MODULI SPACES OF TWISTED HERMITE-EINSTEIN CONNECTIONS OVER K3 SURFACES

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DISSERTATION ABSTRACT

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We study the moduli space $\mathcal{M}$ of twisted Hermite-Einstein connections on a vector bundle over a K3 surface X. We show that the universal bundle $\mathcal{U} \to X \times \mathcal{M}$ can be viewed as a family of stable vector bundles over $\mathcal{M}$ parameterized by X, therefore identifying X with a component of a moduli space of sheaves over $\mathcal{M}$. The proof hinges on a new realization of twisted differential geometry that puts untwisted and twisted bundles on equal footing. Moreover, we use this technique to give a new and streamlined proof that $\mathcal{M}$ is nonempty, compact, and deformation-equivalent to a Hilbert scheme of points on a K3 surface, and that the Mukai map $v^{\perp} \to H^2(\mathcal{M}, \mathbb{Z})$ ($v^{\perp}/\mathbb{Z}v$ when $v^2 = 0$) is a Hodge isometry.

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In memory of my grandparents, Daisy and Bob.

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CHAPTER I

INTRODUCTION

This thesis centers around the relationship between a K3 surface X and moduli spaces of stable sheaves over X. This connection has been studied in great detail over the last several decades using techniques from algebraic geometry by Mukai, Yoshioka, O'Grady, Huybrechts and others; see the survey article [Saw16] for a concise summary of the theory of moduli spaces of sheaves.

The goal of this work is to elucidate foundational results in the theory of moduli spaces of sheaves using techniques from complex geometry and analysis. The central theorem linking these two points of view is the Kobayashi-Hitchin correspondence, which relates the existence of Hermite-Einstein metrics to stable holomorphic structures for vector bundles.

When the first Chern class $c_1(E)$ is non-algebraic, we show that E may be viewed as a holomorphic vector bundle twisted by a Brauer class obtained from a B-field coming from $-c_1(E)/\operatorname{rk}(E)$. The main result we prove is the following theorem.

Theorem 1. *Let X be a K3 surface equipped with a hyperkähler metric g and associated Kähler form ω, and let $E \to X$ a C^∞ hermitian vector bundle with hermitian structure h. Assume that the rank and first Chern class of E are coprime, and that the Mukai vector satisfies the inequality $0 \leq v(E)^2 + 2 < 2\operatorname{rk}(E)$. Then the following hold:*

1. *The moduli space $\mathcal{M} := \mathcal{M}_\omega^{HE}(E, h, B)$ of irreducible B-twisted Hermite-Einstein connections on E is nonempty, compact, and deformation-equivalent to a Hilbert scheme of points on a K3 surface;*

2. *There is a twisted universal bundle $\mathcal{U} \to X \times \mathcal{M}$ that induces a Hodge isometry $v^\perp \to H^2(\mathcal{M}, \mathbb{Z})$ when $v^2 \neq 0$, and $v^\perp/\mathbb{Z}v \to H^2(\mathcal{M}, \mathbb{Z})$ when $v^2 = 0$;*

3. *$\mathcal{U}$ is a family of twisted stable bundles on $\mathcal{M}$ parameterized by X, with stability taken with respect to the hyperkähler structure induced on $\mathcal{M}$ by the hyperkähler metric g. This family identifies X with a connected component of the moduli space of stable sheaves on $\mathcal{M}$ with the topological type of $\mathcal{U}|_{\{x\}\times\mathcal{M}}$.*

Results (1) and (2) are not new, but the proof we give is dramatically shorter than proofs in the literature, which span several papers and often include various restrictions on the rank and first Chern class.

The key to the new proof lies in putting vector bundles twisted by a topologically trivial Brauer class on the same footing as untwisted bundles. An untwisted holomorphic bundle can be understood as a topological bundle with a Dolbeault operator satisfying $\overline{\partial}^2 = 0$; we show that a twisted holomorphic bundle twisted by a topologically trivial Brauer class can also be encoded by a Dolbeault-type operator satisfying $\overline{\partial}^2 = 2\pi i B^{0,2}$. This also allows us to encode twisted Hermite-Einstein connections on a twisted bundle as an ordinary, untwisted connection on an untwisted topological bundle that solves a variant of the classical Hermite-Einstein equations. Chapter III details this correspondence further.

The beauty of this method is that we can deform an untwisted bundle into a twisted bundle and vice-versa, allowing for a faster, more conceptual deformation to a Hilbert scheme. There are five essential operations in deforming the moduli space $\mathcal{M}$ to the Hilbert scheme: tensoring with a line bundle; deforming the polarization; deforming across a twistor family; the Kobayashi-Hitchin correspondence; and a spherical twist. Chapter IV pins down these steps more precisely and contains the proof of part (1) of the Theorem.

In Chapter V we construct the universal bundle $\mathcal{U}$ and address parts (2) and (3) of this theorem. We make use of the deformation to the Hilbert scheme from Chapter IV along with the explicit analytic formulas granted from working over a moduli space of connections.

CHAPTER II

PRELIMINARIES

Differential Geometry of Twisted Vector Bundles

In this section I lay out the foundations of classical differential-geometric structures on twisted vector bundles, which are slightly more subtle than their untwisted counterparts. Most of the material in this section can be found in Perego's recent paper on the twisted Kobayashi-Hitchin correspondence [Per19]; it is presented here because in Chapter III we will see a vast simplification of these definitions in the case of a topologically trivial Brauer class. The standard reference for the theory of twisted sheaves is Căldăraru's thesis [C0̆0].

The definitions of twisted geometric structures mimic the classical definitions, but because twisted bundles do not satisfy the cocycle condition one usually needs to include extra Čech data to get well-defined structures. The primary application of this section will be to a twisted bundle on a K3 surface X; however, in what follows it is no extra cost to work on general complex manifolds.

Let X be a connected complex manifold and $\alpha \in H^2(X, \mathcal{O}_X^*)_{\text{tors}}$ be a Brauer class. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be a good open cover, which is a cover of X by analytic open sets U_i whose finite intersections $U_{ij\ldots k} = U_i \cap U_j \cap \cdots \cap U_k$ are empty or contractible. Also let $\{\alpha_{ijk}\} \in \check{C}^2(\mathfrak{U}, \mathcal{O}^*)$ be a fixed cocycle representative of α.

Definition 2.1.1. A $\{\alpha_{ijk}\}$-*twisted sheaf* $\mathscr{F}$ is a collection of sheaves of $\mathcal{O}$-modules F_i over U_i together with isomorphisms

$$\varphi_{ij} \in \operatorname{Hom}_{\mathcal{O}}(F_j|_{U_{ij}}, F_i|_{U_{ij}})$$

that satsify the twisted cocycle condition

$$\varphi_{ij}\varphi_{jk}\varphi_{ki} = \alpha_{ijk} \cdot 1_{\mathscr{F}}$$

along with the usual conditions $\varphi_{ii} = 1_{F_i}$ and $\varphi_{ij}^{-1} = \varphi_{ji}$. If F_i are coherent sheaves, we call $\mathscr{F}$ coherent, and if F_i are all vector bundles, then we call $\mathscr{F}$ a twisted bundle.

We often relax the notation and refer to these as α-twisted sheaves and bundles. A homomorphism $f\colon \mathscr{E} \to \mathscr{F}$ of twisted sheaves is a sheaf homomorphism $f_i\colon E_i \to F_i$ that intertwine with the gluing maps φ_{ij}.

For the case of a twisted vector bundle $\mathscr{E} = \{E_i, \varphi_{ij}\}$ we allow the local bundles E_i and the transition matrices φ_{ij} to be smooth maps; thus, twisted bundles by convention need not come equipped with a holomorphic structure. We will have more to say in Section 2.1 below.

It is a simple verification that if $\mathscr{F}$ is α-twisted, then the bundle of endomorphisms $\mathrm{End}(\mathscr{F})$ is an untwisted sheaf. More generally, the Hom sheaf $\mathscr{H}om(\mathcal{E}, \mathcal{F})$ from an α_1-twisted sheaf $\mathcal{E}$ to an α_2-twisted sheaf $\mathcal{F}$ is an $\alpha_2\alpha_1^{-1}$-twisted sheaf.

Twisted Connections and Hermitian Metrics

Let $\mathscr{E} = \{E_i, \varphi_{ij}\}$ be an α-twisted vector bundle.

Definition 2.1.2. A *connection* ∇ on $\mathscr{E}$ is a collection $\{\nabla_i, \eta_{ij}\}$ where:

1. ∇_i is a connection on E_i;

2. η_{ij} is a C^∞ complex 1-form on U_{ij};

3. on the overlap U_{ij} we have

$$\nabla_j = \varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}};$$

4. for every i, j, k we have

$$d\log(\alpha_{ijk}) = \eta_{ij} + \eta_{jk} + \eta_{ki}$$

Conditions 3 and 4 look mysterious at first. To understand them, one could first try to define a connection on $\mathscr{E}$ by the usual transformation law $\nabla_j = \varphi_{ij}^* \nabla_i$. However, one would need that $\nabla_i = (\varphi_{ij}\varphi_{jk}\varphi_{ki})^* \nabla_i$, which in general will not be true if the φ_{ij} do not satisfy the usual cocycle condition $\varphi_{ij}\varphi_{jk}\varphi_{ki} = 1$. Instead, with the α-twisted cocycle condition one has

$$(\varphi_{ij}\varphi_{jk}\varphi_{ki})^* \nabla_i = (\alpha_{ijk}\, 1_{\mathscr{E}})^* \nabla_i = \nabla_i + d\log(\alpha_{ijk}) \cdot 1_{\mathscr{E}}\,.$$

Expanding the left using condition 3 would imply that condition 4 is necessary. Note that $\{d\log \alpha_{ijk}\}$ is a Čech 2-cocycle for the sheaf of smooth 1-forms on X, which is acyclic because it admits partitions of unity. Condition (4) says that we have an explicit representation of $\{d \log \alpha_{ijk}\}$ as a coboundary.

In a similar spirit one can define a hermitian structure on a twisted vector bundle as a collection of hermitian metrics H_i on each E_i with the usual formula $H_j = \varphi_{ij}^T H_i \bar{\varphi}_{ij} = \varphi_{ij}^* H_i$, but again one finds an inconsistency with the usual definition: applying this relation three times one finds

$$H_i = (\varphi_{ij}\varphi_{jk}\varphi_{ki})^T H_i \overline{(\varphi_{ij}\varphi_{jk}\varphi_{ki})} = |\alpha_{ijk}|^2 H_i.$$

Agian, the solution is to introduce additional data to fix this inconsistency. The right choice is to multiply one side of the transformation law by a non-vanishing real-valued function ρ_{ij} (defined on U_{ij}) that satisfies

$$\rho_{ij}\rho_{jk}\rho_{ki} = |\alpha_{ijk}|^{-2},$$

which leads us to the next definition.

Definition 2.1.3. A *hermitian structure* on a twisted vector bundle $\mathscr{E} = \{E_i, \varphi_{ij}\}$ is a collection $h = \{H_i, \rho_{ij}\}$ where each H_i is hermitian metric on E_i and ρ_{ij} is a positive-valued function on U_{ij} such that the two equalities hold for all i, j, k:

$$H_j = \rho_{ij}\varphi_{ij}^T H_i \bar{\varphi}_{ij} \tag{2.1.1}$$

$$|\alpha_{ijk}|^{-2} = \rho_{ij}\rho_{jk}\rho_{ki}. \tag{2.1.2}$$

Remark 2.1.4. If h is a hermitian structure on $\mathscr{E}$, then $\mathrm{End}(\mathscr{E})$ inherits an untwisted hermitian structure. This is due to the fact that locally $\mathrm{End}(\mathscr{E}) \cong E_i \otimes E_i^\vee$ inherits the induced hermitian structure $H_i \otimes (H_i^T)^{-1}$. It then follows that the factor ρ_{ij} disappears from equation (2.1.1) in the above definition (with $H_i \otimes (H_i^T)^{-1}$ in place of H_i).

Definition 2.1.5. A connection $\nabla = \{\nabla_i, \eta_{ij}\}$ is hermitian with respect to $h = \{H_i, \rho_{ij}\}$ if each ∇_i is hermitian with respect to the local metrics H_i.

In this thesis I will consistently use the notation $\tilde{\nabla}$ for the connection induced on $\mathrm{End}(\mathscr{E})$ by a connection ∇ on a bundle $\mathscr{E}$. A quick calculation shows that for any 1-form $A \in \mathcal{A}^1(X)$ the action of the induced connection $(\nabla + A \cdot 1_{\mathscr{E}})^{\sim}$ is related to $\tilde{\nabla}$ by

$$(\nabla + A \cdot 1_{\mathscr{E}})^{\sim} R = \tilde{\nabla}(R) - (A + \bar{A})R, \qquad R \in \mathcal{A}^0(\mathrm{End}(E)) \tag{2.1.3}$$

which will be useful in the next lemma.

Lemma 2.1.6. *Suppose* $\nabla = \{\nabla_i, \eta_{ij}\}$ *is a hermitian connection on* $\mathscr{E}$ *with respect to* $\{H_i, \rho_{ij}\}$. *Then* $2\,\mathrm{Re}(\eta_{ij}) = -d \log \rho_{ij}$.

Proof. Assume ∇_i is hermitian with respect to H_i. Using equation (2.1.3) on the connection $\varphi_{ij}^* \nabla_i$ on E_j with the 1-form η_{ij} and the endomorphism H_j, we find

$$\begin{aligned}
(\varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}})^{\sim}(H_j) &= (\varphi_{ij}^* \nabla_i)(H_j) + (\eta_{ij} + \bar{\eta}_{ij})H_j \\
&= (\varphi_{ij}^* \nabla_i)(\rho_{ij} \varphi_{ij}^* H_i) + (\eta_{ij} + \bar{\eta}_{ij})H_j \\
&= d\rho_{ij} \cdot \varphi_{ij}^* H_i + (\eta_{ij} + \bar{\eta}_{ij})H_j \\
&= (\rho_{ij}^{-1} d\rho_{ij})H_j + (\eta_{ij} + \bar{\eta}_{ij})H_j.
\end{aligned}$$

However, $\varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}} = \nabla_j$, and since $\tilde{\nabla}_j(H_j) = 0$, we must have $2\,\mathrm{Re}(\eta_{ij}) = -d \log \rho_{ij}$ as required. □

Curvature and B-fields

Suppose that $\nabla = \{\nabla_i, \eta_{ij}\}$ is a twisted connection on $\mathscr{E}$. One would hope that the local curvatures $\{F_{\nabla_i}\}$ could be glued into a global 2-form, but a quick calculation shows this is not the case. We will apply the well-known formula $F_{\nabla+\gamma} = F_{\nabla} + \tilde{\nabla}(\gamma) + \gamma \wedge \gamma$ for the curvature of a connection shifted by $\gamma \in \mathcal{A}^1(\mathrm{End}(E))$. Using this and the Liebniz rule we see

$$\begin{aligned}
F_{\nabla_j} &= F_{\varphi_{ij}^* \nabla_i} - (\varphi_{ij}^* \nabla_j)^{\sim}(\eta_{ij} \cdot 1_{\mathscr{E}}) + (\eta_{ij} \cdot 1_{\mathscr{E}}) \wedge (\eta_{ij} \cdot 1_{\mathscr{E}}) \\
&= \varphi_{ij}^* F_{\nabla_i} - (d\eta_{ij} \cdot 1_{\mathscr{E}} - \eta_{ij}(\varphi_{ij}^* \nabla_i)^{\sim}(1_{\mathscr{E}})) \\
&= \varphi_{ij}^* F_{\nabla_i} - d\eta_{ij} \cdot 1_{\mathscr{E}}
\end{aligned}$$

To solve this issue observe that $\{d\eta_{ij}\}$ is a Čech 1-cocycle for the sheaf of 2-forms, which is also an acyclic sheaf. Thus, $\{d\eta_{ij}\}$ is expressible as a coboundary via some 0-cochain $\{B_i\}$ for the sheaf of 2-forms, which allows us to write $B_i - B_j = d\eta_{ij}$. Then we have

$$\begin{aligned} F_{\nabla_j} + B_j \cdot 1_{\mathscr{E}} &= \varphi_{ij}^*(F_{\nabla_i} + B_i \cdot 1_{\mathscr{E}}) + (d\eta_{ij} - B_i + B_j) \cdot 1_{\mathscr{E}} \\ &= \varphi_{ij}^*(F_{\nabla_i} + B_i \cdot 1_{\mathscr{E}}). \end{aligned}$$

This shows that the local 2-forms $\{F_{\nabla_i} + B_i \cdot 1_{\mathscr{E}}\}$ glue into a global 2-form valued in $\mathrm{End}(\mathscr{E})$.

Definition 2.1.7. We call a collection of 2-forms $B = \{B_i\}$ satisfying $B_i - B_j = d\eta_{ij}$ a *B-field compatible with* ∇. Similarly, we call the global 2-form $\tilde{F}_\nabla$, given on U_i by $\tilde{F}_{\nabla_i} := F_{\nabla_i} + B_i \cdot 1$, the *curvature with respect to* B, or the B-curvature.

Note that $\tilde{F}_\nabla$ depends on the choice of B-field. To emphasize this dependence I will write $\tilde{F}_B$ or $\tilde{F}_{\nabla,B}$. If $\nabla = \{\nabla_i, \omega_{ij}\}$ is a hermitian connection and B is a B-field consisting purely imaginary 2-forms compatible with ∇, then the B-curvature $\tilde{F}_{\nabla,B}$ is a 2-form with values in $\mathrm{End}(\mathscr{E}, h)$, the bundle of skew-hermitian endomorphisms of $\mathscr{E}$.

Holomorphic Structures

Recall that if E is an ordinary complex vector bundle over X, then a holomorphic structure on E is determined by a Dolbeault operator $\bar{\partial}_E \colon \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ that satisfies $\bar{\partial}_E^2 = 0$, which is the content of the Newlander-Nirenberg theorem. One source of such operators are connections ∇ satisfying $F_\nabla^{0,2} = 0$, and the corresponding Dolbeault operator is $\bar{\partial}_E := \nabla^{0,1}$. To go backwards, fix a hermitian structure h on E and a Dolbeault operator $\bar{\partial}_E$; then one has the Chern connection ∇, which is the unique h-hermitian connection with $\nabla^{0,1} = \bar{\partial}_E$. This story can be generalized to the setting of α-twisted complex bundles.

Definition 2.1.8. We say an α-twisted bundle $\mathscr{E} = \{E_i, \varphi_{ij}\}$ is *holomorphic* if E_i are holomorphic bundles and the φ_{ij} are holomorphic sections of $\mathrm{Hom}(E_j, E_i)$.

Next we discuss holomorphic structures in terms of Dolbeault operators.

Proposition 2.1.9. *If $\mathscr{E} = \{E_i, \varphi_{ij}\}$ is an α-twisted holomorphic vector bundle, then there is a twisted connection $\nabla = \{\nabla_i, \eta_{ij}\}$ such that $\eta_{ij}^{0,1} = 0$ and $\nabla_i^{0,1} = \bar{\partial}$. Conversely, if $\mathscr{E}$ is a complex C^∞ twisted bundle that admits a connection ∇ with $\eta_{ij}^{0,1} = 0$ and $(\nabla_i^{0,1})^2 = 0$, then ∇ furnishes a holomorphic structure on $\mathscr{E}$.*

Proof. Start by fixing an index i. On the bundle E_i we can find a connection ∇_i such that $(\nabla_i^{0,1})^2 = 0$. Now, for all j, choose $(1,0)$-forms η_{ij} such that

$$\eta_{ij} + \eta_{jk} + \eta_{ki} = \partial \log(\alpha_{ijk})$$

which can be done using a partition of unity. (Note α_{ijk} is holomorphic, so d may be replaced by ∂). Then, for any other j such that $U_{ij} \neq \emptyset$, *define* ∇_j by the formula

$$\nabla_j = \varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1 \,.$$

Then, since φ_{ij} is holomorphic, $(\varphi_{ij}^* \nabla_i)^{0,1} = \bar{\partial}$. Together with the assumption that $\eta_{ij}^{0,1} = 0$, we conclude that $\nabla_j^{0,1} = \bar{\partial}$.

We can keep extending like this as long as U_j overlaps with an open set U_k where ∇_k has already been defined. If there is a U_j which does not meet any U_k with ∇_k defined, then choose a connection ∇_j such that $\nabla_j^{0,1}$ is the Dolbeault operator. This proves the first statement. Now assume that $\mathscr{E}$ admits a connection $\nabla = \{\nabla_i, \eta_{ij}\}$ where η_{ij} is type $(1,0)$ and $(\nabla_i^{0,1})^2 = 0$. The Newlander-Nirenberg theorem implies that each E_i is a holomorphic bundle over U_i and that $\nabla_i^{0,1} = \bar{\partial}_{E_i}$. Furthermore, the compatibility on overlaps gives

$$\nabla_j^{0,1} = (\varphi_{ij}^* \nabla_i)^{0,1} - \eta_{ij}^{0,1} \cdot 1_{\mathscr{E}} = (\varphi_{ij}^* \nabla_i)^{0,1}. \tag{2.1.4}$$

In order to see that φ_{ij} is holomorphic we expand $\nabla_i = d + \Gamma_i$ in local coordinates on U_i (here Γ_i is the connection matrix of 1-forms). Since $\nabla_i^{0,1} = \bar{\partial}$, we know $\Gamma_i^{0,1} = 0$. Writing out equation (2.1.4) applied to a local section s of E_j gives

$$\bar{\partial}(s) = \left(\varphi_{ij}^{-1} \nabla_i(\varphi_{ij}(s)) \right)^{0,1} \tag{2.1.5}$$

$$= \left(\varphi_{ij}^{-1} d(\varphi_{ij}(s)) + \varphi_{ij}^{-1} \Gamma_i \varphi_{ij}(s) \right)^{0,1} \tag{2.1.6}$$

$$= \varphi_{ij}^{-1} \bar{\partial}(\varphi_{ij}(s)) \tag{2.1.7}$$

$$= \varphi_{ij}^{-1} \bar{\partial}(\varphi_{ij}) s + \bar{\partial}(s). \tag{2.1.8}$$

From this we see $\bar{\partial}(\varphi_{ij}) = 0$, so φ_{ij} is indeed holomorphic. This proves the second assertion. □

Definition 2.1.10. We will call a connection ∇ as in Proposition 2.1.9 an *integrable connection.*

Next we discuss the Chern connection for a hermitian twisted holomorphic bundle. For a detailed account of Chern connections on untwisted bundles, see sections V.10–12 of Demailly's book [Dem].

Proposition 2.1.11. *Suppose that $\mathscr{E}$ is a holomorphic α-twisted vector bundle equipped with a hermitian structure $h = \{H_i, \rho_{ij}\}$. Then there exists a unique integrable hermitian connection ∇ on $\mathscr{E}$, called the* Chern connection.

Proof. On each U_i we form the Chern connection ∇_i with respect to H_i. We set $\eta_{ij} := -\partial \log \rho_{ij}$, which will ensure the compatibility from Lemma 2.1.6 holds. Using that φ_{ij} is holomorphic and $\eta_{ij}^{0,1} = 0$ we have

$$(\varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}})^{0,1} = \varphi_{ij}^*(\bar{\partial}) = \bar{\partial}.$$

Also, following a similar calculation to the one in the proof of Lemma 2.1.6, we have

$$(\varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}})^{\sim}(H_j) = (d \log \rho_{ij}) H_j - (\partial \log \rho_{ij} + \bar{\partial} \log \rho_{ij}) H_j = 0.$$

which relies on our choice of η_{ij}. This shows that $\varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}}$ is hermitian with respect to H_j. Therefore $\varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}}$ is a Chern connection on E_j for H_j. The uniqueness of the Chern connection for H_j shows that

$$\nabla_j = \varphi_{ij}^* \nabla_i - \eta_{ij} \cdot 1_{\mathscr{E}}$$

by uniqueness of the Chern connection on E_j. This finishes the construction of the Chern connection. Uniqueness follows by the local uniqueness of the Chern connections ∇_i. □

As noted earlier, if B is a purely imaginary B-field, then the B-curvature $\tilde{F}_{\nabla,B}$ is a section of $\mathcal{A}^2(\mathrm{End}(\mathscr{E}, h))$. Since F_{∇_i} has Hodge type $(1,1)$, the Hodge type of $\tilde{F}_{\nabla,B}$ is the same as B.

Hermite-Einstein Connections

Assume that X is now a connected compact Kähler manifold with Kähler metric g and Kähler form ω. Let $L_\omega \colon \bigwedge^* T_{\mathbb{C}}^* M \to \bigwedge^{*+2} T_{\mathbb{C}}^* M$ be the Lefschetz operator and let $\Lambda_\omega \colon \bigwedge^* T_{\mathbb{C}}^* M \to \bigwedge^{*-2} T_{\mathbb{C}}^* M$ be its adjoint.

Definition 2.1.12. A hermitian metric $h = \{H_i, \rho_{ij}\}$ and B-field B on a holomorphic α-twisted bundle $\mathscr{E}$ is called *Hermite-Einstein* if the B-curvature of the associated Chern connection ∇ satisfies the equation

$$i\Lambda_\omega \tilde{F}_{\nabla,B} = \lambda \cdot 1_{\mathscr{E}}$$

This equation can be rewritten as

$$i\tilde{F}_{\nabla,B} \wedge \omega^{n-1} = \frac{1}{n}\lambda\,\omega^n \cdot 1_{\mathscr{E}}$$

by using the Kähler identity $[L_\omega^n, \Lambda_\omega] = nL_\omega^{n-1}$ on 2-forms. Both versions of the Hermite-Einstein equations will be useful at different points throughout this thesis.

The importance of this definition is the Kobayashi-Hitchin correspondence, which will be reviewed in the next section.

Stability for Twisted Bundles

This section defines the notion of stability for twisted bundles on a Kähler surface X with Kähler form ω.

