COMBINATORICS OF THE DOUBLE-DIMER MODEL

by

HELEN K. JENNE

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2020

DISSERTATION APPROVAL PAGE

Student: Helen K. Jenne

Title: Combinatorics of the Double-Dimer Model

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Benjamin Young	Chair
Shabnam Akhtari	Core Member
Ben Elias	Core Member
Nicholas Proudfoot	Core Member
David Sutherland	Institutional Representative

and

Kate Mondloch	Interim Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2020

© 2020 Helen K. Jenne

DISSERTATION ABSTRACT

Helen K. Jenne

Doctor of Philosophy

Mathematics

June 2020

Title: Combinatorics of the Double-Dimer Model

We prove that the partition function for tripartite double-dimer configurations
of a planar bipartite graph satisfies a recurrence related to the Desnanot-Jacobi
identity from linear algebra. A similar identity for the dimer partition function
was established nearly 20 years ago by Kuo. This work was motivated in part
by the potential for applications, including a problem in Donaldson-Thomas
and Pandharipande-Thomas theory, which we will discuss. The proof of our
recurrence requires generalizing work of Kenyon and Wilson; specifically, lifting
their assumption that the nodes of the graph be black and odd or white and even.

CURRICULUM VITAE

NAME OF AUTHOR: Helen K. Jenne

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

    University of Oregon, Eugene, OR
    Whitman College, Walla Walla, WA

DEGREES AWARDED:

    Doctor of Philosophy, Mathematics, 2020, University of Oregon
    Master of Science, Mathematics, 2017, University of Oregon
    Bachelor of Arts, Mathematics and Psychology, 2013, Whitman College

AREAS OF SPECIAL INTEREST:

    Algebraic combinatorics

PROFESSIONAL EXPERIENCE:

    Graduate Employee, University of Oregon, 2014-2020

    Ph.D. Intern, Pacific Northwest National Laboratory, Summer 2015, 2017, 2018, 2019

GRANTS, AWARDS AND HONORS:

    Dissertation Research Fellowship, University of Oregon College of Arts and Sciences, 2019

PUBLICATIONS:

    H. Jenne. Combinatorics of the double-dimer model. (Submitted). *arXiv preprint, arXiv:1911.04079*, 2019.

M. Jelić Milutinović, H. Jenne, A. McDonough, and J. Vega. Matching complexes of trees and applications of the matching tree algorithm. (Submitted). *arXiv preprint, arXiv:1905.10560v2*, 2019.

B. Balof and H. Jenne. Tilings, continued fractions, derangements, scramblings, and *e*. *Journal of Integer Sequences*, 17(2): Article 14.2.7, 2014.

ACKNOWLEDGEMENTS

First and foremost, I thank my advisor, Ben Young, without whom this dissertation would not have been possible. Thank you for your support, patience, and invaluable assistance wading through signs. Your enthusiasm and optimism is contagious, and working with you has been the best part of graduate school. I would also like to thank Gregg Musiker for many useful conversations and Richard Kenyon for helpful comments on this document.

I am also grateful to Emilie Purvine, my mentor at Pacific Northwest National Laboratory, who has been an excellent role model and has helped me develop into the mathematician I am today.

I am deeply appreciative of the support I received from my friends over the last six years. In particular, I would like to thank Siobhan McAlister and Kyle Raze for making Eugene home. I would also like to thank Kira Goldner and Samantha Petti for their friendship, encouragement, and inspiration. And to Keegan Boyle and Sarah Frei, who I relied on tremendously throughout graduate school: I could not have done this without you.

Finally, I thank my family, especially my parents Ralph and Polly, my brother Ben, my aunt Amy, and my aunt Eva, for their love and support.

For my grandmother, Janet Eiden.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

In this dissertation, we prove that under certain conditions the partition function for *double-dimer configurations* satisfies a recurrence related to the Desnanot-Jacobi identity from linear algebra.

In this chapter, we motivate this work. We begin with a discussion of the dimer model, which is the study of dimer configurations. Double-dimer configurations and dimer configurations are related in the sense that a double-dimer configuration is the union of two dimer configurations. The recurrence we prove in this dissertation has a dimer version which was established nearly 20 years ago and motivated the search for our recurrence. The remainder of this chapter is devoted to a discussion of applications of our recurrence.

1.1. The Dimer Model and Kuo Condensation

Let $G = (V_1, V_2, E)$ be a finite bipartite planar graph embedded in the plane with a weight function $w : E \to \mathbb{R}_{>0}$ on edges. A *dimer configuration* (or *perfect matching*) of G is a collection of edges of G that covers each vertex exactly once. A dimer configuration of a grid graph is shown in Figure 1.1.

The *dimer model* is the study of Gibbs measures on the set of a dimer configurations of a graph. The dimer model arose in the 1960's in statistical mechanics [9, 27] and separately in combinatorics through the theory of plane partitions and related objects [1]. Since then many connections to geometry, string theory, random matrix theory, and other areas have been found [11, 20, 21, 23].
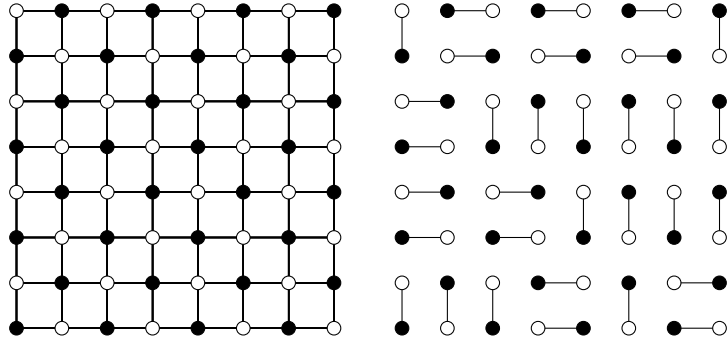
FIGURE 1.1. An 8×8 grid graph G (left) and a dimer configuration of G (right).

We define a probability measure μ on dimer configurations as follows: for a dimer configuration M,

$$\mu(M) = \frac{1}{Z^D(G)} \prod_{e \in M} w(e)$$

where $w(M) := \prod_{e \in M} w(e)$ is referred to as the *weight* of a dimer configuration. The normalization constant $Z^D(G)$ is called the *partition function*, defined to be

$$Z^D(G) = \sum_M w(M),$$

where the sum is over all dimer configurations M of G.

Kasteleyn discovered techniques that made it possible to compute $Z^D(G)$ for large classes of graphs. He defined a matrix K with the property that $|\det(K)| = Z^D(G)$ (see Section 3.2.1). In the case where G has all edge weights equal to 1, $|\det(K)|$ is the number of dimer configurations of G.

The partition function $Z^D(G)$ satisfies an elegant recurrence.

Theorem 1.1.1. *[14, Theorem 5.1] Let $G = (V_1, V_2, E)$ be a planar bipartite graph with a given planar embedding in which $|V_1| = |V_2|$. Let vertices $a, b, c,$ and d appear*

in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G-\{a,b,c,d\}) = Z^D(G-\{a,b\})Z^D(G-\{c,d\}) + Z^D(G-\{a,d\})Z^D(G-\{b,c\}).$$

Kuo proved Theorem 1.1.1 bijectively. There have since been a number of non-bijective proofs. For example, several authors have pointed out that Theorem 1.1.1 follows from certain Pfaffian identities; see for example [7]. In [26], Speyer shows that the six terms in Theorem 1.1.1 are the Plücker coordinates of a point in the totally nonnegative Grassmannian $Gr(2,4)$. Kuo's theorem then follows from a Plücker relation. The connection between dimers and the totally nonnegative Grassmannian is detailed in [6, 18].

Theorem 1.1.1 can also be proven using the *Desnanot-Jacobi identity*, which is also called *Dodgson condensation*.

Theorem 1.1.2 (Desnanot-Jacobi identity)**.** *Let $M = (m_{i,j})_{i,j=1}^n$ be a square matrix, and for each $1 \leq i, j \leq n$, let M_i^j be the matrix that results from M by deleting the ith row and the jth column. Then*

$$\det(M)\det(M_{i,j}^{i,j}) = \det(M_i^i)\det(M_j^j) - \det(M_i^j)\det(M_j^i)$$

To prove Theorem 1.1.1 using the Desnanot-Jacobi identity, let M be the Kasteleyn matrix K of G. The details of this proof are included in the Appendix.

Kuo's proof technique can be considered as a combinatorial interpretation of Dodgson condensation[1], and for this reason it is called *graphical condensation* or *Kuo condensation.*

---

[1] Prior to Kuo's work, Zeilberger gave a bijective proof of Dodgson condensation by interpreting each term in $\det(M) = \sum_{\pi \in S_n} \text{sign}(\pi) \prod_{i=1}^n m_{i,\pi(i)}$ as a matching of n men with n women [30].

Kuo's work has a variety of applications. For example, Kuo uses graphical condensation to give a new proof of the Aztec diamond theorem due to Elkies Kuperberg, Larsen, and Propp [4] and a new proof for MacMahon's generating function for plane partitions that are subsets of a box [19].

There are also applications of Kuo's work to the theory of cluster algebras. Cluster algebras are a class of commutative rings introduced by Fomin and Zelevinsky [5] to study total positivity and dual canonical bases in Lie Theory. The theory of cluster algebras has since been connected to many areas of math, including quiver representations, Teichmüller theory, Poisson geometry, and integrable systems [29]. In [16, 17], Tri Lai and Gregg Musiker study toric cluster variables for the quiver associated to the cone over the del Pezzo surface dP_3, giving algebraic formulas for these cluster variables as Laurent polynomials. Using identities similar to Kuo's Theorem 1.1.1, they give combinatorial interpretations of most of these formulas [16].

The main result of this dissertation is an analogue of Kuo's theorem for *double-dimer configurations.* This work was motivated in part by the potential for similar applications to Kuo's. For example, one application to the theory of cluster algebras is providing combinatorial interpretations of toric cluster variables for the dP_3 quiver in the case where the dimer model is not sufficient [16, Problem 9.1]. In addition, by using both Kuo condensation and the double-dimer analogue of Kuo condensation, we can give a direct proof of a problem in Donaldson-Thomas and Pandharipande-Thomas theory.

1.2. Application to Donaldson-Thomas and Pandharipande-Thomas Theory

Donaldson-Thomas (DT) theory, Pandharipande-Thomas (PT), and Gromov-Witten (GW) theory are branches of enumerative geometry closely related to mirror symmetry and string theory. The DT and GW theories give frameworks for counting curves[2] on a threefold X. One of the conjectures in [20, 21] gives a correspondence between the DT and GW frameworks, which has been proven in special cases, such as when X is toric [22].

PT theory gives a third framework for counting curves when X is a nonsingular projective threefold that is Calabi-Yau. The correspondence between the DT and PT frameworks was first conjectured in [25] and was proven in [2], which is closely related to the work in [28]. Specifically, let X be a toric Calabi-Yau threefold. Define $Z_{DT}(q) = \sum_n I_n q^n$, where I_n counts length n subschemes of X, and $Z_{PT}(q) = \sum_n P_n q^n$, where P_n counts stable pairs on X (see [25]). Bridgeland proved that these generating functions coincide up to a factor of $M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}$, which is the total q-weight of all plane partitions [19].

Theorem 1.2.1. *[2, Theorem 1.1]* $Z_{DT}(q) = Z_{PT}(q)M(-q)$.

The application of Theorem 2.1.1 that we describe relates to Theorem 1.2.1 at the level of the topological vertex. Define $V_{\lambda,\mu,\nu} = q^{c(\lambda,\mu,\nu)} \sum_{\pi} q^{|\pi|}$, where the sum is taken over all plane partitions π *asymptotic to* (λ, μ, ν). Maulik, Nekrasov, Okounkov, and Pandharipande [20, 21] proved that $Z_{DT}(q) = V_{\lambda,\mu,\nu}$ and thus $V_{\lambda,\mu,\nu}$ is called the DT topological vertex. Let $W_{\lambda,\mu,\nu} = q^{c(\lambda,\mu,\nu)} \sum_i d_i q^i$ where d_i is a certain weighted enumeration of *labeled box configurations* of length i [24]. In [24,

---

[2]The frameworks differ in what is meant by a *curve* on X.

Theorem/Conjecture 2] Pandharipande and Thomas conjecture that $W_{\lambda,\mu,\nu}$ is the stable pairs vertex, i.e. that $Z_{PT}(q) = W_{\lambda,\mu,\nu}$.

Conjecture 1.2.2. *[24, Calabi-Yau case of Conjecture 4] $V_{\lambda,\mu,\nu} = W_{\lambda,\mu,\nu}M(-q)$.*

Pandharipande and Thomas remark that a straightforward (but long) approach to this conjecture using DT theory exists [24]. In a forthcoming paper with Gautam Webb and Ben Young [8], we give a direct proof by showing that $V_{\lambda,\mu,\nu}$ is a single dimer model and $W_{\lambda,\mu,\nu}$ is a double-dimer model, and then using Kuo condensation (Theorem 1.1.1) and double-dimer condensation to show that both sides of the above equation satisfy the same recurrence.

1.3. Outline

This dissertation is organized as follows. In Chapter II, we give a precise statement of double-dimer condensation, in addition to collecting relevant background that will be used in later chapters. In this chapter we also give an outline of our proof, including statements of the main results from Chapters III and IV. In Chapter III, we generalize the combinatorial results of [12, 13] that are needed for the proof of double-dimer condensation. We conclude in Chapter IV by proving our condensation theorem.

CHAPTER II

BACKGROUND

In this chapter, we outline the proof of our main result: an analogue of Kuo condensation for the partition function for tripartite double-dimer configurations. We begin in Section 2.1 by defining double-dimer configurations and giving a precise statement of our condensation theorem (Theorem 2.1.1). Then, we review relevant background from Kenyon and Wilson's study of the double-dimer model (Section 2.2) and groves (Section 2.3). We conclude in Section 2.4 with a discussion of the main ideas behind the proof of Theorem 2.1.1 and an outline of Chapters III and IV.

2.1. The Double-Dimer Model and Double-Dimer Condensation

Throughout this dissertation, $G = (V_1, V_2, E)$ is a finite edge-weighted bipartite planar graph embedded in the plane with $|V_1| = |V_2|$. G has a set $\mathbf{N}$ of special vertices called *nodes* on its outer face numbered consecutively in counterclockwise order. A *double-dimer configuration* on $(G, \mathbf{N})$ is a multiset of the edges of G with the property that each vertex in $V \setminus \mathbf{N}$ is the endpoint of exactly two edges, and each vertex in $\mathbf{N}$ is the endpoint of exactly one edge. In other words, it is a configuration of disjoint cycles of length greater than two (called *loops*), doubled edges, and paths connecting the nodes in pairs. Consequently, each double-dimer configuration is associated with a planar pairing of the nodes. For example, in Figure 2.1, the pairing of the nodes is $((1, 8), (3, 4), (5, 2), (7, 6))$.

A double-dimer configuration can be decomposed into two dimer configurations in 2^ℓ ways, where ℓ is the number of loops in the configuration;
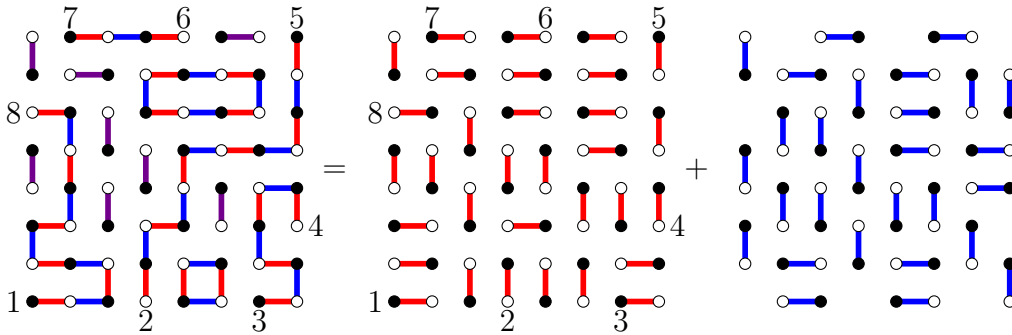
FIGURE 2.1. A double-dimer configuration is the union of two dimer configurations: a dimer configuration (shown in red) of the graph with a subset of the nodes determined by the black/white coloring and a dimer configuration (shown in blue) of the graph with the remaining nodes.

see Figure 2.1. Define a probability measure Pr where the probability of a configuration is proportional to the product of its edge weights times 2^ℓ.

Kenyon and Wilson initiated the study of the double-dimer model in [13], when they showed how to compute the probability that a random double-dimer configuration has a particular planar node pairing. In particular, when σ is a *tripartite pairing*, $\Pr(\sigma)$ is proportional to the determinant of a matrix [12].

A planar pairing σ is a tripartite pairing if the nodes can be divided into three circularly contiguous sets R, G, and B so that no node is paired with a node in the same set (see Figure 2.2). We visualize this by coloring the nodes in the sets red, green, and blue, in which case σ is the unique planar pairing in which like colors are not paired.

We prove that when σ is a tripartite pairing, a similar recurrence to Kuo condensation (Theorem 1.1.1) holds for $Z_\sigma^{DD}(G, \mathbf{N})$, the weighted sum of all double-dimer configurations on $(G, \mathbf{N})$ with pairing σ.

Theorem 2.1.1. *Let $G = (V_1, V_2, E)$ be a finite edge-weighted planar bipartite graph with a set of nodes $\mathbf{N}$. Divide the nodes into three circularly contiguous sets*

8
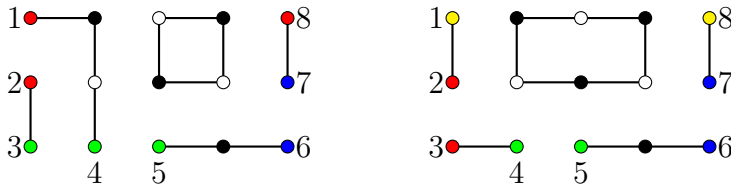
FIGURE 2.2. Two double-dimer configurations on a grid graph. The pairing of the nodes on the left is a tripartite pairing because the nodes can be colored contiguously using three colors so that no pair contains nodes of the same color. The pairing on the right is not a tripartite pairing because four colors are required.

R, G, and B such that $|R|$, $|G|$ and $|B|$ satisfy the triangle inequality and let σ be the corresponding tripartite pairing[1]. Let x, y, w, v be nodes appearing in a cyclic order such that the set $\{x, y, w, v\}$ contains at least one node of each RGB color[2]. If $x, w \in V_1$ and $y, v \in V_2$ then

$$Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) = Z_{\sigma_{xy}}^{DD}(G, \mathbf{N} - \{x, y\})Z_{\sigma_{wv}}^{DD}(G, \mathbf{N} - \{w, v\})$$
$$+ Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\})Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\})$$

where for $i, j \in \{x, y, w, v\}$, σ_{ij} is the unique planar pairing on $\mathbf{N} - \{i, j\}$ in which like colors are not paired together.

We illustrate Theorem 2.1.1 with an example.

Example 2.1.2. If G is a graph with eight nodes colored red, green and blue as shown in Figure 2.3, then $\sigma = ((1, 8), (3, 4), (5, 2), (7, 6))$. If $x = 8, y = 1, w = 2, v =$

---

[1]If $|R|, |G|$, and $|B|$ do not satisfy the triangle inequality, there is no corresponding tripartite pairing σ.

[2]The nodes of G have two colors: the black-white coloring from the bipartite assumption, and the RGB coloring. The coloring we are referring to is often clear from context, but we will sometimes write RGB color to emphasize that we are referring to the red, green, blue coloring of the nodes rather than the black-white coloring.

5, then by Theorem 2.1.1,

$$Z_\sigma^{DD}(G, \mathbf{N}) Z_{\sigma_{1,2,5,8}}^{DD}(G, \mathbf{N} - \{1, 2, 5, 8\}) = Z_{\sigma_{1,8}}^{DD}(G, \mathbf{N} - \{1, 8\}) Z_{\sigma_{2,5}}^{DD}(G, \mathbf{N} - \{2, 5\})$$
$$+ Z_{\sigma_{1,2}}^{DD}(G, \mathbf{N} - \{1, 2\}) Z_{\sigma_{5,8}}^{DD}(G, \mathbf{N} - \{5, 8\})$$

Figure 2.3 gives examples of the double-dimer configurations counted by the partition functions in this equation.
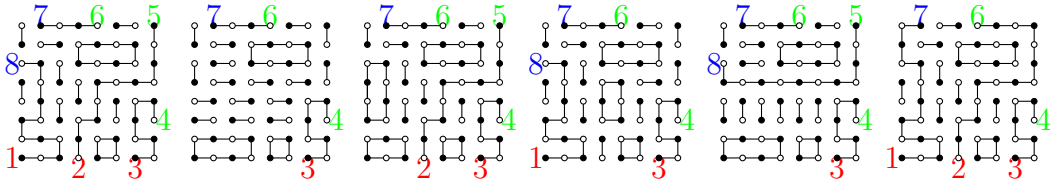
FIGURE 2.3. From left to right: The unique tripartite pairings on the node sets $\mathbf{N}$, $\mathbf{N} - \{1, 2, 5, 8\}$, $\mathbf{N} - \{1, 8\}$, $\mathbf{N} - \{2, 5\}$, $\mathbf{N} - \{1, 2\}$, and $\mathbf{N} - \{5, 8\}$ from Example 2.1.2.

Theorem 2.1.1 is a corollary of a more general theorem, Theorem 2.4.3, in which x, y, w, and z can be any subset of four nodes with an equal number of black and white nodes. The proof of Theorem 2.4.3 is not bijective, like Kuo's proof of Theorem 1.1.1. Instead, we use the Desnanot-Jacobi identity with a matrix that is a generalization of Kenyon and Wilson's matrix for tripartite double-dimer configurations [12].

Before discussing the main ideas behind the proof of Theorem 2.1.1, we give an overview of Kenyon and Wilson's results from [12, 13] that are needed for our work.

2.2. Double-Dimer Pairing Probabilities

Kenyon and Wilson gave explicit formulas for the probability that a random double-dimer configuration has a particular node pairing σ. When σ is a tripartite pairing, this probability is proportional to the determinant of a matrix.

To be more precise, we need to introduce some notation and definitions. Since G is bipartite, we can color its vertices black and white so that each edge connects a black vertex to a white vertex. Let G^{BW} be the subgraph of G formed by deleting the nodes except for the ones that are black and odd or white and even. Define G^{WB} analogously, but with the roles of black and white reversed. Let $G_{i,j}^{BW}$ be the graph G^{BW} with nodes i and j included if and only if they were not included in G^{BW}. For convenience, Kenyon and Wilson assume the nodes alternate in color, so all nodes are black and odd or white and even. (If a graph G does not have this property, we can add edges of weight 1 to each node that has the wrong color to obtain a graph whose double-dimer configurations are in a one-to-one weight-preserving correspondence with double-dimer configurations of G.)

For each planar pairing σ, Kenyon and Wilson showed that the normalized probability

$$\widehat{\Pr}(\sigma) := \Pr(\sigma) \frac{Z^D(G^{WB})}{Z^D(G^{BW})} = \frac{Z_\sigma^{DD}(G, \mathbf{N})}{(Z^D(G^{BW}))^2}$$

that a random double-dimer configuration has pairing σ is an integer-coefficient homogeneous polynomial in the quantities $X_{i,j} := \dfrac{Z^D(G_{i,j}^{BW})}{Z^D(G^{BW})}$ [13, Theorem 1.3].

For example, the normalized probability $\widehat{\Pr}$ that a random double-dimer configuration on six nodes has pairing $((1,2),(3,4),(5,6))$ is

$$\widehat{\Pr} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} = X_{1,4}X_{2,5}X_{3,6} + X_{1,2}X_{3,4}X_{5,6} \tag{2.2.1}$$

11

and the probability that a random double-dimer configuration on eight nodes has the pairing $((1,8),(3,4),(5,2),(7,6))$ (see Figure 2.1) is

$$\widehat{\Pr}\begin{pmatrix}1\\8\end{vmatrix}\begin{matrix}3\\4\end{matrix}\begin{vmatrix}5\\2\end{vmatrix}\begin{matrix}7\\6\end{matrix}\end{pmatrix} = X_{1,8}X_{3,4}X_{5,2}X_{7,6} - X_{1,4}X_{3,8}X_{5,2}X_{7,6} + X_{1,6}X_{3,4}X_{5,8}X_{7,2}$$
$$-X_{1,8}X_{3,6}X_{5,2}X_{7,4} - X_{1,4}X_{3,6}X_{5,8}X_{7,2} + X_{1,6}X_{3,8}X_{5,2}X_{7,4}.$$

Kenyon and Wilson gave an explicit method for computing these polynomials: they proved that

$$\widehat{\Pr}(\sigma) = \sum_{\text{odd-even pairings } \tau} \mathcal{P}_{\sigma,\tau}^{(DD)} X_\tau', \tag{2.2.2}$$

where $X_\tau' = (-1)^{\#\text{ crosses of } \tau} \prod_{i \text{ odd}} X_{i,\tau(i)}$ and showed how to compute the columns of the matrix $\mathcal{P}^{(DD)}$ completely combinatorially [13, Theorem 1.4].

The τth column of $\mathcal{P}^{(DD)}$ is determined by writing the odd-even pairing τ as a linear combination of planar pairings. So if τ is planar, $\mathcal{P}_{\tau,\tau}^{(DD)} = 1$ and $\mathcal{P}_{\sigma,\tau}^{(DD)} = 0$ for all $\sigma \neq \tau$. If τ is nonplanar, we repeatedly apply a *transformation rule* [13, Rule 2] which is a generalization of the following rule for six nodes:

$$14|36|52 \to 14|32|56 + 12|36|52 + 16|34|52 - 12|34|56 - 16|32|54 \tag{2.2.3}$$

This rule tells us, for example, that when $\sigma = ((1,2),(3,4),(5,6))$ and $\tau = ((1,4),(3,6),(5,2))$, $\mathcal{P}_{\sigma,\tau}^{(DD)} = -1$.

When G has six nodes the matrix $\mathcal{P}^{(DD)}$ is

$$
\begin{array}{c}
\\
\begin{array}{cccccc}
\rotatebox{90}{$12\,|\,34\,|\,56$} & \rotatebox{90}{$12\,|\,36\,|\,54$} & \rotatebox{90}{$14\,|\,32\,|\,56$} & \rotatebox{90}{$16\,|\,32\,|\,54$} & \rotatebox{90}{$16\,|\,34\,|\,52$} & \rotatebox{90}{$14\,|\,36\,|\,52$}
\end{array}
\end{array}
$$

$$
\begin{array}{r}
12|34|56 \\
12|36|54 \\
14|32|56 \\
16|32|54 \\
16|34|52
\end{array}
\left(
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{array}
\right).
$$

The first row of the matrix tell us that

$$
\widehat{\mathrm{Pr}}\left(\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right) = X'\left(\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right) - X'\left(\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right)
$$

which agrees with equation $(2.2.1)$, since $((1,4),(3,6),(5,2))$ has three crossings.

Note that equation $(2.2.1)$ could also have been written as

$$
\widehat{\mathrm{Pr}}\left(\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}\right) = \det\begin{pmatrix} X_{1,2} & X_{1,4} & 0 \\ 0 & X_{3,4} & X_{3,6} \\ X_{5,2} & 0 & X_{5,6} \end{pmatrix}.
$$

In general, whenever σ is a tripartite pairing, $\widehat{\mathrm{Pr}}(\sigma)$ is a determinant of a matrix whose entries are $X_{i,j}$ or 0.

Theorem 2.2.1. *[12, Theorem 6.1] Suppose that the nodes are contiguously colored red, green, and blue (a color may occur zero times), and that σ is the (unique) planar pairing in which like colors are not paired together. Let $\sigma(i)$ denote the item that σ pairs with item i. We have*

$$
\widehat{Pr}(\sigma) = \det[1_{i,j \text{ colored differently }} X_{i,j}]_{j=\sigma(1),\sigma(3),\dots,\sigma(2n-1)}^{i=1,3,\dots,2n-1}.
$$

Kenyon and Wilson's discovery that the partition function for double-dimer configurations is proportional to polynomials in the quantities $X_{i,j}$ has an analogue for a different type of subgraph called a *grove*. In [13], Kenyon and Wilson develop their results for groves alongside their double-dimer results; many of the proofs are similar. It turns out that the grove polynomials and double-dimer polynomials are related as formal polynomials and this relationship is essential to the proof of Theorem 2.2.1.

2.3. Groves and Their Connection to the Double-Dimer Model

Definition 2.3.1. [13] If G is a finite edge-weighted planar graph embedded in the plane with a set of nodes, a *grove* is a spanning acyclic subgraph of G such that each component tree contains at least one node. The weight of a grove is the product of the weights of the edges it contains.

The connected components of a grove partition the nodes into a planar partition. If σ is a planar partition of $1, 2, \ldots, n$, let $\Pr(\sigma)$ be the probability that a random grove of G partitions the nodes according to σ. Kenyon and Wilson showed that $\overset{\ldots}{\Pr}(\sigma) := \dfrac{\Pr(\sigma)}{\Pr(1|2|\cdots|n)}$ is an integer-coefficient homogeneous polynomial in the variables $L_{i,j}$[3] [13, Theorem 1.2].

FIGURE 2.4. A grove of a grid graph with 8 nodes. The partition of the nodes is 18|245|3|67.

For example, the normalized probability $\overset{\ldots}{\Pr}(\sigma)$ that a random grove on four nodes partitions the nodes

---

[3]When G is viewed as a resistor network with conductances equal to the edge weights, $L_{i,j}$ is the current that would flow into node j if node i were set to one volt and all other nodes were set to zero volts [13, Appendix A].

according to 1|234 is

$$\dddot{\mathrm{Pr}}(\sigma) = L_{2,3}L_{3,4} + L_{2,3}L_{2,4} + L_{2,4}L_{3,4} + L_{1,3}L_{2,4}.$$

(See [13, Section 1.2]). The probability that a random grove on six nodes has partition 12|34|56 is

$$\begin{aligned}
\dddot{\mathrm{Pr}}(\sigma) &= L_{1,2}L_{3,4}L_{5,6} - L_{1,2}L_{3,5}L_{4,6} + L_{1,3}L_{2,5}L_{4,6} - L_{1,3}L_{2,4}L_{5,6} + L_{1,4}L_{2,6}L_{3,5} \\
&\quad - L_{1,4}L_{2,5}L_{3,6} + L_{1,5}L_{2,4}L_{3,6} - L_{1,5}L_{2,6}L_{3,4}.
\end{aligned}$$

Each monomial in the polynomial $\dddot{\mathrm{Pr}}(\sigma)$ is of the form $L_\tau = \sum_F \prod_{\{i,j\}\in F} L_{i,j}$. The sum is over spanning forests F of the complete graph K_n for which the trees of F span the parts of τ and the product is over edges $\{i,j\}$ of the forest F.

To compute these polynomials, Kenyon and Wilson define a matrix $\mathcal{P}^{(t)}$ with rows indexed by planar partitions and columns indexed by all partitions and show how to compute the columns of this matrix combinatorially. As in the double-dimer case, the τth column of $\mathcal{P}^{(t)}$ is computed by writing the partition τ as a linear combination of planar partitions. So if τ is planar, then $\mathcal{P}^{(t)}_{\tau,\tau} = 1$ and $\mathcal{P}^{(t)}_{\sigma,\tau} = 0$ for all $\sigma \neq \tau$. If τ is nonplanar, the rule is a generalization of the rule for four nodes:

$$13|24 \to 1|234 + 2|314 + 3|124 + 4|123 - 12|34 - 14|23 \tag{2.3.1}$$

This rule tells us, for example, that $P^{(t)}_{12|34,13|24} = -1$.

In general, if a partition is nonplanar, then there will exist nodes $a < b < c < d$ such that a and c belong to one part, and b and d belong to another part. In Kenyon and Wilson's transformation rule, 1, 2, 3, and 4 in equation (2.3.1)

15

are replaced with parts A, B, C and D, which contain the nodes a, b, c and d, respectively.

Rule 2.3.2. [13, Rule 1] Arbitrarily subdivide the part containing a and c into two sets A and C such that $a \in A$ and $c \in C$, and similarly subdivide the part containing b and d into $B \ni b$ and $D \ni d$. Let the remaining parts of the partition be denoted by "rest." Then the transformation rule is

$$AC|BD|\text{rest} \rightarrow A|BCD|\text{rest} + B|ACD|\text{rest} + C|ABD|\text{rest} + D|ABC|\text{rest}$$
$$-AB|CD|\text{rest} - AD|BC|\text{rest}.$$

Remark 2.3.3. If we arbitrarily subdivide the part containing a and c into two sets A and C such that $a \in A$ and $c \in C$, and similarly for the part containing b and d, it is possible to repeat Rule 2.3.2 indefinitely without ever obtaining a linear combination of planar partitions.

For example, consider the partition $1235|46$. One crossing is $a = 1$, $b = 4$, $c = 5$, $d = 6$. If we choose $A = \{1, 2, 3\}$, $B = \{4\}, C = \{5\}$, and $D = \{6\}$, then after applying Rule 2.3.2, all of the partitions are planar. But if we choose $A = \{1, 2\}$, $B = \{4\}$, $C = \{3, 5\}$, and $D = \{6\}$ then after applying Rule 2.3.2 we get

$$12|3456 + 4|12356 + 35|1246 + 6|12345 - 124|356 - 126|345$$

which includes nonplanar partitions. For example, the partition $124|356$ has crossing $a = 1$, $b = 3$, $c = 4$, $d = 6$. If we choose $A = \{1, 2\}$, $B = \{3, 5\}$, $C = \{4\}$, and $D = \{6\}$ then after applying Rule 2.3.2 to $124|356$ we get

$$12|3456 + 35|1246 + 4|12356 + 6|12345 - 1235|46 - 126|345.$$

16

So after applying Rule 2.3.2 twice, all partitions cancel except for the partition 1235|46, which is the partition we started with. We could continue this process indefinitely.

Remark 2.3.3 motivates the following modification of Rule 2.3.2.

Rule 2.3.4. Subdivide the part containing a and c into two sets A and C such that A contains all the items in this part less than b, and C contains all other items. Similarly, subdivide the part containing b and d into two sets B and D so that B contains all items in this part less than c, and D contains all other items. Then the transformation rule is

$$AC|BD|\text{rest} \quad \to \quad A|BCD|\text{rest} + B|ACD|\text{rest} + C|ABD|\text{rest} + D|ABC|\text{rest}$$
$$-AB|CD|\text{rest} - AD|BC|\text{rest}.$$

Applying Rule 2.3.4 repeatedly will result in a linear combination of planar partitions.

We have now presented all the definitions needed to state Kenyon and Wilson's main result for groves.

Theorem 2.3.5. *[13, Theorem 1.2] Any partition τ may be transformed into a formal linear combination of planar partitions by repeated application of Rule 2.3.2, and the resulting linear combination does not depend on the choices made when applying Rule 2.3.2, so that we may write*

$$\tau \to \sum_{\text{planar partitions } \sigma} \mathcal{P}^{(t)}_{\sigma,\tau}\sigma.$$

17

For any planar partition σ, the same coefficients $\mathcal{P}^{(t)}_{\sigma,\tau}$ satisfy the equation

$$\dddot{Pr}(\sigma) = \frac{Pr(\sigma)}{Pr(1|2|3|\cdots|n)} = \sum_{partitions\ \tau} \mathcal{P}^{(t)}_{\sigma,\tau} L_\tau$$

for bipartite edge-weighted planar graphs.

Just as the double-dimer polynomials $\widehat{Pr}(\sigma)$ can be expressed as determinants of a matrix whose entries are $X_{i,j}$ or 0 when σ is a tripartite pairing, the grove polynomials $\dddot{Pr}(\sigma)$ can be expressed as Pfaffians of a matrix whose entries are $L_{i,j}$ or 0 when σ is a partition that is a tripartite pairing.

Theorem 2.3.6. *[12, Theorem 3.1] Let σ be the tripartite pairing partition defined by circularly contiguous sets of nodes R, G, and B, where $|R|, |G|$, and $|B|$ satisfy the triangle inequality. Then*

$$\dddot{Pr}(\sigma) = Pf\begin{pmatrix} 0 & L_{R,G} & L_{R,B} \\ -L_{G,R} & 0 & L_{G,B} \\ -L_{B,R} & -L_{B,G} & 0 \end{pmatrix}.$$

Here $L_{R,G}$ is the submatrix of L whose rows are the red nodes and columns are the green nodes.

Kenyon and Wilson prove their determinant formula (Theorem 2.2.1) by combining Theorem 2.3.6 with the following theorem, which states that the double-dimer polynomials $\widehat{Pr}(\sigma)$ are a specialization of the grove polynomials $\dddot{Pr}(\sigma)$.

18

Theorem 2.3.7. *[13, Theorem 4.2] If a planar partition σ only contains pairs and we make the following substitutions to the grove partition polynomial $\dddot{Pr}(\sigma)$:*

$$L_{i,j} \to \begin{cases} 0, & \text{if } i \text{ and } j \text{ have the same parity,} \\ (-1)^{(|i-j|-1)/2} X_{i,j}, & \text{otherwise,} \end{cases}$$

then the result is $\text{sign}_{OE}(\sigma)$ times the double-dimer pairing polynomial $\widehat{Pr}(\sigma)$, when we interpret σ as a pairing.

The proof of Theorem 2.3.7 requires comparing the double-dimer transformation rule for computing $\widehat{Pr}(\sigma)$ to the grove transformation rule for computing $\dddot{Pr}(\sigma)$. Several observations in this proof are important for our work in Section 3.4, so we include it here.

Proof. [13, Proof of Theorem 4.2] By Rule 2.3.2, when we express a partition τ as a linear combination of planar partitions, any singleton parts of τ show up in each planar partition with nonzero coefficient. Therefore if σ is a planar partition that is a pairing and τ contains at least one singleton part, $\mathcal{P}_{\sigma,\tau}^{(t)} = 0$. Also observe that when we apply Rule 2.3.2 to a partition τ, each of the resulting partitions contains the same number of parts as τ. It follows that if σ is a pairing, and $\mathcal{P}_{\sigma,\tau}^{(t)} \neq 0$ for some partition τ, then τ is also pairing.

