ON ALGEBRAS ASSOCIATED TO FINITE RANKED POSETS AND COMBINATORIAL TOPOLOGY: THE KOSZUL, NUMERICALLY KOSZUL AND COHEN-MACAULAY PROPERTIES

by

TYLER KLOEFKORN

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2014

DISSERTATION APPROVAL PAGE

Student: Tyler Kloefkorn

Title: On Algebras Associated to Finite Ranked Posets and Combinatorial Topology: The Koszul, Numerically Koszul and Cohen-Macaulay Properties

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Brad Shelton	Chairperson
Daniel Dugger	Core Member
Alexander Polishchuk	Core Member
Sergey Yuzvinsky	Core Member
Van Kolpin	Institutional Representative

and

Kimberly Andrews Espy	Vice President for Research and Innovation Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2014

DISSERTATION ABSTRACT

Tyler Kloefkorn

Doctor of Philosophy

Department of Mathematics

June 2014

Title: On Algebras Associated to Finite Ranked Posets and Combinatorial Topology: The Koszul, Numerically Koszul and Cohen-Macaulay Properties

This dissertation studies new connections between combinatorial topology and homological algebra. To a finite ranked poset Γ we associate a finite-dimensional quadratic graded algebra R_Γ. Assuming Γ satisfies a combinatorial condition known as uniform, R_Γ is related to a well-known algebra, the splitting algebra A_Γ. First introduced by Gelfand, Retakh, Serconek and Wilson, splitting algebras originated from the problem of factoring non-commuting polynomials.

Given a finite ranked poset Γ, we ask a standard question in homological algebra: Is R_Γ Koszul? The Koszulity of R_Γ is related to a combinatorial topology property of Γ known as Cohen-Macaulay. One of the main theorems of this dissertation is: If Γ is a finite ranked cyclic poset, then Γ is Cohen-Macaulay if and only if Γ is uniform and R_Γ is Koszul.

We also define a new generalization of Cohen-Macaulay: weakly Cohen-Macaulay. The class of weakly Cohen-Macaulay finite ranked posets includes posets with disconnected open subintervals. We prove: if Γ is a finite ranked cyclic poset, then Γ is weakly Cohen-Macaulay if and only if R_Γ is Koszul.

Finally, we address the notion of numerical Koszulity. We show that there exist algebras R_Γ that are numerically Koszul but not Koszul and give a general construction for such examples.

This dissertation includes unpublished co-authored material.

CURRICULUM VITAE

NAME OF AUTHOR: Tyler Kloefkorn

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, OR
Whittier College, Whittier, CA

DEGREES AWARDED:
Doctor of Philosophy, Mathematics, 2014, University of Oregon
Master of Arts, Mathematics, 2009, University of Oregon
Bachelor of Arts, Mathematics, 2007, Whittier College

AREAS OF SPECIAL INTEREST:
Non-commutative Algebra
Homological Algebra
Combinatorial Topology
Graded Algebras

PROFESSIONAL EXPERIENCE:

Instructor, Oregon Young Scholars Program, University of Oregon, 2012-2013

Graduate Teaching Fellow, University of Oregon, 2007-2014

Intern, The Aerospace Corporation, 2007

GRANTS, AWARDS AND HONORS:

Donald and Darel Stein Graduate Student Teaching Award, Graduate School, University of Oregon, June 2013

Johnson Fellowship, Department of Mathematics, University of Oregon, March 2013

Graduate Student Research Award, Department of Mathematics, University of Oregon, January 2013

Frank W. Anderson Graduate Teaching Award, Department of Mathematics, University of Oregon, June 2012

Dan Kimble First Year Teaching Award, Graduate School, University of Oregon, August 2008

Scholarship, Rose Hills Foundation, June 2008

PUBLICATIONS:

Tyler Kloefkorn and Brad Shelton, *Splitting Algebras: Koszul, Numerically Koszul and Cohen-Macaulay*, submitted

ACKNOWLEDGEMENTS

First and foremost, I would like to thank Brad Shelton. Brad is a remarkable teacher and mentor, and I appreciate his guidance and patience. I would also like to thank Hal Sadofsky, Dan Dugger, Sergey Yuzvinsky and Sasha Polishchuk for their support. Lastly, I wish to thank my colleagues and the staff at the University of Oregon Department of Mathematics.

To Grandma, Mom, Dad, Kristi, Carrie, Ryan and Lindsay

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

CHAPTER I

INTRODUCTION

1.1. History and Significance

The Koszul property originated in work by its namesake, Jean-Louis Koszul. A French geometer, Koszul sought to discover a cohomology theory for Lie algebras. In 1950, he discovered what would later be called the Koszul complex. In 1970, Priddy observed more general applications of the Koszul complex (in particular applications to Steenrod algebras) in his seminal paper, Koszul Resolutions, [17].

Let $\mathbb{F}$ be a field. One of the many equivalent definitions of a Koszul algebra is:

Definition 1.1. A connected graded $\mathbb{F}$-algebra A is Koszul if for all $i \neq j$, $Ext_A^{i,j}(\mathbb{F}, \mathbb{F}) = 0$.

It follows from the above definition that a Koszul algebra is quadratic. We also note that an algebra is Koszul if and only if the trivial module admits linear free resolution. For a complete survey of all things Koszul, see [13].

Koszul algebras are remarkable – they are ubiquitous and often encode geometric data. There are several notable examples. In the setting of algebraic topology, Papadima and Yuzvinsky show that a formal topological space X is rational $K[\pi, 1]$ if and only if the cohomology ring of X, $H^*(X)$, is Koszul; see [15]. In [24], Shelton and Yuzvinsky show that if a hyperplane arrangement $\mathcal{A}$ is supersolvable, then its Orlik-Solomon algebra, $OS(\mathcal{A})$, is Koszul. In the setting of representation theory, Beilinson, Ginzburg and Soergel apply the notion of Koszul duality to category $\mathcal{O}$; see [1].

Further demonstrating its significance, many authors have introduced generalizations of Koszulity. In [2], Berger defined the class of N-Koszul algebras. Cassidy and Shelton introduced the class of $\mathcal{K}_2$ algebras in [5]. In [9], Herscovich studies multi-Koszul algebras.

Numerical Koszulity is an important and a closely related property.

Definition 1.2. Let A be graded connected quadratic $\mathbb{F}$-algebra with quadratic dual algebra $A^!$ and let $H(A,t)$ be the Hilbert series of A. Then A is said to be numerically Koszul if

$$H(A,-t) \cdot H(A^!,t) = 1.$$

Every Koszul algebra is numerically Koszul but, in general, the converse is false. Independently, counterexamples were discovered in [14] and [23].

This dissertation explores new connections between topology and Koszul algebras. More specifically, we study a combinatorial topology property known as Cohen-Macaulay and its relationship to the Koszul property.

We give a brief introduction to Cohen-Macaulay posets. Let $(\Gamma,<)$ be a finite poset with unique minimal element $*$. We say Γ is ranked if for every $b \in \Gamma$, any two maximal chains in $[*,b]$ have the same length. Also, Γ is cyclic if it has a unique maximal element. If $a < b$ in Γ, then $\Delta((a,b))$ denotes the order complex of (a,b).

Definition 1.3. A finite ranked cyclic poset Γ is Cohen-Macaulay relative to $\mathbb{F}$ if for all $a < b$ in Γ, $\tilde{H}^n(\Delta((a,b)),\mathbb{F}) = 0$ for all $n \neq dim\Delta((a,b))$.

Thus, a finite ranked cyclic poset is Cohen-Macaulay if every open interval is, as we say, a cohomology bouquet of spheres (or CBS). We say that the simplicial complex $\Delta(P)$ is Cohen-Macaulay if the poset P is Cohen-Macaulay. The Cohen-Macaulay property has been studied extensively; for a survey of all things Cohen-Macaulay, see [3].

It is worth noting that both the Cohen-Macaulay and Koszul properties are relative to the field $\mathbb{F}$.

We wish to give the reader some context for the results found in this dissertation. Connections between combinatorial topology and homological algebra date back to the work of Reisner and the introduction of face rings (also known as Stanley-Reisner rings) and the Cohen-Macaulay property for rings. A commutative Noetherian local ring is Cohen-Macaulay if its Krull dimension is equal to its depth. The following theorem, due to Reisner, is integral to the study of face rings.

Theorem 1.4. [18] Let P be a finite ranked poset. The face ring of $\Delta(P)$ over $\mathbb{F}$ is Cohen-Macaulay if and only if P is Cohen-Macaulay relative to $\mathbb{F}$.

Later, Eagon and Reiner proved the following.

Theorem 1.5. [6] Let P be a finite ranked poset. The face ring of $\Delta(P)$ over $\mathbb{F}$ has linear free resolution if and only if the Alexander dual of $\Delta(P)$ is Cohen-Macaulay relative to $\mathbb{F}$.

For a more recent and closely related example, we direct the reader to the work of Polo and Woodcock. They proved the following theorem independently.

Theorem 1.6. [16][28] Let P be a finite ranked cyclic poset. The incidence algebra of P over $\mathbb{F}$ is Koszul if and only if P is Cohen-Macaulay relative to $\mathbb{F}$.

We now describe the class of posets and algebras we are interested in studying.

Definition 1.7. Let Γ be a finite ranked poset with unique minimal element $*$. The algebra R_Γ is the $\mathbb{F}$-algebra with degree one generators r_x for all $x \in \Gamma \setminus \{*\}$ and relations

$$r_x \sum_{x \to y} r_y = 0$$

and

$$r_x r_w = 0 \text{ whenever } x \not\to w.$$

We note R_Γ is quadratic.

The algebra R_Γ has a relatively complex history. We give an overview.

We let Γ be as in Defintion 1.7. In [8], Gelfand, Retakh, Serconek and Wilson associate to Γ a connected graded $\mathbb{F}$-algebra A_Γ which is called the *splitting algebra* of Γ; splitting algebras are related to the problem of factoring non-commuting polynomials. Retakh, Serconek and Wilson later showed that if Γ satisfies a combinatorial condition called *uniform*, then an associated graded algebra of the splitting algebra is quadratic. From which it follows that A_Γ is quadratic (c.f. [19]). These authors then asked a standard question in homological algebra: given a finite uniform ranked Γ, is A_Γ Koszul?

If A_Γ is Koszul, one can use the Hilbert series condition of numerical Koszulity to extract combinatorial data from the algebra. Recent work in the area of splitting algebras often focuses on calculating Hilbert series (c.f. [20] and [21]).

It is often difficult to determine if A_Γ is Koszul. In fact, preliminary literature incorrectly asserted that A_Γ is Koszul for all uniform Γ. We thus pass to a related question. Following [19] and assuming Γ is uniform, we filter A_Γ by rank in Γ. We denote the associated graded algebra by grA_Γ. Finally, we study the quadratic dual of grA_Γ, which is denoted by $(grA_\Gamma)^!$. Applying standard techniques, we know that if $(grA_\Gamma)^!$ is Koszul, then so is A_Γ. We then ask: given a finite uniform ranked Γ, is

$(grA_\Gamma)^!$ Koszul? We note in [20] and subsequent papers by the same authors, $(grA_\Gamma)^!$ is denoted by $B(\Gamma)$.

If Γ is uniform, then $R_\Gamma = (grA_\Gamma)^!$. The notation, R_Γ, is from [4]; Cassidy, Phan and Shelton assume Γ is uniform and denote $(grA_\Gamma)^!$ with R_Γ. We emphasize that for our definition of R_Γ, Γ need not be uniform.

The algebra R_Γ and Koszulity of R_Γ are extremely interesting, even if Γ is not uniform and we draw no conclusions about splitting algebras. In [4], Cassidy, Phan and Shelton show, among many things, that there exists a non-Koszul R_Γ. If Γ stems from a geometric object, then the Koszulity of R_Γ gives important combinatorial and topological data. Again, in [4], the authors show that if Γ is the intersection poset of a regular CW complex, then R_Γ is Koszul. Sadofsky and Shelton also study posets associated to regular CW complexes in [25] – they show Koszulity for R_Γ is a topological invariant. Lastly, the following question is of great interest: is every numerically Koszul R_Γ also Koszul?

1.2. Summary of Results

Chapter II contains background notation and definitions.

In chapter III, we study finite ranked uniform posets Γ and R_Γ. The main result of this chapter is as follows.

Theorem 1.8. Let Γ be finite ranked cyclic poset. Then Γ is Cohen-Macaulay if and only if Γ is uniform and R_Γ is Koszul.

In addition, chapter III addresses the notion of numerical Koszulity.

Theorem 1.9. There exist finite ranked uniform posets (including cyclic posets) Γ such that R_Γ is numerically Koszul but not Koszul.

We give a general construction for such examples and use that construction to provide cyclic and non-cyclic examples.

In chapter IV, we study finite ranked (possibly non-uniform) posets Γ and R_Γ. Using combinatorial topology, we define a new generalization of the Cohen-Macaulay property: weakly Cohen-Macaulay. The main theorem of chapter IV is as follows.

Theorem 1.10. Let Γ be finite ranked cyclic poset. Then Γ is weakly Cohen-Macaulay if and only if R_Γ is Koszul.

We find it intriguing that Theorems 1.8 and 1.10 are analogous to results from [3], [16], [18] and [28] in that they connect properties of combinatorial topology and homological algebra.

This dissertation includes unpublished co-authored material in chapters II and III.

CHAPTER II

DEFINITIONS AND NOTATION

This chapter includes unpublished co-authored material. Brad Shelton and I collaborated in writing this entire chapter.

2.1. Ranked Posets and the Order Complex

Definition 2.1. Let Γ be a poset with unique minimal element $*$ and strict order $<$. We say Γ is ranked if for all $b \in \Gamma$, any two maximal chains in $[*, b]$ have the same length. The length of such a maximal chain is then referred to as the *rank* of b and written $rk_\Gamma(b)$. Set $\Gamma_+ = \Gamma \setminus \{*\}$. Let $\Gamma(k)$ be the elements of Γ of rank k.

(1) Γ is pure of rank d if $rk_\Gamma(x) = d$ for every maximal element of Γ.

(2) If Γ is pure, then $\overline{\Gamma}$ is the poset Γ adjoined with a unique maximal element.

(3) If Γ is pure, then Γ' is the poset $\Gamma \setminus \Gamma(rk(\Gamma))$.

(4) Γ_x denotes the interval $[*, x]$ in Γ.

(5) Γ is cyclic if $\Gamma = \Gamma_x$ for some $x \in \Gamma$.

(6) For any $x \in \Gamma$, $S_x(k) = \{y \in \Gamma_x \,|\, rk_\Gamma(y) = rk_\Gamma(x) - k\}$.

For any $a < b$, we say that b covers a, written $b \to a$, if the closed interval $[a, b]$ has order 2, or equivalently $a \in S_b(1)$. This makes Γ into a directed graph that is often referred to as a *layered* graph.

Remark 2.2. The above definition of ranked poset is taken from [8], a fundamental paper in the area of splitting algebras. We note that this definition differs from the traditional one wherein every maximal chain has the same length (c.f. [27]).

We recall the definition of *uniform* from [8].

Definition 2.3. Let Γ be a ranked poset. For $x \in \Gamma$ and $a, b \in S_x(1)$, write $a \sim_x b$ if there exists $c \in S_a(1) \cap S_b(1)$ and extend $\sim_x$ to an equivalence relation on $S_x(1)$. We say that Γ is *uniform* if, for every $x \in \Gamma$, $\sim_x$ has a unique equivalence class.

The notion of the order complex of a finite poset is a standard tool in combinatorial topology and elsewhere. For completeness of exposition, we include the basic definitions.

Definition 2.4. Let Γ be a finite poset with strict order $<$. The *order complex* of Γ, $\Delta(\Gamma)$, is the collection of ordered subsets of Γ:

$$\Delta(\Gamma) = \{(b_0, b_1, \ldots, b_n) \,|\, b_i \in \Gamma \text{ and } \; b_0 < b_1 < \cdots < b_n\}$$

An element $\beta = (b_0, \cdots, b_n)$ in $\Delta(\Gamma)$ is an n-cell (or n-chain) of the complex, $C^n(\Delta(\Gamma))$ denotes the $\mathbb{F}$-vector space generated by the n-cells and $C(\Delta(\Gamma)) = \oplus_n C^n(\Delta(\Gamma))$. Given $x \in \Gamma$, define $u_x : C^n(\Delta(\Gamma)) \to C^{n+1}(\Delta(\Gamma))$ by extending linearly from the formula

$$u_x(b_0, \cdots, b_n) = \begin{cases} (x, b_0, \cdots, b_n) & \text{if } x < b_0 \\ (-1)^{i+1}(b_0, \cdots, b_i, x, b_{i+1}, \cdots, b_n) & \text{if } b_i < x < b_{i+1} \\ (-1)^{n+1}(b_0, \cdots, b_n, x) & \text{if } b_n < x \\ 0 & \text{otherwise} \end{cases}$$

Finally, we set $d_{\Delta(\Gamma)} = d = \sum_x u_x : C(\Delta(\Gamma)) \to C(\Delta(\Gamma))$.

