1})H_2^{-1}, \tag{III.34}$$

which is a nonzero scalar multiple of the idempotent $e(\lambda)$. Clearly, this generates the same two-sided ideal as $\mathcal{J}$. □

For $d \geq 4$, define the embedding $\iota : \mathcal{H}_4 \hookrightarrow \mathcal{H}_d$ by $\iota(H_i) = H_i$, and let $\mathcal{J}_d$ be the two-sided ideal generated by $\iota(\mathcal{J})$.

Theorem III.5.4.

$$\mathcal{I}_d = \mathcal{J}_d \quad \textit{for all } d \geq 4. \tag{III.35}$$

Proof. The following diagram commutes

$$\begin{array}{ccc} \mathcal{H}_4 & \xrightarrow{\ \iota\ } & \mathcal{H}_d \\ {\scriptstyle \pi_4}\big\downarrow & & \big\downarrow{\scriptstyle \pi_d} \\ \mathrm{End}_{\mathcal{U}}(W^{\otimes 4}) & \longrightarrow & \mathrm{End}_{\mathcal{U}}(W^{\otimes d}) \end{array} \tag{III.36}$$

where the bottom map sends $\phi \mapsto \phi \otimes \mathrm{Id}^{\otimes d-4}$. Since $\mathcal{J} \in \ker \pi_4$, $\iota(\mathcal{J}) \in \ker \pi_d$, and so $\mathcal{J}_d \subseteq \mathcal{I}_d$.

Conversely, suppose that $\mathcal{J}_d \neq \mathcal{I}_d$. So, there exists λ that is not a hook shape with $e(\lambda)\mathcal{J}_d = 0$. Since $S(\lambda)$ embeds into $\mathcal{H}_d e(\lambda)$, $\mathcal{J}_d$ acts as 0 on $S(\lambda)$. Since λ is not a hook, its diagram contains the diagram (2^2), so $S(2^2) \hookrightarrow \mathrm{Res}^{\mathcal{H}_d}_{\mathcal{H}_4} S(\lambda)$. Thus, $\mathcal{J}_d$ acts as 0 on $S(2^2)$, a contradiction. □

Remark III.5.5. The technique in Theorem III.5.4 of reducing to the smallest λ that is excluded from the indexing set works generally. In particular, the kernel of the map $\psi : \mathcal{H}_d \twoheadrightarrow TL_d$ is generated by the symmetrizer c_λ for $\lambda = (1^3)$, which is the smallest diagram not in $\Lambda^+(2, d)$.

III.5.2. Robinson-Schenstedt-Knuth Correspondence

Recall that the Robinson-Schenstedt-Knuth (RSK) correspondence establishes a bijection between permutations $w \in S_d$ and pairs of standard Young tableaux (P, Q) each of the same shape $\lambda \in \mathrm{Par}(d)$ (see, e.g. [15]). The tableaux P is called the *insertion tableaux* and is defined by the following algorithm.

Begin with the permutation written in one-line notation: $(w(1), \ldots, w(d))$. Build the sequence of tableaux $\varnothing = P_0, \ldots, P_d = P$, forming P_i from P_{i-1} by inserting the number $w(i)$ as follows. Place $w(i)$ as far to the left on the first row as possible. If it is placed on the end of the row, then we are done. Otherwise, $w(i)$ "bumps" a number out of its box, and that number is placed on the next row, again as far to the left as possible. This algorithm terminates when a number is placed on the end of a row (which may be a new row at the bottom of the diagram).

Two permutations w, w' are in the same two-sided Kazhdan-Lusztig cell in $\mathcal{H}_d$ if they produce tableaux of the same shape under the RSK correspondence. Let C_w denote the canonical basis element (see [29] or [39]). Unlike the case TL_d, which has a basis consisting of monomials in the canonical elements $-U_i = H_i - v$, we do not know of such an elementary basis for STL_d. However,

Theorem III.5.6. *$\{C_w \mid$ shape of $P(w)$ is a hook$\}$ is a basis for STL_d.*

Proof. For any $\lambda \in \mathrm{Par}(d)$ and for any $\mathtt{T} \in \mathrm{Std}(\lambda)$, the collection of C_w such that $P(w) = \mathtt{T}$ is a basis for a submodule of the regular module $\mathcal{H}_d$ that is isomorphic to $S(\lambda)$. Moreover, the collection of all C_w such that the shape of $P(w)$ is λ forms the Wedderburn component in $\mathcal{H}_d$ isomorphic to $\mathrm{End}\, S(\lambda)$. By the definition of STL_d and Theorem III.4.2, we obtain a basis for the algebra by concatenating the bases of each Wedderburn component corresponding to λ of hook shape. ☐

CHAPTER IV

A STATE-SUM FORMULA FOR THE ALEXANDER POLYNOMIAL

IV.1. Semi-normal Representations of the Hecke Algebra

Much of the following is standard (see e.g. [11] or [25, chapters 4 and 5]). We collect some of the definitions here and fix some notation. In what follows, $[r] \in \mathbb{Z}[v, v^{-1}]$ denotes the quantum integer

$$[r] = \frac{v^r - v^{-r}}{v - v^{-1}} \tag{IV.1}$$

for any $r \in \mathbb{Z}$.