When we apply Rule 2.3.2 to a pairing τ, it becomes the following:

If τ is nonplanar, then there will exist items $a < b < c < d$ such that a and c are paired, and b and d are paired. Then

$$ac|bd|\text{rest} \quad \to \quad a|bcd|\text{rest} + b|acd|\text{rest} + c|abd|\text{rest} + d|abc|\text{rest}$$

$$-ab|cd|\text{rest} - ad|bc|\text{rest}.$$

19

Since σ is a pairing, only the last two partitions can contribute to $\mathcal{P}_{\sigma,\tau}^{(t)}$, so we get the following rule

$$ac|bd|\text{rest} \to -ab|cd|\text{rest} - ad|bc|\text{rest}$$

Transforming $14|36|52$ using this abbreviated rule and comparing the result to (2.2.3) completes the proof. $\square$

Now that we have collected the relevant definitions and results from [12, 13], we discuss our proof of Theorem 2.1.1.

2.4. Main Ideas for the Proof of Double-Dimer Condensation

Initially, it seems that Theorem 2.1.1 will follow immediately from combining Theorem 2.2.1 with the Desnanot-Jacobi identity (Theorem 1.1.2). However, we run into some technical obstacles, which we illustrate with an example.

2.4.1. Example

Suppose we wish to prove the equation from Example 2.1.2:

$$Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_{1,2,5,8}}^{DD}(G, \mathbf{N} - \{1, 2, 5, 8\}) = Z_{\sigma_{1,8}}^{DD}(G, \mathbf{N} - \{1, 8\})Z_{\sigma_{2,5}}^{DD}(G, \mathbf{N} - \{2, 5\})$$

$$+ Z_{\sigma_{1,2}}^{DD}(G, \mathbf{N} - \{1, 2\})Z_{\sigma_{5,8}}^{DD}(G, \mathbf{N} - \{5, 8\})$$

where recall that $\sigma = ((1, 8), (3, 4), (5, 2), (7, 6))$. Then the matrix M from Theorem 2.2.1 is

$$M = \begin{pmatrix} X_{1,8} & X_{1,4} & 0 & X_{1,6} \\ X_{3,8} & X_{3,4} & 0 & X_{3,6} \\ X_{5,8} & 0 & X_{5,2} & 0 \\ 0 & X_{7,4} & X_{7,2} & X_{7,6} \end{pmatrix}.$$

20

Since the first row and column of M correspond to nodes 1 and 8, respectively, and the third row and column correspond to nodes 5 and 2, we apply the Desnanot-Jacobi identity with $i = 1$ and $j = 3$:

$$\det(M)\det(M_{1,3}^{1,3}) = \det(M_1^1)\det(M_3^3) - \det(M_1^3)\det(M_3^1).$$

By Theorem 2.2.1,

$$\det(M) = \frac{Z_\sigma^{DD}(G, \mathbf{N})}{(Z^D(G^{BW}))^2}.$$

We also need to prove, for example, that

$$\det(M_3^3) = \frac{Z_{\sigma_{2,5}}^{DD}(G, \mathbf{N} - \{2, 5\})}{(Z^D(G^{BW}))^2} \tag{2.4.1}$$

where

$$M_3^3 = \begin{pmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{pmatrix}.$$

An example of a double-dimer configuration counted by $Z_{\sigma_{2,5}}^{DD}(G, \mathbf{N} - \{2, 5\})$ is shown in Figure 2.5.
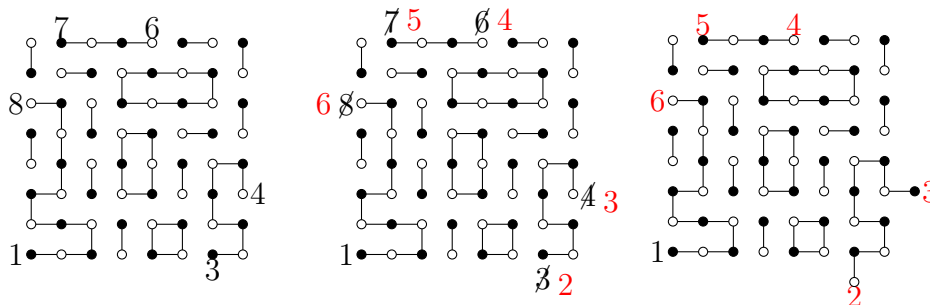
FIGURE 2.5. Shown left is a double-dimer configuration on a grid graph with node set $\mathbf{N} - \{2, 5\}$. We relabel the nodes (center) and add edges of weight 1 to nodes 2 and 3 (right) so that all nodes are black and odd or white and even.

We cannot apply Theorem 2.2.1 to M_3^3 because the nodes are not numbered consecutively. We might hope to resolve this by relabeling the nodes, as shown in Figure 2.5. But since Kenyon and Wilson assume that all nodes are black and odd or white and even, in order to satisfy the assumptions of Kenyon and Wilson's theorem, we need to add edges of weight 1 to nodes 2 and 3. Call the resulting graph $\widetilde{G}$ and let $\widetilde{X}_{i,j} = \dfrac{Z^D(\widetilde{G}_{i,j}^{BW})}{Z^D(\widetilde{G}^{BW})}$. Then by Theorem 2.2.1,

$$\frac{Z_{\sigma_{2,5}}^{DD}(\widetilde{G}, \mathbf{N} - \{2,5\})}{(Z^D(\widetilde{G}^{BW}))^2} = \det(\widetilde{M}),$$

where

$$\widetilde{M} = \begin{pmatrix} \widetilde{X}_{1,6} & 0 & \widetilde{X}_{1,4} \\ \widetilde{X}_{3,6} & \widetilde{X}_{3,2} & 0 \\ 0 & \widetilde{X}_{5,2} & \widetilde{X}_{5,4} \end{pmatrix}.$$

Therefore, to prove equation (2.4.1) it suffices to show

$$(Z^D(G^{BW}))^2 \det(M_3^3) = (Z^D(\widetilde{G}^{BW}))^2 \det(\widetilde{M}). \qquad (2.4.2)$$

Recall that the graph $G_{i,j}^{BW}$ includes the nodes i and j if and only if $G_{i,j}^{BW}$ does not. Since dimer configurations on G with node i removed are in a one-to-one correspondence with dimer configurations on G with an edge of weight 1 added to node i, it is easy to state the relationship between $Z^D(G_{i,j}^{BW})$ and $Z^D(\widetilde{G}^{BW})$. For example, $Z^D(G_{7,4}^{BW}) = Z^D(\widetilde{G}_{5,2}^{BW})$ and $Z^D(G_{1,8}^{BW}) = Z^D(\widetilde{G}^{BW} - \{1,2,3,6\})$ (see Figure 2.5). We will use this observation along with Kuo's Theorem 1.1.1 to verify equation (2.4.2).

Since the nodes of G and $\widetilde{G}$ are all black and odd or white and even, $G^{BW} = G$ and $\widetilde{G}^{BW} = \widetilde{G}$, so we omit the BW superscripts from the computations below.

22

Additionally, we abbreviate $Z^D(G)$ as $Z(G)$.

$$
\begin{aligned}
& Z(\widetilde{G})(Z(G))^3 \det(M_3^3) \\
=\ & Z(\widetilde{G})(Z(G))^3 \begin{vmatrix} X_{1,8} & X_{1,4} & X_{1,6} \\ X_{3,8} & X_{3,4} & X_{3,6} \\ 0 & X_{7,4} & X_{7,6} \end{vmatrix} \\
=\ & Z(\widetilde{G}) \begin{vmatrix} Z(G_{1,8}) & Z(G_{1,4}) & Z(G_{1,6}) \\ Z(G_{3,8}) & Z(G_{3,4}) & Z(G_{3,6}) \\ 0 & Z(G_{7,4}) & Z(G_{7,6}) \end{vmatrix} \\
=\ & Z(\widetilde{G}) \begin{vmatrix} Z(\widetilde{G}-\{1,2,3,6\}) & Z(\widetilde{G}-\{1,2\}) & Z(\widetilde{G}-\{1,2,3,4\}) \\ Z(\widetilde{G}-\{3,6\}) & Z(\widetilde{G}) & Z(\widetilde{G}-\{3,4\}) \\ 0 & Z(\widetilde{G}-\{2,5\}) & Z(\widetilde{G}-\{2,3,4,5\}) \end{vmatrix} \\
=\ & \begin{vmatrix} Z(\widetilde{G})Z(\widetilde{G}-\{1,2,3,6\}) & Z(\widetilde{G})Z(\widetilde{G}-\{1,2\}) & Z(\widetilde{G})Z(\widetilde{G}-\{1,2,3,4\}) \\ Z(\widetilde{G}-\{3,6\}) & Z(\widetilde{G}) & Z(\widetilde{G}-\{3,4\}) \\ 0 & Z(\widetilde{G}-\{2,5\}) & Z(\widetilde{G}-\{2,3,4,5\}) \end{vmatrix} \\
=\ & Z(\widetilde{G})Z(\widetilde{G}-\{1,2,3,6\}) \Big(Z(\widetilde{G})Z(\widetilde{G}-\{2,3,4,5\}) - Z(\widetilde{G}_{3,4})Z(\widetilde{G}_{5,2}) \Big) \\
& -Z(\widetilde{G}_{3,6}) \Big(Z(\widetilde{G}_{1,2})Z(\widetilde{G})Z(\widetilde{G}-\{2,3,4,5\}) - Z(\widetilde{G}_{5,2})Z(\widetilde{G})Z(\widetilde{G}-\{1,2,3,4\}) \Big) \\
=\ & \Big(Z(\widetilde{G}_{1,2})Z(\widetilde{G}_{3,6}) + Z(\widetilde{G}_{1,6})Z(\widetilde{G}_{3,2}) \Big) \Big(Z(\widetilde{G})Z(\widetilde{G}-\{2,3,4,5\}) - Z(\widetilde{G}_{3,4})Z(\widetilde{G}_{5,2}) \Big) \\
& -Z(\widetilde{G}_{3,6}) \Big(Z(\widetilde{G}_{1,2})Z(\widetilde{G})Z(\widetilde{G}-\{2,3,4,5\}) - Z(\widetilde{G}_{5,2})Z(\widetilde{G})Z(\widetilde{G}-\{1,2,3,4\}) \Big) \\
=\ & -Z(\widetilde{G}_{1,2})Z(\widetilde{G}_{3,6})Z(\widetilde{G}_{3,4})Z(\widetilde{G}_{5,2}) + Z(\widetilde{G}_{1,6})Z(\widetilde{G}_{3,2})Z(\widetilde{G})Z(\widetilde{G}-\{2,3,4,5\}) \\
& -Z(\widetilde{G}_{1,6})Z(\widetilde{G}_{3,2})Z(\widetilde{G}_{3,4})Z(\widetilde{G}_{5,2}) + Z(\widetilde{G}_{3,6})Z(\widetilde{G}_{5,2})Z(\widetilde{G})Z(\widetilde{G}-\{1,2,3,4\}) \\
=\ & -Z(\widetilde{G}_{1,2})Z(\widetilde{G}_{3,6})Z(\widetilde{G}_{3,4})Z(\widetilde{G}_{5,2}) - Z(\widetilde{G}_{1,6})Z(\widetilde{G}_{3,2})Z(\widetilde{G}_{3,4})Z(\widetilde{G}_{5,2}) \\
& +Z(\widetilde{G}_{1,6})Z(\widetilde{G}_{3,2}) \Big(Z(\widetilde{G}_{3,2})Z(\widetilde{G}_{5,4}) + Z(\widetilde{G}_{5,2})Z(\widetilde{G}_{3,4}) \Big) \\
& +Z(\widetilde{G}_{3,6})Z(\widetilde{G}_{5,2}) \Big(Z(\widetilde{G}_{1,2})Z(\widetilde{G}_{3,4}) + Z(\widetilde{G}_{1,4})Z(\widetilde{G}_{3,2}) \Big) \\
=\ & Z(\widetilde{G}_{1,6})Z(\widetilde{G}_{3,2})Z(\widetilde{G}_{3,2})Z(\widetilde{G}_{5,4}) + Z(\widetilde{G}_{3,6})Z(\widetilde{G}_{5,2})Z(\widetilde{G}_{1,4})Z(\widetilde{G}_{3,2}) \\
=\ & Z(\widetilde{G}_{3,2}) \begin{vmatrix} Z(\widetilde{G}_{1,6}) & 0 & Z(\widetilde{G}_{1,4}) \\ Z(\widetilde{G}_{3,6}) & Z(\widetilde{G}_{3,2}) & 0 \\ 0 & Z(\widetilde{G}_{5,2}) & Z(\widetilde{G}_{5,4}) \end{vmatrix} = Z(G)(Z(\widetilde{G}))^3 \begin{vmatrix} \widetilde{X}_{1,6} & 0 & \widetilde{X}_{1,4} \\ \widetilde{X}_{3,6} & \widetilde{X}_{3,2} & 0 \\ 0 & \widetilde{X}_{5,2} & \widetilde{X}_{5,4} \end{vmatrix}.
\end{aligned}
$$

23

And thus

$$Z^D(\widetilde{G})(Z^D(G))^3 \det(M_3^3) = Z^D(G)(Z^D(\widetilde{G}))^3 \det(\widetilde{M}),$$

which proves the claim.

When we consider graphs with more nodes, the computations are more involved.

To be able to interpret the minors of Kenyon and Wilson's matrix outside of small examples, we need to lift their assumption that the nodes of the graph be black and odd or white and even. Notice that under the assumption that the nodes of the graph are black and odd or white and even, $X_{i,j} = \dfrac{Z^D(G_{i,j}^{BW})}{Z^D(G^{BW})} = \dfrac{Z^D(G_{i,j})}{Z^D(G)}$. This suggests that the correct generalization of Kenyon and Wilson's matrix will have entries $\dfrac{Z^D(G_{i,j})}{Z^D(G)}$.

.

2.4.2. Outline

The previous remark motivates our approach, which is to define $Y_{i,j} := \dfrac{Z^D(G_{i,j})}{Z^D(G)}$ and $\widetilde{\Pr}(\sigma) = \dfrac{Z_\sigma^{DD}(G, \mathbf{N})}{(Z^D(G))^2}$. When G is a graph with nodes that are either black and odd or white and even, $Z^D(G) = Z^D(G^{BW})$, so $Y_{i,j} = X_{i,j}$ and $\widetilde{\Pr}(\sigma) = \widehat{\Pr}(\sigma)$.

Many of Kenyon and Wilson's results from [12, 13] have analogues in the variables $Y_{i,j}$. Following Kenyon and Wilson's approach, for any black-white pairing ρ, we define

$$Y_\rho' = (-1)^{\#\text{ crosses of } \rho} \prod_{i \text{ black}} Y_{i,\rho(i)}.$$

Note that we work with black-white pairings rather than odd-even pairings since we are not requiring that the nodes are black and odd or white and even. In [12, 13],

black-white pairings and odd-even pairings coincide, so $X_{i,j} = 0$ when i and j have the same parity, which occurs exactly when they are the same color. In our general setting, $Y_{i,j}$ may be nonzero when i and j have the same parity, but if i and j are the same color then there are no dimer configurations of $G_{i,j}$, so $Y_{i,j} = 0$.

Our analogue of Kenyon and Wilson's matrix $\mathcal{P}^{(DD)}$ (see equation (2.2.2)) is $\mathcal{Q}^{(DD)}$. The rows of $\mathcal{Q}^{(DD)}$ are indexed by planar pairings and the columns are indexed by black-white pairings. To prove that $\mathcal{Q}^{(DD)}$ is integer-valued, we show that the columns of this matrix can be computed combinatorially and in Chapter III we prove the following theorem:

Theorem 2.4.1. *Let G be a finite edge-weighted planar bipartite graph with a set of nodes. For any planar pairing σ,*

$$\widetilde{\Pr}(\sigma) = \sum_{\text{black-white pairings } \rho} \mathcal{Q}^{(DD)}_{\sigma,\rho} Y'_\rho,$$

where the coefficients $\mathcal{Q}^{(DD)}_{\sigma,\rho}$ are all integers.

To prove Theorem 2.4.1, we use Kenyon and Wilson [13] as a road map, proving analogues of Lemmas $3.1 - 3.5$ and Theorem 3.6 from [13]. Because we follow their work so closely, before presenting each of our lemmas we state the corresponding lemma from [13]. In some cases the proofs are very similar. In others, substantially more work is required.

In Chapter IV, we use our results from Chapter III to generalize Kenyon and Wilson's determinant formula (Theorem 2.2.1). Before stating our version of their formula, we observe that if we reorder the columns of the matrix from

Theorem 2.2.1 we get

$$\det[1_{i,j \text{ colored diff.}} X_{i,j}]_{j=\sigma(1),\ldots,\sigma(2n-1)}^{i=1,\ldots,2n-1} = \text{sign}_{OE}(\sigma)\det[1_{i,j \text{ colored diff.}} X_{i,j}]_{j=2,\ldots,2n}^{i=1,\ldots,2n-1}$$

where $\text{sign}_{OE}(\sigma)$ is the parity of the permutation $\left(\begin{array}{cccc} \frac{\sigma(1)}{2} & \frac{\sigma(2)}{2} & \cdots & \frac{\sigma(2n-1)}{2} \end{array}\right)$ written in one-line notation.

We prove that

Theorem 2.4.2. *Let G be a finite edge-weighted planar bipartite graph with a set of nodes. Suppose that the nodes are contiguously colored red, green, and blue (a color may occur zero times), and that σ is the (unique) planar pairing in which like colors are not paired together. We have*

$$\widetilde{\text{Pr}}(\sigma) = sign_{OE}(\sigma)\det[1_{i,j \text{ RGB-colored differently}} Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}.$$

where $b_1 < b_2 < \cdots < b_n$ are the black nodes and $w_1 < w_2 < \cdots < w_n$ are the white nodes.

By combining Theorem 2.4.2 with the Desnanot-Jacobi identity, we prove our main result:

Theorem 2.4.3. *Let $G = (V_1, V_2, E)$ be a finite edge-weighted planar bipartite graph with a set of nodes $\mathbf{N}$. Divide the nodes into three circularly contiguous sets R, G, and B such that $|R|, |G|$, and $|B|$ satisfy the triangle inequality and let σ be*

the corresponding tripartite pairing. If $x, w \in V_1$ and $y, v \in V_2$ then

$$
sign_{OE}(\sigma)sign_{OE}(\sigma'_{xywv})Z^{DD}_{\sigma}(G, \mathbf{N})Z^{DD}_{\sigma_{xywv}}(G, \mathbf{N} - \{x, y, w, v\})
$$
$$
= \quad sign_{OE}(\sigma'_{xy})sign_{OE}(\sigma'_{wv})Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\})Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\})
$$
$$
- sign_{OE}(\sigma'_{xv})sign_{OE}(\sigma'_{wy})Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\})Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\})
$$

where for $i, j \in \{x, y, w, v\}$, σ_{ij} is the unique planar pairing on $\mathbf{N} - \{i, j\}$ in which like colors are not paired together, and σ'_{ij} is the pairing after the the node set $\mathbf{N} - \{i, j\}$ has been relabeled so that the nodes are numbered consecutively.

Theorem 2.1.1 follows as a corollary; the additional assumptions in Theorem 2.1.1 lead to a nice simplification of the signs in Theorem 2.4.3.

CHAPTER III

PROOF OF THEOREM 2.4.2.1

Recall that throughout this dissertation each graph is assumed to have a set of nodes $\mathbf{N}$ on its outer face numbered consecutively in counterclockwise order. Kenyon and Wilson [12, 13] assume that the nodes alternate in color so that the black nodes are odd and the white nodes are even. We allow the nodes to have any coloring, as long as $\mathbf{N}$ has an equal number of black and white nodes.

In this chapter we prove Theorem 2.4.1, which is an analogue of Theorem 1.4 from Kenyon and Wilson [13] in this more general setting. We begin with a sign lemma, which is our version of Lemma 3.4 from Kenyon and Wilson (Section 3.1). Then, we show that $\widetilde{\Pr}(\sigma)$ is a homogeneous polynomial in the quantities $Y_{i,j}$, which requires analogues of Lemmas 3.1-3.3 from Kenyon and Wilson (Section 3.2). In Section 3.3 we define the matrix $\mathcal{Q}^{(DD)}$ and in Section 3.4 we show it is integer valued, which concludes the proof of Theorem 2.4.1. Section 3.5 contains the proof of a technical lemma from Section 3.3.

3.1. Lemma 3.4 From Kenyon and Wilson

In this section, we prove an analogue of the following lemma from Kenyon and Wilson [13] for black-white pairings.

Lemma 3.1.1. *[13, Lemma 3.4] For odd-even pairings ρ,*

$$sign_{OE}(\rho) \prod_{(i,j)\in\rho} (-1)^{(|i-j|-1)/2} = (-1)^{\#\ crosses\ of\ \rho}.$$

A *cross* of a pairing ρ is a set of two pairs (a, c) and (b, d) of ρ such that $a < b < c < d$. Recall from Section 2.4 that the sign of an odd-even pairing $\rho = ((1, \rho(1)), (3, \rho(3)), \dots, (2n - 1, \rho(2n - 1)))$ is the parity of the permutation $\left(\frac{\rho(1)}{2} \quad \frac{\rho(2)}{2} \quad \cdots \quad \frac{\rho(2n-1)}{2} \right)$ written in one-line notation.

For our version of this lemma, we need to define the sign of a black-white pairing ρ, which we will denote $\text{sign}_{BW}(\rho)$.

Definition 3.1.2. If ρ is a black-white pairing, then we can write $\rho = ((b_1, w_1), (b_2, w_2), \dots, (b_n, w_n))$, where $b_1 < b_2 < \cdots < b_n$. Let $r : \{w_1, \dots, w_n\} \to \{1, \dots, n\}$ be the map defined by $r(k) = \#\{i : w_i \leq w_k\}$. Then the sign of ρ, denoted $\text{sign}_{BW}(\rho)$, is the parity of the permutation $\sigma_\rho = \left(r(w_1) \quad r(w_2) \quad \cdots \quad r(w_n) \right)$ written in one-line notation.

When ρ is a pairing that is both black-white and odd-even, these signs agree.

Lemma 3.1.3. *If ρ is a black-white pairing that is also odd-even, then $sign_{OE}(\rho) = sign_{BW}(\rho)$.*

The proof of Lemma 3.1.3 is straightforward, but it is postponed to Section 3.1.3 for clarity of exposition.

Lemma 3.1.1 contains factors $(-1)^{(|i-j|-1)/2}$ for each pair (i, j) of ρ. If (b, w) is a pair of a black-white pairing ρ that is not odd-even, it is not necessarily the case that $\frac{|b-w|-1}{2}$ is an integer. Therefore we need a different way to define the sign of a pair.

To motivate this definition, notice that if two nodes of the opposite color b and w have the same parity, it cannot be the case that the nodes between b and w alternate black and white. So we keep track of the number of consecutive nodes of the same color between b and w. Consecutive nodes of the same color appear in

29

pairs. For example, if we have a graph with eight nodes so that nodes $1, 3, 4$, and 6 are black and nodes $2, 5, 7$, and 8 are white, there are two pairs of consecutive nodes of the same color: $(3, 4)$ and $(7, 8)$. Since we frequently use the term pair when describing pairings of the nodes, we will refer to pairs of consecutive nodes as *couples of consecutive nodes* instead.

Definition 3.1.4. If (b, w) is a pair in a black-white pairing, let $a_{b,w}$ be the number of couples of consecutive nodes of the same color in the interval $[\min\{b, w\}, \min\{b, w\} + 1, \ldots, \max\{b, w\}]$.

Note that a triple of consecutive nodes that are all the same color contributes 2 to $a_{b,w}$.

Remark 3.1.5. If (b, w) is a pair in a black-white pairing, then $\dfrac{|b - w| + a_{b,w} - 1}{2}$ is an integer.

Proof. Let $(n_1, n_1 + 1), (n_2, n_2 + 1), \ldots, (n_{2k}, n_{2k} + 1)$ be a complete list of couples of consecutive nodes of the same color in $\mathbf{N}$ so that $n_1 < n_2 < \cdots < n_{2k}$, where it is possible that $n_{i+1} = n_i + 1$. Every time we reach a couple of consecutive nodes, the black nodes and white nodes switch parity. That is, if the black nodes in the interval $[n_\ell + 1, n_\ell + 2, \ldots, n_{\ell+1}]$ are odd, then the black nodes in the interval $[n_{\ell+1} + 1, n_{\ell+1} + 2, \ldots, n_{\ell+2}]$ are even. (Note that these intervals could be length 1). It follows that if b and w are the same parity, then there are an odd number of couples of consecutive nodes in the interval $[\min\{b, w\}, \min\{b, w\} + 1, \ldots, \max\{b, w\}]$. So in this case $\frac{|b-w|+a_{b,w}-1}{2}$ is an integer. If b and w are opposite parity, then there are an even number of couples of consecutive nodes in the interval $[\min\{b, w\}, \min\{b, w\} + 1, \ldots, \max\{b, w\}]$. So $\frac{|b-w|+a_{b,w}-1}{2}$ is an integer in this case as well. $\square$

30

Definition 3.1.6. If (b, w) is a pair in a black-white pairing, let

$$\text{sign}(b, w) = (-1)^{(|b-w|+a_{b,w}-1)/2}.$$

We observe that when the nodes of G alternate black and white, $a_{b,w} = 0$ for all pairs (b, w), so this definition of the sign of a pair agrees with Kenyon and Wilson's definition.

Remark 3.1.7. For the remainder of the paper, we use the following notation. We let

- $(n_1, n_1 + 1), (n_2, n_2 + 1), \ldots, (n_{2k}, n_{2k} + 1)$ be a complete list of couples of consecutive nodes of the same color so that $n_1 < \cdots < n_{2k}$,

- $(s_1, s_1 + 1), (s_2, s_2 + 1), \ldots, (s_k, s_k + 1)$ be a complete list of couples of consecutive black nodes so that $s_1 < \cdots < s_k$, and

- $(u_1, u_1 + 1), (u_2, u_2 + 1), \ldots, (u_k, u_k + 1)$ be a complete list of couples of consecutive white nodes so that $u_1 < \cdots < u_k$.

Note that we could have $n_{i+1} = n_i + 1$, $s_{i+1} = s_i + 1$, or $u_{i+1} = u_i + 1$.

Since we are allowing arbitrary node colorings, many of our results contain a global sign that depends on the order in which the couples of consecutive nodes appear. For example, suppose a node set $\mathbf{N}$ has two couples of consecutive nodes: a couple of consecutive black nodes $(s, s + 1)$ and a couple of consecutive white nodes $(u, u + 1)$. Then the global sign will be 1 if $u < s$ and -1 if $s < u$. To emphasize that this sign only depends on the relative ordering of the couples of consecutive nodes of the same color, we use the notation $\text{sign}_c(\mathbf{N})$.

Definition 3.1.8. Using the notation from Remark 3.1.7, if node 1 is black, define the map $\varphi : \{n_1, n_2, \ldots, n_{2k}\} \to \{1, 2, \ldots, 2k\}$ by

$$
\varphi(n_j) =
\begin{cases}
2i - 1 & \text{if } n_j = u_i \\
2i & \text{if } n_j = s_i
\end{cases}.
$$

Then the image of $\{n_1, n_2, \ldots, n_{2k}\}$ under the map φ can be considered as a permutation in one-line notation:

$$
\sigma_{\mathbf{N}} = \Big(\varphi(n_1) \quad \varphi(n_2) \quad \cdots \quad \varphi(n_{2k}) \Big).
$$

Define $\text{sign}_c(\mathbf{N})$ to be the sign of this permutation. Note that if $u_1 < s_1 < u_2 < s_2 < \cdots < u_k < s_k$ then $\sigma_{\mathbf{N}} = \begin{pmatrix} 1 & 2 & \cdots & 2k \end{pmatrix}$, so $\text{sign}_c(\mathbf{N}) = 1$.

If node 1 is white, define the map $\varphi : \{n_1, n_2, \ldots, n_{2k}\} \to \{1, 2, \ldots, 2k\}$ by

$$
\varphi(n_j) =
\begin{cases}
2i - 1 & \text{if } n_j = s_i \\
2i & \text{if } n_j = u_i
\end{cases}.
$$

As above, the image of $\{n_1, n_2, \ldots, n_{2k}\}$ under the map φ can be considered as a permutation in one-line notation and we define $\text{sign}_c(\mathbf{N})$ to be the sign of this permutation. Note that if $s_1 < u_1 < s_2 < u_2 < \cdots < s_k < u_k$, $\text{sign}_c(\mathbf{N}) = 1$.

In Definition 3.1.8, if node 1 is black, it is possible that $s_k = 2n$. Similarly, if node 1 is white, it is possible that $u_k = 2n$.

Definition 3.1.9. Since the image of $\{n_1, n_2, \ldots, n_{2k}\}$ under the map φ can be considered a permutation in one-line notation, we say that a pair (u_ℓ, s_m) is an

inversion with respect to the node coloring of $\mathbf{N}$ if $(\varphi(u_\ell), \varphi(s_m))$ is an inversion of $\sigma_\mathbf{N}$.

Example 3.1.10. Let $\mathbf{N}$ be a set of nodes where node 1 is black.

- If $\mathbf{N}$ has four couples of consecutive nodes of the same color with $u_1 < s_1 < s_2 < u_2$, then $\sigma_\mathbf{N} = \begin{pmatrix} 1 & 2 & 4 & 3 \end{pmatrix}$, so $\text{sign}_c(\mathbf{N}) = -1$. The pair (s_2, u_2) is an inversion with respect to the node coloring.

- If instead $s_1 < u_1 < s_2 < u_2$, then $\sigma_\mathbf{N} = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}$, so $\text{sign}_c(\mathbf{N}) = 1$. The pairs (s_1, u_1) and (s_2, u_2) are inversions.

Example 3.1.11. Let $\mathbf{N}$ be a set of nodes where node 1 is white. If $\mathbf{N}$ has six couples of consecutive nodes of the same color with $s_1 < s_2 < u_1 < u_2 < u_3 < s_3$, then $\sigma_\mathbf{N} = \begin{pmatrix} 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}$. The pairs (s_2, u_1) and (u_3, s_3) are inversions.

Remark 3.1.12. If node 1 is black, (u_ℓ, s_m) is an inversion with respect to the node coloring when $u_\ell < s_m$ and $\ell > m$. The pair (s_m, u_ℓ) is an inversion when $s_m < u_\ell$ and $m \geq \ell$. If node 1 is white, (u_ℓ, s_m) is an inversion with respect to the node coloring when $u_\ell < s_m$ and $\ell \geq m$. The pair (s_m, u_ℓ) is an inversion when $s_m < u_\ell$ and $m > \ell$.

We have now established the definitions needed to state our version of Kenyon and Wilson's lemma.

Lemma 3.1.13 (analogue of Lemma 3.4 from [13]). *If ρ is a black-white pairing,*

$$sign_c(\mathbf{N}) sign_{BW}(\rho) \prod_{(b,w) \in \rho} sign(b, w) = (-1)^{\# \ crosses \ of \ \rho}.$$

Before proving Lemma 3.1.13, we will prove the following:

Lemma 3.1.14. *There exists a planar black-white pairing ρ such that*

$$sign_{BW}(\rho) \prod_{(b,w)\in\rho} sign(b,w) = sign_c(\mathbf{N}).$$

3.1.1. Proof of Lemma 3.1.14

We will prove Lemma 3.1.14 by induction on k, where $\mathbf{N}$ has $2k$ couples of consecutive nodes of the same color. The following lemma is the base case $k = 1$.

Lemma 3.1.15 (Base case of Lemma 3.1.14). *For any node coloring such that there are exactly two couples of consecutive nodes of the same color, there is a planar black-white pairing ρ such that*

$$sign_{BW}(\rho) \prod_{(i,j)\in\rho} sign(i,j) = sign_c(\mathbf{N}).$$

Proof. Let $(n_1, n_1 + 1), (n_2, n_2 + 1)$ be the list of the couples of consecutive nodes of the same color so that $n_1 < n_2$. There are two cases to consider: Either n_1 and 1 are opposite colors, or n_1 and 1 are the same color.

If n_1 and 1 are opposite colors, the pairing $\rho = ((1,2),(3,4),\dots,(2n-1,2n))$ is black-white. To see this, note that since n_1 and 1 are opposite colors, n_1 is even, so the only pairs of adjacent nodes that are both the same color are of the form $(x, x+1)$, where x is even, or $(2n, 1)$. Since all pairs of ρ are of the form $(i, i+1)$ where i is odd and $i+1$ is even, ρ is a black-white pairing. Since $sign_{BW}(\rho) = 1$, $\prod_{(i,j)\in\rho} sign(i,j) = 1$ and $sign_c(\mathbf{N}) = 1$, the claim holds.

If n_1 and 1 are the same color, the pairing $\rho = ((2n,1),(2,3),\dots,(2n-2,2n-1))$ is black-white. The reasoning is analogous to the previous case: n_1 is odd, so the only pairs of adjacent nodes that are both the same color are of the

form $(x, x+1)$, where x is odd. In this case, $\text{sign}_{BW}(\rho) = (-1)^{n-1}$ and

$$\prod_{(i,j)\in\rho} \text{sign}(i,j) = (-1)^{(2n-1+a_{2n,1}-1)/2} = (-1)^{n-1}(-1)^{a_{2n,1}/2},$$

so $\text{sign}_{BW}(\rho) \prod_{(i,j)\in\rho} \text{sign}(i,j) = (-1)^{a_{2n,1}/2} = -1$. Since $\text{sign}_c(\mathbf{N}) = -1$, the claim holds. $\square$

Definition 3.1.16. Suppose ρ is a black-white pairing. Then recall that we can write $\rho = ((b_1, w_1), (b_2, w_2), \ldots, (b_n, w_n))$, where $b_1 < b_2 < \cdots < b_n$. We say that (w_i, w_j) is an inversion of ρ if $i < j$ and $w_i > w_j$. Note that (w_i, w_j) is an inversion of ρ if and only if $(r(w_i), r(w_j))$ is an inversion of σ_ρ (see Definition 3.1.2).

Remark 3.1.17. An inversion of a planar pairing ρ corresponds to a nesting in the diagram constructed by placing the nodes in order on a line and linking pairs in the upper half-plane (see Figure 3.1). This follows immediately from the four node case, where the only planar pairings are $((1, 2), (3, 4))$ and $((1, 4), (3, 2))$.
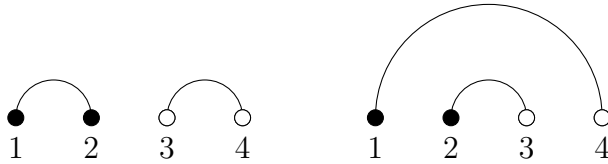
FIGURE 3.1. An inversion of a planar pairing ρ corresponds to a nesting. The pairing $((1, 2), (3, 4))$ has no inversions and its diagram has no nestings. The pairing $((1, 4), (3, 2))$ has one inversion and its diagram has one nesting.

Proof of Lemma 3.1.14. The proof of the lemma is long and technical, so we first identify a few easy cases.

Easy Case 1. If

    − node 1 is black and $u_1 < s_1 < u_2 < s_2 < \cdots < u_k < s_k$, or

– node 1 is white and $s_1 < u_1 < s_2 < u_2 < \cdots < s_k < u_k$,

then the pairing $((1,2),(3,4),\ldots,(2n-1,2n))$ is a black-white pairing and $\mathrm{sign}_c(\mathbf{N}) = 1$, so the claim holds.

Easy Case 2. If

– node 1 is black and $s_1 < u_1 < s_2 < u_2 < \cdots < s_k < u_k$, or

– node 1 is white and $u_1 < s_1 < u_2 < s_2 < \cdots < u_k < s_k$,

then the pairing $\rho = ((2n,1),(2,3),\ldots,(2n-2,2n-1))$ is black-white. In this case,

– $\mathrm{sign}_{BW}(\rho) = (-1)^{n-1}$, and

– $\displaystyle\prod_{(i,j)\in\rho} \mathrm{sign}(i,j) = (-1)^{(2n-1+a_{2n,1}-1)/2} = (-1)^{n-1}(-1)^{a_{2n,1}/2} = (-1)^{n-1}(-1)^k$,

so $\mathrm{sign}_{BW}(\rho) \displaystyle\prod_{(i,j)\in\rho} \mathrm{sign}(i,j) = (-1)^k = \mathrm{sign}_c(\mathbf{N})$.

General case. For the general case, we proceed by induction on the number of couples of consecutive nodes of the same color. The base case is when there are two couples of consecutive nodes of the same color, which is Lemma 3.1.15. Assume the claim holds when we have a set of nodes that has $2(k-1)$ couples of consecutive nodes of the same color and let $\mathbf{N}$ be a set of nodes with $2k$ couples of consecutive nodes of the same color.

Using the notation from Remark 3.1.7, let h be the smallest integer so that n_{h-1} and n_h are different colors. Then $\rho_1 = ((n_{h-1}+1, n_{h-1}+2), \ldots, (n_h-1, n_h))$ is a black-white pairing that contains at least one pair.

Throughout this proof, we will illustrate the main ideas with the example where $\mathbf{N}$ is a set of 12 nodes colored so that nodes 1, 3, 4, 5, 7, and 10 are black, as shown in Figure 3.2. In this example, the couples of consecutive nodes of the same
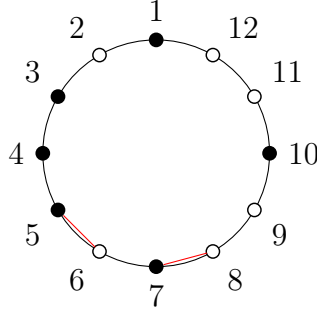
FIGURE 3.2. The pairing ρ_1 when the nodes are colored as shown.

color are $(3,4), (4,5), (8,9)$, and $(11,12)$. Since $n_1 = 3$ and $n_2 = 4$ are black and $n_3 = 8$ is white, $h = 3$. So the pairing ρ_1 is $((5,6),(7,8))$.

