# LINKING FORMS, SINGULARITIES, AND HOMOLOGICAL STABILITY FOR DIFFEOMORPHISM GROUPS OF ODD DIMENSIONAL MANIFOLDS

by

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DISSERTATION ABSTRACT

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Let $n \geq 2$. We prove a homological stability theorem for the diffeomorphism groups of $(4n+1)$-dimensional manifolds, with respect to forming the connected sum with $(2n-1)$-connected, $(4n+1)$-dimensional manifolds that are stably parallelizable. Our techniques involve the study of the action of the diffeomorphism group of a manifold $M$ on the *linking form* associated to the homology groups of $M$. In order to study this action we construct a geometric model for the linking form using the intersections of embedded and immersed $\mathbb{Z}/k$-manifolds. In addition to our main homological stability theorem, we prove several results regarding disjunction for embeddings and immersions of $\mathbb{Z}/k$-manifolds that could be of independent interest.

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# CHAPTER I

# INTRODUCTION

Let $M$ be a smooth manifold and let $\mathrm{Diff}(M)$ denote the group of diffeomorphisms of $M$ topologized in the $C^\infty$-topology. The classifying space, $\mathrm{BDiff}(M)$, occupies a central place in smooth topology. In particular, for any paracompact space $X$ there is a natural bijection between the set of homotopy classes of maps $X \to \mathrm{BDiff}(M)$ and the set of isomorphism classes of principal $\mathrm{Diff}(M)$-bundles over $X$. This bijection reduces the study of principal $\mathrm{Diff}(M)$-bundles to the study of the weak homotopy type of the classifying space $\mathrm{BDiff}(M)$. Furthermore, the cohomology ring $H^*(\mathrm{BDiff}(M))$ contains all characteristic classes for principal $\mathrm{Diff}(M)$-bundles.

For an arbitrary manifold $M$, the space $\mathrm{BDiff}(M)$ is quite complicated and there are very few "universal" results that pertain the homotopy type of $\mathrm{BDiff}(M)$ for all manifolds $M$. A fundamental question is the following: to what extent does the homology group $H_*(\mathrm{BDiff}(M))$ depend on the structure of the underlying manifold $M$? In particular, how does the structure of the group $H_*(\mathrm{BDiff}(M))$ change if we alter the underlying manifold $M$ by a surgery operation? Recently, through the work of M. Weiss and I. Madsen in (28), and the work of S. Galatius and O. Randal-Williams (11), (10), much progress has been made in answering this question when the manifold $M$ is of dimension $2n$. This thesis is primarily concerned with answering this question in the case that the manifold $M$ is of dimension $2n + 1$.

## 1.1. Background and Basic Definitions

Fix a smooth manifold $M$. Let $\mathrm{Diff}^{\partial}(M)$ denote the subgroup of $\mathrm{Diff}(M)$ consisting of all diffeomorphisms of $M$ that restrict to the identity on some neighborhood of the boundary $\partial M$. Our main object of interest is the classifying space of the topological group $\mathrm{Diff}^{\partial}(M)$. For any topological group $G$, the classifying space $BG$ is characterized up to weak homotopy equivalence as the base space of a principal $G$-bundle $EG \longrightarrow BG$ with weakly contractible total space $EG$. We will need to work with a particular model for the classifying space of $\mathrm{Diff}^{\partial}(M)$ which we define below.

**Definition 1.1.1.** Fix a collar embedding $h : (-\infty, 0] \times \partial M \longrightarrow M$ with $h^{-1}(\partial M) = \{0\} \times \partial M$, and fix an embedding $\theta : \partial M \longrightarrow \{0\} \times \mathbb{R}^{\infty}$. We define $\mathrm{EDiff}^{\partial}(M)$ to be the space of all smooth embeddings $\phi : M \longrightarrow (-\infty, 0] \times \mathbb{R}^{\infty}$, for which there exists a real number $\epsilon > 0$ such that $\phi(h(t, x)) = (t, \theta(x))$ when $-\epsilon < t \leq 0$. We then define $\mathrm{BDiff}^{\partial}(M)$ to be the quotient of $\mathrm{EDiff}^{\partial}(M)$ obtained by identifying embeddings that have the same image.

With the above definition, the underlying set of $\mathrm{BDiff}^{\partial}(M)$ is precisely the set of all submanifolds $V \subset (-\infty, 0] \times \mathbb{R}^{\infty}$ that are abstractly diffeomorphic to the manifold $M$ and that have boundary equal to $\theta(\partial M)$. We will generally denote elements of $\mathrm{BDiff}^{\partial}(M)$ by such submanifolds $V$. In this way $\mathrm{BDiff}^{\partial}(M)$ can be thought as a "non-linear" generalization of a Grassmannian manifold.

**Remark 1.1.1.** The definition of $\mathrm{BDiff}^{\partial}(M)$ involved two arbitrary choices: the collar $h$ and the embedding $\theta$. Any two embeddings $\partial M \hookrightarrow \mathbb{R}^{\infty}$ are isotopic. Similarly, any two collar embeddings $(-\infty, 0] \times \partial M \longrightarrow M$ are isotopic as well. It follows that the topology of the space $\mathrm{EDiff}^{\partial}(M)$, and hence the topology of the

space $\mathrm{BDiff}^{\partial}(M)$, does not depend on the choices of embedding $\theta$ and $h$ used in the definition. For this reason we are justified in excluding $h$ and $\theta$ from the notation.

The topological group $\mathrm{Diff}^{\partial}(M)$ acts freely and continuously on $\mathrm{EDiff}^{\partial}(M)$ by pre-composition $(f, \phi) \mapsto \phi \circ f$, and thus $\mathrm{BDiff}^{\partial}(M)$ is identified with the orbit space $\mathrm{EDiff}^{\partial}(M)/\mathrm{BDiff}^{\partial}(M)$. By (4), it follows that the quotient projection $\mathrm{EDiff}^{\partial}(M) \longrightarrow \mathrm{BDiff}^{\partial}(M)$ is a locally trivial fibre-budle with fibre equal to $\mathrm{Diff}^{\partial}(M)$. Furthermore, it is well known that the total space $\mathrm{EDiff}^{\partial}(M)$ is weakly contractible. It follows that our definition of $\mathrm{BDiff}^{\partial}(M)$ given in Definition 1.1.1 coincides with the standard definition of the classifying space $BG$ for a general topological group $G$ given in (32). It then follows that for any paracompact space $X$, there is natural bijection between the set of homotopy classes of maps $X \longrightarrow \mathrm{BDiff}^{\partial}(M)$ and the set of isomorphism classes of principal $\mathrm{Diff}^{\partial}(M)$ fibre bundles over $X$.

We will need to work with certain natural maps connecting the classifying spaces of the diffeomorphism groups of different manifolds. Let $M$ be a smooth manifold of dimension $m$ with non-empty boundary. Let $K$ be an $m$-dimensional manifold with boundary given by the disjoint union $\partial K = \partial_0 K \sqcup \partial_1 K$, where $\partial_0 K = \partial M$ (in other words, $K$ is a cobordism from $\partial M$ to $\partial_1 K$). Let $M \cup_{\partial M} K$ denote the manifold obtained by attaching $K$ to $M$ along $\partial M$.

**Definition 1.1.2.** Let $M$ and $K$ be as above. Choose a collared embedding $\alpha : K \hookrightarrow [0,1] \times \mathbb{R}^{\infty}$ with $\alpha(\partial_i K) \subset \{i\} \times \mathbb{R}^{\infty}$ for $i = 0, 1$, and such that $\alpha|_{\partial_0 K}$ coincides with the embedding $\partial M \hookrightarrow \{0\} \times \mathbb{R}^{\infty}$ used in the definition of $\mathrm{BDiff}^{\partial}(M)$. Let $t_{-1} : (-\infty, 1] \times \mathbb{R}^{\infty} \longrightarrow (-\infty, 0] \times \mathbb{R}^{\infty}$ be the diffeomorphism given by translation in

the first coordinate. We define

$$\_ \cup K : \mathrm{BDiff}^{\partial}(M) \longrightarrow \mathrm{BDiff}^{\partial}(M \cup_{\partial M} K)$$

to be the continuous map given by sending an element $V \in \mathrm{BDiff}^{\partial}(M)$ (which is a submanifold of $(-\infty, 0] \times \mathbb{R}^{\infty}$) to the element of $\mathrm{BDiff}^{\partial}(M \cap_{\partial M} K)$ given by the submanifold $t_{-1}(V \cup \alpha(K))$.

**Remark 1.1.2.** There was an arbitrary choice involved in the construction of the map in the above definition, namely the embedding $\alpha$. Since any two collared embeddings $K \hookrightarrow [0,1] \times \mathbb{R}^{\infty}$ are isotopic, it follows that the homotopy class of the map $\_ \cup K$ in the above definition does not depend on the choice of embedding $\alpha$. It follows that the homotopy class of $\_ \cup K$ is determined entirely by the diffeomorphism class of the cobordism $K$.

A particular case of the above map that we will use is the following. Let $W$ be a closed manifold of dimension $m$ and let $K_W$ denote the manifold obtained by forming the connected sum of $\partial M \times [0,1]$ with $W$. We identify the manifold $M \cup_{\partial M} K_W$ with the connected-sum $M \# W$ and thus the classifying spaces $\mathrm{BDiff}^{\partial}(M \cup_{\partial M} K_W)$ are $\mathrm{BDiff}^{\partial}(M \# W)$ identified. Using these identifications, the map from Definition 1.1.2 yields a map

$$s_W : \mathrm{BDiff}^{\partial}(M) \longrightarrow \mathrm{BDiff}^{\partial}(M \# W). \tag{1.1}$$

## 1.2. Some Fundamental Results

Perhaps the first positive result pertaining to the homomorphism on homology induced by the map $s_W : \mathrm{BDiff}^{\partial}(M) \longrightarrow \mathrm{BDiff}^{\partial}(M \# W)$, is the

homological stability theorem of J. Harer in (13) for the diffeomorphism groups of oriented surfaces. For an integer $g \in \mathbb{N}$, let $L_g$ denote the oriented surface with boundary obtained by removing an open disk from the closed genus-$g$ surface $(S^1 \times S^1)^{\#g}$. Since $L_{g+1} \cong L_g \# (S^1 \times S^1)$, for each integer $g$ (1.1) yields the map $s : \mathrm{BDiff}^{\partial}(L_g) \longrightarrow \mathrm{BDiff}^{\partial}(L_{g+1})$. The main theorem from (13) is the following.

**Theorem 1.2.1** (J. Harer 1985)**.** *The map on homology*

$$s_* : H_\ell(\mathrm{BDiff}^{\partial}(L_g); \mathbb{Z}) \longrightarrow H_\ell(\mathrm{BDiff}^{\partial}(L_{g+1}); \mathbb{Z})$$

*induced by $s$ is an isomorphism when $\ell < \frac{g}{3} - 2$.*

The above theorem implies that the maps in the direct system

$$\cdots \longrightarrow \mathrm{BDiff}^{\partial}(L_{g-1}) \longrightarrow \mathrm{BDiff}^{\partial}(L_g) \longrightarrow \mathrm{BDiff}^{\partial}(L_{g+1}) \longrightarrow \cdots \tag{1.2}$$

induce isomorphisms on the homology groups $H_\ell(\_; \mathbb{Z})$ when $\ell << g$. We say then that the direct system (1.3) satisfies *homological stability* and that Theorem 1.2.3 is a *homological stability theorem*. Significant improvements in the stability range of the above theorem were made years latter, separately by Ivanov (18), Boldsen (6), and Randal-Williams (31). Furthermore, an analogue of Theorem 1.2.1 for non-orientable surfaces was proven by N. Wahl in (40).