Since $\Delta(\Gamma)$ is a simplicial complex, it has a well-defined geometric realization, or *total space* which we will denote $||\Delta(\Gamma)||$. We typically give this topological space a name, say $Y = ||\Delta(\Gamma)||$. We will consistently abuse notation and write $C(Y)$ for

$C(\Delta(\Gamma))$ and d_Y for $d_{\Delta(\Gamma)}$. It is standard that $(C(Y), d_Y)$ is a cochain complex and that $H^n(Y) = H^n(C(Y), d_Y)$. We remind the reader that these cohomology groups are all calculated with coefficients in our base field, $\mathbb{F}$.

Recall from Definition 1.3 that a finite ranked cyclic poset is Cohen-Macaulay relative to a field $\mathbb{F}$ if the order complex of any open subinterval (a, b) has non-zero reduced cohomology only in the degree equal to its dimension.

2.2. The Algebra R_Γ

In this section, and all remaining chapters, Γ denotes a finite ranked poset with unique minimal element $*$. We recall the definition of R_Γ from Chapter I.

Definition 2.5. Let V_Γ be the $\mathbb{F}$-vector space with basis elements r_x, $* \neq x \in \Gamma$. For each $k \geq 0$ and $* \neq x \in \Gamma$, set $r_x(k) = \sum_{y \in S_x(k)} r_y$. Let I_Γ be the quadratic ideal of the free (tensor) algebra $\mathbb{F}(V_\Gamma)$ generated by the elements:

(1) $r_x \otimes r_y$ for all pairs $\{x, y\}$ such that $y \notin S_x(1)$,

(2) $r_x \otimes r_x(1)$ for all x.

Then the algebra R_Γ is a the quadratic $\mathbb{F}$-algebra $\mathbb{F}(V_\Gamma)/I_\Gamma$ and we continue to write r_x for the generators of R_Γ.

The algebra R_Γ can be graded in several convenient ways, but we will only use the standard connected grading $R_\Gamma = \bigoplus_{n \geq 0} R_{\Gamma,n}$ in which the generators r_x have degree one.

Cassidy, Phan, and Shelton proved the following lemma in [4]. This lemma is a very powerful tool; we will use it repeatedly and without further comment. We note that Γ need not be uniform.

Lemma 2.6 ([4], (3.1))**.** Let Γ be a finite ranked poset. Then

$$(R_\Gamma)_+ = \bigoplus_{x \in \Gamma_+} r_x R_\Gamma.$$

Recall from Definition 1.1 that a graded connected $\mathbb{F}$-algebra A is Koszul if for all $i \neq j$, $\mathrm{Ext}_A^{i,j}(\mathbb{F}, \mathbb{F}) = 0$. There are many equivalent ways to define Koszul (c.f. [13]). We will often use the following equivalent definition: a graded connected $\mathbb{F}$-algebra A is Koszul if the trivial right A-module $\mathbb{F}_A$ admits a linear projective resolution.

We will say that a finite ranked poset Γ is Koszul if the algebra R_Γ is Koszul. We warn the reader that this is an abuse of notation since we know from [4] that this definition is dependent on the field $\mathbb{F}$. That is, there are posets Γ such that the property "R_Γ is Koszul" is dependent on the field $\mathbb{F}$.

The Koszul property is closely related to a certain co-chain complex built from the ring R_Γ.

Definition 2.7. Let Γ be a finite ranked poset.

(1) $d_\Gamma = \sum_{* \neq x \in \Gamma} r_x \in R_{\Gamma,1}$. Also let d_Γ denote the function $d_\Gamma : R_\Gamma \to R_\Gamma$ given by left (but never right) multiplication by d_Γ.

(2) For all $n \geq k \geq 0$, set $R_\Gamma(n,k) = \sum_{rk_\Gamma(y)=n+1} r_y R_{\Gamma,n-k}$.

By definition, R_Γ has a spanning set of monomials of the form $r_{b_1} r_{b_2} \cdots r_{b_j}$ where $b_1 \to b_2 \to \cdots \to b_j$. The space $R_\Gamma(n,k)$ is then the span of such monomials for which $rk_\Gamma(b_1) = n+1$ and $rk_\Gamma(b_j) = k+1$. The degree of such a monomial is $n-k+1$.

From the definitions we see at once that $(d_\Gamma)^2 = 0$. In particular, for each $k \geq 0$ we have a cochain complex:

$$\cdots R_\Gamma(n-1,k) \xrightarrow{d_\Gamma} R_\Gamma(n,k) \xrightarrow{d_\Gamma} R_\Gamma(n+1,k) \cdots$$

It is useful to note the folllowing. Let $d_\Gamma^n = \sum_{rk(y)=n+1} r_y$, so that $d_\Gamma = \sum_n d_\Gamma^n$. Then the cochain complex above is the same as:

$$\cdots R_\Gamma(n-1,k) \xrightarrow{d_\Gamma^n} R_\Gamma(n,k) \xrightarrow{d_\Gamma^{n+1}} R_\Gamma(n+1,k) \cdots$$

Definition 2.8. For each $k \geq 0$, we will denote the cohomology of the complex above, $H^n(R_\Gamma(\cdot,k), d_\Gamma)$, by $H_{R_\Gamma}(n,k)$, or more simply as $H_\Gamma(n,k)$.

It is sometimes convenient to augment each of the cochain complexes $R_\Gamma(\cdot,k)$ by defining $\mathbb{F} \to R(k,k)$ via $1 \mapsto d_\Gamma^k$. We denote the cohomology of the augemented complex by $\tilde{H}_\Gamma(n,k)$. Please note that this differs from $H_\Gamma(n,k)$ in cohomology degree k, not 0.

CHAPTER III

FINITE RANKED UNIFORM POSETS AND R_Γ

This chapter includes unpublished co-authored material. Brad Shelton and I collaborated in writing this entire chapter.

We recall important results related to R_Γ. The following result from [4] is extremely useful and will be used repeatedly without further comment.

Theorem 3.1 ([4], (3.5))**.** If the poset Γ is uniform, then Γ is Koszul if and only if Γ_x is Koszul for every $* \neq x \in \Gamma$.

The following important theorem from [4] explains how the internal cohomology groups $H_\Gamma(n,k)$ are related to the Koszul property of the algebra R_Γ.

Theorem 3.2. Assume $\Gamma = \Gamma_x$ is a finite ranked uniform cyclic poset with $rk_\Gamma(x) = d+1$. Then

(1) $H_\Gamma(k,k) = \mathbb{F}$ for all $0 \leq k \leq d$.

(2) $H_\Gamma(n,k) = 0$ if $n = d$ or $d-1$ and $k < n$.

(3) Assume Γ_z is Koszul for every $z < x$. Then Γ is Koszul if and only if $H_\Gamma(n,k) = 0$ for all $0 \leq k < n \leq d-2$.

Proof. This is 3.7 and 3.8 of [4]. □

Remark 3.3. Despite a remark to the contrary in [4], neither the cyclic hypothesis nor the "inductive" hypothesis can be removed from part (3) of 3.2. We give two examples below to illustrate these points.

Example 3.4. Historically, the first known example of a poset for which R_Γ is not Koszul is the poset Γ whose Hasse diagram is shown in Figure 3.1. Let $\Gamma' = \Gamma \setminus \{X\}$.

One sees directly that the element $r_D r_G$ in $R_{\Gamma'}(1,0) = R_\Gamma(1,0)$ represents a non-zero cohomology class in both $H_{\Gamma'}(1,0)$ and $H_\Gamma(1,0)$. Since $R_{\Gamma'}$ is Koszul (because all rank 3 cases are Koszul), this shows that Koszulity, without the cyclic hypothesis, does not guarantee vanishing of cohomology. The fact that $H_\Gamma(1,0)$ is non-zero does, however, prove that R_Γ is not Koszul (by (3) of 3.2).

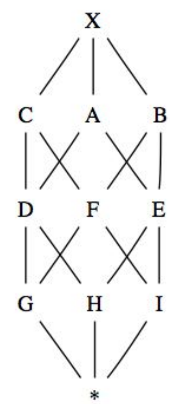

FIGURE 3.1. The poset Γ.

It is rather more complex to show that without the inductive hypothesis vanishing of the cohomology groups $H_{R_\Gamma}(n,k)$ for $k < n$ does not imply the Koszul property. We use the results of [4] to build a fairly straightforward cyclic example.

Example 3.5. Let $\mathbf{Z}$ and $\mathbf{Y}$ be two regular CW complexes pictured to the left in Figure 3.2 and let Ω be the uniform ranked poset whose Hasse diagram is given to the right in Figure 3.2.

We claim that $H_\Omega(n,k) = 0$ for all $0 \le k < n \le 4$, but Ω is not Koszul.

First note that Ω_Z has the form $P \cup \{*, Z\}$, where P is the incidence poset of the CW complex $\mathbf{Z}$. Since $\mathbf{Z}$ is homotopic to S^1, but is pure of dimension 2, Corollary 5.6 of [4] tells us Ω_Z is not Koszul. Hence Ω is not Koszul.

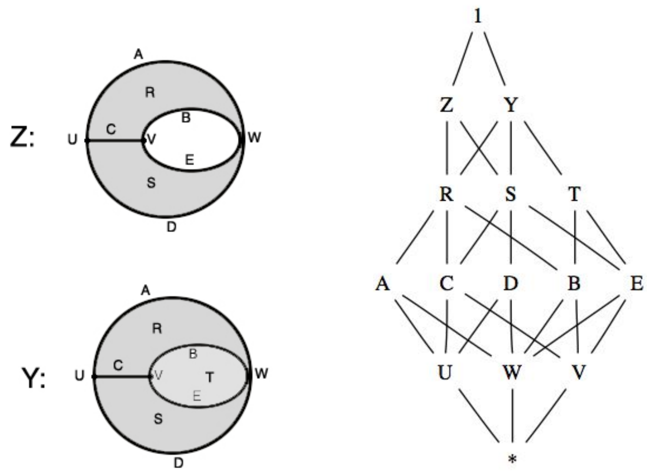

FIGURE 3.2. The regular CW complexes $\mathbf{Z}$ and $\mathbf{Y}$ and the Poset Ω.

On the other hand, Ω_Y has the form $Q \cup \{*, Y\}$, where Q is the incidence poset of $\mathbf{Y}$. Since $\mathbf{Y}$ is a 2-disc, Corollary 5.6 of [4] tells us that Ω_Y is Koszul and then 3.2 tells us that $H_{\Omega_Y}(n,k) = 0$ for all $0 \leq k < n \leq 3$. Since the element Y majorizes every element of Ω of rank at most 3, and since $H_{\Omega_Y}(2,k) = 0$ for $k = 0, 1$, one can see by inspection that $H_\Omega(n,k) = H_{\Omega_Y}(n,k)$ for all $0 \leq k \leq n \leq 2$. Combining this with (2) of 3.2 shows that $H_\Omega(n,k) = 0$ for all $0 \leq k < n \leq 4$, as claimed.

3.1. A Co-Chain Map

Let Y be the total space of the order complex $\Delta(\Gamma \setminus \{*\})$. We define an epimorphism:

$$\Phi_\Gamma : C^n(Y) \to R_\Gamma(n, 0)$$

by extending linearly from the formula:

$$\Phi_\Gamma((b_0, \cdots, b_n)) = \begin{cases} r_{b_n} r_{b_{n-1}} \cdots r_{b_0} & \text{if } rk_\Gamma(b_0) = 1 \\ 0 & \text{otherwise} \end{cases}$$

We note that $\Phi_\Gamma((b_0, \cdots, b_n)) = 0$ unless $rk_\Gamma(b_n) = n+1$, since otherwise either $rk_\Gamma(b_0) \neq 1$ or there is some j for which $b_j \not\to b_{j-1}$ and then $r_{b_j} r_{b_{j-1}} = 0$. It may seem odd to utilize a map that annihilates so much information, but it works.

Lemma 3.6. $\Phi_\Gamma : C(Y) \to R_\Gamma(\cdot, 0)$ is a cochain epimorphism.

Proof. We begin with a preliminary observation. Fix $* \neq a < b$ in Γ. We claim: $\sum\limits_{a<x<b} r_b r_x r_a = 0$ in R_Γ. To see this, first observe that we may eliminate from the sum any x that is not in $S_b(1)$ since for such x, $r_b r_x = 0$. On the other hand, for any $y \in S_b(1)$ for which $a \not< y$, we have $r_y r_a = 0$. Hence we may add such terms to the sum without changing it. Hence the sum is the same as $r_b r_b(1) r_a$, which is 0 by definition. A similar observation is that $\sum\limits_{* \neq x < b} r_b r_x = 0$.

Fix $\beta = (b_0, \cdots, b_n) \in C^n(Y)$. Consider first the case when $rk(b_0) > 1$, in which case $d_\Gamma \Phi_\Gamma(\beta) = d_\Gamma 0 = 0$. Then

$$\Phi_\Gamma(d_Y(\beta)) = \sum_{* \neq x < b_0} \Phi_\Gamma(x, b_0, \cdots, b_n) = \sum_{x < b_0,\, rk(x)=1} r_{b_n} \cdots r_{b_0} r_x.$$

If $rk(b_0) > 2$ then this sum is 0 since every term $r_{b_0} r_x = 0$. If $rk(b_0) = 2$, then this sum is 0 by the observation above. Either way, $\Phi_\Gamma(d_Y(\beta)) = (-1)^{n+1} d_\Gamma \Phi_\Gamma(\beta) = 0$.

Consider the case when $rk(b_0) = 1$. Recalling that $r_y r_{b_n} = 0$ whenever $b_n \not< y$, we get

$$d_\Gamma \Phi_\Gamma(\beta) = d_\Gamma r_{b_n} \cdots r_{b_0} = \sum_{b_n < y} r_y r_{b_n} \cdots r_{b_0}.$$

On the other hand,

$$\Phi_\Gamma(d_Y(\beta)) = \sum_i (-1)^{i+1} \sum_{b_i<x<b_{i+1}} r_{b_n} \cdots r_{b_{i+1}} r_x r_{b_i} \cdots r_{b_0}$$
$$+ (-1)^{n+1} \sum_{b_n<x} r_x r_{b_n} \cdots r_{b_0}.$$

By the observation above, each sum inside the first term of this expression is 0. This shows that $\Phi_\Gamma(d_Y(\beta)) = (-1)^{n+1} d_\Gamma \Phi_\Gamma(\beta)$, as required. $\square$

It is clear that the cochain map Φ_Γ extends to a cochain map between the augmented cochains $\mathbb{F} \to C(Y)$ and $\mathbb{F} \to R_\Gamma(\cdot, 0)$.

Theorem 3.7. Let Γ be a finite ranked uniform poset and $Y = ||\Delta(\Gamma \backslash \{*\})||$. Assume R_Γ is Koszul. Then the cochain map $\Phi_\Gamma : C^n(Y) \to R_\Gamma(n, 0)$ is a quasi-isomorphism. In particular:

$$H^n(Y) = H_\Gamma(n, 0) \text{ for all } n.$$

Proof. Let $R = R_\Gamma$ throughout the proof. Let $d+1$ be the maximal rank of any element of Γ. We prove the theorem by induction on d. The case $d = 0$ is clear. Henceforth we assume $d > 0$. We begin by proving a special case of the theorem. The special case contains substantive extra information.

Lemma 3.8. Let Γ, Y be as in 3.7, with the additional hypothesis that Γ is cyclic, that is $\Gamma = \Gamma_x$ where $rk_\Gamma(x) = d+1$. Let $\Gamma' = \Gamma \backslash \{x\}$, let Z be the $(d-1)$-dimensional closed subspace of Y given by $Z = ||\Delta(\Gamma' \setminus \{*\})|| = ||\Delta((*, x))||$.

(1) $\Phi_\Gamma : C^n(Y) \to R(n, 0)$ is a quasi-isomorphism,

(2) $\tilde{H}^n(Y) = \tilde{H}_\Gamma(n, 0) = 0$ for all n,

(3) $\tilde{H}^n(Z) = 0$ for all $n \neq d-1$,

(4) The map $\tilde{H}^{d-1}(Z) \to R(d,0)$ given by

$$[(b_0, \cdots, b_{d-1})] \mapsto r_x \Phi_{\Gamma'}(b_0, \cdots, b_{d-1}) = r_x r_{b_{d-1}} \cdots r_{b_0}$$

is an isomorphism.

Proof. Since $R = R_\Gamma$ is by assumption Koszul, 3.2 tells us $\tilde{H}_\Gamma(n,0) = 0$ for all n. Since Γ is cyclic, the space Y is contractible and thus $\tilde{H}^n(Y) = 0$ for all n. This proves (2), from which (1) follows trivially.

Let $R' = R_{\Gamma'}$. By induction, $\Phi_{\Gamma'} : C^n(Z) \to R'(n,0)$ is a quasi-isomorphism and $\tilde{H}^n(Z) = \tilde{H}_{\Gamma'}(n,0)$ for all n.

