THE $RO(C_2)$-GRADED COHOMOLOGY OF C_2-SURFACES AND EQUIVARIANT FUNDAMENTAL CLASSES

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2020

DISSERTATION APPROVAL PAGE

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Title: The $RO(C_2)$-graded Cohomology of C_2-surfaces and Equivariant Fundamental Classes

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Degree awarded June 2020

DISSERTATION ABSTRACT

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Doctor of Philosophy

Department of Mathematics

June 2020

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Let C_2 denote the cyclic group of order two. Given a manifold with a C_2-action, we can consider its equivariant Bredon $RO(C_2)$-graded cohomology. We first use a classification due to Dugger to compute the Bredon cohomology of all C_2-surfaces in constant $\mathbb{Z}/2$ coefficients as modules over the cohomology of a point. We show the cohomology depends only on three numerical invariants in the nonfree case, and only on two numerical invariants in the free case. We next develop a theory of fundamental classes for equivariant submanifolds of any dimension in $RO(C_2)$-graded cohomology in constant $\mathbb{Z}/2$ coefficients. We connect these classes back to our initial computations by showing the cohomology of any C_2-surface is generated by fundamental classes, and these classes can be used to easily compute the ring structure. To define fundamental classes we are led to study the cohomology of Thom spaces of equivariant vector bundles. In general the cohomology of the Thom space is not just a shift of the cohomology of the base space, but we show there are still elements that act as Thom classes, and cupping with these classes gives an isomorphism within a certain range.

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Dan, for all of his guidance and input over the years. I would also like to thank my academic siblings, Clover and Eric, for encouraging me to explore the equivariant world.

Many thanks also to my friends and family who have supported me during graduate school and during life in general. A special thanks to my parents, Art and Lydia, for raising me to be hard-working and for always supporting me, even when my work takes me to the other side of the country; to my best friend and person, Emily, for responding to years of texts from me about the pains of graduate school; to my partner, Joe, for always believing in me and for listening to many practice talks about the material in this thesis; and lastly, a sweet thank you to my two kitties, Rhaenys and Fluffy, for always keeping me company and lifting my mood when I work from home.

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CHAPTER I

INTRODUCTION

There has been recent interest in better understanding $RO(C_2)$-graded Bredon cohomology. For example, explicit computations can be found in [2], [9], [11], [5], and [16], and certain freeness and structure theorems can be found in [10] and [13]. The goal of this thesis is twofold: first we compute the $RO(C_2)$-graded Bredon cohomology of a family of C_2-spaces, and next we provide a geometric framework in which to understand these answers.

To say more, our first goal is to compute the $RO(C_2)$-graded cohomology of all C_2-surfaces in constant $\mathbb{Z}/2$ coefficients and to state the answer in a concise and coherent way based on a few properties of the space and its action. We show the answers depend only on three invariants of the C_2-surface in the nonfree case, and two invariants in the free case. Even better, the cohomology can be described formulaically in terms of these invariants. Next we consider C_2-manifolds of any dimension and develop a theory of fundamental classes for equivariant submanifolds in constant $\mathbb{Z}/2$ coefficients. These classes can be defined for both free and nonfree submanifolds. When two submanifolds intersect transversally, the cup product of their classes is given in terms of the fundamental class of their intersection. In order to define these classes, we consider equivariant Thom spaces for real C_2-vector bundles and prove properties of the $RO(C_2)$-graded cohomology of these spaces in constant $\mathbb{Z}/2$ coefficients. We end by showing the $RO(C_2)$-graded cohomology of any C_2-surface can be understood in terms of equivariant fundamental classes of submanifolds.

1.1. Introduction to Bredon Cohomology

In order to state the main results, we first recall some facts about Bredon cohomology; a more detailed exposition of this theory can be found in [8, Section 2], for example. For a finite group G the Bredon cohomology of a G-space is a sequence of abelian groups graded on $RO(G)$, the Grothendieck group of finite-dimensional, real, orthogonal G-representations. When G is the cyclic group of order two, recall any C_2-representation is isomorphic to a direct sum of trivial representations and sign representations. Thus $RO(C_2)$ is a free abelian group of rank two, and the Bredon cohomology of any C_2-space can be regarded as a bigraded abelian group.

We will use the motivic notation $H^{*,*}(X; M)$ for the Bredon cohomology of a C_2-space X with coefficients in a Mackey functor M. The first grading indicates the dimension of the representation, and we will often refer to this grading as the "topological dimension". The second grading indicates the number of sign representations appearing and will be referred to as the "weight". Given any C_2-space X, there is always an equivariant map $X \to pt$ where pt denotes a single point with the trivial C_2-action. This gives a map of bigraded abelian groups $H^{*,*}(pt; M) \to H^{*,*}(X; M)$. If M has the additional structure of a Green functor, then in fact this is a map of bigraded rings, and thus $H^{*,*}(X; M)$ forms a bigraded algebra over the cohomology of a point. In this paper, we will be working with the constant Green functor $\underline{\mathbb{Z}/2}$, and we will write $\mathbb{M}_2$ for the bigraded ring $H^{*,*}(pt; \underline{\mathbb{Z}/2})$. In addition to being a Green functor, $\underline{\mathbb{Z}/2}$ satisfies an additional property ($\text{tr}(1) = 2$) that ensures $H^{*,*}(X; \underline{\mathbb{Z}/2})$ is a bigraded commutative ring.

When working in $\underline{\mathbb{Z}/2}$-coefficients, it is shown in [13] that, as a module over the cohomology of a point, the cohomology of any finite C_2-CW complex decomposes into a direct sum of two types of summands. Specifically, the cohomology is given by a direct sum of free modules and shifted copies of the cohomology of antipodal spheres. We provide an introduction to these two types of modules below.

The Basic Pieces

Since the cohomology of any C_2-space is a bigraded module over a bigraded ring, we can use a grid to record information about the cohomology groups and module structures. For example, the cohomology ring of a point in $\underline{\mathbb{Z}/2}$-coefficients is illustrated on the left-hand grid in the figure below. Each dot represents a copy of $\mathbb{Z}/2$, and the connecting lines indicate properties of the ring structure. For example, the top portion is polynomial in two elements ρ and τ which are in bidegrees $(1,1)$ and $(0,1)$, respectively. A precise description of this ring can be found in Section II. In practice, it is cumbersome to draw the detailed picture, so instead, we draw the abbreviated version shown on the right.

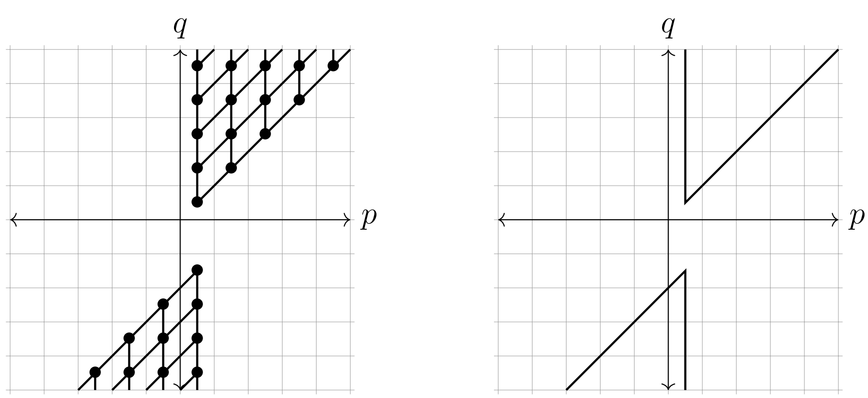


FIGURE 1. The ring $\mathbb{M}_2 = H^{*,*}(pt)$ with $H^{p,q}(pt)$ in spot (p,q).

Let S^n_a denote the C_2-space whose underlying space is S^n and whose C_2-action is given by the antipodal map. Note S^0_a is the free orbit C_2. We denote the cohomology of the space S^n_a by A_n. This $\mathbb{M}_2$-module can be described algebraically as $A_n \cong \tau^{-1}\mathbb{M}_2/(\rho^{n+1})$. In the figure below, we illustrate the module structure of A_0, A_1, and A_2, respectively. As before, the detailed picture is shown on the left, while the abbreviated picture is shown to the right. The dots again indicate a copy of $\mathbb{Z}/2$ while the lines indicate the module structure.

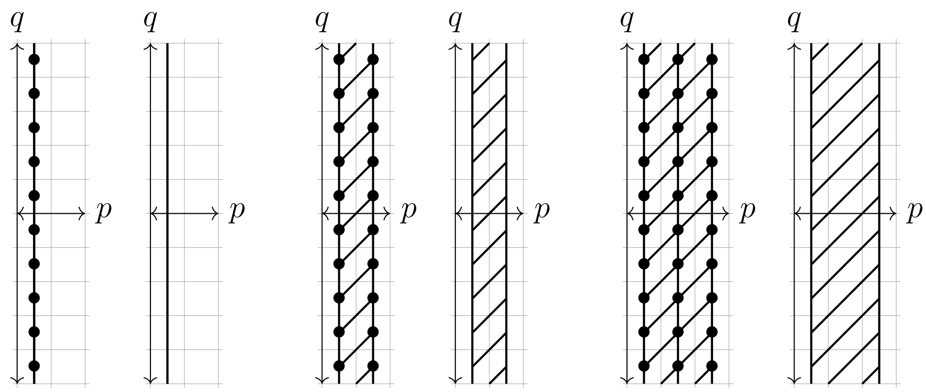


FIGURE 2. The $\mathbb{M}_2$-modules A_0, A_1, and A_2, respectively.

It is shown in [13] that as an $\mathbb{M}_2$-module, the cohomology any finite C_2-CW complex is isomorphic to a direct sum of shifted copies of $\mathbb{M}_2$ and shifted copies of A_j for some values of j. Our first goal is to find the specific decompositions for the cohomology of all C_2-surfaces. We begin by showing a few examples.

1.2. Computational Examples

To give the reader a flavor of the sorts of decompositions that can appear, we provide three examples of C_2-surfaces and state the cohomology of each.

Example 1.2.1. Let X_1 denote the C_2-space whose underlying space is the genus one torus, and whose action is given by the reflection action. The space X_1 is depicted below with the fixed set $X_1^{C_2}$ shown in blue. We will eventually show as an $\mathbb{M}_2$-module

$$H^{*,*}(X_1;\underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2.$$

This module is illustrated below.

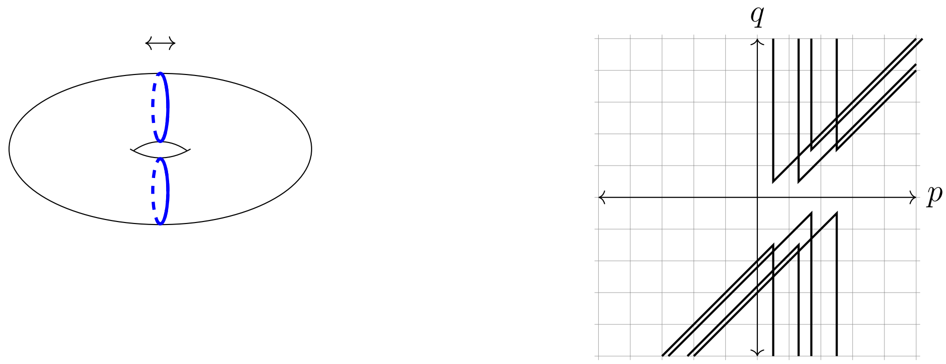


FIGURE 3. The space X_1 and its cohomology.

Observe the cohomology is free over $\mathbb{M}_2$, and there is exactly one generator in topological dimension zero, exactly two generators in topological dimension one, and exactly one generator in topological dimension two (recall p is the topological dimension). This should be unsurprising based on the singular cohomology of the torus, though the weights of these generators are more mysterious.

Example 1.2.2. Let X_2 denote the C_2-surface whose underlying space is the genus 7 torus, and whose C_2-action is given by the rotation action depicted below. The fixed set consists of 8 isolated points that are shown in blue. The cohomology of X_2 is given by

$$H^{*,*}(X_2;\underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus 6} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus 4} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

The module is illustrated below.

Again there is exactly one free generator in topological dimensions zero and two, but there is something more interesting going on in topological dimension one. There are four nonfree summands, and six free summands in weight one.

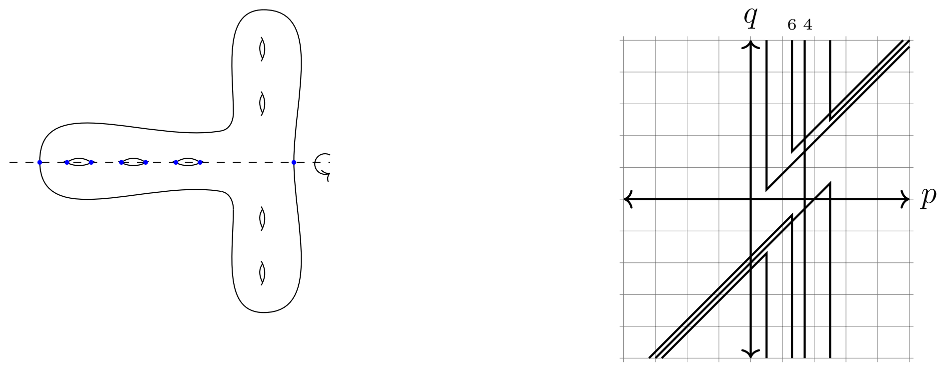


FIGURE 4. The space X_2 and its cohomology.

Example 1.2.3. Our last example X_3 is the C_2-space whose underlying space is $\mathbb{R}P^2$ and whose C_2-action is depicted below. Note the fixed set contains both a fixed circle and a fixed point; this did not happen in the previous examples. The cohomology of this space is given by

$$H^{*,*}(X_3; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2.$$

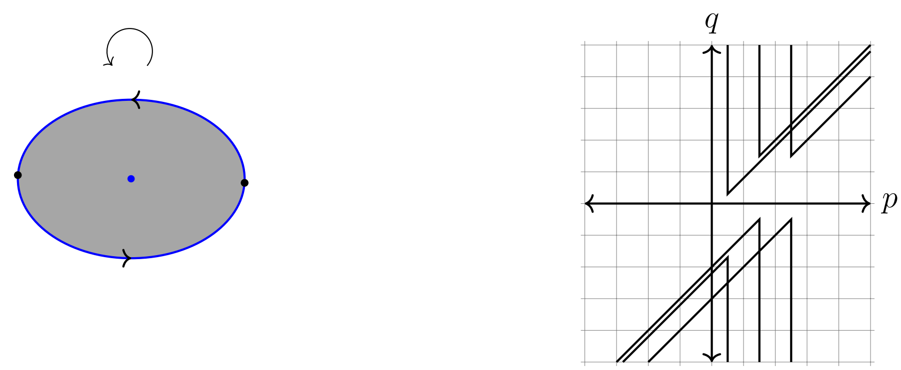


FIGURE 5. The space X_3 and its cohomology.

With some care, one can compute the cohomology of the above spaces using tools given in Section III, but it is not obvious how we could have predicted these answers by just looking at the spaces. What properties are being detected by the cohomology? In order to answer this question, we next define some invariants of C_2-surfaces.

1.3. Invariants of C_2-surfaces

There are a handful of invariants that can be associated to a given C_2-surface. For example, we can count the number of isolated fixed points, or the number of fixed circles. Recall for a nonequivariant surface X the homeomorphism type is entirely determined by two invariants, namely $\dim_{\mathbb{Z}/2} H^1_{sing}(X;\mathbb{Z}/2)$ and whether or not X is orientable. In what follows, we will denote $\dim_{\mathbb{Z}/2} H^1_{sing}(X;\mathbb{Z}/2)$ by $\beta(X)$ and refer to this as the β-*genus* of X.

In [7] it is shown there is a list of invariants that uniquely determines the isomorphism type of a C_2-action on a given surface. We recall two of these invariants below.

Definition 1.3.1. Let X be a C_2-surface. We can associate the following invariants to X:

(i) $F(X)$ is the number of isolated fixed points;

(ii) $C(X)$ is the number of fixed circles.

When there is no ambiguity about the space we are discussing, we will simply write F instead of $F(X)$ and C instead of $C(X)$. There are four other invariants needed in the classification given in [7] that we will not define here, but do note the invariants defined above are not enough to uniquely determine the isomorphism type in general.

In the examples above, we can compute

$$\beta(X_1) = 2, \quad F(X_1) = 0, \quad C(X_1) = 2;$$
$$\beta(X_2) = 14, \quad F(X_2) = 8, \quad C(X_2) = 0;$$
$$\beta(X_3) = 1, \quad F(X_3) = 1, \quad C(X_3) = 1.$$

The first half of this thesis addresses the following questions: how does the cohomology of a C_2-surface relate to the invariants? Better yet, can we find a formula that gives the cohomology of X based on some of these invariants? It may not be apparent how to do this based on the three given examples, but the answer to the latter is yes, as explained below.

1.4. The Answer for Nonfree C_2-surfaces

We now state the decompositions for nontrivial, nonfree C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients. Recall the fixed set of an involution on a surface is always given by a disjoint union of isolated points and copies of S^1. Recall $\mathbb{M}_2$ denotes the cohomology of a point and A_0 denotes the cohomology of the free orbit C_2. We will show the following:

Theorem 1.4.1. *Let X be a nontrivial, nonfree C_2-surface. There are two cases for the $RO(C_2)$-graded Bredon cohomology of X in $\underline{\mathbb{Z}/2}$-coefficients.*

(i) Suppose $C = 0$. Then $H^{*,*}(X;\underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus (\Sigma^{1,1}\mathbb{M}_2)^{\oplus F-2} \oplus (\Sigma^{1,0}A_0)^{\oplus \frac{\beta-F}{2}+1} \oplus \Sigma^{2,2}\mathbb{M}_2$.

(ii) Suppose $C \neq 0$. Then $H^{*,*}(X;\underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus (\Sigma^{1,1}\mathbb{M}_2)^{\oplus F+C-1} \oplus (\Sigma^{1,0}\mathbb{M}_2)^{\oplus C-1} \oplus (\Sigma^{1,0}A_0)^{\oplus \frac{\beta-F}{2}+1-C} \oplus \Sigma^{2,1}\mathbb{M}_2$.

We invite the reader to check that the above formulas match the answers given in the three examples. Note not all of the invariants given in [7] are needed to determine the cohomology. In particular, the module structure of the cohomology in $\underline{\mathbb{Z}/2}$-coefficients does not determine the isomorphism type of a C_2-surface.

Remark 1.4.2. As observed in our examples, there is exactly one summand generated in topological dimension zero, exactly one summand generated in topological dimension two, and some number of summands appearing in topological dimension one. Based on what we know of the singular cohomology of surfaces, this shouldn't be surprising, though, the exact number and type of summands generated in topological dimension one is nonobvious. Note we can recover dimension of the singular cohomology of X by

$$\beta = 2 \cdot \#\,(\Sigma^{1,0}A_0\text{-summands}) + \#\,(\Sigma^{1,0}\mathbb{M}_2\text{-summands}) + \#\,(\Sigma^{1,1}\mathbb{M}_2\text{-summands})\,.$$

1.5. The Answer for Free C_2-surfaces

We need to define one construction in order to state the decomposition for free C_2-surfaces. Given a nontrivial equivariant surface X and a nonequivariant surface Y, we can form the equivariant connected sum $X\#_2Y$ as follows. Let Y' denote the space obtained by removing a small disk from Y. Let D be a disk in X that is disjoint from its conjugate disk σD, and let X' denote the space obtained by removing both of these disks. Choose an isomorphism $f : \partial Y' \to \partial D$. Then the space $X\#_2Y$ is given by

$$[(Y' \times \{0\}) \sqcup (Y' \times \{1\}) \sqcup X']/\sim$$

where $(y, 0) \sim f(y)$ and $(y, 1) \sim \sigma(f(y))$ for $y \in \partial Y'$. Note nonequivariantly $X\#_2Y \cong Y\#X\#Y$. Below is an example where $X = S^2_a$ and $Y = T_1$ is the genus one torus.

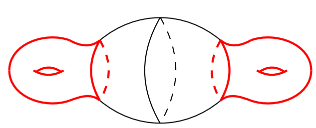


FIGURE 6. $S^2_a\#_2T_1$

In [7], it is shown that there is exactly one free action on the sphere up to equivariant isomorphism, namely the antipodal action, and there are exactly two free actions on the torus, namely the antipodal action and the action given by rotating 180° around an axis through the center of the hole of the torus. Interestingly, it is also shown for every free C_2-surface X there is a surface Y such that X is isomorphic to $Z\#_2 Y$ where Z is either the free C_2-sphere or one of the two free C_2-tori.

We can now state the theorem for free C_2-surfaces. Recall A_n denotes the cohomology of S^n with the antipodal action. We will prove the following:

Theorem 1.5.1. *Let X be a free C_2-surface. There are two cases for the $RO(C_2)$-graded Bredon cohomology of X in $\underline{\mathbb{Z}/2}$-coefficients.*

(i) Suppose X is equivariantly isomorphic to $S^2_a\#_2 Y$. Then $H^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0}A_0)^{\oplus \beta(X)/2} \oplus A_2$.

(ii) Suppose X is equivariantly isomorphic to $Z\#_2 Y$ where Z is a free C_2-torus. Then $H^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0}A_0)^{\oplus \frac{\beta(X)-2}{2}} \oplus A_1 \oplus \Sigma^{1,0}A_1$.

For example, the cohomology of the space $S^2_a\#_2 T_1$ shown above is given by

$$H^{*,*}(S^2_a\#_2 T_1) \cong (\Sigma^{1,0}A_0)^{\oplus 2} \oplus A_2.$$

Remark 1.5.2. It is not clear a priori why the number of summands of $\Sigma^{1,0}A_0$ given in Theorem 1.4.1 and Theorem 1.5.1 is necessarily an integer. This nontrivial fact follows from restrictions on β, F, and C that arise in the classification of C_2-surfaces given in [7].

1.6. Summary of Main Geometric Points

After the initial computations, there are two main topics in this thesis, fundamental classes and the cohomology of Thom spaces. These topics can be further divided into five subtopics:

1. *Nonfree fundamental classes:* Given a nonfree C_2-manifold X and a nonfree C_2-submanifold Y, we prove there exists a fundamental class $[Y] \in H^{n-k,q}(X;\underline{\mathbb{Z}/2})$ where $k = \dim(Y)$ and q is found as follows. Consider the restriction of the equivariant normal bundle of Y in X to the fixed set Y^{C_2}. Over each component of this fixed set, the fibers are C_2-representations, and the isomorphism type of the representation is constant over each component. Thus each component corresponds to a representation of some weight, and the integer q is chosen to be the maximum such weight.

2. *Free fundamental classes:* Given a C_2-manifold X (free or nonfree) and a free submanifold Y, we show there is an infinite family of classes $[Y]_q \in H^{n-k,q}(X;\underline{\mathbb{Z}/2})$ where $k = \dim(Y)$ and q is any integer. These classes satisfy the module relation $\tau \cdot [Y]_q = [Y]_{q+1}$.

3. *Intersection product of fundamental classes:* We show the cup product of two fundamental classes for nonfree, submanifolds Y and Z that intersect transversally is given by a predicted τ-multiple of the fundamental class of the intersection. For free submanifolds, we show $[Y]_r \smile [Z]_s = [Y \cap Z]_{r+s}$.

4. *Restricted Thom isomorphism theorem for nonfree spaces:* Let X be a finite, nonfree C_2-CW complex and $E \to X$ be an n-dimensional C_2-vector bundle whose maximum weight representation over X^{C_2} is q. We show there is a unique class $u_E \in H^{n,q}(E, E-0;\underline{\mathbb{Z}/2})$ that restricts to τ-multiples of the generators of the cohomology of the fibers. We also show cupping with this class gives an isomorphism $H^{f,g}(X;\underline{\mathbb{Z}/2}) \to H^{f+n,g+q}(E, E-0;\underline{\mathbb{Z}/2})$ whenever $g \geq f$.

5. *Thom isomorphism theorem for free spaces:* Let X be a finite, free C_2-CW complex. For an n-dimensional C_2-vector bundle $E \to X$, we show for each integer q there is a unique class $u_{E,q} \in H^{n,q}(E, E-0; \underline{\mathbb{Z}/2})$ that restricts to the nonzero class in the cohomology of the fibers. In this case, cupping with any one of these Thom classes provides an isomorphism from the cohomology of the base space to the cohomology of a shift of the Thom space.

Remark 1.6.1. These equivariant fundamental classes act in many ways just like their nonequivariant analogs, though there are still various subtleties that arise in the equivariant context. For example, one might expect nonfree fundamental classes to always generate free summands and free fundamental classes to string together to form modules similar to A_0. While this is often the case, there are exceptions. As shown in Example 1.7.5 below, there can be free submanifolds whose classes are nonzero for a while, but become zero in high enough weight. This and other subtleties are best explained through a series of examples.

1.7. Examples of Equivariant Fundamental Classes

Before stating the general results, let's consider some examples of C_2-surfaces to determine what properties we might expect of equivariant fundamental classes.

Example 1.7.1. We begin with a simple example. Let X denote the C_2-torus from Example 1.2.1. Recall the cohomology of this space is given by

$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2.$$

Illustrations of the space and this module are given below.

Suppose we wanted to find equivariant submanifolds whose fundamental classes generate these free summands. Take, for example, the fixed circle labeled

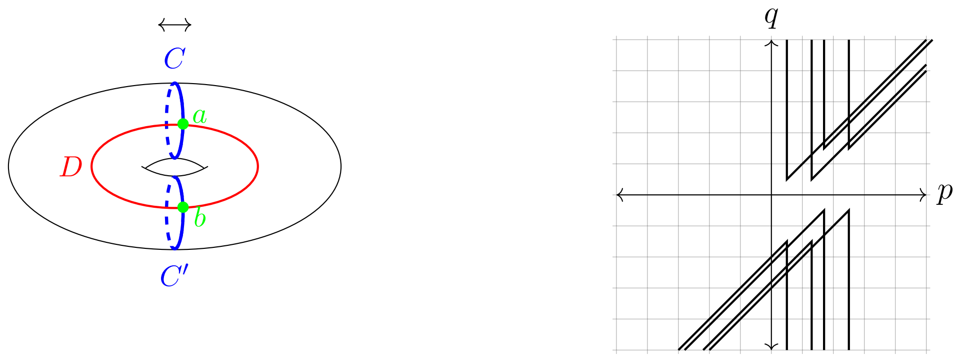


FIGURE 7. The space X and the $\mathbb{M}_2$-module $H^{*,*}(X)$.

C above. Topologically, this is a codimension one submanifold, so we would expect to have a fundamental class in bidegree $(1, q)$ for some q, but how do we determine this value of q? Let's consider a tubular neighborhood of C. Note the tubular neighborhood is equivariantly homeomorphic to $C \times \mathbb{R}^{1,1}$ where $\mathbb{R}^{1,1}$ denotes the sign representation, so one might expect a fundamental class in bidegree $(1, 1)$. Indeed, we will show there is such a class $[C] \in H^{1,1}(X; \underline{\mathbb{Z}/2})$, and furthermore, this class generates a free summand in bidegree $(1, 1)$. We also have a class $[C'] \in H^{1,1}(X; \underline{\mathbb{Z}/2})$ corresponding to the other fixed circle, and we will show $[C] = [C'] + \rho \cdot 1$.

Now consider the circle D that travels around the hole of the torus and is isomorphic to a circle with a reflection action. In this case, the tubular neighborhood of D is homeomorphic to $D \times \mathbb{R}^{1,0}$ where $\mathbb{R}^{1,0}$ is the trivial representation. We expect to get a class in bidegree $(1, 0)$, and indeed such a class $[D]$ exists and generates a free summand.

These two circles intersect at a single fixed point a whose tubular neighborhood is the unit disk in $\mathbb{R}^{2,1}$. The fundamental class $[a]$ generates a free summand in bidegree $(2, 1)$, and furthermore we obtain the relation

$$[C] \smile [D] = [a].$$

The submanifolds C and C' do not intersect, so we also obtain the relation

$$[C] \smile [C] = [C] \smile ([C'] + \rho \cdot 1) = \rho \cdot [C].$$

We conclude

$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2[x, y]/(x^2 = \rho x, y^2 = 0), \quad |x| = (1, 1), \ |y| = (1, 0)$$

as an $\mathbb{M}_2$-algebra.

Remark 1.7.2. In the example above, all of the submanifolds considered had trivial normal bundles, and this led to an obvious choice of bidegree for each fundamental class. Of course, we would like to have fundamental classes for submanifolds whose normal bundles are nontrivial. The next example illustrates this.

Example 1.7.3. Let Y denote the C_2-space given by the projective plane with action induced by the rotation action on the disk as shown in Figure 8 below. The fixed set is again shown in blue. The cohomology of Y as an $\mathbb{M}_2$-module is given by

$$H^{*,*}(Y; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2.$$

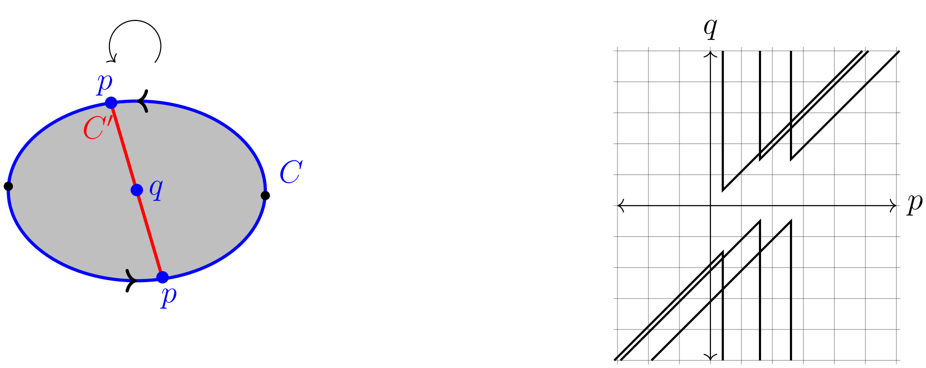


FIGURE 8. The space Y and the $\mathbb{M}_2$-module $H^{*,*}(Y)$.

We again look for equivariant submanifolds. Consider the fixed circle labeled C. The equivariant normal bundle N is given by the nontrivial Möbius bundle over S^1. This bundle is not trivial, but observe every point $x \in C$ has an equivariant neighborhood U_x such that $N|_{U_x} \cong U_x \times \mathbb{R}^{1,1}$. In other words, this bundle is a locally trivial $\mathbb{R}^{1,1}$-bundle, so we get a class $[C] \in H^{1,1}(X; \underline{\mathbb{Z}/2})$.

Now consider the circle C'. This circle is isomorphic to a circle with a reflection action, and again the normal bundle N' is the nontrivial bundle over S^1. Around the fixed point p there is a neighborhood U such that $N'|_U \cong U \times \mathbb{R}^{1,0}$ while around the point q there is a neighborhood V such that $N'|_V \cong V \times \mathbb{R}^{1,1}$. The bundle N' is not a locally trivial W-bundle for any C_2-representation W, so if it even exists, the bidegree of the fundamental class for C' is unclear.

We will show the fundamental class $[C']$ does exist, and its grading is determined by the fibers of the normal bundle over the fixed set. In this case, $N'|_p \cong \mathbb{R}^{1,0}$ while $N'|_q \cong \mathbb{R}^{1,1}$. The class $[C']$ will have grading $(1,k)$ where k is the *maximum* weight representation appearing over the fixed set, so in this case $k = 1$.

The classes $[C]$ and $[C']$ are both in bidegree $(1,1)$, and we will see in Example 7.2.7 that $[C] = [C'] + \rho \cdot 1$. Let's consider their product. One might hope

$$[C] \smile [C'] = [C \cap C'] = [p],$$

but something has gone wrong. The product has bidegree $(2,2)$, while the fundamental class $[p]$ has bidegree $(2,1)$. We will show a slightly modified formula holds. Namely

$$[C] \smile [C'] = \tau \cdot [p]$$

where recall $\tau \in H^{0,1}(pt; \underline{\mathbb{Z}/2})$. This give us the relation

$$[C]^2 = [C]([C'] + \rho \cdot 1) = \tau \cdot [p] + \rho \cdot [C].$$

In Example 7.2.7 we use these classes to conclude as an $\mathbb{M}_2$-algebra

$$H^{*,*}(Y; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2[x, y]/(x^2 = \tau y + \rho x, xy = 0, y^2 = 0), \quad |x| = (1, 1), |y| = (2, 1).$$

Remark 1.7.4. The first two examples give a flavor of fundamental classes for nonfree submanfiolds. It is natural to ask if such things can be defined for free submanifolds. We see this in the next example.

