# TOPICS IN RANDOM WALKS

by

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DISSERTATION ABSTRACT

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We study a family of random walks defined on certain Euclidean lattices that are related to incidence matrices of balanced incomplete block designs. We estimate the return probability of these random walks and use it to determine the asymptotics of the number of balanced incomplete block design matrices. We also consider the problem of collisions of independent simple random walks on graphs. We prove some new results in the collision problem, improve some existing ones, and provide counterexamples to illustrate the complexity of the problem.

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For my wife, Melissa.

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# CHAPTER I

# INTRODUCTION

In this thesis, we investigate attributes of a particular Markov chain known as a random walk. We begin with the intuitive definition of a random walk: if $S$ is a (countable) set, a random walk is envisioned as a walker that starts at some element $o$ of $S$ and chooses where to go next according to some preset distribution. After taking this step, the walker will take another step based on some preset distribution, then another, and so forth. These distributions can (and often do) depend on where the walker is, but they do not depend on the path the walker took to get there. If the walker happens to wander back to some vertex that it has previously visited, its options for how to take its next step are identical to what they were on its previous visit.

For our purposes, a random walk on $S$ is a function $X : \mathbb{N} \to S$ with random outputs. (We use $\mathbb{N}$ to denote the nonnegative integers.) We will typically denote this function by $X_n$. The outputs of the function must be governed by a nonnegative transition density function $p : S \times S \to \mathbb{R}$ with the property that for any $x$,

$$\sum_{y \in S} p(x, y) = 1.$$

This function corresponds to the 'choices' of the imaginary walker; the quantity $p(x, y)$ represents the probability that the walker, standing at $x$, will move next to $y$. For $n \geq 2$, we inductively define $p^n(x, y)$ via convolution; that is,

$$p^{n+1}(x, y) = \sum_{z \in S} p^n(x, z)p(z, y).$$

We then set

$$\mathbb{P}_o(X_n = x) = p^n(o, x)$$

to give the distribution of the function $X$. Our use of the subscript $o$ on the probability symbol $\mathbb{P}$ corresponds to the notion that the random walk is started at $o$. As is typical of probability theory, we will only 'define' the function $X$ in the sense of providing the distribution of $X_n$.

With these definitions, it follows that a random walk is entirely determined by the transition function $p(x, y)$. Hence, to understand the random walk, we will need stipulate how $p(x, y)$ is generated. There are two primary ways that this will occur in this thesis. The 'classical' definition of a random walk usually takes place on some countable subset of a Euclidean lattice $\mathbb{R}^d$. This is generally given by specifying a sequence of random, independent, identically distributed increments $\vec{\xi_i}$ and specifying that

$$X_n = \sum_{i=1}^{n} \xi_i.$$

(The empty sum should be regarded as the zero vector so that the walk starts at the origin.) This definition will be the one used in Chapter II, but it requires the ambient space to be a vector space and thus does not generalize well to graphs.

For our purposes, a graph $G$ consists of a vertex set $v(G)$ and an edge set $e(G) \subset v(G) \times v(G)$ that gives an adjacency relation. We will use $x \sim y$ to denote that $(x, y) \in e(G)$; that is, that there is an edge from $x$ to $y$ in $G$. We will use $d(x)$ to denote the vertex degree of $x$, and will require that $d(x) < \infty$ for all $x \in v(G)$. The simple random walk on $G$ will be defined by the transition density

$$p(x,y) = \begin{cases} 1/d(x), & x \sim y \\ 0, & x \not\sim y \end{cases}$$

which will be used in Chapter III. We note that applying this definition to a Euclidean lattice $\mathbb{Z}^d$ gives a particular instance of the type of random walk on $\mathbb{R}^d$ as previously defined. Specifically, this corresponds to the case where the vector $\xi_i$ can be any lattice vector of length 1, each selected with equal probability.

Finally, we also remark that the above machinery yields the definition of a discrete-time random walk on $S$, which will be the one used by default in the sequel. However, we will occasionally need to refer to a continuous-time random walk on $S$. To construct a (random) function $X : \mathbb{R}_{\geq 0} \to S$, we will employ a sequence of independent, identically distributed mean-1 exponential random variables $W_i$. If $X_n$ is a discrete-time random walk on $G$, we define the continuous-time random walk on $G$ by

$$Y_t = X_{N(t)}$$

where $N(t)$ is the random variable

$$N(t) = \min\left\{ s : \left[\sum_{i=1}^{s} W_i\right] < t \right\}.$$

The interpretation is that the $W_i$ variables represent the wait time between moves of the process $Y_t$. This interpretation allows us to reframe the continuous-time walk in terms of the discrete-time walk.

In Chapter II, we will employ the theory of random walks to derive results about the number of a certain type of combinatorially-defined matrix. We will apply some basic Fourier analysis to the random walk, and will use this analysis to gain information regarding the probability of its return to the origin. This will then immediately yield results about the number of these combinatorially-defined matrices. In Chapter III, we will consider two simultaneous independent random walks on a given graph $G$ and will ask about the probability that they will be in the same place at the same time infinitely often. This question, while easy to state, turns out to have a surprisingly delicate and difficult answer. We will not come close to fully answering the question, but will provide some developments in that direction.

# CHAPTER II

# COUNTING BALANCED INCOMPLETE BLOCK DESIGN INCIDENCE MATRICES

In this chapter, we will relate a random walk on a certain Euclidean lattice to the existence of a matrix that is important to combinatorial design theory. We will then employ well-established techniques involving the random walk to gain knowledge about the combinatorial matrices.

**Definition 2.1.** We say that an $n \times t$ matrix populated with 1's and 0's is an *incidence matrix of a balanced incomplete block design* if there are positive integers $k$ and $\ell$ such that:

- each column has exactly $k$ 1's, and
- each pair of distinct rows has inner product $\ell$, which is independent of the choice of the pair.

We will use BIBD as a shorthand for balanced incomplete block design. It is well-known that the above conditions imply that the number of 1's in each row is a constant, which we will call $r$. The following relations between $n, t, k, r,$ and $\ell$ are also well-known:

$$tk = nr \tag{2.1}$$

$$r(k-1) = \ell(n-1) \tag{2.2}$$

$$tk(k-1) = \ell n(n-1) \tag{2.3}$$

A reference for (2.1) and (2.2) can be found at [DS92, p. 2]; from these, one can easily derive (2.3). We note from these relations that choosing values for the parameters $n, t, k$ forces the values of $r$ and $\ell$, so we will focus our attention on the various possibilities for $n, k, t$.

Our strategy for generating these incidence matrices will be as follows: for a fixed $n$ and $k$, we define $V_{n,k}$ to be the collection of all vectors in $\mathbb{R}^n$ with $k$ 1's and $n-k$ 0's. We will construct a BIBD incidence matrix by concatenating randomly-drawn columns from the collection $V_{n,k}$ and considering whether the inner product condition is satisfied for the randomly-generated matrix.

We now define our random walk and explain its correspondence with BIBD incidence matrices. For an integer $n \geq 2$, we set $d = \binom{n}{2}$; the random walk will occur in $\mathbb{R}^d$, which will be regarded as a set of column vectors. Instead of using the standard index system for coordinates of $\mathbb{R}^d$ (i.e. $1, \ldots, d$), we will take our index set to be the set of all $S \subset \{1, \ldots, n\}$ with $|S| = 2$. When important, we will refer to a lexicographic ordering; that is, for $\vec{x} \in \mathbb{R}^d$,

$$\vec{x} = (x_{\{1,2\}}, x_{\{1,3\}}, \ldots, x_{\{n-2,n\}}, x_{\{n-1,n\}})^T.$$

We define a function $Z : V_{n,k} \to \mathbb{R}^d$ by

$$Z(\vec{y}) = (y_1y_2, y_1y_3, \ldots, y_{n-2}y_n, y_{n-1}y_n)^T.$$

The purpose of this function is that if $Y = [\vec{y}^{(1)} \ldots \vec{y}^{(t)}]$ and $\vec{1}$ is the vector of all ones, then

$$Z(\vec{y}^{(1)}) + \cdots + Z(\vec{y}^{(t)}) = \ell\vec{1}$$

if and only if the inner product between any two rows of $Y$ is $\ell$. This allows us to reframe out constraint about the inner product or rows as one of a sum, which gives us a way to consider a random walk.

**Definition 2.2.** We define our random walk $X_t$ on $\mathbb{Z}^d$ to be the random walk with increments drawn randomly and uniformly from $\{Z(\vec{y}) : \vec{y} \in V_{n,k}\}$.

From the previous discussion, the existence of a BIBD incidence matrix is then equivalent to the entry of the random walk $X_t$ into the diagonal set $\Delta = \{\ell\vec{1} : \ell \in \mathbb{Z}\}$. The random walk $X_t$ is not the ideal random walk to consider, for two reasons: first, the set $\Delta$ is infinite, which makes the probability that $X_t$ enters it a bit complicated. Second, the increments of $X_t$ clearly do not have mean $\vec{0}$, since vectors of the form $\{Z(\vec{y}) : \vec{y} \in V_{n,k}\}$ also have entries that are only 0 and 1.

To fix the issues with $X_t$, we introduce a new random walk, $Y_t$, which is the drift-corrected version of $X_t$. If a vector is chosen uniformly from $\{Z(\vec{y}) : \vec{y} \in V_{n,k}\}$, then the probability of a given coordinate (say, $\{i, j\}$) being 1 is equal to the probability that $y_i = 1$ and $y_j = 1$. This probability is $\binom{n-2}{k-2}/\binom{n}{k} = \frac{k(k-1)}{n(n-1)}$, so to get a centered random walk, we subtract this term from each coordinate of the increments. That is,

$$Y_t = X_t - \frac{k(k-1)}{n(n-1)} t\vec{1}.$$

Since we are interested in the probability that the random walk $X_t$ is equal to $\ell\vec{1}$ for some constant $\ell$, we notice by (2.3) that $\ell = \frac{k(k-1)}{n(n-1)}t$, which implies that $X_t = \ell\vec{1}$ iff $Y_t = 0$. Hence, our tactic will be to estimate the probability that the random walk $Y_t$ returns to $\vec{0}$ after $t$ steps, which we will denote $\mathbb{P}^{(t)}_{n,k}(\vec{0}, \vec{0})$.

Since $n \times t$ matrices populated with columns from $V_{n,k}$ lie in a 1-1 correspondence with paths of the random walk $X_t$ (hence, with $Y_t$), it follows that

$$\frac{\#\text{ BIBD incidence matrices}}{\#\text{ total matrices}} = \frac{\#\text{ return paths of } Y_t \text{ to } \vec{0}}{\#\text{ all paths of } Y_t}.$$

The right-hand side of this equation is precisely the probability that the random walk $Y_t$ returns to 0, which we will denote by $\mathbb{P}_{n,k}^{(t)}(\vec{0},\vec{0})$. (Our random walks in this chapter will be understood to always start at the origin, and the $n, k$ subscript serves only to indicate the preset parameters $n$ and $k$.) The denominator of the left-hand side is $\binom{n}{k}^t$, since there are $\binom{n}{k}$ distinct choices for each of the $t$ columns. Thus,

$$\#\text{ BIBD incidence matrices} = \binom{n}{k}^t \mathbb{P}_{n,k}^{(t)}(\vec{0},\vec{0}) \tag{2.4}$$

so to count the number of BIBD incidence matrices, we need only to find sufficiently accurate estimates on the return probability of the random walk $Y_t$. We will prove a local central limit theorem for the quantity $\mathbb{P}_{n,k}^{(t)}(\vec{0},\vec{0})$, which will yield the following theorem:

**Theorem 2.3.** Let $n, k, t$ be such that $k \geq 2$, $n - k \geq 2$, $t\frac{k}{n} \in \mathbb{Z}$, and $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$. Let $\Psi_{n,k,t}$ be the number of BIBD incidence matrices of dimensions $n \times t$ with $k$ 1's in each column, and let $d = \binom{n}{2}$. If

$$f(n,k) = \frac{2\left(\frac{(n-3)(k-1)}{n-k-1}\right)^n (n-2)\left(\frac{k(k-1)[k(k+1)-2kn+n(n-1)]}{n(n-1)(n-2)(n-3)}\right)^d}{(n-k)(k-1)^2 k},$$

then

$$\Psi_{n,k,t} = [1 + o(1)]\binom{n}{k}^t \frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1} f(n,k)}} \text{ as } t \to \infty.$$

The basic strategy for estimating $\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0})$ will be the standard tactic of using the Fourier inversion formula (see, for instance, [Spi76, P3, p. 57]). Using the characteristic function $\Phi_Y(\vec{\theta})$, defined as

$$\Phi_Y(\vec{\theta}) = \mathbb{E}[e^{i\vec{\theta}\cdot Y_1}] = \sum_{\vec{y}\in V_{n,k}} \binom{n}{k}^{-1} e^{i\vec{\theta}\cdot\left(Z(\vec{y})-\frac{k(k-1)}{n(n-1)}\vec{1}\right)}$$

the return probability can be obtained by using

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta}. \tag{2.5}$$

Since the random walk $Y_t$ is merely a spatially-shifted version of $X_t$, it will also be useful to consider the analogously-defined characteristic function $\Phi_X(\vec{\theta}) = \mathbb{E}[e^{i\vec{\theta}\cdot X_1}]$; we will explore the connections between the two and will switch our focus between $\Phi_X$ and $\Phi_Y$ depending on what is more convenient.

We note that the Fourier inversion formula in (2.5) only holds when $Y_t$ is supported on $\mathbb{Z}^d$, which will occur if and only if $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$; however, it is not necessary to consider the case when $t\frac{k(k-1)}{n(n-1)} \notin \mathbb{Z}$, since by (2.3) we see that no such BIBD incidence matrix can exist. To estimate the integral in (2.5), we will divide $[-\pi,\pi]^d$ into regions where $|\Phi_Y(\vec{\theta})|$ is close to 1 and those where it is not, and provide estimates on $\Phi_Y(\vec{\theta})$ accordingly. As $t$ becomes large, the bulk of the integral will be determined by the regions where $|\Phi_Y(\vec{\theta})|$ is close to 1, and the contributions from the other parts will become negligible.

Before the proof of Theorem 2.3, we note that the restrictions that $k \geq 2$ and $n-k \geq 2$ occur for technical reasons, although if $k=1$, the BIBD incidence matrices are trivial in the sense that the inner product of any two distinct rows of any such matrix is automatically 0. The case where $k=2$ is nearly trivial as well, since a

BIBD incidence matrix with $k = 2$ can only occur when every possible column from $V_{n,k}$ occurs the same number of times. One can see without any advanced tactics that the number of such matrices must then be

$$\Psi_{n,2,t} = \frac{t!}{[(t/d)!]^d}$$

which is asymptotically equivalent to the formula in Theorem 2.3 as shown by Stirling's formula.

We also remark that while in principle the calculation of the return probability of $Y_t$ is just a matter of computing asymptotic values in a local central limit theorem, the walk has a special structure that complicates matters. In particular, the increment set of the walk is not symmetric, and the walk takes place on a sublattice of $\mathbb{R}^d$ which is difficult to specify as a purely combinatorial entity. For these reasons, the common approach of explicitly transforming the walk $Y_t$ to a simple random walk on an integer lattice is challenging here, and we will instead opt for the Fourier-analytic approach as previously outlined.

As a final remark, although we do not carry out these computations here, we note that the estimates used to prove Theorem 2.3 are sufficiently sharp to prove existence results for balanced incomplete block designs. Specifically, for a fixed $n$ and $k$, the return probability in (2.5) could be shown to be positive for sufficiently large suitable $t$ (that is, $t$ where $t\frac{k}{n} \in \mathbb{Z}$ and $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$). This would imply that there exist balanced incomplete block designs with those parameters. This claim would be similar in principle to Wilson's Theorem (see [DS92, Theorem 4.1, p. 7]), which asserts the same result for fixed $k$ and $\ell = 1$ while allowing $n$ to increase to infinity.

The outline of the sections is as follows: in Section 2.1, we give an explicit description of the so-called 'maximal set'; that is, the set where $|\Phi_Y(\vec{\theta})| = 1$. This set has the structure of a number of distinct lines in $\mathbb{R}^d$. In Section 2.2, we discuss how to decompose the integral in (2.5) in terms of this maximal set. In Section 2.3, we provide estimates on the integral contributions far from the maximal set. In Section 2.4, we introduce an important combinatorially-defined matrix $N$ and use it to get bounds on the integral contribution near the maximal set. In Section 2.5, we compute the expression $f(n,k)$ found in the statement of Theorem 2.3. This expression will arise as the determinant of a principal submatrix of the aforementioned matrix $N$. Finally, in Section 2.6 we put all the parts together to prove Theorem 2.3.

## 2.1 Extreme Values of the Characteristic Function

In this section, we seek to understand the set where the characteristic functions $\Phi_X$ and $\Phi_Y$ have maximum absolute value. We begin with the operative definitions:

$$\Lambda_X = \{\vec{\theta} \in [-\pi,\pi]^d : |\Phi_X(\vec{\theta})| = 1\}$$
$$\Lambda_Y = \{\vec{\theta} \in [-\pi,\pi]^d : |\Phi_Y(\vec{\theta})| = 1\}$$

**Proposition 2.4.** The sets $\Lambda_X$ and $\Lambda_Y$ are equal.

*Proof.* Note that $Y_1 = X_1 - \vec{v}$, where $\vec{v}$ is deterministic. Then for any $\vec{\theta}$,

$$\begin{aligned}|\Phi_Y(\vec{\theta})| &= |\mathbb{E}[e^{i\vec{\theta}\cdot(X_1-\vec{v})}]| \\ &= |e^{-i\vec{\theta}\cdot\vec{v}}||\mathbb{E}[e^{i\vec{\theta}\cdot X_1}]| \\ &= |\Phi_X(\vec{\theta})|\end{aligned}$$

which gives the desired result. □

Although $\Lambda_Y$ corresponds to the random walk actually used in the calculation and in the Fourier Inversion formula in (2.5), $\Lambda_X$ corresponds to the walk without the drift correction and is at times more computationally convenient. We note that

$$\vec{\lambda} \in \Lambda_X \Longleftrightarrow e^{i\vec{\lambda}\cdot Z(\vec{x})} = e^{i\vec{\lambda}\cdot Z(\vec{y})} \text{ for all } \vec{x}, \vec{y} \in V_{n,k}$$

which implies that

$$\vec{\lambda} \in \Lambda_X \iff \text{for all } \vec{x}, \vec{y} \in V_{n,k},\ \vec{\lambda} \cdot Z(\vec{x}) \equiv \vec{\lambda} \cdot Z(\vec{y}) \pmod{2\pi}. \tag{2.6}$$

**Proposition 2.5.** If $\vec{\lambda} \in \Lambda_X$ and $\vec{\gamma} \in [-\pi, \pi]^d$, then $\Phi_X(\vec{\lambda} + \vec{\gamma}) = \Phi_X(\vec{\lambda})\Phi_X(\vec{\gamma})$ and $\Phi_Y(\vec{\lambda} + \vec{\gamma}) = \Phi_Y(\vec{\lambda})\Phi_Y(\vec{\gamma})$.

*Proof.* Let $\vec{\lambda} \in \Lambda_X$. By (2.6), we see that $\vec{\lambda} \cdot X_1$ does not depend on the random vector $X_1$, so $e^{i\vec{\lambda}\cdot X_1}$ is a deterministic quantity. Hence,

$$\begin{aligned}\Phi_X(\vec{\lambda} + \vec{\gamma}) &= \mathbb{E}[e^{i(\vec{\lambda}+\vec{\gamma})\cdot X_1}] \\ &= e^{i\vec{\lambda}\cdot X_1}\mathbb{E}[e^{\vec{\gamma}\cdot X_1}]\end{aligned}$$

and since $e^{i\vec{\lambda}\cdot X_1} = \mathbb{E}[e^{i\vec{\lambda}\cdot X_1}]$, the result is shown. The proof of the same statement for $\Phi_Y$ is identical. □

**Remark 2.6.** In particular, we see that $\Lambda_X$ is closed under addition modulo $2\pi$. Moreover, (2.6) shows that $\Lambda_X$ is closed under negation, so it is closed under subtraction as well.

In all the following, we will assume that $k \geq 2$ and $n - k \geq 2$.

**Lemma 2.7.** Let $\epsilon_0 > 0$ and let $\vec{\mu} \in [-\pi, \pi]^d$. Suppose that there exists $\epsilon_0 > 0$ such that for all $\vec{x}, \vec{y} \in V_{n,k}$, there exist $z \in \mathbb{Z}$ and $\epsilon$ with $|\epsilon| < \epsilon_0$ such that $[Z(\vec{x}) \cdot \vec{\mu} - Z(\vec{y}) \cdot \vec{\mu}] = 2\pi z + \epsilon$. Then for any integers $a, b, c, d \in \{1, \ldots, n\}$ there exist $z \in \mathbb{Z}$ and $\epsilon$ with $|\epsilon| < 2\epsilon_0$ such that $[\mu_{\{a,c\}} - \mu_{\{b,c\}}] = [\mu_{\{a,d\}} - \mu_{\{b,d\}}] + 2\pi z + \epsilon$.

The interpretation of this lemma is that if $[Z(\vec{x}) \cdot \vec{\mu} - Z(\vec{y}) \cdot \vec{\mu}] \bmod 2\pi$ is nearly 0 for all $\vec{x}, \vec{y} \in V_{n,k}$, then expressions of the form $[\mu_{\{a,j\}} - \mu_{\{b,j\}}] \bmod 2\pi$ are (nearly) independent of $j$. We also remark that the use of $\{a, c\}$ as an index pair implicitly requires that $a \neq c$; similar constraints exist for the other constants, which we will assume to be satisfied henceforth.

After establishing Lemma 2.7, we obtain a useful corollary by letting $\epsilon_0 \to 0$ and using (2.6):

**Corollary 2.8.** If $\vec{\lambda} \in \Lambda_X$, then for any fixed $a, b$ the expression $\lambda_{\{a,j\}} - \lambda_{\{b,j\}}$ is independent of $j \pmod{2\pi}$.

We remark that the original idea for Corollary 2.8 was communicated by Warwick de Launey in his personal notes ([dL]).

*Proof of Lemma 2.7.* We first define the following vectors in $V_{n,k}$:

$$\vec{x}_1 = (1, 0, 1, 0, \overbrace{1, \ldots, 1}^{k-2}, \overbrace{0, \ldots, 0}^{n-k-2})^T$$

$$\vec{x}_2 = (1, 0, 0, 1, 1, \ldots, 1, 0, \ldots, 0)^T$$

$$\vec{x}_3 = (0, 1, 1, 0, 1, \ldots, 1, 0, \ldots, 0)^T$$

$$\vec{x}_4 = (0, 1, 0, 1, 1, \ldots, 1, 0, \ldots, 0)^T$$

These vectors are identical except in the first four coordinates. For any $\vec{\mu} \in [-\pi, \pi]^d$, we have

$$\vec{\mu} \cdot Z(\vec{x}_1) = \mu_{\{1,3\}} + \sum_{j=5}^{k+2} \mu_{\{1,j\}} + \sum_{j=5}^{k+2} \mu_{\{3,j\}} + \sum_{5 \le i < j \le k+2} \mu_{\{i,j\}}$$

$$\vec{\mu} \cdot Z(\vec{x}_2) = \mu_{\{1,4\}} + \sum_{j=5}^{k+2} \mu_{\{1,j\}} + \sum_{j=5}^{k+2} \mu_{\{4,j\}} + \sum_{5 \le i < j \le k+2} \mu_{\{i,j\}}$$

$$\vec{\mu} \cdot Z(\vec{x}_1) = \mu_{\{2,3\}} + \sum_{j=5}^{k+2} \mu_{\{2,j\}} + \sum_{j=5}^{k+2} \mu_{\{3,j\}} + \sum_{5 \le i < j \le k+2} \mu_{\{i,j\}}$$

$$\vec{\mu} \cdot Z(\vec{x}_1) = \mu_{\{2,4\}} + \sum_{j=5}^{k+2} \mu_{\{2,j\}} + \sum_{j=5}^{k+2} \mu_{\{4,j\}} + \sum_{5 \le i < j \le k+2} \mu_{\{i,j\}}$$

and hence,

$$\begin{aligned} &\vec{\mu} \cdot [Z(\vec{x}_1) - Z(\vec{x}_2)] + \vec{\mu} \cdot [Z(\vec{x}_4) - Z(\vec{x}_3)] \\ &\quad = \vec{\mu} \cdot [Z(\vec{x}_1) - Z(\vec{x}_2) - Z(\vec{x}_3) + Z(\vec{x}_4)] \\ &\quad = \mu_{\{1,3\}} + \mu_{\{2,4\}} - \mu_{\{1,4\}} - \mu_{\{2,3\}}. \end{aligned}$$

Our assumption implies that there exist $z \in \mathbb{Z}$ and $\epsilon_1 \in (-2\epsilon_0, 2\epsilon_0)$ such that

$$[\mu_{\{1,3\}} - \mu_{\{2,3\}}] = [\mu_{\{1,4\}} - \mu_{\{2,4\}}] + \epsilon_1 + 2\pi z$$

by the triangle inequality.

Now, we let $a, b, c, d$ be arbitrary and distinct. We can adjust the previous argument by permuting the coordinates of $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ so that the vectors are identical in all coordinates but $a, b, c, d$, and that those coordinates exhibit a pattern similar to the one found in coordinates $1, 2, 3, 4$ in the original vectors. Repeating the above argument then shows that there exist $z \in \mathbb{Z}$ and $\epsilon_1$ with $|\epsilon_1| < 2\epsilon_0$ such that

$$[\mu_{\{a,c\}} - \mu_{\{b,c\}}] = [\mu_{\{a,d\}} - \mu_{\{b,d\}}] + \epsilon_1 + 2\pi z$$

as desired. □

**Lemma 2.9.** Let $\epsilon_0 > 0$ and $\vec{\mu} \in [-\pi, \pi]^d$. Suppose that there exists $\epsilon_0 > 0$ such that for all $\vec{x}, \vec{y} \in V_{n,k}$, there exist $z \in Z$ and $\epsilon$ with $|\epsilon| < \epsilon_0$ such that $[Z(\vec{x}) \cdot \vec{\mu} - Z(\vec{y}) \cdot \vec{\mu}] = 2\pi z + \epsilon$. Then for all $a, b, c, d$, there exists $z \in \mathbb{Z}$ and $\epsilon$ with $|\epsilon| < 4\epsilon_0$ such that $[\mu_{\{a,b\}} - \mu_{\{c,d\}}] = \frac{2\pi}{k-1} z + \epsilon$.

The interpretation of this lemma is that if $[Z(\vec{x}) \cdot \vec{\mu} - Z(\vec{y}) \cdot \vec{\mu}] \bmod 2\pi$ is nearly 0 for all $\vec{x}, \vec{y} \in V_{n,k}$, then all vector components of $\vec{\mu}$ are nearly constant modulo $\frac{2\pi}{k-1}$. As before, Lemma 2.9 yields a useful corollary obtained by letting $\epsilon_0 \to 0$ and using (2.6):

**Corollary 2.10.** If $\vec{\lambda} \in \Lambda_X$, then all the components of $\vec{\lambda}$ are congruent to one another $\pmod{\frac{2\pi}{k-1}}$.

*Proof of Lemma 2.9.* As before, we define some vectors from $V_{n,k}$:

$$\vec{y}_1 = (1, 0, \overbrace{1, \ldots, 1}^{k-1}, \overbrace{0, \ldots, 0}^{n-k-1})^T$$
$$\vec{y}_2 = (0, 1, 1, \ldots, 1, 0, \ldots, 0)^T$$

These vectors are identical except in the first two coordinates. For any $\vec{\mu}$, we have

$$\vec{\mu} \cdot Z(\vec{y}_1) = \sum_{j=3}^{k+1} \mu_{\{1,j\}}$$
$$\vec{\mu} \cdot Z(\vec{y}_2) = \sum_{j=3}^{k+1} \mu_{\{2,j\}}$$

so by assumption, we then have $z \in \mathbb{Z}$ and $\epsilon_1 \in (-\epsilon_0, \epsilon_0)$ such that

$$\begin{aligned} \vec{\mu} \cdot [Z(\vec{y}_1) - Z(\vec{y}_2)] &= \sum_{j=3}^{k+1} [\mu_{\{1,j\}} - \mu_{\{2,j\}}] \\ &= 2\pi z + \epsilon_1. \end{aligned}$$

Next, we fix some integer $c$ with $3 \leq c \leq n$. For each term in the sum where $j \neq c$, we use Lemma 2.7 to replace $[\mu_{\{1,j\}} - \mu_{\{2,j\}}]$ with $[\mu_{\{1,c\}} - \mu_{\{2,c\}}]$ plus an error term. Executing this replacement for all $j$ shows that there exist $z \in \mathbb{Z}$ and $\epsilon_2$ with $|\epsilon_2| < 2(k-1)\epsilon_0$ such that

$$(k-1)[\mu_{\{1,c\}} - \mu_{\{2,c\}}] = 2\pi z + \epsilon_2.$$

Dividing by $k-1$ then shows that there exists $\epsilon_3$ with $|\epsilon_3| < 2\epsilon_0$ such that

$$[\mu_{\{1,c\}} - \mu_{\{2,c\}}] = \frac{2\pi}{k-1} z + \epsilon_3.$$

We note here that the choices of 1 and 2 in the coordinates of $\mu$ were merely consequences of the construction of the vectors $\vec{y}_1$ and $\vec{y}_2$. For any distinct $a, b, c$, permuting the coordinates of those vectors appropriately shows that there exist $z \in Z$

and $\epsilon_3$ with $|\epsilon_3| < 2\epsilon_0$ such that

$$[\mu_{\{a,c\}} - \mu_{\{b,c\}}] = \frac{2\pi}{k-1} z + \epsilon_3. \tag{2.7}$$

Finally, we let $a, b, c, d$ be distinct. By applying (2.7) twice and using the triangle inequality, we see that there exist $z \in \mathbb{Z}$ and $\epsilon_4$ with $|\epsilon_4| < 4\epsilon_0$ such that

$$\begin{aligned}[\mu_{\{a,b\}} - \mu_{\{c,d\}}] &= [\mu_{\{a,b\}} - \mu_{\{a,d\}}] + [\mu_{\{a,d\}} - \mu_{\{c,d\}}] \\ &= 2\pi z + \epsilon_4\end{aligned}$$

as desired. □

Next, we examine some "building block" vectors that will help to characterize the set $\Lambda_X$. For a fixed $n$ and $k$ and $1 \leq a \leq n$, we define the vector $\vec{\beta}^a$ to be the vector with $\beta^a_{\{i,j\}} = 1$ if $i = a$ or $j = a$ and 0 otherwise. We also define $\vec{\alpha}^a = \vec{1} - \vec{\beta}^a$; that is, $\alpha^a_{\{i,j\}} = 0$ if $i = a$ or $j = a$ and $\alpha^a_{\{i,j\}} = 1$ otherwise.

**Proposition 2.11.** The vectors $\frac{2\pi}{k-1}\vec{\beta}^a$ and $\frac{2\pi}{k-1}\vec{\alpha}^a$ are in $\Lambda_X$. Moreover, so also is $\gamma\vec{1}$ for any real $\gamma$.

*Proof.* In light of (2.6), we wish to show that $\frac{2\pi}{k-1}\vec{\beta}^a \cdot Z(\vec{x})$ and $\frac{2\pi}{k-1}\vec{\alpha}^a \cdot Z(\vec{x})$ do not depend on the choice of $\vec{x} \in V_{n,k}$.