Consider $\mathbf{N}' = \{1, \ldots, |\mathbf{N}| - (n_h - n_{h-1})\}$. Define $\psi : \mathbf{N} - \{n_{h-1}+1, \ldots, n_h\} \rightarrow \mathbf{N}'$ by

$$\psi(\ell) = \begin{cases} \ell & \text{if } \ell \leq n_{h-1} \\ \ell - (n_h - n_{h-1}) & \text{if } \ell > n_h \end{cases} \tag{3.1.1}$$

That is, ψ defines a relabeling of the nodes of $\mathbf{N} - \{n_{h-1}+1, \ldots, n_h\}$ so that node 1 is labeled $1, \ldots$, node n_{h-1} is labeled n_{h-1}, node $n_h + 1$ is labeled $n_{h-1}+1, \ldots$, node $2n$ is labeled $2n - (n_h - n_{h-1})$. Since $\mathbf{N}'$ has $2k-2$ couples of consecutive nodes of the same color, by the induction hypothesis there is a black-white planar pairing ρ_2 of the nodes of $\mathbf{N}'$ such that

$$\text{sign}_{BW}(\rho_2) \prod_{(i,j)\in\rho_2} \text{sign}(i,j) = \text{sign}_c(\mathbf{N}').$$

Let $\psi^{-1}(\rho_2)$ denote the pairing that results from applying ψ^{-1} to each node in ρ_2. That is, $\psi^{-1}(\rho_2)$ is the pairing obtained by returning the nodes of ρ_2 to their original labels in $\mathbf{N}$. Let $\rho = \rho_1 \cup \psi^{-1}(\rho_2)$. Observe that ρ is a planar black-white pairing of $\mathbf{N}$.

37
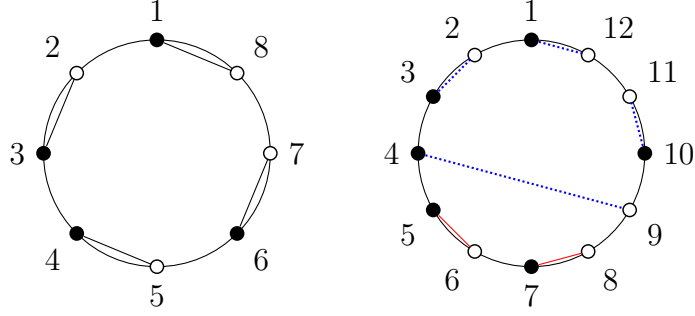
FIGURE 3.3. Left: The pairing ρ_2 of $\mathbf{N}'$ guaranteed by the induction hypothesis. Right: The pairing ρ of $\mathbf{N}$. The pairing ρ_1 is shown with solid red lines and the pairing $\psi^{-1}(\rho_2)$ is shown with dotted blue lines.

In our example, the map ψ defines a relabeling of $\mathbf{N} - \{5,6,7,8\}$ so that node 9 is labeled $5, \ldots,$ node 12 is labeled 8. The node set $\mathbf{N}'$ has two couples of consecutive pairs of the same color. By Lemma 3.1.15, the pairing ρ_2 is $((1,8),(3,2),(5,4),(7,6))$, so the pairing $\psi^{-1}(\rho_2)$ is $((1,12),(3,2),(9,4),(11,10))$ and thus $\rho = ((1,12),(3,2),(5,6),(7,8),(9,4),(11,10))$, as shown in Figure 3.3.

We will next

(1) Compare $\prod\limits_{(i,j)\in\rho_2} \text{sign}(i,j)$ to $\prod\limits_{(i,j)\in\rho} \text{sign}(i,j)$,

(2) Compare $\text{sign}_{BW}(\rho_2)$ to $\text{sign}_{BW}(\rho)$, and

(3) Compare $\text{sign}_c(\mathbf{N}')$ to $\text{sign}_c(\mathbf{N})$.

(1) Comparing $\prod\limits_{(i,j)\in\rho_2} \mathbf{sign}(i,j)$ to $\prod\limits_{(i,j)\in\rho} \mathbf{sign}(i,j)$.

If (i,j) is a pair in ρ that is a pair of ρ_1, then $\text{sign}(i,j) = 1$. If (i,j) is a pair in ρ that is a pair of $\psi^{-1}(\rho_2)$, then consider $(\psi(i),\psi(j))$ (the corresponding pair of ρ_2). If $i,j \leq n_{h-1}$ or $i,j \geq n_h + 1$, then $\text{sign}(i,j) = \text{sign}(\psi(i),\psi(j))$ because

$a_{i,j} = a_{\psi(i),\psi(j)}$. If $i \leq n_{h-1}$ and $j \geq n_h + 1$ then

$$\text{sign}(\psi(i), \psi(j)) = (-1)^{(\psi(j)-\psi(i)+a_{\psi(i),\psi(j)}-1)/2} = (-1)^{(j-(n_h-n_{h-1})-i+a_{i,j}-2-1)/2}$$

$$= (-1)^{(n_h-n_{h-1}+2)/2}\text{sign}(i, j)$$

so

$$\prod_{(i,j)\in\rho_2} \text{sign}(i, j) = \prod_{\substack{(i,j)\in\rho: \\ \min(i,j)\leq n_{h-1} \text{ and} \\ \max(i,j)\geq n_h+1}} (-1)^{(n_h-n_{h-1}+2)/2} \prod_{(i,j)\in\rho} \text{sign}(i, j).$$

In the example, there are two pairs (i, j) with $i \leq n_2$ and $j \geq n_3 + 1$: the pairs $(1, 12)$ and $(4, 9)$.

(2) Comparing $\text{sign}_{BW}(\rho_2)$ to $\text{sign}_{BW}(\rho)$.

Comparing $\text{sign}_{BW}(\rho_2)$ to $\text{sign}_{BW}(\rho)$ requires comparing the number of inversions of ρ to the number of inversions of ρ_2 (see Definition 3.1.16). Since ρ_1 contains only pairs of the form $(i, i + 1)$, ρ_1 contains no inversions. By Remark 3.1.17, inversions in ρ correspond to nestings in the corresponding diagram. Since there are $\frac{n_h-n_{h-1}}{2}$ pairs in ρ_1, ρ has $\frac{n_h-n_{h-1}}{2}$ additional inversions compared to ρ_2 for each pair (i, j) such that $\min(i, j) \leq n_{h-1}$ and $\max(i, j) \geq n_h + 1$. So,

$$\text{sign}_{BW}(\rho_2) = \text{sign}_{BW}(\rho) \prod_{\substack{(i,j)\in\rho: \\ \min(i,j)\leq n_{h-1} \text{ and} \\ \max(i,j)\geq n_h+1}} (-1)^{(n_h-n_{h-1})/2}$$

In the example, since there are two pairs (i, j) with $i \leq n_2$ and $j \geq n_3 + 1$ and the pairing ρ_1 consists of two pairs, there are four more inversions in ρ than in ρ_2.

(3) Comparing $\text{sign}_c(\mathbf{N}')$ to $\text{sign}_c(\mathbf{N})$. We will show that

$$\text{sign}_c(\mathbf{N}') = (-1)^{h-1}\text{sign}_c(\mathbf{N})$$

39

by comparing the number of inversions with respect to the node coloring of $\mathbf{N}$ to the number of inversions in with respect to the node coloring of $\mathbf{N}'$ (see Definition 3.1.9).

Recall the notation from Remark 3.1.7: s_i is the first in a couple of consecutive black nodes in $\mathbf{N}$ and u_i is the first in a couple of consecutive white nodes in $\mathbf{N}$. Define s_i' and u_i' analogously for $\mathbf{N}'$. Recall also that we have the map $\psi : \mathbf{N} - \{n_{h-1} + 1, \ldots, n_h\} \to \mathbf{N}'$ which defines a relabeling of the nodes of $\mathbf{N} - \{n_{h-1} + 1, \ldots, n_h\}$ (see equation (3.1.1)).

First assume node 1 is black and that we have

$$s_1 < \cdots < s_{h-1} < u_1 < \cdots .$$

Inversions with respect to the node coloring of $\mathbf{N}$. By Remark 3.1.12, there are two types of inversions with respect to the node coloring of $\mathbf{N}$.

(1) Nodes x and y in $\mathbf{N}$ such that $x < y$, $x = s_a$, $y = u_b$, and $a \geq b$.

(2) Nodes x and y in $\mathbf{N}$ such that $x < y$, $x = u_a$, $y = s_b$, and $a > b$.

Considering the first type of inversion, there are several cases:

(a) If $a \leq h-2$ and $b > 1$, then $\psi(x) = s_a'$ and $\psi(y) = u_{b-1}'$. Since $a \geq b$, $a \geq b-1$, so in this case there is a corresponding inversion in $\mathbf{N}'$.

(b) If $a \leq h - 2$ and $b = 1$, then $y \notin \mathbf{N} - \{n_{h-1} + 1, \ldots, n_h\}$, so in this case there is not a corresponding inversion in $\mathbf{N}'$.

(c) If $a = h - 1$ and $b \leq h - 1$, then $x \notin \mathbf{N} - \{n_{h-1} + 1, \ldots, n_h\}$, so in this case there is not a corresponding inversion in $\mathbf{N}'$.

(d) If $a > h - 1$ and $b \le a$, then $b > 1$ (since $u_1 < s_a$). In this case, $\psi(x) = s'_{a-1}$ and $\psi(y) = u'_{b-1}$, so there is a corresponding inversion in $\mathbf{N}'$.

Note that (b) gives $h - 2$ inversions in $\mathbf{N}$ that are not in $\mathbf{N}'$ and (c) gives $h - 1$ inversions in $\mathbf{N}$ that are not in $\mathbf{N}'$.

Considering the second type of inversion, since $s_{h-1} < u_1$ we must have $a > h$. In this case, $\psi(x) = u'_{a-1}$ and $\psi(y) = s'_{b-1}$, so there is a corresponding inversion in $\mathbf{N}'$.

In the example, the pairs $(s_1, u_1), (s_2, u_1)$, and (s_2, u_2) are inversions with respect to the node coloring of $\mathbf{N}$. Since $h = 3$, the inversion (s_1, u_1) is in case (b) of the first type and the inversions (s_2, u_1) and (s_2, u_2) are in case (c) of the first type. So in this example, all of the inversions with respect to the node coloring of $\mathbf{N}$ do not have corresponding inversions in $\mathbf{N}'$.

Inversions with respect to the node coloring of $\mathbf{N}'$. Similarly, there are two types of inversions in $\mathbf{N}'$.

(1) Nodes w and z in $\mathbf{N}'$ such that $w < z$, $w = s'_a$, $z = u'_b$, and $a \ge b$.

(2) Nodes w and z in $\mathbf{N}'$ such that $w < z$, $w = u'_a$, $z = s'_b$, and $a > b$.

Considering the first type of inversion, there are two cases:

(a) If $a \le h - 2$, then $\psi^{-1}(w) = s_a$ and $\psi^{-1}(z) = u_{b+1}$.

    (i) If $a \ge b + 1$ then there is a corresponding inversion in $\mathbf{N}$.

    (ii) If $a = b$ there is not a corresponding inversion in $\mathbf{N}$.

(b) If $a \ge h - 1$, then $\psi^{-1}(w) = s_{a+1}$ and $\psi^{-1}(z) = u_{b+1}$, so there is a corresponding inversion in $\mathbf{N}$.

41

We see that case (a)(ii) gives $h - 2$ inversions in $\mathbf{N}'$ that are not in $\mathbf{N}$.

Considering the second type of inversion, since $s'_{h-2} < u'_1$ the only possibility is that $a > h - 1$. In this case, $\psi^{-1}(w) = u_{a+1}$ and $\psi^{-1}(z) = s_{b+1}$, so there is a corresponding inversion in $\mathbf{N}$.

In the example, the only inversion with respect to the node coloring of $\mathbf{N}'$ is (s'_1, u'_1), which is an example of case (a)(ii), so there is not a corresponding inversion in $\mathbf{N}$.

We conclude that in the case where node 1 is black and we have $s_1 < \cdots < s_{h-1} < u_1 < \cdots$, the equation $\mathrm{sign}_c(\mathbf{N}') = (-1)^{h-1}\mathrm{sign}_c(\mathbf{N})$ holds.

Combining this with

$$- \ \mathrm{sign}_{BW}(\rho_2) \prod_{(i,j)\in\rho_2} \mathrm{sign}(i, j) = \mathrm{sign}_c(\mathbf{N}'),$$

$$- \prod_{(i,j)\in\rho_2} \mathrm{sign}(i, j) = \prod_{\substack{(i,j)\in\rho:\\ \min(i,j)\leq n_{h-1} \text{ and}\\ \max(i,j)\geq n_h+1}} (-1)^{(n_h-n_{h-1}+2)/2} \prod_{(i,j)\in\rho} \mathrm{sign}(i, j), \text{ and}$$

$$- \ \mathrm{sign}_{BW}(\rho_2) = \mathrm{sign}_{BW}(\rho) \prod_{\substack{(i,j)\in\rho:\\ \min(i,j)\leq n_{h-1} \text{ and}\\ \max(i,j)\geq n_h+1}} (-1)^{(n_h-n_{h-1})/2},$$

we have

$$\mathrm{sign}_{BW}(\rho) \prod_{(i,j)\in\rho} \mathrm{sign}(i, j) = \mathrm{sign}_c(\mathbf{N}) \cdot (-1)^{h-1} \cdot \prod_{\substack{(i,j)\in\rho:\\ \min(i,j)\leq n_{h-1} \text{ and}\\ \max(i,j)\geq n_h+1}} (-1).$$

So it remains to observe that the number of pairs $(i, j) \in \rho$ such that $\min(i, j) \leq s_{h-1}$ and $\max(i, j) \geq u_1 + 1$ has the same parity as $h - 1$. There are exactly $h - 1$ more black nodes than white nodes in the interval $[1, \ldots, s_{h-1}]$ because there are $h - 1$ black nodes that are not followed by a white node in this interval. So there are $h - 1$ black nodes that must all be paired with a white node

42

with label $\geq u_1 + 1$. Therefore there are at least $h - 1$ pairs $(i, j) \in \rho$ such that $\min(i, j) \leq s_{h-1}$ and $\max(i, j) \geq u_1 + 1$. There may be more than $h - 1$ such pairs, but there must be $h - 1 + 2m$ pairs for some $m \geq 0$.

There are three other cases: when node 1 is white and we have $s_1 < \cdots <$ $s_{h-1} < u_1 < \cdots$, when node 1 is black and we have $u_1 < \cdots < u_{h-1} < s_1 < \cdots$, and when node 1 is white and we have $u_1 < \cdots < u_{h-1} < s_1 < \cdots$. These are omitted because the analyses are nearly identical to the case we just considered.

$\square$

3.1.2. Proof of Lemma 3.1.13

Recall that we want to show that if ρ is a black-white pairing on a graph G with node set $\mathbf{N}$,

$$\operatorname{sign}_c(\mathbf{N})\operatorname{sign}_{BW}(\rho) \prod_{(b,w)\in\rho} \operatorname{sign}(b, w) = (-1)^{\# \text{ crosses of } \rho}. \tag{3.1.2}$$

By Lemma 3.1.14 there is a black-white planar pairing ρ such that

$$\operatorname{sign}_{BW}(\rho) \prod_{(i,j)\in\rho} \operatorname{sign}(i, j) = \operatorname{sign}_c(\mathbf{N}).$$

Since ρ is planar, $(-1)^{\#\text{crosses of }\rho} = 1$, so equation (3.1.2) holds.

To prove equation (3.1.2) holds for all black-white pairings we consider ways we can modify black-white pairings to obtain new black-white pairings and determine how these modifications affect equation (3.1.2).

Definition 3.1.18. Let σ be a (not necessarily black-white) pairing on $\{1, \ldots, 2n\}$, such that x is not paired with y. When we *swap the locations of x and y in σ*

we create a new pairing σ' that is identical to σ except that it contains the pairs $(x, \sigma(y))$ and $(y, \sigma(x))$ rather than $(x, \sigma(x))$ and $(y, \sigma(y))$.

Example 3.1.19. Suppose σ is the pairing $((1,3),(2,4),(5,6))$. If we swap the locations of 3 and 4 in σ we obtain the pairing $\sigma' = ((1,4),(2,3),(5,6))$.

Remark 3.1.20. If ρ is a black-white pairing and ρ' is obtained from ρ by swapping the locations of two nodes of the same color, $\text{sign}_{BW}(\rho') = -\text{sign}_{BW}(\rho)$.

Now we observe that we can obtain any black-white pairing on $\mathbf{N}$ from a given black-white pairing ρ using the following types of swaps:

(1) Swapping the locations of u and $u + 1$ in ρ, where $(u, u + 1)$ is a couple of consecutive white nodes.

(2) Swapping the locations of x and y in ρ, where $x < y$ are white nodes and all ℓ nodes appearing between x and y are black, where $\ell \geq 1$.

To see that these swaps are sufficient, let $w_1, \ldots, w_n$ be the white nodes in increasing order. The swaps described are the adjacent transpositions $(w_1, w_2), (w_2, w_3), \ldots, (w_{n-1}, w_n)$.

We will show that equation (3.1.2) holds after applying each type of swap. This requires a few additional lemmas. Note that the proofs of Lemmas 3.1.21 through 3.1.23 follow immediately from Definition 3.1.6.

Lemma 3.1.21. *Let b be a black node and let $(u, u + 1)$ be a couple of consecutive white nodes. Then $sign(b, u) = -sign(b, u + 1)$.*

Proof. If $b < u$, then $a_{b,u+1} = a_{b,u} + 1$. So

$$\text{sign}(b, u) = (-1)^{(u-b+a_{b,u}-1)/2} = -(-1)^{(u+1-b+a_{b,u+1}-1)/2} = -\text{sign}(b, u + 1).$$

If $b > u + 1$, then $a_{b,u+1} = a_{b,u} - 1$. So

$$\text{sign}(b, u) = (-1)^{(b-u+a_{b,u}-1)/2} = -(-1)^{(b-(u+1)+a_{b,u+1}-1)/2} = -\text{sign}(b, u + 1).$$

$\square$

Lemma 3.1.22. *Assume the nodes x and y with $x < y$ are white and all ℓ nodes between x and y are black, where $\ell \geq 1$. If b is a black node not in the interval $[x + 1, \ldots, y - 1]$, then $\text{sign}(b, x) = (-1)^\ell \text{sign}(b, y)$.*

Proof. If $b < x$, then $a_{b,y} = a_{b,x} + \ell - 1$. Then since $y = x + \ell + 1$,

$$
\begin{aligned}
\text{sign}(b, x) = (-1)^{(x-b+a_{b,x}-1)/2} &= (-1)^{(y-(\ell+1)-b+a_{b,y}-\ell+1-1)/2} \\
&= (-1)^\ell (-1)^{(y-b+a_{b,y}-1)/2} \\
&= (-1)^\ell \text{sign}(b, y).
\end{aligned}
$$

If $b > y$, then $a_{b,y} = a_{b,x} - (\ell - 1)$. Then

$$
\begin{aligned}
\text{sign}(b, x) = (-1)^{(b-x+a_{b,x}-1)/2} &= (-1)^{(b-(y-(\ell+1))+a_{b,y}+(\ell-1)-1)/2} \\
&= (-1)^\ell (-1)^{(b-y+a_{b,y}-1)/2} \\
&= (-1)^\ell \text{sign}(b, y).
\end{aligned}
$$

$\square$

Lemma 3.1.23. *Assume the nodes x and y with $x < y$ are white and all ℓ nodes between x and y are black, where $\ell \geq 1$. If b is a black node in the interval $[x + 1, \ldots, y - 1]$, so $b = x + j$ for some $j \leq \ell$, then $\text{sign}(b, x) = (-1)^{\ell-1} \text{sign}(b, y)$.*

Proof. Since $b = x + j$ and $a_{b,x} = j - 1$, we see that

$$\text{sign}(b, x) = (-1)^{(b-x+a_{b,x}-1)/2} = (-1)^{(j+j-1-1)/2} = (-1)^{j-1}.$$

Using the fact that $y - b = \ell + 1 - j$ and $a_{b,y} = \ell - j$, we have

$$\text{sign}(b, y) = (-1)^{(y-b+a_{b,y}-1)/2} = (-1)^{(\ell+1-j+\ell-j-1)/2} = (-1)^{\ell-j}.$$

So $\text{sign}(b, x) = (-1)^{\ell-1}\text{sign}(b, y)$. □

Remark 3.1.24. The symmetric group S_{2n} acts on the set of pairings on $\{1, \ldots, 2n\}$: the transposition $(i, i+1)$ acts on a pairing ρ by swapping the locations of i and $i + 1$. If i is paired with $i + 1$, acting with $(i, i + 1)$ leaves the pairing fixed; otherwise, $(i, i + 1)$ acts nontrivially and changes the parity of the number of crossings.

Let ρ be a (not necessarily black-white) pairing on $\{1, \ldots, 2n\}$. Let x and y be two nodes such that $x < y$. Assume no node in the interval $[x, y]$ is paired with any other node in this interval. Then

$$(x, y)\rho = (x, x + 1) \cdots (y - 1, y) \cdots (x + 1, x + 2)(x, x + 1)\rho$$

where each transposition of the form $(i, i + 1)$ acts nontrivially.

Lemma 3.1.25. *Let ρ be a (not necessarily black-white) pairing on $\{1, \ldots, 2n\}$. Let x and y be two nodes such that $x < y$ and x is not paired with y. Assume that no node in the interval $[x + 1, \ldots, y - 1]$ is paired with any other node in this interval. Then when the locations of x and y in ρ are swapped,*

46

(1) if x and y were both paired with nodes in the interval $[x + 1, \ldots, y - 1]$, the number of crossings of ρ changes parity,

(2) if exactly one of x and y was paired with a node in the interval $[x + 1, \ldots, y - 1]$, then the number of crossings of ρ does not change parity, and

(3) if neither x nor y was paired with a node in the interval $[x + 1, \ldots, y - 1]$ then the number of crossings of ρ changes parity.

Proof. Let ρ be a pairing on $\{1, \ldots, 2n\}$ and consider $(x, y)\rho$. There are several cases. The strategy is to factor (x, y) into adjacent transpositions and determine which transpositions act nontrivially.

Case 1. If the nodes $\rho(x)$ and $\rho(y)$ are both in the interval $[x + 1, \ldots, y - 1]$, then $(x, y)\rho = (\rho(x), \rho(y))\rho$. Let $a = \min(\rho(x), \rho(y))$ and let $b = \max(\rho(x), \rho(y))$. Then

$$(\rho(x), \rho(y))\rho = (a, a + 1) \cdots (b - 1, b) \cdots (a + 1, a + 2)(a, a + 1)\rho$$

We have written $(\rho(x), \rho(y))$ as a product of an odd number of transpositions of the form $(i, i + 1)$. Since no node in the interval $[a, \ldots, b]$ is paired with any other node in this interval, all these transpositions act nontrivially by Remark 3.1.24. Thus the parity of the number of crossings changes.

Case 2. If exactly one of the nodes $\rho(x)$ or $\rho(y)$ is in the interval $[x + 1, \ldots, y - 1]$, then

$$(x, y)\rho = (x, x + 1) \cdots (y - 1, y) \cdots (x + 1, x + 2)(x, x + 1)\rho$$

and exactly one of these transpositions acts trivially. For if x is paired with $x + k$, then after applying the transposition $(x, x + 1)$ to ρ, $x + 1$ and $x + k$ are paired. Similarly, after applying the transposition $(x + 1, x + 2)$ to $(x, x + 1)\rho$, $x + 2$ and

47

$x + k$ are paired. It follows that the transposition $(x + k - 1, x + k)$ acts trivially. Since an even number of transpositions of the form $(i, i + 1)$ act nontrivially, the parity of the number of crossings does not change.

Case 3. If neither of the nodes $\rho(x)$ and $\rho(y)$ are in the interval $[x + 1, \ldots, y - 1]$, then

$$(x, y)\rho = (x, x + 1) \cdots (y - 1, y) \cdots (x + 1, x + 2)(x, x + 1)\rho$$

so we have written $(\rho(x), \rho(y))$ as a product of an odd number of transpositions of the form $(i, i + 1)$. Since no node in the interval $[x, y]$ is paired with any other node in this interval, all of these transpositions act nontrivially. Thus the parity of the number of crossings changes. $\square$

Now that we have established Lemmas 3.1.21 through 3.1.25 we can show that equation (3.1.2) holds after applying both types of swaps to ρ. By Remark 3.1.20, each swap changes $\text{sign}_{BW}(\rho)$.

(1) **Swapping the locations of u and $u + 1$.**

Let b_1 be the node paired with u and let b_2 be the node paired with $u + 1$. By Lemma 3.1.21, $\text{sign}(b_1, u) = -\text{sign}(b_1, u + 1)$ and $\text{sign}(b_2, u + 1) = -\text{sign}(b_2, u)$. So when we swap the locations of u and $u + 1$, $\prod_{(b,w) \in \rho} \text{sign}(b, w)$ does not change. Since $\text{sign}_{BW}(\rho)$ changes, the sign of the LHS of (3.1.2) changes. Swapping u and $u + 1$ changes $(-1)^{\# \text{ crosses of } \rho}$, so swapping the locations of u and $u + 1$ does not affect equation (3.1.2).

(2) **Swapping the locations of x and y, where $x < y$ are white nodes and all ℓ nodes between x and y are black.**

Case 1. If x and y are both paired with black nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$ then $(-1)^{\# \text{ crosses of } \rho}$ changes sign by Lemma 3.1.25. By Lemma 3.1.23,

$$\text{sign}(\rho(x), x)\text{sign}(\rho(y), y) = ((-1)^{\ell-1})^2\text{sign}(\rho(x), y)\text{sign}(\rho(y), x)$$

so $\prod_{(b,w)\in\rho} \text{sign}(b, w)$ does not change. Since $\text{sign}_{BW}(\rho)$ changes, the sign of the LHS of (3.1.2) changes.

Case 2. If exactly one of x and y is paired with a black node in the interval $[x + 1, x + 2, \ldots, y - 1]$, then $(-1)^{\# \text{ crosses of } \rho}$ does not change sign by Lemma 3.1.25. By Lemmas 3.1.22 and 3.1.23,

$$\text{sign}(\rho(x), x)\text{sign}(\rho(y), y) = (-1)^{\ell-1}(-1)^{\ell}\text{sign}(\rho(x), y)\text{sign}(\rho(y), x)$$

so $\prod_{(b,w)\in\rho} \text{sign}(b, w)$ changes sign. Since $\text{sign}_{BW}(\rho)$ changes, the sign of the LHS of (3.1.2) does not change sign.

Case 3. If neither x nor y is paired with a black node in the interval $[x + 1, x + 2, \ldots, y - 1]$, then $(-1)^{\# \text{ crosses of } \rho}$ changes sign. By Lemma 3.1.22,

$$\text{sign}(\rho(x), x)\text{sign}(\rho(y), y) = ((-1)^{\ell})^2\text{sign}(\rho(x), y)\text{sign}(\rho(y), x)$$

so $\prod_{(b,w)\in\rho} \text{sign}(b, w)$ does not change. Since $\text{sign}_{BW}(\rho)$ changes, the sign of the LHS of (3.1.2) changes.

This completes the proof of Lemma 3.1.13.

We conclude Section 3.1 by proving Lemma 3.1.3, which states that when a black-white pairing ρ is also odd-even, $\text{sign}_{OE}(\rho) = \text{sign}_{BW}(\rho)$.

3.1.3. Proof of Lemma 3.1.3

Before we prove Lemma 3.1.3, we prove the lemma in the case where ρ is planar.

Lemma 3.1.26. *When ρ is a planar black-white pairing,*

$$sign_{OE}(\rho) = sign_{BW}(\rho)$$

Proof. Let ρ be a planar black-white pairing. Recall from Definition 3.1.16 that all black-white pairings can be written $\rho = ((b_1, \rho(b_1)), (b_2, \rho(b_2)), \ldots, (b_n, \rho(b_n)))$, where $b_1 < b_2 < \cdots < b_n$, and we say that $(\rho(b_i), \rho(b_j))$ is an inversion of ρ if $i < j$ and $\rho(b_i) > \rho(b_j)$.

All planar pairings are odd-even, and an inversion of an odd-even pairing is an inversion of the permutation $\left(\frac{\rho(1)}{2} \quad \frac{\rho(3)}{2} \quad \cdots \quad \frac{\rho(2n-3)}{2} \quad \frac{\rho(2n-1)}{2} \right)$. In this proof, will say $(\rho(i), \rho(j))$ is an inversion if $i < j$ and $\rho(i) > \rho(j)$, even though the inversion is actually $(\frac{\rho(i)}{2}, \frac{\rho(j)}{2})$.

We will show that there is a one-to-one correspondence between inversions of ρ when it is considered as a black-white pairing (which we will call black-white inversions) and inversions of ρ when it is considered as an odd-even pairing (which we will call odd-even inversions).

Consider a black-white inversion, that is, some $b_i < b_j$ such that $\rho(b_i) > \rho(b_j)$. There are several cases to consider:

Case 1. b_i, b_j are both odd.

In this case, $b_i = 2k - 1$ and $b_j = 2\ell - 1$ for some $k < \ell$, so $(\rho(b_i), \rho(b_j))$ is an odd-even inversion.

Case 2. b_i, b_j are both even.

Since $b_i < b_j$ and $\rho(b_i) > \rho(b_j)$, (b_j, b_i) is an odd-even inversion.

Case 3. b_i is odd and b_j is even.

There are two subcases to consider. If $\rho(b_j) < b_i$, then it must be the case that $b_j > \rho(b_i)$. To see this, observe that if $b_j < \rho(b_i)$, then $\rho(b_j) < b_i < b_j < \rho(b_i)$, but then we have a crossing, which contradicts the planarity of ρ. So $(b_j, \rho(b_i))$ is an odd-even inversion.

If $\rho(b_j) > b_i$, then $\rho(b_i) > b_j$ (otherwise $b_i < \rho(b_j) < \rho(b_i) < b_j$, so ρ has a crossing). So $(\rho(b_i), b_j)$ is an odd-even inversion.

Case 4. b_i is even and b_j is odd.

If $\rho(b_i) > b_j$, then $\rho(b_j) > b_i$ (otherwise $\rho(b_j) < b_i < b_j < \rho(b_i)$ is a crossing), so $(\rho(b_j), b_i)$ is an odd-even inversion. If $\rho(b_i) < b_j$, then $b_i > \rho(b_j)$ (otherwise $b_i < \rho(b_j) < \rho(b_i) < b_j$ is a crossing), so $(b_i, \rho(b_j))$ is an odd-even inversion.

A similar argument shows that for each odd-even inversion, there is a black-white inversion. Since there is a one-to-one correspondence between odd-even inversions and black-white inversions, $\text{sign}_{OE}(\rho) = \text{sign}_{BW}(\rho)$. □

Lemma 3.1.3. When ρ is a black-white pairing that is also odd-even,

$$\text{sign}_{OE}(\rho) = \text{sign}_{BW}(\rho).$$

Proof. One can get from an odd-even black-white pairing ρ_1 to any other odd-even black white pairing ρ_2 by applying a series of moves, where each move swaps the locations of two nodes of the same color and parity. Since each of these moves changes sign_{OE} and sign_{BW}, the claim follows from Lemma 3.1.26. □

3.2. Lemmas 3.1 and 3.2 From Kenyon and Wilson

Throughout this section, S denotes a balanced subset of nodes (a subset containing an equal number of black and white nodes). The union of a dimer configuration of $G \setminus S$ and a dimer configuration of $G \setminus S^c$ is a double-dimer configuration of $(G, \mathbf{N})$ (see Figure 2.1 in Section 2.1). In [13], Kenyon and Wilson show that $Z^D(G \setminus S)Z^D(G \setminus S^c)$ is a sum of double-dimer partition functions $Z_\pi^{DD}(G, \mathbf{N})$, where the sum is over all pairings π that do not connect nodes in S to nodes in S^c.

Lemma 3.2.1. *[13, Lemma 3.1] If S is a balanced subset of nodes then $Z^D(G \setminus S)Z^D(G \setminus S^c)$ is a sum of double-dimer configurations for all connection topologies π for which π connects no element of S to an element of $S^c := \mathbf{N} \setminus S$. That is,*

$$Z^D(G \setminus S)Z^D(G \setminus S^c) = Z^{DD} \sum_\pi M_{S,\pi} \Pr(\pi),$$

where $M_{S,\pi}$ is 0 or 1 according to whether π connects nodes in S to S^c or not.

This lemma relates the quantity $Z^D(G \setminus S)Z^D(G \setminus S^c)$ to $\Pr(\pi)$. Next, Kenyon and Wilson show that $\dfrac{Z^D(G \setminus S)Z^D(G \setminus S^c)}{(Z^D(G))^2}$ is a determinant in the quantities $X_{i,j}$.

Lemma 3.2.2. *[13, Lemma 3.2] Let S be a balanced subset of $\{1, \ldots, 2n\}$. Then*

$$\frac{Z^D(G \setminus S)Z^D(G \setminus S^c)}{(Z^D(G))^2} = \det[(1_{i,j \in S} + 1_{i,j \notin S}) \times (-1)^{(|i-j|-1)/2} X_{i,j}]_{j=2,4,\ldots,2n}^{i=1,3,\ldots,2n-1}.$$

The combination of these results shows that $\widehat{\Pr}(\pi)$ is a homogeneous polynomial in the $X_{i,j}$, since the matrix M from [13, Lemma 3.1] has full rank [13, Lemma 3.3]. Our analogues of these lemmas have several differences (such as the

52

additional global signs in our version of Lemma 3.2.2, see Lemma 3.2.5), but our proofs are quite similar to their proofs.

We begin with Lemma 3.2.1. For a graph G with node set $\mathbf{N}$ that does not necessarily have the property that all nodes are black and odd or white and even, a statement very similar to Lemma 3.2.1 holds. Let $T \subseteq \mathbf{N}$ be the set of nodes that are odd and white or even and black. Let $\widetilde{G}$ be G with an extra vertex and edge of weight 1 added to each node in T, so all of the nodes in $\widetilde{G}$ are black and odd or white and even. Since $Z^D(\widetilde{G} \setminus S) = Z^D(G \setminus (S \triangle T))$, Lemma 3.2.1 implies the following.

Corollary 3.2.3. *Let S be a balanced subset of nodes. $Z^D(G \setminus (S \triangle T))Z^D(G \setminus (S \triangle T)^c)$ is a sum of double-dimer configurations for all connection topologies π for which π connects no element of S to an element of S^c. That is,*

$$Z^D(G \setminus (S \triangle T))Z^D(G \setminus (S \triangle T)^c) = Z^{DD}(G)\sum_{\pi} M_{S,\pi}\Pr(\pi),$$

where $M_{S,\pi}$ is 0 or 1 according to whether π connects nodes in S to S^c or not.

If $V = S \triangle T$, then $S = V \triangle T$, so we have:

Corollary 3.2.4. *Let V be a balanced subset of nodes. $Z^D(G \setminus V)Z^D(G \setminus V^c)$ is a sum of all connection topologies π for which π connects no elements of $V \triangle T$ to $(V \triangle T)^c$. That is,*

$$Z^D(G \setminus V)Z^D(G \setminus V^c) = Z^{DD}(G)\sum_{\pi} M_{V \triangle T,\pi}\Pr(\pi),$$

where $M_{V \triangle T,\pi}$ is 0 or 1 depending on whether π connects nodes in $V \triangle T$ to $(V \triangle T)^c$.

Corollary 3.2.4 is the version of Lemma 3.2.1 that we will need to prove Theorem 2.4.1.

Our version of [13, Lemma 3.2] is the following.

Lemma 3.2.5. *Let S be a balanced subset of $\mathbf{N} = \{1, \ldots, 2n\}$. Then*

$$\frac{Z^D(G \setminus S) Z^D(G \setminus S^c)}{(Z^D(G))^2} = sign_c(\mathbf{N}) sign(S) \det \left[(1_{i,j \in S} + 1_{i,j \notin S}) \times sign(i,j) Y_{i,j} \right]_{j=w_1,\ldots,w_n}^{i=b_1,\ldots,b_n}$$

(3.2.1)

where $b_1, b_2, \ldots, b_n$ are the black nodes of $\{1, 2, \ldots, 2n\}$ listed in ascending order, $w_1, w_2, \ldots, w_n$ are the white nodes of $\{1, 2, \ldots, 2n\}$ listed in ascending order, $sign(i,j)$ is defined in Definition 3.1.6 and $sign(S) = (-1)^{\# \ crosses \ of \ \rho}$, where ρ is a black-white pairing that does not bridge S to S^c and is planar when restricted to S and planar when restricted to S^c.

Remark 3.2.6. The fact that such a pairing ρ always exists is a consequence of Lemma 3.1.14, which states that for any node coloring there is a planar black-white pairing ρ satisfying $\text{sign}_{BW}(\rho) \prod_{(b,w) \in \rho} sign(b,w) = \text{sign}_c(\mathbf{N})$. Since S is balanced, the existence of a planar black-white pairing of S and a planar black-white pairing of S^c follows.

The proof of Lemma 3.2.5 requires some Kasteleyn theory. The reader familiar with basic facts about Kasteleyn matrices can skip the following section.

3.2.1. Kasteleyn Matrices

Recall that $G = (V_1, V_2, E)$ is a finite edge-weighted bipartite planar graph embedded in the plane. Let $w((i,j))$ denote the weight of an edge $(i,j) \in E$.

Definition 3.2.7. A Kasteleyn (or flat) weighting of G is a choice of sign for each edge with the property that each face with 0 mod 4 edges has an odd number of $-$ signs and each face with 2 mod 4 edges has an even number of $-$ signs.

For the remainder of this section we will let $\sigma : E \to \pm 1$ denote the Kasteleyn weighting of G. A Kasteleyn matrix of G is a signed bipartite adjacency matrix of G. More precisely, define a $|V_1| \times |V_2|$ matrix K by

$$
K_{i,j} = \begin{cases} \sigma((i,j))w((i,j)) & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}
$$

Kasteleyn showed that every bipartite planar graph with an even number of vertices has a Kasteleyn matrix. Moreover, $|\det K|$ is the weighted sum of all dimer configurations of G [10].

The proof of Lemma 3.2.5 uses a few straightforward facts about Kasteleyn weightings. First, we will show that if $G = (V_1, V_2, E)$ has a Kasteleyn weighting σ, and we add edges to G to obtain G', we can choose weights for the added edges to obtain a Kasteleyn weighting σ' of G' with the property that $\sigma'(e) = \sigma(e)$ for all $e \in E$.

