AFFINE CELLULARITY OF FINITE TYPE KLR ALGEBRAS, AND HOMOMORPHISMS BETWEEN SPECHT MODULES FOR KLR ALGEBRAS IN AFFINE TYPE A

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JOSEPH WILLIAM LOUBERT

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Student: Joseph William Loubert

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Alexander Kleshchev	Chair
Jonathan Brundan	Core Member
Victor Ostrik	Core Member
Nicholas Proudfoot	Core Member
Michael Kellman	Institutional Representative

and

Scott L. Pratt	Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Joseph William Loubert

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This thesis consists of two parts. In the first we prove that the Khovanov-Lauda-Rouquier algebras R_α of finite type are (graded) affine cellular in the sense of Koenig and Xi. In fact, we establish a stronger property, namely that the affine cell ideals in R_α are generated by idempotents. This in particular implies the (known) result that the global dimension of R_α is finite.

In the second part we use the presentation of the Specht modules given by Kleshchev-Mathas-Ram to derive results about Specht modules. In particular, we determine all homomorphisms from an arbitrary Specht module to a fixed Specht module corresponding to any hook partition. Along the way, we give a complete description of the action of the standard KLR generators on the hook Specht module. This work generalizes a result of James.

This dissertation includes previously published coauthored material.

CURRICULUM VITAE

NAME OF AUTHOR: Joseph William Loubert

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:
University of Oregon, Eugene, OR
University of Minnesota, Minneapolis, MN

DEGREES AWARDED:
Doctor of Philosophy, Mathematics, 2015, University of Oregon
Master of Science, Mathematics, 2011, University of Oregon
Bachelor of Science (cum laude), Mathematics, 2009, University of Minnesota
Bachelor of Electrical Engineering, 2009, University of Minnesota

AREAS OF SPECIAL INTEREST:
Representation theory, categorification

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2009-2015

GRANTS, AWARDS AND HONORS:

Borsting Graduate Student Award, University of Oregon, 2015

PUBLICATIONS:

A. Kleshchev and J. Loubert, Affine cellularity of Khovanov-Lauda-Rouquier algebras of finite types, *Int. Math. Res. Notices*, to appear; Advanced Access published June 2014, doi:10.1093/imrn/rnu096

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

There is a collection of $\mathbb{Z}$-graded algebras defined by Khovanov and Lauda (14; 15) and Rouquier (32), called *Khovanov-Lauda-Rouquier* algebras. For every Lie type Γ there is a corresponding KLR algebra $R := R(\Gamma)$, categorifying the upper half of the quantum group of type Γ. The theory of proper standard modules for KLR algebras of finite Lie type was established by Kleshchev and Ram (21) and in more generality by McNamara (30). These modules categorify the dual PBW basis. Koenig and Xi introduce the concept of *affine cellular algebras* in (24). This extends the usual theory of finite-dimensional cellular algebras to infinite-dimensional algebras. In particular, one gets a theory of standard modules and proper standard modules for any affine cellular algebra. Kleshchev defines the related notions of *affine highest weight categories* and *affine quasihereditary algebras* in (23). Such algebras have many nice properties; for instance under some natural assumptions these algebras have finite global dimension. The first of our two main theorems in this paper establishes the affine cellularity and affine quasiheredity of KLR algebras of finite types. This work is joint with Alexander Kleshchev.

Let $\mathfrak{S}_n$ denote the symmetric group on n letters, and F any field. Brundan and Kleshchev (1) put a nontrivial $\mathbb{Z}$-grading on the group algebras $F\mathfrak{S}_n$ and Hecke algebras for complex reflection groups of type $G(l, 1, d)$, which include symmetric groups. The Brundan-Kleshchev grading comes from an isomorphism between the Iwahori-Hecke algebras H_n and certain *cyclotomic quotients* R^Λ of KLR algebras in (affine) type A. The Specht modules S^λ were first given a compatible grading by Brundan, Kleshchev and Wang (4). Kleshchev, Mathas, and Ram (20) provide a presentation for the Specht modules as modules over the full KLR algebra R, thus redefining these classical objects purely in the context of the KLR algebras. Our second main theorem determines $\mathrm{Hom}(S^\mu, S^\lambda)$ when λ is a hook.

Affine Cellularity of KLR Algebras of Finite Types

The content of chapter II has already been published as (18). The goal of chapter II is to establish (graded) affine cellularity in the sense of Koenig and Xi (24) for the Khovanov-Lauda-Rouquier algebras R_α of finite Lie type. In fact, we construct a chain of affine cell ideals in R_α which are generated by idempotents. This stronger property is analogous to quasi-heredity for finite dimensional algebras, and by a general result of Koenig and Xi (24, Theorem 4.4), it also implies finiteness of the global dimension of R_α. Thus we obtain a new proof of (a slightly stronger version of) a recent result of Kato (13) and McNamara (30) (see also (3)). As another application, one gets a theory of standard and proper standard modules, cf. (13),(3). It would be interesting to apply this paper to prove the conjectural (graded) cellularity of cyclotomic KLR algebras of finite types.

Our approach is independent of the homological results in (30), (13) and (3) (which relies on (30)). The connection between the theory developed in (3) and this paper is explained in (23). This paper generalizes (19), where analogous results were obtained for finite type A.

We now give a definition of (graded) affine cellular algebra from (24, Definition 2.1). Throughout the paper, unless otherwise stated, we assume that all algebras are ($\mathbb{Z}$)-graded, all ideals, subspaces, etc. are homogeneous, and all homomorphisms are homogeneous degree zero homomorphisms with respect to the given gradings. For this introduction, we fix a noetherian domain k (later on it will be sufficient to work with $k = \mathbb{Z}$). Let A be a (graded) unital k-algebra with a k-anti-involution τ. A (two-sided) ideal J in A is called an *affine cell ideal* if the following conditions are satisfied:

1. $\tau(J) = J$;

2. there exists an affine k-algebra B with a k-involution σ and a free k-module V of finite rank such that $\Delta := V \otimes_k B$ has an A-B-bimodule structure, with the right B-module structure induced by the regular right B-module structure on B;

3. let $\Delta' := B \otimes_k V$ be the B-A-bimodule with left B-module structure induced by the regular left B-module structure on B and right A-module structure defined by

$$(b \otimes v)a = \mathrm{s}(\tau(a)(v \otimes b)), \tag{1.1}$$

where $\mathrm{s} : V \otimes_k B \to B \otimes_k V,\ v \otimes b \to b \otimes v$; then there is an A-A-bimodule isomorphism $\mu : J \to \Delta \otimes_B \Delta'$, such that the following diagram commutes:

$$\begin{tikzcd} J \arrow[r, "\mu"] \arrow[d, "\tau"] & \Delta \otimes_B \Delta' \arrow[d, "v\otimes b\otimes b'\otimes w \mapsto w\otimes\sigma(b')\otimes\sigma(b)\otimes v"] \\ J \arrow[r, "\mu"] & \Delta \otimes_B \Delta'. \end{tikzcd} \tag{1.2}$$

The algebra A is called *affine cellular* if there is a k-module decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ with $\tau(J_l') = J_l'$ for $1 \leq l \leq n$, such that, setting $J_m := \bigoplus_{l=1}^m J_l'$, we obtain an ideal filtration

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$$

so that each J_m/J_{m-1} is an affine cell ideal of A/J_{m-1}.

To describe our main results we introduce some notation referring the reader to the main body of the paper for details. Fix a Cartan datum of finite type, and denote by $\Phi_+ = \{\beta_1, \dots, \beta_N\}$ the set of positive roots, and by Q_+ the positive part of the root lattice. For $\alpha \in Q_+$ we have the KLR algebra R_α with standard idempotents $\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$. We denote by $\Pi(\alpha)$ the set of root partitions of α. This is partially ordered with respect to a certain bilexicographic order '$\leq$'.

To any $\pi \in \Pi(\alpha)$ one associates a proper standard module $\bar{\Delta}(\pi)$ and a word $\boldsymbol{i}_\pi \in \langle I \rangle_\alpha$. We fix a distinguished vector $v_\pi^- \in \bar{\Delta}(\pi)$, and choose a set $\mathfrak{B}_\pi \subseteq R_\alpha$ so that $\{bv_\pi^- \mid b \in \mathfrak{B}_\pi\}$ is a basis of $\bar{\Delta}(\pi)$. We define polynomial subalgebras $\Lambda_\pi \subseteq R_\alpha$—these are isomorphic to tensor products of algebras of symmetric polynomials. We also explicitly define elements $\delta_\pi, D_\pi \in R_\alpha$ and set $e_\pi := D_\pi \delta_\pi$. Then we set

$$I'_\pi := k\text{-span}\{be_\pi \Lambda_\pi D_\pi (b')^\tau \mid b, b' \in \mathfrak{B}_\pi\},$$

$I_\pi := \sum_{\sigma \geq \pi} I'_\sigma$, and $I_{>\pi} = \sum_{\sigma > \pi} I'_\sigma$. Our main results are now as follows

Main Theorem. *The algebra R_α is graded affine cellular with cell chain given by the ideals $\{I_\pi \mid \pi \in \Pi(\alpha)\}$. Moreover, for a fixed $\pi \in \Pi(\alpha)$, we set $\bar{R}_\alpha := R_\alpha / I_{>\pi}$ and $\bar{h} := h + I_{>\pi}$ for any $h \in R_\alpha$. We have:*

(i) $I_\pi = \sum_{\sigma \geq \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha$;

(ii) $\bar{e}_\pi$ *is an idempotent in* $\bar{R}_\alpha$;

(iii) *the map* $\Lambda_\pi \to \bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi,\ f \mapsto \bar{e}_\pi \bar{f} \bar{e}_\pi$ *is an isomorphism of graded algebras;*

(iv) $\bar{R}_\alpha \bar{e}_\pi$ *is a free right* $\bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi$*-module with basis* $\{\bar{b}\bar{e}_\pi \mid b \in \mathfrak{B}_\pi\}$;

(v) $\bar{e}_\pi \bar{R}_\alpha$ *is a free left* $\bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi$*-module with basis* $\{\bar{e}_\pi \bar{D}_\pi \bar{b}^\tau \mid b \in \mathfrak{B}_\pi\}$;

(vi) *multiplication provides an isomorphism*

$$\bar{R}_\alpha \bar{e}_\pi \otimes_{\bar{e}_\pi \bar{R}_\alpha \bar{e}_\pi} \bar{e}_\pi \bar{R}_\alpha \xrightarrow{\sim} \bar{R}_\alpha \bar{e}_\pi \bar{R}_\alpha;$$

(vii) $\bar{R}_\alpha \bar{e}_\pi \bar{R}_\alpha = I_\pi / I_{>\pi}$.

(viii) *For each* $\pi \in \Pi(\alpha)$*, let* X_π *be a homogeneous basis of* Λ_π*. Then*

$$\{b e_\pi f D_\pi (b')^\tau \mid \pi \in \Pi(\alpha),\ b, b' \in \mathfrak{B}_\pi,\ f \in X_\pi\}$$

is a homogeneous k*-basis for* R_α*.*

We give an example that the reader can consult with while reading this paper further.

Example. Let the Cartan datum be of type B_2, and $\alpha = \alpha_1 + 2\alpha_2$. The set $\Pi(\alpha)$ has three elements: $\pi_1 := (\alpha_2^2, \alpha_1)$, $\pi_2 := (\alpha_2, \alpha_1 + \alpha_2)$, and $\pi_3 := (\alpha_1 + 2\alpha_2)$.

For π_1 we have: $\bar{\Delta}(\pi_1) = L(\alpha_2)^{\circ 2} \circ L(\alpha_1)$. The modules $L(\alpha_1)$ and $L(\alpha_2)$ are 1-dimensional with basis elements $v_{\alpha_1}^-$ and $v_{\alpha_2}^-$ respectively, and $v_{\pi_1}^- = \psi_1 \otimes v_{\alpha_2}^- \otimes v_{\alpha_2}^- \otimes v_{\alpha_1}^-$. We have $\mathfrak{B}_{\pi_1} = \{1, \psi_2, \psi_1\psi_2, y_2, \psi_2 y_2, \psi_1\psi_2 y_2\}$. The algebra Λ_{π_1} is $\Lambda_2 \otimes k[y_3]$, where Λ_2 denotes the symmetric polynomials $k[y_1, y_2]^{S_2}$. Finally, $D_{\pi_1} = \psi_1 e(2,2,1)$, $\delta_{\pi_1} = y_2 e(2,2,1)$, and $e_{\pi_1} = \psi_1 y_2 e(2,2,1)$.

For π_2 we have: $\bar{\Delta}(\pi_2) = L(\alpha_2) \circ L(\alpha_1 + \alpha_2)$. The module $L(\alpha_1 + \alpha_2)$ is 1-dimensional with basis element $v_{\alpha_1+\alpha_2}^-$, and $v_{\pi_2}^- = 1 \otimes v_{\alpha_2}^- \otimes v_{\alpha_1+\alpha_2}^-$. We have $\mathfrak{B}_{\pi_2} = \{1, \psi_1, \psi_2\psi_1\}$. The algebra Λ_{π_2} is $k[y_1, y_3]$. Finally, $e_{\pi_2} = D_{\pi_2} = \delta_{\pi_2} = e(2,1,2)$.

For π_3 we have: $\bar{\Delta}(\pi_3) = L(\alpha_1 + 2\alpha_2)$. The module $L(\alpha_1 + 2\alpha_2)$ is 2-dimensional with basis $\{v_{\alpha_1+2\alpha_2}^-, y_3 v_{\alpha_1+2\alpha_2}^-\}$, and $v_{\pi_3}^- = v_{\alpha_1+2\alpha_2}^-$. We have $\mathfrak{B}_{\pi_3} =$

$\{1, y_3\}$. The algebra Λ_{π_3} is $k[y_1]$. Finally, $D_{\pi_3} = \psi_2 e(1,2,2)$, $\delta_{\pi_3} = y_3 e(1,2,2)$, and $e_{\pi_3} = \psi_2 y_3 e(1,2,2)$.

Main Theorem(vii) shows that each affine cell ideal $I_\pi / I_{>\pi}$ in $R_\alpha / I_{>\pi}$ is generated by an idempotent. This, together with the fact that each algebra Λ_π is a polynomial algebra, is enough to invoke (24, Theorem 4.4) to get

Corollary. *If the ground ring k has finite global dimension, then the algebra R_α has finite global dimension.*

This seems to be a slight generalization of (13),(30),(3) in two ways: (13) assumes that k is a field of characteristic zero (and the Lie type is simply-laced), while (30),(3) assume that k is a field; moreover, (13),(30),(3) deal with categories of graded modules only, while our corollary holds for the algebra R_α even as an ungraded algebra.

The paper is organized as follows. Section 2 contains preliminaries needed for the rest of the paper. The first subsection contains mostly general conventions that will be used. Subsection 2.1 goes over the Lie theoretic notation that we employ. We move on in subsection 2.1 to the definition and basic results of Khovanov-Lauda-Rouquier (KLR) algebras. The next two subsections are devoted to recalling results about the representation theory of KLR algebras. Then, in subsection 2.1, we introduce our notation regarding quantum groups, and recall some well-known basis theorems. The next subsection is devoted to the connection between KLR algebras and quantum groups, namely the categorification theorems. Finally, subsection 2.1 contains an easy direct proof of a graded dimension formula for the KLR algebras, cf. (3, Corollary 3.15).

Section 3 is devoted to constructing a basis for the KLR algebras that is amenable to checking affine cellularity. We begin in subsection 2.2 by choosing some special word idempotents and proving some properties they enjoy. Subsection 2.2 introduces the notation that allows us to define our affine cellular structure. This subsection also contains the crucial Hypothesis 2.2.9. Next, in subsection 2.2, we come up with an affine cellular basis in the special case corresponding to a root partition of that is a power of a single root. Finally, we use this in the last subsection to come up with our affine cellular basis in full generality.

In section 4 we show how the affine cellular basis is used to prove that the KLR algebras are affine cellular.

Finally, in section 5 we verify Hypothesis 2.2.9 for all positive roots in all finite types. We begin in subsection 2.4 by recalling some results concerning homogeneous representations. In subsection 2.4 we recall the definition of special Lyndon orders and Lyndon words, which will serve as the special words of subsection 2.2. The next subsection is devoted to verifying Hypothesis 2.2.9 in the special case when the cuspidal representation corresponding to the positive root is homogeneous. We then employ this in subsection 2.4 to show that the hypothesis holds in simply-laced types. Finally, we have subsection 2.4, wherein we verify the hypothesis by hand in the non-symmetric types.

Homomorphisms between Certain Specht Modules

Let H_d be an Iwahori-Hecke algebra for the symmetric group $\mathfrak{S}_d$ with deformation parameter q, over a field F. Define e to be the smallest integer so that $1+q+\cdots+q^{e-1}=0$, setting $e=0$ if no such value exists. To every partition λ of d, there is a corresponding *Specht module* S^λ over H_d. Brundan and Kleshchev show in (1) that H_d is isomorphic to a certain algebra $R_d^{\Lambda_0}$ known as a cyclotomic KLR (Khovanov-Lauda-Rouquier) algebra of type $A_{e-1}^{(1)}$ when $e \neq 0$, and type A_∞ when $e = 0$. The Specht modules are described in (4) as modules over the cyclotomic KLR algebras. This result is extended by Kleshchev, Mathas, and Ram in (20), in which Specht modules for the (full) KLR algebras R_d are explicitly defined in terms of generators and relations. The purpose of chapter III is to use this presentation to completely determine $\mathrm{Hom}_{R_d}(S^\mu, S^\lambda)$ when μ is an arbitrary partition and λ is a hook. Of course, when $e = 0$ there are no nontrivial homomorphisms. Furthermore, our methods do not apply when $e = 2$, so we make the assumption that $e \geq 3$.

To state the main theorem, we need some notation. The element $z^\mu \in S^\mu$ is the standard cyclic generator of weight $\boldsymbol{i}^\mu$, see Definition 3.2.1. When the $\boldsymbol{i}^\mu$-weight space of S^λ is nonempty, it will turn out that its top degree component is one-dimensional. In this case we take $[\sigma_\mu] \in S^\lambda$ to be any non-zero vector of that component. For any weakly decreasing sequence of positive integers $\boldsymbol{a} = (a_1, \dots, a_N)$ we define its *Garnir content* $\mathrm{Gc}(\boldsymbol{a}) \in F$ by

$$\mathrm{Gc}(\boldsymbol{a}) = \gcd\left\{ \binom{a_i}{k} \mid 1 \leq k \leq a_{i+1} - 1,\ 1 \leq i \leq N-1 \right\}$$

with the convention that $\gcd(\varnothing) = 0$.

Main Theorem. *Let μ be an arbitrary partition and $\lambda = (d-k, 1^k)$ for $k \geq 0$. Then $\mathrm{Hom}_{R_d}(S^\mu, S^\lambda)$ is at most one-dimensional, spanned by a map satisfying $z^\mu \mapsto [\sigma_\mu]$, and $\dim \mathrm{Hom}_{R_d}(S^\mu, S^\lambda) = 1$ if and only if one of the following conditions holds:*

1. *there exist* $n \in \{1, \ldots, k+1\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^n$, *and* $0 \leq m < e$ *such that* $\mathrm{Gc}(\boldsymbol{a}) = 0$ *and*

$$\mu = (a_1 e, \ldots, a_{n-1} e, a_n e - m, 1^{k-n+1}),$$

2. *e divides d, there exist* $n \in \{1, \ldots, k\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^n$, *and* $0 \leq m < e$ *such that* $\mathrm{Gc}(\boldsymbol{a}) = 0$ *and*

$$\mu = (a_1 e, \ldots, a_{n-1} e, a_n e - m, 1^{k-n+2}), \text{ or}$$

3. *there exist* $n > k+1$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^n$ *and* $0 \leq m < e$ *such that* $\mathrm{Gc}(\boldsymbol{a}) = 0$ *and*

$$\mu = (a_1 e, \ldots, a_k e, a_{k+1} e - 1, \ldots, a_{n-1} e - 1, a_n e - 1 - m).$$

This theorem generalizes James (11, Theorem 24.4), which corresponds to $k = 0$. Our Specht modules are the dual of the Specht modules defined in (11). We note here that the main theorem also allows us to determine all homomorphisms between Specht modules when the source is a hook, as follows; see (20) for details. For any partition ν we define ν' to be the conjugate partition and S_ν to be the dual of S^ν. There is an automorphism $\mathtt{sgn}$ of R_d such that $S_\nu \cong (S^{\nu'})^{\mathtt{sgn}}$, and so

$$\begin{aligned} \mathrm{Hom}_{R_d}(S^\mu, S^\lambda) &\cong \mathrm{Hom}_{R_d}(S_\lambda, S_\mu) \cong \mathrm{Hom}_{R_d}((S^{\lambda'})^{\mathtt{sgn}}, (S^{\mu'})^{\mathtt{sgn}}) \\ &\cong \mathrm{Hom}_{R_d}(S^{\lambda'}, S^{\mu'}). \end{aligned}$$

Clearly, λ is a hook if and only if λ' is.

Section 2 introduces the notation to be used throughout the paper. In addition, we collect a few elementary facts which will be necessary in later sections.

Section 3 reviews the definitions of the affine and cyclotomic KLR algebras, and their universal Specht modules as introduced in (20). Section 4 provides a study of the structure of the Specht modules S^λ, where λ is a hook. Specifically, there is a basis for S^λ given by standard tableaux on λ, and in Theorem 3.3.3 we determine how the generators of the KLR algebra act on this basis. In section 5, we look at homomorphisms from an arbitrary Specht module S^μ to S^λ, where λ is a hook. In section 6, we consider some examples.

CHAPTER II

AFFINE CELLULARITY OF KLR ALGEBRAS

The content of this chapter has already been published as (18).

Preliminaries and a Dimension Formula

In this section we set up the theory of KLR algebras and their connection to quantum groups following mainly (14) and also (21). Only subsection 2.1 contains some new material.

Generalities

Throughout the paper we work over the ground ring $\mathcal{O}$ which is assumed to be either $\mathbb{Z}$ or an arbitrary field F. Most of the time we work over F and then deduce the corresponding result for $\mathbb{Z}$ using the following standard lemma

Lemma 2.1.1

Let M be a finitely generated $\mathbb{Z}$-module, and $\{x_\alpha\}_{\alpha \in A}$ a subset of M. Then $\{x_\alpha\}$ is a spanning set (resp. basis) of M if and only if $\{1_F \otimes x_\alpha\}$ is a spanning set (resp. basis) of $F \otimes_{\mathbb{Z}} M$ for every field F.

Let q be an indeterminate, $\mathbb{Q}(q)$ the field of rational functions, and $\mathcal{A} := \mathbb{Z}[q, q^{-1}] \subseteq \mathbb{Q}(q)$. Let $^-: \mathbb{Q}(q) \to \mathbb{Q}(q)$ be the $\mathbb{Q}$-algebra involution with $\bar{q} = q^{-1}$, referred to as the *bar-involution*. For $f, g \in \mathbb{Z}[[q, q^{-1}]]$, write $f \leq g$ if and only if $g - f \in \mathbb{Z}_{\geq 0}[[q, q^{-1}]]$.

For a graded vector space $V = \oplus_{n \in \mathbb{Z}} V_n$, with finite dimensional graded components its *graded dimension* is $\dim_q V := \sum_{n \in \mathbb{Z}} (\dim V_n) q^n \in \mathbb{Z}[[q, q^{-1}]]$.

For any graded F-algebra H we denote by H-Mod the abelian category of all graded left H-modules, with morphisms being *degree-preserving* module homomorphisms, which we denote by hom. Let H-mod denote the abelian subcategory of all *finite dimensional* graded H-modules and H-proj denote the additive subcategory of all *finitely generated projective* graded H-modules. Denote the corresponding Grothendieck groups by $[H\text{-mod}]$ and $[H\text{-proj}]$, respectively. These Grothendieck groups are $\mathcal{A}$-modules via $q^m[M] := [M\langle m\rangle]$, where $M\langle m\rangle$ denotes the module obtained by shifting the grading up by m: $M\langle m\rangle_n := M_{n-m}$. For $n \in \mathbb{Z}$, let $\mathrm{Hom}_H(M,N)_n := \mathrm{hom}_H(M\langle n\rangle, N)$ denote the space of homomorphisms of degree n. Set $\mathrm{Hom}_H(M,N) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_H(M,N)_n$.

Lie theoretic data

A *Cartan datum* is a pair $(I,\cdot)$ consisting of a set I and a $\mathbb{Z}$-valued symmetric bilinear form $i,j \mapsto i\cdot j$ on the free abelian group $\mathbb{Z}[I]$ such that $i\cdot i \in \{2,4,6,\dots\}$ for all $i \in I$ and $2(i\cdot j)/(i\cdot i) \in \{0,-1,-2\dots\}$ for all $i \neq j$ in I. Set $a_{ij} := 2(i\cdot j)/(i\cdot i)$ for $i,j \in I$ and define the *Cartan matrix* $A := (a_{ij})_{i,j\in I}$. Throughout the paper, unless otherwise stated, we assume that A has *finite type*, see (12, §4). We have simple roots $\{\alpha_i \mid i \in I\}$, and we identify α_i with i. Let $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$. For $\alpha \in Q_+$, we write $\mathrm{ht}(\alpha)$ for the sum of its coefficients when expanded in terms of the α_i's. Denote by $\Phi_+ \subset Q_+$ the set of *positive* roots, cf. (12, §1.3), and by W the corresponding Weyl group. A total order on Φ_+ is called *convex* if $\beta,\gamma,\beta+\gamma \in \Phi_+$ and $\beta < \gamma$ imply $\beta < \beta+\gamma < \gamma$.

Given $\beta \in \mathbb{Z}[I]$, denote

$$q_\beta := q^{(\beta\cdot\beta)/2},\ [n]_\beta := (q_\beta^n - q_\beta^{-n})/(q_\beta - q_\beta^{-1}),\ [n]_\beta^! := [n]_\beta[n-1]_\beta\dots[1]_\beta.$$

In particular, for $i \in I$, we have $q_i, [n]_i, [n]_i^!$. Let A be a Q_+-graded $\mathbb{Q}(q)$-algebra, $\theta \in A_\alpha$ for $\alpha \in Q_+$, and $n \in \mathbb{Z}_{\geq 0}$. We use the standard notation for quantum divided powers: $\theta^{(n)} := \theta^n/[n]_\alpha^!$.

Denote by $\langle I \rangle := \bigsqcup_{d \geq 0} I^d$ the set of all tuples $\boldsymbol{i} = i_1 \dots i_d$ of elements of I, which we refer to as *words*. We consider $\langle I \rangle$ as a monoid under the concatenation product. If $\boldsymbol{i} \in \langle I \rangle$, we can write it in the form $\boldsymbol{i} = j_1^{m_1} \dots j_r^{m_r}$ for $j_1, \dots, j_r \in I$ such that $j_s \neq j_{s+1}$ for all $s = 1, 2, \dots, r-1$. Denote

$$[\boldsymbol{i}]! := [m_1]_{j_1}^! \dots [m_r]_{j_r}^!. \tag{2.1}$$

For $\boldsymbol{i} = i_1 \dots i_d$ set $|\boldsymbol{i}| := \alpha_{i_1} + \cdots + \alpha_{i_d} \in Q_+$. The symmetric group S_d with simple transpositions $s_1, \dots, s_{d-1}$ acts on I^d on the left by place permutations. The S_d-orbits on I^d are the sets $\langle I \rangle_\alpha := \{\boldsymbol{i} \in I^d \mid |\boldsymbol{i}| = \alpha\}$ parametrized by the elements $\alpha \in Q_+$ of height d.

Khovanov-Lauda-Rouquier algebras

Let A be a Cartan matrix. Choose signs ε_{ij} for all $i, j \in I$ with $a_{ij} < 0$ so that $\varepsilon_{ij}\varepsilon_{ji} = -1$, and define the polynomials $\{Q_{ij}(u,v) \in F[u,v] \mid i, j \in I\}$:

$$Q_{ij}(u,v) := \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } a_{ij} = 0; \\ \varepsilon_{ij}(u^{-a_{ij}} - v^{-a_{ji}}) & \text{if } a_{ij} < 0. \end{cases} \tag{2.2}$$

In addition, fix $\alpha \in Q_+$ of height d. Let $R_\alpha = R_\alpha(\Gamma, \mathcal{O})$ be an associative graded unital $\mathcal{O}$-algebra, given by the generators

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\}$$

and the following relations for all $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle_\alpha$ and all admissible r, t:

$$e(\boldsymbol{i})e(\boldsymbol{j}) = \delta_{\boldsymbol{i},\boldsymbol{j}} e(\boldsymbol{i}), \quad \textstyle\sum_{\boldsymbol{i} \in \langle I \rangle_\alpha} e(\boldsymbol{i}) = 1; \tag{2.3}$$

$$y_r e(\boldsymbol{i}) = e(\boldsymbol{i}) y_r; \qquad y_r y_t = y_t y_r; \tag{2.4}$$

$$\psi_r e(\boldsymbol{i}) = e(s_r \boldsymbol{i}) \psi_r; \tag{2.5}$$

$$y_r \psi_s = \psi_s y_r \qquad (r \neq s, s+1); \tag{2.6}$$

$$(y_t \psi_r - \psi_r y_{s_r(t)}) e(\boldsymbol{i}) = \delta_{i_r, i_{r+1}} (\delta_{t,r+1} - \delta_{t,r}) e(\boldsymbol{i}); \tag{2.7}$$

$$\psi_r^2 e(\boldsymbol{i}) = Q_{i_r, i_{r+1}}(y_r, y_{r+1}) e(\boldsymbol{i}) \tag{2.8}$$

$$\psi_r \psi_t = \psi_t \psi_r \qquad (|r - t| > 1); \tag{2.9}$$

$$\begin{aligned} &(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r) e(\boldsymbol{i}) \\ =&\delta_{i_r, i_{r+2}} \frac{Q_{i_r, i_{r+1}}(y_{r+2}, y_{r+1}) - Q_{i_r, i_{r+1}}(y_r, y_{r+1})}{y_{r+2} - y_r} e(\boldsymbol{i}). \end{aligned} \tag{2.10}$$

The *grading* on R_α is defined by setting:

$$\deg(e(\boldsymbol{i})) = 0, \quad \deg(y_r e(\boldsymbol{i})) = i_r \cdot i_r, \quad \deg(\psi_r e(\boldsymbol{i})) = -i_r \cdot i_{r+1}.$$

In this paper *grading* always means $\mathbb{Z}$-*grading*, ideals are assumed to be homogeneous, and modules are assumed graded, unless otherwise stated.

It is pointed out in (15) and (32, §3.2.4) that up to isomorphism the graded $\mathcal{O}$-algebra R_α depends only on the Cartan datum and α. We refer to the algebra R_α as an *(affine) Khovanov-Lauda-Rouquier algebra*. It is convenient to consider the direct sum of algebras $R := \bigoplus_{\alpha \in Q_+} R_\alpha$. Note that R is non-unital, but it is locally unital since each R_α is unital. The algebra R_α possesses a graded anti-automorphism

$$\tau : R_\alpha \to R_\alpha,\ x \mapsto x^\tau \tag{2.11}$$

which is the identity on generators.

For each element $w \in S_d$ fix a reduced expression $w = s_{r_1} \dots s_{r_m}$ and set $\psi_w := \psi_{r_1} \dots \psi_{r_m}$. In general, ψ_w depends on the choice of the reduced expression of w.

Theorem 2.1.2

(14, Theorem 2.5), (32, Theorem 3.7) The following set is an $\mathcal{O}$-basis of R_α: $\{\psi_w y_1^{m_1} \dots y_d^{m_d} e(\boldsymbol{i}) \mid w \in S_d,\ m_1, \dots, m_d \in \mathbb{Z}_{\geq 0},\ \boldsymbol{i} \in \langle I \rangle_\alpha\}$.