Example 1.7.5. Let Z denote the C_2-space whose underlying space is the genus two torus and whose C_2-action is given by a rotation action with two fixed points, as illustrated below. The cohomology of this space is given by

$$H^{*,*}(Z; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \oplus (\Sigma^{1,0} A_0)^{\oplus 2} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

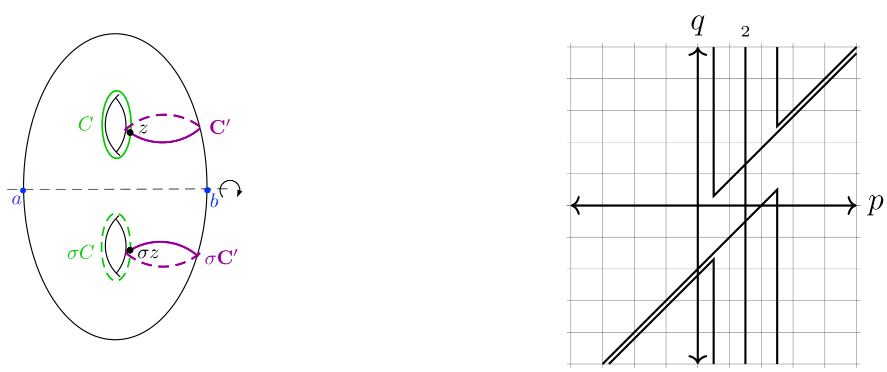


FIGURE 9. The space Z and the $\mathbb{M}_2$-module $H^{*,*}(Z)$.

In this example the cohomology is an infinitely generated, nonfree $\mathbb{M}_2$-module. The free submanifold $C \sqcup \sigma C$ is a codimension one submanifold, so we would expect to have a class $[C \sqcup \sigma C] \in H^{1,?}(Z; \underline{\mathbb{Z}/2})$. The submanifold is free, so there is no fixed set to determine the weight as in the previous examples. We will show there are actually infinitely many fundamental classes, one in each weight. That is, we have classes $[C \sqcup \sigma C]_q \in H^{1,q}(Z; \underline{\mathbb{Z}/2})$ for all integers q that are related via the formula

$$\tau \cdot [C \sqcup \sigma C]_q = [C \sqcup \sigma C]_{q+1}.$$

The submodule generated by these classes corresponds to a $\Sigma^{1,0} A_0$-summand. One can show the fundamental classes of the free submanifold $C' \sqcup \sigma C'$ generate the other summand. Lastly, there are classes $[a], [b] \in H^{2,2}(Z; \underline{\mathbb{Z}/2})$ corresponding to the two fixed points. Either class will generate a free summand in bidegree $(2, 2)$, and the classes are related via $[a] = [b] + \rho^2 \cdot 1$.

We can also choose a point $z \in Z \setminus Z^{C_2}$ and consider the classes $[z \sqcup \sigma z]_q \in H^{2,q}(Z; \underline{\mathbb{Z}/2})$. One might expect these classes to all be zero, but in fact, they are nonzero whenever $q \leq 0$. The intersection product then leads to some interesting relations as shown in Example 7.3.6.

We mention two other nonfree classes one might consider. There are two copies of a circle with a reflection action in the space Z. We can consider the circle that travels around the equator through a and b, and the circle that travels around the perimeter of the picture through a and b. Call these circles D and E, respectively. Then $[D] = \rho \cdot 1$ while $[E] = [C \sqcup \sigma C]_1 + \rho \cdot 1$.

Nonequivariant Fundamental Classes

Before trying to define equivariant fundamental classes, let's recall nonequivariant fundamental classes. Let X be a closed manifold and $Y \subset X$ be a closed, connected submanifold of codimension k. We can define the fundamental class $[Y] \in H^k_{sing}(X; \mathbb{Z}/2)$ using the classical Thom isomorphism theorem from [17].

Consider the normal bundle N of Y in X. Let $U \subset X$ be a tubular neighborhood of Y. By excision

$$H^k_{sing}(X, X - Y; \mathbb{Z}/2) \cong H^k_{sing}(U, U - Y; \mathbb{Z}/2) \cong H^k(N, N - 0; \mathbb{Z}/2).$$

By the Thom isomorphism theorem, the righthand group is $\mathbb{Z}/2$ and generated by the Thom class u_N. Thus there exists a unique nonzero class in $H^k(X, X - Y; \mathbb{Z}/2)$. We now define the fundamental class $[Y] \in H^k(X; \mathbb{Z}/2)$ to be the image of this unique class under the induced map from the inclusion of the pairs $(X, \emptyset) \hookrightarrow (X, X - Y)$. Recall these classes have a nice intersection product. If Y and Z are two submanifolds of X that intersect transversally, then $[Y] \smile [Z] = [Y \cap Z]$.

One goal is to define an equivariant analog to these classes in Bredon cohomology. Given an equivariant submanifold, the above hints that we should consider the normal bundle, and use some fact about the cohomology of the corresponding Thom space. Unfortunately, no direct analog of the Thom isomorphism theorem exists for general C_2-vector bundles in $\underline{\mathbb{Z}/2}$-coefficients; see Example B.1.4 for an example of a vector bundle E such that the cohomology of $(E, E - 0)$ is not just a shift of the cohomology base space. Despite this failure, we can still prove a weaker version of the Thom isomorphism theorem, and this is enough to define fundamental classes.

1.8. The Main Geometric Theorems

We now state the main theorems from the second part of this thesis. We begin with nonfree C_2-vector bundles. Let X be a finite, nonfree C_2-CW complex and let $\pi : E \to X$ be a real n-dimensional C_2-vector bundle (precise definitions can be found in Appendix B). Let $X_1, \ldots, X_r$ denote the connected components of the fixed set X^{C_2}. As explained in Section B, there exist weights $q_1, \ldots, q_r$ such that for all $x \in X_i$, the fiber $E_x \cong \mathbb{R}^{n,q_i}$. Let q be the maximum such weight. We will prove the following where all coefficients are understood to be $\underline{\mathbb{Z}/2}$.

Theorem 1.8.1. *Let X, X_i, $\pi : E \to X$, n, q_i and q be defined as above and let $E' = E - 0$. There exists a unique class $u_E \in H^{n,q}(E, E')$ such that the following holds:*

(i) $\psi(u_E)$ is the singular Thom class, where $\psi : H^{n,k}(E, E') \to H^n_{sing}(E, E')$ is the forgetful map;

(ii) $\mathbb{M}_2 \cdot u_E \cong \Sigma^{n,q}\mathbb{M}_2$, where $\mathbb{M}_2 \cdot u_E$ denotes the submodule generated by u_E;

(iii) For every i and $x \in X_i$, the class u_E restricts to $\tau^{q-q_i}\alpha_x$ where α_x is the generator of $H^{*,*}(E_x, E_x - 0) \cong \tilde{H}^{*,*}(S^{n,q_i})$.

(iv) For every $x \in X \setminus X^{C_2}$, the class u_E restricts to the unique nonzero class in $H^{*,*}(E_{x,\sigma x}, E_{x,\sigma x} - 0) \cong \tilde{H}^{*,*}(S^{n,0} \wedge C_{2+})$ where $E_{x,\sigma x} = \pi^{-1}(\{x, \sigma x\})$.

(v) The map $\phi_E = \pi^*(-) \smile u_E : H^{f,g}(X) \to H^{f+n,g+q}(E, E')$ is an isomorphism if $g \geq f$;

(vi) Suppose $H^{*,*}(X) \cong (\oplus_{i=1}^{c}\Sigma^{k_i,\ell_i}\mathbb{M}_2) \oplus (\oplus_{j=1}^{d}\Sigma^{s_j,0}A_{r_j})$. Then $H^{*,*}(E, E') \cong (\oplus_{i=1}^{c}\Sigma^{k_i+n,\ell_i'}\mathbb{M}_2) \oplus (\oplus_{j=1}^{d}\Sigma^{s_j+n,0}A_{r_j})$ where the weights ℓ_i' satisfy $\ell_i + q \geq \ell_i' \geq 0$;

(vii) If in fact $E_x \cong E_y$ for all $x, y \in X^{C_2}$, then ϕ_E is an isomorphism in all bidegrees and $H^{*,*}(X) \cong H^{*+n,*+q}(E, E')$.

Let Y be a nonfree k-codimensional equivariant submanifold of X and let N denote the equivariant normal bundle. We can construct an equivariant tubuluar neighbrhood of Y, and then use the class u_N to get a class $[Y] \in H^{k,q}(X; \underline{\mathbb{Z}/2})$. As with the nonequivariant classes, we have a nice formula for how these classes multiply. For now, we state a summary. Recall $\tau \in H^{0,1}(pt; \underline{\mathbb{Z}/2})$.

Theorem 1.8.2. *Let X be a nonfree, n-dimensional C_2-manifold, and let Y and Z be two closed, nonfree equivariant submanifolds. Suppose Y intersects Z transversally in the nonequivariant sense and that $Y \cap Z$ is a nonfree submanifold. Then there is a unique integer $j \geq 0$ such that $[Y] \smile [Z] = \tau^j [Y \cap Z]$.*

The exact value of j is dependent on Y, Z, and $Y \cap Z$ and is given explicitly in Theorem 7.2.3.

For free C_2-vector bundles, we will prove something similar. Again the coefficients are understood to be $\underline{\mathbb{Z}/2}$.

Theorem 1.8.3. *Let X be a free, finite C_2-CW complex and let $\pi : E \to X$ be a real C_2-vector bundle and $E' = E - 0$. For every integer q there exists a unique class $u_{E,q} \in H^{n,q}(E, E')$ such that the following holds:*

(i) $\psi(u_{E,q})$ is the singular Thom class, where $\psi : H^{n,q}(E,E') \to H^n_{sing}(E,E')$ is the forgetful map;

(ii) $\tau \cdot u_{E,q} = u_{E,q+1}$;

(iii) For every pair of conjugate points $x, \sigma x \in X$, the class $u_{E,q}$ restricts to the unique nonzero element in $H^{n,q}(E_{x,\sigma x}, E_{x,\sigma x} - 0) \cong \tilde{H}^{n,q}(S^n \wedge C_{2+})$.

(iv) The map $\pi^*(-) \smile u_{E,q} : H^{*,*}(X) \to H^{*+n,*+q}(E,E')$ is an isomorphism for all q. In particular, $H^{*,*}(X) \cong H^{*+n,*}(E,E')$.

Now given a free submanifold, we can use the classes corresponding to the normal bundle to define fundamental classes $[Y]_q \in H^{k,q}(X; \underline{\mathbb{Z}/2})$ for all q. There is also a nice intersection product for these free fundamental classes.

Theorem 1.8.4. *Let X be an n-dimensional C_2-manifold, and suppose Y and Z are equivariant submanifolds that intersect transversally in the nonequivariant sense* *and whose intersection is free. We have the following cases for the product of their fundamental classes.*

- *Suppose Y and Z are nonfree and their fundamental classes have weights q, r, respectively. Then $[Y] \smile [Z] = [Y \cap Z]_{q+r}$.*
- *Suppose Y is nonfree and Z is free. Then for every r, $[Y] \smile [Z]_r = [Y \cap Z]_{q+r}$.*
- *Suppose Y and Z are both free. Then for every r, s, $[Y]_r \smile [Z]_s = [Y \cap Z]_{r+s}$.*

In the last section of this paper, we show these classes give a geometric interpretation for the Bredon cohomology of C_2-surfaces. Specifically, we prove the following theorem.

Theorem 1.8.5. *As an $\mathbb{M}_2$-module, the Bredon cohomology of any C_2-surface is generated by fundamental classes of submanifolds.*

Remark 1.8.6. There are many examples of C_2-surfaces whose Bredon cohomology is infinitely generated as an $\mathbb{M}_2$-module; see Example 1.7.5. This is unsurprising given that the modules $A_n = \tau^{-1}\mathbb{M}_2/(\rho^{n+1})$ are infinitely generated. Though, there is always a finite list of submanifolds whose fundamental classes generate the cohomology, and any free A_i-summand is generated by fundamental classes of free submanifolds.

1.9. Organization

In Chapters II and III we review basics about $RO(C_2)$-graded Bredon cohomology and introduce computational tools that will be used throughout the thesis. In Chapter IV we review the necessary information about C_2-surfaces. The surface computations are then given in Chapters V and VI. In Chapter

VII we define equivariant fundamental classes using the theorems developed in Appendix B. In the last chapter, Chapter VIII, we show the $RO(C_2)$-graded Bredon cohomology of any C_2-surface is generated by equivariant fundamental classes. Finally, Appendix A contains a useful theorem about the cohomology of closed, nonfree C_2-manifolds of any dimension.

CHAPTER II

BACKGROUND ON BREDON COHOMOLOGY

In this section, we review some preliminary facts about $RO(G)$-graded Bredon cohomology in the case of $G = C_2$. Our coefficients are given by Mackey functors, so we provide a definition of a Mackey functor in the case of $G = C_2$ and review the Mackey functor that will be used throughout the thesis. We next review how the cohomology theory is a bigraded theory, and how this lends itself to pictorially representing various module and ring structures.

2.1. Mackey functors

The coefficients of $RO(G)$-graded Bredon cohomology are what is known as a Mackey functor. In general, the definition of a Mackey functor requires some work, and a general exposition of Mackey functors can be found in [16] or in [14]. In the case of $G = C_2$, the definition can be distilled to the following.

Definition 2.1.1. A **Mackey functor** M for $G = C_2$ is the data of

$$M: \quad M(C_2) \underset{p_*}{\overset{p^*}{\rightleftarrows}} M(*) \qquad t^* \circlearrowright M(C_2)$$

where $M(C_2)$ and $M(*)$ are abelian groups, and p^*, p_*, t^* are homomorphisms that satisfy

(i) $(t^*)^2 = id$,

(ii) $t^* \circ p^* = p^*$,

(iii) $p_* \circ t^* = p_*$, and

(iv) $p^* \circ p_* = 1 + t^*$.

Given an abelian group B, we can form the *constant Mackey functor* $\underline{B}$ where $\underline{B}(C_2) = \underline{B}(*) = B$, $t^* = id$, $p_* = 2$, and $p^* = id$. We will be concerned with the following constant Mackey functor.

$$\underline{\mathbb{Z}/2}: \quad \underset{\circlearrowleft_{1}}{\mathbb{Z}/2} \underset{0}{\overset{1}{\leftrightarrows}} \mathbb{Z}/2$$

2.2. Bigraded Theory

For a group G, Bredon cohomology is graded on $RO(G)$, the Grothendieck ring of finite-dimensional, real, orthogonal G-representations. When G is the cyclic group of order two, observe any such C_2-representation V is isomorphic to a direct sum of copies of the trivial representation $\mathbb{R}_{triv}$ and copies of the sign representation $\mathbb{R}_{sgn}$. Up to isomorphism, V is entirely determined by its dimension and the number of sign representations appearing in this decomposition. It follows that $RO(C_2)$ is a rank 2 free abelian group with generators given by $[\mathbb{R}_{triv}]$ and $[\mathbb{R}_{sgn}]$. For brevity, we will write $\mathbb{R}^{p,q}$ for the p-dimensional representation $\mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn}$. We will also write $\mathbb{R}^{p,q}$ for the element of $RO(C_2)$ that is equal to $(p-q)[\mathbb{R}_{triv}] + q[\mathbb{R}_{sqn}]$. When computing cohomology groups, we will write $H^{p,q}(X;M)$ for the cohomology group $H^{\mathbb{R}^{p,q}}(X;M)$. Note some authors have different grading conventions for $RO(C_2)$, and here we are using what is known as the motivic grading.

Given any finite-dimensional, real, orthogonal, G-representation V we can form the one-point compactification $\hat{V}$. Note this new space will be an equivariant sphere which we will denote S^V; such spaces are referred to as *representation* *spheres*. Using these representation spheres, we can form equivariant suspensions. Whenever we have a based G-space X, we can form the V-th suspension of X by

$$\Sigma^V X = S^V \wedge X.$$

Note the basepoint must be a fixed point. Often when working with free spaces we will add a disjoint basepoint in order to form suspensions and cofiber sequences. We use the common notation of X_+ for $X \sqcup \{*\}$ where the disjoint basepoint is understood to be fixed by the action.

An important feature of Bredon cohomology is that we have suspension isomorphisms: given any finite dimensional, real, orthogonal G-representation, there are natural isomorphisms

$$\Sigma^V : \tilde{H}^\alpha(-; M) \to \tilde{H}^{\alpha+V}(\Sigma^V(-); M).$$

Given a cofiber sequence of based G-spaces

$$A \xrightarrow{f} X \to C(f)$$

we can form the Puppe sequence

$$A \to X \to C(f) \to \Sigma^{\mathbf{1}} A \to \Sigma^{\mathbf{1}} C(f) \to \Sigma^{\mathbf{1}} X \to \dots$$

where $\mathbf{1}$ is the one-dimensional trivial representation. From the suspension isomorphism this yields a long exact sequence

$$\tilde{H}^V(A) \leftarrow \tilde{H}^V(X) \leftarrow \tilde{H}^V(C(f)) \leftarrow \tilde{H}^{V-\mathbf{1}}(A) \leftarrow \tilde{H}^{V-\mathbf{1}}(X) \leftarrow \dots$$

for each representation $V \in RO(G)$. We will make use of such long exact sequences throughout this thesis.

When $G = C_2$, we have already discussed how the Bredon cohomology theory is a bigraded theory, and we will carry this notation over when discussing representation spheres and equivariant suspensions. In particular, we will denote $S^{\mathbb{R}^{p,q}}$ by $S^{p,q}$ and for a based space X we will denote $\Sigma^{\mathbb{R}^{p,q}}X$ by $\Sigma^{p,q}X$. Translating the above into this notation, we have natural isomorphisms

$$\Sigma^{p,q} : \tilde{H}^{a,b}(-;M) \to \tilde{H}^{a+p,b+q}(\Sigma^{p,q}(-);M)$$

for all $p, q \geq 0$. Given a cofiber sequence we have long exact sequences

$$\cdots \to \tilde{H}^{p,q}(C(f)) \to H^{p,q}(X) \to H^{p,q}(A) \to \tilde{H}^{p+1,q}(C(f)) \to H^{p+1,q}(X) \to \ldots$$

for each $q \in \mathbb{Z}$.

During our computations, three particular representation spheres will appear often, namely $S^{1,1}$, $S^{2,1}$, and $S^{2,2}$. We include an illustration of these equivariant spheres below in Figure 10. The fixed set is shown in blue while the arrow is used to indicate the action of C_2 on the space.

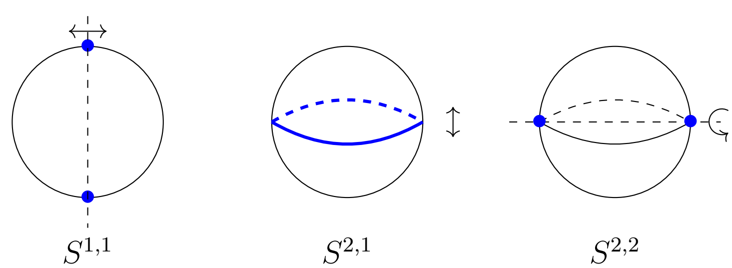


FIGURE 10. Some representation spheres.

2.3. The Cohomology of Orbits

Given any C_2-space X we have an equivariant map $X \to pt$ where pt denotes a single point with the trivial action. On cohomology, this gives a map of rings $H^{*,*}(pt;\underline{\mathbb{Z}/2}) \to H^{*,*}(X;\underline{\mathbb{Z}/2})$. Thus the cohomology of X is a module over the cohomology of a point, which recall we denote $\mathbb{M}_2 = H^{*,*}(pt;\underline{\mathbb{Z}/2})$. In this thesis, we will be computing the cohomology of various spaces as $\mathbb{M}_2$-modules.

Below we describe the cohomology of $pt = C_2/C_2$ as well as the cohomology of the free orbit C_2. These computations have been done many times and are often attributed to unpublished notes of Stong. The computation for coefficients in any constant Mackey functor can be found in [12]. A computation for constant integer coefficients can also be found in Appendix B of [6], and the same methods used there can be used to compute the cohomology of orbits in constant $\mathbb{Z}/2$ coefficients.

In $\underline{\mathbb{Z}/2}$-coefficients, the cohomology of a point is illustrated in the left-hand grid shown in Figure 11. The (p, q) spot on the grid refers to the $\mathbb{R}^{p,q}$-cohomology group. Each dot represents a copy of $\mathbb{Z}/2$, and we adopt the convention that the (p, q) group is plotted up and to right of the (p, q) coordinate. For example, $H^{0,0}(pt; \underline{\mathbb{Z}/2})$ is isomorphic to $\mathbb{Z}/2$, while $H^{1,0}(pt; \underline{\mathbb{Z}/2})$ is zero.

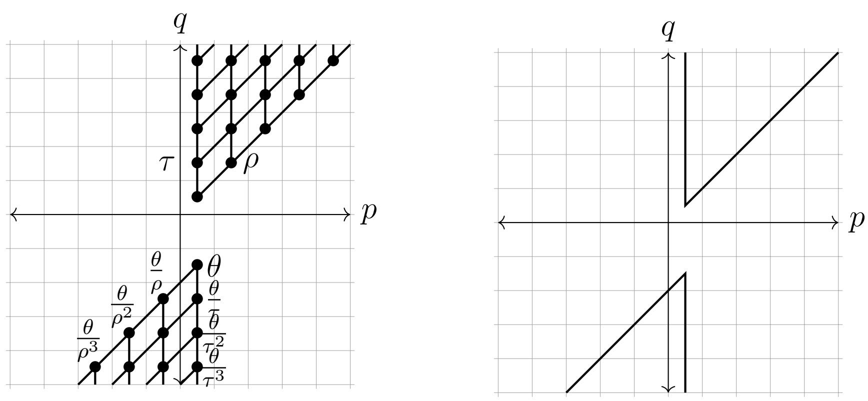


FIGURE 11. The ring $\mathbb{M}_2 = H^{*,*}(pt)$.

We will often refer to the portion of the cohomology in the first quadrant as the "top cone" and refer to the other portion as the "bottom cone". The top cone is polynomial in the elements ρ and τ, where ρ is in bidegree $(1, 1)$ and τ is in bidegree $(0, 1)$. Multiplication by τ is indicated with vertical lines, and multiplication by ρ is indicated with diagonal lines. For example, the nonzero element in $(1, 4)$ is equal to $\rho\tau^3$ which is equal to $\tau^3\rho$. The bottom cone is slightly more complicated. The nonzero element θ in bidegree $(0,-2)$ is divisible by all nonzero elements in the top cone. Explicitly, this means for pairs $i \geq 0$, $j \geq 0$, there exists an element denoted by $\frac{\theta}{\rho^i\tau^j}$ that satisfies $\rho^i\tau^j \cdot \frac{\theta}{\rho^i\tau^j} = \theta$. Note ρ and τ are not invertible elements in the ring; the notation $\frac{\theta}{\tau}, \frac{\theta}{\rho}$ is simply used to keep track of how ρ and τ multiply with these elements.

While doing computations, it is often easier to work with an abbreviated picture, which is given on the right-hand grid in the above figure. It is understood that there is a $\mathbb{Z}/2$ at each spot within the top cone and within the bottom cone with the relations described above.

We also include the cohomology of the free orbit C_2. As a ring, $H^{*,*}(C_2;\underline{\mathbb{Z}/2})$ is isomorphic to $\mathbb{Z}/2[u,u^{-1}]$ where u is in bidegree $(0,1)$. As an $\mathbb{M}_2$-module, $H^{*,*}(C_2)$ is isomorphic to $\tau^{-1}\mathbb{M}_2/(\rho)$. See Figure 12 for the pictorial representation of this module and its abbreviated version. In these module pictures, action by τ is indicated by vertical lines, while action by ρ is indicated by diagonal lines.

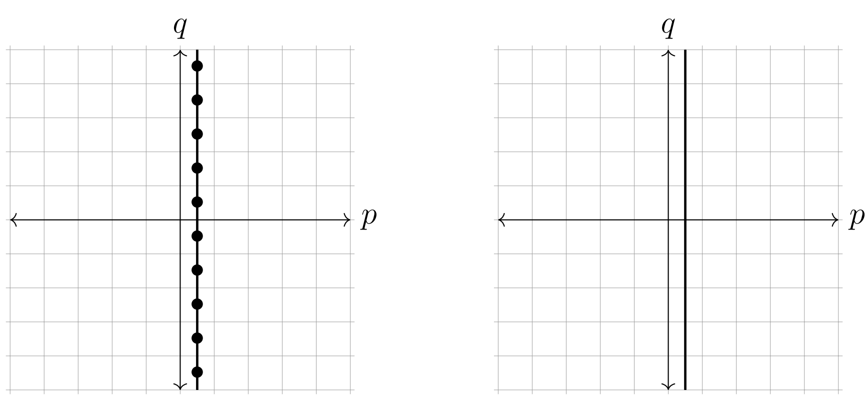


FIGURE 12. The cohomology of C_2 as an $\mathbb{M}_2$-module.

CHAPTER III

COMPUTATIONAL TOOLS

In this section we introduce various computational tools. The first two lemmas relate the Bredon cohomology to the singular cohomology of the quotient space and of the underlying space. We next show how these lemmas can be used to compute the cohomology of the antipodal spheres. After this example, we introduce a lemma that relates the Bredon cohomology to the cohomology of the fixed set via localization. Finally, we end with a few general theorems about the cohomology of finite C_2-CW complexes and general C_2-manifolds in the discussed coefficient system.

The first lemma holds for any constant Mackey functor, and it will be extremely useful in starting computations. We state it below for constant $\mathbb{Z}/2$-coefficients.

Lemma 3.0.1. *(The quotient lemma). Let X be a finite C_2-CW complex. We have the following isomorphisms for all p:*

$$H^{p,0}(X;\underline{\mathbb{Z}/2}) \cong H^{p,0}(X/C_2;\underline{\mathbb{Z}/2}) \cong H^{p}_{sing}(X/C_2;\mathbb{Z}/2).$$

Proof. We have a quotient map $X \to X/C_2$ that induces a map on cohomology $H^{p,0}(X/C_2;\underline{\mathbb{Z}/2}) \to H^{p,0}(X;\underline{\mathbb{Z}/2})$. Note both $H^{p,0}(X;\underline{\mathbb{Z}/2})$ and $H^{p,0}(X/C_2;\underline{\mathbb{Z}/2})$ are integer-graded cohomology theories. These two cohomology theories agree on both orbits C_2 and C_2/C_2 because the coefficients are given by a constant Mackey functor, and thus the first isomorphism follows for any finite C_2-CW complex. The second isomorphism follows because X/C_2 is a trivial C_2-space. □

Lemma 3.0.2. *(ρ-localization). Let X be a finite C_2-CW complex. Then*

$$\rho^{-1}H^{*,*}(X;\underline{\mathbb{Z}/2}) \cong \rho^{-1}H^{*,*}(X^{C_2};\underline{\mathbb{Z}/2}) \cong \rho^{-1}\mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X^{C_2};\mathbb{Z}/2).$$

The proof of this lemma is similar to the proof of the quotient lemma and can be found in [13].

For the next lemma, consider the cofiber sequence

$$S^{0,0} \hookrightarrow S^{1,1} \to C_{2+} \wedge S^{1,0}. \tag{3.0.1}$$

Smashing with any pointed C_2-space X, we obtain the cofiber sequence

$$S^{0,0} \wedge X \hookrightarrow S^{1,1} \wedge X \to C_{2+} \wedge S^{1,0} \wedge X. \tag{3.0.2}$$

The long exact sequence induced by this cofiber sequence relates multiplication by the element ρ to the singular cohomology of the space. Specifically, the long exact sequence is as stated in the following lemma. This statement can be found in [10] and is originally due to [1].

Lemma 3.0.3. *(The forgetful long exact sequence). Let X be a pointed C_2-space. For every integer q, we have a long exact sequence*

$$\longrightarrow \tilde{H}^{p-1,q}(X) \xrightarrow{\rho\cdot} \tilde{H}^{p,q+1}(X) \xrightarrow{\psi} \tilde{H}^p_{sing}(X) \longrightarrow \tilde{H}^{p,q}(X) \longrightarrow$$

where the coefficients are understood to be $\underline{\mathbb{Z}/2}$.

We will refer to the map $\psi : \tilde{H}^{p,q}(X) \to \tilde{H}^p_{sing}(X)$ as the "forgetful map". Note in $\mathbb{M}_2$, the element τ forgets to $1 \in H^0_{sing}(pt)$, while ρ forgets to zero. Indeed, by the exactness of the forgetful long exact sequence, for any X, a given cohomology class forgets to zero if and only if it is the image of ρ.

Example 3.0.4. Let's see how these tools can be used to compute the cohomology of a C_2-space. Note this computation is certainly not a new computation, but instead is done to review a standard fact, as well as to show the reader how we will use the computational tools discussed in this section. Let S^n_a denote the equivariant n-sphere whose C_2-action is given by the antipodal map. We proceed by induction to show

$$H^{*,*}(S^n_a; \underline{\mathbb{Z}/2}) \cong \tau^{-1}\mathbb{M}_2/(\rho^{n+1})$$

as an $\mathbb{M}_2$-module. The module $\tau^{-1}\mathbb{M}_2/(\rho^{n+1})$ can be represented pictorially as shown in Figure 13.

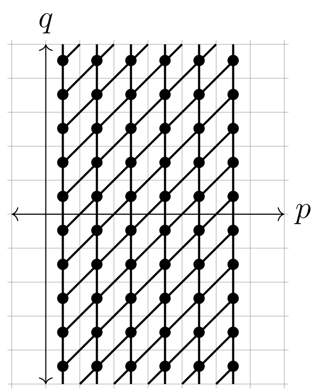


FIGURE 13. The $\mathbb{M}_2$-module $\tau^{-1}\mathbb{M}_2/(\rho^{n+1})$ when $n = 5$.

The base case in our inductive argument is given by $n = 0$, where S^0_a is understood to be the free orbit C_2. This case is done by the comments made in Section II. For the inductive hypothesis, let $n \geq 1$ and suppose $H^{*,*}(S^{n-1}_a) \cong \tau^{-1}\mathbb{M}_2/(\rho^n)$.

Our goal is to compute $H^{*,*}(S^n_a)$ using the inductive hypothesis, so we need a way to relate the cohomology of the $(n-1)$-dimensional antipodal sphere to the cohomology of the n-dimensional antipodal sphere. Consider the cofiber sequence

$$S^{n-1}_{a+} \hookrightarrow S^n_{a+} \to S^{n,0} \wedge C_{2+}$$

where S^{n-1}_a includes as the equator of S^n_a. (Note the disjoint basepoint on the antipodal spheres is needed to run the Puppe sequence, as described in Section

II.) The quotient space S^n_a/S^{n-1}_a is nonequivariantly homeomorphic to $S^n \vee S^n$ and the inherited C_2-action swaps the two copies; this is exactly the C_2-space $S^{n,0}\wedge C_{2+}$. For every integer q, we have a long exact sequence given by

$$\overset{d^{p-1,q}}{\to} \tilde{H}^{p,q}(S^{n,0}\wedge C_{2+}) \to \tilde{H}^{p,q}(S^n_{a+}) \to \tilde{H}^{p,q}(S^{n-1}_{a+}) \overset{d^{p,q}}{\to} \tilde{H}^{p+1,q}(S^{n,0}\wedge C_{2+}) \to$$

Note $\tilde{H}^{*,*}(X_+) = H^{*,*}(X)$, so we can express this long exact sequence as

$$\overset{d^{p-1,q}}{\to} \tilde{H}^{p,q}(S^{n,0}\wedge C_{2+}) \to H^{p,q}(S^n_{a}) \to H^{p,q}(S^{n-1}_{a}) \overset{d^{p,q}}{\to} \tilde{H}^{p+1,q}(S^{n,0}\wedge C_{2+}) \to \quad (3.0.3)$$

In order to find $H^{p,q}(S^n_a)$, we need to understand the differentials

$$d^{p,q} : H^{p,q}(S^{n-1}_a) \to \tilde{H}^{p+1,q}(C_{2+}\wedge S^{n,0})$$

for all (p,q). It is helpful to consider all of these differentials at once. Let

$$d = \bigoplus_{p,q} d^{p,q} : H^{*,*}(S^{n-1}_a) \to H^{*+1,*}(S^{n,0}\wedge C_{2+})$$

be the total differential. Note this differential is a module map, i.e. for every $r \in H^{*,*}(pt)$ and class $\alpha \in H^{*,*}(S^{n-1}_a)$, $d(r\alpha) = rd(\alpha)$.