Lemma 3.2.8. *Let b and w be two vertices of opposite color on a face F of $G = (V_1, V_2, E)$. Let $E' = E \cup \{\widetilde{e}\}$, where $\widetilde{e} \notin E$ is an edge connecting b and w that separates F into two faces and let $G' = (V_1, V_2, E')$. Define $\sigma' : E \cup \{\widetilde{e}\} \to \pm 1$ so that $\sigma'(e) = \sigma(e)$ for all $e \in E$ and choose $\sigma'(\widetilde{e})$ so that one of the faces bounded by $\widetilde{e}$ is flat (i.e., it has an odd number of $-$ signs if it has 0 mod 4 edges, and an even number of $-$ signs otherwise). Then σ' is a Kasteleyn weighting of G'.*

Proof. By assumption, the edge $\widetilde{e}$ separates F into two faces: the face consisting of the edges of a path $\mathcal{Q}$ and the edge $\widetilde{e}$, and the face consisting of the edges of a path $\mathcal{P}$ and the edge $\widetilde{e}$. The path $\mathcal{P}$ consists of 1 mod 4 edges or 3 mod 4 edges. Define

$$
\sigma'(\widetilde{e}) = \begin{cases} \displaystyle\prod_{e \in \mathcal{P}} \sigma(e) & \text{if } \mathcal{P} \text{ has 1 mod 4 edges} \\[2ex] -\displaystyle\prod_{e \in \mathcal{P}} \sigma(e) & \text{if } \mathcal{P} \text{ has 3 mod 4 edges} \end{cases}
$$

and define $\sigma'(e) = \sigma(e)$ for all $e \in E$. Now the face consisting of the path $\mathcal{P}$ and the edge e is flat. Consider the edges of $\mathcal{Q}$. By the assumption that σ is a Kasteleyn weighting,

$$
\prod_{e \in \mathcal{Q}} \sigma(e) = \begin{cases} \displaystyle\prod_{e \in \mathcal{P}} \sigma(e) & \text{if } \mathcal{P} \text{ and } \mathcal{Q} \text{ have the same number of edges mod 4} \\[2ex] -\displaystyle\prod_{e \in \mathcal{P}} \sigma(e) & \text{if } \mathcal{P} \text{ and } \mathcal{Q} \text{ have a different number of edges mod 4} \end{cases}.
$$

So $\sigma'(\widetilde{e}) \displaystyle\prod_{e \in \mathcal{Q}} \sigma'(e) =$

$$
\begin{cases} \left(\displaystyle\prod_{e \in \mathcal{P}} \sigma(e)\right)^2 = 1 & \text{if } \mathcal{P} \text{ and } \mathcal{Q} \text{ have 1 mod 4 edges} \\[2ex] \left(-\displaystyle\prod_{e \in \mathcal{P}} \sigma(e)\right)^2 = 1 & \text{if } \mathcal{Q} \text{ has 1 mod 4 edges and } \mathcal{P} \text{ has 3 mod 4 edges} \\[2ex] -\left(\displaystyle\prod_{e \in \mathcal{P}} \sigma(e)\right)^2 = -1 & \text{if } \mathcal{P} \text{ and } \mathcal{Q} \text{ have 3 mod 4 edges} \\[2ex] -\left(\displaystyle\prod_{e \in \mathcal{P}} \sigma(e)\right)^2 = -1 & \text{if } \mathcal{Q} \text{ has 3 mod 4 edges and } \mathcal{P} \text{ has 1 mod 4 edges} \end{cases}.
$$

Noting that the face F' consisting of $\mathcal{Q}$ and $\widetilde{e}$ has 2 mod 4 edges if $\mathcal{Q}$ has 1 mod 4 edges and 0 mod 4 edges otherwise, we conclude that F' is flat, so σ' is a Kasteleyn weighting of G'. $\qquad\square$

Lemma 3.2.9. *Let $W = \{v_1, \ldots, v_{2m}\}$ be a set of vertices on the outer face of $G = (V_1, V_2, E)$. Pair the vertices of W so that we can add edges $e_1, \ldots, e_m$ connecting the pairs without introducing any edge crossings. Let $E^{(m)} = E \cup \{e_1, \ldots, e_m\}$ and let $G^{(m)} = (V_1, V_2, E^{(m)}, w)$. Define $\sigma_i : E \cup \{e_i\} \to \pm 1$ as in Lemma 3.2.8: $\sigma_i(e) = \sigma(e)$ for all $e \in E$ and $\sigma_i(e_i)$ is chosen so that one of the faces bounded by e_i is flat. By Lemma 3.2.8, σ_i is a Kasteleyn weighting for all $1 \leq i \leq m$. Then $\tau : E \cup \{e_1, \ldots, e_m\} \to \pm 1$ defined by $\tau(e) = \sigma(e)$ for all $e \in E$ and $\tau(e_i) = \sigma_i(e_i)$ for $1 \leq i \leq m$ is a Kasteleyn weighting of $G^{(m)}$.*

Proof. We prove the claim by induction on m. When $m = 1$, there is nothing to show. Assume the claim holds when we add $m - 1$ edges to G. Now suppose we add m edges $e_1, \ldots, e_m$. Choose an "innermost" edge e_j, i.e. an edge with the property that one of its faces is bounded only by edges of G and e_j. By the induction hypothesis, $\tau : E \cup \{e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_m\} \to \pm 1$ defined by $\tau(e) = \sigma(e)$ for all $e \in E$ and $\tau(e_i) = \sigma_i(e_i)$ for $i = 1, 2, \ldots, j - 1, j + 1, \ldots, m$ is a Kasteleyn weighting of $G^{(m-1)} = (V_1, V_2, E \cup \{e_1, \ldots, e_{j-1}, e_{j+1}, \ldots e_m\})$. Since e_j is an innermost edge and $\sigma_j : E \cup \{e_j\} \to \pm 1$ was defined so that when e_j is added to G, one of the faces bounded by e_j is flat, we may apply Lemma 3.2.8 to conclude that $\tau : E \cup \{e_1, \ldots, e_m\} \to \pm 1$ defined by $\tau(e) = \sigma(e)$ for all $e \in E$ and $\tau(e_i) = \sigma_i(e_i)$ for $1 \leq i \leq m$ is a Kasteleyn weighting of $G^{(m)} = (V_1, V_2, E \cup \{e_1, \ldots, e_m\})$. $\square$

We also need to show that if we delete rows and columns from a Kasteleyn matrix of a graph, the resulting submatrix is a Kasteleyn matrix of the corresponding graph.

Lemma 3.2.10. *Let K be a Kasteleyn matrix of G. Let S be a balanced subset of vertices on the outer face of G. Then $K_{\backslash S}$, the submatrix of K formed by deleting the rows and columns from S, is a Kasteleyn matrix of $G \setminus S$.*

To prove this, we need the following lemma and corollary from [15].

Lemma 3.2.11. *[15, Theorem 2.1] If G is a planar bipartite graph with an even number of vertices, there are an even number of faces with $4k$ sides.*

Proof. By Euler's formula, $f - e + v = 2$, where f denotes the number of faces, e denotes the number of edges, and v denotes the number of vertices of G. Divide the contribution to e from each edge evenly between the two incident faces. Then a face with $4k$ sides contributes an odd integer to $f - e$ and a face with $4k + 2$ sides contributes an even integer to $f - e$. So there must be an even number of faces with $4k$ sides. $\square$

Corollary 3.2.12. *[15, Theorem 2.2] Every signed graph with an even number of vertices has an even number of non-flat faces.*

Proof. If each edge of a graph has sign $+1$, the faces which are not flat are exactly the faces with $4k$ sides. By Lemma 3.2.11, there are an even number of faces with $4k$ sides. It remains to observe that every time we change the sign of an edge, we change the flatness of exactly two faces. $\square$

Proof of Lemma 3.2.10. $G \setminus S$ is flat at each internal face because G is flat at each internal face, so it remains to show that it is also flat on the outer face. Since $G \setminus S$ has an even number of vertices, it has an even number of non-flat faces by Corollary 3.2.12, so it must be flat on the outer face. $\square$

3.2.2. Proof of Lemma 3.2.5

Proof of Lemma 3.2.5. Assume there are $2k$ couples of consecutive nodes of the same color. As in Remark 3.1.7, we label the couples of consecutive white nodes $(u_i, u_i + 1)$ and the couples of consecutive black nodes $(s_i, s_i + 1)$ for $1 \leq i \leq k$.

Following the proof of [13, Lemma 3.2], we adjoin to the graph G $2n - 2k$ edges connecting all adjacent nodes except nodes s_i and s_i+1 and nodes u_i and u_i+1. The resulting graph is still bipartite by the assumption that the nodes alternate between black and white except for the nodes s_i and $s_i + 1$ and the nodes u_i and $u_i + 1$. Now add $4k$ more edges as follows. Since G is bipartite, there is a white vertex t_i on the outer face of G between nodes s_i and $s_i + 1$ and a black vertex v_i on the outer face of G between nodes u_i and $u_i + 1$. Add edges connecting nodes s_i and t_i and t_i and $s_i + 1$, and edges connecting nodes u_i and v_i and v_i and $u_i + 1$. Give the $2n - 2k + 4k = 2n + 2k$ edges we have added weight ϵ (and then take the limit $\epsilon \to 0$). Let G' denote the resulting graph.

Given a Kasteleyn weighting of a graph, the signs of edges incident to a vertex may be reversed, and each face will still have a correct number of minus signs. Fix a Kasteleyn weighting of the graph G'. List the vertices from the set $\mathbf{N} \cup \{t_i\}_{i=1}^k \cup \{v_i\}_{i=1}^k$ in counterclockwise order. For each vertex in this list, if the edge from the vertex i to the next vertex in the list j has a minus sign, reverse the signs of all edges incident to vertex j. This ensures that the edges of weight ϵ we added to G have positive sign, with the possible exception of the edge from node $2n$ to 1, which must have sign $-(-1)^{n+k}$ for the outer face to have a correct number of minus signs (because if $n + k$ is even, the outer face has 0 mod 4 edges, and if $n + k$ is odd, the outer face has 2 mod 4 edges).

Let S be a balanced subset of $\{1, \ldots, 2n\}$. Let $(w_1, b_1), \ldots, (w_j, b_j)$ be any noncrossing pairing of the nodes of S, where $w_1, \ldots, w_j$ are the white nodes of S and $b_1, \ldots, b_j$ are the black nodes of S. Adjoin edges of weight W connecting w_i to b_i for $1 \leq i \leq j$. Because of the edges of weight ϵ we adjoined to G, we let the sign

of a new edge of weight W connecting black node b and white node w be

$$\mathrm{sign}(b, w) = (-1)^{(|b-w|+a_{b,w}-1)/2},$$

where recall that $a_{b,w}$ is the number of couples of consecutive nodes of the same color in the interval $[\min\{b, w\}, \min\{b, w\} + 1, \ldots, \max\{b, w\}]$.

Observe that with this choice of signs, when we add any one of the edges $e_i = (b_i, w_i)$ to G' so that it separates the outer face of G' into two faces, one of the faces bounded by e_i is flat. So by Lemma 3.2.9, this is a Kasteleyn weighting.

Let K_W be the Kasteleyn matrix of the resulting graph, with rows and columns ordered so that $b_1, \ldots, b_j$ are the first j rows and $w_1, \ldots, w_j$ are the first j columns. Let $K = K_0$ be the corresponding Kasteleyn matrix when $W = 0$. Then $Z^D(G \setminus S) = \pm [W^j] \det(K_W)$ where $[W^j] \det(K_W)$ denotes the coefficient of W^j in the polynomial $\det(K_W)$. (Because $[W^j] \det(K_W)$ is, up to a sign, the weighted sum of matchings that include all of the edges of weight W, which is exactly the weighted sum of matchings of $G \setminus S$.) Since each term of $\det(K_W)$ has the same sign,

$$\frac{Z^D(G \setminus S)}{Z^D(G)} = \frac{[W^j] \det(K_W)}{[W^0] \det(K_W)}.$$

Next let $K_{\setminus S}$ denote the submatrix of K formed by deleting the rows and columns from S. By Lemma 3.2.10, $K_{\setminus S}$ is a Kasteleyn matrix of $G \setminus S$. The sign of $\det(K_{\setminus S})$ and the sign of $[W^j] \det(K_W)$ differ by the product of the signs of the edges of weight W. So, noting that $[W^0] \det(K_W) = \det(K)$, we have

$$\frac{[W^j] \det(K_W)}{[W^0] \det(K_W)} = \prod_{\ell=1}^{j} \mathrm{sign}(b_\ell, w_\ell) \frac{\det(K_{\setminus S})}{\det(K)}.$$

By Jacobi's determinant identity,

$$\prod_{\ell=1}^{j} \text{sign}(b_\ell, w_\ell) \frac{\det(K_{\backslash S})}{\det(K)} = \prod_{\ell=1}^{j} \text{sign}(b_\ell, w_\ell) \det[K_{b,w}^{-1}]_{w=w_1,\ldots,w_j}^{b=b_1,\ldots,b_j}.$$

So we have

$$\frac{Z^D(G \backslash S)}{Z^D(G)} = \prod_{\ell=1}^{j} \text{sign}(b_\ell, w_\ell) \det[K_{b,w}^{-1}]_{w=w_1,\ldots,w_j}^{b=b_1,\ldots,b_j}. \qquad (3.2.2)$$

Letting $S = \{b, w\}$ in equation (3.2.2), we get

$$Y_{b,w} = \frac{Z^D(G_{b,w})}{Z^D(G)} = \text{sign}(b, w) K_{b,w}^{-1}.$$

From this and equation (3.2.2) we find that when $\rho_1 = (w_1, b_1), \ldots, (w_j, b_j)$ is a noncrossing pairing of the nodes of S and $\rho_2 = (w_{j+1}, b_{j+1}), \ldots, (w_n, b_n)$ is a noncrossing pairing of the nodes of S^c,

$$\frac{Z^D(G \backslash S) Z^D(G \backslash S^c)}{(Z^D(G))^2}$$

$$= \prod_{\ell=1}^{n} \text{sign}(b_\ell, w_\ell) \det\left[\text{sign}(b, w) Y_{b,w}\right]_{w=w_1,\ldots,w_j}^{b=b_1,\ldots,b_j} \det\left[\text{sign}(b, w) Y_{b,w}\right]_{w=w_{j+1},\ldots,w_n}^{b=b_{j+1},\ldots,b_n}$$

$$= \prod_{\ell=1}^{n} \text{sign}(b_\ell, w_\ell) \det\left[(1_{i,j \in S} + 1_{i,j \notin S}) \text{sign}(b, w) Y_{b,w}\right]_{w=w_1,\ldots,w_n}^{b=b_1,\ldots,b_n}$$

which is equation (3.2.1), except for the global sign and the order of the rows and columns (since $w_1, \ldots, w_n$ and $b_1, \ldots, b_n$ are not necessarily in ascending order).

Let $\rho = \rho_1 \cup \rho_2$. Reorder the rows so that the black nodes are in ascending order. For each row swap, make the corresponding column swap. Then ρ pairs the node corresponding to row i with the node corresponding to column i. Since the row swaps and column swaps we have made are in one-to-one correspondence, we

61

have not changed the sign of the determinant. Finally, we need to put the columns in ascending order. The number of swaps required to do this is exactly $\text{sign}_{BW}(\rho)$.

So after reordering the rows and columns so that they are listed in ascending order, the global sign is:

$$\prod_{\ell=1}^{n} \text{sign}(b_\ell, w_\ell)\text{sign}_{BW}(\rho)$$

which is equal to $\text{sign}_{\text{c}}(\mathbf{N})(-1)^{\#\text{ crosses of }\rho}$ by Lemma 3.1.13.

$\square$

3.3. Defining $\mathcal{Q}^{(DD)}$

Let Y' be the vector of monomials Y'_ρ indexed by black-white pairings, where

$$Y'_\rho = (-1)^{\#\text{ crosses of }\rho} \prod_{(i,j)\in\rho} Y_{i,j}.$$

In this section, we define $Q^{(DD)}$, which is the matrix satisfying

$$\widetilde{\Pr}(\sigma) = \sum_{\text{black-white pairings }\rho} \mathcal{Q}^{(DD)}_{\sigma,\rho} Y'_\rho.$$

We begin with a few definitions.

Definition 3.3.1. If σ and τ are two pairings on a set of nodes $\{1, 2, \ldots, 2n\}$, construct the undirected multigraph C with vertex set $\{1, 2, \ldots, 2n\}$ by adding an edge between vertices i and j for each pair (i, j) of σ, and similarly for τ. The *number of components in $\sigma \cup \tau$* is the number of connected components in C. Note that all connected components of C are cycles.

Example 3.3.2. If $\sigma = ((1, 2), (3, 4), (5, 6))$ and $\tau = ((1, 5), (2, 6), (3, 4))$ then there are two components in $\sigma \cup \tau$, as shown in Figure 3.4.
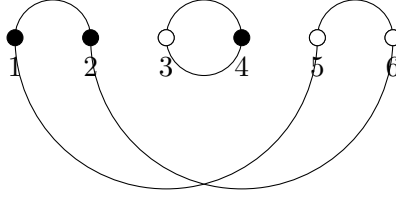
FIGURE 3.4. The multigraph C from Definition 3.3.1 for the pairings
$\sigma = ((1,2),(3,4),(5,6))$ and $\tau = ((1,5),(2,6),(3,4))$.

Definition 3.3.3. If π is an odd-even pairing and ρ is a black-white pairing, let

$$\text{sign}(\pi,\rho) := (-1)^{\#\text{nodes}/2}(-1)^{\#\text{ components in } \pi \cup \rho}\text{sign}_{OE}(\pi)\text{sign}_{BW}(\rho).$$

Definition 3.3.4. Define the matrix $\mathcal{B}_2$ which has rows indexed by planar pairings and columns indexed by black-white pairings by

$$(\mathcal{B}_2)_{\pi,\rho} = \text{sign}(\pi,\rho)2^{\#\text{ components in } \rho \cup \pi}.$$

Let M be the matrix from Corollary 3.2.4 and let D be the vector indexed by balanced sets S with entries $D_S = \dfrac{Z^D(G \setminus S)Z^D(G \setminus S^c)}{(Z^D(G))^2}$ (see Lemma 3.2.5). Following Kenyon and Wilson, we will show that

$$M^T D = \mathcal{B}_2 Y'$$

(Theorem 3.3.5). This result is nontrivial, requiring several lemmas, but once it is established it is nearly immediate that

$$M^T M P = \mathcal{B}_2 Y',$$

63

where P is the vector indexed by planar pairings π with entries $\widetilde{\Pr}(\pi)$ (Corollary 3.3.12). Kenyon and Wilson proved that $M^T M$ is invertible ([12, Theorem 3.3]), so we conclude the section by defining $\mathcal{Q}^{(DD)}$ as $(M^T M)^{-1} \mathcal{B}_2$.

Theorem 3.3.5 (analogue of Lemma 3.5 from [13]). $M^T D = \mathcal{B}_2 Y'$.

In the proof of [13, Lemma 3.5], Kenyon and Wilson use the fact that if the nodes of G are all either black and odd or white and even and π and ρ are odd-even pairings, then there are $2^{\# \text{ comp in } \pi \cup \rho}$ balanced sets S such that π and ρ do not bridge S to S^c (for each component, either put all of its nodes in S or all of its nodes in S^c). Recall from Section 3.2 that $T \subseteq \mathbf{N}$ is the set of nodes that are odd and white or even and black; under Kenyon and Wilson's assumptions, $T = \emptyset$. It turns out that after removing the requirement that the nodes be black and odd or white and even, if π is an odd-even pairing and ρ is a black-white pairing there are still $2^{\# \text{ comp in } \pi \cup \rho}$ sets S such that ρ does not bridge S to S^c and π does not bridge $S \triangle T$ to $(S \triangle T)^c$.

Lemma 3.3.6. *Let π be an odd-even pairing and let ρ be a black-white pairing. For each component of $\pi \cup \rho$ there are exactly two ways to put the nodes in this component into S and S^c so that ρ does not bridge S to S^c and π does not bridge $S \triangle T$ to $(S \triangle T)^c$.*

Proof. We start by placing an initial node a into S or S^c, and then apply the algorithm below until all nodes in the component have been placed into S or S^c.

Algorithm 3.3.7.

Step 1

  (a) If $a \in S \cap T^c$ or $a \in S^c \cap T$:

(i) If $\pi(a) \in T$:

   Put $\pi(a)$ in S^c.

(ii) Else if $\pi(a) \notin T$:

   Put $\pi(a)$ in S.

(b) Else if $a \in S \cap T$ or $a \in S^c \cap T^c$:

   (i) If $\pi(a) \in T$:

      Put $\pi(a)$ in S.

   (ii) Else if $\pi(a) \notin T$:

      Put $\pi(a)$ in S^c.

   Go to Step 2 with $a := \pi(a)$.

Step 2

   If $a \in S$:

      Put $\rho(a)$ in S.

   Else if $a \in S^c$:

      Put $\rho(a)$ in S^c.

   Go to Step 1 with $a := \rho(a)$.

Claim 3.3.8. *The set S described in Algorithm 3.3.7 is well-defined and balanced.*

Proof. We will prove this claim by induction on the number of nodes in a component of $\pi \cup \rho$.

Base Cases. First note that in the case where the nodes alternate between black and white, $T = \emptyset$ or $T = \mathbf{N}$ so the algorithm reduces to putting all of the nodes in a component in S or all of the nodes of a component in S^c, so S is well-defined. Since in this case both pairings are black-white, S is balanced as well.

If there are two nodes in a component, since ρ is a black-white pairing one of the nodes is black and the other is white, so by the previous comment there is nothing to show.

If there are four nodes in a component, since ρ is a black-white pairing two nodes must be black and two nodes must be white. By symmetry, it is enough to consider when nodes 1 and 2 are black and nodes 3 and 4 are white. There are two odd-even pairings: $((1,2),(3,4))$ and $((1,4),(3,2))$ and two black-white pairings: $((1,4),(3,2))$ and $((1,3),(2,4))$.

For example, when $\pi = ((1,2),(3,4))$ and $\rho = ((1,4),(2,3))$ (see Figure 3.5), we start by putting node 1 in S. (We could also start by putting node 1 in S^c.) Then we run the algorithm:

Step 1. Since $1 \notin T$ and $\pi(1) = 2 \in T$, we put $2 \in S^c$.

Step 2. Since $2 \in S^c$ we put $\rho(2) = 3 \in S^c$.

Step 1. Since $3 \in S^c$, $3 \in T$ and $4 \notin T$, we put $4 \in S$.

So we get $S = \{1, 4\}$, which is balanced. To check that S is well-defined, it suffices to show that if we continue the algorithm for one more step, we do not get a contradiction. If we apply Step 2 starting at node 4, we find that we should put $\rho(4) = 1$ in S, as desired.

In Table 3.1 we show the results of applying the algorithm for each possible combination of odd-even pairings π and black-white pairings ρ that results in a
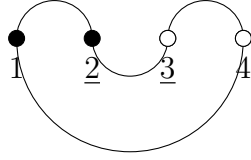
FIGURE 3.5. The diagram of $\pi \cup \rho$ when $\pi = ((1,2),(3,4))$ and $\rho = ((1,4),(2,3))$. The nodes in T are underlined. If we begin the algorithm by putting $1 \in S$, we get $S = \{1,4\}$.

component of size 4. When $\pi = \rho = ((1,4),(3,2))$, there are two components that each have size 2, so this is omitted from the table.

π	ρ	S	start	end	one more step
$((1,2),(3,4))$	$((1,4),(2,3))$	$\{1,\ 4\}$	$1 \in S$	$4 \in S$	$1 \in S$
$((1,2),(3,4))$	$((1,3),(2,4))$	$\{1,\ 3\}$	$1 \in S$	$3 \in S$	$1 \in S$
$((1,4),(3,2))$	$((1,3),(2,4))$	$\{1,\ 2,\ 3,\ 4\}$	$1 \in S$	$3 \in S$	$1 \in S$

TABLE 3.1. The base case for the proof of Claim 3.3.8. This table shows the set S obtained after applying the algorithm to each possible combination of odd-even pairings π and black-white pairings ρ that result in a component of size 4.

In each case, S is balanced, and continuing the algorithm for one more step does not create a contradiction.

Now suppose that a component of $\pi \cup \rho$ has $2n$ nodes, where $2n > 4$. Assume that if a component has fewer than $2n$ nodes, the set S is well-defined and balanced. Let $\mathbf{N}$ denote the set of nodes in this component. There are two cases to consider based on whether or not $\pi|_{\mathbf{N}}$ has a black-white pair.

Case 1. (Illustrated in Figure 3.6). Assume $\pi|_{\mathbf{N}}$ has at least one black-white pair $(a, \pi(a))$. Since ρ is a black-white pairing, $\rho(a)$ and $\rho(\pi(a))$ are opposite color. Consider the black-white pairing $\widetilde{\rho}$ on $\mathbf{N} - \{a, \pi(a)\}$ obtained from ρ by removing the pairs $(a, \rho(a))$ and $(\pi(a), \rho(\pi(a)))$ and adding the pair $(\rho(a), \rho(\pi(a)))$. Let $\widetilde{\pi} = \pi|_{\mathbf{N}-\{a,\pi(a)\}}$. Now $\widetilde{\pi} \cup \widetilde{\rho}$ is a single component with $2n - 2$ nodes. Start the algorithm by putting $\rho(\pi(a)) \in S$. By the induction hypothesis, the set S produced

67

by the algorithm is well-defined and balanced. Note that the fact that S is well-defined means that $\rho(a) \in S$.

Considering the original component of $\pi \cup \rho$, when we start the algorithm at $\rho(\pi(a))$ it proceeds identically as it did with $\widetilde{\pi} \cup \widetilde{\rho}$ until we reach the node $\rho(a)$. Since $\rho(a) \in S$, applying Step 2 of the algorithm we add a to S. (Note that we are guaranteed to be on Step 2 here by the fact that $\rho(a)$ is paired with $\rho(\pi(a))$ in $\widetilde{\rho}$, and the algorithm starts with Step 1.) Since π is odd-even, black-white pairs of π have the property that either both nodes are in T or both are not in T. So after the next step of the algorithm (Step 1) we add $\pi(a)$ to S. Since we added a and $\pi(a)$ to S, S is still balanced. Since $\pi(a) \in S$, continuing the algorithm for one more step would put $\rho(\pi(a)) \in S$, which is consistent.

FIGURE 3.6. Illustration of case 1 of Claim 3.3.8. On the left we have the odd-even pairing π (top) and the black-white pairing ρ (bottom). On the right we have replaced ρ with $\widetilde{\rho}$, the black white pairing on $\mathbf{N} - \{a, \pi(a)\}$ obtained by pairing $\rho(\pi(a))$ with $\rho(a)$.

Case 2. (Illustrated in Figure 3.7). If $\pi|_{\mathbf{N}}$ does not have a black-white pair, then consider a white pair of $\pi|_{\mathbf{N}}$: $(a, \pi(a))$. Let $b = \rho(a)$. Since a is white, $\rho(a)$ must be black, and $(b, \pi(b))$ is a black pair of $\pi|_{\mathbf{N}}$ by the assumption that $\pi|_{\mathbf{N}}$ does not have a black-white pair. Consider the black-white pairing $\widetilde{\rho}$ on $\mathbf{N} - \{a, \pi(a), b, \pi(b)\}$ obtained from ρ by removing the pairs $(a, b), (\pi(a), \rho(\pi(a)))$, and $(\pi(b), \rho(\pi(b)))$ and adding the pair $(\rho(\pi(a)), \rho(\pi(b)))$. Let $\widetilde{\pi} = \pi|_{\mathbf{N}-\{a, \pi(a), b, \pi(b)\}}$. Now $\widetilde{\pi} \cup \widetilde{\rho}$ is a single component with $2n - 4$ nodes. Start the algorithm by putting $\rho(\pi(a)) \in S$. By

68

the induction hypothesis, the set S produced by the algorithm is well-defined and balanced. Note that the fact that S is well defined means that $\rho(\pi(b)) \in S$.

Considering the original component of $\pi \cup \rho$, when we start the algorithm by putting $\rho(\pi(a)) \in S$ it proceeds identically as it did with $\widetilde{\pi} \cup \widetilde{\rho}$ until we reach the node $\rho(\pi(b))$. Since $\rho(\pi(b)) \in S$, applying Step 2 of the algorithm we add $\pi(b)$ to S. Since π is odd-even and contains no black-white pairs, exactly one of $\{b, \pi(b)\}$ is in T. This means that after applying Step 1 we put $b \in S^c$. Then we put $a \in S^c$ (since $\rho(a) = b$) and $\pi(a) \in S$ (since exactly one of $\{a, \pi(a)\}$ is in T). Since we added $\pi(b)$ and $\pi(a)$ to S, S is still balanced. Since $\pi(a) \in S$, continuing the algorithm for one more step puts $\rho(\pi(a)) \in S$, which is consistent.
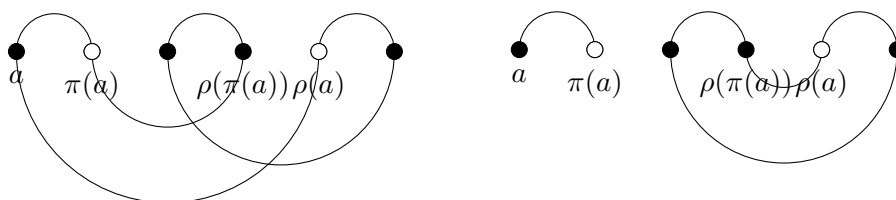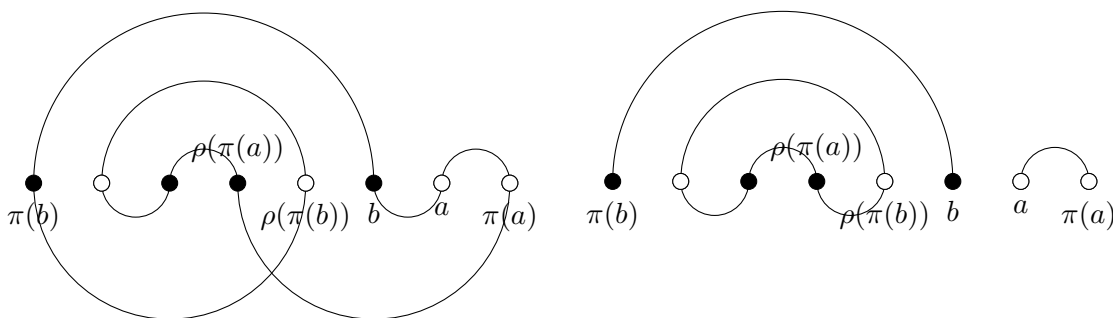

FIGURE 3.7. Illustration of case 2 of Claim 3.3.8. The left image shows the odd-even pairing π (top) and the black-white pairing ρ (bottom). On the right we have replaced ρ with $\widetilde{\rho}$, the black white pairing on $\mathbf{N} - \{a, \pi(a), b, \pi(b)\}$ obtained by pairing $\rho(\pi(a))$ with $\rho(\pi(b))$.

$\square$

Claim 3.3.9. *After applying Algorithm 3.3.7, ρ does not bridge S to S^c and π does not bridge $S\triangle T$ to $(S\triangle T)^c$.*

Proof. By Step 2, for each node a, a and $\rho(a)$ will either both be in S or S^c, so ρ does not bridge S to S^c. To show that a and $\pi(a)$ are either both in $S\triangle T$ or both in $(S\triangle T)^c$, there are several cases to consider.

69

- If a and $\pi(a)$ are both not in T, then they are both placed into S by Step 1(a)(ii) or both placed into S^c by Step 1(b)(ii). In the first case, a and $\pi(a)$ are both in $S \triangle T$, and in the second case a and $\pi(a)$ are both in $(S \triangle T)^c$.

- If $a \in T$ and $\pi(a) \notin T$, then one of $a, \pi(a)$ is placed into S and one is placed into S^c by Step 1(a)(ii) or Step 1(b)(ii). If a is placed into S and $\pi(a)$ is placed into S^c, then a and $\pi(a)$ are both in $(S \triangle T)^c$. The other case is similar.

- If $a \in T$ and $\pi(a) \in T$, then they are both placed into S^c by Step 1(a)(i) or both placed into S by Step 1(b)(i).

- If $a \notin T$ and $\pi(a) \in T$, then one is placed in S and one is placed in S^c by Step 1(a)(i) or Step 1(b)(i).

$\square$

We have shown that the algorithm produces a well-defined balanced set S with the desired properties. We conclude that for each component of $\pi \cup \rho$ there are exactly two ways to put the nodes in this component into S and S^c so that ρ does not bridge S to S^c and π does not bridge $S \triangle T$ to $(S \triangle T)^c$.

$\square$

We need two more facts to prove Theorem 3.3.5.

Lemma 3.3.10. *Let S be a balanced subset of nodes and let $sign(S)$ be defined as in Lemma 3.2.5. Then*

$$sign(S) = (-1)^{\frac{\#nodes}{2}} (-1)^{\#\ comp\ in\ \pi \cup \rho} sign_{OE}(\pi) sign_{BW}(\rho),$$

70

where π is an odd-even pairing such that π does not bridge $S \triangle T$ to $(S \triangle T)^c$ and ρ is a black-white pairing such that ρ does not bridge S to S^c.

The proof of Lemma 3.3.10 is long and technical so we postpone it to Section 3.5 for ease of exposition. The following is an immediate consequence of this lemma.

Corollary 3.3.11. *Let π be an odd-even pairing and let ρ be a black-white pairing. If S_1 and S_2 are sets such that π does not bridge $S_i \triangle T$ to $(S_i \triangle T)^c$ and ρ does not bridge S_i to S_i^c for $i = 1, 2$, then $sign(S_1) = sign(S_2)$.*

Proof of Theorem 3.3.5. Recall from Lemma 3.2.5 that

$$
\begin{aligned}
D_S &= \frac{Z^D(G \setminus S) Z^D(G \setminus S^c)}{(Z^D(G))^2} \\
&= \text{sign}_c(\mathbf{N})\text{sign}(S) \det \left[(1_{i,j \in S} + 1_{i,j \notin S}) \times \text{sign}(i,j) Y_{i,j} \right]_{j=w_1, w_2, \ldots, w_n}^{i=b_1, b_2, \ldots, b_n},
\end{aligned}
$$

where $b_1, b_2, \ldots, b_n$ are the black nodes listed in ascending order and $w_1, w_2, \ldots, w_n$ are the white nodes listed in ascending order.

When we expand the determinant, we get

$$
D_S = \text{sign}_c(\mathbf{N})\text{sign}(S) \sum_{\substack{\text{BW pairings } \rho: \\ \rho \text{ does not bridge} \\ S \text{ to } S^c}} \text{sign}_{BW}(\rho) \prod_{(i,j) \in \rho} \text{sign}(i,j) Y_{i,j}.
$$

By Lemma 3.1.13,

$$
D_S = \text{sign}_c(\mathbf{N})\text{sign}(S) \sum_{\substack{\text{BW pairings } \rho: \\ \rho \text{ does not bridge} \\ S \text{ to } S^c}} \text{sign}_c(\mathbf{N})(-1)^{\# \text{ crosses of } \rho} \prod_{(i,j) \in \rho} Y_{i,j},
$$

71

and thus by definition,

$$D_S = \text{sign}(S) \sum_{\substack{\text{BW pairings } \rho: \\ \rho \text{ does not bridge} \\ S \text{ to } S^c}} Y'_\rho. \qquad (3.3.1)$$

Let π be a planar pairing and let M be the matrix from Corollary 3.2.4. The πth row of $M^T D$ is

$$\sum_{\substack{S \subseteq \{1,2,\ldots,2n\} \\ \pi \text{ does not bridge} \\ S \triangle T \text{ to } (S \triangle T)^c}} D_S.$$

We see that

$$\sum_{\substack{S \subseteq \{1,2,\ldots,2n\}: \\ \pi \text{ does not bridge} \\ S \triangle T \text{ to } (S \triangle T)^c}} D_S = \sum_{\substack{S \subseteq \{1,2,\ldots,2n\}: \\ \pi \text{ does not bridge} \\ S \triangle T \text{ to } (S \triangle T)^c}} \text{sign}(S) \sum_{\substack{\text{BW pairings } \rho: \\ \rho \text{ does not bridge} \\ S \text{ to } S^c}} Y'_\rho$$

$$= \sum_{\text{BW pairings } \rho} \sum_{\substack{S: \, \rho \text{ does not} \\ \text{bridge } S \text{ to } S^c \text{ and} \\ \pi \text{ does not bridge} \\ S \triangle T \text{ to } (S \triangle T)^c}} \text{sign}(S) Y'_\rho.$$

By Lemma 3.3.6 and Corollary 3.3.11,

$$\sum_{\text{BW pairings } \rho} \sum_{\substack{S: \, \rho \text{ does not} \\ \text{bridge } S \text{ to } S^c \text{ and} \\ \pi \text{ does not bridge} \\ S \triangle T \text{ to } (S \triangle T)^c}} \text{sign}(S) Y'_\rho = \sum_{\text{BW pairings } \rho} \text{sign}(\pi, \rho) 2^{\# \text{ comp in } \pi \cup \rho} Y'_\rho.$$

Since this sum is the πth row of $\mathcal{B}_2 Y'$, we have proven the claim.

$\square$

Corollary 3.3.12 (analogue of Theorem 3.6 from [13])**.** $M^T M P = \mathcal{B}_2 Y'$

Proof. Noting that $\widetilde{\Pr}(\pi) = \Pr(\pi)Z^{DD}(G)/(Z^D(G))^2$, we see that by Corollary 3.2.4, $MP = D$. Then, applying Theorem 3.3.5 we get $M^T M P = M^T D = \mathcal{B}_2 Y'$. □

It remains to show that $M^T M$ is invertible. In fact, $M^T M$ is equal to the *meander matrix* $\mathcal{M}_q$ evaluated at $q = 2$.