IV.1.1. Seminormal Representations

This exposition follows [35, section 3], although the results were originally worked out in [19]. For a new point of view and substantial generalization, see [8, section 5].

Let $\mathrm{Tab}(\lambda)$ denote the set of λ-tableaux. These are fillings of the boxes in the Young diagram λ by the numbers $1, \ldots, d$. Let $\mathrm{Std}(\lambda)$ denote the set of *standard* λ-tableaux, namely those that increase across rows and down columns. The symmetric group S_d acts on $\mathrm{Tab}(\lambda)$ via its natural action on the entries, although $\mathrm{Std}(\lambda)$ is not stable under this action. For a tableau $\mathtt{T} \in \mathrm{Tab}(\lambda)$, its *residue sequence* $(i_1, \ldots, i_d) \in \mathbb{Z}^d$ is defined by setting $i_r = b - a$ where the box labeled r in $\mathtt{T}$ appears in row a and column b.

$$\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline \\ \cline{1-1}\end{array} \qquad \mathtt{T} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad s_2\mathtt{T} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Figure IV.1: The partition $\lambda = (3, 1) \in \mathrm{Par}(4)$ and standard tableaux $\mathtt{T}$ and $s_2\mathtt{T}$. Here, $\mathtt{T}$ has residue sequence $(0, 1, -1, 2)$.

Fix a partition $\lambda \in \mathrm{Par}(n)$ and let $S(\lambda)$ be the $\mathbb{C}(v)$-vector space on basis $\{x_{\mathtt{T}} \mid \mathtt{T} \in \mathrm{Std}(\lambda)\}$.

Let $(i_1, \dots, i_d)$ be the residue sequence of $\mathtt{T}$ and define $a_r(\mathtt{T}), b_r(\mathtt{T}) \in \mathbb{C}(v)$ to be

$$a_r(\mathtt{T}) = \frac{v - v^{-1}}{1 - v^{2(i_r - i_{r+1})}}, \qquad b_r(\mathtt{T}) = v^{-1} + a_r(\mathtt{T}). \tag{IV.2}$$

Define actions of the generators $H_1, \dots, H_{d-1}$ of $\mathcal{H}_d$ on $S(\lambda)$ by

$$H_r x_{\mathtt{T}} = a_r(\mathtt{T}) x_{\mathtt{T}} + b_r(\mathtt{T}) x_{s_r\mathtt{T}}, \tag{IV.3}$$

where we interpret $x_{s_r\mathtt{T}} = 0$ if $s_r\mathtt{T}$ is not a standard tableau.

Theorem IV.1.1 (Semi-normal representations)**.** *This action extends to make $S(\lambda)$ into a well-defined $\mathcal{H}_d$-module. Furthermore, the modules $\{S(\lambda) \mid \lambda \in \mathrm{Par}(d)\}$ constitute a complete set of pairwise non-isomorphic irreducible modules for $\mathcal{H}_d$.*

IV.1.2. Sign Sequences and Hook Partitions

For $0 \le \ell \le d-1$, let λ_ℓ be the hook partition $(d-\ell, 1^\ell)$. We refer to ℓ as *leg length.*

Lemma IV.1.2. *Standard tableaux of shape λ_ℓ are in bijection with sign sequences $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{\pm\}^d$ such that $\varepsilon_1 = +$ and ℓ entries equal $-$.*

Proof. Beginning with a standard λ_ℓ-tableau, define $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ by

$$\varepsilon_r = \begin{cases} +, & \text{if } r \text{ appears on the first row} \\ -, & \text{otherwise} \end{cases} \tag{IV.4}$$

Notice that the box labeled 1 has to be in the corner of the hook, so $\varepsilon_1 = +$. Also, ℓ numbers are on the leg of the hook, so there are ℓ entries equal to $-$.

For the inverse, starting with a sign sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ with $\varepsilon_1 = +$ and ℓ other entries equal to $-$, construct a standard tableau recursively, as follows. Place 1 in the corner of the diagram. Now, for each $r > 1$, suppose that the numbers $1, \dots, r-1$ have been placed. Either add r to the end of the first row or at the bottom of the first column, according to whether ε_r is $+$ or $-$, respectively. □

Using this bijection, we can adapt the semi-normal representation to the combinatorics of sign sequences.

Theorem IV.1.3. *The irreducible module $S(\lambda_\ell)$ has basis $\{x_\varepsilon\}$, where ε runs over sign sequences having $\varepsilon_1 = +$ and ℓ other entries equal to $-$. The generators $H_1, \ldots, H_{k-1}$ of $\mathcal{H}_d$ act by*

$$H_r x_\varepsilon = a_r(\varepsilon) x_\varepsilon + b_r(\varepsilon) x_{s_r \varepsilon} \tag{IV.5}$$

where $s_r\varepsilon$ denotes the sign sequence obtained from ε by permuting ε_r and ε_{r+1}, x_ε is interpreted as zero if $\varepsilon_1 = -$, and

$$a_r(\varepsilon) = \begin{cases} v & \textit{if } (\varepsilon_r, \varepsilon_{r+1}) = (+,+) \\ -v^{-1} & \textit{if } (\varepsilon_r, \varepsilon_{r+1}) = (-,-) \\ v^r/[r] & \textit{if } (\varepsilon_r, \varepsilon_{r+1}) = (-,+) \\ -v^{-r}/[r] & \textit{if } (\varepsilon_r, \varepsilon_{r+1}) = (+,-), \end{cases} \tag{IV.6}$$

$$b_r(\varepsilon) = \begin{cases} [r+1]/[r] & \textit{if } (\varepsilon_r, \varepsilon_{r+1}) = (-,+) \\ [r-1]/[r] & \textit{if } (\varepsilon_r, \varepsilon_{r+1}) = (+,-) \\ 0 & \textit{otherwise.} \end{cases} \tag{IV.7}$$