Definition 2.2.1. An α-twisted holomorphic vector bundle $\mathscr{E}$ is ω-*(semi)stable* if for all α-twisted subsheaves $\mathscr{F} \subset \mathscr{E}$,

$$\int_X c_1(\mathrm{Hom}(\mathscr{F}, \mathscr{E})) \wedge \omega\, (\geq)\, 0.$$

Since $\mathscr{E}$ and $\mathscr{F}$ are both twisted by the same amount, $\mathrm{Hom}(\mathscr{E}, \mathscr{F})$ is an untwisted sheaf, so our definition of stability is independent of how Chern classes are defined for twisted sheaves. Also, if $\mathscr{E}$ and $\mathscr{F}$ are untwisted, then this definition reduces to usual ω-slope stability by recalling that

$$c_1(\mathrm{Hom}(F, E)) = \mathrm{rk}(F)c_1(E) - \mathrm{rk}(E)c_1(F).$$

Remark 2.2.2. On surfaces one can replace subsheaves by subbundles. Indeed, by [HL10, Prop. 1.2.6] it suffices to work only with saturated subsheaves of $\mathscr{E}$. Now, if $\mathscr{F}$ is a saturated subsheaf, then [Kob14, Cor. V.5.20] implies that $\mathcal{F}$ is reflexive, and reflexive sheaves on surfaces are vector bundles since their singular locus has complex codimension at least 3.

Slope stable twisted bundles are linked to Hermite-Einstein connections via the Kobayashi-Hitchin correspondence. The classical version of the Kobayashi-Hitchin correspondence began with Narasimhan and Seshadri in [NS64]. In [Don85], Donaldson extended this theory to complex algebraic surfaces, and later Uhlenbeck and Yau [UY86] extended the result to arbitrary compact Kähler manifolds. This story continues with Wang [Wan12], who proved the correspondence holds for twisted bundles. His methods use the theory of gerbes, which are an alternative way of working with twisted sheaves. More recently, Perego [Per19] wrote a proof of the twisted Kobayashi-Hitchin correspondence that closely follows the original of Uhlenbeck and Yau using the more approachable language of Čech cocycles, which is the style adopted in this thesis.

Theorem 2.2.3 (Donadlson, Uhlenbeck, Yau, Wang, Perego)**.** *A twisted holomorphic bundle $\mathscr{E}$ over (X, ω) is ω-stable if and only if it admits an irreducible Hermite-Einstein metric.*

A connection ∇ on $\mathscr{E}$ is *irreducible* if the induced connection

$$\tilde{\nabla}\colon \mathcal{A}^0(\mathrm{End}(\mathscr{E}, h)) \to \mathcal{A}^1(\mathrm{End}(\mathscr{E}, h))$$

has kernel consisting of the constant endomorphisms $c \cdot 1_{\mathscr{E}}$. (Recall that $\mathrm{End}(\mathscr{E})$ and its real subbundle $\mathrm{End}(\mathscr{E}, h)$ are untwisted bundles.) This notion is equivalent to requiring that the holonomy group of E is irreducible, and to the underlying holomorphic bundle being simple; see [Kob14, Prop VII.4.14]. Later in Proposition 4.3.4 I describe how this correspondence works on the level of moduli spaces.

Hyperkähler Structures

This section introduces the notion of hyperkähler structures on a K3 surface. In this thesis a K3 surface is a simply-connected compact complex manifold of dimension 2 with trivial canonical bundle. For background on K3 surfaces I recommend [Huy16]. All K3 surfaces are diffeomorphic to each other, so different K3 surfaces can be viewed as different complex structures on the same smooth manifold M, which can be taken to be the smooth manifold underlying the Fermat quartic

$$\left\{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\} \subset \mathbb{C}P^3.$$

Definition 2.3.1. A *hyperkähler structure* on the K3 manifold M is a triple of real two-forms $\varpi = (\omega_I, \omega_J, \omega_K)$ such that

1. $d\omega_I = d\omega_J = d\omega_K = 0$
2. $\omega_\alpha \wedge \omega_\beta = 0$ for $\alpha \neq \beta \in \{I, J, K\}$
3. $\omega_I^2 = \omega_J^2 = \omega_K^2 > 0$ as 4-forms on M.

From such a triple, we can form a complex 2-form $\sigma = \omega_J + i\omega_K$, which satisfies the relations

$$\sigma \wedge \sigma = 0, \quad \sigma \wedge \bar{\sigma} > 0, \quad d\sigma = 0.$$

I will write $\varpi = (\sigma, \omega)$ to emphasize this view of a hyperkähler structure.

As expected, a hyperkähler structure corresponds with a hyperkähler metric.

Lemma 2.3.2. *Given a hyperkähler structure $\varpi = (\sigma, \omega)$, there is a complex structure on M such that σ generates $\mathcal{A}^{2,0}(M)$ and a hyperkähler metric g with Kähler form ω.*

Proof. By exercise 2.6.10 in [Huy05a], σ generates a complex structure I on M, where the $(1,0)$-forms are the kernel of the map $\wedge\sigma\colon \mathcal{A}^1_{\mathbb{C}}(M) \to \mathcal{A}^3_{\mathbb{C}}(M)$. Once the complex structure is constructed, ω will be type $(1,1)$ from the condition $\sigma \wedge \omega = 0$. A Kähler metric g can then be constructed from ω and I as $g(-,-) = \omega(I(-),-)$. Furthermore, our definition of hyperkähler structure entails $2\omega^2 = \sigma \wedge \bar{\sigma}$. From Corollary 4.B.23 in [Huy05a] (which relies on Yau's Theorem), the Kähler metric g is Ricci-Flat. Therefore, the holonomy group of (M, g) is $SU(2) \cong Sp(1)$, which implies that g is a hyperkähler metric on M. □

The definition of hyperkähler structure given here is a special case of the definition of a hyperkähler structure given in [Huy05b, Def. 2.3].

Associated to a hyperkähler structure ϖ is a 3-plane

$$P_\varpi := \mathbb{R}\langle \omega_I, \omega_J, \omega_K \rangle \subset \mathcal{A}^2_{\mathbb{R}}$$

and its image in cohomology

$$P_{[\varpi]} := \mathbb{R}\langle [\omega_I], [\omega_J], [\omega_K] \rangle \subset H^2(M, \mathbb{R}).$$

We also have real 2-planes $P_\sigma, P_{[\sigma]}$ defined as $\mathbb{R}\langle \omega_J, \omega_K\rangle$ and $\mathbb{R}\langle[\omega_J], [\omega_K]\rangle$, respectively. It is clear that taking cohomology classes yields an isomorphism

$$[-]\colon P_\varpi \xrightarrow{\sim} P_{[\varpi]}. \tag{2.3.9}$$

Notice that these spaces all come with an orientation, and that each space consists of positive forms or classes, i.e. $\alpha^2 > 0$ for all α in these spaces. In particular, $P_{[\sigma]} \in \mathrm{Gr}_2^{\mathrm{po}}(H^2(M,\mathbb{R}))$ is the period of a K3 surface induced by σ.

v-Genericity and Stability

Suppose now we are given a Mukai vector $v = (r, c, s) \in H^*(M, \mathbb{Z})$. In this thesis, a Mukai vector need not be algebraic; that is, c is not restricted to be in the Néron-Severi group. The reason, as we shall see later, is that when c is not algebraic we can interpret a rational multiple of $c^{0,2}$ as a B-field for a possible twisted holomorphic structure on a bundle having Mukai vector v; see Proposition 3.4.7 in Chapter III.

Definition 2.3.3. A hyperkähler structure ϖ is *v-generic* if there are no integral classes $\xi \in H^2(M,\mathbb{Z})$ such that $\xi \in P_{[\varpi]}^\perp$ and

$$-\frac{r^2}{4}\Delta \le \xi^2 < 0,$$

where $\Delta = v^2 + r^2$ is the discriminant of a sheaf with Mukai vector v.

Proposition 2.3.4. *Assume that $\varpi = (\sigma, \omega)$ is a v-generic hyperkähler structure. Then an ω-semistable twisted sheaf $\mathscr{E}$ with $v(\mathscr{E}) = v$ is ω-stable.*

Proof. As noted in [Yos06, Theorem 3.11], the proof in the untwisted case goes through the same with twisted sheaves. □

Twistor Families

Here I discuss notation and known results relating to twistor families for hyperkähler manifolds.

Let g be a hyperkähler metric on a smooth manifold M, with compatible complex structures I, J, K satisfying the quaternion relations. For each point $(a, b, c) \in S^2 \subset \mathbb{R}^3$, $aI + bJ + cK$ is another complex structure compatible with g. The *twistor family* is the holomorphic map

$$f\colon \mathcal{X} \to \mathbb{P}^1$$

where the fiber $f^{-1}(a, b, c)$ is the complex manifold M with the complex structure $aI + bJ + cK$. If instead we parameterize the family by a complex coordinate $t \in \mathbb{P}^1$, the complex structure I_t is given by (see [GS15])

$$I_t = \left(\frac{1 - |t|^2}{1 + |t|^2}\right) I + \left(\frac{2\operatorname{Re}(t)}{1 + |t|^2}\right) J + \left(\frac{2\operatorname{Im}(t)}{1 + |t|^2}\right) K.$$

In particular, $I_0 = I$, $I_1 = J$, and $I_{\sqrt{-1}} = K$.

As a smooth manifold, $\mathcal{X}$ is diffeomorphic to $M \times \mathbb{P}^1$, and hence comes with a smooth (but not holomorphic) map $p\colon \mathcal{X} \to M$. This projection can be used to prove a well-known result characterizing when a 2-form is of type $(1, 1)$ for all complex structures on a hyperkähler manifold. Recall that on a complex manifold with complex structure I, (p, q)-forms bring out a factor of i^{p-q} upon inserting I into each argument of the form. In particular, a 2-form α is type $(1, 1)$ provided that $\alpha(Iv, Iw) = \alpha(v, w)$ for all tangent vectors v, w. The result is the following lemma.

Lemma 2.4.1. *A 2-form α on M is type $(1, 1)$ for each complex structure in the twistor family if and only if $p^*\alpha$ is type $(1, 1)$ on $\mathcal{X}$.*

Proof. At $(x, t) \in \mathcal{X}$, the tangent space is $T_xM \times T_t\mathbb{P}^1$, and the complex structure is $\mathbb{I} := I_t \times I_{\mathbb{P}^1}$. Note that

$$p^*\alpha(\mathbb{I}v, \mathbb{I}w) = \alpha(I_t p_* v, I_t p_* w).$$

If α is type $(1, 1)$ for each $t \in \mathbb{P}^1$, then the right-hand side equals $\alpha(p_*v, p_*w) = p^*\alpha(v, w)$, showing that $p^*\alpha$ is type $(1, 1)$ on $\mathcal{X}$. Conversely, if $p^*\alpha$ is type $(1, 1)$ on $\mathcal{X}$, then the left-hand side equals $p^*\alpha(v, w)$, which implies that α is type $(1, 1)$ for each $t \in \mathbb{P}^1$. □

K3 Surfaces

Now consider the case where M is the underlying smooth manifold of a K3 surface. For another account of hyperkähler structures on K3 surfaces see [Huy16, Ch. VII].

A hyperkähler structure $\varpi = (\sigma, \omega)$ gives rise to a twistor family $\mathcal{X}(\varpi)$. Each complex structure I_t from this twistor family has a corresponding Kähler form ω_t and I_t-holomorphic 2-form σ_t that satsify the same relations as σ_I, ω_I:

$$\sigma_t \wedge \sigma_t = 0, \quad \sigma_t \wedge \bar{\sigma}_t = 2\omega_t \wedge \omega_t > 0, \quad \sigma_t \wedge \omega_t = 0. \tag{2.4.10}$$

With this notation, $\sigma_0 = \sigma$ and $\omega_0 = \omega$. The pair (σ_t, ω_t) for I_t can be described by

$$\begin{aligned} \sigma_t &= \frac{1}{1+|t|^2}\left(\sigma - 2t\omega - t^2\bar{\sigma}\right) \\ \omega_t &= \frac{1}{1+|t|^2}\left((1-|t|^2)\omega + t\bar{\sigma} + \bar{t}\sigma\right), \end{aligned} \tag{2.4.11}$$

which follows from a direct calculation; see [GS15, Lemma 2].

Lemma 2.4.2. *Suppose $\varpi = (\sigma, \omega)$ is a hyperkähler structure on the K3 manifold M. Then any oriented 2-plane $Q \subset P_{[\varpi]}$ defines a unique complex structure on M. These complex structures completely describe all complex structures appearing in the twistor family $\mathcal{X}(\varpi)$. Furthermore, the subspace $H^{1,1} \subset H^2(M, \mathbb{R})$ of real $(1,1)$-classes is equal to $Q^\perp$ (taken with respect to the intersection pairing).*

Proof. Given a positive oriented 2-plane $Q \subset P_{[\varpi]}$, choose an orthonormal basis $[\alpha], [\beta]$ of Q with respect to the intersection pairing. Using equation (2.3.9), there are unique lifts $\alpha, \beta \in P_\varpi$. These lifts automatically satisfy

$$\alpha \wedge \beta = 0, \quad \alpha^2 = \beta^2 > 0,$$

which can be checked by expanding α and β in terms of $\omega_I, \omega_J, \omega_K$. Such expansions have constant coefficients (not functions), so the relations $[\alpha].[\beta] = 0$ and $[\alpha]^2 = [\beta]^2 > 0$ lift directly to α and β. Therefore the complex 2-form $\sigma' = \alpha + i\beta$ satisfies $\sigma' \wedge \sigma' = 0$, $\sigma' \wedge \bar{\sigma'} > 0$, and $d\sigma' = 0$. This means that σ' defines a complex structure on M (cf. the proof of Lemma 2.3.2), and that $P_{[\sigma']} = Q$.

It is straightforward to check that different choices of orthonormal basis for Q multiply σ' by a complex number of unit norm. Therefore, different choices of orthonormal bases lead to the same complex structure on M. Thus, a single oriented 2-plane Q gives rise to a unique complex structure on M. Also, the complex structures in $\mathcal{X}(\varpi)$ arise from the σ_t described in equations (2.4.11). The real and imaginary parts of $[\sigma_t]$ span the oriented 2-plane $P_{[\sigma_t]} \subset P_{[\varpi]}$, and it is clear that these two constructions are inverse to each other.

Finally, note that a cohomology class $\eta \in H^2(M, \mathbb{R})$ is type $(1,1)$ on $X = (M, \sigma')$ if and only if $\eta.[\sigma'] = 0$. Taking real and imaginary parts of this equation leads directly to the condition $\eta \in P^{\perp}_{[\sigma']}$. □

We now use this characterization of the twistor family to prove a useful lemma on when a cohomology class is type $(1,1)$.

Lemma 2.4.3. *Given a positive class $\eta \in H^2(M, \mathbb{R})$, there are precisely two complex structures in the twistor family $\mathcal{X}(\varpi)$ where η is type $(1,1)$.*

Proof. We begin by representing a complex structure in ϖ as a 2-plane $Q \subset P_{[\varpi]}$. Since $Q^{\perp}$ describes $H^{1,1}$, the condition that η is a $(1,1)$-class on X is equivalent to $\eta \in Q^{\perp}$, which, in turn, is equivalent to $Q \subset \eta^{\perp}$. Therefore, the complex structures in $\mathcal{X}(\varpi)$ where η is type $(1,1)$ are constrained by the requirement that $Q \subset P_{[\varpi]} \cap \eta^{\perp}$. When η is a positive class, $\eta^{\perp} \cong \mathbb{R}^{2,19}$, while $P_{[\varpi]} \cong \mathbb{R}^{3,0}$. So, the desired Q must be a subspace of $P_{[\varpi]} \cap \eta^{\perp} \cong \mathbb{R}^{2,0}$. This only leaves two possibilities for Q, which are obtained by choosing an orientation on $P_{[\varpi]} \cap \eta^{\perp}$. □

Elliptic Complexes

This section collects facts about elliptic operators and complexes. In a proper treatment of these ideas it is important to work in the setting of the Hilbert space completions of spaces of forms, but we will neglect this difficulty since it does not affect the results we use. For a more detailed analytic treatment of this material, I recommend [Kob14, Ch VII.2], for the specific case of elliptic complexes for moduli spaces of vector bundles, [DK90, Appendix II] for general results on Sobolev spaces, and [Wel08, Ch. IV] for a treatment of the general theory of elliptic operators and elliptic complexes.

Fix a hyperkähler structure $\varpi = (\omega_I, \omega_J, \omega_K)$ on the K3 manifold M and let ∇ be a Hermite-Einstein connection on E. [1] The content of this section generalizes to higher dimensional Kähler manifolds, though we choose to phrase results for surfaces; we again refer to [Kob14, Ch. VII] for the general case.

Let E and F be two C^∞ vector bundles over a smooth manifold X. From a differential operator $D\colon \mathcal{A}^0(E) \to \mathcal{A}^0(F)$ we will define the *(principal) symbol* $\sigma_D(x,\xi)$. Choose a local trivialization of both bundles, in which case the operator D can be represented as

$$D = \sum_{|\alpha|\leq d} A^\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

where A^α is a matrix-valued function of $x \in X$. We use multi-index notation, which means that for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}$$

and similarly for a tangent vector $\xi \in T_{x,\mathbb{R}}X$,

$$\xi^\alpha = \xi^{\alpha_1}\xi^{\alpha_2}\cdots\xi^{\alpha_n}$$

where $n = \dim_{\mathbb{R}}(X)$. The principal symbol is then obtained by substituting ξ^α in for $\partial^\alpha/\partial x^\alpha$ in the top-degree terms in D: [2]

$$\sigma_D(x,\xi) = \sum_{|\alpha|=d} A^\alpha(x)\xi^\alpha.$$

For a fixed $x \in X$ and nonzero real tangent vector $\xi \in T_xX$, the matrix written above can be viewed as a linear map $E_x \to F_x$. This symbol is independent of the choice of local coordinates on X. We say that the operator D is *elliptic* when the symbol $\sigma_D(x,\xi)$ is an invertible linear map for each $x \in X$ and each nonzero $\xi \in T_xX$.

[1] This connection may also be a twisted Hermite-Einstein connection, which will be defined in Chapter III.

[2] This operation is less strange when considering the Fourier transform of such an operator, which turns differentiation into multiplication by ξ.

Elliptic *complexes* are an extension of this idea. To a complex of differential operators

$$\cdots \longrightarrow \mathcal{A}^0(E_{i-1}) \xrightarrow{D_{i-1}} \mathcal{A}^0(E_i) \xrightarrow{D_i} \mathcal{A}^0(E_{i+1}) \longrightarrow \cdots$$

we associate a symbol sequence of vector spaces for each x, ξ:

$$\cdots \longrightarrow (E_{i-1})_x \xrightarrow{\sigma_{D_{i-1}}(x,\xi)} (E_i)_x \xrightarrow{\sigma_{D_i}(x,\xi)} (E_{i+1})_x \longrightarrow \cdots$$

This sequence is automatically a complex, and if it is exact for each x and nonzero ξ we call the original complex *elliptic*. Elliptic complexes enjoy many wonderful properties: the cohomology spaces $H^i = \ker(D_i)/\operatorname{im}(D_{i-1})$ are finite dimensional, and in each homological dimension there is a generalized Hodge decomposition

$$\mathcal{A}^0(E_i) = \operatorname{im}(D_i^*) \oplus \mathscr{H}^i \oplus \operatorname{im}(D_{i-1}),$$

where $\mathscr{H}^i$ is the space of D-harmonic forms that satisfy $D_{i-1}^*\alpha = D_i\alpha = 0$.

We will be interested in the following elliptic complex, which is sometimes called the *deformation complex*:

$$C^\bullet(E,\nabla) := \left(0 \longrightarrow \mathcal{A}^0(\operatorname{End}(E,h)) \xrightarrow{\tilde{\nabla}} \mathcal{A}^1(\operatorname{End}(E,h)) \xrightarrow{\tilde{\nabla}_+} \mathcal{A}_2^+(\operatorname{End}(E,h)) \longrightarrow 0 \right) \quad (2.5.12)$$

where $\mathcal{A}_+^2$ is the space of global sections of the trivial real line bundle generated by the hyperkähler forms $\omega_I, \omega_J, \omega_K$ and $\tilde{\nabla}_+ = P_+\tilde{\nabla}$ is the derivative followed by projection onto $\mathcal{A}_+^2$. For a 2-form $\eta \in \mathcal{A}^2$, $P_+\eta$ can be described by

$$P_+\eta = \frac{\eta \wedge \omega_I}{\omega_I^2}\omega_I + \frac{\eta \wedge \omega_J}{\omega_J^2}\omega_J + \frac{\eta \wedge \omega_K}{\omega_K^2}\omega_K. \quad (2.5.13)$$

Here, the ratio $\eta \wedge \omega_I/\omega_I^2$ stands for the smooth function f such that $\eta \wedge \omega_I = f\omega^2$, which exists since both ω_I^2 and $\eta \wedge \omega_I$ are top forms and ω_I^2 is non-vanishing. The complex $C^\bullet(E,\nabla)$ is elliptic when ∇ is a Hermite-Einstein connection, see [Kob14, Lem. VII.2.20]. We will write $\mathscr{H}^\bullet(E,\nabla)$ for the harmonic spaces for this complex.

The deformation complex describes the tangent space to the moduli space of Hermite-Einstein connections $\mathcal{M}^{HE}(E,h)$. This moduli space is a quotient of the infinite dimensional space of all Hermite-Einstein connections on E by the infinite dimensional group of unitary reduced gauge transformations

$$\mathscr{G} = U(E,h)/U(1) = \{g \in \mathcal{A}^0(\operatorname{End} E) \mid g^* g = 1_E\}/\{c \cdot 1_E\};$$

more details will be given in Chapter V. At a point $[\nabla] \in \mathcal{M}^{HE}(E,h)$, the tangent space is given by $\mathscr{H}^1(E,\nabla)$. These spaces are isomorphic if a different choice of gauge representative of the class $[\nabla]$ is chosen, so this space is well-defined. Lastly, there is a chain map from $C^\bullet$ to the Dolbeault complex of $\operatorname{End}(E)$ (see [Kob14, Sec. VII.2]) that is an isomorphism in cohomology in degree 1, identifying $T_{[\nabla]}\mathcal{M}^{HE}$ with $\operatorname{Ext}^1(E,E)$.

CHAPTER III

UNTWISTING

Overview

In this section I describe an “untwisting” procedure associating an α-twisted vector bundle $\mathscr{E}$ to a smooth, untwisted bundle E under the assumption that the Brauer class α is *topologically trivial*. This means that α has trivial image under the connecting homomorphism $H^2(X, \mathcal{O}_X^*) \to H^3(X, \mathbb{Z})$ from the exponential sequence.

The idea for this untwisting procedure originated from [HS05] where Huybrechts and Stellari define $\mathrm{ch}(\mathscr{E})$ of a bundle $\mathscr{E}$ twisted by a topologically trivial Brauer class as the Chern character of the associated untwisted bundle. Here we study this untwisting procedure as a method in its own right. I will show that twisted differential-geometric data can be faithfully represented as ordinary differential geometric data on the untwisted bundle. The most important thing to observe is that a twisted holomorphic structure on $\mathscr{E}$ can be encoded by a Dolbeault operator $\overline{\partial}_E$ on E that satisfies $\overline{\partial}_E^2 = 2\pi i B^{0,2} 1_E$ for some B-field B, rather than the ordinary integrability condition $\overline{\partial}_E^2 = 0$. Using this idea it is straightforward to encode a twisted Hermite-Einstein connection as an untwisted connection on an untwisted bundle satisfying a slight variation of the classical Hermite-Einstein equations.

For this section, X is a connected complex manifold, as in Section 2.1. Fix a good open cover $\mathfrak{U}$ of X. In contrast to Section 2.1, a cocycle representative of α need not be fixed.

Topological Untwisting

When a Brauer class is topologically trivial we can lift it to a rational cohomology class, as the next lemma shows.

Lemma 3.2.1. *Given a topologically trivial Brauer class* $\alpha \in H^2(X, \mathcal{O}^*)_{tors}$*, there exists a rational class* $\beta \in H^2(X, \mathbb{Q})$ *such that* $\mathrm{ord}(\alpha) \cdot \beta \in H^2(X, \mathbb{Z})$ *and* $\exp(2\pi i \beta^{0,2}) = \alpha$.

Proof. Consider the morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{\exp} & \mathcal{O}^* & \longrightarrow & 0 \end{array} \tag{3.2.1}$$

where the second row is the exponential sequence with the convention that $\exp\colon f \mapsto \exp(2\pi i f)$. This induces a chain map

$$\begin{array}{ccccccc} H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathbb{Q}) & \longrightarrow & H^2(X,\mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^3(X,\mathbb{Z}) \\ \| & & \downarrow & & \downarrow & & \| \\ H^2(X,\mathbb{Z}) & \longrightarrow & H^2(X,\mathcal{O}) & \xrightarrow{\exp} & H^2(X,\mathcal{O}^*) & \xrightarrow{\delta} & H^3(X,\mathbb{Z}) \end{array} \tag{3.2.2}$$

Since α is topologically trivial, $\delta(\alpha) = 0$. Using exactness of the bottom row, we can find a class $\beta \in H^2(X,\mathbb{Q})$ with $\exp(2\pi i \beta^{0,2}) = \alpha$. Let $r = \operatorname{ord}(\alpha)$. Since $\exp(2\pi i r \beta^{0,2}) = \alpha^r = 1$, $r\beta \in H^2(X,\mathbb{Z})$ from chasing the left-most square above. □

The untwisting procedure from [HS05] begins by fixing a lift $\beta \in H^2(X,\mathbb{Q})$ as in Lemma 3.2.1. Fix a Čech cocycle representative $\{\beta_{ijk}\}$ of β, where $\beta_{ijk}\colon U_{ijk} \to \mathbb{Q}$ is a constant function. This gives a Čech representative $\alpha_{ijk} := \exp(2\pi i \beta_{ijk})$ of α. Next, because the sheaf of C^∞ functions is acyclic, we may view β_{ijk} as smooth functions and choose a Čech 1-cochain $a_{ij} \in \Gamma(U_{ij}, \mathcal{C}^\infty)$ such that $\beta_{ijk} = a_{ij} + a_{jk} + a_{ki}$.

Now suppose that $\mathscr{E}$ is an α-twisted vector bundle given by the data $\{E_i, \varphi_{ij}\}$ with $\varphi_{ij}\varphi_{jk}\varphi_{ki} = \alpha_{ijk} \cdot 1_{\mathscr{E}}$. Let $\psi_{ij} = e^{-2\pi i a_{ij}}\varphi_{ij}$. Then ψ_{ij} satisfies $\psi_{ij}\psi_{jk}\psi_{ki} = 1_E$, so $E = \{E_i, \psi_{ij}\}$ glues into a C^∞ bundle.