By the inductive hypothesis, we already understand the cohomology of the domain. Namely $H^{*,*}(S^{n-1}_a) \cong \tau^{-1}\mathbb{M}_2/(\rho^n)$. Observe the codomain is the cohomology of the space $S^{n,0}\wedge C_{2+} = \Sigma^{n,0}C_{2+}$ which by the suspension isomorphism is given by

$$\tilde{H}^{*,*}(\Sigma^{n,0}C_{2+}) \cong \tilde{H}^{*-n,*}(C_{2+}) = \Sigma^{n,0}H^{*,*}(C_2).$$

Pictorially, the reduced cohomology of $C_{2+}\wedge S^{n,0}$ is just given by taking the cohomology of C_2 and shifting it to the right n units.

It is helpful to illustrate d via the following color-coded picture. The only

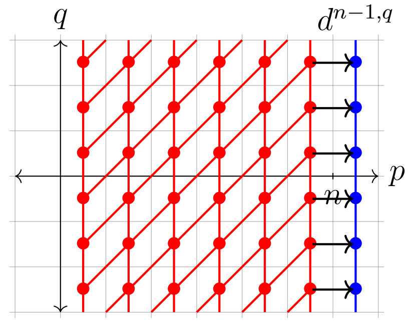


FIGURE 14. The differential $d : H^{*,*}(S_a^{n-1}) \to \tilde{H}^{*+1,*}(S^{n,0} \wedge C_{2+})$.

possible nonzero differentials occur in topological dimension $(n-1)$. Since d is a module map, it commutes with action of τ. The element τ acts on both modules as an isomorphism, so it suffices to find $d^{n-1,q}$ for a single q. Let's consider $q = 0$. By the quotient lemma given in Lemma 3.0.1, we have the following isomorphisms

$$H^{n,0}(S_a^n) \cong H^n_{sing}(S_a^n/C_2) \cong H^n_{sing}(\mathbb{R}P^n) \cong \mathbb{Z}/2.$$

By the exactness of the sequence in 3.0.3, we have the short exact sequence

$$0 \to \text{coker}(d^{n-1,0}) \to H^{n,0}(S_a^n) \to \ker(d^{n,0}) \to 0.$$

Now $H^{n,0}(S_a^{n-1}) = 0$ so $\ker(d^{n,0}) = 0$, and it must be that $\text{coker}(d^{n-1,0}) \cong \mathbb{Z}/2$ and $d^{n-1,0} = 0$. Thus the total differential d is zero.

We now need to solve the extension problem of $\mathbb{M}_2$-modules

$$0 \to \text{coker}(d) \to H^{*,*}(S_a^n) \to \ker(d) \to 0.$$

The kernel and cokernel are illustrated below (they are just the domain and codomain of d, respectively). Note the extension could be trivial, i.e. $H^{*,*}(S_a^n) \cong \text{coker}(d) \oplus \ker(d)$ as $\mathbb{M}_2$-modules, or there could be elements of $\ker(d)$ in topological dimension $(n-1)$ that lift to elements of $H^{n-1,0}(S_a^n)$ with a nontrivial ρ-action, as illustrated below.

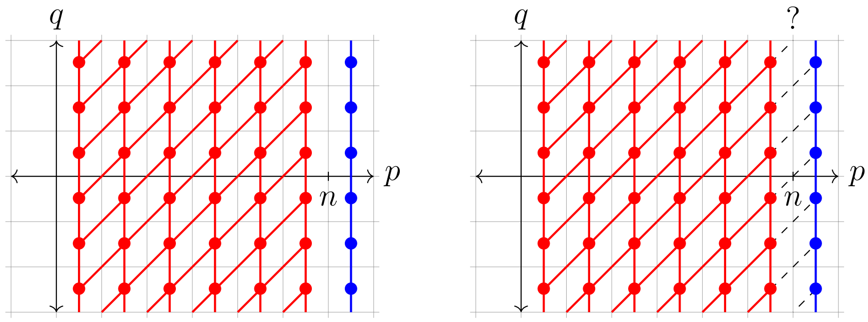


FIGURE 15. The extension problem.

We use a portion of the forgetful long exact sequence to see the extension is indeed nontrivial. Consider

$$\tilde{H}^{n-1,q-1}(S^n_{a+}) \xrightarrow{\cdot\rho} \tilde{H}^{n,q}(S^n_{a+}) \longrightarrow \tilde{H}^n_{sing}(S^n_{a+}) \longrightarrow \tilde{H}^{n,q-1}(S^n_{a+}) \xrightarrow{\cdot\rho} \tilde{H}^{n+1,q}(S^n_{a+})$$

Observe $\tilde{H}^{n+1,q}(S^n_{a+}) = 0$ while $\tilde{H}^{n,q-1}(S^n_{a+}) \cong \mathbb{Z}/2 \cong \tilde{H}^n_{sing}(S^n_{a+})$. Exactness then shows the multiplication by ρ in the left map must be an isomorphism for all q, and we conclude the module structure is the one shown in Figure 13.

Remark 3.0.5. We will be making arguments such as the one in Example 3.0.4 throughout this thesis. We will be less verbose and will use abbreviated pictures in future computations. For example, Figures 14 and 15 would be combined into the single abbreviated figure shown below. If the reader gets confused about the techniques in a future computation, we invite them to return to the above example as a kind of computational tutorial.

During our computations, we will often encounter spaces of the form $Y \times C_2$ where Y is some finite CW-complex. The cohomology of such a space depends entirely on the singular cohomology of Y, as shown in the lemma below.

Lemma 3.0.6. *Let Y be a C_2-space. The cohomology of the free C_2-space $C_2 \times Y$ is given by*

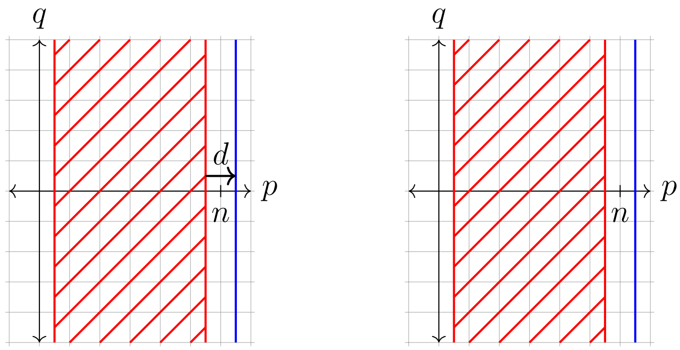


FIGURE 16. The abbreviated pictures for Example 3.0.4.

$$H^{*,*}(Y \times C_2; \underline{\mathbb{Z}/2}) \cong \mathbb{Z}/2[\tau, \tau^{-1}] \otimes_{\mathbb{Z}/2} H^*_{sing}(Y; \mathbb{Z}/2)$$

as an $\mathbb{M}_2$-algebra. If Y is a based C_2-space Y, we also have the isomorphism

$$\tilde{H}^{*,*}(Y \wedge C_{2+}; \underline{\mathbb{Z}/2}) \cong \mathbb{Z}/2[\tau, \tau^{-1}] \otimes_{\mathbb{Z}/2} \tilde{H}^*_{sing}(Y; \mathbb{Z}/2).$$

Proof. From the results in [4], a model representing Bredon cohomology in constant $\underline{\mathbb{Z}/2}$-coefficients is given by $K(\underline{\mathbb{Z}/2}; p, q) \simeq \mathbb{Z}/2\langle S^{p,q}\rangle$ where $\mathbb{Z}/2\langle S^{p,q}\rangle$ has underlying space given by the usual Dold-Thom space of configurations of points in S^p with labels in $\mathbb{Z}/2$, and has C_2-action given by the action on $S^{p,q}$.

Any C_2-equivariant map $Y \times C_2 \to \mathbb{Z}/2\langle S^{p,q}\rangle$ is entirely determined by the restriction to $Y \times \{1\}$ and thus

$$\begin{aligned} H^{p,q}(Y \times C_2; \underline{\mathbb{Z}/2}) &\cong [Y \times C_2, \mathbb{Z}/2\langle S^{p,q}\rangle]_{C_2} \\ &\cong [Y, \mathbb{Z}/2\langle S^p\rangle]_e \\ &\cong H^p_{sing}(Y; \mathbb{Z}/2) \end{aligned}$$

where $[-,-]_{C_2}$ denotes the collection of C_2-equivariant maps up to C_2-equivariant homotopy, and $[-,-]_e$ denotes the collection of nonequivariant homotopy classes of maps. This establishes the isomorphism stated in the lemma as bigraded abelian groups.

For the algebra structure, note the above shows the forgetful map ψ : $H^{p,q}(Y \times C_2) \to H^p_{sing}(Y \times C_2)$ is just the diagonal map, so in particular, nothing is in the kernel of ψ and ρ must act trivially. On the other hand $\psi(\tau) = 1$, so τ must act as an isomorphism on $H^{*,*}(Y \times C_2)$. This shows

$$H^{*,*}(Y \times C_2; \underline{\mathbb{Z}/2}) \cong \mathbb{Z}/2[\tau, \tau^{-1}] \otimes_{\mathbb{Z}/2} H^*_{sing}(Y; \mathbb{Z}/2)$$

as $\mathbb{M}_2$-modules, and the algebra statement follows because ψ is a multiplicative map. The reduced statement is proven similarly. □

If X is a trivial C_2-space, then we can similarly state the Bredon cohomology of X entirely in terms of the singular cohomology of X.

Lemma 3.0.7. *Let X be a trivial finite C_2-CW complex. Then as $\mathbb{M}_2$-algebras*

$$H^{*,*}(X; \underline{\mathbb{Z}/2}) \cong \mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X; \mathbb{Z}/2).$$

Proof. We have a functor $\Psi : \mathcal{T}op \to C_2\text{-}\mathcal{T}op$ that takes a space X and regards it as a trivial C_2-space. For each integer q we can define two cohomology theories on $\mathcal{T}op$ by $(\mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X; \mathbb{Z}/2))^q$ and $H^{*,q}(\Psi(X); \underline{\mathbb{Z}/2})$. As explained in the previous proof,

$$H^p_{sing}(X; \mathbb{Z}/2) = [X, \mathbb{Z}/2\langle S^p \rangle]_e$$

while

$$H^{p,0}(\Psi(X); \underline{\mathbb{Z}/2}) = [\Psi(X), \mathbb{Z}/2\langle S^{p,0} \rangle]_{C_2}.$$

Since both $\Psi(X)$ and $\mathbb{Z}/2\langle S^{p,0} \rangle$ are trivial C_2-spaces, the above is exactly equal to $[X, \mathbb{Z}/2\langle S^p \rangle]_e$. Thus we have a clear map

$$H^*_{sing}(X; \mathbb{Z}/2) = H^{*,0}(\Psi(X); \underline{\mathbb{Z}/2}) \hookrightarrow H^{*,*}(\Psi(X); \underline{\mathbb{Z}/2})$$

which induces a map from the free module

$$\mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X;\mathbb{Z}/2) \to H^{*,*}(\Psi(X);\underline{\mathbb{Z}/2}).$$

Restricting to the q-th grading, we have a map between cohomology theories

$$(\mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X;\mathbb{Z}/2))^q \to H^{*,q}(\Psi(X);\underline{\mathbb{Z}/2}).$$

This map is an isomorphism when $X = pt$, and so these cohomology theories agree for finite CW-complexes for all values of q. This establishes the stated isomorphism as bigraded abelian groups, and the algebra structure follows from noting the map $\mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X;\mathbb{Z}/2) \to H^{*,*}(\Psi(X);\underline{\mathbb{Z}/2})$ is actually an algebra map. □

We next state an important theorem about the cohomology of C_2-manifolds. Here by "C_2-manifold" we mean a piecewise linear manifold with a locally linear C_2-action (we want to ensure the fixed set is a disjoint union of submanifolds). By closed, we simply mean a closed manifold in the nonequivariant sense. The proof of the following theorem is somewhat tedious, so it has been been moved to Appendix A.

Theorem 3.0.8. *Let X be an n-dimensional, closed C_2-manifold with a nonfree C_2-action. Suppose $n-k$ is the largest dimension of submanifold appearing as a component of the fixed set. Then there is exactly one summand of $H^{*,*}(X;\underline{\mathbb{Z}/2})$ of the form $\Sigma^{i,j}\mathbb{M}_2$ where $i \geq n$, and it occurs for $(i,j) = (n,k)$.*

A Structure Theorem

We conclude this section by recalling a fact about the coefficient ring $\mathbb{M}_2$ as well as a structure theorem for the cohomology of finite C_2-CW complexes. The two theorems below can be found in [13].

Theorem 3.0.10 (C. May)**.** *As a module over itself, $\mathbb{M}_2$ is injective.*

The next theorem is a precise statement of the decomposition mentioned in the introduction. Note it is necessary that X is a finite C_2-CW complex in the sense that it only contains finitely many cells. Any closed C_2-manifold can be given the structure of a C_2-CW complex, so in particular, this theorem applies to all closed C_2-manifolds.

Theorem 3.0.11 (C. May)**.** *For any finite C_2-CW complex X, we can decompose the $RO(C_2)$-graded cohomology of X with constant $\underline{\mathbb{Z}/2}$-coefficients as*

$$H^{*,*}(X; \underline{\mathbb{Z}/2}) = (\oplus_i \Sigma^{p_i,q_i} \mathbb{M}_2) \oplus (\oplus_j \Sigma^{p_j,0} A_{n_j})$$

as a module over $\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{Z}/2})$ where A_n denotes the cohomology of the n-sphere with the free antipodal action.

CHAPTER IV

BACKGROUND ON SURFACES WITH AN INVOLUTION

In [7], all C_2-surfaces were classified up to equivariant isomorphism, and furthermore, a language was developed for describing the C_2-structure on a given equivariant surface. These descriptions are essential for many of the computations given in this paper. In this section, we review the needed terms and notations as well as some of the classification theorems. Notice all proofs are omitted in this section, and we direct the curious reader to [7].

4.1. Free Actions

In order to state the classification of free C_2-surfaces, we need to review one construction from equivariant surgery. This definition was first given in the introduction, but we restate it here for reference.

Definition 4.1.1. Let X be a nontrivial C_2-surface and Y be a nonequivariant surface. We can form the **equivariant connected sum** of X and Y as follows. Let Y' denote the space obtained by removing a small disk from Y. Let D be a disk in X that is disjoint from its conjugate disk σD and let X' denote the space obtained by removing both of these disks. Choose an isomorphism $f : \partial Y' \to \partial D$. Then the equivariant connected sum is given by

$$(Y' \times \{0\}) \sqcup (Y' \times \{1\}) \sqcup X']/ \sim$$

where $(y, 0) \sim f(y)$ and $(y, 1) \sim \sigma(f(y))$ for $y \in \partial Y'$. We denote this space by $X \#_2 Y$.

Remark 4.1.2. In our classifications, there are three important examples of equivariant connected sums that warrant their own notation. The first occurs when Y is the projective plane. In this case, we refer to the surgery as "adding dual cross caps" and write $X + [DCC]$ for $X\#_2\mathbb{R}P^2$. The phrase "dual cross caps" arrises from the pictorial representation often used to denote such surgery; see Figure 17 below for an example. Note we can add more than one set of dual cross caps to form $X+[DCC]+[DCC]$ which we will denote $X+2[DCC]$. In general, $X+n[DCC]$ is the space obtained by adding n dual cross caps to the C_2-surface X.

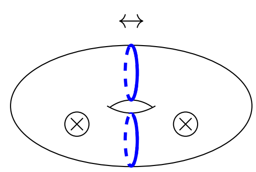


FIGURE 17. Dual cross caps added to a genus one torus with a reflection action. The fixed set is shown in blue.

The other two examples occur when X is one of $S^{2,2}$ or $S^{2,1}$, the two nonfree, nontrivial equivariant spheres. We refer to such spaces as **doubling spaces**, and denote the space $X\#_2 Y$ by $\text{Doub}(Y, 1 : S^{1,1})$ or $\text{Doub}(Y, 1 : S^{1,0})$, respectively. The phrase "doubling" is used to acknowledge that, nonequivariantly, $X\#_2 Y$ is homeomorphic to $Y\#Y$. Note these spaces can also be formed by removing a disk from Y to form Y' and then attaching a copy of Y' to each end of a cylinder of the form $S^{1,1} \times D(\mathbb{R}^{1,1})$ or $S^{1,0} \times D(\mathbb{R}^{1,1})$, respectively, where $D(\mathbb{R}^{1,1})$ is the unit interval in the sign representation. This is why $S^{1,1}$ and $S^{1,0}$ appear in the notation.

Recall from Chapter 3, we write S^n_a for the n-dimensional antipodal sphere. We are now ready to state the main classification theorem for free C_2-surfaces. All statements in the theorem should be followed with "up to equivariant isomorphism". Note by genus of a nonorientable space X we simply mean the

β-genus of X which was defined in the introduction to be the dimension of $H^1_{sing}(X;\mathbb{Z}/2)$. We denote such a space by N_s where s is the genus.

Theorem 4.1.3. *(Classification of free actions).*

(i) There is exactly one free structure on the even genus torus T_{2k} which is given by $S^2_a\#_2T_k$.

(ii) There are exactly two free structures on the odd genus torus T_{2k+1} which are given by $T_1^{anti}\#_2T_k$ and $T_1^{rot}\#_2T_k$.

(iii) There are no free structures on the odd genus non-orientable space N_{2k+1}.

(iv) There is exactly one free structure on the genus two non-orientable space N_2 which is given by $S^2_a+[DCC]$.

(v) There are exactly two free structures on the even genus non-orientable space N_{2k} when $k\geq 2$, which are given by $S^2_a+k[DCC]$ and $T_1^{anti}+(k-1)[DCC]$.

The above completely classifies free C_2-surfaces. We now state various classification theorems for the nonfree C_2-surfaces.

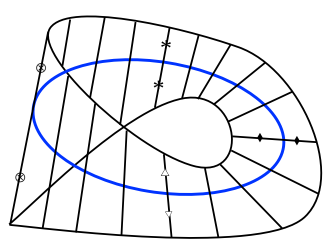
4.2. Nonfree Actions

For the classification of nonfree C_2-surfaces, we need to introduce three other types of equivariant surgery. The first two involve removing conjugate disks in order to attach an equivariant handle. There are two types of handles that can be attached. The first handle is given by $S^{1,1}\times D(\mathbb{R}^{1,1})$ where $D(\mathbb{R}^{1,1})$ is the unit disk in $\mathbb{R}^{1,1}$; we will refer to such a handle as an "$S^{1,1}-$antitube". The second type of handle is given by $S^{1,0}\times D(\mathbb{R}^{1,1})$; we refer to this handle as an "$S^{1,0}-$antitube". We give the precise definitions below.

Definition 4.2.1. Let X be a nontrivial C_2-surface. Form a new space denoted $X + [S^{1,0} - AT]$ as follows. Let D be a disk contained in X that is disjoint from its conjugate disk σD. Remove both disks from X and then attach an $S^{1,0}-$antitube. Whenever we construct such a space, we say we have done $\mathbf{S^{1,0}}$**−surgery**.

We can similarly define $\mathbf{S^{1,1}}$**−surgery** by instead attaching an $S^{1,1}-$antitube.

The third type of surgery involves removing a disk isomorphic to the unit disk in $\mathbb{R}^{2,2}$ and sewing in an equivariant Möbius band. This equivariant Möbius band can be formed as follows. Begin with the nonequivariant Möbius bundle over S^1, and then define an action on the fibers by reflection. In other words, each fiber should be isomorphic to $\mathbb{R}^{1,1}$ (in particular, the zero section is fixed). If we now take the closed unit disk bundle, note the boundary is a copy of S^1_a, as is the boundary of the removed disk $D(\mathbb{R}^{2,2})$. An illustration of this Möbius bundle is shown below. Conjugate points are indicated by matching symbols, while the fixed set is shown in blue.



Definition 4.2.2. Let X be a nontrivial C_2-surface that contains an isolated fixed point p. Then there exists an open disk $p \in D \subset X$ such that $D \cong D(\mathbb{R}^{2,2})$. Remove D and note $\partial D \cong S^1_a$. Now sew in a copy of the Möbius band described above. We denote this new space by $X + [FM]$ and refer to this surgery as **FM−surgery**. (Note "FM" is an abbreviation for "fixed point to Möbius band".)

The complete list of nonfree, nontrivial C_2-surfaces is given in [7], but it turns out we only need the following takeaway from this list in order to do computations in $\underline{\mathbb{Z}/2}$-coefficients.

Theorem 4.2.3. *Let X be a nonfree, nontrivial surface. If X is not isomorphic to a doubling space or to an equivariant sphere, then X can be obtained by doing $S^{1,0}-$, $S^{1,1}-$, or $FM-$ surgery to an equivariant space of lower β-genus.*

CHAPTER V

COMPUTATION FOR FREE SURFACES

In this section, we compute the cohomology of all free C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients. We first compute the cohomology of the two free tori T_1^{anti} and T_1^{rot}. We then utilize the decompositions given in Theorem 4.1.3 together with these initial computations to compute the cohomology of all free C_2-surfaces.

Notation 5.0.1. In this section, all coefficients will be understood to be $\underline{\mathbb{Z}/2}$.

Proposition 5.0.2. *We have the following isomorphisms of* $\mathbb{M}_2$*-modules*

$$H^{*,*}(T_1^{anti}) \cong H^{*,*}(S_a^1) \oplus \Sigma^{1,0} H^{*,*}(S_a^1) \cong H^{*,*}(T_1^{rot}).$$

Proof. Observe another way to define the antipodal torus is by the product $T_1^{anti} = S^{1,1} \times S_a^1$. This gives rise to the cofiber sequence for $X = T_1^{anti}$

$$S_{a+}^1 \hookrightarrow X_+ \to S^{1,1} \wedge S_{a+}^1$$

which is illustrated below in Figure 18. This cofiber sequence gives rise to long

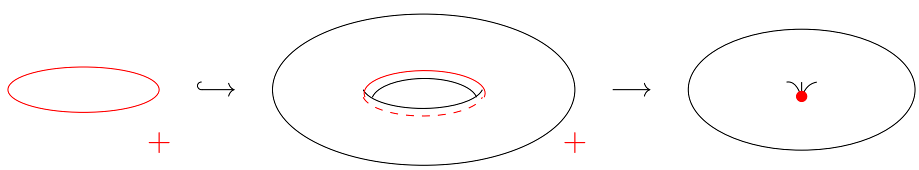


FIGURE 18. The cofiber sequence $S_{a+}^1 \hookrightarrow T_{1+}^{anti} \to S^{1,1} \wedge S_{a+}^1$.

exact sequences on cohomology. Similar to Example 3.0.4, we can organize these long exact sequences into the picture shown in Figure 19 below. By the quotient lemma given in Lemma 3.0.1, $\tilde{H}^{0,0}(X_+) \cong \tilde{H}^0_{sing}((X/C_2)_+) \cong \mathbb{Z}/2$, and so $d^{0,0} = 0$. From the module structure, we conclude $d^{p,q} = 0$ for all p, q.

It remains to solve the extension problem

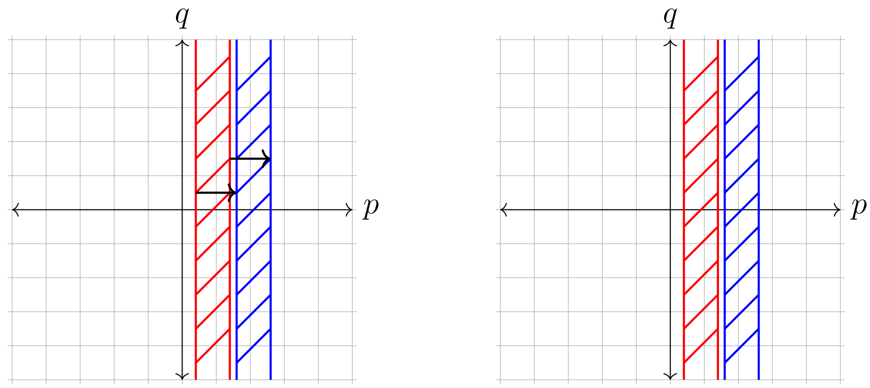


FIGURE 19. The differential $d : \tilde{H}^{*,*}(S^1_a) \to \tilde{H}^{*+1,*}(S^{1,1} \wedge S^1_{a+})$.

$$0 \to \Sigma^{1,1} H^{*,*}(S^1_a) \to H^{*,*}(X) \to H^{*,*}(S^1_a) \to 0,$$

which is shown on the right in Figure 19. The only possibility for a nontrivial extension is for there to exist a class $\alpha \in H^{0,q}(X)$ such that $\rho^2\alpha \in H^{2,q+2}(X)$ is nonzero.

Let $x \in S^{1,1}$ be one of the two fixed points. We have the following equivariant maps

$$S^1_a \cong S^1_a \times \{x\} \hookrightarrow S^1_a \times S^{1,1} = X \xrightarrow{\pi_1} S^1_a$$

where the last map is just projection onto the first factor. Note this composition is the identity, so in particular,

$$(\pi_1)^* : H^{*,*}(S^1_a) \to H^{*,*}(X)$$

is injective. In fact, this is actually an isomorphism in bidegrees $(0, q)$ because both groups have been computed to be $\mathbb{Z}/2$. Thus for every $\alpha \in H^{0,q}(X)$, there exists a class $\beta \in H^{0,q}(S^1_a)$ such that $\alpha = (\pi_1)^*(\beta)$ and

$$\rho^2\alpha = \rho^2(\pi_1)^*(\beta) = (\pi_1)^*(\rho^2\beta) = 0.$$

We conclude the extension in Figure 19 is trivial, and $H^{*,*}(T^{anti}_1) \cong H^{*,*}(S^1_a) \oplus \Sigma^{1,1} H^{*,*}(S^1_a)$. Lastly, observe as $\mathbb{M}_2$-modules, $\Sigma^{1,1} H^{*,*}(S^1_a) \cong \Sigma^{1,0} H^{*,*}(S^1_a)$.

For the other free torus, note $T_1^{rot} = S^{1,0} \times S^1_a$, so we can make similar use of the cofiber sequence

$$S^1_{a+} \hookrightarrow T_1^{rot} \to S^{1,0} \wedge S^1_{a+}$$

to see $H^{*,*}(T_1^{rot}) \cong H^{*,*}(S^1_a) \oplus \Sigma^{1,0} H^{*,*}(S^1_a)$. We leave the details to the reader. □

We have now computed the cohomology of the two free tori and the free C_2-sphere (see Example 3.0.4). By Theorem 4.1.3, all other free C_2-surfaces can be obtained by forming equivariant connected sums with these three spaces. We have the following lemmas on how such surgery affects the cohomology.

Lemma 5.0.3. *Suppose X is a free C_2-surface. If there is a surface Y such that $X \cong S^2_a \#_2 Y$, then*

$$H^{*,*}(X) \cong H^{*,*}(S^2_a) \oplus \left(\Sigma^{1,0} A_0\right)^{\oplus \beta(Y)}.$$

Proof. Let Y' be the space obtained by removing a small disk from Y. Consider the cofiber sequence

$$(Y' \times C_2)_+ \hookrightarrow \left(S^2_a \#_2 Y\right)_+ \to \tilde{S}^2_a \tag{5.0.1}$$

where $\tilde{S}^2_a$ is the "pinched" space appearing in the cofiber sequence

$$C_{2+} \hookrightarrow S^2_{a+} \to \tilde{S}^2_a. \tag{5.0.2}$$

An illustration of this cofiber sequence when $Y = T_1$ is shown below in Figure 20. Note the two red points on the right sphere are identified.

To make use of the cofiber sequence in 5.0.1, we first compute the cohomology of $\tilde{S}^2_a$. We can extend the cofiber sequence in 5.0.2 to

$$S^2_{a+} \to \tilde{S}^2_a \to \Sigma^{1,0} C_{2+}$$

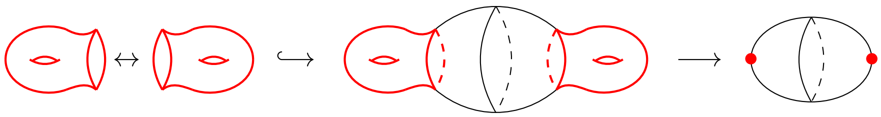


FIGURE 20. An example of the cofiber sequence appearing in 5.0.1. The red points in the right picture are identified.

and then analyze the connecting homomorphism $d : H^{p,q}(S^2_a) \to \tilde{H}^{p+1,q}(\Sigma^{1,0} C_{2+})$. This module map d is shown in Figure 21. By the quotient lemma,

$$\tilde{H}^{0,0}(\tilde{S}^2_a) \cong \tilde{H}^0_{sing}(S^2_a/C_2) = \tilde{H}^0_{sing}(\mathbb{R}P^2) = 0.$$

Thus $d^{0,0}$ must be an isomorphism, and by the module structure, $d^{0,q}$ must be an isomorphism for all q. It follows that $\operatorname{coker}(d) = 0$ and $\tilde{H}^{*,*}(\tilde{S}^2_a)$ is the module given on the right in Figure 21.

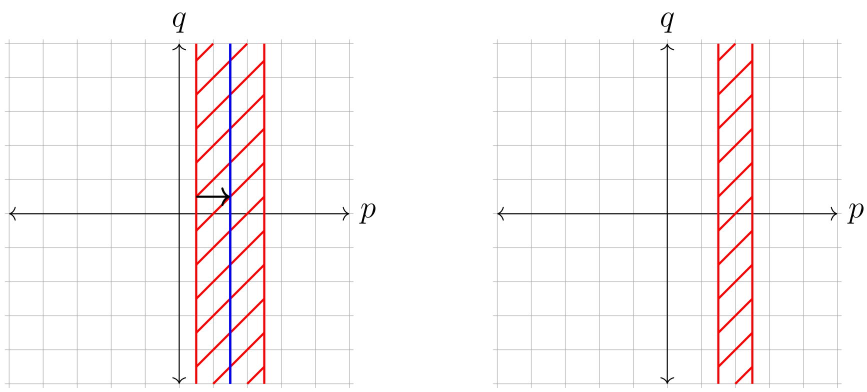


FIGURE 21. The differential $d : \tilde{H}^{*,*}(S^2_{a+}) \to \tilde{H}^{*+1,*}(\Sigma^{1,0} C_{2+})$

We now return to the cofiber sequence given in 5.0.1. The cohomology of $Y \times C_2$ is computed by Lemma 3.0.6, namely

$$H^{*,*}(Y \times C_2) \cong \mathbb{Z}/2[\tau, \tau^{-1}] \otimes_{\mathbb{Z}/2} H^*_{sing}(Y) \cong A_0 \oplus (\Sigma^{1,0} A_0)^{\oplus \beta(Y)}.$$

The cofiber sequence will thus give rise to the long exact sequences whose differentials are shown in Figure 22.