Fix $a$. First, suppose that $x_a = 1$. The vector $Z(\vec{x})$ will have exactly $k-1$ coordinates of the form $\{a, \cdot\}$ whose entry is 1, corresponding to the pairings of the $a^{th}$ coordinate of $\vec{x}$ with the other $k-1$ coordinates whose entry is 1. Since there are a total of $\binom{k}{2}$ 1's in $Z(\vec{x})$, the rest are found in coordinates not of the form $\{a, \cdot\}$. If on the other hand $x_a = 0$, then all the components of $Z(\vec{x})$ of the form $\{a, \cdot\}$ will be 0, and the $\binom{k}{2}$ 1's will all be found elsewhere.

To see that $\frac{2\pi}{k-1}\vec{\beta}^a \in \Lambda_X$, we note that if $x_a = 1$, then $Z(\vec{x}) \cdot \frac{2\pi}{k-1}\vec{\beta}^a = (k-1)\frac{2\pi}{k-1} \equiv 0 \pmod{2\pi}$, and if $x_a = 0$, then $Z(\vec{x}) \cdot \frac{2\pi}{k-1}\vec{\beta}^a = 0$. Next, to see that $\frac{2\pi}{k-1}\vec{\alpha}^a \in \Lambda_X$, we note that if $x_a = 1$, then $Z(\vec{x}) \cdot \frac{2\pi}{k-1}\vec{\alpha}^a = \left(\binom{k}{2} - (k-1)\right)\frac{2\pi}{k-1} \equiv \binom{k}{2}\frac{2\pi}{k-1} \pmod{2\pi}$, and if $x_a = 0$, then $Z(\vec{x}) \cdot \frac{2\pi}{k-1}\vec{\alpha}^a = \binom{k}{2}\frac{2\pi}{k-1}$. Finally, we observe that for any $\vec{x} \in V_{n,k}$,

$$Z(\vec{x}) \cdot \gamma\vec{1} = \binom{k}{2}\gamma \tag{2.8}$$

as desired. □

Using these vectors, we arrive at the desired full characterization of $\Lambda_X$.

**Lemma 2.12.** Suppose that $\vec{\lambda} \in [-\pi, \pi]^d$ and $\vec{\lambda} \in \Lambda_X$. Then there exist $\gamma \in [0, 2\pi)$ and integers $m_i \in [0, k-1)$ such that

$$\vec{\lambda} = \gamma\vec{1} + m_1\frac{2\pi}{k-1}\vec{\alpha}^1 + \sum_{j=3}^{n} m_j\frac{2\pi}{k-1}\vec{\beta}^j.$$

Moreover, this representation of $\vec{\lambda}$ is unique.

**Remark 2.13.** This decomposition of $\Lambda_X(= \Lambda_Y)$ shows that the set is made up of a number of distinct 1-dimensional sets, all of which are parallel to the vector $\vec{1}$.

*Proof of Lemma 2.12.* Let $\lambda \in \Lambda_X$. First, suppose that $\lambda_{\{1,2\}} \equiv \gamma \not\equiv 0 \pmod{2\pi}$. By Remark 2.6 and Proposition 2.11, we can subtract $\gamma\vec{1}$ from $\lambda$ to obtain a new vector $\vec{\theta}$, still in $\Lambda_X$, for which $\theta_{\{1,2\}} \equiv 0 \pmod{2\pi}$. Hence, we will assume that $\lambda_{\{1,2\}} \equiv 0 \pmod{2\pi}$. By Corollary 2.10, this implies that $\lambda_{\{a,b\}} \equiv 0 \pmod{\frac{2\pi}{k-1}}$ for all $\{a, b\}$.

Next, we suppose that $\lambda_{\{1,j\}} \not\equiv 0 \pmod{2\pi}$ for some $j \geq 3$. Since $\frac{2\pi}{k-1}\vec{\beta}^j \in \Lambda_X$ by Proposition 2.11, then by Remark 2.6 we can subtract a requisite number of copies (where the number is an integer between 0 and $k-2$, inclusively) of $\frac{2\pi}{k-1}\vec{\beta}^j$ to obtain a new vector $\vec{\theta}$ for which $\theta_{\{1,j\}} \equiv 0 \pmod{2\pi}$. Moreover, for $j \geq 3$, each vector

$\vec{\beta}^j$ has precisely one nonzero component of the form $\beta^j_{\{1,a\}}$ (namely, $\beta^j_{\{1,j\}}$), which implies that this same reduction can be applied to each $j \geq 3$ simultaneously. Hence, we will assume that $\lambda_{\{1,j\}} \equiv 0 \pmod{2\pi}$ for all $j \geq 2$, since our previous reduction established that $\lambda_{\{1,2\}} \equiv 0 \pmod{2\pi}$.

From here, we observe that if $\lambda_{\{1,j\}} \equiv 0 \pmod{2\pi}$ for all $j$, then $\vec{\lambda}$ is an integer multiple of $\frac{2\pi}{k-1}\vec{\alpha}^1$. To see this, we notice the following: if $j \geq 4$, then by Corollary 2.8 we must have

$$\lambda_{\{2,j\}} - \lambda_{\{2,3\}} \equiv \lambda_{\{1,j\}} - \lambda_{\{1,3\}} \pmod{2\pi},$$

which we assumed to be 0. In particular, this implies that all coordinates of the form $\lambda_{\{2,j\}}$ are constant modulo $2\pi$. Further, if $3 \leq i < j \leq n$, then again by Corollary 2.8 we have

$$\lambda_{\{i,j\}} - \lambda_{\{1,j\}} \equiv \lambda_{\{2,i\}} - \lambda_{\{1,2\}} \pmod{2\pi}.$$

From these relations and the assumption that $\lambda_{\{1,j\}} \equiv 0 \pmod{2\pi}$ for all $j$, we see that $\lambda_{\{i,j\}} \equiv \lambda_{\{2,j\}} \equiv \lambda_{\{2,3\}} \pmod{2\pi}$ whenever $1 < i < j \leq n$. Since we also knew that these terms were all equivalent to $0 \pmod{\frac{2\pi}{k-1}}$, this shows that $\vec{\lambda} = m_1 \frac{2\pi}{k-1}\vec{\alpha}^1$ for some integer $m_1 \in [0, k-1)$, as desired.

Finally, to see the uniqueness of this expression, suppose that

$$\begin{aligned} \gamma\vec{1} + m_1\frac{2\pi}{k-1}\vec{\alpha}^1 + \sum_{j=3}^{n} m_j \frac{2\pi}{k-1}\vec{\beta}^j \\ \equiv \delta\vec{1} + p_1\frac{2\pi}{k-1}\vec{\alpha}^1 + \sum_{j=3}^{n} p_j \frac{2\pi}{k-1}\vec{\beta}^j \mod 2\pi \end{aligned} \tag{2.9}$$

for some $\gamma, \delta \in [0, \frac{2\pi}{k-1})$ and integers $m_j, p_j \in [0, k-1)$. Of the vectors $\vec{1}, \vec{\alpha}^1, \vec{\beta}^3, \ldots \vec{\beta}^n$, the only vector with a nonzero $\{1,2\}$ coordinate is $\vec{1}$; hence, we must have $\gamma = \delta$. For

$j \geq 3$, $\vec{\beta}^j$ is the only term in the sum other than $\vec{1}$ with a nonzero entry in the $\{1, j\}$ coordinate; this implies that $m_j = p_j$ for $j \geq 3$. By subtracting off the terms in (2.9) that are already known to be equal, we obtain

$$m_1 \frac{2\pi}{k-1} \vec{\alpha}^1 \equiv p_1 \frac{2\pi}{k-1} \vec{\alpha}^1 \mod 2\pi$$

which shows that $m_1 = p_1$, as desired. □

### 2.2 Anatomy of the Integral

Having worked in the previous section to obtain a full characterization of the set $\Lambda_Y$, our next goal is to explain how we will decompose the integral in (2.5). The ultimate goal of this section will be to work toward the decompositions found in (2.21) and (2.22). These expression will require a good deal of technical setup. The outline of this section is as follows: first, Lemma 2.14 and Proposition 2.15 will explore the nature of the multi-set $\{\Phi_Y(\vec{\lambda})^t : \vec{\lambda} \in \Lambda_Y\}$. Next, we will discuss how we separate the region $[-\pi, \pi]^d$ into smaller pieces, culminating with (2.20). Finally, we will combine these two ideas to obtain (2.21) and (2.22).

We begin with the multi-set $\{\Phi_Y(\vec{\lambda})^t : \vec{\lambda} \in \Lambda_Y\}$ and will first consider the case where $t = 1$.

**Lemma 2.14.** Let $\vec{\lambda} = \gamma\vec{1} + m_1 \frac{2\pi}{k-1} \vec{\alpha}^1 + \sum_{j=3}^n m_j \frac{2\pi}{k-1} \vec{\beta}^j$ be in $\Lambda_Y$, and define $S(\vec{\lambda}) = m_1 - \sum_{j=3}^n m_j$. Then $\Phi_Y(\vec{\lambda}) = e^{i \frac{2\pi k}{n} S(\vec{\lambda})}$.

*Proof.* We make the three relevant computations in $\mathbb{Z}^d$, where $d = \binom{n}{2}$. We are only concerned with the value of these calculations modulo $2\pi$. The first computation is straightforward from the definition of $X_1$.

$$\begin{aligned}\vec{1}\cdot Y_1 &= \vec{1}\cdot\left(X_1 - \frac{k(k-1)}{n(n-1)}\vec{1}\right)\\ &= \binom{k}{2} - \frac{\binom{k}{2}}{\binom{n}{2}}\binom{n}{2}\\ &= 0\end{aligned}$$

Next,

$$\begin{aligned}\frac{2\pi}{k-1}\vec{\alpha}^1\cdot Y_1 &= \frac{2\pi}{k-1}\vec{\alpha}^1\cdot\left(X_1 - \frac{k(k-1)}{n(n-1)}\vec{1}\right)\\ &\equiv \frac{2\pi}{k-1}\binom{k}{2} - \frac{2\pi k}{n(n-1)}\left[\binom{n}{2} - (n-1)\right]\\ &= \pi k - \pi k + \frac{2\pi k}{n}\end{aligned}$$

The calculation that $\vec{\alpha}^1 \cdot X_1 \equiv \binom{k}{2}$ comes from (2.8). To compute $\vec{\alpha}^1 \cdot \vec{1}$, we notice that all $\binom{n}{2}$ of the coordinates of $\vec{\alpha}^1$ are 1 except for the $n-1$ coordinates whose indices possess a 1. Finally,

$$\begin{aligned}\frac{2\pi}{k-1}\vec{\beta}^j\cdot Y_1 &= \frac{2\pi}{k-1}\vec{\beta}^j\cdot\left(X_1 - \frac{k(k-1)}{n(n-1)}\vec{1}\right)\\ &\equiv 0 - \frac{2\pi k}{n(n-1)}(n-1)\end{aligned}$$

Here again, the computation that $\frac{2\pi}{k-1}\vec{\beta}^j \cdot X_1 \equiv 0$ was carried out in the proof of Proposition 2.11, while the observation that $\vec{\beta}^j \cdot \vec{1} = (n-1)$ comes from the fact that all coordinates of $\vec{\beta}^j$ are 0 except for the $n-1$ coordinates whose indices possess a $j$. The desired conclusion is now immediate. □

Our next goal is to investigate the nature of the multi-set

$$\{\Phi_Y(\vec{\lambda}) : \vec{\lambda} \in \Lambda_Y\}.$$

Since the set $\Lambda_Y$ is always necessarily infinite, we define a set

$$\Lambda_Y^{\square} = \left\{ m_1 \frac{2\pi}{k-1} \vec{\alpha}^1 + \sum_{j=3}^{n} m_j \frac{2\pi}{k-1} \vec{\beta}^j : m_i \in \mathbb{Z} \cap [0, k-1) \right\} \tag{2.10}$$

by eliminating the $\gamma\vec{1}$ component of $\Lambda_Y$. We also define the set

$$\Lambda_Y^{\star} = \left\{ \vec{\lambda} \in [-\pi, \pi)^d : \vec{\lambda} \equiv \vec{\lambda}^{\square} \pmod{2\pi} \text{ for some } \vec{\lambda}^{\square} \in \Lambda_Y^{\square} \right\}. \tag{2.11}$$

We note that each vector in $\Lambda_Y^{\square}$ has a unique representative in $[-\pi, \pi)^d$. Lemma 2.14 shows that for any $\vec{\lambda} \in \Lambda_Y$ and any $\gamma$, we have $\Phi_Y(\vec{\lambda} + \gamma\vec{1}) = \Phi_Y(\vec{\lambda})$. Therefore, in order to understand the nature of the multi-set $\{\Phi_Y(\vec{\lambda}) : \vec{\lambda} \in \Lambda_Y\}$, it suffices to consider the multi-set $\{\Phi_Y(\vec{\lambda}) : \vec{\lambda} \in \Lambda_Y^{\star}\}$. This is particularly useful since $\Lambda_Y$ consists of several subsets parallel to $\vec{1}$, whence the set $\Lambda_Y^{\star}$ consists of one representative vector for each distinct diagonal component. It is easy to see that $|\Lambda_Y^{\star}| = (k-1)^{n-1}$.

We remark here that since

$$Y_t = X_t - \frac{k(k-1)}{n(n-1)} t\vec{1}$$

and $X_t \in \mathbb{Z}^d$, the random walk $Y_t$ is supported on the lattice $\mathbb{Z}^d$ if and only if $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$. Hence, the Fourier Inversion Formula in (2.5) only applies when $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$, and when this is is not the case the return probability is trivially 0. This constraint that $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$ corresponds to the BIBD constraint in (2.3). We also note by the

BIBD constraint in (2.1) that we must have $t\frac{k}{n} \in \mathbb{Z}$ as well, though this requirement manifests in a more subtle way than the necessity that $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$. For certain choices of $k$ and $n$, such as $k = 3$ and $n = 5$, it holds that $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$ implies that $t\frac{k}{n} \in \mathbb{Z}$. For other choices, such as $k = 3$ and $n = 6$, this is not the case. Our next lemma will eventually be used to show how a positive return probability of the $Y_t$ intrinsically requires that $t\frac{k}{n} \in \mathbb{Z}$.

**Proposition 2.15.** Suppose $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$.

- If $t\frac{k}{n} \in \mathbb{Z}$, then the multi-set $\{\Phi_Y(\vec{\lambda})^t : \vec{\lambda} \in \Lambda_Y^\star\}$ consists only of the number 1, repeated $(k-1)^{n-1}$ times.
- If $t\frac{k}{n} \notin \mathbb{Z}$, then the multi-set $\{\Phi_Y(\vec{\lambda})^t : \vec{\lambda} \in \Lambda_Y^\star\}$ consists of all the powers of a certain root of unity, each appearing the same number of times; consequently, the sum of these roots is zero.

*Proof.* Suppose that $t\frac{k}{n} \in \mathbb{Z}$. By Lemma 2.14, we have

$$\Phi_Y(\vec{\lambda})^t = e^{i2\pi t\frac{k}{n}S(\vec{\lambda})}$$

and since $t\frac{k}{n}S(\vec{\lambda}) \in \mathbb{Z}$, it follows that $\Phi_Y(\vec{\lambda})^t = 1$ for all $\vec{\lambda} \in \Lambda_Y$.

Next, suppose that $t\frac{k}{n} \notin \mathbb{Z}$, but that $t\frac{k(k-1)}{n(n-1)} = j$ with $j \in \mathbb{Z}$. In this case, we have $t\frac{k}{n} = \frac{j(n-1)}{k-1}$. We can express this in a reduced form; i.e. $t\frac{k}{n} = \frac{a}{b}$ with $b|(k-1)$, $b \neq 1$, and $a$ relatively prime to $b$. By examining (2.10), we see that the multi-set $\{S(\vec{\lambda}) \bmod (k-1) : \lambda \in \Lambda_Y^\star\}$ consists of the numbers in $\{0, \ldots, k-2\}$, counted $(k-1)^{n-2}$ times each. Since

$$\Phi_Y(\vec{\lambda}) = \exp\left(2\pi i\frac{a}{b}S(\vec{\lambda})\right)$$

and $b|(k-1)$, it follows that the multi-set $\{\Phi_Y(\vec{\lambda}) : \vec{\lambda} \in \Lambda_Y^\star\}$ consists of all the $b^{th}$ roots of unity, each having the same number of appearances. $\square$

We now seek to break up the integral $(2\pi)^{-d} \int_{[-\pi,\pi]^d} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta}$ into manageable pieces. We define the set

$$\Lambda_0 = \left\{ \vec{\lambda} \in \mathbb{R}^d : \lambda_{\{a,b\}} \equiv \lambda_{\{c,d\}} \pmod{2\pi/(k-1)} \text{ for all } a,b,c,d \in \{1,\dots,n\} \right\}$$

and note by Corollary 2.10 that $\Lambda_X \subset \Lambda_0$. For $\delta > 0$, we divide the set of equivalence classes modulo $2\pi\mathbb{Z}^d$, which we will regard as $[-\pi,\pi]^d$, into three regions:

$$R_A^\delta = \{\vec{\lambda} + \vec{\zeta} : \vec{\lambda} \in \Lambda_X \text{ and } |\zeta_{\{i,j\}}| < \delta \text{ for all } i,j\}$$
$$R_B^\delta = \{\vec{\lambda} + \vec{\zeta} : \vec{\lambda} \in \Lambda_0 \setminus \Lambda_X \text{ and } |\zeta_{\{i,j\}}| < \delta \text{ for all } i,j\}$$
$$R_C^\delta = \mathbb{R}^d \setminus (R_A^\delta \cup R_B^\delta)$$

We first prove some needed results about the disjointness of these regions.

**Lemma 2.16.** If $\delta < \frac{\pi}{2(k-1)}$, the regions $R_A^\delta$ and $R_B^\delta$ are disjoint.

*Proof.* Suppose that $R_A^\delta$ and $R_B^\delta$ are not disjoint. Then there are vectors $\vec{\lambda}^1, \vec{\lambda}^2, \vec{\zeta}^1, \vec{\zeta}^2$ such that $\vec{\lambda}^1 + \vec{\zeta}^1 \equiv \vec{\lambda}^2 + \vec{\zeta}^2$ with $\vec{\lambda}^1 \in \Lambda_X, \vec{\lambda}^2 \in \Lambda_0 \setminus \Lambda_X$, and $|\zeta^i_{\{a,b\}}| < \delta$ for $i = 1,2$ and all choices of $a,b$. We can equivalently replace the vectors $\vec{\zeta}^1, \vec{\zeta}^2$ by a single vector $\vec{\zeta} = \vec{\zeta}^2 - \vec{\zeta}^1$, so that

$$\vec{\lambda}^1 \equiv \vec{\lambda}^2 + \vec{\zeta} \pmod{2\pi} \tag{2.12}$$

with $|\zeta_{\{a,b\}}| < 2\delta$ for all coordinates $\{a,b\}$. By subtracting $\lambda^1_{\{1,2\}}\vec{1}$ from both sides of (2.12), we have

$$\vec{\lambda}^1 - \lambda^1_{\{1,2\}}\vec{1} \equiv (\vec{\lambda}^2 - \lambda^1_{\{1,2\}}\vec{1}) + \vec{\zeta} \pmod{2\pi}. \tag{2.13}$$

Next, we set $\vec{\theta}^1 = \vec{\lambda}^1 - \lambda^1_{\{1,2\}}\vec{1}$. By Proposition 2.11, we know that $\lambda^1_{\{1,2\}}\vec{1} \in \Lambda_X$; hence, it follows by Remark 2.6 that $\vec{\theta}^1 \in \Lambda_X$. Moreover, the coordinates $\theta^1_{\{a,b\}}$ of $\vec{\theta}^1$ are all integer multiples of $\frac{2\pi}{k-1}$. Similarly, we observe from (2.12) that taken modulo $\frac{2\pi}{k-1}$, the coordinates of $(\vec{\lambda}^2 - \lambda^1_{\{1,2\}}\vec{1})$ are all equivalent to some constant $c$ with $|c| < 2\delta$. Accordingly, we set $\vec{\theta}^2 = \vec{\lambda}^2 - \lambda^1_{\{1,2\}}\vec{1} - c\vec{1}$; Proposition 2.11 implies that $\vec{\theta}^2 \in \Lambda_0 \setminus \Lambda_X$, and therefore, the components $\theta^2_{\{a,b\}}$ of $\vec{\theta}^2$ are all integer multiples of $\frac{2\pi}{k-1}$. Equation (2.13) then becomes

$$\vec{\theta}^1 \equiv \vec{\theta}^2 + (\vec{\zeta} + c\vec{1}) \pmod{2\pi}$$

where each component of $\vec{\zeta} + c\vec{1}$ is an integer multiple of $\frac{2\pi}{k-1}$. Since $\vec{\theta}^1 \in \Lambda_X$ and $\vec{\theta}^2 \notin \Lambda_X$, there is a coordinate $\{a,b\}$ for which $\theta^1_{\{a,b\}} \not\equiv \theta^2_{\{a,b\}} \pmod{2\pi}$; since they are supported on $\frac{2\pi}{k-1}\mathbb{Z}$, their residues modulo $2\pi$ must differ by at least $\frac{2\pi}{k-1}$. Because $|\theta^1_{\{a,b\}} - \theta^2_{\{a,b\}}| = |\zeta_{\{a,b\}} + c|$, it follows that $|\zeta_{\{a,b\}} + c| \geq \frac{2\pi}{k-1}$. We also have $|\zeta_{\{a,b\}} + c| \leq |\zeta_{\{a,b\}}| + |c| < 4\delta$, so it must be the case that $4\delta \geq \frac{2\pi}{k-1}$. □

**Lemma 2.17.** Suppose $\delta < \frac{\pi}{2(k-1)}$ and that $\vec{\mu}^1 \equiv \vec{\mu}^2 \pmod{2\pi}$ with $\vec{\mu}^1, \vec{\mu}^2 \in R^\delta_A$. Let $\vec{\mu}^1 = \vec{\lambda}^1 + \vec{\zeta}^1$ and $\vec{\mu}^2 = \vec{\lambda}^2 + \vec{\zeta}^2$, and using the notation of Lemma 2.12 let $\vec{\lambda}^1$ be defined by coefficients $\gamma^1, m^1_i$ and let $\vec{\lambda}^2$ be defined by coefficients $\gamma^2, m^2_i$. Then for all $i$, it must follow that $m^1_i = m^2_i$.

**Remark 2.18.** The purpose of this lemma is to show that while expressions of vectors in $R^\delta_A$ are certainly not unique, they are unique up to the diagonal components of $\Lambda_X$, which are determined by the coefficients $m_i$. We will eventually want to decompose $R^\delta_A$ into a collection of tubes, and it will be important that these tubes are disjoint, which is what is proved by this lemma.

*Proof of Lemma 2.17.* Suppose that

$$\vec{\zeta}^1 + \gamma^1 \vec{1} + m_1^1 \frac{2\pi}{k-1} \vec{\alpha}^1 + \sum_{j=3}^{n} m_j^1 \frac{2\pi}{k-1} \vec{\beta}^j$$
$$\equiv \vec{\zeta}^2 + \gamma^2 \vec{1} + m_1^2 \frac{2\pi}{k-1} \vec{\alpha}^1 + \sum_{j=3}^{n} m_j^2 \frac{2\pi}{k-1} \vec{\beta}^j \pmod{2\pi}. \tag{2.14}$$

We first examine the $\{1,2\}$ coordinate of this relationship. All of the vectors $\vec{\alpha}^1$ and $\vec{\beta}^3, \ldots, \vec{\beta}^n$ have 0 in the $\{1,2\}$ position, whence (2.14) yields

$$\zeta^1_{\{1,2\}} + \gamma^1 = \zeta^2_{\{1,2\}} + \gamma^2 + 2\pi z$$

for some $z \in \mathbb{Z}$. Rearranging this yields

$$\gamma^1 - \gamma^2 = 2\pi z + \zeta^2_{\{1,2\}} - \zeta^1_{\{1,2\}}. \tag{2.15}$$

Next, we examine the $\{1,j\}$ coordinate for $j \geq 3$. Of $\vec{\alpha}^1, \vec{\beta}^3, \ldots, \vec{\beta}^n$, the only vector with a nonzero $\{1,j\}$ coordinate is $\vec{\beta}^j$; hence, (2.14) becomes

$$\zeta^1_{\{1,j\}} + \gamma^1 + m_j^1 \frac{2\pi}{k-1} \equiv \zeta^2_{\{1,j\}} + \gamma^2 + m_j^2 \frac{2\pi}{k-1} \pmod{2\pi}.$$

Rearranging this and using (2.15) shows that for some $z' \in \mathbb{Z}$,

$$2\pi z' + (m_j^1 - m_j^2) \frac{2\pi}{k-1} = (\zeta^1_{\{1,2\}} - \zeta^2_{\{1,2\}}) + (\zeta^2_{\{1,j\}} - \zeta^1_{\{1,j\}}).$$

The left-hand side is supported on $\frac{2\pi}{k-1}\mathbb{Z}$, while the right is at most $4\delta < \frac{2\pi}{k-1}$ by the triangle inequality. Hence, the right-hand side is 0. This implies that

$$z'(k-1) = m_j^2 - m_j^1$$

and since $|m_j^2 - m_j^1| < k-1$ we must have $m_j^2 = m_j^1$.

Finally, by subtracting off the terms $m_j^1 \frac{2\pi}{k-1}\vec{\beta}^1$ and $m_j^2 \frac{2\pi}{k-1}\vec{\beta}^2$ with $j \geq 3$ from (2.14), we see that

$$\vec{\zeta}^1 + \gamma^1\vec{1} + m_1^1 \frac{2\pi}{k-1}\vec{\alpha}^1 + \equiv \vec{\zeta}^2 + \gamma^2\vec{1} + m_1^2 \frac{2\pi}{k-1}\vec{\alpha}^1 \pmod{2\pi}$$

and examining the $\{2,3\}$ coordinate shows that

$$\zeta_{\{2,3\}}^1 + \gamma^1 + m_1^1 \frac{2\pi}{k-1} \equiv \zeta_{\{2,3\}}^2 + \gamma^2 + m_1^2 \frac{2\pi}{k-1} \pmod{2\pi}.$$

An argument identical to the one made for the $\{1,j\}$ coordinate above shows that $m_1^1 = m_1^2$. □

We now discuss the full anatomy of the integral used in the Fourier inversion formula. For convenience of notation, we define

$$I_{n,k}(t) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta}.$$

Here, the parameter $n$ is implicitly involved in determining $d = \binom{n}{2}$. When $\delta < \frac{\pi}{2(k-1)}$, by Lemma 2.16 we have

$$(2\pi)^d I_{n,k}(t) = \int_{R_A^\delta} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} + \int_{R_B^\delta \cup R_C^\delta} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} \tag{2.16}$$

which is motivated by segregating the region where $|\Phi_Y(\vec{\theta})^t|$ is close to 1 (that is, $R_A^\delta$) from those where it is not.

To further analyze the integral over $R_A^\delta$, we recall from Remark 2.13 that $\Lambda_Y(= \Lambda_X)$ consists of a disjoint union of dimension 1 subsets of $[-\pi,\pi]^d$, all parallel to the vector $\vec{1}$. Accordingly, the region $R_A^\delta$ consists of a disjoint union of 'tubes' surrounding lines parallel to the vector $\vec{1}$. We formalize this notion by defining the following sets, where $\vec{\lambda}$ is a fixed vector in $\Lambda_X$:

$$T_{\vec{\lambda}}^\delta = \{\vec{\lambda} + \gamma\vec{1} + \vec{\zeta} : \gamma \in [0, 2\pi) \text{ and } |\zeta_{\{i,j\}}| < \delta \text{ for all } i, j\}. \tag{2.17}$$

This definition sets $T_{\vec{\lambda}}^\delta$ as the 'tube' in $[-\pi,\pi)^d$ that contains the vector $\vec{\lambda}$. We remark that in the case that $\vec{\lambda} + \gamma\vec{1} + \vec{\zeta} \notin [-\pi,\pi)^d$, we can add or subtract multiples of $2\pi$ in each coordinate to find its representative in $[-\pi,\pi)^d$.

From here, we can re-express $R_A^\delta$ as a union of the pieces $T_{\vec{\lambda}}^\delta$: namely,

$$R_A^\delta = \bigcup_{\vec{\lambda}\in\Lambda_Y^\star} T_{\vec{\lambda}}^\delta \tag{2.18}$$

where $\Lambda_Y^\star$ is defined as in (2.11). We recall that $|\Lambda_Y^\star| = (k-1)^{n-1}$, and Lemma 2.17 shows that this is a disjoint union when $\delta < \frac{2\pi}{k-1}$.

We now use (2.18) to reconsider the integral in (2.16), which yields

$$(2\pi)^d I_{n,k}(t) = \sum_{\vec{\lambda}\in\Lambda_Y^\star} \int_{T_{\vec{\lambda}}^\delta} \Phi_Y(\vec{\theta})^t \,\mathrm{d}\vec{\theta} + \int_{R_B^\delta \cup R_C^\delta} \Phi_Y(\vec{\theta})^t \,\mathrm{d}\vec{\theta}. \tag{2.19}$$

We note that $\vec{0} \in \Lambda_Y^\star$ and so we consider the nonzero vectors $\vec{\lambda} \in \Lambda_Y^\star$. Proposition 2.5 implies that if $\vec{\theta} = \vec{\lambda} + \gamma\vec{1} + \vec{\zeta}$, then

$$\Phi_Y(\vec{\theta}) = \Phi_Y(\vec{\lambda})\Phi_Y(\vec{\zeta})$$

since $\Phi_Y(\gamma\vec{1}) = 1$ as implied by the proof of Lemma 2.14. Hence, it follows that

$$\int_{T^\delta_{\vec{\lambda}}} \Phi_Y(\vec{\theta})\,\mathrm{d}\vec{\theta} = \Phi_Y(\vec{\lambda}) \int_{T^\delta_{\vec{0}}} \Phi_Y(\vec{\theta})\,\mathrm{d}\vec{\theta}$$

whence (2.19) becomes

$$(2\pi)^d I_{n,k}(t) = \left(\sum_{\vec{\lambda}\in\Lambda_Y^\star} \Phi_Y(\vec{\lambda})^t\right) \int_{T^\delta_{\vec{0}}} \Phi_Y(\vec{\theta})^t\,\mathrm{d}\vec{\theta} + \int_{R^\delta_B \cup R^\delta_C} \Phi_Y(\vec{\theta})^t\,\mathrm{d}\vec{\theta}. \tag{2.20}$$

Finally, we note by Proposition 2.15 that if $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$ but $t\frac{k}{n} \notin \mathbb{Z}$, then the sum in the parentheses of (2.20) is 0 and we have

$$(2\pi)^d I_{n,k}(t) = \int_{R^\delta_B \cup R^\delta_C} \Phi_Y(\vec{\theta})^t\,\mathrm{d}\vec{\theta}. \tag{2.21}$$

On the other hand, if $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$ and $t\frac{k}{n} \in \mathbb{Z}$, then by Proposition 2.15, (2.20) becomes

$$(2\pi)^d I_{n,k}(t) = (k-1)^{n-1} \int_{T^\delta_{\vec{0}}} \Phi_Y(\vec{\theta})^t\,\mathrm{d}\vec{\theta} + \int_{R^\delta_B \cup R^\delta_C} \Phi_Y(\vec{\theta})^t\,\mathrm{d}\vec{\theta}. \tag{2.22}$$

Later, we will observe that as $t$ and $\delta$ vary in a certain way together, the integral over $R^\delta_B \cup R^\delta_C$ approaches zero in both (2.21) and (2.22). This corresponds to the fact that a balanced incomplete block design cannot exist unless $t\frac{k}{n} \in \mathbb{Z}$, which is shown by (2.1).