The inverse generators $H_1^{-1}, \ldots, H_{d-1}^{-1}$ act by

$$H_r^{-1} x_\varepsilon = \bar{a}_r(\varepsilon) x_\varepsilon + b_r(\varepsilon) x_{s_r \varepsilon} \tag{IV.8}$$

where $\bar{a}_r(\varepsilon)$ is obtained from $a_r(\varepsilon)$ by replacing v by v^{-1}.

Proof. This is just a translation of Theorem IV.1.1 using the bijection from Lemma IV.1.2. Given a sign sequence $\varepsilon \in \{\pm\}^k$ having $\varepsilon_1 = +$ and ℓ other entries equal to $-$, construct the corresponding standard tableau, and let $(i_1, \ldots, i_d)$ be its residue sequence. We have $i_1 = 0$, and for $1 \leq r < d$,

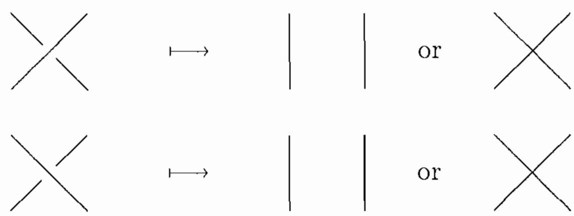
$$i_{r+1} = \begin{cases} i_r + 1 & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+,+) \\ i_r - 1 & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-,-) \\ i_r + r & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (-,+) \\ i_r - r & \text{if } (\varepsilon_r, \varepsilon_{r+1}) = (+,-). \end{cases} \tag{IV.9}$$

Given this, the formulae (IV.6)–(IV.7) are easily deduced from (IV.2). Finally the formula (IV.8) is easily deduced from (IV.5) since $H_r^{-1} = H_r - (v - v^{-1})$. □

IV.2. Construction of the State-sum

Begin with a word $\alpha \in \Sigma_d^\bullet$ in the braid generators $\sigma_1, \ldots, \sigma_{d-1}$ and their inverses, which we picture as a diagram drawn up the page as the word is read from right to left.

Construct *permutation diagrams* from the braid diagram by replacing each crossing by one of two resolutions:



Definition IV.2.1. A permutation diagram $\mathbf{x}$ is *admissible* if

(P1) the first (leftmost) strand goes straight through without crossing any other strands, and

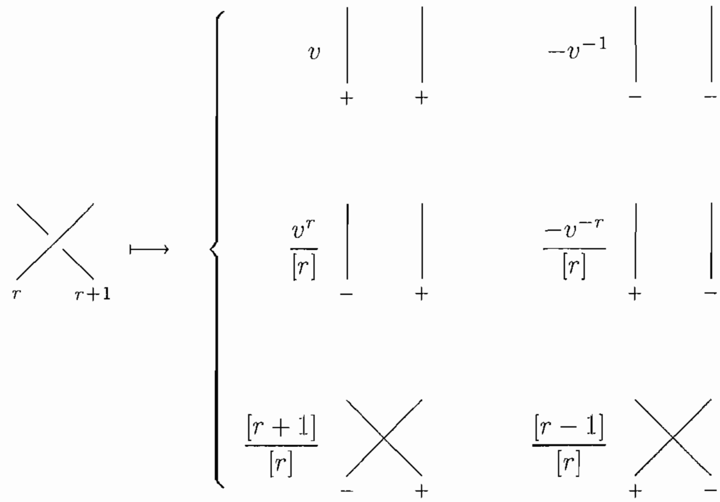
(P2) the underlying permutation is the identity.

A *state* is a pair $(\mathbf{x}, \varepsilon)$, where $\mathbf{x}$ is an admissible permutation diagram and ε is an assignment of a sign $\pm$ to each strand such that

(S1) the first (leftmost) sign is $+$, and

(S2) no two strands of the same sign cross.

Let $\mathcal{S}(\alpha)$ denote the set of states for α. To a state $(\mathbf{x}, \varepsilon) \in \mathcal{S}(\alpha)$, we associate a weight $M(\alpha, \mathbf{x}, \varepsilon) \in \mathbb{C}(v)$ defined by multiplying together certain scalars, one for each resolved crossing. The scalar associated to a positive crossing of strands in positions r and $r+1$ is given in (IV.10). For a negative crossing, replace v by v^{-1} in each expression.



(IV.10)

Define $A(\alpha) \in \mathbb{C}(v)$ by

$$A(\alpha) = \frac{1}{[d]} \sum_{(\mathbf{x},\varepsilon) \in \mathcal{S}(\alpha)} \langle \varepsilon \rangle M(\alpha, \mathbf{x}, \varepsilon) \tag{IV.11}$$

where $\langle \varepsilon \rangle \in \{\pm 1\}$ is the product of the signs $\varepsilon_1, \dots, \varepsilon_d$ attached to the strands.