Since $X \cong S^2_a \#_2 Y$, $X/C_2 \cong \mathbb{R}P^2 \# Y$. We can again use Lemma 3.0.1 to see

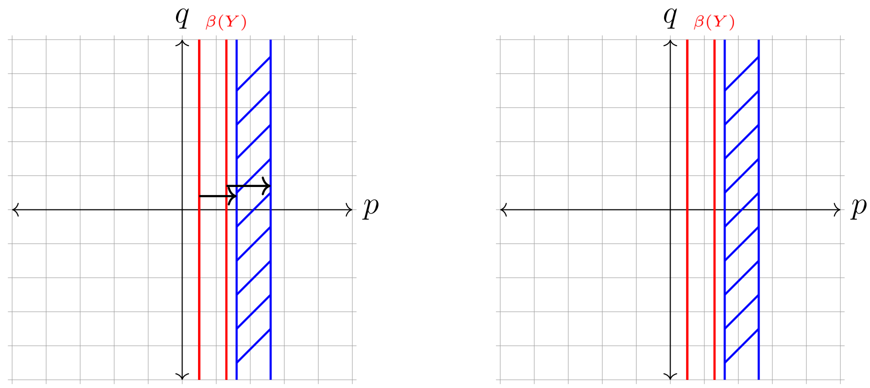


FIGURE 22. The differential $d : \tilde{H}^{*,*}((Y' \times C_2)_+) \to \tilde{H}^{*+1,*}(\tilde{S}^2_a)$

$$H^{0,0}(X) \cong H^0_{sing}(X/C_2) \cong \mathbb{Z}/2, \quad \text{and}$$

$$H^{1,0}(X) \cong H^1_{sing}(X/C_2) \cong (\mathbb{Z}/2)^{\beta(X)/2+1} = (\mathbb{Z}/2)^{\beta(Y)+1}.$$

Thus $d = 0$ and we must solve the extension problem shown on the right in Figure 22. To do so, consider the following map of cofiber sequences where the left vertical map is collapsing the two nonequivariant components of $Y' \times C_2$ to two points.

$$\begin{array}{ccccc} (Y' \times C_2)_+ & \hookrightarrow & X_+ & \longrightarrow & \tilde{S}^2_a \\ \downarrow & & \downarrow q & & \downarrow \\ C_{2+} & \hookrightarrow & S^2_{a+} & \longrightarrow & \tilde{S}^2_a \end{array}$$

As we've shown, both cofiber sequences induce long exact sequences on cohomology where the differential is zero. Thus we have the following commutative diagram where the rows are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{H}^{*,*}(\tilde{S}^2_a) & \longrightarrow & H^{*,*}(X) & \longrightarrow & H^{*,*}(Y' \times C_2) & \longrightarrow & 0 \\ & & id \uparrow & & q^* \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \tilde{H}^{*,*}(\tilde{S}^2_a) & \longrightarrow & H^{*,*}(S^2_a) & \longrightarrow & H^{*,*}(C_2) & \longrightarrow & 0 \end{array}$$

From exactness, the middle map must be injective. In particular, for all nonzero $\alpha \in H^{0,q}(S^2_a)$, $\rho^2\alpha \neq 0$ and so $\rho^2 q^*(\alpha) = q^*(\rho^2\alpha) \neq 0$. We conclude the extension in Figure 22 must be the nontrivial extension given by $H^{*,*}(S^2_a) \oplus (\Sigma^{1,0}A_0)^{\oplus\beta(Y)}$, as desired. (While this can be seen by making an appropriate choice of basis, it may be faster to apply the structure theorem given in 3.0.11. The above shows ρ^2 induces an isomorphism on bidegrees $(0,q)$, and the only possibility based on this fact and the groups we've computed is for the decomposition to be the described nontrivial extension.) □

Lemma 5.0.4. *Suppose X is a free C_2-surface. If there is a surface Y such that $X \cong T\#_2 Y$ where T is a free C_2-torus, then*

$$H^{*,*}(X) \cong H^{*,*}(T) \oplus (\Sigma^{1,0}A_0)^{\oplus \beta(Y)}.$$

Proof. We use the cofiber sequence

$$(Y' \times C_2)_+ \hookrightarrow (T\#_2 Y)_+ \to \tilde{T}$$

where $\tilde{T}$ is the space appearing in the cofiber sequence

$$C_{2+} \hookrightarrow T_+ \to \tilde{T}.$$

The proof will follow similarly to the proof of Lemma 5.0.3. We leave the details to the reader. □

Let A_n denote the cohomology of antipodal n-sphere, i.e. $A_n = \tau^{-1}\mathbb{M}_2/(\rho^{n+1})$. We can summarize the results of these lemmas in the following theorem.

Theorem 5.0.5. *Let X be a free C_2-surface. Then there are two cases for the $RO(C_2)$-graded Bredon cohomology of X in $\underline{\mathbb{Z}/2}$-coefficients.*

(i) Suppose X is equivariantly isomorphic to $S^2_a\#_2 Y$. Then $H^{*,*}(X) \cong (\Sigma^{1,0}A_0)^{\oplus \beta(X)/2} \oplus A_2$.

(ii) Suppose X is equivariantly isomorphic to $T\#_2 Y$ where T is a free C_2-torus. Then $H^{*,*}(X) \cong (\Sigma^{1,0}A_0)^{\oplus \frac{\beta(X)-2}{2}} \oplus A_1 \oplus \Sigma^{1,0}A_1$.

Proof. This follows almost immediately from Lemma 5.0.3 and Lemma 5.0.4. It remains to check we have the appropriate number of summands of $\Sigma^{1,0}A_0$ in each case. Since X is a free C_2-surface, $\chi(X) = 2\chi(X/C_2)$ and so $\beta(X) = 2\beta(X/C_2) - 2$. Observe in (i), $X/C_2 \cong Y\#\mathbb{R}P^2$ so

$$\beta(X) = 2\beta(X/C_2) - 2 = 2(\beta(Y) + 1) - 2 = 2\beta(Y).$$

In (ii), $X/C_2 \cong Y\#T_1$ or $X/C_2 \cong Y\#(\mathbb{R}P^2\#\mathbb{R}P^2)$ depending on the action on T. In either case,

$$\beta(X) = 2\beta(X/C_2) - 2 = 2(\beta(Y) + 2) - 2 = 2\beta(Y) + 2.$$

In both (i) and (ii), solving for $\beta(Y)$ yields the desired number of summands. □

We have now completed the computation for free C_2-surfaces in the given coefficient system. The next section handles nonfree C_2-surfaces.

CHAPTER VI

COMPUTATION FOR NONFREE SURFACES

In this section, we compute the cohomology of all nonfree C_2-surfaces in coefficients given by the constant Mackey functor $\underline{\mathbb{Z}/2}$. We first prove some lemmas about how the various equivariant surgeries discussed in Section III affect the cohomology of a C_2-surface. Utilizing Theorem 4.2.3 which states that all C_2-surfaces can be realized by doing such surgery to a simpler space, we then prove Theorem 1.4.1.

Notation 6.0.1. In this section, all coefficients are understood to be $\underline{\mathbb{Z}/2}$, unless stated otherwise. Given a C_2-surface X, we will often use F and C to denote the number of isolated fixed points and the number of fixed circles, respectively. Whenever there is some ambiguity, such as when we are working with multiple C_2-surfaces at once, we will write $F(X)$ and $C(X)$ for the corresponding values.

Lemma 6.0.2. *Let X be a C_2-surface. Suppose X is isomorphic to $Y + [S^{1,1} - \mathrm{AT}]$ for some free C_2-surface Y. Then*

$$\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0}A_0)^{\frac{\beta(Y)+2}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

Proof. Suppose Y is a free C_2-surface. In Section V we computed the cohomology of all such surfaces. Since τ acts invertibly on the modules A_n, we can regard these modules as $\tau^{-1}\mathbb{M}_2 \cong \mathbb{F}_2[\tau, \tau^{-1}, \rho]$-modules. If we ignore the action of ρ (in other words, regard as just $\mathbb{F}_2[\tau, \tau^{-1}]$-modules), observe the cohomology of all free C_2-surfaces can be described as

$$H^{*,*}(Y) \cong A_0 \oplus (\Sigma^{1,0}A_0)^{\oplus\frac{\beta(Y)+2}{2}} \oplus \Sigma^{2,0}A_0$$

where as before $A_0 \cong \tau^{-1}\mathbb{M}_2/(\rho)$. We now consider the following cofiber sequence for X

$$S^{1,1} \hookrightarrow X \to \tilde{Y}$$

where $\tilde{Y}$ is the pinched space appearing in the cofiber sequence

$$C_{2+} \hookrightarrow Y_+ \to \tilde{Y}.$$

We can compute the cohomology of $\tilde{Y}$ by extending to the cofiber sequence

$$Y_+ \to \tilde{Y} \to \Sigma^{1,0}C_{2+}. \tag{6.0.1}$$

For this computation, it suffices to understand $\tilde{H}^{*,*}(\tilde{Y})$ as a $\mathbb{F}_2[\tau, \tau^{-1}]$-module. The differential is shown in Figure 23.

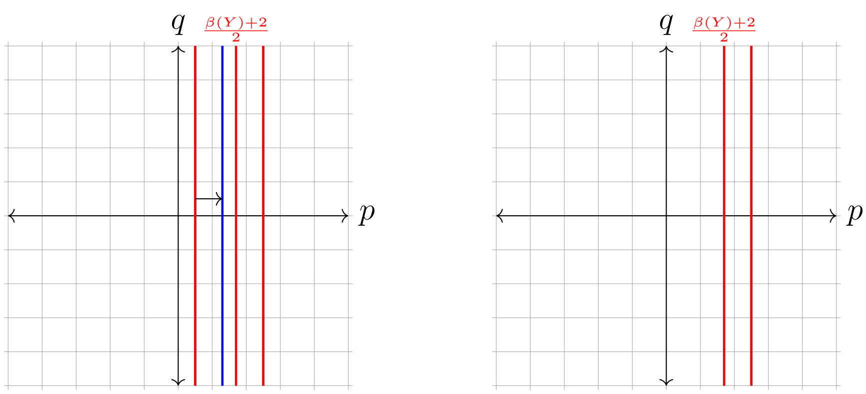


FIGURE 23. The differential $\tilde{H}^{*,*}(Y_+) \to \tilde{H}^{*,*}(\Sigma^{1,0}C_{2+})$.

Now $\tilde{H}^{0,0}(\tilde{Y}) \cong \tilde{H}^0_{sing}(\tilde{Y}/C_2)$ by Lemma 3.0.1, and $\tilde{H}^0_{sing}(\tilde{Y}/C_2) = 0$. Hence $d^{0,0}$ must be an isomorphism, and by the module structure, $d^{0,q}$ must be an isomorphism for all q. We conclude

$$\tilde{H}^{*,*}(\tilde{Y}) \cong (\Sigma^{1,0}A_0)^{\oplus \frac{\beta(Y)+2}{2}} \oplus \Sigma^{2,0}A_0$$

as $\mathbb{F}_2[\tau, \tau^{-1}]$-modules.

We now return to the cofiber sequence $S^{1,1} \hookrightarrow X \to \tilde{Y}$. The differential is shown in Figure 24, where we have omitted possible ρ actions in the picture of $\tilde{H}^{*,*}(\tilde{Y})$. The differential must be nonzero for if it were zero, the answer for the

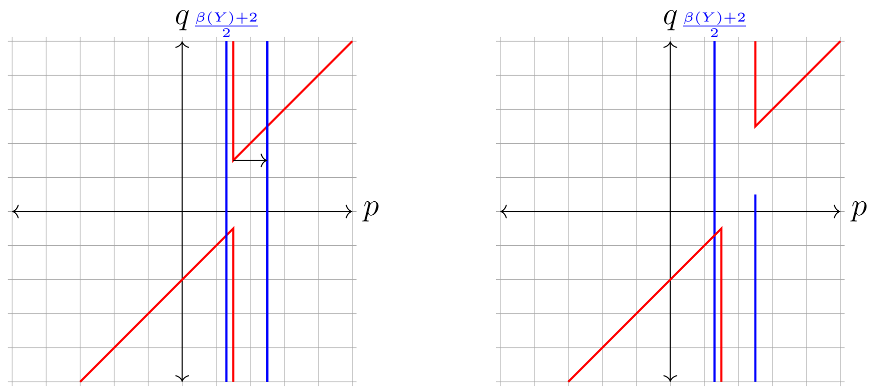


FIGURE 24. The differential $\tilde{H}^{*,*}(S^{1,1}) \to \tilde{H}^{*+1,*}(\tilde{Y})$.

cohomology of X would not have a free summand generated in dimension $(2,2)$ as is required by Theorem 3.0.8. Using the module structure, we thus know $d^{p,q}$ for all (p,q), and it remains to solve the extension problem given to the right in Figure 24. We can conclude by Theorem 3.0.8 (or by the forgetful long exact sequence) that the extension must be nontrivial, and we arrive at the desired answer. □

Lemma 6.0.3. *Let X be a C_2-surface that is isomorphic to $Y + [S^{1,0} - \mathrm{AT}]$ for some free C_2-surface Y. Then*

$$\tilde{H}^{*,*}(X; \underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0} A_0)^{\frac{\beta(Y)+2}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2.$$

Proof. The proof will follow as in the proof of Lemma 6.0.2 except now using the cofiber sequence $S^{1,0} \hookrightarrow X \to \tilde{Y}$. We leave the details to the reader. □

The following two lemmas state the cohomology of doubling spaces. The definition of these spaces can be found in Remark 4.1.2

Lemma 6.0.4. *Suppose X is homeomorphic to* $\mathrm{Doub}(Y, 1 : S^{1,1})$ *for some nonequivariant surface Y. Then*

$$\tilde{H}^{*,*}(X; \underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0} A_0)^{\beta(Y)} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

Proof. Consider the cofiber sequence

$$S^{1,1} \hookrightarrow X \to C_{2+} \wedge Y.$$

By Lemma 3.0.6 the cohomology of $C_{2+} \wedge Y$ is given by

$$\tilde{H}^{p,q}(C_{2+} \wedge Y) \cong \mathbb{Z}/2[\tau, \tau^{-1}] \otimes_{\mathbb{Z}/2} \tilde{H}^*_{sing}(Y) \cong (\Sigma^{1,0} A_0)^{\oplus \beta(Y)} \oplus \Sigma^{2,0} A_0.$$

The picture of the differential in the cofiber sequence is shown on the left in Figure 25. The differential cannot be zero for if it were, the answer for the cohomology of X would violate Theorem 3.0.8. Thus $d^{1,1}$ must be an isomorphism, and we can use the module structure to determine $d^{p,q}$ for all p, q.

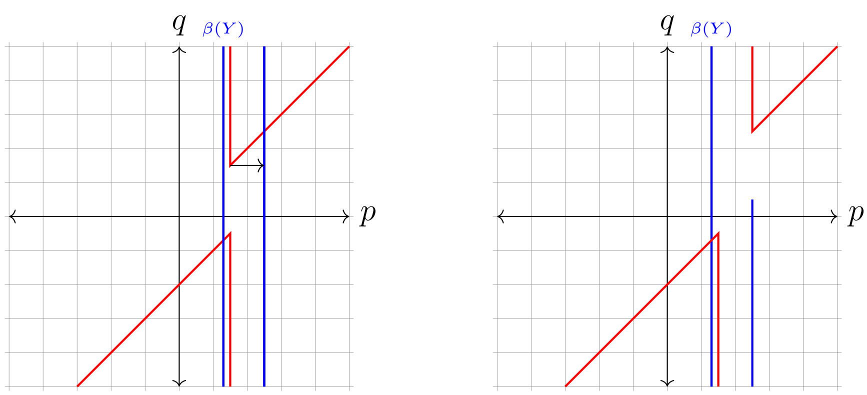


FIGURE 25. The differential $\tilde{H}^{*,*}(S^{1,1}) \to \tilde{H}^{*+1,*}(C_{2+} \wedge Y)$.

It remains to solve the extension problem

$$0 \to \mathrm{coker}(d) \to \tilde{H}^{*,*}(X) \to \ker(d) \to 0$$

which is shown on the right in Figure 25; note the kernel is shown in red, and the cokernel is shown in blue. Using either Theorem 3.0.8 or the forgetful long exact sequence, we can see the extension must be nontrivial, and in particular,

$$\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0}A_0)^{\beta(Y)} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

□

Lemma 6.0.5. *Suppose X is isomorphic to* $\mathrm{Doub}(Y, 1\colon S^{1,0})$ *for some nonequivariant surface Y. Then*

$$\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0}A_0)^{\beta(Y)} \oplus \Sigma^{2,1}\mathbb{M}_2.$$

Proof. The proof follows similarly to the proof of Lemma 6.0.4 using the cofiber sequence

$$S^{1,0} \hookrightarrow X \to C_{2+} \wedge Y.$$

We leave the details to the reader. □

We are now ready to prove the main theorem about the cohomology of nonfree, nontrivial C_2-surfaces. The proof will make use of the classification given in Theorem 4.2.3. We will induct on the β-genus of the surface and explore the various cases for what type of surgery is needed to construct a given C_2-surface from a C_2-surface of lower β-genus.

Theorem 6.0.6. *Let X be a nontrivial, nonfree C_2-surface. Let F denote the number of isolated fixed points, C denote the number of fixed circles, and β denote the β-dimension. There are two cases for the reduced $RO(C_2)$-graded Bredon cohomology of X in $\underline{\mathbb{Z}/2}$-coefficients:*

(i) Suppose $C = 0$. Then

$$\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,1}\mathbb{M}_2)^{\oplus F-2} \oplus (\Sigma^{1,0}A_0)^{\oplus \frac{\beta+2-F}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2$$

(ii) Suppose $C \neq 0$. Then

$$\begin{aligned}\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2}) \cong \left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C-1} &\oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F+C-1} \\ &\oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta+2-(F+2C)}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2\end{aligned}$$

Proof. We begin with case (i) and proceed by induction on the β-genus of X. If $\beta(X) = 0$, then X must be an equivariant sphere. The only equivariant sphere with isolated fixed points is $S^{2,2}$. By the suspension theorem, the reduced cohomology of $S^{2,2}$ is $\Sigma^{2,2}\mathbb{M}_2$, which matches the decomposition given in (i).

Now suppose $\beta(X) > 0$. By Theorem 4.2.3, we know X is either a doubling space, or X can be obtained by doing $S^{1,0}-$, $S^{1,1}-$, or $FM-$surgery to a C_2-surface of lower β-genus. If X is a doubling space, then we are done by Lemma 6.0.4. If X is obtained by doing equivariant surgery, then X must be isomorphic to $Y + [S^{1,1} - \mathrm{AT}]$ for some Y by consideration of the fixed set. If Y is free, then by Lemma 6.0.2 the cohomology of X is given by

$$\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2}) \cong (\Sigma^{1,0}A_0)^{\frac{\beta(Y)+2}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

We just need to check there are the appropriate number of summands of each module. Observe $F(X) = 2$ and $\beta(X) = \beta(Y) + 2$, so indeed

$$F(X) - 2 = 0, \text{ and } \tfrac{\beta(X)+2-F(X)}{2} = \tfrac{\beta(Y)+2}{2}$$

as desired.

If Y is nonfree, then by consideration of the fixed set of X, $F(Y) \neq 0$ while $C(Y) = 0$. Thus by induction, the cohomology of Y is given by

$$\tilde{H}^{*,*}(Y) \cong \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)-2} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-F(Y)}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2. \tag{6.0.2}$$

Consider the cofiber sequence

$$S^{1,1} \hookrightarrow X \to \tilde{Y}, \tag{6.0.3}$$

where as before $\tilde{Y}$ is the space appearing in the cofiber sequence

$$C_{2+} \hookrightarrow Y_+ \to \tilde{Y}.$$

Since Y has at least one fixed point, we claim $\tilde{Y}$ is homotopy equivalent to $Y \vee S^{1,1}$. To see why, let's be more careful with how we construct $\tilde{Y}$. Let $y \in Y^{C_2}$ be a chosen fixed point. Since Y has only isolated fixed points, there is a disk D in Y such that $y \in D$ and $D \cong D(\mathbb{R}^{2,2})$. Let $x \in D$ be an interior point that is not fixed, and include C_2 into Y as $\{x, \sigma x\}$. Let γ be a path from x to y contained in the interior of D such that when we quotient to $\tilde{Y}$

$$\mathrm{im}(\gamma) \cup \mathrm{im}(\sigma\gamma) \cong S^{1,1}.$$

See Figure 26 for an illustration of the image of γ in $\tilde{Y}$.

There is a homotopy from $\tilde{D} = \mathrm{cof}(C_2 \hookrightarrow D)$ to $D \vee S^{1,1}$ that keeps the boundary of D fixed, and thus can be extended to all of $\tilde{Y}$ to see $\tilde{Y} \simeq Y \vee S^{1,1}$. See the illustration below of the homotopy equivalence for $\tilde{D}$. Note the fixed set is in blue and the copy of $S^{1,1}$ is shown in red.

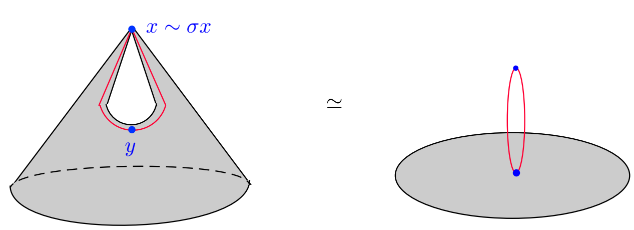


FIGURE 26. $\tilde{D} \simeq D \vee S^{1,1}$

Using the homotopy discussed above, we see that

$$\tilde{H}^{*,*}(\tilde{Y}) \cong \tilde{H}^{*,*}(Y) \oplus \Sigma^{1,1}\mathbb{M}_2.$$

The differential associated to the cofiber sequence in 6.0.3 is shown below in Figure 27. The number of summands of each of the blue submodules is omitted (note, in particular, the number of summands of $\Sigma^{1,0}A_0$ could be 0).

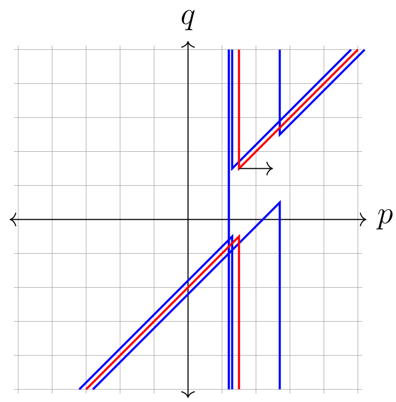


FIGURE 27. The differential $\tilde{H}^{*,*}(S^{1,1}) \to \tilde{H}^{*,*}(\tilde{Y})$.

The differential must be zero because the generator of $\Sigma^{1,1}\mathbb{M}_2$ maps to the trivial group. Hence we must solve the extension problem

$$0 \to \tilde{H}^{*,*}(\tilde{Y}) \to \tilde{H}^{*,*}(X) \to \tilde{H}^{*,*}(S^{1,1}) \to 0.$$

The above splits since $\tilde{H}^{*,*}(S^{1,1})$ is a free $\mathbb{M}_2$-module, and so

$$\begin{aligned}\tilde{H}^{*,*}(X) &\cong \tilde{H}^{*,*}(\tilde{Y}) \oplus \tilde{H}^{*,*}(S^{1,1}) \\ &\cong \tilde{H}^{*,*}(Y) \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus 2}.\end{aligned}$$

Putting this together with the isomorphism in 6.0.2, the cohomology of X is given by

$$\tilde{H}^{*,*}(X) \cong \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)-2+2} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-F(Y)}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2. \qquad (6.0.4)$$

It remains to check there are the appropriate number of summands of each module. Adding an $S^{1,1}-$antitube increases both the number of isolated fixed points and the

β-genus by 2, which can be written as

$$F(X) = F(Y) + 2, \text{ and } \quad \beta(X) = \beta(Y) + 2.$$

The number of $(\Sigma^{1,1}\mathbb{M}_2)$-summands and the number of $(\Sigma^{1,0}A_0)$-summands in 6.0.4 can thus be written as

$$F(Y) - 2 + 2 = F(X) - 2, \text{ and}$$
$$\tfrac{\beta(Y)+2-F(Y)}{2} = \tfrac{\beta(X)-2+2-(F(X)-2)}{2} = \tfrac{\beta(X)+2-F(X)}{2},$$

respectively. We have completed the proof for case (i).

We now consider the case when $C(X) \neq 0$. We again proceed by induction on the β-genus of X. If $\beta(X) = 0$, then X must be an equivariant sphere, and the only equivariant sphere with a fixed circle is $S^{2,1}$. By the suspension theorem, the reduced cohomology of $S^{2,1}$ is $\Sigma^{2,1}\mathbb{M}_2$, which matches the decomposition given in (ii).

Now suppose $\beta(X) > 0$. If X is a doubling space, we are done by Lemma 6.0.5. We can thus assume X is obtained by doing equivariant surgery to a C_2-surface Y of lower β-genus. Since we know X^{C_2} contains at least one fixed circle, we can assume X is obtained by doing $S^{1,0}-$ or $FM-$ surgery to an equivariant surface.

Let's first assume $X \cong Y + [S^{1,0} - AT]$. If Y is a free C_2-surface, then we are done after applying Lemma 6.0.3 and noting $C(X) = 1$ and $\beta(X) = \beta(Y) + 2$. Thus suppose Y is a nonfree C_2-surface and consider the cofiber sequence

$$S^{1,0} \hookrightarrow X \to \tilde{Y}.$$

Since Y^{C_2} is nonempty, we can make a similar argument as before to see $\tilde{Y} \simeq Y \vee S^{1,1}$ and so

$$\tilde{H}^{*,*}(\tilde{Y}) \cong \tilde{H}^{*,*}(Y) \oplus \Sigma^{1,1}\mathbb{M}_2.$$

If Y^{C_2} contains at least one fixed circle, then we know the cohomology of Y by induction. If Y^{C_2} contains only isolated fixed points, then we know the cohomology of Y by case (i). Thus there are two possibilities for the differential appearing in the long exact sequence associated to the cofiber sequence above; both are shown below in Figure 28. The case illustrated on the left is when Y^{C_2} contains at least

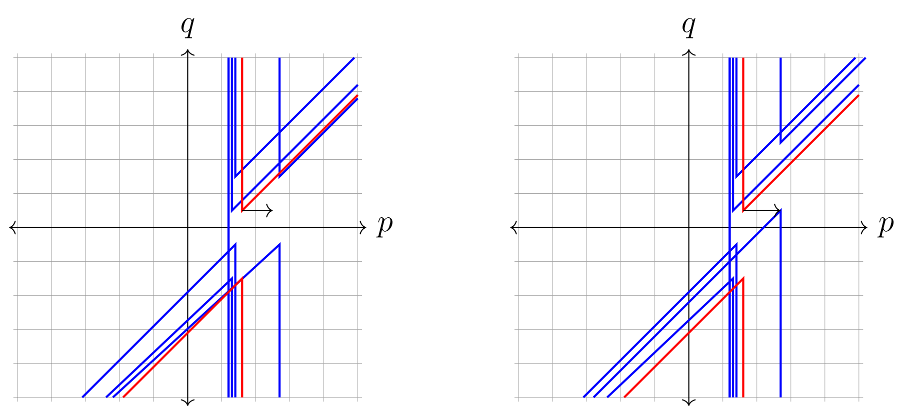


FIGURE 28. The two cases for the differential $\tilde{H}^{*,*}(S^{1,0}) \to \tilde{H}^{*,*}(\tilde{Y})$.

one fixed circle. In this case, we immediately see the differential must be 0 and conclude the extension is trivial because the kernel is a free $\mathbb{M}_2$-module. Thus

$$\tilde{H}^{*,*}(X) \cong \tilde{H}^{*,*}(Y) \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2,$$

which by induction gives

$$\begin{aligned}\tilde{H}^{*,*}(X) \cong &\Big(\left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C(Y)-1} \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)+C(Y)-1} \\ &\oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-(F(Y)+2C(Y))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2 \Big) \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2 \\ \cong &\left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C(Y)} \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)+C(Y)} \\ &\oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-(F(Y)+2C(Y))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2.\end{aligned}$$

Recall $X \cong Y + [S^{1,0} - AT]$, and so

$$F(Y) = F(X),\ C(Y) = C(X) - 1,\ \text{and}\ \beta(Y) = \beta(X) - 2.$$

By making the above substitutions, we arrive at the desired answer for case (ii).

The case on the right in Figure 28 is slightly more complicated. The differential cannot be zero in this case for if it were, the answer for the cohomology of X would violate Theorem 3.0.8. Noting $d^{1,0}$ must be nonzero and using the module structure to determine $d^{p,q}$ for all (p,q), we solve the extension problem

$$0 \to \operatorname{coker}(d) \to \tilde{H}^{*,*}(X) \to \ker(d) \to 0$$

which is illustrated below in Figure 29. The kernel of d is shown in red while the cokernel is shown in blue.

From Theorem 3.0.8, we have that $\Sigma^{2,1}\mathbb{M}_2$ must be a summand of $\tilde{H}^{*,*}(X)$, and so the extension is nontrivial. In particular, the extension must be given by

$$\tilde{H}^{*,*}(X) \cong \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)-2} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-F(Y)}{2}} \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2$$

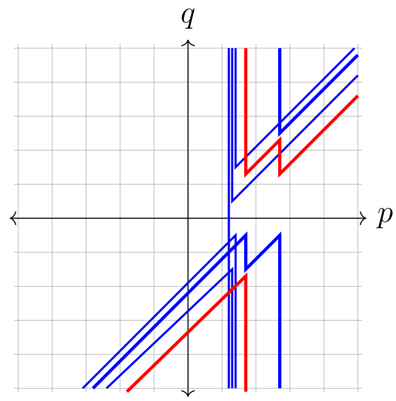


FIGURE 29. The extension problem.

where the last two summands arise from the nontrivial extension. It remains to check there are the appropriate number of summands of each submodule. Recall $X \cong Y + [S^{1,0} - AT]$ where Y^{C_2} consisted only of isolated fixed points. Thus

$$F(Y) = F(X),\ C(X) = 1,\ \text{and } \beta(Y) = \beta(X) - 2.$$

Making the substitution for $F(Y)$ and $\beta(Y)$ and noting $C(X) - 1 = 0$, we arrive at the desired decomposition given in (ii).

The only remaining case is when $X \cong Y + [FM]$ for some C_2-surface Y of lower β-genus. Let D be an equivariant closed neighborhood containing the attached Möbius band so that $D \simeq M \simeq S^{1,0}$. Consider the following cofiber sequence

$$D \hookrightarrow Y + [FM] \to Y.$$

Notice Y^{C_2} must be nonempty in order to do $FM-$surgery, so we know the cohomology of Y either from induction or from part (i). There are two cases for the cohomology of Y depending on whether or not Y^{C_2} contains a fixed circle. Similar to when $X \cong Y + [S^{1,0} - AT]$, this yields two cases for the differentials appearing in the long exact sequences associated to the above cofiber sequence. Though, note the cohomology of Y will now be appearing rather than the cohomology of $\tilde{Y}$.

The two cases for the differential are shown in Figure 30 below. We only include the relevant portion of the cohomology of Y in the picture, noting the other summands cannot be in the image of the differential for degree reasons as in Figure 28. The picture on the left shows the case when the fixed set of Y contains at least

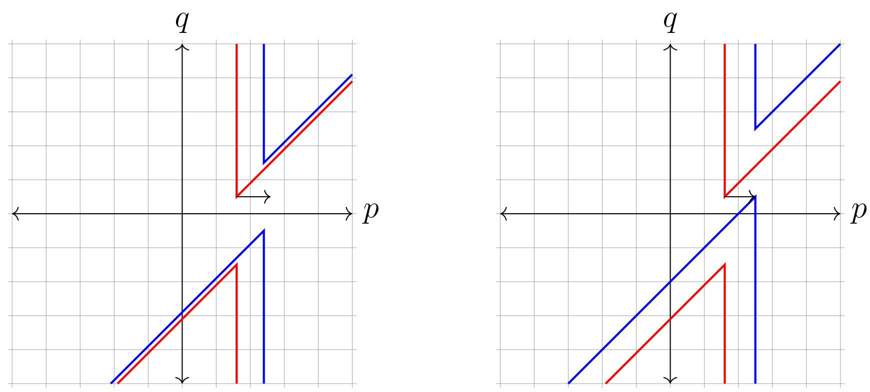


FIGURE 30. The two cases for the differential $d : \tilde{H}^{*,*}(S^{1,0}) \to \tilde{H}^{*,*}(Y)$.

one fixed circle. In this case, we quickly see the differential must be zero and the extension must be trivial. Thus

$$\tilde{H}^{*,*}(X) \cong \tilde{H}^{*,*}(Y) \oplus \Sigma^{1,0}\mathbb{M}_2.$$

By induction, we have

$$\begin{aligned}
\tilde{H}^{*,*}(X) \cong & \Big(\left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C(Y)-1} \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)+C(Y)-1} \\
& \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-(F(Y)+2C(Y))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2 \Big) \oplus \Sigma^{1,0}\mathbb{M}_2 \\
\cong & \left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C(Y)} \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)+C(Y)-1} \\
& \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-(F(Y)+2C(Y))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2.
\end{aligned}$$

Recall $X \cong Y + [FM]$, and so

$$F(Y) = F(X) + 1,\ C(Y) = C(X) - 1,\ \text{and}\ \beta(Y) = \beta(X) - 1.$$

These substitutions will give the decomposition stated in (ii).