**Remark 1.2.1.** In (8) it was proven that the diffeomorphism groups $\mathrm{Diff}^{\partial}(L_g)$ have contractible path components when $g > 1$. It follows that $\mathrm{BDiff}^{\partial}(L_g)$ is homotopy equivalent to the classifying space of the mapping-class group $\pi_0(\mathrm{Diff}^{\partial}(L_g))$ when $g > 1$. The proof of Theorem 1.2.1 makes use of these facts and for this reason,

Theorem 1.2.1 is usually stated in terms of the mapping-class groups of surfaces, rather than of the diffeomorphism groups.

The above theorem has many formal similarities to other classical homological stability theorems regarding direct systems of groups. For example, for large classes of rings $R$, the direct system of classifying spaces of the general linear groups, $BGl_n(R) \to BGl_{n+1}(R) \to BGl_{n+2}(R) \to \cdots$, is known to satisfy homological stability (see (20)). Also the direct system, $B\Sigma_n \to B\Sigma_{n+1} \to B\Sigma_{n+2} \to \cdots$ of classifying spaces of the symmetric groups satisfies homological stability as well (see (29)). The proof of Theorem 1.2.1 shares many formal similarities with the proofs of these above mentioned homological stability results.

In (28) I. Madsen and M. Weiss identify the homology of the limiting space $\operatorname{colim}_{g\to\infty} \mathrm{BDiff}^{\partial}(L_g)$. Let $\mathbb{C}P^{\infty}_{-1}$ denote the Thom-spectrum associated to the formal inverse of the canonical complex line bundle $\gamma^1_{\mathbb{C}}$ over $\mathbb{C}P^{\infty}$. We will need to consider the infinite loopspace $\Omega^{\infty}\mathbb{C}P^{\infty}_{-1}$ associated to the spectrum $\mathbb{C}P^{\infty}_{-1}$. We will denote by $\Omega^{\infty}_0\mathbb{C}P^{\infty}_{-1}$ the path-component of $\Omega^{\infty}\mathbb{C}P^{\infty}_{-1}$ containing the constant loop. The following is the main theorem from (28).

**Theorem 1.2.2** (Madsen-Weiss 2002)**.** *There is a homological equivalence,*

$$\operatorname{colim}_{g\to\infty} \mathrm{BDiff}^{\partial}(L_g) \xrightarrow{\cong} \Omega^{\infty}_0\mathbb{C}P^{\infty}_{-1}.$$

The cohomology of the space $\Omega^{\infty}_0\mathbb{C}P^{\infty}_{-1}$ can be readily computed and thus the above theorem, together with Theorem 1.2.3, enables a calculation of the cohomology (and homology) groups of the space $\mathrm{BDiff}^{\partial}(L^{2n}_g)$ in a dimensional range when $g$ is large. With rational coefficients there is an isomorphism $H^*(\Omega^{\infty}_0\mathbb{C}P^{\infty}_{-1};\mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]$, where $\kappa_i \in H^{2i}(\Omega^{\infty}_0\mathbb{C}P^{\infty}_{-1};\mathbb{Q})$ for $i \geq 0$. The

classes $\kappa_i$ are known as the *kappa classes* and were studied by Miller, Morita, and Mumford in (25), (26), and (27). This theorem of Madsen and Weiss proved the long-standing *Mumford Conjecture* from (27), which conjectured the isomorphism $H^*(\Omega_0^\infty \mathbb{C}P^\infty_{-1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots, ]$.

**Remark 1.2.2.** If one views the symmetric group $\Sigma_g$ as the diffeomorphism group of the zero-dimensional manifold consisting of $g$-many points, then the theorem of Barratt, Priddy, Quillen and Segal, establishing the homological equivalence $\operatorname{colim}_{g\to\infty} B\Sigma_g \times \mathbb{Z} \xrightarrow{\simeq} QS^0$ (see (2)), can be viewed as an analogue of the Madsen-Weiss theorem for zero-dimensional manifolds. The proof of theorem 1.2.4 has formal similarities to the theorem of Barrat, Priddy, Quillen, and Segal as well.

We now discuss some recent results pertaining to the diffeomorphism groups of high dimensional manifolds analogous to those from the previous section. Let $n \in \mathbb{N}$ be an integer an let $M$ be a manifold of dimension $2n$ with non-empty boundary. We will consider the map $s : \mathrm{BDiff}^\partial(M) \longrightarrow \mathrm{BDiff}^\partial(M \# (S^n \times S^n))$ and the direct system

$$\cdots \longrightarrow \mathrm{BDiff}^\partial(M \#(S^n \times S^n)^{\#g}) \longrightarrow \mathrm{BDiff}^\partial(M \#(S^n \times S^n)^{\#(g+1)}) \longrightarrow \cdots \tag{1.3}$$

obtained by iterating this map $s$.

The idea of studying manifolds of dimension $2n$ up to connected sum with factors of $S^n \times S^n$ can be traced back to Wall in his study of simply connected 4-manifolds in (38). This idea was taken further by M. Kreck in (22) where he gives a diffeomorphism classification of manifolds of dimension $2n \geq 6$, up to connected sum with factors of $S^n \times S^n$.

In order to state the main theorems regarding the direct system (1.3), we will need to define a generalized notion of genus for manifolds of dimension $2n$. For each $g \in \mathbb{N}$, let $L_g^{2n}$ denote the manifold obtained by deleting an open disk from $(S^n \times S^n)^{\#g}$. We define $g(M)$ to be the largest integer $g \in \mathbb{N}$ such that there exists an embedding of $L_g^{2n}$ into $M$. The following result is the main theorem from (11).

**Theorem 1.2.3** (Galatius, Randal-Williams 2014)**.** *Let $M$ be a simply connected, compact manifold of dimension $2n \geq 6$, with non-empty boundary. Let $g(M) \geq g$. Then the map on integral homology*

$$s_* : H_\ell(\mathrm{BDiff}^\partial(M); \mathbb{Z}) \longrightarrow H_\ell(\mathrm{BDiff}^\partial(M \# (S^n \times S^n)); \mathbb{Z})$$

*induced by $s$ is an isomorphism when $\ell \leq \frac{1}{2}(g-3)$ and an epimorphism when $\ell \leq \frac{1}{2}(g-1)$.*

The above theorem implies that for any simply connected manifold $M$ of dimension $2n \geq 6$ with $\partial M \neq \emptyset$, the maps in the direct system (1.3) induce isomorphisms on the homology groups $H_\ell(\_; \mathbb{Z})$ when $g >> \ell$. Thus the above theorem is a homological stability theorem analogous to Theorem 1.2.1 from the previous section.

In addition to the homological stability theorem described in the previous paragraphs, in (10) the authors identify the homological type of the limiting space $\operatorname{colim}_{g\to\infty} \mathrm{BDiff}^\partial(L_g^{2n})$. Let $\gamma_{2n} \longrightarrow BO(2n)$ denote the universal $2n$-dimensional vector bundle, let $\theta_n : B\theta_n \longrightarrow BO(2n)$ denote the *$n$-connective cover*, and let $\mathsf{MT}\theta_n$ denote the Thom-spectrum associated to the formal inverse of the pull-back bundle $\theta_n^*(\gamma_{2n}) \longrightarrow B\theta_n$.

**Theorem 1.2.4** (Galatius, Randal-Williams 2014)**.** *Let $n \geq 3$. Then there is a homological equivalence,* $\operatorname{colim}_{g\to\infty} \mathrm{BDiff}^{\partial}(L_g^{2n}) \xrightarrow{\simeq} \Omega_0^{\infty}\mathsf{MT}\theta_n$.

The cohomology of the space $\Omega_0^{\infty}\mathsf{MT}\theta_n$ can be readily computed (it has been completely determined rationally) and thus the above theorem, together with Theorem 1.2.3, enables a calculation of the homology groups of the space $\mathrm{BDiff}^{\partial}(L_g^{2n})$ in a dimensional range when $g$ is large. The proof of Theorem 1.2.4 was largely inspired by the proof of the Madsen-Weiss theorem and uses many of the same techniques.

## 1.3. Statement of Our Main Results

The results about the diffeomorphism groups discussed in the previous sections all pertain solely to manifolds of even dimension. Indeed, the results of Galatius and Randal-Williams from (11) rely heavily on techniques from the surgery theory of Kreck (22) and Wall (35) specialized to the category of manifolds of dimension $2n \geq 6$. The homological type of the diffeomorphism groups of manifolds of odd dimension have until now been largely untouched and there are very few existing results in the literature. In this thesis we prove a homological stability theorem for the diffeomorphism groups of manifolds of odd dimension. We extend the techniques used by Galatius and Randal-Williams to the category of odd dimensional manifolds, and thus also set the stage to approach an odd dimensional version of Theorem 1.2.4.

Let $M$ be a $(4n+1)$-dimensional manifold with non-empty boundary, where $n \geq 2$. For any closed $(4n+1)$-dimensional manifold $W$, we consider the map

$\mathrm{BDiff}^{\partial}(M) \longrightarrow \mathrm{BDiff}^{\partial}(M\#W)$ from (1.1) and the direct system

$$\mathrm{BDiff}^{\partial}(M) \longrightarrow \mathrm{BDiff}^{\partial}(M\#W) \longrightarrow \cdots \longrightarrow \mathrm{BDiff}^{\partial}(M\#W^{\#g}) \longrightarrow \cdots \tag{1.4}$$

obtained by iteration of this map. The main result of this thesis is a theorem concerning the homological stability of the above direct system.

**Theorem 1.3.1.** *Let $n \geq 2$ and let $M$ be a 2-connected, $(4n+1)$-dimensional, compact manifold with non-empty boundary. Let $W$ be a closed, $(4n+1)$-dimensional manifold that satisfies the following conditions:*

- *$W$ is $(2n-1)$-connected,*
- *$W$ is stably parallelizable,*
- *the homology group $H_{2n}(W;\mathbb{Z})$ has no 2-torsion.*

*Then the group $H_\ell(\mathrm{BDiff}^{\partial}(M\#W^{\#g});\mathbb{Z})$ is independent of the integer $g$ if $g \geq 2\ell+3$. In particular, the direct system (1.4) satisfies homological stability.*

**Remark 1.3.1.** The case of Theorem 1.3.1 when $W$ is a product of spheres $S^{2n} \times S^{2n+1}$ follows as a special case of (30, Theorem 1.3) by the same author of this thesis.

The proof of the above theorem draws heavily from the classification of $(2n-1)$-connected, $(4n+1)$-dimensional manifolds of Wall in (37). In order to describe the techniques used in the proof of Theorem 1.3.1, we must describe this classification theorem of Wall. Let us first fix some notation that we will use throughout the paper. Let $\mathcal{W}_{4n+1}$ denote the set of all $(2n-1)$-connected, $(4n+1)$-dimensional, compact manifolds. Let $\bar{\mathcal{W}}_{4n+1} \subset \mathcal{W}_{4n+1}$ denote the subset of those

manifolds that are closed, let $\mathcal{W}^S_{4n+1} \subset \mathcal{W}_{4n+1}$ denote the subset of those manifolds that are stably-parallelizable, and let $\bar{\mathcal{W}}^S_{4n+1}$ denote the intersection $\mathcal{W}^S_{4n+1} \cap \bar{\mathcal{W}}_{4n+1}$. In order to prove Theorem 1.3.1, we will need to analyze the diffeomorphism invariants associated to elements of $\mathcal{W}_{4n+1}$. For $M \in \mathcal{W}_{4n+1}$, let $\pi^\tau_{2n}(M) \leq \pi_{2n}(M)$ denote the torsion subgroup. The primary diffeomorphism invariant associated to $M$ is the *linking form*, which is a skew-symmetric, bilinear pairing

$$b : \pi^\tau_{2n}(M) \otimes \pi^\tau_{2n}(M) \longrightarrow \mathbb{Q}/\mathbb{Z}, \tag{1.5}$$

which is non-singular in the case that $M$ is closed. For $n \geq 2$, the classification of manifolds in $\mathcal{W}_{4n+1}$ was studied by Wall in (37). Recall that two closed manifolds $M_1$ and $M_2$ are said to be *almost diffeomorphic* if there exists a homotopy sphere $\Sigma$ such that $M_1 \# \Sigma$ is diffeomorphic to $M_2$. It follows from Wall's classification theorem (37, Theorem 7), that two elements $M_1, M_2 \in \bar{\mathcal{W}}^S_{4n+1}$ are almost diffeomorphic if and only if there exists an isomorphism, $\pi^\tau_{2n}(M_1) \xrightarrow{\cong} \pi^\tau_{2n}(M_2)$ that preserves the linking form $b$. Furthermore, given any finite abelian group $G$ equipped with a non-singular, skew-symmetric bilinear form $b' : G \otimes G \longrightarrow \mathbb{Q}/\mathbb{Z}$, there exists a manifold $M \in \bar{\mathcal{W}}^S_{4n+1}$ and an isomorphism of forms, $(\pi^\tau_{2n}(M),\ b) \cong (G,\ b')$.