In view of the theorem, we have a polynomial subalgebra

$$P_d = \mathcal{O}[y_1, \dots, y_d] \subseteq R_\alpha. \tag{2.12}$$

Let $\gamma_1, \dots, \gamma_l$ be elements of Q_+ with $\gamma_1 + \dots + \gamma_l = \alpha$. Then we have a natural embedding

$$\iota_{\gamma_1,\dots,\gamma_l} : R_{\gamma_1} \otimes \dots \otimes R_{\gamma_l} \hookrightarrow R_\alpha \tag{2.13}$$

of algebras, whose image is the *parabolic subalgebra* $R_{\gamma_1,\ldots,\gamma_l} \subseteq R_\alpha$. This is not a unital subalgebra, the image of the identity element of $R_{\gamma_1} \otimes \cdots \otimes R_{\gamma_l}$ being

$$1_{\gamma_1,\ldots,\gamma_l} = \sum_{\boldsymbol{i}^{(1)} \in \langle I \rangle_{\gamma_1},\ldots,\boldsymbol{i}^{(l)} \in \langle I \rangle_{\gamma_l}} e(\boldsymbol{i}^{(1)} \ldots \boldsymbol{i}^{(l)}).$$

An important special case is where $\alpha = d\alpha_i$ is a multiple of a simple root, in which case we have that $R_{d\alpha_i}$ is the d^{th} nilHecke algebra H_d generated by $\{y_1,\ldots,y_d,\psi_1,\ldots,\psi_{d-1}\}$ subject to the relations

$$\psi_r^2 = 0 \tag{2.14}$$

$$\psi_r\psi_s = \psi_s\psi_r \qquad \text{if } |r-s| > 1 \tag{2.15}$$

$$\psi_r\psi_{r+1}\psi_r = \psi_{r+1}\psi_r\psi_{r+1} \tag{2.16}$$

$$\psi_r y_s = y_s\psi_r \qquad \text{if } s \neq r, r+1 \tag{2.17}$$

$$\psi_r y_{r+1} = y_r\psi_r + 1 \tag{2.18}$$

$$y_{r+1}\psi_r = \psi_r y_r + 1. \tag{2.19}$$

The grading is so that $\deg(y_r) = \alpha_i \cdot \alpha_i$ and $\deg(\psi_r) = -\alpha_i \cdot \alpha_i$. Note that here the elements ψ_w do not depend on a choice of reduced decompositions.

Let $w_0 \in \mathfrak{S}_d$ be the longest element, and define the following elements of H_d:

$$\delta_d := y_2 y_3^2 \ldots y_d^{d-1}, \quad e_d := \psi_{w_0}\delta_d.$$

It is known that

$$e_d\psi_{w_0} = \psi_{w_0}, \tag{2.20}$$

and in particular e_d is an idempotent, see for example (14, §2.2). The following is a special case of our main theorem for the case where $\alpha = d\alpha_i$, which will be used in its proof. It is known that the center $Z(H_d)$ consists of the symmetric polynomials $\mathcal{O}[y_1, \ldots, y_d]^{\mathfrak{S}_d}$.

Theorem 2.1.3

(19, Theorem 4.16) Let X be a $\mathcal{O}$-basis of $\mathcal{O}[y_1, \ldots, y_d]^{\mathfrak{S}_d}$ and let $\mathfrak{B}$ be a basis of $\mathcal{O}[y_1, \ldots, y_d]$ as an $\mathcal{O}[y_1, \ldots, y_d]^{\mathfrak{S}_d}$-module. Then $\{b e_d f \psi_{w_0} (b')^\tau \mid b, b' \in \mathfrak{B}, f \in X\}$ is a $\mathcal{O}$-basis of H_d.

Basic representation theory of R_α

By (14), every irreducible graded R_α-module is finite dimensional, and there are finitely many irreducible R_α-modules up to isomorphism and grading shift. For $\boldsymbol{i} \in \langle I \rangle_\alpha$ and $M \in R_\alpha$-Mod, the *$\boldsymbol{i}$-word space* of M is $M_{\boldsymbol{i}} := e(\boldsymbol{i})M$. We have a decomposition of (graded) vector spaces $M = \bigoplus_{\boldsymbol{i} \in \langle I \rangle_\alpha} M_{\boldsymbol{i}}$. We say that $\boldsymbol{i}$ is a *word of M* if $M_{\boldsymbol{i}} \neq 0$.

We identify in a natural way:

$$[R\text{-mod}] = \bigoplus_{\alpha \in Q_+} [R_\alpha\text{-mod}], \quad [R\text{-proj}] = \bigoplus_{\alpha \in Q_+} [R_\alpha\text{-proj}].$$

Recall the anti-automorphism τ from (2.11). This allows us to introduce the left R_α-module structure on the graded dual of a finite dimensional R_α-module M—the resulting left R_α-module is denoted $M^\circledast$. On the other hand, given any left R_α-module M, denote by M^τ the right R_α-module with the action given by $mx = \tau(x)m$ for $x \in R_\alpha, m \in M$. Following (15, (14)), define the *Khovanov-Lauda* *pairing* to be the $\mathcal{A}$-linear pairing

$$(\cdot,\cdot) : [R_\alpha\text{-proj}] \times [R_\alpha\text{-proj}] \to \mathcal{A} \cdot \prod_{i \in I} \prod_{a=1}^{m_i} \frac{1}{(1-q_i^{2a})}$$

such that $([P],[Q]) = \dim_q (P^\tau \otimes_{R_\alpha} Q)$.

Let $\alpha, \beta \in Q_+$. Recalling the isomorphism $\iota_{\alpha,\beta} : R_\alpha \otimes\ R_\beta \to R_{\alpha,\beta} \subseteq R_{\alpha+\beta}$, consider the functors

$$\mathrm{Ind}_{\alpha,\beta} := R_{\alpha+\beta} 1_{\alpha,\beta} \otimes_{R_{\alpha,\beta}} ? : R_{\alpha,\beta}\text{-Mod} \to R_{\alpha+\beta}\text{-Mod},$$

$$\mathrm{Res}_{\alpha,\beta} := 1_{\alpha,\beta} R_{\alpha+\beta} \otimes_{R_{\alpha+\beta}} ? : R_{\alpha+\beta}\text{-Mod} \to R_{\alpha,\beta}\text{-Mod}\,.$$

For $M \in R_\alpha$-mod and $N \in R_\beta$-mod, we denote $M \circ N := \mathrm{Ind}_{\alpha,\beta}(M \boxtimes N)$. The functors of induction define products on the Grothendieck groups $[R\text{-mod}]$ and $[R\text{-proj}]$ and the functors of restriction define coproducts on $[R\text{-mod}]$ and $[R\text{-proj}]$. These products and coproducts make $[R\text{-mod}]$ and $[R\text{-proj}]$ into twisted unital and counital bialgebras (14, Proposition 3.2).

Let $i \in I$ and $n \in \mathbb{Z}_{>0}$. As explained in (14, §2.2), the algebra $R_{n\alpha_i}$ has a representation on the polynomials $F[y_1, \ldots, y_n]$ such that each y_r acts as multiplication by y_r and each ψ_r acts as the divided difference operator $\partial_r : f \mapsto \frac{{}^{s_r}f - f}{y_r - y_{r+1}}$. Let $P(i^{(n)})$ denote this representation of $R_{n\alpha_i}$ viewed as a graded $R_{n\alpha_i}$-module with grading defined by

$$\deg(y_1^{m_1} \cdots y_n^{m_n}) := (\alpha_i \cdot \alpha_i)(m_1 + \cdots + m_n - n(n-1)/4).$$

By (14, §2.2), the left regular $R_{n\alpha_i}$-module decomposes as $P(i^n) \cong [n]_i^! \cdot P(i^{(n)})$. In particular, $P(i^{(n)})$ is projective. Set

$$\theta_i^{(n)} := \mathrm{Ind}_{\alpha,n\alpha_i}(? \boxtimes P(i^{(n)})) : R_\alpha\text{-Mod} \to R_{\alpha+n\alpha_i}\text{-Mod},$$
$$(\theta_i^*)^{(n)} := \mathrm{Hom}_{R'_{n\alpha_i}}(P(i^{(n)}), ?) : R_{\alpha+n\alpha_i}\text{-Mod} \to R_\alpha\text{-Mod},$$

where $R'_{n\alpha_i} := 1 \otimes R_{n\alpha_i} \subseteq R_{\alpha,n\alpha_i}$. These functors induce $\mathcal{A}$-linear maps on the corresponding Grothendieck groups:

$$\theta_i^{(n)} : [R_\alpha\text{-proj}] \to [R_{\alpha+n\alpha_i}\text{-proj}], \quad (\theta_i^*)^{(n)} : [R_{\alpha+n\alpha_i}\text{-mod}] \to [R_\alpha\text{-mod}].$$

Cuspidal and standard modules

Standard module theory for R_α has been developed in (21; 9; 5; 30). Here we follow the most general approach of McNamara (30). Fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N}$ of the longest element $w_0 \in W$. This gives a convex total order on the positive roots

$$\Phi_+ = \{\beta_1 > \cdots > \beta_N\},$$

with $\beta_{N+1-k} = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$.

To every positive root $\beta \in \Phi_+$ of the corresponding root system Φ, one associates a *cuspidal module* $L(\beta)$. This irreducible module is uniquely determined by the following property: if $\delta, \gamma \in Q_+$ are non-zero elements such that $\beta = \delta + \gamma$ and $\mathrm{Res}_{\delta,\gamma} L(\beta) \neq 0$, then δ is a sum of positive roots less than β and γ is a sum of positive roots greater than β.

A standard argument involving the Mackey Theorem from (14) and convexity as in the proof of (3, Lemma 2.11), yields:

Lemma 2.1.4

Let $\beta \in \Phi_+$ and $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$. All composition factors of $\mathrm{Res}_{a_1\beta,\ldots,a_n\beta} L(\beta)^{\circ(a_1+\cdots+a_n)}$ are of the form $L(\beta)^{\circ a_1} \boxtimes \cdots \boxtimes L(\beta)^{\circ a_n}$.

Let $\alpha \in Q_+$. A tuple $\pi = (p_1, \ldots p_N) \in \mathbb{Z}_{\geq 0}^N$ is called a *root partition of* α if $p_1\beta_1 + \cdots + p_N\beta_N = \alpha$. We also use the notation $\pi = (\beta_1^{p_1}, \ldots, \beta_N^{p_N})$. For example, if $\alpha = n\beta$ for $\beta \in \Phi_+$, we have a root partition $(\beta^n) \in \Pi(\alpha)$. Denote by $\Pi(\alpha)$ the set of all root partitions of α. This set has two total orders: $\leq_l$ and $\leq_r$ defined as follows: $(p_1, \ldots, p_N) <_l (s_1, \ldots, s_N)$ (resp. $(p_1, \ldots, p_N) <_r (s_1, \ldots, s_N)$) if there exists $1 \leq k \leq N$ such that $p_k < s_k$ and $p_m = s_m$ for all $m < k$ (resp. $m > k$). Finally, we have a *bilexicographic partial order*:

$$\pi \leq \sigma \iff \pi \leq_l \sigma \text{ and } \pi \leq_r \sigma \qquad (\pi, \sigma \in \Pi(\alpha)). \tag{2.21}$$

The following lemma is implicit in (30); see also (3, Lemma 2.5).

Lemma 2.1.5

Let $\beta \in \Phi_+$ and $p \in \mathbb{Z}_{\geq 0}$. Given any $\pi \in \Pi(p\beta)$, we have $\pi \geq (\beta^p)$.

For a root partition $\pi = (p_1, \ldots, p_N) \in \Pi(\alpha)$ as above, set $\mathtt{sh}(\pi) := \sum_{k=1}^N (\beta_k \cdot \beta_k) p_k(p_k - 1)/4$, and define the corresponding *proper standard module*

$$\bar{\Delta}(\pi) := L(\beta_1)^{\circ p_1} \circ \cdots \circ L(\beta_N)^{\circ p_N} \langle \mathtt{sh}(\pi) \rangle. \tag{2.22}$$

For $\pi = (\beta_1^{p_1}, \ldots, \beta_N^{p_N})$, we denote

$$\mathrm{Res}_\pi := \mathrm{Res}_{p_1\beta_1,\ldots,p_N\beta_N}.$$

Theorem 2.1.6

(30) For any convex order there exists a cuspidal system $\{L(\beta) \mid \beta \in \Phi_+\}$. Moreover:

(i) For every $\pi \in \Pi(\alpha)$, the proper standard module $\bar{\Delta}(\pi)$ has irreducible head; denote this irreducible module $L(\pi)$.

(ii) $\{L(\pi) \mid \pi \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible R_α-modules up to isomorphism.

(iii) $L(\pi)^{\circledast} \simeq L(\pi)$.

(iv) $[\bar{\Delta}(\pi) : L(\pi)]_q = 1$, and $[\bar{\Delta}(\pi) : L(\sigma)]_q \neq 0$ implies $\sigma \leq \pi$.

(v) $L(\beta)^{\circ n}$ is irreducible for every $\beta \in \Phi_+$ and every $n \in \mathbb{Z}_{>0}$.

(vi) $\mathrm{Res}_\pi \bar{\Delta}(\sigma) \neq 0$ implies $\sigma \geq \pi$, and $\mathrm{Res}_\pi \bar{\Delta}(\pi) \simeq L(\beta_1)^{\circ p_1} \boxtimes \cdots \boxtimes L(\beta_N)^{\circ p_N}$.

Note that the algebra $R_\alpha(F)$ is defined over $\mathbb{Z}$, i.e. $R_\alpha(F) \simeq R_\alpha(\mathbb{Z}) \otimes_{\mathbb{Z}} F$. We will use the corresponding indices when we need to distinguish between modules defined over different rings. The following result shows that cuspidal modules are also defined over $\mathbb{Z}$:

Lemma 2.1.7

Let $\beta \in \Phi_+$, and $v \in L(\beta)_{\mathbb{Q}}$ be a non-zero homogeneous vector. Then $L(\beta)_{\mathbb{Z}} := R_\beta(\mathbb{Z}) \cdot v \subset L(\beta)_{\mathbb{Q}}$ is an $R_\beta(\mathbb{Z})$-invariant lattice such that $L(\beta)_{\mathbb{Z}} \otimes_{\mathbb{Z}} F \simeq L(\beta)_F$ as $R_\beta(F)$-modules for any field F.

Proof. Note using degrees that $L(\beta)_{\mathbb{Z}}$ is finitely generated over $\mathbb{Z}$, hence it is a lattice in $L(\beta)_{\mathbb{Q}}$. Furthermore $\mathrm{ch}_q\, L(\beta)_{\mathbb{Z}} \otimes_{\mathbb{Z}} F = \mathrm{ch}_q\, L(\beta)_{\mathbb{Q}}$, whence by definition of the cuspidal modules, all composition factors of $\mathrm{ch}_q\, L(\beta)_{\mathbb{Z}} \otimes_{\mathbb{Z}} F$ are of the form

$L(\beta)_F$. But there is always a multiplicity one composition factor in a reduction modulo p of any irreducible module over a KLR algebra, thanks to (17, Lemma 4.7). $\square$

Quantum groups

Following (27, Section 1.2), we define the algebra $'\mathbf{f}$ to be the free $\mathbb{Q}(q)$-algebra with generators $'\theta_i$ for $i \in I$ (our q is Lusztig's v^{-1}, in keeping with the conventions of (14)). This algebra is Q_+-graded by assigning the degree α_i to $'\theta_i$ for each $i \in I$, so that $'\mathbf{f} = \oplus_{\alpha \in Q_+} {'\mathbf{f}_\alpha}$. If $x \in {'\mathbf{f}_\alpha}$, we write $|x| = \alpha$. For $\boldsymbol{i} = (i_1, \dots, i_n) \in \langle I \rangle$, write $'\theta_{\boldsymbol{i}} := {'\theta_{i_1}} \dots {'\theta_{i_n}}$. Then $\{'\theta_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ is a basis for $'\mathbf{f}_\alpha$. In particular, each $'\mathbf{f}_\alpha$ is finite dimensional. Consider the graded dual $'\mathbf{f}^* := \oplus_{\alpha \in Q_+} ({'\mathbf{f}_\alpha})^*$. We consider words $\boldsymbol{i} \in \langle I \rangle$ as elements of $'\mathbf{f}^*$, so that $\boldsymbol{i}({'\theta_{\boldsymbol{j}}}) = \delta_{\boldsymbol{i},\boldsymbol{j}}$. That is to say, $\{\boldsymbol{i} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ is the basis of $'\mathbf{f}^*_\alpha$ dual to the basis $\{'\theta_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$.

Let ${}_{\mathcal{A}}'\mathbf{f}$ be the $\mathcal{A}$-subalgebra of $'\mathbf{f}$ generated by $\{({'\theta_i})^{(n)} \mid i \in I, n \in \mathbb{Z}_{\geq 0}\}$. This algebra is Q_+-graded by ${}_{\mathcal{A}}'\mathbf{f} = \oplus_{\alpha \in Q_+} {}_{\mathcal{A}}'\mathbf{f}_\alpha$, where ${}_{\mathcal{A}}'\mathbf{f}_\alpha := {}_{\mathcal{A}}'\mathbf{f} \cap {'\mathbf{f}_\alpha}$. Given $\boldsymbol{i} = j_1^{r_1} \dots j_m^{r_m} \in \langle I \rangle$ with $j_n \neq j_{n+1}$ for $1 \leq n < m$, denote $'\theta_{(\boldsymbol{i})} := ({'\theta_{j_1}})^{(r_1)} \dots ({'\theta_{j_m}})^{(r_m)} \in {}_{\mathcal{A}}'\mathbf{f}$. Then

$$\{'\theta_{(\boldsymbol{i})} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \tag{2.23}$$

is an $\mathcal{A}$-basis of ${}_{\mathcal{A}}'\mathbf{f}_\alpha$. We also define ${}_{\mathcal{A}}'\mathbf{f}^* := \{x \in {'\mathbf{f}^*} \mid x({}_{\mathcal{A}}'\mathbf{f}) \subseteq \mathcal{A}\}$, and assign it the induced Q_+-grading. For every $\alpha \in Q_+$, the $\mathcal{A}$-module ${}_{\mathcal{A}}'\mathbf{f}^*_\alpha$ is free with basis

$$\{[\boldsymbol{i}]!\boldsymbol{i} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \tag{2.24}$$

dual to (2.23).

There is a *twisted* multiplication on ${}'\mathbf{f}\otimes{}'\mathbf{f}$ given by $(x\otimes y)(z\otimes w) = q^{-|y|\cdot|z|}xz\otimes yw$ for homogeneous $x, y, z, w \in {}'\mathbf{f}$. Let $r\colon {}'\mathbf{f} \to {}'\mathbf{f}\otimes{}'\mathbf{f}$ be the algebra homomorphism determined by $r({}'\theta_i) = {}'\theta_i \otimes 1 + 1 \otimes {}'\theta_i$ for all $i \in I$. By (27, Proposition 1.2.3) there is a unique symmetric bilinear form $(\cdot,\cdot)$ on ${}'\mathbf{f}$ such that $(1,1) = 1$ and

$$({}'\theta_i, {}'\theta_j) = \frac{\delta_{i,j}}{1-q_i^2} \quad \text{for } i, j \in I,$$

$$(xy, z) = (x \otimes y, r(z)),$$

$$(x, yz) = (r(x), y \otimes z),$$

where the bilinear form on ${}'\mathbf{f} \otimes {}'\mathbf{f}$ is given by $(x \otimes x', y \otimes y') = (x, y)(x', y')$.

Define $\mathbf{f}$ to be the quotient of ${}'\mathbf{f}$ by the radical of $(\cdot,\cdot)$. Denote the image of ${}'\theta_i$ in $\mathbf{f}$ by θ_i. The Q_+-grading on ${}'\mathbf{f}$ descends to a Q_+-grading on $\mathbf{f}$ with $|\theta_i| = i$. Let ${}_{\mathcal{A}}\mathbf{f}$ be the $\mathcal{A}$-subalgebra of $\mathbf{f}$ generated by $\theta_i^{(n)}$ for $i \in I, n \in \mathbb{Z}_{\geq 0}$. This algebra is Q_+-graded by ${}_{\mathcal{A}}\mathbf{f}_\alpha := {}_{\mathcal{A}}\mathbf{f} \cap \mathbf{f}_\alpha$. Given $\boldsymbol{i} = j_1^{r_1} \dots j_m^{r_m} \in \langle I \rangle$ with $j_n \neq j_{n+1}$ for $1 \leq n < m$, denote $\theta_{\boldsymbol{i}} := \theta_{j_1}^{r_1} \dots \theta_{j_m}^{r_m}$ and

$$\theta_{(\boldsymbol{i})} := \theta_{j_1}^{(r_1)} \dots \theta_{j_m}^{(r_m)} \in {}_{\mathcal{A}}\mathbf{f}. \tag{2.25}$$

We recall the definition of the *PBW basis* of ${}_{\mathcal{A}}\mathbf{f}$ from (27, Part VI). Recall that a reduced decomposition $w_0 = s_{i_1} \dots s_{i_N}$ yields a total order on the positive roots $\Phi_+ = \{\beta_1 > \dots > \beta_N\}$, with $\beta_{N+1-k} = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$. Now, embed ${}_{\mathcal{A}}\mathbf{f}$ into the upper half of the full quantum group via $\theta_i \mapsto E_i$ and take the braid group generators $T_i := T''_{i,+}$ from (27, 37.1.3). For $1 \leq k \leq N$, we define

$$E_{\beta_{N+1-k}} := T_{i_1} \dots T_{i_{k-1}}(\theta_{i_k}) \in {}_{\mathcal{A}}\mathbf{f}_{\beta_{N+1-k}}.$$

For a sequence $\pi = (p_1, \ldots p_N) \in \mathbb{Z}_{\geq 0}^N$, we set

$$E_\pi := E_{\beta_1}^{(p_1)} \ldots E_{\beta_N}^{(p_N)}$$

and also define

$$l_\pi := \prod_{r=1}^{N} \prod_{s=1}^{p_k} \frac{1}{1 - q_{\beta_r}^{2s}}. \tag{2.26}$$

The next theorem now gives a PBW basis of ${}_{\mathcal{A}}\mathbf{f}_\alpha$.

Theorem 2.1.8

The set $\{E_\pi \mid \pi \in \Pi(\alpha)\}$ is an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}_\alpha$. Furthermore:

$$(E_\pi, E_\sigma) = \delta_{\pi,\sigma} l_\pi.$$

Proof. This follows from Corollary 41.1.4(b), Propositions 41.1.7, 38.2.3, and Lemma 1.4.4 of Lusztig (27). □

Consider the graded dual $\mathbf{f}^* := \oplus_{\alpha \in Q_+} \mathbf{f}_\alpha^*$. The map $r^* : \mathbf{f}^* \otimes \mathbf{f}^* \to \mathbf{f}^*$ gives $\mathbf{f}^*$ the structure of an associative algebra. Let

$$\kappa : \mathbf{f}^* \hookrightarrow {}'\mathbf{f}^* \tag{2.27}$$

be the map dual to the quotient map $\xi : {}'\mathbf{f} \twoheadrightarrow \mathbf{f}$. Set ${}_{\mathcal{A}}\mathbf{f}^* := \{x \in \mathbf{f}^* \mid x({}_{\mathcal{A}}\mathbf{f}) \subseteq \mathcal{A}\}$ with the induced Q_+-grading. Given $i \in I$, we denote by $\theta_i^* : \mathbf{f}^* \to \mathbf{f}^*$ the dual map to the map $\mathbf{f} \to \mathbf{f},\ x \mapsto x\theta_i$. Then the divided power $(\theta_i^*)^{(n)} : \mathbf{f}^* \to \mathbf{f}^*$ is dual to the map $x \mapsto x\theta_i^{(n)}$. Clearly $(\theta_i^*)^{(n)}$ stabilizes ${}_{\mathcal{A}}\mathbf{f}^*$. For $\beta \in \Phi_+$, define $E_\beta^* \in {}_{\mathcal{A}}\mathbf{f}_\beta^*$ to be dual to E_β. We define

$$(E^*_\beta)^{\langle m\rangle} := q_\beta^{m(m-1)/2}(E^*_\beta)^m \quad \text{and} \quad E^*_\pi := (E^*_{\beta_1})^{\langle p_1\rangle}\dots(E^*_{\beta_N})^{\langle p_N\rangle} \tag{2.28}$$

for $m \geq 0$, and any sequence $\pi = (p_1, \dots p_N) \in \mathbb{Z}^N_{\geq 0}$. The next well-known result gives the *dual PBW basis* of ${}_{\mathcal{A}}\mathbf{f}^*$.

Theorem 2.1.9

The set $\{E^*_\pi \mid \pi \in \Pi(\alpha)\}$ is the $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}^*_\alpha$ dual to the PBW basis of Theorem 2.1.8.

Proof. It easily follows from the properties of the Lusztig bilinear form, the definition of the product on $\mathbf{f}^*$ and Theorem 2.1.8 that the linear functions $(E_\pi, -)$ and $l_\pi E^*_\pi$ on $\mathbf{f}$ are equal. It remains to apply Theorem 2.1.8 one more time. $\square$

Example 2.1.10

Let $C = A_2$, and $w_0 = s_1 s_2 s_1$. Then $E_{\alpha_1+\alpha_2} = T''_{1,+}(E_2) = E_1E_2 - qE_2E_1$, and, switching back to θ's, the PBW basis of ${}_{\mathcal{A}}\mathbf{f}_{\alpha_1+\alpha_2}$ is $\{\theta_2\theta_1, \theta_1\theta_2 - q\theta_2\theta_1\}$. Using the defining properties of Lusztig's bilinear form, one can easily check that $(E_{\alpha_2}E_{\alpha_1}, E_{\alpha_2}E_{\alpha_1}) = \frac{1}{(1-q^2)^2}$, $(E_{\alpha_1+\alpha_2}, E_{\alpha_1+\alpha_2}) = \frac{1}{(1-q^2)}$, and $(E_{\alpha_2}E_{\alpha_1}, E_{\alpha_1+\alpha_2}) = 0$. Finally the dual basis is $\{(12), (21) + q(12)\}$.

Categorification of ${}_{\mathcal{A}}\mathbf{f}$ and ${}_{\mathcal{A}}\mathbf{f}^*$

Now we state the fundamental categorification theorem proved in (14; 15), see also (32). We denote by $[R_0]$ the class of the left regular representation of the trivial algebra $R_0 \cong F$.

Theorem 2.1.11

There is a unique $\mathcal{A}$-linear isomorphism $\gamma : {}_{\mathcal{A}}\mathbf{f} \xrightarrow{\sim} [R\text{-proj}]$ such that $1 \mapsto [R_0]$ and $\gamma(x\theta_i^{(n)}) = \theta_i^{(n)}(\gamma(x))$ for all $x \in {}_{\mathcal{A}}\mathbf{f}$, $i \in I$, and $n \geq 1$. Under this isomorphism:

(1) $\gamma({}_{\mathcal{A}}\mathbf{f}_\alpha) = [R_\alpha\text{-proj}]$;

(2) the multiplication ${}_{\mathcal{A}}\mathbf{f}_\alpha \otimes {}_{\mathcal{A}}\mathbf{f}_\beta \to {}_{\mathcal{A}}\mathbf{f}_{\alpha+\beta}$ corresponds to the product on $[R\text{-proj}]$ induced by the exact functor $\mathrm{Ind}_{\alpha,\beta}$;

(3) for $\boldsymbol{i} \in \langle I \rangle_\alpha$ we have $\gamma(\theta_{\boldsymbol{i}}) = [R_\alpha e(\boldsymbol{i})]$;

(4) for $x, y \in {}_{\mathcal{A}}\mathbf{f}$ we have $(x, y) = (\gamma(x), \gamma(y))$.

Let M be a finite dimensional graded R_α-module. Define the *q-character* of M as follows:

$$\mathrm{ch}_q\, M := \sum_{\boldsymbol{i} \in \langle I \rangle_\alpha} (\dim_q M_{\boldsymbol{i}})\boldsymbol{i} \in {}_{\mathcal{A}}'\mathbf{f}^*.$$

The q-character map $\mathrm{ch}_q : R_\alpha\text{-mod} \to {}_{\mathcal{A}}'\mathbf{f}^*$ factors through to give an $\mathcal{A}$-linear map from the Grothendieck group

$$\mathrm{ch}_q : [R_\alpha\text{-mod}] \to {}_{\mathcal{A}}'\mathbf{f}^*. \tag{2.29}$$

We now state a dual result to Theorem 2.1.11, see (21, Theorem 4.4).

Theorem 2.1.12

There is a unique $\mathcal{A}$-linear isomorphism $\gamma^* : [R\text{-mod}] \xrightarrow{\sim} {}_{\mathcal{A}}\mathbf{f}^*$ with the following properties:

(1) $\gamma^*([R_0]) = 1$;

(2) $\gamma^*((\theta_i^*)^{(n)}(x)) = (\theta_i^*)^{(n)}(\gamma^*(x))$ for all $x \in [R\text{-mod}],\ i \in I,\ n \geq 1$;

(3) the following triangle is commutative:

$$\begin{tikzcd} & {}_{\mathcal{A}}'\mathbf{f}^* & \\ {[R\text{-mod}]} \arrow[ur, "\mathrm{ch}_q"] \arrow[rr, "\gamma^*"] & & {}_{\mathcal{A}}\mathbf{f}^* \arrow[ul, "\kappa"'] \end{tikzcd}$$

(4) $\gamma^*([R_\alpha\text{-mod}]) = {}_{\mathcal{A}}\mathbf{f}^*_\alpha$ for all $\gamma \in Q_+$;

(5) under the isomorphism γ^*, the multiplication ${}_{\mathcal{A}}\mathbf{f}^*_\alpha \otimes {}_{\mathcal{A}}\mathbf{f}^*_\beta \to {}_{\mathcal{A}}\mathbf{f}^*_{\alpha+\beta}$ corresponds to the product on $[R\text{-mod}]$ induced by $\mathrm{Ind}_{\alpha,\beta}$;

We conclude with McNamara's result on the categorification of the dual PBW-basis (see also (13) for simply laced Lie types):

Lemma 2.1.13

For every $\pi \in \Pi(\alpha)$ we have $\gamma^*([\bar{\Delta}(\pi)]) = E^*_\pi$.

Proof. By (30, Theorem 3.1(1)), we have $\gamma^*([L(\beta)]) = E^*_\beta$ for all $\beta \in \Phi_+$. The general case then follows from Theorem 2.1.12(5) and the definition (2.22) of $\bar{\Delta}(\pi)$.
$\square$

A dimension formula

In this section we obtain a dimension formula for R_α, which can be viewed as a combinatorial shadow of the affine quasi-hereditary structure on it. The idea of the proof comes from (2, Theorem 4.20). An independent but much less elementary proof can be found in (3, Corollary 3.15).

Recall the element $\theta_{(\boldsymbol{i})} \in {}_{\mathcal{A}}\mathbf{f}$ from (2.25) and the scalar $[\boldsymbol{i}]! \in \mathcal{A}$ from (2.1). We note that Lemma 2.1.14 and Theorem 2.1.15 *do not require the assumption that the Cartan matrix A is of finite type*, adopted elsewhere in the paper.