### 2.3 Bounds Far from the Maximal Set

Having established our decomposition of the integral, we now desire to estimate the integral terms that appear in (2.21) and (2.22). The region $R_A$ is the set that is "near" $\Lambda_X$ and will contribute the bulk of the integral, so our goal is to provide upper bounds for the integrand on the regions $R_B^\delta$ and $R_C^\delta$ to show that their contribution is negligible when compared to that of $R_A^\delta$. We begin with the integrand on the region $R_B^\delta$.

**Lemma 2.19.** Suppose $\delta < k^{-2}\binom{n}{k}^{-2}\left[\frac{1}{6\cdot 96^2}\left(\frac{2\pi}{k-1}\right)^4\right]$. Then if $\vec{\mu} \in R_B^\delta$, we have

$$|\Phi_X(\vec{\mu})| \leq 1 - \binom{n}{k}^{-1}\left[\frac{1}{96}\left(\frac{2\pi}{k-1}\right)^2\right].$$

**Remark 2.20.** The essential point is that the bound holds when $\delta$ is sufficiently small in a manner that depends only on the preset and fixed parameters $n$ and $k$. In the sequel, we will allow $\delta \to 0$ and the exact threshold for when the bound takes place will not be of importance.

**Remark 2.21.** Our previous assumptions on $n$ and $k$ are that $k \geq 2$ and $n - k \geq 2$. We notice that in the particular case where $k = 2$, the set $R_B^\delta$ is empty. This is because the defining characteristic of $\Lambda_0$ simply reduces to all coordinates being congruent to one another modulo $2\pi$; hence, taken modulo $2\pi$ the vector is a multiple of $\vec{1}$. By Proposition 2.11, vectors which satisfy this condition are necessarily in $\Lambda_X$, implying that $\Lambda_X = \Lambda_0$ in this case. Since $R_B^\delta$ is empty, the bound in Lemma 2.19 vacuously holds in this case, so we will assume that $k \geq 3$ in the proof.

*Proof of Lemma 2.19.* Let $\vec{x}, \vec{y} \in V_{n,k}$; if $\vec{\lambda} \in \Lambda_0$, then $|Z(\vec{x}) \cdot \vec{\lambda} - Z(\vec{y}) \cdot \vec{\lambda}| \in \frac{2\pi}{k-1}\mathbb{Z}$. Hence, taken modulo $2\pi$, the possible values of $|Z(\vec{x}) \cdot \vec{\lambda} - Z(\vec{y}) \cdot \vec{\lambda}|$ are $\{0, \frac{2\pi}{k-1}, \dots, \frac{(k-2)2\pi}{k-1}\}$. If $\vec{\lambda} \notin \Lambda_X$, then by (2.6) there exist $\vec{x}, \vec{y}$ so that modulo $2\pi$, we have $|Z(\vec{x}) \cdot \vec{\lambda} - Z(\vec{y}) \cdot \vec{\lambda}| \neq 0$. Hence, for $\vec{\lambda} \in \Lambda_0 \setminus \Lambda_X$,

$$\begin{aligned} |\Phi_X(\vec{\lambda})| &= \left| \frac{1}{\binom{n}{k}} \sum_{\vec{x} \in V_{n,k}} e^{i\vec{\lambda} \cdot Z(\vec{x})} \right| \\ &\leq \frac{1}{\binom{n}{k}} \left[ \left| e^{i\vec{\lambda} \cdot Z(\vec{x})} + e^{i\vec{\lambda} \cdot Z(\vec{y})} \right| + \left| \sum_{\vec{w} \neq \vec{x}, \vec{y}} e^{i\vec{\lambda} \cdot Z(\vec{w})} \right| \right] \end{aligned}$$

and since $|e^{ia} + e^{ib}|^2 = 2 + 2\cos(a-b)$, we have

$$|\Phi_X(\vec{\lambda})| \leq \frac{1}{\binom{n}{k}} \left[ \sqrt{2 + 2\cos\left(\frac{2\pi}{k-1}\right)} + \binom{n}{k} - 2 \right]. \tag{2.23}$$

We note that

$$\sqrt{x} \leq 1 + x/4$$

and that

$$\cos\left(\frac{2\pi}{k-1}\right) \leq 1 - \frac{\left(\frac{2\pi}{k-1}\right)^2}{2} + \frac{\left(\frac{2\pi}{k-1}\right)^4}{24}$$

so substituting these into (2.23) yields

$$|\Phi_X(\vec{\lambda})| \leq 1 - \frac{1}{\binom{n}{k}} \left[ \frac{\left(\frac{2\pi}{k-1}\right)^2}{4} - \frac{\left(\frac{2\pi}{k-1}\right)^4}{48} \right]. \tag{2.24}$$

We also note that when $k \geq 3$,

$$\left(\frac{2\pi}{k-1}\right)^4 < 11 \left(\frac{2\pi}{k-1}\right)^2$$

and applying this to (2.24) gives

$$|\Phi_X(\vec{\lambda})| \le 1 - \frac{1}{\binom{n}{k}} \left[ \frac{1}{48} \left( \frac{2\pi}{k-1} \right)^2 \right]. \tag{2.25}$$

Now, let $\vec{\mu} = \vec{\lambda} + \vec{\zeta}$, where $|\zeta_{\{i,j\}}| < \delta$ for all $i, j$. Since $\Phi_X(\vec{\mu}) = \binom{n}{k}^{-1} \sum_{\vec{x} \in V_{n,k}} e^{i\vec{\mu} \cdot Z(\vec{x})}$, by the triangle inequality and the fact that $|\cos(a+b) - \cos(a)| \le |b|$, we have

$$\begin{aligned}
&|\operatorname{Re}(\Phi_X(\vec{\lambda}+\vec{\zeta})) - \operatorname{Re}(\Phi_X(\vec{\lambda}))| \\
&\qquad = \binom{n}{k}^{-1} \left| \sum_{\vec{x} \in V_{n,k}} \Big( \cos((\vec{\lambda}+\vec{\zeta}) \cdot Z(\vec{x})) - \cos(\vec{\lambda} \cdot Z(\vec{x})) \Big) \right| \\
&\qquad \le \binom{n}{k}^{-1} \sum_{\vec{x} \in V_{n,k}} \left| \vec{\zeta} \cdot Z(\vec{x}) \right|.
\end{aligned}$$

We note that $|\vec{\zeta} \cdot Z(\vec{x})| \le \binom{k}{2}\delta < k^2\delta$, since the vector $Z(\vec{x})$ is 1 in exactly $\binom{k}{2}$ coordinates and is 0 elsewhere. Since $|V_{n,k}| = \binom{n}{k}$, this shows that

$$|\operatorname{Re}(\Phi_X(\vec{\mu})) - \operatorname{Re}(\Phi_X(\vec{\lambda}))| \le k^2\delta$$

and that in particular,

$$|\operatorname{Re}(\Phi_X(\vec{\mu}))| \le |\operatorname{Re}(\Phi_X(\vec{\lambda}))| + k^2\delta. \tag{2.26}$$

An identical argument with sines instead of cosines shows that

$$|\operatorname{Im}(\Phi_X(\vec{\mu}))| \le |\operatorname{Im}(\Phi_X(\vec{\lambda}))| + k^2\delta. \tag{2.27}$$

By (2.26) and (2.27), we have

$$\begin{aligned}|\Phi_X(\vec{\mu})|^2 &= |\operatorname{Re}(\Phi_X(\vec{\mu}))|^2 + |\operatorname{Im}(\Phi_X(\vec{\mu}))|^2 \\ &\leq |\operatorname{Re}(\Phi_X(\vec{\lambda}))|^2 + |\operatorname{Im}(\Phi_X(\vec{\lambda}))|^2 + 4k^2\delta + 2k^4\delta^2\end{aligned}$$

and since our assumptions on $\delta$ imply that $\delta < k^2$, we employ the estimate

$$\begin{aligned}|\Phi_X(\vec{\mu})| &\leq \sqrt{|\Phi_X(\vec{\lambda})|^2 + 6k^2\delta} \\ &\leq |\Phi_X(\vec{\lambda})| + \sqrt{6k^2\delta}.\end{aligned}$$

Putting this together with (2.25) and our assumptions on $\delta$ gives

$$|\Phi_X(\vec{\mu})| \leq 1 - \binom{n}{k}^{-1}\left[\frac{1}{48}\cdot\left(\frac{2\pi}{k-1}\right)^2\right] + \binom{n}{k}^{-1}\left[\frac{1}{96}\left(\frac{2\pi}{k-1}\right)^2\right]$$

as desired. □

Next, we seek to find a bound for the integrand on the region $R_C^\delta$, which will be achieved with the use of Lemma 2.9

**Lemma 2.22.** Suppose $\delta < 4$. Then if $\vec{\mu} \in R_C^\delta$, we have

$$|\Phi_X(\vec{\mu})| \leq 1 - \binom{n}{k}^{-1}\frac{11}{48}\left(\frac{\delta}{4}\right)^2.$$

*Proof.* For $x \in \mathbb{R}$ and $y, \epsilon_0 > 0$, we say that

$$|x| \bmod y < \epsilon_0$$

if there exist $z \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$ such that $x = yz + \epsilon$ and $|\epsilon| < \epsilon_0$. Its negation is denoted

$$|x| \bmod y \geq \epsilon_0$$

and signifies that for every $z \in \mathbb{Z}$ and $\epsilon \in \mathbb{R}$, if $x - yz = \epsilon$, then $|\epsilon| > \epsilon_0$.

Suppose $\vec{\mu} \in R_C^\delta$; then there must exist a choice of $a, b, c, d$ such that $|\mu_{\{a,b\}} - \mu_{\{c,d\}}| \bmod \frac{2\pi}{k-1} \geq \delta$. To see this, we suppose that for every choice of $a, b, c, d$, we have $|\mu_{\{a,b\}} - \mu_{\{c,d\}}| \bmod \frac{2\pi}{k-1} < \delta$. In this case, we form vectors $\vec{\zeta}$ and $\vec{\lambda}$ by setting $\zeta_{\{a,b\}}$ to be the $\frac{2\pi}{k-1}$-residue of $\mu_{\{a,b\}} - \mu_{\{1,2\}}$ and $\vec{\lambda} = \vec{\mu} - \vec{\zeta}$. It follows that $\vec{\mu} = \vec{\lambda} + \vec{\zeta}$ with $\vec{\lambda} \in \Lambda_0$ and $|\zeta_{\{a,b\}}| < \delta$ for all $\{a, b\}$, which immediately implies that $\vec{\mu}$ is either in $R_A^\delta$ or $R_B^\delta$.

Since there exists a choice of $a, b, c, d$ such that $|\mu_{\{a,b\}} - \mu_{\{c,d\}}| \bmod \frac{2\pi}{k-1} \geq \delta$, we see by Lemma 2.9 that there are vectors $\vec{x}, \vec{y} \in V_{n,k}$ for which $|Z(\vec{x}) \cdot \vec{\mu} - Z(\vec{y}) \cdot \vec{\mu}| \bmod 2\pi \geq \delta/4$. This condition implies that

$$\cos(Z(\vec{x}) \cdot \vec{\mu} - Z(\vec{y}) \cdot \vec{\mu}) \leq \cos(\delta/4). \tag{2.28}$$

When computing $\Phi_X(\vec{\mu})$, we use the same calculations that led to (2.23) and (2.24), but with $\delta/4$ in place of $\frac{2\pi}{k-1}$ as indicated by (2.28), to obtain

$$|\Phi_X(\vec{\mu})| \leq 1 - \binom{n}{k}^{-1} \left[ \frac{(\delta/4)^2}{4} - \frac{(\delta/4)^4}{48} \right].$$

Then if $\delta < 4$, we have

$$|\Phi_X(\vec{\mu})| \leq 1 - \binom{n}{k}^{-1} \frac{11}{48} \left( \frac{\delta}{4} \right)^2$$

as desired. □

Having established our bounds on the integrands on regions $R_B^\delta$ and $R_C^\delta$, we are now prepared to bound the corresponding integrals in (2.21) and (2.22). The previous lemmas give rise to the following upper bound on the regions of the integral that are far from $\Lambda_X$.

**Proposition 2.23.** When $\delta < k^{-2}\binom{n}{k}^{-2}\left[\frac{1}{6\cdot 96^2}\left(\frac{2\pi}{k-1}\right)^4\right]$,

$$\left|(2\pi)^{-d}\int_{R_B^\delta \cup R_C^\delta} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta}\right| < \exp\left(-\binom{n}{k}^{-1}\frac{11}{768}t\delta^2\right).$$

*Proof.* We remark that since $|\Phi_Y(\vec{\mu})| = |\Phi_X(\vec{\mu})|$ as shown in the proof of Proposition 2.4, the bounds in Lemmas 2.19 and 2.22 apply to $|\Phi_Y(\vec{\mu})|$ as well. The assumption on $\delta$ implies that both Lemmas 2.19 and 2.22 apply. Moreover, when this assumption on $\delta$ holds, it is easy to verify that the upper bound given in Lemma 2.22 is larger than the upper bound given in Lemma 2.19. Putting those estimates together yields

$$\begin{aligned}\left|(2\pi)^{-d}\int_{R_B^\delta \cup R_C^\delta} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta}\right| &\le (2\pi)^{-d}\int_{R_B^\delta \cup R_C^\delta} |\Phi_Y(\vec{\theta})|^t \, \mathrm{d}\vec{\theta} \\ &< \left[1 - \binom{n}{k}^{-1}\frac{11}{48}\left(\frac{\delta}{4}\right)^2\right]^t \\ &\le \exp\left(-\binom{n}{k}^{-1}\frac{11}{768}t\delta^2\right).\end{aligned}$$

□

### 2.4 Bounds Near the Maximal Set

We now seek to analyze the integrand in the region $R_A^\delta$. By considering (2.22), we see that our primary concern will be to determine bounds for the integral on the region $T_{\vec{0}}^\delta \subset R_A^\delta$. We first define some combinatorial terms; for $j \in \mathbb{Z}^+$ with $j \le n$, we set

$$C_j = \frac{\prod_{i=0}^{j-1}(k-i)}{\prod_{i=0}^{j-1}(n-i)}$$

and we note that if $j \leq k$, then

$$C_j = \frac{\binom{k}{j}}{\binom{n}{j}}$$

whereas if $j > k$ then $C_j = 0$. (Although the $C_j$ terms depend on both parameters $n$ and $k$, we will opt to omit this from the notation.)

We first observe a pair of computations that will be referenced several times:

**Proposition 2.24.** With $C_2, C_3, C_4$ defined as above, and with $k \geq 2$, $n-k \geq 2$, and $d = \binom{n}{2}$,

$$1 + 2(n-2) + \binom{n-2}{2} = d \tag{2.29}$$

and

$$C_2 + 2(n-2)C_3 + \binom{n-2}{2}C_4 = d \cdot C_2^2\,. \tag{2.30}$$

We also define a $d \times d$ matrix $N$. We regard the indices of $N$ in the same way that we regard the indices of $\mathbb{R}^d$; that is, its indices are sets of the form $\{a,b\}$ with $1 \leq a < b \leq n$. Entries in the matrix $N$ will be denoted by $N_{\{a,b\},\{c,d\}}$. We define these entries in terms of the aforementioned combinatorial coefficients $C_j$, as follows:

$$N_{\{a,b\},\{c,d\}} = \begin{cases} C_2 - C_2^2, & |\{a,b\} \cap \{c,d\}| = 2 \\ C_3 - C_2^2, & |\{a,b\} \cap \{c,d\}| = 1 \\ C_4 - C_2^2, & |\{a,b\} \cap \{c,d\}| = 0 \end{cases} \tag{2.31}$$

This makes $N$ a real, symmetric matrix.

**Proposition 2.25.** With $N$ as defined in (2.31) and with $k \geq 2$ and $n - k \geq 2$, we have $N\vec{1} = \vec{0}$ and $\vec{1}^T N = \vec{0}^T$.

*Proof.* We will show that the sum of the columns of $N$ is $\vec{0}$. For a fixed $\{a, b\}$, we consider coordinates of the form $\{c, d\}$. Exactly one coordinate (namely, $\{a, b\}$) has $|\{a, b\} \cap \{c, d\}| = 2$, exactly $2(n-2)$ coordinates have $|\{a, b\} \cap \{c, d\}| = 1$, and exactly $\binom{n-2}{2} = \frac{(n-2)(n-3)}{2}$ coordinates have $|\{a, b\} \cap \{c, d\}| = 0$. The proposition then amounts to showing that

$$(C_2 - C_2^2) + (C_3 - C_2^2) \cdot 2(n-2) + (C_4 - C_2^2) \cdot \binom{n-2}{2} = 0$$

which follows immediately from (2.29) and (2.30). The equation $\vec{1}^T N = \vec{0}^T$ then follows from the symmetry of $N$. □

To motivate the construction of the matrix $N$, we let $\vec{\xi}$ be an element of $V_{n,k} \in \mathbb{R}^n$ and we recall that $Z(\vec{\xi}) = (\xi_1\xi_2, \xi_1\xi_3, \ldots, \xi_{n-1}\xi_n)$. We also recall that the random walk $Y_t$ has increments of the form $Z(\vec{\xi}) - C_2\vec{1}$ where $\xi$ is chosen randomly and uniformly from the elements in $V_{n,k}$. For $\vec{\mu} \in [-\pi, \pi]^d$, we will be interested in computing and estimating quantities of the form

$$\mathbb{E}\left[\left(\vec{\mu} \cdot (Z(\vec{\xi}) - C_2\vec{1})\right)^p\right] \tag{2.32}$$

for $p = 1, 2, 3, 4$. The purpose of constructing $N$ is the following proposition:

**Proposition 2.26.** Let $\vec{\mu} \in [-\pi, \pi]^d$. Then

$$\mathbb{E}\left[\left(\vec{\mu} \cdot (Z(\vec{\xi}) - C_2\vec{1})\right)^2\right] = \vec{\mu}^T N \vec{\mu}\,.$$

*Proof.* The left term is

$$\mathbb{E}\left[\left(\vec{\mu}\cdot(Z(\vec{\xi})-C_2\vec{1})\right)^2\right]$$
$$=\mathbb{E}\left[\left(\sum_{\{a,b\}}\mu_{\{a,b\}}(\xi_a\xi_b-C_2)\right)^2\right]$$
$$=\sum_{\{a,b\},\{c,d\}}\mu_{\{a,b\}}\mu_{\{c,d\}}\mathbb{E}\left[(\xi_a\xi_b-C_2)(\xi_c\xi_d-C_2)\right]$$

where the last sum is taken over all ordered pairs of coordinate sets. To prove the result, we must show that this quadratic form agrees with the entries of $N$; that is, that $\mathbb{E}[(\xi_a\xi_b-C_2)(\xi_c\xi_d-C_2)]$ is given by the coefficients of $N$ in (2.31).

We first consider the case where $|\{a,b\}\cap\{c,d\}|=2$; that is, $\{c,d\}=\{a,b\}$. Here,

$$\mathbb{E}[(\xi_a\xi_b-C_2)(\xi_a\xi_b-C_2)]=\mathbb{E}[\xi_a\xi_b-2C_2\xi_a\xi_b+C_2^2] \tag{2.33}$$

since all vectors in $V_{n,k}$ have entries that are either 0 or 1. The product $\xi_a\xi_b$ will be 1 if $\xi_a=1$ and $\xi_b=1$; otherwise, it will be 0. Of the $\binom{n}{k}$ vectors in $V_{n,k}$, there are $\binom{n-2}{k-2}$ vectors which have $\xi_a=1$ and $\xi_b=1$, corresponding to the ways to select the locations for the remaining $k-2$ 1's from the remaining $n-2$ possible positions. Hence, the probability that $\xi_a\xi_b$ is 1 is $\binom{n-2}{k-2}/\binom{n}{k}=C_2$, from which it follows that

$$\mathbb{E}[\xi_a\xi_b]=C_2\,. \tag{2.34}$$

Substituting this into (2.33) gives

$$\mathbb{E}[(\xi_a\xi_b-C_2)(\xi_a\xi_b-C_2)]=C_2-C_2^2$$

which agrees with the corresponding coefficient of $N$.

Next, we consider the case where $|\{a,b\} \cap \{c,d\}| = 1$ by considering an index pair of the form $\{a,b\}, \{a,c\}$. In this case,

$$\mathbb{E}[(\xi_a\xi_b - C_2)(\xi_a\xi_c - C_2)] = \mathbb{E}[\xi_a\xi_b\xi_c - C_2\xi_a\xi_b - C_2\xi_a\xi_c + C_2^2]\,. \tag{2.35}$$

By analyzing the first term in a fashion similar to our discussion of (2.34), we see that $\mathbb{E}[\xi_a\xi_b\xi_c] = \binom{n-3}{k-3}/\binom{n}{k} = C_3$. Using this and (2.34) in (2.35) shows that

$$\mathbb{E}[(\xi_a\xi_b - C_2)(\xi_a\xi_c - C_2)] = C_3 - C_2^2$$

which again agrees with the corresponding coefficient of $N$.

Finally, we consider the case where $|\{a,b\} \cap \{c,d\} = 0|$; that is, $a, b, c, d$ are all distinct. Here,

$$\mathbb{E}[(\xi_a\xi_b - C_2)(\xi_c\xi_d - C_2)] = \mathbb{E}[\xi_a\xi_b\xi_c\xi_d - C_2\xi_a\xi_b - C_2\xi_c\xi_d + C_2^2] \tag{2.36}$$

and as before, the expectation of the first term is $\binom{n-4}{k-4}/\binom{n}{k} = C_4$, whence (2.36) becomes

$$\mathbb{E}[(\xi_a\xi_b - C_2)(\xi_c\xi_d - C_2)] = C_4 - C_2^2$$

which also agrees with the corresponding entry of $N$. □

**Corollary 2.27.** The matrix $N$ is positive semidefinite.

*Proof.* This is immediate from Proposition 2.26 since the expectation term is nonnegative. □

**Remark 2.28.** The process $Y_t$ was defined as being the process $X_t$ with a drift correction, which corresponds to the calculation in (2.34). That calculation shows that the term in (2.32) is 0 when $p = 1$. We have now calculated the term when $p = 2$; we will choose to estimate, rather than to compute, the terms with $p = 3$ and $p = 4$.

**Lemma 2.29.** Let $\delta > 0$. Then there is a function $\varepsilon_1 : T_{\vec{0}}^{\delta} \to \mathbb{R}$ such that for all $\vec{\mu} \in T_{\vec{0}}^{\delta}$, we have

$$\mathrm{Re}(\Phi_Y(\vec{\mu})) = e^{-\frac{1}{2}\vec{\mu}^T N \vec{\mu}}(1 + \varepsilon_1(\vec{\mu})) \tag{2.37}$$

and $|\varepsilon_1(\vec{\mu})| < \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}d^2\delta^2}$. Moreover,

$$|\,\mathrm{Im}(\Phi_Y(\vec{\mu}))| \le \frac{(d\delta)^3}{6}\,. \tag{2.38}$$

Further, if $d\delta < 1$, then for $\vec{\mu} \in T_{\vec{0}}^{\delta}$ we have

$$\mathrm{Re}(\Phi_Y(\vec{\mu})) \ge \frac{1}{3}. \tag{2.39}$$

*Proof.* For this proof, we will mimic the proof of Lemma 3.1 in [dLL10]. Since $\vec{\mu} \in T_{\vec{0}}^{\delta}$, we can write

$$\vec{\mu} = \gamma\vec{1} + \vec{\zeta} \tag{2.40}$$

where $|\zeta_{\{i,j\}}| < \delta$ for all $\{i, j\}$. We begin with the remainder bounds on Taylor polynomials for $e^z$. If $a \ge 0$ and $b$ is real, we have

$$\left| e^{-a} - \sum_{s=0}^{j} \frac{(-a)^j}{s!} \right| \le \min\left\{ \frac{2|a|^j}{j!}, \frac{|a|^{j+1}}{(j+1)!} \right\}, \tag{2.41}$$

$$\left| e^{ib} - \sum_{s=0}^{j} \frac{(ib)^s}{s!} \right| \le \min\left\{ \frac{2|b|^j}{j!}, \frac{|b|^{j+1}}{(j+1)!} \right\}. \tag{2.42}$$

For a reference, one can find (2.41) as [Bil95, equation 26.4]; (2.42) is proved similarly. Using (2.41) with $j = 1$ shows that

$$\left| e^{-\frac{1}{2}\vec{\mu}^T N \vec{\mu}} - \left(1 - \frac{1}{2}\vec{\mu}^T N \vec{\mu}\right) \right| \leq \frac{1}{8}(\vec{\mu}^T N \vec{\mu})^2. \tag{2.43}$$

By (2.40) and Proposition 2.25, we note that

$$\begin{aligned} \vec{\mu}^T N \vec{\mu} &= (\gamma \vec{1}^T + \vec{\zeta}^T) N (\gamma \vec{1} + \vec{\zeta}) \\ &= \vec{\zeta}^T N \vec{\zeta}. \end{aligned}$$

We note from the triangle inequality that

$$|\vec{\zeta}^T N \vec{\zeta}| \leq \sum_{\{a,b\},\{c,d\}} |\zeta_{\{a,b\}} \zeta_{\{c,d\}} N_{\{a,b\},\{c,d\}}|$$

and we observe that all coefficients of $N$ have absolute value at most 1 since $0 \leq C_j < 1$ for $j = 2, 3, 4$. Since the components of $\vec{\zeta}$ are bounded by $\delta$, it follows that

$$|\vec{\mu}^T N \vec{\mu}| < \sum_{\{a,b\},\{c,d\}} \delta^2 = d^2 \delta^2 \,. \tag{2.44}$$

Using this in conjunction with (2.43) establishes that

$$\left| e^{-\frac{1}{2}\vec{\mu}^T N \vec{\mu}} - \left(1 - \frac{1}{2}\vec{\mu}^T N \vec{\mu}\right) \right| \leq \frac{1}{8} d^4 \delta^4. \tag{2.45}$$

Next, let $\vec{y}$ be any vector in $V_{n,k}$. For convenience of notation, we set $W(\vec{y}) = Z(\vec{y}) - C_2\vec{1}$. Using (2.42) with $j = 3$ implies that

$$\begin{aligned}&\left|e^{i\vec{\mu}\cdot W(\vec{y})} - \left[1 + i\vec{\mu}\cdot W(\vec{y}) - \frac{1}{2}(\vec{\mu}\cdot W(\vec{y}))^2 - \frac{i}{6}(\vec{\mu}\cdot W(\vec{y}))^3\right]\right| \\ &\qquad \leq \frac{1}{24}(\vec{\mu}\cdot W(\vec{y}))^4\end{aligned}$$

Using this with the fact that $|\operatorname{Re}(z)| < |z|$ for any $z \in \mathbb{C}$, we see that

$$\left|\operatorname{Re}(e^{i\vec{\mu}\cdot W(\vec{y})}) - \left[1 - \frac{1}{2}(\vec{\mu}\cdot W(\vec{y}))^2\right]\right| \leq \frac{1}{24}(\vec{\mu}\cdot W(\vec{y}))^4. \tag{2.46}$$

We now let $\vec{\xi}$ be a random, uniformly-chosen element of $V_{n,k}$. From (2.46), we see that

$$\begin{aligned}&\left|\mathbb{E}\left[\operatorname{Re}(e^{i\vec{\mu}\cdot W(\vec{\xi})})\right] - \mathbb{E}\left[1 - \frac{1}{2}(\vec{\mu}\cdot W(\xi))^2\right]\right| \\ &\qquad \leq \mathbb{E}\left|\operatorname{Re}(e^{i\vec{\mu}\cdot W(\vec{\xi})}) - \left[1 - \frac{1}{2}(\vec{\mu}\cdot W(\vec{\xi}))^2\right]\right| \\ &\qquad \leq \frac{1}{24}\mathbb{E}[(\vec{\mu}\cdot W(\vec{\xi}))^4].\end{aligned} \tag{2.47}$$

Since Re is linear, we have $\mathbb{E}[\operatorname{Re}(e^{i\vec{\mu}\cdot W(\vec{\xi})})] = \operatorname{Re}(\Phi_Y(\vec{\mu}))$. Hence, (2.47) and Proposition 2.26 combine to yield

$$\left|\operatorname{Re}(\Phi_Y(\vec{\mu})) - \left[1 - \frac{1}{2}\vec{\mu}^T N\vec{\mu}\right]\right| \leq \frac{1}{24}\mathbb{E}[(\vec{\mu}\cdot W(\vec{\xi}))^4]. \tag{2.48}$$

To obtain a preliminary bound on $\operatorname{Im}(\Phi_Y(\vec{\mu}))$, we set $j = 2$ in (2.42) to obtain

$$\left|e^{i\vec{\mu}\cdot W(\vec{y})} - \left[1 + i\vec{\mu}\cdot W(\vec{y}) - \frac{1}{2}(\vec{\mu}\cdot W(\vec{y}))^2\right]\right| \leq \frac{1}{6}|\vec{\mu}\cdot W(\vec{y})|^3$$

and since $|\operatorname{Im}(z)| < |z|$, we have

$$\left|\operatorname{Im}(e^{i\vec{\mu}\cdot W(\vec{y})}) - \vec{\mu}\cdot W(\vec{y})\right| \leq \frac{1}{6}|\vec{\mu}\cdot W(\vec{y})|^3.$$

Using the same argument as for the real part, we see that if $\vec{\xi}$ is a random, uniformly-chosen element of $V_{n,k}$,

$$|\operatorname{Im}(\Phi_Y(\vec{\mu})) - \mathbb{E}[\vec{\mu}\cdot W(\vec{y})]| \leq \frac{1}{6}\mathbb{E}[|\vec{\mu}\cdot W(\vec{y})|^3]$$

and by Remark 2.28 we have $\mathbb{E}[\vec{\mu}\cdot W(\vec{y})] = 0$, so it follows that

$$|\operatorname{Im}(\Phi_Y(\vec{\mu}))| \leq \frac{1}{6}\mathbb{E}[|\vec{\mu}\cdot W(\vec{y})|^3]. \tag{2.49}$$

To prove (2.37) and (2.38), we need to bound the expectations in (2.48) and (2.49). For any $\vec{y} \in V_{n,k}$, we have $\vec{1}\cdot Z(\vec{y}) = \binom{k}{2}$, and $\vec{1}\cdot C_2\vec{1} = \frac{k(k-1)}{n(n-1)}\frac{n(n-1)}{2} = \binom{k}{2}$; hence, using $\vec{\mu} = \gamma\vec{1} + \vec{\zeta}$ from (2.40) shows that

$$\vec{\mu}\cdot W(y) = (\gamma\vec{1} + \vec{\zeta})\cdot(Z(\vec{y}) - C_2\vec{1}) = \vec{\zeta}\cdot W(y)\,.$$

For any $\vec{y} \in V_{n,k}$, the components of $W(\vec{y})$ all have absolute value at most 1; this follows from the fact that components of $Z(\vec{y})$ are either 1 or 0 and that $0 < C_2 < 1$. Since the components of $\vec{\zeta}$ have absolute value at most $\delta$, by the triangle inequality we have

$$|\vec{\mu}\cdot W(\vec{y})| \leq \sum_{\{a,b\}} |\zeta_{\{a,b\}}| \leq d\delta. \tag{2.50}$$

Combining (2.50) with (2.49) yields (2.38). Likewise, using (2.50) with (2.48) shows that

$$\left|\mathrm{Re}(\Phi_Y(\vec{\mu})) - \left[1 - \frac{1}{2}\vec{\mu}^T N\vec{\mu}\right]\right| \le \frac{(d\delta)^4}{24}$$

and combining this with (2.45) via the triangle inequality gives

$$\left|\mathrm{Re}(\Phi_Y(\vec{\mu})) - e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}\right| \le \frac{(d\delta)^4}{6}.$$

Dividing both sides by $e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}$ yields

$$\left|\frac{\mathrm{Re}(\Phi_Y(\vec{\mu}))}{e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}} - 1\right| \le \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}\vec{\mu}^T N\vec{\mu}}$$

and by (2.44), we see that

$$\left|\frac{\mathrm{Re}(\Phi_Y(\vec{\mu}))}{e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}} - 1\right| \le \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}d^2\delta^2}.$$

Therefore, we have

$$\mathrm{Re}(\Phi_Y(\vec{\mu})) = e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}\left[\frac{\mathrm{Re}(\Phi_Y(\vec{\mu}))}{e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}}\right] = e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}(1 + \varepsilon_1(\vec{\mu}))$$

where

$$|\varepsilon_1(\vec{\mu})| \le \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}d^2\delta^2}$$

which establishes (2.37).