Theorem IV.2.2. *Let L be an oriented link, and let $\alpha \in \Sigma_d^\bullet$ represent a braid in $\mathcal{B}_d$ such that $\widehat{\alpha} = L$. Then, $A(\alpha)$ is a polynomial in $v - v^{-1}$, and*

$$A(\alpha) = \nabla(L),$$

where $\nabla(L)$ is the Conway-normalized Alexander polynomial with $z = v - v^{-1}$.

Proof. To avoid confusion when switching between the generators H_r and $T_r = vH_r$ of $\mathcal{H}_d$, let us write $\varphi : \mathcal{B}_d \twoheadrightarrow \mathcal{H}_d^\times$ for the group homomorphism given by $\varphi(\sigma_r) = H_r$ and $\phi : \mathcal{B}_d \twoheadrightarrow \mathcal{H}_d^\times$ for the one with $\phi(\sigma_r) = T_r$. Formula (7.2) in [24] gives the Alexander polynomial for a link L as

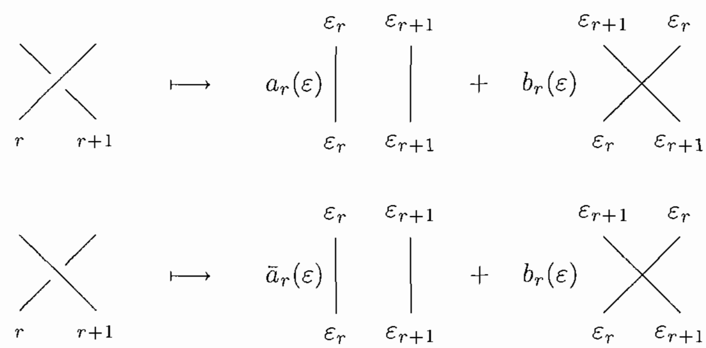
$$\nabla(L) = (-1)^{d-1} \left(\frac{1}{q}\right)^{(e(\alpha)-d+1)/2} \frac{1-q}{1-q^d} \sum_{j=0}^{d-1} (-1)^j \chi_{d-1-j}\big(\phi(\alpha)\big) \tag{IV.12}$$

where $\chi_\ell = \operatorname{tr} \circ \rho_\ell$ is the character of $\mathcal{H}_d$ arising from the irreducible representation $\rho_\ell : \mathcal{H}_d \to \operatorname{End}(S(\lambda_\ell))$ indexed by the hook partition $(d-\ell, 1^\ell)$, and e is the exponent sum of α. Put $q = v^2$

and $T_r = vH_r$ for each r, so that $\phi(\alpha) = v^{e(\alpha)}\varphi(\alpha)$. Reindex the sum over $\ell = d - 1 - j$ to get

$$\begin{aligned} \nabla(L) &= (-1)^{d-1}\left(\frac{1}{v}\right)^{e(\alpha)-d+1} \frac{1-v^2}{1-v^{2d}} \sum_{\ell=0}^{d-1} (-1)^{d-1-\ell} \chi_\ell\big(v^e \varphi(\alpha)\big) \\ &= \frac{1}{[d]} \sum_{\ell=0}^{d-1} (-1)^{\ell} \chi_\ell\big(\varphi(\alpha)\big). \end{aligned} \tag{IV.13}$$

Now we compute χ_ℓ by using the semi-normal form for $S(\lambda_\ell)$. The action of the generators H_r, H_r^{-1} on x_ε from Theorem IV.1.3 are pictured in (IV.14).

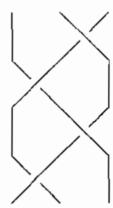


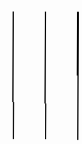
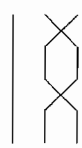
(IV.14)

Moreover, if $r = 1$, the second term on the right hand side should be omitted. Only diagonal entries of the matrix $\rho_\ell(\varphi(\alpha))$ contribute to the trace. Hence, for each ℓ we need only consider those permutation diagrams that represent the identity permutation and whose first strand goes straight through. The theorem follows on comparing formulas (IV.6) and (IV.7) with (IV.10). □

IV.3. An Example

Let's use the braid presentation $\alpha = \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$ for the figure-eight knot, pictured below with its six possible states.



$\mathbf{x}$	ε	$M(\alpha, \mathbf{x}, \varepsilon)$		
	$+ + +$	$v^{-1} \cdot v \cdot v^{-1} \cdot v$	$=$	1
	$+ + -$	$\frac{-v^2}{[2]} \cdot v \cdot \frac{-v^2}{[2]} \cdot v$	$=$	$\frac{v^6}{[2]^2}$
	$+ - +$	$\frac{v^{-2}}{[2]} \cdot (-v^{-1}) \cdot \frac{v^{-2}}{[2]} \cdot (-v^{-1})$	$=$	$\frac{v^{-6}}{[2]^2}$
	$+ - -$	$(-v) \cdot (-v^{-1}) \cdot (-v) \cdot (-v^{-1})$	$=$	1
	$+ + -$	$\frac{[3]}{[2]} \cdot (-v^{-1}) \cdot \frac{[1]}{[2]} \cdot v$	$=$	$\frac{-[3]}{[2]^2}$
	$+ - +$	$\frac{[1]}{[2]} \cdot v \cdot \frac{[3]}{[2]} \cdot (-v^{-1})$	$=$	$\frac{-[3]}{[2]^2}$