Define the cochain complex K to be $0 \to R(d,0) \to 0$ with the term $R(d,0)$ in degree d. We note that $R(n,0) = R'(n,0)$ for all $n < d$, and $R'(d,0) = 0$. Moreover, the maps d_Γ and $d_{\Gamma'}$ coincide on the spaces $R(n,0)$ for $n < d-1$. Hence we have short exact sequence of cochains:

$$0 \to K \to R(\cdot,0) \to R'(\cdot,0) \to 0$$

The associated long exact sequence in cohomology, together with (2), yields $H_{\Gamma'}(n,0) = H_\Gamma(n,0) = 0$ for all $n < d-1$. Furthermore, $H_{\Gamma'}(d-1,0)$ is isomorphic to $R(d,0)$ via the connecting homomorphism. Composing the connecting homomorphism with the isomorphism $\Phi_{\Gamma'} : \tilde{H}^{d-1}(Z) \to \tilde{H}_{\Gamma'}(d-1,0)$ gives exactly the map given in (4). This proves (3) and (4). $\square$

We return to proving the general case of the Theorem. Let $\Gamma(d+1) = \{y_1, \ldots, y_s\}$ and set $\Omega = \Gamma \setminus \Gamma(d+1)$. We define closed subspaces of Y: $Y_i = ||\Delta((*, y_i])||$ for

$1 \leq i \leq s$ and $Z = ||\Delta(\Omega \setminus \{*\})||$. Note that for each i, $Z \cap Y_i = ||\Delta((*, y_i))||$, so that Lemma 3.8 applies to the pair $(Y_i, Z \cap Y_i)$.

Consider the relative cochain complex $C(Y, Z)$. The basis elements of $C^{n+1}(Y, Z)$ are those $n+1$-cells $(b_0, \cdots, b_{n+1})$ in $C^{n+1}(Y)$ for which $b_{n+1} = y_i$ for some i, in which case $(b_0, \cdots, b_n)$ is in $C^n(Z \cap Y_i)$. Hence there is a vector space isomorphism $\zeta : \oplus_i C^n(Z \cap Y_i) \to C^{n+1}(Y, Z)$ given by mapping $(b_0, \cdots, b_n)$ in $C^n(Z \cap Y_i)$ to $(b_0, \ldots, b_n, y_i)$. We also define the isomorphism $\zeta : \mathbb{F}^s \to C^0(Y, Z)$ by $\zeta(e_i) = (y_i)$. Finally, define an augmentation $\mathbb{F}^s \to \oplus_i C^0(Z \cap Y_i)$ via $e_i \mapsto \sum\limits_{b<y_i} (b)$ in $C^0(Z \cap Y_i)$.

Using the fact that each y_i is maximal in Γ, it is a straightforward calculation to see that ζ is a degree $+1$ cochain map between the augmented cochain complex $\mathbb{F}^s \to \oplus_i C(Z \cap Y_i)$ and the complex $C(Y, Z)$. Hence ζ is an isomorphism of cochain complexes. Thus $H^{n+1}(Y, Z) = \bigoplus_i \tilde{H}^n(Z \cap Y_i)$ for all n. By Lemma 3.8, we then have $H^n(Y, Z) = 0$ for all $n < d$. Since $R(d, 0) = \bigoplus_i r_{y_i} R(d-1, 0) = \bigoplus_i R_{\Gamma_{y_i}}(d, 0)$, 3.8 also shows us that $H^d(Y, Z)$ is isomorphic to $R(d, 0)$, via the map $[\beta = (b_0, \cdots, b_{d-1}, y_i)] \mapsto r_{y_i} r_{b_{d-1}} \cdots r_{b_0} = \Phi_\Gamma(\beta)$.

Let K be the cochain complex $0 \to R(d, 0) \to 0$, concentrated in degree d. Exactly as in the proof of 3.8 we have a short exact sequence of cochain complexes: $0 \to K \to R(\cdot, 0) \to R_\Omega(\cdot, 0) \to 0$. For any $n < d$, let $\hat{\Phi}_\Gamma$ be the restriction of Φ_Γ to $C^n(Y, Z)$. By the note just after the definition of Φ_Γ, we see $\hat{\Phi}_\Gamma(C^n(Y, Z)) = 0$ for all $n < d$.

Using the last observation, we see that we have a commutative diagram of cochain complexes:

$$\begin{tikzcd}
0 \arrow[r] & C(Y,Z) \arrow[r] \arrow[d, "\hat{\Phi}_\Gamma"] & C(Y) \arrow[r] \arrow[d, "\Phi_\Gamma"] & C(Z) \arrow[r] \arrow[d, "\Phi_\Omega"] & 0 \\
0 \arrow[r] & K \arrow[r] & R(\cdot,0) \arrow[r] & R_\Omega(\cdot,0) \arrow[r] & 0
\end{tikzcd}$$

By induction, Φ_Ω is a quasi-isomorphism. By the previous paragraph, $\hat{\Phi}_\Gamma$ is a quasi-isomorphism. Thus Φ_Γ is a quasi-isomorphism. This completes the proof of Theorem 3.7. □

3.2. Connection to a Theorem of Retakh, Serconek and Wilson

This section is a brief digression in order make a connection between our methods and a very good result: Proposition 3.2.1 of [20]. The basic idea of this section is to see just how far one can push the techniques of the previous section without the Koszul hypothesis. We will prove a weaker version of 3.7, from which we get a corollary that is equivalent to 3.2.1 of [20].

Theorem 3.9. Let $\Gamma = \Gamma_x$ be a finite ranked uniform cyclic poset with $rk_\Gamma(x) = d+1$ and set $\Gamma' = \Gamma \setminus \{x\}$. Let Z be the total space of the order complex $\Delta(\Gamma' \setminus \{*\})$. Then:

(1) The cochain epimorphism $\Phi_{\Gamma'} : C^n(Z) \to R_{\Gamma'}(n,0)$, as described in the previous section, induces an isomorphism in cohomology in degree $d-1$, that is

$$H^{d-1}(Z) \cong H_{\Gamma'}(d-1,0).$$

(2) The map $\tilde{H}^{d-1}(Z) \to R_\Gamma(d,0)$ given by

$$[(b_0, \cdots, b_{d-1})] \mapsto r_x \Phi_{\Gamma'}(b_0, \cdots, b_{d-1}) = r_x r_{b_{d-1}} \cdots r_{b_0}$$

is an isomorphism.

Proof. We prove both parts of the theorem by induction on d. The case $d = 0$ is trivial, as there is nothing to prove. The case $d = 1$ is essentially trivial (and is anyways covered by 3.8). We assume $d \geq 2$.

Let $\Gamma'(d) = \{y_1, \ldots, y_s\}$ be the elements of rank d. Set $\Omega' = \Gamma' \setminus \{y_1, \ldots, y_s\}$. Let Ω be the poset obtained by adjoining to Ω' a unique new maximal element $\bar{1}$ (so that, in particular, $\Omega' = \Omega \setminus \{\bar{1}\}$). In order to apply induction to the pair (Ω,Ω') we need to observe that Ω is a ranked uniform cyclic poset. It is ranked because every maximal element of Ω' has rank $d-1$. It is cyclic by construction. The fact that Ω is uniform is Lemma 2.3 of [4]. Let $W = ||\Delta(\Omega' \setminus \{*\})||$.

For each y_i, let $Z_i = ||\Delta((*, y_i])||$. Then $Z_i \cap W = ||\Delta((*, y_i))||$. Since $(\Gamma')_{y_i} = [*, y_i]$ is uniform and cyclic, we may also apply the inductive hypothesis to the pairs $(Z_i, Z_i \cap W)$.

Exactly as in the proof of 3.7 we have a degree $+1$ cochain isomorphism ζ between the augmented cochain complex $\mathbb{F}^s \to \bigoplus_i C^{n-1}(Z_i \cap W)$ and the cochain complex $C^n(Z, W)$. Also exactly as in that proof we have a commutative diagram of cochain compexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(Z,W) & \longrightarrow & C(Z) & \longrightarrow & C(W) & \longrightarrow & 0 \\
 & & \Big\downarrow \hat{\Phi}_{\Gamma'} & & \Big\downarrow \Phi_{\Gamma'} & & \Big\downarrow \Phi_{\Omega'} & & \\
0 & \longrightarrow & K' & \longrightarrow & R_{\Gamma'}(\cdot,0) & \longrightarrow & R_{\Omega'}(\cdot,0) & \longrightarrow & 0
\end{array}$$

where K' is the cochain complex $0 \to R_{\Gamma'}(d-1,0) \to 0$ with the nonzero term in degree $d-1$. Consider the final terms of the associated diagram of long exact cohomology sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{d-2}(W) \arrow[r] \arrow[d, "\Phi_{\Omega'}"] & H^{d-1}(Z,W) \arrow[r] \arrow[d, "\hat{\Phi}_{\Gamma'}"] & H^{d-1}(Z) \arrow[r] \arrow[d, "\Phi_{\Gamma'}"] & 0 \\
\cdots \arrow[r] & H_{\Omega'}(d-2,0) \arrow[r] & R_{\Gamma'}(d-1,0) \arrow[r] & H_{\Gamma'}(d-1,0) \arrow[r] & 0
\end{tikzcd}$$

The first downward arrow in this diagram is an isomorphism by induction. Using the isomorphism ζ and induction again we have $H^{d-1}(Z,W) = \bigoplus_i H^{d-2}(Z_i \cap W) = \bigoplus_i R_{\Gamma'_{y_i}}(d-1,0) = R_{\Gamma'}(d-1,0)$. Hence the second downward map is also an isomorphism. Hence the final downward map is also an isomorphism.

This completes the inductive proof of part (1) of the theorem. Part (2) follows immediately from part (1), exactly as in the proof of part (3) of Lemma 3.8. □

Definition 3.10. For each $k \geq 0$ we set $\Gamma^{>k} = \{y \in \Gamma \,|\, rk_\Gamma(y) > k\} \cup \{*\}$.

We note that $\Gamma^{>0} = \Gamma$ and that $\Gamma^{>k}$ is uniform if Γ is uniform. The rank function on $\Gamma^{>k} \setminus \{*\}$ is $rk_{\Gamma^{>k}}(a) = rk_\Gamma(a) - k$. It is also clear that for all $0 \leq j \leq n$, $R_{\Gamma^{>k}}(n,j) = R_\Gamma(n+k, j+k)$ and furthermore that $H_{\Gamma^{>k}}(n,j) = H_\Gamma(n+k,j+k)$. Using this notation we get the following corollary to Theorem 3.9. This corollary is an exact restatement of Proposition 3.2.1 from [20] in our notation. (Remark: there is a typographical error in 3.2.1 of [20], which uses H^{n-2} instead of $\tilde{H}^{n-2}$. It is clear that reduced cohomology was intended by the authors.)

Corollary 3.11. Let Γ be a finite ranked uniform poset and $* \neq v \in \Gamma$ an element of rank $d+1$. Then for any $0 \leq k \leq d-1$,

$$\dim(r_v R_\Gamma(d-1,k)) = \dim(\tilde{H}^{d-k-1}(\Delta(\Gamma_v^{>k} \setminus \{*, v\})).$$

Proof. $r_v R_\Gamma(d-1,k) = R_{\Gamma_v}(d,k) = R_{\Gamma_v^{\geq k}}(d-k,0)$. Apply (2) of 3.9. □

As in [20], this formula easily yields a closed formula for the Hilbert series of R_Γ. We will return to that formula in Section 8.

3.3. The First Main Theorem

Definition 3.12. Let Γ be a finite ranked poset. For any $a < b$ let $X_\Gamma(a,b) = X(a,b)$ be the total space of the order complex $\Delta((a,b))$.

We note that the dimension of $X(a,b)$ is $rk_\Gamma(b) - rk_\Gamma(a) - 2$. This is consistent with the definition: $\dim(\Delta(\emptyset)) = -1$. We also take as a (standard) convention $\tilde{H}^n(\Delta(\emptyset)) = 0$ for $n \neq -1$ and $\tilde{H}^{-1}(\Delta(\emptyset)) = \mathbb{F}$ (cf. [28]). We note that using the above notation, a cyclic poset Γ is Cohen-Macaulay if and only if

$$(*) \qquad \tilde{H}^n(X(a,b)) = 0 \text{ for all } a < b \in \Gamma \text{ and all } n \neq dim(X(a,b)).$$

We are now prepared to restate and then prove Theorem **??**.

Theorem 3.13. Let Γ be a finite ranked cyclic poset. Then Γ is Cohen-Macaulay if and only if Γ is uniform and the algebra R_Γ is Koszul.

Proof. The proof of 3.13 will proceed by induction on the rank of Γ. We first prove three technical lemmas.

Lemma 3.14. If the uniform ranked poset Γ is Koszul, then the poset $\Gamma^{>k}$ is Koszul for all $k \geq 0$.

Proof. Fix $k \geq 0$ and suppose $\Gamma^{>k}$ is not Koszul. Then for some $x \in \Gamma^{>k}$, $(\Gamma^{>k})_x$ is not Koszul. Choose such an x of minimal rank.

Note that $(\Gamma^{>k})_x = (\Gamma_x)^{>k} := \Gamma_x^{>k}$. By minimality of x, $\Gamma_y^{>k}$ is Koszul for every $y < x$ in $\Gamma^{>k}$. So by 3.2, $H_{\Gamma_x^{>k}}(n, j) \neq 0$ for some $0 \leq j < n$. But $H_{\Gamma_x^{>k}}(n, j) = H_{\Gamma_x}(n + k, j + k)$. This contradicts Lemma 2.7, since Γ_x is Koszul. □

Lemma 3.15. Let $\Gamma = \Gamma_b$ be a cyclic uniform ranked poset. Set $\Gamma' = \Gamma \backslash \{b\}$, $\Omega = \Gamma^{>1}$ and $\Omega' = \Omega \setminus \{b\}$. Let Y and Z be the total spaces of the order complexes of Γ' and Ω' respectively. Then for all $n > 0$,

$$H^n(Y, Z) = \bigoplus_{a \in \Gamma(1)} \tilde{H}^{n-1}(X(a, b)).$$

Proof. An n-cell $\beta = (b_0, \cdots, b_n)$ is a basis element of $C^n(Y, Z)$ if and only if it is not an n-cell of Z, which happens precisely when b_0 has rank 1 in Γ. For each $a \in \Gamma(1)$ let C_a^n be the F-span of those β for which $b_0 = a$. Since each such a is minimal, $d_Y : C_a^n \to C_a^{n+1}$. Thus we have a cochain complex decomposition: $C^*(Y, Z) = \bigoplus_{a \in \Gamma(1)} C_a^*$.

For $n > 0$, let $\zeta : C_a^n \to C^{n-1}(X(a, b))$ be the isomorphism defined by $(a, b_1, \cdots, b_n) \mapsto (b_1, \cdots, b_n)$. Similarly define $\zeta : C_a^0 \to \mathbb{F}$ by $(a) \mapsto 1$. It is clear that we have defined a degree -1 cochain complex isomorphism between C_a^* and the augmented cocomplex $\mathbb{F} \to C^*(X(a, b))$.

The statement of the lemma is now clear. □

Our last lemma relates the property of being uniform to the property that the topological spaces $X(a, b)$ are connected.

Lemma 3.16. Let Γ be a finite ranked poset which satisfies:

$$(**) \qquad X(a,b) \text{ is connected for all } a<b \text{ with } rk_\Gamma(b) - rk_\Gamma(a) \geq 3.$$

Then for all $a<b$, the interval $[a,b]$ is uniform (as a ranked poset with unique minimal element a). In particular Γ is uniform.

Proof. Choose $a<b$ in Γ. We proceed by induction on $rk_\Gamma(b) - rk_\Gamma(a)$. We may also assume $rk_\Gamma(b) - rk_\Gamma(a) \geq 3$, since otherwise $[a,b]$ is automatically uniform.

For any $a<c<b$, the interval $[a,c]$ is uniform by induction. Therefore, it remains only to check the definition of uniform against the element b itself. Returning to the definition of uniform (relative to $[a,b]$) we define $S_{[a,b]}(1) = \{c \mid a<c<b \text{ and } rk_\Gamma(b) - rk_\Gamma(c) = 1\}$. The equivalence relation on $S_{[a,b]}(1)$ is defined by transitive extension from the definition: $c_1 \sim_{[a,b]} c_2$ if there exists $a \leq u$ with $u<c_1$, $u<c_2$ and $rk_\Gamma(b) - rk_\Gamma(u) = 2$. Let $[c_1], [c_2], \ldots [c_r]$ be the distinct equivalence classes of $\sim_{[a,b]}$.

It remains only to prove $r=1$, so let us assume $r>1$. For each $1 \leq i \leq r$, let $U_i = (a,[c_i]] := \{x \in \Gamma \mid a<x \leq f \text{ for some } f \in [c_i]\}$. Then $(a,b) = \cup_i U_i$.