Finally, to establish (2.39), we note from (2.37) that

$$\mathrm{Re}(\Phi_Y(\vec{\mu})) \ge e^{-\frac{1}{2}\vec{\mu}^T N\vec{\mu}}\left(1 - \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}(d\delta)^2}\right)$$

and by (2.44) and the assumption that $(d\delta) < 1$, we see that

$$\begin{aligned}\mathrm{Re}(\Phi_Y(\vec{\mu})) &\geq \frac{1 - \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}(d\delta)^2}}{e^{\frac{1}{2}(d\delta)^2}} \\ &\geq \frac{1 - \sqrt{e}/6}{\sqrt{e}} \\ &\geq 1/3\end{aligned}$$

as desired. □

## 2.5 The Submatrix Determinant

We now reconsider the $d \times d$ matrix $N$ as defined in (2.31). As implied by Proposition 2.25 this matrix is singular. Our primary concern in the upcoming calculations will not be $N$, but its $(d-1) \times (d-1)$ principal submatrix obtained by removing the row and column with index $\{n-1, n\}$. We will denote this submatrix by $M$. We will need to discuss the corresponding subspace $\mathbb{R}^{d-1} \subset \mathbb{R}^d$, so we specify that if our enumeration of the coordinates of $\mathbb{R}^d$ is

$$\{1,2\}, \{1,3\}, \ldots, \{n-2, n\}, \{n-1, n\}$$

then the coordinates of $\mathbb{R}^{d-1}$ are enumerated as

$$\{1,2\}, \{1,3\}, \ldots, \{n-2, n\}$$

to correspond to our definition of $M$.

**Lemma 2.30.** With $M, N$ as previously defined,

$$2\pi \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M\vec{\mu}}\,\mathrm{d}\vec{\mu} \leq \int_{T^{\delta}_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} \leq 2\pi \int_{[-2\delta,2\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M\vec{\mu}}\,\mathrm{d}\vec{\mu}.$$

*Proof.* We begin by reparametrizing the middle integral. We define a region better suited for the upcoming reparametrization:

$$S^{\delta}_{\vec{0}} = \left\{\vec{\gamma}\vec{1} + \vec{\zeta} : \gamma \in [0, 2\pi), |\zeta_{\{i,j\}}| < \delta \text{ for all } i, j \text{ and } \zeta_{\{n-1,n\}} = 0\right\}$$

where as always, coordinates of $S^{\delta}_{\vec{0}}$ are understood to be taken modulo $2\pi$. We note from (2.17) that $S^{\delta}_{\vec{0}} \subset T^{\delta}_{\vec{0}}$ is clear. We also claim that $T^{\delta}_{\vec{0}} \subset S^{2\delta}_{\vec{0}}$. To see this, we let $\gamma\vec{1} + \vec{\zeta} \in T^{\delta}_{\vec{0}}$; if we set $\vec{\zeta}' = \vec{\zeta} - \zeta_{\{n-1,n\}}\vec{1}$ and $\gamma' = \gamma + \zeta_{\{n-1,n\}}$, then we have $\gamma\vec{1} + \vec{\zeta} = \gamma'\vec{1} + \vec{\zeta}'$, and the latter is in $S^{2\delta}_{\vec{0}}$ by the triangle inequality. From the relation that $S^{\delta}_{\vec{0}} \subset T^{\delta}_{\vec{0}} \subset S^{2\delta}_{\vec{0}}$, it follows that

$$\int_{S^{\delta}_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} \leq \int_{T^{\delta}_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} \leq \int_{S^{2\delta}_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta}. \tag{2.51}$$

To reparametrize the integral, we define a function $g : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\begin{aligned}
g(\vec{\mu})_{\{1,2\}} &= \nu_{\{1,2\}} + \nu_{\{n-1,n\}}\\
g(\vec{\nu})_{\{1,3\}} &= \nu_{\{1,3\}} + \nu_{\{n-1,n\}}\\
&\vdots\\
g(\vec{\nu})_{\{n-2,n\}} &= \nu_{\{n-2,n\}} + \nu_{\{n-1,n\}}\\
g(\vec{\nu})_{\{n-1,n\}} &= \nu_{\{n-1,n\}}.
\end{aligned}$$

It is easy to see that the Jacobian determinant of this transformation is 1, and that

$$g\left([-\delta,\delta]^{d-1}\times[0,2\pi)\right)=S^{\delta}_{\vec{0}}\,.$$

For convenience of notation, we write $\vec{\nu}^0 = (\nu_{\{1,2\}},\ldots,\nu_{\{n-2,n\}},0)^T$ and we set $\vec{\theta} = g(\vec{\nu})$, so that $\vec{\theta} = \vec{\nu}^0 + \nu_{\{n-1,n\}}\vec{1}$. From Proposition 2.25, we see that

$$\begin{aligned}\vec{\theta}^T N\vec{\theta} &= (\vec{\nu}^0 + \nu_{\{n-1,n\}}\vec{1})^T N(\vec{\nu}^0 + \nu_{\{n-1,n\}}\vec{1})\\ &= (\vec{\nu}^0)^T N\vec{\nu}^0.\end{aligned}$$

By applying the change of variables formula to the integral, we obtain

$$\int_{S^{\delta}_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} = \int_0^{2\pi}\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta} e^{-\frac{t}{2}(\vec{\nu}^0)^T N\vec{\nu}^0}\,\mathrm{d}\nu_{\{1,2\}}\ldots\,\mathrm{d}\nu_{\{n-2,n\}}\;\mathrm{d}\nu_{\{n-1,n\}}$$

and since the rightmost integrand no longer involves $\nu_{\{n-1,n\}}$, we can integrate that variable to get

$$\int_{S^{\delta}_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} = 2\pi\int_{-\delta}^{\delta}\cdots\int_{-\delta}^{\delta} e^{-\frac{t}{2}(\vec{\nu}^0)^T N\vec{\nu}^0}\,\mathrm{d}\nu_{\{1,2\}}\ldots\,\mathrm{d}\nu_{\{n-2,n\}}\,. \qquad (2.52)$$

Next, we consider the subspace $\mathbb{R}\times\cdots\times\mathbb{R}\times\{\vec{0}\}$ of $\mathbb{R}^d$ and the function $h : \mathbb{R}\times\cdots\times\mathbb{R}\times\{\vec{0}\}\to\mathbb{R}^{d-1}$ given by $h((\nu_{\{1,2\}},\ldots,\nu_{\{n-2,n\}},0)^T) = (\nu_{\{}1,2\},\ldots,\nu_{\{n-2,n\}})$; we also set $\vec{\mu} = h(\vec{\nu})$. (We introduce this notation only so that we have a convenient way to distinguish between vectors in $\mathbb{R}^d$ and in $\mathbb{R}^{d-1}$). Since the $\{n-1,n\}$ component of $\vec{\nu}^0$ is 0, we have $(\vec{\nu}^0)^T N\vec{\nu}^0 = \vec{\mu}^T M\vec{\mu}$. Applying the change of variables formula to

(2.52) then yields

$$\int_{S_{\vec{0}}^{\delta}} e^{-\frac{t}{2}\vec{\theta}^T N \vec{\theta}}\, \mathrm{d}\vec{\theta} = 2\pi \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M \vec{\mu}}\, \mathrm{d}\vec{\mu}\,.$$

Using this on the left and right of (2.51) completes the proof. □

For a variety of reasons, it will be important for us to know that the matrix $M$ is nonsingular. We can of course deduce this fact by computing the determinant (which we will do eventually anyway) and arguing that it is nonzero. However, the determinant computation is rather lengthy and difficult, and it is possible to argue the nonsingularity of $M$ at an elementary level while bypassing the need to compute $\det(M)$ at all. We will accomplish this task in Lemma 2.32. Though the result of Lemma 2.32 is redundant with that of Lemma 2.40, the method is quite different, and we present it for interest's sake.

We will first require a small bit of machinery. Let $\sigma$ be a transposition of the set $\{1, \ldots, n\}$; that is, $\sigma$ is a permutation of the elements of $\{1, \ldots, n\}$ that swaps two elements. This permutation induces a permutation on the collection of subsets of $\{1, \ldots, n\}$ of size 2 in a natural way, given by $\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$. This permutation on 2-element sets also has cycle length 2 and can therefore be represented with a symmetric $d \times d$ permutation matrix $P_\sigma$.

**Proposition 2.31.** Let $\sigma$ be a transposition of $S = \{1, \ldots, n\}$ and let $P_\sigma$ be the matrix corresponding to the induced permutation of 2-element subsets of $S$. Then $P_\sigma N = N P_\sigma$.

*Proof.* Since the induced permutation has cycle length 2, it follows that $P_\sigma = P_\sigma^T = P_\sigma^{-1}$. Using the notation of (2.31), the $\{a, b\}, \{c, d\}$ entry of $P_\sigma N P_\sigma$ is

$N_{\{\sigma(a),\sigma(b)\},\{\sigma(c),\sigma(d)\}}$. Since

$$\begin{aligned} N_{\{\sigma(a),\sigma(b)\},\{\sigma(c),\sigma(d)\}} &= |\{\sigma(a),\sigma(b)\} \cap \{\sigma(c),\sigma(d)\}| \\ &= |\{a,b\} \cap \{c,d\}| \\ &= N_{\{a,b\},\{c,d\}} \end{aligned}$$

the desired result follows. □

**Lemma 2.32.** The matrix $M$ is nonsingular.

*Proof.* Suppose that there is a nonzero vector $\vec{x} \in \mathbb{R}^{d-1}$ for which $M\vec{x} = \vec{0}$. We will derive a contradiction by constructing a vector $\vec{x}^0 \in \mathbb{R}^d$ formed by appending a zero to $\vec{x}$; the contradiction will come from considering $N\vec{x}^0$. By construction, all coordinates of $N\vec{x}^0$ are 0 except possibly the last coordinate.

If the last coordinate of $N\vec{x}^0$ is 0, i.e. $N\vec{x}^0 = \vec{0}$, then for any real $a \in \mathbb{R}$ we have $(a\vec{x}^0)^T N(a\vec{x}^0) = 0$. By Proposition 2.26, it follows that $\mathbb{E}[(a\vec{x}^0 \cdot Y_1)^2] = 0$, whence $a\vec{x}^0 \cdot Y_1 = 0$ almost surely. By (2.6), this implies that $a\vec{x}^0 \in \Lambda_Y$. Since this holds for any real $a$, Lemma 2.17 implies that $\vec{x}^0 = c\vec{1}$ for some real $c$. By construction, the last coordinate of $\vec{x}^0$ is 0; therefore we must have $\vec{x}^0 = \vec{0}$, which contradicts our assumption that $\vec{x} \neq 0$.

Suppose instead that the last coordinate of $N\vec{x}^0$ is nonzero. Without loss of generality, we can assume that $N\vec{x}^0 = \vec{e}_{\{n-1,n\}}$, which is the vector whose $\{n-1,n\}$ coordinate is 1 and all other coordinates are 0. Let $\{i,j\}$ be arbitrary; we will show that $\vec{e}_{\{i,j\}}$ is in the range of $M$. First, assume that $|\{i,j\} \cap \{n-1,n\}| = 0$; let $\sigma$ be the transposition that swaps $n-1$ and $i$, and let $\tau$ be the transposition that swaps $n$ and $j$. We let $P_\sigma, P_\tau$ denote the respective $d \times d$ matrices of the induced permutations on 2-element subsets of $\{1, \ldots, n\}$. Since

$$P_\tau P_\sigma N \vec{x}^0 = P_\tau P_\sigma \vec{e}_{\{n-1,n\}}$$

and since

$$P_\tau P_\sigma \vec{e}_{\{n-1,n\}} = P_\tau \vec{e}_{\{i,n\}} = \vec{e}_{\{i,j\}}$$

then from Proposition 2.31 it follows that

$$N(P_\tau P_\sigma \vec{x}^0) = \vec{e}_{\{i,j\}}.$$

In the case that $|\{i,j\} \cap \{n-1,n\}| = 1$, we can repeat the same argument as above using only a single permutation matrix instead of two. Thus, every $\vec{e}_{\{i,j\}}$ is in the range of $N$, and $N$ is therefore invertible. This contradicts the singularity of $N$ proven in Proposition 2.25 and completes the proof. □

The nonsingularity of $M$ yields some useful corollaries:

**Corollary 2.33.** The matrix $M$ is positive definite.

*Proof.* By Corollary 2.27, we know that $N$ is positive semidefinite; hence, its eigenvalues are all nonnegative. By Cauchy's interlace theorem (see, for example, [Hwa04]), the eigenvalues of $M$ are also all nonnegative. But by Lemma 2.32 we see that their product, $\det(M)$, is nonzero. This means that each eigenvalue is strictly positive and that $M$ is therefore positive definite. □

Since $M$ is positive definite, there is a unique symmetric, positive definite matrix $P$ such that $P^2 = M$. In an upcoming integral computation, we will need to understand the set

$$P[-\delta,\delta]^{d-1} = \left\{ P\vec{\mu} : \vec{\mu} \in \mathbb{R}^{d-1} \text{ and } |\mu_{\{i,j\}}| < \delta \text{ for all } i,j \right\}.$$

Rather than actually computing this set, it will suffice for us to bound it.

**Proposition 2.34.** There exist positive constants $D_1, D_2$ which depend only on $n$ and $k$ such that for all $\delta > 0$,

$$[-D_1\delta, D_1\delta]^{d-1} \subset P[-\delta,\delta]^{d-1} \subset [-D_2\delta, D_2\delta]^{d-1}.$$

*Proof.* Since $P$ is positive definite, the linear transformation corresponding to $P$ maps the box $[-1,1]^{d-1}$ to some nondegenerate subset of $\mathbb{R}^{d-1}$. Therefore, there are constants $D_1$ and $D_2$ such that

$$[-D_1, D_1]^{d-1} \subset P[-1,1]^{d-1} \subset [-D_2, D_2]^{d-1}.$$

These constants depend on the matrix $P$, which is defined in terms of the matrix $M$, which depends only on the constants $n$ and $k$. We scale these sets by a factor of $\delta$ and exploit the linearity of the transformation associated to matrix $P$ to obtain

$$[-D_1\delta, D_1\delta]^{d-1} \subset P[-\delta,\delta]^{d-1} \subset [-D_2\delta, D_2\delta]^{d-1}$$

as desired. □

**Remark 2.35.** The salient detail of Proposition 2.34 is that $D_1$ and $D_2$ do not depend on $\delta$.

The remainder of this section is dedicated to computing $\det(M)$. The first step toward this goal is finding a convenient expression of $N$ in terms of elementary matrices. We remark here that at several points in the upcoming calculations, we will refer to $1 \times 1$ matrices, to their entries, and to their determinants interchangeably.

Fix $r \in \mathbb{N}$. We will denote the $r \times r$ identity matrix by $I_r$. We will define $\vec{x}_r$ to be the vector in $\mathbb{R}^r$ with all entries 1; i.e.

$$\vec{x}_r = (1, \ldots, 1)^T. \tag{2.53}$$

We will also define $y_r \in \mathbb{R}^r$ to be the vector with the last two entries 1 and all other entries 0; i.e.

$$\vec{y}_r = (0, \ldots, 0, 1, 1)^T. \tag{2.54}$$

We collect some useful computations involving these vectors:

$$\vec{x}_r \vec{x}_r^T = \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \end{bmatrix} \tag{2.55}$$

$$\vec{x}_r^T \vec{x}_r = r \tag{2.56}$$

$$\vec{y}_r \vec{y}_r^T = \begin{bmatrix} 0 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & 1 & 1 \\ 0 & \ldots & 0 & 1 & 1 \end{bmatrix} \tag{2.57}$$

$$\vec{x}_r^T \vec{y}_r = \vec{y}_r^T \vec{x}_r = \vec{y}_r^T \vec{y}_r = 2 \tag{2.58}$$

We recall from the discussion immediately preceding Proposition 2.11 that $\vec{\beta}^a$ is defined by $\beta^a_{\{i,j\}} = 1$ if $i = a$ or $j = a$ and $\beta^a_{\{i,j\}} = 0$ otherwise. We let $\vec{\chi}^a$ be the vector obtained by truncating the $\{n-1, n\}$ coordinate from $\vec{\beta}^a$, so that $\vec{\chi}^a \in \mathbb{R}^{d-1}$. We define a $(d-1) \times n$ matrix $Q$ by

$$Q = \begin{bmatrix} \vec{\chi}^1 & \vec{\chi}^2 & \dots & \vec{\chi}^n \end{bmatrix}. \tag{2.59}$$

For example, if $n = 5$, then

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{1,5\} \\ \{2,3\} \\ \{2,4\} \\ \{2,5\} \\ \{3,4\} \\ \{3,5\} \end{matrix}$$

where the labels to the right denote the standard coordinate enumeration. We note that of all the vectors $\vec{\beta}^a$, the only ones that had a (now removed) 1 in the $\{n-1, n\}$ coordinate are $\vec{\beta}^{n-1}$ and $\vec{\beta}^n$.

The primary importance of the matrix $Q$ is the computation of the $(d-1) \times (d-1)$ matrix $QQ^T$, which can be found by examining the inner products of rows $\{a, b\}$ and

$\{c, d\}$ of $Q$:

$$(QQ^T)_{\{a,b\},\{c,d\}} = \begin{cases} 2, & |\{a,b\} \cap \{c,d\}| = 2 \\ 1, & |\{a,b\} \cap \{c,d\}| = 1 \\ 0, & |\{a,b\} \cap \{c,d\}| = 0. \end{cases} \tag{2.60}$$

Comparing this computation with (2.31) sheds light on why $Q$ is a useful matrix. We will also need to consider the $n \times n$ matrix $QQ^T$, which is given by the formula

$$Q^T Q = (n-2)I_n + \vec{x}_n \vec{x}_n^T - \vec{y}_n \vec{y}_n^T. \tag{2.61}$$

To see this, we consider the inner products of columns of the matrix $Q$. The inner product of any column with itself is the number of 1's in that column, which is $n-1$ for all but the last two columns and is $n-2$ for the last two columns; these agree with the diagonal entries of the sum in (2.61). Similarly, the inner product of distinct columns $i$ and $j$ is 1, corresponding to the 1 found in the $\{i, j\}$ row of each column. The exception is if $i = n-1$ and $j = n$ (or vice versa), where the inner product is 0. These entries are also given by the sum in (2.61).

We also make note of the following computation, to be used when computing $\det(M)$:

$$\vec{x}_{d-1}^T Q = (n-1)\vec{x}_n^T - \vec{y}_n^T. \tag{2.62}$$

This follows from fact the every column in $Q$ has $n-1$ entries equal to 1, except for the last two, which have only $n-2$ entries equal to 1.

We are ready to express our matrix $M$ of interest in terms of these constituent parts:

**Proposition 2.36.** With matrices $M$, $I_{d-1}$, $\vec{x}_{d-1}$, $Q$, and coefficients $C_i$ as previously defined, and with $a_1 = C_2 - 2C_3 + C_4$, $a_2 = C_4 - C_2^2$, and $a_3 = C_3 - C_4$,

$$M = a_1 I_{d-1} + a_2 \vec{x}_{d-1} \vec{x}_{d-1}^T + a_3 Q Q^T. \tag{2.63}$$

*Proof.* Let $R = a_1 I_{d-1} + a_2 \vec{x}_{d-1} \vec{x}_{d-1}^T + a_3 Q Q^T$. We will verify that these entries of $R$ agree with the entries in (2.31) by using (2.55) and (2.60). A coordinate pair of the form $\{a, b\}, \{a, b\}$ (i.e. one on the diagonal of $R$) receives a contribution from all three parts of the sum in (2.63):

$$\begin{aligned} R_{\{a,b\},\{a,b\}} &= a_1 + a_2 + 2a_3 \\ &= C_2 - C_2^2. \end{aligned}$$

A coordinate pair of the form $\{a, b\}, \{a, c\}$ (i.e. exactly one shared component) does not receive a contribution from the identity matrix in (2.63), so

$$\begin{aligned} R_{\{a,b\},\{a,c\}} &= a_2 + a_3 \\ &= C_3 - C_2^2. \end{aligned}$$

Finally, a coordinate pair of the form $\{a, b\}, \{c, d\}$ (i.e. no shared components) receives a contribution only from the $\vec{x}_{d-1} \vec{x}_{d-1}^T$ term in (2.63):

$$\begin{aligned} R_{\{a,b\},\{c,d\}} &= a_2 \\ &= C_4 - C_2^2. \end{aligned}$$

□

The useful characterization of $M$ in Proposition 2.36 will allow us to compute the determinant of $M$ when combined with the following lemmas:

**Lemma 2.37** (Matrix Determinant Lemma)**.** Let $W$ be an invertible $r \times r$ matrix and let $U, V$ be $r \times s$ matrices. Then

$$\det(W + UV^T) = \det(W)\det(I_s + V^T W^{-1} U)\,.$$

*Proof.* See [Har97, Theorem 18.1]. □

**Lemma 2.38** (Generalized Sherman-Morrison-Woodbury Identity)**.** Let $W$ be an invertible $r \times r$ matrix and for $i = 1, \ldots, L$ let $U_i, V_i$ be $r \times s$ matrices. Define the $Ls \times Ls$ matrix $X$ by

$$X = \begin{bmatrix} I_s + V_1^T W^{-1} U_1 & V_1^T W^{-1} U_2 & \ldots & V_1^T W^{-1} U_L \\ V_2^T W^{-1} U_1 & I_s + V_2^T W^{-1} U_2 & \ldots & V_2^T W^{-1} U_L \\ \vdots & \vdots & \ddots & \vdots \\ V_L^T W^{-1} U_1 & V_L^T W^{-1} U_2 & \ldots & I_s + V_L^T W^{-1} U_L \end{bmatrix}.$$

If $X$ is invertible, then the matrix $\left(W + \sum_{i=1}^L U_i V_i^T\right)$ is invertible, and its inverse is given by

$$\left(W + \sum_{i=1}^L U_i V_i^T\right)^{-1} = W^{-1} - W^{-1} [\, U_1 \quad \ldots \quad U_L \,] X^{-1} [\, V_1^T \quad \ldots \quad V_L^T \,]^T W^{-1}\,.$$

*Proof.* See [Bat08]. □

In particular, with $L = 1$ in Lemma 2.38, we obtain the following:

**Lemma 2.39** (Woodbury Matrix Identity)**.** Let $W$ be an invertible $r \times r$ matrix and let $U, V$ be $r \times s$ matrices. Define $X = I_s + V^T W^{-1} U$. If $X$ is invertible, then $W + UV^T$ is invertible, and

$$(W + UV^T)^{-1} = W^{-1} - W^{-1} U X^{-1} V^T W^{-1} .$$

The basic strategy for computing $\det(M)$ will be to use the Matrix Determinant Lemma several times to trade the products $QQ^T$ and $\vec{x}_{d-1}\vec{x}_{d-1}^T$ for their lower-rank counterparts, $Q^T Q$ and $\vec{x}_{d-1}^T \vec{x}_{d-1}$. Executing this plan will require use of the generalized Sherman-Morrison-Woodbury and Woodbury Matrix Identities.

We are nearly ready to compute $\det(M)$. We first remark that if $k = 2$, we have $C_3 = C_4 = 0$ and therefore $a_3 = 0$ in Lemma 2.36. For technical reasons, this will require us to approach the computation differently when $k = 2$. However, the formula given in the calculation will still hold in this case, even though the method of proof is different.

**Lemma 2.40.** The $(d-1) \times (d-1)$ matrix $M$ has

$$\det(M) = \frac{2\left(\frac{(n-3)(k-1)}{n-k-1}\right)^n (n-2)\left(\frac{(k-1)k[k(k+1)-2kn+n(n-1)]}{n(n-1)(n-2)(n-3)}\right)^d}{(n-k)k(k-1)^2} . \qquad (2.64)$$

*Proof.* We first assume that $k \geq 3$. Recalling the definitions of $a_1, a_2$, and $a_3$ in Proposition 2.36, we have

$$a_3 = C_3 - C_4 = \frac{k(k-1)(k-2)}{n(n-1)(n-2)}\left(1 - \frac{k-3}{n-3}\right)$$

so $a_3 > 0$. Similarly,

$$\begin{aligned} a_1 &= C_2 - 2C_3 + C_4 \\ &= \frac{k(k-1)}{n(n-1)}\left(1 - 2\frac{k-2}{n-2} + \frac{(k-2)(k-3)}{(n-2)(n-3)}\right) \end{aligned}$$

and since

$$\begin{aligned} 0 &< [(n-3)-(k-2)]^2 + (n-k-1) \\ &= (n-3)(n-2) - 2(k-2)(n-3) + (k-2)(k-3) \end{aligned}$$

it follows that $a_1 > 0$ as well. We define a constant $w$ that will appear in several places:

$$w = \frac{a_3}{a_1}(n-2) + 1 \tag{2.65}$$

Since $a_3 > 0$ and $a_1 > 0$, it follows that $w \geq 1$.

Starting with the decomposition in Proposition 2.36, we set

$$E = a_1 I_{d-1} + a_3 QQ^T \tag{2.66}$$

so that we have

$$M = E + a_2 \vec{x}_{d-1} \vec{x}_{d-1}^T \,.$$

Once we have shown that $E$ is invertible, by the Matrix Determinant Lemma we will have

$$\det(M) = \det(E)(1 + a_2 \vec{x}_{d-1}^T E^{-1} \vec{x}_{d-1}). \tag{2.67}$$

This breaks the computation of $\det(M)$ into two smaller computations; we will handle the computation of $1 + \vec{x}_{d-1}^T E^{-1} \vec{x}_{d-1}$ first. Since $E = a_1 I_{d-1} + a_3 QQ^T$, so long as the matrix

$$G = I_n + \frac{a_3}{a_1} Q^T Q$$

is invertible, applying the Woodbury Matrix Identity to (2.66) yields

$$E^{-1} = a_1^{-1} I_{d-1} - a_1^{-2} a_3 Q \left( I_n + \frac{a_3}{a_1} Q^T Q \right)^{-1} Q^T. \tag{2.68}$$

We recall from (2.61) that

$$Q^T Q = (n-2) I_n + \vec{x}_n \vec{x}_n^T - \vec{y}_n \vec{y}_n^T$$

so using $w$ as in (2.65), we have

$$G = w I_n + \frac{a_3}{a_1} \vec{x}_n \vec{x}_n^T - \frac{a_3}{a_1} \vec{y}_n \vec{y}_n^T. \tag{2.69}$$

To argue that $G$ is invertible (hence, that $E$ is), and to compute $G^{-1}$, we use the generalized Sherman-Morrison-Woodbury Identity on (2.69). Here, the matrix $X$ in Lemma 2.38 is the $2 \times 2$ matrix which can be computed using (2.56) and (2.58):

$$X = \begin{bmatrix} 1 + \frac{1}{w} \frac{a_3}{a_1} \vec{x}_n^T \vec{x}_n & -\frac{1}{w} \frac{a_3}{a_1} \vec{x}_n^T \vec{y}_n \\ \frac{1}{w} \frac{a_3}{a_1} \vec{x}_n^T \vec{y}_n & 1 - \frac{1}{w} \frac{a_3}{a_1} \vec{y}_n^T \vec{y}_n \end{bmatrix} = \begin{bmatrix} 1 + n \frac{a_3}{a_1 w} & -2 \frac{a_3}{a_1 w} \\ 2 \frac{a_3}{a_1 w} & 1 - 2 \frac{a_3}{a_1 w} \end{bmatrix} \tag{2.70}$$

We note that

$$\begin{aligned} \det(X) &= \left( 1 + n \frac{a_3}{a_1 w} \right) \left( 1 - 2 \frac{a_3}{a_1 w} \right) + 4 \left( \frac{a_3}{a_1 w} \right)^2 \\ &= \left( \frac{a_3}{a_1 w} \right)^2 \left( \left( \frac{a_1 w}{a_3} + n \right) \left( \frac{a_1 w}{a_3} - 2 \right) + 4 \right). \end{aligned}$$

Since $\frac{a_1 w}{a_3} = n - 2 + \frac{a_1}{a_3} \geq 2$, it follows that this determinant is nonzero. Hence, $X$ is invertible, which implies that $G$ is invertible, and therefore $E$ is invertible, justifying the use of (2.67).