Lemma 3.3.13. *[13, Lemma 3.3] Let M be the matrix from Lemma 3.2.1. Then $M^T M = \mathcal{M}_2$, where $\mathcal{M}_2$ is a matrix with rows and columns indexed by planar pairings, with entries*

$$(\mathcal{M}_2)_{\sigma,\tau} = 2^{\# \ comp \ in \ \sigma \cup \tau}$$

Since the only difference between the matrix from Lemma 3.2.1 and the matrix from Corollary 3.2.4 is the ordering of the rows, Lemma 3.3.13 applies to the matrix M from Corollary 3.2.4 as well.

Definition 3.3.14. Since $\mathcal{M}_2$ is invertible (see [3]), define

$$\mathcal{Q}^{(DD)} = \mathcal{M}_2^{-1} \mathcal{B}_2$$

Since $P = \mathcal{Q}^{(DD)} Y'$, $\mathcal{Q}^{(DD)}$ is the matrix of the Y' polynomials: for a given planar pairing π, the πth row of $\mathcal{Q}^{(DD)}$ gives the polynomial $\widetilde{\Pr}(\pi)$.

We will next prove that $\mathcal{Q}^{(DD)}$ is integer-valued, which will complete the proof of Theorem 2.4.1.

3.4. Proof That $\mathcal{Q}^{(DD)}$ Is Integer-Valued

To show that $\mathcal{Q}^{(DD)}$ is integer-valued, we prove that we can compute the columns combinatorially using a transformation rule closely related to the transformation rule for groves.

Recall that Kenyon and Wilson's proof of Theorem 2.3.7 in Section 2.3 established the following transformation rule:

If a pairing ρ is nonplanar, then there will exist items $a < b < c < d$ such that a and c are paired, and b and d are paired. Then the transformation rule is

$$ac|bd|\text{rest} \to -ab|cd|\text{rest} - ad|bc|\text{rest} \tag{3.4.1}$$

Rule 3.4.1. For a black-white pairing ρ, repeatedly apply (3.4.1) until we have written ρ as a linear combination of planar pairings. Then multiply each planar pairing σ in this sum by $\text{sign}_{OE}(\sigma)\text{sign}_{BW}(\rho)$.

The fact that Rule 3.4.1 is well-defined follows from Theorem 2.3.5.

Let $\widetilde{\mathcal{Q}}$ be the matrix obtained by the procedure from Rule 3.4.1, so the (σ,ρ)th entry of $\widetilde{\mathcal{Q}}$ is the product of $\text{sign}_{OE}(\sigma)\text{sign}_{BW}(\rho)$ with the coefficient of σ when ρ is written as a linear combination of planar pairings using (3.4.1). That is,

$$\widetilde{\mathcal{Q}}_{\sigma,\rho} = \text{sign}_{OE}(\sigma)\text{sign}_{BW}(\rho)\mathcal{P}^{(t)}_{\sigma,\rho}. \tag{3.4.2}$$

We will show that
$$\mathcal{M}_2\widetilde{\mathcal{Q}}\mathbf{e}_i = \mathcal{B}_2\mathbf{e}_i$$

for all i. This shows that $\mathcal{M}_2\widetilde{\mathcal{Q}} = \mathcal{M}_2\mathcal{Q}^{(DD)}$, which proves that $\widetilde{\mathcal{Q}} = \mathcal{Q}^{(DD)}$ since $\mathcal{M}_2$ is invertible.

Let ρ be a black-white pairing. Recall from Definition 3.3.4 that $(\mathcal{B}_2)_{\pi,\rho} = \text{sign}(\pi,\rho)2^{\#\text{ comp in }\pi\cup\rho}$. So to show that $\mathcal{M}_2\widetilde{\mathcal{Q}}\mathbf{e}_i = \mathcal{B}_2\mathbf{e}_i$ we need to show that for each planar pairing π,

$$\text{sign}(\pi,\rho)2^{\#\text{ comp in }\pi\cup\rho} = \sum_{\text{planar pairings }\sigma} \mathcal{P}^{(t)}_{\sigma,\rho}\text{sign}_{OE}(\sigma)\text{sign}_{BW}(\rho)2^{\#\text{ comp in }\pi\cup\sigma}.$$

By Definition 3.3.3,

$$\text{sign}(\pi,\rho) = (-1)^{\#\text{nodes}/2}(-1)^{\#\text{ comp in }\pi\cup\rho}\text{sign}_{OE}(\pi)\text{sign}_{BW}(\rho).$$

So, it suffices to prove the following.

Lemma 3.4.2. *Let ρ be a pairing (not necessarily black-white). Then for any planar pairing π,*

$$sign_{OE}(\pi)(-1)^{C_\rho}(-1)^{\#nodes/2}2^{C_\rho} = \sum_{\text{planar pairings }\sigma} \mathcal{P}^{(t)}_{\sigma,\rho}sign_{OE}(\sigma)2^{C_\sigma}. \qquad (3.4.3)$$

Here, C_ρ denotes the number of components in $\pi \cup \rho$ and C_σ denotes the number of components in $\pi \cup \sigma$.

Lemma 3.4.2 requires one additional lemma.

Lemma 3.4.3. *Let π be a pairing and let ρ be a pairing with nodes $a < b < c < d$ that form a crossing in ρ. Let ρ_1 be the pairing obtained from ρ by replacing the pairs (a,c) and (b,d) with (a,b) and (c,d) and let ρ_2 be the pairing obtained from ρ by replacing the pairs (a,c) and (b,d) with (a,d) and (b,c). Then either*

(1) $\pi \cup \rho$ has one more component than both $\pi \cup \rho_1$ and $\pi \cup \rho_2$,

75

(2) $\pi \cup \rho_1$ has one more component than $\pi \cup \rho$, and $\pi \cup \rho_2$ and $\pi \cup \rho$ have the same number of components, or

(3) $\pi \cup \rho_2$ has one more component than $\pi \cup \rho$, and $\pi \cup \rho_1$ and $\pi \cup \rho$ have the same number of components.

Proof. Observe that either a, b, c, and d are all in the same component of $\pi \cup \rho$ or a and c are in the same component and b and d are in a different component. If a and c are in the same component and b and d are in a different component, then pairing a with b and c with d merges these two components. Similarly, pairing a with d and b with c merges these two components.

If a, b, c and d are in the same component, then we consider the following path in $\pi \cup \rho$:

$$c - a - \pi(a) - \rho(\pi(a)) - \cdots \tag{3.4.4}$$

This path reaches b or d before it reaches c since by assumption a, b, c, d are all in the same component. If it reaches b before d, then in ρ_1, a and b are in a different component than c and d. This is because path (3.4.4) is replaced with

$$b - a - \pi(a) - \rho(\pi(a)) - \cdots - b,$$

so $\pi \cup \rho_1$ has one more component than $\pi \cup \rho$. In ρ_2, a, b, c, and d are all in the same component, because path (3.4.4) is replaced with

$$d - a - \pi(a) - \rho(\pi(a)) - \cdots - b - c,$$

so $\pi \cup \rho_2$ and $\pi \cup \rho$ have the same number of components. If the path reaches d before b, then in ρ_2, a and d are in a different component than b and c, so $\pi \cup \rho_2$

has one more component than $\pi \cup \rho$. In ρ_1, a, b, c, and d are in the same component, so $\pi \cup \rho_1$ and $\pi \cup \rho$ have the same number of components. $\qquad \square$

Proof of Lemma 3.4.2. We will prove the claim by induction on the number of crossings in ρ.

Base Case. When ρ has 0 crossings, equation (3.4.3) becomes

$$\text{sign}_{OE}(\pi)(-1)^{C_\rho}(-1)^{\#\text{nodes}/2}2^{C_\rho} = \mathcal{P}^{(t)}_{\rho,\rho}\text{sign}_{OE}(\rho)2^{C_\rho}.$$

Recall from Section 2.3 that when ρ is planar, $\mathcal{P}^{(t)}_{\rho,\rho} = 1$. So the above equation is equivalent to

$$\text{sign}_{OE}(\pi)(-1)^{C_\rho}(-1)^{\#\text{nodes}/2}\text{sign}_{OE}(\rho) = 1. \tag{3.4.5}$$

First suppose $\rho = \pi$. Since $(-1)^{\#\text{ comp in }\pi \cup \pi} = (-1)^{\#\text{nodes}/2}$, equation (3.4.5) holds. We can obtain any planar pairing from any other planar pairing by a sequence of moves, where each move consists of swapping the locations of two nodes of the same parity. So we will show that when ρ is a planar pairing, x and y are two nodes of the same parity, and ρ' is the pairing obtained from ρ by swapping the locations of x and y, replacing ρ with ρ' does not change the left hand side of equation (3.4.5). Since $\text{sign}_{OE}(\rho) = -\text{sign}_{OE}(\rho')$, we must show that $(-1)^{\#\text{ comp in }\pi \cup \rho} = -(-1)^{\#\text{ comp in }\pi \cup \rho'}$.

If x and $\rho(x)$ are in a different component than y and $\rho(y)$ in $\pi \cup \rho$, then $\pi \cup \rho'$ has one fewer component than $\pi \cup \rho$. If $x, \rho(x), y$, and $\rho(y)$ are all in the same component in $\pi \cup \rho$, then without loss of generality assume that x and y are both even, so $\rho(x)$ and $\rho(y)$ are both odd, and consider the following path in $\pi \cup \rho$:

$$\rho(x) - x - \pi(x) - \rho(\pi(x)) - \cdots.$$

77

Since ρ and π are both odd-even, segments $n \frown \rho(n)$ go from an odd node to an even node. Since $\rho(y)$ is odd and y is even, this means that we must reach the node $\rho(y)$ before the node y. Therefore we have the path

$$\rho(x) - x - \pi(x) - \rho(\pi(x)) - \cdots - \rho(y) - y - \cdots$$

When we replace the pairs $(x, \rho(x)), (y, \rho(y))$ with $(x, \rho(y))$ and $(y, \rho(x))$, this path is replaced with

$$\rho(y) - x - \pi(x) - \rho(\pi(x)) - \cdots - \rho(y)$$

so $(x, \rho(y))$ and $(y, \rho(x))$ are in different components of $\pi \cup \rho'$. We conclude that equation (3.4.5) holds for all planar pairings ρ.

Now assume that equation (3.4.3) holds for pairings ρ with $\leq k$ crossings.

Let ρ be a pairing with $k + 1$ crossings. Let $a < b < c < d$ be nodes that form a crossing in ρ. Let ρ_1 be the pairing obtained by replacing the pairs (a, c) and (b, d) with (a, b) and (c, d) and let ρ_2 be the pairing obtained by replacing the pairs (a, c) and (b, d) with (a, d) and (b, c). We claim that both ρ_1 and ρ_2 have fewer than $k + 1$ crossings. Observe that if a chord connecting two nodes n_1 and n_2 crosses the chord connecting a and b in ρ_1, it also crosses the chord connecting a and c or the chord connecting b and d in ρ. Similarly, if a chord connecting two nodes crosses the chord connecting c and d in ρ_1, it also crosses the chord connecting a and c or the chord connecting b and d in ρ. It follows that ρ_1 has at least one less crossing than ρ. A similar argument shows that ρ_2 has at least one less crossing than ρ.

By the induction hypothesis,

$$\mathrm{sign}_{OE}(\pi)(-1)^{C_{\rho_1}}(-1)^{\#\mathrm{nodes}/2} 2^{C_{\rho_1}} = \sum_{\text{planar pairings } \sigma} \mathcal{P}^{(t)}_{\sigma,\rho_1} \mathrm{sign}_{OE}(\sigma) 2^{C_\sigma}$$

78

and

$$\text{sign}_{OE}(\pi)(-1)^{C_{\rho_2}}(-1)^{\#\text{nodes}/2}2^{C_{\rho_2}} = \sum_{\substack{\text{planar pairings } \sigma}} \mathcal{P}^{(t)}_{\sigma,\rho_2}\text{sign}_{OE}(\sigma)2^{C_\sigma}.$$

By the transformation rule (3.4.1), $\mathcal{P}^{(t)}_{\sigma,\rho_1} + \mathcal{P}^{(t)}_{\sigma,\rho_2} = -\mathcal{P}^{(t)}_{\sigma,\rho}$, so we have

$$\sum_{\substack{\text{planar} \\ \text{pairings } \sigma}} \mathcal{P}^{(t)}_{\sigma,\rho}\text{sign}_{OE}(\sigma)2^{C_\sigma} = -\text{sign}_{OE}(\pi)(-1)^{\#\text{nodes}/2}\left((-1)^{C_{\rho_1}}2^{C_{\rho_1}} + (-1)^{C_{\rho_2}}2^{C_{\rho_2}}\right).$$

By Lemma 3.4.3 there are three cases to consider:

(1) $\pi \cup \rho$ has one more component than both $\pi \cup \rho_1$ and $\pi \cup \rho_2$,

(2) $\pi \cup \rho_1$ has one more component than $\pi \cup \rho$, and $\pi \cup \rho_2$ and $\pi \cup \rho$ have the same number of components, and

(3) $\pi \cup \rho_2$ has one more component than $\pi \cup \rho$, and $\pi \cup \rho_1$ and $\pi \cup \rho$ have the same number of components.

Case (1). Since $C_{\rho_i} - C_\rho = -1$ for $i = 1, 2$,

$$\begin{aligned}
(-1)^{C_{\rho_1}}2^{C_{\rho_1}} + (-1)^{C_{\rho_2}}2^{C_{\rho_2}} &= -(-1)^{C_\rho} \cdot \frac{1}{2} \cdot 2^{C_\rho} + -(-1)^{C_\rho} \cdot \frac{1}{2} \cdot 2^{C_\rho} \\
&= (-1)^{C_\rho}2^{C_\rho}\left(-\frac{1}{2} - \frac{1}{2}\right) \\
&= -(-1)^{C_\rho}2^{C_\rho}.
\end{aligned}$$

79

Cases (2) and (3). We will only include the proof for case (2), since case (3) is completely analogous. Since $C_{\rho_1} - C_\rho = 1$ and $C_{\rho_2} - C_\rho = 0$,

$$
\begin{aligned}
(-1)^{C_{\rho_1}} 2^{C_{\rho_1}} + (-1)^{C_{\rho_2}} 2^{C_{\rho_2}} &= -(-1)^{C_\rho} 2 \cdot 2^{C_\rho} + (-1)^{C_\rho} 2^{C_\rho} \\
&= (-1)^{C_\rho} 2^{C_\rho}(-2 + 1) \\
&= -(-1)^{C_\rho} 2^{C_\rho}.
\end{aligned}
$$

So in all cases,

$$
-\mathrm{sign}_{OE}(\pi)(-1)^{\#\mathrm{nodes}/2}\left((-1)^{C_{\rho_1}} 2^{C_{\rho_1}} + (-1)^{C_{\rho_2}} 2^{C_{\rho_2}}\right) = \mathrm{sign}_{OE}(\pi)(-1)^{\#\mathrm{nodes}/2}(-1)^{C_\rho} 2^{C_\rho},
$$

and thus

$$
\mathrm{sign}_{OE}(\pi)(-1)^{C_\rho}(-1)^{\#\mathrm{nodes}/2} 2^{C_\rho} = \sum_{\text{planar pairings } \sigma} \mathcal{P}^{(t)}_{\sigma,\rho} \mathrm{sign}_{OE}(\sigma) 2^{C_\sigma}.
$$

$\square$

This proves that

$$
(\mathcal{Q}^{(DD)})_{\pi,\rho} = \mathrm{sign}_{OE}(\pi)\mathrm{sign}_{BW}(\rho)\mathcal{P}^{(t)}_{\pi,\rho}, \tag{3.4.6}
$$

which completes the proof of Theorem 2.4.1. Its full statement is below.

Theorem 2.4.1. Any black-white pairing ρ can be transformed into a formal linear combination of planar pairings by repeated application of Rule 3.4.1, and the resulting linear combination does not depend on the choices we made when

80

applying Rule 3.4.1, so that we may write

$$\rho \to \sum_{\text{planar pairings } \sigma} \mathcal{Q}_{\sigma,\rho}^{(DD)} \sigma.$$

For any planar pairing σ, these same coefficients $\mathcal{Q}_{\sigma,\rho}^{(DD)}$ satisfy the equation

$$\widetilde{\Pr}(\sigma) := \frac{Z_\sigma^{DD}(G, \mathbf{N})}{(Z^D(G))^2} = \sum_{\text{black-white pairings } \rho} \mathcal{Q}_{\sigma,\rho}^{(DD)} Y_\rho'.$$

Remark 3.4.4. The fact that the resulting linear combination does not depend on the choices we made when applying Rule 3.4.1 is an immediate consequence of Theorem 2.3.5.

3.5. Another Characterization of sign(S)

In this section, we prove Lemma 3.3.10, which was key in establishing Theorem 3.3.5.

Lemma 3.3.10. Let S be a balanced subset of nodes. Recall that $\text{sign}(S) := (-1)^{\# \text{ crosses of } \rho}$, where ρ is a black-white pairing that does not bridge S to S^c, and $\rho|_S$ and $\rho|_{S^c}$ are planar (see Lemma 3.2.5). Let π be an odd-even pairing such that π does not bridge $S \triangle T$ to $(S \triangle T)^c$ and let ρ be a black-white pairing such that ρ does not bridge S to S^c. Then

$$\text{sign}(S) = (-1)^{\frac{\#\text{nodes}}{2}}(-1)^{\# \text{ comp in } \pi \cup \rho}\text{sign}_{OE}(\pi)\text{sign}_{BW}(\rho). \tag{3.5.1}$$

Proving Lemma 3.3.10 requires

(1) proving that such pairings π and ρ always exist,

(2) proving that equation (3.5.1) is well-defined, and

(3) proving that equation (3.5.1) holds.

We will postpone the proof of (1) because the fact that such pairings π and ρ always exist will follow quickly from the proofs of (2) and (3).

3.5.1. Proof That Equation (3.5.1) is Well-Defined

The strategy of the proof is to define local moves that allow us to get from a pair (π_1, ρ_1) such that π_1 does not bridge $S \triangle T$ to $(S \triangle T)^c$ and ρ_1 does not bridge S to S^c to any other pair (π_2, ρ_2) with this property, and to show that these moves do not change the right hand side of equation (3.5.1).

Specifically, we will define two types of local moves. First, we define moves that modify π by swapping the locations of two nodes of the same parity under certain conditions but leave ρ fixed, called moves of type A_{OE}. Next, we define moves that modify ρ by swapping the locations of two nodes of the same color under similar conditions but leave π fixed, called moves of type A_{BW}.

In order to describe the conditions under which we can swap the locations of two nodes, we need the following definition.

Definition 3.5.1. We call a pair of nodes $(a, \pi(a))$ a *transition pair* if exactly one of the nodes $a, \pi(a)$ is in T.

Definition 3.5.2. Suppose π is an odd-even pairing and ρ is a black-white pairing. Let a and b be two nodes of the same parity. If

- a and b are in different components,

- a and b are the same color and a path from a to b contains an even number of transition pairs, or

82

– a and b are different colors and a path from a to b contains an odd number of transition pairs,

let π' be the pairing obtained from π by swapping the locations of a and b in π. We say that (π', ρ) and (π, ρ) differ by a move of type A_{OE}. See Figure 3.8 for an example.

If a and b are two nodes in the same component of $\pi \cup \rho$, there are two paths from a to b. We note that moves of type A_{OE} are well-defined because the algorithm in Lemma 3.3.6 is well-defined, so the parity of the number of transition pairs is independent of the path.
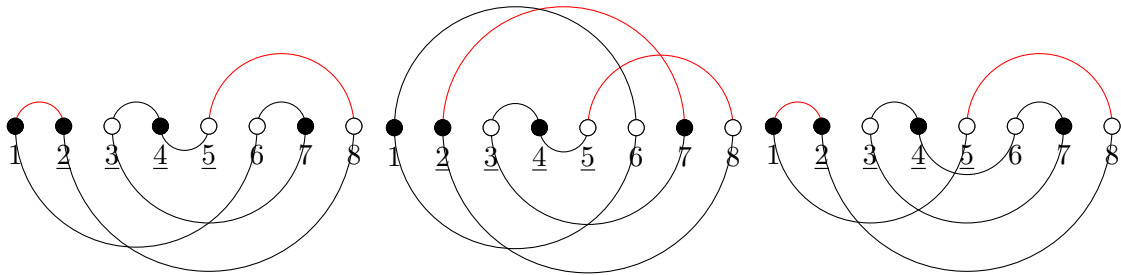
FIGURE 3.8. *Left*: The diagram of $\pi \cup \rho$, where $\pi = ((1, 2), (3, 4), (5, 8), (7, 6))$ and $\rho = ((1, 6), (2, 8), (4, 5), (7, 3))$. Nodes that are in T are underlined and arcs between transition pairs are red. *Center*: Since 1 and 7 are two nodes of the same parity and color and a path from 1 to 7 contains an even number of transition pairs, if we let $\pi' = ((1, 6), (3, 4), (5, 8), (7, 2))$ then (π, ρ) and (π', ρ) differ by a move of type A_{OE}. *Right*: Since 1 and 4 are two nodes of the same color and a path from 1 to 4 contains an even number of transition pairs, if we let $\rho' = ((1, 5), (2, 8), (4, 6), (7, 3))$ then (π, ρ) and (π, ρ') differ by a move of type A_{BW}.

Definition 3.5.3. Let π be an odd-even pairing and let ρ be a black-white pairing. Suppose a and b are the same color and either a and b are in different components, or a path in $\pi \cup \rho$ from a to b contains an even number of transition pairs.

83

Suppose we swap the locations of a and b in ρ to obtain the pairing ρ'. Then we say that (π, ρ') and (π, ρ) differ by a *move of type A_{BW}*. See Figure 3.8 for an example.

Lemma 3.5.4. *Let π, π' be odd-even pairings and let ρ be a black-white pairing such that (π, ρ) and (π', ρ) differ by a move of type A_{OE}. Then the number of components in $\pi \cup \rho$ and the number of components in $\pi' \cup \rho$ differ by one.*

Proof. If a and b are in different components of $\pi \cup \rho$, swapping the locations of a and b in π merges these two components, so the number of components decreases by one.

If a and b are in the same component, without loss of generality assume that node a is white. Consider the following path from a to b, which starts by traversing the edge connecting a to $\pi(a)$:

$$a - \pi(a) - \cdots - b.$$

We claim that we always reach b before $\pi(b)$. This follows from the observation that because ρ is black-white and π is odd-even, a path in $\pi \cup \rho$ alternates between black and white nodes unless a pair $(d, \pi(d))$ in the path is a transition pair. So since our path starts at a white node by traversing the edge in π, if we consider an edge $d \frown \pi(d)$ of the path, d is white and $\pi(d)$ is black if and only if we traverse this edge after passing through an even number of transition pairs. So, if we were to reach $\pi(b)$ before b, b is black if and only if there are an even number of transition pairs between a and b, a contradiction since a is white. It follows that we must reach b before $\pi(b)$.

Thus we have the following path in $\pi \cup \rho$:

$$a - \pi(a) - \cdots - b - \pi(b)$$

When we replace the pairs $(a, \pi(a))$ and $(b, \pi(b))$ in π with $(a, \pi(b))$ and $(b, \pi(a))$ to obtain π' the middle portion of the path above $\pi(a) - \cdots - b$ becomes a new component, so the number of components increases by one. $\qquad \square$

Corollary 3.5.5. *A move of type A_{OE} does not change the right hand side of equation (3.5.1).*

Proof. If (π, ρ) and (π', ρ) differ by a move of type A_{OE}, then $(-1)^{\# \text{ comp in } \pi \cup \rho} = -(-1)^{\# \text{ comp in } \pi' \cup \rho}$ by Lemma 3.5.4 and $\text{sign}_{OE}(\pi) = -\text{sign}_{OE}(\pi')$, so replacing π with π' does not change the right hand side of equation (3.5.1). $\qquad \square$

Corollary 3.5.6. *A move of type A_{BW} does not change the right hand side of equation (3.5.1).*

Proof. The proof that a move of type A_{BW} changes the number of components in $\pi \cup \rho$ by one is analogous to the proof of Lemma 3.5.4. The claim follows as it did in the proof of Corollary 3.5.5. $\qquad \square$

Proof that equation (3.5.1) is well-defined. By Corollaries 3.5.5 and 3.5.6, moves of type A_{OE} and type A_{BW} do not change the right hand side of equation (3.5.1). So to prove that the formula for $\text{sign}(S)$ is well-defined, it suffices to show that these two types of moves are enough to get from a pair (π_1, ρ_1) such that π_1 does not bridge $S \triangle T$ to $(S \triangle T)^c$ and ρ_1 does not bridge S to S^c to any other pair (π_2, ρ_2) with this property.

We can get from any pairing of nodes in S to any other pairing of nodes in S using moves of type A_{BW} because type A_{BW} moves allow us to exchange any nodes of the same color in S. By the same reasoning, we can get from any pairing of nodes in S^c to any other pairing of nodes in S^c. So, if ρ and ρ' are two pairings that both do not bridge S to S^c, then we can get from ρ to ρ' using a sequence of moves of type A_{BW}.

Similarly, we can get from any odd-even pairing of nodes in $S \triangle T$ to any other odd-even pairing of nodes in $S \triangle T$ by swapping nodes of the same parity in $S \triangle T$. We can also get from any odd-even pairing of nodes in $(S \triangle T)^c$ to any other odd-even pairing of nodes in $(S \triangle T)^c$. So if π and π' are two odd-even pairings that both do not bridge $S \triangle T$ to $(S \triangle T)^c$, then we can get from π to π' using a sequence of moves of type A_{OE}.

We have thus shown if we have two pairs of pairings (π_1, ρ_1) and (π_2, ρ_2) such that π_i is odd-even and does not bridge $S \triangle T$ to $(S \triangle T)^c$ and ρ_i is black-white and does not bridge S to S^c, that the right hand side of equation (3.5.1) is unchanged when we replace (π_1, ρ_1) with (π_2, ρ_2).

$\square$

3.5.2. Proof That Equation (3.5.1) Holds

The argument that equation (3.5.1) holds is relatively long, so we break it into three sections: an introduction, where we explain the proof strategy, several lemmas, and the proof itself.

3.5.2.1. Overview of Proof

First assume that S is a balanced set of size $2j$ such that there is a planar black-white pairing ρ that does not bridge S to S^c. Although it may not be obvious that such a set always exists, recall from Lemma 3.1.14 that regardless of the node coloring of $\mathbf{N}$, there exists a planar black-white pairing ρ of $\mathbf{N}$. So we choose S to be $2j$ of the arcs of ρ.

Then by definition, $\text{sign}(S) = (-1)^{\#\text{ crosses of } \rho} = 1$. Let $\pi = \rho$. Since π is odd-even and black-white, for all pairs in π, either both nodes of the pair are in T or both are not in T, so π does not bridge $S\triangle T$ to $(S\triangle T)^c$. Since $\pi = \rho$, $(-1)^{\frac{\#\text{nodes}}{2}} = (-1)^{\#\text{ comp in } \pi\cup\rho}$. Also, $\text{sign}_{OE}(\pi) = \text{sign}_{BW}(\rho)$ by Lemma 3.1.26, so equation (3.5.1) holds.

We can obtain any balanced set of size $2j$ from S by making a sequence of the following types of replacements:

(1) Replace $x \in S$ with $x + 1 \in S^c$, where $(x, x + 1)$ is a couple of consecutive nodes of the same color. (Or replace $x + 1 \in S$ with $x \in S^c$).

(2) Replace $x \in S$ with $y \in S^c$, where $x < y$ are the same color and all ℓ nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$ are the opposite color of x and y ($\ell \geq 1$). (Or replace $y \in S$ with $x \in S^c$).

Therefore it suffices to show the following. Assume we're given a balanced set S, an odd-even pairing π that does not bridge $S\triangle T$ to $(S\triangle T)^c$, and a black-white pairing ρ that does not bridge S to S^c such that $\rho|_S$ and $\rho|_{S^c}$ are planar. After making either of the above two types of replacements to obtain S', we can construct an odd-even pairing π' that does not bridge $S'\triangle T$ to $(S'\triangle T)^c$ and a black-white pairing ρ' that does not bridge S' to S'^c such that $\rho'|_{S'}$ and $\rho'|_{S'^c}$ are

87

planar. After replacing S, π, ρ in equation (3.5.1) with S', π' and ρ', equation (3.5.1) still holds.

3.5.2.2. Lemmas

Lemma 3.5.7. *Let S be a balanced subset of nodes. Let x and y be two nodes of the same color and opposite parity with $x < y$ such that $x \in S$ and $y \notin S$. Let ρ be a black-white pairing such that ρ does not bridge S to S^c and let π be an odd-even pairing such that π does not bridge $S \triangle T$ to $(S \triangle T)^c$. Let $S' = S \setminus \{x\} \cup \{y\}$ and let ρ' be the pairing obtained by swapping the locations of x and y in ρ. Then*

(a) if $\pi(x) = y$,

    (i) π does not bridge $S' \triangle T$ to $(S' \triangle T)^c$, and

    (ii) when ρ is replaced with ρ', the right hand side of equation (3.5.1) changes sign.

(b) if $\pi(x) \neq y$, let π' be the pairing obtained from π by pairing x with y, $\pi(x)$ with $\pi(y)$, and leaving the remaining pairs the same. Then

    (i) π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$.

    (ii) when ρ is replaced with ρ' and π is replaced with π', the right hand side of equation (3.5.1) changes sign.

Proof. We will first prove part (a). The fact that π does not bridge $S' \triangle T$ to $(S' \triangle T)^c$ follows from the observation that since $\pi(x) = y$, both x and y are in $S \triangle T$ or both are in $(S \triangle T)^c$. If both x, y are in $S \triangle T$ then since we assumed $x \in S$ and $y \notin S$, y must be in T, so both x, y are in $(S' \triangle T)^c$. So π does not bridge $S' \triangle T$ to $(S' \triangle T)^c$.

88

Since we obtained ρ' from ρ by swapping the locations of x and y,

$\text{sign}_{BW}(\rho') = -\text{sign}_{BW}(\rho)$. The number of components in $\pi \cup \rho$ is the same as the number of components in $\pi \cup \rho'$ because when we replace ρ with ρ' the path $\pi(\rho(x)) - \rho(x) - x - y - \rho(y)$ is replaced with $\pi(\rho(x)) - \rho(x) - y - x - \rho(y)$. So the right hand side of equation (3.5.1) changes sign.

Next, we prove part (b). The proof of (i) relies on the observation that since x and y are the same color but opposite parity, exactly one of the nodes x, y is in T. This implies that x and y are both in $S \triangle T$ or both in $(S \triangle T)^c$ and that x and y are both in $S' \triangle T$ or both in $(S' \triangle T)^c$.

Since x and y are both in $S \triangle T$ or both in $(S \triangle T)^c$, $\pi(x)$ and $\pi(y)$ are both in $S \triangle T$ or both in $(S \triangle T)^c$. Since neither $\pi(x)$ nor $\pi(y)$ is x or y, $\pi(x)$ and $\pi(y)$ are both in $S' \triangle T$ or both in $(S' \triangle T)^c$. We conclude that π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$.

For the proof of (ii), first note that pairing x and y and $\pi(x)$ with $\pi(y)$ is the same as swapping the locations of y and $\pi(x)$. It follows that $\text{sign}_{OE}(\pi') = -\text{sign}_{OE}(\pi)$, and since $\text{sign}_{BW}(\rho') = -\text{sign}_{BW}(\rho)$, it remains to show that the number of components in $\pi' \cup \rho'$ and the number of components in $\pi \cup \rho$ differ by 1.

By letting $a = \pi(x)$ and $b = y$ in Definition 3.5.2, we see that (π', ρ) and (π, ρ) differ by a move of type A_{OE}. If $\pi(x)$ and y are in different components, this is clear, since $\pi(x)$ and y have the same parity. If $\pi(x)$ and y are in the same component, we must show that they are the same color if and only if there are an even number of transition pairs between them. This is because

    $-$ y and x are the same color,

– a path from y to x contains an odd number of transition pairs (since $x \in S$ and $y \notin S$), and

– x and $\pi(x)$ are the same color if and only if $(x, \pi(x))$ is a transition pair.

So, by Lemma 3.5.4, the number of components in $\pi' \cup \rho$ and the number of components in $\pi \cup \rho$ differ by one. Then, since $\pi'(x) = y$, by the proof of part (a), the number of components in $\pi' \cup \rho'$ is the same as the number of components in $\pi' \cup \rho$.

We conclude that when ρ is replaced with ρ' and π is replaced with π' the right hand side of equation (3.5.1) changes sign. $\square$

Lemma 3.5.8. *Let $S \subseteq \mathbf{N}$ be a balanced set. Let x, y be nodes of the same color such that $x \in S$, $y \in S^c$, $x < y$ and all ℓ nodes in the interval $[x+1, x+2, \ldots, y-1]$ are the opposite color of x and y ($\ell \geq 1$).*

Let ρ be a black-white pairing such that ρ does not bridge S to S^c and $\rho|_S$ and $\rho|_{S^c}$ are planar.

(1) *If $\rho(x)$ is not in the interval $[x + 1, \ldots, y - 1]$ and there is a node in this interval that is in S, let k be the smallest integer such that $x + k$ is in S and let ρ' be the pairing obtained from ρ by replacing the pairs $(x, \rho(x))$ and $(x + k, \rho(x + k))$ with the pairs $(x, x + k)$ and $(\rho(x), \rho(x + k))$. Then $\rho'|_S$ and $\rho'|_{S^c}$ are planar. Also, replacing ρ with ρ' does not change the right hand side of equation (3.5.1).*

(2) *If $\rho(y)$ is not in the interval $[x + 1, \ldots, y - 1]$ and there is a node in this interval that is in S^c, let k be the smallest integer such that $y - k$ is in S^c and let ρ' be the pairing obtained from ρ by replacing the pairs $(y, \rho(y))$ and*

$(y - k, \rho(y - k))$ *with the pairs* $(y, y - k)$ *and* $(\rho(y), \rho(y - k))$. *Then* $\rho'|_S$ *and* $\rho'|_{S^c}$ *are planar. Also, replacing* ρ *with* ρ' *does not change the right hand side of equation (3.5.1).*

Proof. Since the proofs of (1) and (2) are completely analogous, we will only prove (1).

We first show that $\rho'|_S$ is planar. Since we chose the smallest integer k such that $x + k$ is in S, there are no chords connecting two nodes in S that cross the chord $x \frown (x + k)$. We need to check that there are no chords connecting two nodes in S that cross the chord $\rho(x) \frown \rho(x + k)$. If there was such a crossing, that means that there is a node $a \in S$ such that one of the following holds:

(1) $a < \rho(x + k) < \rho(a) < \rho(x)$,

(2) $\rho(x + k) < a < \rho(x) < \rho(a)$,

(3) $a < \rho(x) < \rho(a) < \rho(x + k)$, or

(4) $\rho(x) < a < \rho(x + k) < \rho(a)$.

We use the facts that if $a > x$ then $a > x + k$ (since otherwise $a \in S^c$, a contradiction) or, similarly, if $\rho(a) > x$ then $\rho(a) > x + k$, to show that if any of the inequalities in (1), (2), (3), or (4) hold, then $\rho|_S$ is not planar.

For example, in case (1), if $a > x$ then $a > x + k$. So we have

$$x + k < a < \rho(x + k) < \rho(a),$$

which contradicts that $\rho|_S$ is planar. If $a < x$ then there are two cases. If $\rho(a) < x$, we have $a < \rho(x + k) < \rho(a) < x + k$. If instead $\rho(a) > x$, we have $a < x < \rho(a) < \rho(x)$. In both cases, we have a contradiction.

91

In case (2), if $a > x$, then we have $x < a < \rho(x) < \rho(a)$. If $a < x$ and $\rho(a) < x$, then $a < \rho(x) < \rho(a) < x$. If $a < x$ and $\rho(a) > x$, then $\rho(x + k) < a < x + k < \rho(a)$. In all cases, we have a contradiction.

Case (3) is similar to case (2), and case (4) is similar to case (1).

We conclude that $\rho'|_S$ is planar. Since $\rho|_{S^c}$ was planar and the nodes $x, x + k, \rho(x), \rho(x + k)$ are all in S, $\rho'|_{S^c}$ is also planar.

Next, we observe that the number of components in $\pi \cup \rho$ and the number of components in $\pi \cup \rho'$ differ by 1. This is because to obtain the pairing ρ' from ρ, we swapped the locations of x and $\rho(x + k)$. Since x and $\rho(x + k)$ are both in S and both the same color, (π, ρ') and (π, ρ) differ by a move of type A_{BW}. So by Corollary 3.5.6, the number of components in $\pi \cup \rho$ and the number of components in $\pi \cup \rho'$ differ by 1. Since $\text{sign}(\rho') = -\text{sign}(\rho)$, replacing ρ with ρ' does not change the right hand side of equation (3.5.1). $\qquad\square$

The following useful observation is immediate from the definitions.

Remark 3.5.9. Let σ be a pairing such that x and y are two nodes that are not paired in σ, and let σ' be the pairing obtained by swapping the locations of x and y in σ. Suppose S is a balanced subset of nodes such that $x \in S$ and $y \in S^c$. Let $S' = (S \setminus \{x\}) \cup \{y\}$. If σ does not bridge S to S^c, then σ' does not bridge S' to S'^c.

3.5.2.3. Proof That Equation (3.5.1) Holds

Throughout this proof, we assume that we are given a balanced set S, an odd-even pairing π that does not bridge $S \triangle T$ to $(S \triangle T)^c$, and a black-white pairing ρ that does not bridge S to S^c and is planar when restricted to S and when restricted to S^c.

For both types of replacements described in the overview of the proof, we will make the replacement to obtain S', and then construct a black-white pairing ρ' that does not bridge S' to S'^c such that $\rho|_{S'}$ and $\rho|_{S'^c}$ are planar and an odd-even pairing π' that does not bridge $S' \triangle T$ to $(S' \triangle T)^c$. We will show that after replacing S, π, ρ in equation (3.5.1) with S', π' and ρ', equation (3.5.1) still holds.

(1) Replace $x \in S$ with $x + 1 \in S^c$.