Since $X(a,b)$ is connected, the poset (a,b) must be connected as a graph. Since each U_i is a union of maximal intervals in (a,b), the various U_i can not all be disjoint. So we may assume $U_1 \cap U_2 \neq \emptyset$. Choose $y \in U_1 \cap U_2$. Then $y<c_1'$ and $y<c_2'$ for some $c_1' \in [c_1]$ and $c_2' \in [c_2]$. By induction, $[y,b]$ is uniform. This implies $c_1' \sim_{[y,b]} c_2'$, which is clearly a contradiction to $c_1' \not\sim_{[a,b]} c_2'$, since $[y,b] \subset [a,b]$. Hence $r=1$ and $[a,b]$ is uniform. $\square$

We can now complete the proof of 3.13.

We first prove the claim that if Γ is uniform and Koszul then condition (*) holds. Fix $a < b$ in Γ. Let k be the rank of a and $d+1$ be the rank of b. Then $dim(X(a,b)) = d-k-1$ and so there is nothing to prove unless $d > k+1$.

Let Γ', Ω, Ω', Y and Z be as in Lemma 3.15. By 3.14, Ω and Ω' are Koszul.

If $k = 0$, that is $a = *$, then $X(a,b) = Y$ and by (3) of 3.8 we have $\tilde{H}^n(X(a,b)) = 0$ for all $n < dim(X(a,b))$.

Assume $k = 1$. Consider the short exact sequence of cochain complexes

$$0 \to C(Y,Z) \to C(Y) \to C(Z) \to 0$$

and associated long exact sequence

$$(***) \qquad \cdots \to \tilde{H}^{n-1}(Z) \to H^n(Y,Z) \to \tilde{H}^n(Y) \to \cdots$$

By 3.8, $\tilde{H}^n(Y) = 0$ for $n < dim(Y) = d-1$ and $\tilde{H}^{n-1}(Z) = 0$ for $n-1 < dim(Z) = d-2$. Hence $H^n(Y,Z) = 0$ for $n < d-1$. By 3.15, $\tilde{H}^{n-1}(X(a,b))$ is a summand of $H^n(Y,Z)$ and we obtain $\tilde{H}^{n-1}(X(a,b)) = 0$ for $n-1 < dim(X(a,b)) = d-2$, as required.

Finally, consider the case $k > 1$. In this case $a \in \Omega$ with $rk_\Omega(a) > 0$. By induction on d, we immediately get $\tilde{H}^n(X(a,b)) = 0$ for $n < dim(X(a,b))$. This completes the proof of the first half.

We now turn to the converse. Assume condition (*) holds. By Lemma 3.16, Γ is uniform. We proceed to prove that Γ is Koszul, again by induction on $d+1$, the rank of Γ. If $d = 0$ there is nothing to prove. We assume $d > 0$.

Again, let Γ', Ω, Ω' and Y be as in Lemma 3.15. By induction, the posets Γ', Ω and Ω' are all Koszul, since the hypothesis (*) is true for all three and all three only have elements of rank at most d.

For any $* \neq a < b$ in Γ, Γ_a also satisfies (*) and thus, by induction Γ_a is Koszul. Hence by 3.2 it suffices to prove

$$H_\Gamma(n,k) = 0 \text{ for all } 0 \leq k < n \leq d-2.$$

Since Ω is cyclic, Koszul, and rank d, Theorem 3.2 tells us that for all $0 < k < n \leq d-2$:

$$H_\Gamma(n,k) = H_{\Gamma'}(n,k) = H_{\Omega'}(n-1,k-1) = H_\Omega(n-1,k-1) = 0.$$

By Theorem 3.7 and (*), for all $0 < n \leq d-2$,

$$H_\Gamma(n,0) = H_{\Gamma'}(n,0) = H^n(Y) = 0.$$

This completes the proof of 3.13. $\square$

The following corollary is 3.13 together with 3.1.

Corollary 3.17. Let Γ be a finite ranked poset. For all elements $x \in \Gamma$ of maximal rank, Γ_x is Cohen-Macaulay if and only if Γ is uniform and R_Γ is Koszul.

3.4. A Few Examples

We first discuss three types of examples that were of specific interest to us, as they represent different areas where we struggled with the Koszul question in the past.

Example 3.18. Let Γ be the infinite ranked poset of all partitions of non-negative integers. We identify a partition with its Young diagram, and then the order on Γ is $\subset$. The unique minimal element is the partition (0). For any integer n and any partition $\lambda \vdash n$, the finite poset Γ_λ is a modular lattice, hence Cohen-Macaulay (cf. [3]). Thus R_{Γ_λ} is Koszul. This result was proved earlier by T. Cassidy and Shelton using a modification of the techniques of [4].

Example 3.19. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be an arrangement of hyperplanes in an ambient vector space V (over an arbitrary field, not necessarily $\mathbb{F}$). Then the intersection lattice Γ associated to $\mathcal{A}$ is semi-modular and therefore Cohen-Macaulay over any field (cf. [3]). Hence R_Γ is Koszul. This class of examples includes, in particular, the lattice of all subspaces of a finite dimensional vector space over a finite field.

Example 3.20. Let Γ be the incidence poset of any finite regular CW complex, let X be the total space of the CW complex and let $\bar{\Gamma} = \Gamma \cup \{\emptyset\}$, where $\emptyset$ is uniquely minimal. If Γ is pure (all maximal cells have the same dimension), then we set $\hat{\Gamma} = \Gamma \cup \{\emptyset, X\}$ where X is uniquely maximal.

It is well known that for any $Z \in \bar{\Gamma}$, $\bar{\Gamma}_Z$ is Cohen-Macaulay over any field. Hence $R_{\bar{\Gamma}}$ is Koszul over any field. This theorem was first proved in both [22] and [4].

The poset $\hat{\Gamma}$ may well not be Cohen-Macaulay. The second main result of [4], Theorem 5.3, is an exact description, in combinatorial-topological terms, of when the algebras $R_{\hat{\Gamma}}$ are Koszul (these conditions include uniform, expressed as a topological condition). In the paper [25] it was further shown that the conditions under which $\hat{\Gamma}$ is Koszul are topological invariants rather than just combinatorial invariants. However, the conditions of that theorem can, with some small effort, be translated directly into the statement that $\hat{\Gamma}$ is Cohen-Macaulay, thereby relating that theorem directly to

3.13. Since Cohen-Macaulay is known to be a topological invariant, Theorem 3.13 also recaptures the results of [25].

There is an explicit example in [4] (Example 5.9 and Theorem 5.10) of a pure 3-dimensional regular CW complex X that is contractible, and yet $\hat{\Gamma}$ is not Koszul. The proof of this fact was somewhat detailed. But one can see by inspection, in the notation of that example, that the open interval (C_4, X) is not connected as a graph. That is enough show that $\hat{\Gamma}$ is not Cohen-Macaulay and conclude that the algebra is not Koszul.

We now give two specific examples. The first demonstrates that we can readily build posets Γ such that the Koszulity of R_Γ is field dependent. The second shows that if Γ is a lattice, then R_Γ need not be Koszul.

Example 3.21. We give $\mathbb{R}P^2$ a regular CW structure with four faces, eight edges and five vertices. Let Σ be the intersection poset of the CW complex. Similar to the above example, we set $\hat{\Sigma} = \Sigma \cup \{*, X\}$; $\hat{\Sigma}$ is shown in Figure 3.3. Then $\Delta((*, X))$ is the

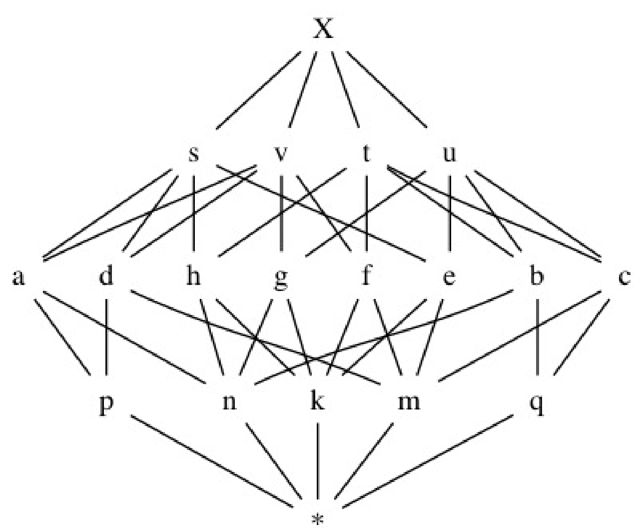

FIGURE 3.3. The poset $\hat{\Sigma}$.

barycentric subdivision of the CW structure on $\mathbb{R}P^2$, thus $\|\Delta((*, X))\|$ is homotopic to $\mathbb{R}P^2$. If $\mathbb{F}$ is a field of characteristic two, then $H^k(\Delta((*, X)), \mathbb{F}) = \mathbb{F}$ for all $0 \leq k \leq n$. Hence Σ is not Cohen-Macaulay relative to $\mathbb{F}$. We conclude that the $\mathbb{F}$-algebra R_Σ is not Koszul. On the other hand, if $\mathcal{K}$ is a field with characteristic different from two, then $H^k(\Delta((*, X)), \mathcal{K}) = \mathcal{K}$ for $k = 0$ and $k = n$ odd, and zero otherwise. Thus Σ is Cohen-Macaulay relative to $\mathcal{K}$ and the $\mathcal{K}$-algebra R_Σ is Koszul.

Example 3.22. Let Π be the finite ranked poset shown in Figure 3.4. By inspection,

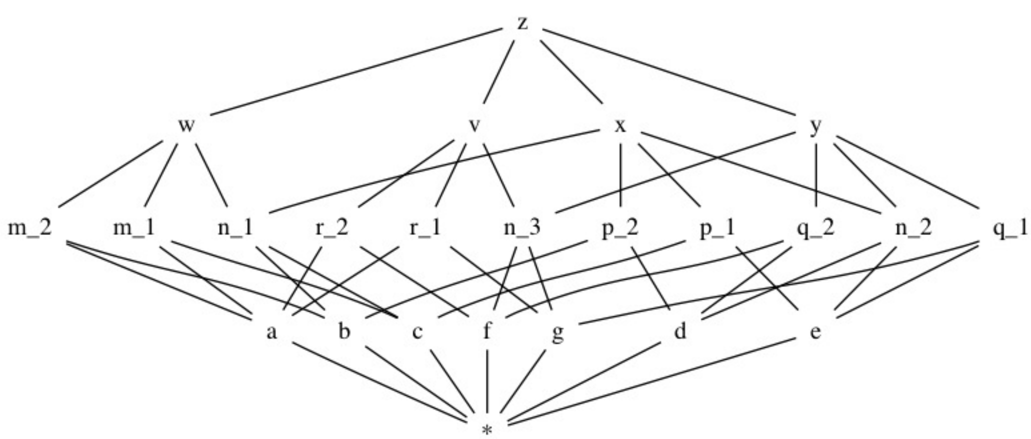

FIGURE 3.4. The poset Π.

Π is a lattice. The interval $(*, z)$ is the incidence poset of a regular CW complex on a pinched $S^1 \times I$. Thus $\|\Delta(*, z)\|$ is homotopic to S^1. We conclude R_Π is not Koszul.

3.5. Uniform and Cohen-Macaulay Posets

We borrow some nice notation from [20].

Definition 3.23. For any $a \in \Gamma$ and $1 \leq i \leq rk_\Gamma(a)$ we set

$$\Gamma_{a,i} = \{w < a \,|\, rk_\Gamma(a) - rk_\Gamma(w) \leq i - 1\} = \Gamma^{> rk_\Gamma(a) - i} \cap (*, a)$$

Note that $\Gamma_{a,i}$ is a subposet of $[*, a)$ and the dimension of $\Delta(\Gamma_{a,i})$ is $i-2$. It is helfpul to note that $\Gamma_{a,1} = \emptyset$, $\Gamma_{a,2} = S_1(a)$ and $\Gamma_{a,rk_\Gamma(a)} = (*, a)$. Also, we remark that $\Delta(\Gamma_{x,k})$ has dimension $k-2$.

We give an equivalent definition of uniform.

Proposition 3.24. Let Γ be a finite ranked poset. Then Γ is uniform if and only if for all $x \in \Gamma_+$ of rank at least three, $\Gamma_{x,3}$ is connected as a graph.

Similarly, we give an equivalent definition of Cohen-Macaulay.

Theorem 3.25. Let Γ be a finite ranked cyclic poset. Then Γ is Cohen-Macaulay if and only if for all $x \in \Gamma_+$ and all $rk_\Gamma(x) \geq k > n$, $\tilde{H}^{n-2}(\Delta(\Gamma_{x,k})) = 0$.

Proof. Assume Γ is Cohen-Macaulay and let $x \in \Gamma_+$. By Theorem 3.13, R_Γ is Koszul, from which it follows that R_{Γ_x} is Koszul. Applying Theorem 3.2, we see $H_{\Gamma_x}(m,j) = 0$ for all $rk_\Gamma(x) > m > j \geq 0$. Theorem 3.7 together with Lemma 3.14 tell us that $H_{\Gamma_x}(m,j) = H^{m-j}(\Delta(\Gamma_{x,rk_\Gamma(x)-j}))$ for all $rk_\Gamma(x) > m \geq j \geq 0$, which completes the proof of the forward direction.

For the reverse direction, we proceed by induction on the rank of $\Gamma = \Gamma_b$. For a poset of rank one, there is nothing to show. We then assume Γ has rank $d+1$ with $d > 0$.

By the inductive hypothesis, Γ_a is Cohen-Macaulay for all $a < b$ in Γ. Let $m \leq rk_\Gamma(b)$. We observed that $\Delta(\Gamma_{b,m-1})$ is a closed subspace of $\Delta(\Gamma_{b,m})$. We obtain the standard long exact sequence

$$\cdots \to \tilde{H}^n(\Delta(\Gamma_{b,m-1})) \to \tilde{H}^{n+1}(\Delta(\Gamma_{b,m}), \Delta(\Gamma_{b,m-1})) \to \tilde{H}^{n+1}(\Delta(\Gamma_{b,m})) \to \cdots .$$

Then, by assumption, $\tilde{H}^n(\Delta(\Gamma_{b,m}), \Delta(\Gamma_{b,m-1})) = 0$ for all $n \leq m-2$. Similar to 3.15, we get

$$H^n(\Delta(\Gamma_{b,m}), \Delta(\Gamma_{b,m-1})) = \bigoplus_{a \in S_b(m-1)} \tilde{H}^{n-1}(\Delta((a,b))).$$

This implies

$$\bigoplus_{a \in S_b(m-1)} \tilde{H}^j(\Delta((a,b))) = 0$$

for all $j \leq m-3$. This implies Γ is Cohen-Macaulay. □

3.6. On Numerical Koszulity

The Hilbert series of an $\mathbb{N}$-graded $\mathbb{F}$-algebra $R = \oplus_i R_i$ is the power series $H(R,t) = \sum_i \dim(R_i)t^i$. It is well known that if R is a quadratic and Koszul algebra with quadratic dual algebra $R^!$ (cf. [13]), then $H(R,-t)H(R^!,t) = 1$. We say that a quadratic algebra is *numerically Koszul* if it satisfies this equation.

Given a finite ranked uniform poset Γ, the quadratic dual of the algebra R_Γ will be denoted here as A'_Γ. This algebra was first described in [8] as a deformation of another important quadratic algebra A_Γ, the *splitting algebra* of the poset Γ. The algebras A'_Γ and A_Γ have the same Hilbert series, which was computed in [21] and then recalculated in terms of order complexes in [20]. We record here Theorem 4.1.1 of [20], translated into our notation. (Remark: as with Theorem 3.2.1 of [20], Theorem 4.1.1 has a typographical error. The theorem must use reduced cohomology, not cohomology.)

Definition 3.26. The reduced Euler characteristic of a space X, relative to $\mathbb{F}$, is

$$\tilde{\chi}(X) = \sum_i (-1)^i \dim(\tilde{H}^i(X))$$

where cohomology is calculated with coefficients in $\mathbb{F}$.

Note that $\tilde{\chi}(\Delta(\emptyset)) = -1$.

Theorem 3.27. ([20], 4.1.1) Let Γ be a uniform finite ranked poset. Then:

$$H(A_\Gamma, t)^{-1} = 1 + \sum_{i \geq 1} \sum_{\substack{a \in \Gamma \\ rk_\Gamma(a) \geq i}} \tilde{\chi}(\Delta(\Gamma_{a,i}))t^i$$

For further use we also record the Hilbert series of R_Γ from 3.1.1 of [20] (again, correcting for reduced cohomology).

Theorem 3.28. ([20], 3.1.1) Let Γ be a uniform finite ranked poset. Then:

$$H(R_\Gamma, -t) = 1 + \sum_{i \geq 1} \sum_{\substack{a \in \Gamma \\ rk_\Gamma(a) \geq i}} (-1)^{i-2} \dim \tilde{H}^{i-2}(\Delta(\Gamma_{a,i}))t^i$$

Definition 3.29. Let Γ be a finite uniform ranked poset.