By inverting the $2 \times 2$ matrix $X$ and applying the generalized Sherman Morrison-Woodbury Identity, after some algebra we have

$$G^{-1} = \frac{1}{w}\Bigg( I_n - \frac{1}{(\frac{a_1 w}{a_3} + n)(\frac{a_1 w}{a_3} - 2) + 4} \times \begin{bmatrix} \vec{x}_n & -\vec{y}_n \end{bmatrix} \begin{bmatrix} \frac{a_1 w}{a_3} - 2 & 2 \\ -2 & \frac{a_1 w}{a_3} + n \end{bmatrix} \begin{bmatrix} \vec{x}_n^T \\ \vec{y}_n^T \end{bmatrix} \Bigg) \tag{2.71}$$

giving us an explicit formula for $G^{-1}$. By (2.68), this also gives an explicit formula for $E^{-1}$. The right half of the computation in (2.67) can be rewritten using (2.68) to obtain

$$\begin{aligned} &1 + a_2 \vec{x}_{d-1}^T E^{-1} \vec{x}_{d-1} \\ &\quad = 1 + \frac{a_2}{a_1} \vec{x}_{d-1}^T \vec{x}_{d-1} - \frac{a_2 a_3}{a_1^2} \vec{x}_{d-1}^T Q G^{-1} Q^T \vec{x}_{d-1} \end{aligned}$$

and by using (2.62) to replace $\vec{x}_{d-1}^T Q$ and $Q^T \vec{x}_{d-1}$ we have

$$\begin{aligned} &1 + a_2 \vec{x}_{d-1}^T E^{-1} \vec{x}_{d-1} \\ &\quad = 1 + \frac{a_2}{a_1} \vec{x}_{d-1}^T \vec{x}_{d-1} - \frac{a_2 a_3}{a_1^2} [(n-1)\vec{x}_n^T - \vec{y}_n^T] G^{-1} [(n-1)\vec{x}_n - \vec{y}_n] \end{aligned} \tag{2.72}$$

which can be computed due to the explicit formula for $G^{-1}$ given in (2.71). For convenience of notation, we set

$$U = \begin{bmatrix} \vec{x}_n & -\vec{y}_n \end{bmatrix}$$

$$H = \begin{bmatrix} \frac{a_1 w}{a_3} - 2 & 2 \\ -2 & \frac{a_1 w}{a_3} + n \end{bmatrix}$$

$$V^T = \begin{bmatrix} \vec{x}_n^T \\ \vec{y}_n^T \end{bmatrix}$$

since these matrices appear in the more complicated portion of $G^{-1}$. To expand the product in (2.72), we observe four useful calculations that make use of (2.56) and (2.58):

$$\vec{x}_n^T U H V^T \vec{x}_n = (n-2)\left(\frac{a_1 w}{a_3}(n+2) - 2n\right)$$

$$\vec{x}_n^T U H V^T \vec{y}_n = 2(n-2)\left(\frac{a_1 w}{a_3} - 2\right)$$

$$\vec{y}_n^T U H V^T \vec{x}_n = 2(n-2)\left(\frac{a_1 w}{a_3} - 2\right)$$

$$\vec{y}_n^T U H V^T \vec{y}_n = 8 - 4n$$

Using these calculations in (2.72), along with (2.56) and (2.58) again and a great deal of algebra, we have

$$\begin{aligned} &1 + a_2 \vec{x}_{d-1}^T E^{-1} \vec{x}_{d-1} \\ &\quad = 1 + \frac{a_2}{a_1}\Bigg[d - 1 - \frac{w-1}{w}\Bigg\{n^2 - 3 - \frac{n-2}{(\frac{a_1 w}{a_3} + n)(\frac{a_1 w}{a_3} - 2) + 4} \\ &\qquad \times \left((n-3)(n^2+n-4) + \frac{a_1}{a_3}(n-1)(n+3)\right)\Bigg\}\Bigg]. \end{aligned} \tag{2.73}$$

This yields a formula for the second factor on the right-hand side of (2.67).

To find a formula the first factor on the right-hand side of (2.67), we seek to compute $\det(E)$. To accomplish this, we will use the Matrix Determinant Lemma on (2.66). From (2.61), we have

$$\begin{aligned}\det(E) &= a_1^{d-1} \det\left(I_{d-1} + \frac{a_3}{a_1} QQ^T\right) \\ &= a_1^{d-1} \det\left(I_n + \frac{a_3}{a_1} Q^T Q\right) \\ &= a_1^{d-1} \det\left(wI_n + \frac{a_3}{a_1} \vec{x}_n \vec{x}_n^T - \frac{a_3}{a_1} \vec{y}_n \vec{y}_n^T\right).\end{aligned}$$

We set

$$F = wI_n + \frac{a_3}{a_1} \vec{x}_n \vec{x}_n^T \tag{2.74}$$

and we note that if $F$ is invertible, then by the Matrix Determinant Lemma, we have

$$\det(E) = a_1^{d-1} \det(F) \left(1 - \frac{a_3}{a_1} \vec{y}_n^T F^{-1} \vec{y}_n\right). \tag{2.75}$$

To establish that $F$ is invertible and to compute $F^{-1}$, we use the Woodbury Matrix Identity on (2.74). Because

$$1 + \frac{a_3}{a_1 w} \vec{x}_n^T \vec{x}_n = 1 + \frac{a_3 n}{a_1 w} > 0$$

it follows from Lemma 2.39 that $F$ is invertible, so that the use of (2.75) is indeed justified. Moreover, from this lemma we obtain

$$F^{-1} = \frac{1}{w}\left(I_n - \frac{1}{\frac{a_1 w}{a_3} + n} \vec{x}_n \vec{x}_n^T\right).$$

We combine this with (2.58) and find, after some algebra, that

$$1 - \frac{a_3}{a_1}\vec{y}_n^T F^{-1}\vec{y}_n = 1 - \frac{a_3}{wa_1}\left(2 - \frac{4}{\frac{a_1 w}{a_3} + n}\right). \tag{2.76}$$

To find $\det(F)$, we apply the Matrix Determinant Lemma to (2.74) to see that

$$\begin{aligned}\det(F) &= w^n \det\left(I_n + \frac{a_3}{a_1 w}\vec{x}_n\vec{x}_n^T\right)\\ &= w^{n-1}\left(w + \frac{a_3}{a_1}n\right).\end{aligned} \tag{2.77}$$

By substituting the results of (2.77) and (2.76) into (2.75) and simplifying, we find that

$$\det(E) = a_1^{d-1}w^{n-2}\left(2w^2 - w - 2\frac{a_3}{a_1}(w-1)\right). \tag{2.78}$$

From here, (2.78) and (2.73) yield the two factors of $\det(M)$ in (2.67). We multiplying these together and substitute the definition of $w$ in (2.65). Then, we substitute the values of $a_1, a_2, a_3$ in Proposition 2.36; following this, using the definition of the $C_i$ constants and simplifying yields (2.64).

Finally, in the case where $k = 2$, we note that (2.64) reduces to the particularly simple expression

$$\det(M) = \frac{1}{d^d}. \tag{2.79}$$

When $k = 2$, the coefficients $a_1, a_2, a_3$ are

$$\begin{aligned}a_1 &= C_2 = d^{-1},\\ a_2 &= -C_2^2 = -d^{-2},\\ a_3 &= 0.\end{aligned}$$

The preceding proof does not work since $a_3$ appears in many denominators. To verify that the formula in (2.79) still holds, we reconsider the decomposition of $M$ in Proposition 2.36. In this case,

$$M = a_1 I_{d-1} + a_2 \vec{x}_{d-1} \vec{x}_{d-1}^T \tag{2.80}$$

so the determinant is much more straightforward than the case where $k \geq 3$. In particular, since $a_1 \neq 0$ we can apply the Matrix Determinant Lemma to (2.80). This gives

$$\begin{aligned}
\det(M) &= \det(a_1 I_{d-1}) \left(1 + \frac{a_2}{a_1} \vec{x}_{d-1}^T \vec{x}_{d-1}\right) \\
&= (d^{-1})^{d-1} \left(1 - \frac{d^{-2}}{d^{-1}}(d-1)\right) \\
&= d^{-d}
\end{aligned}$$

which matches (2.79) and completes the proof. □

## 2.6 Proof of Main Theorem

Our next task is to find suitable lower and upper bounds for the integral used to compute $\mathbb{P}_{n,k}^{(t)}(\vec{0}, \vec{0})$. With $D_1$ and $D_2$ defined as in Proposition 2.34, we define two quantities of interest:

$$\begin{aligned}
L(n,k,t,\delta) &= [1 + t^2 (d\delta)^6]^{-1/2} \left[1 - \frac{1}{3}(d\delta)^4\right]^t [1 - e^{-\frac{1}{2}t(D_1\delta)^2}]^{(d-1)/2} \\
U(n,k,t,\delta) &= \left[1 + \frac{1}{4}(d\delta)^6\right]^{t/2} \left[1 + \frac{1}{3}(d\delta)^4\right]^t [1 - e^{-t(2D_2\delta)^2}]^{(d-1)/2}
\end{aligned}$$

**Theorem 2.41.** Suppose that $\delta < k^{-2}\binom{n}{k}^{-2}\left[\frac{1}{6\cdot 96^2}\left(\frac{2\pi}{k-1}\right)^4\right]$, and let $t \geq 2$ be any integer. If $t\frac{k(k-1)}{n(n-1)}$ is not an integer, then

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) = 0. \tag{2.81}$$

If $t\frac{k(k-1)}{n(n-1)}$ is an integer but $t\frac{k}{n}$ is not, then

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) \leq \exp\left(-\binom{n}{k}^{-1}\frac{11}{768}t\delta^2\right). \tag{2.82}$$

Finally, if both $t\frac{k(k-1)}{n(n-1)}$ and $t\frac{k}{n}$ are integers with $t < 2(d\delta)^{-3}$, then

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) \leq \frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1}\det(M)}}U(n,k,t,\delta) + e^{-\binom{n}{k}^{-1}\frac{11}{768}t\delta^2} \tag{2.83}$$

and

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) \geq \frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1}\det(M)}}L(n,k,t,\delta) - e^{-\binom{n}{k}^{-1}\frac{11}{768}t\delta^2}. \tag{2.84}$$

**Remark 2.42.** In the sequel, $\delta$ will be chosen to vary with $t$ in such a way that $t\delta^2$ diverges to infinity. This will cause the bound in (2.82) to tend to 0, which reflects the fact that a balanced incomplete block design can only exist when $t\frac{k}{n}$ is an integer as shown in (2.1). The terms $U(n,k,t,\delta)$ and $L(n,k,t,\delta)$ will also approach 1, which will cause (2.83) and (2.84) to yield the asymptotics for the return probability of the random walk $Y_t$. This will then give the asymptotics for the number of balanced incomplete block designs as $t$ increases.

**Remark 2.43.** Since $k \geq 2$ and $n-k \geq 2$, we have $\binom{n}{k} \geq \binom{n}{2} = d$; hence, our assumption on $\delta$ implies in particular that $\delta < d^{-1}$, which will be referenced throughout the proof.

*Proof of Theorem 2.41.* We first consider the case where $t\frac{k(k-1)}{n(n-1)}$ is not an integer. From the definitions of $X_t$ and $Y_t$, since $X_t$ is supported on $\mathbb{Z}^d$ then it is trivially only possible to have $Y_t = \vec{0}$ if $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$, which establishes (2.81).

When $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$, we recall from (2.5) that

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \Phi_Y(\vec{\theta})^t \, d\vec{\theta}.$$

We first suppose that $t\frac{k}{n} \notin \mathbb{Z}$. From (2.21), in this case we have

$$\mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) = (2\pi)^{-d} \int_{R_B^\delta \cup R_C^\delta} \Phi_Y(\vec{\theta})^t \, d\vec{\theta}$$

whence Proposition 2.23 gives rise to (2.82). If instead $t\frac{k}{n} \in \mathbb{Z}$, (2.22) implies that

$$\left| \mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) - (2\pi)^{-d}(k-1)^{n-1} \int_{T^\delta_{\vec{0}}} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} \right|$$
$$= \left| (2\pi)^{-d} \int_{R_B^\delta \cup R_C^\delta} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} \right|$$

so that Proposition 2.23 yields

$$\left| \mathbb{P}^{(t)}_{n,k}(\vec{0},\vec{0}) - (2\pi)^{-d}(k-1)^{n-1} \int_{T^\delta_{\vec{0}}} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} \right| \le e^{-\binom{n}{k}^{-1} \frac{11}{768} t\delta^2} .$$

Therefore, to prove (2.83) and (2.84), it will suffice to show that

$$\frac{L(n,k,t,\delta)}{\sqrt{(2\pi t)^{d-1}\det(M)}} \le (2\pi)^{-d} \int_{T^\delta_{\vec{0}}} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} \le \frac{U(n,k,t,\delta)}{\sqrt{(2\pi t)^{d-1}\det(M)}} . \tag{2.85}$$

Moreover, since $\Phi_Y(-\vec{\theta})$ and $\Phi_Y(\vec{\theta})$ are complex conjugates, we have

$$\int_{T_{\vec{0}}^{\delta}} \Phi_Y(\vec{\theta})^t \, \mathrm{d}\vec{\theta} = \int_{T_{\vec{0}}^{\delta}} \mathrm{Re}(\Phi_Y(\vec{\theta})^t) \, \mathrm{d}\vec{\theta}. \tag{2.86}$$

Our strategy will be to relate $\mathrm{Re}(\Phi_Y(\vec{\theta})^t)$ to $[\mathrm{Re}(\Phi_Y(\vec{\theta}))]^t$ by using Lemma 2.45.

Let $t \geq 2$ be an integer. For a complex number $z = a + bi$ with $a > 0$, we set $\beta(z) = b/a$ and $\alpha(z, t) = 1 - \binom{t}{2}\beta(z)^2$. From Lemma 2.29, for $\vec{\theta} \in T_{\vec{0}}^{\delta}$ we have

$$|\beta(\Phi_Y(\vec{\theta}))| \leq \frac{(d\delta)^3/6}{1/3} = \frac{(d\delta)^3}{2} \tag{2.87}$$

Since by hypothesis $t < 2(d\delta)^{-3}$, it follows that $\binom{t}{2}\beta(\Phi_Y(\vec{\theta}))^2 \leq \binom{t}{2}\frac{(d\delta)^6}{4} < \frac{1}{2}$, whence $\alpha(\Phi_Y(\vec{\theta}), t) > \frac{1}{2}$. In particular, since $\mathrm{Re}(\Phi_Y(\vec{\theta})) > 0$ by (2.39) and since $\alpha(\Phi_Y(\vec{\theta}), t) > 0$, we can make full use of Lemma 2.45. From (2.95) and (2.87) we have

$$\mathrm{Re}(\Phi_Y^t(\vec{\theta})) \leq \Big[\mathrm{Re}(\Phi_Y(\vec{\theta}))\Big]^t \left(1 + \frac{(d\delta)^6}{4}\right)^{t/2} \tag{2.88}$$

and if $\beta$ and $\alpha$ denote $\beta(\Phi_Y(\vec{\theta}))$ and $\alpha(\Phi_Y(\vec{\theta}), t)$ respectively, then from (2.98) we have

$$\begin{aligned} &\mathrm{Re}(\Phi_Y(\vec{\theta})^t) \\ &\quad \geq \Big[\mathrm{Re}(\Phi_Y(\vec{\theta}))\Big]^t \left(1 + \beta^2\right)^{t/2} \left(1 + t^2 \left[\frac{\beta}{\alpha}\right]^2\right)^{-\frac{1}{2}} \\ &\quad \geq \Big[\mathrm{Re}(\Phi_Y(\vec{\theta}))\Big]^t \left(1 + t^2 \left[\frac{\beta}{\alpha}\right]^2\right)^{-1/2}. \end{aligned} \tag{2.89}$$

Since $\alpha(\Phi_Y(\vec{\theta}), t) \geq 1/2$ and $\beta(\Phi_Y(\vec{\theta})) \leq (d\delta)^3/2$, it follows that

$$\left[\frac{\beta(\Phi_Y(\vec{\theta}))}{\alpha(\Phi_Y(\vec{\theta}), t)}\right]^2 \leq (d\delta)^6$$

so (2.88) and (2.89) combine to give

$$\begin{aligned}
&[1 + t^2(d\delta)^6]^{-1/2} \int_{T^\delta_{\vec{0}}} \left[\mathrm{Re}(\Phi_Y(\vec{\theta}))\right]^t \, \mathrm{d}\vec{\theta} \\
&\qquad \leq \int_{T^\delta_{\vec{0}}} \mathrm{Re}(\Phi_Y(\vec{\theta})^t) \, \mathrm{d}\vec{\theta} \\
&\qquad \leq \left[1 + \frac{(d\delta)^6}{4}\right]^{t/2} \int_{T^\delta_{\vec{0}}} \left[\mathrm{Re}(\Phi_Y(\vec{\theta}))\right]^t \, \mathrm{d}\vec{\theta}. \qquad (2.90)
\end{aligned}$$

The inequality (2.90) grants us the ability to consider $\left[\mathrm{Re}(\Phi_Y(\vec{\theta}))\right]^t$ instead of $\mathrm{Re}(\Phi_Y(\vec{\theta})^t)$ in our calculations. From Lemma 2.29, we see that there exists a function $\varepsilon_1 : T^\delta_{\vec{0}} \to \mathbb{R}$ such that for $\vec{\theta} \in T^\delta_{\vec{0}}$,

$$\left[\mathrm{Re}(\Phi_Y(\vec{\theta}))\right]^t = e^{-\frac{1}{2}\vec{\theta}^T N \vec{\theta}} (1 + \varepsilon_1(\vec{\theta}))^t$$

and $|\varepsilon_1(\vec{\theta})| < \frac{1}{6}(d\delta)^4 e^{\frac{1}{2}(d\delta)^2}$. Since our assumptions imply that $d\delta < 1$, it follows that $e^{\frac{1}{2}(d\delta)^2} < 2$, so $|\varepsilon_1(\vec{\theta})| < \frac{1}{3}(d\delta)^4$. Hence, we have

$$e^{-\frac{t}{2}\vec{\theta}^T N \vec{\theta}} \left[1 - \frac{1}{3}(d\delta)^4\right]^t \leq \left[\mathrm{Re}(\Phi_Y(\vec{\theta}))\right]^t \leq e^{-\frac{t}{2}\vec{\theta}^T N \vec{\theta}} \left[1 + \frac{1}{3}(d\delta)^4\right]^t$$

and substituting these bounds into (2.90) gives

$$\begin{aligned}
&[1+t^2(d\delta)^6]^{-1/2}\left[1-\frac{1}{3}(d\delta)^4\right]^t \int_{T^\delta_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} \\
&\quad\le \int_{T^\delta_{\vec{0}}} \mathrm{Re}(\Phi_Y(\vec{\theta})^t)\,\mathrm{d}\vec{\theta} \\
&\quad\le \left[1+\frac{(d\delta)^6}{4}\right]^{t/2}\left[1+\frac{1}{3}(d\delta)^4\right]^t \int_{T^\delta_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta}. \qquad (2.91)
\end{aligned}$$

To verify (2.85) (and thus complete the proof), by (2.91) and (2.86) it suffices to show that

$$\begin{aligned}
&\frac{[1-e^{-\frac{1}{2}t(D_1\delta)^2}]^{(d-1)/2}}{\sqrt{\det(M)}}(2\pi)\left(\frac{2\pi}{t}\right)^{(d-1)/2} \\
&\quad\le \int_{T^\delta_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} \\
&\quad\le \frac{[1-e^{-t(2D_2\delta)^2}]^{(d-1)/2}}{\sqrt{\det(M)}}(2\pi)\left(\frac{2\pi}{t}\right)^{(d-1)/2} \qquad (2.92)
\end{aligned}$$

so we now turn our attention to the integral in the middle.

By Lemma 2.30, we first notice that that we can replace the Gaussian integral in (2.92) with a different one:

$$2\pi\int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M\vec{\mu}}\,\mathrm{d}\vec{\mu} \le \int_{T^\delta_{\vec{0}}} e^{-\frac{t}{2}\vec{\theta}^T N\vec{\theta}}\,\mathrm{d}\vec{\theta} \le 2\pi\int_{[-2\delta,2\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M\vec{\mu}}\,\mathrm{d}\vec{\mu}$$

The main purpose of this exchange is that while $N$ is only positive semidefinite (by Lemma 2.27), $M$ is positive definite (by Corollary 2.33); additionally, the limits on the leftmost and rightmost integrals are easier to manage. We recall from the discussion preceding Proposition 2.34 and its preceding discussion that there exists a symmetric, positive definite matrix $P$ for which $P^2 = M$. Hence,

$$2\pi \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M \vec{\mu}}\, \mathrm{d}\vec{\mu} = 2\pi \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}(P\vec{\mu})^T (P\vec{\mu})}\, \mathrm{d}\vec{\mu}$$

so if we use the change of variables $\vec{\lambda} = P\vec{\mu}$, we have

$$2\pi \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M \vec{\mu}}\, \mathrm{d}\vec{\mu} = \frac{2\pi}{\det(P)} \int_{P[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\lambda}^T \vec{\lambda}}\, \mathrm{d}\vec{\lambda}$$

and similarly,

$$2\pi \int_{[-2\delta,2\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M \vec{\mu}}\, \mathrm{d}\vec{\mu} = \frac{2\pi}{\det(P)} \int_{P[-2\delta,2\delta]^{d-1}} e^{-\frac{t}{2}\vec{\lambda}^T \vec{\lambda}}\, \mathrm{d}\vec{\lambda}.$$

Since the integrand is positive, using Proposition 2.34 gives

$$\begin{aligned}\frac{2\pi}{\det(P)} \int_{[-D_1\delta,D_1\delta]^{d-1}} & e^{-\frac{t}{2}\vec{\lambda}^T \vec{\lambda}}\, \mathrm{d}\vec{\lambda} \\ &< \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M \vec{\mu}}\, \mathrm{d}\vec{\mu} \\ &< \frac{2\pi}{\det(P)} \int_{[-2D_2\delta,2D_2\delta]^{d-1}} e^{-\frac{t}{2}\vec{\lambda}^T \vec{\lambda}}\, \mathrm{d}\vec{\lambda}.\end{aligned}$$

Making one last change of variables with $\vec{\nu} = \sqrt{t}\vec{\lambda}$ on the upper and lower bounds yields

$$\begin{aligned}\frac{2\pi}{\det(P)}(\sqrt{t})^{-(d-1)} \int_{[-D_1\sqrt{t}\delta,D_1\sqrt{t}\delta]^{d-1}} & e^{-\frac{1}{2}\vec{\nu}^T \vec{\nu}}\, \mathrm{d}\vec{\nu} \\ &< \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M \vec{\mu}}\, \mathrm{d}\vec{\mu} \\ &< \frac{2\pi}{\det(P)}(\sqrt{t})^{-(d-1)} \int_{[-2D_2\sqrt{t}\delta,2D_2\sqrt{t}\delta]^{d-1}} e^{-\frac{1}{2}\vec{\nu}^T \vec{\nu}}\, \mathrm{d}\vec{\lambda}.\end{aligned}$$

Since $\vec{\nu}^T\vec{\nu} = \sum \nu_{\{i,j\}}^2$, we can regard the integrals in the lower and upper bounds as the product of $d-1$ integrals of the form $\int e^{-\frac{1}{2}x^2}\,\mathrm{d}x$. Using the estimates in Lemma 2.46 gives

$$\begin{aligned}
&\frac{2\pi}{\det(P)}(\sqrt{t})^{-(d-1)}\left(\sqrt{2\pi(1-e^{-\frac{1}{2}t(D_1\delta)^2})}\right)^{d-1} \\
&\qquad < \int_{[-\delta,\delta]^{d-1}} e^{-\frac{t}{2}\vec{\mu}^T M\vec{\mu}}\,\mathrm{d}\vec{\mu} \\
&\qquad < \frac{2\pi}{\det(P)}(\sqrt{t})^{-(d-1)}\left(\sqrt{2\pi(1-e^{-t(2D_2\delta)^2})}\right)^{d-1}
\end{aligned}$$

and since $\det(P) = \sqrt{\det(M)}$, this yields (2.92) and completes the proof. □

*Proof of Theorem 2.3.* The main point of the proof is to allow $t$ and $\delta$ to vary in such a way that in (2.83) and (2.84), the $U$ and $L$ terms tend to 1, while the error terms in (2.82), (2.83), and (2.84) tend to 0. For a fixed $n$ and $k$, we claim that setting $\delta = t^{-5/12}$ will accomplish this.

We first note that for sufficiently large $t$, $\delta$ is arbitrarily small and thus $\delta < k^{-2}\binom{n}{k}^{-2}\left[\frac{1}{6\cdot 96^2}\left(\frac{2\pi}{k-1}\right)^4\right]$ eventually holds. Similarly, since $(d\delta)^{-3} = d^{-3}t^{5/4}$, for sufficiently large $t$ we have $t < 2(d\delta)^{-3}$. This allows all parts of Theorem 2.41 to be used.

We turn our attention to the terms in square brackets in $L$ and $U$. Since $t^2\delta^6 = t^{-1/2}$, it follows that $[1+t^2(d\delta)^6]^{-1/2} \to 1$ as $t \to \infty$. For any constant $C$ that does not depend on $t$, we have

$$(1+Ct^{-5/3})^t = e^{Ct^{-2/3}}[1+o(1)]$$

which tends to 1 as $t \to \infty$. Since $\frac{d^4}{3}$ does not depend on $t$, it follows that $\left[1-\frac{1}{3}(d\delta)^4\right]^t \to 1$ and $\left[1+\frac{1}{3}(d\delta)^4\right]^t \to 1$ as $t \to \infty$. Since $t\delta^2 = t^{1/6}$ and $D_1, D_2, d$ do not depend on $t$, it follows that $[1-e^{-\frac{1}{2}t(D_1\delta)^2}]^{(d-1)/2} \to 1$ and $[1-e^{-t(2D_2\delta)^2}]^{(d-1)/2} \to 1$ as $t \to \infty$. Finally, for $C$ that does not depend on $t$ we have

$$(1 + Ct^{-5/2})^t = e^{Ct^{-3/2}}[1 + o(1)]$$

which tends to 1 as $t \to \infty$; since $\frac{d^6}{4}$ does not depend on $t$, it follows that $\left[1 + \frac{1}{4}(d\delta)^6\right]^{t/2} \to 1$ as $t \to \infty$.

Putting the above pieces together, we have now shown that as $t \to \infty$, $L(n, k, t, t^{-5/12}) \to 1$ and $U(n, k, t, t^{-5/12}) \to 1$. Hence, (2.83) and (2.84) imply that if $t$ is such that $t\frac{k}{n} \in \mathbb{Z}$ and $t\frac{k(k-1)}{n(n-1)} \in \mathbb{Z}$,

$$\liminf_{t\to\infty} \frac{\mathbb{P}^{(t)}_{n,k}(\vec{0}, \vec{0})}{\left[\frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1}\det(M)}}\right]} \geq \lim_{t\to\infty} \left[ L(n, k, t, t^{-5/12}) - \frac{e^{-\binom{n}{k}^{-1}\frac{11}{768}t^{1/6}}}{\left[\frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1}\det(M)}}\right]} \right] = 1$$

and

$$\limsup_{t\to\infty} \frac{\mathbb{P}^{(t)}_{n,k}(\vec{0}, \vec{0})}{\left[\frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1}\det(M)}}\right]} \leq \lim_{t\to\infty} \left[ U(n, k, t, t^{-5/12}) - \frac{e^{-\binom{n}{k}^{-1}\frac{11}{768}t^{1/6}}}{\left[\frac{(k-1)^{n-1}}{\sqrt{(2\pi t)^{d-1}\det(M)}}\right]} \right] = 1.$$

Combining these inequalities with (2.4) and the calculation of $\det(M)$ in Lemma 2.40 completes the proof. □

## 2.7 Conclusion

In this chapter, for fixed values $n$ and $k$ we have developed a non-symmetric random walk in $\mathbb{R}^{\binom{n}{2}}$. We have related this walk to the existence of balanced incomplete block design incidence matrices with $n$ columns and $k$ occurrences of 1 per each column. From there, we obtained estimates on the return probability of the random walk. We then exploited the relationship between the walk and the incidence matrices to calculate the asymptotic number of incidence matrices with the given parameters as the number of columns increases.

The basic strategy is one adopted in principle from [dLL10], where these analogous tasks were completed for partial Hadamard matrices instead of BIBD incidence matrices. However, these projects were vastly different in two key areas. First, the maximal set of the Hadamard walk characteristic function had a significantly different structure than the one given for the BIBD walk characteristic function in (2.12). In particular, the maximal set for the partial Hadamard walk characteristic function was a zero-dimensional subset of $\mathbb{R}^{\binom{n}{2}}$, whereas the maximal set for the BIBD walk characteristic function was a one-dimensional subset of $\mathbb{R}^{\binom{n}{2}}$. This corresponds to the fundamental difference that the partial Hadamard walk was supported on a $\binom{n}{2}$-dimensional sublattice of $\mathbb{R}^{\binom{n}{2}}$, whereas the BIBD walk is actually supported on an $\left(\binom{n}{2}-1\right)$-dimensional sublattice of $\mathbb{R}^{\binom{n}{2}}$.

The second key difference between the partial Hadamard walk and the BIBD walk rested in a computation of a second moment. Specifically, finding the return probabilities of each walk required computation of the quantity $\mathbb{E}\left[(\vec{\mu}\cdot Y_1)^2\right]$, where $\vec{\mu}\in\mathbb{R}^{\binom{n}{2}}$ and $Y_1$ represented a single step of the respective random walks. In both cases, it was computed that

$$\mathbb{E}\left[(\vec{\mu} \cdot Y_1)^2\right] = \vec{\mu}^T N \vec{\mu}$$

for some $\binom{n}{2} \times \binom{n}{2}$ matrix $N$. In the BIBD walk, $N$ was the combinatorially-defined matrix given in (2.31), which required significant further analysis and a lengthy computation of its principal minor. In the partial Hadamard walk, $N$ was instead the identity matrix $I_d$, which simplified some of the calculations and entirely avoided the need for a discussion such as that in Section 2.5.

While we believe that counting the incidence matrices of balanced incomplete block designs is of independent interest, we note here that a related and well-studied problem in combinatorial design theory is that of the number of isomorphism classes of balanced incomplete block designs. These designs are typically regarded as a set of elements (called points) and a multi-set of subsets (called blocks) of these points. Each BIBD incidence matrix corresponds to a balanced incomplete block design, though this correspondence is not one-to-one. Permuting different columns of a BIBD incidence matrix will yield a different incidence matrix, but these correspond to the same underlying design. The isomorphism classes of the underlying designs correspond to certain permissible permutations of the rows of the incidence matrices. Translating the equivalence classes of the underlying designs to equivalence classes of the incidence matrices is a nontrivial combinatorial problem that we hope to consider in the future. Specifically, we hope to relate Theorem 2.3 to these equivalence classes of incidence matrices to see if anything can be learned about the number of the underlying design isomorphism classes.

Finally, as first remarked in [dLL10], we point out that this general strategy of relating a random walk to the existence of combinatorial designs can be applied to other types of designs as well, such as difference matrices.

## 2.8 Supplementary Material

In this section, we state and prove inequalities that relate $\mathrm{Re}(z^t)$ with $\mathrm{Re}(z)^t$. These statements and their proofs are nearly identical to those found in the appendix of [dLL10]. Our first lemma is a variant of the Neyman-Pearson Lemma.