Lemma 2.1.14

Let $V^1, \dots, V^m \in R_\alpha$-mod, and let $v^n := \gamma^*([V^n]) \in {}_{\mathcal{A}}\mathbf{f}^*_\alpha$ for $n = 1, \dots, m$. Assume that $\{v^1, \dots, v^m\}$ is an $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}^*_\alpha$. Let $\{v_1, \dots, v_m\}$ be the dual basis of ${}_{\mathcal{A}}\mathbf{f}_\alpha$. Then for every $\boldsymbol{i} \in \langle I \rangle_\alpha$, we have

$$\theta_{(\boldsymbol{i})} = \sum_{n=1}^{m} \frac{\dim_q V^n_{\boldsymbol{i}}}{[\boldsymbol{i}]!} v_n.$$

Proof. Recall the map κ from (2.27) dual to the natural projection $\xi : {}'\mathbf{f} \twoheadrightarrow \mathbf{f}$. By Theorem 2.1.12 we have for any $1 \leq n \leq m$:

$$\kappa(v^n) = \kappa(\gamma^*([V^n])) = \mathrm{ch}_q\left([V^n]\right) = \sum_{\boldsymbol{i} \in \langle I \rangle_\alpha} (\dim_q V^n_{\boldsymbol{i}}) \boldsymbol{i} = \sum_{\boldsymbol{i} \in \langle I \rangle_\alpha} \frac{\dim_q V^n_{\boldsymbol{i}}}{[\boldsymbol{i}]!} [\boldsymbol{i}]! \boldsymbol{i}.$$

Recalling (2.23) and (2.24), $\{{}'\theta_{(\boldsymbol{i})} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ and $\{[\boldsymbol{i}]!\boldsymbol{i} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ is a pair of dual bases in ${}'_{\mathcal{A}}\mathbf{f}_\alpha$ and ${}'_{\mathcal{A}}\mathbf{f}^*_\alpha$. So, using our expression for $\kappa(v^n)$, we can now get by dualizing:

$$\begin{aligned}
\theta_{(\boldsymbol{i})} &= \xi({}'\theta_{(\boldsymbol{i})}) = \sum_{n=1}^{m} v^n(\xi({}'\theta_{(\boldsymbol{i})})) v_n \\
&= \sum_{n=1}^{m} \kappa(v^n)({}'\theta_{(\boldsymbol{i})}) v_n = \sum_{n=1}^{m} \sum_{\boldsymbol{j} \in \langle I \rangle_\alpha} \frac{\dim_q V^n_{\boldsymbol{j}}}{[\boldsymbol{j}]!} [\boldsymbol{j}]! \boldsymbol{j}({}'\theta_{(\boldsymbol{i})}) v_n \\
&= \sum_{n=1}^{m} \sum_{\boldsymbol{j} \in \langle I \rangle_\alpha} \frac{\dim_q V^n_{\boldsymbol{j}}}{[\boldsymbol{j}]!} \delta_{\boldsymbol{i},\boldsymbol{j}} v_n = \sum_{n=1}^{m} \frac{\dim_q V^n_{\boldsymbol{i}}}{[\boldsymbol{i}]!} v_n,
\end{aligned}$$

as required. $\square$

Theorem 2.1.15

With the assumptions of Lemma 2.1.14, for every $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle_\alpha$, we have

$$\dim_q \left(e(\boldsymbol{i}) R_\alpha e(\boldsymbol{j})\right) = \sum_{n,k=1}^{m} (\dim_q V_{\boldsymbol{i}}^n)(\dim_q V_{\boldsymbol{j}}^k)(v_n, v_k).$$

In particular,

$$\dim_q \left(R_\alpha\right) = \sum_{n,k=1}^{m} (\dim_q V^n)(\dim_q V^k)(v_n, v_k).$$

Proof. Theorem 2.1.11(3) shows that $[R_\alpha e(\boldsymbol{i})] = \gamma(\theta_{\boldsymbol{i}}) = \gamma([\boldsymbol{i}]!\theta_{(\boldsymbol{i})})$. Using the definitions and Theorem 2.1.11(4), we have

$$\begin{aligned} \dim_q \left(e(\boldsymbol{i}) R_\alpha e(\boldsymbol{j})\right) &= \dim_q \left((R_\alpha e(\boldsymbol{i}))^\tau \otimes_{R_\alpha} R_\alpha e(\boldsymbol{j})\right) \\ &= ([R_\alpha e(\boldsymbol{i})], [R_\alpha e(\boldsymbol{j})]) = ([\boldsymbol{i}]!\theta_{(\boldsymbol{i})}, [\boldsymbol{j}]!\theta_{(\boldsymbol{j})}). \end{aligned}$$

Now, by Lemma 2.1.14 we see that

$$([\boldsymbol{i}]!\theta_{(\boldsymbol{i})}, [\boldsymbol{j}]!\theta_{(\boldsymbol{j})}) = \Big(\sum_{n=1}^{m} (\dim_q V_{\boldsymbol{i}}^n) v_n, \sum_{k=1}^{n} (\dim_q V_{\boldsymbol{j}}^k) v_k \Big),$$

which implies the theorem. $\square$

Recall the scalar l_π from (2.26), the module $\bar{\Delta}(\pi)$ from (2.22), and PBW-basis elements E_π from §2.1.

Corollary 2.1.16

For every $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle_\alpha$, we have

$$\dim_q \left(e(\boldsymbol{i}) R_\alpha e(\boldsymbol{j})\right) = \sum_{\pi \in \Pi(\alpha)} (\dim_q \bar{\Delta}(\pi)_{\boldsymbol{i}})(\dim_q \bar{\Delta}(\pi)_{\boldsymbol{j}}) l_\pi.$$

In particular,

$$\dim_q (R_\alpha) = \sum_{\pi \in \Pi(\alpha)} (\dim_q \bar{\Delta}(\pi))^2 l_\pi.$$

Proof. By Lemma 2.1.13, we have $\gamma^*(\bar{\Delta}(\pi)) = E_\pi^*$ for all $\pi \in \Pi(\alpha)$. Moreover, by Theorem 2.1.9, $\{E_\pi^* \mid \pi \in \Pi(\alpha)\}$ and $\{E_\pi \mid \pi \in \Pi(\alpha)\}$ is a pair of dual bases in ${}_{\mathcal{A}}\mathbf{f}_\alpha^*$ and ${}_{\mathcal{A}}\mathbf{f}_\alpha$. Finally, $(E_\pi, E_\sigma) = \delta_{\pi,\sigma} l_\pi$ by Theorem 2.1.8. It remains to apply Theorem 2.1.15. $\square$

Affine Cellular Structure

Throughout this section we fix $\alpha \in Q_+$ and a total order $\leq$ on the set $\Pi(\alpha)$ of root partitions of α, which refines the bilexicographic partial order (2.21).

Some special word idempotents

Recall from Section 2.1 that for each $\beta \in \Phi_+$, we have a cuspidal module $L(\beta)$. Every irreducible R_α-module L has a word space $L_{\boldsymbol{i}}$ such that the lowest degree component of $L_{\boldsymbol{i}}$ is one-dimensional, see for example (17, Lemma 2.30) or (3, Lemma 4.5) for two natural choices. From now on, for each $\beta \in \Phi_+$ we make an arbitrary choice of such word $\boldsymbol{i}_\beta$ for the cuspidal module $L(\beta)$.

For $\pi = (\beta_1^{p_1}, \ldots, \beta_N^{p_N}) \in \Pi(\alpha)$, define

$$\boldsymbol{i}_\pi := \boldsymbol{i}_{\beta_1}^{p_1} \ldots \boldsymbol{i}_{\beta_N}^{p_N},$$
$$I_\pi := \sum_{\sigma \geq \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha,$$
$$I_{>\pi} := \sum_{\sigma > \pi} R_\alpha e(\boldsymbol{i}_\sigma) R_\alpha,$$

the sums being over $\sigma \in \Pi(\alpha)$. We also consider the (non-unital) embedding of algebras:

$$\iota_\pi := \iota_{p_1\beta_1,\ldots,p_N\beta_N} : R_{p_1\beta_1} \otimes \cdots \otimes R_{p_N\beta_N} \hookrightarrow R_\alpha,$$

whose image is the parabolic subalgebra

$$R_\pi := R_{p_1\beta_1,\ldots,p_N\beta_N}.$$

Lemma 2.2.1

If a two-sided ideal J of R_α contains all idempotents $e(\boldsymbol{i}_\pi)$ with $\pi \in \Pi(\alpha)$, then $J = R_\alpha$.

Proof. If $J \neq R_\alpha$, let I be a maximal left ideal containing J. Then $R_\alpha/I \cong L(\pi)$ for some π. Then $e(\boldsymbol{i}_\pi)L(\pi) \neq 0$, which contradicts the assumption that $e(\boldsymbol{i}_\pi) \in J$. This argument proves the lemma over any field, and then it also follows for $\mathbb{Z}$. $\square$

Lemma 2.2.2

Let $\pi \in \Pi(\alpha)$ and $e \in R_\alpha$ a homogeneous idempotent. If $eL(\sigma) = 0$ for all $\sigma \leq \pi$, then $e \in I_{>\pi}$.

Proof. Let I be any maximal (graded) left ideal containing $I_{>\pi}$. Then $R_\alpha/I \cong L(\sigma)$ for some $\sigma \in \Pi(\alpha)$ such that $\sigma \leq \pi$. Indeed, if we had $\sigma > \pi$ then by definition $e(\boldsymbol{i}_\sigma) \in I_{>\pi} \subseteq I$, and so $e(\boldsymbol{i}_\sigma)L(\sigma) = e(\boldsymbol{i}_\sigma)(R_\alpha/I) = 0$, which is a contradiction.

We have shown that e is contained in every maximal left ideal containing $I_{>\pi}$. By a standard argument, explained in (19, Lemma 5.8), we conclude that $e \in I_{>\pi}$. □

Corollary 2.2.3

Suppose that $\alpha = p\beta$ for some $p \geq 1$ and $\beta \in \Phi_+$. Let $\boldsymbol{i} \in \langle I \rangle_\alpha$. If $e(\boldsymbol{i})L(\beta^p) = 0$, then $e(\boldsymbol{i}) \in I_{>(\beta^p)}$.

Proof. This follows from Lemma 2.1.5 together with Proposition 2.2.2. □

Lemma 2.2.4

Let $\pi = (\beta_1^{p_1} \dots \beta_N^{p_N}) \in \Pi(\alpha)$. Then $R_\pi \subseteq I_\pi$.

Proof. By Lemma 2.2.1, we have

$$R_{p_n\beta_n} = \sum_{\pi^{(n)} \in \Pi(p_n\beta_n)} R_{p_n\beta_n} e(\boldsymbol{i}_{\pi^{(n)}}) R_{p_n\beta_n}$$

for all $n = 1, \dots, N$. Therefore the image of ι_π equals

$$\sum R_\pi e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}}) R_\pi \tag{2.30}$$

where the sum is over all $\pi^{(1)} \in \Pi(p_1\beta_1), \dots, \pi^{(N)} \in \Pi(p_N\beta_N)$. Fix $\pi^{(n)} \in \Pi(p_n\beta_n)$ for all $n = 1, \dots, N$. If $\pi^{(n)} = (\beta_n^{p_n})$ for every n, then $\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}} = \boldsymbol{i}_\pi$, and the corresponding term of (2.30) is in I_π by definition.

Let us now assume that $\pi^{(k)} \neq (\beta_k^{p_k})$ for some k. In view of Lemma 2.1.5, we have $\pi^{(k)} > (\beta_k^{p_k})$. For any $\sigma \in \Pi(\alpha)$ we have $e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}})\bar{\Delta}(\sigma) \subseteq \mathrm{Res}_\pi \bar{\Delta}(\sigma)$, and by Theorem 2.1.6(vi), if $\sigma < \pi$ then $\mathrm{Res}_\pi \bar{\Delta}(\sigma) = 0$. Furthermore, for $\sigma = \pi$ we have, by Theorem 2.1.6(vi) applied again

$$\begin{aligned} e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}})\bar{\Delta}(\pi) &= e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}})\mathrm{Res}_\pi \bar{\Delta}(\pi) \\ &= e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}})(\bar{\Delta}(\beta_1^{p_1}) \boxtimes \cdots \boxtimes \bar{\Delta}(\beta_N^{p_N})) \\ &\subseteq (\mathrm{Res}_{\pi^{(1)}} \bar{\Delta}(\beta_1^{p_1})) \boxtimes \cdots \boxtimes (\mathrm{Res}_{\pi^{(N)}} \bar{\Delta}(\beta_N^{p_N})), \end{aligned}$$

which is zero since $\mathrm{Res}_{\pi^{(k)}} \bar{\Delta}(\beta_k^{p_k}) = 0$ by Theorem 2.1.6(vi) again. We have shown that for all $\sigma \leq \pi$ we have $e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}})\bar{\Delta}(\sigma) = 0$, and consequently $e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}})L(\sigma) = 0$. Applying Lemma 2.2.2, we have that $e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(N)}}) \in I_{>\pi} \subseteq I_\pi$. $\square$

The following result will often allow us to reduce to the case of a smaller height.

Proposition 2.2.5

Let $\gamma_1, \dots, \gamma_m \in Q_+$, $1 \leq k \leq m$, and $\pi_0 \in \Pi(\gamma_k)$. Assume that $\pi \in \Pi(\gamma_1 + \cdots + \gamma_m)$ is such that all idempotents from the set

$$E = \{e(\boldsymbol{i}_{\pi^{(1)}} \dots \boldsymbol{i}_{\pi^{(m)}}) \mid \pi^{(n)} \in \Pi(\gamma_n) \text{ for all } n = 1, \dots, m \text{ and } \pi^{(k)} > \pi_0\}$$

annihilate the irreducible modules $L(\sigma)$ for all $\sigma \leq \pi$. Then

$$\iota_{\gamma_1,\dots,\gamma_m}(R_{\gamma_1} \otimes \cdots \otimes R_{\gamma_{k-1}} \otimes I_{>\pi_0} \otimes R_{\gamma_{k+1}} \otimes \cdots \otimes R_{\gamma_m}) \subseteq I_{>\pi}. \quad (2.31)$$

Proof. We may assume that $\gamma_k \neq 0$ since otherwise $I_{>\pi_0} = 0$, and the result is clear. By Lemma 2.2.1, we have $R_{\gamma_n} = \sum_{\pi^{(n)} \in \Pi(\gamma_n)} R_{\gamma_n} e(\boldsymbol{i}_{\pi^{(n)}}) R_{\gamma_n}$ for all $n =$

$1, \ldots, m$, and by definition, we have $I_{>\pi_0} = \sum_{\pi^{(k)} > \pi_0} R_{\gamma_k} e(\boldsymbol{i}_{\pi^{(k)}}) R_{\gamma_k}$. Therefore the left hand side of (2.31) equals $\sum_{e \in E} R_{\gamma_1,\ldots,\gamma_m} e R_{\gamma_1,\ldots,\gamma_m}$. The result now follows by applying Lemma 2.2.2. $\square$

Recall from Lemma 2.2.4 that $\mathrm{im}(\iota_\pi) \subseteq I_\pi$.

Corollary 2.2.6

Let $\pi = (\beta_1^{p_1}, \ldots, \beta_N^{p_N}) \in \Pi(\alpha)$ and $1 \leq k \leq N$. Then

$$\iota_\pi(R_{p_1\beta_1} \otimes \cdots \otimes R_{p_{k-1}\beta_{k-1}} \otimes I_{>(\beta_k^{p_k})} \otimes R_{p_{k+1}\beta_{k+1}} \otimes \cdots \otimes R_{p_N\beta_N}) \subseteq I_{>\pi}.$$

In particular, the composite map $R_\pi \xrightarrow{\iota_\pi} I_\pi \longrightarrow I_\pi / I_{>\pi}$ factors through the quotient $R_{p_1\beta_1}/I_{>(\beta_1^{p_1})} \otimes \cdots \otimes R_{p_N\beta_N}/I_{>(\beta_N^{p_N})}$.

Proof. Apply Proposition 2.2.5 with $m = N$, $\gamma_n = p_n\beta_n$, for $1 \leq n \leq N$, $\pi_0 = (\beta_k^{p_k})$, and $\pi = \pi$. We have to prove that any $e = e(\boldsymbol{i}_{\pi^{(1)}} \ldots \boldsymbol{i}_{\pi^{(N)}}) \in E$ annihilates all $L(\sigma)$ for $\sigma \leq \pi$. We prove more, namely that e annihilates $\bar{\Delta}(\sigma)$ for all $\sigma \leq \pi$. By Theorem 2.1.6(vi):

$$\begin{aligned} e\bar{\Delta}(\sigma) = e\mathrm{Res}_\pi \bar{\Delta}(\sigma) &= e\delta_{\pi,\sigma}(L(\beta_1)^{\circ p_1} \boxtimes \cdots \boxtimes L(\beta_1)^{\circ p_1}) \\ &= \delta_{\pi,\sigma}\, e(\boldsymbol{i}_{\pi^{(1)}}) L(\beta_1)^{\circ p_1} \boxtimes \cdots \boxtimes e(\boldsymbol{i}_{\pi^{(N)}}) L(\beta_N)^{\circ p_N}, \end{aligned}$$

which is zero since

$$e(\boldsymbol{i}_{\pi^{(k)}}) L(\beta_k)^{\circ p_k} = e(\boldsymbol{i}_{\pi^{(k)}}) \mathrm{Res}_{\pi^{(k)}} L(\beta_k)^{\circ p_k} = 0$$

by Theorem 2.1.6(vi) again. $\square$

Corollary 2.2.7

For $\beta \in \Phi_+$ and $a, b, c \in \mathbb{Z}_{\geq 0}$ we have

$$\iota_{a\beta,b\beta,c\beta}(R_{a\beta} \otimes I_{>(\beta^b)} \otimes R_{c\beta}) \subseteq I_{>(\beta^{a+b+c})}.$$

Proof. We apply Proposition 2.2.5 with $m = 3$, $k = 2$, $\gamma_1 = a\beta$, $\gamma_2 = b\beta$, $\gamma_3 = c\beta$, $\pi_0 = (\beta^b)$ and $\pi = (\beta^{a+b+c})$. Pick an idempotent $e = e(\boldsymbol{i}_{\pi(1)}\boldsymbol{i}_{\pi(2)}\boldsymbol{i}_{\pi(3)}) \in E$. Since π is the minimal element of $\Pi((a + b + c)\beta)$, it suffices to prove that $eL(\pi) = 0$. Note that $eL(\pi) = e\mathrm{Res}_{a\beta,b\beta,c\beta}L(\pi)$, so using Lemma 2.1.4, we just need to show that $e(L(\beta)^{\circ a} \boxtimes L(\beta)^{\circ b} \boxtimes L(\beta)^{\circ c}) = 0$. But

$$e(L(\beta)^{\circ a} \boxtimes L(\beta)^{\circ b} \boxtimes L(\beta)^{\circ c}) = e(\boldsymbol{i}_{\pi(1)})L(\beta)^{\circ a} \boxtimes e(\boldsymbol{i}_{\pi(2)})L(\beta)^{\circ b} \boxtimes e(\boldsymbol{i}_{\pi(3)})L(\beta)^{\circ c}$$

is zero, since $e(\boldsymbol{i}_{\pi(2)})L(\beta)^{\circ b} = e(\boldsymbol{i}_{\pi(2)})\mathrm{Res}_{\pi(2)}L(\beta)^{\circ b} = 0$ by Theorem 2.1.6(vi). $\square$

Repeated application of Corollary 2.2.7 gives the following result.

Corollary 2.2.8

For $\beta \in \Phi_+$ and $p \in \mathbb{Z}_{>0}$ we have

$$\iota_{\beta,\dots,\beta}(R_\beta \otimes \cdots \otimes I_{>(\beta)} \otimes \cdots \otimes R_\beta) \subseteq I_{>(\beta^p)}.$$

Basic notation concerning cellular bases

Let β be a fixed positive root of height d. Recall that we have made a choice of $\boldsymbol{i}_\beta$ so that in the word space $e(\boldsymbol{i}_\beta)L(\beta)$ of the cuspidal module, the lowest degree part is 1-dimensional. We fix its spanning vector v_β^- defined over $\mathbb{Z}$, see Lemma 2.1.7. Similarly, the highest degree part is spanned over $\mathbb{Z}$ by some v_β^+.

We consider the element of the symmetric group $w_{\beta,r} \in \mathfrak{S}_{pd}$

$$w_{\beta,r} := \prod_{k=1}^{d}((r-1)d+k, rd+k).$$

which permutes the rth and the $(r+1)$st 'd-blocks'. Now define

$$\psi_{\beta,r} := \psi_{w_{\beta,r}} \in R_{p\beta}.$$

Moreover, for $u \in \mathfrak{S}_p$ with a fixed reduced decomposition $u = s_{r_1} \dots s_{r_m}$, define the elements

$$w_{\beta,u} := w_{\beta,r_1} \dots w_{\beta,r_m} \in \mathfrak{S}_{pd},$$

$$\psi_{\beta,u} := \psi_{\beta,r_1} \dots \psi_{\beta,r_m} \in R_{p\beta}.$$

In Section 2.4, we will explicitly define homogeneous elements

$$\delta_\beta,\ D_\beta,\ y_\beta\ \in\ e(\boldsymbol{i}_\beta)R_\beta e(\boldsymbol{i}_\beta)$$

and $e_\beta := D_\beta \delta_\beta$ so that the following hypothesis is satisfied:

Hypothesis 2.2.9

We have:

(i) $e_\beta^2 - e_\beta \in I_{>(\beta)}$.

(ii) δ_β, D_β and y_β are τ-invariant.

(iii) $\delta_\beta v_\beta^- = v_\beta^+$ and $D_\beta v_\beta^+ = v_\beta^-$,

(iv) y_β has degree $\beta \cdot \beta$ and commutes with δ_β and D_β,

(v) The algebra $(e_\beta R_\beta e_\beta + I_{>(\beta)})/I_{>(\beta)}$ is generated by $e_\beta y_\beta e_\beta + I_{>(\beta)}$.

(vi) $\iota_{\beta,\beta}(D_\beta \otimes D_\beta)\psi_{\beta,1} = \psi_{\beta,1}\iota_{\beta,\beta}(D_\beta \otimes D_\beta)$.

From now on until we verify it in Section 2.4, we will work under the assumption that Hypothesis 2.2.9 holds. It turns out that this hypothesis is sufficient to construct affine cellular bases.

Lemma 2.2.10

$R_\beta e_\beta R_\beta + I_{>(\beta)} = R_\beta$

Proof. This follows as in the proof of Lemma 2.2.1 using $e_\beta L(\beta) \neq 0$. $\square$

Fix $p \in \mathbb{Z}_{>0}$, and define the element

$$v^-_{(\beta^p)} = \psi_{\beta,w_0} \otimes (v^-_\beta)^{\otimes p} \in \bar{\Delta}(\beta^p),$$

where $w_0 \in \mathfrak{S}_p$ is the longest element. Using Lemma 2.1.7, we can choose a set

$$\mathfrak{B}_{(\beta^p)} \subseteq R_{p\beta}$$

of elements defined over $\mathbb{Z}$ such that

$$\{bv^-_{(\beta^p)} \mid b \in \mathfrak{B}_{(\beta^p)}\}$$

is an $\mathcal{O}$-basis of $L(\beta)^{\circ p}_{\mathcal{O}}$. Define the elements

$$y_{\beta,r} := \iota_{(r-1)\beta,\beta,(p-r)\beta}(1 \otimes y_\beta \otimes 1) \in R_{p\beta} \qquad (1 \leq r \leq p).$$

Further, define the elements of $R_{p\beta}$

$$e_{\beta^{\boxtimes p}} := \iota_{\beta,\dots,\beta}(e_\beta, \dots, e_\beta), \tag{2.32}$$

$$\delta_{(\beta^p)} := y_{\beta,2} y_{\beta,3}^2 \dots y_{\beta,p}^{p-1} \iota_{\beta,\dots,\beta}(\delta_\beta \otimes \dots \otimes \delta_\beta), \tag{2.33}$$

$$D_{(\beta^p)} := \psi_{\beta,w_0} \iota_{\beta,\dots,\beta}(D_\beta \otimes \dots \otimes D_\beta), \tag{2.34}$$

$$e_{(\beta^p)} := D_{(\beta^p)} \delta_{(\beta^p)} = \psi_{\beta,w_0} y_{\beta,2} y_{\beta,3}^2 \dots y_{\beta,p}^{p-1} e_{\beta^{\boxtimes p}}. \tag{2.35}$$

It will be proved in Corollary 2.2.24 that $e_{(\beta^p)}^2 - e_{(\beta^p)} \in I_{>(\beta^p)}$ generalizing part (i) of Hypothesis 2.2.9. It is easy to see, as in (19, Lemma 2.4), that there is always a choice of a reduced decompositon of w_0 such that

$$\psi_{\beta,w_0}^\tau = \psi_{\beta,w_0}. \tag{2.36}$$

We have the algebras of polynomials and the symmetric polynomials:

$$P_{(\beta^p)} = \mathcal{O}[y_{\beta,1}, \dots, y_{\beta,p}] \quad \text{and} \quad \Lambda_{(\beta^p)} = P_{(\beta^p)}^{\mathfrak{S}_p} \tag{2.37}$$

While it is clear that the $y_{\beta,r}$ commute, we do not yet know that they are algebraically independent, but this will turn out to be the case. For now, one can interpret $\Lambda_{(\beta^p)}$ as the algebra generated by the elementary symmetric functions in $y_{\beta,1}, \dots, y_{\beta,p}$. Note using Hypothesis 2.2.9(iv) that

$$\dim_q \Lambda_{(\beta^p)} \leq \prod_{s=1}^{p} \frac{1}{1 - q_\beta^{2s}}. \tag{2.38}$$

Given $\alpha \in Q_+$ of height d and a root partition $\pi = (\beta_1^{p_1}, \dots, \beta_N^{p_N}) \in \Pi(\alpha)$ we define the parabolic subgroup

$$\mathfrak{S}_\pi := \mathfrak{S}_{p_1 \mathrm{ht}(\beta_1)} \times \cdots \times \mathfrak{S}_{p_N \mathrm{ht}(\beta_N)} \subseteq \mathfrak{S}_d,$$

and we denote by $\mathfrak{S}^\pi$ the set of minimal left coset representatives of $\mathfrak{S}_\pi$ in $\mathfrak{S}_d$. Set

$$\mathfrak{B}_\pi := \{\psi_w \iota_\pi (b_1 \otimes \cdots \otimes b_N) \mid w \in \mathfrak{S}^\pi,\ b_n \in \mathfrak{B}_{(\beta_n^{p_n})} \text{ for } n = 1, \dots, N\}.$$

Using the natural embedding of $L(\beta_1)^{\circ p_1} \boxtimes \cdots \boxtimes L(\beta_N)^{\circ p_N} \subseteq \bar{\Delta}(\pi)$, we define the element

$$v_\pi^- = v_{(\beta_1^{p_1})}^- \otimes \cdots \otimes v_{(\beta_N^{p_N})}^- \in \bar{\Delta}(\pi)$$

which belongs to the word space of $\bar{\Delta}(\pi)$ corresponding to the word

$$\boldsymbol{i}_\pi := \boldsymbol{i}_{\beta_1}^{p_1} \dots \boldsymbol{i}_{\beta_N}^{p_N}.$$

From definitions we have

<u>Lemma 2.2.11</u>

Let $\pi \in \Pi(\alpha)$. Then $\{b v_\pi^- \mid b \in \mathfrak{B}_\pi\}$ is a basis for $\bar{\Delta}(\pi)$.

Define

$$\delta_\pi := \iota_\pi(\delta_{(\beta_1^{p_1})} \otimes \cdots \otimes \delta_{(\beta_N^{p_N})}),$$
$$D_\pi := \iota_\pi(D_{(\beta_1^{p_1})} \otimes \cdots \otimes D_{(\beta_N^{p_N})}),$$
$$e_\pi := \iota_\pi(e_{(\beta_1^{p_1})} \otimes \cdots \otimes e_{(\beta_N^{p_N})}) = D_\pi \delta_\pi,$$
$$\Delta(\pi) := ((R_\alpha e_\pi + I_{>\pi})/I_{>\pi})\langle \deg(v_\pi^-)\rangle,$$
$$\Delta'(\pi) := ((e_\pi R_\alpha + I_{>\pi})/I_{>\pi})\langle \deg(v_\pi^+)\rangle,$$
$$\Lambda_\pi := \iota_\pi(\Lambda_{(\beta_1^{p_1})} \otimes \cdots \otimes \Lambda_{(\beta_N^{p_N})}).$$

Note by (2.38) and (2.26) that

$$\dim_q \Lambda_\pi \leq l_\pi. \tag{2.39}$$

Choose also a homogeneous basis X_π for Λ_π. The following lemma is a consequence of Hypothesis 2.2.9(ii),(vi) and (2.36).

Lemma 2.2.12

We have $D_\pi^\tau = D_\pi$ and $\delta_\pi^\tau = \delta_\pi$.

Powers of a single root

Throughout this subsection $\beta \in \Phi_+$ and $p \in \mathbb{Z}_{>0}$ are fixed. Define $\alpha := p\beta$, and $\sigma := (\beta^p) \in \Pi(\alpha)$.

Define $\bar{R}_\alpha := R_\alpha/I_{>\sigma}$, and given $r \in R_\alpha$ write $\bar{r}$ for its image in $\bar{R}_\alpha$. The following proposition is the main result of this subsection.

Proposition 2.2.13

We have that

(i) $\{\bar{b}\bar{f}\bar{e}_\sigma \mid b \in \mathfrak{B}_\sigma, f \in X_\sigma\}$ is an $\mathcal{O}$-basis for $\Delta(\sigma)$.

(ii) $\{\bar{e}_\sigma \bar{f}\bar{D}_\sigma \bar{b}^\tau \mid b \in \mathfrak{B}_\sigma, f \in X_\sigma\}$ is an $\mathcal{O}$-basis for $\Delta'(\sigma)$.

(iii) $\{\bar{b}\bar{e}_\sigma \bar{f}\bar{D}_\sigma (\bar{b}')^\tau \mid b, b' \in \mathfrak{B}_\sigma, f \in X_\sigma\}$ is an $\mathcal{O}$-basis for $\bar{R}_\alpha$.

(iv) The elements $\bar{y}_{\beta,1}, \ldots, \bar{y}_{\beta,p}$ are algebraically independent.

The proof of the Proposition will occupy this subsection. It goes by induction on $p\,\mathrm{ht}(\beta) = \mathrm{ht}(\alpha)$. If β a simple root, then $R_\alpha = \bar{R}_\alpha$ is exactly the nil-Hecke algebra, and we are done by Theorem 2.1.3. For the rest of the section, we assume the Proposition holds with $\sigma = (\gamma^s) \in \Pi(s\gamma)$ whenever $\gamma \in \Phi_+$ and $s\,\mathrm{ht}(\gamma) < p\,\mathrm{ht}(\beta)$ and prove that it also holds for $\sigma = (\beta^p)$. We shall also assume that $\mathcal{O} = F$ is a field, and then use Lemma 2.1.1 to lift to $\mathbb{Z}$-forms.

Lemma 2.2.14

Assume that $p = 1$. Then Proposition 2.2.13 holds.

Proof. Since $L(\beta)$ is the unique simple module in $\bar{R}_\beta$-mod and

$$\mathrm{Hom}_{\bar{R}_\beta}(\Delta(\beta), L(\beta)) = e_\beta L(\beta) = F v_\beta^-$$

is one-dimensional by Hypothesis 2.2.9, it follows that $\Delta(\beta)$ is the projective cover of $L(\beta)$ in $\bar{R}_\beta$-mod under the map $\bar{e}_\beta \mapsto v_\beta^-$. All composition factors of $\Delta(\beta)$ are isomorphic to $L(\beta)$. Therefore, lifting the basis $\{b v_\beta^- \mid b \in \mathfrak{B}_\beta\}$ of $L(\beta)$ to $\Delta(\beta)$ we see that $\Delta(\beta)$ is spanned by

$$\{\bar{b}\varphi(\bar{e}_\beta) \mid b \in \mathfrak{B}_\beta, \varphi \in \mathrm{End}_{\bar{R}_\beta}(\Delta(\beta))\}.$$

By Hypothesis 2.2.9(v), $\mathrm{End}_{\bar{R}_\beta}(\Delta(\beta)) \simeq \bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is generated by $\bar{e}_\beta \bar{y}_\beta \bar{e}_\beta = \bar{y}_\beta \bar{e}_\beta$. Thus

$$\Delta(\beta) = F\text{-span}\{\bar{b}\bar{f}\bar{e}_\beta \mid b \in \mathfrak{B}_\beta, f \in X_\beta\}.$$

Analogously, $\Delta'(\beta)$ is the projective cover of $L(\beta)^\tau$ as right $\bar{R}_\beta$-modules under the map $\bar{e}_\beta \mapsto v_\beta^+$. As above, lifting the basis $\{v_\beta^+ D_\beta b^\tau \mid b \in \mathfrak{B}_\beta\}$ of $L(\beta)^\tau$ to $\Delta'(\beta)$ we see that

$$\Delta'(\beta) = F\text{-span}\{\bar{e}_\beta \bar{f} \bar{D}_\beta \bar{b}^\tau \mid b \in \mathfrak{B}_\beta, f \in X_\beta\}.$$

Therefore by Lemma 2.2.10 and Hypothesis 2.2.9(iv),

$$\bar{R}_\beta = \bar{R}_\beta \bar{e}_\beta \bar{R}_\beta = F\text{-span}\{\bar{b}\bar{e}_\beta \bar{f} \bar{D}_\beta (\bar{b}')^\tau \mid b, b' \in \mathfrak{B}_\beta, f \in X_\beta\}.$$

Let $\pi = (\beta_1^{p_1}, \ldots, \beta_N^{p_N}) > (\beta)$. By definition and (14, Proposition 2.16) we have

$$\begin{aligned} I_\pi &= R_\beta e(\boldsymbol{i}_\pi) R_\beta + I_{>\pi} \\ &= \sum_{u,v \in \mathfrak{S}^\pi} \psi_u R_\pi e(\boldsymbol{i}_\pi) R_\pi \psi_v^\tau + I_{>\pi} \subseteq \sum_{u,v \in \mathfrak{S}^\pi} \psi_u R_\pi \psi_v^\tau + I_{>\pi}, \end{aligned}$$

because $e(\boldsymbol{i}_\pi) \in R_\pi$. The opposite inclusion follows from Lemma 2.2.4.