Definition 3.2.2. With the fixed Čech data $\{\beta_{ijk}\}$ and $a := \{a_{ij}\}$, the bundle $E = \{E_i, \psi_{ij}\}$ will be called E the *untwisting* of $\mathscr{E}$ by a.

Simplifications in the Topologically Trivial Case

There are two main observations that allow for simplifications of the definitions given in Section 2.1. The first is that the functions $\beta_{ijk}\colon U_{ijk} \to \mathbb{Q}$ can be taken to be constant, and therefore the same is true of the α_{ijk}. Second, notice that if $\beta_{ijk} = \delta(a)_{ijk}$ then $\beta_{ijk} = \delta(\operatorname{Re}(a))_{ijk}$ since β_{ijk} are real. Thus, we can assume that the functions a_{ij} are real-valued. Many of the properties in this chapter hold only under these two assumptions; greater care would be needed if one chooses a cocycle representative of α with non-constant holomorphic functions.

Twisted connections. The 1-forms η_{ij} form a Čech cocycle because $d \log \alpha_{ijk}$ now vanishes.

Hermitian Structures. Since α_{ijk} has unit modulus, the functions ρ_{ij} for a hermitian metric $\{H_i, \rho_{ij}\}$ can be taken to be 1. Indeed, from the compatibility relation

$$\rho_{ij}\rho_{jk}\rho_{ki} = |\alpha_{ijk}|^{-2} = 1,$$

it follows that the functions $\log(\rho_{ij})$ form a Čech 1-cocycle for the sheaf of smooth real-valued functions on X. Since this sheaf is acyclic, we can choose smooth functions $g_i\colon U_i \to \mathbb{R}$ with $\log(\rho_{ij}) = g_i - g_j$, or $\rho_{ij} = e^{g_i} e^{-g_j}$. Then the local metrics $H_i' = e^{g_i} H_i$ form a twisted hermitian metric with $\rho_{ij}' = 1$. From now on we assume hermitian metrics have $\rho_{ij} = 1$.

Chern Connections. The Chern connection $\nabla = \{\nabla_i, \eta_{ij}\}$ must satisfy $\eta_{ij} = 0$, as we prove in the next lemma.

Lemma 3.3.1. *Let $\nabla = \{\nabla_i, \eta_{ij}\}$ be a connection on $\mathscr{E}$ and $h = \{H_i\}$ a hermitian structure on $\mathscr{E}$.*

1. *If ∇ is hermitian, then η_{ij} is a purely imaginary 1-form.*
2. *If ∇ is integrable, then $\eta_{ij}^{0,1} = 0$.*
3. *If ∇ is the Chern connection on $\mathscr{E}$ for h, then $\eta_{ij} = 0$.*

Proof. The first is a consequence of requiring that every ∇_i is hermitian: using the compatibility condition in Lemma 2.1.6 we see that $2\,\mathrm{Re}(\eta_{ij}) = -\partial \log(1) = 0$.

The second property is just the definition of an integrable connection. The third follows from the other two: for the Chern connection η_{ij} must simultaneously be imaginary and satisfy $\eta_{ij}^{0,1} = 0$. But then $\eta_{ij} = \eta_{ij}^{1,0} = -\overline{\eta_{ij}^{0,1}} = 0$. □

Transporting Structures

It is natural to expect that twisted geometric structures on $\mathscr{E}$ correspond to untwisted geometric structures on E, especially in light of the simplifications noted above. This section collects results along these lines.

Proposition 3.4.1. *Suppose that $\nabla = \{\nabla_i, \eta_{ij}\}$ is a twisted connection on $\mathscr{E}$. Then ∇ glues into an untwisted connection on E if and only if $\eta_{ij} = 2\pi i da_{ij}$. Conversely, if ∇ is any connection on E, then $\nabla_i := \nabla|_{U_i}$ defines a twisted connection $\{\nabla_i, 2\pi i da_{ij}\}$ on $\mathscr{E}$.*

Proof. Recall that $\psi_{ij} = e^{-2\pi i a_{ij}}\varphi_{ij}$. The statement that ∇ glues into a connection on E is equivalent to the equality

$$\nabla_j = \psi_{ij}^* \nabla_i \tag{3.4.3}$$

for each i, j. It follows that

$$\psi_{ij}^* \nabla_i = \varphi_{ij}^* \nabla_i - 2\pi i da_{ij} \cdot 1_{\mathscr{E}}$$

Thus, equation (3.4.3) holds if and only if

$$\eta_{ij} \cdot 1_{\mathscr{E}} = \varphi_{ij}^* \nabla_i - \nabla_j = 2\pi i da_{ij} \cdot 1_{\mathscr{E}}$$

For the second statement, note that ∇ being a globally defined connection says that the ∇_i must satisfy equation (3.4.3), and the above calculations show exactly that $\{\nabla|_{U_i}, 2\pi i da_{ij}\}$ is a twisted connection on $\mathscr{E}$. □

These kinds of connections will be called *untwistable*. Proposition 3.4.1 gives a very strong condition for when a connection is untwistable. If a connection $\nabla = \{\nabla_i, \eta_{ij}\}$ has $\eta_{ij} \neq 2\pi i da_{ij}$, then ∇ can be shifted by a 1-cochain of 1-forms to produce an untwistable connection. This is summarized in the next Proposition.

Proposition 3.4.2. *If $\nabla = \{\nabla_i, \eta_{ij}\}$ is a twisted connection, then one can choose local 1-forms $r_i \in \mathcal{A}^1(U_i)$ to ensure that $\widehat{\nabla} = \{\nabla_i - r_i \cdot 1_{\mathscr{E}}, 2\pi i da_{ij}\}$ is an untwistable connection on $\mathscr{E}$.*

Proof. Observe that $\{\eta_{ij} - 2\pi i da_{ij}\}$ is a Čech 1-cocycle for the sheaf of smooth 1-forms (which is acyclic), so we may choose 1-forms r_i on U_i such that $\eta_{ij} = 2\pi i da_{ij} + r_i - r_j$. Define $\widehat{\nabla} = \nabla_i - r_i \cdot 1_{\mathscr{E}}$.

Then

$$\begin{aligned}\varphi_{ij}^*\widehat{\nabla}_i &= \varphi_{ij}^*\nabla_i - r_i \cdot 1_{\mathscr{E}} \\ &= \nabla_j + (\eta_{ij} - r_i)\cdot 1_{\mathscr{E}} \\ &= \nabla_j - r_j \cdot 1_{\mathscr{E}} + 2\pi i da_{ij} \cdot 1_{\mathscr{E}} \\ &= \widehat{\nabla}_j + 2\pi i da_{ij} \cdot 1_{\mathscr{E}}\end{aligned}$$

and hence $\widehat{\nabla} = \{\nabla_i - r_i\, 1_{\mathscr{E}}, 2\pi i da_{ij}\}$ defines an untwistable connection on $\mathscr{E}$. □

Untwisting Curvature

As explained in section 2.1, in general one needs to choose a B-field in order to define the curvature of a twisted connection as a global section of $\mathcal{A}^2(\mathrm{End}(E))$. However, in the topologically trivial setting we will see that there is a natural way to define a B-field.

First notice that when ∇ is an untwistable connection its curvature is already a well-defined global quantity with no B-field required.

Proposition 3.4.3. *Suppose $\nabla = \{\nabla_i, 2\pi i da_{ij}\}$ is an untwistable connection. Then F_∇ is a globally-defined 2-form with values in* $\mathrm{End}(E)$ *without needing to choose a B-field.*

Proof. The connections on $\mathscr{E}$ and E are both given by $\{\nabla_i\}$, but glue according to φ_{ij} and ψ_{ij} respectively. However, their curvatures transform according to the adjoint action of GL, and since φ_{ij} and ψ_{ij} differ by a scalar they have the same adjoint action. Consequently, the curvatures defined by the twisted connection are identical to the curvatures defined by the untwisted connection, which is already a global quantity. □

Suppose now that ∇ is not untwistable. Following Proposition 3.4.2, choose local 1-forms r_i on U_i such that $\eta_{ij} = 2\pi i da_{ij} + r_i - r_j$ and set $\widehat{\nabla}_i := \nabla_i - r_i$. The local curvatures then satisfy the equation

$$F_{\widehat{\nabla}_i} = F_{\nabla_i} - dr_i \cdot 1_{\mathscr{E}}, \tag{3.4.4}$$

and the curvature of $\widehat{\nabla}$ is globally-defined. [1] As a consequence, we can take the B-field $B = \{-dr_i\}$ and notice that the B-curvature with this B-field coincides with the curvature of $\widehat{\nabla}$:

$$\tilde{F}_{\nabla_i} = F_{\nabla_i} + B_i \cdot 1_{\mathscr{E}} = F_{\nabla_i} - dr_i \cdot 1_{\mathscr{E}} = F_{\widehat{\nabla}}. \tag{3.4.5}$$

Integrable Connections

For what follows we need to recall the process of passing between Čech and de Rham cohomology class representatives. Keep fixed the choice of a good open cover $\mathfrak{U}$. We compare two resolutions of the sheaf of smooth functions $\mathcal{C}^\infty_X$, one being the de Rham resolution and the other being the Čech resolution. Taking the Čech resolution of each sheaf in the de Rham resolution gives the following grid of sheaves:

$$\begin{array}{ccccccc}
\check{C}^0(\mathcal{C}^\infty) & \xrightarrow{d} & \check{C}^0(\mathcal{A}^1) & \xrightarrow{d} & \check{C}^0(\mathcal{A}^2) & \xrightarrow{d} & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\check{C}^1(\mathcal{C}^\infty) & \xrightarrow{d} & \check{C}^1(\mathcal{A}^1) & \xrightarrow{d} & \check{C}^1(\mathcal{A}^2) & \xrightarrow{d} & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\check{C}^2(\mathcal{C}^\infty) & \xrightarrow{d} & \check{C}^2(\mathcal{A}^1) & \xrightarrow{d} & \check{C}^2(\mathcal{A}^2) & \xrightarrow{d} & \cdots \\
\downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

Starting with a 2-cocycle $\{\beta_{ijk}\}$ of the constant sheaf $\mathbb{Q}$ regarded as a 2-cocycle in the sheaf of smooth functions (which starts at the bottom left corner of the above diagram), we zig-zag from bottom left corner to the top right corner making choices along the way. This gives us the following data: a 1-cochain of smooth functions $\{a_{ij}\}$, a 0-cochain of 1-forms $\{c_i\}$, and a 0-cochain of 2-forms $\{B_i\}$ satisfying the following identities:

$$\begin{aligned}
a_{ij} + a_{jk} + a_{ki} &= \beta_{ijk} \\
c_i - c_j &= da_{ij} \\
B_i &= dc_i.
\end{aligned} \tag{3.4.6}$$

[1] This equation shows that topological information, such as Chern classes, may be defined using connections on the untwisted bundle E.

The notation is chosen to match with the previous sections, so β_{ijk} and a_{ij} agree with their previous incarnations. Observe that $B_i - B_j = d^2 a_{ij} = 0$, so B is a closed real-valued 2-form on X that represents of β.

I want to point out that all of these choices are made *without* referencing the twisted bundle. Furthermore, notice that the above process works in reverse: beginning with a global 2-form B representing the cohomology class β, following the zig-zag in reverse produces c_i, a_{ij}, and β_{ijk}.

Note that an integrable connection ∇ is not untwitstable unless the Brauer class α is trivial. Indeed, if $\eta_{ij}^{0,1} = 0$, then it follows that $\bar{\partial} a_{ij} = 0$ from the condition $\eta_{ij} = 2\pi i d a_{ij}$. This means that the cocycle β_{ijk} is a Čech coboundary when viewed as a cocycle for $\mathcal{O}_X$; in other words, $\beta^{0,2} \in H^2(X, \mathcal{O}_X)$ is trivial, making $\alpha = \exp(2\pi i \beta^{0,2})$ trivial as well.

While an integral connection may not be untwistable in general, the shifted connection from Proposition 3.4.2 still produces an interesting connection on E.

Proposition 3.4.4. *Suppose a holomorphic α-twisted bundle $\mathscr{E}$ untwists to E by a. Then there exists a connection $\widehat{\nabla}$ on E such that $F^{0,2}_{\widehat{\nabla}} = 2\pi i B^{0,2} \cdot 1$ for some real closed 2-form B representing the cohomology class $\beta \in H^2(X, \mathbb{Q})$.*

Proof. Fix a hermitian structure h on $\mathscr{E}$ and let $\nabla = \{\nabla_i, 0\}$ be the associated Chern connection. We set

$$r_i := -2\pi i c_i.$$

The 1-forms r_i satisfy $2\pi i d a_{ij} + r_j - r_j = 0$ by virtue of equations (3.4.6). It follows that $\widehat{\nabla}_i := \nabla_i - r_i \cdot 1_E$ is untwistable since $\eta_{ij} = 0$. We get that

$$F_{\widehat{\nabla}_i} = F_{\nabla_i} + 2\pi i B \cdot 1_E \,. \tag{3.4.7}$$

But since ∇ is integrable, $F^{0,2}_{\nabla_i} = 0$, and we see that $F^{0,2}_{\widehat{\nabla}_i} = 2\pi i B^{0,2} \cdot 1_E$. This proves the proposition. ☐

Proposition 3.4.4 gives several corollaries.

Corollary 3.4.5. *Let B be the 2-form representative of β constructed in equation* (3.4.6)*. The $2\pi iB$-curvature of the Chern connection coincides with the curvature of the untwistable connection $\widehat{\nabla}$.*

Proof. This follows immediately from equation (3.4.7) and the definition of $\tilde{F}_{\nabla,2\pi iB}$. □

Corollary 3.4.6. *The first Chern class of the untwisted bundle E satisfies*

$$c_1(E)^{0,2} = -\operatorname{rk}(E)\beta^{0,2}.$$

Proof. This follows by taking the trace and cohomology class of $F^{0,2}_{\widehat{\nabla}} = 2\pi iB^{0,2}$. □

Next we see how to reverse the untwisting procedure starting with a connection ∇ on E with $F^{0,2}_{\nabla} = 2\pi iB^{0,2}1_E$.

Proposition 3.4.7. *Suppose a smooth bundle E admits a connection $\hat{\nabla}$ such that $F^{0,2}_{\widehat{\nabla}} = 2\pi iB^{0,2}$. Then there exists a holomorphic α-twisted bundle $\mathscr{E}$ untwisting to E.*

Proof. Suppose that E has transition matrices ψ_{ij} on the open cover $\mathfrak{U}$. With the smooth functions a_{ij} we can define $\varphi_{ij} := e^{2\pi i a_{ij}}\psi_{ij}$, which produces an α-twisted bundle $\mathscr{E} = \{E_i, \varphi_{ij}\}$ that a priori is only a C^∞ twisted bundle. However, we do know that $\{\widehat{\nabla}_i, 2\pi i da_{ij}\}$ is a connection on $\mathscr{E}$. Setting $\nabla_i = \widehat{\nabla}_i - 2\pi i c_i^{0,1}$ produces another twisted connection $\nabla = \{\nabla_i, \eta_{ij}\}$ with

$$\eta_{ij} := 2\pi i\partial a_{ij}$$

which can be checked with the help of the equations (3.4.6). Clearly $\eta_{ij}^{0,1} = 0$, and we also have that

$$F^{0,2}_{\nabla_i} = F^{0,2}_{\widehat{\nabla}_i} - 2\pi iB^{0,2} \cdot 1_E = 0.$$

Hence, ∇ is an integrable twisted connection, furnishing $\mathscr{E}$ with a holomorphic structure. □

We have begun to see an equivalence between untwisted bundles with a special kind of connection and twisted holomorphic bundles, with the equivalence given by untwisting. Let $\mathcal{V}(\alpha)$ be the category of $\{\alpha_{ijk}\}$-twisted holomorphic vector bundles with twisted sheaf homomorphisms.

Let $V(B)$ be the category of smooth vector bundles E admitting a connection $\widehat{\nabla}$ satisfying

$$F^{0,2}_{\widehat{\nabla}} = 2\pi i B^{0,2},$$

with morphisms given by holomorphic[2] sections of $\mathrm{Hom}(E,F)$. Note that these categories depend on the choices of representatives α_{ijk} and B of α and β respectively, but recall that these choices are all dependent on the initial choice of β_{ijk} (as well as the open cover $\mathfrak{U}$) through equations (3.4.6).

Theorem 3.4.8. *Let X be a connected complex manifold and α a topologically trivial Brauer class. Then the untwisting functor $U\colon \mathcal{V}(\alpha) \to V(B)$ is an equivalence of categories.*

Proof. Proposition 3.4.7 shows that the untwisting functor is essentially surjective. To prove that it is fully faithful we first note that $\mathrm{Hom}(\mathscr{E},\mathscr{F}) \cong \mathscr{F} \otimes \mathscr{E}^\vee$, and this in turn is isomorphic to $F \otimes E^\vee$, which can be seen by noting that the transition matrices for $\mathscr{F} \otimes \mathscr{E}^\vee$ are identical to the transition matrices of $F \otimes E^\vee$. Consequently there is an isomorphism of bundles

$$\mathrm{Hom}(\mathscr{E},\mathscr{F}) \cong \mathrm{Hom}(E,F). \tag{3.4.8}$$

It remains to be seen that this isomorphism respects the holomorphic structures on these bundles. Choose integrable twisted connections $\nabla_{\mathscr{E}}, \nabla_{\mathscr{F}}$ corresponding to untwisted connections $\widehat{\nabla}_E$, $\widehat{\nabla}_F$ whose $(0,1)$-components square to $2\pi i B^{0,2}$, and observe that

$$(\nabla_{\mathscr{E}})^{0,1}_i = (\widehat{\nabla}_E)^{0,1}_i + 2\pi i c^{0,1}_i.$$

The final term is the same for $\mathscr{F}$, and therefore the $(0,1)$-parts of the connections $\nabla_{\mathscr{F}} \otimes \nabla_{\mathscr{E}^\vee}$ and $\widehat{\nabla}_F \otimes \widehat{\nabla}_{E^\vee}$ agree. Thus, holomorphic sections on both sides of equation (3.4.8) are the same. Since the holomorphic sections on either side are the Hom sets in these categories, we see that U is fully faithful. ∎

[2] If E and F have connections $\widehat{\nabla}_E, \widehat{\nabla}_F$ such that $F^{0,2}_{\widehat{\nabla}_E} = F^{0,2}_{\widehat{\nabla}_F} = 2\pi i B^{0,2}$, then the induced connection on $\mathrm{Hom}(E,F)$ has $F^{0,2}_{\widehat{\nabla}} = 0$, so we can make sense of holomorphic maps between such bundles.

Assume that (X, ω) is a compact connected Kähler manifold. Recall from the proof of Proposition 3.4.4 that untwisting the Chern connection $\nabla = \{\nabla_i, 0\}$ produces a connection $\widehat{\nabla}$ given locally by $\widehat{\nabla}_i = \nabla_i + 2\pi i dc_i$, and from Corollary 3.4.5 we know that the curvature $F_{\widehat{\nabla}}$ agrees with the B-curvature $\tilde{F}_{\nabla, 2\pi i B}$. Applying this to a Hermite-Einstein metric h, we immediately have the following proposition.

Proposition 3.4.9. *Suppose B is a real, closed 2-form on X representing a cohomology class β. If h is a Hermite-Einstein metric on a twisted bundle $\mathscr{E}$ with respect to the B-field $2\pi i B$, then the resulting untwisted connection $\widehat{\nabla}$ on E satisfies the twisted Hermite-Einstein equations:*

$$\begin{aligned} F^{0,2}_{\widehat{\nabla}} &= 2\pi i B^{0,2} \\ i\Lambda_\omega F_{\widehat{\nabla}} &= \lambda\, 1_E \end{aligned} \tag{3.4.9}$$

Conversely, any smooth hermitian bundle (E, h) with hermitian connection satisfying these equations gives rise to a twisted bundle $\mathscr{E}$ with a twisted Hermite-Einstein metric.

Later it will be helpful to relax the condition that λ in the Hermite-Einstein equations be a constant. When $\lambda\colon X \to \mathbb{R}$ is a real-valued smooth function on X we call the metric (or connection) a B-twisted *weak* Hermite-Einstein metric (connection). For untwisted metrics it is well-known that a weak Hermite-Einstein metric can be made into a Hermite-Einstein metric by a conformal mapping $h \mapsto e^f h$, where f is a smooth function on X. The next lemma shows that this is still true for B-twisted metrics without needing to adjust the B-field.

Lemma 3.4.10. *Let E be a C^∞ bundle on X with hermitian structure h, and assume h is a weak Hermite-Einstein metric with B-field 2-form B. Then there is a conformal change in metric $h' = e^f h$ which is Hermite-Einstein with respect to the same B-field B.*

Proof. Let ∇ be the h-hermitian connection on the untwisted bundle E satisfying equations (3.4.9) and define $\nabla' = \nabla + \partial f$. It is straightforward to check that ∇' is hermitian with respect to h' (which is the result of untwisting the twisted Chern connection for h'). Then the curvature $F_{\nabla'}$ is related to F_∇ by

$$F_{\nabla'} = F_\nabla + \partial\bar{\partial} f.$$

Since $\partial\bar{\partial}f$ is of type $(1,1)$, we see that $F^{0,2}_{\nabla'} = F^{0,2}_{\nabla} = 2\pi i B^{0,2}$.

For the other equation we have $i\Lambda_\omega F_{\nabla'} = \lambda(x) + i\Lambda\partial\bar{\partial}f$. The Kähler identities show that $i\Lambda_\omega\partial\bar{\partial}f = -\Delta_{\bar{\partial}}f$, and elliptic operator theory shows that $C^\infty(X,\mathbb{R}) = \mathrm{Im}(\Delta_{\bar{\partial}}) \oplus \mathbb{R}$. Using this decomposition we can choose the function f so that $\lambda = \Delta_{\bar{\partial}}(f) + \lambda_0$ for $\lambda_0 \in \mathbb{R}$. We then get $i\Lambda_\omega F_{\nabla'} = \lambda_0 1_E$ as desired. □

We now understand that the structure of an ω-slope stable twisted holomorphic bundle $\mathscr{E}$ is encoded by a C^∞ bundle E that admits a B-twisted Hermite-Einstein connection. For the rest of this thesis a *twisted Hermite-Einstein connection* will mean a hermitian connection on an untwisted bundle E satisfying the twisted Hermite-Einstein equations (3.4.9).

Remark 3.4.11. We will end this section with a compactification of the Hermite-Einstein equations on a K3 surface. The content of this remark is interesting but will not be used in later sections, so the reader may feel free to skip this remark. When we have a hyperkähler structure $\varpi = (\omega_I, \omega_J, \omega_K)$, the Hermite-Einstein equations can be rewritten into the following equivalent form:

$$iF_\nabla \wedge (\omega_I, \omega_J, \omega_K) = \left(\frac{1}{2}\lambda\omega_I^2 1_E,\ -2\pi B\wedge\omega_J,\ -2\pi B\wedge\omega_K\right) \tag{3.4.10}$$

This follows from splitting the equation $F_\nabla \wedge \sigma_I = 2\pi i B \wedge \sigma_I$ into the hermitian and skew-hermitian parts and remembering that $\sigma_I = \omega_J + i\omega_K$ for a hyperkähler structure.

We can make one more interesting simplification. Both $B\wedge\omega_J$ and $B\wedge\omega_K$ are top degree forms, and hence both can be expressed as a function multiple of ω_J^2 and ω_K^2. If we assume that the B-field is a harmonic representative of $-c_1(E)/rk(E)$ with respect to the hyperkähler metric g induced by ϖ, then both $B\wedge\omega_J$ and $B\wedge\omega_K$ are also harmonic, which is a consequence of the Lefschetz operators L_{ω_J} and L_{ω_K} preserving harmonic forms. Therefore, $B\wedge\omega_J$ and $B\wedge\omega_K$ are both constant multiples of ω_J^2 and ω_K^2. After absorbing minus signs and multiples of 2π, the Hermite-Einstein equations now take the form

$$iF_\nabla \wedge \varpi = \frac{1}{2}(\lambda_I, \lambda_J, \lambda_K)\omega_I^2 \tag{3.4.11}$$

where the symbol $iF_\nabla \wedge \varpi$ means wedging with each component of ϖ. If we *define* $\varpi^2 = \omega_I^2 = \omega_J^2 = \omega_K^2$ then we can write this as

$$iF_\nabla \wedge \varpi = \frac{1}{2}\lambda\varpi^2 1_E \tag{3.4.12}$$

where $\lambda = (\lambda_I, \lambda_J, \lambda_K)$, which is a rather compact way of expressing the Hermite-Einstein equations for a hyperkähler structure. Equation (3.4.12) contains the information about the twisted holomorphic structure of E as well as the slope stability information held by λ_I.

On the other hand, given a hyperkähler structure ϖ, a hermitian bundle E, and a hermitian connection ∇ satisfying equations (3.4.12), we can immediately deduce

$$F_\nabla \wedge \sigma_I = 2\pi i \left(-\frac{\lambda_J + i\lambda_K}{4\pi}\bar{\sigma}\right) \wedge \sigma_I 1_E.$$

Thus, we merely need to choose a real 2-form B having the property $B^{0,2} = -(\lambda_J + i\lambda_K)\bar{\sigma}/4\pi$, and $B = -(\lambda_J\omega_J + \lambda_K\omega_K)/2\pi$ will meet this requirement. Thus, the twisted Hermite-Einstein equations are equivalent to the compact form in equation (3.4.12).

Deformations Over a Twistor Family

So far we have been working in the general context of Kähler manifolds, but now we turn to K3 surfaces. As before, M is the smooth manifold underlying the Fermat quartic K3, and throughout this section we assume that a hyperkähler structure $\varpi = (\sigma, \omega)$ has been fixed.

The next proposition is not strictly needed in proving the results of the next few chapters, but it illustrates an important point in the philosophy of this thesis. Because twisted bundles are now represented by Dolbeault operators obeying $\overline{\partial}_E^2 = 2\pi i B^{0,2}$, it is a matter of turning on a B-field to move from an untwisted bundle to a twisted one. This can be achieved for stable bundles using a twistor family.