**Lemma 2.44.** Let $\lambda_0, \ldots, \lambda_n$, $A_0, \ldots, A_n$ be positive real numbers and $B_n, \ldots, B_n$ be real numbers. Then

$$\min_{0 \leq s \leq n} \left( \frac{B_s}{A_s} \right) \leq \frac{\sum_{s=0}^{n} \lambda_s B_s}{\sum_{s=0}^{n} \lambda_s A_s} \leq \max_{0 \leq s \leq n} \left( \frac{B_s}{A_s} \right).$$

**Lemma 2.45.** Let $t \geq 2$ be a positive integer, and let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Set

$$\alpha(z,t) = 1 - \binom{t}{2} \left[ \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \right]^2.$$

Then

$$\left[ \mathrm{Re}(z^t) \left( 1 + \left[ \frac{\mathrm{Im}(z^t)}{\mathrm{Re}(z^t)} \right]^2 \right)^{1/2} \right]^2 = \left[ \left( 1 + \left[ \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \right]^2 \right)^{t/2} \mathrm{Re}(z)^t \right]^2. \tag{2.93}$$

Further, if $\alpha(z,t) > 0$, then all the following hold:

$$\mathrm{Re}(z^t) > 0, \tag{2.94}$$

$$\mathrm{Re}(z^t) = \mathrm{Re}(z)^t \left( 1 + \left[ \frac{\mathrm{Im}(z^t)}{\mathrm{Re}(z^t)} \right]^2 \right)^{-1/2} \left( 1 + \left[ \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \right]^2 \right)^{t/2}, \tag{2.95}$$

$$\operatorname{Re}(z^t) \le \operatorname{Re}(z)^t \left(1 + \left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right)^2\right)^{t/2}, \tag{2.96}$$

$$\left[\frac{\operatorname{Im}(z^t)}{\operatorname{Re}(z^t)}\right]^2 \le \left(\frac{t}{\alpha(z,t)}\right)^2 \left[\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right]^2, \tag{2.97}$$

and

$$\operatorname{Re}(z^t) \ge \operatorname{Re}(z)^t \left(1 + \left[\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right]^2\right)^{t/2} \left(1 + \left[\frac{t}{\alpha(z,t)}\right]^2 \left[\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right]^2\right)^{-1/2}. \tag{2.98}$$

**Lemma 2.46.** Let $\rho$ be a positive real number. Then

$$\sqrt{2\pi(1 - e^{-\rho^2/2})} \le \int_{-\rho}^{\rho} e^{-\frac{1}{2}x^2}\,\mathrm{d}x \le \sqrt{2\pi(1 - e^{-\rho^2})}.$$

*Proof of Lemma 2.44.* Let $s_0$ and $s_1$ be such that

$$\frac{B_{s_0}}{A_{s_0}} = \min_{0 \le s \le n} \left\{\frac{B_s}{A_s}\right\}$$

and

$$\frac{B_{s_1}}{A_{s_1}} = \max_{0 \le s \le n} \left\{\frac{B_s}{A_s}\right\}.$$

Then

$$\begin{aligned}
\frac{B_{s_0}}{A_{s_0}} &= \frac{\sum \lambda_s A_s (B_{s_0}/A_{s_0})}{\sum \lambda_s A_s} \\
&\le \frac{\sum \lambda_s B_s}{\sum \lambda_s A_s} \\
&\le \frac{\sum \lambda_s A_s (B_{s_1}/A_{s_1})}{\sum \lambda_s A_s} \\
&= \frac{B_{s_1}}{A_{s_1}}.
\end{aligned}$$

□

*Proof of Lemma 2.45.* We begin by computing $|z|^{2t}$ in two different ways:

$$(|z|^2)^t = \left(\mathrm{Re}(z)^2\left[1+\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)^2\right]\right)^t$$
$$= \left(\mathrm{Re}(z)^t\left[1+\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)^2\right]^{t/2}\right)^2$$
$$(|z^t|)^2 = \mathrm{Re}(z^t)^2\left[1+\left(\frac{\mathrm{Im}(z^t)}{\mathrm{Re}(z^t)}\right)^2\right]$$
$$= \left(\mathrm{Re}(z^t)\left[1+\left(\frac{\mathrm{Im}(z^t)}{\mathrm{Re}(z^t)}\right)^2\right]^{1/2}\right)^2$$

The equality of these expressions yields (2.93).

To show the remaining claims, we will show first that if $\alpha(z,t) > 0$, then $\mathrm{Re}(z^t) > 0$. For technical reasons, we split this consideration up into four cases which are based on the residue of $t$ mod 4 and use the binomial theorem. Though we will only need the computations of $\mathrm{Re}(z^t)$ to prove (2.95), we will require the computations of $\mathrm{Im}(z^t)$ for the proof of (2.97), so we record them both at this time. In what follows, we set $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$.

If $t = 4x$ for $x \in \mathbb{Z}$, then using the convention that $\binom{4x}{4x+1} = 0$, we have

$$\mathrm{Re}(z^t) = a^{4x}\sum_{s=0}^{x}\left\{\binom{4x}{4s}\left(\frac{b}{a}\right)^{4s} \times\left[1-\frac{(4x-4s)(4x-4s-1)}{(4s+1)(4s+2)}\left(\frac{b}{a}\right)^2\right]\right\} \tag{2.99}$$

and

$$\operatorname{Im}(z^t) = a^{4x} \sum_{s=0}^{x} \left\{ \binom{4x}{4s+1} \left(\frac{b}{a}\right)^{4s+1} \right.$$
$$\left. \times \left[ 1 - \frac{(4x-4s-1)(4x-4s-2)}{(4s+2)(4s+3)} \left(\frac{b}{a}\right)^2 \right] \right\}. \qquad (2.100)$$

One can check that if $z = 4x + y$ with $y \in \{0, 1, 2, 3\}$, then valid decompositions of $\operatorname{Re}(z^t)$ and $\operatorname{Im}(z^t)$ can be obtained by replacing every instance of $4x$ in (2.99) and (2.100) with $4x + y$. The essential detail is that regardless of the residue of $t$ mod 4, by adopting the convention that $\binom{t}{t+1} = 0$ we can write

$$\operatorname{Re}(z^t) = a^t \sum_{s=0}^{\lfloor t/4 \rfloor} \left\{ \binom{t}{4s} \left(\frac{b}{a}\right)^{4s} \right.$$
$$\left. \times \left[ 1 - \frac{(t-4s)(t-4s-1)}{(4s+1)(4s+2)} \left(\frac{b}{a}\right)^2 \right] \right\}, \qquad (2.101)$$

and

$$\operatorname{Im}(z^t) = a^t \sum_{s=0}^{\lfloor t/4 \rfloor} \left\{ \binom{t}{4s+1} \left(\frac{b}{a}\right)^{4s+1} \right.$$
$$\left. \times \left[ 1 - \frac{(t-4s-1)(t-4s-2)}{(4s+2)(4s+3)} \left(\frac{b}{a}\right)^2 \right] \right\}. \qquad (2.102)$$

Examining the terms in (2.101), we note that since we assume $a > 0$, every term is positive except potentially the terms in square brackets. However, regardless of what $s$ is, by inspection each term in square brackets is at least $1 - \binom{t}{2}\left(\frac{b}{a}\right)^2 = \alpha(z, t)$, which we also assume to be positive. Hence, $\operatorname{Re}(z^t) > 0$ in every case, which establishes (2.94).

We continue under the assumption that $\alpha(z,t) > 0$. To prove (2.95), we notice that $\left(1 + \left[\frac{\mathrm{Im}(z^t)}{\mathrm{Re}(z^t)}\right]^2\right)^{1/2}$ and $\left(1 + \left[\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right]^2\right)^{t/2}$ are both clearly positive, that $\mathrm{Re}(z)^t$ is positive by assumption, and that $\mathrm{Re}(z^t)$ is positive by (2.94). Hence, taking square roots in (2.93) and solving for $\mathrm{Re}(z^t)$ yields (2.95), and (2.96) is an immediate consequence thereof.

To prove (2.97), we employ Lemma 2.44. Again, let $z = a + bi$ and assume that $t = 4x + y$ with $x \in \mathbb{Z}$ and $y \in \{0, 1, 2, 3\}$. For $s \in \{0, \ldots, x\}$, we set

$$\begin{aligned}
\lambda_s &= \binom{t}{4s}\left(\frac{b}{a}\right)^{4s}, \\
A_s &= \left[1 - \frac{(t-4s)(t-4s-1)}{(4s+1)(4s+2)}\left(\frac{b}{a}\right)^2\right], \\
B_s &= \left(\frac{t-4s}{4s+1}\right)\left[1 - \frac{(t-4s-1)(t-4s-2)}{(4s+2)(4s+3)}\left(\frac{b}{a}\right)^2\right].
\end{aligned}$$

From (2.101) and (2.102), it follows that

$$\mathrm{Re}(z^t) = a^t \sum_{s=0}^{\lfloor t/4 \rfloor} \lambda_s A_s, \tag{2.103}$$

$$\mathrm{Im}(z^t) = a^t \left(\frac{b}{a}\right) \sum_{s=0}^{\lfloor t/4 \rfloor} \lambda_s B_s. \tag{2.104}$$

We note that each term $B_s$ is at most $t$, and each term $A_s$ is at least $\left[1 - \frac{t(t-1)}{2}\left(\frac{b}{a}\right)^2\right] = \alpha(z,t)$. Hence,

$$\max_{0 \le s \le \lfloor t/4 \rfloor} \left\{\frac{B_s}{A_s}\right\} \le \frac{t}{\alpha(z,t)}$$

and therefore by Lemma 2.44 and equations (2.103) and (2.104),

$$\left(\frac{\mathrm{Im}(z^t)}{\mathrm{Re}(z^t)}\right)^2 = \left(\frac{b}{a}\right)^2 \left(\frac{\sum \lambda_s B_s}{\sum \lambda_s A_s}\right)^2 \le \left(\frac{b}{a}\right)^2 \left(\frac{t}{\alpha(z,t)}\right)^2$$

which gives (2.97). Finally, to obtain (2.98), we substitute (2.97) into (2.95). □

*Proof of Lemma 2.46.* Using the standard trick of multiplying two copies of the integral together, using Fubini's Theorem, and converting to polar coordinates, we have

$$\int_0^{\rho} 2\pi r e^{-\frac{1}{2}r^2}\,\mathrm{d}r < \int_{[-\rho,\rho]^2} e^{-\frac{1}{2}(x^2+y^2)}\,\mathrm{d}y\;\mathrm{d}x < \int_0^{\sqrt{2}\rho} 2\pi r e^{-\frac{1}{2}r^2}\,\mathrm{d}r$$

so computing the left and right integrals and taking square roots gives the result. □

# CHAPTER III

# COLLISIONS OF INDEPENDENT RANDOM WALKS ON GRAPHS

Given an infinite, locally finite graph $G$, we let $X_n$ and $X'_n$ denote two independent simple random walks on $G$ (started at the same distinguished vertex $o$). After a certain number of steps, one can ask about the probability that $X_n$ and $X'_n$ have collided; that is, $\mathbb{P}(X_n = X'_n)$. However, a more interesting and more delicate question is this: with what probability does the event $\{X_n = X'_n \text{ infinitely often}\}$ occur?

This question was first posed by George Pólya, who was primarily concerned with the case where the graphs were Euclidean lattices $\mathbb{Z}^d$. Because Euclidean lattices are highly structured, the problem simply amounted to computing whether the event $\{X_n = o \text{ infinitely often}\}$ occurred for a single walk $X_n$. In other words, for $\mathbb{Z}^d$, the event $\{X_n = X'_n \text{ i.o.}\}$ had probability 1 if and only if the graph was recurrent, which for $\mathbb{Z}^d$ was known to be true if and only if $d \leq 2$.

In [KP04], Krishnapur and Peres considered the comb graph $\text{Comb}(\mathbb{Z}, \mathbb{Z})$ obtained by removing all horizontal edges from $\mathbb{Z}^2$ except those on the $x$-axis. This graph is recurrent, since it is a subgraph of $\mathbb{Z}^2$; however, two random walks on $\text{Comb}(\mathbb{Z}, \mathbb{Z})$ have the property that $\mathbb{P}(X_n = X_n \text{ i.o.}) = 0$. This result was surprising because it was the first bounded-degree graph for which the question of infinite collisions of two independent random walks was not equivalent to the question of the recurrence of the graph. This discovery marked the beginning of a new line of investigation into the structural properties of graphs that govern the quantity $\mathbb{P}(X_n = X'_n \text{ i.o.})$.

This question of infinite collisions on a graph was later partially answered by [BPS10], which developed a criterion in terms of certain Green's functions of a graph. The importance of this criterion is when it is satisfied (for a particular graph), it follows $\mathbb{P}(X_n = X'_n \text{ i.o.}) = 1$. Other aspects of the quantity $\mathbb{P}(X_n = X'_n \text{ i.o.})$ have been investigated by various authors, including the importance of the ambient time parameter. Additionally, some work has been done on the analogous question regarding three independent simple random walks on a graph.

The outline of this chapter is as follows: in Section 3.1, we discuss a number of counterexamples that show the complexity and nuance of the infinite collision question. In Section 3.2, we prove that if $\text{Comb}(\mathbb{Z}, \mathbb{Z})$ is truncated to retain only vertices $(x, y)$ with $y \leq C|x|^{1-\epsilon}$ for $C, \epsilon > 0$, then two walks will collide infinitely often almost surely. In Section 3.3, we show that the Green's function criterion given in [BPS10] is stable under certain types of graph mappings known as rough isometries. In Section 3.4, we give a complete answer to the collision question for certain types of well-structured graphs. Finally, in Section 3.5, we show that the analogous question of having four independent random walks collide simultaneously has a trivial answer when the underlying graph has bounded degree.

We remark that the work in Section 3.2 is redundant with some existing literature. Additional results regarding truncations of $\text{Comb}(\mathbb{Z}, \mathbb{Z})$ were developed independently in both [CWZ08] and [BPS10], both of which were published after the development of the material in Section 3.2. After these developments, the strongest results in this topic were proved in [CC10]. Each of the four results show that certain truncations of $\text{Comb}(\mathbb{Z}, \mathbb{Z})$ have the property that $\mathbb{P}(X_n = X'_n \text{ i.o.}) = 1$. The result in [CWZ08] permits truncations that retain vertices $(x, y)$ with $y \leq |x|^{1/5-\epsilon}$. The result in Section 3.2 requires only that $y \leq C|x|^{1-\epsilon}$. The result in [BPS10] relaxes this assumption further to permit that $y \leq C|x|$, while the result in [CC10] permits that $y \leq C|x|\log(|x|)$. Although the main result of Section 3.2 is now obsolete, we provide it here because it was discovered independently of the results in [BPS10] and [CC10], and because the method of proof of Theorem 3.16 differs significantly from the methods in [CWZ08], [BPS10], or [CC10].

Finally, we remark here that a result of Section 3.4 is an improvement on previous work in [CWZ08]. In Theorem 5 of [CC10], Chen and Chen assert that if a graph is quasi-transitive and of sub-exponential growth, then the collision property does not depend on whether the ambient time parameter is discrete or continuous. This fact is established in Corollary 3.40, but without the need for the assumption that the graph is of sub-exponential growth.

### 3.1 Counterexamples in Collision Theory

In this section, we will attempt to illustrate the complexity of the infinite collision question with a number of counterexamples. In Claims (3.3) and (3.4), we explore a transient graph of unbounded degree for which two independent walkers collide infinitely often with probability 1. (This graph first appeared in this context in [KP04]). In Claim (3.7), we explore a transient graph for which the probability of infinite collisions is strictly between 0 and 1; this result contrasts with Proposition 2.1 of [BPS10], which shows that this phenomenon is not possible for a recurrent graph. In Claim (3.9), we explore a transient graph for which the continuous-time and discrete-time collision properties differ. We also remark that this graph represents a case where adding a single edge changes the collision property (in discrete time); the question of whether such a phenomenon can occur in the recurrent case remains open.

**Definition 3.1.** Let $G$ be the graph defined by taking vertices corresponding to $\{1, 2, \dots\}$ and adding $2^n$ paths of length 2 between vertices $n$ and $n+1$ as shown in Figure 3.1. (We emphasize that this graph is not an original construction and that it appeared first in [KP04].)

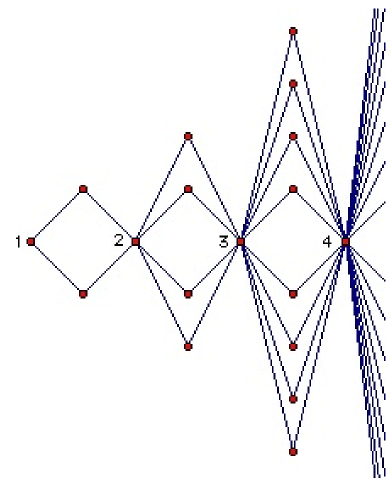


FIGURE 3.1: The graph $G$.

**Remark 3.2.** If $X_n$ is a simple random walk on $G$, then $X_{2n}$ is a biased lazy random walk on $\{1, 2, \dots\}$. For $k \geq 2$, it is clear that

$$\mathbb{P}_k(X_2 = j) = \begin{cases} \frac{1}{2}, & j = k \\ \frac{1}{6}, & j = k-1 \\ \frac{1}{3}, & j = k+1. \end{cases}$$

**Claim 3.3.** Two independent discrete-time simple random walks on $G$ started at vertex 1 will collide infinitely many times with probability 1.

*Proof.* Let $Y_n$ be a random walk on $\mathbb{Z}$ started at 1 with transition probability

$$\mathbb{P}_k(Y_1 = j) = \begin{cases} \frac{1}{2}, & j = k \\ \frac{1}{6}, & j = k-1 \\ \frac{1}{3}, & j = k+1, \end{cases}$$

so that $Y_n$ is a lazy $p \uparrow q \downarrow$ random walk on $\mathbb{Z}$. We can condition $Y_n$ never to reach the vertex 0. Specifically, let $T_0 = \inf\{n \geq 0 : Y_n = 0\}$; it is easy to compute that $\mathbb{P}(T_0 = \infty) = 1/2$.

Let $Y_n'$ be an independent copy of the same chain, and define the analogous stopping time $T_0'$. An easy calculation shows that $Y_n - Y_n'$ is a martingale with bounded increments. Since this martingale clearly does not converge to a limit, it follows from Theorem 4.3.1 of [Dur96] that $\limsup(Y_n - Y_n') = \infty$ and $\liminf(Y_n - Y_n') = -\infty$. In particular, because the martingale is supported on a discrete set and has bounded increments, it is equal to 0 infinitely often, meaning that $Y_n = Y_n'$ infinitely often a.s.

Now, let $A_n$ denote the event that $Y_n = Y_n'$ only finitely often. Then

$$\begin{aligned} \mathbb{P}(A_n) &\geq \mathbb{P}(A_n | \, T_0 = T_0' = \infty) \cdot \mathbb{P}(T_0 = T_0' = \infty) \\ &= \mathbb{P}(A_n | \, T_0 = T_0' = \infty) \cdot (1/2)^2. \end{aligned}$$

Hence $\mathbb{P}(A_n | \, T_0 = T_0' = \infty)$ is 0, meaning that $Y_n = Y_n'$ infinitely often even if conditioned never to move to the left of 0.

Finally, if $X_n$ and $X_n'$ are independent copies of a simple random walk on $G$, we can consider the walks $X_{2n}$ and $X_{2n}'$. The walks $X_{2n}$ and $X_{2n}'$ are supported on the vertex set $\{1, 2, \dots\} \subset v(G)$ and their distributions are the same as those of $Y_n$

and $Y'_n$ conditioned not to move to the left of 1. It follows that $X_{2n} = X'_{2n}$ infinitely often. $\square$

**Claim 3.4.** Two independent continuous-time simple random walks on $G$ started at vertex 1 will collide infinitely many times with probability 1.

**Remark 3.5.** It is perhaps unsurprising that the continuous-time question would have the same result as the discrete-time question; however, the trick of considering $X_{2n}$ as a lazy $p \uparrow q \downarrow$ random walk no longer works in the continuous-time case, so the proof of this result is more involved. Moreover, Claim 3.9 below will establish the existence of a graph for which the discrete-time and continuous-time questions have different answers, which will show that the difference between the two environments is indeed somewhat delicate.

*Proof of Claim 3.4.* We consider a (discrete-time) random walk $Y_k$ on the set $\mathbb{Z} \cup (\mathbb{Z} + \frac{5}{12})$ with transition probabilities as follows:

$$\mathbb{P}_z(Y_1 = j) = \begin{cases} \frac{2}{3}, & j = z + \frac{5}{12} \\ \frac{1}{3}, & j = z - \frac{7}{12} \end{cases} \qquad (z \in \mathbb{Z})$$

$$\mathbb{P}_{z+5/12}(Y_1 = j) = \begin{cases} \frac{1}{2}, & j = z \\ \frac{1}{2}, & j = z + 1 \end{cases} \qquad (z \in \mathbb{Z})$$

The walk $Y_{2k}$ is supported on $\mathbb{Z}$, and it is easy to see that

$$\mathbb{P}_z(Y_2 = j) = \begin{cases} \frac{1}{3}, & j = z + 1 \\ \frac{1}{6}, & j = z - 1 \\ \frac{1}{2}, & j = z \end{cases}$$

which is the same transition probability as the walk from the proof of Claim 3.3. Again, there is a positive probability that $Y_k$ will never move to the left of vertex 1 which we will use momentarily.

It is easy to check that $Y_k$ is a submartingale with $\mathbb{E}[Y_{k+1}|\mathcal{F}_k] = Y_k + 1/12$. Let $Y_k$ and $Y_k'$ be independent copies of the same walk, and let $B_k$ be an independent sequence of i.i.d. variables with $\mathbb{P}(B_k = 0) = \mathbb{P}(B_k = 1) = 1/2$; then $M_k = Y_k B_k - Y_k'(1 - B_k)$ forms a martingale with bounded increments. Intuitively, we throw a coin to decide which walk is allowed to move, and this move will have an expected increment of $+\frac{1}{12}$; the coin throw determines whether this gain is added or subtracted. By reasoning exactly like that in the proof of Claim 3.3, it follows that $M_k = 0$ infinitely often. Further, there is a positive probability that neither walk will move to the left of vertex 1 at any point. Thus, a conditioning argument identical to that in the proof of Claim 3.3 shows that the two walks collide infinitely often almost surely even if conditioned never to move to the left of 1.

We now seek to relate this walk to one on $G$; we let the vertices $\{1, 2, \dots\} \subset v(G)$ correspond to the same integers in the submartingale, and we let the vertices in the paths of length 2 correspond to the non-integer vertices in this walk. To model the continuous-time walks on $G$, we consider (independent) walkers $X_t$ and $X_t'$ on $G$. We inductively define a sequence of stopping times by $T_k = \inf\{t \geq T_{k-1} : Y_t \neq X_{T_{k-1}}$ or $Y_t' \neq X'_{T_{k-1}}\}$. Finally, we define new processes $Z_k = X_{T_k}$ and $Z_k' = X'_{T_k}$, which completes the discretization of this problem. With these definitions, $Z_k$ and $Z_k'$ are discrete-time simple random walks on $G$; at each time $k$, a fair coin is flipped and a corresponding walker is allowed to take a step while the other walker does nothing. By using the vertex association above, we see that $Z_k$ and $Z_k'$ project onto walks $\hat{Y}_k$ and $\hat{Y}_k'$ on the set $\mathbb{Z} \cup (\mathbb{Z} + \frac{5}{12})$. Their transition probabilities are identical to those of $Y_k$ conditioned never to move to the left of 1, and the two projections therefore collide infinitely often almost surely.

It is not true that $\hat{Y}_k = \hat{Y}'_k$ implies that $Z_k = Z'_k$, since the vertices at $\mathbb{Z}+\frac{5}{12}$ have multiple preimages in $G$. However, if $\hat{Y}_k = \hat{Y}'_k$ somewhere in the vertex set $\mathbb{Z}+\frac{5}{12}$, then to induce a collision $Z_k = Z'_k$, it is sufficient that the next two moves not be made by the same walk (but rather, each walk takes one step), and that each walk takes its next step in the same direction. The probability of this occurring 1/4, and since there are infinitely many opportunities for this to occur, it will occur infinitely often almost surely. □

**Definition 3.6.** Let $H_1$ be the graph formed by taking two disjoint copies of $G$ and identifying the two copies of vertex 1 as shown in Figure 3.2.

The idea of this construction was due to a question asked by Jon Wherry during a seminar talk ([Whe]).

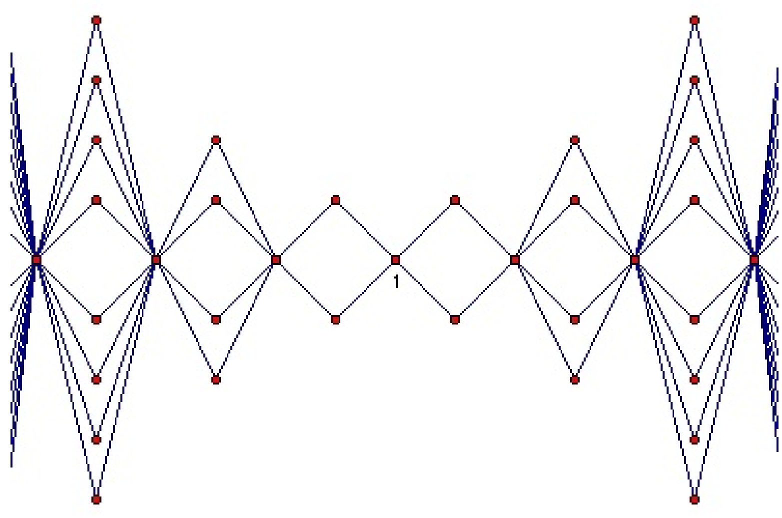


FIGURE 3.2: The graph $H_1$.

**Claim 3.7.** Two independent discrete-time simple random walks on $H_1$ started at the identified vertex will collide infinitely often with probability 1/2.

*Proof.* We model a simple random walk on $H_1$ by modifying a simple random walk on $G$. Let $X_n$ be a simple random walk on $G$ and let $B_n$ be a sequence of independent fair coin flips, i.e. $\mathbb{P}(B_n = 0) = \mathbb{P}(B_n = 1) = 1/2$. Let $R_n = \sum_{k=1}^{n} \mathbb{1}_{X_n=1}$ denote the number of visits of $X_n$ to vertex 1. Then $Y_n = (X_n, B_{R_n})$ models a simple random walk on $H_1$; at each visit to the wedge point, the walk chooses which of the two copies of $G$ it will visit next, where the choice between the two copies of $G$ corresponds to the coin value $B_{R_n}$.

Let $X'_n, B'_n, R'_n$, and $Y'_n$ all be defined analogously and independently of their unprimed counterparts. Then for $Y_n = Y'_n$, clearly we must have $X_n = X'_n$ and $B_{R_n} = B'_{R'_n}$. Claim 3.3 shows that $X_n = X'_n$ infinitely often with probability 1. Walkers $X_n$ and $X'_n$ will almost surely have a last visit to vertex 1, whence $B_{R_n}$ and $B'_{R'_n}$ converge almost surely. Since they are independent, they converge to the same value with probability 1/2. Thus, $\mathbb{P}(Y_n = Y_n \text{ i.o.}) = 1/2$. □

**Definition 3.8.** Let $H_2$ be a modification of $G$ formed by adding a single edge between vertices 1 and 2 as shown in Figure 3.3.

**Claim 3.9.** Consider two independent random walkers on $H_2$ started at 1. With a discrete time parameter, the two walkers will collide infinitely often with positive probability that is strictly less than 1; with a continuous time parameter, the two walkers will collide infinitely often with probability 1.

*Proof.* The difference between these two cases is one of periodicity. In the discrete time parameter, by the transience of the graph there is a positive probability that neither walker will ever traverse the new edge. If we condition on this event, we see

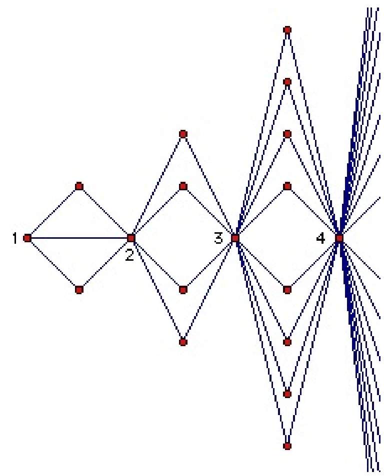


FIGURE 3.3: The graph $H_2$.

that their distribution is equal to that of the simple random walk on $G$; hence, on this event of positive probability, the walkers collide infinitely often. On the other hand, there is a positive probability that one walker will use the edge exactly once and the other will never use it. The walker who traversed the new edge will eventually be on the vertex set $\{1, 2, \dots\}$ only at odd times and on the midpoints of the paths between them only at even times. The other walker will eventually do the opposite.

In the continuous time parameter, the periodicity phenomenon is no longer an obstruction. It should seem intuitively reasonable that the two walks collide infinitely often; what follows is a technical proof.

Our strategy here will be the same as in the proof of Claim 3.4. We wish to relate this walk to one on some discrete subset of $\mathbb{R}$, but will have to adjust some nodes and probabilities to account for the added edge. We maintain the same vertices and transition probabilities on $(-\infty, 0] \cup [\frac{29}{12}, \infty)$ while adjusting those on the interior of $(0, \frac{29}{12})$ in a way that relates to the graph $H_2$ and maintains an expected increment of $\frac{1}{12}$. We will replace vertex 1 with a vertex at $\frac{37}{24}$ and we will replace vertex $\frac{17}{12}$ with vertex $\frac{27}{16}$; all mappings from $G$ will respect these replacements. We will also adjust a few transition probabilities, as follows:

$$\mathbb{P}_{37/24}(Y_1 = j) = \begin{cases} \frac{58}{99}, & j = \frac{27}{16} \\ \frac{4}{33}, & j = \frac{5}{12} \\ \frac{29}{99}, & j = 2 \end{cases}$$

$$\mathbb{P}_{27/16}(Y_1 = j) = \begin{cases} \frac{1}{2}, & j = \frac{37}{24} \\ \frac{1}{2}, & j = 2 \end{cases}$$

$$\mathbb{P}_{2}(Y_1 = j) = \begin{cases} \frac{1}{7}, & j = \frac{37}{24} \\ \frac{2}{7}, & j = \frac{27}{16} \\ \frac{4}{7}, & j = \frac{29}{12} \end{cases}$$

Any vertex whose transition probabilities are not listed above is assumed to have the same probabilities as in the proof of Claim 3.4. Given that a walk started at $\frac{37}{24}$ does not move to the left, it takes one step to the right with probability $\frac{2}{3}$ and it takes two steps to the right with probability $\frac{1}{3}$. Hence, the projections of random walks on $H_2$ onto our new random walk preserves probabilities if we condition that our walk never moves to to the left of $\frac{37}{24}$. There is a nonzero chance that both walks will fail to move to the left of $\frac{37}{24}$, whence we apply the same argument as made in Claim 3.4 to show that two walks on $H_2$ meet infinitely often almost surely. □

**Remark 3.10.** In [BPS10], the authors asked whether adding or removing a finite number of edges and vertices from a graph could alter its collision properties. Graphs $G$ and $H_2$ differ only by a single edge, yet their discrete-time collision properties differ.

**Remark 3.11.** Barlow, Peres and Sousi also remarked that in the discrete-time environment, three independent simple random walks on $\mathbb{Z}$ collide infinitely often. This is false in the continuous-time environment, since the joint distribution of the three independent walks is the same as that of a simple random walk on $\mathbb{Z}^3$, as seen by identifying each independent walk with a dimension in $\mathbb{Z}^3$.

### 3.2 Truncations of the Comb Graph

In [KP04], Krishnapur and Peres proved that on $\text{Comb}(\mathbb{Z}, \mathbb{Z})$, defined as the subgraph of $\mathbb{Z}^2$ with all horizontal edges off the $x$-axis removed, two independent simple random walks collide only finitely often. We will show that for certain truncations of $\text{Comb}(\mathbb{Z}, \mathbb{Z})$, two independent simple random walks collide infinitely often.

**Definition 3.12.** Let $G \subset \text{Comb}(\mathbb{Z})$ be a connected subgraph of the comb that is symmetric about the $y$-axis and does not include any vertices below the $x$-axis. Let $v(G)$ and $e(G)$ represent the vertex and edge sets of $G$, respectively. For a fixed $n$, we define branch $n$ to be the subgraph of $G$ with vertex set $\{(n, y) : (n, y) \in v(G)\}$ and edge set $\{((n, a), (n, b)) : |a - b| = 1\}$. We define the height of branch $n$ to be $H(n) = \sup\{x : (n, x) \in v(G)\}$.

**Remark 3.13.** The assumptions that $G$ must be symmetric about the $y$-axis and have vertices only on or above the $x$-axis are merely for simplicity and are not essential to the proof.