We use the classification result discussed above to specify certain elements of $\bar{\mathcal{W}}^S_{4n+1}$. For each integer $k \geq 2$, fix a manifold $W_k \in \bar{\mathcal{W}}^S_{4n+1}$ whose linking-form $(\pi^\tau_{2n}(W_k),\ b)$ is given by the data,

$$\pi_{2n}(W_k) = \mathbb{Z}/k \oplus \mathbb{Z}/k, \qquad b(\sigma,\sigma) = b(\rho,\rho) = 0, \qquad b(\sigma,\rho) = -b(\rho,\sigma) = \tfrac{1}{k} \mod 1, \tag{1.6}$$

where $\langle \rho, \sigma \rangle$ is the standard basis for $\mathbb{Z}/k \oplus \mathbb{Z}/k$. It follows from (37, Theorem 7) and the classification of skew symmetric bilinear forms over $\mathbb{Q}/\mathbb{Z}$ in (38, Lemma 7), that any element $M \in \bar{\mathcal{W}}^S_{4n+1}$ is diffeomorphic to a manifold of the form

$$W_{k_1} \# \cdots \# W_{k_l} \# (S^{2n} \times S^{2n+1})^{\# g} \# \Sigma, \tag{1.7}$$

where $\Sigma$ is a homotopy sphere.

**Remark 1.3.2.** It follows from these classification results, (37, Theorem 7) and (38, Lemma 7), that if $k$ and $\ell$ are relatively prime, then $W_k \# W_\ell \cong W_{k \cdot \ell}$. In this way, the (almost) diffeomorphism classification of $\bar{\mathcal{W}}^S_{4n+1}$ mirrors the classification of finitely generated abelian groups. Thus it will suffice to restrict our attention to the manifolds $W_k$ in the case that $k = p^j$ for a prime number $p$. By the connected-sum decomposition (1.7) it follows that the manifolds $W_{p^j}$ are indecomposable.

Now, let $M$ be a $(4n+1)$-dimensional manifold with non-empty boundary. For each integer $k \geq 2$, let

$$s_k : \mathrm{BDiff}^\partial(M) \longrightarrow \mathrm{BDiff}^\partial(M \# W_k) \tag{1.8}$$

denote the map from (1.1). We will refer to this map as the $k$-th stabilization map. Let $r_k(M)$ be the quantity defined by,

$$r_k(M) = \max\{g \in N \mid \text{there exists an embedding, } W_k^{\# g} \setminus D^{4n+1} \longrightarrow M\}. \tag{1.9}$$

For each $k$ we may think of $r_k(M)$ as a generalized version of the genus of a surface or an analogue of the quantity $g(M)$ used in the statement of Theorem 1.2.3. Using the diffeomorphism classification for manifolds in $\bar{\mathcal{W}}^S_{4n+1}$ described in Section III,

the following result, combined with (30) implies Theorem 1.3.1. This is the main homological stability result that we prove in this paper.

**Theorem 1.3.2.** *For $n \geq 2$, let $M$ be a 2-connected, compact, $(4n+1)$-dimensional manifold with non-empty boundary. If $k > 2$ is an odd integer, then the map on homology induced by (1.8),*

$$(s_k)_* : H_\ell(\mathrm{BDiff}^\partial(M);\, \mathbb{Z}) \longrightarrow H_\ell(\mathrm{BDiff}^\partial(M \# W_k);\, \mathbb{Z})$$

*is an isomorphism if $2\ell \leq r_k(M) - 3$ and an epimorphism when $2\ell \leq r_k(M) - 1$.*

## 1.4. Methodology

In this section we give an overview of the methods used in the proof of Theorem 1.3.2. To prove our homological stability theorem, for each integer $k \geq 2$ we construct a highly connected, semi-simplicial space $X_\bullet(M)_k$, which admits an action of the topological group $\mathrm{Diff}^\partial(M)$ that is transitive on the zero-simplicies. The semi-simplicial space $X_\bullet(M)_k$ is defined roughly as follows. Let $W_k'$ denote the manifold with boundary obtained from $W_k$ by removing an open disk. The space of $p$-simplices, $X_p(M)_k$, is defined to be the space of ordered $(p+1)$-tuples $(\phi_0, \ldots, \phi_p)$ where $\phi_i : W_k' \longrightarrow M$ is an embedding for $i = 0, \ldots, p$ and $\phi_j(W_k') \cap \phi_l(W_k') = \emptyset$ if $j \neq l$. With this definition, a $p$-simplex in $X_p(M)_k$ can be viewed as a particular way of splitting a connected-sum factor of $W_k^{\#(p+1)}$ off of $M$.

The majority of the technical work of this paper is devoted to proving that if $M$ is 2-connected and $k$ is odd, then the geometric realization $|X_\bullet(M)_k|$ is $\frac{1}{2}(r_k(M) - 4)$-connected. This is established in Section VIII and uses all of the techniques developed throughout the rest of the thesis. In Section IX it is

shown how high connectivity of $|X_\bullet(M)_k|$ implies Theorem 1.3.2. This uses a spectral sequence an argument essentially identical to (9, Section 5.2) and thus, the majority of the technical work of this paper is devoted to proving that $|X_\bullet(M)_k|$ is $\frac{1}{2}(r_k(M)-4)$-connected.

**Remark 1.4.1.** This method of proving a homological stability theorem by construction of a highly connected semi-simplicial space (or simplicial complex) analogous to $X_\bullet(M)_k$, is a fairly standard method and has been used to prove homological stability theorems in many other contexts. A general overview of this method of proving homological stability is given in (39) and many examples are treated there as well.

The semi-simplicial space $X_\bullet(M)_k$ is very difficult to study directly. In order to prove that its geometric realization is $\frac{1}{2}(r_k(M)-4)$-connected, we must compare it to an auxiliary simplicial complex $L(\pi^\tau_{2n}(M))_k$ whose definition is based on an algebraic structure and thus can be analyzed via combinatorial methods. A $p$-simplex in the simplicial complex $L(\pi^\tau_{2n}(M))_k$ is defined to be a set of $(p+1)$-many, pairwise orthogonal morphisms of linking forms $(\pi^\tau_{2n}(W'_k),\ b) \longrightarrow (\pi^\tau_{2n}(M),\ b)$, which mimic the pairwise disjoint embeddings $W'_k \to M$ from the semi-simplicial space $X_\bullet(M)_k$. In Section 4.3, we prove that the geometric realization $|L(\pi^\tau_{2n}(M))_k|$ is $\frac{1}{2}(r_k(M)-4)$-connected (see Theorem 4.3.2) using the simplicial techniques developed in Chapter II and the algebraic properties of bilinear forms defined on finite groups. This result is similar to the classical homological stability theorem of R. Charney from (7).

To compare $|X_\bullet(M)_k|$ to $L(\pi^\tau_{2n}(M))_k$, we consider a map $|X_\bullet(M)_k| \longrightarrow |L(\pi^\tau_{2n}(M))_k|$ induced by sending an embedding $\varphi : W'_k \longrightarrow M$, which represents a 0-simplex in $X_\bullet(M)_k$, to its induced morphism of linking forms,

$\varphi_* : (\pi^\tau_{2n}(W'_k),\ b) \longrightarrow (\pi^\tau_{2n}(M),\ b)$, which represents a vertex in $L(\pi^\tau_{2n}(M))_k$. With the high-connectivity of $L(\pi^\tau_{2n}(M))_k$ established, to prove that $|X_\bullet(M)_k|$ is highly connected it will suffice to prove that the above map induces an injection on homotopy groups $\pi_j(\_)$ when $j \leq \frac{1}{2}(r_k(M) - 4)$. This requires several new geometric constructions which are of independent interest. In particular, we need a technique for realizing morphisms $(\pi^\tau_{2n}(W'_k),\ b) \longrightarrow (\pi^\tau_{2n}(M),\ b)$ by actual embeddings $W'_k \to M$.

To solve the problem of realizing morphisms of linking forms by actual embeddings of manifolds, we will need a suitable geometric model for the linking form based on $\mathbb{Z}/k$-manifolds and their intersections. Roughly, an $m$-dimensional $\mathbb{Z}/k$-manifold is a manifold with singularities whose local structure at the singularity is of the form $\mathbb{R}^{n-1} \times C(\langle k \rangle)$, where $\langle k \rangle$ denotes the set of $k$ distinct points and $C(\langle k \rangle)$ is the cone formed over that set. $\mathbb{Z}/k$-manifolds can be used to represent cycles in homology groups with $\mathbb{Z}/k$-coefficients and thus they arise for us in our analysis of the linking form. Sections V through VII, as well as the appendices, are devoted to developing the intersection theory of $\mathbb{Z}/k$-manifolds.

Our $\mathbb{Z}/k$-manifold methods are largely inspired by the work of Morgan and Sullivan in (33). However, this thesis contains several (to our knowledge) new results regarding $\mathbb{Z}/k$-manifolds, and other types of manifolds with singularities, that don't currently exist in the literature. These new results/techniques include:

- An $h$-principle for immersions of $\mathbb{Z}/k$-manifolds into a smooth (non-singular) manifold (Appendix 11.2).

- A technique for eliminating the self-intersections of an immersion of a $\mathbb{Z}/k$-manifold (Appendix 11.4).

– A method for modifying the intersections of embedded $\mathbb{Z}/k$-manifolds when a certain bordism invariant vanishes (Appendix **??**). In the case that the intersection in question is a zero dimensional manifold, this technique can be viewed as a $\mathbb{Z}/k$-version of the *Whitney-trick* (24, Theorem 6.6) used in the proof of the h-cobordism theorem.

These techniques enable us to get a handle geometrically on the linking form and they are all necessary to analyze the map which compares $X_\bullet(M)_k$ and $L(M)_k$. The main place where these singularity techniques are employed is in the proof of Lemma 8.1.2.

**Remark 1.4.2.** Our main homological stability result requires the integer $k$ to be odd. The source of this restriction on the integer $k$ is the technical result Theorem 10.5.1 and Theorem 11.6.1. If these theorems could be upgraded to include the case that $k$ is even, then Theorem 1.3.2 could be upgraded to include the case where $k$ is even as well.

It is also desirable to have a result analogous Theorem 1.3.1 for manifolds of dimension $4n + 3$. The key technical result in this paper for which the condition that our manifolds be $(4n + 1)$-dimensional is required is Theorem 5.4.1 and Corollary 11.5.3 (which is used to prove Theorem 5.4.1). If a version of Theorem 5.4.1 were to be extended to apply to manifolds of dimension $4n + 3$, then an analogue of the main result of this paper could be obtained for $(4n + 3)$-dimensional manifolds. However, the diffeomorphism classification of highly-connected manifolds of dimension $4n + 3$ (see (37)) is more involved than the classification in the dimension $4n + 1$ case, and so some other difficulties beyond Theorem 5.4.1 arise as well.