Now, we calculate the sum, minding the signs associated to each state and the global rescaling.

$$A(\alpha) = \frac{1}{[3]}\left(2 - \frac{v^6 + v^{-6}}{[2]^2} + \frac{2[3]}{[2]^2}\right) = -v^2 + 3 - v^{-2} = 1 - (v - v^{-1})^2$$

CHAPTER V

COMBINATORIAL PROOF

In this chapter we prove Theorem IV.2.2 directly from formulas (IV.10) and (IV.11). Since the function A is defined on the set of words in the braid generators, we must show that A is well-defined on the braid group.

V.1. Braid Group Invariance

For a permutation diagram $\mathbf{x}$, let $\pi(\mathbf{x}) \in S_d$ denote the underlying permutation. Let $\mathcal{P}_\pi(\alpha) = \{\mathbf{x} \text{ permutation diagram satisfying } (\mathbf{P1}) \mid \pi(\mathbf{x}) = \pi\}$. Notice that $\mathcal{P}_1(\alpha) = \mathcal{P}(\alpha)$, the admissible diagrams. We have immediately:

$$\begin{aligned} \mathcal{P}(\alpha\beta) &= \coprod_{\pi \in S_d} \left\{\mathbf{xy} \mid \mathbf{x} \in \mathcal{P}_\pi(\alpha) \text{ and } \mathbf{y} \in \mathcal{P}_{\pi^{-1}}(\beta)\right\} \\ &\cong \coprod_{\pi \in S_d} \mathcal{P}_\pi(\alpha) \times \mathcal{P}_{\pi^{-1}}(\beta) \end{aligned} \tag{V.1}$$

With signs attached to the strands in the diagrams, we have the analogous definition of $\mathcal{S}_\pi(\alpha)$ and the decomposition:

$$\mathcal{S}(\alpha\beta) \cong \coprod_{\pi \in S_d} \mathcal{S}_\pi(\alpha) \times \mathcal{S}_{\pi^{-1}}(\beta). \tag{V.2}$$

We prove a Lemma now that shows that the function A is invariant under concatenation for certain pairs of braid words. For $\pi \in S_d$, define

$$A_\pi(\alpha) = \sum_{(\mathbf{x},\varepsilon) \in \mathcal{S}_\pi(\alpha)} \langle \varepsilon \rangle M(\alpha, \mathbf{x}, \varepsilon) \tag{V.3}$$

Lemma V.1.1. *Suppose that $\beta, \beta' \in \Sigma_d^\bullet$ satisfy $A_\pi(\beta) = A_\pi(\beta')$ for all $\pi \in S_d$. Then, $A(\alpha\beta) = A(\alpha\beta')$ and $A(\beta\alpha) = A(\beta'\alpha)$ for any $\alpha \in \Sigma_d^\bullet$.*

Proof. Let $\mathcal{S}_\pi(\alpha,\beta) = \mathcal{S}_\pi(\alpha) \times \mathcal{S}_{\pi^{-1}}(\beta)$.

$$A(\alpha\beta) = \frac{1}{[d]} \sum_{\pi \in S_d} \sum_{\mathcal{S}_\pi(\alpha,\beta)} \langle \varepsilon\nu \rangle M(\alpha\beta, \mathbf{xy}, \varepsilon\nu) \tag{V.4}$$

$$= \frac{1}{[d]} \sum_{\pi \in S_n} \sum_{(\mathbf{x},\varepsilon) \in \mathcal{S}_\pi(\alpha)} \langle \varepsilon \rangle M(\alpha, \mathbf{x}, \varepsilon) \sum_{(\mathbf{y},\nu) \in \mathcal{S}_{\pi^{-1}}(\beta)} \langle \nu \rangle M(\beta, \mathbf{y}, \nu) \tag{V.5}$$

$$= \frac{1}{[d]} \sum_{\pi \in S_n} \sum_{(\mathbf{x},\varepsilon) \in \mathcal{S}_\pi(\alpha)} \langle \varepsilon \rangle M(\alpha, \mathbf{x}, \varepsilon)\, A_{\pi^{-1}}(\beta) \tag{V.6}$$

This last expression is equal to $A(\alpha\beta')$, using the fact that $A_\pi(\beta) = A_\pi(\beta')$. The proof that $A(\beta\alpha) = A(\beta'\alpha)$ is completely analogous and is omitted. □

Proposition V.1.2. *If $\alpha, \beta \in \Sigma_d^\bullet$ represent the same group element in $\mathcal{B}_d$, then $A(\alpha) = A(\beta)$.*

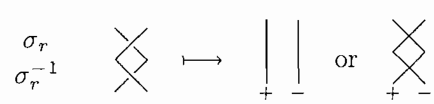
Proof. **Identity.** $A(1\alpha) = A(\alpha) = A(\alpha 1)$.

This is clear since A is defined by resolving crossings in the braid diagram. For the remainder of the proof, we use Lemma V.1.1, checking cases according to permutations π.