Proposition 3.5.1. *Suppose that E admits an ω-(poly)stable σ-holomorphic structure. Then E deforms as a twisted ω_t-(poly)stable holomorphic bundle over the twistor family $\mathcal{X}(\varpi)$.*

Proof. The (untwisted) Kobayashi-hitchin correspondence implies that E admits a Hermite-Einstein connection ∇, which satisfies

$$F_\nabla \wedge \sigma = 0$$
$$iF_\nabla \wedge \omega = \frac{\lambda}{2}\omega^2 \cdot 1_E\,. \tag{3.5.13}$$

Let $B := -\frac{\lambda}{4\pi}\omega$ and $\lambda_t := \frac{1-|t|^2}{1+|t|^2}\lambda$. We claim that ∇ satisfies the following equations:

$$F_\nabla \wedge \sigma_t = 2\pi i B \wedge \sigma_t$$
$$iF_\nabla \wedge \omega_t = \frac{\lambda_t}{2}\omega_t^2 \cdot 1_E$$

This can be checked using equations (2.4.11) for σ_t and ω_t. Note that we need both of equations (3.5.13) to deduce these equations. Using the twisted Kobayashi-Hitchin correspondence in reverse, we learn that E is (poly)stable for $t \neq 0$.

Irreducibility of ∇ is independent of the complex structure on X since it is a purely differential-geometric condition. So, if the bundle is stable, it remains so over the deformation. In either case (stable or polystable), ∇ endows E with an ω_t-stable twisted holomorphic structure for each t, which we denote by E_t.

We now construct a twisted bundle $\mathcal{E} \to \mathcal{X}$ that restricts to E_t on $M \times \{t\}$. Let $\mathcal{E} := p^*E$, which is a smooth bundle over $\mathcal{X}$, and endow $\mathcal{E}$ with the pullback connection $p^*\nabla$. Since $F_\nabla - 2\pi i B$ is type $(1,1)$ for each σ_t, Lemma 2.4.1 implies that $p^*F_\nabla - 2\pi i p^*B$ is type $(1,1)$ on $\mathcal{X}$. By Proposition 3.4.7, $\mathcal{E}$ corresponds to an α-twisted vector bundle over $\mathcal{X}$, where the Brauer class is $\alpha = \exp(p^*B) \in H^2(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}})$. Furthermore, $(\mathcal{E}, p^*\nabla)|_{M\times\{t\}}$ is just E_t equipped with ∇. Thus, the set of E_t's with ∇ gives rise to a deformation $\mathcal{E}$ of E. $\square$

Remark 3.5.2. The proof of Proposition 3.5.1 refines a result mentioned by Huybrechts and Schröer, [HS03, Proposition 2.3], where it was noted that if $\lambda = 0$ then a bundle E as in the above proposition is hyperholomorphic. Indeed, when $\lambda = 0$, the B-field is zero, so the bundle E remains untwisted over the entire twistor family. When $\lambda \neq 0$, only the projectivized bundle $\mathbb{P}(E)$ deforms, in which case the bundle is called *projectively hyperholomorphic*. Our interpretation instead says that the bundle E deforms as a twisted bundle, which can be thought of as a new characterization of projectively hyperholomorphic.

CHAPTER IV

DEFORMATION TO THE HILBERT SCHEME

Overview

The goal of this chapter is to prove part (1) of Theorem 1, that the moduli space $\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ of B_E-twisted Hermite-Einstein connections on E is nonempty, compact, and deformation-equivalent to a Hilbert scheme (Theorem 4.6.1 below).

Before outlining this chapter we recall some of our conventions. M is the smooth manifold underlying all K3 surfaces. Mukai vectors are triples $(r, c, s) \in H^*(M, \mathbb{Z})$ and need not be algebraic. We will always be working with the Mukai pairing on $H^*(X, \mathbb{Z})$, which will be written as either $\langle \alpha, \beta \rangle$ or $\alpha.\beta$. It is given by

$$\langle \alpha, \beta \rangle = -(\alpha_0.\beta_4) + (\alpha_2.\beta_2) - (\alpha_4.\beta_0)$$

where $(\,.\,)$ denotes the intersection pairing on $H^*(M, \mathbb{Z})$. When a complex 2-form σ defines a complex structure on M, we will sometimes write $\operatorname{Pic}(\sigma)$ to mean the Picard group of the associated K3 surface.

Let $v = (r, c_0, s)$ be the Mukai vector of a hermitian bundle (E, h_E), and assume that r and c_0 are coprime and $0 \leq v^2 + 2 < 2r$. Also suppose that $\varpi = (\sigma, \omega)$ a v-generic hyperkähler structure (in the sense of Definition 2.3.3) on the K3 manifold M. The condition $v^2 + 2 < 2r$ ensures that that the moduli space $\mathcal{M}^s_\omega(v)$ of ω-slope-stable sheaves with Mukai vector v consists entirely of vector bundles, which is proven in Lemma 4.4.2 below. Also fix a real closed 2-form B_E representing $-c_1(E)/\operatorname{rk}(E)$.

We will begin by constructing a C^∞ hermitian line bundle (L, h_L) equipped with a twisted Hermite-Einstein metric; see Lemmas 4.2.2 and 4.2.3 for constructing L and Proposition 4.2.4 for the Hermite-Einstein connection. Tensoring with L yields an isomorphism

$$\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E) \xrightarrow{\sim} \mathcal{M}^{HE}_{\sigma,\omega}(E \otimes L, h_E \otimes h_L, B_E + B_L)$$

where B_L is a B-field for L and the new Mukai vector $v' = (r, c, 1)$. We will select L to ensure that c is primitive, positive, and not algebraic with respect to σ. This final property of c is crucial in deforming the Kähler form from ω to ω' so that the twistor family $\mathcal{X}(\sigma, \omega')$ contains a K3 surface whose Picard group is generated by c with c being an ample class; see Proposition 4.3.3. The deformation of ω to ω' will be done within a single chamber of the σ-Kähler cone, thus preserving slope stability and hence producing an isomorphism

$$\mathcal{M}^{HE}_{\sigma,\omega}(E \otimes L, h_E \otimes h_L, B_E + B_L) \cong \mathcal{M}^{HE}_{\sigma,\omega'}(E \otimes L, h_E \otimes h_L, B_E + B_L).$$

We then perform the aforementioned twistor rotation on both the underlying K3 surface and the moduli space. That the moduli space inherits the hyperkähler structure necessary for twistor rotation is detailed in Proposition 4.3.6 below. We will let (σ_t, ω'_t) be the complex structure and Kähler form arising from the twistor rotation beginning at the hyperkähler structure (σ, ω').

At this point we will have demonstrated a deformation-equivalence

$$\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E) \sim \mathcal{M}^{HE}_{\sigma_t,\omega'_t}(E \otimes L, h_E \otimes h_L, B_E + B_L).$$

Due to $c = c_1(E \otimes L)$ being algebraic with respect to σ_t, the latter moduli space consists of untwisted Hermite-Einstein connections, and the Kobayashi-Hitchin correspondence allows us to identify this moduli space with the moduli space $\mathcal{M}^s_{\sigma_t,\omega'_t}(E \otimes L)$ of ω'_t-stable holomorphic structures on $E \otimes L$. Using the fact that any two C^∞ bundles with equal Chern characters are C^∞-isomorphic (Proposition 4.4.1 below) we can identify $\mathcal{M}^s_{\sigma_t,\omega'_t}(E \otimes L)$ with $\mathcal{M}^s_{\sigma_t,\omega'_t}(r, c, 1)$, the moduli space of stable bundles with Mukai vector $(r, c, 1)$. Changing the polarization from ω'_t to c, dualizing, and applying the spherical twist (Proposition 4.5.3) finally establishes an isomorphism

$$\mathcal{M}^{HE}_{\sigma_t,\omega'_t}(E \otimes L, h_E \otimes h_L, B_E + B_L) \cong X^{[n]}_{\sigma_t}.$$

This will finish our proof that $\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ is deformation-equivalent to a Hilbert scheme on a K3 surface. We also learn that $\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ is nonempty and compact, the latter following from the property that c is a v-generic polarization.

Tensoring with a Line Bundle

Our first goal will be to construct the C^∞ line bundle L discussed in the section introduction. We begin with a lattice-theoretic lemma.

Lemma 4.2.1. *Let $x \in H^2(M, \mathbb{Z})$ be a primitive class, and let p, q be arbitrary integers. Then there exists a class $y \in H^2(M, \mathbb{Z})$ with $x.y = p$ and $(y)^2 = 2q$.*

Proof. We begin by fixing an isometry $H^2(M, \mathbb{Z}) \cong U \oplus U \oplus U \oplus (-E_8)^{\oplus 2}$. Taking the standard basis e, f of U satisfying $e^2 = f^2 = 0$, $e.f = 1$, we write an element of $H^2(M, \mathbb{Z})$ in the first two copies of U as a four-tuple $(\alpha, \beta, \gamma, \delta)$.

Let $(x)^2 = 2d$. Since x is primitive, it is unique up to the action of $\mathrm{O}(\Lambda)$ among primitive vectors with square $2d$. Thus there is some $g \in \mathrm{O}(\Lambda)$ for which $g(x) = (1, d, 0, 0) \in U \oplus U$. Take y to be the vector such that $g(y) = (0, p, 1, q) \in U \oplus U$. $\square$

Lemma 4.2.2. *Suppose $v = (r, c_0, s)$ is primitive and r is coprime to c_0. Then there exists a class $\ell \in H^2(M, \mathbb{Z})$ such that $v.\exp(\ell) = (r, c_1, 1)$, where c_1 is primitive.*

Proof. Let $c_0 = kc_0'$ with c_0' primitive and k an integer. We know that $\gcd(r, k) = 1$, so choose integers a, b such that $ak + br = 1 - s$. By Lemma 4.2.1 there is an ℓ with $c_0'.\ell = a$ and $\ell^2 = 2b$. It follows that $kc_0'.y + ry^2/2 = 1 - s$, whence

$$v.e^\ell = (r, c_0 + r\ell, 1).$$

Using the notation in the proof of Lemma 4.2.1, $c_1 := c_0 + r\ell$ is primitive because

$$g(c_1) = (k, kd - rb, r, ra) \in U \oplus U$$

is primitive. $\square$

Next is another short calculation that will ensure that the first Chern class of our bundle is not algebraic.

Lemma 4.2.3. *Let X be a K3 surface and let $c_1 \in H^2(X, \mathbb{Z})$ be a primitive vector with $c_1^2 \neq 0$. Then there exists a class $\ell' \in H^2(X, \mathbb{Z})$ such that $c_1.\ell' = 0$, $(\ell')^2 = 0$, and $c_1 + r\ell' \notin \mathrm{Pic}(X)$.*

Proof. Right away, if c_1 is not algebraic, then we take $\ell' = 0$ and we are done. So, we assume that c_1 is algebraic. Under this assumption it follows that $c_1 + r\ell' \in \text{Pic}(X)$ if and only if $\ell' \in \text{Pic}(X)$, so we now seek an isotropic $\ell' \notin \text{Pic}(X) \cap c_1^{\perp}$.

Let $c_1^2 = 2d \neq 0$. We identify $H^2(X, \mathbb{Z})$ with $U^3 \oplus (-E_8)^2$ in a way that sends c_1 to the vector $e_1 + df_1$, where e_1, f_1 are the standard basis for the first copy of U. Then $c_1^{\perp}$ is identified with $(-d) \oplus U^2 \oplus (-E_8)^2$, where $(-d)$ is the one-dimensional lattice $\mathbb{Z}$ whose generator has length $-d$. Next, note that $\text{Pic}(X) \cap c^{\perp}$ can have rank at most 20, while $c_1^{\perp}$ has rank 21 since $c_1^2 \neq 0$. Since we can find a basis of $c_1^{\perp}$ consisting of isotropic vectors, we will certainly be able to ensure one of these isotropic basis elements, call it ℓ', does not lie in $\text{Pic}(X)$. □

Summarizing the last two lemmas, we now have the following proposition.

Proposition 4.2.4. *Let X be a K3 surface and $v = (r, c_0, s)$ a primitive Mukai vector with coprime r and c_0. Also suppose that E is a smooth vector bundle with Mukai vector v. Then there exists a smooth line bundle L such that $v(E \otimes L) = (r, c, 1)$, c is primitive, and c is not algebraic on X.*

Proof. Combining Lemmas 4.2.2 and 4.2.3, we can find a topological line bundle L with $c_1(L) = \ell + \ell'$. The properties of ℓ and ℓ' imply that

$$
\begin{aligned}
v(E \otimes L) &= (r, c_0, s)e^{\ell+\ell'} \\
&= (r, c, 1)
\end{aligned}
$$

where, with the notation of the previous lemmas,

$$c = c_0 + r(\ell + \ell') = c_1 + r\ell'.$$

The properties of ℓ' ensure that c is not algebraic. □

Line bundles always admit Hermite-Einstein connections when the bundle is holomorphic. When the bundle is not holomorphic, that is, when $c_1(L)^{0,2} \neq 0$, we can construct a twisted Hermite-Einstein metric on L.

One usually prefers to think of line bundles as naturally untwisted objects, since any Brauer class α that twists a line bundle must necessarily be trivial. This is because the twisted cocycle condition implies α_{ijk} is a coboundary when the transition matrices φ_{ij} are rank one. However, it is crucial that we make use of this twisted structure on line bundles to construct a twisted Hermite-Einstein connection on L, as will be seen in the next proposition.

Proposition 4.2.5. *Let g be the hyperkähler metric determined from a hyperkähler structure $\varpi = (\sigma, \omega)$ (see Lemma 2.3.2). Let (E, h) be a hermitian bundle on M and L an arbitrary smooth line bundle on M. Then L admits a B_L-twisted Hermite-Einstein metric h_L, where B_L is the g-harmonic representative of $-c_1(L)$. Consequently, tensoring with L yields an isomorphism*

$$\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E) \xrightarrow{\sim} \mathcal{M}^{HE}_{\sigma,\omega}(E \otimes L, h_E \otimes h_L, B_E + B_L).$$

Proof. First, fix a hermitian structure h_L and hermitian connection ∇_0 on L. Since $iF_{\nabla_0}/2\pi$ is an ordinary 2-form on X that represents $c_1(L)$, $iF_{\nabla_0}/2\pi = -B_L + d\alpha$ for some real 1-form α. Then set $\nabla_L := \nabla_0 + 2\pi i\alpha$. This connection is still hermitian and its curvature satisfies $iF_{\nabla_L}/2\pi = -B_L$.

It now follows that ∇_L satisfies the twisted Hermite-Einstein equations. Indeed, Λ_ω preserves harmonicity, and hence it takes iF_{∇_L} to $i\Lambda_\omega F_{\nabla_L} \in \mathscr{H}^0 = \mathbb{R}$. The other equation, $F^{0,2}_{\nabla_L} = 2\pi i B^{0,2}_L$, is trivially satisfied since $F_{\nabla_L} = 2\pi i B_L$.

From here it is a straightforward check that if ∇ is a B-twisted Hermite-Einstein connection on E, then $\nabla \otimes \nabla_L := \nabla \otimes 1_L + 1_E \otimes \nabla_L$ is $(B + B_L)$-twisted on $E \otimes L$ with respect to the induced hermitian structure $h_E \otimes h_L$. This operation is invertible with inverse given by tensoring with $L^\vee$. Furthermore, tensoring with ∇_L preserves irreducibility of ∇.

Finally, we note that a gauge orbit of ∇ is mapped onto the gauge orbit of $\nabla \otimes \nabla_L$. This follows from the isomorphism in Lemma 4.2.6 below and by noting that the map $\nabla \mapsto \nabla \otimes \nabla_L$ commutes with the action of $U(E, h)$ and $U(E \otimes L, h \otimes h_L)$. □

Lemma 4.2.6. *Let (E, h_E) be a smooth hermitian bundle and (L, h_L) a smooth hermitian line bundle. Then the map $\Phi\colon U(E, h_E) \to U(E \otimes L, h_E \otimes h_L)$ defined by $g \mapsto g \otimes 1_L$ is an isomorphism.*

Proof. We will break this down into two steps. First we show that the map $\Phi\colon \mathrm{End}(E) \to \mathrm{End}(E \otimes L)$ is an isomorphism, and then that Φ preserves unitary endomorphisms.

To see that Φ is an isomorphism consider the sequence of maps

$$\mathrm{End}(E) \xrightarrow{\Phi} \mathrm{End}(E \otimes L) \xrightarrow{\Phi'} \mathrm{End}(E \otimes L \otimes L^\vee) \overset{\mathrm{tr}_L}{\cong} \mathrm{End}(E) \xrightarrow{\Phi} \mathrm{End}(E \otimes L)$$

where $\mathrm{tr}_L \colon L \otimes L^\vee \xrightarrow{\sim} \mathcal{O}$ is the usual trace map, and Φ' is defined as Φ but with $L^\vee$ in place of L. It is straightforward to verify that $\mathrm{tr}_L \circ \Phi' \circ \Phi = 1_{\mathrm{End}(E)}$ and $\Phi \circ \mathrm{tr}_L \circ \Phi' = 1_{\mathrm{End}(E \otimes L)}$.

Now let $g \in U(E, h_E)$ be a unitary endomorphism. Working locally on M, g is represented by an h_E-unitary matrix. Since L is a line bundle, $g \otimes 1_L$ is represented by the same matrix as g. Furthermore, h_L is locally represented as a positive, real-valued scalar function, which does not change how the conjugate-transpose of g is computed. (Recall that when H is a hermitian matrix defining a hermitian structure on $\mathbb{C}^n$, the adjoint of an $n \times n$ matrix A is computed as $A^* = H^{-1}\overline{A}^T H$, which is invariant under multiplying H by a nonzero real number.) □

Deforming the Kähler Form

We now focus on deforming the Kähler form in a hyperkähler structure $\varpi = (\sigma, \omega)$. The goal is to deform ω and keep σ fixed so that a twistor rotation around $\varpi' = (\sigma, \omega')$ contains a point where the Picard group is generated by c.

Before embarking on this I will explain the idea behind the deformation. The goal is to find a hyperkähler structure $\varpi' = (\sigma, \omega')$ and a twistor rotation (σ_t, ω'_t) through the twistor family of ϖ' such that the class c is algebraic for σ_t (see Lemma 2.4.3) and $\mathrm{Pic}(\sigma_t) = \mathbb{Z}c$. (See equations (2.4.11) for the definition of σ_t.) Certainly $\mathrm{Pic}(\sigma_t)$ will contain $\mathbb{Z}c$, but for some choices of ω' it may be larger. If $\mathrm{Pic}(\sigma_t)$ is larger than $\mathbb{Z}c$, then we will be able to find a nonzero $\alpha \in \mathrm{Pic}(\sigma_t)$ orthogonal to c. This produces an affine[1] line $\alpha + \mathbb{R}c$, which we will show below must intersect $P^\perp_{[\varpi']}$. Turning this idea on its head, for fixed c we ask for Kähler forms ω' where $(\alpha + \mathbb{R}c) \cap P^\perp_{[\varpi']} = \emptyset$ for all $\alpha \in c^\perp \cap H^2(M, \mathbb{Z})$. This can be thought of a genericity condition on hyperkähler structures or positive oriented 3-planes that is similar in spirit to the idea of a generic twistor line from [Huy16, Ch. 7.3].

[1] Affine here means that the line does not pass through the origin.

Lemma 4.3.1. *Let $c \in H^2(M,\mathbb{Z})$ be a class with $c^2 > 0$, and let $P_{[\sigma_t]} \subset P_{[\varpi]}$ be the 2-plane associated to a complex structure σ_t in the twistor family for ϖ where c is algebraic. Suppose that* $\mathrm{Pic}(\sigma_t) \cap c^\perp$ *contains a nonzero class α. Then $(\alpha + \mathbb{R}c) \cap P_{[\varpi]}^\perp \neq \emptyset$.*

Proof. Recall from Lemma 2.4.3 that the complex structures σ_t in the twistor family where c is algebraic are determined by the constraint $P_{[\sigma_t]} = c^\perp \cap P_{[\varpi]}$. Taking the perpendicular complement, we get a description of all of the $(1,1)$-classes:

$$H^{1,1}(\sigma_t) = P_{[\sigma_t]}^\perp = P_{[\varpi]}^\perp + \mathbb{R}c.$$

Since $\alpha \in H^{1,1}(\sigma_t)$, we can write $\alpha = \beta + rc$ with $\beta \in P_{[\varpi]}^\perp$ and $r \in \mathbb{R}$. Then $\beta = \alpha - rc$ lies in $(\alpha + \mathbb{R}c) \cap P_{[\varpi]}^\perp$. □

Lemma 4.3.2. *Let $v = (r, c, s)$ be a Mukai vector, let $\varpi = (\sigma, \omega)$ be a v-generic hyperkähler structure, and assume that $c^2 > 0$ and c is not algebraic with respect to σ. Then there exists a v-generic hyperkähler structure $\varpi' = (\sigma, \omega')$ such that $P_{[\varpi']}^\perp \cap (\alpha + \mathbb{R}c) = \emptyset$ for all $\alpha \in H^2(M,\mathbb{Z}) \cap c^\perp$. Moreover, the class $[\omega']$ can be taken close to $[\omega]$, so that ω and ω' define the same stability condition.*

Proof. Fix $\alpha \in H^2(M,\mathbb{Z}) \cap c^\perp$, and let $\ell_\alpha = \alpha + \mathbb{R}c$ be its corresponding affine line. We will also consider the real 2-plane $L_\alpha = \mathbb{R}\alpha + \mathbb{R}c$ spanned by ℓ_α.

We start by describing the problematic α resulting in $\ell_\alpha \cap P_{[\sigma]}^\perp \neq \emptyset$. First note that $c \in L_\alpha$, and that $c \notin P_{[\sigma]}^\perp$, for otherwise c would be algebraic with respect to σ. Therefore, we know $L_\alpha \cap P_{[\sigma]}^\perp \neq L_\alpha$. It follows that $L_\alpha \cap P_{[\sigma]}^\perp$ has dimension 0 or 1 (as a vector space). If this dimension is zero, then it immediately follows that $\ell_\alpha \cap P_{[\sigma]}^\perp = \emptyset$. On the other hand, if the dimension is 1, the line $L_\alpha \cap P_{[\sigma]}^\perp$ may either be parallel to ℓ_α (leaving $\ell_\alpha \cap P_{[\sigma]}^\perp = \emptyset$ again) or, more generically, will meet ℓ_α. We then see that a nonzero class $\alpha \in H^2(M,\mathbb{Z}) \cap c^\perp$ will be problematic when $\ell_\alpha \cap P_{[\sigma]}^\perp = \{z_\alpha\}$ for some nonzero vector z_α.

Each problematic α leads to a hyperplane section $K_\sigma \cap z_\alpha^\perp$ of the Kähler cone for σ. Consider the complement of their union

$$Z = \mathcal{K}_\sigma \setminus {\bigcup_\alpha}' z_\alpha^\perp$$

(the prime on the union indicating only union over the problematic α). Since each $z_\alpha^\perp$ is codimension 1 and since K_σ is an open cone in $H^{1,1}(\sigma)$, the Baire category theorem implies that Z is dense in $\mathcal{K}_\sigma$. Since we assumed that ϖ is v-generic, the Kähler class $[\omega]$ does not lie on a v-wall in the Kähler cone. Thus, density of Z allows us to find a Kähler class $[\omega_0'] \in Z$ as close as we like to $[\omega]$. Since the v-walls are locally finite in K_σ ([HL10, Lem. 4.C.2]) we can ensure $[\omega_0']$ lies in the interior of the same v-chamber as $[\omega]$. Therefore, $[\omega_0']$ can be taken to be v-generic.

We now use Yau's theorem in the following form: for a compact Kähler manifold X with given volume form vol compatible with the natural orientation, there is a unique Kähler metric g' with Kähler form ω' such that $(\omega')^{\dim X} = \mathrm{vol}$ with prescribed Kähler class $[\omega_0'] \in H^2(X, \mathbb{R})$. We take $\mathrm{vol} = \frac{1}{2}\sigma \wedge \bar{\sigma}$ and $[\omega_0']$ as our Käher class. This ensures that $2(\omega')^2 = \sigma \wedge \bar{\sigma}$, which entails that $\varpi' = (\sigma, \omega')$ is a hyperkähler structure. We also know that $[\omega'] = [\omega_0']$, so $[\omega']$ lies in the same v-chamber as $[\omega]$.

Having chosen $[\omega'] \in Z$, we see that $[\omega'].z_\alpha \neq 0$ for problematic α. Hence, $P_{[\varpi']}^\perp$ will not intersect any ℓ_α for $\alpha \in H^2(M, \mathbb{Z}) \cap c^\perp$; either such an ℓ_α fails to intersect $P_{[\sigma]}^\perp$, or, if it does, then ℓ_α will fail to intersect $[\omega']^\perp$. Either way, we can guarantee $\ell_\alpha \cap P_{[\varpi']}^\perp = \emptyset$. □

The next proposition combines the last two lemmas.

Proposition 4.3.3. *Let $v = (r, c, s)$ be a Mukai vector, and let $\varpi = (\sigma, \omega)$ be a v-generic hyperkähler structure. Assume that c is primitive, $c^2 > 0$, and that c is not algebraic with respect to σ. Then there is a twistor rotation (σ_t, ω_t') of the hyperkähler structure ϖ' from Lemma 4.3.2 such that $\mathrm{Pic}(\sigma_t) = \mathbb{Z}c$ and such that c is ample.*

Proof. From Lemma 2.4.3, we know there are two places in the twistor family of ϖ' such that c is algebraic with respect to the complex structure σ_t, and the two complex structures are determined by the two possible orientations of the 2-plane $c^\perp \cap P_{[\varpi']}$. Precisely one of these orientations ensures that c is in the positive cone. To see this, suppose that $c^\perp \cap P_{[\varpi']}$ is spanned by classes α and β with $\alpha^2 = \beta^2 = 1$ and $\alpha.\beta = 0$, and let $\gamma \in P_{[\varpi']}$ be orthogonal to α and β so that the orientation of $P_{[\varpi']}$ is realized by the ordering $\langle \alpha, \beta, \gamma \rangle$. The two complex structures are given by $\sigma_1 = \alpha + i\beta$ and $\sigma_2 = \beta + i\alpha$. The Kähler forms for σ_1 and σ_2 are γ and $-\gamma$, respectively. If $\gamma.c > 0$, then c is in the positive cone of σ_1, and if $\gamma.c < 0$ then c is in the positive cone for σ_2. Either way, it is possible to choose a complex structure in the twistor family of ϖ' for which c is algebraic and lies in the positive cone.