The final case is $X \cong Y + [FM]$ where Y^{C_2} consists only of isolated fixed points. We return to the differential shown on the right in Figure 30. Observe on the quotient level, doing $FM-$surgery removes a disk. Thus X/C_2 is Y with a disk removed, so $H^2_{sing}(X/C_2) = 0$. By Lemma 3.0.1, $H^{2,0}(X) \cong H^2_{sing}(X/C_2)$, and so we conclude $d^{1,0}$ must be an isomorphism. We now must solve the extension problem shown below. The extension problem is solved after applying Theorem 3.0.8, and

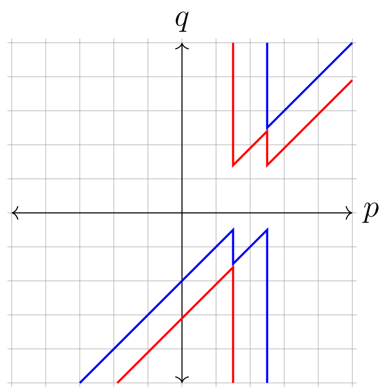


FIGURE 31. The extension problem.

we conclude

$$\begin{aligned}\tilde{H}^{*,*}(X) &\cong \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)-2} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-F(Y)}{2}} \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2 \\ &\cong \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)-1} \oplus \left(\Sigma^{1,0}A_0\right)^{\oplus \frac{\beta(Y)+2-F(Y)}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2.\end{aligned}$$

Observe $F(Y) = F(X) + 1$, $C(X) = 1$, and $\beta(Y) = \beta(X) - 1$. Making these substitutions will finish the proof. $\square$

CHAPTER VII

EQUIVARIANT FUNDAMENTAL CLASSES

We now employ Theorems B.2.1 and B.3.1 to define fundamental classes for C_2-submanifolds. We prove these classes forget to the usual singular fundamental classes, and the product of these fundamental classes is given in terms of the intersection of the submanifolds in the transverse case. We also show a handful of examples involving submanifolds of C_2-surfaces.

7.1. Nonequivariant Fundamental Classes

Recall for singular cohomology, we can define fundamental classes using the Thom isomorphism theorem. Let X be a closed manifold and $Y \subset X$ be a closed submanifold. If we are working with $\mathbb{Z}/2$-coefficients, all vector bundles over Y are orientable. In particular, if $\pi : N \to Y$ is the normal bundle of Y in X, then the Thom isomorphism theorem guarantees a unique class $u \in H^{n-k}(N, N-0)$ known as the Thom class such that

$$\pi^*(-) \smile u : H^j(Y) \to H^{n-k+j}(N, N-0)$$

is an isomorphism. There exists a tubular neighborhood U of Y in X, and by excision we have the following isomorphism

$$H^{n-k}(N, N-0) \cong H^{n-k}(U, U-Y) \cong H^{n-k}(X, X-Y).$$

Thus there is a unique nonzero class in $H^{n-k}(X, X-Y)$ corresponding to the Thom class. We can now define $[Y] \in H^{n-k}(X)$ to be the image of this unique class under the induced map from the inclusion of pairs $(X, \emptyset) \hookrightarrow (X, X-Y)$.

We will often denote these singular classes by $[Y]_{sing}$ to distinguish them from the Bredon cohomology fundamental classes defined below.

7.2. Fundamental Classes for Nonfree Submanifolds

We prove facts about the cohomology of Thom spaces in Theorem B.2.1 and B.3.1, and these theorems are enough to transfer the definitions given above from the singular world into the equivariant world.

Let X be a closed C_2-manifold and Y be a closed C_2-submanifold. For now, suppose both Y and X are nonfree, and suppose topologically Y is k-dimensional and X is n-dimensional. Let $N \to Y$ be the normal bundle of Y in X, and let q be the maximum weight of N over Y^{C_2} as in Theorem B.2.1. By this theorem, we are guaranteed a unique class $u_N \in H^{n-k,q}(N, N-0)$. Let U be an equivariant tubular neighborhood of Y. Using this neighborhood and excision

$$H^{n-k,q}(N, N-0) \cong H^{n-k,q}(U, U-Y) \cong H^{n-k,q}(X, X-Y).$$

We are now ready for the following definition.

Definition 7.2.1. Let Y, X, n, k, and q be defined as above. The unique nonzero class in $H^{n-k,q}(X, X-Y)$ corresponding to the Thom class u_N in the above isomorphism is denoted by $[Y]_{rel}$ and referred to as **the relative fundamental class of Y in X**. Furthermore, the image of this class in $H^{n-k,q}(X)$ under the induced map by the inclusion of the pair $(X, \emptyset) \hookrightarrow (X, X-Y)$ will be denoted $[Y]$ and referred to as **the fundamental class of Y in X**.

A simple corollary of property (i) of Theorem B.2.1 relates these equivariant fundamental classes to the nonequivariant fundamental classes.

Corollary 7.2.2. *Suppose $Y \subset X$ as above. Then $\psi([Y]) = [Y]_{sing}$ where ψ is the forgetful map to singular cohomology.*

This corollary allows us to prove the following statement about the product of fundamental classes of submanifolds whose intersection is nonequivariantly transverse. For notational simplicity, we say a submanifold X is codimension (k, q) if it has topological codimension k and if the maximum weight of the normal bundle over the fixed set is q, as in Theorem B.2.1.

Theorem 7.2.3. *Let X be a nonfree, n-dimensional C_2-manifold, and suppose Y and Z are two closed, nonfree, equivariant submanifolds of codimensions (k, q) and (ℓ, r), respectively, that intersect transversally. Suppose further $Y \cap Z$ is a nonfree submanifold. Let w be the maximum weight over the fixed set of the normal bundle of $Y \cap Z$ in X. Then*

$$[Y] \smile [Z] = \tau^{(q+r)-w}[Y \cap Z].$$

Proof. We first prove $w \leq q + r$. Let $N_{Y\cap Z}$, N_Y, and N_Z denote the normal bundles of $Y \cap Z$, Y, and Z in X, respectively. Let $W_1, \ldots, W_m$ be the connected components of the fixed set $(Y \cap Z)^{C_2}$. For $x \in W_j$ the fiber is given by

$$(N_{Y\cap Z})_x \cong \mathbb{R}^{k+\ell,w_j}$$

for some integer w_j. By Definition 7.2.1, the fundamental class $[Y \cap Z]$ has bidegree $(k + \ell, w)$ where $w = \max\{w_1, \ldots, w_m\}$.

Fix j and $x \in W_j$. Note $(Y \cap Z)^{C_2} = Y^{C_2} \cap Z^{C_2}$ so $W_r \subset Y_s \cap Z_t$ where Y_s and Z_t are connected components of Y^{C_2} and Z^{C_2}. Suppose

$$(N_Y)_x \cong \mathbb{R}^{k,q_s} \text{ and } (N_Z)_x \cong \mathbb{R}^{\ell,r_t}.$$

Since Y and Z intersect transversally,

$$(N_{Y\cap Z})_x \cong (N_Y)_x \oplus (N_Z)_x \cong \mathbb{R}^{k,q_s} \oplus \mathbb{R}^{\ell,r_t} \cong \mathbb{R}^{k+\ell,q_s+r_t}.$$

Thus $w_j = q_s + r_t$ and

$$w = \max_{1\le j\le m}\{w_j\} \le \max_{s,t}\{q_s + r_t\} = q + r,$$

as desired.

Now consider the relative fundamental classes

$$[Y]_{rel} \in H^{k,q}(X, X-Y),\ [Z]_{rel} \in H^{\ell,r}(X, X-Z),$$

$$[Y \cap Z]_{rel} \in H^{k+\ell,w}(X, X-(Y\cap Z)).$$

We have the following commutative diagram involving the forgetful map that allows us to relate these equivariant relative classes to singular relative classes.

$$\begin{array}{ccc} H^{k,q}(X,X-Y)\otimes H^{\ell,r}(X,X-Z) & \xrightarrow{\smile} & H^{k+\ell,q+r}(X,X-(Y\cap Z)) \\ {\scriptstyle \psi\otimes\psi}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi}\big\downarrow{\scriptstyle \cong} \\ H^k_{sing}(X,X-Y)\otimes H^\ell_{sing}(X,X-Z) & \xrightarrow{\smile} & H^{k+\ell}_{sing}(X,X-(Y\cap Z)) \end{array}$$

We have shown $q + r \ge w$, so the left and right vertical maps are isomorphisms by Corollary B.2.2. Let's consider the of image of the various relative classes under these maps. We have

$$\begin{array}{ccc} [Y]_{rel}\otimes [Z]_{rel} & \dashrightarrow & \tau^{(q+r)-w}\,[Y\cap Z]_{rel} \\ \big\downarrow & & \big\downarrow \\ [Y]_{rel,sing}\otimes [Z]_{rel,sing} & \longmapsto & [Y\cap Z]_{rel,sing} \end{array}$$

Note the vertical images in the above follow from Theorem B.2.1, while the bottom is a classical fact about singular cohomology. Now the dashed line follows from the commutativity of the above the diagram and the fact that the right vertical map is an isomorphism.

We get the statement for fundamental classes after considering the below diagram and using the above fact that $[Y]_{rel} \smile [Z]_{rel} = \tau^{q+r-w}[Y \cap Z]_{rel}$.

$$\begin{array}{ccc} H^{n-k,q_k}(X) \otimes H^{n-\ell,q_\ell}(X) & \xrightarrow{\smile} & H^{2n-k-\ell,q_k+q_\ell}(X) \\ \uparrow & & \uparrow \\ H^{n-k,q_k}(X, X-Y) \otimes H^{n-\ell,q_\ell}(X, X-Z) & \xrightarrow{\smile} & H^{2n-k-\ell,q_k+q_\ell}(X, X-(Y \cap Z)) \end{array}$$

□

Before defining fundamental classes for free submanifolds, we give a few examples.

Example 7.2.4 (*Fundamental class of a fixed point*)**.** Suppose X is a nonfree, closed, connected C_2-manifold and $x \in X^{C_2}$ is a fixed point. Let D be a tubular neighborhood of this fixed point, so $D \cong \mathbb{R}^{n,k}$ for some k. Then there is a class $[x] \in H^{n,k}(X)$, and since this class forgets to the singular class $[x] \in H^n_{sing}(X)$, it is necessarily nonzero.

The classes for fixed points provide some insight into Theorem 3.0.8. Choose a point $x \in X^{C_2}$ that is in a component of X^{C_2} of the smallest topological codimension k. Let $D \cong \mathbb{R}^{n,k}$ be a neighborhood of this point. It is shown in Corollary A.0.2 that the map $q : X \to X/(X \setminus D) \cong S^{n,k}$ induces a split injection on Bredon cohomology. By choosing a smaller neighborhood $x \in D' \subset D$ such that $q(D') \cong D'$, one can check $q^*([q(x)]) = [x]$. Now the class $[q(x)] \in H^{*,*}(S^{n,k}) \cong \mathbb{M}_2 \oplus \Sigma^{n,k}\mathbb{M}_2$ forgets to something nonzero, so it must generate the free summand in bidegree (n,k). Thus the class $[x]$ generates a free summand in bidegree (n,k) in $H^{*,*}(X)$.

Example 7.2.5. Consider the one-dimensional C_2-manifold $S^{1,1}$ whose cohomology is shown in the figure below.

There are two fixed points, so there are two fundamental classes $[a]$, $[b] \in H^{1,1}(S^{1,1})$. By our above discussion, $[a]$ and $[b]$ are both nonzero. We thus have

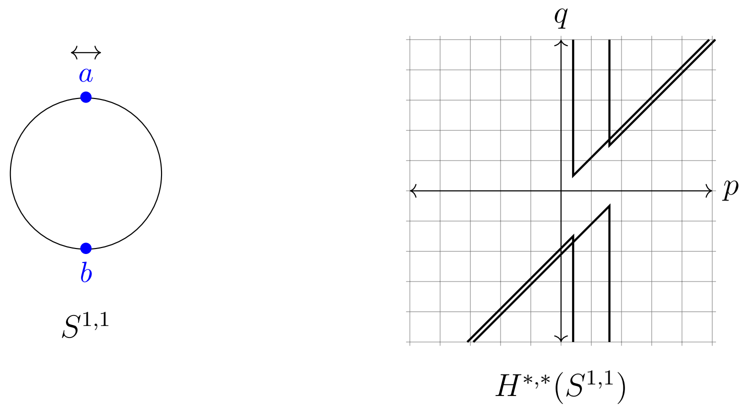


FIGURE 32. $S^{1,1}$ and its cohomology

three nonzero elements appearing in $H^{1,1}(S^{1,1})$, namely $[a]$, $[b]$, and $\rho \cdot 1$. We would like to determine the dependence relation between these classes.

Since $\psi([a]) \neq 0$ and $\psi([b]) \neq 0$, $[a] \neq \rho$ and $[b] \neq \rho$. To show $[a] \neq [b]$, let $a \in U \subset S^{1,1}$ where $U \cong D(\mathbb{R}^{1,1})$ and let $\iota_U : U \hookrightarrow S^{1,1}$ be the inclusion. Consider the following commutative diagram where the rows are exact:

$$\begin{array}{ccccc} H^{1,1}(S^{1,1}, S^{1,1} - \{a\}) & \longrightarrow & H^{1,1}(S^{1,1}) & \longrightarrow & H^{1,1}(S^{1,1} - \{a\}) \\ \downarrow \iota_U^* & & \downarrow \iota_U^* & & \downarrow \iota_U^* \\ H^{1,1}(U, U - \{a\}) & \longrightarrow & H^{1,1}(U) & \longrightarrow & H^{1,1}(U - \{a\}) \end{array}$$

By excision, the left vertical map is an isomorphism. Now $U - \{a\} \simeq C_2$, so $H^{1,1}(U - \{a\}) = 0$, and the bottom left map is a surjection. Hence, the composition of the top left map and the middle vertical map must be nonzero, and so $\iota_U^*([a]) \neq 0$.

On the other hand, we have another commutative diagram where again the rows are each part of a long exact sequence for a pair:

$$\begin{array}{ccc} H^{1,1}(S^{1,1}, S^{1,1} - \{b\}) & \longrightarrow & H^{1,1}(S^{1,1}) \\ \downarrow \iota_U^* & & \downarrow \iota_U^* \\ H^{1,1}(U, U) & \longrightarrow & H^{1,1}(U) \end{array}$$

The above shows $\iota_U^*([b]) = 0$. Thus $[a] \neq [b]$, and it must be that the three nonzero elements are pairwise distinct. Lastly, since we are working over $\mathbb{Z}/2$, the only possibility for a dependance relation is

$$[a] + [b] + \rho \cdot 1 = 0.$$

By Theorem 7.2.3, $[a] \cdot [b] = [\{a\} \cap \{b\}] = 0$. Using the dependance relation above we obtain $[a]^2 = [a]([b] + \rho) = \rho[a]$, and we have recovered the following isomorphism of $\mathbb{M}_2$-algebras:

$$H^{*,*}(S^{1,1}) \cong \mathbb{M}_2[x]/(x^2 = \rho x), \quad |x| = (1,1).$$

Example 7.2.6. Consider the C_2-surface $X \cong S^{2,1} + [S^{1,0} - AT]$ which can be depicted as a torus with a reflection action as shown below. The fixed set is shown in blue. By Theorem 6.0.6, the cohomology of X is

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2$$

as shown on the grid below.

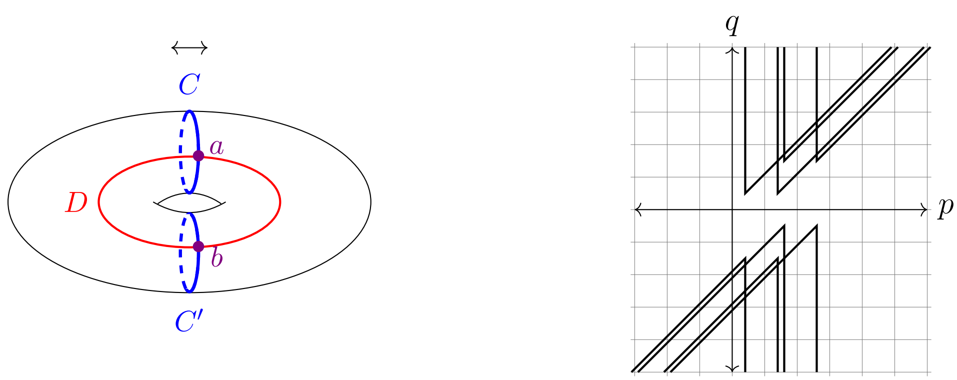


FIGURE 33. A C_2-torus and its cohomology

Notice $[C]$, $[C'] \in H^{1,1}(X)$, $[D] \in H^{1,0}(X)$, and $[a]$, $[b] \in H^{2,1}(X)$. All of these fundamental classes forget to nonzero singular classes, so they must be nonzero.

Now in $H^{1,1}(X)$, we have four nonzero elements $[C]$, $[C']$, $\tau[D]$, and $\rho \cdot 1$. Let's determine the dependence relations between these classes.

The classes $[C]$, $[C']$, and $\tau[D]$ forget to nonzero classes and thus cannot equal $\rho \cdot 1$. Using a neighborhood of C similar to Example 7.2.5, we can conclude $[C] \neq [C']$ and so $[C]+[C'] \neq 0$. Based on the decomposition for $H^{*,*}(X)$ and the forgetful long exact sequence, $\dim(\ker(\psi)) = 1$ in bidegree $(1,1)$. We have that $\psi(\rho) = \psi([C]+[C']) = 0$, so it must be that $[C]+[C'] = \rho \cdot 1$, and $\{[C], \rho, \tau[D]\}$ forms a basis for $H^{1,1}(X)$.

Using Theorem 7.2.3, we have the following multiplicative relations

$$[C][C'] = 0, \quad [C][D] = [a], \quad [C'][D] = [b], \quad [C]^2 = [C]([C'] + \rho \cdot 1) = \rho \cdot [C].$$

We can now state the cohomology of X as an algebra over $\mathbb{M}_2$:

$$H^{*,*}(X) \cong \mathbb{M}_2[x,y]/(x^2 = \rho x, y^2 = 0), \quad |x| = (1,1), |y| = (1,0).$$

(Here $x = [C]$ and $y = [D]$.)

Example 7.2.7. Consider the C_2-surface Y whose underlying space is the projective plane. We can form $\mathbb{R}P^2$ by identifying antipodal points on the boundary of D^2, and the space Y will inherit the action from the rotation action on D^2, as depicted below. Again, the fixed set is shown in blue. The cohomology of this space is given by

$$H^{*,*}(Y) \cong \mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2$$

as shown on the grid below. Let's consider the submanifold C and its normal bundle N_C. The circle C is fixed, and for every $x \in N_C$, the fiber is given by $(N_C)_x \cong \mathbb{R}^{1,1}$. Thus we have a class $[C] \in H^{1,1}(Y)$. The submanifold C' has two fixed points p, q, and $(N_{C'})_p \cong \mathbb{R}^{1,0}$ while $(N_{C'})_q \cong \mathbb{R}^{1,1}$. Thus we also have a class

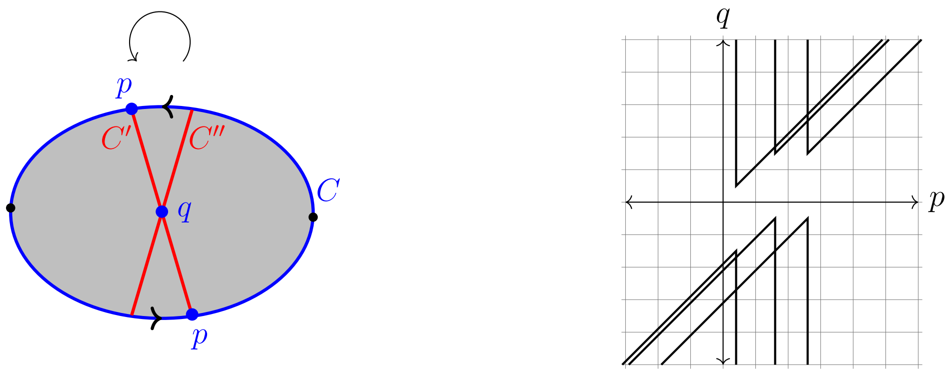


FIGURE 34. The space Y and its cohomology

$[C'] \in H^{1,1}(Y)$ and similarly a class $[C''] \in H^{1,1}(Y)$. By considering neighborhoods, we also see $[p] \in H^{2,1}(Y)$ while $[q] \in H^{2,2}(Y)$.

It is clear $[C'] = [C'']$. We have the following multiplicative relations given by Theorem 7.2.3

$$[C][C'] = \tau \cdot [C \cap C'] = \tau[p], \; [C']^2 = [C'][C''] = [C' \cap C''] = [q], \text{ and}$$

$$[p][C'] = [p][C''] = [\{p\} \cap C''] = 0.$$

Let D be a neighborhood of the isolated fixed point q such that $D \cong D(\mathbb{R}^{2,2})$. Similar to Example 7.2.5, we can show $\iota_D^*([q]) \neq 0$ while $\iota_D^*([p]) = 0$. Thus $\iota_D^*(\tau[p]) = 0$ and $\tau[p] \neq [q]$. We show $[q] = \tau[p] + \rho[C']$.

Note $[C]$ and $[C']$ both forget to the same nonequivariant nonzero class, so in particular, $[C] \neq \rho$ and $[C'] \neq \rho$. Also, since $[C][C'] = \tau[p] \neq [q]$ while $[C']^2 = [q]$, we see that $[C] \neq [C']$. Thus it must be that $[C] = [C'] + \rho$, and this also shows $[q] = [C']^2 = [C']([C] + \rho) = \tau[p] + \rho[C']$.

Taking $x = [C']$, $y = [p]$, we can now state the cohomology of Y as an $\mathbb{M}_2$-algebra:

$$H^{*,*}(Y) \cong \mathbb{M}_2[x, y]/(x^2 = \tau y + \rho x, y^2 = 0, xy = 0), \quad |x| = (1, 1), |y| = (2, 1).$$

7.3. Free Fundamental Classes

We now provide a definition for fundamental classes of free submanifolds of a given C_2-manifold. Note the manifold may or may not be free itself. Let X be a C_2-manifold of dimension n and let Y be a free, equivariantly connected submanifold of dimension k. By equivariantly connected, we mean the space cannot be covered by two nonempty, disjoint, equivariant open sets. Using excision and Theorem B.3.1, we are guaranteed a unique nonzero element in $H^{n-k,q}(X, X-Y)$ for every integer q.

Definition 7.3.1. Let X, Y, n, and k be as in the above. For every integer q, the unique nonzero element in $H^{n-k,q}(X, X-Y)$ is denoted $[Y]_{q,rel}$ and referred to as the **relative fundamental class of Y in X of weight q**. The image of this relative class under the map induced by the inclusion of pairs $(X, \emptyset) \hookrightarrow (X, X-Y)$ is denoted $[Y]_q$ and referred to as the **fundamental class of Y in X of weight q**.

There are a few immediate consequences of Theorem B.3.1 and the definition above.

Lemma 7.3.2. *Let $Y \subset X$ be as above. The following hold for all integers q:*

(a) $\psi([Y]_q) = [Y]_{sing}$*;*

(b) $\tau[Y]_q = [Y]_{q+1}$*;*

(c) If $Y \cong Z \times C_2$ for some connected nonequivariant submanifold Z, then $\rho \cdot [Y]_q = 0$ for all q.

Proof. Parts (a) and (b) follow from properties (i) and (ii) of Theorem B.3.1. For (c), let U be a tubular neighborhood of Y. By property (iv) of Theorem B.4.3,

$H^{*,*}(U, U - 0) \cong H^{*-k,*}(Y)$. By Lemma 3.0.6, every element in $H^{*,*}(Y)$ has ρ-torsion, and thus every element in $H^{*,*}(U, U-0)$ has ρ-torsion. In particular, $\rho \cdot [Y]_{q,rel} = 0$ which implies $\rho \cdot [Y]_q = 0$. □

We also have the following lemma about products involving free classes.

Lemma 7.3.3. *Let X be an n-dimensional C_2-manifold, and suppose Y and Z are equivariant submanifolds that intersect transversally in the nonequivariant sense and whose intersection is free. We have the following cases for the product of their fundamental classes.*

- *Suppose Y and Z are nonfree and their fundamental classes have weights q, r, respectively. Then $[Y] \smile [Z] = [Y \cap Z]_{q+r}$.*
- *Suppose Y is nonfree and Z is free. Then for every r, $[Y] \smile [Z]_r = [Y \cap Z]_{q+r}$.*
- *Suppose Y and Z are both free. Then for every r, s, $[Y]_r \smile [Z]_s = [Y \cap Z]_{r+s}$.*

Proof. This follows by forgetting to singular cohomology and using property (i) of Theorem B.3.1 and Corollary B.2.2 when one of the submanifolds is nonfree. We leave the details to the reader. □

We now discuss three examples involving these free fundamental classes.

Example 7.3.4 (Fundamental class of conjugate points)**.** Let X be a closed, connected, n-dimensional C_2-manifold with a non-fixed point $x \in X$. Consider the set of conjugate points $\{x, \sigma x\}$ and note this is isomorphic to the free orbit C_2. We show if X is free, then $[x, \sigma x]_q \neq 0$ for all q. If X is nonfree, let $p \in X^{C_2}$ be a fixed point whose fundamental class generates a free summand in bidegree (n, k). We show $[x, \sigma x]_q \neq 0$ only for $q \leq k-2$, and explicitly, $[x, \sigma x]_q = \frac{\theta}{\tau^{q-k+2}}[p]$.

To see this, observe the space $X - \{x, \sigma x\}$ is a punctured n-manifold, so $H^j_{sing}(X - \{x, \sigma x\}) = 0$ for $j \geq n$. Using the forgetful long exact sequence, we see that

$$\rho : H^{j,q}(X - \{x, \sigma x\}) \to H^{j+1,q+1}(X - \{x, \sigma x\})$$

must be an isomorphism whenever $j \geq n$ and surjective when $j = n - 1$.

Assume X is free. Then $X - \{x, \sigma x\}$ is also free and $H^{*,*}(X - \{x, \sigma x\})$ is a direct sum of shifted copies of A_r by Corollary **??**. By the above comments, it must be that $H^{j,q}(X - \{x, \sigma x\}) = 0$ for $j \geq n$. On the other hand, $H^n_{sing}(X) = \mathbb{Z}/2$ and so by the forgetful long exact sequence, $H^{n,q}(X) \neq 0$ for some q, and since τ acts invertibly on the cohomology of free spaces, $H^{n,q}(X) \neq 0$ for all q. Thus by the long exact sequence for the pair, $[x, \sigma x]_q \neq 0$ for all q.

If X is nonfree, then $X - \{x, \sigma x\}$ is also nonfree. Considering the structure theorem and the properties of the ρ action discussed above, it must be that no summands in $H^{*,*}(X - \{x, \sigma x\})$ are generated in topological dimension j for $j \geq n$ and all antipodal summands $\Sigma^{s,0} A_r$ must be concentrated in topological dimension less than n. Thus there is a sufficiently small ℓ such that $H^{n,q}(X - \{x, \sigma x\}) = 0$ whenever $q \leq \ell$.

Fix q such that $q \leq \ell$ and $q \leq k - 2$. Consider the long exact sequence below:

$$\longrightarrow H^{n,q}(X, X - \{x, \sigma x\}) \longrightarrow H^{n,q}(X) \longrightarrow H^{n,q}(X - \{x, \sigma x\}) \longrightarrow$$

The left group is $\mathbb{Z}/2$, and since the second map is surjective, there must be at most one nonzero element in $H^{n,q}(X)$. One such element is $\frac{\theta}{\tau^{q-k+2}}[p]$, and so the only option is $[x, \sigma x]_q = \frac{\theta}{\tau^{q-k+2}}[p]$. Now if this holds for some q, then it must hold for all $q \leq k - 2$ by the action of τ. Lastly for $q > k - 2$,

$$[x, \sigma x]_q = \tau^{q-(k-2)}[x, \sigma x]_{k-2} = \tau^{q-(k-2)} \cdot \theta[p] = 0.$$

Example 7.3.5. Let's consider the free torus where the action is orientation reversing; this space was denoted T_1^{anti} earlier in the paper. For notational simplicity, let $X = T_1^{anti}$. An illustration of the space and its cohomology are shown below.

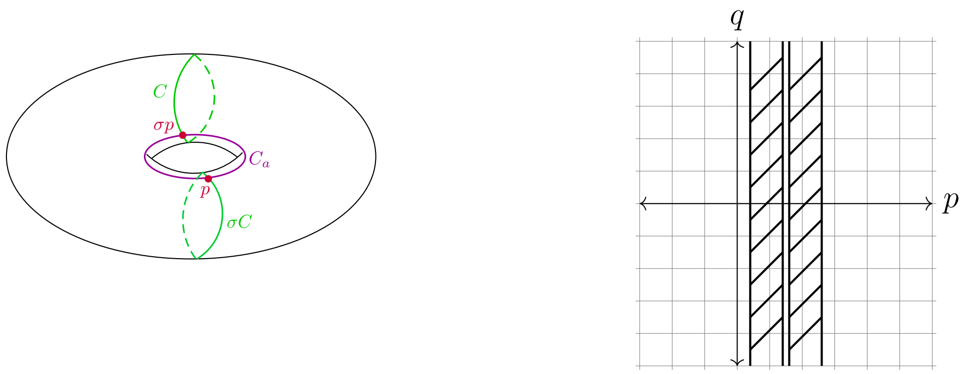


FIGURE 35. A free C_2-torus and its cohomology

There are four families of cohomology classes of interest: $[C_a]_q$, $[C \sqcup \sigma C]_q$, $[p \sqcup \sigma p]_q$, and $[X]_q$. Observe the classes $[C_a]_q$ and $[X]_q$ forget to nonzero classes in singular cohomology, so $[C_a]_q$ and $[X]_q$ are nonzero for all q. Using the long exact sequence for the pair $(X, X - (C \sqcup \sigma C))$, one can check $[C \sqcup \sigma C]_q \neq 0$ for all q. Now $\psi([C \sqcup \sigma C]_q) = 0$ so it must be that $[C \sqcup \sigma C]_q$ is in the image of ρ, and the only possibility is that $[C \sqcup \sigma C]_q = \rho \cdot [X]_{q-1} = \rho\tau^{q-1} \cdot 1$. By part (c) of Lemma 7.3.2, $\rho \cdot [C \sqcup \sigma C]_q = 0$ and in particular this implies $\rho \cdot [C \sqcup \sigma C]_1 = \rho^2 \cdot 1 = 0$. There is yet another way to see $\rho^2 \cdot 1 = 0$. By perturbing C, we can find a submanifold $C' \sqcup \sigma C'$ such that $[C' \sqcup \sigma C']_q = [C \sqcup \sigma C]_q$ and the transverse intersection $(C' \sqcup \sigma C') \cap (C \sqcup \sigma C)$ is empty.

Applying Theorem 7.3.3, we have that $[C_a]_r \cdot [C \sqcup \sigma C]_s = [p \sqcup \sigma p]_{r+s}$ for all r, s. We could have seen this already using the module structure and the above fact that $[C \sqcup \sigma C]_1 = \rho \cdot 1$, but it is nice to be able to recover the relation using fundamental classes.