For $n = 1, \ldots, N$, define

$$B_n := \{b e_{(\beta_n^{p_n})} f D_{\beta_n^{p_n}} (b')^\tau \mid b, b' \in \mathfrak{B}_{(\beta_n^{p_n})}, f \in X_{(\beta_n^{p_n})}\}.$$

By part (iii) of the induction hypothesis, for $n = 1, \dots, N$, the image of B_n in $\bar{R}_{p_n\beta_n}$ is a basis. Let

$$B_\pi := \{\iota_\pi(b_1 \otimes \cdots \otimes b_N) \mid b_1 \in \mathfrak{B}_{(\beta_1^{p_1})}, \dots, b_N \in \mathfrak{B}_{(\beta_N^{p_N})}\}.$$

By Corollary 2.2.6 and definitions from Section 2.2,

$$\begin{aligned} R_\pi + I_{>\pi} &= F\text{-span}\{\iota_\pi(r_1 \otimes \cdots \otimes r_N) \mid r_n \in B_n \text{ for } n = 1, \dots, N\} + I_{>\pi} \\ &= F\text{-span}\{be_\pi f D_\pi(b')^\tau \mid b, b' \in B_\pi, f \in X_\pi\} + I_{>\pi} \end{aligned}$$

and therefore

$$I_\pi = F\text{-span}\{\psi_u b e_\pi f D_\pi(b')^\tau \psi_v^\tau \mid u, v \in \mathfrak{S}^\pi, b, b' \in B_\pi, f \in X_\pi\} + I_{>\pi}.$$

By definition of $\mathfrak{B}_\pi$ we have

$$I_\pi = F\text{-span}\{be_\pi f D_\pi(b')^\tau \mid b, b' \in \mathfrak{B}_\pi, f \in X_\pi\} + I_{>\pi}, \tag{2.40}$$

$$R_\beta = \sum_{\pi \in \Pi(\beta)} F\text{-span}\{be_\pi f D_\pi(b')^\tau \mid b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}. \tag{2.41}$$

Using (2.39) and the equality $\deg(D_\pi) = 2\deg(v_\pi^-)$ for all $\pi \in \Pi(\beta)$, we get

$$\begin{aligned}
\dim_q(R_\beta) &= \sum_{\pi\in\Pi(\beta)} \dim_q(F\text{-span}\{be_\pi f D_\pi (b')^\tau \mid b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}) \\
&\le \sum_{\pi\in\Pi(\beta)} \Big(\sum_{b\in\mathfrak{B}_\pi} q^{\deg(b)}\Big) \dim_q(\Lambda_\pi) q^{\deg(D_\pi)} \Big(\sum_{b\in\mathfrak{B}_\pi} q^{\deg(b)}\Big) \\
&\le \sum_{\pi\in\Pi(\beta)} \Big(\sum_{b\in\mathfrak{B}_\pi} q^{\deg(bv_\pi^-)}\Big)^2 l_\pi \\
&= \sum_{\pi\in\Pi(\beta)} \dim_q(\bar{\Delta}(\pi))^2 l_\pi = \dim_q(R_\beta),
\end{aligned}$$

by Corollary 2.1.16. The inequalities are therefore equalities, and this implies that the spanning set $\{be_\pi f D_\pi (b')^\tau \mid \pi \in \Pi(\beta), b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}$ of R_β is a basis and $\dim_q \Lambda_\pi = l_\pi$ for all π. These yield (iii) and (iv) of Proposition 2.2.13 in our special case $p = 1$.

To show (i) and (ii), we have already noted that the claimed bases span $\Delta(\beta)$ and $\Delta'(\beta)$, respectively. We now apply part (iii) to see that they are linearly independent. □

Corollary 2.2.15

We have

(i) $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is a polynomial algebra in the variable $\bar{y}_\beta \bar{e}_\beta$.

(ii) $\Delta(\beta)$ is a free right $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$-module with basis $\{\bar{b}\bar{e}_\beta \mid b \in \mathfrak{B}_\beta\}$.

(iii) $\Delta'(\beta)$ is a free left $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$-module with basis $\{\bar{e}_\beta \bar{D}_\beta \bar{b}^\tau \mid b \in \mathfrak{B}_\beta\}$.

Proof. By the lemma, we have Proposition 2.2.13 for $p = 1$. Now, (i) follows from parts (i) and (iv) of the proposition. The remaining statements follow from parts (i) and (ii) of the proposition. □

Corollary 2.2.16

In the Grothendieck group, we have $[\Delta(\beta)] = [L(\beta)]/(1 - q_\beta^2)$.

Lemma 2.2.17

Up to a degree shift, $\Delta(\beta)^{\circ p} \cong \bar{R}_{p\beta}\bar{e}_{\beta^{\boxtimes p}}$.

Proof. By Corollary 2.2.8 we have a map

$$\Delta(\beta)^{\boxtimes p} \to \mathrm{Res}_{\beta,\dots,\beta}(\bar{R}_{p\beta}\bar{e}_{\beta^{\boxtimes p}}),\ \bar{e}_\beta^{\otimes p} \mapsto \bar{e}_{\beta^{\boxtimes p}}.$$

By Frobenius reciprocity, we obtain a map

$$\mu : \Delta(\beta)^{\circ p} \to \bar{R}_{p\beta}\bar{e}_{\beta^{\boxtimes p}},\ 1_{\beta,\dots,\beta} \otimes \bar{e}_\beta^{\otimes p} \mapsto \bar{e}_{\beta^{\boxtimes p}}.$$

We now show that $I_{>(\beta^p)}\Delta(\beta)^{\circ p} = 0$. It is enough to prove that $\mathrm{Res}_\pi \Delta(\beta)^{\circ p} = 0$ for all $\pi > (\beta^p)$. Since all composition factors of $\Delta(\beta)$ are isomorphic to $L(\beta)$, it follows that all composition factors of $\Delta(\beta)^{\circ p}$ are isomorphic to $L(\beta)^{\circ p} \cong L(\beta^p)$. By Theorem 2.1.6(vi), $\mathrm{Res}_\pi(L(\beta)^{\circ p}) = 0$, which proves the claim. Since $e_{\beta^{\boxtimes p}} 1_{\beta,\dots,\beta} \otimes \bar{e}_\beta^{\otimes p} = 1_{\beta,\dots,\beta} \otimes \bar{e}_\beta^{\otimes p}$, we obtain a map

$$\nu : \bar{R}_{p\beta}\bar{e}_{\beta^{\boxtimes p}} \to \Delta(\beta)^{\circ p},\ \bar{e}_{\beta^{\boxtimes p}} \mapsto 1_{\beta,\dots,\beta} \otimes \bar{e}_\beta^{\otimes p}.$$

The homomorphisms μ, ν map the evident cyclic generators to each other, and so are inverse isomorphisms. $\square$

Lemma 2.2.18

There exists an endomorphism of $\Delta(\beta) \circ \Delta(\beta)$ which sends $1_{\beta,\beta} \otimes (\bar{e}_\beta \otimes \bar{e}_\beta)$ to $\psi_{\beta,1} 1_{\beta,\beta} \otimes (\bar{e}_\beta \otimes \bar{e}_\beta)$.

Proof. Apply the Mackey theorem to $\mathrm{Res}_{\beta,\beta}(\Delta(\beta)\circ\Delta(\beta))$. We get a short exact sequence of $R_\beta \boxtimes R_\beta$-modules

$$0 \to \Delta(\beta)\boxtimes\Delta(\beta) \to \mathrm{Res}_{\beta,\beta}(\Delta(\beta)\circ\Delta(\beta)) \to (\Delta(\beta)\boxtimes\Delta(\beta))\langle -\beta\cdot\beta\rangle \to 0,$$

where $\psi_{\beta,1}1_{\beta,\beta}\otimes(\bar{e}_\beta\otimes\bar{e}_\beta) \in \mathrm{Res}_{\beta,\beta}(\Delta(\beta)\circ\Delta(\beta))$ is a preimage of the standard generator of $(\Delta(\beta)\boxtimes\Delta(\beta))\langle -\beta\cdot\beta\rangle$.

We now show that this is actually a sequence of $\bar{R}_\beta \boxtimes \bar{R}_\beta$-modules. It is sufficient to show that for any $\pi > (\beta)$, we have that

$$\mathrm{Res}_{\pi,\beta}\circ\mathrm{Res}_{\beta,\beta}(\Delta(\beta)\circ\Delta(\beta)) = 0 = \mathrm{Res}_{\beta,\pi}\circ\mathrm{Res}_{\beta,\beta}(\Delta(\beta)\circ\Delta(\beta)).$$

We show the first equality, the second being similar. All composition factors of $\Delta(\beta)$ are isomorphic to $L(\beta)$, so all composition factors of $\Delta(\beta)\circ\Delta(\beta)$ are isomorphic to $L(\beta)\circ L(\beta)$, and thus all composition factors of $\mathrm{Res}_{\beta,\beta}(\Delta(\beta)\circ\Delta(\beta))$ are isomorphic to $L(\beta)\boxtimes L(\beta)$. Theorem 2.1.6 now tells us that $\mathrm{Res}_\pi(L(\beta)) = 0$ for all $\pi > (\beta)$.

By the projectivity of $\Delta(\beta)$ as $\bar{R}_\beta$-module, the short exact sequence splits, giving the required endomorphism by Frobenius reciprocity. □

Corollary 2.2.19

$\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}} = \bar{e}_{\beta^{\boxtimes 2}}\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}$.

Proof. Let φ be the endomorphism of $\Delta(\beta)\circ\Delta(\beta)$ constructed in Lemma 2.2.18, regarded as an endomorphism of $\bar{R}_{2\beta}\bar{e}_{\beta^{\boxtimes 2}}$ by Lemma 2.2.17. Then

$$\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}} = \varphi(\bar{e}_{\beta^{\boxtimes 2}}) = \varphi(\bar{e}^2_{\beta^{\boxtimes 2}}) = \bar{e}_{\beta^{\boxtimes 2}}\varphi(\bar{e}_{\beta^{\boxtimes 2}}) = \bar{e}_{\beta^{\boxtimes 2}}\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}},$$

as required. $\square$

Corollary 2.2.20

We have $\bar{e}_{\beta^{\boxtimes p}}\bar{e}_{(\beta^p)}\bar{e}_{\beta^{\boxtimes p}} = \bar{e}_{(\beta^p)}$.

Proof. Follows from (2.35), Corollary 2.2.19 and Hypothesis 2.2.9. $\square$

Lemma 2.2.21

The set $\{\bar{e}_{\beta^{\boxtimes p}}\bar{y}_{\beta,1}^{a_1}\dots\bar{y}_{\beta,p}^{a_p}\bar{\psi}_{\beta,w}\bar{e}_{\beta^{\boxtimes p}} \mid w \in \mathfrak{S}_p, a_1,\dots,a_p \geq 0\}$ gives a linear basis of $\bar{e}_{\beta^{\boxtimes p}}\bar{R}_\alpha\bar{e}_{\beta^{\boxtimes p}}$.

Proof. The elements above are linearly independent by Lemmas 2.2.17 and 2.2.18, and Corollary 2.2.15. We use Frobenius reciprocity, Corollary 2.2.16, and (3, Lemma 2.11) to see that

$$\begin{aligned}
\dim_q \operatorname{End}_{R_\alpha}(\Delta(\beta)^{\circ p}) &= \dim_q \operatorname{Hom}_{R_{\beta,\dots,\beta}}(\Delta(\beta)^{\boxtimes p}, \operatorname{Res}_{\beta,\dots,\beta}\Delta(\beta)^{\circ p}) \\
&\leq [\operatorname{Res}_{\beta,\dots,\beta}\Delta(\beta)^{\circ p} : L(\beta)^{\boxtimes p}] \\
&= [\operatorname{Res}_{\beta,\dots,\beta}L(\beta)^{\circ p} : L(\beta)^{\boxtimes p}]/(1-q_\beta^2)^p \\
&= q_\beta^{-\frac{1}{2}p(p-1)}[p]^!_\beta/(1-q_\beta^2)^p.
\end{aligned}$$

By the formula for the Poincaré polynomial of $\mathfrak{S}_p$, we have shown that

$$\dim_q \bar{e}_{\beta^{\boxtimes p}}\bar{R}_\alpha\bar{e}_{\beta^{\boxtimes p}} \leq \frac{\sum_{w\in\mathfrak{S}_p} q_\beta^{-2l(w)}}{(1-q_\beta^2)^p},$$

showing that the proposed basis also spans. $\square$

The next two lemmas are proved using ideas that already appeared in the proofs of (3, Lemmas 3.7, 3.9).

<u>Lemma 2.2.22</u>

We have that

$$
\begin{aligned}
\bar{\psi}_{\beta,r}^2 \bar{e}_{\beta^{\boxtimes p}} &= 0, && \text{for } 1 \leq r \leq p-1,\\
\bar{\psi}_{\beta,r}\bar{\psi}_{\beta,s}\bar{e}_{\beta^{\boxtimes p}} &= \bar{\psi}_{\beta,s}\bar{\psi}_{\beta,r}\bar{e}_{\beta^{\boxtimes p}}, && \text{for } |r-s|>1, \text{ and}\\
\bar{\psi}_{\beta,r}\bar{\psi}_{\beta,r+1}\bar{\psi}_{\beta,r}\bar{e}_{\beta^{\boxtimes p}} &= \bar{\psi}_{\beta,r+1}\bar{\psi}_{\beta,r}\bar{\psi}_{\beta,r+1}\bar{e}_{\beta^{\boxtimes p}}, && \text{for } 1 \leq r \leq p-2.
\end{aligned}
$$

Proof. We use Lemma 2.2.17 to identify $\bar{R}_{p\beta}\bar{e}_{\beta^{\boxtimes p}}$ with $\Delta(\beta)^{\circ p}$. It is enough to prove the first relation in the case $p = 2$. The Mackey theorem analysis in the proof of Lemma 2.2.18 shows that, as a graded vector space

$$
(\Delta(\beta) \circ \Delta(\beta))_{\boldsymbol{i}_\beta^2} = e(\boldsymbol{i}_\beta^2) \otimes (\Delta(\beta) \boxtimes \Delta(\beta)) \oplus \psi_{\beta,1} e(\boldsymbol{i}_\beta^2) \otimes (\Delta(\beta) \boxtimes \Delta(\beta)). \tag{2.42}
$$

The vector $\bar{e}_\beta \in \Delta(\beta)_{\boldsymbol{i}_\beta}$ is of minimal degree, and thus $\psi_{\beta,1}e(\boldsymbol{i}_\beta^2) \otimes (\bar{e}_\beta \otimes \bar{e}_\beta)$ is of minimal degree in $(\Delta(\beta) \circ \Delta(\beta))_{\boldsymbol{i}_\beta^2}$. The degree of $\psi_{\beta,1}^2 e(\boldsymbol{i}_\beta^2) \otimes (\bar{e}_\beta \otimes \bar{e}_\beta)$ is smaller by $\beta \cdot \beta$, so the vector is zero.

The second relation is clear from the definitions. To prove the third relation, it is sufficient to consider $p = 3$. Let $w_r := w_{\beta,r}$, and set $w_0 := w_1 w_2 w_1$. Using the defining relations of $R_{3\beta}$, we deduce that $(\psi_{\beta,2}\psi_{\beta,1}\psi_{\beta,2} - \psi_{\beta,1}\psi_{\beta,2}\psi_{\beta,1})e(\boldsymbol{i}_\beta^3) \otimes (\bar{e}_\beta \otimes \bar{e}_\beta \otimes \bar{e}_\beta)$ is an element of degree $3\deg(v_\beta^-) - 6\beta \cdot \beta$ in $S := \sum_{w<w_0} \psi_w e(\boldsymbol{i}_\beta^3) \otimes (\Delta(\beta) \boxtimes \Delta(\beta) \boxtimes \Delta(\beta))$, where $<$ denotes the Bruhat order. By a Mackey theorem analysis as in the proof of Lemma 2.2.18, we see that

$$
S = \sum_{w \in \{1, w_1, w_2, w_1 w_2, w_2 w_1\}} \psi_w e(\boldsymbol{i}_\beta^3) \otimes (\Delta(\beta) \boxtimes \Delta(\beta) \boxtimes \Delta(\beta)).
$$

The lowest degree of an element in S is therefore $3\deg(v_\beta^-) - 4\beta\cdot\beta$, and the third relation is proved. $\square$

Lemma 2.2.23

There exists a unique choice of $\varepsilon_\beta = \pm 1$ such that

$$\begin{aligned}
\bar\psi_{\beta,r}\bar y_{\beta,s}\bar e_{\beta^{\boxtimes p}} &= \bar y_{\beta,s}\bar\psi_{\beta,r}\bar e_{\beta^{\boxtimes p}}, && \text{for } s \neq r, r+1,\\
\bar\psi_{\beta,r}\varepsilon_\beta\bar y_{\beta,r+1}\bar e_{\beta^{\boxtimes p}} &= (\varepsilon_\beta\bar y_{\beta,r}\bar\psi_{\beta,r}+1)\bar e_{\beta^{\boxtimes p}}, && \text{for } 1\leq r<p, \text{ and}\\
\varepsilon_\beta\bar y_{\beta,r+1}\bar\psi_{\beta,r}\bar e_{\beta^{\boxtimes p}} &= (\bar\psi_{\beta,r}\varepsilon_\beta\bar y_{\beta,r}+1)\bar e_{\beta^{\boxtimes p}}, && \text{for } 1\leq r<p.
\end{aligned}$$

Proof. The first relation is clear from the definitions. It is enough to prove the remaining relations for $p = 2$. Using the defining relations of $R_{2\beta}$ and a Mackey theorem analysis as in the proof of Lemma 2.2.18, we deduce that

$$\begin{aligned}
(\bar\psi_{\beta,1}\bar y_{\beta,2} - \bar y_{\beta,1}\bar\psi_{\beta,1})\bar e_{\beta^{\boxtimes 2}} &\in \sum_{w<w_{\beta,1}} \psi_w e(\boldsymbol{i}_\beta^2)\otimes(\Delta(\beta)\boxtimes\Delta(\beta))\\
&= e(\boldsymbol{i}_\beta^2)\otimes(\Delta(\beta)\boxtimes\Delta(\beta)),
\end{aligned}$$

and the only vector of the correct degree is $\bar e_{\beta^{\boxtimes 2}}$. Therefore (working over $\mathbb{Z}$) we must have that

$$(\bar\psi_{\beta,1}\bar y_{\beta,2} - \bar y_{\beta,1}\bar\psi_{\beta,1})\bar e_{\beta^{\boxtimes 2}} = c_+\bar e_{\beta^{\boxtimes 2}}$$

for some $c_+\in\mathbb{Z}$. Similarly, we obtain

$$(\bar\psi_{\beta,1}\bar y_{\beta,1} - \bar y_{\beta,2}\bar\psi_{\beta,1})\bar e_{\beta^{\boxtimes 2}} = c_-\bar e_{\beta^{\boxtimes 2}}$$

for some $c_- \in \mathbb{Z}$. We compute

$$\begin{aligned}
(\bar{\psi}_{\beta,1}\bar{y}_{\beta,1}\bar{y}_{\beta,2} - \bar{y}_{\beta,1}\bar{y}_{\beta,2}\bar{\psi}_{\beta,1})\bar{e}_{\beta^{\boxtimes 2}} &= (\bar{y}_{\beta,2}\bar{\psi}_{\beta,1} + c_-)\bar{y}_{\beta,2} - \bar{y}_{\beta,2}(\bar{\psi}_{\beta,1}\bar{y}_{\beta,2} - c_+) \\
&= (c_- + c_+)\bar{y}_{\beta,2}\bar{e}_{\beta^{\boxtimes 2}} \\
(\bar{\psi}_{\beta,1}\bar{y}_{\beta,1}\bar{y}_{\beta,2} - \bar{y}_{\beta,1}\bar{y}_{\beta,2}\bar{\psi}_{\beta,1})\bar{e}_{\beta^{\boxtimes 2}} &= (\bar{y}_{\beta,1}\bar{\psi}_{\beta,1} + c_+)\bar{y}_{\beta,1} - \bar{y}_{\beta,1}(\bar{\psi}_{\beta,1}\bar{y}_{\beta,1} - c_-) \\
&= (c_+ + c_-)\bar{y}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}
\end{aligned}$$

and since $\bar{y}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}$ and $\bar{y}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}$ are linearly independent by Lemma 2.2.21, we must have $c_- = -c_+$. We now fix a prime p and extend scalars to $\mathbb{F}_p$. Suppose that $\varepsilon_\beta = 0 \in \mathbb{F}_p$, so that

$$\begin{aligned}
\bar{\psi}_{\beta,1}\bar{y}_{\beta,2}\bar{e}_{\beta^{\boxtimes 2}} &= \bar{y}_{\beta,1}\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}} \\
\bar{\psi}_{\beta,1}\bar{y}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}} &= \bar{y}_{\beta,2}\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}.
\end{aligned}$$

Define S to be the submodule of $\Delta(\beta) \circ \Delta(\beta)$ generated by $\bar{y}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}$ and $\bar{y}_{\beta,2}\bar{e}_{\beta^{\boxtimes 2}}$. The above equations show that the endomorphism defined by right multiplication by $\bar{\psi}_{\beta,1}\bar{e}_{\beta^{\boxtimes 2}}$ leaves S invariant. On the other hand, $\Delta(\beta) \circ \Delta(\beta)/S \cong L(\beta) \circ L(\beta)$ is irreducible. Since the endomorphism algebra of an irreducible module is one dimensional, we have a contradiction. Therefore $\varepsilon_\beta \neq 0$ when reduced modulo any prime, i.e. $\varepsilon_\beta = \pm 1$. $\square$

Corollary 2.2.24

The homomorphism from the nilHecke algebra H_p determined by

$$\zeta : H_p \to \bar{e}_{\beta^{\boxtimes p}}\bar{R}_\alpha\bar{e}_{\beta^{\boxtimes p}}, y_r \mapsto \varepsilon_\beta\bar{y}_{\beta,r}\bar{e}_{\beta^{\boxtimes p}}, \psi_r \mapsto \bar{\psi}_{\beta,r}\bar{e}_{\beta^{\boxtimes p}}$$

is an isomorphism. Under this isomorphism the idempotent $e_p \in H_p$ is mapped onto $\bar{e}_\sigma$.

Proof. Using Lemmas 2.2.22 and 2.2.23, we see that the map exists. By Lemma 2.2.21, the map is an isomorphism. The second statement now follows using Corollary 2.2.20. $\square$

Corollary 2.2.25

Given $f \in \Lambda_\sigma$, $\bar{f}$ commutes with $\bar{\delta}_\sigma$, $\bar{e}_\sigma$, and $\bar{e}_\sigma \bar{D}_\sigma$.

Proof. It follows directly from Hypothesis 2.2.9(iv) and the definitions that δ_σ commutes with every element of P_σ, and in particular with every element of the subalgebra Λ_σ. Denote by w_0 the longest element of $\mathfrak{S}_p$. Then by Corollaries 2.2.19 and 2.2.24

$$\begin{aligned}
\bar{e}_\sigma \bar{D}_\sigma &= \bar{\psi}_{\beta,w_0} \bar{y}_{\beta,2} \dots \bar{y}_{\beta,p}^{p-1} \bar{e}_{\beta^{\boxtimes p}} \bar{\psi}_{\beta,w_0} \iota(D_\beta \otimes \dots \otimes D_\beta) \\
&= (\bar{e}_{\beta^{\boxtimes p}} \psi_{\beta,w_0} \bar{e}_{\beta^{\boxtimes p}})(\bar{y}_{\beta,2} \dots \bar{y}_{\beta,p}^{p-1})(\bar{e}_{\beta^{\boxtimes p}} \psi_{\beta,w_0} \bar{e}_{\beta^{\boxtimes p}}) \iota(D_\beta \otimes \dots \otimes D_\beta) \\
&= \zeta(\psi_{w_0})(\bar{y}_{\beta,2} \dots \bar{y}_{\beta,p}^{p-1}) \zeta(\psi_{w_0}) \iota(D_\beta \otimes \dots \otimes D_\beta)
\end{aligned}$$

Any $f \in \Lambda_\sigma$ commutes with $\iota(D_\beta \otimes \dots \otimes D_\beta)$ by Hypothesis 2.2.9(iv). It is well known that the center of the nilHecke algebra H_p is given by the symmetric functions Λ_p. In particular, every element of Λ_p commutes with ψ_{w_0}. Let $g \in \Lambda_p$ be such that $\zeta(g) = \bar{f} \bar{e}_{\beta^{\boxtimes p}}$. Then $\zeta(\psi_{w_0}) \bar{f} = \zeta(\psi_{w_0} g) = \zeta(g \psi_{w_0}) = f \zeta(\psi_{w_0})$. This implies the claim. $\square$

We can now finish the proof of Proposition 2.2.13. Corollary 2.2.24 provides an isomorphism $H_p \cong \mathrm{End}_{R_\alpha}(\bar{R}_\alpha \bar{e}_{\beta^{\boxtimes p}})$ under which the idempotent e_p corresponds to right multiplication by $\bar{e}_\sigma$. But e_p is a primitive idempotent, so the image

$\bar{R}_\alpha \bar{e}_\sigma = \Delta(\sigma)$ of this endomorphism is an indecomposable projective $\bar{R}_\alpha$-module. We may identify

$$\mathrm{End}_{R_\alpha}(\Delta(\sigma)) \cong \bar{e}_\sigma \bar{R}_\alpha \bar{e}_\sigma = \zeta(e_p H_p e_p) = \zeta(\Lambda_p e_p) = \bar{e}_\sigma \bar{\Lambda}_\sigma \bar{e}_\sigma \cong \Lambda_\sigma, \tag{2.43}$$

where the action of Λ_σ on $\Delta(\sigma) = \bar{R}_\alpha \bar{e}_\sigma$ is given by right multiplication which makes sense in view of Corollary 2.2.25. Therefore $\Delta(\sigma) \twoheadrightarrow L(\sigma), \bar{e}_\sigma \mapsto v_\sigma^-$ is a projective cover in $\bar{R}_\alpha$-mod. Furthermore, since $\bar{R}_\alpha$-mod has only one irreducible module, every composition factor of $\Delta(\sigma)$ is isomorphic to $L(\sigma)$ with an appropriate degree shift. We can lift the basis $\{bv_\sigma^- \mid b \in \mathfrak{B}_\sigma\}$ for $L(\sigma)$ to the set $\{\bar{b}\bar{e}_\sigma \mid b \in \mathfrak{B}_\sigma\} \subseteq \Delta(\sigma)$. Using the basis X_σ for Λ_σ, we get a basis $\{\bar{b}\bar{f}\bar{e}_\sigma \mid b \in \mathfrak{B}_\sigma, f \in X_\sigma\}$ for $\Delta(\sigma)$.

Similarly, $\Delta'(\sigma) \twoheadrightarrow L(\sigma)^\tau, \bar{e}_\sigma \mapsto v_\sigma^+$ is a projective cover in $\bar{R}_\alpha^{op}$-mod. It is immediate that $\{v_\sigma^+ D_\sigma b^\tau \mid b \in \mathfrak{B}_\sigma\}$ is a basis of $L(\sigma)^\tau$. Lifting as above, we have that $\{\bar{e}_\sigma \bar{f} \bar{D}_\sigma \bar{b}^\tau \mid b \in \mathfrak{B}_\sigma, f \in X_\sigma\}$ is a basis for $\Delta'(\sigma)$.

Finally, applying the multiplication map and Corollary 2.2.25 we have that $\{\bar{b}\bar{e}_\sigma \bar{f} \bar{D}_\sigma \bar{(b')}^\tau \mid b, b' \in \mathfrak{B}_\sigma, f \in X_\sigma\}$ spans $\bar{R}_\alpha$. Therefore by induction,

$$R_\alpha = F\text{-span}\{\bar{b}\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{(b')}^\tau \mid \pi \in \Pi(\alpha), b, b' \in \mathfrak{B}_\pi, f \in X_\pi\},$$

and comparing graded dimensions with Corollary 2.1.16 as in the proof of Lemma 2.2.14, this set is therefore a basis.

General case

In this section we use the results of the previous subsections to obtain affine cellular bases of the KLR algebras of finite type. Fix $\alpha \in Q_+$ and $\pi =$

$(\beta_1^{p_1}, \dots, \beta_N^{p_N}) \in \Pi(\alpha)$. Define $\bar{R}_\alpha := R_\alpha / I_{>\pi}$, and write $\bar{r} \in \bar{R}_\alpha$ for the image of an element $r \in R_\alpha$.

We begin with some easy consequences of the previous section.

Corollary 2.2.26

We have

(i) Given $f \in \Lambda_\pi$, $\bar{f}$ commutes with $\bar{\delta}_\pi$, $\bar{e}_\pi$, and $\bar{e}_\pi \bar{D}_\pi$.

(ii) Up to a grading shift, $\Delta(\pi) \cong \Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_N^{p_N})$.

(iii) The map $\Lambda_\pi \to \mathrm{End}_{\bar{R}_\alpha}(\Delta(\pi))$ sending f to right multiplication by $\bar{e}_\pi \bar{f} \bar{e}_\pi$ is an isomorphism of algebras.

(iv) The map $\Lambda_\pi \to \mathrm{End}_{\bar{R}_\alpha}(\Delta'(\pi))$ sending f to left multiplication by $\bar{e}_\pi \bar{f} \bar{e}_\pi$ is an isomorphism of algebras.

Proof. Claim (i) follows directly from Corollary 2.2.25 and the definitions.

The proof of claim (ii) is similar to that of Lemma 2.2.17. To be precise, by Corollary 2.2.6 we have a map

$$\Delta(\beta_1^{p_1}) \boxtimes \cdots \boxtimes \Delta(\beta_N^{p_N}) \to \mathrm{Res}_\pi \Delta(\pi),\ \bar{e}_{(\beta_1^{p_1})} \otimes \cdots \otimes \bar{e}_{(\beta_N^{p_N})} \mapsto \bar{e}_\pi,$$

which by Frobenius reciprocity determines a homomorphism

$$\mu : \Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_N^{p_N}) \to \Delta(\pi),\ 1_\pi \otimes (\bar{e}_{(\beta_1^{p_1})} \otimes \cdots \otimes \bar{e}_{(\beta_N^{p_N})}) \mapsto \bar{e}_\pi.$$

We now claim that $I_{>\pi}(\Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_N^{p_N})) = 0$. It is enough to prove that $\mathrm{Res}_\sigma(\Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_N^{p_N})) = 0$ for all $\sigma > \pi$. By exactness of induction, it follows that $\Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_N^{p_N})$ has an exhaustive filtration by $L(\beta_1^{p_1}) \circ \cdots \circ L(\beta_N^{p_N}) = \bar{\Delta}(\pi)$. By Theorem 2.1.6(vi), $\mathrm{Res}_\sigma(\bar{\Delta}(\pi)) = 0$, which proves the claim.

Since

$$e_\pi 1_\pi \otimes (\bar{e}_{(\beta_1^{p_1})} \otimes \cdots \otimes \bar{e}_{(\beta_N^{p_N})}) = 1_\pi \otimes (\bar{e}_{(\beta_1^{p_1})} \otimes \cdots \otimes \bar{e}_{(\beta_N^{p_N})}),$$

we obtain a map

$$\nu : \Delta(\pi) \to \Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_N^{p_N}),\ \bar{e}_\pi \mapsto 1_\pi \otimes (\bar{e}_{(\beta_1^{p_1})} \otimes \cdots \otimes \bar{e}_{(\beta_N^{p_N})}).$$

The homomorphisms μ, ν map the evident cyclic generators to each other, and so are inverse isomorphisms.