Inverse. $A(\sigma_r^{\pm 1}\sigma_r^{\mp 1}) = A(\varnothing)$.

Let $(\mathbf{x}, \varepsilon) \in \mathcal{S}_\pi(\sigma_r\sigma_r^{-1})$, where $\pi = \pi(\mathbf{x})$. We must show that $A_\pi(\sigma_r\sigma_r^{-1}) = 1$ when $\pi = 1$ and 0 otherwise.

(Case $\pi = 1$). If the signs on strands r and $r+1$ match, then the only available diagrams have the crossings $\sigma_r\sigma_r^{-1}$ resolved straight through. If $(\varepsilon_r, \varepsilon_{r+1}) = (+,+)$, then $M(\sigma_r\sigma_r^{-1}, \mathbf{x}, \varepsilon) = v \cdot v^{-1} = 1$. And with $(-,-)$, $M(\sigma_r\sigma_r^{-1}, \mathbf{x}, \varepsilon) = (-v^{-1}) \cdot (-v) = 1$. If $\varepsilon_r \neq \varepsilon_{r+1}$, then there are two possible resolutions that form states, as illustrated.

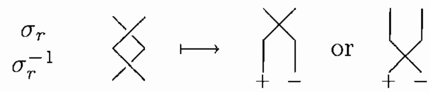


With $(\varepsilon_r, \varepsilon_{r+1}) = (+,-)$, we calculate the contribution of the two states is

$$\frac{-v^{-r}}{[r]} \cdot \frac{-v^{r}}{[r]} + \frac{[r+1]}{[r]} \cdot \frac{[r-1]}{[r]},$$

which equals 1, using Lemma V.2.3. With $(\varepsilon_r, \varepsilon_{r+1}) = (-,+)$, a similar calculation establishes equality.

(Case $\pi = (r\ r+1)$). With $\varepsilon_r = \varepsilon_{r+1}$, $\mathcal{S}_\pi = \varnothing$, so $A_\pi(\sigma_r\sigma_r^{-1}, \mathbf{x}, \varepsilon) = 0$. If the $\varepsilon_r \neq \varepsilon_{r+1}$, then there are two possible resolutions that form states, as illustrated.



With $(\varepsilon_r, \varepsilon_{r+1}) = (+, -)$, we calculate the contribution from the two states is

$$\frac{[r-1]}{[r]} \cdot \frac{-v^r}{[r]} + \frac{v^r}{[r]} \cdot \frac{[r-1]}{[r]} = 0.$$

The calculation with $(\varepsilon_r, \varepsilon_{r+1}) = (+, -)$ is analogous and is omitted.

Braid relations. $A(\sigma_r\sigma_s) = A(\sigma_s\sigma_r)$ when $|r-s| > 1$. This is a simple calculation only using the commutativity of the ring $\mathbb{C}(v)$.

$A(\sigma_r\sigma_{r+1}\sigma_r) = A(\sigma_{r+1}\sigma_r\sigma_{r+1})$. Let $\mathbf{x}$ (resp., $\mathbf{y}$) denote a permutation diagram for $\sigma_r\sigma_{r+1}\sigma_r$ (resp., $\sigma_{r+1}\sigma_r\sigma_{r+1}$). Again, we organize by cases $\pi = \pi(\mathbf{x}) = \pi(\mathbf{y})$. In Figure V.2, the possible permutation diagrams are shown with labels suggesting the possible signs on those strands. For example, the label $(a, b, \bar{b})$ indicates that possible signs are $(+,+,-)$, $(+,-,+)$, $(-,+,-)$, or $(-,-,+)$. Permutations are named by how they act on strands r, $r+1$, and $r+2$. Notice that for $\pi = (3\ 2\ 1)$, there are no ways to assign signs, as each pair of strands crosses.

We illustrate one calculation (of twenty): $\pi = 1$ and $(\varepsilon_r, \varepsilon_{r+1}, \varepsilon_{r+2}) = (-,-,+)$. These signs match each of the patterns except $(a, \bar{a}, c)$, so it suffices to verify that the following equation holds:

$$(-v^{-1})\frac{v^{r+1}}{[r+1]}(-v^{-1}) + 0 = \frac{v^{r+1}}{[r+1]}(-v^{-1})\frac{v^{r+1}}{[r+1]} + \frac{[r+2]}{[r+1]}\frac{v^r}{[r]}\frac{[r]}{[r+1]}$$

This is equivalent to showing that $v^{r-1}[r+1] = -v^{2r+1} + v^r[r+2]$, which follows from Lemma V.2.3. □

V.2. Markov Moves

Now that we have established that $A : \mathcal{B} \to \mathbb{C}(v)$ is well-defined, we check that A is invariant under the Markov moves.