Since X is free, the action of τ on the cohomology of X is invertible, and it is easier to describe the cohomology as a $\tau^{-1}\mathbb{M}_2$-algebra. Note this also encodes the $\mathbb{M}_2$-algebra structure. As a $\tau^{-1}\mathbb{M}_2$-algebra we have recovered the following isomorphism where $x = [C_a]$:

$$H^{*,*}(T_1^{anti}) \cong \tau^{-1}\mathbb{M}_2[x]/(\rho^2 \cdot 1 = 0, x^2 = 0), \ |x| = (1,1).$$

Example 7.3.6. We do one more example that has both nonfree and free fundamental classes. Consider $X = S^{2,2}\#_2 T_1$ which can be depicted as a genus two torus with a rotation action, as shown below. The cohomology of this space is given by

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus (\Sigma^{1,0}A_0)^{\oplus 2} \oplus \Sigma^{2,2}\mathbb{M}_2$$

which is shown in the grid below.

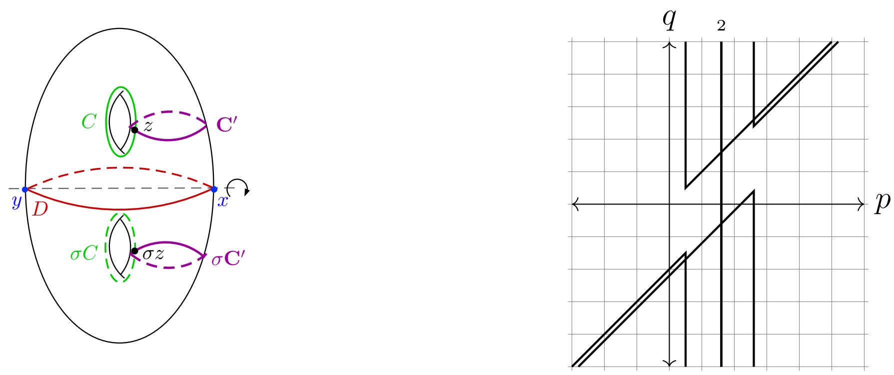


FIGURE 36. The space X and its cohomology.

Let's consider the nonfree fundamental classes $[D] \in H^{1,1}(X)$, $[x], [y] \in H^{2,2}(X)$, and the free fundamental classes, $[C \sqcup \sigma C]_q, [C' \sqcup \sigma C']_q \in H^{1,q}(X)$ and $[z \sqcup \sigma z]_q \in H^{2,q}(X)$. Note $[C \sqcup \sigma C]_q$ and $[C' \sqcup \sigma C']_q$ forget to different nonzero classes, so both are nonzero, and they are not equal. These two families of classes therefore give rise to the two $\Sigma^{1,0}A_0$ summands appearing. One can also check $[D] = \rho$ and $[x] + [y] = \rho^2 = \rho \cdot [D]$, using arguments as in the previous examples.

From Example 7.3.4, $[z \sqcup \sigma z]_q = \frac{\theta}{\tau^q}[p]$ for $q \leq 0$. This gives the multiplicative relation

$$[C \sqcup \sigma C]_r \smile [C' \sqcup \sigma C']_s = [z \sqcup \sigma z]_{r+s} = \frac{\theta}{\tau^{r+s}}[p]$$

for $r + s \leq 0$.

CHAPTER VIII

EQUIVARIANT FUNDAMENTAL CLASSES AND C_2-SURFACES

It is easy to check the singular cohomology in $\mathbb{Z}/2$-coefficients of any surface is generated by fundamental classes. In the previous examples, we saw the analogous statement held for the Bredon cohomology of a handful of C_2-surfaces. In this section, we show, in fact, the Bredon cohomology of any C_2-surface is generated by fundamental classes.

Notation 8.0.1. As before, the coefficients are understood to be $\underline{\mathbb{Z}/2}$ in this section.

We begin by defining the precise property we will be proving:

Definition 8.0.2. Let X be a C_2-manifold. Suppose there exist a (possibly empty) collection of nonfree equivariant submanifolds $Y_1, \dots Y_n$ and a (possibly empty) collection of free equivariant submanifolds $F_1, \dots, F_m$ such that the corresponding fundamental classes generate $H^{*,*}(X)$ as an $\mathbb{M}_2$-module, i.e.

$$\mathbb{M}_2\{[Y_1], \dots, [Y_n], [F_1]_q, \dots, [F_m]_q \colon q \in \mathbb{Z}\} = H^{*,*}(X).$$

Then we say $H^{*,*}(X)$ is **generated by fundamental classes**.

Our goal is to show if X is any C_2-surface, then $H^{*,*}(X)$ is generated by fundamental classes. We begin with free surfaces, spheres, and doubling spaces, and then we consider surfaces obtained by doing surgery to such spaces and apply Theorem 4.2.3.

Lemma 8.0.3. *Suppose X is a free C_2-surface. Then $H^{*,*}(X)$ is generated by fundamental classes.*

Proof. Let X be a free C_2-surface. By Theorem 4.1.3, there are two cases for the isomorphism type of X. Either $X \cong S^2_a \#_2 Y$ or $X \cong T\#_2 Y$ for some surface Y where T is one of the two free tori.

First suppose $X \cong S^2_a \#_2 Y$. Ignoring the ρ-action (so viewing $H^{*,*}(X)$ as an $\mathbb{Z}/2[\tau, \tau^{-1}]$-module), we have the following isomorphism from Theorem 5.0.5:

$$H^{*,*}(X) \cong A_0 \oplus (\Sigma^{1,0} A_0)^{\oplus \beta(X)/2+1} \oplus \Sigma^{2,0} A_0$$

Observe $\{[X]_q \colon q \in \mathbb{Z}\}$ will generate the classes corresponding to the summand A_0, while if $x \in X$ is any point, $[x \sqcup \sigma x]_q$ will generate the classes corresponding to the summand $\Sigma^{2,0} A_0$ as shown in Example 7.3.4. Next we find $(\beta(X)/2 + 1)$ free one-dimensional submanifolds of X to generate the classes appearing in topological dimension one.

Note $\beta(X) = 2\beta(Y)$, and since Y is a surface, there are $\beta(Y)$ circles in Y whose fundamental classes give a basis for $H^1_{sing}(Y)$. Let $r = \beta(Y)$ and call these circles $C_1, \ldots, C_r$. We can assume the disk removed from Y to form $S^2_a \#_2 Y$ does not intersect any of these submanifolds. Then for each $i = 1, \ldots, r$, $C_i \sqcup \sigma C_i$ is an equivariant submanifold of X.

Nonequivariantly, $X \cong Y \# Y$ and so the singular classes $[C_i \sqcup \sigma C_i]_{sing}$ are nonzero, and furthermore

$$\{[C_1 \sqcup \sigma C_1]_{sing}, \ldots, [C_r \sqcup \sigma C_r]_{sing}\}$$

is a linearly independent set in $H^1_{sing}(X)$. Using the forgetful map, the set

$$\{[C_1 \sqcup \sigma C_1]_q, \ldots, [C_r \sqcup \sigma C_r]_q\}$$

must be a linearly independent set in $H^{1,q}(X)$ for all q.

We have found a linearly independent set with $\beta(Y)$ elements, but we need $\beta(X)/2 + 1 = \beta(Y) + 1$ elements to give a basis for the $(\beta(Y) + 1)$-dimensional vector space $H^{1,q}(X)$. Let E an equatorial copy of S^1_a contained in S^2_a such that E is also a submanifold of X. Consider the following long exact sequence for the pair $(X, X-E)$:

$$\begin{array}{l} H^{1,q}(X,X-E) \longrightarrow H^{1,q}(X) \longrightarrow H^{1,q}(X-E) \\ \hookrightarrow H^{2,q}(X,X-E) \longrightarrow H^{2,q}(X) \longrightarrow H^{2,q}(X-E) \end{array}$$

Let U be a tubular neighborhood of E. Then by Theorem B.4.3

$$H^{*,*}(X,X-E) \cong H^{*,*}(U,U-E) \cong H^{*-1,*}(E).$$

On the other hand, $X-E$ is isomorphic to $C_2 \times Y'$ where Y' is the space obtained by removing a disk from Y. In particular $X-E \simeq (\vee_{\beta(Y)} S^1) \times C_2$ and $H^{2,q}(X-E) = 0$. Thus the above portion of the long exact sequence becomes

$$\begin{array}{l} H^{1,q}(X,X-E) \longrightarrow (\mathbb{Z}/2)^{\beta(Y)+1} \longrightarrow (\mathbb{Z}/2)^{\beta(Y)} \\ \hookrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \end{array}$$

By considering dimensions, we see that the top left map must be nonzero, and $[E]_q \neq 0$ for all q.

Lastly, note ψ applied to any nontrivial linear combination of the classes $[C_i \sqcup \sigma C_i]_q$ is nonzero, while $\psi([E]_q) = 0$ because E bounds a nonequivariant submanifold in X. Thus $[E]_q$ cannot be a linear combination of these classes, and

$$\{[C_1 \sqcup \sigma C_1]_q, \ldots, [C_r \sqcup \sigma C_r]_q, [E]_q\}$$

is a basis of fundamental classes for $H^{1,q}(X)$. We conclude $H^{*,*}(X)$ is generated by fundamental classes.

The case when $X \cong T\#_2 Y$ is similar. Again applying Theorem 5.0.5 and viewing $H^{*,*}(X)$ as an $\mathbb{Z}/2[\tau,\tau^{-1}]$-module, we have

$$H^{*,*}(X) \cong A_0 \oplus (\Sigma^{1,0} A_0)^{\oplus \beta(X)/2+1} \oplus \Sigma^{2,0} A_0.$$

In this case, $\beta(X) = 2\beta(Y) + 2$, and so we need to find $\beta(Y) + 2$ one-dimensional equivariant fundamental classes. Take the $\beta(Y)$ classes $[C_i \sqcup \sigma C_i]_q$ coming from Y as before, and then take the two equivariant classes in T as described in Example 7.3.5. These classes will forget to a linearly independent set of fundamental classes in singular cohomology, and thus must be linearly independent in $H^{1,q}(X)$. We again conclude $H^{*,*}(X)$ is generated by fundamental classes. □

Lemma 8.0.4. *Suppose X is a C_2-sphere. Then $H^{*,*}(X)$ is generated by fundamental classes.*

Proof. There are exactly four C_2-spheres up to equivariant isomorphism: S^2_a, $S^{2,0}$, $S^{2,1}$, and $S^{2,2}$. Note S^2_a was handled in the above theorem. When X is $S^{2,0}$, $S^{2,1}$, or $S^{2,2}$, we need only take the classes $[X]$ and $[p]$ where $p \in X$ is some fixed point. These classes will generate $H^{*,*}(X)$. □

Lemma 8.0.5. *Suppose X is a doubling space. Then $H^{*,*}(X)$ is generated by fundamental classes.*

Proof. There are two cases, either $X \cong S^{2,2}\#_2 Y$ or $X \cong S^{2,1}\#_2 Y$. The proof for both cases is very similar, so we only provide details for the former. Also, the techniques used are similar to those used in the proof of Lemma 8.0.3, so we only provide an outline for the case when $X \cong S^{2,2}\#_2 Y$, leaving the details to the reader.

Using Theorem 6.0.6, the cohomology of $S^{2,2}\#_2 Y$ is given by

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus (\Sigma^{1,0} A_0)^{\oplus \beta(Y)} \oplus \Sigma^{2,2}\mathbb{M}_2.$$

The class $[X]$ will generate the $\mathbb{M}_2$-summand appearing in bidegree $(0,0)$ while $[p]$ where $p \in X$ is any fixed point is a generator for the $\mathbb{M}_2$-summand appearing in bidegree $(2,2)$. Thus we must find $\beta(Y)$ one-dimensional submanifolds whose classes will generate the A_0-summands appearing in topological degree one. Using submanifolds of Y, we can create a basis of fundamental classes for the space $H^{1,q}(X)$ as in the proof of Lemma 8.0.3. □

Our next goal is to show this property holds for all nonfree surfaces. To do so, we make use of Theorem 4.2.3, which states if X is a nontrivial C_2-surface that is not free, not isomorphic to a sphere, and not isomorphic to a doubling space, then up to equivariant isomorphism, X can be constructed by doing $S^{1,1}-$, $S^{1,0}-$, or $FM-$surgery to a C_2-surface of lower β-genus. We first prove a lemma that will be helpful in what follows.

Lemma 8.0.6. *Let X be a nonfree C_2-surface. Suppose $C \subset X$ is a connected, one-dimensional, nonfree submanifold whose normal bundle is given by $C \times \mathbb{R}^{1,1}$. Let $q \in C \subset X$ be a fixed point. Then $\iota_q^*([C]) = \rho$ where $\iota_q : \{q\} \hookrightarrow X$ is the inclusion map.*

Proof. We make use of the geometric interpretation of $\rho \in H^{1,1}(pt)$ in order to prove this lemma. From the results in [4], a model representing Bredon cohomology in constant $\underline{\mathbb{Z}/2}$-coefficients is given by $K(\underline{\mathbb{Z}/2}; p, q) \simeq \mathbb{Z}/2\langle S^{p,q}\rangle$ where $\mathbb{Z}/2\langle S^{p,q}\rangle$ has underlying space given by the usual Dold-Thom space of configurations of points in S^p with labels in $\mathbb{Z}/2$, and has C_2-action given by the action on $S^{p,q}$.

Thus the element ρ can be realized as an element of the homotopy class of based maps $S^0 \to \mathbb{Z}/2\langle S^{1,1}\rangle$. Specifically, the map $\rho : S^0 \to \mathbb{Z}/2\langle S^{1,1}\rangle$ is given by the inclusion of the fixed set $S^0 \hookrightarrow S^{1,1}$ followed by the canonical map $\iota : S^{1,1} \to \mathbb{Z}/2\langle S^{1,1}\rangle$ that takes each point to the corresponding point in $\mathbb{Z}/2\langle S^{1,1}\rangle$ with label one.

Let U be a tubular neighborhood of $C \simeq S^{1,\delta}$ so $U \cong S^{1,\delta} \times \mathbb{R}^{1,1}$ where $\delta \in \{0,1\}$. By excision and the suspension isomorphism

$$H^{1,1}(X, X-C) \cong H^{1,1}(U, U-C) \cong H^{1,1}(U, \partial U) \cong \tilde{H}^{1,1}(S^{1,1} \wedge S^{1,\delta}_+) \cong \tilde{H}^{0,0}(S^{1,\delta}_+).$$

The nonzero element $\alpha \in \tilde{H}^{0,0}(S^{1,\delta}_+)$ is given by a map $\tilde{\alpha} : S^{1,\delta}_+ \to S^0$ that crushes $S^{1,\delta}$ to the non-basepoint in S^0 followed by the inclusion $\iota : S^0 \to \mathbb{Z}/2\langle S^0\rangle$. Hence the generator of $\tilde{H}^{1,1}(S^{1,1} \wedge S^{1,\delta}_+)$ is given by

$$\Sigma^{1,1}(\iota \circ \tilde{\alpha}) = \Sigma^{1,1}\iota \circ \Sigma^{1,1}\tilde{\alpha} = \iota \circ \Sigma^{1,1}\tilde{\alpha}.$$

Thus we investigate $\Sigma^{1,1}\tilde{\alpha}$ which is illustrated in the figure below.

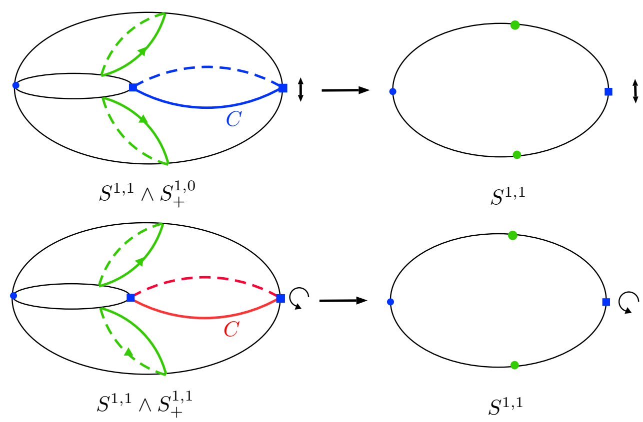


FIGURE 37. The $(1,1)$-suspension of the map $S^{1,\delta}_+ \to S^0$. The conjugate green circles in the left-hand pictures are mapped to the conjugate green points in the right-hand pictures.

If q is one of the fixed points in C illustrated by the blue squares in Figure 37, then precomposing with the inclusion $\iota_q : \{q\} \to S^{1,1} \wedge S^{1,\delta}_+$ maps q to the non-basepoint fixed point of $S^{1,1}$. Thus post-composing with the canonical inclusion $S^{1,1} \hookrightarrow \mathbb{Z}/2\langle S^{1,1}\rangle$ will exactly yield the map ρ as described above. This shows $(\iota_q)^* : \tilde{H}^{1,1}(S^{1,1} \wedge S^{1,\delta}_+) \to H^{1,1}(q)$ takes the unique nonzero element to ρ. The commutative diagram below then shows $\iota_q^*([C]) = \rho$:

$$\begin{array}{ccccc} \tilde{H}^{1,1}(S^{1,1} \wedge S^{1,\delta}_{+}) & \overset{\sim}{=\!=\!=} & H^{1,1}(X, X - C) & \longrightarrow & H^{1,1}(X) \\ & \searrow^{\iota_q^*} & \downarrow \iota_q^* & \swarrow^{\iota_q^*} & \\ & & H^{1,1}(q) & & \end{array}$$

□

We now investigate how doing surgery introduces new fundamental classes.

Lemma 8.0.7. *Let Y be a closed C_2-surface such that $H^{*,*}(Y)$ is generated by fundamental classes. If $X = Y + [S^{1,0} - AT]$, then $H^{*,*}(X)$ is also generated by fundamental classes.*

Proof. The surface X contains a fixed circle, so by Theorem 6.0.6

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2 \oplus \dots$$

where the remaining modules appearing in the decomposition depend on Y. As usual, the class $[X]$ will generate the free summand appearing in bidegree $(0,0)$ while if $x \in X$ is any point contained in a fixed circle, the class $[x]$ will generate a free summand in bidegree $(2,1)$. We just need to find various one-dimensional submanifolds of X that generate the remainder of the cohomology. The procedure to find such submanifolds will depend on the isomorphism type of Y.

Suppose $Y_1, \dots, Y_n$ are nonfree one-dimensional equivariant submanifolds of Y and $F_1, \dots F_m$ are free one-dimensional submanifolds of Y whose fundamental classes together with $[Y]$ and $[y]$ generate $H^{*,*}(Y)$. Let $U_1, \dots, U_n, V_1, \dots, V_m$ be equivariant tubular neighborhoods of $Y_1, \dots, Y_n, F_1, \dots F_m$, respectively. In order to form $X = Y + [S^{1,0} - AT]$, we must remove disjoint conjugate disks D, σD from Y, and we may assume without loss of generality that these disks and the tubular neighborhoods are chosen in a way such that $(D \cup \sigma D) \cap U_i$ and $(D \cup \sigma D) \cap V_j$ are empty for all i, j.

Let Y' be the space obtained by removing $D \sqcup \sigma D$ from Y. We would like to relate the cohomology of Y to the cohomology of X, and we will use the cohomology of the space Y' as a stepping stone from $H^{*,*}(Y)$ to $H^{*,*}(X)$.

Observe Y and X can be realized as the homotopy pushouts of the following diagrams, respectively.

$$\begin{array}{ccc} \partial(D \sqcup \sigma D) & \xrightarrow{\iota} & Y' \\ \big\downarrow{\scriptstyle \pi} & & \\ C_2 & & \end{array} \qquad\qquad \begin{array}{ccc} \partial(D \sqcup \sigma D) & \xrightarrow{\iota} & Y' \\ \big\downarrow{\scriptstyle \pi} & & \\ S^{1,0} & & \end{array}$$

Using the diagram on the left, for each q we have a long exact sequence as shown below:

$$H^{0,q}(Y') \oplus H^{0,q}(C_2) \longrightarrow H^{0,q}(\partial(D \sqcup \sigma D)) \longrightarrow H^{1,q}(Y) \longrightarrow H^{1,q}(Y') \oplus H^{1,q}(C_2)$$

The map $\pi^* : H^{0,q}(C_2) \to H^{0,q}(\partial(D \sqcup \sigma D))$ is an isomorphism because on the level of spaces

$$C_2 \hookrightarrow \partial(D \sqcup \sigma D) \xrightarrow{\pi} C_2$$

is the identity, so π^* is an injective map from $\mathbb{Z}/2$ to $\mathbb{Z}/2$. Thus the leftmost map is surjective, and the rightmost map is injective by exactness. Though $H^{1,q}(C_2) = 0$ so the map $H^{1,q}(Y) \to H^{1,q}(Y')$ is injective. The inclusion $Y' \hookrightarrow X$ induces a map $H^{1,q}(X) \to H^{1,q}(Y')$; this map is often not injective, but for each submanifold C from our list, we have the following commutative diagram:

$$\begin{array}{ccccc} H^{1,q}(Y, Y - C) & \xrightarrow{\cong} & H^{1,q}(Y', Y' - C) & \xleftarrow{\cong} & H^{1,q}(X, X - C) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ H^{1,q}(Y) & \hookrightarrow & H^{1,q}(Y') & \longleftarrow & H^{1,q}(X) \end{array}$$

Note the top horizontal maps are isomorphisms due to excision: all three of these groups are isomorphic to $H^{1,q}(U, U - C)$ where U is the chosen tubular neighborhood of C (note this is why we chose the disks and neighborhoods to be disjoint). The bottom left horizontal map is injective from the above discussion.

In particular, this commutative diagram holds for $C = Y_1, \ldots, Y_n$ and $C = F_1, \ldots, F_m$. Hence, the image of each of the classes $[Y_i]$ and $[F_j]_k$ in $H^{1,*}(X)$ under the right horizontal map is equal to the image of the respective class $[Y_i]$ or $[F_j]_k$ in $H^{1,*}(Y)$ under the left horizontal map. The injectivity of the left map shows the fundamental classes $[Y_i], [F_j]_k$ inherit no new relations in the cohomology of X that were not present in the cohomology of Y. Intuitively, this is unsurprising: attaching a handle should not introduce dependence relations, and the above formalizes this intuition.

There are three cases for how the cohomology of X differs from the cohomology of Y. First, suppose Y already contains a fixed oval. Then by Theorem 6.0.6

$$H^{*,*}(Y) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C(Y)-1} \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(Y)+C(Y)-1} \oplus \left(\Sigma^{1,0}A_0\right)^{\frac{\beta(Y)+2-(F(Y)+2C(Y))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2 \tag{8.0.1}$$

while

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus \left(\Sigma^{1,0}\mathbb{M}_2\right)^{\oplus C(X)-1} \oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(X)+C(X)-1} \oplus \left(\Sigma^{1,0}A_0\right)^{\frac{\beta(X)+2-(F(X)+2C(X))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2$$

Since $X = Y + [S^{1,0} - AT]$, we have the following relations

$$F(X) = F(Y), \quad C(X) = C(Y) + 1, \quad \beta(X) = \beta(Y) + 2$$

that show

$$H^{*,*}(X) \cong H^{*,*}(Y) \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{1,0}\mathbb{M}_2. \tag{8.0.2}$$

Recall the classes $[X]$ and $[x]$ generate free summands in topological degrees zero and two, respectively, while the classes $[Y_i], [F_j]_q$ generate any summands appearing in topological dimension one coming from $H^{*,*}(Y)$ by the discussion above. Thus it suffices to find two new fundamental classes in $H^{1,*}(X)$.

There is an obvious choice for one of the submanifolds, namely the fixed circle contained in the attached handle. Let C_1 be this circle and note $[C_1] \in H^{1,1}(X)$. For the other submanifold, let $p \in C_1$ be a fixed point, and choose a point s contained in another fixed circle. It is shown in [7] that we can construct a path γ from p to s such that

$$C_\gamma := \mathrm{im}(\gamma) \cup \mathrm{im}(\sigma\gamma) \cong S^{1,1}$$

and such that C_γ and C_1 intersect at the single point p. Let U be a tubular neighborhood of C_γ. Over each fixed point the normal fiber is a fixed interval, so $[C_\gamma] \in H^{1,0}(X)$. The two classes are in the correct bidegrees; we next show they are linearly independent from the classes coming from Y.

We begin with $[C_\gamma]$. By construction C_1 and C_γ intersect at a single fixed point, so

$$[C_1][C_\gamma] = [p] \neq 0.$$

On the other hand, C_1 does not intersect any of the other submanifolds Y_i, F_j, and so for any $\mathbb{M}_2$-linear combination of these fundamental classes

$$[C_1] \cdot (\Sigma_i a_i [Y_i] + \Sigma_j b_j [F_j]_q) = 0.$$

Hence, it must be that $[C_\gamma]$ is not in the $\mathbb{M}_2$-span of these classes. By the isomorphism in 8.0.1, the fact that $H^{*,*}(Y)$ is generated by fundamental classes, and consideration of degrees, it must be that

$$\begin{aligned}\dim\left(H^{1,0}(Y)\right) &= \dim\left(\mathbb{M}_2\{[Y_i],[F_j]_q,[Y],[y]\}\cap H^{1,0}(Y)\right)\\ &= \dim\left(\mathbb{M}_2\{[Y_i],[F_j]_q\}\cap H^{1,0}(Y)\right).\end{aligned}$$

We have already remarked that

$$\dim\left(\mathbb{M}_2\cdot\{[Y_i],[F_j]_q\}\cap H^{1,0}(Y)\right) = \dim\left(\mathbb{M}_2\cdot\{[Y_i],[F_j]_q\}\cap H^{1,0}(X)\right).$$

By the isomorphism in 8.0.2,

$$\dim\left(H^{1,0}(Y)\right) = \dim\left(H^{1,0}(X)\right) - 1.$$

We just proved $[C_\gamma]$ is not in the $\mathbb{M}_2$-span of the classes $[Y_i]$, $[F_j]_q$, so by dimensions it must now follow that

$$\mathbb{M}_2\cdot\{[C_\gamma],[Y_i],[F_j]_q\}\cap H^{1,0}(X) = H^{1,0}(X).$$

Thus any generator of the new summand $\Sigma^{1,0}\mathbb{M}_2$ is a linear combination of fundamental classes.

Returning to C_1, we can apply Lemma 8.0.6 to see $\iota_p^*([C_1])$ is nonzero. Since $\{p\}$ does not intersect any of the submanifolds Y_i, F_j, $\iota_p^*([Y_i]) = \iota_p^*([F_j]_q) = 0$. For degree reasons, $\iota_p^*([C_\gamma])$ is also zero, and thus any $\mathbb{M}_2$-combination of these classes must be in the kernel of ι_p. We conclude $[C_1]$ is not in the $\mathbb{M}_2$-span of any of these classes. Again using our isomorphisms and degree arguments we can say

$$\dim\left(\mathbb{M}_2\{[C_1],[C_\gamma],[Y_i],[F_j]_q,[X]\}\cap H^{1,1}(X)\right) = \dim\left(H^{1,1}(X)\right).$$

We conclude any generator of the new summand $\Sigma^{1,1}\mathbb{M}_2$ is in this span. Thus

$$H^{*,*}(X) = \mathbb{M}_2\{[C_1],[C_\gamma],[Y_i],[F_j]_q,[X],[x]\}$$

as desired. This completes the proof in the case that Y already contains a fixed oval.

Next suppose the fixed set of Y contains only isolated fixed points. Then by Theorem 6.0.6

$$H^{*,*}(Y) \cong \mathbb{M}_2 \oplus (\Sigma^{1,1}\mathbb{M}_2)^{\oplus F(Y)-2} \oplus (\Sigma^{1,0}A_0)^{\frac{\beta(Y)+2-F(Y)}{2}} \oplus \Sigma^{2,2}\mathbb{M}_2$$

while

$$\begin{aligned} H^{*,*}(X) \cong \mathbb{M}_2 &\oplus \left(\Sigma^{1,1}\mathbb{M}_2\right)^{\oplus F(X)+C(X)-1} \\ &\oplus \left(\Sigma^{1,0}A_0\right)^{\frac{\beta(X)+2-(F(X)+2C(X))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2 \end{aligned}$$

We have the relations

$$F(X) = F(Y), \quad C(X) = 1, C(Y) = 0, \quad \beta(X) = \beta(Y) + 2.$$

Hence the number of $\Sigma^{1,0}\mathbb{M}_2$ and $\Sigma^{1,0}A_0$ summands match, and we just need to find two new classes that generate the two additional $\Sigma^{1,1}\mathbb{M}_2$-summands.

As in the previous case, we need to find two new fundamental classes. Again let C_1 be the attached (and now only) fixed oval, noting $[C_1] \in H^{1,1}(X)$. Let γ be a path from a fixed point $p \in C_1$ to an isolated fixed point s, choosing the path γ such that

$$C_\gamma := \mathrm{im}(\gamma) \cup \mathrm{im}(\sigma\gamma) \cong S^{1,1}.$$

and such that C_γ and C_1 only intersect at a single point. Let U be a tubular neighborhood of the circle C_γ, and note that the fiber over p is isomorphic to $\mathbb{R}^{1,0}$, while the fiber over s is isomorphic to $\mathbb{R}^{1,1}$, so it must be that $[C_\gamma] \in H^{1,1}(X)$ as well.

We can do the same tricks as before to conclude $[C_1]$ and $[C_\gamma]$ generate the rest of the cohomology of X. Namely, their nontrivial product shows $[C_\gamma]$ is not an $\mathbb{M}_2$-combination of our current classes. To see $[C_1]$ is not in the span of the classes given by Y_i, F_j, C_γ, choose another point on the fixed oval $p' \neq p$ and use the map $\iota_{p'}$. This will complete the proof in this case.

Lastly, suppose Y is a free C_2-surface. Then by Theorem 5.0.5, ignoring the action of ρ,

$$H^{*,*}(Y) \cong A_0 \oplus (\Sigma^{1,0}A_0)^{\frac{\beta(Y)+2}{2}} \oplus \Sigma^{2,1}A_0$$

while by Theorem 6.0.6

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus (\Sigma^{1,0}A_0)^{\frac{\beta(X)+2-(F(X)+2C(X))}{2}} \oplus \Sigma^{2,1}\mathbb{M}_2$$

We now have the relations

$$F(X) = F(Y) = 0, \quad C(X) = 1, C(Y) = 0 \quad \beta(X) = \beta(Y) + 2.$$

The number of summands generated in topological dimension one is the same in $H^{*,*}(X)$ as in $H^{*,*}(Y)$. Thus, by adding $[X]$ and $[x]$ to our list of classes $[Y_i], [F_j]_q$, we will have found a collection of fundamental classes that generate the cohomology of X. □

Lemma 8.0.8. *Let Y be a nontrivial C_2-surface such that $H^{*,*}(Y)$ is generated by fundamental classes. If $X = Y + [S^{1,1} - AT]$, then $H^{*,*}(X)$ is also generated by fundamental classes.*

Proof. The proof is similar to that of the previous lemma. Instead of having a fixed circle in the attached handle, we have a circle $C_1 \cong S^{1,1}$. The circle C_1 is still two-sided and its fundamental class is contained in bidegree $(1, 1)$. In the case where Y has an isolated fixed point, the other class will be given by a circle C_γ where γ

is a path from a fixed point on C_1 to this other isolated fixed point. To see this circle is two-sided, note the two fixed points in C_γ are isolated fixed points in Y, so the fibers over these points in the normal bundle are both isomorphic to $\mathbb{R}^{1,1}$. There is only one action on the Möbius bundle over $S^{1,1}$ and the fibers over the fixed points in this bundle are $\mathbb{R}^{1,0}$ and $\mathbb{R}^{1,1}$. In the case where Y only has fixed ovals, the other class will given by a circle C_γ where γ is a path from a fixed point on C_1 to a point on a fixed oval. This circle must be one-sided because the fibers of the normal bundle over the two fixed points are different representations. Lastly, in the case where Y is free, no other classes besides $[X]$ and $[x]$ will be needed. □

We are now ready to prove the main theorem of this section.

Theorem 8.0.9. *Let X be a C_2-surface. Then $H^{*,*}(X;\underline{\mathbb{Z}/2})$ is generated by fundamental classes of equivariant submanifolds.*

Proof. If X is trivial, then by Lemma 3.0.7

$$H^{*,*}(X) \cong \mathbb{M}_2 \otimes_{\mathbb{Z}/2} H^*_{sing}(X).$$

Since $H^*_{sing}(X)$ is generated by fundamental classes, it immediately follows that $H^{*,*}(X)$ is generated by fundamental classes.