We use claim (ii) to identify $\Delta(\pi)$ with $\Delta(\beta_1^{p_1}) \circ \ldots \Delta(\beta_N^{p_N})$. As noted in the proof of claim (ii), $\Delta(\pi)$ has an exhaustive filtration by

$$\bar{\Delta}(\pi) = \oplus_{w \in \mathfrak{S}^\pi} \psi_w 1_\pi \otimes (\bar{\Delta}(\beta_1^{p_1}) \boxtimes \cdots \boxtimes \bar{\Delta}(\beta_N^{p_N})).$$

By Theorem 2.1.6(vi), $\mathrm{Res}_\pi \bar{\Delta}(\pi)$ picks out the summand corresponding to $w = 1$. Therefore $\mathrm{Res}_\pi \Delta(\pi) \cong \Delta(\beta_1^{p_1}) \boxtimes \cdots \boxtimes \Delta(\beta_N^{p_N})$. Applying Frobenius reciprocity and (2.43), we obtain

$$\begin{aligned}
\mathrm{End}_{R_\alpha}(\Delta(\pi)) &= \mathrm{Hom}_{R_\alpha}(\Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_n^{p_N}), \Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_n^{p_N})) \\
&\simeq \mathrm{Hom}_{R_\pi}(\Delta(\beta_1^{p_1}) \boxtimes \cdots \boxtimes \Delta(\beta_n^{p_N}), \mathrm{Res}_\pi(\Delta(\beta_1^{p_1}) \circ \cdots \circ \Delta(\beta_n^{p_N}))) \\
&\simeq \mathrm{End}_{R_\pi}(\Delta(\beta_1^{p_1}) \boxtimes \cdots \boxtimes \Delta(\beta_N^{p_N})) \\
&\simeq \mathrm{End}_{R_{p_1\beta_1}}(\Delta(\beta_1^{p_1})) \otimes \cdots \otimes \mathrm{End}_{R_{p_1\beta_N}}(\Delta(\beta_1^{p_N})) \\
&\simeq \Lambda_{(\beta_1^{p_1})} \otimes \cdots \otimes \Lambda_{(\beta_N^{p_N})} \simeq \Lambda_\pi.
\end{aligned}$$

This proves claim (iii), and claim (iv) is shown similarly. □

Proposition 2.2.27

We have that

(i) $\{\bar{b}\bar{f}\bar{e}_\pi \mid b \in \mathfrak{B}_\pi, f \in X_\pi\}$ is an $\mathcal{O}$-basis for $\Delta(\pi)$,

(ii) $\{\bar{e}_\pi \bar{f}\bar{D}_\pi \bar{b}^\tau \mid b \in \mathfrak{B}_\pi, f \in X_\pi\}$ is an $\mathcal{O}$-basis for $\Delta'(\pi)$, and

(iii) $\{\bar{b}\bar{e}_\pi \bar{f}\bar{D}_\pi (\bar{b}')^\tau \mid b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}$ is an $\mathcal{O}$-basis for $\bar{I}_\pi$.

Proof. For $n = 1, \ldots, N$, define

$$B_n := \{\bar{b}\bar{f}\bar{e}_{(\beta_n^{p_n})} \mid b \in \mathfrak{B}_{(\beta_n^{p_n})}, f \in X_{(\beta_n^{p_n})}\}.$$

By Proposition 2.2.13, B_n is a basis of $\Delta(\beta_n^{p_n})$ for each $n = 1, \ldots, N$. Let $\bar{\iota}_\pi$: $\bar{R}_{p_1\beta_1} \otimes \cdots \otimes \bar{R}_{p_N\beta_N} \to \bar{R}_\alpha$ be the map induced by ι_π, as in Corollary 2.2.6. Using (14, Proposition 2.16), and computing as in the proof of Lemma 2.2.14, we have

$$\begin{aligned}\Delta(\pi) &= \sum_{w \in \mathfrak{S}^\pi} \bar{\psi}_w \bar{R}_\pi \bar{e}_\pi = \sum_{w \in \mathfrak{S}^\pi} \bar{\psi}_w \bar{\iota}_\pi(\Delta(\beta_1^{p_1}) \otimes \cdots \otimes \Delta(\beta_N^{p_N})) \\ &= \mathcal{O}\text{-span}\{\bar{\psi}_w \bar{\iota}_\pi(b_1 \otimes \cdots \otimes b_N) \mid w \in \mathfrak{S}^\pi, b_n \in B_n\} \\ &= \mathcal{O}\text{-span}\{\bar{b}\bar{f}\bar{e}_\pi \mid b \in \mathfrak{B}_\pi, f \in X_\pi\}.\end{aligned}$$

We have shown that the set in (i) spans $\Delta(\pi)$. A similar argument shows that the set in (ii) spans $\Delta'(\pi)$. Now, applying the multiplication map $\Delta(\pi) \otimes \Delta'(\pi) \twoheadrightarrow \bar{I}_\pi$ and using Corollary 2.2.26(i) yields the spanning set of (iii). Letting π vary over $\Pi(\alpha)$, we have

$$R_\alpha = \sum_{\pi \in \Pi(\alpha)} \mathcal{O}\text{-span}\{b e_\pi f D_\pi (b')^\tau \mid b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}.$$

Using (2.39) and the equality $\deg(D_\pi) = 2\deg(v_\pi^-)$ for all $\pi \in \Pi(\alpha)$, we get

$$\begin{aligned}
\dim_q(R_\alpha) &= \sum_{\pi\in\Pi(\alpha)} \dim_q(\mathcal{O}\text{-span}\{be_\pi f D_\pi(b')^\tau \mid b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}) \\
&\leq \sum_{\pi\in\Pi(\alpha)} \Big(\sum_{b\in\mathfrak{B}_\pi} q^{\deg(b)}\Big) \dim_q(\Lambda_\pi) q^{\deg(D_\pi)} \Big(\sum_{b\in\mathfrak{B}_\pi} q^{\deg(b)}\Big) \\
&\leq \sum_{\pi\in\Pi(\alpha)} \Big(\sum_{b\in\mathfrak{B}_\pi} q^{\deg(bv_\pi^-)}\Big)^2 l_\pi \\
&= \sum_{\pi\in\Pi(\alpha)} \dim_q(\bar{\Delta}(\pi))^2 l_\pi = \dim_q(R_\alpha),
\end{aligned}$$

by Corollary 2.1.16. The inequalities are therefore equalities, and this implies that the spanning set $\{be_\pi f D_\pi(b')^\tau \mid \pi \in \Pi(\alpha), b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}$ of R_α is a basis and $\dim_q \Lambda_\pi = l_\pi$ for all π.

To show (i) and (ii), we have already noted that the claimed bases span $\Delta(\beta)$ and $\Delta'(\beta)$, respectively. We now apply part (iii) to see that they are linearly independent. □

Corollary 2.2.28

The set $\{be_\pi f D_\pi(b')^\tau \mid \pi \in \Pi(\alpha),\ b, b' \in \mathfrak{B}_\pi, f \in X_\pi\}$ is an $\mathcal{O}$-basis for R_α.

Proof. Apply Proposition 2.2.27(iii) and the fact that the filtration by the ideals I_π exhausts R_α, which follows from Lemma 2.2.1. □

Affine Cellularity

Recall the notion of an affine cellular algebra from the introduction. In this section, we fix $\alpha \in Q_+$ and prove that R_α is affine cellular over $\mathbb{Z}$ (which then implies that it is affine cellular over any k).

For any $\pi \in \Pi(\alpha)$, we define

$$I'_\pi := \mathbb{Z}\text{-span}\{be_\pi \Lambda_\pi D_\pi (b')^\tau \mid b, b' \in \mathfrak{B}_\pi\}.$$

By Corollary 2.2.28, we have $R_\alpha = \oplus_{\pi\in\Pi(\alpha)} I'_\pi$. Moreover, $\tau(I'_\pi) = I'_\pi$. Indeed, δ_π commutes with elements of Λ_π in view of Hypothesis 2.2.9(iv). So by Lemma 2.2.12, we have

$$\begin{aligned}\tau(I'_\pi) &= \mathbb{Z}\text{-span}\{b' D_\pi^\tau \Lambda_\pi^\tau \delta_\pi^\tau D_\pi^\tau b^\tau \mid b, b' \in \mathfrak{B}_\pi\}\\ &= \mathbb{Z}\text{-span}\{b' D_\pi \delta_\pi \Lambda_\pi D_\pi b^\tau \mid b, b' \in \mathfrak{B}_\pi\} = I'_\pi.\end{aligned}$$

By Proposition 2.2.27, we have $I_\pi = \oplus_{\sigma\geq\pi} I'_\sigma$, and we have a nested family of ideals $(I_\pi)_{\pi\in\Pi(\alpha)}$. To check that R_α is affine cellular, we need to verify that $\bar{I}_\pi := I_\pi/I_{>\pi}$ is an affine cell ideal in $\bar{R}_\alpha := R_\alpha/I_{>\pi}$. As usual we denote $\bar{x} := x + I_{>\pi} \in \bar{R}_\alpha$ for $x \in R_\alpha$.

The affine algebra B in the definition of a cell ideal will be the algebra Λ_π, with the automorphism σ being the identity map. The $\mathbb{Z}$-module V will be the formal free $\mathbb{Z}$-module V_π on the basis $\mathfrak{B}_\pi$. By Corollary 2.2.26(i) and Proposition 2.2.27, the following maps are isomomorphisms of Λ_π-modules.

$$\begin{aligned}&\eta_\pi : V_\pi \otimes_{\mathbb{Z}} \Lambda_\pi \to \Delta(\pi), b \otimes f \mapsto \bar{b}\bar{f}\bar{e}_\pi,\\ &\eta'_\pi : \Lambda_\pi \otimes_{\mathbb{Z}} V_\pi \to \Delta'(\pi), f \otimes b \mapsto \bar{e}_\pi \bar{f} \bar{D}_\pi \bar{b}^\tau.\end{aligned}$$

This allows us to endow $V_\pi \otimes_{\mathbb{Z}} \Lambda_\pi$ with a structure of an (R_α, Λ_π)-bimodule and $\Lambda_\pi \otimes_{\mathbb{Z}} V_\pi$ with a structure of an (Λ_π, R_α)-bimodule.

In view of Corollary 2.2.26(iii),(iv) we see that $\Delta(\pi)$ (resp. $\Delta'(\pi)$) is a right (resp. left) Λ_π-module, and so we may define an R_α-bimodule homomorphism

$$\nu_\pi : \Delta(\pi) \otimes_{\Lambda_\pi} \Delta'(\pi) \to I_\pi / I_{>\pi}, \bar{r}\bar{e}_\pi \otimes \bar{e}_\pi \bar{r}' \mapsto \bar{r}\bar{e}_\pi \bar{r}'.$$

By Proposition 2.2.27, ν_π is an isomorphism. Let $\mu_\pi := \nu_\pi^{-1}$. This will be the map μ in the definition of a cell ideal.

Theorem 2.3.1

The above data make R_α into an affine cellular algebra.

Proof. To verify that $\bar{I}_\pi$ is a cell ideal in $\bar{R}_\alpha$, we first check that our (Λ_π, R_α)-bimodule structure on $\Lambda_\pi \otimes_{\mathbb{Z}} V_\pi$ comes from our (R_α, Λ_π)-bimodule structure on $V_\pi \otimes_{\mathbb{Z}} \Lambda_\pi$ via the rule (1.1). Let $\mathrm{s}_\pi : V_\pi \otimes_{\mathbb{Z}} \Lambda_\pi \xrightarrow{\sim} \Lambda_\pi \otimes_{\mathbb{Z}} V_\pi$ be the swap map. This is equivalent to the fact that the composition map

$$\varphi : \Delta'(\pi) \xrightarrow{(\eta'_\pi)^{-1}} \Lambda_\pi \otimes_{\mathbb{Z}} V_\pi \xrightarrow{\mathrm{s}_\pi^{-1}} V_\pi \otimes_{\mathbb{Z}} \Lambda_\pi \xrightarrow{\eta_\pi} \Delta(\pi) = \Delta(\pi)^\tau, \tag{2.44}$$

is an isomorphism of right R_α-modules. We already know that this is an isomorphism of $\mathbb{Z}$-modules, and so it suffices to check that

$$\varphi(\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau \bar{r}) = \bar{r}^\tau \varphi(\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau)$$

for all $f \in \Lambda_\pi$, $c \in \mathfrak{B}_\pi$, and $r \in R_\alpha$. Note that $\varphi(\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau) = \bar{c}\bar{f}\bar{e}_\pi$. So we have to check

$$\varphi(\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau \bar{r}) = \bar{r}^\tau \bar{c} \bar{f} \bar{e}_\pi. \tag{2.45}$$

By Proposition 2.2.27(ii) we can find $\{f_b \mid b \in \mathfrak{B}_\pi\} \subseteq \Lambda_\pi$ such that

$$\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau \bar{r} = \sum_{b \in \mathfrak{B}_\pi} \bar{e}_\pi \bar{f}_b \bar{D}_\pi \bar{b}^\tau. \tag{2.46}$$

Also, by Corollary 2.2.26(i), we have

$$\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau = \bar{f} \bar{e}_\pi \bar{D}_\pi \bar{c}^\tau = \bar{e}_\pi \bar{D}_\pi \bar{f} \bar{c}^\tau$$

Using this and the τ-invariance of D_π and δ_π, we get (2.45) as follows:

$$\begin{aligned}
\bar{r}^\tau \bar{c} \bar{f} \bar{e}_\pi = \bar{r}^\tau \bar{c} \bar{f} \bar{e}_\pi^2 = \bar{r}^\tau \bar{c} \bar{f} \bar{D}_\pi^\tau \bar{\delta}_\pi^\tau \bar{D}_\pi^\tau \bar{\delta}_\pi^\tau &= (\bar{\delta}_\pi \bar{D}_\pi \bar{\delta}_\pi \bar{D}_\pi \bar{f} \bar{c}^\tau \bar{r})^\tau \\
&= (\bar{\delta}_\pi \bar{e}_\pi \bar{D}_\pi \bar{f} \bar{c}^\tau \bar{r})^\tau = (\bar{\delta}_\pi \bar{e}_\pi \bar{f} \bar{D}_\pi \bar{c}^\tau \bar{r})^\tau = (\bar{\delta}_\pi \sum_{b \in \mathfrak{B}_\pi} \bar{e}_\pi \bar{f}_b \bar{D}_\pi \bar{b}^\tau)^\tau \\
&= (\sum_{b \in \mathfrak{B}_\pi} \bar{\delta}_\pi \bar{D}_\pi \bar{\delta}_\pi \bar{D}_\pi \bar{f}_b \bar{b}^\tau)^\tau = \sum_{b \in \mathfrak{B}_\pi} \bar{b} \bar{f}_b \bar{D}_\pi \bar{\delta}_\pi \bar{D}_\pi \bar{\delta}_\pi = \sum_{b \in \mathfrak{B}_\pi} \bar{b} \bar{f}_b \bar{e}_\pi^2 \\
&= \sum_{b \in \mathfrak{B}_\pi} \bar{b} \bar{f}_b \bar{e}_\pi,
\end{aligned}$$

which equals the left hand side of (2.45) by definition of φ.

To complete the proof, it remains to verify the commutativity of (1.2). This is equivalent to

$$\tau \circ \nu_\pi \circ (\eta_\pi \otimes \eta'_\pi)((b \otimes f) \otimes (f' \otimes b')) = \nu_\pi \circ (\eta_\pi \otimes \eta'_\pi)((b' \otimes f') \otimes (f \otimes b))$$

for all $b, b' \in \mathfrak{B}_\pi$ and $f, f' \in \Lambda_\pi$. The left hand side equals

$$\tau \circ \nu_\pi(\bar{b}\bar{f}\bar{e}_\pi \otimes \bar{e}_\pi \bar{f}' \bar{D}_\pi(\bar{b}')^\tau) = \tau(\bar{b}\bar{f}\bar{e}_\pi \bar{f}' \bar{D}_\pi(\bar{b}')^\tau) = \tau(\bar{b}\bar{e}_\pi \bar{f}\bar{f}' \bar{D}_\pi(\bar{b}')^\tau)$$
$$= \bar{b}'\bar{D}_\pi \bar{f}' \bar{f} \bar{e}_\pi^\tau \bar{b}^\tau = \bar{b}'\bar{D}_\pi \bar{f}' \bar{f} \bar{\delta}_\pi \bar{D}_\pi \bar{b}^\tau = \bar{b}'\bar{D}_\pi \delta_\pi \bar{f}' \bar{f} \bar{D}_\pi \bar{b}^\tau = \bar{b}' \bar{e}_\pi \bar{f}' \bar{f} \bar{D}_\pi \bar{b}^\tau$$
$$= \bar{b}'\bar{f}'\bar{e}_\pi \bar{f} \bar{D}_\pi \bar{b}^\tau = \nu_\pi(\bar{b}'\bar{f}'\bar{e}_\pi \otimes \bar{e}_\pi \bar{f} \bar{D}_\pi \bar{b}^\tau),$$

which equals $\nu_\pi \circ (\eta_\pi \otimes \eta'_\pi)((b' \otimes f') \otimes (f \otimes b))$, as required. □

Verification of the Hypothesis

In this section we verify Hypothesis 2.2.9 for all finite types. In ADE types (with one exception) this can be do using the theory of homogeneous representations developed in (22). This theory is reviewed in the next subsection. We use the cuspidal modules of (9).

Throughout the section β is a positive root, and $\bar{R}_\beta := R_\beta / I_{>(\beta)}$, $\bar{r} := r + I_{>(\beta)}$ for $r \in R_\beta$.

Homogeneous representations

In this section we assume that the Cartan matrix A is symmetric. In this subsection we fix $\alpha \in Q_+$ with $d = \mathrm{ht}(\alpha)$. A graded R_α-module is called *homogeneous* if it is concentrated in one degree. Let $\boldsymbol{i} \in \langle I \rangle_\alpha$. We call $s_r \in S_d$ an *admissible transposition* for $\boldsymbol{i}$ if $a_{i_r, i_{r+1}} = 0$. The *word graph* G_α is the graph with the set of vertices $\langle I \rangle_\alpha$, and with $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle_\alpha$ connected by an edge if and only if $\boldsymbol{j} = s_r \boldsymbol{i}$ for some admissible transposition s_r for $\boldsymbol{i}$. A connected component C of G_α is called *homogeneous* if for some $\boldsymbol{i} = (i_1, \dots, i_d) \in C$ the following condition holds:

$$\begin{aligned} &\text{if } i_r = i_s \text{ for some } r < s \text{ then there exist } t, u \\ &\text{such that } r < t < u < s \text{ and } a_{i_r,i_t} = a_{i_r,i_u} = -1. \end{aligned} \tag{2.47}$$

Theorem 2.4.1

(22, Theorems 3.6, 3.10, (3.3)) Let C be a homogeneous connected component of G_α. Let $L(C)$ be the vector space concentrated in degree 0 with basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \in C\}$ labeled by the elements of C. The formulas

$$\begin{aligned} &1_{\boldsymbol{j}} v_{\boldsymbol{i}} = \delta_{\boldsymbol{i},\boldsymbol{j}} v_{\boldsymbol{i}} \qquad (\boldsymbol{j} \in \langle I \rangle_\alpha,\ \boldsymbol{i} \in C), \\ &y_r v_{\boldsymbol{i}} = 0 \qquad (1 \le r \le d,\ \boldsymbol{i} \in C), \\ &\psi_r v_{\boldsymbol{i}} = \begin{cases} v_{s_r \boldsymbol{i}} & \text{if } s_r \boldsymbol{i} \in C, \\ 0 & \text{otherwise;} \end{cases} \qquad (1 \le r < d,\ \boldsymbol{i} \in C) \end{aligned}$$

define an action of R_α on $L(C)$, under which $L(C)$ is a homogeneous irreducible R_α-module. Furthermore, $L(C) \not\cong L(C')$ if $C \neq C'$, and every homogeneous irreducible R_α-module, up to a degree shift, is isomorphic to one of the modules $L(C)$.

We need to push the theory of homogeneous modules a little further. In Proposition 2.4.3 below we give a presentation for a homogeneous module as a cyclic modules generated by a word vector. Let C be a homogeneous component of G_α and $\boldsymbol{i} \in C$. An element $w \in \mathfrak{S}_d$ is called $\boldsymbol{i}$-*admissible* if it can be written as $w = s_{r_1} \dots s_{r_b}$, where s_{r_a} is an admissible transposition for $s_{r_{a+1}} \dots s_{r_b} \boldsymbol{i}$ for all $a = 1, \dots, b$. We denote the set of all $\boldsymbol{i}$-admissible elements by $\mathfrak{D}_{\boldsymbol{i}}$.

Lemma 2.4.2

Let C be a homogeneous component of G_α and $\boldsymbol{i} \in C$. Then $\{\psi_w v_{\boldsymbol{i}} \mid w \in \mathfrak{D}_{\boldsymbol{i}}\}$ is a basis of $L(C)$.

Proof. Note that if w, w' are admissible elements, then $w = w'$ if and only if $w\boldsymbol{i} = w'\boldsymbol{i}$. Indeed, it suffices to prove that $w\boldsymbol{i} = \boldsymbol{i}$ implies $w = 1$, which follows from the property (2.47). The lemma follows. □

Proposition 2.4.3

Let C be a homogeneous component of G_α and $\boldsymbol{i} \in C$. Let $J(\boldsymbol{i})$ be the left ideal of R_α generated by

$$\{y_r, 1_{\boldsymbol{j}}, \psi_w 1_{\boldsymbol{i}} \mid 1 \le r \le d,\ \boldsymbol{j} \in \langle I \rangle_\alpha \setminus \boldsymbol{i},\ w \in \mathfrak{S}_d \setminus \mathfrak{D}_{\boldsymbol{i}}\}. \tag{2.48}$$

Then $R_\alpha/J_\alpha \simeq L(C)$ as (graded) left R_α-modules.

Proof. Note that the elements in (2.48) annihilate the vector $v_{\boldsymbol{i}} \in L(C)$, which generates $L(C)$, whence we have a (homogeneous) surjection

$$R_\alpha/J_\alpha \twoheadrightarrow L(C),\ h + J_\alpha \mapsto h v_{\boldsymbol{i}}.$$

To prove that this surjection is an isomorphism it suffices to prove that the dimension of R_α/J_α is at most $\dim L(C) = |C|$, which follows easily from Lemma 2.4.2. □

Special Lyndon orders

Recall the theory of standard modules reviewed in §2.1. We now specialize to the case of a *Lyndon* convex order on Φ_+ as studied in (21). For this we first need to fix a total order '$\leq$' on I. This gives rise to a lexicographic order '$\leq$' on the set $\langle I \rangle$. In particular, each finite dimensional R_α-module has its (lexicographically) highest word, and the highest word of an irreducible module determines the irreducible module uniquely up to an isomorphism. This leads to the natural notion of *dominant words* (called good words in (21)), namely the elements of $\langle I \rangle_\alpha$ which occur as highest words of finite dimensional R_α-modules.

The dominant words of cuspidal modules are characterized among all dominant words by the property that they are *Lyndon words*, so we refer to them as *dominant Lyndon words*. There is an explicit bijection

$$\Phi_+ \to \{\text{dominant Lyndon words}\}, \ \beta \mapsto \boldsymbol{i}_\beta,$$

uniquely determined by the property $|\boldsymbol{i}_\beta| = \beta$. Note that this notation $\boldsymbol{i}_\beta$ will be consistent with the same notation used in §2.2.

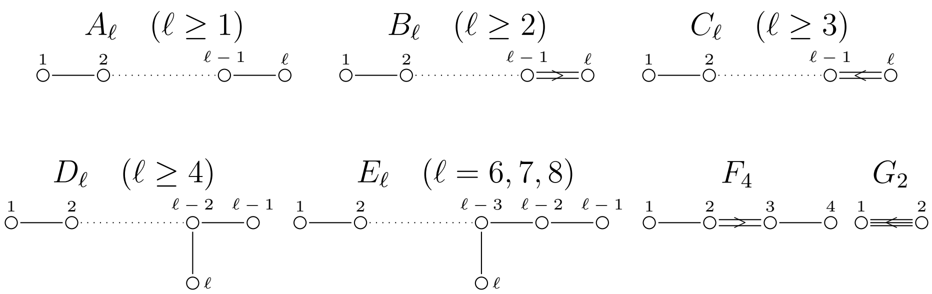
Setting $\beta \leq \gamma$ if and only if $\boldsymbol{i}_\beta \leq \boldsymbol{i}_\gamma$ for $\beta, \gamma \in \Phi_+$ defines a total order on Φ_+ called a *Lyndon order*. It is known that each Lyndon order is convex, and the theory of standard modules for Lyndon orders, developed in (21), fits into the general theory described in §2.1. However, working with Lyndon orders allows us to be a little more explicit. In particular, given a a root partition $\pi = (p_1, \dots, p_N) \in \Pi(\alpha)$, set

$$\boldsymbol{i}_\pi := \boldsymbol{i}_{\beta_1}^{p_1} \dots \boldsymbol{i}_{\beta_N}^{p_N} \in \langle I \rangle_\alpha. \tag{2.49}$$

Lemma 2.4.4

(21, Theorem 7.2) Let $\pi \in \Pi(\alpha)$. Then $\boldsymbol{i}_\pi$ is the highest word of $L(\pi)$.

From now on, we fix the notation for the Dynkin diagrams as follows:



Also, we choose the signs ε_{ij} as in §2.1 and the total order $\leq$ on I so that $\varepsilon_{ij} = 1$ and $i < j$ if the corresponding labels i and j satisfy $i < j$ as integers.

Homogeneous roots

We stick with the choices made in §2.4. Throughout the subsection, we assume that the Cartan matrix is of ADE type and $\beta \in \Phi_+$ is such that $\boldsymbol{i}_\beta$ is homogeneous. Let $d := \mathrm{ht}(\beta)$. The module $L(\beta)$ is concentrated in degree 0, and each of its word spaces is one dimensional. Set $\mathfrak{D}_\beta := \mathfrak{D}_{\boldsymbol{i}_\beta}$. Then we can take $\mathfrak{B}_\beta = \{\psi_w e(\boldsymbol{i}_\beta) \mid w \in \mathfrak{D}_\beta\}$. Let $\delta_\beta = D_\beta = e(\boldsymbol{i}_\beta)$, and define $y_\beta := y_d e(\boldsymbol{i}_\beta)$. All parts of Hypothesis 2.2.9 are trivially satisfied, except (v). In the rest of this subsection we verify Hypothesis 2.2.9(v).

<u>Lemma 2.4.5</u>

Let $w \in \mathfrak{S}_d \setminus \mathfrak{D}_\beta$. Then $\psi_w P_d e(\boldsymbol{i}_\beta) \subseteq I_{>(\beta)}$.

Proof. We have $\psi_w = \psi_{r_1} \dots \psi_{r_m}$ for a reduced decomposition $w = s_{r_1} \dots s_{r_m}$. Let k be the largest index such that s_{r_k} is not an admissible transposition of $s_{r_{k+1}} \dots s_{r_m} \boldsymbol{i}_\beta$. By Theorem 2.4.1, $s_{r_k} \dots s_{r_m} \boldsymbol{i}_\beta$ is not a word of $L(\beta)$. So by

Corollary 2.2.3,

$$\psi_{r_k} \dots \psi_{r_m} P_d e(\boldsymbol{i}_\beta) = e(s_{r_k} \dots s_{r_m} \boldsymbol{i}_\beta) \psi_{r_k} \dots \psi_{r_m} P_d e(\boldsymbol{i}_\beta) \subseteq I_{>(\beta)},$$

whence $\psi_w P_d e(\boldsymbol{i}_\beta) \subseteq I_{>(\beta)}$. □

<u>Lemma 2.4.6</u>

Given $1 \leq r, s \leq d$, we have $(y_s - y_r) e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$.

Proof. We prove by induction on $s = 1, \dots, d$ that $(y_s - y_r) e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$ for all $1 \leq r \leq s$. The base case $s = 1$ is trivial. Let $s > 1$, and write $\boldsymbol{i}_\beta = (i_1, \dots, i_d)$. If $i_r \cdot i_s = 0$ for all $1 \leq r < s$, then

$$(i_s, i_1, i_2, \dots, i_{s-1}, i_{s+1}, \dots, i_d)$$

is a word of $L(\beta)$. On the other hand, Lemma 2.4.4 says that $\boldsymbol{i}_\beta$ is the largest word of $L(\beta)$ and so $i_s < i_1$. But then $\boldsymbol{i}_\beta$ is not a Lyndon word, which is a contradiction. Thus there exists some $r < s$ with $i_r \cdot i_s \neq 0$. Since the Cartan matrix is assumed to be of ADE type, either $i_r \cdot i_s = -1$ or $i_r = i_s$. In the second case, by homogeneity (2.47) we can find $r < r' < s$ with $i_{r'} \cdot i_s = -1$. This shows that the definition $t := \max\{r \mid r < s \text{ and } i_r \cdot i_s = -1\}$ makes sense. Once again by homogeneity we must have that $i_r \cdot i_s = 0$ for any r with $t < r < s$. Therefore, using defining relations in R_α, we get

$$(\psi_{s-1} \dots \psi_t)(\psi_t \dots \psi_{s-1}) e(\boldsymbol{i}_\beta) = \pm (y_s - y_t) e(\boldsymbol{i}_\beta).$$

On the other hand, the cycle $(t, t + 1, \ldots, s)$ is not an element of $\mathfrak{D}_\beta$. By Lemma 2.4.5 we must have $\psi_t \ldots \psi_{s-1} e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$. This shows that $(y_s - y_t)e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$, and therefore by induction that $(y_s - y_r)e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$ for every r with $1 \leq r \leq s$. □

Recall the notation $\bar{R}_\beta := R_\beta / I_{>(\beta)}$ and $\bar{r} := r + I_{>(\beta)} \in \bar{R}_\beta$ for $r \in R_\beta$.

Corollary 2.4.7

We have that $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is generated by $\bar{y}_\beta$.

Proof. By Theorem 2.1.2, an element of $e_\beta R_\beta e_\beta$ is a linear combination of terms of the form $\psi_w y_1^{a_1} \ldots y_d^{a_d} e(\boldsymbol{i}_\beta)$ such that $w\boldsymbol{i}_\beta = \boldsymbol{i}_\beta$. If $w \notin \mathfrak{D}_\beta$, then $\psi_w e_\beta \in I_{>(\beta)}$ by Lemma 2.4.5. Otherwise, Lemma 2.4.2 shows that $w = 1$. Therefore, $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is spanned by terms of the form $\bar{y}_1^{a_1} \ldots \bar{y}_d^{a_d} \bar{e}_\beta$. In view of Lemma 2.4.6, we see that $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is generated by $\bar{y}_\beta = \bar{y}_d$. □

Types ADE

Throughout the subsection, we assume again that the Cartan matrix is of *ADE* type. By (9), with a correction made in (3, Lemma A7), if $\beta \in \Phi_+$ is any positive root, except the highest root in type E_8, then $\boldsymbol{i}_\beta$ is homogeneous. We have proved in the previous subsection that Hypothesis 2.2.9 holds in this case.

Now, we deal with the highest root

$$\theta := 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

in type E_8. By (3, Example A.5), the corresponding Lyndon word is

$$\boldsymbol{i}_\theta = 12345867564534231234586756458.$$

Define the positive roots

$$\theta_1 := \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8, \tag{2.50}$$

$$\theta_2 := \alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8. \tag{2.51}$$

Then the root partition (θ_1, θ_2) is a minimal element of $\Pi(\theta) \setminus \{(\theta)\}$. Moreover, $\boldsymbol{i}_{\theta_2} = 1234586756453423$ and $\boldsymbol{i}_{\theta_1} = 1234586756458$. Indeed, one sees by inspection that these words are highest words in the corresponding homogeneous representations and are Lyndon. Finally, we have $\boldsymbol{i}_\theta = \boldsymbol{i}_{\theta_2}\boldsymbol{i}_{\theta_1}$.

Denote by v_{θ_1} and v_{θ_2} non-zero vectors in the $\boldsymbol{i}_{\theta_1}$- and $\boldsymbol{i}_{\theta_2}$-word spaces in the homogeneous modules $L(\theta_1)$ and $L(\theta_2)$, respectively. Note that $L(\theta_1) \boxtimes L(\theta_2)$ is naturaly a submodule of $L(\theta_1) \circ L(\theta_2)$, so we can consider $v_{\theta_1} \otimes v_{\theta_2}$ as a cyclic vector of $L(\theta_1) \circ L(\theta_2)$, and similarly $v_{\theta_2} \otimes v_{\theta_1}$ as a cyclic vector of $L(\theta_2) \circ L(\theta_1)$. By definition, $L(\theta_1) \circ L(\theta_2)$ is the proper standard module $\bar{\Delta}(\theta_1, \theta_2)$, and let v_{θ_1,θ_2} be the image of $v_{\theta_1} \otimes v_{\theta_2}$ under the natural projection $\bar{\Delta}(\theta_1, \theta_2) \twoheadrightarrow L(\theta_1, \theta_2)$. Denote by $w(\theta)$ the element of $\mathfrak{S}_{29}$ which sends $(1, \dots, 29)$ to $(17, \dots, 29, 1, \dots, 16)$. The following has been established in (3), see especially (3, Theorem A.9, Proof), but we sketch its very easy proof for the reader's convenience.