By Lemma 4.3.2, both of these complex structures have Picard group equal to $\mathbb{Z}c$, and therefore neither of these K3 surfaces can contain a rational curve $\mathbb{P}^1 \cong C \subset X$, for such a curve would produce the line bundle $\mathcal{O}_X(C)$ with degree -2, which is impossible.

For K3 surfaces, the Kähler cone is determined by those classes $\alpha \in H^{1,1}(X, \mathbb{R})$ such that α is in the positive cone and $\alpha.C > 0$ for all rational curves $\mathbb{P}^1 \cong C \subset X$ by [Huy16, Thm. 8.5.2]. Since we have no rational curves, the class c is automatically Kähler. Since c is also integral, the Kodaira Embedding Theorem implies that c is ample. □

This finishes the description of the deformation of ω to ω'. However, I do not know of an explicit method for taking an ω-Hermite-Einstein connection ∇ and producing an ω'-Hermite-Einstein connetion ∇', even if ω and ω' are cohomologous. Thus I will make use of the Kobayashi-Hitchin correspondence to identify the ω-Hermite-Einstein moduli space with the ω'-Hermite-Einstein moduli space. The next proposition describes the Kobayashi-Hitchin correspondence on the level of moduli spaces.

Proposition 4.3.4. *Let E be a smooth bundle with hermitian structure h, and let (X, ω) be a compact connected Kähler manifold. Then the twisted Kobayashi-Hitchin correspondence induces a complex-analytic isomorphism between the moduli space $\mathcal{M}^{HE}_\omega(E, h, B)$ of B-twisted ω-Hermite-Einstein connections on E and the moduli space $\mathcal{M}^{st}_\omega(E, \alpha)$ of α-twisted ω-stable holomorphic structures on E, where $\alpha = \exp(2\pi i B^{0,2})$.*

Proof. To simplify notation let $\mathcal{M}^{HE}$ and $\mathcal{M}^{st}$ be the two moduli spaces in question.

Though never explicitly said, Kobayashi essentially proves this in his book [Kob14]. We will sketch the roadmap through this book (chapter 7 in particular) to prove this proposition. Furthermore, the central results and arguments from this book go through exactly the same when E is a smooth bundle arising from untwisting a twisted bundle. This is because $\mathrm{End}(E)$ is naturally an untwisted bundle, and any twisted connection on E gives rise to an untwisted connection on $\mathrm{End}(E)$. The arguments and results of [Kob14] are based on the induced connections on $\mathrm{End}(E)$, so the arguments go through the same for the twisted connections.

The twisted Kobyashi-Hitchin correspondence (Theorem 2.2.3) gives a set bijection

$$\begin{aligned}\Phi\colon \mathcal{M}^{HE} &\to \mathcal{M}^{st}\\ [\nabla] &\mapsto [\nabla^{0,1}]\end{aligned}$$

By analyzing the local models for each moduli space we will see that this turns into a complex-analytic isomorphism.

Fix a connection ∇ representing a point $[\nabla] \in \mathcal{M}^{HE}$. A local model of the moduli space near $[\nabla]$ is obtained by taking a neighborhood of origin $\eta = 0$ in the slice

$$S_\nabla = \left\{ \nabla + \eta \;\middle|\; \begin{array}{ll} \eta \in \mathcal{A}^1(\mathrm{End}(E,h)), & \Lambda_\omega(\tilde{\nabla}\eta + \eta \wedge \eta) = 0, \\ \tilde{\nabla}\eta + \eta\wedge\eta \in \mathcal{A}^{1,1}, & \tilde{\nabla}^*\eta = 0 \end{array} \right\}. \tag{4.3.1}$$

The first condition says $\nabla + \eta$ is hermitian; the next two conditions come from writing down the twisted Hermite-Einstein equations (3.4.9) for $\nabla + \eta$, expanding $F_{\nabla+\eta} = F_\nabla + \tilde{\nabla} + \eta \wedge \eta$, and using the fact that ∇ also satisfies the same equations; and the third condition says that S_∇ is perpendicular to the gauge orbit $U(E,h)\cdot\nabla$. If we linearize these equations we get

$$T_\nabla S_\nabla = \left\{ \eta \in \mathcal{A}^1(\mathrm{End}(E,h)) \mid \tilde{\nabla}\eta \in \mathcal{A}^{1,1}(\mathrm{End}(E,h)),\, \Lambda_\omega \tilde{\nabla}\eta = 0,\, \tilde{\nabla}^*\eta = 0 \right\}. \tag{4.3.2}$$

The set on the right is exactly the harmonic space $\mathscr{H}^1(E,\nabla)$ of the complex $C^\bullet(E,\nabla)$ from Section 2.5. Consequently, we can identify $T_{[\nabla]}\mathcal{M}^{HE}$ with $\mathscr{H}^1 \cong H^1(\mathrm{End}(E))$.

Similarly, at a holomorphic structure $\overline{\partial}_E$, a slice for the action of $GL(E)$ on the space $\mathscr{H}_B$ of B-twisted holomorphic structures is given by

$$S_{\overline{\partial}_E} = \{\overline{\partial}_E + \gamma \mid \gamma \in \mathcal{A}^{0,1}(\mathrm{End}(E)),\, \overline{\partial}_{\mathrm{End}(E)}(\gamma) + \gamma\wedge\gamma = 0,\, \overline{\partial}^*_{\mathrm{End}(E)}(\gamma) = 0\}.$$

Linearizing these equations gives

$$T_{\overline{\partial}_E} S_{\overline{\partial}_E} = \left\{ \gamma \in \mathcal{A}^{0,1}(\mathrm{End}(E,h)) \mid \overline{\partial}_E(\gamma) = 0,\, \overline{\partial}^*(\gamma) = 0 \right\}.$$

The latter set is exactly the harmonic space $\mathscr{H}^{0,1}$ of the Dolbeault complex for $\overline{\partial}_{\mathrm{End}(E)}$. Consequently, we can identify $T_{[\overline{\partial}_E]}\mathcal{M}^{st}$ with $\mathscr{H}^{0,1} \cong H^{0,1}(\mathrm{End}(E))$.

Now, when Φ is considered as a map $\Phi\colon \mathcal{A}^1(\mathrm{End}(E,h)) \to \mathcal{A}^{0,1}(\mathrm{End}(E))$, the derivative $d\Phi$ is exactly the map $\eta \mapsto \eta^{0,1}$. Kobayashi shows that $d\Phi$ induces a linear isomorphism on the harmonic spaces

$$d\Phi\colon \mathscr{H}^1 \xrightarrow{\sim} \mathscr{H}^{0,1}. \tag{4.3.3}$$

(see [Kob14, Section 7.2]). We note that $\Phi(S_\nabla)$ is not exactly $S_{\overline{\partial}_E}$, but the isomorphism (4.3.3) shows that $\Phi(S_\nabla)$ is also a slice to the $GL(E)$-orbit through $\overline{\partial}_E$.

It remains to be seen that $d\Phi$ commutes with the complex structures on $\mathscr{H}^1$ and $\mathscr{H}^{0,1}$. The complex structure on $\mathscr{H}^1$ is defined through the complex structure I induced by σ on M via

$$\mathcal{I}\eta = -\eta(I(-)) = -i\eta^{0,1} + i\eta^{0,1}, \tag{4.3.4}$$

while the complex structure on $\mathscr{H}^{0,1}$ is multiplication by i. It is now immediate that $d\Phi$ commutes with these complex structures, and hence $\Phi\colon \mathcal{M}^{HE} \to \mathcal{M}^{st}$ is a holomorphic bijection. Since holomorphic bijections are automatically biholomorphic ([Huy05a, Prop. 1.1.13]) we are done. □

Proposition 4.3.5. *Let E be a smooth bundle with Mukai vector v, and let $\varpi = (\sigma, \omega)$ and $\varpi' = (\sigma, \omega')$ be two v-generic hyperkähler structures on M such that ω and ω' define the same stability condition. If E admits a twisted Hermite-Einstein metric h with B-field 2-form B, then there exists a complex analytic isomorphism*

$$\mathcal{M}^{HE}_{\varpi}(E,h,B) \xrightarrow{\sim} \mathcal{M}^{HE}_{\varpi'}(E,h,B).$$

Proof. Using the Kobayashi-Hitchin correspondence in the form of Proposition 4.3.4, we have the chain of isomorphisms

$$\mathcal{M}^{HE}_{\varpi}(E,h,B) \cong \mathcal{M}^{st}_{\varpi}(E,\alpha) \cong \mathcal{M}^{st}_{\varpi'}(E,\alpha) \cong \mathcal{M}^{HE}_{\varpi'}(E,h,B).$$

Note that the B-field remains fixed since ϖ and ϖ' share the same holomorphic structure σ. □

Proposition 4.3.6. *Let (σ, ω) be any v-generic hyperkähler structure on the K3 manifold M and E a smooth bundle with Mukai vector v. The hyperkähler metric g corresponding to (σ, ω) induces a hyperkähler metric on $\mathcal{M} := \mathcal{M}^{HE}_{\sigma,\omega}(E, h, B)$. Moreover, the complex structures $\mathcal{I}_t$ in the twistor family of $\mathcal{M}$ all arise from complex structures I_t in the twistor family $\mathcal{X}(\sigma, \omega)$. Furthermore, the twistor rotation $(\mathcal{M}, \mathcal{I}_t)$ is isomorphic to the moduli space $\mathcal{M}^{HE}_{\sigma_t,\omega_t}(E, h, B)$. Also, the three hyperkähler forms $\Omega_I, \Omega_J, \Omega_K$ on $\mathcal{M}$ are given by*[2]

$$\Omega_\ell(\eta, \xi) = 2 \int_M \operatorname{tr}(\eta \wedge \xi) \wedge \omega_\ell, \qquad \ell = I, J, K.$$

Proof. The statement that g induces a hyperkähler metric on $\mathcal{M}$ can be found in [IN90, Thm. 2.17], and in [Kob14, Ch. VII.6] Kobayashi shows that the Kähler forms are the Ω_ℓ described above.

A similar computation to the one performed in Proposition 3.5.1 shows that the underlying set of the moduli space is unaffected from the Kähler rotation. The local models S_∇ are also preserved by the hyperkähler rotation, which can be seen from analyzing equation (4.3.1).

Any complex structure I in the twistor family for g defines a complex structure on $\mathscr{H}^1$ by the formula

$$\mathcal{I}(\eta) = -\eta(I(-)). \tag{4.3.5}$$

So, the complex structure $\mathcal{I}'$ on $\mathcal{M}^{HE}_{\sigma_t,\omega_t}(E, h, B)$ is given by $\mathcal{I}'(\eta) = -\eta(I_t(-))$, which is exactly the formula for $\mathcal{I}_t$. Hence these two moduli spaces are the same. ☐

Miscellaneous Results on Vector Bundles

We now collect some results on vector bundles that aid in the proof of Theorem 4.6.1.

Proposition 4.4.1. *If E and E' are two C^∞ complex vector bundles over the K3 manifold with the same rank and Chern classes, then there is a C^∞ isomorphism $E \xrightarrow{\sim} E'$.*

Proof. The result is standard for $r = 1$, so we assume $r \geq 2$. The Chern character gives an isomorphism $\operatorname{ch}\colon K_{top}(M) \to H^*(M, \mathbb{Z})$, which is true because we are working on K3 surfaces. It follows that the bundles E and E' have the same classes in K_{top}, which implies that $E \oplus \mathcal{O}^\ell =$

[2] The factor of 2 ensures this Kähler form corresponds with the L^2 metric on $\mathscr{H}^1$.

$E' \oplus \mathcal{O}^\ell$ for some topologically trivial bundle $\mathcal{O}^\ell$. Since $r \geq 2$, [Hus94, Theorem 9.1.5] implies that E and E' are topologically isomorphic. Finally, any two C^∞ vector bundles that are topologically isomorphic are smoothly isomorphic, see [Hir94, Theorem 3.5]. □

Lemma 4.4.2. *Let $v = (r, c, s)$ be a primitive Mukai vector satisfying $v^2 + 2 < 2r$ and let H be a v-generic polarization on a K3 surface X. Then any H-slope-stable sheaf E is locally free.*

Proof. I would like to thank Daniel Huybrechts for explaining this result. Suppose E an ω-stable sheaf with Mukai vector v that is not locally free. Let $E^{\vee\vee}$ be the reflexive hull, which is locally free, and let T be the cokernel of the inclusion $E \hookrightarrow E^{\vee\vee}$. The singular set of E is at least codimension 2 since E is torsion-free, so T is supported on a finite set of points, and $v(T) = (0, 0, n)$ for some integer $n \geq 1$.

Next we show that $E^{\vee\vee}$ is also H-slope stable. To see this, suppose $F \subset E^{\vee\vee}$ is a subsheaf with $0 < \mathrm{rk}(F) < \mathrm{rk}(E^{\vee\vee})$ and consider the diagram below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F \cap E & \longrightarrow & F & \longrightarrow & T' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & E^{\vee\vee} & \longrightarrow & T & \longrightarrow & 0 \end{array} \tag{4.4.6}$$

Since $T' \subset T$, T' is also supported at a finite number of points, and thus $v(T') = (0, 0, m)$ for some integer $m \leq n$. So, the top row of the diagram gives us $\mathrm{rk}(F \cap E) = \mathrm{rk}(F)$ as well as $\deg(F \cap E) = \deg(F)$, so $\mu(F \cap E) = \mu(F)$. Likewise, $\mu(E) = \mu(E^{\vee\vee})$, so the fact that E is stable gives

$$\mu(F) = \mu(F \cap E) < \mu(E) = \mu(E^{\vee\vee}).$$

Finally, we see that $v(E^{\vee\vee}) = (r, c, s + n)$, so

$$v(E^{\vee\vee})^2 + 2 = v(E)^2 + 2 - 2rn < 2r - 2rn \leq 0$$

by our assumption that $v^2 + 2 < 2r$ and $n \geq 1$. This implies that $E^{\vee\vee}$ belongs to the smooth locus of a moduli space of sheaves with negative dimension, a contradiction. Therefore E must be locally free. □

Proposition 4.4.3. *Let E be a C^∞ bundle on M with primitive Mukai vector $v = (r, c, s)$ satisfying $v^2 + 2 < 2r$ and let $\varpi = (\sigma, \omega)$ be a v-generic hyperkähler structure. Assume further that the moduli space $\mathcal{M}^s_\varpi(v)$ is nonempty. Then $\mathcal{M}^s_\varpi(E)$ is isomorphic to $\mathcal{M}^s_\varpi(v)$.*

Proof. By Lemma 4.4.2, any sheaf in $\mathcal{M}^s_\varpi(v)$ is locally free, and by Proposition 4.4.1 its underlying smooth bundle is smoothly isomorphic to E. This produces a holomorphic structure on E, and thus $\mathcal{M}^s_\varpi(E)$ is also nonempty.

Let $F\colon \mathcal{M}^s_\varpi(E) \to \mathcal{M}^s_\varpi(v)$ be the map sending a holomorphic structure $\overline{\partial}_E$ to the holomorphic bundle $[(E, \overline{\partial}_E)]$. Surjectivity of F follows by the observations in the previous paragraph. This map is also seen to be injective, for if two holomorphic structures $\overline{\partial}_E, \overline{\partial}'_E$ map to the same holomorphic bundle, then we get a bundle endomorphism $f \in GL(E)$ commuting with $\overline{\partial}_E$ and $\overline{\partial}'_E$; hence $[\overline{\partial}_E] = [\overline{\partial}'_E]$ in $\mathcal{M}^s_\varpi(E)$. This follows from the fact that $\mathcal{M}^s_\varpi(E)$ is a quotient of a space of $\overline{\partial}$ operators modulo $GL(E)$.

Finally, note that the tangent spaces to both moduli spaces are $H^1(\mathrm{End}(E))$. It is then seen that F induces a holomorphic bijection, which must be a complex-analytic isomorphism. □

Hilbert Schemes of Points

We now recall some properties of the Hilbert scheme $X^{[n]}$ of n points on a K3 surface. We identify $X^{[n]}$ with the moduli space of c-stable sheaves $\mathcal{M}^s_c(1, 0, 1-n)$ by associating a subscheme of length n with its ideal sheaf $\mathcal{I}_n$. Later we will encounter the moduli space $\mathcal{M}^s_c(1, c, r)$, which is isomorphic to $X^{[n]}$ by tensoring with $\mathcal{O}_X(-c)$, where $n := \frac{c^2}{2} - r + 1$. Note that the assumption $v^2 + 2 \geq 0$ ensures n is nonnegative. We let $\mathcal{Z} \subset X \times X^{[n]}$ be the universal subscheme and $\mathcal{I}_\mathcal{Z}(c) = \mathcal{I}_\mathcal{Z} \otimes \pi_2^*\mathcal{O}_X(c)$ for the ideal sheaf of $\mathcal{Z}$ twisted by c. We also let $\Delta \subset X \times X$ be the main diagonal. Note that $\mathcal{I}_\Delta$ is the kernel for the spherical twist. For a review of Fourier-Mukai transforms, see [Huy06, Ch. 5].

Lemma 4.5.1. *Let X be a K3 surface with* $\mathrm{Pic}(X) = \mathbb{Z}c$ *with c ample and with $c^2 < 4r - 2$. Every sheaf of the form $\mathcal{I}_n(c)$ is globally generated and acyclic.*

Proof. The reduced Hilbert polynomial of $\mathcal{I}_n(c)$ is given by

$$p_{c,\mathcal{I}_n(c)} = \frac{m^2}{2} + m + \mathrm{const},$$

and since $p_{c,\mathcal{O}_X} = m^2/2 + \text{const}$, we have $p_{c,\mathcal{O}_X} < p_{c,\mathcal{I}_n(c)}$. Therefore,

$$H^2(\mathcal{I}_n(c)) \cong \operatorname{Hom}(\mathcal{I}_n(c), \mathcal{O})^* = 0.$$

We next look at vanishing of H^1. Markman proves in [Mar01, Corollary 34] that the locus of sheaves in $\mathcal{M}^s_H(r,c,s)$ having nonzero H^1 is empty when $\operatorname{Pic}(X) = \mathbb{Z}H$ and

$$\dim(\mathcal{M}^s_H(r,c,s)) \leq 2(r+s). \tag{4.5.7}$$

Our moduli space has dimension $c^2 - 2r + 2$ which is less than $2(r+1)$ by our assumption that $v^2 + 2 < 2r$. Therefore $H^1(\mathcal{I}_n(c)) = 0$ for all $\mathcal{I}_n(c) \in \mathcal{M}^s_c(1,c,r)$.

For global generation, note that $h^0(\mathcal{I}_n(c)) = \chi(\mathcal{I}_n(c)) = r+1$. Suppose for sake of contradiction that the evaluation map $\text{ev}\colon \mathcal{O}^{r+1} \to \mathcal{I}_n(c)$ is not surjective, and let R be the cokernel. Choose a point $P \in \operatorname{supp}(R)$ and a surjection $R \twoheadrightarrow \mathcal{O}_P$. Now let K be the kernel of the composite map $\mathcal{I}_n(c) \to R \to \mathcal{O}_P$, from which we get the short exact sequence

$$0 \to K \to \mathcal{I}_n(c) \to \mathcal{O}_P \to 0. \tag{4.5.8}$$

With $P_{c,K}$ being the nonreduced Hilbert polynomial with respect to c, we have $P_{c,K} = P_{c,\mathcal{I}_n(c)} - 1$, so any sheaf destablizing K would destablize $I_n(c)$, which is absurd. Therefore, K is Gieseker stable with Mukai vector $(1, c, r-1)$. On one hand, using the assumption that $v^2 + 2 < 2r$, we see that $\mathcal{M}^s_c(1,c,r-1)$ also satisfies the condition in equation (4.5.7) and conclude that $H^1(K) = 0$. On the other hand, all sections of F factor through K, so $h^0(K) = h^0(I_n(c))$, and we see from the long exact sequence in cohomology arising from the sequence (4.5.8) that $\mathbb{C} \hookrightarrow H^1(K)$. This is a contradiction, so it must be that R is the zero sheaf. We conclude that $\mathcal{I}_n(c)$ is globally generated. □

The next lemma establishes that twisting $\mathcal{I}_{\mathcal{Z}}(c)$ around $\mathcal{O}$ leaves us with another family of sheaves over $X \times X^{[n]}$.

Lemma 4.5.2. *Keep the assumptions from Lemma 4.5.1. The convolution $\mathcal{I}_\Delta \circ \mathcal{I}_{\mathcal{Z}}(c)$ is a sheaf (and not a complex) flat over $X^{[n]}$.*

Proof. We will demonstrate this by appealing to [Huy06, Lem. 3.31] in the following form: if a complex $\mathcal{Q} \in D^b(X \times X^{[n]})$ has the property that every derived restriction $i_F^*\mathcal{Q}$ to a fiber $X \times \{F\}$ of the projection map $X \times X^{[n]} \to X^{[n]}$ is a sheaf, then $\mathcal{Q}$ is concentrated in degree zero (that is, $\mathcal{Q}$ is a sheaf) and it is flat over $X^{[n]}$. We will apply this to $\mathcal{Q} = \mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(c)$, so to prove the lemma it is enough to prove that $i_F^*\mathcal{Q}$ is a sheaf for every $F \in X^{[n]}$.

We begin by recasting $i_F^*\mathcal{Q}$ in a more useful form. Let $\pi_{12}, \pi_{13}, \pi_{23}$ be the projections from $X \times X \times X^{[n]}$. By definition,

$$\mathcal{Q} = \pi_{13*}\left(\pi_{12}^*\mathcal{I}_\Delta \otimes \pi_{23}^*\mathcal{I}_\mathcal{Z}(c)\right).$$

We first base change around the diagram

$$\begin{array}{ccc} X \times X & \xrightarrow{i_F} & X \times X \times X^{[n]} \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_{13}} \\ X & \xrightarrow{i_F} & X \times X^{[n]} \end{array} \tag{4.5.9}$$

to get

$$\begin{aligned} i_F^*\mathcal{Q} &= \pi_{1*}i_F^*\left(\pi_{12}^*\mathcal{I}_\Delta \otimes \pi_{23}^*\mathcal{I}_\mathcal{Z}(c)\right) \\ &= \pi_{1*}\left(i_F^*\pi_{12}^*\mathcal{I}_\Delta \otimes i_F^*\pi_{23}^*\mathcal{I}_\mathcal{Z}(c)\right) \end{aligned}$$

Next, note that $\pi_{12} \circ i_F = 1$ and $\pi_{23} \circ i_F = i_F \circ \pi_2$ using projections onto the second factor in place of the first in diagram (4.5.9). We are then left with

$$i_F^*\mathcal{Q} = \pi_{1*}\left(I_\Delta \otimes \pi_2^*\left(i_F^* I_Z(c)\right)\right). \tag{4.5.10}$$

The right hand side we recognize as the spherical twist. Since $\mathcal{I}_\mathcal{Z}(c)$ is flat over $X^{[n]}$, the derived restriction is an honest restriction. But we know that the spherical twist of a sheaf produces a sheaf because each fiber of $\mathcal{I}_\mathcal{Z}(c)$ satisfies the conclusions of Lemma 4.5.2. Thus the lemma cited above applies, so the lemma is proven. □

Under the same hypotheses the spherical twist of an ideal sheaf $\mathcal{I}_n(c)$ is a stable vector bundle, which will be demonstrated in the next proposition. Combined with the previous lemma, this will produce a classifying map to the moduli space of stable bundles.

Proposition 4.5.3. *Let X be a K3 surface with $\mathrm{Pic}(X) = \mathbb{Z}c$ with c ample and satisfying $c^2 < 4r - 2$. Then the spherical twist of an ideal sheaf $\mathcal{I}_n(c) \in X^{[n]}$ is a stable vector bundle with Mukai vector $(r, -c, 1)$. Consequently, the family $\mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(c)$ induces an isomorphism*

$$\phi \colon X^{[n]} \to \mathcal{M}_c^s(r, -c, 1). \tag{4.5.11}$$

Proof. The spherical twist of $\mathcal{I}_n(c)$ is the sheaf E appearing in the short exact sequence

$$0 \to E \to \mathcal{O}_X^{r+1} \to \mathcal{I}_n(c) \to 0. \tag{4.5.12}$$

(This makes use of the global generation from Lemma 4.5.1.) We first prove that E is c-slope stable. Suppose for the sake of contradiction that this is not the case, and let $F \subset E$ be the maximal destablizing subsheaf ([HL10, Def. 1.3.6]); F is c-semistable by definition. First observe that there is an inclusion $F \hookrightarrow E \hookrightarrow \mathcal{O}^{r+1}$, and since both F and $\mathcal{O}^{r+1}$ are semistable we must have $\mu(F) \leq 0$. Suppose next that $\mu(F) = 0$. One can then find a c-stable $F' \subset F$ also having slope 0 by taking the first nonzero sheaf in the Jordan-Hölder filtration for F. Then, because $F' \to \mathcal{O}^r$ is a non-trivial morphism, $0 \neq \mathrm{Hom}(F', \mathcal{O}^r) = \mathrm{Hom}(F', \mathcal{O})^r$, so there is a nonzero morphism $F' \to \mathcal{O}$. Since both F and $\mathcal{O}$ are stable of the same slope, this map is an isomorphism, and hence E contains $\mathcal{O}$ as a subsheaf. This would furnish a nonzero section of E; however, the long exact cohomology sequence arising from the sequence (4.5.12) implies that $H^0(E) = 0$. This is a contradiction, so it must be that $\mu(F) < 0$.