Lemma V.2.1. *The state-sum formula in (IV.11) descends to a well-defined link invariant: $A(\alpha) = A(\beta)$ if α and β are related by the Markov moves.*

Proof. In order to show invariance under the cyclic move (II.2), note that there is a bijection

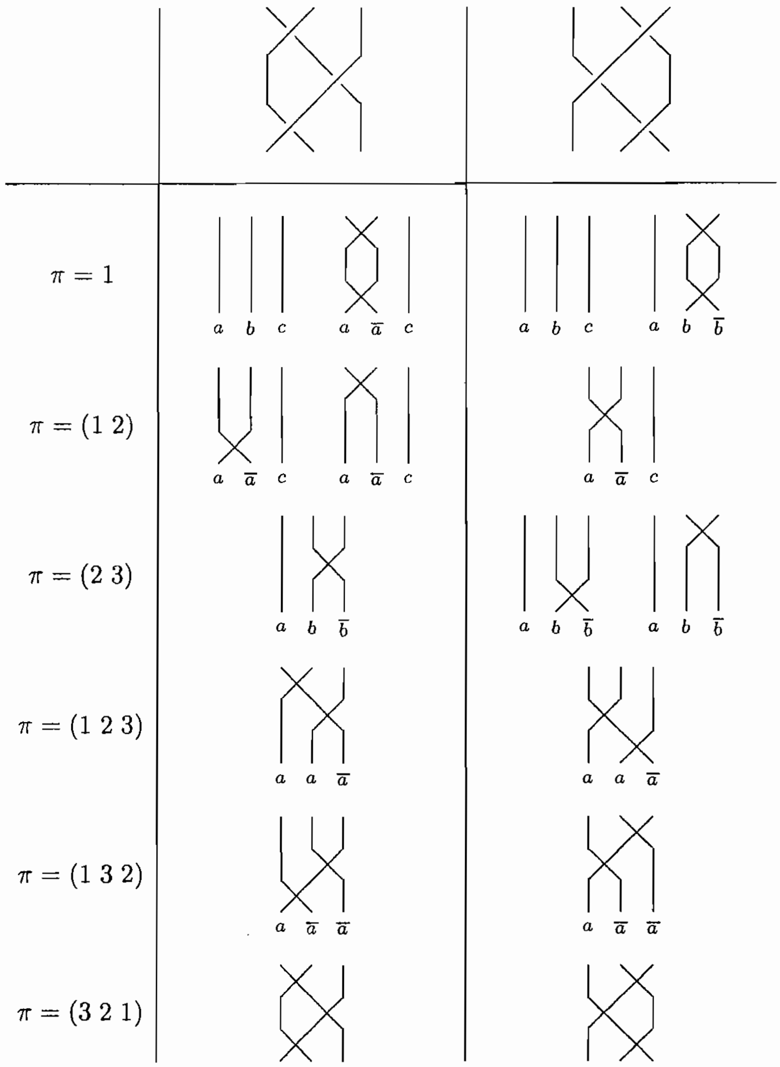


Figure V.1: Possible states $\mathbf{x}$ and $\mathbf{y}$, organized according to π.

between states for $\alpha\beta$ and $\beta\alpha$, defined as follows.

Any admissible permutation diagram of $\alpha\beta$ looks like the concatenation $\mathbf{xy}$, where $\mathbf{x}$ (resp., $\mathbf{y}$) is a permutation diagram for α (resp., β). The diagrams $\mathbf{x}$ and $\mathbf{y}$ may not be admissible, but $\pi(\mathbf{x})\pi(\mathbf{y}) = \pi(\mathbf{xy}) = 1$. So, $\mathbf{yx}$ is admissible, as well, since left- and right-inverses agree in a group. For any state $(\mathbf{xy}, \varepsilon) \in \mathcal{S}(\alpha\beta)$, $(\mathbf{yx}, \pi(\mathbf{y}).\varepsilon)$ is a state for $\beta\alpha$, where $\pi(\mathbf{y}).\varepsilon$ denotes the signs permuted by $\pi(\mathbf{y})$.

Beginning with a state for $\beta\alpha$, and repeating the procedure, only this time, twisting the signs by $\pi(\mathbf{x})$, we construct an inverse. Now, since $M(\alpha\beta, \mathbf{xy}, \varepsilon) = M(\beta\alpha, \mathbf{yx}, \pi(\mathbf{y}).\varepsilon)$ and

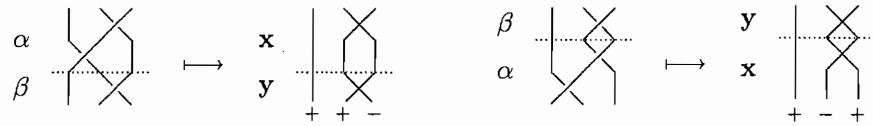


Figure V.2: A state $(\mathbf{xy}, \varepsilon)$ for $\alpha\beta$ and the corresponding state $(\beta\alpha, \pi(\mathbf{y}).\varepsilon)$ for $\beta\alpha$.

$\langle\varepsilon\rangle = \langle\pi(\mathbf{y}).\varepsilon\rangle$ for each state of $\alpha\beta$, we obtain the equality $A(\alpha\beta) = A(\beta\alpha)$.

To prove invariance under the stabilization move (II.3), observe that the crossing σ_d must be resolved straight through in order to form an admissible permutation diagram. Assume that $\varepsilon_{d-1} = +$ (the other case is similar). Then, for each state $(\mathbf{x}, \varepsilon)$ for α, either sign on strand d forms a state for $\alpha\sigma_d$. Call them $(\mathbf{y}, \varepsilon_\pm)$, respectively.