<u>Lemma 2.4.8</u>

The multiplicity of the highest word $\boldsymbol{i}_\theta$ in $L(\theta)$ is one. Moreover, there is a non-zero vector v_θ in the θ-word space of $L(\theta)$ and homogeneous R_θ-module maps

$$\mu : L(\theta_1, \theta_2)\langle 1\rangle \to L(\theta_2) \circ L(\theta_1),\ v_{\theta_1,\theta_2} \mapsto \psi_{w(\theta)}(v_{\theta_2} \otimes v_{\theta_1}),$$

$$\nu : L(\theta_2) \circ L(\theta_1) \to L(\theta),\ v_{\theta_2} \otimes v_{\theta_1} \mapsto v_\theta,$$

such that the sequence

$$0 \to L(\theta_1, \theta_2)\langle 1\rangle \xrightarrow{\mu} L(\theta_2) \circ L(\theta_1) \xrightarrow{\nu} L(\theta) \to 0$$

is exact. Finally,

$$\mathrm{ch}_q\, L(\theta) = (\mathrm{ch}_q\, L(\theta_2) \circ \mathrm{ch}_q\, L(\theta_1) - q\, \mathrm{ch}_q\, L(\theta_1) \circ \mathrm{ch}_q\, L(\theta_2))/(1 - q^2).$$

Proof. By (21, Theorem 7.2(ii)), the multiplicity of the word $\boldsymbol{i}_{\theta_1}\boldsymbol{i}_{\theta_2}$ in $L(\theta_1) \circ L(\theta_2)$ is 1. Moreover, an explicit check shows that the multiplicity of $\boldsymbol{i}_\theta$ in $L(\theta_1) \circ L(\theta_2)$ is q. We conclude using Theorem 2.1.6 and the minimality of (θ_1, θ_2) in $\Pi(\theta) \setminus \{(\theta)\}$ that the standard module $L(\theta_1) \circ L(\theta_2)$ is uniserial with head $L(\theta_1, \theta_2)$ and socle $L(\theta)\langle 1\rangle$. The result follows from these observations since $L(\theta_1, \theta_2)$ is $\circledast$-self-dual and $(L(\theta_1) \circ L(\theta_2))^{\circledast} \simeq L(\theta_2) \circ L(\theta_1)\langle -1\rangle$ in view of (25, Theorem 2.2). $\square$

Consider the parabolic subgroup $\mathfrak{S}_{\mathrm{ht}(\theta_2)} \times \mathfrak{S}_{\mathrm{ht}(\theta_1)} \subseteq \mathfrak{S}_d$ and define

$$\mathfrak{D}_{\theta_2,\theta_1} := \{(w_2, w_1) \in \mathfrak{S}_{\mathrm{ht}(\theta_2)} \times \mathfrak{S}_{\mathrm{ht}(\theta_1)} \mid w_2 \in \mathfrak{D}_{\theta_2},\ w_1 \in \mathfrak{D}_{\theta_1}\}.$$

With this notation we finally have:

Lemma 2.4.9

The cuspidal module $L(\theta)$ is generated by a degree 0 vector v_θ subject only to the relations:

$$(e(\boldsymbol{j}) - \delta_{\boldsymbol{j},\boldsymbol{i}_\theta})v_\theta = 0, \quad \text{for all } \boldsymbol{j} \in \langle I \rangle_\theta, \tag{2.52}$$

$$y_r v_\theta = 0, \quad \text{for all } r = 1, \dots, \mathrm{ht}(\theta), \tag{2.53}$$

$$\psi_w v_\theta = 0, \quad \text{for all } w \in (\mathfrak{S}_{\mathrm{ht}(\theta_2)} \times \mathfrak{S}_{\mathrm{ht}(\theta_1)}) \setminus \mathfrak{D}_{\theta_2,\theta_1}, \tag{2.54}$$

$$\psi_{w(\theta)} v_\theta = 0. \tag{2.55}$$

Proof. The theorem follows easily from Proposition 2.4.3 applied to homogeneous modules $L(\theta_1)$ and $L(\theta_2)$, and Lemma 2.4.8. □

We now define $\delta_\theta = D_\theta = e(\boldsymbol{i}_\theta)$, and $y_\theta = y_{\mathrm{ht}(\theta)} e(\boldsymbol{i}_\theta)$. All parts of Hypothesis 2.2.9 are trivially satisfied, except (v). We now verify Hypothesis 2.2.9(v).

Lemma 2.4.10

We have

$$\iota_{\theta_2,\theta_1}(I_{>(\theta_2)} \otimes R_{\theta_1} + R_{\theta_2} \otimes I_{>(\theta_1)}) \subseteq I_{>(\theta)}.$$

Proof. Apply Proposition 2.2.5 twice with $m = 2$, $\gamma_1 = \theta_2$, $\gamma_2 = \theta_1$, $\pi = (\theta)$, and either $k = 1$ and $\pi_0 = (\theta_2)$, or $k = 2$ and $\pi_0 = (\theta_1)$. □

Lemma 2.4.11

We have that $\bar{e}_\theta \bar{R}_\theta \bar{e}_\theta$ is generated by $\bar{y}_\theta$.

Proof. By Theorem 2.1.2, an element of $e_\theta R_\theta e_\theta$ is a linear combination of terms of the form $\psi_w y_1^{a_1} \dots y_d^{a_d} e(\boldsymbol{i}_\theta)$ such that $w\boldsymbol{i}_\theta = \boldsymbol{i}_\theta$. If $w \in (\mathfrak{S}_{\mathrm{ht}(\theta_2)} \times \mathfrak{S}_{\mathrm{ht}(\theta_1)}) \setminus$

$\mathfrak{D}_{\theta_2,\theta_1}$, then $\psi_w e_\theta \in I_{>(\theta)}$ by Lemmas 2.4.10 and 2.4.5. So we may assume that $w = uv$ with $u \in \mathfrak{S}^{\mathrm{ht}(\theta_2),\mathrm{ht}(\theta_1)}$, $v \in \mathfrak{D}_{\theta_2,\theta_1}$. It is easy to check that the only such permutation that fixes $\boldsymbol{i}_\theta$ is the identity. We therefore see that $\bar{e}_\theta \bar{R}_\theta \bar{e}_\theta$ is generated by $\bar{y}_1, \ldots, \bar{y}_{\mathrm{ht}(\theta)}$.

Note that $\mathrm{ht}(\theta_2) = 16$ and $\mathrm{ht}(\theta_1) = 13$. Using the cases $\beta = \theta_2$ and $\beta = \theta_1$ proved above and Lemma 2.4.10, we have that $(y_r - y_s)e(\boldsymbol{i}_\theta) \in I_{>(\theta)}$ if $1 \leq r, s \leq 16$ or $17 \leq r, s \leq 29$. It remains to show that $(y_r - y_s)e(\boldsymbol{i}_\theta) \in I_{>(\theta)}$ for some $1 \leq r \leq 16$ and $17 \leq s \leq 29$. Let $w \in \mathfrak{S}_{29}$ be the cycle $(27, 26, \ldots, 16)$. By considering words and using Corollary 2.2.3, one can verify that

$$\psi_w^\tau \psi_w e(\boldsymbol{i}_\theta) \equiv (y_{16} - y_{27})e(\boldsymbol{i}_\theta) \pmod{I_{>(\theta)}}.$$

On the other hand, by the formula for the character of $L(\theta)$ from Lemma 2.4.8, we have that $w\boldsymbol{i}_\theta$ is not a word of $L(\theta)$. Therefore, by Corollary 2.2.3, we have that $\psi_w e(\boldsymbol{i}_\theta) \in I_{>(\theta)}$, so $(y_{16} - y_{27})e(\boldsymbol{i}_\theta) \in I_{>(\theta)}$, and we are done. $\square$

Non-symmetric types

Now we deal with non-symmetric Cartan matrices, i.e. Cartan matrices of $BCFG$ types.

<u>Lemma 2.4.12</u>

Suppose that $\delta_\beta, D_\beta \in e(\boldsymbol{i}_\beta)R_\beta e(\boldsymbol{i}_\beta)$ have been chosen so that Hypothesis 2.2.9(iii) is satisfied. If the minimal degree component of $e(\boldsymbol{i}_\beta)R_\beta e(\boldsymbol{i}_\beta)$ is spanned by D_β, then Hypothesis 2.2.9(i) and (vi) are satisfied.

Proof. Since $D_\beta \delta_\beta D_\beta$ has the same degree as D_β, the assumption above implies that $D_\beta \delta_\beta D_\beta$ is proportional to D_β. Acting on v_β^+ and using Hypothesis 2.2.9(iii) gives $D_\beta \delta_\beta D_\beta = D_\beta$, which upon multiplication by δ_β on the right gives the property $e_\beta^2 = e_\beta$, which is even stronger than (i).

To see (vi), we look at the lowest degree component in $e(\boldsymbol{i}_\beta \boldsymbol{i}_\beta) R_{2\beta} e(\boldsymbol{i}_\beta \boldsymbol{i}_\beta)$ using (21, Lemma 5.3(ii)) and commutation relations in the algebra $R_{2\beta}$. $\square$

It will be clear in almost all cases that the condition of Lemma 2.4.12 will be satisfied, and moreover Hypothesis 2.2.9(ii) and (iv) are easy to verify by inspection. This leaves Hypothesis 2.2.9(v) to be shown in each case.

Type B_l

The set of positive roots is broken into two types. For $1 \le i \le j \le l$ we have the root $\alpha_i + \cdots + \alpha_j$, and for $1 \le i < j \le l$ we have the root $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l$.

Let $\beta := \alpha_i + \cdots + \alpha_j$. Then $\boldsymbol{i}_\beta := (i, \ldots, j)$, and the irreducible module $L(\beta)$ is one-dimensional with character $\boldsymbol{i}_\beta$. Define $\delta_\beta := D_\beta := e(\boldsymbol{i}_\beta)$ and $y_\beta := y_d e(\boldsymbol{i}_\beta)$. Using Corollary 2.2.3 one sees that $\psi_r e_\beta \in I_{>(\beta)}$ for all r, which by Theorem 2.1.2 shows that $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_1, \ldots, \bar{y}_d]\bar{e}_\beta$. This also shows that for $1 \le r \le d$ we have the elements of $I_{>(\beta)}$:

$$\psi_r^2 e_\beta = \begin{cases} (y_r - y_{r+1}^2) e_\beta, & \text{if } j = l \text{ and } r = d-1 \\ (y_r - y_{r+1}) e_\beta, & \text{otherwise.} \end{cases}$$

It follows that $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_\beta]\bar{e}_\beta$, and thus $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is generated by $\bar{e}_\beta \bar{y}_\beta \bar{e}_\beta$.

Consider $\beta := \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l$. In this case, $\boldsymbol{i}_\beta = (i, \ldots, l, l, \ldots, j)$, and $\mathrm{ch}_q\, L(\beta) = (q + q^{-1})\boldsymbol{i}_\beta$. Define $\delta_\beta := y_{l-i+2}e(\boldsymbol{i}_\beta)$, $D_\beta := \psi_{l-i+1}e(\boldsymbol{i}_\beta)$, and $y_\beta = y_1 e(\boldsymbol{i}_\beta)$. Using Corollary 2.2.3, one sees that $\psi_r e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$ for $r \neq l-i+1$. It is also clear that $\psi_{l-i+1}e_\beta = 0$, and therefore by Theorem 2.1.2, $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_1, \ldots, \bar{y}_d]e_\beta$. We also have the following elements of $I_{>(\beta)}$:

$$\psi_r^2 e(\boldsymbol{i}_\beta) = \begin{cases} (y_r - y_{r+1})e(\boldsymbol{i}_\beta), & \text{for } 1 \leq r \leq l-i-1; \\ (y_{l-i} - y_{l-i+1}^2)e(\boldsymbol{i}_\beta), & \text{for } r = l-i; \\ (y_{l-i+3} - y_{l-i+2}^2)e(\boldsymbol{i}_\beta), & \text{for } r = l-i+2; \\ (y_{r+1} - y_r)e(\boldsymbol{i}_\beta) & \text{for } l-i+3 \leq r \leq d-1. \end{cases}$$

Taken together, these show that $\bar{R}_\beta \bar{e}(\boldsymbol{i}_\beta) = F[\bar{y}_{l-i+1}, \bar{y}_{l-i+2}]\bar{e}(\boldsymbol{i}_\beta)$. Multiplying on both sides by $\bar{e}_\beta$ and using the KLR / nil-Hecke relations, we have

$$\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta = F[\bar{y}_{l-i+1} + \bar{y}_{l-i+2}, \bar{y}_{l-i+1}\bar{y}_{l-i+2}]\bar{e}_\beta.$$

Furthermore, $(y_{l-i+1} + y_{l-i+2})\psi_{l-i+1}e(\boldsymbol{i}_\beta) = \psi_{l-i+1}\psi_{l-i}^2\psi_{l-i+1}e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$, and so in fact

$$\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta = F[\bar{y}_{l-i+1}^2]\bar{e}_\beta = F[\bar{y}_1]\bar{e}_\beta.$$

Type C_l

The set of positive roots is broken into three types. For $1 \leq i \leq j \leq l$ we have the root $\alpha_i + \cdots + \alpha_j$, for $1 \leq i < j < l$ we have the root $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-1} + \alpha_l$, and for $1 \leq i < l$ we have the root $2\alpha_i + \cdots + 2\alpha_{l-1} + \alpha_l$.

Consider $\beta = \alpha_i + \cdots + \alpha_j$. Then $\boldsymbol{i}_\beta = (i, \ldots, j)$ and $\mathrm{ch}_q\, L(\beta) = \boldsymbol{i}_\beta$. Define $\delta_\beta = D_\beta := e(\boldsymbol{i}_\beta)$. Define $y_\beta := y_1 e(\boldsymbol{i}_\beta)$. Using Corollary 2.2.3 one sees that

$\psi_r e_\beta \in I_{>(\beta)}$ for all r, which by Theorem 2.1.2 shows that $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_1, \dots, \bar{y}_d]\bar{e}_\beta$. This also shows that for $1 \le r \le d$ we have the elements of $I_{>(\beta)}$:

$$\psi_r^2 e_\beta = \begin{cases} (y_r^2 - y_{r+1})e_\beta, & \text{if } j = l \text{ and } r = d-1 \\ (y_r - y_{r+1})e_\beta, & \text{otherwise.} \end{cases}$$

Consequently, $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is generated by $\bar{e}_\beta \bar{y}_\beta \bar{e}_\beta$.

Consider $\beta = \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{l-1} + \alpha_l$. Then $\boldsymbol{i}_\beta = (i, \dots, l-1, l, l-1, \dots, j)$ and $\mathrm{ch}_q\, L(\beta) = (i, \dots, l-1, l, l-1, \dots, j)$. Define $\delta_\beta = D_\beta := e(\boldsymbol{i}_\beta)$, and $y_\beta := y_1 e(\boldsymbol{i}_\beta)$. Using Corollary 2.2.3 one sees that $\psi_r e_\beta \in I_{>(\beta)}$ for all r, which by Theorem 2.1.2 shows that $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_1, \dots, \bar{y}_d]\bar{e}_\beta$. This also shows that for $1 \le r \le d$ we have the elements of $I_{>(\beta)}$:

$$\psi_r^2 e_\beta = \begin{cases} (y_r - y_{r+1})e_\beta, & \text{for } 1 \le r \le l-i-1; \\ (y_{l-i}^2 - y_{l-i+1})e_\beta, & \text{for } r = l-i; \\ (y_{l-i+2}^2 - y_{l-i+1})e_\beta, & \text{for } r = l-i+1; \\ (y_{r+1} - y_r)e_\beta & \text{for } l-i+2 \le r \le d-1. \end{cases}$$

It follows that $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_{l-i}, \bar{y}_{l-i+2}]\bar{e}_\beta$. Furthermore, by the relation (2.10),

$$(y_{l-i} + y_{l-i+2})e_\beta = (\psi_{l-i+1}\psi_{l-i}\psi_{l-i+1} - \psi_{l-i}\psi_{l-i+1}\psi_{l-i})e_\beta \in I_{>(\beta)}$$

and therefore $\bar{R}_\beta \bar{e}_\beta = F[\bar{y}_{l-i}]e_\beta = F[\bar{y}_\beta]\bar{e}_\beta$.

Consider $\beta = 2\alpha_i + \dots + 2\alpha_{l-1} + \alpha_l$. Then $\boldsymbol{i}_\beta = (i, \dots, l-1, i, \dots, l)$ and

$$\mathrm{ch}_q\, L(\beta) = q((i, \dots, l-1) \circ (i, \dots, l-1)) \cdot (l).$$

Let $w \in \mathfrak{S}_d$ be the permutation that sends $(1, 2, \ldots, d)$ to $(l - i + 1, \ldots, d - 1, 1, \ldots, l - i, d)$, and define $D_\beta := \psi_w e(\boldsymbol{i}_\beta)$. Define also $\delta_\beta := y_{d-1}e(\boldsymbol{i}_\beta)$ and $y_\beta := y_d e(\boldsymbol{i}_\beta)$. Set $\gamma = \alpha_i + \cdots + \alpha_{l-1}$. Since $I_{>(\gamma^2)}$ is generated by idempotents $e(\boldsymbol{i})$ with $\boldsymbol{i} > \boldsymbol{i}_\gamma^2$, and $\boldsymbol{i}_\beta = \boldsymbol{i}_\gamma^2 i_l$ is the highest word of L_β, we see that

$$\iota_{2\gamma,\alpha_l}(I_{>(\gamma^2)} \otimes R_{\alpha_l}) \subseteq I_{>(\beta)}.$$

Let $\mu : \bar{R}_{2\gamma} \boxtimes R_{\alpha_l} \to \bar{R}_\beta$ be the induced map. Note that every word of $L(\beta)$ ends with l, so that $\psi_u e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$ unless $u \in \mathfrak{S}_{d-1,1}$, by Corollary 2.2.3. Therefore, applying (2.43) in the type A case of (γ^2) (which has already been verified), we obtain

$$\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta = \mu(\bar{e}_{(\gamma^2)} \bar{R}_{2\gamma} \bar{e}_{(\gamma^2)} \otimes R_{\alpha_l}) = \bar{e}_\beta \mathcal{O}[\bar{y}_{l-i} + \bar{y}_{2l-2i}, \bar{y}_{l-i}\bar{y}_{2l-2i}, \bar{y}_d]\bar{e}_\beta.$$

Furthermore,

$$(\bar{y}_{l-i} + \bar{y}_{2l-2i})\bar{e}(\boldsymbol{i}_\beta) = \bar{\psi}_{l-i} \ldots \bar{\psi}_{2l-2i-1}\bar{\psi}_{2l-2i}^2\bar{\psi}_{2l-2i-1} \ldots \bar{\psi}_{l-i}\bar{e}(\boldsymbol{i}_\beta) = 0$$

and $(\bar{y}_{2l-2i}^2 - \bar{y}_d)\bar{e}(\boldsymbol{i}_\beta) = \bar{\psi}_{2l-2i}^2\bar{e}(\boldsymbol{i}_\beta) = 0$. Thus $\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta$ is generated by $\bar{e}_\beta \bar{y}_d \bar{e}_\beta$.

Type F_4

We write $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 \in \Phi_+$. If $c_4 = 0$, then this root lies in a subsystem of type B_3 with the *same order* as in section 2.4 and we are done.

If $\beta = \alpha_i + \cdots + \alpha_j$ for some $1 \leq i \leq j \leq 4$, then $\boldsymbol{i}_\beta = (i, \ldots, j)$ and $\mathrm{ch}_q L(\beta) = (i, \ldots, j)$. In this case we take $D_\beta = \delta_\beta = e(\boldsymbol{i}_\beta)$, and set $y_\beta = y_{\mathrm{ht}(\beta)}e(\boldsymbol{i}_\beta)$.

The following table shows the choice of data for the remaining roots, except for the highest root $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, which we discuss separately. In each of these cases, the hypotheses may be verified by employing the same methods used above. For example, in each case either Hypothesis 2.2.9(i)-(iv),(vi) may be verified directly or with the help of Lemma 2.4.12 when it applies.

$\boldsymbol{i}_\beta$	D_β	δ_β	y_β
2343	$e(\boldsymbol{i}_\beta)$	$e(\boldsymbol{i}_\beta)$	$y_3 e(\boldsymbol{i}_\beta)$
12343	$e(\boldsymbol{i}_\beta)$	$e(\boldsymbol{i}_\beta)$	$y_5 e(\boldsymbol{i}_\beta)$
23434	$\psi_3\psi_2\psi_4\psi_3 e(\boldsymbol{i}_\beta)$	$y_5 e(\boldsymbol{i}_\beta)$	$y_1 e(\boldsymbol{i}_\beta)$
123432	$e(\boldsymbol{i}_\beta)$	$e(\boldsymbol{i}_\beta)$	$y_5 e(\boldsymbol{i}_\beta)$
123434	$\psi_4\psi_3\psi_5\psi_4 e(\boldsymbol{i}_\beta)$	$y_6 e(\boldsymbol{i}_\beta)$	$y_2 e(\boldsymbol{i}_\beta)$
1234323	$e(\boldsymbol{i}_\beta)$	$e(\boldsymbol{i}_\beta)$	$y_5 e(\boldsymbol{i}_\beta)$
1234342	$\psi_4\psi_3\psi_5\psi_4 e(\boldsymbol{i}_\beta)$	$y_6 e(\boldsymbol{i}_\beta)$	$y_2 e(\boldsymbol{i}_\beta)$
12343423	$\psi_4\psi_3\psi_5\psi_4 e(\boldsymbol{i}_\beta)$	$y_6 e(\boldsymbol{i}_\beta)$	$y_8 e(\boldsymbol{i}_\beta)$
123434233	$\psi_4\psi_3\psi_5\psi_4\psi_8 e(\boldsymbol{i}_\beta)$	$y_6 y_9 e(\boldsymbol{i}_\beta)$	$y_7 e(\boldsymbol{i}_\beta)$
1234342332	$\psi_4\psi_3\psi_5\psi_4\psi_8 e(\boldsymbol{i}_\beta)$	$y_6 y_9 e(\boldsymbol{i}_\beta)$	$y_{10} e(\boldsymbol{i}_\beta)$

Consider now $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$, where $\boldsymbol{i}_\beta = (12343123432)$. Let $w \in \mathfrak{S}_{11}$ be the permutation that sends $(1, \dots, 11)$ to $(6, 7, 8, 9, 10, 1, 2, 3, 4, 5, 11)$, and set $D_\beta = \psi_w e(\boldsymbol{i}_\beta)$. Let $\delta_\beta = y_{10} e(\boldsymbol{i}_\beta)$, and $y_\beta = y_{11} e(\boldsymbol{i}_\beta)$. Define $\gamma = \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4$. There is a map $\mu : \bar{R}_{2\gamma} \boxtimes R_{\alpha_2} \to \bar{R}_\beta$. This map is not surjective, but one can show that

$$\bar{e}(\boldsymbol{i}_\beta)\bar{R}_\beta \bar{e}(\boldsymbol{i}_\beta) = \mu(\bar{e}(\boldsymbol{i}_\gamma^2)\bar{R}_{2\gamma}\bar{e}(\boldsymbol{i}_\gamma^2) \otimes R_{\alpha_2})$$

and thus

$$\bar{e}_\beta \bar{R}_\beta \bar{e}_\beta = \mu(\bar{e}_{(\gamma^2)} \bar{R}_{2\gamma} \bar{e}_{(\gamma^2)} \otimes R_{\alpha_2}) = \mu(\mathcal{O}[\bar{y}_5 + \bar{y}_{10}, \bar{y}_5\bar{y}_{10}]\bar{e}_{(\gamma^2)} \otimes R_{\alpha_2})$$
$$= \mathcal{O}[\bar{y}_5 + \bar{y}_{10}, \bar{y}_5\bar{y}_{10}, \bar{y}_{11}]\bar{e}_\beta.$$

We also compute (cf. (30, §5)):

$$(\bar{y}_5 + \bar{y}_{10})\bar{e}(\boldsymbol{i}_\beta) = -\bar{\psi}_5\bar{\psi}_6\bar{\psi}_7\bar{\psi}_8\bar{\psi}_9\bar{\psi}_{10}^2\bar{\psi}_9\bar{\psi}_8\bar{\psi}_7\bar{\psi}_6\bar{\psi}_5\bar{e}(\boldsymbol{i}_\beta),$$

which is zero because it contains the word (12341234323), and this is not a word of $L(\beta)$. Since $\bar{y}_{11}\bar{e}(\boldsymbol{i}_\beta) = \bar{y}_{10}^2\bar{e}(\boldsymbol{i}_\beta)$, we see that $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta = \mathcal{O}[\bar{y}_{11}]\bar{e}_\beta$, as required.

Type G_2

$\beta = \alpha_1$: $\boldsymbol{i}_\beta = (1)$, $D_\beta = \delta_\beta = e(\boldsymbol{i}_\beta)$, and $y_\beta = y_1 e(\boldsymbol{i}_\beta)$.

$\beta = \alpha_2$: $\boldsymbol{i}_\beta = (2)$, $D_\beta = \delta_\beta = e(\boldsymbol{i}_\beta)$, and $y_\beta = y_1 e(\boldsymbol{i}_\beta)$.

$\beta = \alpha_1 + \alpha_2$: $\boldsymbol{i}_\beta = (12)$, $D_\beta = \delta_\beta = e(\boldsymbol{i}_\beta)$, and $y_\beta = y_1 e(\boldsymbol{i}_\beta)$.

$\beta = 2\alpha_1 + \alpha_2$: $\boldsymbol{i}_\beta = (112)$, $D_\beta = \psi_1 e(\boldsymbol{i}_\beta)$, $\delta_\beta = y_2 e(\boldsymbol{i}_\beta)$, and $y_\beta = (y_1 + y_2)e(\boldsymbol{i}_\beta)$.

$\beta = 3\alpha_1 + \alpha_2$: $\boldsymbol{i}_\beta = (1112)$, $D_\beta = \psi_1\psi_2\psi_1 e(\boldsymbol{i}_\beta)$, $\delta_\beta = y_2 y_3^2 e(\boldsymbol{i}_\beta)$, and $y_\beta = y_1 y_2 y_3 e(\boldsymbol{i}_\beta)$.

Let μ be the composition $R_{3\alpha_1} \boxtimes R_{\alpha_2} \hookrightarrow R_\beta \twoheadrightarrow \bar{R}_\beta$. If $w \notin \mathfrak{S}_{3,1}$, then $\psi_w e(\boldsymbol{i}_\beta) \in I_{>(\beta)}$, and so μ is surjective. Furthermore, $\bar{e}_\beta = \mu(e_{(\alpha_1^3)} \otimes 1)$. Thus $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta = \mathcal{O}[\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4]^{\mathfrak{S}_{3,1}}\bar{e}_\beta$. Since $(\bar{y}_3^3 - \bar{y}_4)\bar{e}(\boldsymbol{i}_\beta) = \bar{\psi}_3^2\bar{e}(\boldsymbol{i}_\beta) = 0$, we have that $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta$ is generated by $\mathcal{O}[\bar{y}_1, \bar{y}_2, \bar{y}_3]^{\mathfrak{S}_3}\bar{e}_\beta$. Observe using (19, Theorem 4.12(i)) that

$$(\bar{y}_1 + \bar{y}_2 + \bar{y}_3)\bar{e}_\beta = \bar{e}_\beta\bar{\psi}_1\bar{\psi}_2\bar{\psi}_3^2\bar{e}_\beta \in I_{>(\beta)}$$

and

$$((\bar{y}_1+\bar{y}_2+\bar{y}_3)^2-(\bar{y}_1\bar{y}_2+\bar{y}_1\bar{y}_3+\bar{y}_2\bar{y}_3))\bar{e}_\beta=\bar{e}_\beta\bar{\psi}_2\bar{\psi}_3^2\bar{e}_\beta\in I_{>(\beta)}.$$

Therefore $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta$ is generated by $\bar{y}_1\bar{y}_2\bar{y}_3$.

$\beta=3\alpha_1+2\alpha_2$: $\boldsymbol{i}_\beta=(11212)$, $D_\beta=\psi_1\psi_3\psi_2\psi_4\psi_1\psi_3 e(\boldsymbol{i}_\beta)$, $\delta_\beta=y_2y_4^2e(\boldsymbol{i}_\beta)$, and $y_\beta=y_1y_2y_4e(\boldsymbol{i}_\beta)$. We first prove

Claim: If $w\neq 1$ then $e(\boldsymbol{i}_\beta)\psi_w e_\beta\in I_{>(\beta)}$.

This is clearly true unless w is one of the twelve permutations that stabilizes the word $\boldsymbol{i}_\beta$. Of these, six produce a negative degree. Since D_β spans the smallest degree component of $e(\boldsymbol{i}_\beta)R_\beta e(\boldsymbol{i}_\beta)$ we the Claim holds for these six permutations. Two of the remaining six permutations end with the cycle (12). Since $\psi_1D_\beta=0$, this implies that the Claim holds for them too. Finally, reduced decompositions for the remaining non-identity permutations may be chosen so that $e(\boldsymbol{i}_\beta)\psi_w\in I_{>(\beta)}$ by Lemmas 2.2.2 and 2.4.4.

Now we combine the Claim with Theorem 2.1.2 to see that $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta$ is generated by $\bar{e}_\beta\mathcal{O}[\bar{y}_1,\dots,\bar{y}_5]\bar{e}_\beta$. Next, by a word argument and quadratic relations, $(y_3-y_2^3)e(\boldsymbol{i}_\beta),(y_5-y_4^3)e(\boldsymbol{i}_\beta)\in I_{>(\beta)}$. Thus $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta=\bar{e}_\beta\mathcal{O}[\bar{y}_1,\bar{y}_2,\bar{y}_4]\bar{e}_\beta$. This can then be seen to be equal to $\mathcal{O}[\bar{y}_1,\bar{y}_2,\bar{y}_4]^{\mathfrak{S}_3}\bar{e}_\beta$, arguing as in the case of the root $3\alpha_1+\alpha_2$ above. Again as in the case of the root $3\alpha_1+\alpha_2$, one then shows using specific elements of $I_{>(\beta)}$that $(\bar{y}_1+\bar{y}_2+\bar{y}_4)\bar{e}_\beta=0$ and $(\bar{y}_1\bar{y}_2+\bar{y}_1\bar{y}_4+\bar{y}_2\bar{y}_4)\bar{e}_\beta=0$, so that $\bar{e}_\beta\bar{R}_\beta\bar{e}_\beta=\mathcal{O}[\bar{y}_1\bar{y}_2\bar{y}_4]e_\beta$.

CHAPTER III

HOMOMORPHISMS BETWEEN SPECHT MODULES

Preliminaries

Lie theoretic notation

We collect some notation that will be used in the sequel; the interested reader is refered to (20) for more details. Let $e \in \{3, 4, \ldots\}$ and $I := \mathbb{Z}/e\mathbb{Z}$. Let Γ be the quiver with vertex set I, with a directed edge from i to j if $j = i - 1$. Thus Γ is a quiver of type $A^{(1)}_{e-1}$. We denote the simple roots by $\{\alpha_i \mid i \in I\}$, and define $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ to be the positive part of the root lattice. For $\alpha \in Q_+$ let $\mathrm{ht}(\alpha)$ be the *height of* α. That is, $\mathrm{ht}(\alpha)$ is the sum of the coefficients when α is expanded in terms of the α_i's.

Let $\mathfrak{S}_d$ be the symmetric group on d letters and let $s_r = (r, r+1)$, for $1 \leq r < d$, be the simple transpositions of $\mathfrak{S}_d$. Then $\mathfrak{S}_d$ acts on the left on the set I^d by place permutations. If $\boldsymbol{i} = (i_1, \ldots, i_d) \in I^d$ then its *weight* is $|\boldsymbol{i}| := \alpha_{i_1} + \cdots + \alpha_{i_d} \in Q_+$. The $\mathfrak{S}_d$-orbits on I^d are the sets

$$\langle I \rangle_\alpha := \{\boldsymbol{i} \in I^d \mid \alpha = |\boldsymbol{i}|\}$$

parametrized by all $\alpha \in Q_+$ of height d.