Assume X is nontrivial. We proceed by induction on the β-genus of X. If the β-genus is zero, then we are done by Lemma 8.0.4. For the inductive hypothesis, let $k \geq 1$ and assume the statement holds for all surfaces of β-genus less than k.

Let X be a surface of β-genus k. If X is a free C_2-surface or a doubling space, then we are done by Lemmas 8.0.3 and 8.0.5. Thus suppose X is nonfree and not a doubling space. By Theorem 4.2.3, there are three cases for X: the surface X is isomorphic to a space given by doing $S^{1,1}-$, $S^{1,0}-$, or $FM-$ surgery to a surface of lower β-genus.

Combining the first and second cases, suppose $X \cong Y + [S^{1,\epsilon} - AT]$ where $\epsilon \in \{0, 1\}$. Note $\beta(Y) = \beta(X) - 2$, so we can apply the inductive hypothesis to conclude $H^{*,*}(Y)$ is generated by fundamental classes. We are then done after applying either Lemma 8.0.7 or Lemma 8.0.8.

The remaining case is $X \cong Y + [FM]$. There are four subcases based on the fixed set of Y. First, suppose Y contains a two-sided fixed circle. Note Y also contains an isolated fixed point, so the circle must be non-separating, i.e. the complement of this circle is connected. Thus we can do surgery around this fixed circle to see $Y \cong Z + [S^{1,0} - AT]$ where Z is some other C_2-surface with $\beta(Z) = \beta(Y) - 2 = \beta(X) - 3$. Let $W = Z + [FM]$, and note $\beta(W) = \beta(X) - 2$, so the inductive hypothesis implies $H^{*,*}(W)$ is generated by fundamental classes. We are done after noting $X \cong W + [S^{1,0} - AT]$ and applying Lemma 8.0.7.

Second, suppose Y contains at least three isolated fixed points. From [7] there is a path γ between two of the points such that the image of γ and its conjugate path form a copy of $S^{1,1}$. As before, this copy of $S^{1,1}$ must non-separating, so we can do surgery to see $Y \cong Z + [S^{1,1} - AT]$ for some other C_2-surface Z. As in the previous case, we are done after noting $X \cong (Z + [FM]) + [S^{1,1} - AT]$, applying the inductive hypothesis to $Z + [FM]$, and then applying Lemma 8.0.8.

Third, suppose Y is isomorphic to a doubling space $\text{Doub}(Z, 1 : 1, 1)$. In this case, we can find fundamental classes for X by hand. By substituting $\beta(X) = 2\beta(Z) + 1$ and $C(X) = F(X) = 1$ into Theorem 6.0.6, we have that

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2 \oplus (\Sigma^{1,0}A_0)^{\oplus\beta(Z)} \oplus \Sigma^{2,1}\mathbb{M}_2.$$

Let C be the fixed circle contained in the attached Möbius band and $Z_1, \ldots, Z_{\beta(Z)}$ be the circles whose fundamental classes generate $H^1_{sing}(Z)$. The classes $[C]$, $[Z_1 \sqcup$

$\sigma Z_1]$, ..., $[Z_{\beta(Z)} \sqcup \sigma Z_{\beta(Z)}]$, $[X]$, $[x]$ will generate $H^{*,*}(X)$ where $x \in X^{C_2}$ is a point on the fixed circle. We leave the details to the reader.

Lastly, suppose Y is not a doubling space, contains no two-sided fixed ovals, and has at most two isolated fixed points. In this case, we need more specific details from the classification given in [7]. There are two cases for the isomorphism class of X. The two cases are

$$X \cong S^{2,2} + \tfrac{C(X)+F(X)-2}{2}[S^{1,1} - AT] + C(X)[FM], \quad \text{or}$$

$$X \cong W + \tfrac{C(X)+F(X)}{2}[S^{1,1} - AT] + C(X)[FM],$$

where W is some free C_2-surface. In both cases, fundamental classes can be found by hand; we provide an outline of how to find such classes.

Suppose X is given by the top isomorphism. For notational simplicity, let $r = C(X)$ and $s = F(X)$ (note $s = 0$ or $s = 1$). By Theorem 6.0.6,

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus (\Sigma^{1,0}\mathbb{M}_2)^{\oplus r-1} \oplus (\Sigma^{1,1}\mathbb{M}_2)^{\oplus r+s-1} \oplus \Sigma^{2,1}\mathbb{M}_2.$$

Let $C_1, \dots, C_r$ be the fixed circles contained in the attached Möbius bands. The circles give r classes in $H^{1,1}(X)$, and the forgetful map shows these classes are linearly independent. Next, for $j = 1, \dots, r$, fix a point $p_j \in C_j$. Let γ_{1j} be a path from p_1 to p_j such that $C_{1j} = \operatorname{im}(\gamma_{1j}) \cup \operatorname{im}(\sigma\gamma_{1j})$ is isomorphic to $S^{1,1}$ for $j = 2, \dots, r$. These circles will give $(r-1)$ linearly independent classes in $H^{1,0}(X)$. If $s = 0$, the classes given by $C_1, \dots, C_r, C_{12}, \dots, C_{1r}$ together with $[X]$ and $[x]$ where $x \in X$ is a fixed point contained in a fixed circle will generate $H^{*,*}(X)$. If $s = 1$, we can construct one more circle D using a path from p_1 to the isolated fixed point. The class $[D] \in H^{1,1}(X)$ together with the other classes will generate $H^{*,*}(X)$.

For the second case, note $\beta(X) = \beta(W) + 2C + F$ and thus by Theorem 6.0.6 the cohomology of X is

$$H^{*,*}(X) \cong \mathbb{M}_2 \oplus (\Sigma^{1,0}\mathbb{M}_2)^{C-1} \oplus (\Sigma^{1,1}\mathbb{M}_2)^{F+C-1} \oplus (\Sigma^{1,0}A_0)^{\oplus \beta(W)/2+1} \oplus \Sigma^{2,1}\mathbb{M}_2.$$

To find equivariant classes, construct nonfree circles as in the previous case, and also include the free generators for $H^{1,*}(W)$ (note we saw in the proof of Lemma 8.0.3 there are $\beta(W)/2+1$ such generators).

We have exhausted all cases, and we conclude $H^{*,*}(X)$ is generated by fundamental classes for all C_2-surfaces. □

APPENDIX A

A THEOREM FOR C_2-MANIFOLDS

In this appendix we provide a proof of Theorem 3.0.8 which is given as Theorem A.0.1 below. Here by "manifold" we mean a piecewise linear manifold, and by C_2-action, we mean a locally linear C_2-action. Note this is sufficient to guarantee the fixed set is a disjoint union of submanifolds.

Theorem A.0.1. *Let X be an n-dimensional, closed C_2-manifold with a nonfree C_2-action. Suppose $n-k$ is the largest dimension of submanifold appearing as a component of the fixed set. Then there is exactly one summand of $\tilde{H}^{*,*}(X;\underline{\mathbb{Z}/2})$ of the form $\Sigma^{i,j}\mathbb{M}_2$ where $i \geq n$, and it occurs for $(i,j)=(n,k)$.*

Proof. If X is a trivial space, then this follows immediately from Lemma 3.0.7 and facts about the singular cohomology of closed n-manifolds in $\mathbb{Z}/2$-coefficients. Thus assume X is nontrivial. We first show there is a unique summand generated in topological dimension n, we then show it must be free, and lastly we argue it must be in weight k.

From the structure theorem given in Theorem 3.0.11, the cohomology of X must have a direct sum decomposition given by

$$H^{*,*}(X) \cong (\oplus_i \Sigma^{m_i,k_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{r_j} A_{s_j}). \tag{A.0.1}$$

Consider the following portion of the forgetful long exact sequence for X:

$$H^{p-1,q}(X) \xrightarrow{\rho} H^{p,q+1}(X) \xrightarrow{\psi} H^p_{sing}(X) \longrightarrow H^{p,q}(X)$$

Since X is a closed n-manifold, $H^p_{sing}(X)=0$ for $p>n$ while $H^n_{sing}(X)\cong\mathbb{Z}/2$. By exactness, it must be that $H^{p,q}(X)=\operatorname{im}(\rho)$ for $p>n$, and when $p=n$, there are two possibilities: either $H^{n,q}(X) = \operatorname{im}(\rho)$ or $H^{n,q}(X)/\operatorname{im}(\rho) \cong \mathbb{Z}/2$. Returning to A.0.1, this immediately implies $m_i, r_j \leq n$ for all i, j. We claim this also implies there are either zero summands or one summand generated in topological dimension n; that is there is at most one i or j such that $m_i = n$ or $r_j = n$. Indeed, if there were two or more summands generated in topological dimension n, then there would exist a sufficiently large q such that $\dim\left(H^{n,q}(X)/\operatorname{im}(\rho)\right) \geq 2$.

To see there is a summand generated in topological dimension n, pick a point $x \in X^{C_2}$ that is contained in a connected component of dimension $n-k$, where recall $n-k$ is the maximum dimension. There exists an open equivariant disk D such that $x \in D \subset X$ and $D \cong \mathbb{R}^n$ nonequivariantly. By consideration of the fixed set, we see that $D \cong D(\mathbb{R}^{n,k})$ where $D(\mathbb{R}^{n,k})$ denotes the unit disk in $\mathbb{R}^{n,k}$. Consider the quotient map

$$q : X \to X/(X-D) \cong S^{n,k}.$$

We have the following commutative square involving the forgetful map:

$$\begin{array}{ccc} \tilde{H}^{n,k}(S^{n,k}) & \xrightarrow{\psi} & \tilde{H}^{n}_{sing}(S^{n,k}) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\ \tilde{H}^{n,k}(X) & \xrightarrow{\psi} & \tilde{H}^{n}_{sing}(X) \end{array}$$

Recall $\tilde{H}^{*,*}(S^{n,k}) \cong \Sigma^{n,k}\mathbb{M}_2$ by the suspension isomorphism, so the top map is an isomorphism. The right vertical map is also an isomorphism because X is a closed n-manifold. By commutativity of the square, the forgetful map $\psi : \tilde{H}^{n,k}(X) \to \tilde{H}^{n}_{sing}(X)$ must be nonzero. Returning to the forgetful long exact sequence above, we see $H^{n,k}(X)/\operatorname{im}(\rho) \cong \mathbb{Z}/2$ and there is indeed exactly one summand generated in topological dimension n.

There are thus two options for the cohomology of X. Either

$$H^{*,*}(X) \cong (\oplus_i \Sigma^{m_i,k_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{r_j} A_{s_j}) \oplus \Sigma^n A_b; \text{ or} \tag{A.0.2}$$

$$H^{*,*}(X) \cong (\oplus_i \Sigma^{m_i,k_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{r_j} A_{s_j}) \oplus \Sigma^{n,c}\mathbb{M}_2. \tag{A.0.3}$$

where in both equations $m_i, r_j < n$. We show the first case cannot happen.

Suppose to the contrary the cohomology of X is given by A.0.2. Let p be a nonfixed point and consider the punctured space $X - \{p, \sigma p\} \simeq X - \{D(p), \sigma D(p)\}$ where $D(p)$ is a small open disk around p that does not intersect its conjugate disk $\sigma D(p)$. Note $X - \{D(p), \sigma D(p)\}$ is an n-manifold with boundary, so

$$H^j_{sing}(X - \{D(p), \sigma D(p)\}) = 0 \text{ for } j \geq n.$$

We can put a C_2-CW structure on $X - \{D(p), \sigma D(p)\}$ with no cells of dimension greater than n. The map $q : X - \{D(p), \sigma D(p)\} \to (X - \{D(p), \sigma D(p)\})/C_2$ will be a cellular map that induces a levelwise surjective map on the cellular chain complexes. Using this map of chain complexes and the above fact, a diagram chase shows

$$H^j_{sing}((X - \{D(p), \sigma D(p)\})/C_2) = 0 \text{ for } j \geq n.$$

By the quotient lemma given in 3.0.1, it follows that $H^{j,0}(X - \{p, \sigma p\}) = 0$ for $j \geq n$.

Consider the pair $(X, X - \{p, \sigma p\})$. Note

$$H^{*,*}(X, X - \{p, \sigma p\}) \cong \tilde{H}^{*,*}(X/(X - \{D(p), \sigma D(p)\})) \cong \tilde{H}^{*,*}(C_{2+} \wedge S^n) \cong \Sigma^n A_0.$$

We have the following diagram where the rows are exact:

$$\begin{array}{ccccc} H^{n,0}(X, X - \{p, \sigma p\}) & \longrightarrow & H^{n,0}(X) & \longrightarrow & H^{n,0}(X - \{p, \sigma p\}) \\ \downarrow \psi & & \downarrow \psi & & \downarrow \psi \\ H^n_{sing}(X, X - \{p, \sigma p\}) & \longrightarrow & H^n_{sing}(X) & \longrightarrow & H^n_{sing}(X - \{p, \sigma p\}) \end{array}$$

The right-hand groups are both zero by the above discussion, so the left horizontal maps are both surjective. The middle vertical map is surjective based on the decomposition given in A.0.2, while the left vertical map is given by the diagonal map

$$\tilde{H}^{*,*}(C_{2+} \wedge S^n) \cong \tilde{H}^*_{sing}(S^n) \to \tilde{H}^*_{sing}(S^n \vee S^n) \cong \tilde{H}^*_{sing}(S^n) \oplus \tilde{H}^*_{sing}(S^n).$$

Thus we have the following commutative diagram coming from the left square where Δ is the diagonal map and ∇ is the fold map.

$$\begin{array}{ccc} \mathbb{Z}/2 & \twoheadrightarrow & H^{n,0}(X) \\ \downarrow \Delta & & \downarrow \psi \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \xrightarrow{\nabla} & \mathbb{Z}/2 \end{array}$$

We have arrived at a contradiction: going around the diagram one way is zero, while the other way is nonzero. We conclude the cohomology of X must have a decomposition as in A.0.3. In particular, there is a unique free summand in topological dimension n, and furthermore there are no other summands generated in topological dimension greater than or equal to n.

We now show this free summand is generated in weight k. Let's reconsider the quotient map

$$q : X \to X/(X - D) \cong S^{n,k}.$$

Let $s \geq k$. We have the following map between the forgetful long exact sequences for X and $S^{n,k}$.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & \tilde{H}^{n-1,s-1}(S^{n,k}) & \xrightarrow{\rho} & \tilde{H}^{n,s}(S^{n,k}) & \xrightarrow{\psi} & \tilde{H}^{n}_{sing}(S^{n,k}) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \cong\downarrow{\scriptstyle q^*} & & \\
\dots & \longrightarrow & \tilde{H}^{n-1,s-1}(X) & \xrightarrow{\rho} & \tilde{H}^{n,s}(X) & \xrightarrow{\psi} & \tilde{H}^{n}_{sing}(X) & \longrightarrow & \dots
\end{array}$$

Recall $\tilde{H}^{*,*}(S^{n,k}) \cong \Sigma^{n,k}\mathbb{M}_2$ by the suspension isomorphism. We provide an illustration of this cohomology below for reference. The above map of long exact

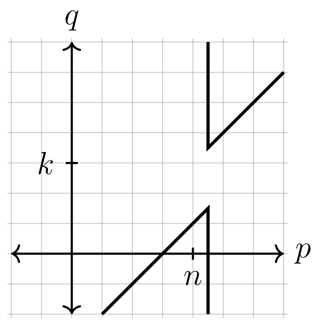


FIGURE 38. The reduced cohomology of $S^{n,k}$.

sequences is thus

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow[\cong]{\psi} & \mathbb{Z}/2 \\
\downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \cong\downarrow{\scriptstyle q^*} \\
\tilde{H}^{n-1,s-1}(X) & \xrightarrow{\rho} & \tilde{H}^{n,s}(X) & \xrightarrow{\psi} & \mathbb{Z}/2
\end{array}$$

The square on the right shows

$$q^* : \tilde{H}^{n,s}(S^{n,k}) \to \tilde{H}^{n,s}(X)$$

is injective for all $s \geq k$. If we let $a \in \tilde{H}^{n,k}(S^{n,k})$ be the generator, the exactness also shows the nonzero element q^*a is not in the image of ρ.

Returning to decomposition given in A.0.3, we can write q^*a as an $\mathbb{M}_2$-combination of generators of the summands. Observe a τ-multiple of the generator of the summand $\Sigma^{n,c}\mathbb{M}_2$ must appear in this linear combination since otherwise q^*a would be in the image of ρ. Thus the weight c must be less than or equal to k.

To see $c = k$, note ρ-localization will yield a generator in $\tilde{H}^{n-c}_{sing}(X^{C_2})$. Since $n-k$ is the largest dimension appearing in the fixed set, it must be that $n-c \leq n-k$ or $c \geq k$. We conclude $c = k$, as desired. □

We end by mentioning one corollary of the above proof.

Corollary A.0.2. *Let X be an n-dimensional nonfree C_2-manifold and let $x \in X^{C_2}$ be a point in a component of the fixed set of smallest codimension k. Then the map $q : X \to S^{n,k}$ that collapses the complement of a small disk around x to a point induces a split injection.*

Proof. In the proof above, we showed q^*a where a is the generator of $\tilde{H}^{*,*}(S^{n,k})$ generates a free summand of $H^{*,*}(X)$. This implies the map is injective, and it is split because $\mathbb{M}_2$ is self-injective. □

APPENDIX B

C_2-VECTOR BUNDLES AND THOM ISOMORPHISM THEOREMS

In this appendix, we provide some background on C_2-vector bundles and then prove the Thom isomorphism theorems given as Theorem 1.4.1 and Theorem 1.5.1 in the introduction. Another approach to the Thom isomorphism is given in [3] and uses a grading system larger than the usual $RO(G)$-grading. Presumably there is a connection between the two approaches, but we haven't investigated this.

The two main theorems in this appendix are broken into two parts. The first part focuses on the existence of the Thom classes and their relations to the cohomology of the fibers. The second part focuses on the map given by cupping with the Thom class.

Notation B.0.1. All coefficients in this section are understood to be $\underline{\mathbb{Z}/2}$. Given a vector bundle E, we will often write E' for the complement of the zero-section.

B.1. Background

We begin by reviewing C_2-vector bundles. The following can be found in Section 1 of [15].

Definition B.1.1. Let X be a C_2-space. A $\mathbf{C_2}$**-vector bundle** over X is the data of a nonequivariant vector bundle $\pi : E \to X$ such that E is a C_2-space. Furthermore, C_2 should act on E via vector bundle maps over the action of C_2 on X. Explicitly, the following diagram should commute where σ denotes the action of C_2

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\sigma} & X \end{array}$$

and for each $x \in X$ the restriction of the action to the fibers

$$E_x \xrightarrow{\sigma} E_{\sigma x}$$

should be a linear map.

Many of the constructions for nonequivariant bundles exist for C_2-vector bundles. For example, any C_2-vector bundle can be given a C_2-invariant Euclidean metric that allows us to define the unit disk bundle and the unit sphere bundle, which in turn allows us to define the Thom space. Also, given an equivariant map $f : Y \to X$ and a vector bundle $E \to X$ we can form the pullback bundle $f^*E \to Y$. When working with these pullback bundles, the following important fact is still true; see [15] for a proof.

Lemma B.1.2. *Let X and Y be C_2-CW complexes. Suppose $f, g : Y \to X$ are equivariantly homotopic and $E \to X$ is a C_2-vector bundle. Then $f^*E \cong g^*E$ as C_2-vector bundles over Y.*

This allows us to prove the following.

Lemma B.1.3. *Let $E \to X$ be a finite dimensional C_2-vector bundle over a C_2-CW complex. Suppose x, y are two fixed points contained in the same connected component of X^{C_2}. Then the fibers E_x and E_y are isomorphic as C_2-representations.*

Proof. Consider the maps $j_x \colon * \hookrightarrow X$ and $j_y \colon * \hookrightarrow X$ which include the point as x and y, respectively. Since x and y are in the same connected component of the fixed set, these two inclusions are homotopic. Thus $j_x^*E \cong j_y^*E$ which implies $E_x \cong E_y$. □

While many of the constructions and basic lemmas carry over from nonequivariant vector bundle theory, issues arise when we start considering cohomology. In particular, there is no direct analog of the Thom isomorphism theorem in Bredon cohomology that holds for general vector bundles in $\underline{\mathbb{Z}/2}$-coefficients, as seen in the following example.

Example B.1.4. Let $E \to S^{1,1}$ be the nontrivial one-dimensional bundle over $S^{1,1}$. An illustration of the disk bundle is shown below. As usual, the fixed set is shown in blue, while conjugate points are indicated by matching symbols. In this

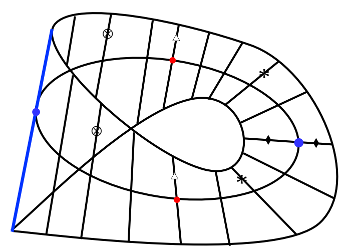


FIGURE 39. The Möbuis bundle over $S^{1,1}$.

example, there are two components of the fixed set of the base space, both of which are isolated points. Over one point, the fiber is isomorphic to the C_2-representation $\mathbb{R}^{1,0}$; over the other point, the fiber is isomorphic to the C_2-representation $\mathbb{R}^{1,1}$. We have

$$H^{*,*}(E, E-0) \cong H^{*,*}(DE, SE) \cong \tilde{H}^{*,*}(DE/SE)$$

where DE and SE are the unit disk and unit sphere bundle, respectively. Now DE/SE is a familiar space: the underlying space is the projective plane, so in particular, it is a C_2-surface with exactly one fixed circle and one fixed point. The cohomology is given by Theorem 6.0.6 to be

$$\tilde{H}^{*,*}(DE/SE) \cong \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2.$$

On the other hand, the cohomology of the base space is given by the suspension isomorphism to be

$$H^{*,*}(S^{1,1}) \cong \mathbb{M}_2 \oplus \Sigma^{1,1}\mathbb{M}_2.$$

We see the cohomology of the Thom space is not a shift of the cohomology of the base space, but there are still similarities. There are the same number of free summands, and both summands are shifted by one topological dimension. There is also a unique class in $H^{*,*}(E, E-0)$ that generates a free summand and has topological dimension equal to the dimension of the bundle. Note the weight of this class corresponds to the maximum weight representation over the fixed set. This class will be the Thom class described in Theorem B.2.1.

B.2. The Theorem for Nonfree Bundles, Part I

There is no direct analog of the Thom isomorphism theorem, but the above example hints there may still be connections between the cohomology of the base space and the cohomology of the Thom space. Indeed, we prove the existence of a unique class that acts similarly to the singular Thom class for bundles over nonfree, finite C_2-CW complexes. In this part, we show the class generates a free summand and restricts to τ-multiples of the generators of the cohomology of the fibers. In a later subsection, we show cupping with this class gives an isomorphism within a certain range.

We begin with some setup. Let X be a finite, nonfree C_2-CW complex and $E \to X$ be an n-dimensional C_2-vector bundle. Let $X_1, \ldots, X_m$ denote the connected components of the fixed set X^{C_2}. By Lemma B.1.3, for each X_i there is an integer q_i such that for all $x \in X_i$, $E_x \cong \mathbb{R}^{n,q_i}$. Let $q = \max\{q_1, \dots q_m\}$ be the largest weight. We now restate and prove the first parts of Theorem 1.4.1.

Theorem B.2.1 (Nonfree Bundles, Part I)**.** *Let X, $\pi : E \to X$, n, X_i, q_i and q be defined as above and let $E' = E - 0$. There exists a unique class $u_E \in H^{n,q}(E, E')$ such that the following holds:*

(i) *$\psi(u_E)$ is the singular Thom class, where $\psi : H^{n,q}(E, E') \to H^n_{sing}(E, E')$ is the forgetful map;*

(ii) *$\mathbb{M}_2 \cdot u_E \cong \Sigma^{n,q}\mathbb{M}_2$, where $\mathbb{M}_2 \cdot u_E$ denotes the submodule generated by u_E;*

(iii) *For every i and $x \in X_i$, the class u_E restricts to $\tau^{q-q_i}\alpha_x$ where α_x is the generator of $H^{*,*}(E_x, E_x - 0) \cong \tilde{H}^{*,*}(S^{n,q_i})$;*

(iv) *For every $x \in X \setminus X^{C_2}$, the class u_E restricts to the unique nonzero class in $H^{n,q}(E_{x,\sigma x}, E_{x,\sigma x} - 0) \cong \tilde{H}^{n,q}(S^{n,0} \wedge C_{2+})$ where $E_{x,\sigma x} = \pi^{-1}(\{x, \sigma\}) \cong C_2 \times \mathbb{R}^n$.*

Proof. The proof of (i) and (ii) will be shown together, followed by a short proof of (iii) and then (iv). Begin by considering the fixed set of the vector bundle E. Based on the definition of C_2-vector bundles, E^{C_2} maps to X^{C_2}, and over each component X_i, we have a vector bundle $E_i = (\pi^{-1}(X_i))^{C_2}$ of dimension $n - q_i$. By the nonequivariant Thom isomorphism theorem,

$$H^{k+n-q_i}_{sing}(E_i, E_i') \cong H^k_{sing}(X_i).$$

In particular, $H^{\ell}_{sing}(E_i, E_i') = 0$ for all $\ell < n - q_i$. Now $n - q$ is the smallest vector bundle rank amongst the components E^{C_2}. Since

$$H^{\ell}_{sing}(E^{C_2}, (E')^{C_2}) \cong \textstyle\bigoplus_{i=1}^m H^{\ell}_{sing}(E_i, E_i'),$$

we see that $H^{\ell}_{sing}(E^{C_2}, (E')^{C_2}) = 0$ for $\ell < n - q$. We will be using this fact throughout the proof.

Since X is a finite C_2-CW complex, the Thom space DE/SE is also a finite C_2-CW complex. Thus Theorem 3.0.11 holds, and the cohomology of the pair (E, E') must decompose as

$$H^{*,*}(E, E') \cong (\oplus_i \Sigma^{k_i,\ell_i} \mathbb{M}_2) \oplus (\oplus_j \Sigma^{s_j,0} A_{r_j}).$$

We now use the forgetful long exact sequence to connect the above decomposition to the singular cohomology of (E, E'). Based on the summands above, for each i there is a class in bidegree (k_i, ℓ_i) that is not the in the image of ρ and thus forgets to a distinct nonzero class in $H^{k_i}_{sing}(E, E')$. Similarly for each j there is a class in bidegree $(s_j, 0)$ that is not in the image of ρ and thus forgets to a distinct nonzero class in $H^{s_j}_{sing}(E, E')$. The Thom isomorphism theorem applied to $H^*_{sing}(E, E')$ implies $H^k_{sing}(E, E') = 0$ if $k < n$ and $H^n_{sing}(E, E') = \mathbb{Z}/2$. We conclude $k_i \geq n$ and $s_j \geq n$, and even better, exactly one of these integers must be n. Thus there are really two cases for this decomposition, both with $k_i, s_j > n$:

$$H^{*,*}(E, E') \cong \Sigma^{n,c} \mathbb{M}_2 \oplus (\oplus_i \Sigma^{k_i,\ell_i} \mathbb{M}_2) \oplus (\oplus_j \Sigma^{s_j,0} A_{r_j}), \quad \text{or} \tag{B.2.1}$$

$$H^{*,*}(E, E') \cong (\oplus_i \Sigma^{k_i,\ell_i} \mathbb{M}_2) \oplus \Sigma^{n,0} A_d \oplus (\oplus_j \Sigma^{s_j,0} A_{r_j}). \tag{B.2.2}$$

Consider the following portion of the forgetful long exact sequence for (E, E'):

$$\xrightarrow{\rho} H^{n,q}(E, E') \xrightarrow{\psi} H^n_{sing}(E, E') \xrightarrow{g} H^{n,q-1}(E, E') \xrightarrow{\rho}$$

There is a unique nonzero class $u \in H^n_{sing}(E, E')$, namely the Thom class of E. Thus either

(a) there is a nonzero class $\alpha \in H^{n,q}(E, E')$ such that $\psi(\alpha) = u$; or

(b) $g(u) = \beta \neq 0$.

Suppose to the contrary (b) holds. If we had a decomposition as in B.2.2, then the forgetful map would be nonzero in topological dimension n for all weights which violates (b). Thus it must be that the cohomology of X has a direct sum decomposition as in B.2.1. Based on this decomposition, the forgetful map ψ : $H^{n,j}(E, E') \to H^n_{sing}(E, E')$ is nonzero for all $j \geq c$ where c is the weight of the free summand in topological dimension n. Since we are still in case (b), it must be that $q < c$. Now by the ρ-localization given in Lemma 3.0.2, $H^{n-c}_{sing}(E^{C_2}, (E-0)^{C_2}) \cong \mathbb{Z}/2$, but $n - c < n - q$ and this contradicts the discussion in the first paragraph of this proof. We conclude (b) is false, so it must be that (a) holds.

We have shown there exists a class $\alpha \in H^{n,q}(E, E')$ such that $\psi(\alpha) = u$. This class α is our candidate for u_E in statement (i) of the theorem. Pick an index j such that $q_j = q$ and let $y \in X_j$. The inclusion $\iota : (E_y, E'_y) \hookrightarrow (E, E')$ induces a map between the forgetful long exact sequences. The relevant portion is shown below:

$$\begin{array}{ccccc} H^{n,q}(E,E') & \xrightarrow{\psi} & H^n_{sing}(E,E') & \xrightarrow{g} & H^{n,q-1}(E,E') \\ \downarrow{\scriptstyle \iota^*} & & {\scriptstyle \iota^*}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \iota^*} \\ H^{n,q}(E_y,E'_y) & \xrightarrow[\cong]{\psi} & H^n_{sing}(E_y,E'_y) & \longrightarrow & H^{n,q-1}(E_y,E'_y) \end{array}$$

The middle vertical map is an isomorphism by the nonequivariant Thom isomorphism theorem. The bottom left map is an isomorphism because $H^{n,q}(E_y, E'_y) \cong \tilde{H}^{n,q}(S^{n,q})$. The top left map is surjective from the above discussion. Thus the element $\alpha \in H^{n,q}(E, E')$ has the property that $\iota^*(\alpha)$ is the generator of $\tilde{H}^{*,*}(S^{n,q}) \cong \Sigma^{n,q}\mathbb{M}_2$. Hence $\theta\alpha \neq 0$ because $\theta\iota^*(\alpha) \neq 0$. In [13] it is shown that θ detects free submodules and furthermore $\mathbb{M}_2$ is self-injective. It must be that $\mathbb{M}_2 \cdot \alpha \cong \Sigma^{n,q}\mathbb{M}_2$, and this submodule splits off as a summand. We have shown (i) and (ii) hold.

For (iii), let $x \in X_i$. The statement follows from the below diagram and noting $H^{*,*}(E_x, E'_x) \cong \tilde{H}^{*,*}(S^{n,q_i}) \cong \Sigma^{n,q_i}\mathbb{M}_2$ (recall, $q \geq q_i$).

$$\begin{array}{ccc} H^{n,q}(E,E') & \xrightarrow{\psi} & H^n_{sing}(E,E') \\ \downarrow{\scriptstyle \iota^*} & & {\scriptstyle \iota^*}\downarrow{\scriptstyle \cong} \\ H^{n,q}(E_x,E'_x) & \xrightarrow[\cong]{\psi} & H^n_{sing}(E_x,E'_x) \end{array}$$

For (iv), let $x \in X \setminus X^{C_2}$. The fiber $E_{x,\sigma x}$ is isomorphic to $C_2 \times \mathbb{R}^n$ and thus $H^{*,*}(E_{x,\sigma x}, E'_{x,\sigma x}) \cong \tilde{H}^{*,*}(C_{2+} \wedge S^{n,0})$. Consider the following diagram:

$$\begin{array}{ccc} H^{n,q}(E,E') & \xrightarrow{\psi} & H^n_{sing}(E,E') \\ \downarrow{\scriptstyle \iota^*} & & {\scriptstyle \iota^*}\downarrow \\ H^{n,q}(E_{x,\sigma x},E'_{x,\sigma x}) & \xrightarrow{\psi} & H^n_{sing}(E_{x,\sigma x},E'_{x,\sigma x}) \end{array}$$

The bottom map is the diagonal map. Similarly from the singular Thom isomorphism theorem, the right vertical map is also the diagonal map. Thus the left vertical map is nonzero, and $\iota^*(u_E)$ is the unique nonzero element. □

The following corollary is helpful for understanding the fundamental classes in Section VII.