(1) Let $v \in \Gamma$ and $i \leq rk_\Gamma(v)$. We say that the pair (v, i) is *good* if

$$\tilde{\chi}(\Delta(\Gamma_{v,i})) = (-1)^{i-2} \dim \tilde{H}^{i-2}(\Delta(\Gamma_{v,i}))$$

and *bad* if the equality does not hold.

(2) We define the *numerical Koszul defect* of Γ to be

$$\begin{aligned} \mathrm{NKD}(\Gamma) &= H(R_\Gamma, -t) - H(A_\Gamma, t)^{-1} \\ &= \sum_{(v,i)\ bad} \Big[(-1)^{i-2} \dim \tilde{H}^{i-2}(\Delta(\Gamma_{v,i})) - \tilde{\chi}(\Delta(\Gamma_{v,i}))\Big]\, t^i \end{aligned}$$

Since A_Γ and $A'_\Gamma = R^!_\Gamma$ have the same Hilbert series for any uniform Γ, we see that the ring R_Γ is numerically Koszul if and only if $\mathrm{NKD}(\Gamma) = 0$.

We record the following trivial observation which we will use frequently: if Γ is uniform, then for any $v \in \Gamma$, $(v,1)$, $(v,2)$, $(v,3)$ are always good ($(v,3)$ is good because uniformity implies the graph $\Gamma_{v,3}$ is connected). And $(v,4)$ is good if and only if $H^1(\Delta(\Gamma_{(v,4)})) = 0$. This immediately gives the following simple theorem.

Theorem 3.30. If Γ is a finite uniform ranked poset and no element of Γ has rank bigger than 4, then R_Γ is Koszul if and only if R_Γ is numerically Koszul.

Proof. Only one direction needs to be proved. Assume R_Γ is numerically Koszul. Fix any $a < b$ in Γ. If $dim(X(a,b)) \leq 1$ then it is clear from uniform that $\tilde{H}^n(X(a,b))$ is non-zero only for $n = dim(X(a,b))$. So assume $dim(X(a,b)) = 2$ (the maximum possible such dimension). This can only happen if $rk_\Gamma(b) = 4$ and $a = *$. But by hypothesis, $(b,4)$ is good. Since $\Gamma_{b,4} = (*,b)$, we have $\tilde{H}^n(X(*,b)) = 0$ for $n \neq 2$. We have shown Γ_x is Cohen-Macaulay for all x of rank four. Thus Γ is Koszul. $\square$

We now describe a construction for combining two uniform finite ranked posets, over which the NKD will be additive. This will allow us to construct examples that are numerically Koszul but not Koszul.

For the time being, let Γ and Ω be two finite ranked uniform posets with minimal elements $*_\Gamma$ and $*_\Omega$ respectively. Let X_Γ and X_Ω be the total spaces of the respective order complexes $\Delta(\Gamma)$ and $\Delta(\Omega)$. Fix elements $v \in \Gamma(1)$ and $v' \in \Omega(1)$.

Definition 3.31. With notation as above we set

$$\Gamma \vee_{(v,v')} \Omega = \Gamma \cup \Omega / (*_\Gamma \sim *_\Omega, v \sim v')$$

We identify Γ and Ω as subsets of $\Gamma \vee_{(v,v')} \Omega$. The set inherits an order from Γ and Ω wherein two elements are related if and only if they are related in either Γ or in Ω. We denote the unique minimal element by $*$ and the common image of v and v' by $\bar{v}$.

It is clear that the poset $\Gamma \vee_{(v,v')} \Omega$ inherits the uniform property from Γ and Ω. The following lemma is obvious.

Lemma 3.32. Let notation be as above. Then the total space of the order complex of the poset $\Gamma \vee_{(v,v')} \Omega$ is $X_\Gamma \vee X_\Omega$.

Lemma 3.33. Let notation be as above. Then

$$\mathrm{NKD}(\Gamma \vee_{(v,v')} \Omega) = \mathrm{NKD}(\Gamma) + \mathrm{NKD}(\Omega).$$

Proof. To ease the notation, let $\Theta = \Gamma \vee_{(v,v')} \Omega$. For any $a \in \Theta$ we see that $\Theta_{a,i} = \Gamma_{a,i}$ if $a \in \Gamma$ and $\Theta_{a,i} = \Omega_{a,i}$ if $a \in \Omega$. In particular, $\Theta_{\bar{v},1} = \emptyset$. Thus, by 3.27 we have the following decomposition:

$$\begin{aligned} H(A_\Theta, t)^{-1} &= 1 + \sum_{\substack{a \in \Gamma \\ rk_\Gamma(a) \geq i \geq 1}} \tilde{\chi}(\Delta(\Gamma_{a,i}))t^i \\ &\quad + \sum_{\substack{a \in \Omega \\ rk_\Gamma(a) \geq i \geq 1}} \tilde{\chi}(\Delta(\Omega_{a,i}))t^i - \tilde{\chi}(\Theta_{\bar{v},1})t \\ &= H(A_\Gamma, t)^{-1} + H(A_\Omega, t)^{-1} + t - 1 \end{aligned}$$

Exactly the same calculation for $H(R_\Theta, -t)$ yields

$$H(R_\Theta, -t) = H(R_\Gamma, -t) + H(R_\Omega, -t) + t - 1$$

Subtracting gives the required equation. □

Lemma 3.34. Let Γ be the uniform ranked poset shown to the right in Figure 3.5. Then $\mathrm{NKD}(\Gamma) = -t^5$. In particular Γ is not Koszul.

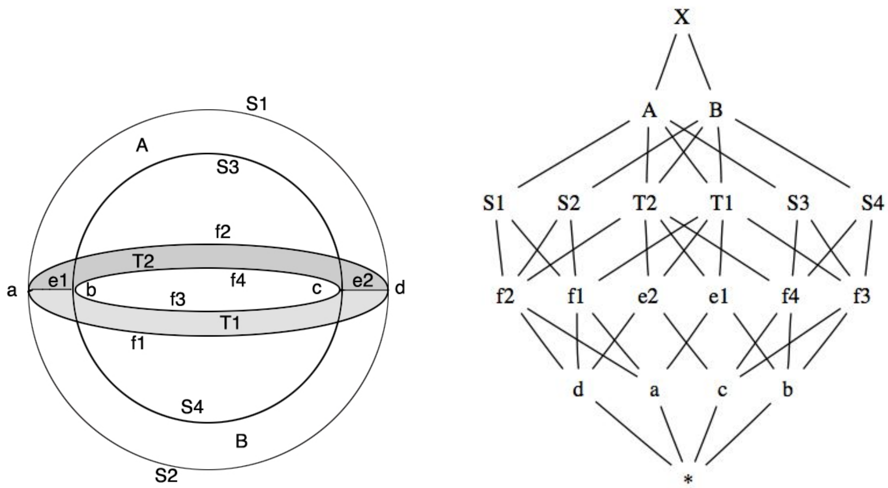

FIGURE 3.5. The regular CW complex $S^2 \times I$ and the poset Γ.

Proof. The poset Γ is of the form $P \cup \{*, X\}$, where P is the incidence poset of a regular CW complex realization of $S^2 \times I$. The picture to the left in Figure 3.5 labels the CW complex P (to the best of our abilities). Because P is a CW complex, the only pairs (v, i) that might be bad are the pairs $(X, 5)$ and $(X, 4)$.

The pair $(X, 5)$ is bad. To see this note $\Gamma_{X,5} = (*, X) = P$ and hence $||\Delta(\Gamma_{X,5})|| = S^2 \times I$. This 3-dimensional space is homotopic to S^2 and has non-zero reduced cohomology only in degree 2. Hence the pair $(X, 5)$ is bad and contributes $-t^5$ to $\mathrm{NKD}(\Gamma)$.

The pair $(X, 4)$ is good. To see this, first apply 3.15 to get

$$H^n(\Delta(\Gamma_{X,5}), \Delta(\Gamma_{X,4})) = \bigoplus_{v \in \Gamma(1)} \tilde{H}^{n-1}(\Delta(\,(v, X)\,)).$$

Since $S^2 \times I$ is a manifold with boundary and each 0-cell in P is on the boundary, the spaces $\Delta(\,(v, X)\,)$ are all homeomorphic to 2-discs and thus have no reduced cohomology. I.e. $H^n(\Delta(\Gamma_{X,5}), \Delta(\Gamma_{X,4})) = 0$ for all n. Since $\tilde{H}^n(\Delta(\Gamma_{X,5})) = 0$ for all

$n \neq 2$, the usual long exact sequence for relative cohomology tells us $\tilde{H}^n(\Delta(\Gamma_{X,4})) = 0$ for all $n \neq 2$. Thus $(X, 4)$ is good.

This proves the lemma. □

Lemma 3.35. Let Ω be the the uniform ranked poset shown in Figure 3.6. Then $\mathrm{NKD}(\Omega) = t^5$. In particular Ω is not Koszul.

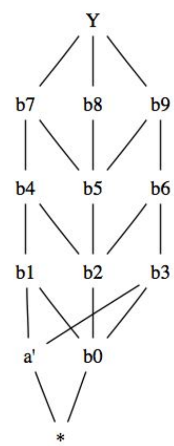

FIGURE 3.6. The poset Ω.

Proof. We see at once that Ω is not Koszul because the interval (a', Y) is not connected as a graph. By direct inspection, every pair $(v, 4)$ is good, because every pair $(v, 4)$ corresponds to a contractible space with no reduced cohomology. This leaves $(Y, 5)$ as the only pair that can be bad. Since $\Delta(\Omega_{Y,5}) = \Delta((*, Y))$ is 3-dimensional but homotopic to S^1 (by inspection), the pair $(Y, 5)$ is bad and contributes exactly t^5 to $\mathrm{NKD}(\Omega)$, as claimed. □

We now see easily that a numerically Koszul algebra in our class need not be Koszul, as promised in Theorem 1.9.

Theorem 3.36. Let Γ be as in 3.34 and Ω as in 3.35. Set $\Theta = \Gamma \vee_{(a,a')} \Omega$. Then the algebra R_Θ is not Koszul, but it is numerically Koszul.

Proof. The intervals $(*, Y)$ and $(*, X)$ show that the respective posets Γ and Ω are not Cohen-Macauley. Hence by Corollary 3.17, R_Θ is not Koszul.

By 3.33, 3.34 and 3.35, $\mathrm{NKD}(\Theta) = \mathrm{NKD}(\Gamma) + \mathrm{NKD}(\Omega) = -t^5 + t^5 = 0$. Thus Θ is numerically Koszul. $\square$

It is clear that any number of examples could be constructed in this fashion, but the resulting examples are not very satisfying, as they are far from being cyclic. Fortunately, through much more ad-hoc methods we were able to obtain the following cyclic example.

Theorem 3.37. Let Γ be the uniform ranked poset shown in Figure 3.7. Then Γ is numerically Koszul, but not Koszul.

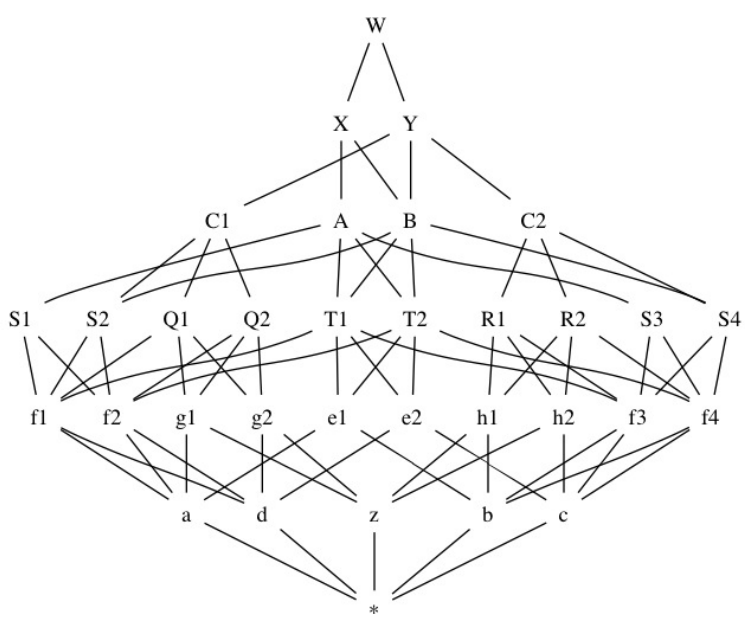

FIGURE 3.7. The poset Γ.

Proof. We first list several useful observations about Γ. $[*, X]$ is the poset from 3.34. In particular, $(*, X)$ is the incidence poset of a regular CW complex realization of $S^2 \times I$; we will refer to this CW complex as $\|X\|$. Let $\|Y\|$ be the three-dimensional

CW complex drawn (to the best of our abilities) in Figure 3.8. $\|Y\|$ has three 3-cells B, $C1$, and $C2$ and we note that B is also a 3-cell of $\|X\|$. $\|Y\|$ has a singular point at z and is homotopic to S^1. $(*, Y)$ in Γ is the incidence poset of $\|Y\|$. $(*, W)$ in Γ is not the incidence poset of a CW complex.

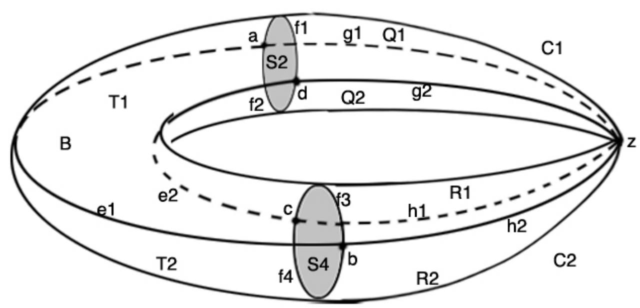

FIGURE 3.8. The CW complex $\|Y\|$.

By construction, Γ is uniform. Γ is not Cohen-Macaulay because the open interval (z, Y) is not connected as a graph, By 3.13, R_Γ is not Koszul.

To see that Γ is numerically Koszul, we need to examine the pairs $(A, 4)$, $(B, 4)$, $(C1, 4)$, $(C2, 4)$, $(X, 4)$, $(X, 5)$, $(Y, 4)$, $(Y, 5)$, $(W, 4)$, $(W, 5)$, and $(W, 6)$.

Due to the regular CW structures, the realizations of the order complexes of $\Gamma_{A,4}$, $\Gamma_{B,4}$, $\Gamma_{C1,4}$, and $\Gamma_{C2,4}$ are each homeomorphic to 2-spheres. We conclude $(A, 4)$, $(B, 4)$, $(C1, 4)$, and $(C2, 4)$ are good. 3.34 tells us $(X, 4)$ is good and $(X, 5)$ is bad. $(Y, 5)$ is bad because $\|\Delta(\Gamma_{Y,5})\|$ is homotopic to S^1.

Before calculating the remaining pairs, we will state a useful observation. Let V be a topological space and let U_1, U_2 be closed subsets of V such that $U_1 \cup U_2 = V$. If $U_1 \cap U_2$ is contractible, then $\text{cone}(U_1) \cup \text{cone}(U_2)$ is contractible.

Claim: $(Y,4)$ is good. We apply 3.15 to get

$$(**) \qquad H^n(\Delta(\Gamma_{Y,5}),\Delta(\Gamma_{Y,4})) = \bigoplus_{t\in\Gamma_Y(1)} \tilde{H}^{n-1}(\Delta((t,Y)))$$

for all $n \geq 0$. The intervals $(a,Y),(d,Y),(b,Y)$ and (c,Y) are isomorphic as posets and we can use the above useful observation to see that the order complex of each interval is contractible. The realization of the order complex of (z,Y) is homotopic to S^0. From (**), we conclude that $H^n(\Delta(\Gamma_{Y,5}),\Delta(\Gamma_{Y,4}))$ is one-dimensional for $n=1$ and is zero otherwise.

We now apply the standard long exact cohomology sequence related to relative cohomology for $\Delta(\Gamma_{Y,5})$ and $\Delta(\Gamma_{Y,4})$. Recalling that $\tilde{H}^n(\Delta(\Gamma_{Y,5}))$ is one-dimensional for $n=1$ and is zero otherwise, we see that $\tilde{H}^1(\Delta(\Gamma_{Y,4}))=0$ as required.

(W,n) is good for all $4 \leq n \leq 6$ because $\|\Delta(\Gamma_{W,n})\|$ is contractible by the above useful observation.

Finally, we compute

$$\begin{aligned}
\mathrm{NKD}(\Gamma) &= \sum_{(V,i)\ bad} \left[(-1)^{i-2}\dim \tilde{H}^{i-2}(\Delta(\Gamma_{V,i})) - \tilde{\chi}(\Delta(\Gamma_{V,i}))\right] t^i \\
&= [(-1)^3 \dim \tilde{H}^3(\Delta(\Gamma_{X,5})) - \tilde{\chi}(\Delta(\Gamma_{X,5}))]t^5 \\
&+[(-1)^3 \dim \tilde{H}^3(\Delta(\Gamma_{Y,5})) - \tilde{\chi}(\Delta(\Gamma_{Y,5}))]t^5 \\
&= [-\tilde{\chi}(\Delta(\Gamma_{X,5})) - \tilde{\chi}(\Delta(\Gamma_{Y,5}))]t^5 = [(-1)-(-1)]t^5 = 0.
\end{aligned}$$

Thus R_Γ is numerically Koszul and this completes our proof. □

CHAPTER IV

FINITE RANKED POSETS AND R_Γ

We begin with an illustrative example.