**Lemma 3.14.** Let $\mathcal{F}_n$, $n \geq 0$ be a filtration with $\mathcal{F}_0 = \{A_n, n \geq 1\}$ a sequence of events with $A_n \in \mathcal{F}_n$ for all integers $n \geq 1$. Then

$$\{A_n \text{ i.o.}\} = \left\{ \sum_{n=1}^{\infty} \mathbb{P}(A_n | \mathcal{F}_{n-1}) = \infty \right\} .$$

*Proof.* See Corollary 4.3.2 of [Dur96]. □

**Lemma 3.15.** Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_1| = \infty$ and let $S_n = X_1 + \cdots + X_n$. Let $a_n$ be a sequence of positive numbers with $a_n/n$ increasing. Then $\sum_n \mathbb{P}(|X_1| \geq a_n) < \infty$ implies that $\limsup_n |S_n|/a_n = 0$.

*Proof.* See Theorem 1.8.9 of [Dur96]. □

**Theorem 3.16.** If $G$ has the property that $H(n) \leq C|n|^{1-\epsilon}$ for some $\epsilon > 0$, then $G$ has the infinite collision property.

*Proof.* We let $Y_t$ and $Y_t'$ denote the two (independent) copies of the simple random walk on $G$, both started at $(0, 0)$, and we let $X_t$ and $X_t'$ denote the projection of these walks onto the $x-$axis. We inductively define a sequence of stopping times by $T_0 = 0$ and

$$T_m = \inf \left\{ t : t > T_{m-1} \text{ and } \left[ X_t \neq X_{T_{m-1}} \text{ or } X_t' \neq X_{T_{m-1}}' \right] \right\} .$$

This sequence of times advances each time one of the copies of the walk changes its $x$-coordinate. (It is also possible that both walks change their $x$-coordinates simultaneously.) It is easy to see that each $T_m$ is a stopping time, that $m < n$ implies $T_m < T_n$ almost surely, and that for each $m$, $T_m < \infty$ almost surely.

With these stopping times the process $Z_m = X_{T_m} - X_{T_m}'$ is an unbiased, bounded-increment walk on $\mathbb{Z}$, since at each advancement of $T_m$, one or both of the walkers takes an unbiased step in either a positive or negative direction. As such, $Z_m$ crosses

0 infinitely many times with probability 1. We define a subsequence $\{S_n\} \subset \{T_m\}$ by $S_0 = 0$ and $S_n = \inf\{t : t > S_{n-1} \text{ and } Z_t = 0\}$ to represent the $n^{th}$ occurrence that at least one walker has changed $x$-coordinates and the two $x$-coordinates are now the same.

Let $\{\mathcal{G}_t\}$ be the natural filtration defined by the two walks. Let $A_m$ be the event of a collision in the interval $[T_m, T_{m+1})$. Since $T_m$ is a stopping time, we can define $\mathcal{F}_m = \mathcal{G}_{T_m}$, the usual $\sigma$-algebra corresponding to a stopping time. Our goal will be to provide a lower bound for $\mathbb{P}(A_m|\mathcal{F}_m)$. By Lemma 3.14, to show that $G$ has infinite collisions it suffices to show that

$$\sum_{m=1}^{\infty} \mathbb{P}(A_m|\mathcal{F}_m) = \infty \text{ almost surely.} \tag{3.1}$$

Here, we define some quantities of interest: We note that for each fixed (deterministic) $n$, there is some (random) $p$ such that $S_n = T_p$. We define the random variable $M_n$ to be this $p$. Conversely, for a fixed (deterministic) $m$, depending on the $\omega \in \Omega$ there may or may not be some (random) $r$ such that $S_r = T_m$. We let $G_m$ be the event that $T_m = S_r$ from some $r$, and on the event $G_m$ we define the random variable $N_m$ to be this $r$.

We consider time $S_n$, which is equal to $T_m$ for some (random) $m = M_n \geq n$. At time $S_n$, one walker is positioned on the $x$-axis, and the other is either at or directly above the same location. Suppose that $Y_{S_n}$ lies on the $x-$axis and that $Y'_{S_n}$ lies above the $x-$axis (on the same branch, by the definition of $S_n$). These assumptions are only for clarity of exposition and do not affect the estimates that follow. Because the underlying graph is bipartite, it is not possible for the walks to move 'past' each other without colliding. Hence, for a fixed $m$, provided that $T_m = S_n$ for some $n$, in order for a collision to occur in time interval $[T_m, T_{m+1})$ it is sufficient for $Y_t$ to first choose to step above the $x$-axis and then to visit the extreme high end of the branch before returning to the $x$-axis. The latter condition amounts simply to a Gambler's Ruin consideration. Hence, on the event $G_m$, by hypothesis we have

$$\begin{aligned}\mathbb{P}(A_m|\mathcal{F}_m) &\geq \frac{1}{3}\cdot\frac{1}{1+H(X_{T_m})}\mathbb{1}_{G_m}\\ &\geq \frac{1}{3}\cdot\frac{1}{1+|X_{T_m}|^{1-\epsilon}}\mathbb{1}_{G_m}.\end{aligned} \tag{3.2}$$

We begin with the left side of (3.1) and seek to use the estimate derived in (3.2). Let $B_m$ denote the event $\{|X_{T_m}| \leq \sqrt{2m\log(m)}\}$. Then

$$\begin{aligned}\sum_{m=1}^{\infty}\mathbb{P}(A_m|\mathcal{F}_m) &\geq \sum_{m=1}^{\infty}\frac{1}{3}\cdot\frac{1}{1+|X_{T_m}|^{1-\epsilon}}\mathbb{1}_{G_m}\mathbb{1}_{B_m}\\ &\geq \sum_{m=1}^{\infty}\frac{1}{3}\cdot\frac{\mathbb{1}_{G_m}\mathbb{1}_{m\leq N_m^{2+\epsilon}}}{1+\left(\sqrt{2m\log(m)}\right)^{1-\epsilon}}\mathbb{1}_{B_m}.\end{aligned} \tag{3.3}$$

We note that for any fixed $m$,

$$\mathbb{1}_{G_m} = \sum_{n=1}^{\infty}\mathbb{1}_{m=M_n}$$

and therefore (3.3) and Fubini's Theorem show that

$$\begin{aligned}\sum_{m=1}^{\infty}\mathbb{P}(A_m|\mathcal{F}_m) &\geq \sum_{m=1}^{\infty}\frac{1}{3}\cdot\frac{\mathbb{1}_{m\leq N_m^{2+\epsilon}}}{1+\left(\sqrt{2m\log(m)}\right)^{1-\epsilon}}\mathbb{1}_{B_m}\sum_{n=1}^{\infty}\mathbb{1}_{m=M_n}\\ &\geq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{3}\cdot\frac{\mathbb{1}_{m=M_n}\mathbb{1}_{m\leq N_m^{2+\epsilon}}}{1+\left(\sqrt{2m\log(m)}\right)^{1-\epsilon}}\mathbb{1}_{B_m}.\end{aligned} \tag{3.4}$$

For a fixed $\omega \in \Omega$ and for fixed values $n$ and $m$, the definitions of $N_m$ and $M_n$ imply that $m = M_n$ if and only if $n = N_m$. Therefore, (3.4) can be reframed as

$$\begin{aligned}\sum_{m=1}^{\infty} \mathbb{P}(A_m|\mathcal{F}_m) &\geq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{3} \cdot \frac{\mathbb{1}_{m=M_n}\mathbb{1}_{m\leq n^{2+\epsilon}}}{1+\left(\sqrt{2m\log(m)}\right)^{1-\epsilon}}\mathbb{1}_{B_m} \\ &\geq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \frac{1}{1+\left(\sqrt{2n^{2+\epsilon}\log(n^{2+\epsilon})}\right)^{1-\epsilon}} \\ &\qquad \times \sum_{m=1}^{\infty} \mathbb{1}_{m=M_n}\mathbb{1}_{M_n\leq n^{2+\epsilon}}\mathbb{1}_{B_m}. \end{aligned} \tag{3.5}$$

Now, we wish to consider the three indicator variables; we will first consider $\mathbb{1}_{B_m}$. We recall that $T_m$ represents the sequence of times where either of the two processes $X_t$ or $X_t'$ takes a step. Therefore, $X_{T_m}$ is a delayed random walk, since it does not necessarily take a step at every value $T_m$. We define the variable $N_m$ to be the number of times that $X_t$ (as opposed to $X_t'$) has moved by time $T_m$; clearly, $N_m \leq m$ almost surely. When $X_{T_m}$ does move, it is a simple random walk on $\mathbb{Z}$, so the law of the iterated logarithm implies that with probability 1, it eventually holds that $|X_{T_m}| < \sqrt{2N_m\log(N_m)} \leq \sqrt{2m\log(m)}$. Therefore, as $m \to \infty$, the indicator $\mathbb{1}_{B_m}$ converges to 1 almost surely.

Next, we turn our attention to the first two indicator variables in (3.5). The presence of the first indicator means that for a fixed $m$, there exists $n$ such that $T_m = S_n$. The interpretation is that $m$ is the number of times that the random walks have changed $x$-coordinates, and $n$ is the number of times that the walks have had the same $x$-coordinate. We claim that with probability 1,

$$\{m = M_n\} \subset \{M_n \leq n^{2+\epsilon}\} \quad \text{for sufficiently large } m. \tag{3.6}$$

To see this, we regard $m$ as the number of steps taken by the random walk $X_{T_m} - X'_{T_m}$, and $n$ as the number of its returns to 0. The process $Z_m = X_{T_m} - X'_{T_m}$ is not a true random walk on $\mathbb{Z}$, but rather a random walk that on certain time increments takes two steps simultaneously (corresponding to the occasion that both $X_{T_m}$ and $X'_{T_m}$ change at the same time). Since this $Z_m$ is an accelerated random walk on $\mathbb{Z}$, the lengths of times between its returns to 0 are shorter than those of a standard random walk. This can be seen via a coupling argument where we consider each time value where both $X_t$ and $X'_t$ move simultaneously. On such times, we can insert an additional value into the time index, and we use this new time value to flip a coin to impose an order of the two moves. Doing this transforms the sample paths of the process $Z_m$ to sample paths of a genuine simple random walk. Return times in the simple random walk are then necessarily longer than those of $Z_m$ (almost surely), since extra time increments were added to the process $Z_m$.

For a fixed $n$, we note that the quantity $M_n$ measures the number of steps taken by the process $Z_m$ before its $n^{th}$ visit to 0. We define $P_n = M_n - M_{n-1}$ to denote the length of the $n^{th}$ excursion of $Z_m$ from 0. We wish to compare these quantities to those of a simple random walk, so we define $\alpha_n$ to be the number of steps taken by a simple random walk on $\mathbb{Z}$ before it returns to 0, and we define $\beta_n = \alpha_n - \alpha_{n-1}$ to be the excursion lengths. Our previous analysis shows that we can establish a single probability space on which $M_n, P_n, \alpha_n$, and $\beta_n$ are all defined and for which $P_n \leq \beta_n$ almost surely (and consequently, $M_n \leq \alpha_n$ almost surely). The increments $\beta_n$ are i.i.d. variables for which it is well known (see, for example, [Dur96, Equation 3.4, p. 199]) that there is some constant $C$ such that

$$\mathbb{P}(\beta_1 \geq r) \sim Cr^{-1/2}.$$

Hence, if we set $a_n = n^{2+\epsilon}$, we observe that $\mathbb{P}(\beta_1 \geq a_n) \sim Cn^{-(2+\epsilon)/2}$ is summable. By Lemma 3.15, this implies that $\limsup_n \alpha_n/n^{2+\epsilon} = 0$ almost surely. In particular, with probability 1 it is true that $\alpha_n < n^{2+\epsilon}$ for sufficiently large $n$, and since $M_n \leq \alpha_n$ we conclude that with probability 1, $M_n \leq n^{2+\epsilon}$ for sufficiently large $n$ as well. Since $M_n = m$ if and only if $N_m = n$ and the mapping $n \mapsto M_n$ is increasing, it follows that $\mathbb{1}_{M_n \leq n^{2+\epsilon}} = \mathbb{1}_{m \leq N_m^{2+\epsilon}}$, which also converges to 1 almost surely as $m \to \infty$.

Finally, to handle the leftmost indicator of (3.5), we note that for any fixed $n$, the sum

$$\sum_{m=1}^{\infty} \mathbb{1}_{m=M_n}$$

is equal to 1 (almost surely). Thus, combining all our previous analysis shows that for almost all $\omega \in \Omega$, for sufficiently large $m$ and $n$ we have

$$\mathbb{1}_{B_m} = 1$$

and

$$\mathbb{1}_{m=M_n} \mathbb{1}_{M_n \leq n^{2+\epsilon}} = \mathbb{1}_{m=M_n}.$$

Define $M_\star(\omega)$ and $N_\star(\omega)$ so that these conditions hold when $m \geq M_\star$ and $n \geq N_\star$. Then the sum in (3.5) is bounded by

$$\sum_{m=1}^{\infty} \mathbb{P}(A_m|\mathcal{F}_m) \geq \sum_{n=N_\star}^{\infty} \frac{1}{3} \cdot \frac{1}{C_\epsilon n^{1-\frac{1}{2}\epsilon-\frac{1}{2}\epsilon^2} (\log(n))^{\frac{1}{2}-\frac{1}{2}\epsilon}} \sum_{m=M_\star}^{\infty} \mathbb{1}_{m=M_n}. \tag{3.7}$$

For sufficiently large $n$ (specifically, for $n$ such that $M_n \geq M_\star$), the sum $\sum_{m=M_\star}^{\infty} \mathbb{1}_{m=M_n}$ is equal to 1 and the right-hand side of (3.7) therefore diverges. By Lemma 3.14 the proof is complete. □

We remark here that a theorem similar to Theorem 3.16 was proved in [CWZ08]; there, the authors required that $H(n) \leq |n|^{1/5-\epsilon}$. Likewise, two results stronger than Theorem 3.16 have been shown. In [BPS10], the authors used the Green's function criterion to prove the result for the case where $H(n) \leq C|n|$. Finally, in [CC10], the authors proved the result for the case where $H(n) \leq C|n|\log(|n|)$.

### 3.3 Stability of the Green's Function Criterion

In this section, we will recall the Green's function criterion for sufficiency of infinite collisions of two independent simple random walks on $G$ due to Barlow, Peres, and Sousi in [BPS10]. Our aim is to show that this criterion is preserved under rough isometries of graphs. In particular, this criterion is preserved when performing basic graph operations such as adding or removing finitely many vertices or edges.

In all of the following, any graph $G$ will be assumed to be connected, recurrent, and to have uniformly bounded degree. We will use $v(G)$ to refer to the vertex set of $G$ and $e(G)$ to refer to the (directed) edge set of $G$. Our goal is to develop results for graphs with undirected edges, but our consideration of directed edge sets is a technical convenience. To resolve the distinction, we will require that if $(x, y) \in e(G)$, then $(y, x) \in e(G)$, so that the edge set is a symmetric set of undirected edges.

We will abuse notation by using $d(x)$ to denote the degree of vertex $x$ and $d(x, y)$ to denote the graph distance between vertices $x$ and $y$. We will denote the transition density function by $p(x, y)$, and we will use the symmetric Green's function, i.e. $\mathcal{G}(x, y) := \sum_n p^n(x, y)/d(y)$. An exhaustion of a graph $G$ will be an increasing sequence of finite subgraphs $B_n$ (i.e., $B_n \subset B_{n+1}$) such that $\bigcup_n B_n = G$. We will use $\mathcal{G}_{B_n}(x, y)$ to refer to the Green's function of the walk that is killed upon leaving $B_n$;

that is,

$$\mathcal{G}_{B_n}(x, y) = \sum_{t=0}^{\infty} \mathbb{P}_x \left( X_n = y \text{ and } \tau_{B_n^c} > n \right)$$
$$= \mathbb{E}_x \left[ \text{number of visits to } y \text{ before exiting } B_n \right].$$

**Definition 3.17.** A graph $G$ with a distinguished vertex $o$ satisfies the Green's function criterion if there exists an exhaustion $B_n$ of $G$ and a uniform constant $C < \infty$ such that for all $x \in v(B_n)$,

$$\mathcal{G}_{B_n}(x, x) \leq C\mathcal{G}_{B_n}(o, o).$$

For shorthand, we will say that such a graph satisfies GFC.

**Theorem 3.18.** If $G$ satisfies GFC, then two simple independent random walks on $G$ will collide infinitely often almost surely.

*Proof.* See Theorem 3.1 of [BPS10]. □

The goal of this section is to prove the following theorem:

**Theorem 3.19.** Let $\varphi_1 : v(G) \to v(G')$ be a rough isometry. If $G'$ satisfies GFC, then so also does $G$.

We will prove that rough isometries form an equivalence class on graphs, so it will follow from Theorem 3.19 that for roughly isometric graphs $G$ and $G'$, $G'$ satisfies GFC if and only if $G$ does. Rough isometries are a rather broad class of mappings which preserve the global structure of graphs while potentially radically altering the local structure. In particular, the following can all be realized as corollaries of Theorem 3.19:

**Corollary 3.20.** If $G$ satisfies GFC, then the following graph operations yield a graph which is roughly isometric to $G$ and hence has infinite collisions:

- adding an edge between existing vertices,
- removing an edge between vertices (as long as this removal does not disconnect the graph),
- adding a new vertex and connecting it with a single edge to an existing vertex,
- removing a vertex and any edges connected to it (as long as this removal does not disconnect the graph),
- identifying two vertices and deleting any resulting loops,

or any finite combinations thereof.

*Proof.* This corollary follows from Theorem 3.19 since each of the operations are rough isometries. □

Although the results in Theorem 3.19 and Corollary 3.20 are not necessarily surprising, we remark that the question of whether these operations can change the collision properties in general remain open, even for recurrent, bounded-degree graphs. We note that we provided a transient, unbounded-degree counterexample to this phenomenon in Section 3.1 (see Remark 3.10).

It will take a good deal of machinery to work up to the proof of Theorem 3.19. We will use $\mathscr{R}(A \leftrightarrow B)$ to denote the effective resistance between disjoint sets $A$ and $B$; for a definition of this concept, see Section 9.4 of [LPW09]. In this expression, we will often abuse notation and write $\mathscr{R}(x \leftrightarrow A)$ to mean $\mathscr{R}(\{x\} \leftrightarrow A)$. The following equality, which can be found (for instance) in Lemma 9.6 of [LPW09], will be of use to us:

$$\mathcal{G}_{B_n}(x,x) = \mathscr{R}(x \leftrightarrow B_n^c). \tag{3.8}$$

To analyze these Green's functions, we will use tools that are well-suited for discussing effective resistances. We will recall a number of preliminary notions; these can all be found within Section 2.4 of [LP13]. Let $G$ be any finite graph. The space of functions on the vertices is a real Hilbert space with inner product

$$\langle f, g\rangle = \sum_{x \in v(G)} f(x)g(x)$$

and the space of *antisymmetric* functions on the (directed) edges is a real Hilbert space with inner product

$$\langle \theta, \phi\rangle = \frac{1}{2}\sum_{c \in e(G)} \theta(c)\phi(c).$$

We denote these real Hilbert spaces by $\ell^2(V)$ and $\ell^2_-(E)$, respectively. Next, we define two operators between these spaces. For any directed edge $c$, let $c^+$ denote its head and $c^-$ denote its tail. Let $\nabla : \ell^2(V) \to \ell^2_-(E)$ and $\nabla^* : \ell^2_-(E) \to \ell^2(V)$ be given by

$$(\nabla f)(c) = f(c^-) - f(c^+),$$
$$(\nabla^*\theta)(x) = \sum_{c^- = x} \theta(c)\,.$$

Let $A$ and $Z$ be disjoint subsets of $v(G)$. We call $\theta \in \ell^2_-(E)$ a flow from $A$ to $Z$ if the function $\nabla^*\theta(x)$ is 0 off of $A$ and $Z$, nonnegative on $A$, and nonpositive on $Z$. We say that $\theta$ is a unit flow if $\sum_{a\in A}\nabla^*\theta(a) = 1$; this condition implies that $\sum_{z\in Z}\nabla^*\theta(z) = -1$. If for every cycle $c_1, \ldots, c_n$ of directed edges we have

$\sum_{i=1}^{n} \theta(e_i) = 0$, then we say $\theta$ is the unit current flow. Although this is a different characterization of the unit current flow than is typical in the literature, one can show that it is equivalent to the typical definition; see, for instance, Proposition 9.4 of [LPW09]. For a fixed $A$ and $B$, one can also show that this unit current flow exists and is unique, which are immediate consequences of the typical definition of the unit current flow. We define the energy of a flow by $\mathscr{E}(\theta) = ||\theta||^2 = \langle \theta, \theta \rangle$.

**Remark 3.21.** The inner product of $\ell^2_-(E)$ is often defined with an inclusion of edge conductances or resistances. However, such considerations will not be necessary for our purposes, since we consider only simple random walks which correspond to all edge conductances (and resistances) being 1.

**Remark 3.22.** Our setup so far, as well as many theorems to come, will only explicitly deal with finite graphs. Though our considerations will be on infinite graphs, all applications of theorems will be to the case of random walk that is killed upon exiting some finite set. Consequently, all functions considered in $\ell^2(V)$ and $\ell^2_-(E)$ will be finitely supported, and the theorems will still apply.

There are two main notions we will need to reference throughout the course of this proof. Let $\mathscr{C}(A \leftrightarrow Z) = 1/\mathscr{R}(A \leftrightarrow Z)$; this is the effective conductance between $A$ and $Z$.

**Lemma 3.23** (Thomson's Principle)**.** If $A$ and $Z$ are disjoint vertex sets on a finite graph $G$, then

$$\mathscr{R}(A \leftrightarrow Z) = \min\left\{||\theta||^2\right\}$$

where the minimum is taken over unit flows from $A$ to $Z$. The minimizer in this expression is the unit current flow.

*Proof.* See, for instance, Theorem 9.10 of [LPW09]. □

**Lemma 3.24** (Dirichlet's Principle)**.** If $A$ and $Z$ are disjoint vertex sets on a finite graph $G$, then

$$\mathscr{C}(A \leftrightarrow Z) = \min \left\{ ||\nabla F||^2 \right\}$$

where the minimum is taken over functions $F \in \ell(V)$ for which $F|_A = 1$ and $F|_Z = 0$.

*Proof.* See, for instance, Exercise 2.13 of [LP13]. □

Next, we will define a rough isometry and discuss some basic lemmas.

**Definition 3.25.** Let $\varphi$ be a function from $v(G)$ to $v(G')$. We say that $\varphi$ is a rough isometry if there exist constants $\alpha \geq 1$, $\beta \geq 0$ such that

- for all $x, y \in v(G)$, $\alpha^{-1} d_G(x, y) - \beta \leq d_{G'}(\varphi x, \varphi y) \leq \alpha d_G(x, y) + \beta$, and
- for all $x' \in v(G')$, $d_{G'}(x, \varphi G) \leq \beta$.

We will use prime notation for vertices in $G'$; e.g., $x' \in v(G')$.

**Lemma 3.26.** If $\varphi_1 : v(G) \to v(G')$ is a rough isometry with constants $\alpha, \beta$, then there exists a rough isometry $\varphi_2 : v(G') \to v(G)$ with constants $\alpha, 3\alpha\beta$. Moreover, for any $x \in v(G)$, we have $d_G(x, \varphi_2(\varphi_1 x)) \leq 2\alpha\beta$. We call $\varphi_2$ a rough inverse of $\varphi_1$.

**Remark 3.27.** We will reserve the notation $\varphi^{-1}$ for preimages of sets under $\varphi$. When necessary, we will always distinguish between a rough isometry and its rough inverse by $\varphi_1$ and $\varphi_2$ instead.

*Proof of Lemma 3.26.* For each $x' \in v(G')$, there is at least one $x \in v(G)$ for which $d_{G'}(x', \varphi_1 x) \leq \beta$; choose any one of these and set $\varphi_2 x' = x$. Then for any $x', y' \in v(G')$

and corresponding $x = \varphi_2 x', y = \varphi_2 y'$:

$$\begin{aligned} d_G(x, y) &\le \alpha d_{G'}(\varphi_1 x, \varphi_1 y) + \beta\alpha \\ &\le \alpha[d_{G'}(\varphi_1 x, x') + d_{G'}(x', y') + d_{G'}(y', \varphi_1 y)] + \beta\alpha \\ &\le \alpha d_{G'}(x', y') + 3\alpha\beta. \end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned} d_G(x, y) &\ge \alpha^{-1} d_{G'}(\varphi_1 x, \varphi_1 y) - \beta/\alpha \\ &\ge \alpha^{-1}[d_{G'}(x', y') - d_{G'}(x', \varphi_1 x) - d_{G'}(\varphi_1 y, y')] - \beta/\alpha \\ &\ge \alpha^{-1} d_{G'}(x', y') - 3\beta\alpha^{-1}. \end{aligned} \tag{3.10}$$

Putting (3.9) and (3.10) together gives the first part of the definition. For the second, pick any $x \in v(G)$; we must exhibit some $y' \in v(G')$ for which $d_G(x, \varphi_2 y') \le 3\alpha\beta$. We claim that $\varphi_1 x$ works in place of $y$. Note that

$$d_G(x, \varphi_2(\varphi_1 x)) \le \alpha d_{G'}(\varphi_1 x, \varphi_1(\varphi_2(\varphi_1 x))) + \beta\alpha$$

but that $\varphi_2(\varphi_1 x)$ is by definition some vertex $y$ for which $d_{G'}(\varphi_1 x, \varphi_1 y) \le \beta$. Hence, $d_G(x, \varphi_2(\varphi_1 x)) \le 2\alpha\beta$. This establishes the final statement in the lemma, but is in particular also less than $3\alpha\beta$. □

**Lemma 3.28.** If $\varphi_1 : v(G_1) \to v(G_2)$ and $\varphi_2 : v(G_2) \to v(G_3)$ are both rough isometries, then so also is their composition $\varphi_2 \circ \varphi_1$.

*Proof.* Let $\alpha_1, \beta_1$ denote the constants for $\varphi_1$ and let $\alpha_2, \beta_2$ denote the constants for $\varphi_2$. The constants for the composition will be $\alpha_1\alpha_2$ and $\alpha_2\beta_1 + 2\beta_2$.

First, for any $x, y \in v(G_1)$ we have

$$\begin{aligned} d_{G_3}(\varphi_2\varphi_1 x, \varphi_2\varphi_1 y) &\leq \alpha_2 d_{G_2}(\varphi_1 x, \varphi_2 y) + \beta_2 \\ &\leq \alpha_2 \left[\alpha_1 d_{G_1}(x, y) + \beta_1\right] + \beta_2 \\ &= \alpha_1\alpha_2 d_{G_1}(x, y) + \alpha_2\beta_1 + \beta_2 \end{aligned}$$

and

$$\begin{aligned} d_{G_3}(\varphi_2\varphi_1 x, \varphi_2\varphi_1 y) &\geq \alpha_2^{-1} d_{G_2}(\varphi_1 x, \varphi_1 y) - \beta_2 \\ &\geq \alpha_2^{-1}\left[\alpha_1^{-1} d_{G_1}(x, y) - \beta_1\right] - \beta_2 \\ &= [\alpha_1\alpha_2]^{-1} d_{G_1}(x, y) - \alpha_2^{-1}\beta_1 - \beta_2 \end{aligned}$$

which verifies the first condition.

For the second condition, choose any $z \in v(G_3)$. There is some $y \in v(G_2)$ for which $d_{G_3}(\varphi_2 y, z) \leq \beta_2$; similarly, there is some $x \in v(G_1)$ for which $d_{G_1}(\varphi_1 x, y) \leq \beta_1$. Then,

$$\begin{aligned} d_{G_3}(\varphi_2\varphi_1 x, z) &\leq d_{G_3}(\varphi_2\varphi_1 x, \varphi_2 y) + d_{G_3}(\varphi_2 y, z) \\ &\leq \left[\alpha_2 d_{G_2}(\varphi_1 x, y) + \beta_2\right] + \beta_2 \\ &\leq \alpha_2\beta_1 + 2\beta_2 \end{aligned}$$

which verifies the second condition. □

We say that graphs $G$ and $H$ are roughly isometric if there exists a rough isometry $\varphi : G \to H$.

**Corollary 3.29.** Rough isometry is an equivalence relation on graphs.

*Proof.* From Lemmas 3.26 and 3.28, we obtain the following: □

**Lemma 3.30.** Let $G'$ have a distinguished vertex $o'$. If $\varphi : v(G) \to v(G')$ is a rough isometry, then there exists a rough isometry $\varphi_0 : v(G) \to v(G')$ such that $o' \in \varphi_0(v(G))$.

*Proof.* Let $\alpha, \beta$ be the constants for $\varphi$. There is some $z \in v(G)$ for which $d_{G'}(\varphi z, o') \leq \beta$. Construct a new map $\varphi_0 : v(G) \to v(G')$ by $\varphi_0(x) = \varphi(x)$ for all $x \neq z$, and for which $\varphi_0(z) = o'$. We claim that this map is a rough isometry with constants $\alpha, 2\beta$.

To check the first condition, let $x, y \in v(G)$. If neither $x$ nor $y$ is equal to $z$, then the inequality holds since $\varphi$ is a rough isometry. Hence, it suffices to assume $y = z$. We have

$$\begin{aligned} d_{G'}(\varphi_0 x, \varphi_0 z) &\leq d_{G'}(\varphi_0 x, \varphi z) + d_G(\varphi z, \varphi_0 z) \\ &\leq \alpha d_G(x, y) + \beta + \beta \end{aligned}$$

and on the other side,

$$\begin{aligned} d_{G'}(\varphi_0 x, \varphi_0 z) &\geq d_{G'}(\varphi_0 x, \varphi z) - d_G(\varphi_0 z, \varphi z) \\ &\geq \alpha^{-1} d_G(x, z) - \beta - \beta \end{aligned}$$

as desired.

To check the second condition, choose any $x' \in v(G')$. First, if $d_{G'}(x', o') > 2\beta$, then

$$d_{G'}(x', \varphi z) \geq d_{G'}(x', o') - d_{G'}(\varphi z, o') > 2\beta - \beta$$

whence there is some *other* vertex $y \neq z$ for which $d_{G'}(x', \varphi y) \leq \beta$. Since $\varphi_0(y) = \varphi(y)$ for this vertex, we have $d_{G'}(x', \varphi_0 y) \leq \beta$. Second, if $d_{G'}(x', o') \leq 2\beta$, then since $\varphi_0(z) = o'$, we are done. □

Next, we attempt to bridge the gap between effective resistances and rough isometries.

**Lemma 3.31.** If $G$ and $G'$ are infinite graphs with bounded degree and $\varphi : G \to G'$ is a rough isometry, then there is a universal constant $K > 0$ (depending only on $\varphi$) such that for all $f \in \ell_0(v(G'))$,

$$||\nabla f||_{G'}^2 \geq K||\nabla(f \circ \varphi)||_G^2.$$

*Proof.* See Theorem 3.10 in [Woe00]. □

**Lemma 3.32.** Suppose $\varphi : v(G) \to v(G')$ is a rough isometry. Then there is a constant $M$, depending only on $\varphi$, such that whenever $x \in v(G)$ and $A' \subset v(G')$ satisfy $\varphi(x) \in A'$ and $\varphi^{-1}(A') \subset A$, then $\mathscr{R}(\varphi x \leftrightarrow A'^c) \leq M\mathscr{R}(x \leftrightarrow A^c)$.