We will treat both of these extraordinary cases, where $k$ is even or the manifolds are of dimension $4n+3$, in a sequel to this paper.

## 1.5. Organization

Chapter II is a recollection of some basic definitions and results about simplicial complexes and semi-simplicial spaces. In Chapter III we describe the classification of $(2n-1)$-connected, $(2n+1)$-dimensional manifolds in terms of linking forms. In Chapter IV we define the primary semi-simplicial space $X_\bullet(M)_k$. Then define the simplicial complex of linking forms $L(\pi^\tau_{2n}(M)_k)$, and prove that it is highly-connected. In Chapters V, VI, and VII we give the necessary background on $\mathbb{Z}/k$-manifolds used in the proof of Theorem 1.3.2. In these three sections we state all of the necessary technical results regarding the intersections of immersions and embeddings of $\mathbb{Z}/k$-manifolds, but we put off most of the difficult proofs until Appendix X and XI. In Chapter VIII we prove that the geometric realization $|X_\bullet(M)_k|$ is highly connected. In Chapter IX we show how high-connectivity of $|X_\bullet(M)_k|$ implies Theorem 1.3.2. In Appendix X and Appendix XI, we prove several technical results regarding the intersections of immersions and embeddings of $\mathbb{Z}/k$-manifolds that were used earlier in the paper.

# CHAPTER II

# SIMPLICIAL TECHNIQUES

In this section we recall a number of simplicial techniques that we will need to use throughout the paper. We will need to consider a variety of different simplicial complexes and semi-simplicial spaces.

## 2.1. Cohen-Macaulay Complexes

Let $X$ be a simplicial complex. Recall that the *link* of a simplex $\sigma < X$, denoted by $\mathrm{lk}_X(\sigma)$, is defined to be the subcomplex of $X$ consisting of all simplices $\zeta$ disjoint from $\sigma$, for which there exists a simplex $\xi$ such that both $\sigma$ and $\zeta$ are faces of $\xi$.

We now present a key definition that will be used throughout the paper.

**Definition 2.1.1.** A simplicial complex $X$ is said to be *weakly Cohen-Macaulay* of dimension $n$ if it is $(n-1)$-connected and the link of any $p$-simplex is $(n-p-2)$-connected. In this case we write $\omega CM(X) \geq n$. The complex $X$ is said to be *locally weakly Cohen-Macaulay* of dimension $n$ if the link of any simplex is $(n-p-2)$-connected (but no global connectivity is required on $X$ itself). In this case we shall write $lCM(X) \geq n$.

We have some basic results about the links of simplices and Cohen-Macaulay complexes. The following two lemmas are taken directly from (11, Lemmas 2.1 and 2.3).

**Lemma 2.1.1.** *If $\omega CM(X) \geq n$ and $\sigma < X$ is a $p$-simplex, then $\omega CM(\mathrm{lk}_X(\sigma)) \geq n-p-1$.*

**Definition 2.1.2.** We say that a simplicial map $f : X \longrightarrow Y$ is *simplexwise injective* if its restriction to each simplex of $X$ is injective, i.e. if $v, v'$ are adjacent vertices in $X$, then $h(v) \neq h(v')$.

**Lemma 2.1.2.** *Let $f : X \longrightarrow Y$ be a simplicial map of simplicial complexes. Then the following conditions are equivalent.*

*i. $f$ is simplexwise injective,*

*ii. $f(\mathrm{lk}_X(\sigma)) \subset \mathrm{lk}_Y(f(v))$ for all vertices $v \in X$,*

*iii. $f(\mathrm{lk}_X(v)) \subset \mathrm{lk}_Y(f(v))$ for all vertices $v \in X$,*

*iv. the image of any 1-simplex in $X$ is a (non-degeneate) 1-simplex in $Y$.*

The next theorem is proven in (11, Section 2.1) and is a generalization of the "Coloring Lemma" of Hatcher and Wahl from (15, Lemma 3.1). We include the proof here since the result is so fundamental to us.

**Theorem 2.1.3.** *Let $X$ be a simplicial complex with $lCM(X) \geq n$, let $f : \partial I^n \to |X|$ be a map which is simplicial with respect to some PL triangulation of $\partial I^n$, and $h : I^n \to |X|$ be a null-homotopy of $f$. Then the triangulation of $\partial I^n$ extends to a PL triangulation of $I^n$, and $h$ is homotopic relative $\partial I^n$, to a simplicial map $g : I^n \to |X|$ with the property that $g(\mathrm{lk}_{I^n}(v)) \leq \mathrm{lk}_X(g(v))$ for any interior vertex $v \in \mathrm{Int}(I^n)$. In particular, $g$ is simplexwise injective on the interior of $I^n$.*

*Proof.* Since $|X|$ is in particular $(n-1)$-connected, we may extend $f$ to a map $h : I^n \longrightarrow |X|$ which, by the simplicial approximation theorem, may be assumed simplicial with respect to some PL triangulation of $I^n$ extending the given one on $\partial I^n$.

Now, let us say that a simplex $\sigma \in I^n$ is *bad* if every vertex $v \in \sigma$ is contained in a 1-simplex $\{v, v'\} \subset \sigma$ with $h(v) = h(v')$. We will describe a procedure which replaces the simplicial map $h : I^n \longrightarrow X$, by changing both $h$ and the triangulation of $I^n$, to a new map $g : I^n \longrightarrow X$ with no bad simplices in the interior of $I^n$.

If all bad simplices are contained in $\partial I^n$, then we are done. If not, let $\sigma < I^n$ be a bad simplex not contained in $\partial I^n$, of maximal dimension $p$. By the definition of bad simplex, the integer $p$ must be greater than zero. Furthermore, we must have $h(\mathrm{lk}_{I^n}(\sigma)) \subset \mathrm{lk}_X(h(\sigma))$, since otherwise we could join a simplex in $\mathrm{lk}_{I^n}(\sigma)$ to $\sigma$ and get a new bad simplex of larger dimension. Now $|\sigma| \subset I^n$, so $h$ restricts to a map

$$\partial I^{n-p} \cong \mathrm{lk}_{I^n}(\sigma) \longrightarrow \mathrm{lk}_X(h(\sigma)).$$

By badness of $\sigma$, the image $h(\sigma)$ must be a simplex of dimension less than or equal to $p-1$, since otherwise $h|_\sigma$ would be injective. Since $\omega CM(X) \geq n$ and $\dim(h(\sigma)) \leq p-1$, it follows that

$$\omega CM(\mathrm{lk}_X(h(\sigma)) \geq n - (p-1) - 1 = n - p,$$

and in particular $\mathrm{lk}_X(h(\sigma))$ is $(n-p-1)$-connected, so $h|_{\mathrm{lk}_{I^n}(\sigma)}$ extends to a PL map

$$I^{n-p} \cong C(\mathrm{lk}_{I^n}(\sigma)) \xrightarrow{\hat{h}} \mathrm{lk}_X(h(\sigma))$$

where $C(\mathrm{lk}_{I^n}(\sigma))$ denotes he cone over $\mathrm{lk}_{I^n}(\sigma)$. By induction on $n$, we may assume that $\hat{h}$ is simplicial with respect to a PL triangulation of $C(\mathrm{lk}_{I^n}(\sigma))$ which extends the triangulation of $\mathrm{lk}_{I^n}(\sigma)$, and such that all bad simplices of $\hat{h}$ are in $\partial I^{n-p} =$

$\mathrm{lk}(\sigma)$. We may extend this map by forming the join with $h|_{\partial\sigma}$ to get a map

$$\sigma \star \mathrm{lk}_{I^n}(\sigma) \cong (\partial\sigma) \star (C(\mathrm{lk}_{I^n}(\sigma)) \xrightarrow{\hat{h}} X,$$

which we may finally extend to $I^n$ by setting it equal to $h$ outside of $\sigma \star \mathrm{lk}_{I^n}(\sigma) \subset I^n$. The new map $\hat{h}$ has fewer bad simplices of dimension $p$. By continuing this process we eliminate all interior bad simplices in finitely many steps. This completes the proof of the theorem. □

Next we prove a result (Corollary 2.1.4) which will be employed several times in Section VIII. This result, along with the property defined in Definition 2.1.3, abstracts and isolates the key technique used in the proof of (11, Lemma 5.4).

**Definition 2.1.3.** Let $f : X \longrightarrow Y$ be a simplicial map. The map $f$ is said to have the *link lifting property* if for any vertex $y \in Y$, the following condition holds: given any subcomplex $K \leq X$ with $f(K) \leq \mathrm{lk}_Y(y)$, there exists a vertex $x \in X$ with $f(x) = y$ such that $K \leq \mathrm{lk}_X(x)$.

**Corollary 2.1.4.** *Let $X$ and $Y$ be simplicial complexes and let $f : X \longrightarrow Y$ be a simplicial map. Suppose that the following conditions are met:*

*i. $f$ has the link lifting property,*

*ii. $lCM(Y) \geq n$.*

*Then the induced map $|f|_* : \pi_j(|X|) \longrightarrow \pi_j(|Y|)$ is injective for all $j \leq n-1$. Furthermore, suppose that in addition to properties i. and ii., $f$ satisfies*

*iii. $f(\mathrm{lk}_X(\zeta)) \leq \mathrm{lk}_Y(f(\zeta))$ for all simplices $\zeta < X$.*

*Then it follows that $lCM(X) \geq n$.*

*Proof.* For $l+1 \leq n$, let $h : \partial I^{l+1} \longrightarrow |X|$ be a map which is simplicial with respect to some PL triangulation of $\partial I^{l+1}$, and let $H : I^{l+1} \longrightarrow |Y|$ be a null-homotopy of the composition $|f| \circ h$, i.e. $H|_{\partial I^{l+1}} = |f| \circ h$. To prove that $|f|_* : \pi_l(|X|) \longrightarrow \pi_l(|Y|)$ is injective for all $l \leq n-1$, it will suffice to construct a lift $\widehat{H}$ of $H$ that makes the diagram

$$\begin{array}{ccc} \partial I^{l+1} & \xrightarrow{h} & |X| \\ \downarrow & \overset{\widehat{H}}{\nearrow} & \downarrow{\scriptstyle |f|} \\ I^{l+1} & \xrightarrow{H} & |Y| \end{array}$$

commute. Since $lCM(Y) \geq n$, by Theorem 2.1.3 there exists a PL triangulation of $I^{l+1}$ that extends the chosen PL triangulation on $\partial I^{l+1}$. Furthermore, we may arrange that the map $H$ satisfy $H(\mathrm{lk}_{I^l}(x)) \leq \mathrm{lk}_Y(H(x))$ for any interior vertex $x \in \mathrm{Int}(I^{l+1})$ (without altering the original definition of $H$ on the boundary $\partial I^{l+1}$). We construct the lift $\widehat{H}$ by inductively choosing lifts of each vertex in $\mathrm{Int}(I^{l+1})$ as follows.