Corollary B.2.2. *Let X, $E \to X$, n, and q be defined as above. Then the forgetful map $\psi : H^{n,t}(E,E') \to H^n_{sing}(E,E')$ is an isomorphism for all $t \geq q$.*

Proof. Fix $t \geq q$. By parts (i) and (ii) of the above theorem, the forgetful map $\psi : H^{n,t}(E,E') \to H^n_{sing}(E,E')$ is surjective. Namely, the element $\tau^{t-q} \cdot u_E$ forgets to the unique nonzero class in $H^n_{sing}(E,E')$. We show $H^{n,t}(E,E') \cong \mathbb{Z}/2$ to conclude this map is an isomorphism.

In the previous proof, it was shown

$$H^{*,*}(E,E') \cong \Sigma^{n,q}\mathbb{M}_2 \oplus (\oplus_i \Sigma^{k_i,\ell_i}\mathbb{M}_2) \oplus (\oplus_j \Sigma^{s_j,0} A_{r_j})$$

where $k_i, s_j > n$. The summands in the last family contribute no nonzero elements to $H^{n,t}(E, E')$, but it is possible a priori that an $\mathbb{M}_2$-multiple of a generator of some free summand is in bidegree (n, t). In other words, the lattice point (n, t) might fall in the bottom cone of some other $\mathbb{M}_2$. Assume to the contrary this happens.

All lattice points (x, y) in the bottom cone of a summand generated at (k_i, ℓ_i) satisfy the inequality $y-(\ell_i-2) \leq x-k_i$. Thus $t-(\ell_i-2) \leq n-k_i$ or $k_i-\ell_i+2 \leq n-t$. Now $n - t \leq n - q$, so we have $k_i - \ell_i < n - q$. From the first paragraph of the previous proof and ρ-localization, $k_i - \ell_i \geq n - q$ for all i, so this is a contradiction. We conclude $H^{n,t}(E, E') \cong \mathbb{Z}/2$, and the proof is complete. □

B.3. The Theorem for Free Bundles, Part I

We now prove an analog to the above for free bundles. This was given as parts (i)-(iii) of Theorem 1.5.1 in the introduction. Let X be an equivariantly connected free C_2-CW complex. By equivariantly connected, we simply mean X cannot be covered by two nonempty, disjoint, equivariant open subsets.

Theorem B.3.1. *Let X be as above. Suppose $E \to X$ is an n-dimensional equivariant vector bundle. Then for every integer q, there exists a unique class $u_{E,q} \in H^{n,q}(E, E')$ such that the following holds:*

(i) $\psi(u_{E,q})$ is the singular Thom class, where $\psi : H^{n,q}(E, E') \to H^n_{sing}(E, E')$ is the forgetful map;

(ii) $\tau \cdot u_{E,q} = u_{E,q+1}$;

(iii) For every pair of conjugate points $x, \sigma x \in X$, the class $u_{E,q}$ restricts to the unique nonzero element in $H^{n,q}(E_{x,\sigma x}, E'_{x,\sigma x}) \cong \tilde{H}^{n,q}(S^n \wedge C_{2+})$.

Proof. Note $H^{*,*}(E,E') \cong \tilde{H}^{*,*}(DE/SE)$ where DE and SE are the unit disk and sphere bundles, respectively. Since X is a finite CW-complex and E is a finite dimensional vector bundle, the Thom space DE/SE is also a finite CW-complex, and hence a finite C_2-CW complex. The space DE/SE has exactly one fixed point, so Corollary **??** implies there are integers s_j, r_j such that

$$H^{*,*}(E,E') \cong \tilde{H}^{*,*}(DE/SE) \cong \oplus_j \Sigma^{s_j,0} A_{r_j}. \tag{B.3.3}$$

We find restrictions on the integers s_j. By the nonequivariant Thom isomorphism theorem, $H^k_{sing}(E,E') = 0$ for $k < n$. Based on the isomorphism in B.3.3, for each j and integer q, the forgetful map

$$\psi : H^{s_j,q}(E,E') \to H^{s_j}_{sing}(E,E')$$

must be nonzero by the forgetful long exact sequence. Hence $s_j \geq n$ for all j.

Consider the following portion of the forgetful long exact sequence for some integer q:

$$\ldots \xrightarrow{\rho} H^{n,q}(E,E') \xrightarrow{\psi} H^n_{sing}(E,E') \longrightarrow H^{n,q-1}(E,E') \xrightarrow{\rho} \ldots.$$

Based on the decomposition in B.3.3, $H^{n,q}(E,E') \cong H^{n,q-1}(E,E')$. Now if X is connected in the nonequivariant sense, then $H^n_{sing}(E,E') \cong \mathbb{Z}/2$ and so by exactness of the above, it must be that $H^{n,q}(E,E') \cong \mathbb{Z}/2$ or $H^{n,q-1}(E,E') \cong \mathbb{Z}/2$. Whichever is given by exactness, we can then conclude $H^{n,i}(E,E') \cong \mathbb{Z}/2$ for all integers i. If $X \cong C_2 \times Y$ for some nonequivariant connected space Y, then the vector bundle E must also have two nonequivariant connected components. In particular, $E \cong C_2 \times F$ where $F = \pi^{-1}(Y)$ is an n-dimensional vector bundle over Y. In this case $H^{n,i}(E, E-0) \cong H^n_{sing}(F, F-0) \cong \mathbb{Z}/2$ for all i by Lemma 3.0.6.

We have shown in either case $H^{n,q}(E, E-0) \cong \mathbb{Z}/2$ for all q. If we let $u_{E,q}$ be the unique nonzero element in $H^{n,q}(E, E-0)$, then properties (i) and (ii) immediately follow. Lastly, property (iii) holds by considering the following diagram:

$$\begin{array}{ccc} H^{n,q}(E,E') & \longrightarrow & H^{n,q}(E_{x,\sigma x}, E'_{x,\sigma x}) \cong \tilde{H}^{n,q}(S^n \wedge C_{2+}) \\ \downarrow & & \hookdownarrow \\ H^n_{sing}(E,E') & \longrightarrow & H^n_{sing}(E_{x,\sigma x}, E'_{x,\sigma x}) \cong \tilde{H}^n_{sing}(S^n \wedge C_{2+}) \end{array}$$

If E is nonequivariantly connected, then the left vertical map is an isomorphism, and the bottom horizontal map is injective by the singular Thom isomorphism theorem. If $E \cong F \sqcup F$, then the the left vertical map is injective, and the bottom horizontal map is an isomorphism by the singular Thom isomorphism theorem. In either case, the top horizontal map is nonzero, and the result follows after noting both groups are $\mathbb{Z}/2$. □

B.4. Part II of the Theorems

In the nonequivariant setting, cupping with the Thom class gives an isomorphism from the cohomology of the base space to a shift of the cohomology of the Thom space. It is natural to ask what happens when we cup with the equivariant Thom classes defined in the previous subsections. We explore that question now, first introducing some definitions and notation.

Notation and Terminology

Given an $\mathbb{M}_2$-module that is isomorphic to a direct sum of shifts of free modules and modules of the form $A_r = \tau^{-1}\mathbb{M}_2/(\rho^{r+1})$, we will refer to the A_r-summands as "antipodal summands". Given an antipodal summand of the form

$\Sigma^{s,0}A_r$, we can associate the tuple $(s;r)$, where an antipodal summand with tuple $(s;r)$ begins in topological dimension s and ends in topological dimension $(s+r)$. Lastly for an $\mathbb{M}_2$-module V and an element $v \in V$, we will write $wt(v)$ for the weight of v.

Definition B.4.1. Let X be a finite C_2-CW complex and $\pi : E \to X$ be an n-dimensional vector bundle. If X is nonfree, let $u_E \in H^{n,q}(E,E')$ be the Thom class and define $\phi_E : H^{*,*}(X) \to H^{*+n,*+q}(E,E')$ to be the map $\phi_E(x) = \pi^*(x) \smile u_E$. If X is free, let $\phi_{E,q}$ denote the degree (n,q) map $\phi_{E,q}(x) = \pi^*(x) \smile u_{E,q}$.

Here are the two main theorems we will prove in this subsection:

Theorem B.4.2 (Nonfree Bundles, Part II)**.** *Let X, $E \to X$ be as in Theorem B.2.1. We can add the following properties to this theorem:*

- *(v) The map ϕ_E is an isomorphism in bidegrees (f,g) where $g \geq f$;*
- *(vi) Suppose $H^{*,*}(X) \cong (\oplus_{i=1}^{c}\Sigma^{k_i,\ell_i}\mathbb{M}_2) \oplus (\oplus_{j=1}^{d}\Sigma^{s_j,0}A_{r_j})$. Then $H^{*,*}(E,E') \cong (\oplus_{i=1}^{c}\Sigma^{k_i+n,\ell_i'}\mathbb{M}_2) \oplus (\oplus_{j=1}^{d}\Sigma^{s_j+n,0}A_{r_j})$. where the weights ℓ_i' satisfy $\ell_i + q \geq \ell_i' \geq 0$;*
- *(vii) If in fact $E_x \cong E_y$ for all $x, y \in X^{C_2}$, then ϕ_E is an isomorphism in all bidegrees and $H^{*,*}(X) \cong H^{*+n,*+q}(E,E')$.*

Theorem B.4.3 (Free Bundles, Part II)**.** *Let X and $E \to X$ be as in Theorem B.3.1. The following property can be added to the theorem:*

- *(iv) The map $\phi_{E,j} : H^{*,*}(X) \to H^{*+n,*+j}(E,E')$ is an isomorphism for all j. In particular, $H^{*,*}(E,E') \cong H^{*+n,*}(X)$.*

We prove these theorems in a sequence of lemmas. We begin with trivial bundles, and as in singular cohomology, the isomorphism follows easily from the suspension isomorphism.

Lemma B.4.4. *Let X be a finite C_2-CW complex and $E = \mathbb{R}^{n,q} \times X$ be a trivial C_2-vector bundle over X. If X is nonfree, then ϕ_E is an isomorphism. If X is free, then $\phi_{E,j}$ is an isomorphism for all j*

Proof. Note $H^{*,*}(E, E') \cong \tilde{H}^{*,*}(X_+ \wedge S^{n,q})$ and so we have the suspension isomorphism

$$\Sigma^{n,q} : H^{*,*}(X) \to \tilde{H}^{*+n,*+q}(X_+ \wedge S^{n,q}) \cong H^{*,*}(E, E').$$

This agrees with the map ϕ_E if X is nonfree, and thus ϕ_E is an isomorphism. If X is free, the above agrees with the map $\phi_{E,q}$. In this case the cohomology of (E, E') and X are both $\tau^{-1}\mathbb{M}_2$-modules. By property (ii) of Theorem B.3.1, $u_{E,j} = \tau^{j-q} u_{E,q}$ so $\phi_{E,j} = \tau^{j-q}\phi_{E,q}$. We see that $\phi_{E,j}$ is a composition of isomorphisms, and thus an isomorphism for all j. □

To prove the main theorems, we will choose a cellular filtration for X such that each successive space is obtained by attaching a single equivariant cell. This will require us to understand how the Thom class behaves when restricted to the boundary of our trivial and nontrivial cells. The following lemmas address these questions.

Lemma B.4.5. *Let X be a finite C_2-CW complex and $E \to X$ be an n-dimensional C_2-vector bundle. Suppose A is a subcomplex of X and let $E_A = E|_A$.*

(i) Suppose X is nonfree and the maximum weight over X^{C_2} is q. If A is nonfree and the maximum weight over A^{C_2} is q_A, then the Thom class u_E restricts to $\tau^{q-q_A} u_{E_A}$. If A is free, then u_E restricts to $u_{E_A,q}$.

(ii) Suppose X is free. Then the Thom class $u_{E,j}$ restricts to $u_{E_A,j}$ for all j.

Proof. Both (i) and (ii) follow by considering how the Thom classes restrict to the fibers and the uniqueness of these classes given in Theorem B.2.1. □

Lemma B.4.6. *Let E be an n-dimensional bundle over the trivial sphere $S^{j,0}$ with $j \geq 1$. Then the map ϕ_E is an isomorphism in all bidegrees.*

Proof. We proceed by induction on j beginning with $j = 1$. The base space is trivial and connected, so there is a q such that $E_x \cong \mathbb{R}^{n,q}$ for all $x \in S^{1,0}$. Cover $S^{1,0}$ with two contractible open sets U_1 and U_2 such that $U_1 \cap U_2$ is homotopic to S^0. Let $E_i = E|_{U_i}$ and $E_{12} = E|_{U_1 \cap U_2}$. Note $E_i \cong \mathbb{R}^{n,q} \times U_i$ and similarly $E_{12} \cong \mathbb{R}^{n,q} \times (U_1 \cap U_2)$. Consider the following map between Mayer-Vietoris sequences:

$$\begin{array}{ccccccccc}
\longrightarrow & H^{f-1,g}(U_1 \cap U_2) & \longrightarrow & H^{f,g}(S^{1,0}) & \longrightarrow & H^{f,q}(U_1) \oplus H^{f,g}(U_2) & \longrightarrow \\
& {\scriptstyle \phi_{E_{12}}} \downarrow {\scriptstyle \cong} & & {\scriptstyle \phi_E} \downarrow & & {\scriptstyle \phi_{E_1} \oplus \phi_{E_2}} \downarrow {\scriptstyle \cong} & \\
\longrightarrow & H^{f-1,g}(E_{12}, E'_{12}) & \longrightarrow & H^{f,g}(E, E') & \longrightarrow & H^{f,q}(E_1, E'_1) \oplus H^{f,g}(E_2, E'_2) & \longrightarrow
\end{array}$$

Note the diagram commutes by Lemma B.4.5. Now the outer vertical maps and the previous and following vertical maps not shown are isomorphisms by Lemma B.4.4. By the five-lemma, the middle vertical map must also be an isomorphism.

For the inductive step. Let $j \geq 2$ and cover $S^{j,0}$ with two contractible open sets U_1, U_2 such that $U_1 \cap U_2 \simeq S^{j-1,0}$. We can again use the map between Mayer-Vietoris sequences now with Lemma B.4.4 and the inductive hypothesis to prove the claim. □

Lemma B.4.7. *Suppose Y is a finite CW complex and consider the free space $X = C_2 \times Y$. Let $E \to X$ be an n-dimensional vector bundle over X. Then $\phi_{E,j}$ is an isomorphism for all j.*

Proof. Let $C_2 = \{0,1\}$ and $F = \pi^{-1}(\{1\} \times Y)$. Then $F \to Y$ is a nonequivaraint bundle and $E \cong C_2 \times F$. We have fold maps $\nabla : X \to Y$ and $\nabla : E \to F$. On singular cohomology, note $\nabla^* = \Delta$ where Δ is the diagonal map. Consider the following commutative diagram

$$\begin{array}{ccc} H^{*,*}(X) & \xrightarrow{\phi_{E,0}} & H^{*+n,*}(E,E') \\ \big\downarrow \psi & & \big\downarrow \psi \\ H^{*}_{sing}(X) & \xrightarrow{\phi_E} & H^{*+n,*}_{sing}(E,E') \\ \Delta \big\uparrow & & \Delta \big\uparrow \\ H^{*}_{sing}(Y) & \xrightarrow{\phi_F} & H^{*+n,*}(F,F') \end{array}$$

From the proof of Lemma 3.0.6, the image of $H^{*,*}(X)$ and $H^{*,*}(E,E')$ under the forgetful map is the same as the image of $H^{*}_{sing}(Y)$ and $H^{*,*}(F,F')$ under the diagonal map, respectively. The bottom map is an isomorphism by the singular Thom isomorphism theorem, and thus the top map is an isomorphism as well.

We have shown $\phi_{E,0}$ is an isomorphism. Since $H^{*,*}(X)$ and $H^{*,*}(E,E')$ are both $\tau^{-1}\mathbb{M}_2$-modules and $\phi_{E,j} = \tau^j \phi_{E,0}$, we see that $\phi_{E,j}$ must be an isomorphism for all j. □

In what follows, we will need to understand how powers of τ act on the cohomology of a space. The result below tells us this action is an isomorphism in a certain range.

Lemma B.4.8. *Let X be a finite C_2-CW complex. For $k > 0$, action by τ^k gives an isomorphism $\tau^k : H^{f,g}(X) \to H^{f,g+k}(X)$ if $g \geq f$.*

Proof. By inspection this holds for modules of the form $\Sigma^{s,0}A_r$ and $\Sigma^{p,q}\mathbb{M}_2$ where $p \geq q$. The statement then follows because $H^{*,*}(X)$ is isomorphic to a direct sum of such modules by Theorem 3.0.11. □

We are now ready to prove part (v) of Theorem B.4.2.

Lemma B.4.9. *Let X be a finite, nonfree C_2-CW complex and $\pi : E \to X$ be an n-dimensional vector bundle with maximum weight q over X^{C_2}. Then the map ϕ_E is an isomorphism in bidegrees (f, g) such that $g \geq f$.*

Proof. Fix a C_2-CW structure on X. Choose a cellular filtraton

$$A_0 \subset A_1 \subset \cdots \subset A_n = X$$

such that A_0 is the zero skeleton and each A_{i+1} is obtained by adding a single equivariant cell to A_i. Let $E_i = E|_{A_i}$. We inductively prove the statement holds for each ϕ_{E_i}. The weight of the class u_{E_i} may change as i changes, and thus let $q_i = wt(u_{E_i})$. Note $q_i \leq q$ for all i.

The zero skeleton A_0 is a disjoint union of fixed points x_i and free orbits $\{y_j, \sigma y_j\}$. The restricted bundle E_0 is a disjoint union of trivial vector bundles of the form $V_j = \{x_i\} \times \mathbb{R}^{n,k_j}$ and $V_\ell = \{y_\ell, \sigma y_\ell\} \times \mathbb{R}^n$. Let $k = \max\{k_j\}$. In this case, ϕ_{E_0} is just the sum of $\tau^{k-k_j}\phi_{V_j} = \phi_{V_j}\tau^{k-k_j}$ and $\phi_{V_j,k}$, each of which is an isomorphism in the described range by Lemmas B.4.4 and B.4.8.

Now for $i \geq 0$ assume ϕ_{E_i} is an isomorphism in the described range. The zero skeleton A_0 must be nonfree since X is nonfree, and thus A_i is nonfree. There are two cases based on the cell we are attaching to form A_{i+1}. First, assume the cell is a trivial cell of the form $C_2/C_2 \times D^j$ for some $j \geq 1$. Let $\tilde{V} = D^j - \partial D^j$ and let $\tilde{U} \subset D^j$ be a small neighborhood of the boundary of D^j such that $\tilde{U} \simeq \partial D^j$ and $\tilde{U} \cap \tilde{V} \simeq S^{j-1}$. Now let $U \subset A_{i+1}$ be the open set consisting of A_i and the image of $\tilde{U}$ under the attaching map, and let V be the image of $\tilde{V}$ in A_{i+1}. Note $U \simeq A_i$, $V \simeq \tilde{V} \simeq pt$, and $U \cap V \simeq S^{j-1,0}$.

Our plan is to consider the maps between Mayer-Vietoris sequences as in Lemma B.4.6. To do so, we introduce some notation. Let $G_1 = E_i|_U$, $G_2 = E_i|_V$, and $G_{12} = E_i|_{U\cap V}$. Note the weights of the corresponding Thom classes are less than or equal to the weight of $u_{E_{i+1}}$. Let $g_k = q_{i+1} - wt(u_{G_k})$ and $g_{12} = q_{i+1} - wt(u_{G_{12}})$. Now consider the following map between Mayer-Vietoris sequences where $g \geq f$.

$$\begin{array}{ccccc}
\ldots \longrightarrow & H^{f-1,g}(U\cap V) & \longrightarrow & H^{f,g}(A_{i+1}) & \longrightarrow \\
& \Big\downarrow {\scriptstyle \tau^{g_{12}}\phi_{G_{12}}} \cong & & \Big\downarrow {\scriptstyle \phi_{E_{i+1}}} & \\
\ldots \longrightarrow & H^{f-1+n,g+q}(G_{12},G'_{12}) & \longrightarrow & H^{f+n,g+q}(E_{i+1},E'_{i+1}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{f,g}(U)\oplus H^{f,g}(V) & \longrightarrow & H^{f,g}(U\cap V) & \longrightarrow \ldots \\
& \Big\downarrow {\scriptstyle \tau^{g_1}\phi_{G_1}\oplus\tau^{g_2}\phi_{G_2}} \cong & & \Big\downarrow {\scriptstyle \tau^{g_{12}}\phi_{G_{12}}} \cong & \\
\longrightarrow & H^{f+n,g+q}(G_1,G'_1)\oplus H^{f+n,g+q}(G_2,G'_2) & \longrightarrow & H^{f+n,g+q}(G_{12},G'_{12}) & \longrightarrow \ldots
\end{array}$$

The upper left and bottom right vertical maps are given by $\tau^{g_{12}}\phi_{G_{12}}$ which is equal to $\phi_{G_{12}}\tau^{g_{12}}$. By Lemma B.4.8, $\tau^{g_{12}}$ is an isomorphism since $g \geq f > f-1$. By Lemma B.4.6, $\phi_{G_{12}}$ is an isomorphism on $\tau^{g_{12}}H^{f-1,g}(X)$. Similarly the bottom left vertical map and the previous vertical map that is not pictured are isomorphisms by the inductive hypothesis and Lemma B.4.4. The five-lemma now implies the upper right vertical map is an isomorphism.

This completes the case when the attached cell is trivial. The argument is similar when the attached cell is of the form $C_2 \times D^j$. Again let U be a neighborhood of A_i such that $U \simeq A_i$, and now let V be the image of $C_2 \times (D^j - \partial D^j)$. The intersection will be homotopic to $C_2 \times S^{j-1}$. We can again use Mayer-Vietoris and the five-lemma, now replacing $\phi_{G_{12}}$ with $\phi_{G_{12},q}$ and ϕ_{G_2} with $\phi_{G_2,q}$. This will complete the proof in the case when the attached cell is free. $\square$

Corollary B.4.10. *Let X be a nonfree, finite C_2-CW complex and $E \to X$ be an n-dimensional C_2-vector bundle. Suppose there is a q such that for every $x \in X^{C_2}$, $E_x \cong \mathbb{R}^{n,q}$. Then ϕ_E is an isomorphism in all bidegrees.*

Proof. In the previous proof, the values of q_i will be constant, and there will be no powers of τ needed in the maps between the Mayer-Vietoris sequences. In particular, one will have that maps $\phi_{G_{12}}$ (or $\phi_{G_{12},q}$ if the attached cell is free) and $\phi_{G_1} \oplus \phi_{G_2}$ (or $\phi_{G_1} \oplus \phi_{G_2,q}$ if the attached cell is free) are isomorphisms for all f, g, and by the five-lemma $\phi_{E_{i+1}}$ will be an isomorphism for all f, g. $\square$

We now provide a proof of the main theorems, beginning with the statement for nonfree bundles.

Proof of Theorem B.4.2. Parts (v) and (vii) were done in Lemma B.4.9 and Corollary B.4.10. The proof of the isomorphism in (vi) is entirely algebraic, and follows from Theorem B.5.3 below. $\square$

Proof of Theorem B.4.3. We can put a cellular filtration on the free C_2-CW complex X as in the proof of Lemma B.4.9. The proof will then follow similarly, though now we will only be attaching free cells. By taking $\phi_{E,0}$, we obtain the isomorphism $H^{*,*}(X) \cong H^{*+n,*}(E, E')$. $\square$

B.5. Algebra Proof

We conclude this appendix by stating and proving a theorem about maps between nice $\mathbb{M}_2$-modules that are isomorphisms in a certain range. This theorem will imply property (vi) of Theorem B.4.2.

Notation and Terminology

We say an $\mathbb{M}_2$-module is "nice" if it is a direct sum of finitely many copies of shifted free modules and shifted copies of $A_r = \tau^{-1}\mathbb{M}_2/(\rho^{r+1})$ for various values of r, and furthermore, if all shifts are given by actual representations, i.e. the shifts are given by (p, q) where $p \geq q \geq 0$. We will refer to the A_r-summands as "antipodal summands". Given an antipodal summand of the form $\Sigma^{s,0}A_r$, we can associate the tuple $(s; r)$; note an antipodal summand with tuple $(s; r)$ begins in topological dimension s and ends in topological dimension $(s + r)$. Given an $\mathbb{M}_2$-module V and an element $v \in V$ we will write $wt(v)$ for the weight of v. When considering a single bidegree, we will write $V^{f,g}$ for the elements of V in bidegree (f, g).

In the proof, we will consider certain quotients, submodules, and localizations of nice $\mathbb{M}_2$-modules in order to detect the properties of free versus antipodal summands. For an $\mathbb{M}_2$-module M let $T(M) = \{m \in M : \rho^i m = 0 \text{ for some } i\}$. If M is a nice module, note $T(M)$ consists of the antipodal summands and the bottom cones of free summands. We can then consider the quotient $M/T(M)$ which is isomorphic to a direct sum of top cones, one for each free summand in M. We will denote such quotients by $\tilde{M}$. If we further quotient to form $\tilde{M}/\operatorname{im}(\rho)$, we obtain a module isomorphic to a direct sum of shifts of the module $\mathbb{Z}/2[\tau]$, one for each free summand. We can also consider the localization $\tau^{-1}T(M)$ which is isomorphic to the antipodal summands of M.

We begin by proving a lemma about $\mathbb{Z}/2[t]$-modules which will be useful when considering the quotient $\tilde{M}/\operatorname{im}(\rho)$.

Lemma B.5.2. *Consider the graded polynomial ring $R = \mathbb{Z}/2[t]$ where $|t| = 1$. Let M be a finitely generated, free R-module with R-basis $\{\alpha_1, \ldots, \alpha_m\}$. Suppose N is* *another finitely generated, free R-module, and $\phi : M \to N$ is a degree q map such that $\phi : M^g \to N^{g+q}$ is an isomorphism whenever $g \geq g_0$ for some integer g_0. Then there exists an R-basis $\{\beta_1, \ldots, \beta_m\}$ for N such that $|\alpha_i| + q \geq |\beta_i|$.*

Proof. There exist integers $j_1, \ldots, j_m$ such that $g = |t^{j_i}\alpha_i|$ is larger than g_0 and constant for all i. The elements $t^{j_i}\alpha_i$ form a linearly independent set in M^g and thus the images $\phi(t^{j_i}\alpha_i)$ form a linearly independent set in N^{g+q}. This implies there are at least m free summands in N. We can similarly use ϕ^{-1} in the range it exists to show there can be at most m free summands in N. We conclude N has an R-basis consisting of m elements.

We proceed by induction on m to show there is a basis $\{\beta_1, \ldots, \beta_m\}$ for N such that $|\alpha_i| + q \geq |\beta_i|$. This is clear if the bases consist of exactly one element since otherwise ϕ would be zero. For the inductive hypothesis, suppose we can find such a basis whenever we have a map that is an isomorphism in sufficiently high degrees between finitely generated, free R-modules of rank $m-1$. Choose some R-basis $\{b_1, \ldots, b_m\}$ for N and suppose $\phi(\alpha_1) = \sum_i \epsilon_i t^{j_i} b_i$ where $\epsilon_i \in \mathbb{Z}/2$. Let b_k be a basis element of maximal degree such that the coefficient ϵ_i is nonzero. Reorder the set so that this element is now b_1. We can factor out powers of t to obtain

$$\phi(\alpha_1) = t^{j_1}\left(b_1 + \textstyle\sum_{i>1} \epsilon_i t^{j_i - j_1} b_i\right).$$

Let $\beta_1 = b_1 + \sum_{i>1} \epsilon_i t^{j_i - j_1} b_i$ and note the set $\{\beta_1, b_2, \ldots, b_m\}$ is still an R-basis for N. Furthermore, we obtain a map

$$\langle \alpha_2, \ldots, \alpha_m \rangle \xrightarrow{\phi} N \twoheadrightarrow N/\langle \beta_1 \rangle$$

that is still an isomorphism in the desired range. By the inductive hypothesis, there exists an R-basis $\beta_2', \ldots, \beta_m'$ for the quotient such that $|\beta_i'| \leq |\alpha_i| + q$. Let β_i be a lift of β_i' to N. Then $\beta_1, \ldots, \beta_m$ will be a basis for N such that $|\beta_i| \leq |\alpha_i| + q$. □

Theorem B.5.3. *Let V and W be two nice $\mathbb{M}_2$-modules such that V has c free summands generated in bidegrees (k_i, ℓ_i) and d antipodal summands with tuples $(s_j; r_j)$. Suppose $\phi : V \to W$ is an (n, q)-degree $\mathbb{M}_2$-module map such that $\phi : V^{f,g} \to W^{f+n,g+q}$ is an isomorphism whenever $g \geq f$. Then W has exactly c free summands generated in bidegrees $(k_i + n, \ell_i')$ such that $\ell_i + q_i \geq \ell_i' \geq 0$ and exactly d antipodal summands with tuples $(s_j + n; r_j)$.*

Proof. One readily checks the restrictions $\phi : T(V) \to T(W)$ and $\phi : \mathrm{im}(\rho) \to \mathrm{im}(\rho)$ are isomorphisms in the given range. The map ϕ then descends to a map

$$\tilde{\phi} : (\tilde{V}/\mathrm{im}(\rho))^{f,g} \to (\tilde{W}/\mathrm{im}(\rho))^{f+n,g+q}$$

that is still an isomorphism when $g \geq f$. The quotients $\tilde{V}/\mathrm{im}(\rho)$ and $\tilde{W}/\mathrm{im}(\rho)$ are isomorphic to a direct sum of shifts of the module $\mathbb{Z}/2[\tau]$, one for each free summand in V and W, respectively. Explicitly

$$\tilde{V}/\mathrm{im}(\rho) \cong \textstyle\bigoplus_{i=1}^{c} \Sigma^{k_i,\ell_i}\mathbb{Z}/2[\tau].$$

Fix a topological dimension k and consider $(\tilde{V}/\mathrm{im}(\rho))^{k,*}$ and $(\tilde{W}/\mathrm{im}(\rho))^{k+n,*}$. These are both finitely generated, free $\mathbb{Z}/2[\tau]$-modules and the map

$$\tilde{\phi} : (\tilde{V}/\mathrm{im}(\rho))^{k,g} \to (\tilde{W}/\mathrm{im}(\rho))^{k+n,g+q}$$

is an isomorphism whenever $g \geq k$. By Lemma B.5.2, there are respective bases $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_m\}$ for $(\tilde{V}/\mathrm{im}(\rho))^{k,*}$ and $(\tilde{W}/\mathrm{im}(\rho))^{k+n,*}$ such that $wt(\beta_i) \leq wt(\alpha_i) + q$. Lifts of the elements α_i and β_i to V and W will generate the free summands in bidegrees $(k, wt(\alpha_i))$ and $(k+n, wt(\beta_i))$ in V and W, respectively. This proves the statement about the free summands.

We next consider the antipodal summands. Consider the localization $\tau^{-1}T(V)$. Let $F = \mathbb{Z}/2[\tau, \tau^{-1}]$ and note $\tau^{-1}T(V)$ is a finitely generated $F[\rho]$-

module. By the decomposition of V into summands as an $\mathbb{M}_2$-module, we can say explicitly

$$\tau^{-1}T(V) \cong \bigoplus_{j=1}^{d} \Sigma^{s_j,0} F[\rho]/(\rho^{r_j+1}).$$

Similarly $\tau^{-1}T(W)$ is isomorphic to a direct sum of shifts of $F[\rho]/(\rho^{r+1})$, one for each antipodal summand of W.

Note ϕ restricts to a map $\phi : T(V) \to T(W)$ which then localizes to give a map $\tau^{-1}\phi\colon \tau^{-1}T(V) \to \tau^{-1}T(W)$ of finitely generated $F[\rho]$-modules. Since ϕ was an isomorphism in bidegrees (f, g) such that $g \geq f$, the map $\tau^{-1}\phi$ is guaranteed to be an isomorphism in these bidegrees. Though, now that τ is invertible, we see that $\tau^{-1}\phi$ is an isomorphism in all bidegrees. We conclude $\tau^{-1}T(V) \cong \tau^{-1}T(W)$ and W must have exactly d antipodal summands with tuples $(s_j + n; r_j)$. $\square$

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