This observation shows that $c_1(F) = -nc$ for some $n \geq 1$. But since $c_1(E) = -c$, we have $\deg(F) \leq \deg(E) < 0$. Combining these inequalities with the inequalities $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we see that

$$\deg(F)\,\mathrm{rk}(E) < \deg(E)\,\mathrm{rk}(F).$$

This implies that $\mu(F) < \mu(E)$, a contradiction. Thus, E must be a c-stable sheaf with Mukai vector $(r, -c, 1)$. This Mukai vector satisfies the hypotheses of Lemma 4.4.2, and therefore E must also be locally free.

Next, the family $\mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(c)$ is a sheaf over $X \times X^{[n]}$ whose fibers are the c-stable bundles E just considered, and thus the universal property of $\mathcal{M} := \mathcal{M}_c^s(r, -c, 1)$ gives a classifying map $\phi \colon X^{[n]} \to \mathcal{M}$ sending a closed point represented by $\mathcal{I}_n(c)$ to the bundle E from above.

The functor $FM_{\mathcal{I}_\Delta}\colon D^b(X^{[n]}) \to D^b(\mathcal{M})$ is an equivalence and therefore the map ϕ is injective on closed points. Viewing ϕ as a holomorphic map, we see that it is a holomorphic injection. However, holomorphic injections are open embeddings, and therefore the image of ϕ is an open compact subset of $\mathcal{M}$.

Next, note that the assumption $\mathrm{Pic}(X) = \mathbb{Z}c$ with $c^2 > 0$ implies that the set of walls of the Kähler cone of X is empty, which tells us that every polarization, in particular c, is v-generic. Therefore, $\mathcal{M} = \mathcal{M}_c^{ss}(v)$ is compact, and it is smooth since it is also a moduli space of stable sheaves. Moreover, by results of Kaledin, Lehn, and Sorger [KLS06], $\mathcal{M}$ is connected. Therefore, $\phi(X^{[n]})$ must equal $\mathcal{M}$, and we are done since holomorphic bijections are biholomorphisms. □

Proof of Deformation-Equivalence to the Hilbert Scheme

Theorem 4.6.1. *Let (E, h_E) be a smooth bundle over the K3 manifold M with $v(E) = (r, c_0, s)$ such that r and c_0 are coprime and $0 \le v(E)^2 + 2 < 2r$, and let $\varpi = (\sigma, \omega)$ be a $v(E)$-generic hyperkähler structure and B_E a B-field 2-form on M. Then the moduli space $\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ is nonempty, compact, and deformation-equivalent to a Hilbert scheme $S^{[n]}$, with $n = (v^2+2)/2$.*

Proof. We will describe how the propositions and lemmas in this chapter fit together to prove this result. Beginning with a C^∞ bundle E with Mukai vector (r, c_0, s), choose a smooth line bundle L on M as in Proposition 4.2.4 so that $v(E \otimes L) = (r, c, 1)$, with c being primitive and not algebraic on the initial K3 surface defined by σ. Note that the condition $0 \le v^2 + 2 < 2r$ is preserved by tensoring with L, and this constraint implies implies that $c^2 \ge 2r \ge 2$. Therefore we can deform the Kähler form ω to ω' following Proposition 4.3.3 so that there is a twistor rotation (σ_t, ω'_t) of the hyperkähler structure ϖ' so that $\mathrm{Pic}(\sigma_t) = \mathbb{Z}c$ with c ample.

We begin from the Hilbert scheme and apply the spherical twist from Proposition 4.5.3:

$$X^{[n]}_{\sigma_t} \cong \mathcal{M}^s_{\sigma_t,c}(1, c, r) \cong \mathcal{M}^s_{\sigma_t,c}(r, -c, 1).$$

This latter moduli space is a moduli space of untwisted vector bundles, since the Mukai vector satisfies the hypotheses of Lemma 4.4.2. Hence, sending a bundle to its dual gives an isomorphism $\mathcal{M}^s_{\sigma_t,c}(r, -c, 1) \cong \mathcal{M}^s_{\sigma_t,c}(r, c, 1)$. Since $\mathrm{Pic}(\sigma_t) = \mathbb{Z}c$ with $c^2 > 0$, there are no v-walls in the Kähler cone of X_{σ_t}, so c and ω_t' both define the same stability condition. This gives an identification

$$\mathcal{M}^s_{\sigma_t,c}(r,c,1) = \mathcal{M}^s_{\sigma_t,\omega_t'}(r,c,1).$$

Next, Proposition 4.4.3 lets us view this moduli space as a moduli space of stable structures on $E \otimes L$, giving

$$\mathcal{M}^s_{\sigma_t,\omega_t'}(r,c,1) \cong \mathcal{M}^s_{\sigma_t,\omega_t'}(E \otimes L).$$

Combining this with the untwisted Kobayashi-Hitchin correspondence (Proposition 4.3.4 with $B = 0$ and $\alpha = 1$) we see that the latter moduli space is isomorphic to $\mathcal{M}^{HE}_{\sigma_t,\omega_t'}(E \otimes L, h_E \otimes h_L)$. Summarizing, we have a complex analytic isomorphism

$$X^{[n]}_{\sigma_t} \cong \mathcal{M}^{HE}_{\sigma_t,\omega_t'}(E \otimes L, h_E \otimes h_L). \tag{4.6.13}$$

Starting from the other direction we tensor E with L, and we get isomorphisms coming from Propositions 4.2.5 and 4.3.5:

$$\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E) \cong \mathcal{M}^{HE}_{\sigma,\omega}(E \otimes L, h_E \otimes h_L, B_E + B_L) \cong \mathcal{M}^{HE}_{\sigma,\omega'}(E \otimes L, h_E \otimes h_L, B_E + B_L).$$

We then see from equation (4.6.13) and Proposition 4.3.6 that the moduli space $\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ is deformation-equivalent to $X^{[n]}_{\sigma_t}$, with the twistor family of ϖ' serving as the deformation. $\square$

CHAPTER V

UNIVERSAL BUNDLES

Overview

In this chapter I describe the universal bundle over $X \times \mathcal{M}$, where $\mathcal{M}$ is a moduli space of twisted Hermite-Einstein connections on $\mathcal{M}$. As in the previous chapter, the main strategy is to use the deformation to the Hilbert scheme of Picard rank one from the previous chapter and utilize results for moduli spaces of sheaves over such K3 surfaces.

As always, fix C^∞ bundle E with hermitian metric h over X and a hyperkähler structure (σ, ω) with corresponding hyperkähler metric g on the K3 manifold M. Also fix a B-field 2-form B representing $-c_1(E)/\operatorname{rk}(E)$. Let $\mathscr{A}^{HE} = \mathscr{A}^{HE}_{\sigma,\omega}(E, h, B)$ be the space of irreducible Hermite-Einstein connections on E with respect to this data. The moduli space $\mathcal{M} := \mathcal{M}^{HE}_{\sigma,\omega}(E, h, B)$ is a quotient of $\mathscr{A}^{HE}$ by the group of reduced gauge transformations $\mathscr{G} = U(E, h)/U(1)$.

We will construct a universal bundle over $X \times \mathcal{M}$ by forming a $\mathscr{G}$-equivariant bundle over $X \times \mathscr{A}^{HE}$. Let π_1, π_2 be the projections to X and $\mathscr{A}^{HE}$. The pullback $\pi_1^* E$ and its natural connection $\mathbb{A}$, the *tautological connection*, would be ideal for this task were it to be $\mathscr{G}$-equivariant. The tautological connection is defined by the property that $\mathbb{A}|_{X \times \{\nabla\}} = \nabla$ on E, and that it is flat in the $\{x\} \times \mathscr{A}^{HE}$-direction.

The problem is that $\mathscr{G}$ does not necessarily *act* on E. To clarify, we think of $U(E, h)$ as acting on $X \times \mathscr{A}^{HE}$ trivially in the X-coordinate and as its usual action on the $\mathscr{A}^{HE}$-coordinate, $\nabla \mapsto g \circ \nabla \circ g^{-1}$. The action of $U(E, h)$ on $\pi_1^* E = E \times \mathscr{A}^{HE}$ is only slightly more involved: a gauge transformation g acts on a point $(x, v, \nabla) \in E \times \mathscr{A}^{HE}$ by

$$(x, v, \nabla) \mapsto (x, g_x(v), g \circ \nabla \circ g^{-1}),$$

where a point in E has been represented by (x, v) in a local frame for some $x \in X$ and $v \in \mathbb{C}^r$. That this is well-defined is a consequence of g being a gauge transformation, which is by definition compatible with a change in frame. The action of $U(E, h)$ enjoys the property of being proper [Kob14, Prop 7.1.14], however it is far from free: the scalars $U(1) = e^{i\theta} 1_E$ leave every connection fixed. In order to get a free action it is customary to divide $U(E, h)$ by $U(1)$. This poses no problem for $X \times \mathscr{A}^{HE}$, though it creates an issue for $\pi_1^* E$ since scalars act non-trivially on the fibers of E.

The solution to this is to use the *determinant line bundle* $\mathscr{L} \to \mathscr{A}^{HE}$ associated to a family of elliptic operators. For an account of the determinant line bundle see Donaldson's paper [Don87]. The determinant line bundle was studied by Quillen, Atiyah and Singer, and Bismut and Freed, among others. Bismut and Freed in [BF86a, BF86b] showed that one can associate to a family of elliptic operators a determinant line bundle $\mathscr{L}$ over the space of hermitian connections that comes equipped with a natural metric and connection, the *Bismut-Freed* connection, which we denote by $\nabla_{\mathscr{L}}$. For a suitable family of operators $U(1)$ will act on $\mathscr{L}$ with weight -1. This ensures that $E \boxtimes \mathscr{L}$ has a trivial $U(1)$-action, so that $E \boxtimes \mathscr{L}$ admits a $\mathscr{G}$-action. More details will be given in Section 5.2.

The universal bundle $\mathcal{U} := E \boxtimes \mathscr{L}/\mathscr{G}$ and its quotient connection $\nabla_{\mathcal{U}}$ will then be used to define the Mukai map $\theta \colon v^{\perp} \to H^2(\mathcal{M}, \mathbb{Z})$ (or $v^{\perp}/Zv$ if $v^2 = 0$). O'Grady proved in [O'G97] that θ is a Hodge isometry over an algebraic K3 surface with Picard rank one. Combining this with the deformation to the Hilbert scheme from the previous chapter will prove part (2) of Theorem 1; see Theorem 5.4.4 below.

Last, I will show that the "wrong-way" slice of $\nabla_{\mathcal{U}}|_{\{x\}\times\mathcal{M}}$ is an irreducible twisted Hermite-Einstein connection on $\mathcal{U}|_{\{x\}\times\mathcal{M}}$. This will show that we can identify the original K3 surface X with a component of the moduli space of twisted sheaves over $\mathcal{M}$ with the same topological type as $\mathcal{U}|_{\{x\}\times\mathcal{M}}$; see Corollary 5.5.4 below. This result is similar to results obtained by Reede and Zhang [RZ19], who prove this for certain moduli spaces of rank zero sheaves whose first Chern class generates $\mathrm{Pic}(X)$, as well as for $X^{[n]}$ for a general K3 surface X.

In what follows we will be dealing with differential forms on $X \times \mathscr{A}^{HE}$ and $X \times \mathcal{M}$. These differential forms can be decomposed using the Künneth decomposition

$$\bigwedge^k (X \times \mathscr{A}^{HE}) = \bigoplus_{i+j=k} \bigwedge^i X \otimes \bigwedge^j \mathscr{A}^{HE}.$$

Furthermore, both X and $\mathscr{A}^{HE}$ have complex structures, the latter being equipped with the complex structure from equation (4.3.4) interpreted as an endomorphism of $T_{\nabla}\mathscr{A}^{HE}$. With both the Hodge decomposition and Künneth decomposition we will use a subscript of H or K to indicate which decomposition is being referred to: $(p,q)_K$ for Künneth, $(p,q)_H$ for Hodge.

In the rest of this chapter we will sometimes use the notation $F(\nabla)$ for the curvature of a connection, especially when the connection being referred to is on a bundle over $\mathscr{A}^{HE}$ or a similarly large space. We will also write $F(E)$ for a bundle E when the connection is clear from context.

Determinant Line Bundles

In this section we recall some results on the determinant line bundle $\mathscr{L}$. First we justify the ellipticity of a relevant differential operator.

Lemma 5.2.1. *Let X be a compact complex manifold with a hermitian metric and E a smooth hermitian bundle over X admitting a B-twisted holomorphic structure $\overline{\partial}_E$ such that $\overline{\partial}_E^2 = 2\pi i B^{0,2}$ for a real closed 2-form B on X. Then the differential operator*

$$\overline{\partial}_E \oplus \overline{\partial}_E^* \colon \bigoplus \mathcal{A}^{0,2k}(E) \to \bigoplus \mathcal{A}^{0,2k+1}(E)$$

is elliptic. Moreover, considered as a family of operators indexed by $\mathscr{A}^{HE}$, this family is $U(E,h)$-equivariant.

Proof. We prove ellipticity by relating the principal symbol $\sigma_x(\overline{\partial}_E \oplus \overline{\partial}_E^*)$ at a point $x \in X$ to a principal symbol known to be invertible. The untwisting theory from Section 3.1 tells us that, near x, $\overline{\partial}_E$ differs from the Dolbeault operator $\overline{\partial}_X^r$ of a trivial bundle $\mathcal{O}_X^r$ by a 1-form. Since the principal symbol is unaffected by (differential-)degree 0 terms, the symbols $\sigma_x(\overline{\partial}_E)$ and $\sigma_x(\overline{\partial}_X^r)$ are the same. Using additivity and duality properties of principal symbols, we conclude that $\sigma_x(\overline{\partial}_E \oplus \overline{\partial}_E^*)$ is a direct sum of r copies of $\sigma_x(\overline{\partial}_X \oplus \overline{\partial}_X^*)$, which is invertible since $\overline{\partial}_X \oplus \overline{\partial}_X^*$ is elliptic.

The operators $\overline{\partial}_E \oplus \overline{\partial}_E^*$ can be viewed as a family of elliptic operators indexed by $\nabla \in \mathscr{A}^{HE}$ since ∇ gives[1] $\overline{\partial}_E := \nabla^{0,1}$. The $U(E,h)$-action on a connection ∇ coincides with the $GL(E)$-action on $\nabla^{0,1} = \overline{\partial}_E$. This means that a unitary gauge transformation g sends the $\overline{\partial}$-operator associated with ∇ to the $\overline{\partial}$-operator associated with $g(\nabla)$, which means that this family is $U(E,h)$-equivariant. □

[1] This may seem like we are throwing away a lot of information, but $\nabla^{0,1}$ determines $\nabla^{1,0}$ for unitary connections; see [Dem, Sec. V.10].

Remark 5.2.2. There is likely to be a generalization of this lemma to coupling Dirac operators to twisted connections on α-twisted bundles. The operator $\overline{\partial}_X \oplus \overline{\partial}_X^*$ is the Dirac operator for the standard spin structure on a complex manifold with $c_1(X) = 0$, and a similar statement likely holds for more general spin structures on other manifolds.

The next lemma illustrates that we can always find a complementary Mukai vector to a given Mukai vector v that will aid in the construction of our determinant line bundle.

Lemma 5.2.3. *For any Mukai vector $v = (r, c, s) \in H^*(X, \mathbb{Z})$ with r and c coprime, there exists a C^∞ line bundle G with $v.v(G) = -1$ and $v(G)^2 = -2$.*

Proof. Begin by writing $c = kc_0$ for c_0 primitive. Then, using $\gcd(k, r) = 1$, choose integers a, b with $ak - br = s - 1$. By Lemma 4.2.1 we get a class c' with $c_0.c' = a$ and $(c')^2 = 2(b - 1)$. We then take G to be the line bundle representing c', so $v(G) = (1, c', b)$. The properties $v.v(G) = -1$ and $v(G)^2 = -2$ are then immediate. □

Fix a twisted Hermite-Einstein connection ∇_G on G. (See the proof of Proposition 4.2.5 for constructing ∇_G.) Given a connection $\nabla \in \mathscr{A}^{HE}$, $\overline{\partial}_E := \nabla^{0,1}$ is a Dolbeault operator that defines a B-twisted holomorphic structure on E, and likewise $\overline{\partial}_G := \nabla_G^{0,1}$ defines a B_G-twisted holomorphic structure on G. The bundle $E \otimes G^\vee$ is therefore also a twisted holomorphic bundle with Dolbeault operator given by

$$\overline{\partial}_{E\otimes G^\vee} = \overline{\partial}_E \otimes 1_{G^\vee} + 1_E \otimes \overline{\partial}_{G^\vee}. \tag{5.2.1}$$

Then, applying Lemma 5.2.1 to the bundle $E \otimes G^\vee$ we see that

$$D_\nabla := \overline{\partial}_{E\otimes G^\vee} \oplus \overline{\partial}^*_{E\otimes G^\vee} : \mathcal{A}^{0,0}(E \otimes G^\vee) \oplus \mathcal{A}^{0,2}(E \otimes G^\vee) \to \mathcal{A}^{0,1}(E \otimes G^\vee) \tag{5.2.2}$$

is an elliptic operator. We view this as a $U(E, h)$-equivariant family of elliptic operators indexed by $\mathscr{A}^{HE}$. The $U(E, h)$-equivariance is a consequence of the equivariance noted in Lemma 5.2.1 and the fact that $U(E, h)$ ignores any data pertaining to G. Thus we have the determinant line bundle $\mathscr{L} \to \mathscr{A}^{HE}$ whose fiber over a point $\nabla \in \mathscr{A}^{HE}$ is

$$\mathscr{L}_\nabla = \bigwedge^{\mathrm{top}} \ker(D_\nabla) \otimes \Big(\bigwedge^{\mathrm{top}} \ker(D^*_\nabla)\Big)^\vee. \tag{5.2.3}$$

Proposition 5.2.4. *The determinant line bundle $\mathscr{L} \to \mathscr{A}^{HE}$ associated to the family of elliptic operators in equation* (5.2.2) *has the following properties:*

1. *$\mathscr{L}$ admits an action of the unitary gauge group $U(E,h)$. The action of $U(E,h)$ induces an action of $U(1)$ on $\mathscr{L}$ of weight -1.*

2. *The curvature of the Bismut-Freed connection $\nabla_{\mathscr{L}}$ on $\mathscr{L}$ is given by*

$$F(\mathscr{L}) = 2\pi i \left\{ \int_X \mathrm{ch}(\mathbb{A} \otimes \pi_1^* \nabla_{G^\vee}) \,\mathrm{td}(X) \right\}_{(2)}, \tag{5.2.4}$$

   where the integral is interpreted as integration along the fibers of the projection $\pi_2 \colon X \times \mathscr{A}^{HE} \to \mathscr{A}^{HE}$ and $\{\}_{(2)}$ indicates the degree 2 part of this differential form. The Bismut-Freed connection is $U(E,h)$-invariant.

Proof. For details of the action of the Gauge group on $\mathscr{L}$, see the paper by Freed [Fre18]. Note that the action of a scalar preserves the fiber of $\mathscr{L}$, and the weight of the action of $U(1)$ on $\mathscr{L}_\nabla$ is seen to be $\mathrm{Ind}(D_\nabla) = \dim(\ker(D_\nabla)) - \dim(\ker(D_\nabla^*))$ from equation (5.2.3). From the Atiyah-Singer index theorem,

$$\mathrm{Ind}(D_\nabla) = -v.v(G) = -1.$$

For (2), see [BF86b] for the curvature formula for the Bismut-Freed connection for a general family of Dirac operators, or [Don87] for the specific case of the determinant line bundle over the space of connections on a vector bundle. For our setting, a K3 surface admits only a single spin structure since $H^2(X, \mathbb{Z}/2) = 0$. The Dirac operators are exactly the operators in equation (5.2.1).

To obtain our formula for the curvature from theirs, we note that our line bundle $\mathscr{L}$ can be viewed as the restriction of a similarly defined determinant line bundle over the space of connections $\mathscr{A}^{HE}(E \otimes G^\vee, B_E - B_G)$ on $E \otimes G^\vee$ along the inclusion $\mathscr{A}^{HE}(E) \to \mathscr{A}^{HE}(E \otimes G^\vee)$ sending $\nabla \mapsto \nabla \otimes \nabla_{G^\vee}$. The tautological connection on $\pi_1^*(E \otimes G^\vee)$ restricts to $\mathbb{A} \otimes \pi_1^* \nabla_{G^\vee}$ under this map, thus giving our formula for $F(\mathscr{L})$.

For $U(E,h)$-invariance, see [Fre18, Remark 19]. □

We now turn equation (5.2.4) into a more explicit form. According to [DK90, Ch. 5.2.3], the tautological connection $\mathbb{A}$ has the following Künneth components, which will be useful in evaluating $F(\mathscr{L})$. In the equations below, $u, v \in T_x X$ and $\eta, \xi \in T_\nabla \mathscr{A}^{HE} \subset \mathcal{A}^1(\mathrm{End}(E,h))$.

$$\begin{aligned} F(\mathbb{A})|_{\nabla,x}(u,v) &= F(\nabla)_x(u,v) \\ F(\mathbb{A})|_{\nabla,x}(\eta,v) &= \eta_x(v) \\ F(\mathbb{A})|_{\nabla,x}(\eta,\xi) &= 0. \end{aligned} \tag{5.2.5}$$

Here, $\eta(v)$ is the contraction of a 1-form with a tangent vector to the K3 surface.

The calculation of $F(\mathscr{L})$ can be viewed as a warm-up for computing the Mukai map, which will be carried out in a similar manner in the appendix.

Proposition 5.2.5. *For $\eta, \xi \in T_\nabla \mathscr{A}_B$, the determinant line bundle $\mathscr{L}$ associated with the family of elliptic operators in equation* (5.2.2) *is given by*

$$F(\mathscr{L})|_\nabla(\eta,\xi) = \frac{1}{4\pi^2}\int_X \mathrm{tr}\left[(\eta\wedge\xi - \xi\wedge\eta)\wedge F_\nabla\right] + \frac{1}{2\pi^2}\int_X \mathrm{tr}\left[\eta\wedge\xi\right]\wedge \mathrm{tr}(F_{\nabla_{G^\vee}}). \tag{5.2.6}$$

Proof. To evaluate $F(\mathscr{L})$ on a pair of tangent vectors $\eta, \xi \in T_\nabla \mathscr{A}^{HE} \subset \mathcal{A}^1(\mathrm{End}(E,h))$, we contract $\mathrm{ch}(\mathbb{A}\otimes\pi_1^*\nabla_G)$ with η and ξ and integrate the resulting top form over X. We proceed by working out the $(4,2)_K$ component of this characteristic class. Expand the Chern character to third order in $\mathcal{F} = F(\mathbb{A}\otimes\pi_1^*\nabla_G)$, since no higher terms will contribute to the $(4,2)_K$ component:

$$\mathrm{ch}(\mathbb{A}\otimes\pi_1^*\nabla_G)\,\mathrm{td}(X) = \mathrm{tr}\left[1 + \frac{i\mathcal{F}}{2\pi} - \frac{\mathcal{F}^2}{8\pi^2} - \frac{i\mathcal{F}^3}{24\pi^3} + \cdots\right]\mathrm{td}(X)$$

The terms of type $(4,2)_K$ from this expression are

$$-\frac{i}{24\pi^3}(\mathcal{F}^3)_{4,2}\,\mathrm{td}(X)_{0,0} + \frac{i}{2\pi}\mathcal{F}_{0,2}\,\mathrm{td}(X)_{4,0}.$$

The second term is easier to calculate. Note that

$$\mathcal{F}_{0,2} = F(\mathbb{A})_{0,2}\otimes 1_{\pi_1^*G} + 1_{\pi_1^*E}\otimes F(\pi_1^*\nabla_G)_{0,2} = 0$$

by equation (5.2.5) and since the pullback of a form from X can only have type $(*,0)_K$.

The other term contains the most information. The $(4,2)_K$ part of $\mathcal{F}^3$ will only come from cyclic permutations of $\mathcal{F}_{2,0}\wedge\mathcal{F}_{2,0}\wedge\mathcal{F}_{0,2}$ (all of which are zero for the same reason above) and from cyclic permutations of $\mathcal{F}_{1,1} \wedge \mathcal{F}_{1,1} \wedge \mathcal{F}_{2,0}$. There will be three such terms, all of which are equal after applying the trace operator. Hence, we arrive at

$$\{\mathrm{ch}(\mathbb{A} \otimes \pi_1^* \nabla_G)\, \mathrm{td}(X)\}_{4,2} = -\frac{i}{8\pi^3} \operatorname{tr}(\mathcal{F}_{1,1} \wedge \mathcal{F}_{1,1} \wedge \mathcal{F}_{2,0}). \tag{5.2.7}$$

By definition of integration over the fiber of π_2,

$$F(\mathscr{L})|_\nabla(\eta, \xi) = 2\pi i \int_X \iota_\eta \iota_\xi \big[\, \mathrm{ch}(\mathbb{A} \otimes \pi_1^* \nabla_{G^\vee})\, \mathrm{td}(X)\big]_{4,2}. \tag{5.2.8}$$

To proceed we make use the fact that the contraction operation ι_η is an antiderivation of degree -1 on $\bigwedge^* \mathscr{A}^{HE}$, as well as the two identities

$$\iota_\eta \mathcal{F}_{2,0} = \iota_\xi \mathcal{F}_{2,0} = 0, \qquad \iota_\eta \iota_\xi \mathcal{F}_{1,1} = 0.$$

Using equation (5.2.7) we arrive at the following equation:

$$F(\mathscr{L})|_\nabla(\eta, \xi) = \frac{1}{4\pi^2} \int_X \operatorname{tr}\Big[\iota_\eta \mathcal{F}_{1,1} \wedge \iota_\xi \mathcal{F}_{1,1} \wedge \mathcal{F}_{2,0} - \iota_\xi \mathcal{F}_{1,1} \wedge \iota_\eta \mathcal{F}_{1,1} \wedge \mathcal{F}_{2,0}\Big].$$

Now observe that $\iota_\eta \mathcal{F}_{1,1} = \eta$ since tensoring $\mathbb{A}$ by $\pi_1^* \nabla_G$ does not affect the middle equation in (5.2.5). Also, $\mathcal{F}_{2,0}|_\nabla = F_\nabla \otimes 1_{\pi_1^* G^\vee} + 1_{\pi_1^* E} \otimes F_{\pi_1^* \nabla_G^\vee}$, so

$$\begin{aligned} F(\mathscr{L})|_\nabla(\eta, \xi) =& \frac{1}{4\pi^2} \int_X \operatorname{tr}\Big[(\eta \wedge \xi - \xi \wedge \eta) \wedge (F_\nabla \otimes 1_{\pi_1^* G^\vee})\Big] \\ &+ \frac{1}{4\pi^2} \int_X \operatorname{tr}\Big[(\eta \wedge \xi - \xi \wedge \eta) \wedge (1_{\pi_1^* E} \otimes F_{\pi_1^* \nabla_{G^\vee}})\Big]. \end{aligned} \tag{5.2.9}$$

We note that tr[...] here represents the trace operator on the bundle $\mathrm{End}(\pi_1^* E \otimes \pi_1^* G^\vee)$. Therefore, the first term in equation (5.2.9) picks up a factor of 1 from tracing out $1_{\pi_1^* G^\vee}$, and the trace in the second term splits in two. We arrive at

$$F(\mathscr{L})|_\nabla(\eta, \xi) = \frac{1}{4\pi^2} \int_X \operatorname{tr}\Big[(\eta \wedge \xi - \xi \wedge \eta) \wedge F_\nabla\Big] + \frac{1}{2\pi^2} \int_X \operatorname{tr}(\eta \wedge \xi) \wedge \operatorname{tr}(F_{\nabla_{G^\vee}}),$$

which is the desired result. □

Corollary 5.2.6. *The curvature $F(\mathscr{L})$ is $\mathscr{G}$-invariant, and thus descends to a 2-form on $\mathcal{M}$.*

Proof. We will show that a gauge transformation $g \in U(E,h)$ preserves the form, and the $\mathscr{G}$-equivariance will follow by the fact that scalars act trivially on connections and tangent vectors to $\mathscr{A}^{HE}$.