Suppose that $\widehat{H}$ has already been defined on a full subcomplex $K \leq I^{l+1}$ (we may assume that $\partial I^{l+1} \leq K$). Let $v \in I^{l+1}$ be a vertex in the compliment of $K$. Let $\langle K, v \rangle$ denote the full subcomplex of $I^{l+1}$ generated by the vertices of $K$ and $v$. We will use the link lifting property of $f$ to extend the domain of $\widehat{H}$ to $\langle K, v \rangle$. Consider the subcomplex $K' := K \cap \mathrm{lk}_{I^l}(v)$. We have $H(K') \leq \mathrm{lk}_Y(H(v))$ (recall that by applying Theorem 2.1.3, we arranged for $H$ to have this property in the above paragraph). By the link lifting property of $f$, we may then choose a vertex $\hat{v} \in Y$ with $f(\hat{v}) = H(v)$, such that $\widehat{H}(K') \leq \mathrm{lk}_X(\hat{v})$. We then define $\widehat{H}(v) = \hat{v}$. The fact that $\widehat{H}(K') \leq \mathrm{lk}_X(\hat{v})$, implies that the definition $\widehat{H}(v) = \hat{v}$ determines a well defined simplicial map from $\langle K, v \rangle$, that extends the definition of $\widehat{H}$ on $K$. By repeating this process, we can extend the lift $\widehat{H}$ over all of $I^{l+1}$ inductively. This

establishes the existence of the lift $\widehat{H}$. It follows that $|f|_* : \pi_l(|X|) \longrightarrow \pi_l(|Y|)$ is injective for all $l < n$.

Assume now that in addition to properties i. and ii. we have $f(\mathrm{lk}_X(\sigma)) \leq \mathrm{lk}_Y(f(\sigma))$ for all simplices $\sigma < X$. We will show that $lCM(X) \geq n$. Let $\zeta \leq X$ be a $p$-simplex. Since $f$ has the link lifting property, it follows that the map

$$f|_{\mathrm{lk}_X(\zeta)} : \mathrm{lk}_X(\zeta) \longrightarrow \mathrm{lk}_Y(f(\zeta)) \tag{2.1}$$

obtained by restricting $f$ has the link lifting property as well. Since $lCM(Y) \geq n$, it follows from (9, Lemma 2.2) that $lCM[\mathrm{lk}_Y(f(\zeta))] \geq n - p - 1$. It follows from the result proven in the previous paragraph that the map induced by (2.1) on $\pi_j(\_)$ is injective for $j \leq n - p - 2$. Since $\mathrm{lk}_Y(f(\zeta))$ is $(n - p - 2)$-connected, it follows that $\mathrm{lk}_X(\zeta)$ is $(n - p - 2)$-connected as well. This proves that $lCM(X) \geq n$ and completes the proof of the result. □

**Remark 2.1.1.** The main technical challenge in this paper will be to prove that a certain simplicial map (see (8.2) and Section 8.1) has the link lifting property. This is established in the proof of Lemma 8.1.2 but it uses the geometric techniques regarding $\mathbb{Z}/k$-manifolds developed throughout Sections V, VI, VII and in the appendix.

We will also need the useful following proposition, which was proven in (11, Section 2.1). We will use it in the proof of Theorem 4.3.2.

**Proposition 2.1.5.** *Let $X$ be a simplicial complex and let $Y \subset X$ be a full subcomplex. Let $n$ be an integer with the property that for each $p$-simplex $\sigma < X$, the complex $Y \cap \mathrm{lk}_X(\sigma)$ is $(n - p - 1)$-connected. Then the inclusion $|Y| \hookrightarrow |X|$ is $n$-connected.*

## 2.2. Topological Flag Complexes

We will need to work with a certain class of semi-simplicial spaces called *topological flag complexes* (see (**?** , Definition 6.1)).

**Definition 2.2.1.** Let $X_\bullet$ be a semi-simplicial space. We say that $X_\bullet$ is a *topological flag complex* if for each integer $p \geq 0$,

i. the map $X_p \longrightarrow (X_0)^{\times(p+1)}$ to the $(p+1)$-fold product (which takes a $p$-simplex to its $(p+1)$ vertices) is a homeomorphism onto its image, which is an open subset,

ii. a tuple $(v_0, \ldots, v_p) \in (X_0)^{\times(p+1)}$ lies in the image of $X_p$ if and only if $(v_i, v_j) \in X_1$ for all $i < j$.

If $X_\bullet$ is a topological flag complex, we may denote any $p$-simplex $x \in X_p$ by a $(p+1)$-tuple $(x_0, \ldots, x_p)$ of zero-simplices.

**Definition 2.2.2.** Let $X_\bullet$ be a topological flag complex and let $x = (x_0, \ldots, x_p) \in X_p$ be a $p$-simplex. The *link* of $x$, denoted by $X_\bullet(x) \subset X_\bullet$, is defined to be the sub-semi-simplicial space whose $l$-simplices are given by the space of all ordered lists $(y_0, \ldots, y_l) \in X_l$ such that the list $(x_0, \ldots, x_p, y_0, \ldots, y_\ell) \in (X_0)^{\times(p+\ell+2)}$, is a $(p+\ell+1)$-simplex.

It is easily verified that the link $X_\bullet(x)$ is a topological flag complex as well. The topological flag complex $X_\bullet$ is said to be *weakly Cohen-Macaulay* of dimension $n$ if its geometric realization is $(n-1)$-connected and if for any $p$-simplex $x \in X_p$, the geometric realization of the link $|X_\bullet(x)|$ is $(n-p-2)$-connected. In this case we write $\omega CM(X_\bullet) \geq n$.

The main result from this section is a result about the discretization of a topological flag complex.

**Definition 2.2.3.** Let $X_\bullet$ be a semi-simplicial space. Let $X^\delta_\bullet$ be the semi-simplicial set defined by setting $X^\delta_p$ equal to the discrete topological space with underlying set equal to $X_p$, for each integer $p \geq 0$. We will call the semi-simplicial set $X^\delta_\bullet$ the *discretization* of $X_\bullet$.

The following theorem is proven by repackaging several results from (11). In particular, the proof is basically the same as the proof of (11, Theorem 5.5). We provide a sketch of the proof here and we provide references to the key technical lemmas employed from (11).

**Theorem 2.2.1.** *Let $X_\bullet$ be a topological flag complex and suppose that $\omega CM(X^\delta_\bullet) \geq n$. Then the geometric realization $|X_\bullet|$ is $(n-1)$-connected.*

*Proof Sketch.* For integers $p, q \geq 0$, let $Y_{p,q} = X_{p+q+1}$ be toplogized as a subspace of the product $(X_0)^{\times p} \times (X^\delta_0)^{\times q}$. The assignment $[p,q] \mapsto Y_{p,q}$ defines a bi-semi-simplicial space with augmentations

$$\varepsilon : Y_{\bullet,\bullet} \longrightarrow X_\bullet, \quad \delta : Y_{\bullet,\bullet} \longrightarrow X^\delta_\bullet.$$

This doubly augmented bi-semi-simplicial space is analogous to the one considered in (11, Definition 5.6). Let $\iota : X^\delta_\bullet \longrightarrow X_\bullet$ be the map induced by the identity. By (11, Lemma 5.7), there exists a homotopy of maps,

$$|\iota| \circ |\delta| \simeq |\epsilon| : |Y_{\bullet,\bullet}| \longrightarrow |X_\bullet|. \tag{2.2}$$

For each integer $p$, consider the map

$$|Y_{p,\bullet}| \longrightarrow X_p \tag{2.3}$$

induced by $\epsilon$. By how $Y_{\bullet,\bullet}$ was constructed, it follows from (11, Proposition 2.8) that for each $p$, (2.3) is a *Serre-microfibration*. For any $x \in X_p$, the fibre over $x$ is equal to the space $|X^{\delta}_{\bullet}(x)|$, where $X^{\delta}_{\bullet}(x)$ is the link of the $p$-simplex $x$, as defined in Definition 2.2.2. Since $\omega CM(X^{\delta}_{\bullet}) \geq n$, this implies that the fibre of (2.3) over any $x \in X_p$ is $(n-p-2)$-connected. Using the fact that this map is a Serre-microfibration, (11, Proposition 2.6) then implies that (2.3) is $(n-p-1)$-connected. It then follows by (11, Proposition 2.7) that the map

$$|\epsilon| : |Y_{\bullet,\bullet}| \longrightarrow |X_{\bullet}| \tag{2.4}$$

is $(n-1)$-connected. The homotopy from (2.2) implies that the map $|\iota| : |X^{\delta}_{\bullet}| \longrightarrow |X_{\bullet}|$ induces a surjection on homotopy groups $\pi_j(\_)$ for all $j \leq n-1$. The proof of the theorem then follows from the fact that $|X^{\delta}_{\bullet}|$ is $(n-1)$-connected by hypothesis.

□

## 2.3. Transitive Group Actions

In order to prove our homological stability theorem, we will need to consider groups acting on simplicial spaces and simplicial complexes. We will need a technique for determining when such actions are transitive. For the lemma that follows, let $X_{\bullet}$ be a topological flag complex, let $G$ be a topological group, and let

$$G \times X_{\bullet} \longrightarrow X_{\bullet}, \quad (g, \sigma) \mapsto g \cdot \sigma$$

be a continuous group action.

**Lemma 2.3.1.** *Let $G$ and $X_\bullet$ be as above and suppose that the following conditions hold:*

- *for any 1-simplex $(v, w) \in X_1$, there exists $g \in G$ such that $g \cdot v = w$,*
- *for any two vertices $x, y$ that lie on the same path-component of $X_0$, there exists $g \in G$ such that $g \cdot x = y$,*
- *the geometric realization $|X_\bullet|$ is path-connected.*

*Then for any two vertices $x, y \in X_0$, there exists $g \in G$ such that $g \cdot x = y$.*

*Proof.* We define an equivalence relation on the elements of $X_0$ by setting $x \sim y$ if there exists $g \in G$ such that $g \cdot x = y$. Since $G$ is a group (and thus every element has a multiplicative inverse), it follows that this relation is indeed an equivalence relation, i.e. it is transitive, reflexive, and symmetric. By transitivity of the relation, it follows from the in the statement of the lemma that $x \sim y$ if there exists some zig-zag of edges connecting $x$ and $y$. It also follows that $x \sim y$ if $x$ and $y$ lie on the same path component of $X_0$.

Let $v, w \in X_0$ be any two zero simplices. We will prove that there exists $g \in G$ such that $g \cdot v = w$. Since the geometric realization $|X_\bullet|$ is path-connected, it follows that there exists a vertex $v'$ in the path component containing $v$ and a vertex $w'$ in the path component containing $w$, such that $v'$ and $w'$ are connected by a zig-zag of edges. We have $v \sim v' \sim w' \sim w$, and thus $v \sim w$. This concludes the proof of the lemma. □

There is a similar result for simplicial complexes which is proven in essentially the same way as the previous lemma. We state the proposition without proof.

**Proposition 2.3.2.** *Let $K$ be a simplicial complex and let $G \times K \longrightarrow K$ be a group action. Suppose $|K|$ is path-connected and that for any edge $\{v, w\} \leq K$, there exists $g \in G$ such that $g \cdot x = y$. Then for any two vertices $x, y \in K$, there exists $g \in G$ such that $g \cdot x = y$.*

# CHAPTER III

# LINKING FORMS AND ODD-DIMENSIONAL MANIFOLDS

## 3.1. Linking Forms

The basic algebraic structure that we will encounter is that of a bilinear form on a finite abelian group. For $\epsilon = \pm 1$, a pair $(\mathbf{M}, b)$ is said to be a ($\epsilon$-symmetric) *linking form* if $\mathbf{M}$ is a finite abelian group and $b : \mathbf{M} \otimes \mathbf{M} \longrightarrow \mathbb{Q}/\mathbb{Z}$ is an $\epsilon$-symmetric bilinear map. A morphism between linking forms is defined to be a group homomorphism $f : \mathbf{M} \longrightarrow \mathbf{N}$ such that $b_{\mathbf{M}}(x, y) = b_{\mathbf{N}}(f(x), f(y))$ for all $x, y \in \mathbf{M}$. We denote by $\mathcal{L}_\epsilon$ the category of all $\epsilon$-symmetric linking forms. By forming direct sums, $\mathcal{L}_\epsilon$ obtains the structure of an *additive category.*

**Notational Convention 3.1.1.** We will usually denote linking forms by their underlying abelian group. We will always denote the bilinear map by $b$. If more than one linking form is present, we will decorate $b$ with a subscript so as to eliminate ambiguity.