Partitions

Let $\mathscr{P}_d$ be the set of all partitions of d and put $\mathscr{P} := \bigsqcup_{d\geq 0} \mathscr{P}_d$. The *Young diagram* of the partition $\mu \in \mathscr{P}$ is

$$\{(a, b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid 1 \leq b \leq \mu_a\}.$$

The elements of this set are the *nodes of* μ. More generally, a *node* is any element of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$.

To each node $A = (a, b)$ we associate its *residue*, which is the following element of $I = \mathbb{Z}/e\mathbb{Z}$:

$$\mathrm{res} A = (b - a) \pmod{e}. \tag{3.1}$$

An *i-node* is a node of residue i. Define the *residue content of* μ to be

$$\mathrm{cont}(\mu) := \sum_{A\in\mu} \alpha_{\mathrm{res}A} \in Q_+. \tag{3.2}$$

Denote

$$\mathscr{P}_\alpha := \{\mu \in \mathscr{P} \mid \mathrm{cont}(\mu) = \alpha\} \qquad (\alpha \in Q_+).$$

A node $A \in \mu$ is a *removable node (of μ)* if $\mu \setminus \{A\}$ is (the diagram of) a partition. A node $B \notin \mu$ is an *addable node (for μ)* if $\mu \cup \{B\}$ is a partition. We use the notation

$$\mu_A := \mu \setminus \{A\}, \qquad \mu^B := \mu \cup \{B\}.$$

Let $\mu \in \mathscr{P}_d$. A *μ-tableau* $\mathtt{T}$ is obtained from the diagram of μ by inserting the integers $1, \dots, d$ into the nodes, allowing no repeats. If the node $A = (a, b) \in \mu$ is occupied by the integer r in $\mathtt{T}$ then we write $r = \mathtt{T}(a, b)$ and set $\mathrm{res}_{\mathtt{T}}(r) = \mathrm{res}A$. The *residue sequence* of $\mathtt{T}$ is

$$\boldsymbol{i}(\mathtt{T}) = (i_1, \dots, i_d) \in I^d, \tag{3.3}$$

where $i_r = \mathrm{res}_{\mathtt{T}}(r)$ is the residue of the node occupied by r in $\mathtt{T}$ $(1 \leq r \leq d)$.

A μ-tableau $\mathtt{T}$ is *row-strict* (resp. *column-strict*) if its entries increase from left to right (resp. from top to bottom) along the rows (resp. columns) of each component of $\mathtt{T}$. A μ-tableau $\mathtt{T}$ is *standard* if it is row- and column-strict. Let $\mathrm{St}(\mu)$ be the set of standard μ-tableaux.

Let $\mu \in \mathscr{P}$, $i \in I$, and A be a removable i-node of μ. We set

$$d_A(\mu) = \#\left\{ \begin{array}{c} \text{addable } i\text{-nodes of} \\ \mu \\ \text{strictly below } A \end{array} \right\} - \#\left\{ \begin{array}{c} \text{removable } i\text{-nodes of} \\ \mu \\ \text{strictly below } A \end{array} \right\}. \tag{3.4}$$

Given $\mu \in \mathscr{P}_d$ and $\mathtt{T} \in \mathrm{St}(\mu)$, the *degree* of $\mathtt{T}$ is defined in (4, section 3.5) inductively as follows. If $d = 0$, then $\mathtt{T}$ is the empty tableau $\varnothing$, and we set $\deg(\mathtt{T}) := 0$. Otherwise, let A be the node occupied by d in $\mathtt{T}$. Let $\mathtt{T}_{<d} \in \mathrm{St}(\mu_A)$ be the tableau obtained by removing this node and set

$$\deg(\mathtt{T}) := d_A(\mu) + \deg(\mathtt{T}_{<d}). \tag{3.5}$$

The group $\mathfrak{S}_d$ acts on the set of μ-tableaux from the left by acting on the entries of the tableaux. Let $\mathtt{T}^\mu$ be the μ-tableau in which the numbers $1, 2, \dots, d$

appear in order from left to right along the successive rows, working from top row to bottom row.

Set

$$\boldsymbol{i}^{\mu} := \boldsymbol{i}(\mathtt{T}^{\mu}). \tag{3.6}$$

For each μ-tableau $\mathtt{T}$ define permutations $w^{\mathtt{T}} \in \mathfrak{S}_d$ by the equation

$$w^{\mathtt{T}}\mathtt{T}^{\mu} = \mathtt{T}. \tag{3.7}$$

Binomial coefficients

In this section, we state some elementary theorems about binomial coefficients that will be useful later. Let p be a fixed prime.

Definition 3.1.1

For $n \in \mathbb{Z}_{>0}$ we define

1. $\nu_p(n) = \max\{i \mid p^i \text{ divides } n\}$
2. $\ell_p(n) = \min\{i \mid p^i > n\}$.

We also set $\ell_p(0) = -\infty$.

The following can be easily derived from (11, Lemma 22.4).

Lemma 3.1.2

For any $a, b \in \mathbb{Z}_{>0}$, one has

$$p \left| \binom{a}{k} \right. \text{ for } k = 1, \dots, b \qquad \Leftrightarrow \qquad \nu_p(a) \geq \ell_p(b).$$

Recall the definition of $\mathrm{Gc}(a_1, \dots, a_N)$ from the introduction. Then

Corollary 3.1.3

We have $p \mid \mathrm{Gc}(a_1, \dots, a_N)$ if and only if

$$\nu_p(a_i) \geq \ell_p(a_{i+1} - 1) \text{ for } 1 \leq i \leq N - 1.$$

Shuffles

In this section, we fix $e \in \{0, 3, 4, 5, \dots\}$ and $I := \mathbb{Z}/e\mathbb{Z}$. Given $a, b \in \mathbb{Z}_{\geq 0}$ we define

$$\mathrm{Sh}(a, b) := \{\sigma \in \mathfrak{S}_{a+b} \mid \sigma(1) < \cdots < \sigma(a) \text{ and } \sigma(a+1) < \cdots < \sigma(a+b)\}.$$

For $\boldsymbol{i} \in I^a, \boldsymbol{i}' \in I^b$ write $\boldsymbol{i}\boldsymbol{i}' \in I^{a+b}$ for their concatenation, and define

$$\mathrm{Shuf}(\boldsymbol{i}, \boldsymbol{i}') := \{\sigma \cdot \boldsymbol{i}\boldsymbol{i}' \mid \sigma \in \mathrm{Sh}(a, b)\},$$

$$\mathrm{Sh}(\boldsymbol{j}; \boldsymbol{i}, \boldsymbol{i}') := \{\sigma \in \mathrm{Sh}(a, b) \mid \sigma \cdot \boldsymbol{i}\boldsymbol{i}' = \boldsymbol{j}\}, \text{ and}$$

$$H(\boldsymbol{i}) := \langle s_m \mid s_m \boldsymbol{i} = \boldsymbol{i} \rangle < \mathfrak{S}_a.$$

For $i \in I$, define the elements of I^a

$$S^+(i, a) := (i, i+1, \dots, i+a-1), \tag{3.8}$$

$$S^-(i, a) := (i, i-1, \dots, i-a+1). \tag{3.9}$$

Fix $j, k \in I$, $a, b \in \mathbb{Z}_{\geq 0}$, and write $S^+ := S^+(j, a)$ and $S^- := S^-(k, b)$. If $\boldsymbol{i} \in \mathrm{Shuf}(S^+, S^-)$ then $\boldsymbol{i}$ cannot contain three or more equal adjacent indices. In this case $H(\boldsymbol{i})$ is an elementary 2-subgroup of $\mathfrak{S}_{a+b}$.

Proposition 3.1.4

For every $\boldsymbol{i} \in \mathrm{Shuf}(S^+, S^-)$, there is a unique element $\sigma_{\boldsymbol{i}} \in \mathrm{Sh}(\boldsymbol{i}; S^+, S^-)$ of minimal length. Furthermore, $\mathrm{Sh}(\boldsymbol{i}; S^+, S^-) = \{h\sigma_{\boldsymbol{i}} \mid h \in H(\boldsymbol{i})\}$.

Proof. Write $S^+ = (j_1, \dots, j_a)$ and $S^- = (k_1, \dots, k_b)$, and for every integer $d \geq 1$ introduce the following notation. For $\sigma \in \mathfrak{S}_d$, define $E\sigma \in \mathfrak{S}_{d-1}$ by

$$E\sigma(r) = \begin{cases} \sigma(r), & \text{for } r = 1, \dots, \sigma^{-1}(d) - 1; \\ \sigma(r+1), & \text{for } r = \sigma^{-1}(d), \dots, d-1. \end{cases}$$

Similarly, for $\boldsymbol{i} = (i_1, \dots, i_d) \in I^d$, define $E\boldsymbol{i} = (i_1, \dots, i_{d-1})$.

The proposition will be proved by induction on $a + b$. If $a + b = 1$ or 2, or if $a = 0$ or $b = 0$ the claim follows immediately. Suppose $a + b \geq 3$, with $a, b \geq 1$, and assume the induction hypothesis. Fix $\boldsymbol{i} \in \mathrm{Shuf}(S^+, S^-)$. We distinguish several cases based on the values of i_{a+b}, i_{a+b-1}, j_a, and k_b.

Case 1: $i_{a+b} = j_a$ and either $j_a \neq k_b$ or $i_{a+b-1} = i_{a+b} - 1$. Given $\sigma \in \mathrm{Sh}(\boldsymbol{i}; S^+, S^-)$, we must have $\sigma(a) = a + b$. Then the map

$$\mathrm{Sh}(\boldsymbol{i}; S^+, S^-) \to \mathrm{Sh}(E\boldsymbol{i}; ES^+, S^-),\ \sigma \mapsto E\sigma$$

is a bijection, and moreover $\ell(E\sigma) = \ell(\sigma) - b$ and $H(\boldsymbol{i}) \cong H(E\boldsymbol{i})$. We define $\sigma_{\boldsymbol{i}}$ by $E\sigma_{\boldsymbol{i}} = \sigma_{E\boldsymbol{i}}$, which completes the induction in this case.

Case 2: $i_{a+b} = k_b$ and either $j_a \neq k_b$ or $i_{a+b-1} = i_{a+b} + 1$. Here the map

$$\mathrm{Sh}(\boldsymbol{i}; S^+, S^-) \to \mathrm{Sh}(E\boldsymbol{i}; S^+, ES^-),\ \sigma \to E\sigma$$

gives a bijection as above.

Case 3: $i_{a+b} = i_{a+b-1} = j_a = k_b$. We must either have $\sigma(a) = a + b$ and $\sigma(a + b) = a + b - 1$ or $\sigma(a) = a + b - 1$ and $\sigma(a + b) = a + b$. Thus, we get a well-defined map

$$\mathrm{Sh}(\boldsymbol{i}; S^+, S^-) \to \mathrm{Sh}(EE\boldsymbol{i}; ES^+, ES^-),\ \sigma \mapsto EE\sigma$$

which is two-to-one, with $s_{a+b-1}\sigma$ and σ having the same image. This proves the claim, since $H(\boldsymbol{i}) \cong H(EE\boldsymbol{i}) \times \langle s_{a+b-1} \rangle$. $\square$

KLR Algebras and Universal Specht Modules

KLR algebras

Let $\mathcal{O}$ be a commutative ring with identity and $\alpha \in Q_+$. There is a unital $\mathcal{O}$-algebra $R_\alpha = R_\alpha(\mathcal{O})$ called a *Khovanov-Lauda-Rouquier (KLR) algebra*, first defined in (14; 15; 32). We follow the notations and conventions of (20), so R_α is generated by elements

$$\{e(\boldsymbol{i}) \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots, \psi_{d-1}\}, \tag{3.10}$$

subject to some explicit relations.

The algebras R_α have $\mathbb{Z}$-gradings determined by setting $e(\boldsymbol{i})$ to be of degree 0, y_r of degree 2, and $\psi_r e(\boldsymbol{i})$ of degree $-a_{i_r,i_{r+1}}$ for all r and $\boldsymbol{i} \in \langle I \rangle_\alpha$.

For a graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, with finite dimensional graded components its *graded dimension* is $\dim_q V := \sum_{n\in\mathbb{Z}} (\dim V_n) q^n \in \mathbb{Z}[[q, q^{-1}]]$.

Fix $\mu \in \mathscr{P}_\alpha$. In this section we define the Specht module S^μ over R_α, mostly following (20). A node $A = (x, y) \in \mu$ is called a *Garnir node* of μ if also $(x, y+1) \in \mu$. We define the *Garnir belt* of A to be the set $\mathbf{B}^A$ of nodes of μ containing A and all nodes directly to the right of A, along with the node directly below A and all nodes directly to the left of this node. Explicitly,

$$\mathbf{B}^A = \{(x, z) \in \mu \mid y \leq z \leq \mu_x\} \cup \{(x+1, z) \in \mu \mid 1 \leq z \leq y\}.$$

We define an A-*brick* to be a set of e successive nodes in the same row

$$\{(w, z), (w, z+1), \ldots, (w, z+e-1)\} \subseteq \mathbf{B}^A$$

such that $\mathrm{res}(z, w) = \mathrm{res}(A)$. Let $k \geq 0$ be the number of bricks in $\mathbf{B}^A$. We label the bricks

$$B_1^A, B_2^A, \ldots, B_k^A$$

going from left to right along row x, and then from left to right along row $x+1$. Define C^A to be the set of nodes in row x of $\mathbf{B}^A$ not contained in any A-brick, and D^A to be the set of nodes in row $x+1$ of $\mathbf{B}^A$ not contained in any A-brick. Define f to be the number of A-bricks in row x of $\mathbf{B}^A$.

Let $u, u+1, \ldots, v$ be the values in $\mathbf{B}^A$ in the standard tableau $\mathtt{T}^\mu$. We obtain a new tableau $\mathtt{T}^A$ by placing the numbers $u, u+1 \ldots, v$ from left to right in the following order: D^A, B_1^A, B_2^A, $\ldots$, B_k^A, C^A. The rest of the numbers are placed in the same positions as in T^μ. Define $\boldsymbol{i}^A := \boldsymbol{i}(\mathtt{T}^A)$.

Assume that $k > 0$, and let $n = \mathtt{T}^A(A)$. Define

$$w_r^A = \prod_{z=n+re-e}^{n+re-1} (z, z+e) \in \mathfrak{S}_d \qquad (1 \le r < k).$$

Informally, w_r^A swaps the bricks B_r^A and B_{r+1}^A. The elements $w_1^A, w_2^A, \dots, w_{k-1}^A$ are Coxeter generators of the group

$$\mathfrak{S}^A := \langle w_1^A, w_2^A, \dots, w_{k-1}^A \rangle \cong \mathfrak{S}_k.$$

By convention, if $k = 0$ we define $\mathfrak{S}^A$ to be the trivial group.

Recall that f is the number of A-bricks in row x of $\mathbf{B}^A$. Let $\mathtt{D}^A$ be the set of minimal length left coset representatives of $\mathfrak{S}_f \times \mathfrak{S}_{k-f}$ in $\mathfrak{S}^A \cong \mathfrak{S}_k$. Define the elements of R_α

$$\sigma_r^A := \psi_{w_r^A} e(\boldsymbol{i}^A) \quad \text{and} \quad \tau_r^A := (\sigma_r^A + 1) e(\boldsymbol{i}^A).$$

Let $u \in \mathtt{D}^A$ with reduced expression $u = w_{r_1}^A \dots w_{r_a}^A$. Since every element of $\mathtt{D}^A$ is fully commutative, the element $\tau_u^A := \tau_{r_1}^A \dots \tau_{r_a}^A$ does not depend upon this reduced expression. We now define the *Garnir element* to be

$$g^A := \sum_{u \in \mathtt{D}^A} \tau_u^A \psi^{\mathtt{T}^A} \in R_\alpha.$$

<u>Definition 3.2.1</u>

Let $\alpha \in Q_+$, $d = \mathrm{ht}(\alpha)$, and $\mu \in \mathscr{P}_\alpha$. Define the following left ideals of R_α.

(i) $J_1^\mu = \langle e(\boldsymbol{j}) - \delta_{\boldsymbol{j}, \boldsymbol{i}^\mu} \mid \boldsymbol{j} \in \langle I \rangle_\alpha \rangle$;

(ii) $J_2^\mu = \langle y_r \mid r = 1, \dots, d \rangle$;

(iii) $J_3^\mu = \langle \psi_r \mid$ r and $r+1$ appear in the same row of $\mathtt{T}^\mu \rangle$;

(iv) $J_4^\mu = \langle g^A \mid$ Garnir nodes $A \in \mu \rangle$.

Let $J^\mu = J_1^\mu + J_2^\mu + J_3^\mu + J_4^\mu$ and define the *universal graded Specht module* $S^\mu := R_\alpha / J^\mu \langle \deg(\mathtt{T}^\mu) \rangle$. Define $z^\mu := 1 + J^\mu \in S^\mu$.

Specht Modules Corresponding to Hooks

For the rest of the paper, we assume that $e \geq 3$. In this section we fix two integers $d \geq k \geq 0$. We set $\lambda := (d-k, 1^k)$ and $\alpha := \mathrm{cont}(\lambda)$. We write Sh^λ for the image of $\mathrm{Sh}(d-k-1, k)$ under the embedding $\mathfrak{S}_{d-1} \to \mathfrak{S}_d$ determined by $s_i \mapsto s_{i+1}$. For $\sigma \in \mathrm{Sh}^\lambda$, we define $[\sigma] := \psi_\sigma z^\lambda \in S^\lambda$ of weight $\boldsymbol{i}_\sigma := \sigma \boldsymbol{i}^\lambda$. Note that $\mathrm{Sh}^\lambda = \{\sigma \in \mathfrak{S}_d \mid \sigma \mathtt{T}^\lambda$ is standard$\}$. Therefore, by (20, Corollary 6.24), the set $\{[\sigma] \mid \sigma \in \mathrm{Sh}^\lambda\}$ is a basis of S^λ.

<u>Remark 3.3.1</u>

Recall that in order to define the element $\psi_\sigma \in R_\alpha$ we needed to fix a reduced decomposition for σ. However, every $\sigma \in \mathrm{Sh}^\lambda$ is fully commutative, and so the element ψ_σ is independent of this choice. Thus $[\sigma] \in S^\lambda$ only depends on σ.

<u>Definition 3.3.2</u>

Given $\sigma \in \mathrm{Sh}^\lambda$, the strands in the braid diagram of σ beginning at positions $2, 3, \ldots, d-k$ are called *arm strands*, and we define $\mathrm{Arm}(\sigma) = \{\sigma(2), \ldots, \sigma(d-k)\}$. Similarly, the strands beginning at positions $d-k+1, \ldots, d$ are called *leg strands*, and $\mathrm{Leg}(\sigma) = \{\sigma(d-k+1), \ldots, \sigma(d)\}$

The following theorem tells us how the KLR generators act on the standard basis of S^λ.

Theorem 3.3.3

Let $\sigma \in \mathrm{Sh}^\lambda$ and write $\boldsymbol{i}_\sigma = (i_1, \ldots, i_d)$. Then

1. For $\boldsymbol{j} \in \langle I \rangle_\alpha$, $e(\boldsymbol{j})[\sigma] = \delta_{\boldsymbol{i}_\sigma, \boldsymbol{j}}[\sigma]$.

2. For $1 \leq r \leq d$,

$$y_r[\sigma] = \begin{cases} -[s_r\sigma], & \text{if } i_r = i_{r+1},\ r \in \mathrm{Leg}(\sigma), \text{ and } r+1 \in \mathrm{Arm}(\sigma) \\ [s_{r-1}\sigma], & \text{if } i_{r-1} = i_r,\ r-1 \in \mathrm{Leg}(\sigma), \text{ and } r \in \mathrm{Arm}(\sigma) \\ 0, & \text{otherwise.} \end{cases}$$

3. We have $\psi_1[\sigma] = 0$, and for $2 \leq r \leq d-1$,

$$\psi_r[\sigma] = \begin{cases} [s_r\sigma], & \text{if } r \in \mathrm{Arm}(\sigma) \text{ and } r+1 \in \mathrm{Leg}(\sigma), \text{ or if } i_r \not- i_{r+1} \\ [s_{r+1}s_r\sigma], & \text{if } r \in \mathrm{Leg}(\sigma),\ r+1, r+2 \in \mathrm{Arm}(\sigma), \text{ and } i_r \to i_{r+1} \\ -[s_rs_{r+1}\sigma], & \text{if } r-1, r \in \mathrm{Leg}(\sigma),\ r+1 \in \mathrm{Arm}(\sigma), \text{ and } i_r \leftarrow i_{r+1} \\ [s_rs_{r-1}\sigma], & \text{if } r-1 \in \mathrm{Leg}(\sigma),\ r, r+1 \in \mathrm{Arm}(\sigma), \text{ and } i_r = i_{r-1} \\ -[s_{r-1}s_r\sigma], & \text{if } r, r+1 \in \mathrm{Leg}(\sigma),\ r+2 \in \mathrm{Arm}(\sigma), \text{ and } i_{r+1} = i_{r+2} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) Immediate from the definitions.

(ii) We define $\mathrm{Sh}^\lambda(\boldsymbol{i}) = \{\tau \in \mathrm{Sh}^\lambda \mid \mathrm{wt}([\tau]) = \boldsymbol{i}\}$. By Proposition 3.1.4 there is a unique element $\tau \in \mathrm{Sh}^\lambda(\boldsymbol{i}_\sigma)$ of minimal length, and $\sigma = h\tau$ for some $h \in H(\boldsymbol{i}_\sigma)$. Choosing any reduced decomposition $h = s_{r_1} \ldots s_{r_a}$, it follows that $[\sigma] = \psi_{r_1} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}}]$. Moreover $\deg([\tau]) - \deg([\sigma]) = 2a$, and so $[\tau]$ is the unique vector of weight $\boldsymbol{i}_\sigma$ with largest degree. In particular, $y_s[\tau] = 0$ for $1 \le s \le d$. Since $H(\boldsymbol{i}_\sigma)$ is commutative, at most one of $r-1$ and r is an element of $\{r_1, \ldots, r_a\}$.

Suppose that $r \in \{r_1, \ldots, r_a\}$. This is equivalent to $i_r = i_{r+1}$, $r \in \mathrm{Leg}(\sigma)$ and $r+1 \in \mathrm{Arm}(\sigma)$. Since the ψ_{r_i} commute with each other, we may assume that $r = r_1$. We compute

$$\begin{aligned} y_r[\sigma] &= y_r \psi_r \psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}}] \\ &= (\psi_r y_{r+1} - 1)\psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}}] \\ &= \psi_r \psi_{r_2} \ldots \psi_{r_a} y_{r+1}[\sigma_{\boldsymbol{i}}] - \psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}}] \\ &= -\psi_{r_2} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}}] = -[s_r\sigma]. \end{aligned}$$

Similarly, $r-1 \in \{r_1, \ldots, r_a\}$ is equivalent to $i_{r-1} = i_r$, $r-1 \in \mathrm{Leg}(\sigma)$, and $r \in \mathrm{Arm}(\sigma)$. An argument similar to the above shows that $y_r[\sigma] = [s_{r-1}\sigma]$. If neither r nor $r-1$ is among $\{r_1, \ldots, r_a\}$, then y_r commutes with each ψ_{s_r}, and thus $y_r \psi_{s_{r_1}} \ldots \psi_{r_a}[\sigma_{\boldsymbol{i}}] = \psi_{r_1} \ldots \psi_{r_a} y_r[\tau] = 0$.

(iii) Observe that $\mathrm{wt}([\sigma])$ begins with either $(0, 1, \ldots)$ or $(0, e-1, \ldots)$. By weight, we must have $\psi_1[\sigma] = 0$. We break down the rest of the proof into many cases.

Case 1: Assume $r \in \mathrm{Arm}(\sigma)$ and $r+1 \in \mathrm{Leg}(\sigma)$. Clearly $\psi_r[\sigma] = [s_r\sigma]$.

Case 2: Assume $r \in \mathrm{Leg}(\sigma)$, $r+1 \in \mathrm{Arm}(\sigma)$. Then $[\sigma] = \psi_r[s_r\sigma]$, and so $\psi_r[\sigma] = \psi_r^2[s_r\sigma]$. If $i_r \not\to i_{r+1}$ then $\psi_r^2[s_r\sigma] = [s_r\sigma]$, and if $i_r = i_{r+1}$ then $\psi_r^2[s_r\sigma] = 0$.

If $i_r \to i_{r+1}$, then $\psi_r^2[s_r\sigma] = (y_r - y_{r+1})[s_r\sigma]$. Since $r \in \mathrm{Arm}(s_r\sigma)$, part (ii) of this theorem shows that if $y_r[s_r\sigma] \neq 0$ then $r-1 \in \mathrm{Leg}(s_r\sigma)$ and $i_{r-1} = i_{r+1}$. This is seen to be impossible by considering weights, and so $y_r[s_r\sigma] = 0$. Similarly, $r+1 \in \mathrm{Leg}(s_r\sigma)$, so part (ii) says that $y_{r+1}[s_r\sigma] = 0$ unless $r+2 \in \mathrm{Arm}(s_r\sigma)$

and $i_r = i_{r+2}$. But $r+2 \in \mathrm{Arm}(s_r\sigma)$ if and only if $r+2 \in \mathrm{Arm}(\sigma)$, in which case $i_{r+2} = i_{r+1} + 1 = i_r$. We then conclude that $y_{r+1}[s_r\sigma] = -[s_{r+1}s_r\sigma]$. To summarize, if $i_r \to i_{r+1}$, then

$$\psi_r[\sigma] = \begin{cases} [s_{r+1}s_r\sigma], & \text{if } r+2 \in \mathrm{Arm}(\sigma) \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose $i_r \leftarrow i_{r+1}$. An argument similar to the one above shows that

$$\psi_r[\sigma] = \begin{cases} -[s_r s_{r+1}\sigma], & \text{if } r-1 \in \mathrm{Leg}(\sigma) \\ 0, & \text{otherwise.} \end{cases}$$

Case 3: Assume $r, r+1 \in \mathrm{Arm}(\sigma)$. We prove the following statement by induction on t: If $\tau \in \mathrm{Sh}^\lambda$ satisfies $t, t+1 \in \mathrm{Arm}(\tau)$ we have

$$\psi_t[\tau] = \begin{cases} [s_t s_{t+1}\sigma], & \text{if } t-1 \in \mathrm{Leg}(\tau) \text{ and } i_{t-1} = i_t \\ 0, & \text{otherwise.} \end{cases}$$

The base case of $t = 1$ has been shown above.

We proceed with the induction, fixing $t > 1$, $\tau \in \mathrm{Sh}^\lambda$ with $t, t+1 \in \mathrm{Arm}(\tau)$ and assuming the claim holds for all smaller values of t. Let u be the largest value of $\mathrm{Leg}(\tau)$ which is less than t. If no such value exists, then $\psi_t[\tau] = \psi_t\psi_\tau z^\lambda = \psi_\tau\psi_t z^\lambda = 0$. Otherwise, there is a reduced expression for τ beginning with

$s_u \dots s_{t-1}s_t$. Write $\tau' = s_t s_{t-1} \dots s_u \tau$. Then

$$\begin{aligned}\psi_t[\tau] &= \psi_t(\psi_u \dots \psi_{t-2})\psi_{t-1}\psi_t[\tau'] = (\psi_u \dots \psi_{t-2})\psi_t\psi_{t-1}\psi_t[\tau'] \\ &= (\psi_u \dots \psi_{t-2})(\psi_{t-1}\psi_t\psi_{t-1} + \delta_{i_t,i_u})[\tau'] \\ &= (\psi_u \dots \psi_t)\psi_{t-1}[\tau'] + \delta_{i_t,i_u}(\psi_u \dots \psi_{t-2})[\tau'].\end{aligned}$$

Assume that $t-1 \in \mathrm{Arm}(\tau)$. Then the first term is zero by induction. Furthermore, the second term is zero unless $i_t = i_u$ and (by induction) $u = t-2$, $t-3 \in \mathrm{Leg}(\tau)$, and $i_{t-3} = i_{t-1}$. But since $i_{t-1} = i_t - 1$ and $i_{t-3} = i_u + 1$, this is impossible. Therefore the second term is zero as well.

Now suppose that $u = t-1$. We can see that the first term is zero using an argument similar to the above. If the second term appears, it is visibly nonzero. We have thus proved the induction step.

Applying this to case 3, we have

$$\psi_r[\sigma] = \begin{cases} [s_r s_{r-1}\sigma], & \text{if } r-1 \in \mathrm{Leg}(\sigma), \text{ and } i_r = i_{r-1} \\ 0, & \text{otherwise.} \end{cases}$$

Case 4: Assume $r, r+1 \in \mathrm{Leg}(\sigma)$. This situation is entirely analogous to case 3, except now induction runs backwards from $t = d-1$ to $t = 2$. We find that

$$\psi_r[\sigma] = \begin{cases} -[s_{r-1}s_r\sigma], & \text{if } r+2 \in \mathrm{Arm}(\sigma), \text{ and } i_{r+1} = i_{r+2} \\ 0, & \text{otherwise.} \end{cases}$$

□

Corollary 3.3.4

Suppose that $e(\boldsymbol{i})S^\lambda \neq 0$. Then

$$\{v \in e(\boldsymbol{i})S^\lambda \mid y_r v = 0 \ \text{ for } r = 1, \ldots, d\} = \mathcal{O}[\sigma_{\boldsymbol{i}}].$$

Proof. By Theorem 3.3.3, we have $y_r[\sigma_{\boldsymbol{i}}] = 0$ for $r = 1, \ldots, d$. Suppose given

$$v = \sum_{\sigma \in \mathrm{Sh}^\lambda} c_\sigma[\sigma] \in e(\boldsymbol{i})S^\lambda,$$

and let $h = s_{r_1} \ldots s_{r_m} \in H(\boldsymbol{i})$ be a reduced expression for an element of maximal length for which $c_{h\sigma_{\boldsymbol{i}}} \neq 0$. Apply Theorem 3.3.3 to obtain $y_{r_1} \ldots y_{r_m} v = (-1)^m c_{h\sigma_{\boldsymbol{i}}}[\sigma_{\boldsymbol{i}}] \neq 0$. Therefore v is killed by every y_r if and only if $v \in \mathcal{O}[\sigma_{\boldsymbol{i}}]$. □

Homomorphisms into S^λ

We again fix two integers $d > k \geq 0$, $\lambda := (d - k, 1^k)$ and $\alpha := \mathrm{cont}(\lambda)$. We also fix $\mu = (\mu_1, \ldots, \mu_N)$ a partition of d. The goal of this section is to determine $\mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda)$, and the answer is given in Theorem 3.4.1. In order for a nonzero homomorphism to exist $\boldsymbol{i}^\mu$ must be a weight of S^λ. We assume this, so in particular we are able to define $\sigma_\mu := \sigma_{\boldsymbol{i}^\mu}$. This determines a mapping from μ to λ in which the node containing r in $\mathtt{T}^\mu$ is sent to the node containing r in $\sigma_\mu \mathtt{T}^\lambda$. We define *arm nodes* and *leg nodes* of μ to be the nodes which under this map are sent to the arm or the leg of λ, respectively.

Theorem 3.4.1

There is a homomorphism $S^\mu \to S^\lambda$ satisfying $z^\mu \mapsto v$ for $v \neq 0$ if and only if

1. there exist $c \in \{1, \ldots, k+1\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^c$, and $0 \leq m < e$ such that $v \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Gc}(\boldsymbol{a}))[\sigma_\mu]$ and

$$\mu = (a_1 e, \ldots, a_{c-1} e, a_c e - m, 1^{k-c+1}),$$

2. e divides d, there exist $c \in \{1, \ldots, k\}$, $\boldsymbol{a} \in (\mathbb{Z}_{>0})^c$, and $0 \leq m < e$ such that $v \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Gc}(\boldsymbol{a}))[\sigma_\mu]$ and

$$\mu = (a_1 e, \ldots, a_{c-1} e, a_c e - m, 1^{k-c+2}), \text{ or}$$

3. $N > k+1$, there exist $\boldsymbol{a} \in (\mathbb{Z}_{>0})^N$ and $0 \leq m < e$ such that $v \in \mathrm{Ann}_{\mathcal{O}}(\mathrm{Gc}(\boldsymbol{a}))[\sigma_\mu]$ and

$$\mu = (a_1 e, \ldots, a_k e, a_{k+1} e - 1, \ldots, a_{N-1} e - 1, a_N e - 1 - m).$$

The proof of this will follow from several lemmas. Recall the ideals J_i^μ of Definition 3.2.1.