Calculate the contribution to the state-sum for each such pair of states:

$$\begin{aligned} \langle\varepsilon_+\rangle M(\alpha\sigma_d, \mathbf{y}, \varepsilon_+) &+ \langle\varepsilon_-\rangle M(\alpha\sigma_d, \mathbf{y}, \varepsilon_-) \\ &= \left(v - \frac{-v^{-n}}{[d]}\right)\langle\varepsilon\rangle M(\alpha, \mathbf{x}, \varepsilon) \\ &= \frac{[d+1]}{[d]}\langle\varepsilon\rangle M(\alpha, \mathbf{x}, \varepsilon) \end{aligned} \tag{V.7}$$

Now, $A(\alpha\sigma_d) = A(\alpha)$ follows once the global normalizations are included. □

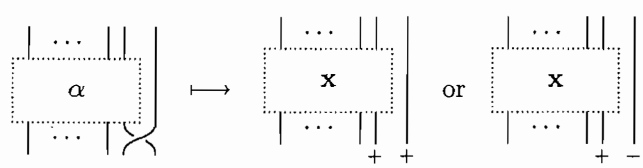


Figure V.3: The states $(\mathbf{y}, \varepsilon_+)$ and $(\mathbf{y}, \varepsilon_-)$ for $\alpha\sigma_n$ associated to the state $(\mathbf{x}, \varepsilon)$ for α.

Theorem V.2.2. *For any link L and any choice of braid word w whose closure is L,*

$$A(w) = \Delta(L).$$

Proof. By Lemma V.2.1, the function A is a link invariant, so it suffices to show that it agrees with the Alexander-Conway polynomial. There are two things to show: the skein relation and the Conway normalization.

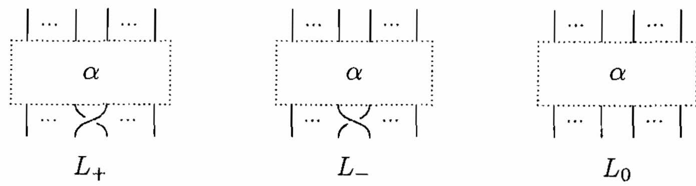


Figure V.4: The Conway skein relation on links: $\Delta(L_+) - \Delta(L_-) = (v - v^{-1})\Delta(L_0)$

For the skein relation, it suffices to show that

$$A(\alpha\sigma_r) - A(\alpha\sigma_r^{-1}) = (v - v^{-1})A(\alpha). \tag{V.8}$$

Let $\mathbf{x}$ be any permutation diagram for α, and let $\mathbf{y}_+$ and $\mathbf{y}_-$ be permutation diagrams for $\alpha\sigma_r$ and $\alpha\sigma_r^{-1}$, respectively. Now, consider the permutation $\pi = \pi(\mathbf{x})$. If $\pi = 1$, so $\mathbf{x}$ is admissible, then $\mathbf{y}_\pm$ must be resolved straight through to form states. For any such state with sign sequences with $(\varepsilon_r, \varepsilon_{r+1})$ equal to $(+,+)$ or $(-,-)$, we verify that

$$M(\alpha\sigma_r, \mathbf{y}_+, \varepsilon) - M(\alpha\sigma_r^{-1}, \mathbf{y}_-, \varepsilon) = (v - v^{-1})M(\alpha, \mathbf{x}, \varepsilon).$$

When the signs on strands r and $r+1$ are different, we calculate

$$M(\alpha\sigma_r, \mathbf{y}_+, \varepsilon) - M(\alpha\sigma_r^{-1}, \mathbf{y}_-, \varepsilon) = \left(\frac{v^r}{[r]} - \frac{v^{-r}}{[r]}\right) M(\alpha, \mathbf{x}, \varepsilon) \tag{V.9}$$

$$= (v - v^{-1})M(\alpha, \mathbf{x}, \varepsilon). \tag{V.10}$$

Finally, consider the case where $\pi = (r\ r+1)$, so $\mathbf{x}$ is not admissible. There are no states for α with such a permutation diagram. Any state for $\alpha\sigma_r^{\pm 1}$ must have $\mathbf{y}_\pm$ with a crossing of strands r and $r+1$. The contribution to the sum when $(\varepsilon_r, \varepsilon_{r+1}) = (+,-)$ is

$$M(\alpha\sigma_r, \mathbf{y}_+, \varepsilon) - M(\alpha\sigma_r^{-1}, \mathbf{y}_-, \varepsilon) = \left(\frac{[r+1]}{[r]} - \frac{[r+1]}{[r]}\right) M(\alpha, \mathbf{x}, \varepsilon) = 0.$$

The case $(\varepsilon_r, \varepsilon_{r+1}) = (-,+)$ gives 0 as well.

The Conway normalization for the Alexander polynomial has $\nabla(\textit{unknot}) = 1$. A braid representative is the trivial braid on one strand. There is only one state available, carrying the sign $+$. The weight involves the empty product, hence gives value 1. □

Lemma V.2.3. *The following identities hold for quantum integers:*

$$[r][s] = \sum_{k=1}^{\min(r,s)} [r+s+1-2k] \qquad r,s \geq 0 \tag{V.11}$$

$$[r+s][r-s] = [r]^2 - [s]^2 \tag{V.12}$$

$$v^r[s] + v^{-s}[r] = [r+s] \tag{V.13}$$

Proof. The proofs are elementary and follow from the definition of quantum integers. □

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