Example 4.1. Let Θ be the ranked poset shown in Figure 4.1. This poset was

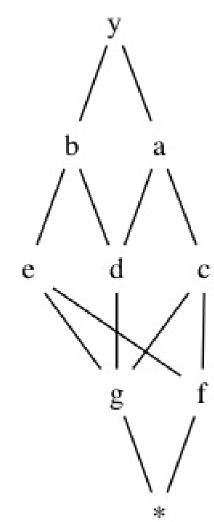

FIGURE 4.1. The poset Θ.

first introduced by Cassidy and it was studied extensively in [10]. We make several observations. By inspection, we see that Θ is uniform. Also, by inspection, we know Θ is not Cohen-Macaulay; $\|\Delta(\Theta_{y,4})\|$ is homotopic to S^1. By Theorem 3.13, R_Θ is not Koszul.

Let Θ^* be the dual poset of Θ, as shown in Figure 4.2. We again make several observations. Since $\Theta^*_{n,3}$ is disconnected as a graph, Θ^* is not uniform and Θ^* is not Cohen-Macaulay. Also, $\|\Delta({\Theta^*}_{x,4})\| = \|\Delta(\Theta_{y,4})\|$ is homotopic to S^1.

We may not apply Theorem 3.13 to determine if R_{Θ^*} is Koszul or not. Nonetheless, we claim R_{Θ^*} is Koszul. We use techniques similar to (3.3) of [4]; we will

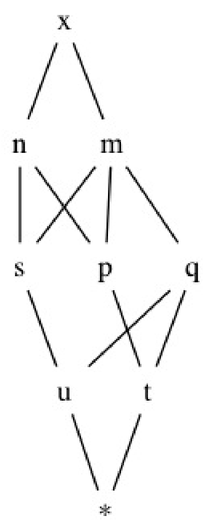

FIGURE 4.2. The poset Θ^*.

soon make these techniques more precise. We only need to show $rann_{R_{\Theta^*}}(r_m + r_n)$ is linearly generated. We see $(r_s + r_p + r_q)r_u = r_q r_u = -r_q r_t$. Then, by inspection,

$$rann_{R_{\Theta^*}}(r_m + r_n) = (r_s + r_p + r_q)R_\Theta \oplus \bigoplus_{z \in \Theta_+} r_z R_{\Theta^*}.$$

We reiterate to the reader: Θ^* is not Cohen-Macaulay, nonetheless R_{Θ^*} is Koszul.

Let Γ be an arbitrary (possibly non-uniform) finite ranked poset with unique minimal element $*$ and rank $m + 1$. In this chapter, we wish to find necessary and sufficient conditions on Γ that make R_Γ Koszul.

4.1. The Right Annihilator Condition

In this section, we establish algebraic results that generalize section 3 of [4]. We remind the reader of the following fact from [4]: for a uniform Γ, the Koszul property for R_Γ is equivalent to a condition on right annihilators of certain elements of R_Γ.

First, we need some notation and some new combinatorial objects.

Definition 4.2. Let $u \in R_\Gamma$. Define the linearly generated right annihilator of u, $L_\Gamma(u) := (rann_{R_\Gamma}(u))_1 R_\Gamma$.

Definition 4.3. Let $1 \le n \le m+1$ and $S \subset \Gamma(n)$. Then

$$r_S := \sum_{z\in S} r_z.$$

For all $x \in \Gamma_+$ we write $r_{\{x\}} = r_x$. Suppose $u, v \in \Gamma_+$. We write $u \sim^S v$ if for all

$$q = \sum_{z\in\Gamma_+} q_z r_z, \qquad (q_z \in \mathbb{F})$$

$r_S q = 0$ implies $q_u = q_v$.

We note $\sim^S$ defines an equivalence relation on Γ_+. Equivalence classes will be denoted by $[-]^S$ and thus

$$r_{[z_0]^S} = \sum_{z\in[z_0]^S} r_z.$$

Proposition 4.4. Let n and S be given as in Definition 4.3. Then

$$L_\Gamma(r_S) = \bigoplus_{[z_0]^S} r_{[z_0]^S} R_\Gamma.$$

Proof. Suppose

$$r_S \cdot \sum_{z\in\Gamma_+} q_z r_z = 0.$$

Then

$$\sum_{z\in\Gamma_+} q_z r_z = \sum_{[z_0]^S} q_{[z_0]^S} r_{[z_0]^S} \in \bigoplus_{[z_0]^S} r_{[z_0]^S} R_\Gamma.$$

We also need to show $r_{[z_0]^S} \in rann_{R_\Gamma}(r_S)$ for all $[z_0]^S$. If $z_0 \in \Gamma \setminus \Gamma(n-1)$, then $r_{[z_0]^S} = r_{z_0}$ and $r_S r_{z_0} = 0$. Now assume $z_0 \in \Gamma(n-1)$. We partition S into two sets:

$S_1 = \{s \in S : s \to y \text{ for some } y \in [z_0]^S\}$ and S_2 is the compliment of S_1 in S. Then we compute

$$r_S \cdot r_{[z_0]^S} = (r_{S_1} + r_{S_2}) r_{[z_0]^S} = r_{S_1} r_{[z_0]^S} = 0$$

and this completes our proof. $\square$

Definition 4.5. Let n and S be given as in Definition 4.3. Let $U = \{u \in \Gamma(n-1) : u < s \text{ for some } s \in S\}$. We define

$$A_\Gamma(r_S) := \bigoplus_{[z_0]^S \subseteq U} r_{[z_0]^S} R_\Gamma,$$

$$B_\Gamma(r_S) := \bigoplus_{z \in \Gamma(n-1) \setminus U} r_z R_\Gamma$$

and

$$H_\Gamma(n-1) := \bigoplus_{z \in \Gamma_+ \setminus \Gamma(n-1)} r_z R_\Gamma.$$

It is apparent that

$$L_\Gamma(r_S) = A_\Gamma(r_S) \oplus B_\Gamma(r_S) \oplus H_\Gamma(n-1).$$

Definition 4.6. Let Γ be a finite ranked cyclic poset with rank $m+1$ and $\Gamma = \Gamma_x$. We define

$$T_{m+1}(\Gamma) = \{\Gamma(m+1)\} = \{\{x\}\}$$

and recursively define for $m \geq i \geq 1$

$$T_i(\Gamma) = \{[z_0]^S \subset \Gamma_+ | S \in T_{i+1} \text{ and there exists } s \in S \text{ such that } s \to z_0\}.$$

Let

$$T(\Gamma) = \bigcup_{i=1}^{m+1} T_i(\Gamma).$$

Definition 4.7. Let Γ be a finite ranked poset. Define

$$\mathcal{T}(\Gamma) = \bigcup_{x \in \Gamma_+} T(\Gamma_x).$$

Remark 4.8. By (3.3) from [4], we see that if Γ is uniform, then $\mathcal{T}(\Gamma)$ consists of sets of the form $\Gamma_x(n)$ where $x \in \Gamma_+$ and $1 \leq n \leq rk_\Gamma(x)$.

The following lemma is (3.3) of [4], without the uniform hypothesis.

Theorem 4.9. Let Γ be a finite ranked poset. The algebra R_Γ is Koszul if and only if for all $S \in \mathcal{T}(\Gamma)$, $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$.

Proof. For the forward direction, assume R_Γ is Koszul. Then the right R_Γ-modules $(R_\Gamma)_+$ and $\mathbb{F}_{R_\Gamma} = R_\Gamma/(R_\Gamma)_+$ are Koszul. Let $S \in \mathcal{T}(\Gamma)$, so there is $x \in \Gamma_+$ and $1 \leq n \leq rk_\Gamma(x)$ with $S \in T_n(\Gamma_x)$. By (reverse) induction on n, we will simultaneously prove $rann_{R_\Gamma}(r_S) = L(r_S)$ and $r_S R_\Gamma$ is a Koszul module.

If $n = rk_\Gamma(x)$, then $S = \{x\}$. By definition of R_Γ, $rann_{R_\Gamma}(r_x) = L_\Gamma(r_x)$. $r_x R_\Gamma$ is a direct summand of $(R_\Gamma)_+$ by Lemma 2.6, thus it is a Koszul module.

We now assume $n < rk_\Gamma(x)$. Then there exists $U \in T_{n+1}(\Gamma_x)$, $z_0 \in \Gamma_x(n)$ and $u \in U$ such that $u \to z_0$ and $S = [z_0]^U$. By induction, $rann_{R_\Gamma}(r_U) = L(r_U)$. We then have a short exact sequence

$$0 \to L_\Gamma(r_U) \to R_\Gamma \to r_U R_\Gamma \to 0.$$

Again, by induction, $r_U R_\Gamma$ is Koszul. Thus $L_\Gamma(r_U)$ is a Koszul module. The module $r_S R_\Gamma$ is a direct summand of $L(r_U)$ and therefore, $r_S R_\Gamma$ is a Koszul module. Now

$rann_{R_\Gamma}(r_S)$ is linearly generated and $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$. This completes the proof of the forward direction.

We now prove the reverse direction. Let $\mathcal{F}$ be the set of right ideals in R_Γ of the form

$$I = \bigoplus_{i=1}^{m} r_{S_i} R_\Gamma$$

where $S_1, \ldots, S_m \in \mathcal{T}(\Gamma)$ are pairwise disjoint. We include the zero ideal in $\mathcal{F}$ and observe $(R_\Gamma)_+ \in \mathcal{F}$. To prove that R_Γ is Koszul, we show that $\mathcal{F}$ is a Koszul filtration (c.f. [12]).

Let

$$0 \neq I = \bigoplus_{i=1}^{m} r_{S_i} R_\Gamma \in \mathcal{F}.$$

Then set

$$J = \bigoplus_{i=1}^{m-1} r_{S_i} R_\Gamma$$

and observe $J \in \mathcal{F}$ and $I = J + r_{S_m} R_\Gamma$. Also, the right ideal $(r_{S_m} : J) = \{p \in R_\Gamma : r_{S_m} p \in J\}$ (the conductor of r_{S_m} into J) is equal to $rann_{R_\Gamma}(r_{S_m})$. Since $rann_{R_\Gamma}(r_{S_m}) = L_\Gamma(r_{S_m})$, $(r_{S_m} : J) \in \mathcal{F}$. We conclude $\mathcal{F}$ is a Koszul filtration, which completes our proof. $\square$

Remark 4.10. We reiterate that if $S \in \mathcal{T}(\Gamma)$ is a singleton, then $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$, even if R_Γ is not Koszul.

The following lemma is (3.5) from [4], again, without the uniform hypothesis.

Theorem 4.11. Let Γ be a finite ranked poset. Then R_Γ is Koszul if and only if R_{Γ_x} is Koszul for all $x \in \Gamma_+$.

Proof. Let $x \in \Gamma_+$ and $S \in T(\Gamma_x)$. By Theorem 2.6,

$$rann_{R_\Gamma}(r_S) \cap R_{\Gamma_x} = rann_{R_{\Gamma_x}}(r_S).$$

The theorem then follows from Theorem 4.9. $\square$

Remark 4.12. We often use the following fact, which is Theorem 4.9 together with Theorem 4.11. Suppose Γ is a finite ranked cyclic poset with $\Gamma = \Gamma_x$. Set $\Omega = \Gamma \setminus \{x\}$ and assume R_Ω is Koszul. Then R_Γ is Koszul if and only if for all $S \in T(\Gamma)$, $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$.

Corollary 4.13. Let Γ be a finite ranked poset with rank less than or equal to three. Then R_Γ is Koszul.

Example 4.14. We return to Θ^* in Example 4.1. The theorems above, along with the computations from Example 4.1, show R_{Θ^*} is Koszul.

Definition 4.15. Let $1 \leq n \leq m+1$ and $S \subset \Gamma(n)$. We set

$$\Gamma_S = \{a \in \Gamma : a \leq s \text{ for some } s \in S\} = \bigcup_{s \in S} [*, s].$$

Additionally, for $0 \leq k \leq n-1$, we set

$$\Gamma(S,k) = \Gamma_S \cap \Gamma^{\geq n-k}.$$

We say S is linked if $\Gamma(S,1)$ is connected as a graph. We say $S' \subseteq S$ is maximally linked relative to S if S' is maximal amongst linked subsets of S.

The collection of all maximally linked subsets of S forms a partition of S.

Lemma 4.16. Let n and S be given as in Definition 4.15. Then $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$ if and only if for all maximally linked subsets $S' \subseteq S$, $rann_{R_\Gamma}(r_{S'}) = L_\Gamma(r_{S'})$.

Proof. Due to the observation immediately following Definition 4.5, it is enough to prove the lemma for $\Gamma = \Gamma_S$. We therefore assume $S = \Gamma(n)$. We enumerate the maximally linked subsets of S: $S_1, \ldots S_p$. For all $i = 1, \ldots, p$, set $U_i = \{t \in \Gamma(n-1) : t < s \text{ for some } s \in S_i\}$.

For the forward direction, suppose towards a contradiction that for some $k \in \{1, \ldots, p\}$, there exists $A \in rann_{R_\Gamma}(r_{S_k}) \setminus L_\Gamma(r_{S_k})$. Then $A = A_1 + A_2$ with

$$A_1 \in \bigoplus_{z \in U_k} r_z R_\Gamma \quad \text{and} \quad A_2 \in \bigoplus_{z \in \Gamma_+ \setminus U_k} r_z R_\Gamma = B_\Gamma(r_{S_k}) \oplus H_\Gamma(n-1).$$

Clearly $r_{S_k} A_2 = 0$, thus $r_{S_k} A_1 = 0$. Also, A_1 is nonzero and $A_1 \notin r_{U_k} R_\Gamma = A_\Gamma(r_{S_k})$; if not, then $A \in L_\Gamma(r_{S_k})$. We compute

$$r_S A_1 = (r_{S_k} + r_{\Gamma(n) \setminus S_k}) A_1 = r_{S_k} A_1 = 0.$$

Therefore, $A_1 \in rann_{R_\Gamma}(r_S)$. Also, we see $A_1 \notin A_\Gamma(r_S) = \bigoplus_{i=1}^p r_{U_i} R_\Gamma$ since $A_1 \notin r_{U_k} R_\Gamma$. Thus $A_1 \notin L_\Gamma(r_S)$, which is a contradiction.

For the converse, we assume $rann_{R_\Gamma}(r_{S_i}) = L_\Gamma(r_{S_i})$ for all $i = 1, \ldots, p$. Carefully applying Definition 4.5, we compute

$$\begin{aligned}
rann_{R_\Gamma}(r_S) &= \bigcap_{i=1}^{p} rann_{R_\Gamma}(r_{S_i}) \\
&= \bigcap_{i=1}^{p} L_\Gamma(r_{S_i}) \\
&= \bigoplus_{i=1}^{p} A_\Gamma(r_{S_i}) \oplus H_\Gamma(n-1) \\
&= \bigoplus_{[z_0]^S} r_{[z_0]^S} R_\Gamma \\
&= L_\Gamma(r_S),
\end{aligned}$$

and this completes our proof. $\square$

Lemma 4.17. Let n and S be given as in Definition 4.15 and assume S is linked. Then $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$ if and only if for all $0 \leq k \leq n-3$, $H^{n-2}(R_{\Gamma_S}(\bullet, k), d_{\Gamma_S}) = 0$.

Proof. Again, it is enough to prove the claim for $\Gamma = \Gamma_S$. We set $S = \Gamma(n)$. Then $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$ if and only if

$$rann_{R_\Gamma}(r_S) = r_{\Gamma(n-1)} R_\Gamma \oplus H_\Gamma(n-1).$$

This equality holds if and only if

$$R_\Gamma(n-3, k) \xrightarrow{r_{\Gamma(n-1)}\cdot} R_\Gamma(n-2, k) \xrightarrow{r_S\cdot} R_\Gamma(n-1, k)$$

is exact for all $0 \leq k \leq n-3$. $\square$

Finally, we define

$$M_n(\Gamma) = \bigcup_{S \in T_n(\Gamma)} \{S' : S' \text{ is maximally linked relative to } S\}$$

and

$$M(\Gamma) = \bigcup_{n=1}^{m+1} M_n(\Gamma).$$

Remark 4.18. Using Remark 4.8, we see that if Γ is cyclic and uniform then, $M_n(\Gamma) = \{\Gamma(n)\}$ for all $1 \leq n \leq m+1$.

We combine the definition of $M(\Gamma)$ with Lemmas 4.16 and 4.17 arrive at the following theorem.