Suppose we replace $x \in S$ with $x + 1 \in S^c$ to obtain S'. There are two cases to consider based on whether or not $\pi(x) = x + 1$. In both cases, we let ρ' be the pairing obtained by swapping the locations of x and $x + 1$ in ρ. By Remark 3.5.9, ρ' is a black-white pairing that does not bridge S' to S'^c. Also note that since $\rho|_S$ and $\rho|_{S^c}$ are planar, $\rho'|_{S'}$ and $\rho'|_{S'^c}$ are planar.

Case 1. If $\pi(x) = x + 1$, π does not bridge $S' \triangle T$ to $(S' \triangle T)^c$ and when we replace ρ with ρ', the right hand side of equation (3.5.1) changes sign by Lemma 3.5.7. Since we swapped the locations of x and $x + 1$ in ρ to obtain ρ', $(-1)^{\text{\# of crosses of } \rho} = -(-1)^{\text{\# of crosses of } \rho'}$. So equation (3.5.1) holds.

Case 2. If $\pi(x) \neq x + 1$, let π' be the pairing obtained from π by pairing x with $x + 1$, $\pi(x)$ with $\pi(x + 1)$, and leaving the remaining pairs the same. By Lemma 3.5.7, π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$ and when we replace π with π' and ρ with ρ', the right hand side of equation (3.5.1) changes sign. As in Case 1, $(-1)^{\text{\# of crosses of } \rho} = -(-1)^{\text{\# of crosses of } \rho'}$, so equation (3.5.1) holds.

(2) Replace x with y, where $x < y$ are the same color and all ℓ nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$ are the opposite color of x and y ($\ell \geq 1$).

Suppose we replace $x \in S$ with $y \in S^c$ to obtain S'. There are several cases to consider based on whether x and y are paired with nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$.

Case 1. We first consider the case when both x and y are paired with a node in the interval $[x + 1, x + 2, \ldots, y - 1]$.

Construction of ρ'. Let $\rho^{(1)}$ be the pairing obtained by swapping the locations of x and y. By Remark 3.5.9, $\rho^{(1)}$ does not bridge S' to S'^c.

We observe that if $\ell > 2$, at least one of the pairings $\rho^{(1)}|_{S'}$, $\rho^{(1)}|_{S'^c}$ is not planar. To see this, observe that since $\rho|_S$ and $\rho|_{S^c}$ are planar, the nodes in the interval $[x+1, \ldots, \rho(x)-1]$ are in S^c and the nodes in the interval $[\rho(y)+1, \ldots, y-1]$ are in S (see Figure 3.9).

Suppose towards a contradiction that $\rho^{(1)}|_{S'}$ and $\rho^{(1)}|_{S'^c}$ are planar. Since $\rho^{(1)}|_{S'^c}$ is planar, all nodes in the interval $[x + 1, \ldots, \rho(y) - 1]$ are in S'. This means that either

(1) $\rho(y) = x + 1$, or

(2) $\rho(x) = x + 1$ and $\rho(y) = x + 2$.

If (1) holds, there is at least one node in the interval $[\rho(y) + 1, \ldots, y - 1]$ other than $\rho(x)$. By the observation in the previous paragraph, this node is in S'. If it is in the interval $[\rho(x) + 1, \ldots, y - 1]$ its chord crosses the $\rho(x) \frown y$ chord, contradicting the assumption that $\rho^{(1)}|_{S'}$ is planar. If it is in the interval $[x + 1, \ldots, \rho(x) - 1]$ it crossed the $\rho(x) \frown x$ chord, contradicting the planarity of $\rho|_S$. If (2) holds, there is at least one node in the interval $[\rho(y)+1, \ldots, y-1]$, this node is in S', and its chord crosses the $\rho(x) \frown y$ chord, contradicting the assumption that $\rho^{(1)}|_{S'}$ is planar.

Observe that since ρ pairs x and y with nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$, any crossings in $\rho^{(1)}|_{S'}$ must involve nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$.
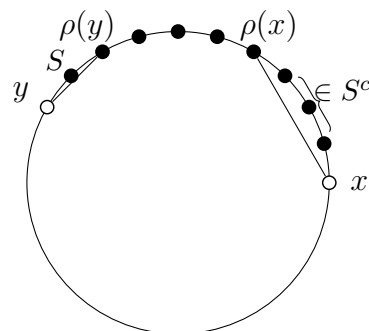

FIGURE 3.9. A possible configuration of the nodes in Case 1.

We claim that we can undo the crossings in $\rho^{(1)}|_{S'}$ one at a time without changing the right hand side of equation (3.5.1). To prove the claim, we will describe a procedure for constructing $\rho^{(m+1)}$ from $\rho^{(m)}$ so that $\rho^{(m+1)}|_{S'}$ has one fewer crossing than $\rho^{(m)}|_{S'}$.

Procedure 3.5.10. (Illustrated in Figure 3.10). Let $\rho^{(1)}$ be the pairing described above, so $\rho^{(1)}(y) = \rho(x)$. For $m \geq 1$, we construct $\rho^{(m+1)}$ as follows. Choose the smallest numbered node $i_m \in S'$ with $i_m > \rho^{(m)}(y)$ such that $i_m \frown \rho^{(m)}(i_m)$ crosses the chord $y \frown \rho^{(m)}(y)$. Since i_m and $\rho^{(m)}(y)$ are the same color and both in S', we can swap the locations of i_m and $\rho^{(m)}(y)$ in $\rho^{(m)}$ to obtain $\rho^{(m+1)}$, and this is a move of type A_{BW}. By Corollary 3.5.6, replacing $\rho^{(m)}$ with $\rho^{(m+1)}$ does not change the right hand side of equation (3.5.1). We claim that $\rho^{(m+1)}|_{S'}$ has exactly one fewer crossing than $\rho^{(m)}|_{S'}$.

Swapping the locations of i_m and $\rho^{(m)}(y)$ undoes the crossing of the chords $i_m \frown \rho^{(m)}(i_m)$ and $y \frown \rho^{(m)}(y)$. Next observe that since all the nodes $x + 1, x + 2, \ldots, y - 1$ are the same color, any chord that crosses the chord $i_m \frown y$ must have also crossed the chord $y \frown \rho^{(m)}(y)$. By the minimality of i_m, any chord that crossed the chord $y \frown \rho^{(m)}(y)$ (other than the chord $i_m \frown \rho^{(m)}(i_m)$) must also cross the chord $i_m \frown y$.

So we just need to check that pairing $\rho^{(m)}(y)$ with $\rho^{(m)}(i_m)$ did not create any crossings or undo any additional crossings. We claim that a black-white chord $a \frown \rho^{(m)}(a)$ with $a, \rho^{(m)}(a) \in S'$ crosses $\rho^{(m)}(y) \frown \rho^{(m)}(i_m)$ if and only if it crossed $i_m \frown \rho^{(m)}(i_m)$. This follows from the observation that neither one of $a, \rho^{(m)}(a)$ can be in the interval $[\rho^{(m)}(y) + 1, \ldots, i_m - 1]$ by the minimality of i_m.

Note that if $\rho^{(m)}$ does not bridge S' to S'^c, then $\rho^{(m+1)}$ does not bridge S' to S'^c. We repeat Procedure 3.5.10 until we have a pairing $\rho^{(n)}$ such that $\rho^{(n)}|_{S'}$ is planar.
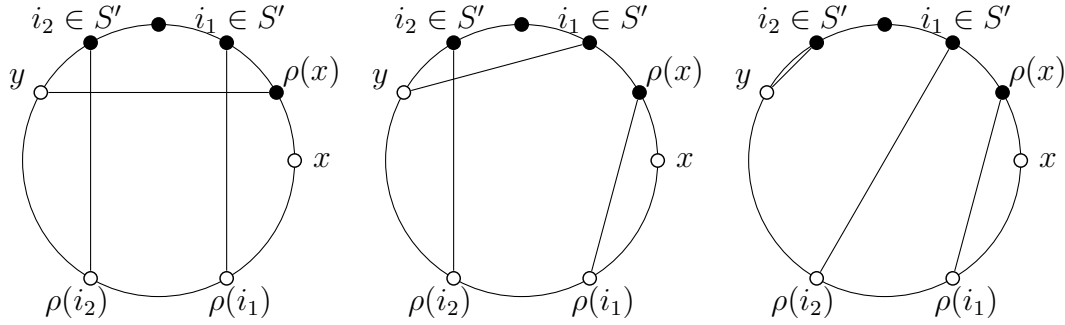
FIGURE 3.10. Illustration of the procedure for undoing the crossings in $\rho^{(1)}|_{S'}$. Left: Choose the smallest node i_1 in S' greater than $\rho^{(1)}(y) = \rho(x)$ whose chord crosses the chord $y \frown \rho(x)$. Center: Swap the locations of i_1 and $\rho(x)$ in $\rho^{(1)}$ to obtain $\rho^{(2)}$. Right: Repeat this procedure to obtain $\rho^{(2)}$.

Similarly, we can undo the crossings in $\rho^{(n)}|_{S'^c}$ one at a time without changing the right hand side of equation (3.5.1). The resulting pairing is ρ'.

Construction of π' and analysis of equation (3.5.1).

We break into subcases based on the parity of ℓ.

Case 1a. ℓ is odd

Analysis of LHS of (3.5.1). Since x and y are both paired with black nodes in the interval $[x+1, x+2, \ldots, y-1]$, $(-1)^{\# \text{ crosses of } \rho^{(1)}} = -(-1)^{\# \text{ crosses of } \rho}$ by Lemma 3.1.25. We will show that when we undo crossings to obtain ρ' as described, we apply Procedure 3.5.10 an odd number of times. Recall that every node between x and $\rho(x)$ is in S^c and every node between $\rho(y)$ and y is in S. It follows that $\rho(x) < \rho(y)$ or $\rho(x) = \rho(y) + 1$. Putting these facts together, we see that every node in S'^c crosses the $x \frown \rho(y)$ chord, and every node in S' crosses the $y \frown \rho(x)$ chord. Since there are an odd number of nodes in $\{x+1, \ldots, y-1\} \setminus \{\rho(x), \rho(y)\}$, we must apply

96

Procedure 3.5.10 an odd number of times. We conclude that $(-1)^{\#\text{ crosses of }\rho'} = (-1)^{\#\text{ crosses of }\rho}$.

Construction of π' and analysis of RHS of (3.5.1). Since ℓ is odd, x and y are the same parity, so we let π' be the pairing obtained by swapping the locations of x and y. We claim that π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$. Since x and y are the same parity and the same color, either both of x, y are in T or neither x nor y are in T. Since x, y are either both in T or both not in T, exactly one of x, y is in $S \triangle T$. So by Remark 3.5.9, π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$.

Also, $\pi' \cup \rho^{(1)}$ has the same number of components as $\pi \cup \rho$ because when we replace π with π' and ρ with $\rho^{(1)}$, the path $\cdots - \pi(x) - x - \rho(x) - \cdots$ in $\pi \cup \rho$ is replaced with $\cdots - \pi(y) - x - \rho(y) - \cdots$ in $\pi' \cup \rho^{(1)}$ and the path $\cdots - \pi(y) - y - \rho(y) - \cdots$ in $\pi \cup \rho$ is replaced with $\cdots - \pi(x) - y - \rho(x) - \cdots$.

Since we applied Procedure 3.5.10 an odd number of times and each application of Procedure 3.5.10 is a move of type A_{BW}, by Lemma 3.5.4,

$$(-1)^{\#\text{ comp in }\pi' \cup \rho'} = -(-1)^{\#\text{ comp in }\pi' \cup \rho^{(1)}} = -(-1)^{\#\text{ comp in }\pi \cup \rho}.$$

Since $\text{sign}_{BW}(\rho^{(1)}) = -\text{sign}_{BW}(\rho)$ and $\text{sign}_{BW}(\rho^{(m+1)}) = -\text{sign}_{BW}(\rho^{(m)})$, $\text{sign}_{BW}(\rho') = \text{sign}_{BW}(\rho)$. Finally, since $\text{sign}_{OE}(\pi') = -\text{sign}_{OE}(\pi)$, we conclude that equation (3.5.1) holds when π is replaced with π' and ρ is replaced with ρ'.

Case 1b. ℓ is even

Analysis of LHS of (3.5.1). As in Case 1a, $(-1)^{\#\text{ crosses of }\rho^{(1)}} = -(-1)^{\#\text{ crosses of }\rho}$. If $\ell = 2$ then we let $\rho' = \rho^{(1)}$ and both $\rho'|_{S'}$ and $\rho'|_{S'^c}$ are planar. If $\ell > 2$, then we will show that when we undo crossings in $\rho^{(1)}$ to obtain ρ' we apply Procedure 3.5.10 an even number of times. The reasoning is analogous to the ℓ is odd case: the claim

97

follows from the fact that there are an even number of nodes in $\{x+1, \ldots, y-1\} \setminus \{\rho(x), \rho(y)\}$. We conclude that $(-1)^{\# \text{ crosses of } \rho'} = -(-1)^{\# \text{ crosses of } \rho}$.

Construction of π' and analysis of RHS of (3.5.1). We break into cases based on whether $\pi(x) = y$ or $\pi(x) \neq y$. If $\pi(x) = y$, we let $\pi' = \pi$. If $\pi(x) \neq y$, we let π' be the pairing obtained from π by pairing x with y, $\pi(x)$ with $\pi(y)$, and leaving the remaining pairs the same. In both cases π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$ and $\text{sign}_{OE}(\pi)(-1)^{\# \text{ comp in } \pi \cup \rho} = \text{sign}_{OE}(\pi')(-1)^{\# \text{ comp in } \pi' \cup \rho^{(1)}}$ by Lemma 3.5.7. Since we applied Procedure 3.5.10 an even number of times and each application of Procedure 3.5.10 is a move of type A_{BW}, by Lemma 3.5.4, $(-1)^{\# \text{ comp in } \pi' \cup \rho^{(1)}} = (-1)^{\# \text{ comp in } \pi' \cup \rho'}$. Finally, since $\text{sign}_{BW}(\rho^{(1)}) = -\text{sign}_{BW}(\rho)$ and $\text{sign}_{BW}(\rho^{(m+1)}) = -\text{sign}_{BW}(\rho^{(m)})$, $\text{sign}_{BW}(\rho') = -\text{sign}_{BW}(\rho)$. We conclude that when ρ is replaced with ρ' and π is replaced with π', the right hand side of equation (3.5.1) changes sign. Thus equation (3.5.1) holds.

Case 2. We next consider the case where exactly one of x or y is paired with a black node in the interval $[x+1, x+2, \ldots, y-1]$.

Without loss of generality, suppose that x is the node that is paired with a black node in the interval $[x+1, \ldots, y-1]$. There are two subcases to consider.

Case 2a. If one of the ℓ nodes between x and y is in S^c, then let k be the smallest integer such that $y-k$ is in S^c and let ρ' be the pairing obtained by pairing y with $y-k$ and $\rho(y)$ with $\rho(y-k)$. By Lemma 3.5.8, $\rho'|_S$ and $\rho'|_{S^c}$ are planar, and replacing ρ with ρ' does not change the right hand side of equation (3.5.1).

To show that replacing ρ with ρ' does not change the left hand side of equation (3.5.1), we must show that $(-1)^{\# \text{ crosses of } \rho'} = (-1)^{\# \text{ crosses of } \rho}$. This follows from the observations that:

- since $\rho|_{S^c}$ is planar, the chords $(y - k) \frown \rho(y - k)$ and $y \frown \rho(y)$ do not cross, and

- a chord $a \frown \rho(a)$ crosses exactly one of $(y - k) \frown \rho(y - k)$, $y \frown \rho(y)$ if and only if it crosses exactly one of $\rho(y - k) \frown \rho(y)$, $(y - k) \frown y$.

Thus we have reduced Case 2a to Case 1, where both x and y are paired with nodes in the interval $[x + 1, x + 2, \ldots, y - 1]$.

Case 2b. If all of the ℓ nodes between x and y are in S (this includes the case where the only node between x and y is $x + 1$), then since $\rho|_S$ is planar, x is paired with $x + 1$. When we swap the locations of x and y to obtain $\rho^{(1)}$, $\rho^{(1)}|_{S'^c}$ is planar but $\rho^{(1)}|_{S'}$ is not planar. In fact, every node between $x + 1$ and y is in S (and therefore in S') and crosses the $y \frown (x + 1)$ chord. As in Case 1, we obtain ρ' by applying Procedure 3.5.10 to undo the crossings in $\rho^{(1)}|_{S'}$, and this does not change the right hand side of equation (3.5.1). We break into cases based on whether ℓ is odd or ℓ is even before constructing π'.

Case 2bi. ℓ is odd

Since exactly one of x and y is paired with a node in the interval $[x + 1, x + 2, \ldots, y - 1]$, $(-1)^{\text{\# crosses of } \rho^{(1)}} = (-1)^{\text{\# crosses of } \rho}$ by Lemma 3.1.25. We claim that when we undo crossings to obtain ρ', there are an even number of crossings to undo. This is because every node between $x + 1$ and y crosses the $(x + 1) \frown y$ chord, and since ℓ is odd there are an even number of such nodes. So $(-1)^{\text{\# crosses of } \rho'} = (-1)^{\text{\# crosses of } \rho}$.

We let π' be the pairing obtained by swapping the locations of x and y. By the type of arguments used in Case 1, π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$, $(-1)^{\text{\# comp in } \pi' \cup \rho'} = (-1)^{\text{\# comp in } \pi \cup \rho}$, and $\text{sign}_{BW}(\rho') = -\text{sign}_{BW}(\rho)$. We conclude that equation (3.5.1) holds when π is replaced with π' and ρ is replaced with ρ'.

Case 2bii. ℓ is even

As in Case 2bi, $(-1)^{\# \text{ crosses of } \rho^{(1)}} = (-1)^{\# \text{ crosses of } \rho}$. When we undo crossings to obtain ρ', there are an odd number of crossings to undo, so $(-1)^{\# \text{ crosses of } \rho'} = -(-1)^{\# \text{ crosses of } \rho}$.

We break into cases based on whether $\pi(x) = y$ or $\pi(x) \neq y$. If $\pi(x) = y$, we let $\pi' = \pi$. If $\pi(x) \neq y$, we let π' be the pairing obtained from π by pairing x with y, $\pi(x)$ with $\pi(y)$, and leaving the remaining pairs the same. In both cases π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$, and $\text{sign}_{OE}(\pi)(-1)^{\# \text{ comp in } \pi \cup \rho} = \text{sign}_{OE}(\pi')(-1)^{\# \text{ comp in } \pi' \cup \rho^{(1)}}$. By the type of arguments used in Case 1, $(-1)^{\# \text{ comp in } \pi' \cup \rho^{(1)}} = -(-1)^{\# \text{ comp in } \pi' \cup \rho'}$ and $\text{sign}_{BW}(\rho') = \text{sign}_{BW}(\rho)$. We conclude that when when ρ is replaced with ρ' and π is replaced with π', the right hand side of equation (3.5.1) changes sign. Thus equation (3.5.1) holds.

Case 3. Finally, we observe that we can reduce the case where neither x nor y is paired with a black node in the interval $[x+1, x+2, \ldots, y-1]$ to the case where exactly one of x or y is paired with a black node in the interval $[x+1, \ldots, y-1]$.

First assume that at least one of the ℓ nodes between x and y is in S. Choose the smallest integer k such that $x+k$ is in S. Let ρ' be the pairing that pairs x with $x+k$ and $\rho(x)$ with $\rho(x+k)$. By Lemma 3.5.8, $\rho'|_S$ is planar and $\rho|_{S^c}$ are planar and replacing ρ to ρ' does not change the right hand side of equation (3.5.1). The argument that $(-1)^{\# \text{ crosses of } \rho'} = (-1)^{\# \text{ crosses of } \rho}$ is the same as the argument in Case 2a.

Finally, if all of the ℓ nodes between x and y are in S^c, pair y with $x+\ell$. The argument then proceeds identically.

3.5.3. Proof That (π, ρ) Exists

We conclude by proving the existence of an odd-even pairing π and a black-white pairing ρ such that π does not bridge $S \triangle T$ to $(S \triangle T)^c$ and ρ does not bridge S to S^c.

Recall that in Section 3.5.2.1 we showed that for all j there is a balanced set S of size $2j$ with a planar black-white pairing ρ that does not bridge S to S^c, and by choosing $\pi = \rho$ we also have an odd-even pairing π that does not bridge $S \triangle T$ to $(S \triangle T)^c$.

We also showed that any balanced set of size $2j$ can be obtained from S by making a sequence of replacements of the form (1) and (2) from Section 3.5.2.1. Additionally, we showed that given an odd-even pairing π and a black-white pairing ρ such that π does not bridge $S \triangle T$ to $(S \triangle T)^c$ and ρ does not bridge S to S^c, and a set S' obtained from S by making a replacement of the form (1) or (2), we can modify π and ρ to obtain π' and ρ' so that π' does not bridge $S' \triangle T$ to $(S' \triangle T)^c$ and ρ' does not bridge S' to S'^c.

We conclude that for each balanced subset S, there is an odd-even pairing π and a black-white pairing ρ with the desired properties.

CHAPTER IV

PROOF OF DOUBLE-DIMER CONDENSATION

In this chapter we prove our condensation theorem (Theorem 2.4.3). We begin by generalizing Kenyon and Wilson's determinant formula. Next, we use our determinant formula and the Desnanot-Jacobi identity to prove Theorem 2.4.3. We conclude with a special case of this theorem where all signs are positive (Theorem 2.1.1).

4.1. Kenyon and Wilson's Determinant Formula

In this section, we prove our generalization of Kenyon and Wilson's determinant formula for tripartite pairings. Recall the statement of their theorem from Section 2.2:

Theorem 2.2.1. [12, Theorem 6.1] Suppose that the nodes are contiguously colored red, green, and blue (a color may occur zero times), and that σ is the (unique) planar pairing in which like colors are not paired together. We have

$$\widehat{\Pr}(\sigma) = \text{sign}_{OE}(\sigma) \det[1_{i,j \text{ colored differently}} X_{i,j}]_{j=2,4\ldots,2n}^{i=1,3,\ldots,2n-1}.$$

Recall from Section 2.3 that the proof of Theorem 2.2.1 requires two of Kenyon and Wilson's results from their study of groves. The first is the fact that when σ is a partition that is a tripartite pairing, $\overset{\ldots}{\Pr}(\sigma)$ can be expressed as a Pfaffian (Theorem 2.3.6). The second is Theorem 2.3.7, which states that when σ is a pairing, the double-dimer polynomials are a specialization of the grove polynomials.

102

Theorem 2.3.7. [13, Theorem 4.2] If a planar partition σ only contains pairs and we make the following substitutions to the grove partition polynomial $\overset{...}{\mathrm{Pr}}(\sigma)$:

$$
L_{i,j} \to
\begin{cases}
0, & \text{if } i \text{ and } j \text{ have the same parity,} \\
(-1)^{(|i-j|-1)/2} X_{i,j}, & \text{otherwise,}
\end{cases}
$$

then the result is $\mathrm{sign}_{OE}(\sigma)$ times the double-dimer pairing polynomial $\widehat{\mathrm{Pr}}(\sigma)$, when we interpret σ as a pairing.

We prove Theorem 2.4.2 (our version of Theorem 2.2.1) similarly. We can use Theorem 2.3.6 as stated, but we need an analogue of Theorem 2.3.7 that allows us to obtain our polynomials $\widetilde{\mathrm{Pr}}(\sigma)$ from the grove polynomials.

Theorem 4.1.1. *If a planar partition σ only contains pairs and we make the following substitutions to the grove partition polynomial $\overset{...}{\mathrm{Pr}}(\sigma)$:*

$$
L_{i,j} \to
\begin{cases}
0, & \text{if } i \text{ and } j \text{ are the same color,} \\
\mathrm{sign}(i,j) Y_{i,j}, & \text{otherwise,}
\end{cases}
$$

then the result is $\mathrm{sign}_c(\mathbf{N}) \mathrm{sign}_{OE}(\sigma) \widetilde{\mathrm{Pr}}(\sigma)$.

Proof. We proved in Section 3.4 that $\mathcal{Q}_{\pi,\rho}^{(DD)} = \mathrm{sign}_{OE}(\pi) \mathrm{sign}_{BW}(\rho) \mathcal{P}_{\pi,\rho}^{(t)}$ (see equation (3.4.6)). Recall that the matrix $\mathcal{Q}^{(DD)}$ gives coefficients for the monomials $Y_\rho = \prod_{(i,j)\in\rho} Y_{i,j}$ weighted by $(-1)^{\#\text{ crosses of } \rho}$ and by Lemma 3.1.13,

$$
\mathrm{sign}_c(\mathbf{N}) \mathrm{sign}_{BW}(\rho) \prod_{(i,j)\in\rho} \mathrm{sign}(i,j) = (-1)^{\#\text{ crosses of } \rho}.
$$

The theorem follows. $\qquad\square$

In addition to Theorems 2.3.6 and 4.1.1, we need the following lemma.

Lemma 4.1.2. *Let* $\mathbf{N}$ *be a set of* $2n$ *nodes and let* $(n_1, n_1 + 1), \ldots, (n_{2k}, n_{2k} + 1)$ *be a complete list of couples of consecutive nodes of the same color. Define* $(-1)^{i>j}$ *to be* -1 *if* $i > j$, *and* 1 *otherwise, and let*

$$M = [(-1)^{i>j} sign(i,j) Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n},$$

where $b_1 < b_2 < \cdots < b_n$ *are the black nodes listed in increasing order and* $w_1 < w_2 < \cdots < w_n$ *are the white nodes listed in increasing order. Then* M *is a block matrix where within each block, the signs of the entries are staggered in a checkerboard pattern.*

Furthermore, let t *be the total number of rows and columns of* M *that we need to multiply by* -1 *to obtain a matrix with entries whose signs are staggered in a checkerboard pattern where the upper left entry is positive. If node 1 is black,*

$$(-1)^t = sign_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$$

and if node 1 is white,

$$(-1)^t = (-1)^n sign_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}.$$

Proof. We will first prove the claim that M is a block matrix where within each block, the signs of the entries are staggered in a checkerboard pattern.

We begin with an example. Suppose we have 20 nodes colored so there are four couples of consecutive nodes of the same color: $(4, 5), (8, 9), (13, 14)$, and $(17, 18)$. Then M is the matrix shown below. We see that the blocks of M

correspond to consecutive nodes of the same color. More precisely, the last column in a block corresponds to a white node that precedes at least two consecutive black nodes. The first column in the next block corresponds to the first white node after these consecutive black nodes. Similarly, the nodes corresponding to the last row in a block and the first row in the next block are separated by consecutive white nodes.

$$
\left(
\begin{array}{cccc|cc|cccc}
Y_{1,2} & -Y_{1,4} & Y_{1,5} & -Y_{1,7} & -Y_{1,10} & Y_{1,12} & Y_{1,15} & -Y_{1,17} & Y_{1,18} & -Y_{1,20} \\
-Y_{3,2} & Y_{3,4} & -Y_{3,5} & Y_{3,7} & Y_{3,10} & -Y_{3,12} & -Y_{3,15} & Y_{3,17} & -Y_{3,18} & Y_{3,20} \\
\hline
-Y_{6,2} & Y_{6,4} & -Y_{6,5} & Y_{6,7} & Y_{6,10} & -Y_{6,12} & -Y_{6,15} & Y_{6,17} & -Y_{6,18} & Y_{6,20} \\
Y_{8,2} & -Y_{8,4} & Y_{8,5} & -Y_{8,7} & -Y_{8,10} & Y_{8,12} & Y_{8,15} & -Y_{8,17} & Y_{8,18} & -Y_{8,20} \\
-Y_{9,2} & Y_{9,4} & -Y_{9,5} & Y_{9,7} & Y_{9,10} & -Y_{9,12} & -Y_{9,15} & Y_{9,17} & -Y_{9,18} & Y_{9,20} \\
Y_{11,2} & -Y_{11,4} & Y_{11,5} & -Y_{11,7} & -Y_{11,10} & Y_{11,12} & Y_{11,15} & -Y_{11,17} & Y_{11,18} & -Y_{11,20} \\
-Y_{13,2} & Y_{13,4} & -Y_{13,5} & Y_{13,7} & Y_{13,10} & -Y_{13,12} & -Y_{13,15} & Y_{13,17} & -Y_{13,18} & Y_{13,20} \\
Y_{14,2} & -Y_{14,4} & Y_{14,5} & -Y_{14,7} & -Y_{14,10} & Y_{14,12} & Y_{14,15} & -Y_{14,17} & Y_{14,18} & -Y_{14,20} \\
-Y_{16,2} & Y_{16,4} & -Y_{16,5} & Y_{16,7} & Y_{16,10} & -Y_{16,12} & -Y_{16,15} & Y_{16,17} & -Y_{16,18} & Y_{16,20} \\
\hline
-Y_{19,2} & Y_{19,4} & -Y_{19,5} & Y_{19,7} & Y_{19,10} & -Y_{19,12} & -Y_{19,15} & Y_{19,17} & -Y_{19,18} & Y_{19,20}
\end{array}
\right).
$$

Since in the matrix above, row i does not correspond to node i, we introduce the following notation. We define the map $B : \{1, 2, \ldots, n\} \rightarrow \{b_1, \ldots, b_n\}$ by letting $B(i)$ be the node corresponding to row i. Similarly, we define $W : \{1, 2, \ldots, n\} \rightarrow \{w_1, \ldots, w_n\}$ by letting $W(j)$ be the node corresponding to column j. In the example above, $B(4) = 8$ and $W(8) = 17$.

We will show that M has the form

$$
\begin{array}{c c}
 & \begin{array}{cccc} W(j)<s_1 & s_1<W(j)<s_2 & \cdots & s_{k-1}<W(j)<s_k \end{array} \\
\begin{array}{c} B(i)<u_1 \\ u_1<B(i)<u_2 \\ \vdots \\ u_{k-1}<B(i)<u_k \end{array} &
\left[
\begin{array}{cccc}
A_{1,1} & A_{1,2} & \cdots & A_{1,k} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k,1} & A_{k,2} & \cdots & A_{k,k}
\end{array}
\right]
\end{array},
$$

where the notation $B(i) < u_1$ and $W(j) < s_1$ means that the block $A_{1,1}$ has rows i such that $B(i) < u_1$ and columns j such that $W(j) < s_1$. In each block $A_{i,j}$, the signs of the entries are staggered in a checkerboard pattern. Note that a block could be empty.

We first show that within a block, rows i and $i+1$ have opposite sign. There are two cases to consider:

(1) $B(i+1) - B(i) = 2$, and

(2) $B(i+1) - B(i) = 1$.

These are the only cases because if $B(i+1) - B(i) > 2$, then there is at least one couple of consecutive white nodes between $B(i)$ and $B(i+1)$, so rows i and $i+1$ are in different blocks.

In case (1), there is not a couple of consecutive nodes of the same color between $B(i)$ and $B(i+1)$, so $a_{B(i),w} = a_{B(i+1),w}$ for all w. It follows immediately from the definition $\mathrm{sign}(b,w) = (-1)^{(|b-w|+a_{b,w}-1)/2}$ that $\mathrm{sign}(B(i+1),w) = -\mathrm{sign}(B(i),w)$ unless $B(i) < w < B(i+1)$. But if $B(i) < w < B(i+1)$, the sign $(-1)^{b>w}$ flips. So in case (1), rows i and $i+1$ have opposite sign.

106

In case (2), $(B(i), B(i+1))$ is a couple of consecutive black nodes, so $|a_{B(i+1),w} - a_{B(i),w}| = 1$. If $B(i+1) > w$,

$$\text{sign}(B(i+1), w) = (-1)^{(B(i+1)-w+a_{B(i+1),w}-1)/2} = (-1)^{(B(i)+1-w+a_{B(i),w}+1-1)/2}$$

$$= -\text{sign}(B(i), w).$$

The case where $B(i+1) < w$ is completely analogous.

We conclude that within a block, rows i and $i+1$ have opposite sign. The proof that within a block columns j and $j+1$ have opposite sign is identical. So, within each block, the signs of the entries are staggered in a checkerboard pattern.

Since M is a block matrix where the signs of each block are staggered in a checkerboard pattern, we can always choose rows and columns to multiply by -1 so that the signs of the matrix entries are staggered in a checkerboard pattern and the upper left entry is positive. Let t be the total number of rows and columns we need to multiply by -1. We claim that if node 1 is black, $(-1)^t = \text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$ and if node 1 is white, $(-1)^t = (-1)^n \text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$.

We will prove the claim by induction on n, where $2n$ is the total number of nodes. The base case is when there are 4 nodes. In this case, $(-1)^n = 1$. We check all possible node colorings in Table 4.1.

Now assume the claim holds when there are $2n - 2$ nodes and suppose that $|\mathbf{N}| = 2n$. Choose the largest nodes $x, x+1$ such that $x, x+1$ are different colors.

Let $\mathbf{N}' = \{1, \ldots, 2n-2\}$. Define $\psi : \mathbf{N} - \{x, x+1\} \to \mathbf{N}'$ by

$$\psi(\ell) = \begin{cases} \ell & \text{if } \ell < x \\ \ell - 2 & \text{if } \ell > x + 1 \end{cases}.$$

107

black nodes	M	t	$\text{sign}_{\text{c}}(\mathbf{N})$	$(-1)^{\sum \lfloor \frac{n_i}{2} \rfloor}$
1, 2	$\begin{pmatrix} -Y_{1,3} & Y_{1,4} \\ Y_{2,3} & -Y_{2,4} \end{pmatrix}$	2	-1	-1
1, 3	$\begin{pmatrix} Y_{1,2} & -Y_{1,4} \\ -Y_{3,2} & Y_{3,4} \end{pmatrix}$	0	1	1
1, 4	$\begin{pmatrix} Y_{1,2} & -Y_{1,3} \\ Y_{4,2} & -Y_{4,3} \end{pmatrix}$	1	1	-1
3, 4	$\begin{pmatrix} Y_{3,1} & -Y_{3,2} \\ -Y_{4,1} & Y_{4,2} \end{pmatrix}$	0	-1	-1
2, 4	$\begin{pmatrix} -Y_{2,1} & Y_{2,3} \\ Y_{4,1} & -Y_{4,3} \end{pmatrix}$	2	1	1
2, 3	$\begin{pmatrix} -Y_{2,1} & -Y_{2,4} \\ Y_{3,1} & Y_{3,4} \end{pmatrix}$	1	1	-1

TABLE 4.1. The base case for the proof of Lemma 4.1.2. When a graph has 4 nodes, $(-1)^t = \text{sign}_{\text{c}}(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$.

That is, ψ defines a relabeling of the nodes of $\mathbf{N} - \{x, x + 1\}$ so that node 1 is labeled 1, ..., node $x - 1$ is labeled $x - 1$, node $x + 2$ is labeled $x, \ldots$, node $2n$ is labeled $2n - 2$.

Recall that $(n_1, n_1 + 1), \ldots, (n_{2k}, n_{2k} + 1)$ is a complete list of couples of consecutive nodes of the same color in $\mathbf{N}$. Let $(n_1', n_1' + 1), \ldots, (n_{2j}', n_{2j}' + 1)$ be a complete list of couples of consecutive nodes of the same color in $\mathbf{N}'$.

Let M' denote the matrix corresponding to $\mathbf{N}'$. Let t' denote the total number of rows and columns we need to multiply by -1 to get a matrix $M'_{(1)}$ with entries whose signs are staggered in a checkerboard pattern so that the upper left entry is positive. By the induction hypothesis,

$$(-1)^{t'} = \text{sign}_{\text{c}}(\mathbf{N}')(-1)^{\sum_{i=1}^{2j} \lfloor \frac{n_i'}{2} \rfloor}$$

if node 1 is black and

$$(-1)^{t'} = (-1)^{n-1}\text{sign}_\text{c}(\mathbf{N}')(-1)^{\sum\limits_{i=1}^{2j}\lfloor\frac{n_i'}{2}\rfloor}$$

if node 1 is white.

There are several cases to consider based on whether or not $\mathbf{N}$ and $\mathbf{N}'$ have the same number of couples of consecutive nodes of the same color. In each case, we will assume that node $2n$ is white. When $2n$ is black, the argument is completely analogous.

Each case will involve two steps:

(i) comparing $\text{sign}_\text{c}(\mathbf{N}')$ to $\text{sign}_\text{c}(\mathbf{N})$ and $(-1)^{\sum\lfloor\frac{n_i'}{2}\rfloor}$ to $(-1)^{\sum\lfloor\frac{n_i}{2}\rfloor}$, and

(ii) comparing t to t'.

Case 1. In the first case, $\mathbf{N}'$ has the same number of couples of consecutive nodes of the same color as $\mathbf{N}$. There are two ways this can occur: $x + 1 < 2n$, or $x + 1 = 2n$.

Case 1(a). $x + 1 < 2n$

We first assume that node 1 is black.

(i) Comparing $\text{sign}_\text{c}(\mathbf{N})$ to $\text{sign}_\text{c}(\mathbf{N}')$ and $(-1)^{\sum\lfloor\frac{n_i'}{2}\rfloor}$ to $(-1)^{\sum\lfloor\frac{n_i}{2}\rfloor}$.

Since $\mathbf{N}'$ has the same number of couples of consecutive nodes as $\mathbf{N}$, $\text{sign}_\text{c}(\mathbf{N}) = \text{sign}_\text{c}(\mathbf{N}')$. Since we assumed that x and $x + 1$ are the largest nodes such that x and $x + 1$ are different colors and $\mathbf{N}'$ has the same number of couples of consecutive nodes as $\mathbf{N}$, node $x - 1$ and all nodes in the interval $[x + 1, \ldots, 2n]$ are white. Since each node in the interval $[x + 1, \ldots, 2n]$ of $\mathbf{N}$ is white, each node in the interval $[x - 1, \ldots, 2n - 2]$ of $\mathbf{N}'$ is white, and node 1 is black in both $\mathbf{N}$ and $\mathbf{N}'$ we

have

$$(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n'_i}{2} \rfloor} = (-1)^{2n-x-1}(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}.$$

We conclude that

$$\mathrm{sign}_c(\mathbf{N'})(-1)^{\sum_{i=1}^{2j} \lfloor \frac{n'_i}{2} \rfloor} = (-1)^{2n-x-1}\mathrm{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2j} \lfloor \frac{n_i}{2} \rfloor}. \qquad (4.1.1)$$

(ii) Comparing t to t'.