*Proof.* Let $K$ be as in Lemma 3.31, and let $x, A, A'$ be as in the hypotheses. By Dirichlet's Principle, $\mathscr{C}(\varphi x \leftrightarrow A'^c) = ||\nabla F||_{G'}^2$ for some function $F$ which is 1 on $\varphi x$ and is 0 on $A'^c$. The function $F \circ \varphi$ is 1 on $x$ and is 0 on $A^c$, since by assumption elements not in $A$ cannot map into $A'$. By Lemma 3.31, there is a constant $K$ depending only on $\varphi$ for which $K||\nabla(F \circ \varphi)||_G^2 \leq ||F||_{G'}^2$. Using Dirichlet's Principle once again shows that

$$K \cdot \mathscr{C}(x \leftrightarrow A^c) \leq K||\nabla(F \circ \varphi)||_G^2 \leq ||\nabla F||_{G'}^2 = \mathscr{C}(\varphi x \leftrightarrow A'^c).$$

and thus,

$$\mathscr{R}(\varphi x \leftrightarrow A'^c) \leq \frac{1}{K}\mathscr{R}(x \leftrightarrow A^c)$$

as desired. □

**Lemma 3.33.** If $A$ and $B$ are disjoint vertex sets in a finite graph $G$ for which there is a path of length $k$ connecting a vertex in $A$ to a vertex in $B$, then $\mathscr{R}(A \leftrightarrow B) \leq k$.

*Proof.* Assume the path to be loopless. Consider the unit flow from $A$ to $B$ which is 1 on the assumed path. This flow has energy $k$, so Thomson's Principle gives the desired result. □

We denote the ball of radius $R$ around the vertex $x$ by $B(x, R) = \{z \in v(G) : d(z, x) \leq R\}$.

**Lemma 3.34.** Let $S$ be a finite, connected set of vertices in $G$, and fix some vertex $o \in S$. Let $L$ be some nonnegative integer for which $B(o, L) \subset S$, and define $T = \{x \in v(G) : B(x, L) \subset S\}$. If $L \leq \mathscr{R}(o \leftrightarrow T^c)$, then

$$\mathscr{R}(o \leftrightarrow S^c) \leq 4\mathscr{R}(o \leftrightarrow T^c).$$

*Proof.* For a set $S$, define its outer boundary $\partial S$ by $\{x \in v(G) : d(x, S) = 1\}$. Note that $\partial S \subset S^c$. By Dirichlet's Principle, we have $\mathscr{C}(x \leftrightarrow S^c) = \mathscr{C}(x \leftrightarrow \partial S)$, since the minimizing function $F$ in $\mathscr{C}(x \leftrightarrow \partial S)$ will be 0 on all of $S^c$. Hence, $\mathscr{R}(x \leftrightarrow S^c) = \mathscr{R}(x \leftrightarrow \partial S)$.

Let $i$ be the unit current flow from $o$ to $\partial T$. For each vertex $t \in \partial T$, we have $\nabla^* i(t) \leq 0$, and collectively they satisfy $\sum_{t \in \partial T} \nabla^* i(t) = -1$. For each $t \in \partial T$, we let

$\theta_t$ denote the unit current flow from $t$ to $\partial S$. We define a new function by

$$\Theta = i + \sum_{t \in \partial T} |\nabla^* i(t)| \cdot \theta_t.$$

We see that $\Theta$ is a unit flow from $o$ to $\partial S$, as follows: for any $s \in \partial T$, the term $\nabla^* \theta_t(s)$ vanishes except when $t = s$, so $\nabla^* \Theta(s) = \nabla^* i(s) + |\nabla^* i(s)| = 0$. We also have $\nabla^* \Theta(o) = \nabla^* i(o) = 1$. For all other $x \in S$ besides $o$ and those in $\partial T$, the terms $\nabla^* i(x)$ and $\nabla^* \theta_t(x)$ are all 0, so $\nabla^* \Theta(x) = 0$ as well.

Note that every element of $\partial T$ is within $L$ steps of $\partial S$; it follows by Lemma 3.33 that $L \geq \mathscr{R}(t \leftrightarrow S^c) = ||\theta_t||^2$. Then by our assumption that

$$L \leq \mathscr{R}(o \leftrightarrow T^c) = ||i||^2$$

we have

$$\begin{aligned} ||\Theta|| &\leq ||i|| + \sum_{t \in \partial T} |\nabla^* i(t)| \cdot ||\theta_t|| \\ &\leq ||i|| + \sqrt{L} \left[ \sum_{t \in \partial T} |\nabla^* i(t)| \right] \\ &= ||i|| + \sqrt{L} \\ &\leq 2||i||. \end{aligned}$$

Thomson's Principle shows that

$$\mathscr{R}(o \leftrightarrow S^c) \leq ||\Theta||^2 \leq 4||i^2|| = 4\mathscr{R}(o \leftrightarrow T^c). \qquad \square$$

**Lemma 3.35.** Let $G$ be finite with $S \subset v(G)$ and let $x, y$ be distinct points not contained in $S$. Set $D = d(x, y)$; then

$$\mathscr{R}(x \leftrightarrow S^c) \leq 4 \max\{\mathscr{R}(y \leftrightarrow S^c), D\}.$$

*Proof.* By Thomson's Principle, there exists $i \in \ell^2_-(E)$ such that $\mathscr{R}(y \leftrightarrow S^c) = ||i||^2$. There is a directed path of length $D$ from $x$ to $y$; define a unit flow $\theta$ from $x$ to $y$ by assigning each edge in the directed path a value of 1. This flow has energy $\mathscr{E}(\theta) = ||\theta||^2 = D$. Adding the two flows gives a unit flow from $x$ to $S^c$, so Thomson's Principle shows that $\mathscr{R}(x \leftrightarrow S^c) \leq ||i + \theta||^2$. But $||i + \theta|| \leq ||i|| + ||\theta|| \leq 2 \max\{||i||, \sqrt{D}\}$, so we have

$$\mathscr{R}(x \leftrightarrow S^c) \leq ||i + \theta||^2 \leq 4 \max\{||i||^2, D\}$$

as desired. □

We arrive now at the proof of the main result.

*Proof of Theorem 3.19.* By Lemma 3.30, we can assume without loss of generality that there is some vertex (call it $o$) for which $\varphi_1 o = o'$. Let $\alpha, \beta$ denote the constants for $\varphi_1$, and let $\varphi_2$ denote a rough inverse of $\varphi_1$ as in Lemma 3.26. Set $L = \alpha(1+5\alpha\beta)$. We note that there exists an $N$ such that $n \geq N$ implies $B(o', L) \subset B_n$, so we assume without loss of generality that $B(o', L) \subset B_1$. Further, since $\mathscr{R}(o' \leftrightarrow (B'_n)^c)$ diverges to infinity, it is eventually more than $L$, so we can assume without loss of generality that $L \leq \mathscr{R}(o' \leftrightarrow (B'_1)^c)$. These two assumptions will eventually allow us to use Lemma 3.34. For each $n$, we define $T'_n$ by $T'_n = \{x' \in v(G') : B(x', L) \subset B'_n\}$. Note that $T'_n$ is also an exhaustion of $G'$. Define an exhaustion of $G$ by $B_n = \varphi_1^{-1}(T'_n)$.

Next, we seek to show that $\varphi_2^{-1}(B_n) \subset B'_n$. Suppose $x' \notin B'_n$; then $d_{G'}(x', T'_n) \geq L$. For any $y \in B_n$, we have $d_{G'}(x', \varphi_1 y) \geq L$. Using the constants for $\varphi_2$ obtained from Lemma 3.26, we see that

$$\begin{aligned} d_G(\varphi_2 x', \varphi_2(\varphi_1 y)) &\geq \alpha^{-1} d_{G'}(x', \varphi_1 y) - 3\alpha\beta \\ &\geq \alpha^{-1} L - 3\alpha\beta \\ &= 1 + 5\alpha\beta - 3\alpha\beta \\ &= 1 + 2\alpha\beta. \end{aligned}$$

However, the second claim in Lemma 3.26 implies that

$$\begin{aligned} d_G(\varphi_2 x', y) &\geq d_G(\varphi_2 x', \varphi_2(\varphi_1 y)) - d_G(y, \varphi_2(\varphi_1 y)) \\ &\geq [1 + 2\alpha\beta] - [2\alpha\beta] \\ &\geq 1. \end{aligned}$$

Thus, $\varphi_2 x' \neq y$, which shows that $\varphi_2^{-1}(B_n) \subset B'_n$, as desired.

Now, let $x \in B_n$. Assume first that $d_G(x, B_n^c) > 3\alpha\beta$. Then there is some $y' \in v(G')$ for which $d_G(x, \varphi_2 y') \leq 3\alpha\beta$; this implies that $\varphi_2 y' \in B_n$. By Lemma 3.35, we have

$$\mathscr{R}(x \leftrightarrow B_n^c) \leq 4 \max\{\mathscr{R}(\varphi_2 y' \leftrightarrow B_n^c), 3\alpha\beta\}. \tag{3.11}$$

Since $\varphi_2^{-1}(B_n) \subset B'_n$, by Lemma 3.32 there is a universal constant $M_2$ depending only on $\varphi_2$ such that

$$\mathscr{R}(\varphi_2 y' \leftrightarrow B_n^c) \leq M_2 \cdot \mathscr{R}(y' \leftrightarrow (B'_n)^c). \tag{3.12}$$

The GFC assumption yields that

$$\mathscr{R}(y' \leftrightarrow (B'_n)^c) \leq C \cdot \mathscr{R}(o' \leftrightarrow (B'_n)^c) \tag{3.13}$$

so putting (3.11), (3.12), and (3.13) together yields

$$\mathscr{R}(x \leftrightarrow B_n^c) \leq 4 \max\{M_2 C \cdot \mathscr{R}(o' \leftrightarrow (B'_n)^c), 3\alpha\beta\}. \tag{3.14}$$

We assumed first that $d_G(x, B_n^c) > 3\alpha\beta$; if this is not true, then by Lemma 3.33 we still have $\mathscr{R}(x \leftrightarrow B_n^c) \leq 3\alpha\beta$, so this estimate holds in either case.

Next, our earlier assumptions about the exhaustion imply that the hypotheses of Lemma 3.34 are satisfied with $S = B'_n$ and $T = T'_n$, so we have

$$\mathscr{R}(o' \leftrightarrow (B'_n)^c) \leq 4\mathscr{R}(o' \leftrightarrow (T'_n)^c). \tag{3.15}$$

Since $B_n$ is the preimage of $T'_n$ under $\varphi_1$, then again by Lemma 3.32 there is some universal constant $M_1$ for which

$$\mathscr{R}(o' \leftrightarrow (T'_n)^c) \leq M_1 \cdot \mathscr{R}(o \leftrightarrow B_n^c) \tag{3.16}$$

since $\varphi_1 o = o'$. Combining (3.14) with (3.15) and (3.16) shows that

$$\mathscr{R}(x \leftrightarrow B_n^c) \leq 4 \max\{4 M_1 M_2 C \cdot \mathscr{R}(o \leftrightarrow B_n^c), 3\alpha\beta\}.$$

As $n \uparrow \infty$, the terms $\mathscr{R}(o \leftrightarrow B_n^c)$ increase to infinity. Since none of the constants $M_1$, $M_2$, or $C$ depends on $x$ or on $n$, there is some constant $K$ so that $K[4M_1M_2C\mathscr{R}(o \leftrightarrow$

$B_n^c)] \geq 3\alpha\beta$ for all $n$. Therefore,

$$\mathscr{R}(x \leftrightarrow B_n^c) \leq 16KM_1M_2C \cdot \mathscr{R}(o \leftrightarrow B_n^c)$$

which, when combined with (3.8), completes the proof. □

### 3.4 Quasi-transitive Graphs

In this section, we provide a complete answer to the question of infinite collisions in the case of quasi-transitive graphs. While this result is not necessarily surprising, it does nevertheless yield Corollary 3.40 as a consequence, which is an improvement to an existing theorem in [CWZ08].

**Definition 3.36.** We say that a graph $G$ is quasi-transitive if there exists a finite set $S = \{v_1, \ldots, v_n\}$ of vertices such that for any $x \in V(G)$, there exists a bijection $\phi : G \to G$ which preserves edge relations and for which $\phi(x) \in S$.

For notation's sake, if $x, y$ are neighboring vertices of $G$, we will write $x \sim y$. We will use $d(x)$ to denote the degree of vertex $x$. If for the map $\phi$ in the definition of quasi-transitivity we have $\phi(x) = v_k$, we will write $x \approx v_k$. If $G$ is quasi-transitive then its graph has uniformly bounded degree; we call this bound $M$. We will abuse notation below when the meaning is clear; for instance, $\mathbb{P}_x$ may refer to starting eiter one or two simple random walks at vertex $x$.

**Theorem 3.37.** If $G$ is quasi-transitive, then two independent continuous-time simple random walks on $G$ will collide infinitely often a.s. if $G$ is recurrent and will collide finitely often a.s. if $G$ is transient.

**Theorem 3.38.** If $G$ is quasi-transitive, then two independent discrete-time simple random walks on $G$ will collide infinitely often a.s. if $G$ is recurrent and will collide finitely often a.s. if $G$ is transient.

**Remark 3.39.** The two claims are separated because the technical details involved in their proofs are slightly different.

*Proof of Theorem 3.37.* Let $X_t$ and $X'_t$ denote the two (independent) random walks started at vertex $o$; let $T$ denote the amount of time the two walks spend at the same vertex, and let $N$ denote the number of meetings between them. We will show that for a quasi-transitive graph $G$,

$$G \text{ is recurrent} \iff \mathbb{E}_o T = \infty \iff \mathbb{E}_o N = \infty \iff N = \infty \text{ a. s.}$$

To show the first implication, we observe that

$$\begin{aligned}\mathbb{E}_o T &= \mathbb{E}_o \left[ \int_0^\infty \sum_{x \in v(G)} \mathbb{1}_{X_t = x} \mathbb{1}_{X'_t = x} \, \mathrm{d}t \right] \\ &= \int_0^\infty \sum_{x \in v(G)} \left[ p^t(o,x) \right]^2 \, \mathrm{d}t \\ &= \int_0^\infty \sum_{x \in v(G)} p^t(o,x) \cdot p^t(x,o) \frac{d(x)}{d(o)} \, \mathrm{d}t \,.\end{aligned}$$

Since $G$ is quasi-transitive, it has bounded degree; suppose that $d(x) \leq M$ for all $x$. Chapman-Kolmogorov shows that

$$\sum_{x \in v(G)} p^t(o,x) p^t(x,o) = p^{2t}(o,o)$$

so we have

$$\frac{1}{d(o)} \cdot \int_0^\infty p^{2t}(o,o)\,\mathrm{d}t \le \mathbb{E}_o T \le \frac{M}{d(o)} \cdot \int_0^\infty p^{2t}(o,o)\,\mathrm{d}t\,.$$

The left and right sides of this expression diverge to infinity if and only if $G$ is recurrent, which establishes that recurrence is equivalent to $\mathbb{E}_o T = \infty$.

Next, let $T_n$ denote the time that the two walks spend together during their $n^{th}$ meeting. Then

$$T = \sum_{n=1}^{N} T_n\,.$$

It is easy to see that the variables $T_n$ are i.i.d., so we have

$$\mathbb{E}_o T = [\mathbb{E}_o N]\,[\mathbb{E}_o T_n]$$

and one easily computes that $\mathbb{E}_o T_n = 1/2$, establishing that $\mathbb{E}_o T = \infty$ if and only if $\mathbb{E}_o N = \infty$.

Finally, we seek to show that $\mathbb{E}_o N = \infty$ if and only if $N = \infty$ almost surely. The implication $(\Leftarrow)$ is obvious. To show $(\Rightarrow)$, suppose that $\mathbb{P}_o(N < \infty) > 0$. This implies that there is some vertex $z$ for which $\mathbb{P}_z(X_n \neq X'_n \text{ for all } n) = \delta_1 > 0$; in words, there is some vertex that has a positive probability of being the site of the last collision. Without loss of generality, suppose that $z \approx v_1$.

For $k, j \in \{1, \ldots, n\}$, define $B_{k,j}$ to be the event that two independent simple random walks started at the same $x \approx v_k$ will have their next meeting at some $y \approx v_j$. In the continuous time parameter environment, we have $\mathbb{P}_x(B_{k,1}) > 0$ for all $k$. This can be seen as follows: there is a path $\gamma$ of length $L$ from $x$ to a vertex $y \approx v_1$. Assume $\gamma$ has no loops. There is a nonzero probability that:

– Immediately after meeting at $x$, the first $L$ steps will all be taken by the walk $X_t$, and these $L$ steps will involve $X_t$ traveling along the path $\gamma$ and ending at $y$.

– The next $L$ steps after that will all be taken by the walk $X_t'$, which will also traverse path $\gamma$ and ending at $y$.

Since $\gamma$ has no loops, the two walkers do not meet any time strictly between the beginning and end of these $2L$ steps. This event has nonzero probability, and there are only finitely many starting state classes $v_k$ to consider, so the probability $\mathbb{P}_x(B_{k,j})$ is uniformly bounded below.

It follows that $\mathbb{P}_x(N = 1) \geq \delta_1\delta_2$ since it is sufficient for the two walkers to meet next at some $y \approx v_1$, and to then never meet again. In particular,

$$\mathbb{P}_x(N > 1) \leq 1 - \delta_1\delta_2 \tag{3.17}$$

for all $x$. Let $T_K$ denote the time of the $K^{th}$ collision. By the Strong Markov Property,

$$\mathbb{P}_o(N \geq K + 2) = \mathbb{E}_o\left[\mathbb{1}_{N \geq K}\mathbb{P}_{X_{T_K}}(N \geq 2)\right] \tag{3.18}$$

and combining (3.17) with (3.18) shows that

$$\mathbb{P}_o(N \geq K + 2) \leq \mathbb{P}_o(N \geq K) \cdot (1 - \delta_1\delta_2) \,.$$

so inductively, we have

$$\mathbb{P}_o(N \geq 2k) \leq (1 - \delta_1\delta_2)^k$$

for any integer $k$. This geometric bound implies that $\mathbb{E}_o[N] < \infty$, which shows that $N = \infty$ a.s. if and only if $\mathbb{E}_o N = \infty$. Since we have already established that $\mathbb{E}_o N = \infty$ if and only if $G$ is recurrent, the proof is complete. □

*Proof of Theorem 3.38.* Let $X_t$, $X'_t$ be two independent discrete-time simple random walks on $G$ started at the same vertex (call it $o$). Let $N$ denote the number of collisions between $X_t$ and $X'_t$; note that

$$N = \sum_{t=1}^{\infty} \sum_{x \in V(G)} \mathbb{1}_{X_t = x} \mathbb{1}_{X'_t = x} \,.$$

We will show that

$$G \text{ is recurrent } \iff \mathbb{E}_o N = \infty \iff N = \infty \text{ a. s.}$$

First, we observe that

$$\begin{aligned} \mathbb{E}_o[N] &= \mathbb{E}_o \left[ \sum_{t=0}^{\infty} \sum_{x \in V(G)} \mathbb{1}_{X_t = x} \mathbb{1}_{X'_t = x} \right] \\ &= \sum_{t=0}^{\infty} \sum_{x \in V(G)} \mathbb{P}_o(X_t = x)^2 \\ &= \sum_{t=0}^{\infty} \sum_{x \in V(G)} \mathbb{P}_o(X_t = x) \cdot \mathbb{P}_x(X_t = o) \cdot \frac{d(x)}{d(o)} \,. \end{aligned}$$

Chapman-Kolmogorov gives

$$\sum_{t=0}^{\infty} \sum_{x \in V(G)} \mathbb{P}_o(X_t = x) \mathbb{P}_x(X_t = 0) = \sum_{t=0}^{\infty} \mathbb{P}_o(X_{2t} = o)$$

and since $\frac{1}{d(o)} \leq \frac{d(x)}{d(o)} \leq \frac{M}{d(o)}$, we have

$$\frac{1}{d(o)} \sum_{t=0}^{\infty} \mathbb{P}_o(X_{2t} = o) \leq \mathbb{E}_o[N] \leq \frac{M}{d(o)} \sum_{t=0}^{\infty} \mathbb{P}_o(X_{2t} = o).$$

The left and right sides diverge to infinity if and only if $G$ is recurrent, so $\mathbb{E}_o N = \infty$ if and only if $G$ is recurrent.

Next, we will again show that if $E_o[N] = \infty$, then $N$ is infinite almost surely. Suppose that $\mathbb{P}_o(N < \infty) > 0$. This implies that there is some vertex $z$ for which $\mathbb{P}_z(X_n \neq X'_n \text{ for all } n) = \delta_1 > 0$; in words, there is some vertex that has a positive probability of being the site of the last collision. Without loss of generality, suppose that $z \approx v_1$.

This proof will not be exactly the same as that of Theorem 3.37. In particular, if we define $B_{k,j}$ to be the event that two independent simple random walks started at $x \approx v_k$ will have their next meeting $y \approx v_j$, it need not be the case that $\mathbb{P}_x(B_{k,1}) \geq 0$ for all $k$. For example, on the quasi-transitive graph shown in Figure 3.4, the event $B_{3,1}$ has probability zero, since two random walks started at a vertex $x \approx v_3$ *must* meet on the next step at the vertex directly below $x$. On that graph, there is no chance that they will meet next at a vertex $y \approx v_1$.

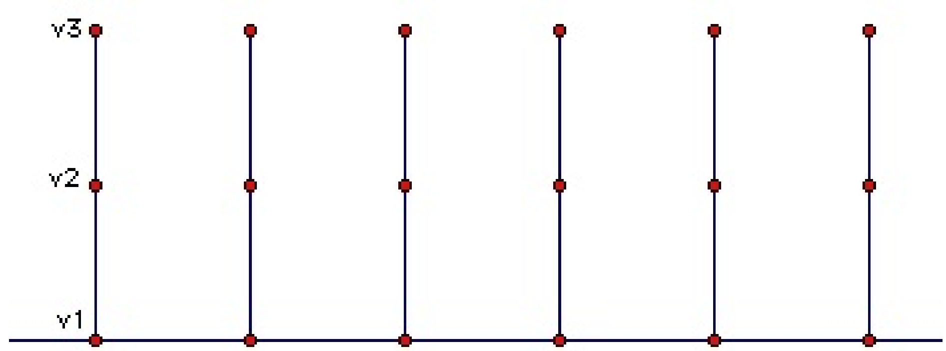


FIGURE 3.4: A certain quasi-transitive truncation of Comb($\mathbb{Z}, \mathbb{Z}$).

To adjust for this, we reconsider a length $L$ loopless path $\gamma : x \to y$ with $x \approx v_k$ and $y \approx v_1$. Suppose that $\gamma = (x, a_1, a_2, \ldots, a_{L-1}, y)$. We specify a length $L+2$ path for $X_t$ by $(x, a_1, a_2, \ldots, a_{L-1}, y, a_{L-1}, y)$ and one for $X'_t$ by $(x, a_1, x, a_1, a_2, \ldots, a_{L-1}, y)$. We observe that after starting, there are exactly three meetings between the two walkers given these paths. Using this analysis, we see that if we redefine $B_{k,j}$ to be the event that two walks started at $x \approx v_k$ have their *third* next meeting at some $y \approx v_j$, that $\mathbb{P}_x(B_{k,j}) \geq \delta_3 > 0$ uniformly.

We now change our bounds to mimic the proof of Theorem 3.37. We have $\mathbb{P}_x(N = 3) \geq \delta_1\delta_3$, since it is sufficient for the two walkers to have their third next meeting at some $y \approx v_1$ and then to never meet again. Thus,

$$\mathbb{P}_x(N > 3) \leq 1 - \delta_1\delta_3 \tag{3.19}$$

for all $x$. Let $T_K$ denote the time of the $K^{th}$ collision. By the Strong Markov Property,

$$\mathbb{P}_o(N \geq K + 4) = \mathbb{E}_o\left[\mathbb{1}_{N\geq K}\mathbb{P}_{X_{T_K}}(N \geq 4)\right] \tag{3.20}$$

and combining (3.19) with (3.20) yields

$$\mathbb{P}_o(N \geq K + 4) \leq \mathbb{P}_0(N \geq K) \cdot (1 - \delta_1\delta_3)$$

so inductively, we have

$$\mathbb{P}_o(N \geq 4k) \leq (1 - \delta_1\delta_3)^k$$

for any integer $k$. Thus, in the discrete time environment we can still establish a geometric bound, which implies that $\mathbb{E}_o[N] < \infty$. Hence, we have preserved the claim that if $\mathbb{E}_o[N] = \infty$, then $\mathbb{P}_o(N < \infty) = 0$. Since $\mathbb{E}_o[N] = \infty$ is equivalent to the recurrence of $G$, the proof is complete. □

**Corollary 3.40.** If $G$ is quasi-transitive, then collision properties do not depend on the time parameter of the walk.

This corollary strengthens Theorem 5 of [CWZ08], which requires the additional assumption that $G$ be of sub-exponential growth.

**Remark 3.41.** In general, it may be the case that discrete-time collision properties and continuous-time collision properties differ, as was demonstrated by counterexample in Claim 3.9.

### 3.5 Quadruple-collisions

All our previous efforts have been directed toward considering the case of two independent simple random walks on some graph $G$. Some work has been done (for example, in [KP04] and [CC10]) on the case of three independent simple random walks on $G$. In such schemes, one considers random walks $X_n^{(1)}, X_n^{(2)}, X_n^{(3)}$ and considers the probability $\mathbb{P}(X_n^{(1)} = X_n^{(2)} = X_n^{(3)} \text{ i.o.})$. In this section, we show that when $G$ is of bounded degree, the analogous problem of four (or more) walkers colliding simultaneously has a trivially negative result.

Let $G$ be a locally finite graph with vertex set $v(G)$, and let $X_t^1, \ldots, X_t^n$ be jointly independent discrete-time simple random walks on $G$ started at some distinguished vertex $o$.

**Theorem 3.42.** On any graph $G$ of bounded degree, four simple independent random walks will collectively meet only finitely many times (a.s.). That is, the set $\{t : X_t^1 = X_t^2 = X_t^3 = X_t^4\}$ is almost surely finite.

**Lemma 3.43.** For any simple random walk on an infinite graph of bounded degree, there exists a constant $C$ such that the transition density satisfies

$$\sup_{x,y} p^{(t)}(x, y) \leq C/\sqrt{t}$$

for $t \geq 1$.

*Proof.* See Corollary 14.6 of [Woe00]. □

*Proof of Theorem 3.42.* Let $M$ denote the maximum degree of all vertices $v \in G$, and let $N = \sum_{t=0}^{\infty} \mathbb{1}\{X_t^1 = X_t^2 = X_t^3 = X_t^4\}$ denote the number of meetings between all four walks strictly after time $t = 0$. In what follows, $\mathbb{E}_o$ will denote that all four walks are started at vertex $o$.

$$\begin{aligned}
\mathbb{E}_o[N] &= \mathbb{E}_o\left[\sum_{t=1}^{\infty} \sum_{y \in v(G)} \prod_{i=1}^{4} \mathbb{1}_{X_t^i = v}\right] \\
&= \sum_{t=1}^{\infty} \sum_{y \in v(G)} \left[p^{(t)}(o, y)\right]^4 \\
&\leq \sum_{t=1}^{\infty} \frac{C^3}{t^{3/2}} \sum_{y \in v(G)} p^{(t)}(o, y)
\end{aligned}$$

The inner sum is equal to 1, so the entire summation is finite. Since the expected number of meetings is finite, the number of meetings is finite almost surely. □

**Corollary 3.44.** If $n \geq 4$, then $n$ simple independent random walks on $G$ will collectively meet only finitely many times (almost surely).

### 3.6 Conclusion

In this chapter, we have explored a number of facets of the study of collisions of simple independent random walks on a graph. We remark here that despite these developments, some of the most basic questions remain open. The following two questions in particular are quite basic, yet answers (positive or negative) have thus far remained elusive:

**Question 3.45.** Let $G, G'$ be connected, bounded-degree, recurrent graphs with the same vertex set such that the edge set of $G'$ differs from that of $G$ by only one edge. Must the quantity $\mathbb{P}(X_n = X_n' \text{ i.o.})$ be the same for both $G$ and $G'$?

**Question 3.46.** Let $G$ be a connected, bounded-degree, recurrent graph. Is the quantity $\mathbb{P}(X_n = X_n' \text{ i.o.})$ independent of whether the ambient time medium is discrete or continuous?

We note that without the assumption of recurrence, each of these questions has a negative answer, as discussed in Section 3.1. However, these counterexamples were particularly messy; they required a graph that was transient, of unbounded degree, and weakly aperiodic. Moreover, the phenomena exhibited by these graphs were not extreme in the sense that while the quantity $\mathbb{P}(X_n = X_n' \text{ i.o.})$ changed, it did not go from 1 to 0 or vice versa; rather, it went from 1 to some number strictly between 0 and 1. As observed in Proposition 2.1 of [BPS10], if the underlying graph is recurrent, then $\mathbb{P}(X_n = X_n' \text{ i.o.}) \in \{0, 1\}$. This would imply that if the answer to either Questions 3.45 or 3.46 were negative, then the change in $\mathbb{P}(X_n = X_n' \text{ i.o.})$ would be from 0 to 1 or vice versa. Such a finding would be conterintuitive, but has not yet been proven to be impossible.

We end with some interesting observations: if $G$ is a graph with vertex and edge sets $v(G)$ and $e(G)$, respectively, then there are a number of reasonable ways that one can define the Cartesian product $G \times G$. We will use the obvious vertex set, i.e. $v(G \times G) = v(G) \times v(G)$. Our first edge set, which is a subset of $v(G \times G) \times v(G \times G)$, will be governed by the edge relation

$$(x, y) \sim (z, w) \Longleftrightarrow x \sim y \text{ and } y \sim w.$$

It can be shown that a simple discrete-time random walk on $G \times G$ with this edge set corresponds to two independent simple discrete-time random walks on $G$. Hence, the collision property $\mathbb{P}(X_n = X'_n \text{ i.o.})$ can be reframed as the probability that the single walk on $G \times G$ enters the diagonal set $\Delta = \{(x, x) : x \in v(G)\}$ infinitely often.

Similarly, we can define the Cartesian product as a network (graph with edge weights called conductances) rather than just a graph. With $v(G \times G) = v(G) \times v(G)$, we define the edge relation by

$$(x, y) \sim (z, y) \Longleftrightarrow x \sim z$$
$$(x, y) \sim (z, w) \Longleftrightarrow y \sim w$$

with conductances $c[(x, y), (z, y)] = d(y)$ and $c[(x, y), (x, w)] = d(x)$. Here, $d$ denotes the degree of a vertex. If a continuous-time random walk is performed on $G \times G$ where steps are taken proportionally to the edge weights, then this walk can be shown to correspond to two simple independent continuous-time walks on the underlying graph $G$. Hence, the collision property can again be reframed as the probability of infinite entry into the diagonal set $\Delta$.

As long as $G$ is either bipartite, or is strongly aperiodic (i.e. every set is part of a short odd cycle), then the two graph constructions are themselves roughly isometric. In the case that $G$ is weakly aperiodic, such as the graph defined in Definition 3.8, then the two constructions are not roughly isometric. We suspect this may be why the graph in Definition 3.8 serves as a counterexample to Question 3.46, but have thus far been unable to find a general proof that verifies this. Similarly, if $G$ and $G'$ are roughly isometric graphs, and both are either bipartite or weakly aperiodic, then one can show that their product graphs $G \times G$ and $G' \times G'$ (using either construction) are roughly isometric. We suspect that weak aperiodicity is what makes the counterexample in Remark 3.10 possible, but have thus far been unable to prove this claim. These observations lead us to believe that Questions 3.45 and 3.46 could potentially have affirmative answers if in addition the graphs are assumed to be bipartite or strongly aperiodic.

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