Theorem 4.19. Let Γ be a finite ranked poset with unique minimal element $*$ and rank $m+1$. Assume $\Gamma = \Gamma_x$. Set $\Omega = \Gamma \setminus \{x\}$ and assume R_Ω is Koszul. The following are equivalent:

1. R_Γ is Koszul;

2. for all $S \in M(\Gamma \setminus \{x\})$, $rann_{R_\Gamma}(r_S) = L_\Gamma(r_S)$;

3. for all $1 < n \leq m$, $S \in M_n(\Gamma \setminus \{x\})$ and $0 \leq k \leq n-3$, $H^{n-2}(R_{\Gamma_S}(\bullet, k), d_{\Gamma_S}) = 0$.

4.2. A Spectral Sequence Associated to $\Delta(\Gamma \setminus \{*\})$

Assume Γ has rank $m+1$ and let $Y = \|\Delta(\Gamma \setminus \{*\})\|$. We start by defining an unusual filtration on $C^\bullet(Y)$. Let $F_pC^\bullet(Y)$ be an increasing filtration on $C^\bullet(Y)$ given by

$$F_pC^n(Y) = \{(\alpha_0, \ldots, \alpha_n) \in C^n(Y) : rk_\Gamma(\alpha_n) \geq (m+1) - p\}.$$

Note: $F_pC^n(Y)$ consists of all n co-chains of Y that emanate from the top $p+1$ layers of Γ. Also, observe

$$0 = F_{-1}C^\bullet(Y) \subset F_0C^\bullet(Y) \subset \cdots \subset F_{m-1}C^\bullet(Y) \subset F_mC^\bullet(Y) = C^\bullet(Y).$$

Let $E^n_{p,q}$ denote the associated cohomology spectral sequence with

$$E^0_{p,q} = F_pC^{p+q}(Y)/F_{p-1}C^{p+q}(Y) \qquad \text{and} \qquad E^1_{p,q} = H^{p+q}(E^0_{p*}).$$

This spectral sequence is bounded since $E^0_{p,q} = 0$ if $p < 0$, $p+q < 0$, or $q > -2p+m$. The differentials are maps $d^r_E : E^r_{p,q} \to E^r_{p-r,q+r+1}$ induced by the differential d_Y on $C^\bullet(Y)$.

Proposition 4.20. Let Γ be a ranked poset with unique minimal element $*$ and rank $m+1$. Let $E^r_{p,q}$ be the spectral sequence of the filtration $F_\bullet$ on $C^\bullet(Y)$. For all p and q, $E^{m+1}_{p,q} = E^\infty_{p,q}$. Also $E^1_{p,q} \implies H^{p+q}(Y)$.

Proof. The equality holds because d^{m+1}_E is the zero map on $E^{m+1}_{p,q}$. The convergence statement follows from Theorem 5.5.1 of [29]. □

The E^1 page of our spectral sequence can be written in terms of reduced cohomology of open intervals of Γ.

Proposition 4.21. Let Γ be a finite ranked poset with unique minimal element $*$ and rank $m+1$. Let $E^r_{p,q}$ be the spectral sequence of the filtration $F_\bullet$ on $C^\bullet(Y)$. For all p and q,

$$E^1_{p,q} = \bigoplus_{rk_\Gamma(x)=(m+1)-p} \tilde{H}^{p+q-1}(\Delta((*,x))).$$

Proof. We fix p and identify two chain complexes: $(E^0_{p,*}, d^0_E)$ and

$$\left(\bigoplus_{rk_\Gamma(x)=(m+1)-p} \tilde{C}^\bullet(\Delta((*,x))), \bigoplus_{rk_\Gamma(x)=(m+1)-p} d_{\Delta((*,x))} \right).$$

For convenience of notation, elements of $\tilde{C}^\bullet(\Delta((*,x)))$ will be denoted by sums of chains of the form $(\alpha_0,\ldots,\alpha_n)_x$. Define a map

$$f : E^0_{p,q} = F_pC^{p+q}(Y)/F_{p-1}C^{p+q}(Y) \to \bigoplus_{rk_\Gamma(x)=(m+1)-p} \tilde{C}^{p+q-1}(\Delta((*,x)))$$

via $(\alpha_0) \mapsto (1)_{\alpha_0}$ and if $p+q>0$

$$(\alpha_0,\ldots,\alpha_{p+q}) \mapsto (\alpha_0,\ldots,\alpha_{p+q-1})_{\alpha_{p+q}},$$

and extend linearly. It is easy to see that f is bijective. We claim f is a co-chain map. Let $(\alpha_0,\ldots,\alpha_{p+q}) \in E^0_{p,q}$ and note $rk_\Gamma(\alpha_{p+q}) = m+1-p$. We compute

$$\begin{aligned}\bigoplus_{rk_\Gamma(x)=(m+1)-p} d_{\Delta((*,x))}\left(f(\alpha_0,\ldots,\alpha_{p+q})\right) &= \bigoplus_{rk_\Gamma(x)=(m+1)-p} d_{\Delta((*,x))}\left((\alpha_0,\ldots,\alpha_{p+q-1})_{\alpha_{p+q}}\right) \\ &= d_{\Delta((*,\alpha_{p+q}))}\left(\alpha_0,\ldots,\alpha_{p+q-1}\right)\end{aligned}$$

and on the other hand

$$
\begin{aligned}
f(d_E^0(\alpha_0,\dots,\alpha_{p+q})) &= f\left(\sum_{x<\alpha_0}(x,\alpha_0,\dots,\alpha_{p+q})\right)\\
&+ f\left(\sum_{i=0}^{p+q-1}(-1)^{i+1}\sum_{\alpha_i<x<\alpha_{i+1}}(\alpha_0,\dots,\alpha_i,x,\alpha_{i+1},\dots,\alpha_{p+q})\right)\\
&= \sum_{x<\alpha_0}(x,\alpha_0,\dots,\alpha_{p+q-1})_{\alpha_{p+q}}\\
&+ \sum_{i=0}^{p+q-1}(-1)^{i+1}\sum_{\alpha_i<x<\alpha_{i+1}}(\alpha_0,\dots,\alpha_i,x,\alpha_{i+1},\dots,\alpha_{p+q-1})_{\alpha_{p+q}}\\
&= d_{\Delta((*,\alpha_{p+q}))}(\alpha_0,\dots,\alpha_{p+q-1}).
\end{aligned}
$$

It follows that f is a co-chain isomorphism. □

We now make an observation related to Theorems 3.7 and 3.13.

Corollary 4.22. Let Γ be a finite ranked cyclic poset with unique minimal element $*$ and rank $m+1$. Let $E_{p,q}^r$ be the spectral sequence of the filtration $F_\bullet$ on $C^\bullet(Y)$. Assume Γ is Cohen-Macaulay. Then $E_{p,q}^1 = 0$ unless $q = -2p+m$. Moreover, $E_{p,q}^2 = E_{p,q}^\infty$.

Definition 4.23. For $0 \le n \le m$, define

$$S^n(\Gamma_+) = \bigoplus_{rk_\Gamma(x)=n+1} \tilde{H}^{n-1}(\Delta((*,x))).$$

For $n > 0$, we denote elements in $\tilde{H}^{n-1}(\Delta((*,x)))$ with linear combinations of equivalence classes of the form $[x_1 \leftarrow \cdots \leftarrow x_n]_x$. This notation requires: $x_n \leftarrow x$. Elements in $\tilde{H}^{-1}(\Delta((*,x)))$ will be denoted by scalar multiples of $[1]_x$.

Define $d_{S^n(\Gamma_+)} : S^n(\Gamma_+) \to S^{n+1}(\Gamma_+)$ via

$$[1]_x \mapsto \bigoplus_{x \leftarrow y} [x]_y$$

and for $n > 0$

$$[x_1 \leftarrow \cdots \leftarrow x_n]_x \mapsto \bigoplus_{x \leftarrow y} (-1)^n [x_1 \leftarrow \cdots \leftarrow x_n \leftarrow x]_y,$$

and extend linearly.

Proposition 4.24. $d_{S^n(\Gamma_+)} : S^n(\Gamma_+) \to S^{n+1}(\Gamma_+)$ is well-defined.

Proof. If $n = 0$, the map is clearly well-defined. Assume $n > 0$. It is sufficient to observe the inclusion

$$\begin{aligned} d_{S^n(\Gamma_+)} &\Big(im(d^{n-2}_{\Delta((*,x))} : C^{n-2}(\Delta((*,x))) \to C^{n-1}(\Delta((*,x)))) \Big) \\ &\subset im(d^{n-1}_{\Delta((*,z))} : C^{n-1}(\Delta((*,z))) \to C^{n}(\Delta((*,z)))). \end{aligned}$$

for all $x \leftarrow z$. □

Proposition 4.25. $(S^\bullet(\Gamma_+), d_{S^\bullet(\Gamma_+)})$ is a co-chain complex.

Proof. Let $[1]_x \in \tilde{H}^{-1}(\Delta((*,x)))$. Then

$$\begin{aligned} d_{S^1(\Gamma_+)} \circ d_{S^0(\Gamma_+)}([1]_x) &= d_{S^1(\Gamma_+)} \left(\bigoplus_{x \leftarrow y} [x]_y \right) \\ &= \bigoplus_{x < z} \left(\sum_{x \leftarrow y \leftarrow z} [x \leftarrow y]_z \right) \\ &= \bigoplus_{x < z} [0]_z \end{aligned}$$

since

$$d^0_{\Delta((*,z))}(x) = -\sum_{x \leftarrow y \leftarrow z} (x,y).$$

for all $x < z$.

Assume $n > 0$. Let $[x_1 \leftarrow \cdots \leftarrow x_n]_x \in \tilde{H}^{n-1}(\Delta((*,x)))$. We compute

$$\begin{aligned}
&d_{S^{n+1}(\Gamma_+)} \circ d_{S^n(\Gamma_+)}([x_1 \leftarrow \cdots \leftarrow x_n]_x) \\
&= d_{S^{n+1}(\Gamma_+)}\left(\bigoplus_{x \leftarrow y} (-1)^n [x_1 \leftarrow \cdots \leftarrow x_n \leftarrow x]_y\right) \\
&= \bigoplus_{x<z}\left(\sum_{x \leftarrow y \leftarrow z} (-1)^{2n+1}[x_1 \leftarrow \cdots \leftarrow x_n \leftarrow x \leftarrow y]_z\right) \\
&= \bigoplus_{x<z} [0]_z.
\end{aligned}$$

The last equality holds because

$$d^n_{\Delta((*,z))}(x_1,\ldots,x_n,x) = \sum_{x \leftarrow y \leftarrow z} (-1)^{n+1}(x_1,\ldots,x_n,x,y).$$

for all $x < z$. □

We note $S^{p+q}(\Gamma_+)$ is $E^1_{p,q}$ for $q = -2p+m$. In fact, $d_{S^n(\Gamma_+))}$ is the differential d^1_E on $E^1_{p,-2p+m}$.

4.3. An Isomorphism of Co-chain Complexes

Definition 4.26. Define $\Psi_{\Gamma_+} : S^n(\Gamma_+) \to R_\Gamma(n,0)$ via

$$[1]_x \mapsto r_x$$

and for $n > 0$

$$[x_1 \leftarrow \cdots \leftarrow x_n]_x \mapsto r_x r_{x_n} \cdots r_{x_1},$$

and extend linearly.

Proposition 4.27. $\Psi_{\Gamma_+} : S^n(\Gamma_+) \to R_\Gamma(n, 0)$ is well-defined.

Proof. The map is clearly well-defined for $n = 0$. We then assume $n > 0$ and let $rk_\Gamma(x) = n + 1$. Similar to the above proofs, it suffices to show:

$$\Psi_{\Gamma_+}\left(im(d^{n-2}_{\Delta((*,x))} : C^{n-2}(\Delta(*, x)) \to C^{n-1}(\Delta((*, x))))\right) = 0.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in C^{n-2}(\Delta((*, x)))$. Since $n - 1 = dim\Delta((*, x))$, $[d^{n-2}_{\|\Delta((*,x))\|}(\alpha)]_x$ is

$$\sum_{a \leftarrow \alpha_1} [a \leftarrow \alpha_1 \leftarrow \cdots \leftarrow \alpha_{n-1}]_x,$$

$$\sum_{\alpha_{n-2} \leftarrow a} [\alpha_1 \leftarrow \cdots \leftarrow \alpha_{n-1} \leftarrow a]_x,$$

or

$$\sum_{\alpha_i \leftarrow a \leftarrow \alpha_{i+1}} [\alpha_1 \leftarrow \cdots \leftarrow \alpha_i \leftarrow a \leftarrow \alpha_{i+1} \leftarrow \cdots \leftarrow a_{n-1}]_x$$

for some $i \in \{1, \ldots, n-2\}$. Using an observation made in the proof of Lemma 3.6, we see that Ψ_{Γ_+} evaluated at any of the above elements is zero in R_Γ. $\square$

The following theorem is an improvement of 3.2.1 of [20]. It is also an improvement of Theorem 3.7. Unlike Theorem 3.7, we need not assume R_Γ is Koszul.

Theorem 4.28. $\Psi_{\Gamma_+} : S^\bullet(\Gamma_+) \to R_\Gamma(\bullet, 0)$ is a co-chain isomorpshim.

Proof. Let $[1]_x \in \tilde{H}^{-1}(\Delta((*,x)))$. Then

$$\Psi_{\Gamma_+}(d_{S^0(\Gamma_+))}([1]_x) = \Psi_{\Gamma_+}\left(\bigoplus_{x\leftarrow y}[x]_y\right) = \left(\sum_{x\leftarrow y} r_y\right) r_x$$

and

$$d_\Gamma(\Psi_{\Gamma_+}([1]_x) = d_\Gamma(x) = \left(\sum_{z\in\Gamma_+} r_z\right) r_x = \left(\sum_{x\leftarrow y} r_y\right) r_x.$$

Assume $n > 0$. Let $[x_1 \leftarrow \cdots \leftarrow x_n]_x \in \tilde{H}^{n-1}(\Delta((*,x)))$. We compute

$$\begin{aligned}\Psi_{\Gamma_+}(d_{S^n(\Gamma_+))}([x_1 \leftarrow \cdots \leftarrow x_n]_x)) &= \Psi_{\Gamma_+}\left(\bigoplus_{x\leftarrow y}(-1)^n[x_1 \leftarrow \cdots \leftarrow x_n \leftarrow x]_y\right) \\ &= (-1)^n\left(\sum_{y\to x} r_y\right) r_x r_{x_n}\cdots r_{x_1}\end{aligned}$$

and

$$\begin{aligned}d_\Gamma(\Psi_{\Gamma_+}([x_1 \leftarrow \cdots \leftarrow x_n]_x)) &= d_\Gamma(r_x r_{x_n}\cdots r_{x_1}) \\ &= \left(\sum_{y\to x} r_y\right) r_x r_{x_n}\cdots r_{x_1}.\end{aligned}$$

We conclude that Ψ_Γ is a chain map.

It is clear that Ψ_{Γ_+} is an epimorphism. It remains to show that Ψ_{Γ_+} is injective. We will proceed by induction on the rank of Γ. If Γ has rank one ($m = 0$), then

$$dim_{\mathbb{F}} R_\Gamma(0,0) = \|\Gamma_+\| = \sum_{x\in\Gamma_+} dim_{\mathbb{F}}\tilde{H}^{-1}(\Delta((*,x))) = dim_{\mathbb{F}} S^0(\Gamma)$$

which proves the base case.

We now assume the theorem is true for all posets of rank m. It suffices to prove the result for cyclic posets of rank $m+1$, since

$$R_\Gamma(m,0) = \bigoplus_{rk_\Gamma(z)=m+1} R_{\Gamma_z}(m,0).$$

Thus, we set $\Gamma = \Gamma_x$ and note $S^m(\Gamma_+) = \tilde{H}^{m-1}(\Delta((*,x)))$. The poset $\Gamma_{<x}$ has rank m and thus $\Psi_{(\Gamma_{<x})_+} : S^\bullet((\Gamma_{<x})_+) \to R_{\Gamma_{<x}}(\bullet,0)$ is a co-chain isomorphism. It is apparent that $S^n(\Gamma_+) = S^n((\Gamma_{<x})_+)$ and $R_\Gamma(n,0) = R_{\Gamma_{<x}}(n,0)$ for $0 \le n \le m-1$. The maps $d_{S^n(\Gamma_+)}$ and $d_{S^n((\Gamma_{<x})_+)}$ coincide for $0 \le n \le m-2$. Similarly, the maps d_Γ and $d_{\Gamma_{<x}}$ coincide for $0 \le n \le m-2$. Hence, to prove Ψ_{Γ_+} is injective, it suffices to prove $\tilde{H}^{m-1}(\Delta((*,x))) \simeq R_{\Gamma_x}(m,0)$.