We first claim that the differential $g_*\colon T_\nabla \mathscr{A}^{HE} \to T_{g(\nabla)}\mathscr{A}^{HE}$ is also given by conjugation on the endomorphism parts of tangent vectors $\eta, \xi \in T_\nabla \mathscr{A}^{HE}$, and furthermore that it takes the subspace $T_\nabla S_\nabla$ of the slice to $T_{g(\nabla)}S_{g(\nabla)}$. To see this, take a path $\nabla + t\eta$ in the space of all connections. Then it follows that

$$g_*\eta = \left.\frac{d}{dt}\right|_{t=0} (g \circ (\nabla + t\eta) \circ g^{-1}) = g \circ \eta \circ g^{-1}.$$

We also see that the induced action on the connection $\tilde{\nabla}$ on $\operatorname{End}(E)$ is given by $g(\tilde{\nabla}) = \tilde{g} \circ \tilde{\nabla} \circ \tilde{g}^{-1}$, where $\tilde{g}$ is the induced action on $\operatorname{End}(E)$ by conjugation. These two observations make it straightforward to verify that g_* preserves all of the equations defining $T_\nabla S_\nabla$ from equation (4.3.2). Therefore, the formula above for $g_*\eta$ requires no additional modifications to define a map $T_\nabla S_\nabla \to T_{g(\nabla)}S_{g(\nabla)}$.

It then remains to check that $F(\mathscr{L})_{g(\nabla)}(g_*\eta, g_*\xi) = F(\mathscr{L})_\nabla(\eta,\xi)$ using the formula obtained in Proposition 5.2.5. To this end, note that g acts as $\eta \wedge \xi \mapsto g(\eta \wedge \xi)g^{-1}$ and $F_\nabla \mapsto gF_\nabla g^{-1}$. Hence, every term inside a $\operatorname{tr}[\dots]$ in equation (5.2.6) also gets transformed by conjugation with g (except the term with $F_{\nabla_{G^\vee}}$, which is left alone). The cyclic property of trace then shows we can cancel the factors of g and g^{-1}, giving the desired result. □

Curvature of a Quotient Connection

We will need to calculate curvature of a quotient connection since the universal bundle $\mathcal{U}$ carries a quotient connection $\nabla_{\mathcal{U}}$. This is a retelling of the explanation found in [DK90, Ch. 5.2.3], which we include for the convenience of the reader. Let G be a Lie group acting on a smooth manifold $\hat{Y}$ and let $\pi\colon \hat{Y} \to Y = \hat{Y}/G$ be the quotient map. When the group acts properly and freely Y is also a smooth manifold. Now, if $\hat{E} \to \hat{Y}$ is a smooth bundle that admits a G-action which carries fibers linearly to fibers, then $E := \hat{E}/G$ is a smooth bundle over Y.

Next, suppose that $\hat{E}$ admits a G-invariant connection, $\hat{\nabla}$, and that we are given a connection H (viewed as a horizontal distribution) with connection 1-form θ in the principal G-bundle $\pi\colon \hat{Y} \to Y$ with curvature form $\Theta = d\theta + [\theta \wedge \theta]$. Then there exists a connection ∇ on E

defined by

$$\widehat{(\nabla_u s)} := \hat{\nabla}_{\hat{u}} \hat{s}. \tag{5.3.10}$$

This means that, given $u \in TY$ and $s \in \Gamma(E)$, we take a θ-horizontal lift $\hat{u}$ of u, a G-invariant lift $\hat{s}$ of s, and apply $\hat{\nabla}$ to $\hat{s}$ along $\hat{u}$. The resulting section $\hat{\nabla}_{\hat{u}} \hat{s}$ of $\hat{E}$ will be G-invariant since $\hat{\nabla}$ is, and therefore it corresponds with a section of E, which we define to be $\nabla_u s$.

The curvature of the quotient connection ∇ can be computed in terms of the curvature of $\widehat{\nabla}$ and the curvature of H. To this end we let $B \in \mathcal{A}^1_{\hat{Y}}(\mathrm{End}(\hat{E}))$ be the 1-form such that

$$\widehat{\nabla} = \pi^* \nabla + B.$$

(This B is entirely unrelated to the B-field.) It is immediate from equation (5.3.10) that the 1-form B vanishes on horizontal vectors. Now, the horizontal subspaces $H_{\hat{y}}$, $\hat{y} \in \hat{Y}$ are the kernel of $\theta_{\hat{y}}$ viewed as a map $T_{\hat{y}} \hat{Y} \to \mathrm{Lie}(G)$. Viewing B as a map $T\hat{Y} \to \mathrm{End}(\hat{E})$ we may then factor $B = \Phi \circ \theta$ for some linear map $\Phi\colon \mathrm{Lie}(G) \to \mathrm{End}(\hat{E})$. The curvature $F(\nabla)$ of the quotient connection evaluated on $u, v \in T_y Y$ is then

$$F(\nabla)(u,v) = F(\hat{\nabla})(\hat{u}, \hat{v}) - \Phi(\Theta(u,v)), \tag{5.3.11}$$

where $\hat{u}, \hat{v} \in T_{\hat{y}} \hat{Y}$ are H-horizontal lifts of u and v, and $\pi(\hat{y}) = y$.

We will be applying this construction to the following data:

$$\hat{Y} = X \times \mathscr{A}^{HE}$$

$$\hat{E} = E \boxtimes \mathscr{L}$$

$$\hat{\nabla} = \mathbb{A} \otimes \pi_2^* \nabla_{\mathscr{L}} = \mathbb{A} \otimes 1_{\pi_2^* \mathscr{L}} + 1_{\pi_1^* E} \otimes \pi_2^* \nabla_{\mathscr{L}}$$

with $\mathscr{G} = U(E,h)/U(1)$ as our Lie group. Part 1 of Proposition 5.2.4 ensures that $\mathscr{G}$ acts on $E \boxtimes \mathscr{L}$. We denote the quotient bundle and connection by $(\mathcal{U}, \nabla_U)$.

Before moving on we take a moment to describe the connection 1-form θ and its curvature Θ. At a point $\nabla \in \mathscr{A}^{HE}$, according to [DK90, Ch. 5.2],

$$\begin{aligned} \theta_{\nabla}(\eta) &= -G_{\nabla}(\tilde{\nabla}^{*}\eta) \\ \Theta_{\nabla}(\eta,\xi) &= 2G_{\nabla}(\{\eta,\xi\}) \end{aligned} \tag{5.3.12}$$

where G_{∇} is the Green's operator[2]for the elliptic complex $C^{\bullet}(E,\nabla)$ from equation (2.5.12) in degree 0 and $\{\eta,\xi\}$ is the tensor product of the Riemannian metric g on X with the Lie bracket on $\mathrm{End}(E,h)$. Explicitly, if we write $\eta = \sum_i a_i \otimes A_i$ with a_i a real 1-form on X and A_i a skew-hermitian matrix (depending on $x \in X$), and similarly $\xi = \sum_j b_j \otimes B_j$, then the value of the pairing $\{\eta,\xi\}$ at a point $x \in X$ is

$$\{\eta,\xi\}|_x = \sum_{ij} g(a_i,b_j)|_x [A_i(x), B_j(x)] \tag{5.3.13}$$

where $[A_i, B_j]$ is the commutator of A_i and B_j.

Mukai Map and Hodge Structures

We next turn to proving that the Mukai map θ (to be defined below) is a Hodge isometry between $v^{\perp} \subset H^*(X,\mathbb{Z})$ (or $v^{\perp}/\mathbb{Z}v$ if $v^2 = 0$) and $H^2(\mathcal{M},\mathbb{Z})$. We start by defining the map $\theta\colon H^*(X,\mathbb{Z}) \to H^2(\mathcal{M},\mathbb{Q})$ by

$$\theta(\beta) = \left\{ \int_X \tilde{\beta} \wedge \mathrm{ch}(\mathcal{U}^{\vee})\sqrt{\mathrm{td}(X \times \mathcal{M})} \right\}_{(2)} \tag{5.4.14}$$

where $\tilde{\beta}$ is a de Rham representative of $\beta \in H^*(X,\mathbb{Z})$. An explicit formula is given in the next lemma.

[2]See references mentioned in Section 2.5 on elliptic complexes for details on the Green's operator. We will only need the fact that G_{∇} is a linear operator.

Lemma 5.4.1. *Let $\beta = (\beta_0, \beta_2, \beta_4) \in H^*(X, \mathbb{Z})$ be any vector. Then the formula for the 2-form $\theta(\beta)$ applied to $\eta, \xi \in T_{[\nabla]}\mathcal{M}$ is given by*

$$\begin{aligned}\theta(\beta)(\eta,\xi) = \frac{i}{2\pi}\langle \beta, v(E)\rangle F(\mathscr{L})(\eta,\xi) - \frac{i}{2\pi}\int_X \beta_4 \operatorname{tr}(\Phi\Theta(\eta,\xi)) \\ - \frac{1}{4\pi^2}\int_X \beta_2 \operatorname{tr}\left[\eta\wedge\xi + F_\nabla \wedge \Phi\Theta(\eta,\xi)\right] - \frac{i}{2\pi}\int_X \beta_0 \operatorname{tr}(\Phi\Theta(\eta,\xi)) \operatorname{vol}_g \\ + \frac{i}{8\pi^3}\int_X \beta_0 \operatorname{tr}\left[\frac{1}{2}F_\nabla \wedge (\eta\wedge\xi - \xi\wedge\eta) + \frac{1}{2}F_\nabla^2 \wedge \Phi\Theta(\eta,\xi)\right]\end{aligned} \tag{5.4.15}$$

where $\langle\, , \rangle$ is the Mukai pairing, and where Φ and Θ are as in equation (5.3.12).

Proof. Postponed to the appendix. □

One immediate consequence of this calculation is that θ is independent of the choice of G and $\mathscr{L}$ when restricted to $v^\perp$. Indeed, if $\beta \in v^\perp$ the only term involving $F(\mathscr{L})$ vanishes. Furthermore, the map $\Phi\colon \operatorname{Lie}(\mathscr{G}) \to \operatorname{End}(E \boxtimes \mathscr{L})$ is independent of $\mathscr{L}$ because of the natural isomorphism $\operatorname{End}(E \boxtimes \mathscr{L}) \cong \pi_1^* \operatorname{End}(E)$.

Hodge Structures

We next clarify the Hodge structures on $v^\perp$ and on $H^2(\mathcal{M}, \mathbb{Z})$. Classically a K3 surface X with holomorphic 2-form σ has a Hodge structure on $H^2(X, \mathbb{Z})$ given by $H^{2,0}(X) = \mathbb{C}[\sigma]$. With the B-field B_E associated with a Brauer class α we can define a twisted Hodge structure on $H^2(X, \mathbb{Z})$. The multiform $e^{-B_E}\sigma = \sigma - B_E \wedge \sigma$ defines a weight-2 Hodge structure on $H^*(X, \mathbb{Z})$ by setting[3] $H^{2,0} = \mathbb{C}e^{-[B_E]}[\sigma]$. This Hodge structure only depends on α and not the choices of lift under the exponential map or de Rham representative. For more details on this Hodge structure see [Huy09]. In order for E to admit a B-twisted Hermite-Einstein connection it is necessary that $e^{-[B_E]}[\sigma].v = 0$, which follows by tracing and taking the cohomology class of the equation $F_\nabla \wedge \sigma = 2\pi i B \wedge \sigma$. Thus, because $e^{-[B_E]}[\sigma] \in v_{\mathbb{C}}^\perp$, the Hodge structure descends to $v^\perp$.

On the other side, $H^2(\mathcal{M}, \mathbb{Z})$ has a natural Hodge structure whose $(2,0)$ component is given by $\mathbb{C}[\Sigma]$, where Σ is the complex 2-form $\Sigma = \Omega_J + i\Omega_K$ and Ω_J and Ω_K are the real 2-forms described in Proposition 4.3.6.

[3] The unsavory minus sign appearing here is necessary. One could remove it at the cost of writing $F_\nabla \wedge \sigma = -2\pi i B^{0,2}$.

We next show that θ maps $e^{-[B_E]}[\sigma]$ to $[\Sigma]$. Note that this calculation happens in place on the moduli space $\mathcal{M} = \mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ without needing to deform to the Hilbert scheme. Later this will be used to directly relate the Hodge structures on $v^\perp$ (or $v^\perp/\mathbb{Z}v$) and $H^2(\mathcal{M}, \mathbb{Z})$ during a twistor deformation.

Proposition 5.4.2. *The Mukai map sends the generator $e^{-[B_E]}[\sigma]$ of the Hodge structure on $v^\perp$ to a scalar multiple of the generator Σ of the Hodge structure on $H^2(\mathcal{M}, \mathbb{Z})$.*

Proof. Apply the formula from Lemma 5.4.1 to $e^{-B}\sigma = \sigma - B \wedge \sigma$. Since this has no degree 0 term, the formula for $\theta(e^{-B}\sigma)$ dramatically simplifies to

$$\theta(e^{-B}\sigma)(\eta, \xi) = \frac{i}{2\pi} \int_X B \wedge \sigma \operatorname{tr}(\Phi\Theta(\eta, \xi)) - \frac{1}{4\pi^2} \sigma \wedge \operatorname{tr}\Big[\eta \wedge \xi + F_\nabla \wedge \Phi\Theta(\eta, \xi)\Big]. \tag{5.4.16}$$

Then, since ∇ is Hermite-Einstein, we can replace $F_\nabla \wedge \sigma$ with $2\pi i B \wedge \sigma$ in the second term. Upon doing so the terms involving $\Phi\Theta$ cancel, and we are left with

$$\theta(e^{-B}\sigma)(\eta, \xi) = -\frac{1}{4\pi^2} \int_X \sigma \wedge \operatorname{tr}(\eta \wedge \xi),$$

which is directly proportional to Σ. □

Next we will work through the reductions to the Hilbert scheme that appeared in Chapter III to relate θ on $\mathcal{M}^{HE}_{\sigma,\omega}(E, h_E, B_E)$ to the Mukai map on $X^{[n]}$. Recall that, broadly speaking, there are five steps in this reduction: tensor E with a line bundle; deform the Kähler structure; deform across the twistor family to a Picard-rank one K3 surface; the Kobayashi-Hitchin correspondence; and the spherical twist. Using these steps we can lift results about θ on the Hilbert scheme of points proved by O'Grady [O'G97] to an arbitrary K3 surface. We will use some of the notation from the proof of Theorem 4.6.1.

Proposition 5.4.3. *The Mukai map $\theta\colon v^\perp \to H^2(\mathcal{M}, \mathbb{Z})$ (or $v^\perp/\mathbb{Z}v$ when $v^2 = 0$) takes integral classes to integral classes. Furthermore, it is a linear isomorphism that intertwines the Mukai pairing on $v^\perp$ and the Beauville-Bogomolov form on $H^2(\mathcal{M}, \mathbb{Z})$.*

Proof. First we assume that $\mathcal{M} = \mathcal{M}^s_c(v)$ is a moduli space of sheaves with Mukai vector $v = (r, -c, 1)$ with c primitive and ample and that $\operatorname{Pic}(X) = \mathbb{Z}c$, along with the assumption that

$v^2+2 < 2r$. Under these conditions we recall from Proposition 4.5.3 that the family $\mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(C)$ on $X \times X^{[n]}$ induces an isomorphism $\phi\colon X^{[n]} \to \mathcal{M}$. Furthermore, this classifying map also satisfies

$$\mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(c) = (1 \times \phi)^* \mathcal{U} \otimes \pi_2^* L$$

for some line bundle over $X^{[n]}$, again from the universal property of $\mathcal{M}$. We take the liberty of absorbing L into $\mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(c)$ and just write

$$\mathcal{I}_\Delta \circ \mathcal{I}_\mathcal{Z}(c) = (1 \times \phi)^* \mathcal{U}.$$

This equality of Fourier-Mukai kernels gives us the following commuting diagram.

$$\begin{array}{ccccc} H^*(X,\mathbb{Z}) & \xrightarrow{v(\mathcal{U}^\vee)} & H^*(\mathcal{M},\mathbb{Q}) & \xrightarrow{\deg 2} & H^2(\mathcal{M},\mathbb{Q}) \\ \downarrow{\scriptstyle v(I_\Delta^\vee)} & & \downarrow{\scriptstyle \phi^*} & & \downarrow{\scriptstyle \phi^*} \\ H^*(X,\mathbb{Z}) & \xrightarrow{v(I_\mathcal{Z}^\vee)} & H^*(X^{[n]},\mathbb{Q}) & \xrightarrow{\deg 2} & H^2(X^{[n]},\mathbb{Q}) \end{array} \tag{5.4.17}$$

The maps labeled with Mukai vectors are cohomological Fourier-Mukai transforms (see [Huy06, Ch. 5.2]). The horizontal compositions are identified with the Mukai map when restricted to either $(r,-c,1)^\perp$ or $(1,c,r)^\perp$. Also, the spherical twist maps the Mukai vector $(r,-c,1)$ to the Mukai vector $(1,c,r)$ and preserves the Mukai pairing on $H^2(X,\mathbb{Z})$, and therefore maps $(r,-c,1)^\perp$ to $(1,c,r)^\perp$. By work of O'Grady [O'G97], the Mukai map on the bottom row has image exactly $H^2(X^{[n]},\mathbb{Z})$ when restricted to $v^\perp$ (or $v^\perp/\mathbb{Z}v$), so the same is true of the Mukai map on the top row. Similarly, the properties of being bijective and preserving the pairings also transfer over.

The next step is to take the dual bundle. Let $\mathcal{M}_1 = \mathcal{M}_c^s(r,-c,1)$, $\mathcal{U}_1$, and θ_1 be the moduli space, universal bundle, and Mukai map for the Mukai vector $(r,-c,1)$, and let and $\mathcal{M}_2 = \mathcal{M}_c^s(r,c,1)$, $\mathcal{U}_2$, and θ_2 be the same for $(r,c,1)$. The bundle $\mathcal{U}_2^\vee$ is a family of stable bundles each with Mukai vector $(r,-c,1)$, and thus the universal property of $\mathcal{M}_1$ gives a map $\mathcal{D}\colon \mathcal{M}_2 \to \mathcal{M}_1$ sending a bundle to its dual. Furthermore, over $X \times \mathcal{M}_2$ we have

$$\mathcal{U}_2^\vee = (1 \times \mathcal{D})^* \mathcal{U}_1 \otimes \pi_2^* L$$

for some line bundle on $\mathcal{M}_2$. As in the last paragraph we absorb L into $\mathcal{U}_2$, and we have a similar diagram that transfers the desired properties from θ_1 to θ_2:

$$\begin{array}{ccccc} H^*(X,\mathbb{Z}) & \xrightarrow{v(\mathcal{U}_1^\vee)} & H^*(\mathcal{M}_1,\mathbb{Q}) & \xrightarrow{\deg 2} & H^2(\mathcal{M}_1,\mathbb{Q}) \\ \big\downarrow (-)^\vee & & \big\downarrow \mathcal{D}^* & & \big\downarrow \mathcal{D}^* \\ H^*(X,\mathbb{Z}) & \xrightarrow{v(\mathcal{U}_2^\vee)} & H^*(\mathcal{M}_2,\mathbb{Q}) & \xrightarrow{\deg 2} & H^2(\mathcal{M}_2,\mathbb{Q}) \end{array} \tag{5.4.18}$$

Here, the dual map on $H^*(X,\mathbb{Z})$ sends $(\beta_0,\beta_2,\beta_4) \mapsto (\beta_0,-\beta_2,\beta_4)$.

After dualizing, we change polarizations from c to ω_t'. There is also another change of polarization later on from ω' back to ω. In both cases, the change in polarization does not change the universal bundle nor its Chern classes, so the Mukai map is the same before and after the polarization change.

The next critical step in generalizing from the Hilbert scheme is the twistor rotation. Under the twistor rotation only the complex structure on $\mathcal{U}$ varies, which will not affect the Chern character of $\mathcal{U}$ nor the Mukai map. (However, the twistor rotation does affect the Hodge structures, but this is addressed by Proposition 5.4.2.)

Lastly, we study the effect of tensoring the initial bundle E by the line bundle L from Proposition 4.2.5 on the Mukai map. The Mukai vector $v(E)$ becomes $v(E\otimes L) = v(E)\wedge \mathrm{ch}(L)$, so if we also tensor the line bundle G with L we preserve the relation $v(E).v(G) = -1$ (since wedging with $\mathrm{ch}(L)$ is an automorphism of the Mukai lattice $H^*(X,\mathbb{Z})$). The family of elliptic operators in equation (5.2.2) also changes quite predictably. Under tensoring by L, $\overline{\partial}_E \mapsto \overline{\partial}_E \otimes \overline{\partial}_L$, so

$$\overline{\partial}_E \otimes \overline{\partial}_{G^\vee} \mapsto \overline{\partial}_E \otimes \overline{\partial}_{G^\vee} \otimes (\overline{\partial}_L \otimes \overline{\partial}_{L^\vee}).$$

However, $\overline{\partial}_L \otimes \overline{\partial}_{L^\vee} = \overline{\partial}_{\mathrm{End}(L)} = \overline{\partial}_X$, since $L\otimes L^\vee \cong \mathrm{End}(L) \cong \mathcal{O}_X$. Under the natural isomorphism $E\otimes G^\vee \otimes \mathcal{O}_X \to E\otimes G^\vee$, the factor of $\overline{\partial}_X$ drops out. This shows that the family of elliptic operators $D_{\nabla\otimes\nabla_L}$ is isomorphic to the original family D_∇, showing that the determinant line bundle $\mathscr{L}$ does not change with tensoring with L. As such, $E\boxtimes\mathscr{L}$ becomes $(E\boxtimes\mathscr{L})\otimes\pi_1^*L$. Then, using equation (5.3.11) together with the definition of the Mukai map in equation (5.4.14), we see that

$$\begin{aligned}\theta_{E\otimes L}(\beta) &= \int_X \beta \wedge \operatorname{tr}\left[\exp\left(F(E \boxtimes \mathscr{L}) - \Phi\Theta + 1_{E\boxtimes\mathscr{L}} \otimes F(L)\right)\right]\sqrt{\operatorname{td}(X \times \mathcal{M})} \\ &= \int_X \beta \wedge \exp(F(L)) \operatorname{tr}\left[\exp\left(F(E \boxtimes \mathscr{L}) - \Phi\Theta\right)\right]\sqrt{\operatorname{td}(X \times \mathcal{M})} \\ &= \theta_E(\beta \wedge \operatorname{ch}(L)).\end{aligned}$$

Thus, the two Mukai maps differ by an automorphism of the Mukai lattice, and again the desired properties of the Mukai map transfer through this isomorphism. □

The last two propositions combine to show part (2) of Theorem 1, which is summarized below. In this theorem $\mathcal{M} = \mathcal{M}^{HE}_{\sigma,\omega}(E, h, B)$.

Theorem 5.4.4. *Let X be a K3 surface with hyperkähler structure (σ, ω), and let E be a bundle with Mukai vector $v = (r, c, s)$. Assume that v is primitive, r is coprime to c, and $0 \leq v^2 + 2 < 2r$. Then the Mukai map $\theta\colon v^\perp \to H^2(\mathcal{M}, \mathbb{Z})$ ($v^\perp/\mathbb{Z}v$ when $v^2 = 0$) is a Hodge isometry, where $v^\perp$ is given the Hodge structure generated by $e^{-B}\sigma$ and $H^2(\mathcal{M}, \mathbb{Z})$ is given the Hodge structure generated by Σ, the induced holomorphic 2-form on $\mathcal{M}$ by σ.*

Wrong Way Slices

We now turn to the restriction of $(\mathcal{U}, \nabla_{\mathcal{U}})$ to a slice $\{x\} \times \mathcal{M}$ for $x \in X$. We will show that $\nabla_{\mathcal{U}}|_{\{x\}\times\mathcal{M}}$ is an irreducible twisted Hermite-Einstein connection with respect to the hyperkähler structure induced on $\mathcal{M}$ from X. We will shorten notation by writing $\mathcal{U}|_x$ in place of $\mathcal{U}|_{\{x\}\times\mathcal{M}}$. The proof will be in two stages: first we prove that this connection is Hermite-Einstein, and then we leverage our deformation to the Hilbert scheme to prove that it is irreducible.

Proposition 5.5.1. *The bundle $\mathcal{U}|_x$ is a twisted polystable bundle with respect to the hyperkähler structure (Σ, Ω) induced on $\mathcal{M}$ from the hyperkähler structure (σ, ω) on X with B-field proportional to $F(\mathscr{L})$.*

Proof. We prove this by showing the connection $\nabla_{\mathcal{U}}|_x$ is weak Hermite-Einstein and appealing to Lemma 3.4.10 and the twisted Kobayashi-Hitchin correspondence.