For $\mathbf{M}$ a linking form and $\mathbf{N} \leq \mathbf{M}$ a subgroup, $\mathbf{N}$ automatically inherits the structure of a sub-linking form of $\mathbf{M}$ by restricting $b_{\mathbf{M}}$ to $\mathbf{N}$. We will denote by $\mathbf{N}^\perp \leq \mathbf{M}$ the *orthogonal compliment* to $\mathbf{N}$ in $\mathbf{M}$. Two sub-linking forms $\mathbf{N}_1, \mathbf{N}_2 \leq \mathbf{M}$ are said to be *orthogonal* if $\mathbf{N}_1 \leq \mathbf{N}_2^\perp$, $\mathbf{N}_2 \leq \mathbf{N}_1^\perp$, and $\mathbf{N}_1 \cap \mathbf{N}_2 = 0$. If $\mathbf{N}_1, \mathbf{N}_2 \leq \mathbf{M}$ are orthogonal sub-linking forms, we let $\mathbf{N}_1 \perp \mathbf{N}_2 \leq \mathbf{M}$ denote the sub-linking form given by the sum $\mathbf{N}_1 + \mathbf{N}_2$. If $\mathbf{M}_1$ and $\mathbf{M}_2$ are two linking forms, the (external) direct sum $\mathbf{M}_1 \oplus \mathbf{M}_2$ obtains the structure of a linking form in a

natural way by setting

$$b_{\mathbf{M}_1\oplus\mathbf{M}_2}(x_1+x_2,\ y_1+y_2) = b_{\mathbf{M}_1}(x_1,y_1)+b_{\mathbf{M}_1}(x_2,y_2) \quad \text{for } x_1,y_1\in\mathbf{M}_1,\quad x_2,y_2\in\mathbf{M}_2. \tag{3.1}$$

We will always assume that the direct sum $\mathbf{M}_1\oplus\mathbf{M}_2$ is equipped with the linking form structure given by (3.1). An element $\mathbf{M}\in\mathcal{O}b(\mathcal{L}_\epsilon)$ is said to be *non-singular* if the duality homomorphism

$$T:\mathbf{M}\longrightarrow \mathrm{Hom}_{\mathrm{Ab}}(\mathbf{M},\mathbb{Q}/\mathbb{Z}),\quad x\mapsto b(x,\_) \tag{3.2}$$

is an isomorphism of abelian groups.

We will mainly need to consider the category $\mathcal{L}_\epsilon$ in the case where $\epsilon=-1$. We denote by $\mathcal{L}^s_{-1}$ the full subcategory of $\mathcal{L}_{-1}$ consisting of linking forms that are *strictly skew symmetric*, or in other words $\mathcal{L}^s_{-1}$ is the category of all linking forms $\mathbf{M}$ for which $b_{\mathbf{M}}(x,x)=0$ for all $x\in\mathbf{M}$ (even in the case when $x$ is an element of order 2).

We proceed to define certain basic, non-singular elements of $\mathcal{L}^s_{-1}$ as follows.

**Definition 3.1.1.** For a positive integer $k\geq 2$, let $\mathbf{W}_k$ denote the abelian group $\mathbb{Z}/k\oplus\mathbb{Z}/k$. Let $\rho$ and $\sigma$ denote the standard generators $(1,0)$ and $(0,1)$ respectively. We then let $b:\mathbf{W}_k\longrightarrow\mathbb{Q}/\mathbb{Z}$ be the $-1$-symmetric bilinear form determined by the values

$$b(\rho,\sigma)=-b(\sigma,\rho)=\tfrac{1}{k},\qquad b(\rho,\rho)=b(\sigma,\sigma)=0. \tag{3.3}$$

With $b$ defined in this way, it follows that $\mathbf{W}_k$ is a non-singular object of $\mathcal{L}^s_{-1}$. It follows easily that if $k$ and $\ell$ are relatively prime, then $\mathbf{W}_k\oplus\mathbf{W}_\ell$ and $\mathbf{W}_{k\cdot\ell}$ are

isomorphic as objects of $\mathcal{L}^s_{-1}$. For $g \geq 2$ an integer, we will let $\mathbf{W}_k^g$ denote the $g$-fold direct sum $(\mathbf{W}_k)^{\oplus g}$.

For $k \in \mathbb{N}$, let $C_k$ denote the cyclic subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the element $1/k \mod 1$. Any group homomorphism $h : \mathbf{W}_k \longrightarrow \mathbb{Q}/\mathbb{Z}$ must factor through the inclusion $C_k \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Hence, it follows that the duality map from (3.2) induces an isomorphism of abelian groups,

$$\mathbf{W}_k \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Ab}}(\mathbf{W}_k, C_k). \tag{3.4}$$

**Lemma 3.1.1.** *Let $k \geq 2$ be a positive integer and let $\mathbf{M} \in \mathcal{O}b(\mathcal{L}^s_{-1})$. Then any morphism*

$$f : \mathbf{W}_k \longrightarrow \mathbf{M}$$

*is split injective and there is an orthogonal direct sum decomposition, $f(\mathbf{W}_k) \perp f(\mathbf{W}_k)^{\perp} = \mathbf{M}$.*

*Proof.* Let $x$ and $y$ denote the elements of $\mathbf{M}$ given by $f(\rho)$ and $f(\sigma)$ respectively where $\rho$ and $\sigma$ are the standard generators of $\mathbf{W}_k$. Let $T : \mathbf{M} \longrightarrow \mathrm{Hom}(\mathbf{M}, \mathbb{Q}/\mathbb{Z})$ denote the duality map from (3.2). Since both $x$ and $y$ have order $k$, it follows that the homomorphisms

$$b(x, \_),\ b(y, \_) : \mathbf{M} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

factor through the inclusion $C_k \hookrightarrow \mathbb{Q}/\mathbb{Z}$. Define a group homomorphism (which is not a morphism of linking forms) by the formula

$$\varphi : \mathbf{M} \longrightarrow \mathbf{W}_k, \quad \varphi(z) = b(x, z) \cdot \rho + b(y, z) \cdot \sigma.$$

It is clear that the kernel of $\varphi$ is the orthogonal compliment $f(\mathbf{W}_k)^{\perp}$ and that the morphism $f : \mathbf{W}_k \longrightarrow \mathbf{M}$ gives a section of $\varphi$. This completes the proof. $\square$

The following theorem is a specialization of the classification theorem of Wall from (38, Lemma 7). The classification of objects of $\mathcal{L}^s_{-1}$ is analogous to the classification of finite abelian groups.

**Theorem 3.1.2.** *Let $\mathbf{M} \in \mathcal{O}b(\mathcal{L}^s_{-1})$ be non-singular. Then there is an isomorphism,*

$$\mathbf{M} \;\cong\; \mathbf{W}_{p_1^{n_1}}^{\ell_1} \oplus \cdots \oplus \mathbf{W}_{p_r^{n_r}}^{\ell_r}$$

*where $p_j$ is a prime number and $\ell_j$ and $n_j$ are positive integers for $j = 1, \ldots, r$. Furthermore, the above direct sum decomposition is unique up to isomorphism.*

## 3.2. The Homological Linking Form

For what follows, let $M$ be a manifold of dimension $2s+1$. Let $H_s^{\tau}(M;\mathbb{Z}) \leq H_s(M;\mathbb{Z})$ denote the torsion subgroup of $H_s(M;\mathbb{Z})$. Following (37), the *homological linking form* $\tilde{b} : H_s^{\tau}(M;\mathbb{Z}) \otimes H_s^{\tau}(M;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is defined as follows. Let $x, y \in \tau H_s(M;\mathbb{Z})$ and suppose that $x$ has order $r > 1$. Represent $x$ by a chain $\xi$ and let $\partial\zeta = r \cdot \xi$. Then if $y$ is represented by the chain $\chi$, we define

$$\tilde{b}(x,y) = \tfrac{1}{r}[\zeta \cap \chi] \mod 1, \tag{3.5}$$

where $\zeta \cap \chi$ denotes the algebraic intersection number associated to the two chains (after being deformed so as to meet transversally). It is proven in (37, Page 274) that $\tilde{b}$ is $(-1)^{s+1}$-symmetric. We refer the reader to (37) for further details on this construction.

Let $\pi_s^\tau(M) \leq \pi_s(M)$ denote the torsion component of the homotopy group $\pi_s(M)$. Using the homological linking form and the Hurewicz homomorphism $h : \pi_s(M) \longrightarrow H_s(M)$, we can define a similar bilinear pairing

$$b : \pi_s^\tau(M) \otimes \pi_s^\tau(M) \longrightarrow \mathbb{Q}/\mathbb{Z}; \quad b(x,y) = \tilde{b}(h(x),\ h(y)). \tag{3.6}$$

The pair $(\pi_s^\tau(M), b)$ is a $(-1)^{s+1}$-symmetric linking form in the sense of Section 3.1 and we will refer to it as the *homotopical linking form* associated to $M$. In the case that $M$ is $(s-1)$-connected, the homotopical linking form is isomorphic to the homological linking form by the *Hurewicz theorem*.

## 3.3. The Classification Theorem

We are mainly interested in manifolds which are $(4n+1)$-dimensional with $n \geq 2$. In this case the homological (and homotopical) linking form is anti-symmetric. It follows from this that $b(x,x) = 0$ whenever $x$ is of odd order. The following lemma of Wall from (37) implies that for $(4n+1)$-dimensional manifolds for $n \geq 2$, the linking form is *strictly* skew symmetric.

**Lemma 3.3.1.** *For $n \geq 2$, let $M$ be a $(2n-1)$-connected, $(4n+1)$-dimensional manifold. Then $b(x,x) = 0$ for all $x \in \pi_{2n}^\tau(M)$.*

It follows from Lemma 3.3.1 that if $M$ is a $(2n-1)$-connected, $(4n+1)$-dimensional manifold (i.e. $M \in \mathcal{W}_{4n+1}$), then the homotopical linking form $(\pi_{2n}^\tau(M), b)$ is an object of the category $\mathcal{L}_-^s$. If $M$ is closed (or has boundary a homotopy sphere), then $(\pi_{2n}^\tau(M), b)$ is non-singular. The following theorem is a specialization of Wall's classification theorem (37, Theorem 7).

**Theorem 3.3.2.** *For $n \geq 2$, two manifolds $M_1, M_2 \in \bar{\mathcal{W}}^S_{4n+1}$ are almost diffeomorphic if and only if:*

*i. There is an isomorphism of $\mathbb{Q}$-vector spaces, $\pi_{2n}(M_1) \otimes \mathbb{Q} \cong \pi_{2n}(M_2) \otimes \mathbb{Q}$.*

*ii. There is an isomorphism of linking forms, $(\pi^\tau_{2n}(M_1),\ b) \cong (\pi^\tau_{2n}(M_2),\ b)$.*

*Furthermore, given any $\mathbb{Q}$-vector space $V$ and non-singular linking form $\mathbf{M} \in \mathcal{L}^s_-$, there exists an element $M \in \bar{\mathcal{W}}^S_{4n+1}$ such that, $\pi_{2n}(M) \otimes \mathbb{Q} \cong V$ and $(\pi^\tau_{2n}(M),\ b) \cong (\mathbf{M},\ b_{\mathbf{M}})$.*

Using the above classification theorem and the classification of skew symmetric linking forms from Theorem 3.1.2, we may specify certain basic manifolds.

**Definition 3.3.1.** For each integer $k \geq 2$, fix a manifold $W_k \in \bar{\mathcal{W}}^S_{4n+1}$ which satisfies:

(a) the homotopical linking form associated to $W_k$ is isomorphic to $\mathbf{W}_k$,

(b) $\pi_{2n}(W_k) \otimes \mathbb{Q} = 0$.