Comparing the parity of t and t' is a multi-step process. Recall that we obtained $M'_{(1)}$ from M' by multiplying t' rows and columns of M'. We start by returning the nodes of $M'_{(1)}$ to their original labels to obtain $M'_{(2)}$. Then, we add the row and column corresponding to nodes x and $x+1$ to $M'_{(2)}$ to get $M'_{(3)}$. Finally, we let $\widetilde{M}$ be the matrix obtained from M by doing all the row and column multiplications we did to M' to get $M'_{(1)}$, and consider the relationship between $\widetilde{M}$ and $M'_{(3)}$.

Previously we defined the map $B : \{1, 2, \ldots, n\} \to \{b_1, \ldots, b_n\}$ by letting $B(i)$ be the node corresponding to row i of M and we defined $W : \{1, 2, \ldots, n\} \to \{w_1, \ldots, w_n\}$ by letting $W(j)$ be the node corresponding to column j of M. It will be convenient to let $R := B^{-1}$ and $C := W^{-1}$, so for example $R(6)$ is the row corresponding to the black node 6, and $C(7)$ is the column corresponding to the white node 7.

We define B', W', R' and C' analogously for M'.

Because this portion of the proof is long, we will illustrate the main ideas with an example. Let G be a graph with 8 nodes where nodes $1, 3, 4$ and 6 are colored black (see Figure 4.1). In this example, $x = 6$. So $\mathbf{N'} = \{1, 2, 3, 4, 5, 6\}$

110

where nodes $1, 3,$ and 4 are black. This means that

$$M' = \begin{pmatrix} Y_{1,2} & Y_{1,5} & -Y_{1,6} \\ -Y_{3,2} & -Y_{3,5} & Y_{3,6} \\ Y_{4,2} & Y_{4,5} & -Y_{4,6} \end{pmatrix}.$$

To obtain

$$M'_{(1)} = \begin{pmatrix} Y_{1,2} & -Y_{1,5} & Y_{1,6} \\ -Y_{3,2} & Y_{3,5} & -Y_{3,6} \\ Y_{4,2} & -Y_{4,5} & Y_{4,6} \end{pmatrix}$$

we multiply the second and third columns of M by -1, so $t' = 2$.

In general, to get from M' to $M'_{(1)}$, either we multiply all of the columns in a block or none of the columns in a block, because within each block, the signs of the entries are staggered in a checkerboard pattern. The same is true for the rows.
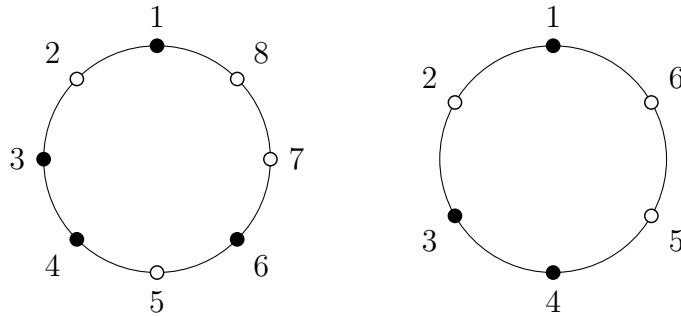

FIGURE 4.1. Shown left is an example of a possible node coloring $\mathbf{N}$ that could occur in Case 1(a) of the proof of Lemma 4.1.2. Nodes 6 and 7 are deleted from $\mathbf{N}$ and relabeled to obtain $\mathbf{N}'$, which is shown right.

Return the nodes of $M'_{(1)}$ to their original labels. Next, we return the nodes to their original labels (equivalently, we apply the map ψ^{-1}) to get $M'_{(2)}$. Note that the only entries that are affected are the entries in the columns corresponding to nodes $\psi(x+2), \ldots, \psi(2n)$.

In the example, we return node 6 to its original label of 8, resulting in the matrix

$$M'_{(2)} = \begin{pmatrix} Y_{1,2} & -Y_{1,5} & Y_{1,8} \\ -Y_{3,2} & Y_{3,5} & -Y_{3,8} \\ Y_{4,2} & -Y_{4,5} & Y_{4,8} \end{pmatrix}.$$

Add the row and column corresponding to nodes x and $x + 1$ to $M'_{(2)}$.
Now, add to $M'_{(2)}$ the column corresponding to node $x + 1$ (i.e. the column with entries $(-1)^{i>x+1}\text{sign}(i, x + 1)Y_{i,x+1})$ in between the columns corresponding to nodes $x - 1$ and $x + 2$. Also add the row corresponding to node x as the last row. Change the sign of the entries in the new column in the rows of M' that we multiplied by -1. Similarly, change the sign of the entries of the new row in the columns that we multiplied by -1. Call the resulting matrix $M'_{(3)}$.

In the example, we get

$$M'_{(3)} = \begin{pmatrix} Y_{1,2} & -Y_{1,5} & -Y_{1,7} & Y_{1,8} \\ -Y_{3,2} & Y_{3,5} & Y_{3,7} & -Y_{3,8} \\ Y_{4,2} & -Y_{4,5} & -Y_{4,7} & Y_{4,8} \\ -Y_{6,2} & Y_{6,5} & Y_{6,7} & -Y_{6,8} \end{pmatrix},$$

where note that we changed the sign of entries $Y_{6,\psi^{-1}(5)} = Y_{6,5}$ and $Y_{6,\psi^{-1}(6)} = Y_{6,8}$ because we multiplied the columns of M' corresponding to nodes 5 and 6 by -1.

Since we changed the signs of entries in the row $R(x)$ and the column $C(x + 1)$ as described above, $M'_{(3)}$ is a block matrix with checkerboard blocks with the following additional properties:

(1) All columns strictly to the left of column $C(x + 1)$ and all rows strictly above row $R(x)$ are in the same block.

(2) The jth entry of $C(x-1)$ and $C(x+2)$ have opposite sign because they were adjacent in M', which is checkerboard.

(3) All columns strictly to the right of $C(x+1)$ are in the same block(s).

(4) $C(x+1)$ is either in same block as $C(x+2)$ or in the same block as $C(x-1)$.

(5) $R(x)$ is either in the same block as all other rows, or in its own block.

Compare $\widetilde{M}$ to $M'_{(3)}$ and conclusion. Observe that if $i < x$ and $j > x+1$, then

$$\text{sign}(\psi(i), \psi(j)) = (-1)^{(\psi(j)-\psi(i)+a_{\psi(i),\psi(j)}-1)/2} = (-1)^{(j-i-2+a_{i,j}-1)/2} = -\text{sign}(i,j),$$

(4.1.2)

so the entries in the columns $C(x+2), \ldots, C(2n)$ in M are opposite in sign compared to the entries in columns $C'(\psi(x+2)), \ldots, C'(\psi(2n))$ in M'.

Returning to our example, we see that

$$M = \begin{pmatrix} Y_{1,2} & Y_{1,5} & -Y_{1,7} & Y_{1,8} \\ -Y_{3,2} & -Y_{3,5} & Y_{3,7} & -Y_{3,8} \\ Y_{4,2} & Y_{4,5} & -Y_{4,7} & Y_{4,8} \\ -Y_{6,2} & -Y_{6,5} & Y_{6,7} & -Y_{6,8} \end{pmatrix} \text{ and } M' = \begin{pmatrix} Y_{1,2} & Y_{1,5} & -Y_{1,6} \\ -Y_{3,2} & -Y_{3,5} & Y_{3,6} \\ Y_{4,2} & Y_{4,5} & -Y_{4,6} \end{pmatrix},$$

so indeed each entry in column $C(8) = 4$ has sign opposite of the corresponding entry of column $C(6) = 3$.

Now let $\widetilde{M}$ be the matrix M obtained by doing all of the t' row and column multiplications we did to M' to obtain $M'_{(1)}$. In our example,

$$\widetilde{M} = \begin{pmatrix} Y_{1,2} & -Y_{1,5} & -Y_{1,7} & -Y_{1,8} \\ -Y_{3,2} & Y_{3,5} & Y_{3,7} & Y_{3,8} \\ Y_{4,2} & -Y_{4,5} & -Y_{4,7} & -Y_{4,8} \\ -Y_{6,2} & Y_{6,5} & Y_{6,7} & Y_{6,8} \end{pmatrix}.$$

113

Since we changed the signs of entries in the row $R(x)$ and the column $C(x+1)$ as described in the previous step, by equation (4.1.2) $\widetilde{M}$ is identical to $M'_{(3)}$ except for the columns $C(x+2), \ldots, C(2n)$. Combining this fact with observations (1), (2), and (3) about $M'_{(3)}$ above, we conclude that that $\widetilde{M}$ is checkerboard except possibly for the row $R(x)$ and column $C(x+1)$. Both the entries in $R(x)$ and the entries in $C(x+1)$ alternate in signs, so it remains to determine whether or not we need to multiply $R(x)$ and/or $C(x+1)$ by -1.

Since $(-1)^{x>x+1} = 1$ and $\text{sign}(x+1, x) = 1$, the entry $(R(x), C(x+1))$ of $\widetilde{M}$ is positive. Also, since in a matrix with checkerboard entries, the entry $(R(x), C(2n))$ has positive sign, and all nodes $x+1, \ldots, 2n$ are white, the entry $(R(x), C(x+1))$ of the final checkerboard matrix we get after multiplying $R(x)$ and/or $C(x+1)$ by -1 has positive sign if and only if x is odd.

This means that x is odd if and only if we must multiply both $R(x)$ and $C(x+1)$ by -1 or neither by -1 to achieve a checkerboard matrix. We conclude that x is odd if and only if the parity of t is the same as the parity of t'. Since x is odd if and only if $(-1)^{2n-x-1} = 1$, by equation (4.1.1), t has the same parity as

$$\text{sign}_\text{c}(\mathbf{N})(-1)^{\sum_{i=1}^{2j} \lfloor \frac{n_i}{2} \rfloor},$$

as desired.

When node 1 is white, the argument is very similar, but we have

$$(-1)^{\sum_{i=1}^{2j} \lfloor \frac{n'_i}{2} \rfloor} = (-1)^{2n-x}(-1)^{\sum_{i=1}^{2j} \lfloor \frac{n_i}{2} \rfloor}$$

since $(8, 1)$ is a couple of consecutive nodes of the same color. It follows that

$$(-1)^{n-1}\text{sign}_c(\mathbf{N}')(-1)^{\sum\limits_{i=1}^{2j}\lfloor\frac{n_i'}{2}\rfloor} = (-1)^n(-1)^{2n-x-1}\text{sign}_c(\mathbf{N})(-1)^{\sum\limits_{i=1}^{2j}\lfloor\frac{n_i}{2}\rfloor}.$$

The rest of the argument is identical.

Case 1(b). $x + 1 = 2n$.

If $\mathbf{N}'$ has the same number of couples of consecutive nodes as $\mathbf{N}$ and $x + 1 = 2n$, there are two possibilities: either nodes $2n - 2$ and 1 are black, or nodes $2n - 2$ and 1 are white.

We first assume that node 1 is black.

(i) Comparing $\text{sign}_c(\mathbf{N})$ to $\text{sign}_c(\mathbf{N}')$ and $(-1)^{\sum\lfloor\frac{n_i'}{2}\rfloor}$ to $(-1)^{\sum\lfloor\frac{n_i}{2}\rfloor}$.

Since $\mathbf{N}'$ has the same number of couples of consecutive nodes as $\mathbf{N}$ and $x + 1 = 2n$, $\text{sign}_c(\mathbf{N}) = \text{sign}_c(\mathbf{N}')$ and

$$(-1)^{\sum\limits_{i=1}^{2k}\lfloor\frac{n_i'}{2}\rfloor} = (-1)^{\sum\limits_{i=1}^{2k}\lfloor\frac{n_i}{2}\rfloor},$$

so

$$(-1)^{\sum\limits_{i=1}^{2k}\lfloor\frac{n_i}{2}\rfloor}\text{sign}_c(\mathbf{N}) = (-1)^{\sum\limits_{i=1}^{2k}\lfloor\frac{n_i'}{2}\rfloor}\text{sign}_c(\mathbf{N}').$$

(ii) Comparing t to t'.

In this case ψ is the identity map, so $M'_{(1)} = M'_{(2)}$. We obtain $M'_{(3)}$ as described in Case 1(a). By the same reasoning as in Case 1(a), all columns to the left of column $C(2n)$ and all rows above row $R(2n-1)$ are in the same block. $C(2n)$ is either in the same block as the other columns or in its own block, and similarly for row $R(2n - 1)$.

Let $\widetilde{M}$ be the matrix M obtained by doing all of the t' multiplications we did to M' to obtain $M'_{(3)}$. Since ψ is the identity map, $\widetilde{M} = M'_{(3)}$. It remains to determine whether or not we need to multiply $R(2n-1)$ and/or $C(2n)$ by -1.

Since $(-1)^{2n-1>2n} = 1$ and $\mathrm{sign}(2n-1, 2n) = 1$, we need to multiply both $C(2n)$ and $R(2n-1)$ or neither in order for $\widetilde{M}$ to be checkerboard. So t has the same parity as t' and therefore the same parity as

$$(-1)^{\sum\limits_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor} \mathrm{sign}_{\mathrm{c}}(\mathbf{N}).$$

This proves the claim when node 1 is black. If node 1 is white, the only difference is that

$$(-1)^{\sum\limits_{i=1}^{2k} \lfloor \frac{n'_i}{2} \rfloor} = -(-1)^{\sum\limits_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$$

because node $2n$ was the first in a couple of consecutive white nodes in $\mathbf{N}$, and in $\mathbf{N}'$, node $2n-2$ is the first in a couple of consecutive white nodes. It follows that

$$(-1)^{n-1}(-1)^{\sum\limits_{i=1}^{2k} \lfloor \frac{n'_i}{2} \rfloor} \mathrm{sign}_{\mathrm{c}}(\mathbf{N}') = (-1)^{n}(-1)^{\sum\limits_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor} \mathrm{sign}_{\mathrm{c}}(\mathbf{N}).$$

The rest of the proof is the same.

Case 2. In the second case, $\mathbf{N}'$ has two fewer couples of consecutive nodes of the same color compared to $\mathbf{N}$. Again, there are two ways this can occur: $x+1 < 2n$, or $x+1 = 2n$.

Case 2(a). $x+1 < 2n$

We first assume that node 1 is black. As in Case 1(a), we illustrate the main ideas with an example. Let G be a graph with 8 nodes where nodes $1, 2, 5$ and 6 are

colored black (see Figure 4.2). In this example, $x = 6$, so $\mathbf{N}' = \{1, 2, 3, 4, 5, 6\}$ where nodes $1, 2$, and 5 are black.

(i) Comparing $\text{sign}_c(\mathbf{N})$ to $\text{sign}_c(\mathbf{N}')$ and $(-1)^{\sum \lfloor \frac{n_i'}{2} \rfloor}$ to $(-1)^{\sum \lfloor \frac{n_i}{2} \rfloor}$.

Since we assumed $\mathbf{N}'$ has two fewer couples of consecutive nodes of the same color compared to $\mathbf{N}$, nodes $x - 1$ and x are both black.
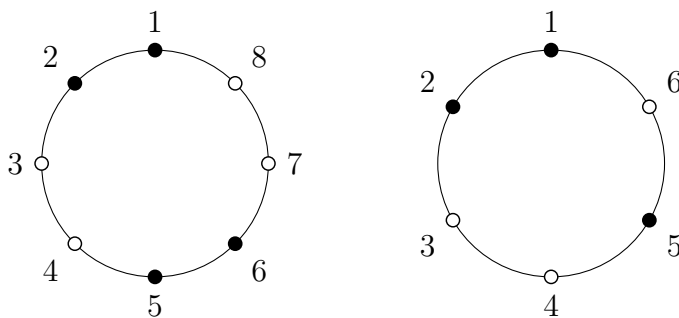

FIGURE 4.2. Shown left is an example of a possible node coloring $\mathbf{N}$ that could occur in Case 2(a) of the proof of Lemma 4.1.2. Nodes 6 and 7 are deleted from $\mathbf{N}$ and relabeled to obtain $\mathbf{N}'$, which is shown right.

Recall that $(s_i, s_i + 1)$ denotes a couple of consecutive black nodes of the same color and $(u_i, u_i + 1)$ denotes a couple of consecutive white nodes of the same color. By our assumptions, we have

$$\cdots < s_k < u_{k-(2n-x-2)} < \cdots < u_{k-1} < u_k.$$

When we remove nodes x and $x + 1$, there is a one-to-one correspondence between the inversions with respect to the node coloring of $\mathbf{N}$ and the inversions with respect to the node coloring of $\mathbf{N}'$ except for the inversions in $\mathbf{N}$ of the form (s_k, u_i) for $k - (2n - x - 2) \le i \le k$. Thus we have that

$$\text{sign}_c(\mathbf{N}) = (-1)^{2n-x-1} \text{sign}_c(\mathbf{N}').$$

In our example, $s_1 < u_1 < s_2 < u_2$ in $\mathbf{N}$ and $s_1' < u_1'$ in $\mathbf{N}'$. There is one fewer inversion in $\mathbf{N}'$ compared to $\mathbf{N}$ and so $\text{sign}_c(\mathbf{N}) = -\text{sign}_c(\mathbf{N}')$.

Next we compare $(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor}$ to $(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$. We begin with our example. Nodes 5 and 7 (nodes $x-1$ and $x+1$, respectively) are in $\{n_1, \ldots, n_{2k}\}$. In $\mathbf{N}'$, the black node $\psi(5) = 5$ is adjacent to the white node $\psi(8) = 6$ and the couple of consecutive white nodes $(7,8)$ is not replaced by a new couple of consecutive nodes, so

$$(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor} = (-1)^{\lfloor \frac{1}{2} \rfloor}(-1)^{\lfloor \frac{3}{2} \rfloor}(-1)^{\lfloor \frac{5}{2} \rfloor}(-1)^{\lfloor \frac{7}{2} \rfloor} = 1$$

while $(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor} = (-1)^{\lfloor \frac{1}{2} \rfloor}(-1)^{\lfloor \frac{3}{2} \rfloor} = -1$.

In general, since the nodes $x+1, \ldots, 2n$ are white and $x-1$ is black, the node $x-1$ and all nodes in the interval $[x+1, \ldots, 2n-1]$ are equal to n_i for some i. In $\mathbf{N}'$, all nodes in the interval $[\psi(x+2), \ldots, \psi(2n-1)]$ are equal to n_i' for some i.

From the observations that

 — to obtain $\mathbf{N}'$ we deleted nodes x and $x+1$ from $\mathbf{N}$,

 — $\psi(x-1)$ is adjacent to the white node $\psi(x+2)$ in $\mathbf{N}'$, and

 — $\psi(y) = y - 2$ for $y > x + 1$,

we get

$$(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor} = (-1)^{2n-x-2}(-1)^{\lfloor \frac{x-1}{2} \rfloor}(-1)^{\lfloor \frac{x+1}{2} \rfloor}(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}.$$

It follows that

$$(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor} = (-1)^{x+1}(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor},$$

so we conclude that

$$\text{sign}_c(\mathbf{N}')(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor} = \text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}.$$

(ii) Comparing t to t'. This portion of the proof is similar in structure to part (ii) of Case 1(a). In our example,

$$M' = \begin{pmatrix} -Y_{1,3} & Y_{1,4} & -Y_{1,6} \\ Y_{2,3} & -Y_{2,4} & Y_{2,6} \\ Y_{5,3} & -Y_{5,4} & Y_{5,6} \end{pmatrix}.$$

We multiply all three columns and the last row of M' by -1 to obtain $M'_{(1)}$ and return node 6 to its original label of 8 to obtain $M'_{(2)}$, so

$$M'_{(1)} = \begin{pmatrix} Y_{1,3} & -Y_{1,4} & Y_{1,6} \\ -Y_{2,3} & Y_{2,4} & -Y_{2,6} \\ Y_{5,3} & -Y_{5,4} & Y_{5,6} \end{pmatrix} \text{ and } M'_{(2)} = \begin{pmatrix} Y_{1,3} & -Y_{1,4} & Y_{1,8} \\ -Y_{2,3} & Y_{2,4} & -Y_{2,8} \\ Y_{5,3} & -Y_{5,4} & Y_{5,8} \end{pmatrix}.$$

Add the column and row corresponding to nodes $x+1$ and x to $M'_{(2)}$. Now, add the column corresponding to node $x + 1$ immediately to the left of the column corresponding to the node $x + 2$ in $M'_{(2)}$. Also add the row corresponding to node x as the last row. Change the sign of the entries in the new column in rows $R(a)$ if $R'(\psi(a))$ was a row we multiplied by -1. Similarly, change the sign of the entries in the new row in columns $C(b)$ if $C'(\psi(b))$ was a column we multiplied by -1, and call the resulting matrix $M'_{(3)}$, which is a block matrix with checkerboard blocks with properties (1)-(5) from Case 1(a). In our example,

$$M'_{(3)} = \begin{pmatrix} Y_{1,3} & -Y_{1,4} & Y_{1,7} & Y_{1,8} \\ -Y_{2,3} & Y_{2,4} & -Y_{2,7} & -Y_{2,8} \\ Y_{5,3} & -Y_{5,4} & Y_{5,7} & Y_{5,8} \\ Y_{6,3} & -Y_{6,4} & Y_{6,7} & Y_{6,8} \end{pmatrix}.$$

Compare $\widetilde{M}$ to the entries of $M'_{(3)}$ and conclusion. Observe that if $i < x$ and $j > x + 1$ then

$$\text{sign}(\psi(i), \psi(j)) = (-1)^{(\psi(j) - \psi(i) + a_{\psi(i), \psi(j)} - 1)/2} = (-1)^{(j - 2 - i + a_{i,j} - 2 - 1)/2} = \text{sign}(i, j),$$

so unlike in Case 1(a), the entries in the columns $C(x + 2), \ldots, C(2n)$ are the same sign in M as the entries in columns $C'(\psi(x + 2)), \ldots, C'(\psi(2n))$ in M'.

Returning to our example, we see that the entries in column $C(8) = 4$ have the same signs as the entries in column $C(6) = 3$, as

$$M = \begin{pmatrix} -Y_{1,3} & Y_{1,4} & Y_{1,7} & -Y_{1,8} \\ Y_{2,3} & -Y_{2,4} & -Y_{2,7} & Y_{2,8} \\ Y_{5,3} & -Y_{5,4} & -Y_{5,7} & Y_{5,8} \\ -Y_{6,3} & Y_{6,4} & Y_{6,7} & -Y_{6,8} \end{pmatrix} \text{ and } M' = \begin{pmatrix} -Y_{1,3} & Y_{1,4} & -Y_{1,6} \\ Y_{2,3} & -Y_{2,4} & Y_{2,6} \\ Y_{5,3} & -Y_{5,4} & Y_{5,6} \end{pmatrix}.$$

Let $\widetilde{M}$ be the matrix M obtained by doing all of the t' multiplications we did to M' to obtain $M'_{(3)}$. We see that $\widetilde{M} = M'_{(3)}$, so $\widetilde{M}$ is checkerboard except for the columns $C(x + 2), \ldots, C(2n)$ and also possibly the row $R(x)$ and/or the column $C(x + 1)$.

There are two cases to consider. In the first case, $C(x + 1)$ is not in the same block as the first column to its left, so we need to multiply $C(x + 1)$ by -1. Then, since $C(x + 1)$ is in the same block as $C(x + 2), \ldots, C(2n)$, we need to multiply the remaining $2n - x - 1$ columns by -1 as well. So we have done $t' + 2n - x$ total multiplications. It remains to consider whether or not we need to multiply row $R(x)$ by -1. Recall from Case 1(a) that after we are finished multiplying rows and columns and have obtained a checkerboard matrix, the entry $(R(x), C(x + 1))$ must

120

have positive sign if and only if x is odd. Since $(-1)^{x > x+1} = 1$, $\text{sign}(x+1, x) = 1$, and we multiplied $C(x+1)$ by -1, we multiply $R(x)$ by -1 if and only if x is odd. Therefore if x is odd, we have done $t' + 2n - x + 1$ multiplications, and if x is even, we have done $t' + 2n - x$ multiplications. We have thus shown that t has the same parity as t'.

If $C(x+1)$ is in the same block as the first column to its left, we do not need to multiply $C(x+1)$ by -1 but we still need to multiply the remaining $2n - x - 1$ columns by -1. So we have done $t' + 2n - x - 1$ total multiplications. Since we did not multiply $C(x+1)$ by -1, we multiply $R(x)$ by -1 if and only if x is even. Therefore, if x is even, we have done $t' + 2n - x$ total multiplications and if x is odd we have done $t' + 2n - x + 1$ multiplications. Again, t has the same parity as t'.

In both cases, t has the same parity as $\text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}$, which completes the proof when node 1 is black.

When node 1 is white, we have

$$\cdots < s_k < u_{k-(2n-x-1)} < \cdots < u_{k-1} < u_k$$

but (s_k, u_k) is not an inversion since node 1 is white. So we still have $\text{sign}_c(\mathbf{N}) = (-1)^{2n-x-1}\text{sign}_c(\mathbf{N}')$. Since $(2n, 1)$ is a couple of consecutive white nodes,

$$(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor} = (-1)^{2n-x-1}(-1)^{\lfloor \frac{x-1}{2} \rfloor}(-1)^{\lfloor \frac{x+1}{2} \rfloor}(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor}.$$

It follows that

$$(-1)^{\sum_{i=1}^{2k-2} \lfloor \frac{n_i'}{2} \rfloor} = (-1)^x(-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor},$$

So

$$\text{sign}_c(\mathbf{N}')(-1)^{\sum_{i=1}^{2k-2}\lfloor\frac{n_i'}{2}\rfloor} = -\text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k}\lfloor\frac{n_i}{2}\rfloor}.$$

We conclude that

$$(-1)^{n-1}\text{sign}_c(\mathbf{N}')(-1)^{\sum_{i=1}^{2k-2}\lfloor\frac{n_i'}{2}\rfloor} = (-1)^n\text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2k}\lfloor\frac{n_i}{2}\rfloor}.$$

The rest of the argument is the same.

Case 2(b). $x + 1 = 2n$

If $\mathbf{N}'$ has two fewer couples of consecutive nodes of the same color and compared to $\mathbf{N}$ and $x + 1 = 2n$, it must be the case that nodes $2n - 1$ and $2n - 2$ are both black and node 1 is white.

(i) Comparing $\text{sign}_c(\mathbf{N})$ to $\text{sign}_c(\mathbf{N}')$ and $(-1)^{\sum\lfloor\frac{n_i'}{2}\rfloor}$ to $(-1)^{\sum\lfloor\frac{n_i}{2}\rfloor}$.

Removing nodes $2n$ and $2n - 1$ does not remove any inversions with respect to the node coloring of $\mathbf{N}$ (recall that (s_k, u_k) is not an inversion when node 1 is white). Thus $\text{sign}_c(\mathbf{N}) = \text{sign}_c(\mathbf{N}')$.

Next observe that

$$(-1)^{\sum_{i=1}^{2k-2}\lfloor\frac{n_i'}{2}\rfloor} = (-1)^{\lfloor\frac{2n}{2}\rfloor}(-1)^{\lfloor\frac{2n-2}{2}\rfloor}(-1)^{\sum_{i=1}^{2k}\lfloor\frac{n_i}{2}\rfloor} = -(-1)^{\sum_{i=1}^{2k}\lfloor\frac{n_i}{2}\rfloor}.$$

Since node 1 is white, we have

$$(-1)^{n-1}\text{sign}_c(\mathbf{N}')(-1)^{\sum_{i=1}^{2j}\lfloor\frac{n_i'}{2}\rfloor} = (-1)^n\text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2j}\lfloor\frac{n_i}{2}\rfloor}.$$

(ii) Comparing t to t'.

This argument is identical to (ii) in Case 1(b), and we conclude that t has the same parity as t', and therefore the same parity as $(-1)^n\text{sign}_c(\mathbf{N})(-1)^{\sum_{i=1}^{2j}\lfloor\frac{n_i}{2}\rfloor}$. $\qquad\square$

Now that we have established Lemma 4.1.2, the proof of Theorem 2.4.2 is very similar to Kenyon and Wilson's proof of Theorem 2.2.1.

Theorem 2.4.2. Suppose that the nodes are contiguously colored red, green, and blue (a color may occur zero times), and that σ is the (unique) planar pairing in which like colors are not paired together. We have

$$\widetilde{\Pr}(\sigma) = \mathrm{sign}_{OE}(\sigma) \det[1_{i,j \text{ RGB-colored differently}} Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n},$$

where $b_1 < b_2 < \ldots < b_n$ are the black nodes listed in increasing order and $w_1 < w_2 < \ldots < w_n$ are the white nodes listed in increasing order.

Proof. Without loss of generality[1], we may assume that when we list the nodes in counterclockwise order starting with the red ones, they are in the order $1, 2, \ldots, 2n$. Combining Theorems 4.1.1 and 2.3.6 immediately gives a Pfaffian formula for the double-dimer model. For example, let G be a graph with eight nodes where nodes 1, 3, 4, and 6 are black. Assume the nodes are colored red, green and blue as shown in Figure 4.3, so $\sigma = ((1,8),(3,4),(5,6),(7,2))$. Then by Theorem 2.3.6,

---

[1] We can renumber the nodes while preserving their cyclic order in without changing the global sign of the Pfaffian in Theorem 2.3.6. This is because if we move the last row and column to be the first row and column, the sign of the Pfaffian changes. But since the entries above the diagonal must be non-negative, we negate the new first row and column and the Pfaffian changes sign again.
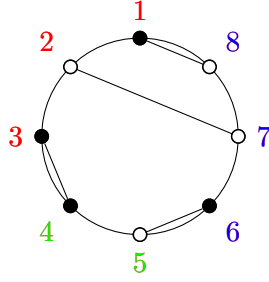
FIGURE 4.3. The example from the proof of Theorem 2.4.2. When G has eight nodes colored red, green, and blue as shown, the corresponding tripartite pairing is $((1, 8), (3, 4), (5, 6), (7, 2))$.

$$
\ddot{\mathrm{Pr}}(18|34|56|72) = \begin{pmatrix}
0 & 0 & 0 & L_{1,4} & L_{1,5} & L_{1,6} & L_{1,7} & L_{1,8} \\
0 & 0 & 0 & L_{2,4} & L_{2,5} & L_{2,6} & L_{2,7} & L_{2,8} \\
0 & 0 & 0 & L_{3,4} & L_{3,5} & L_{3,6} & L_{3,7} & L_{3,8} \\
-L_{4,1} & -L_{4,2} & -L_{4,3} & 0 & 0 & L_{4,6} & L_{4,7} & L_{4,8} \\
-L_{5,1} & -L_{5,2} & -L_{5,3} & 0 & 0 & L_{5,6} & L_{5,7} & L_{5,8} \\
-L_{6,1} & -L_{6,2} & -L_{6,3} & -L_{6,4} & -L_{6,5} & 0 & 0 & 0 \\
-L_{7,1} & -L_{7,2} & -L_{7,3} & -L_{7,4} & -L_{7,5} & 0 & 0 & 0 \\
-L_{8,1} & -L_{8,2} & -L_{8,3} & -L_{8,4} & -L_{8,5} & 0 & 0 & 0
\end{pmatrix}.
$$

$$(4.1.3)$$

So making the substitution in Theorem 4.1.1 expresses $\widetilde{\mathrm{Pr}}(18|34|56|72)$ as a

Pfaffian, up to a global sign.

Presently, we explain how we can obtain a determinant formula from this

Pfaffian formula. We make the substitution $L_{i,j} \to 0$ when i and j are both black or

both white and we reorder the rows and columns so the black nodes are listed first.

In the example, the above matrix becomes

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & L_{1,5} & L_{1,7} & L_{1,8} \\
0 & 0 & 0 & 0 & 0 & L_{3,5} & L_{3,7} & L_{3,8} \\
0 & 0 & 0 & 0 & -L_{4,2} & 0 & L_{4,7} & L_{4,8} \\
0 & 0 & 0 & 0 & -L_{6,2} & -L_{6,5} & 0 & 0 \\
0 & 0 & L_{2,4} & L_{2,6} & 0 & 0 & 0 & 0 \\
-L_{5,1} & -L_{5,3} & 0 & L_{5,6} & 0 & 0 & 0 & 0 \\
-L_{7,1} & -L_{7,3} & -L_{7,4} & 0 & 0 & 0 & 0 & 0 \\
-L_{8,1} & -L_{8,3} & -L_{8,4} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Simultaneous swaps of two different rows and corresponding columns changes the sign of the Pfaffian. Assuming the graph has $2k$ couples of consecutive nodes of the same color, we claim that the number of swaps needed so that the black nodes are listed first has the same parity as

$$\frac{n(n-1)}{2} + \sum_{i=1}^{2k} \left\lfloor \frac{n_i}{2} \right\rfloor,$$

if node 1 is black. If node 1 is white, the number of swaps needed has the same parity as

$$\frac{n(n+1)}{2} + \sum_{i=1}^{2k} \left\lfloor \frac{n_i}{2} \right\rfloor.$$

To prove this, we will first show that the number of node swaps needed to get from a node coloring with $2k$ couples of consecutive nodes of the same color to a node coloring that alternates black and white has the same parity as $\sum_{i=1}^{2k} \left\lfloor \frac{n_i}{2} \right\rfloor$.

We will prove this by induction on k. When $k = 0$, 0 swaps are needed, so the claim holds trivially. Assume the claim holds when $\mathbf{N}$ has $2(k-1)$ couples of consecutive nodes of the same color and suppose we have a set of nodes that has $2k$ couples of consecutive nodes of the same color. Let h be the smallest integer so

125

that n_{h-1} and n_h are different colors. Then n_{h-1} and n_h are the same parity and there are an even number of nodes in the interval $[n_{h-1}+1,\ldots,n_h]$, which alternate in color. If we swap n_h with $n_h - 1$, $n_h - 2$ with $n_h - 3$, ..., $n_{h-1}+2$ with $n_{h-1}+1$, we will have done $\frac{n_h - n_{h-1}}{2}$ swaps and we will have a node coloring with $2(k-1)$ couples of consecutive nodes of the same color. If n_h and n_{h-1} are both even then $\frac{n_h - n_{h-1}}{2}$ clearly has the same parity as $\left\lfloor \frac{n_h}{2} \right\rfloor + \left\lfloor \frac{n_{h-1}}{2} \right\rfloor$. If n_h and n_{h-1} are both odd then by writing $\frac{n_h - n_{h-1}}{2} = \frac{n_h - 1 - (n_{h-1}-1)}{2}$ we see that $\frac{n_h - n_{h-1}}{2}$ and $\left\lfloor \frac{n_h}{2} \right\rfloor + \left\lfloor \frac{n_{h-1}}{2} \right\rfloor$ have the same parity. By the induction hypothesis, the number of swaps needed to get to a node coloring that alternates black and white has the same parity as $\sum\limits_{\substack{1 \leq i \leq 2k \\ i \neq h, h-1}} \left\lfloor \frac{n_i}{2} \right\rfloor$.

The claim follows.

Assume node 1 is black. If there are no couples of consecutive nodes of the same color, the number of swaps needed to put the black nodes first is

$$1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$$

because the third node requires 1 swap, the fifth node requires 2 swaps, the seventh node requires 3 swaps, ..., and the $(2n-1)$st node requires $n-1$ swaps. So if there are $2k$ couples of consecutive nodes of the same color, since $\sum\limits_{i=1}^{2k} \left\lfloor \frac{n_i}{2} \right\rfloor$ node swaps are needed to get to a node coloring that alternates black and white, the number of swaps needed so that the black nodes are listed first has the same parity as

$$\frac{n(n-1)}{2} + \sum_{i=1}^{2k} \left\lfloor \frac{n_i}{2} \right\rfloor .$$

If node 1 is white, the number of swaps needed to put the black nodes first is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

126

because the second node requires 1 swap, the fourth node requires 2 swaps, . . . , and the $(2n)$th node requires n swaps. So the number of swaps needed so that the black nodes are listed first has the same parity as

$$\frac{n(n+1)}{2} + \sum_{i=1}^{2k} \left\lfloor \frac{n_i}{2} \right\rfloor.$$

Next, observe that after the rows and columns have been sorted, the matrix has the form

$$\begin{pmatrix} 0 & \pm L_{B,W} \\ \mp L_{W,B} & 0 \end{pmatrix}$$

where B represents the black nodes, W the white nodes, and the signs of the entries in $\pm L_{B,W}$ are $+$ if the black node has a smaller label than the white node and $-$ otherwise. The Pfaffian of this matrix is the determinant of the upper right submatrix times $(-1)^{\frac{n(n-1)}{2}}$. To summarize, after making the substitution $L_{i,j} \to 0$ when i and j are both black or both white and sorting the rows and columns so the black nodes are listed first,

$$\mathrm{Pf} \begin{pmatrix} 0 & L_{R,G} & L_{R,B} \\ -L_{G,R} & 0 & L_{G,B} \\ -L_{B,R} & -L_{B,G} & 0 \end{pmatrix} = (-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor} \det\left(L_{B,W}\right),$$

when node 1 is black. When node 1 is white,

$$\mathrm{Pf} \begin{pmatrix} 0 & L_{R,G} & L_{R,B} \\ -L_{G,R} & 0 & L_{G,B} \\ -L_{B,R} & -L_{B,G} & 0 \end{pmatrix} = (-1)^n (-1)^{\sum_{i=1}^{2k} \lfloor \frac{n_i}{2} \rfloor} \det\left(L_{B,W}\right).$$

In the example, after this substitution and reordering, the Pfaffian of matrix (4.1.3) is equal to

$$(-1)^{1+3} \det \begin{pmatrix} 0 & L_{1,5} & L_{1,7} & L_{1,8} \\ 0 & L_{3,5} & L_{3,7} & L_{3,8} \\ -L_{4,2} & 0 & L_{4,7} & L_{4,8} \\ -L_{6,2} & -L_{6,5} & 0 & 0 \end{pmatrix}$$

because $n_1 = 3$ and $n_2 = 7$.