We have a commutative diagram

$$\begin{tikzcd}
\displaystyle\bigoplus_{rk_\Gamma(y)=m} \tilde{H}^{m-2}(\Delta((*,y))) \arrow[r, "d_{S^{m-1}(\Gamma_+)}"] \arrow[d, "\Psi_{(\Gamma_{<x})_+}"'] & \tilde{H}^{m-1}(\Delta((*,x))) \arrow[d, "\Psi_{\Gamma_+}"] \\
\displaystyle\bigoplus_{rk_\Gamma(y)=m} R_{\Gamma_y}(m-1,0) \arrow[r, "d_\Gamma"'] & R_{\Gamma_x}(m,0)
\end{tikzcd}$$

By the inductive hypothesis, $\Psi_{(\Gamma_{<x})_+}$ is injective. We will use the notation $\alpha = (\alpha_1, \cdots, \alpha_m)$ for chains in $C^{m-1}(\Delta((*,x)))$. Suppose

$$\sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha [\alpha_1 \leftarrow \cdots \leftarrow \alpha_m]_x \in \tilde{H}^{m-1}(\Delta((*,x))) = S^m(\Gamma_+)$$

is in the kernel of Ψ_{Γ_+}. Thus

$$\sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha r_x r_{\alpha_m} \cdots r_{\alpha_1} = r_x \left(\sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha r_{\alpha_m} \cdots r_{\alpha_1} \right) = 0.$$

We set $\Omega = (\Gamma_{<x})'$. Thus, by definition of R_Γ,

$$\sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha r_{\alpha_m} \cdots r_{\alpha_1} = r_x(1) \sum_{\beta \in C^{m-2}(\Delta(\Omega_+))} q_\beta r_{\beta_{m-1}} \cdots r_{\beta_1}$$

where $\beta = (\beta_1, \ldots, \beta_{m-1})$. We then observe

$$\begin{aligned}
&\Psi_{(\Gamma_{<x})_+}\left(\sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha [\alpha_1 \leftarrow \cdots \leftarrow \alpha_{m-1}]_{\alpha_m}\right) \\
&= \sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha r_{\alpha_m} \cdots r_{\alpha_1} \\
&= r_x(1) \sum_{\beta \in C^{m-2}(\Delta(\Omega_+))} q_\beta r_{\beta_{m-1}} \cdots r_{\beta_1} \\
&= \Psi_{(\Gamma_{<x})_+}\left(\sum_{\beta \in C^{m-2}(\Delta(\Omega_+))} \left(\bigoplus_{\beta_{m-1} \leftarrow y} q_\beta [\beta_1 \leftarrow \cdots \leftarrow \beta_{m-1}]_y\right)\right).
\end{aligned}$$

The map $\Psi_{(\Gamma_{<x})_+}$ is an isomorphism and therefore

$$\begin{aligned}
&\sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha [\alpha_1 \leftarrow \cdots \leftarrow \alpha_{m-1}]_{\alpha_m} \\
&= \sum_{\beta \in C^{m-2}(\Delta(\Omega_+))} \left(\bigoplus_{\beta_{m-1} \leftarrow y} q_\beta [\beta_1 \leftarrow \cdots \leftarrow \beta_{m-1}]_y\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&d_{S^{m-1}(\Gamma_+)}\left(\sum_{\beta \in C^{m-2}(\Delta(\Omega_+))} \left(\bigoplus_{\beta_{m-1} \leftarrow y} [\beta_1 \leftarrow \cdots \leftarrow \beta_{m-1}]_y\right)\right) \\
&= \sum_{\alpha \in C^{m-1}(\Delta((*,x)))} q_\alpha [\alpha_1 \leftarrow \cdots \leftarrow \alpha_m]_x.
\end{aligned}$$

And finally,

$$d_{S^{m-1}(\Gamma_+)}\left(\sum_{\beta\in C^{m-2}(\Delta(\Omega_+))}\left(\bigoplus_{\beta_{m-1}\leftarrow y}[\beta_1\leftarrow\cdots\leftarrow\beta_{m-1}]_y\right)\right)=[0]_x$$

in $\tilde{H}^{m-1}(\Delta((*,x)))$. We conclude $\Psi_{\Gamma_+}:\tilde{H}^{m-1}(\Delta((*,x)))\to R_{\Gamma_x}(m,0)$ is an isomorphism. This completes our proof. □

Remark 4.29. Let Γ be a finite ranked poset. Suppose Γ_x is Cohen-Macaulay for all maximal rank x in Γ. Using the remark at the end of Section 4.2 and Corollary 4.22, we see that the cohomology of $S^\bullet(\Gamma_+)$ is the cohomology of $C^\bullet(\Delta(\Gamma_+))=C^\bullet(Y)$.

We end this section with several corollaries. We recall the definition of $\Gamma^{>i}$ from Section 3.2

Corollary 4.30. Let $0\le i<m$. Then $\Psi_{(\Gamma^{>i})_+}:S^\bullet((\Gamma^{>i})_+)\to R_\Gamma(\bullet,i)$ is a co-chain isomorpshim.

The following corollary is Corollary 3.11, without the uniform hypothesis.

Corollary 4.31. Let Γ be a finite ranked poset and $*\neq v\in\Gamma$ an element of rank $d+1$. Then for any $0\le k\le d-1$,

$$\dim(r_vR_\Gamma(d-1,k))=\dim(\tilde{H}^{d-k-1}(\Delta(\Gamma_v^{>k}\setminus\{*,v\}))).$$

Recall the definition of $\Gamma_{a,i}$ from Section 3.2 The following corollary is (3.1.1) from [20], without the uniform hypothesis.

Corollary 4.32. Let Γ be a finite ranked poset. Then:

$$H(R_\Gamma,-t)=1+\sum_{i\ge1}\sum_{\substack{a\in\Gamma\\ rk_\Gamma(a)\ge i}}(-1)^{i-2}\dim\tilde{H}^{i-2}(\Delta(\Gamma_{a,i}))t^i$$

Finally, we recall the notation $\Gamma(S,k)$ from Definition 4.15.

Corollary 4.33. Let $0 \le k < n \le m+1$ and $S \subset \Gamma(n)$. Then $\Psi_{\Gamma(S,k)} : S^{\bullet}(\Gamma(S,k)) \to R_{\Gamma_S}(\bullet + n - k - 1, n - k - 1)$ is a co-chain isomorphism.

4.4. The Second Main Theorem

Suppose Γ is cyclic. We remind the reader of several facts. By Theorem 3.25, Γ is Cohen-Macaulay if for all $x \in \Gamma_+$ and all $rk_\Gamma(x) \ge l > n$, $\tilde{H}^{n-2}(\Delta(\Gamma_{x,l})) = 0$. Also, from Theorem 3.13, Γ is Cohen-Macaulay if and only if Γ is uniform and R_Γ is Koszul. Recall again the notation $\Gamma(S,k)$ from Definition 4.15 and note that the dimension of $\Delta(\Gamma(S,k))$ is k.

Definition 4.34. Let Γ be a finite ranked cyclic poset. Then Γ is weakly Cohen-Macaulay if for all $x \in \Gamma_+$, $0 \le k < n \le rk_\Gamma(x) - 1$ and $S \in M_n(\Gamma_x \setminus \{x\})$,

$$H^{k-1}(S^{\bullet}(\Gamma(S,k))) = 0.$$

Remark 4.35. Every finite ranked cyclic poset of rank three (or less) is weakly Cohen-Macaulay.

The following fact shows that weakly Cohen-Macaulay is a slightly weaker condition than Cohen-Macaulay.

Proposition 4.36. Let Γ be a finite ranked cyclic poset. Then Γ is Cohen-Macaulay if and only Γ is uniform and weakly Cohen-Macaulay.

Proof. In proving both directions, Γ is uniform. Thus, if $x \in \Gamma_+$, Γ_x is uniform. By Remark 4.18, $M_n(\Gamma_x \setminus \{x\}) = \{\Gamma_x(n)\}$ for all $1 \le n \le rk_\Gamma(x) - 1$.

For the forward direction, assume Γ is Cohen-Macaulay and let $x \in \Gamma_+$ and $0 \leq k < n \leq rk_\Gamma(x) - 1$. Set $S = \Gamma_x(n)$. Then, by Corollary 4.33, $H^{k-1}(S^\bullet(\Gamma(S,k)))$ is isomorphic to $H^{k-1}(R_{\Gamma_S}(\bullet + n - k - 1, n - k - 1), d_{\Gamma_S})$. This is isomorphic to $H^{k-1}(R_{\Gamma_x}(\bullet + n - k - 1, n - k - 1), d_{\Gamma_x})$. Since Γ_x is Cohen-Macaulay, R_{Γ_x} is Koszul. Therefore, the above cohomology is zero, as desired.

For the reverse direction, we will proceed by induction on rank of Γ and show that R_Γ is Koszul. If Γ has rank one, there is nothing to prove. We assume Γ has rank $m+1$ with $m > 0$. Set $\Gamma = \Gamma_z$. By Lemma 3.2, it remains to show $H^n(R_\Gamma(\bullet, k)) = 0$ for all $0 \leq k < n \leq m - 2$. Let $S = \Gamma(n+2)$. By Corollary 4.33, $H^n(R_\Gamma(\bullet, k))$ is isomorphic to $H^{n-k}(S^\bullet(\Gamma(S, n-k+1))) = 0$ and this completes our proof. □

Theorem 4.37. Let Γ be a finite ranked cyclic poset. Then Γ is weakly Cohen-Macaulay if and only if R_Γ is Koszul.

Proof. We will proceed by induction on the rank, $m+1$, of Γ. The theorem is true for all Γ of rank one. Assume $m > 0$ and that the theorem is true for all cyclic posets of rank less than or equal to m. The theorem follows from Theorem 4.19 and Corollary 4.33. □

Corollary 4.38. Let Γ be a finite ranked poset. Then for all elements x in Γ of maximal rank, Γ_x is weakly Cohen-Macaulay if and only if R_Γ is Koszul.

4.5. A Few Examples and Remarks

Example 4.39. The poset Θ^* from Example 4.1 is weakly Cohen-Macaulay. By inspection, $S^0({\Theta^*}_+) = \mathbb{F}[1]_u + \mathbb{F}[1]_t$, $S^1({\Theta^*}_+) = \mathbb{F}[u]_q$, and $S^2({\Theta^*}_+) = S^3({\Theta^*}_+) = 0$. Also, the map $d_{S^0({\Theta^*}_+)} : S^0({\Theta^*}_+) \to S^1({\Theta^*}_+)$ is surjective. We remark that the dual of a weakly Cohen-Macaulay cyclic poset need not be weakly Cohen-Macaulay.

Example 4.40. Let Γ and Ω be finite ranked posets with equal rank. Recall the notation $\Gamma \vee_{(v,v')} \Omega$ from Definition 3.31. If Γ and Ω are weakly Cohen-Macaulay, then so is $\overline{\Gamma \vee_{(v,v')} \Omega}$. In particular, if Γ and Ω are Cohen-Macaulay, then $\overline{\Gamma \vee_{(v,v')} \Omega}$ is weakly Cohen-Macaulay.

Example 4.41. The class of weakly Cohen-Macaulay posets includes many non-Cohen-Macaulay lattices. For example the lattice Λ, given in Figure 4.3, is weakly Cohen-Macaulay but not Cohen-Macaulay.

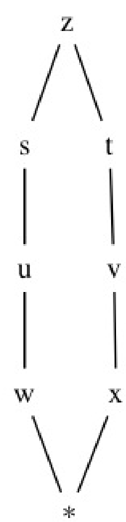

FIGURE 4.3. The poset Λ.

We remark that not all lattices are weakly Cohen-Macaulay; see Example 3.22.

Remark 4.42. Again let Γ and Ω be finite ranked posets with equal rank and unique minimal elements $*_\Gamma$ and $*_\Omega$. We define

$$\Gamma \vee \Omega = \Gamma \cup \Omega / (*_\Gamma \sim *_\Omega).$$

We denote the unique minimal element of $\Gamma \vee \Omega$ by $*$. If Γ and Ω are weakly Cohen-Macaulay, then so is $\overline{\Gamma \vee \Omega}$. In particular, if Γ and Ω are Cohen-Macaulay, then $\overline{\Gamma \vee \Omega}$ is weakly Cohen-Macaulay. Figure 4.3 above gives an example of this construction.

We can give a more general construction of weakly Cohen-Macaulay posets.

Remark 4.43. Suppose Γ is pure and Γ_x is Cohen-Macaulay for all maximal x in Γ. Then $\overline{\Gamma}$ is weakly Cohen-Macaulay if and only if for all $S \in M_{rk(\Gamma)}(\Gamma)$, $\overline{\Gamma_S}$ is Cohen-Macaulay.

REFERENCES CITED

[1] Alexander Beilinson, Victor Ginzburg and Wolfgang Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996), no. 2, 473–527.

[2] Roland Berger, *Koszulity for nonquadratic algebras*, J. Algebra **239** (2001), no. 2, 705–734.

[3] Anders Björner, Adriano M. Garsia, and Richard P. Stanley, *An introduction to Cohen-Macaulay partially ordered sets*, Ordered sets (Banff, Alta., 1981), NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., vol. 83, Reidel, Dordrecht, 1982, 583–615.

[4] Thomas Cassidy, Christopher Phan, and Brad Shelton, *Noncommutative Koszul algebras from combinatorial topology*, J. Reine Angew. Math. **646** (2010), 45–63.

[5] Thomas Cassidy, Brad Shelton, *Generalizing the notion of Koszul algebra*, Math. Z. **260** (2008), no. 1, 93–114.

[6] John A. Eagon and Victor Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, J. Pure Appl. Algebra **130** (1998), 265–275.

[7] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, Shirlei Serconek, and Robert Lee Wilson, *Hilbert series of quadratic algebras associated with pseudo-roots of noncommutative polynomials*, J. Algebra **254** (2002), no. 2, 279–299.

[8] Israel Gelfand, Vladimir Retakh, Shirlei Serconek, and Robert Lee Wilson, *On a class of algebras associated to directed graphs*, Selecta Math. (N.S.) **11** (2005), no. 2, 281–295.

[9] Estanislao Herscovich, *On the multi-Koszul property for connected algebras*, Documenta Math. **18** (2013), 1301–1347.

[10] David Nacin, *The minimal non-Koszul* $A(\Gamma)$, http://arxiv.org/pdf/1204.1534v2.pdf.

[11] Peter Orlik and HiroakiTerao, *Arrangements of Hyperplanes*, vol. 300, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992.

[12] Dmitri Piontkovski, *Noncommutative Koszul filtrations*, http://arxiv.org/pdf/math/0301233v1.pdf.

[13] Alexander Polishchuk and Leonid Positselski, *Quadratic algebras*, University Lecture Series, vol. 37, American Mathematical Society, Providence, RI, 2005.

[14] Leonid Positselski, *Relation between the Hilbert series of dual quadratic algebras does not imply Koszulity*, Functional Anal. and its Appl. **29** (1995), no. 3, 83–87.

[15] Stefan Papadima and Sergey Yuzvinsky, On rational $K[\pi, 1]$ spaces and Koszul algebras, J. Pure Appl. Algebra **144** (1999), 157–167.

[16] Patrick Polo, *On Cohen-Macaulay posets, Koszul algebras and certain modules associated to Schubert varieties*, Bull. London Math. Soc. **27** (1995), no. 5, 425–434.

[17] Stewart Priddy, Koszul resolutions, Trans. Amer. math. Soc. **152** (1970), 39-60.

[18] Gerald Allen Reisner, *Cohen-Macaulay quotients of polynomial rings*, Advances in Math. **21** (1976), no. 1, 30–49.

[19] Vladimir Retakh, Shirlei Serconek, and Robert Lee Wilson, *On a class of Koszul algebras associated to directed graphs*, J. Algebra, **304** (2006), no. 2, 1114?1129.

[20] ———, *Hilbert series of algebras associated to directed graphs and order homology*, J. Pure Appl. Algebra **216** (2012), no. 6, 1397–1409.

[21] ———, *Hilbert series of algebras associated to directed graphs*, J. Algebra **312** (2007), no. 1, 142–151.

[22] ———, *Koszulity of splitting algebras associated with cell complexes*, J. Algebra **323** (2010), no. 4, 983–999.

[23] Jan-Erik Roos, *On the characterization of Koszul algebras. Four counter-examples* Comptes Rendus Acad. Sci. Paris ser. **321** (1995), no. 1, 15–20.

[24] Brad Shelton and Sergey Yuzvinsky, *Koszul algebras from graphs and hyperplane arrangements*, J. London Math. Soc. **56** (1997), no. 2, 477-490.

[25] Hal Sadofsky and Brad Shelton, *The Koszul property as a topological invariant and measure of singularities*, Pacific J. Math. **252** (2011), no. 2, 473–486.

[26] Richard P. Stanley, *Cohen-Macaulay rings and constructible polytopes*, Bull. Amer. Math. Soc. **81** (1975), 133–135.

[27] Richard P. Stanley, *Enumerative Combinatorics, Volume I*, Cambridge University Press, New York, NY, 2012.

[28] D. Woodcock, *Cohen-Macaulay complexes and Koszul rings*, J. London Math. Soc. (2) **57** (1998), no. 2, 398–410.

[29] Charles A. Weibel, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.