It follows from Theorem 3.3.2 that every element of $\bar{\mathcal{W}}^S_{4n+1}$ is almost diffeomorphic (i.e. diffeomorphic up to connect-sum with a homotopy sphere) to the connected sum of copies of $W_k$ and copies of $S^{2n} \times S^{2n+1}$. The manifolds $W_k$ are the subject of our main result, Theorem 1.3.2.

**Remark 3.3.1.** The closed, stably parallelizable manifolds $W_k \in \bar{\mathcal{W}}^S_{4n+1}$ are uniquely determined by conditions (a) and (b) up to *almost diffeomorphism*. For each $k$, let $W'_k$ denote the manifold obtained from $W_k$ by removing an open disk. It

follows from (37, Theorem 7) that $W_k'$ is determined by conditions (a) and (b) up to diffeomorphism.

# CHAPTER IV

# THE PRIMARY COMPLEXES

## 4.1. The Complex of Connected-Sum Decompositions

Fix integers $k, n \geq 2$. Let $W_k$ denote the closed $(4n+1)$-dimensional manifold $W_k$ defined in Section 3.3. We will make a slight alteration of $W_k$ as follows. Let $W_k'$ denote the manifold obtained from $W_k$ by removing an open disk. Choose an oriented embedding

$$\alpha : \{1\} \times D^{4n} \longrightarrow \partial W_k'.$$

We then define $\bar{W}_k$ to be the manifold obtained by attaching $[0,1] \times D^{4n}$ to $W_k'$ by the embedding $\alpha$, i.e.

$$\bar{W}_k := ([0,1] \times D^{4n}) \cup_\alpha W_k'. \tag{4.1}$$

Let $M$ be a $(4n+1)$-dimensional manifold with non-empty boundary. Fix an embedding

$$a : [0,\infty) \times \mathbb{R}^{4n} \longrightarrow M$$

with $a^{-1}(\partial M) = \{0\} \times \mathbb{R}^{4n}$.

**Definition 4.1.1.** Let $M$ and $a : [0,\infty) \times \mathbb{R}^{4n} \longrightarrow M$ be as above and let $k \geq 2$ be an integer. We define a semi-simplicial space $X_\bullet(M,a)_k$ as follows:

(i) Let $X_0(M,a)_k$ be the set of pairs $(\phi, t)$, where $t \in \mathbb{R}$ and $\phi : \bar{W}_k \to M$ is an embedding that satisfies the following condition: there exists $\epsilon > 0$ such that for $(s,z) \in [0,\epsilon) \times D^{4n} \subset \bar{W}_k$, the equality $\phi(s,z) = a(s, z + te_1)$ is satisfied ($e_1 \in \mathbb{R}^{4n}$ denotes the first basis vector).

(ii) For an integer $p \geq 0$, $X_p(M,a)_k$ is defined to be the set of ordered $(p+1)$-tuples

$$((\phi_0, t_0), \ldots, (\phi_p, t_p)) \in (X_0(M,a)_k)^{\times(p+1)}$$

such that $t_0 < \cdots < t_p$ and $\phi_i(\bar{W}_k) \cap \phi_j(\bar{W}_k) = \emptyset$ whenever $i \neq j$.

iii. For each $p$, the space $X_p(M,a)_k$ is topologized in the $C^\infty$-topology as a subspace of the product $(\mathrm{Emb}(\bar{W}_k, M) \times \mathbb{R})^{\times(p+1)}$.

iv. The assignment $[p] \mapsto X_p(M,a)_k$ makes $X_\bullet(M,a)_k$ into a semi-simplicial space where the $i$-th face map $X_p(M,a)_k \to X_{p-1}(M,a)_k$ is given by

$$((\phi_0, t_0, \ldots, (\phi_p, t_p)) \mapsto ((\phi_0, t_0, \ldots, \widehat{(\phi_i, t_i)}), \ldots, (\phi_p, t_p)).$$

It is easy to verify that $X_\bullet(M,a)_k$ is a topological flag complex. For any 0-simplex $(\phi, t) \in X_0(M,a)_k$, it follows from condition i. that the number $t$ is determined by the embedding $\phi$. For this reason we will usually drop the number $t$ when denoting elements of $X_0(M,a)_k$.

We now state a consequence of connectivity of the geometric realization $|X_\bullet(M,a)_k|$, using Lemma 2.3.1. This is essentially the same as (9, Proposition 4.4) and so we only give a sketch of the proof.

**Proposition 4.1.1** (Transitivity)**.** *For $n \geq 2$, let $M$ be a $(4n+1)$-dimensional manifold with non-empty boundary. Let $k \geq 2$ be an integer, and let $\phi_0$ and $\phi_1$ be elements of $X_0(M,a)_k$. Suppose that the geometric realization $|X_\bullet(M,a)_k|$ is path connected. Then there exists a diffeomorphism $\psi : M \xrightarrow{\cong} M$, isotopic to the identity when restricted to the boundary, such that $\psi \circ \phi_0 = \phi_1$.*

*Proof Sketch.* Let $a : [0, \infty) \times \mathbb{R}^{4n} \longrightarrow M$ be the embedding used in the definition of $X_\bullet(M, a)$. Let $\mathrm{Diff}(M, a)$ denote the group of diffeomorphisms $\psi : M \longrightarrow M$ with

$$\psi(a([0, \infty) \times \mathbb{R}^{4n})) \subset a([0, \infty) \times \mathbb{R}^{4n})$$

and such that $\psi|_{\partial M}$ is isotopic to the identity. This group acts on $X_\bullet(M, a)_k$, and by Lemma 2.3.1 it will suffice to show that for $(\phi_0, \phi_1) \in X_1(M, a)_k$, there exists $\psi \in \mathrm{Diff}(M, a)$ such that $\psi \circ \phi_0 = \phi_1$. Let $U \subset M$ be a collar neighborhood of the boundary of $M$, that contains $a([0, \infty) \times \mathbb{R}^{4n})$. The union $\phi_0(\bar{W}_k) \cup \phi_1(\bar{W}_k) \cup U$ is diffeomorphic to manifold

$$W_k \# (\partial M \times [0, 1]) \# W_k. \tag{4.2}$$

To find the desired diffeomorphism $\psi$, it will suffice to construct a diffeomorphism of (4.2), that is isotopic to the identity on the first boundary component, is equal to the identity on the second boundary component, and that permutes the two embedded copies of $W_k'$, that come from the two connected-sum factors. Such a diffeomorphism can be constructed "by hand" using the same procedure that was employed in the proof of (9, Proposition 4.4). We leave the details of this construction to the reader. □

The next proposition is proven in the same way as (9, Corollary 4.5), using Proposition 4.2.1.

**Proposition 4.1.2** (Cancelation)**.** *Let $M$ and $N$ be $(4n+1)$-dimensional manifolds with non-empty boundaries, equipped with a specified identification, $\partial M = \partial N$. For $k \geq 2$, suppose that there exists a diffeomorphism $M \# W_k \xrightarrow{\cong} N \# W_k$, equal*

*to the identity when restricted to the boundary. Then if* $|X_\bullet(M\#W_k, a)_k|$ *is path-connected, there exists a diffeomorphism* $M \xrightarrow{\cong} N$ *which is equal to the identity when restricted to the boundary.*

The main theorem that we will need is the following. Recall from (1.9) the *k-rank* $r_k(M)$, of a $(4n+1)$-dimensional manifold $M$.

**Theorem 4.1.3.** *Let* $n, k \geq 2$ *be integers with* $k$ *odd. Let* $M$ *be a* 2*-connected,* $(4n+1)$*-dimensional manifold with non-empty boundary. Let* $g \in \mathbb{N}$ *be an integer such that* $r_k(M) \geq g$*. Then the geometric realization* $|X_\bullet(M, a)_k|$ *is* $\frac{1}{2}(g-4)$*-connected.*

The majority of the technical work of thesis is devoted to proving the above theorem. The proof will finally be given in Section VIII using techniques developed throughout the rest of the paper. Once this theorem is proven, the proof of the main theorem of this thesis, Theorem 1.3.2, follows by a spectral sequence argument given in Section IX.

## 4.2. The Complex of Linking Forms

The topological flag complex $X_\bullet(M, a)_k$ introduced in the previous section is very difficult to study directly. Indeed, for each integer $p \geq 0$, the space $X_p(M, a)_k$ is a space of codimension-zero embeddings, and relatively little is known about the homotopy type of such spaces of embeddings in general. In order to prove Theorem 4.1.3, we will need to compare $X_\bullet(M, a)_k$ to as simplicial complex, whose definition is more algebraic than $X_\bullet(M, a)_k$, and thus can be analyzed by discrete and combinatorial methods. Below we define a simplicial complex, analogous to the one from (11, Definition 3.1), that is based on the linking forms introduced in Section 3.1.

**Definition 4.2.1.** Let $\mathbf{M} \in \mathcal{O}b(\mathcal{L}_{-1})$ be a linking form and let $k \geq 2$ be a positive integer. We define $L(\mathbf{M})_k$ to be the simplicial complex whose vertices are given by morphisms $f : \mathbf{W}_k \longrightarrow \mathbf{M}$ of linking forms. The set $\{f_0, \ldots, f_p\}$ is a $p$-simplex if the sub-linking forms $f_i(\mathbf{W}_k) \leq \mathbf{M}$ are pairwise orthogonal.

Suppose that $\sigma = \{f_0, \ldots, f_p\}$ is a $p$-simplex in $L(\mathbf{M})_k$. Let $\mathbf{M}' \leq \mathbf{M}$ denote the sub-linking-form given by the orthogonal compliment $[\sum f_i(\mathbf{W}_k)]^{\perp}$. It follows from the definition of the link of a simplex that there is an isomorphism of simplicial complexes,

$$\mathrm{lk}_{L(\mathbf{M})_k}(\sigma) \cong L(\mathbf{M}')_k. \tag{4.3}$$

Below are two formal consequences of path connectivity of $L(\mathbf{M})_k$. They are proven in the exact same way as (11, Propositions 3.3 and 3.4).

**Proposition 4.2.1** (Transitivity)**.** *If $|L(\mathbf{M})_k|$ is path-connected and $f_0, f_1 : \mathbf{W}_k \to \mathbf{M}$ are morphisms of linking forms, then there is an automorphism of linking forms $h : \mathbf{M} \to \mathbf{M}$ such that $f_1 = h \circ f_0$.*

*Proof.* The group of linking form automorphisms $\mathrm{Aut}(\mathbf{M})$ acts on $L(\mathbf{M})_k$ by post-composition with morphisms $f : \mathbf{W}_k \longrightarrow \mathbf{M}$. In order to prove the proposition, by Proposition 2.3.2 it will suffice to show that given two morphisms $f_0, f_1 : \mathbf{W}_k \to \mathbf{M}$ such that $f_0(\mathbf{W}_k)$ and $f_1(\mathbf{W}_k)$ are orthogonal, there exists $\varphi \in \mathrm{Aut}(\mathbf{M})$ such that $\varphi \circ f_0 = f_1$. So, let $\mathbf{M}' \leq \mathbf{M}$ denote $[f_0(\mathbf{W}_k) \perp f_1(\mathbf{W}_k)]^{\perp}$. By Proposition 3.1.1 there is an orthogonal splitting $\mathbf{M} = f_0(\mathbf{W}_k) \perp \mathbf{M}'$. We then define $\varphi$ to be the morphism determined by the data:

$$\varphi(f_i(\sigma)) = f_{1-i}(\sigma), \quad \varphi(f_i(\rho)) = f_{i-1}(\rho), \quad \varphi(v) = v \quad \text{for all } v \in \mathbf{M}',$$

where $i = 0, 1$. Clearly $\varphi$ has the desired property. This concludes the proof. □

**Proposition 4.2.2** (Cancellation)**.** *Suppose that* $\mathbf{M}$ *and* $\mathbf{N}$ *are linking forms and there is an isomorphism* $\mathbf{M} \oplus \mathbf{W}_k \cong \mathbf{N} \oplus \mathbf{W}_k$*. If* $|L(\mathbf{M} \oplus \mathbf{W}_k)_k|$ *is path-connected, then there is also an isomorphism* $\mathbf{M} \cong \mathbf{N}$*.*

## 4.3. High-Connectivity of the Linking Complex

In this section we prove a connectivity result for the complex $L(\mathbf{M})_k$ introduced in the previous section. We will need to defined a notion of rank analogous to (1.9) for skew-symmetric linking forms.