Next we do the substitution $L_{i,j} \to \mathrm{sign}(i,j)Y_{i,j}$. The result is the matrix

$$M = [1_{i,j \text{ RGB-colored differently}}(-1)^{i>j}\mathrm{sign}(i,j)Y_{i,j}]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}$$

where $b_1 < b_2 < \cdots < b_n$ are the black nodes listed in increasing order and $w_1 < w_2 < \cdots < w_n$ are the white nodes listed in increasing order. By Theorem 4.1.1,

$$\widetilde{\mathrm{Pr}}(\sigma) = \mathrm{sign}_{OE}(\sigma)\mathrm{sign}_{\mathrm{c}}(\mathbf{N})(-1)^{\sum_{i=1}^{2k}\lfloor \frac{n_i}{2} \rfloor}\det(M)$$

if node 1 is black and

$$\widetilde{\mathrm{Pr}}(\sigma) = \mathrm{sign}_{OE}(\sigma)\mathrm{sign}_{\mathrm{c}}(\mathbf{N})(-1)^n(-1)^{\sum_{i=1}^{2k}\lfloor \frac{n_i}{2} \rfloor}\det(M)$$

is node 1 is white. By Lemma 4.1.2, M is a block matrix where within each block, the signs of the entries are staggered in a checkerboard pattern. Next, we multiply rows and columns of M by -1 so that the signs of the matrix entries are staggered in a checkerboard pattern and the upper left entry is positive. Call the resulting

matrix $\widetilde{M}$. By Lemma 4.1.2,

$$\widetilde{\Pr}(\sigma) = \mathrm{sign}_{OE}(\sigma)\det(\widetilde{M})$$

regardless of whether node 1 is black or white. Then, if we multiply every other row by -1 and every other column by -1, the signs of all matrix entries are positive and the determinant is unchanged. We conclude that

$$\widetilde{\Pr}(\sigma) = \mathrm{sign}_{OE}(\sigma)\det[1_{i,j\ \text{RGB-colored differently}}\,Y_{i,j}]_{j=w_1,w_2,...,w_n}^{i=b_1,b_2,...,b_n}.$$

Returning to our example, we find that

$$\widetilde{\Pr}(18|34|56|72) = \mathrm{sign}_{OE}(18|34|56|72)\det\begin{pmatrix} 0 & Y_{1,5} & Y_{1,7} & Y_{1,8} \\ 0 & Y_{3,5} & Y_{3,7} & Y_{3,8} \\ Y_{4,2} & 0 & Y_{4,7} & Y_{4,8} \\ Y_{6,2} & Y_{6,5} & 0 & 0 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 0 & Y_{1,5} & Y_{1,7} & Y_{1,8} \\ 0 & Y_{3,5} & Y_{3,7} & Y_{3,8} \\ Y_{4,2} & 0 & Y_{4,7} & Y_{4,8} \\ Y_{6,2} & Y_{6,5} & 0 & 0 \end{pmatrix}.$$

$\square$

Before proceeding to the proof of Theorem 2.4.3, we make several remarks about the matrix from Theorem 2.4.2, which we will denote M.

Remark 4.1.3. Recall that from the proof of Lemma 3.2.5 that $Y_{i,j} = \mathrm{sign}(i,j)K_{i,j}^{-1}$, where K is the Kasteleyn matrix of G. Thus the entries of M are either 0 or signed entries of the inverse Kasteleyn matrix of G.

In Kenyon and Wilson's matrix (see Theorem 2.2.1), $X_{i,j} = \text{sign}(i,j)K_{i,j}^{-1}$, where K is the Kasteleyn matrix of G^{BW} rather than the Kasteleyn matrix of G.

Although the nonzero entries of M are signed entries of K^{-1}, on a graph with more than two vertices it is never the case that $M = K^{-1}$. This is because M and K are in general different sizes: M is a $n \times n$ matrix, where n is the number of nodes, while K is a $|V_1| \times |V_2|$ matrix, where V_1 is the number of black vertices and V_2 is the number of white vertices.

Remark 4.1.4. By Cramer's rule, the entries of M^{-1} are ratios of double-dimer partition functions.[2] More precisely, Cramer's rule says

$$M^{-1} = \frac{1}{\det(M)} \left[\det(M_j^i) \right]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}$$

Note that while the (i,j)th entry of M is Y_{b_i,w_j}, the (i,j)th entry of M^{-1} is the determinant of the matrix M with row j and column i removed. Applying Theorem 2.4.2,

$$\det(M_j^i) = \frac{\text{sign}(\sigma_{ji}) Z_{\sigma_{ji}}^{DD}(G, \mathbf{N} - \{b_j, w_i\})}{(Z(G))^2},$$

where σ_{ji} denotes the unique tripartite pairing on the node set $\mathbf{N} - \{b_j, w_i\}$. So,

$$M^{-1} = \frac{\text{sign}(\sigma)}{Z_\sigma^{DD}(G, \mathbf{N})} [\text{sign}(\sigma_{ji}) Z_{\sigma_{ji}}^{DD}(G, \mathbf{N} - \{b_j, w_i\})]_{j=w_1,w_2,\ldots,w_n}^{i=b_1,b_2,\ldots,b_n}.$$

4.2. Proof of Theorem 2.4.3

Now that we have proven Theorem 2.4.2, Theorem 2.4.3 follows quickly from the proof method described in Section 2.4.

---

[2]We thank Gregg Musiker for making this observation.

Theorem 2.4.3. Divide the nodes into three circularly contiguous sets R, G, and B such that $|R|, |G|$, and $|B|$ satisfy the triangle inequality and let σ be the corresponding tripartite pairing. Let x, w be black nodes and let y, v be white nodes. Then

$$\text{sign}_{OE}(\sigma)\text{sign}_{OE}(\sigma'_{xywv})Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\})$$
$$= \text{sign}_{OE}(\sigma'_{xy})\text{sign}_{OE}(\sigma'_{wv})Z_{\sigma_{xy}}^{DD}(G, \mathbf{N} - \{x, y\})Z_{\sigma_{wv}}^{DD}(G, \mathbf{N} - \{w, v\})$$
$$- \text{sign}_{OE}(\sigma'_{xv})\text{sign}_{OE}(\sigma'_{wy})Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\})Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\})$$

where for $i, j \in \{x, y, w, v\}$, σ_{ij} is the unique planar pairing on $\mathbf{N} - \{i, j\}$ in which like colors are not paired together, and σ'_{ij} is the pairing after the the node set $\mathbf{N} - \{i, j\}$ has been relabeled so that the nodes are numbered consecutively.

Proof. First we assume that all pairings in the theorem statement exist. Let

$$M = [1_{i,j \text{ RGB-colored differently}} Y_{i,j}]_{j=w_1, w_2, \ldots, w_n}^{i=b_1, b_2, \ldots, b_n}$$

and let r_x and r_w denote the rows corresponding to nodes x and w, respectively.

We first move the columns corresponding to y and v (i.e. the columns with entries $Y_{i,y}$ and $Y_{i,v}$, respectively) to the columns r_x and r_w. We observe that we can do this without exchanging the column with entries $Y_{i,y}$ with the column with entries $Y_{i,v}$. For example, if $r_x < c_y < r_w < c_v$, we swap column c_y with column $c_y - 1$, then column $c_y - 1$ with column $c_y - 2, \ldots$, and column $r_x + 1$ with column r_x. Next, we swap column c_v with column $c_v - 1, \ldots$, and column $r_w + 1$ with column r_w. If instead $c_y < c_v < r_x < r_w$, we swap column c_v with column $c_v + 1, \ldots$,

and column $r_w - 1$ with column r_w before swapping c_y with column $c_y + 1, \ldots,$ and column $r_x - 1$ with column r_x.

Without loss of generality we assume that we move the column with entries $Y_{i,y}$ to column r_x and the column with entries $Y_{i,y}$ to column r_w to obtain the matrix $\widetilde{M}$. Let s_y denote the number of column swaps we make in the process of moving the column with entries $Y_{i,y}$. Let s_v denote the number of column swaps we make in the process of moving the column with entries $Y_{i,v}$. Note that s_y and s_v are well-defined up to parity. Note also that after making these swaps, the columns are still in ascending order, aside from the columns with entries $Y_{i,y}$ and $Y_{i,v}$.

By the Desnanot-Jacobi identity,

$$\det(\widetilde{M})\det(\widetilde{M}^{r_x,r_w}_{r_x,r_w}) = \det(\widetilde{M}^{r_x}_{r_x})\det(\widetilde{M}^{r_w}_{r_w}) - \det(\widetilde{M}^{r_x}_{r_w})\det(\widetilde{M}^{r_w}_{r_x}), \qquad (4.2.1)$$

where recall that M^t_s is the matrix M with row s and column t removed.

We apply Theorem 2.4.2 to each term in equation (4.2.1). First consider $\det(\widetilde{M}^{r_x}_{r_x})$. In order to apply Theorem 2.4.2 we must reorder the columns. Since we have removed the column r_x which had entries $Y_{i,y}$, s_v column swaps will put the columns in the correct (ascending) order. This follows from the previous observation that we moved the columns corresponding to y and v without exchanging the column with entries $Y_{i,y}$ with the column with entries $Y_{i,v}$.

We must also relabel the nodes $\mathbf{N} - \{x, y\}$ so that they are numbered consecutively. Recall that σ_{xy} denotes the unique planar pairing of $\mathbf{N} - \{x, y\}$ in which like colors are not paired together. When we relabel $\mathbf{N} - \{x, y\}$ we relabel σ_{xy} as well. Call the resulting node set $\mathbf{N}'$ and the resulting pairing σ'_{xy}. Then by

Theorem 2.4.2,

$$\det(\widetilde{M}_{r_x}^{r_x}) = (-1)^{s_v}\mathrm{sign}_{OE}(\sigma'_{xy})\frac{Z^{DD}_{\sigma'_{xy}}(G,\mathbf{N}')}{(Z^D(G))^2},$$

and thus

$$\det(\widetilde{M}_{r_x}^{r_x}) = (-1)^{s_v}\mathrm{sign}_{OE}(\sigma'_{xy})\frac{Z^{DD}_{\sigma_{xy}}(G,\mathbf{N}-\{x,y\})}{(Z^D(G))^2}.$$

Similarly, we have

$$\det(\widetilde{M}_{r_w}^{r_w}) = (-1)^{s_y}\mathrm{sign}_{OE}(\sigma'_{wv})\frac{Z^{DD}_{\sigma_{wv}}(G,\mathbf{N}-\{w,v\})}{(Z^D(G))^2},$$

$$\det(\widetilde{M}_{r_x}^{r_w}) = (-1)^{s_y}\mathrm{sign}_{OE}(\sigma'_{xv})\frac{Z^{DD}_{\sigma_{xv}}(G,\mathbf{N}-\{x,v\})}{(Z^D(G))^2},\quad\text{and}\qquad(4.2.2)$$

$$\det(\widetilde{M}_{r_w}^{r_x}) = (-1)^{s_v}\mathrm{sign}_{OE}(\sigma'_{yw})\frac{Z^{DD}_{\sigma_{yw}}(G,\mathbf{N}-\{y,w\})}{(Z^D(G))^2}.$$

It follows that the right hand side of equation (4.2.1) is

$$(-1)^{s_y}(-1)^{s_v}\Bigg(\mathrm{sign}_{OE}(\sigma'_{xy})\mathrm{sign}_{OE}(\sigma'_{wv})\frac{Z^{DD}_{\sigma_{xy}}(G,\mathbf{N}-\{x,y\})}{(Z^D(G))^2}\frac{Z^{DD}_{\sigma_{wv}}(G,\mathbf{N}-\{w,v\})}{(Z^D(G))^2}$$

$$-\mathrm{sign}_{OE}(\sigma'_{xv})\mathrm{sign}_{OE}(\sigma'_{yw})\frac{Z^{DD}_{\sigma_{xv}}(G,\mathbf{N}-\{x,v\})}{(Z^D(G))^2}\frac{Z^{DD}_{\sigma_{yw}}(G,\mathbf{N}-\{y,w\})}{(Z^D(G))^2}\Bigg).$$

Applying Theorem 2.4.2 to the left hand side of equation (4.2.1), we have

$$\det(\widetilde{M}) = (-1)^{s_y}(-1)^{s_v}\mathrm{sign}_{OE}(\sigma)\frac{Z^{DD}_{\sigma}(G,\mathbf{N})}{(Z^D(G))^2},\quad\text{and}$$

$$\det(\widetilde{M}_{r_x,r_w}^{r_x,r_w}) = \mathrm{sign}_{OE}(\sigma'_{xywv})\frac{Z^{DD}_{\sigma_{xywv}}(G,\mathbf{N}-\{x,y,w,v\})}{(Z^D(G))^2}.\qquad(4.2.3)$$

We conclude that

$$\text{sign}_{OE}(\sigma)\text{sign}_{OE}(\sigma'_{xywv})Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\})$$

$$= \text{sign}_{OE}(\sigma'_{xy})\text{sign}_{OE}(\sigma'_{wv})Z_{\sigma_{xy}}^{DD}(G, \mathbf{N} - \{x, y\})Z_{\sigma_{wv}}^{DD}(G, \mathbf{N} - \{w, v\})$$

$$-\text{sign}_{OE}(\sigma'_{xv})\text{sign}_{OE}(\sigma'_{wy})Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\})Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\}).$$

It is not necessarily the case that the pairings $\sigma_{xy}, \sigma_{wv}, \sigma_{xv}, \sigma_{wy}$, and σ_{xywv} all exist. First consider the case where one of the pairings $\sigma_{xy}, \sigma_{wv}, \sigma_{xv}, \sigma_{wy}$ does not exist. Without loss of generality, assume that σ_{xy} does not exist. This means that the number of nodes of different colors in $\mathbf{N} - \{x, y\}$ do not satisfy the triangle inequality. Then $\det(\widetilde{M}_{r_x}^{r_x}) = 0$ since every black-white pairing contains an RGB-monochromatic pair. There are two possibilities in this case: either the theorem statement holds trivially, or

$$Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) = Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\})Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\}).$$

$$(4.2.4)$$

Since the numbers of nodes of different colors in $\mathbf{N} - \{x, y\}$ do not satisfy the triangle inequality, without loss of generality, we may assume there are more red nodes than the combined number of blue and green nodes in $\mathbf{N} - \{x, y\}$. Since we assumed that in $\mathbf{N}$, $|R|, |G|$, and $|B|$ satisfy the triangle inequality, it follows that $|R| = |G| + |B|$ in $\mathbf{N}$. Assuming without loss of generality that when we list the nodes in counterclockwise order starting with the red ones, they are in the order $1, 2, \ldots, 2n$, this means σ is the pairing $((1, 2n), (2, 2n - 1), \ldots, (n, n + 1))$. It must be the case that x and y are both green or blue, so if either w or v is green, then σ_{xywv} does not exist and either σ_{xv} or σ_{wv} does not exist, so the equality holds

trivially. If both w and v are red, then σ_{xv}, σ_{wv} and σ_{xywv} all exist. In this case, the rest of the proof proceeds as above and we have

$$\text{sign}_{OE}(\sigma)\text{sign}_{OE}(\sigma_{xywv})Z_\sigma^{DD}(G,\mathbf{N})Z_{\sigma_{xywv}}^{DD}(G,\mathbf{N}-\{x,y,w,v\})$$
$$= -\text{sign}_{OE}(\sigma_{xv})\text{sign}_{OE}(\sigma_{wy})Z_{\sigma_{xv}}^{DD}(G,\mathbf{N}-\{x,v\})Z_{\sigma_{wy}}^{DD}(G,\mathbf{N}-\{w,y\}).$$

Recall that inversions in a planar pairing correspond to nestings (see Remark 3.1.17). Because σ is the pairing $((1,2n),(2,2n-1),\ldots,(n,n+1))$, $\text{sign}(\sigma_{xv}) = \text{sign}(\sigma_{wy})$ and $\text{sign}(\sigma_{xywv}) = \text{sign}(\sigma) \cdot (-1)^{n-1} \cdot (-1)^{n-2}$. Equation (4.2.4) follows.

If σ_{xywv} does not exist, then this means that the numbers of nodes of different colors in $\mathbf{N} - \{x,y,w,v\}$ do not satisfy the triangle inequality. Without loss of generality, we may assume there are more red nodes than the combined number of blue and green nodes in $\mathbf{N} - \{x,y,w,v\}$. By the same reasoning as above, $\det(\widetilde{M}_{r_x,r_w}^{r_x,r_w}) = 0$. There are two possibilities. There are either four more red nodes than the combined number of blue and green nodes in $\mathbf{N} - \{x,y,w,v\}$, or there are two more red nodes than the combined number of blue and green nodes in $\mathbf{N} - \{x,y,w,v\}$. If there are four more red nodes than the combined number of blue and green nodes, the equation holds trivially. If there are two more red nodes than the combined number of blue and green nodes, there are two possibilities. If any one of x,y,w, or v is red, then the equation holds trivially. If all of x,y,w, and v are green or blue, then in the original node set $\mathbf{N}$, $|R|+2 = |G|+|B|$. So each of

135

the pairings $\sigma_{xy}, \sigma_{wv}, \sigma_{xv}$, and σ_{wy} are $((1, 2n - 2), (2, 2n - 1), \ldots)$. Then we have

$$\text{sign}_{OE}(\sigma_{xy})\text{sign}_{OE}(\sigma_{wv})Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\})Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\})$$

$$= \text{sign}_{OE}(\sigma_{xv})\text{sign}_{OE}(\sigma_{wy})Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\})Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\}).$$

Since inversions in a planar pairing correspond to nestings, all pairings have the same sign. So in this case,

$$Z^{DD}_{\sigma_{xy}}(G, \mathbf{N} - \{x, y\})Z^{DD}_{\sigma_{wv}}(G, \mathbf{N} - \{w, v\}) = Z^{DD}_{\sigma_{xv}}(G, \mathbf{N} - \{x, v\})Z^{DD}_{\sigma_{wy}}(G, \mathbf{N} - \{w, y\}).$$

$\square$

Remark 4.2.1. To simplify the expression in Theorem 2.4.3, it suffices to know the RGB-coloring of the nodes x, y, w, v.

Without loss of generality, assume that when we list the nodes in counterclockwise order starting with the red ones, they are in the order $1, 2, \ldots, 2n$. Let $|RG(\sigma)|$ be the number of red-green pairs in σ. Define $|GB(\sigma)|$ and $|RB(\sigma)|$ similarly. Assume that $|RG(\sigma)|, |GB(\sigma)|$, and $|RB(\sigma)|$ are all nonzero.

If the set of nodes $\{x, y\}$ contains one red node and one blue node, then σ_{xy} has one fewer red-blue pair than σ, but the number of red-green and green-blue pairs is the same (see Figure 4.4). By Remark 3.1.17, to determine the relationship between $\text{sign}_{OE}(\sigma)$ and $\text{sign}_{OE}(\sigma'_{xy})$ it suffices to count the number of nestings in the diagram of σ that involve a red-blue pair (n_r, n_b). There is one nesting for each red-blue pair other than (n_r, n_b), one nesting for each red-green pair, and one nesting for each green-blue pair (see Figure 4.5). Therefore,

$$\text{sign}(\sigma'_{xy}) = \text{sign}(\sigma) \cdot (-1)^{|RG(\sigma)|} \cdot (-1)^{|GB(\sigma)|} \cdot (-1)^{|RB(\sigma)|-1}.$$
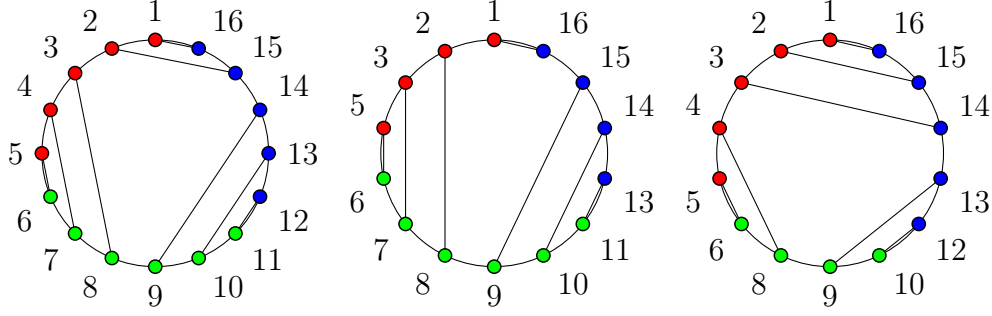
136

FIGURE 4.4. The tripartite pairing on $\mathbf{N} - \{4, 12\}$ (shown center) has one fewer red-blue pair than the tripartite pairing on $\mathbf{N}$ (shown left). The tripartite pairing on $\mathbf{N} - \{7, 11\}$ (shown right) has one fewer red-green pair, one fewer green blue pair, and one more red-blue pair compared to the pairing on $\mathbf{N}$.

If the set of nodes $\{x, y\}$ contains one red node and one green node, then σ_{xy} has one fewer red-green pair than σ, but the number of red-blue and green-blue pairs is the same. So we count the number of nestings in the diagram of σ that involve a red-green pair (n_r, n_g). There is one nesting for each red-green pair other than (n_r, n_g), and one nesting for each red-blue pair. Therefore,

$$\mathrm{sign}_{OE}(\sigma'_{xy}) = \mathrm{sign}_{OE}(\sigma) \cdot (-1)^{|RB(\sigma)|} \cdot (-1)^{|RG(\sigma)|-1}.$$

Similarly, if the set of nodes $\{x, y\}$ contains one green node and one blue node, then

$$\mathrm{sign}_{OE}(\sigma'_{xy}) = \mathrm{sign}_{OE}(\sigma) \cdot (-1)^{|RB(\sigma)|} \cdot (-1)^{|GB(\sigma)|-1}.$$

If both x and y are green nodes, then σ_{xy} has one fewer red-green pair, one fewer green-blue pair, and one more red-blue pair, as shown in Figure 4.4. Removing a red-green pair from σ removes $|RB(\sigma)| + |RG(\sigma)| - 1$ nestings. Then, removing a green-blue pair removes $|RB(\sigma)| + |GB(\sigma)| - 1$ nestings. After these pairs have been removed, adding a red-blue pair results in $|RB(\sigma)| + |GB(\sigma)| - 1 +$
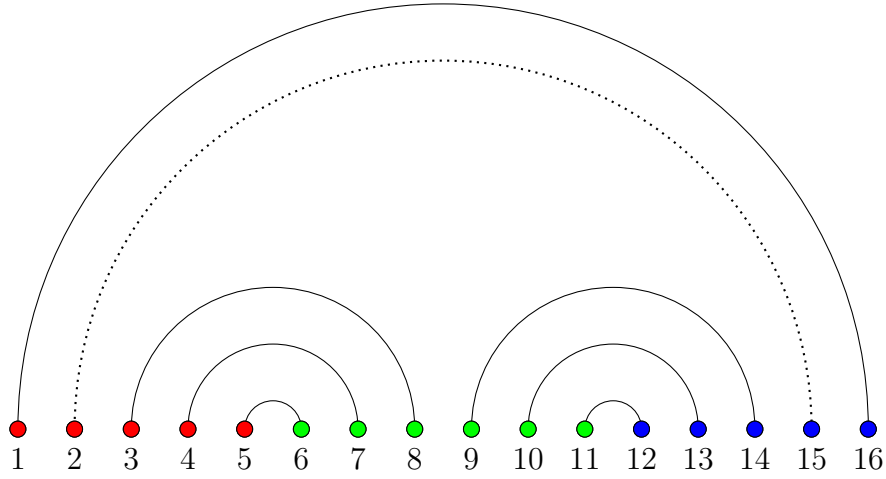
FIGURE 4.5. When a red-blue pair is removed from σ, the number of nestings in the diagram of σ decreases by $|RB(\sigma)| - 1 + |RG(\sigma)| + |GB(\sigma)|$.

$|RG(\sigma)| - 1$ additional nestings. Therefore,

$$\text{sign}(\sigma'_{xy}) = \text{sign}(\sigma) \cdot (-1)^{|RB(\sigma)|}.$$

If both x and y are red nodes, σ_{xy} has one fewer red-blue pair, one fewer red-green pair, and one more green-blue pair. Removing a red-blue pair from σ removes $|RG(\sigma)| + |GB(\sigma)| + |RB(\sigma)| - 1$ nestings. Then, removing a red-green pair removes $|RB(\sigma)| - 1 + |RG(\sigma)| - 1$ nestings. After these pairs have been removed, adding a green-blue pair results in $|RB(\sigma)| - 1 + |GB(\sigma)|$ additional nestings. Thus

$$\text{sign}(\sigma'_{xy}) = \text{sign}(\sigma) \cdot (-1)^{|RB(\sigma)|}.$$

Similarly, if both x and y are blue,

$$\text{sign}(\sigma'_{xy}) = \text{sign}(\sigma) \cdot (-1)^{|RB(\sigma)|}.$$

If we assume that the nodes x, y, w, v alternate black and white and the set $\{x, y, w, v\}$ contains at least one node of each RGB color, we can use Remark 4.2.1 to show that the all the signs in Theorem 2.4.3 are positive.

Theorem 2.1.1. Divide the nodes into three circularly contiguous sets R, G, and B such that $|R|, |G|$ and $|B|$ satisfy the triangle inequality and let σ be the corresponding tripartite pairing. Let x, y, w, v be nodes appearing in a cyclic order such that the set $\{x, y, w, v\}$ contains at least one node of each RGB color. If x and w are both black and y and v are both white, then

$$
\begin{aligned}
Z_\sigma^{DD}(G, \mathbf{N}) Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) &= Z_{\sigma_{xy}}^{DD}(G, \mathbf{N} - \{x, y\}) Z_{\sigma_{wv}}^{DD}(G, \mathbf{N} - \{w, v\}) \\
&\quad + Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\}) Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\}).
\end{aligned}
$$

Proof. Without loss of generality, assume that when we list the nodes in counterclockwise order starting with the red ones, they are in the order $1, 2, \ldots, 2n$. Assume also that one of the nodes x, y, w, v is red, two are green, and one is blue. The other cases are very similar. By the assumption that the nodes are in cyclic order, there are two possibilities[3]:

(i) One of the sets $\{x, y\}, \{w, v\}$ consists of a red node and a green node and the other consists of a green node and a blue node. Also, one of the sets $\{x, v\}, \{y, w\}$ consists consists of one red node and one blue node, and the other consists of two green nodes.

(ii) One of the sets $\{x, v\}, \{y, w\}$ consists of a red node and a green node and the other consists of a green node and a blue node. Also, one of the sets

---

[3]The assumption that the nodes x, y, w, v are in cyclic order is required. Otherwise, it would be possible for x to be red, y to be green, w to be blue, and v to be green. In this case, the sets $\{x, y\}$ and $\{x, v\}$ consist of one red node and one green node, and the sets $\{w, v\}$ and $\{y, w\}$ consists of one green node and one blue node.

$\{x, y\}, \{w, v\}$ consists consists of one red node and one blue node, and the other consists of two green nodes.

We only prove case (i), as case (ii) is essentially the same. By Remark 4.2.1,

$$\text{sign}_{OE}(\sigma'_{xy})\text{sign}_{OE}(\sigma'_{wv}) = (-1)^{|RB(\sigma)|}(-1)^{|RG(\sigma)|-1}(-1)^{|RB(\sigma)|}(-1)^{|GB(\sigma)|-1}$$

$$= (-1)^{|RG(\sigma)|}(-1)^{|GB(\sigma)|}, \text{ and}$$

$$\text{sign}_{OE}(\sigma'_{xv})\text{sign}_{OE}(\sigma'_{wy}) = (-1)^{|RG(\sigma)|}(-1)^{|GB(\sigma)|}(-1)^{|RB(\sigma)|-1}(-1)^{|RB(\sigma)|}$$

$$= -(-1)^{|RG(\sigma)|}(-1)^{|GB(\sigma)|}.$$

Since we can obtain σ_{xywv} by first removing the nodes x, y (which removes a red-green pair) and then removing the nodes w, v (which removes a green-blue pair),

$$\text{sign}_{OE}(\sigma'_{xy})\text{sign}_{OE}(\sigma'_{wv}) = (-1)^{|RB(\sigma)|}(-1)^{|RG(\sigma)|-1}(-1)^{|RB(\sigma)|}(-1)^{|GB(\sigma)|-1}$$

$$= (-1)^{|RG(\sigma)|}(-1)^{|GB(\sigma)|}.$$

Thus by Theorem 2.4.3,

$$Z_\sigma^{DD}(G, \mathbf{N})Z_{\sigma_{xywv}}^{DD}(G, \mathbf{N} - \{x, y, w, v\}) = Z_{\sigma_{xy}}^{DD}(G, \mathbf{N} - \{x, y\})Z_{\sigma_{wv}}^{DD}(G, \mathbf{N} - \{w, v\})$$

$$+ Z_{\sigma_{xv}}^{DD}(G, \mathbf{N} - \{x, v\})Z_{\sigma_{wy}}^{DD}(G, \mathbf{N} - \{w, y\}).$$

$\square$

APPENDIX

ANOTHER PROOF OF KUO CONDENSATION

Recall Kuo's recurrence from Section 1.1:

Theorem 1.1.1. [14, Theorem 5.1] Let $G = (V_1, V_2, E)$ be a planar bipartite graph with a given planar embedding in which $|V_1| = |V_2|$. Let vertices $a, b, c,$ and d appear in a cyclic order on a face of G. If $a, c \in V_1$ and $b, d \in V_2$, then

$$Z^D(G)Z^D(G - \{a, b, c, d\}) = Z^D(G - \{a, b\})Z^D(G - \{c, d\}) + Z^D(G - \{a, d\})Z^D(G - \{b, c\}).$$

We give a non-bijective proof of Theorem 1.1.1 in the case where $a, b, c,$ and d are on the outer face of G using the Desnanot-Jacobi identity and the Kasteleyn matrix. This proof uses ideas from the proof of [13, Lemma 3.2].

Proof. Fix a Kasteleyn weighting of the graph G and let K be the corresponding Kasteleyn matrix, with rows and columns ordered so that the first two rows correspond to vertices a and c and the first two columns correspond to vertices b and d. By the Desnanot-Jacobi identity (Theorem 1.1.2),

$$\frac{\det(K^{1,2}_{1,2})}{\det(K)} = \frac{\det(K^1_1)\det(K^2_2) - \det(K^2_1)\det(K^2_2)}{(\det(K))^2} \tag{A.0.1}$$

where recall that K^j_i is the matrix obtained from K by deleting the ith row and the jth column.

Let $i \in \{a, c\}$ and $j \in \{b, d\}$. Adjoin an edge of weight W connecting i to j so that it separates the outer face of G into two faces. Choose the sign of $e_{i,j}$ so that one of the faces bounded by $e_{i,j}$ is flat. Let K_W denote the Kasteleyn matrix

K with the additional entry corresponding to the edge $e_{i,j}$. By Lemma 3.2.9, K_W is a Kasteleyn matrix of the graph $G \cup \{e_{i,j}\}$.

Observe that $Z^D(G-\{i,j\}) = \pm[W]\det(K_W)$, where $[W]\det(K_W)$ denotes the coefficient of W in the polynomial $\det(K_W)$. (This is because $[W]\det(K_W)$ is, up to a sign, the weighted sum of dimer configurations that include the edge of weight W, which is exactly the weighted sum of dimer configurations of $G-\{i,j\}$.) Similarly, $Z^D(G) = \pm[W^0]\det(K_W)$. Since each term of $\det(K_W)$ has the same sign,

$$\frac{Z^D(G-\{i,j\})}{Z^D(G)} = \frac{[W]\det(K_W)}{[W^0]\det(K_W)}. \tag{A.0.2}$$

Let r_i denote the row corresponding to vertex i and let c_j denote the column corresponding to vertex j. By Lemma 3.2.10, $K_{r_i}^{c_j}$ is a Kasteleyn matrix of $G - \{i,j\}$. The sign of $\det(K_{r_i}^{c_j})$ and the sign of $[W^j]\det(K_W)$ differ by the sign of the edge $e_{i,j}$. So, noting that $[W^0]\det(K_W) = \det(K)$, we have

$$\frac{[W]\det(K_W)}{[W^0]\det(K_W)} = \text{sign}(e_{i,j})\frac{\det(K_{r_i}^{c_j})}{\det(K)}. \tag{A.0.3}$$

Combining equations (A.0.2) and (A.0.3), we have

$$\begin{aligned}
\frac{\det(K_1^1)}{\det(K)} &= \text{sign}(e_{a,b})\frac{Z^D(G-\{a,b\})}{Z^D(G)}, \\
\frac{\det(K_2^2)}{\det(K)} &= \text{sign}(e_{c,d})\frac{Z^D(G-\{c,d\})}{Z^D(G)}, \\
\frac{\det(K_1^2)}{\det(K)} &= \text{sign}(e_{a,d})\frac{Z^D(G-\{a,d\})}{Z^D(G)}, \text{ and} \\
\frac{\det(K_2^1)}{\det(K)} &= \text{sign}(e_{b,c})\frac{Z^D(G-\{b,c\})}{Z^D(G)}.
\end{aligned}$$

142

By repeating the preceding argument with two edges of weight W connecting vertices a to b and c to d, we find that

$$\frac{Z^D(G - \{a, b, c, d\})}{Z^D(G)} = \text{sign}(e_{a,b})\text{sign}(e_{c,d})\frac{\det(K_{1,2}^{1,2})}{\det(K)}.$$

So by equation (A.0.1),

$$\frac{\text{sign}(e_{a,b})\text{sign}(e_{c,d})Z^D(G - \{a, b, c, d\})}{Z^D(G)}$$
$$= \frac{\text{sign}(e_{a,b})\text{sign}(e_{c,d})Z^D(G - \{a, b\})Z^D(G - \{c, d\})}{(Z^D(G))^2}$$
$$- \frac{\text{sign}(e_{a,d})\text{sign}(e_{b,c})Z^D(G - \{a, d\})Z^D(G - \{b, c\})}{(Z^D(G))^2}.$$

Next, we multiply each term in the equation above by $\text{sign}(e_{a,b})\text{sign}(e_{c,d})$ so that the first two terms have positive sign. Finally, we observe that when we adjoin edges connecting a to b, b to c, c to d, and d to a as previously described, the outer face of the modified graph has 4 edges, so

$$\text{sign}(e_{a,b})\text{sign}(e_{c,d})\text{sign}(e_{a,d})\text{sign}(e_{b,c}) = -1,$$

which completes the proof.

$\square$

REFERENCES CITED

[1] David M. Bressoud, *Proofs and confirmations*, MAA Spectrum, Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 1999.

[2] Tom Bridgeland, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), no. 4, 969–998.

[3] P. Di Francesco, O. Golinelli, and E. Guitter, *Meanders and the Temperley-Lieb algebra*, Commun. Math. Phys. **186** (1997), no. 1, 1–59.

[4] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp, *Alternating-Sign matrices and Domino Tilings (Part I)*, J. Algebraic Combin. **1** (1992), no. 2, 111–132.

[5] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[6] Chris Fraser, Thomas Lam, and Ian Le, *From dimers to webs*, Trans. Amer. Math. Soc. **371** (2019), no. 9, 6087–6124.

[7] Markus Fulmek, *Graphical condensation, overlapping Pfaffians and superpositions of matchings*, Electron. J. Comb. **17** (2010).

[8] Helen Jenne, Gautam Webb, and Ben Young, *Work in progress*.

[9] Pieter W. Kasteleyn, *The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice*, Physica **27** (1961), no. 12, 1209–1225.

[10] ———, *Graph theory and crystal physics*, Graph Theory and Theoretical Physics, Academic Press, London, 1967, pp. 43–110.

[11] Richard W. Kenyon, Andrei Okounkov, and Scott Sheffield, *Dimers and amoebae*, Ann. of Math. (2) **163** (2006), no. 3, 1019–1056.

[12] Richard W. Kenyon and David B. Wilson, *Combinatorics of tripartite boundary connections for trees and dimers*, Electron. J. Comb. **16** (2009), no. 1.

[13] ———, *Boundary partitions in trees and dimers*, Trans. Amer. Math. Soc. **363** (2011), no. 3, 1325–1364.

[14] Eric H. Kuo, *Applications of graphical condensation for enumerating matchings and tilings*, Theoret. Comput. Sci. **319** (2004), no. 1-3, 29–57.

144

[15] Greg Kuperberg, *Symmetries of plane partitions and the permanent-determinant method*, J. Comb. Theory **68** (1994), 115–151.

[16] Tri Lai and Gregg Musiker, *Beyond Aztec castles: Toric Cascades in the dP₃ quiver*, Comm. Math. Phys. **356** (2017), no. 3, 823–881.

[17] _____ , *Dungeons and Dragons: Combinatorics for the dP₃ quiver*, arXiv preprint arXiv:11805.09280 (2018).

[18] Thomas Lam, *Dimers, webs, and positroids*, J. Lond. Math. Soc **92** (2015), no. 3, 633–656.

[19] Percy A. MacMahon, *Combinatory analysis*, Cambridge University Press, Cambridge, UK, 1915-16.

[20] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, *Gromov-Witten theory and Donaldson-Thomas theory, I*, Compos. Math. **142** (2006), no. 5, 1263–1285.

[21] _____ , *Gromov-Witten theory and Donaldson-Thomas theory, II*, Compos. Math. **142** (2006), no. 5, 1286–1304.

[22] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Invent. Math. **186** (2011), no. 2, 435–479.

[23] Andrei Okounkov, Nikolai Reshetikhin, and Cumrun Vafa, *Quantum Calabi-Yau and classical crystals*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 597–618.

[24] Rahul Pandharipande and Richard P. Thomas, *The 3-fold vertex via stable pairs*, Geom. Topol. **13** (2009), no. 4, 1835–1876.

[25] _____ , *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), no. 2, 407–447.

[26] David E Speyer, *Variations on a theme of Kasteleyn, with application to the totally nonnegative Grassmannian*, Electron. J. Comb. **23** (2016), no. 2.

[27] H. N. V. Temperley and Michael E. Fisher, *Dimer problem in statistical mechanics-an exact result*, Phil. Mag. **6** (1961), no. 68, 1061–1063.

[28] Yukinobu Toda, *Curve counting theories via stable objects I. DT/PT correspondence*, J. Amer. Math. Soc. **23** (2010), no. 4, 1119–1157.

[29] Lauren K. Williams, *Cluster algebras: an introduction*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 1, 1–26.

[30] Doron Zeilberger, *Dodgon's Determinant-Evaluation Rule proved by Two-Timing Men and Women*, Electron. J. Comb. **4** (1997), no. 2.