Let $\eta, \xi \in T_{[\nabla]}\mathcal{M}$ be tangent vectors viewed as ∇-harmonic 1-forms valued in $\operatorname{End}(E, h)$. When viewed this way we can consider η, ξ as either tangent vectors to $\mathcal{M}$ or $\mathscr{A}^{HE}$, and this gives a natural horizontal lift of the tangent vectors up to $\{x\} \times \mathscr{A}^{HE}$. Since we are evaluating on tangent vectors from $\mathcal{M}$ we only need to worry about the $(0,2)_K$-piece of $F(\mathcal{U})$. Using equations (5.3.11) and (5.2.5), we see that the tautological connection entirely drops out, leaving us with

$$\begin{aligned} F(\mathcal{U}|_x)(\eta,\xi) &= F(\mathbb{A} \boxtimes \nabla_{\mathscr{L}})|_x(\eta,\xi) - \Phi\Theta(\eta,\xi) \\ &= 1_{p^*E}|_x \otimes F(\mathscr{L})(\eta,\xi) - \Phi\Theta(\eta,\xi) \end{aligned}$$

We first focus on the $(0,2)_H$ type of $F(\mathcal{U}|_x)$. The complex structure $\mathcal{I}$ on $\mathcal{M}$ from equation (4.3.4) can be expressed as

$$\mathcal{I}(\eta) = -\sum_i I(a_i) \otimes A_i,$$

where we have expanded $\eta = \sum_i a_i \otimes A_i$ with 1-forms a_i and skew-hermitian matrices A_i. Using this with the formula for Θ given in equation (5.3.12), we have

$$\Theta(\mathcal{I}\eta, \mathcal{I}\xi) = \sum_{ij} g(I(a_i), I(b_j))[A_i, B_j].$$

The complex structure I on X is compatible with g, so we have $g(I(a_i), I(b_j)) = g(a_i, b_j)$, and thus $\Theta(\mathcal{I}\eta, \mathcal{I}\xi) = \Theta(\eta,\xi)$. This means that Θ is a $(1,1)_H$ form on $\mathcal{M}$. From this, we note that

$$F(\mathcal{U}|_x)^{0,2}(\eta,\xi) = F(\mathscr{L})^{0,2}(\eta,\xi)\, 1_{\mathcal{U}} \tag{5.5.19}$$

from which we can interpret $F(\mathscr{L})$ as a B-field for $\mathcal{U}$ up to factors of $2\pi i$. (Recall that $F(\mathscr{L})$ is a 2-form on $\mathcal{M}$ by Corollary 5.2.6.)

Next we look at the other Hermite-Einstein equation. I have found that it is easiest to work with the form of this equation given in equations (3.4.9). We will use a simple fact from complex geometry that the Lefschetz operator Λ_{Ω_I} is calculated on a 2-form Π by

$$\Lambda_{\Omega_I}\Pi = \sum_{i=1}^{\dim \mathcal{M}} \Pi(\xi_i, \mathcal{I}(\xi_i))$$

where $\{\xi, \mathcal{I}(\xi_i)\}$ is a real orthonormal basis for $T_{[\nabla]}\mathcal{M}$ with respect to the metric associated with the Kähler form Ω_I (see [Huy05a, Exercise 1.2.10]).

Observe that $\Theta(\eta, \mathcal{I}(\eta))$ vanishes for any $\eta \in T_{[\nabla]}\mathcal{M}$. To see this, write $\eta = \eta^{1,0} + \eta^{0,1}$ and $\mathcal{I}\eta = -i\eta^{1,0} + i\eta^{0,1}$. It follows that

$$\{\eta, \mathcal{I}\eta\} = -i\{\eta^{1,0}, \eta^{1,0}\} + i\{\eta^{1,0}, \eta^{0,1}\} - i\{\eta^{0,1}, \eta^{1,0}\} + i\{\eta^{0,1}, \eta^{0,1}\}.$$

The first and last terms vanish by antisymmetry of $\{,\}$, while the second and third terms vanish because $(1, 0)$ forms and $(0, 1)$-forms on X are orthogonal with respect to the metric g. We conclude that $\{\eta, \mathcal{I}\eta\} = 0$, which implies that $\Phi\Theta(\eta, \mathcal{I}\eta) = 0$ by equation (5.3.12). This tells us that $\Phi\Theta$ drops out after applying Λ_{Ω_I}. We are left with

$$i\Lambda_{\Omega_I} F(\mathcal{U}|_x) = i\Lambda_{\Omega_I} F(\mathscr{L})\, 1_{\mathcal{U}}.$$

Since $i\Lambda_{\Omega_I} F(\mathscr{L})$ is a real-valued function on $\mathcal{M}$, we recognize that $\nabla_{\mathcal{U}}|_x$ solves the weak twisted Hermite-Einstein equations. □

Next we calculate $\mathrm{Ext}^*(\mathcal{U}|_x, \mathcal{U}|_y)$ for $x, y \in X$ by deforming to the Hilbert scheme.

Proposition 5.5.2. *For $x, y \in X$,*

$$\mathrm{Ext}^*_{\mathcal{M}}(\mathcal{U}|_x, \mathcal{U}|_y) \cong \mathrm{Ext}^*_X(\mathcal{I}_x, \mathcal{I}_y) \otimes H^*(\mathbb{P}^n),$$

where $\mathcal{I}_x$ is the ideal sheaf of $x \in X$.

Proof. We first calculate this in the special case where $\mathrm{Pic}(X) = \mathbb{Z}c$ with c ample and $\mathcal{M} = \mathcal{M}^s_c(r, -c, 1)$. Let $F' \colon D^b(X) \to D^b(\mathcal{M})$ be the Fourier-Mukai functor associated with $\mathcal{U}$, $F \colon D^b(X) \to D^b(X^{[n]})$ the functor associated with $\mathcal{I}_Z(c)$, and R the right adjoint of F. As we noted in the proof of Proposition 5.4.3, $\mathcal{I}_\Delta \circ \mathcal{I}_Z(c) = (1 \times \phi)^*\mathcal{U}$, where $\phi \colon \mathcal{M} \to X^{[n]}$ is the isomorphism induced by the spherical twist. Thus, on the level of functors, $\phi^* \circ F' = F \circ T_{\mathcal{O}}$.

Replacing $\mathcal{U}|_x$ by $F'(\mathcal{O}_x)$ and pulling back to $X^{[n]}$, we have

$$\begin{aligned}\operatorname{Ext}^*_{\mathcal{M}}(\mathcal{U}|_x, \mathcal{U}|_y) &= \operatorname{Ext}^*_{\mathcal{M}}(F'(\mathcal{O}_x), F'(\mathcal{O}_y)) \\ &= \operatorname{Ext}^*_{X^{[n]}}(F(\mathcal{I}_x), F(\mathcal{I}_y)) \\ &= \operatorname{Ext}^*_X(RF(\mathcal{I}_x), \mathcal{I}_y).\end{aligned}$$

In [Add16, Theorem 2], Addington[4] proves that F is a $\mathbb{P}^n$-functor, which means that $RF \cong 1_X \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2n+2]$, or, more compactly, $RF \cong - \otimes H^*(\mathbb{P}^n)$. This finishes the calculation in this special case. Note that

$$\operatorname{Ext}^*_X(\mathcal{I}_x, \mathcal{I}_y) \cong \operatorname{Ext}^*_X(\mathcal{O}_x, \mathcal{O}_y)$$

using the spherical twist, and also note that the right-hand side vanishes in all degrees when $x \neq y$.

Next we generalize from this special case again following the deformation to the Hilbert scheme. The dual map $\mathcal{D}\colon \mathcal{M}^s_c(r,c,1) \to \mathcal{M}^s_c(r,-c,1)$ gives the equality $\mathcal{D}^*\mathcal{U}_{(r,-c,1)} = \mathcal{U}^\vee_{(r,c,1)}$ (again up to a line bundle, which can be ignored for this calculation). Thus,

$$\begin{aligned}\operatorname{Ext}^*_{\mathcal{M}(r,c,1)}(\mathcal{U}_{(r,c,1)}|_x, \mathcal{U}_{(r,c,1)}|_y) &= \operatorname{Ext}^*_{\mathcal{M}(r,c,1)}(\mathcal{U}_{(r,c,1)}|_y^\vee, \mathcal{U}_{(r,c,1)}|_x^\vee) \\ &\cong \operatorname{Ext}^*_{\mathcal{M}(r,-c,1)}(\mathcal{U}_{(r,-c,1)}|_y, \mathcal{U}_{(r,-c,1)}|_x).\end{aligned}$$

Since these Ext groups vanish when $x \neq y$ we can relabel x and y in the last line of this equation to get

$$\operatorname{Ext}^*_{\mathcal{M}(r,c,1)}(\mathcal{U}_{(r,c,1)}|_x, \mathcal{U}_{(r,c,1)}|_y) \cong \operatorname{Ext}^*_X(\mathcal{I}_x, \mathcal{I}_y) \otimes H^*(\mathbb{P}^n).$$

The Kobayashi-Hitchin correspondence then gives

$$\operatorname{Ext}^*_{\mathcal{M}^{HE}(E\otimes L)}(\mathcal{U}^{HE}|_x, \mathcal{U}^{HE}|_y) \cong \operatorname{Ext}^*_X(\mathcal{I}_x, \mathcal{I}_y) \otimes H^*(\mathbb{P}^n).$$

where $\mathcal{U}^{HE}$ is the universal bundle on $X \times \mathcal{M}^{HE}$ constructed earlier in this chapter.

[4] This result was also proved with a different method by Markman and Mehrotra [MM15, Theorem 1.1].

The next step is to deform over the twistor line from the picard-rank one K3 surface to the initial K3 surface. To do this we use the isomorphism

$$\mathrm{Ext}^*_{\mathcal{M}^{HE}(E\otimes L)}(\mathcal{U}^{HE}|_x, \mathcal{U}^{HE}|_y) \cong H^*(\mathcal{M}^{HE}(E\otimes L), \mathcal{U}^{HE}|_y \otimes \mathcal{U}^{HE}|_x^\vee)$$

This quantity is invariant under the twistor deformation by a result of Verbitsky [Ver96, Corollary 8.1]. This result says that for a hyperholomorphic bundle $\mathcal{V}$ over a hyperkähler manifold the cohomology groups $H^*(\mathcal{V})$ are independent of the complex structure chosen from the twistor family. From equation (5.5.19) we see that $F(\mathcal{U}^{HE}|_x)^{0,2}$ is independent of x. Thus, $F(\mathcal{U}^{HE}|_y \otimes \mathcal{U}^{HE}|_x^\vee)^{0,2} = 0$ for any x, y and even any complex structure in the twistor family for $\mathcal{M}^{HE}(E\otimes L) = \mathcal{M}^{HE}_{\sigma_t,\omega'_t}(E\otimes L)$. So, this curvature is type $(1,1)_H$ for each complex structure for the twistor family of $\mathcal{M}^{HE}(E\otimes L)$. This means that $\mathcal{U}|_y \otimes \mathcal{U}|_x^\vee$ is hyperholomorphic, so Verbitsky's result applies, giving the isomorphism

$$\mathrm{Ext}^*_{\sigma,\omega'}(\mathcal{U}^{HE}|_x, \mathcal{U}^{HE}|_y) = \mathrm{Ext}^*_{\sigma_t,\omega'_t}(\mathcal{U}^{HE}|_x, \mathcal{U}^{HE}|_y)$$

where $\mathrm{Ext}^*_{\sigma,\omega'}$ is the Ext group calculated with reference to the holomorphic structure on $\mathcal{M}^{HE}_{\sigma,\omega'}$, which corresponds with the complex structure on the K3 surface associated with σ. Since we have shown the result applies to the right-hand side, we now have it for the left-hand side.

The second to last step in our roadmap is deforming ω to ω'. This does not affect the complex structure on the K3 surface determined by σ, and it does not affect the holomorphic structure on $\mathcal{U}^{HE}|_y \otimes \mathcal{U}^{HE}|_x^\vee$. Thus the proposition holds for the universal bundle over the moduli space $\mathcal{M}^{HE}_{\sigma,\omega}(E\otimes L, B_E + B_L)$. Lastly, tensoring E by the line bundle L does not change $\mathcal{U}|_y \otimes \mathcal{U}|_x^\vee$, so the proposition holds for the universal bundle $\mathcal{U}^{HE}$ over the initial K3 and moduli space. □

Corollary 5.5.3. *The slice $\mathcal{U}|_x$ is stable with respect to the hyperkähler structure (Σ, Ω) on $\mathcal{M}$ induced from the hyperkähler structure (σ, ω) on X.*

Proof. Polystability follows from Proposition 5.5.1, and Proposition 5.5.2 gives

$$\mathrm{Hom}(\mathcal{U}|_x, \mathcal{U}|_x) = \mathrm{Hom}(\mathcal{I}_x, \mathcal{I}_x) = \mathbb{C}$$

since $\mathcal{I}_x$ is a stable sheaf. □

As a final corollary we have Part (3) of Theorem 1.

Corollary 5.5.4. *The K3 surface X embeds as a connected component of the moduli space $\tilde{\mathcal{M}}$ of twisted sheaves on $\mathcal{M}$ with the toplogical type of $\mathcal{U}|_x$.*

Proof. Since $\mathcal{U}|_x$ is stable for all x we get a map $f\colon X \to \tilde{\mathcal{M}}$ sending x to $\mathcal{U}|_x$. By Proposition 5.5.2, we know that $\operatorname{Hom}_{\mathcal{M}}(\mathcal{U}|_x, \mathcal{U}|_y) = \operatorname{Hom}_X(\mathcal{I}_x, \mathcal{I}|_y) = 0$ when $x \neq y$, so f is injective. It also follows that

$$T_{\mathcal{U}|_x}\tilde{\mathcal{M}} = \operatorname{Ext}^1_{\mathcal{M}}(\mathcal{U}|_x, \mathcal{U}|_x) \cong \operatorname{Ext}^1_X(\mathcal{I}_x, \mathcal{I}_x) \cong \operatorname{Ext}^1_X(\mathcal{O}_x, \mathcal{O}_x) = T_x X$$

where we have used the spherical twist for the second to last isomorphism. This shows that f is an isomorphism on tangent spaces and hence a local isomorphism, and in particular f is an open embedding. However, we also know that $\tilde{\mathcal{M}}$ is Hausdorff, and the image $f(X)$ is compact, so $f(X)$ is also closed. Thus, X embeds as a connected component. $\square$

APPENDIX

MUKAI MAP CALCULATION

This appendix is devoted to the calculation for proving Lemma 5.4.1. Here, $\beta \in H^*(X, \mathbb{Z})$. First we note that $\mathrm{td}_1(\mathcal{M}) = 0$ since it is a hyperkähler manifold. Since we are seeking terms of type $(4,2)_K$, only $\mathrm{td}_0(\mathcal{M}) = 1$ contributes to $\theta(\beta)$, and thus we may omit it from the calculation. To ease the notation slightly we introduce $\mathcal{F} := F(\mathbb{A} \otimes \nabla_{\mathscr{L}})$. We begin by expanding the Chern character of $\mathcal{U}^\vee$ to third order; anything higher will not contribute to the degree $(4,2)_K$ component.

$$\begin{aligned}\theta(\beta) = \int_X \beta \wedge \mathrm{tr}\Big[1 - (\mathcal{F} - \Phi\Theta) + \frac{1}{2}(\mathcal{F} - \Phi\Theta)^2 \\ - \frac{1}{3!}(\mathcal{F} - \Phi\Theta)^3 + \dots\Big]\left(\mathrm{td}_0(X) + \frac{1}{2}\,\mathrm{td}_2(X)\right)\end{aligned} \tag{A.1.1}$$

The alternating signs appear since $\mathrm{tr}(F(\mathcal{U}^\vee)) = -\,\mathrm{tr}(F(\mathcal{U}))$. We have also substituted $F(\mathcal{U}) = \mathcal{F} - \Phi\Theta$ following equation (5.3.11). Next we split β into components $(\beta_0, \beta_2, \beta_4)$ and collect terms of type $(4,2)_K$. We have

$$\begin{aligned}\theta(\beta) = -\int_X \beta_0\,\mathrm{tr}\left[\frac{1}{3!}(\mathcal{F} - \Phi\Theta)^3_{4,2}\,\mathrm{td}_0(X) + \frac{1}{2}\,\mathrm{td}_2(X)(\mathcal{F} - \Phi\Theta)_{0,2}\right] \\ + \int_X \beta_2 \wedge \mathrm{tr}\left[\frac{1}{2}(\mathcal{F} - \Phi\Theta)^2_{2,2}\right]\mathrm{td}_0(X) - \int_X \beta_4 \wedge \mathrm{tr}\,(\mathcal{F} - \Phi\Theta)_{0,2}\,\mathrm{td}_0(X).\end{aligned} \tag{A.1.2}$$

We now proceed to break down these terms into smaller degree pieces. Here is the first set of reductions:

$$\begin{aligned}(\mathcal{F} - \Phi\Theta)^3_{4,2} &= (\mathcal{F}^3)_{4,2} - 3(\mathcal{F}^2)_{4,0}\Phi\Theta \\ (\mathcal{F} - \Phi\Theta)^2_{2,2} &= (\mathcal{F}^2)_{2,2} - 2\mathcal{F}_{2,0}\Phi\Theta. \\ (\mathcal{F} - \Phi\Theta)_{0,2} &= 1_{p^*E} \otimes F(\mathscr{L}) - \Phi\Theta\end{aligned} \tag{A.1.3}$$

It is important to keep in mind while using the above identities that $\Phi\Theta$ is of type $(0,2)_K$. We will also use the expansion

$$\mathcal{F} = F(\mathbb{A}) \otimes 1_{\mathscr{L}} + 1_{\pi_1^* E} \otimes F(\mathscr{L}) \tag{A.1.4}$$

The next step is to use eq. (A.1.4) in conjunction with eq. (5.2.5) to evaluate $\mathcal{F}_{2,0}$, $\mathcal{F}_{1,1}$, and $\mathcal{F}_{0,2}$. The original formula (5.4.14) is to be evaluated on a pair of tangent vectors $\eta, \xi \in T_{[\nabla]}\mathcal{M}$, so we will replace $F(\mathbb{A})_{2,0} = F_\nabla$ as we have done in the first equation of eq. (5.2.5). To break down the identities (A.1.3) further, we make use of the following set of identities:

$$\begin{aligned} \mathcal{F}_{2,0} &= F_\nabla \otimes 1_{\mathscr{L}} \\ (\mathcal{F}^2)_{4,0} &= (\mathcal{F}_{2,0})^2 = F_\nabla^2 \otimes 1_{\mathscr{L}} \\ (\mathcal{F}^2)_{2,2} &= F(\mathbb{A})_{1,1}^2 \otimes 1_{\mathscr{L}} + 2F_\nabla \otimes F(\mathscr{L}) \\ (\mathcal{F}^3)_{4,2} &= 3F_\nabla \wedge F(\mathbb{A})_{1,1}^2 \otimes 1_{\mathscr{L}} + 3F_\nabla^2 \otimes F(\mathscr{L}). \end{aligned} \tag{A.1.5}$$

Note that these equalities are only true after taking the trace of both sides; the cyclic property of trace is necessary to collect the like terms from expanding $\mathcal{F}^2$ and $\mathcal{F}^3$. With the identities in eqs. (A.1.3) and (A.1.5), eq. (A.1.2) becomes

$$\begin{aligned} \theta(\beta) = &-\int_X \beta_0 \Big\{ \operatorname{tr}\left[\frac{1}{2}F_\nabla \wedge F(\mathbb{A})_{1,1}^2 + \frac{1}{2}F_\nabla^2 \otimes F(\mathscr{L}) - \frac{1}{2}F_\nabla^2 \wedge \Phi\Theta\right] \operatorname{td}_0(X) \\ &\qquad + \frac{1}{2}\operatorname{tr}\left[1_{p^*E} \otimes F(\mathscr{L}) - \Phi\Theta\right] \operatorname{td}_2(X) \Big\} \\ &+ \int_X \beta_2 \operatorname{tr}\left[\frac{1}{2}F(\mathbb{A})_{1,1}^2 + F_\nabla \wedge F(\mathscr{L}) - F_\nabla \wedge \Phi\Theta\right] \operatorname{td}_0(X) \\ &- \int_X \beta_4 \operatorname{tr}\left[1_{p^*E} \otimes F(\mathscr{L}) - \Phi\Theta\right] \operatorname{td}_0(X) \end{aligned} \tag{A.1.6}$$

where we have traced out the factors of $1_{\mathscr{L}}$. We next trace out factors of $1_{\pi_1^* E}$ (which give a factor of $\operatorname{rk}(E)$) and regroup all terms involving $F(\mathscr{L})$. We also take the liberty of setting $\operatorname{td}_0(X) = 1$ and $\frac{1}{2}\operatorname{td}_2(X) = \operatorname{vol}_g$ in the resulting formula. We are left with

$$\begin{aligned} \theta(\beta) = &-\int_X \beta_0 \Big\{ \operatorname{tr}\left[\frac{1}{2}F_\nabla \wedge F(\mathbb{A})_{1,1}^2 - \frac{1}{2}F_\nabla^2 \wedge \Phi\Theta\right] - \operatorname{tr}(\Phi\Theta)\operatorname{vol}_g \Big\} \\ &+ \int_X \beta_2 \operatorname{tr}\left[\frac{1}{2}F(\mathbb{A})_{1,1}^2 - F_\nabla \wedge \Phi\Theta\right] + \beta_4 \operatorname{tr}(\Phi\Theta) \\ &+ \int_X F(\mathscr{L}) \Big\{ -\beta_0\left[\frac{1}{2}\operatorname{tr}(F_\nabla^2) + \operatorname{rk}(E)\operatorname{vol}_g\right] + \beta_2 \operatorname{tr}(F_\nabla) - \beta_4 \operatorname{rk}(E) \Big\} \end{aligned} \tag{A.1.7}$$

The term in braces next to $F(\mathscr{L})$ we recognize as the Mukai paring between β_0 and $v(E, \nabla)$, keeping in mind that the factors of $i/2\pi$ are implicitly in the factors of F_∇. Here, $v(E, \nabla)$ is a de

Rham representative of the Mukai vector $v(E)$ coming from ∇. We now reinsert[1] the factors of $i/2\pi$ to get

$$\theta(\beta)(\eta,\xi) = \int_X \beta_0 \left\{ \frac{i}{8\pi^3} \operatorname{tr}\left[\frac{1}{2} F_\nabla \wedge F(\mathbb{A})^2_{1,1} - \frac{1}{2} F^2_\nabla \wedge \Phi\Theta\right] + \frac{i}{2\pi} \operatorname{tr}(\Phi\Theta) \operatorname{vol}_g \right\}$$
$$+ \int_X \frac{i}{2\pi} \beta_4 \operatorname{tr}(\Phi\Theta) - \frac{1}{4\pi^2} \beta_2 \operatorname{tr}\left[\frac{1}{2} F(\mathbb{A})^2_{1,1} - F_\nabla \wedge \Phi\Theta\right] + \frac{i}{2\pi} \int_X F(\mathscr{L}) \langle \beta, v(E,\nabla)\rangle \tag{A.1.8}$$

Next we will contract with two tangent vectors $\eta, \xi \in T_{[\nabla]}\mathcal{M}$. The definition of integration over the fiber calls for us to calculate $\theta(\beta)(\eta,\xi)$ by first contracting with ξ and then η. Notationally, this looks like $\theta(\beta)(\eta,\xi) = \int i_\eta i_\xi(\cdots)$, where i_ξ denotes the contraction of the tangent vector ξ with a differential form (from $\mathcal{M}$). We note the following identities that will be used:

$$i_\eta i_\xi F(\mathbb{A})^2_{1,1} = \eta \wedge \xi - \xi \wedge \eta$$
$$i_\eta i_\xi \Phi\Theta = -\Phi\Theta(\eta,\xi) \tag{A.1.9}$$
$$i_\eta i_\xi F(\mathscr{L}) = -F(\mathscr{L})(\eta,\xi)$$

The first follows from (5.2.5) while the other two follow from exchanging the order of inputs to a 2-form. The $F(\mathbb{A})^2_{1,1}$ term appears twice, once on its own and once multiplied by F_∇. When it appears on its own we can use the cyclic property of trace and the fact that η and ξ are differential 1-forms on X to write $\operatorname{tr}(\xi\wedge\eta) = -\operatorname{tr}(\eta\wedge\xi)$, which allows us to make the simplification

$$\frac{1}{2} \operatorname{tr}(\eta \wedge \xi - \xi \wedge \eta) = \operatorname{tr}(\eta \wedge \xi).$$

The other term involving F_∇ will not simplify in this manner. We also note that $F(\mathscr{L})(\eta,\xi)$ is independent of $x \in X$ since the dependence of η and ξ on x is already integrated out; see the formula for $F(\mathscr{L})(\eta,\xi)$ in (5.2.6). We then have the desired result:

$$\theta(\beta)(\eta,\xi) = \frac{i}{2\pi} \langle \beta, v(E)\rangle F(\mathscr{L})(\eta,\xi) - \frac{i}{2\pi} \int_X \beta_4 \operatorname{tr}(\Phi\Theta(\eta,\xi))$$
$$- \frac{1}{4\pi^2} \int_X \beta_2 \operatorname{tr}\left[\eta \wedge \xi + F_\nabla \wedge \Phi\Theta(\eta,\xi)\right] - \frac{i}{2\pi} \int_X \beta_0 \operatorname{tr}(\Phi\Theta(\eta,\xi)) \operatorname{vol}_g \tag{A.1.10}$$
$$+ \frac{i}{8\pi^3} \int_X \beta_0 \operatorname{tr}\left[\frac{1}{2} F_\nabla \wedge (\eta \wedge \xi - \xi \wedge \eta) + \frac{1}{2} F^2_\nabla \wedge \Phi\Theta(\eta,\xi)\right].$$

[1]Since the factors of $i/2\pi$ were essentially absorbed into the curvature $\mathcal{F} - \Phi\Theta$, we reinsert as many factors of $i/2\pi$ as there are curvature terms.

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