Lemma 3.4.2

A nonzero $v \in S^\lambda$ satisfies $(J_1^\mu + J_2^\mu + J_3^\mu)v = 0$ if and only if $v \in \mathcal{O}[\sigma_\mu]$, and all leg nodes of μ are in the first column of its Young diagram.

Proof. Assume that $(J_1^\mu + J_2^\mu + J_3^\mu)v = 0$. Corollary 3.3.4 implies that $v = \gamma[\sigma_\mu]$ for some $\gamma \in \mathcal{O}$. By considering weights we see that $(1,2)$ cannot be a leg node. Suppose that $(a,b) \neq (1,2)$ is a leg node of μ with $b \geq 2$, and define $r = \mathtt{T}^\mu(a,b)$. Then since $J_3^\mu v = 0$ we must have $\psi_{r-1} v = 0$. Furthermore $(a, b-1)$ must be an arm node, for in $\boldsymbol{i}^\mu = (i_1, \ldots, i_d)$ we have $i_r = i_{r-1} + 1$, whereas if they were both leg nodes we would have $i_r = i_{r-1} - 1$. By Theorem 3.3.3 we have $\psi_{r-1}\gamma[\sigma_\mu] = \gamma[s_{r-1}\sigma_\mu] \neq 0$, a contradiction.

Conversely, assume that $v = \gamma[\sigma_\mu]$ and all leg nodes of μ are in the first column. Suppose that r and $r+1$ are in the same row of $\mathtt{T}^\mu$, and so in particular $r+1$ is an arm node. If the node of μ containing r in $\mathtt{T}^\mu$ is a leg node then by considering Theorem 3.3.3 we see that $\psi_r v = 0$ unless the node containing $r-1$ is also a leg node. But then the nodes containing $r-1$ and r are both in the first column of $\mathtt{T}^\mu$ forcing $r+1$ (and every subsequent node) to also be in the first column. This contradicts the assumption that r and $r+1$ appear in the same row of T^μ, and so necessarily $\psi_r v = 0$.

If the node of μ containing r in $\mathtt{T}^\mu$ is an arm node, then write $\boldsymbol{i}^\mu = (i_1, \ldots, i_d)$. Referring again to Theorem 3.3.3 we see that $\psi_r v = 0$ unless the node containing $r-1$ is a leg node, and furthermore $i_{r-1} = i_r$. But if the the node containing $r-1$ is a leg node, then by assumption it is in the first column. Once again the assumption that r and $r+1$ are in the same row of T^μ forces the node containing $r-1$ to be directly to the left of the node containing r. But in this case $i_{r-1} = i_r - 1$ and once again we see that $\psi_r v = 0$. □

Proposition 3.4.3

If there is a nonzero element $v \in S^\lambda$ with $J^\mu v = 0$, then the leg nodes of μ are precisely $(2,1), (3,1), \ldots, (k+1,1)$.

Proof. Let $v = \gamma[\sigma_\mu]$ be such an element. By Lemma 3.4.2, the leg nodes appear in the first column of μ. Suppose that $(a,1)$ is a leg node. Then $a \neq 1$, and so we may consider the Garnir node $A = (a-1,1)$. Define $r = \mathtt{T}^\mu(a-1,1)$ and $s = \mathtt{T}^\mu(a,1)$. It is clear that $g^A = \psi^{T^A} = \psi_{(r,r+1,\ldots,s)}$ where $(r, r+1, \ldots, s)$ is a cycle of the entire Garnir belt. In particular, if $a \geq 3$ and $(a-1, 1)$ is not a leg node, then $g^A[\sigma_\mu] = [(r, r+1, \ldots, s)\sigma_\mu]$. But this is a nonzero basis element by Lemma 3.4.2, contradicting $J_4^\mu v = 0$. This contradiction shows that for every leg node $(a, 1)$ with $a \geq 3$, the node $(a-1, 1)$ is also a leg node, which is equivalent to the statement of the proposition. $\square$

As a consequence of this proposition, we see that the leg nodes of μ are mapped to exactly the same nodes of λ, and therefore that $\sigma_\mu \mathtt{T}^\lambda$ is the tableau obtained by moving all of the arm nodes of $\mathtt{T}^\mu$ to the first row. This suggests the following definition.

<u>Definition 3.4.4</u>

Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition of d with $N \geq k+1$. Define $\mathtt{T}_\mu^\lambda$ to be the λ-tableau obtained as follows. Let $S = \{\mathtt{T}^\mu(A) \mid A \in \mu \setminus \lambda\}$. We fill in the rows of λ from left to right and from top to bottom using the following procedure. Let $A \in \lambda$ and suppose we have filled in all of the nodes to the left of A or in a previous row. If $A \in \mu$ then define $\mathtt{T}_\mu^\lambda(A) = \mathtt{T}^\mu(A)$. Otherwise define $\mathtt{T}_\mu^\lambda(A)$ to be the smallest element of S, and then delete this value from S. Repeat this process until all nodes have been filled. Furthermore, define $\sigma_\mu^\lambda := w^{\mathtt{T}_\mu^\lambda} \in \mathfrak{S}_n$.

If we have a nonzero homomorphism $S^\mu \to S^\lambda$, then Corollary 3.3.4 tells us it must send z^μ to a multiple of $[\sigma_\mu]$. On the other hand, Proposition 3.4.3 requires the image of z^μ to be a multiple of $[\sigma_\mu^\lambda]$. Since $[\sigma_\mu]$ and $[\sigma_\mu^\lambda]$ are elements of an $\mathcal{O}$-basis of S^λ, a necessary condition for a homomorphism is that $[\sigma_\mu] = [\sigma_\mu^\lambda]$. The next two results tell us when this happens.

Lemma 3.4.5

Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition of d such that $N \geq k+1$. Then $[\sigma_\mu^\lambda]$ has weight $\boldsymbol{i}^\mu$ if and only if μ is of one of the following three forms.

1. $N = k+1$, and there exist $c \in \{1, \ldots, k+1\}$, $a_1, \ldots, a_c \in \mathbb{Z}_{>0}$ and $0 \leq m < e$ such that $\mu = (a_1 e, \ldots, a_{c-1}e, a_c e - m, 1^{k-c+1})$.
2. $N > k+1$ and $\mu_{k+1} = 1$, and there exist $c \in \{1, \ldots, k\}$, $a_1, \ldots, a_c \in \mathbb{Z}_{>0}$ and $0 \leq m < e$ satisfying $m + c \equiv k+2 \pmod e$ such that $\mu = (a_1 e, \ldots, a_{c-1}e, a_c e - m, 1^{k-c+1}, 1)$.
3. $N > k+1$ and $\mu_{k+1} > 1$, and there exist $a_1, \ldots, a_N \in \mathbb{Z}_{>0}$ and $0 \leq m < e$ such that $\mu = (a_1 e, \ldots, a_k e, a_{k+1}e - 1, \ldots, a_{N-1}e - 1, a_N e - 1 - m)$.

Proof.

Define $(i_1, \ldots, i_d) := \mathrm{wt}([\sigma_\mu^\lambda])$ and $(j_1, \ldots, j_d) := \boldsymbol{i}^\mu$. Define $r_x = \mathtt{T}^\mu(x+1, 1)$ for $x = 1, 2, \ldots, k$. Since $r_x = \mathtt{T}^\mu(x+1,1) = \mathtt{T}_\mu^\lambda(x+1,1)$, it is immediate that $i_{r_x} = j_{r_x}$ for $x = 1, \ldots, k$. Letting $S = \{1, 2, \ldots, d\} \setminus \{r_1, \ldots, r_k\}$, we therefore have that $\mathrm{wt}([\sigma_\mu^\lambda]) = \boldsymbol{i}^\mu$ if and only if $i_s = j_s$ for all $s \in S$.

For $p = 1, 2, \ldots, n-k$, let s_p be the p^{th} element of the set S (under the usual ordering). It is clear that $s_p = \mathtt{T}_\mu^\lambda(1, p)$. In particular, we see that $i_{s_{p+1}} = i_{s_p} + 1$ for $p = 1, \ldots, d-k-1$. Since $j_{s_1} = i_{s_1}$, we have $\mathrm{wt}([\sigma_\mu^\lambda]) = \boldsymbol{i}^\mu$ if and only if $j_{s_{p+1}} = j_{s_p} + 1$ for $p = 1, \ldots, d-k-1$.

Set $r_0 = 0$. Note that the condition $r_1 \neq r_0 + 1$ is automatically satisfied, so $x_0 = \max\{x \mid r_x \neq r_{x-1}+1 \text{ where } 1 \leq x \leq k\}$ is well defined. We divide S as follows

$$\begin{aligned}(s_1, s_2, \ldots, s_{n-k}) = (&1,2, \ldots, r_1 - 1,\\ &r_1 + 1, \ldots, r_2 - 1,\\ &\ldots,\\ &r_{x_0-1} + 1, \ldots, r_{x_0} - 1,\\ &r_k + 1, \ldots, d).\end{aligned}$$

Now, $(r_{x-1} + 1, \ldots, r_x - 1) = (\mathtt{T}^\mu(x, 2), \ldots, \mathtt{T}^\mu(x, \mu_x))$ for $x \leq x_0$, so it is clear that $j_{p+1} = j_p + 1$ when $r_{x-1} + 1 \leq p \leq r_x - 2$. Furthermore, for $x < x_0$ we have $r_x - 1 = \mathtt{T}^\mu(x, \mu_x)$ while $r_x + 1 = \mathtt{T}^\mu(x + 1, 2)$. If we let $i = \mathrm{res}(x, 1)$, then $j_{r_x-1} = \mathrm{res}(x, \mu_x) \equiv i + \mu_x - 1 \pmod e$, and $j_{r_x+1} = \mathrm{res}(x + 1, 2) = i$. Thus $j_{r_x+1} = j_{r_x-1} + 1$ if and only if $\mu_x = a_x e$ for some $a_x \in \mathbb{Z}_{>0}$.

At this point we have shown that $i_p = j_p$ for $p = 1, \ldots, r_k$ if and only if for every $x < x_0$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e$.

Suppose that $N = k + 1$. If $r_k = d$, then we set $c = x_0$ choose a_c, m so that $\mu_c = a_c e - m$. This clearly is of the form in part (i) of the Lemma. Otherwise, $x_0 = k$ and $r_k + 1, \ldots, d$ are all on row $k + 1$ of $\mathtt{T}^\mu$. The same analysis as above shows that $[\sigma^\lambda_\mu]$ has weight $\boldsymbol{i}^\mu$ if and only if for every $x \leq k$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e$. In this case we set $c = k + 1$ and choose a_c, m as required. We have established part (i) of the Lemma.

Suppose that $N > k + 1$ and $\mu_{k+1} = 1$. Set $c = x_0$ choose a_c, m so that $\mu_c = a_c e - m$. Then $(r_k + 1, \ldots, d) = (\mathtt{T}^\mu(k + 2, 1), \ldots, \mathtt{T}^\mu(N, 1))$. Let $i = \mathrm{res}(c, 1)$. Then $j_{r_c-1} = \mathrm{res}(c, \mu_c) \equiv i + \mu_c - 1 \pmod e$, and $j_{r_k+1} \equiv i + k + 1 - c \pmod e$.

Thus $j_{r_k+1} = j_{r_{x_0}-1} + 1$ if and only if $\mu_c \equiv k+1-c \pmod{e}$. But $\mu_c = a_c e - m$, so this is equivalent to $c + m \equiv k+1 \pmod{e}$. This is exactly the requirement in part (ii) of the Lemma. Finally, suppose that $N \geq k+3$. Then $r_k + 1 = \mathtt{T}^\mu(k+2, 1)$ and $r_k + 2 = \mathtt{T}^\mu(k+3, 1)$. In this case we see that $j_{r_k+2} = j_{r_k+1} - 1 \neq j_{r_k+1} + 1$. Thus part (ii) of the Lemma is finished.

Finally, suppose that $N > k+1$ and $\mu_{k+1} > 1$. Then $x_0 = k$. Once again the analysis of the $N = k+1$ case shows that $i_p = j_p$ for $p = 1, \ldots, r_k + 1$ if and only if for every $x \leq k$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e$. Furthermore, a similar analysis shows that $i_p = j_p$ for $p = 1, \ldots, d$ if and only if additionally for every $k+1 \leq x < N$ there exist $a_x \in \mathbb{Z}_{>0}$ such that $\mu_x = a_x e - 1$. This is exactly the situation of part (iii). $\square$

Corollary 3.4.6

If μ is as in Lemma 3.4.5, then $(J_2^\mu + J_3^\mu)[\sigma_\mu^\lambda] = 0$. In particular, $[\sigma_\mu^\lambda] = [\sigma_\mu]$.

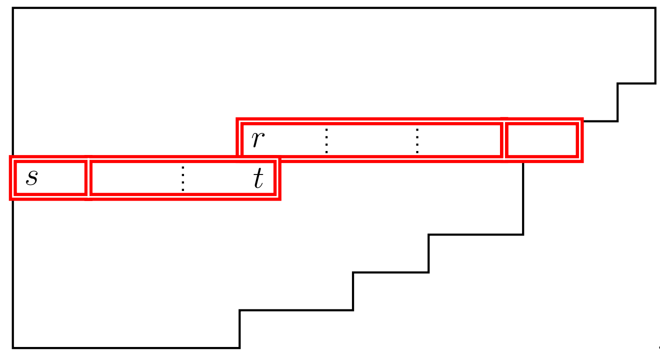
Proof. Let $(i_1, \ldots, i_d) = \mathrm{wt}([\sigma_\mu^\lambda])$. It is clear that in each of the cases above if $i_r = i_{r+1}$ then $r \in \mathrm{Arm}(\sigma_\mu^\lambda)$ and $r+1 \in \mathrm{Leg}(\sigma_\mu^\lambda)$. This implies that $J_2^\mu[\sigma_\mu^\lambda] = 0$. In turn, this implies that $[\sigma_\mu^\lambda] = [\sigma_\mu]$, by Corollary 3.3.4. Lemma 3.4.2 now shows us that $J_3^\mu[\sigma_\mu^\lambda] = 0$. $\square$

As expected, the Garnir relations are the most difficult to verify.

Lemma 3.4.7

Suppose that μ has one of the forms in Lemma 3.4.5 and let $A = (x, y) \in \mu$ be a Garnir node. Define r, s, and t to be the values in $\mathtt{T}^\mu$ of the nodes $(x, y), (x+1, 1)$, and $(x+1, y)$ respectively so that the

Garnir belt is as in the following picture

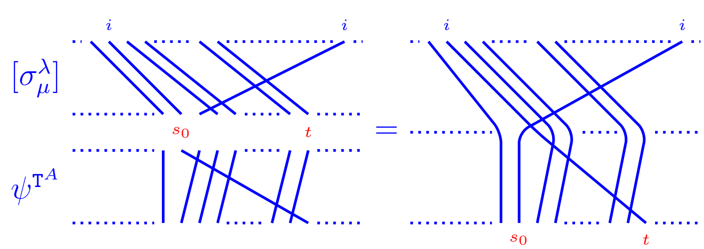


.

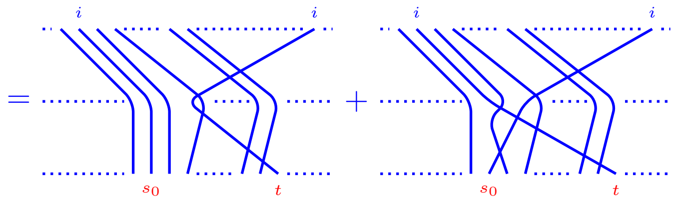
Then

$$\psi^{\mathtt{T}^A}[\sigma^\lambda_\mu] = \begin{cases} [(t+1,t,\dots,s)\sigma^\lambda_\mu], & \text{if } x \le k,\ y \equiv 0 \pmod{e}, \text{ and } y < \mu_{x+1} \\ [(r,r+1,\dots,s)\sigma^\lambda_\mu], & \text{if } x \le k,\ y \equiv 1 \pmod{e}, \text{ and } y > 1 \\ [\sigma^\lambda_\mu], & \text{if } x > k \text{ and } y \equiv 0 \pmod{e} \\ 0, & \text{otherwise} \end{cases}$$

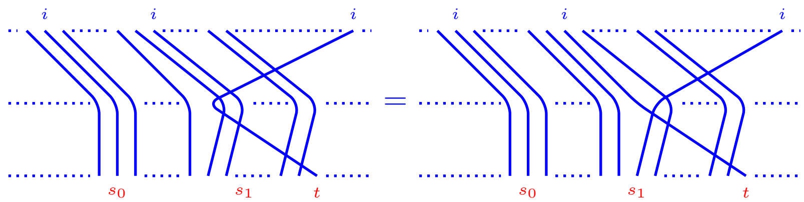
Proof. We begin with the case $x \le k$ and $y \equiv 0 \pmod{e}$. Define $s_p = s + pe$ for $p \in \mathbb{N}$. Using braid diagrams, we compute $\psi^{\mathtt{T}^A}[\sigma^\lambda_\mu]$.



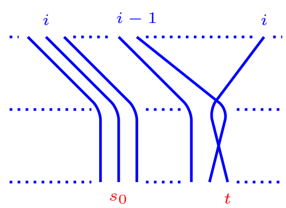
(3.11)



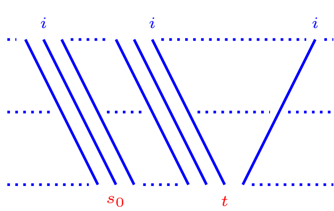
The second term is zero by Theorem 3.3.3(iii), and the first is equal to



using the defining relations of R_α and Theorem 3.3.3(ii). We note that this is of the same form as (3.11) above. An easy induction argument allows us to conclude that this is equal to

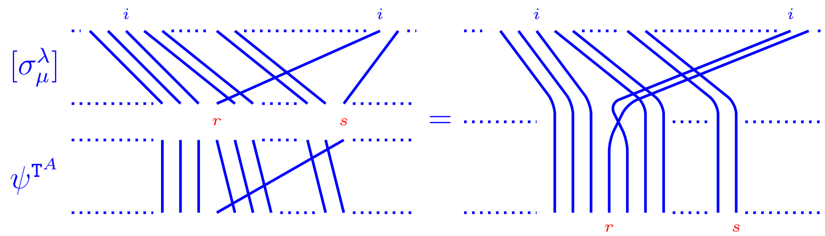


Given our assumptions in this case, it is easy to see that the condition $t+1 \in \mathrm{Arm}(\sigma_\mu^\lambda)$ is equivalent to $y < \mu_{x+1}$. If this holds, then Theorem 3.3.3 says the above element is equal to

 $= [(t+1, t, \ldots, s)\sigma_\mu^\lambda].$

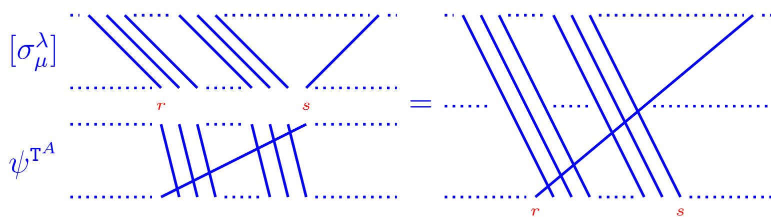
as claimed. Otherwise, we get zero.

We next consider the case that $A = (x, y)$ with $1 \le x \le k$ and $y = 1$. If $x = 1$, then $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = 0$ by the Garnir relation for the node $(1,1) \in \lambda$. Otherwise A is a leg node, and so is $(x+1,1)$. Therefore, $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ takes the following form.



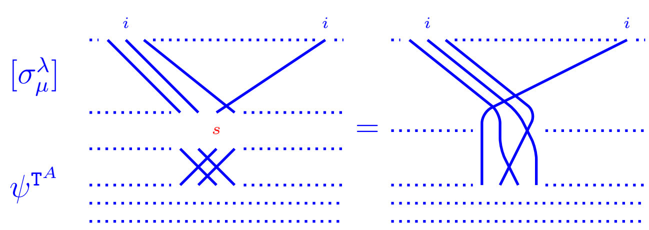
which is zero by Theorem 3.3.3(iii).

Suppose now that $A=(x,y)$ with $1 \le x \le k$, $y \equiv 1 \pmod e$, and $y \ne 1$. Here, $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ takes the following form.



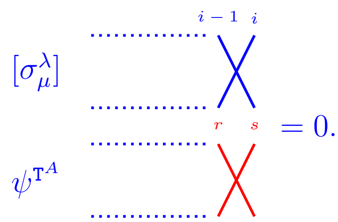
which is $[(r, r+1, \dots, s)\sigma_\mu^\lambda]$.

Consider now the case $A=(x,y) \in \mu$ is a Garnir node with $1 \le x \le k$, and $y \not\equiv 0, 1 \pmod e$. If μ is as in Lemma 3.4.5(i) or (ii) then the requirement that A is Garnir and $y \not\equiv 1 \pmod e$ together imply that $x \le c$. Therefore $\mu_x = a_x e$, and it is easy to deduce from this that C^A and D^A each contain at least two nodes. The portion of $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ showing the crossing between C^A and D^A is depicted in the following picture

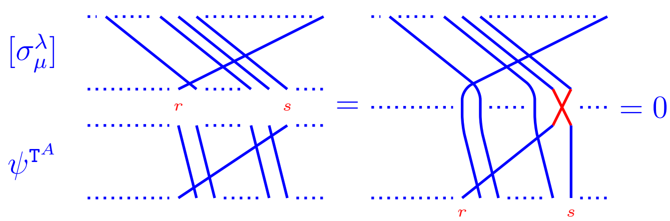


which is zero by Theorem 3.3.3(iii).

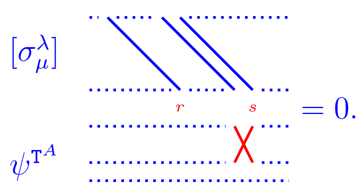
We next check the special case that $A = (k+1, 1)$. If μ is as in Lemma 3.4.5(ii), then the picture we have is as follows.



If μ is of the third type, then the picture is as follows



Suppose now that $A = (x, y)$ with $x > k$ and $y \not\equiv 0 \pmod{e}$. Furthermore assume that $A \neq (k+1, 1)$. We see that C^A is nonempty, and so $\psi^{\mathtt{T}^A}[\sigma^\lambda_\mu]$ is as in the following picture



Finally, if we choose $A = (x, y)$ with $x > k$ and $y \equiv 0 \pmod{e}$ one may easily see that $\mathtt{T}^A = \mathtt{T}^\mu$. Clearly then $\psi^{\mathtt{T}^A} = 1$, and so $\psi^{\mathtt{T}^A}[\sigma^\lambda_\mu] = [\sigma^\lambda_\mu]$. $\square$

Lemma 3.4.8

Suppose that μ is as in Lemma 3.4.5 and let $A \in \mu$ be a Garnir node. Let r, s, t be as in Lemma 3.4.7. Then for $1 \le f \le a_{x+1} - 1$ we have

$$g^A[\sigma_\mu^\lambda] = \begin{cases} \binom{a_x}{f}[(t+1, t, \dots, s)\sigma_\mu^\lambda], & \text{if } A = (x, fe) \text{ with } x \le k \\ \binom{a_x}{f}[(r, r+1, \dots, s)\sigma_\mu^\lambda], & \text{if } A = (x, fe+1) \text{ with } x \le k \\ \binom{a_x}{f}[\sigma_\mu^\lambda], & \text{if } A = (x, fe) \text{ with } x > k \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = 0$ there is nothing to show, so assume otherwise.

Recall the definitions of σ_p^A and τ_p^A from Section 3.2. In each of the three cases in Lemma 3.4.7 for which $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] \neq 0$, one sees that multiplying σ_p^A with $\psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ has the effect of shuffling e arm strands past e arm strands. By Theorem 3.3.3, we have that $\sigma_p^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = 0$. Thus $\tau_p^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ for all p. This furthermore implies that $\tau_w^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda]$ for all $w \in \mathfrak{S}^A$. Therefore

$$g^A[\sigma_\mu^\lambda] = \sum_{u \in \mathtt{D}^A} \tau_u^A \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \sum_{u \in \mathtt{D}^A} \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda] = \left|\mathtt{D}^A\right| \psi^{\mathtt{T}^A}[\sigma_\mu^\lambda].$$

The problem is reduced to calculating $\left|\mathtt{D}^A\right|$ in each of the three cases above, which follows immediately from the definitions. □

Proof of Theorem 3.4.1 By the discussion preceding Lemma 3.4.5, any homomorphism $S^\mu \to S^\lambda$ is of the form $z^\mu \mapsto \gamma[\sigma_\mu^\lambda]$ for some $\gamma \in \mathcal{O}$. By the discussion preceding Lemma 3.4.5 if there is a nonzero homomorphism, then $[\sigma_\mu] = [\sigma_\mu^\lambda]$. Lemma 3.4.5 tells us for which partitions this happens.

Suppose now that μ is as in Lemma 3.4.5. Corollary 3.4.6 ensures that $(J_1^\mu + J_2^\mu + J_3^\mu)\gamma[\sigma_\mu^\lambda] = 0$. By Lemma 3.4.8, $J_4^\mu\gamma[\sigma_\mu^\lambda] = 0$ if and only if for each $x = 1, \ldots, N-1$, we have $\binom{a_x}{f}\gamma = 0$ for $f = 1, \ldots, a_{x+1} - 1$. But $\mathrm{Gc}(\boldsymbol{a})$ is the greatest common divisor of these binomial coefficients. Therefore $\gamma[\sigma_\mu^\lambda]$ satisfies the Garnir relations if and only if $\mathrm{Gc}(\boldsymbol{a})\gamma = 0$. This is exactly the claim of the theorem.

Examples

We now present some examples of Theorem 3.4.1. We will restrict our attention to the case that $\mathcal{O}$ is a torsion-free algebra over either $\mathbb{Z}$ or $\mathcal{F}_p$ for some prime $p \geq 3$. In this case for any $r \in \mathbb{Z}$ the submodule $\mathrm{Ann}_{\mathcal{O}}(r)$ is either (0) or $\mathcal{O}$, and so in Theorem 3.4.1 all of our homomorphism spaces will be free $\mathcal{O}$-modules. In particular, we can talk about their dimensions.

In characteristic 0 the situation simplifies. The following corollary covers in particular Hecke algebras over $\mathbb{C}$ at a root of unity.

Corollary 3.5.1

Let μ be an arbitrary partition, and $\lambda = (d-k, 1^k)$. Then $\mathrm{Hom}(S^\mu, S^\lambda)$ is at most one-dimensional, and $\dim \mathrm{Hom}(S^\mu, S^\lambda) = 1$ if and only if one of the following holds:

1. $\mu = \lambda$;
2. e divides d and $\mu = (d-k-1, 1^{k+1})$;
3. $k = 0$ and there exist $n \geq 0$, $a > 0$, and $0 \leq m < e$ such that

$$\mu = (ae-1, (e-1)^n, m);$$

4. $k \geq 1$, there exist $0 \leq n < k$, $a \in \mathbb{Z}_{>0}$, and $1 \leq m \leq e$ such that

$$\mu = (ae, e^n, m, 1^{k-n-1});$$

5. $k \geq 1$, there exist $n > k+1$, $a \in \mathbb{Z}_{>0}$ and $1 \leq m < e$ such that

$$\mu = (ae, e^{k-1}, (e-1)^{n-k-1}, m);$$

6. $k \geq 2$, e divides d, there exist $0 \leq n \leq k-2$, $a \in \mathbb{Z}_{>0}$, and $1 \leq m \leq e$ such that

$$\mu = (ae, e^n, m, 1^{k-n}).$$

For the rest of the section we assume that $\mathcal{O}$ is an $\mathcal{F}_p$-algebra, with $p \geq 3$, and we furthermore assume that $e = p$. This applies in the case of symmetric groups in positive characteristic.

The trivial module

Let $\lambda = (d)$. The module S^λ is referred to as the *trivial* module. It is one-dimensional over $\mathcal{O}$ having basis z^λ. We recover the following graded version of the result of James (11, Theorem 24.4).

Corollary 3.5.2

Let $\mu = (\mu_1, \ldots, \mu_N)$ be a partition of d. Then $\mathrm{Hom}_{R_\alpha}(S^\mu, S^\lambda) = 0$ unless there exist $a_1, \ldots, a_N \in \mathbb{Z}_{>0}$ and $m \in \{0, \ldots, p-1\}$ such that $\mu = (a_1p - 1, \ldots, a_{N-1}p - 1, a_Np - 1 - m)$, and furthermore $\nu_p(a_x) \geq$

$\ell_p(a_{x+1}-1)$ for $1 \le x < N$. In this case,

$$\dim_q \operatorname{Hom}_{R_\alpha}(S^\mu, S^\lambda) = q^r \text{ where } r = N - \left\lceil \frac{N+m}{p} \right\rceil.$$

Proof. Theorem 3.4.1(i) can only occur if $\mu = \lambda$, and Theorem 3.4.1(ii) never occurs because there are no valid choices for c. Therefore Theorem 3.4.1(iii) and Corollary 3.1.3 tell us the form that μ must take.

To calculate the degree of the homomorphisms, we calculate the degrees of $\mathtt{T}^\lambda$ and $\mathtt{T}^\mu$. For any partition ν with M parts, we have $\deg(\mathtt{T}^\nu) = \sum_{x=1}^{M} \lfloor \frac{\nu_x}{p} \rfloor$. In particular we get $\deg(\mathtt{T}^\mu) = \sum_{x=1}^{N} a_x - N$ and $\deg(\mathtt{T}^\lambda) = \sum_{x=1}^{N} a_x - \lceil \frac{N+m}{p} \rceil$, and

$$\deg(T^\lambda) - \deg(\mathtt{T}^\mu) = N - \left\lceil \frac{N+m}{p} \right\rceil.$$

□

The standard module

Let $\lambda = (d-1, 1)$. Then S^λ is referred to as the *standard* or *reflection* module. We have the following Corollary.

Corollary 3.5.3

For any partition $\mu \neq \lambda$, $\dim \operatorname{Hom}_{R_\alpha}(S^\mu, S^\lambda) = 0$ with the following exceptions:

1. if there are $a_1, a_2 \in \mathbb{Z}_{>0}$ and $0 \le m < p$ such that

$$\mu = (a_1 p, a_2 p - m),$$

and futhermore $\nu_p(a_1) \geq \ell_p(a_2 - 1)$, then

$$\dim_q \operatorname{Hom}_{R_\alpha}(S^\mu, S^\lambda) = \begin{cases} 1, & \text{if } p \mid d-1 \\ q, & \text{otherwise;} \end{cases}$$

2. if p divides d and $\mu = (d-2, 1, 1)$, then $\dim_q \operatorname{Hom}_{R_\alpha}(S^\mu, S^\lambda) = q$;
3. if there are $0 \leq m < p$, $N \geq 3$, and $a_1, \ldots, a_N \in \mathbb{Z}_{>0}$ such that

$$\mu = (a_1 p, a_2 p - 1, a_3 p - 1, \ldots, a_{N-1} p - 1, a_N p - 1 - m),$$

and furthermore $\nu_p(a_i) \geq \ell_p(a_{i+1} - 1)$ for $i = 1, \ldots, N-1$, then $\dim_q \operatorname{Hom}_{R_\alpha}(S^\mu, S^\lambda) = q^r$ where

$$r = N - \left\lceil \frac{N+m}{p} \right\rceil + \begin{cases} -1, & \text{if } p \mid d-1 \\ 1, & \text{if } p \mid d \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Everything follows directly from Theorem 3.4.1 and Corollary 3.1.3 except the degree calculations. The degree in cases (i) and (ii) is straightforward to calculate, and for case (iii) we can make the following calculations.

$$\deg(\mathtt{T}^{\mu}) = \sum_{x=1}^{N} a_x - N + 1$$

$$\deg(\mathtt{T}^{\lambda}) = \sum_{x=1}^{N} a_x - \left\lceil \frac{N+m}{p} \right\rceil$$

$$\deg([\sigma_\mu]) = \deg(\mathtt{T}^{\lambda}) + \begin{cases} 0, & \text{if } p \mid d-1 \\ 2, & \text{if } p \mid d \\ 1, & \text{otherwise.} \end{cases}$$

□

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