**Definition 4.3.1.** Let $\mathbf{M}$ be a linking form and let $k \geq 2$ be a positive integer. We define the *k-rank* of $\mathbf{M}$ to be the quantity, $r_k(\mathbf{M}) = \max\{g \in \mathbb{N} \mid \text{there exists a morphism } \mathbf{W}_k^g \to \mathbf{M}\}$. We then define the *stable k-rank* of $\mathbf{M}$ to be the quantity, $\bar{r}_k(M) = \max\{r_k(\mathbf{M} \oplus \mathbf{W}_k^g) - g \mid g \in \mathbb{N}.\}$

**Corollary 4.3.1.** *Let* $f : \mathbf{W}_k^g \longrightarrow \mathbf{M}$ *be a morphism of linking forms. Then* $\bar{r}_k(f(\mathbf{W}_k)^\perp) \geq \bar{r}_k(\mathbf{M}) - g$*.*

*Proof.* This follows immediately from the orthogonal splitting $f(\mathbf{W}_k^g) \perp f(\mathbf{W}_k^g)^\perp = \mathbf{M}$ and the definition of the stable $k$-rank. □

The main result that we will prove about the above complex is the following theorem. The proof is very similar to the proof of (11, Theorem 3.2).

**Theorem 4.3.2.** *Let* $g, k \in \mathbb{N}$ *and let* $\mathbf{M} \in \mathcal{O}b(\mathcal{L}_-^s)$ *be a linking form with* $\bar{r}_k(\mathbf{M}) \geq g$*. Then the geometric realization* $|L(\mathbf{M})_k|$ *is* $\frac{1}{2}(g-4)$*-connected and* $lCM(L(\mathbf{M})_k) \geq \frac{1}{2}(g-1)$*.*

The proof of Theorem 4.3.2 follows the same inductive argument as the proof of (11, Theorem 3.2). We will need two key algebraic results (Proposition 4.3.3

and Corollary 4.3.4) given below which are analogous to (11, Proposition 4.1 and Corollary 4.2).

**Proposition 4.3.3.** *Let $k, g \in \mathbb{N}$ with $k \geq 2$. Let $\mathrm{Aut}(\mathbf{W}_k^{g+1})$ act on $\mathbf{W}_k^{g+1}$, and consider the orbits of elements of $\mathbf{W}_k \oplus 0 \leq \mathbf{W}_k^{g+1}$. We then have $\mathrm{Aut}(\mathbf{W}_k^{g+1}) \cdot (\mathbf{W}_k \oplus \mathbf{0}) = \mathbf{W}_k^{g+1}$.*

*Proof.* We will prove that for any $v \in \mathbf{W}_k^{g+1}$, there is an automorphism $\varphi : \mathbf{W}_k^{g+1} \longrightarrow \mathbf{W}_k^{g+1}$ such that $v \in \varphi(\mathbf{W}_k \oplus \mathbf{0})$. An element $v \in \mathbf{W}_k^{g+1}$ is said to be *primitive* if the subgroup $\langle v \rangle \leq \mathbf{W}_k^{g+1}$ generated by $v$, splits as a direct summand. Every element of $\mathbf{W}_k^{g+1}$ is the integer multiple (reduced $\mod k$) of a primitive element. Hence it will suffice to prove the statement in the case that $v$ is a primitive element.

So, let $v \in \mathbf{W}_k^{g+1}$ be a primitive element. Since the linking form $\mathbf{W}_k^{g+1}$ is non-singular and $v$ is primitive, it follows that there exists $w \in \mathbf{W}_k^{g+1}$ such that $b(w, v) = \frac{1}{k} \mod 1$. We may then define a morphism $f : \mathbf{W}_k \longrightarrow \mathbf{W}_k^{g+1}$ by setting $f(\sigma) = v$ and $f(\rho) = w$, where $\sigma$ and $\rho$ are the standard generators of $\mathbf{W}_k$. Consider the orthogonal splitting $f(\mathbf{W}_k) \perp f(\mathbf{W}_k)^{\perp} = \mathbf{W}_k^{g+1}$. Since both $\mathbf{W}_k^{g+1}$ and $f(\mathbf{W}_k)$ are non-singular, it follows that the orthogonal compliment $f(\mathbf{W}_k)^{\perp}$ is nonsingular as well. It then follows from the classification theorem (Theorem 3.1.2) that there exists an isomorphism $h : \mathbf{W}_k^g \xrightarrow{\cong} f(\mathbf{W}_k)^{\perp}$ (according to Theorem 3.1.2, there is only one such way, up to isomorphism, to write $\mathbf{W}_k^{g+1}$ as the direct sum of $\mathbf{W}_k$ with another non-singular linking form). The morphism given by the direct sum of maps

$$\varphi := f \oplus h : \mathbf{W}_k \oplus \mathbf{W}_k^{g+1} \longrightarrow f(\mathbf{W}) \perp f(\mathbf{W})^{\perp},$$

is an isomorphism such that $v \in \varphi(\mathbf{W}_k \oplus \mathbf{0})$. This concludes the proof of the proposition. □

**Corollary 4.3.4.** *Let* $\mathbf{M}$ *be a linking form with* $r_k(\mathbf{M}) \geq g$ *and let* $\varphi : \mathbf{M} \longrightarrow C_k$ *be a group homomorphism. Then* $r_k(\mathrm{Ker}(\varphi)) \geq g-1$. *Similarly if* $\bar{r}_k(\mathbf{M}) \geq g$ *then* $\bar{r}_k(\mathrm{Ker}(\varphi)) \geq g-1$.

*Proof.* Since $r_k(\mathbf{M}) \geq g$, there is a morphism $f : \mathbf{W}_k^g \longrightarrow \mathbf{M}$. Consider the group homomorphism given by

$$\varphi \circ f : \mathbf{W}_k^g \longrightarrow C_k.$$

Since $\mathbf{W}_k^g$ is non-singular, there exists $v \in \mathbf{W}_k^g$ such that $\varphi \circ f(x) = b(v,x)$ for all $x \in \mathbf{W}_k^g$. By Proposition 4.3.3, there exists an automorphism $h : \mathbf{W}_k^g \longrightarrow \mathbf{W}_k^g$ such that $h^{-1}(v)$ is in the sub-module $\mathbf{W}_k \oplus \mathbf{0} \leq \mathbf{W}_k^g$. It follows that the submodule $\mathbf{0} \oplus \mathbf{W}_k^{g-1}$ is contained in the kernel of the homomorphism given by the composition,

$$\mathbf{W}_k^g \xrightarrow{\quad h \quad} \mathbf{W}_k^g \xrightarrow{\quad f \quad} \mathbf{M} \xrightarrow{\quad \varphi \quad} C_k.$$

This implies that $f(h(\mathbf{0} \oplus \mathbf{W}_k^{g-1}))$ is contained in the kernel of $\varphi$ and thus $r_k(\mathrm{Ker}(\varphi)) \geq g-1$.

Now suppose that $\bar{r}_k(\mathbf{M}) \geq g$ and let $\varphi : \mathbf{M} \longrightarrow C_k$ be given. It follows that $r_k(\mathbf{M} \oplus \mathbf{W}_k^j) \geq g$ for some integer $j \geq 0$. Consider the map $\bar{\varphi}$ given by the composition,

$$\mathbf{M} \oplus \mathbf{W}_k^j \xrightarrow{\quad \mathrm{proj}_{\mathbf{M}} \quad} \mathbf{M} \xrightarrow{\quad \varphi \quad} C_k.$$

By the result proven in the first paragraph, $r_k(\mathrm{Ker}(\bar{\varphi})) \geq g-1$. Clearly we have $\mathrm{Ker}(\bar{\varphi}) = \mathrm{Ker}(\varphi) \oplus \mathbf{W}_k^j$. It then follows that $\bar{r}_k(\mathrm{Ker}(\varphi)) \geq g-1$. This completes the proof of the corollary. □

The next proposition yields the first non-trivial case of Theorem 4.3.2. Compare with (11, Proposition 4.3)

**Proposition 4.3.5.** *If $\bar{r}_k(\mathbf{M}) \geq 2$, then $L(\mathbf{M})_k \neq \emptyset$. If $\bar{r}_k(\mathbf{M}) \geq 4$, then $L(\mathbf{M})_k$ is connected.*

*Proof.* Let us first make the slightly stronger assumption that $r_k(\mathbf{M}) \geq 4$. It follows that there exists some morphism $f_0 : \mathbf{W}_k \longrightarrow \mathbf{M}$ such that $r_k(f_0(\mathbf{W}_k)^\perp) \geq 3$. Given any morphism $f : \mathbf{W}_k \longrightarrow \mathbf{M}$, we have a homomorphism of abelian groups $f_0(\mathbf{W}_k)^\perp \longrightarrow \mathbf{M} \longrightarrow f(\mathbf{W}_k)$, where the first map is the inclusion and the second is orthogonal projection. The kernel of this map is the intersection $f_0(\mathbf{W}_k)^\perp \cap f(\mathbf{W}_k)^\perp$. Since $\mathbf{W}_k = \mathbb{Z}/k \oplus \mathbb{Z}/k \cong C_k \oplus C_k$ (as an abelian group), it follows from Corollary 4.3.4 that $r_k(f_0(\mathbf{W}_k)^\perp \cap f(\mathbf{W}_k)^\perp) \geq 1$. Thus, we can find a morphism

$$f' : \mathbf{W}_k \longrightarrow f_0(\mathbf{W}_k)^\perp \cap f(\mathbf{W}_k)^\perp.$$

It follows that the sets $\{f_0, f\}$ and $\{f_0, f'\}$ are both 1-simplices, and so there is a path of length 2 from $f$ to $f'$.

Now suppose that $\bar{r}_k(\mathbf{M}) \geq 4$. We then have an isomorphism of linking forms $\mathbf{M} \oplus \mathbf{W}_k^j \cong \mathbf{N} \oplus \mathbf{W}_k^j$ for some $j$ where $r_k(\mathbf{N}) \geq 4$. By the first paragraph, $L(\mathbf{N} \oplus \mathbf{W}_k^j)_k$ is connected for all $j \geq 0$, and so we may apply Proposition 4.2.2 inductively to deduce that $\mathbf{M} \cong \mathbf{N}$ and thus $r_k(\mathbf{M}) \geq 4$. We then apply the result of the first paragraph to conclude that $L(\mathbf{M})_k$ is connected.

If $\bar{r}_k(\mathbf{M}) \geq 2$ we may write $\mathbf{M} \oplus \mathbf{W}_k^j \cong \mathbf{N} \oplus \mathbf{W}_k^j$ for some integer $j$ and linking form $\mathbf{N}$ such that $r_k(\mathbf{N}) \geq 2$. We may then inductively apply Proposition 4.2.2 to obtain an isomorphism $f : \mathbf{M} \oplus \mathbf{W}_k \xrightarrow{\cong} \mathbf{N} \oplus \mathbf{W}_k$. The linking form $\mathbf{M}$ is then isomorphic to the kernel of the orthogonal projection, $\mathbf{N} \oplus \mathbf{W}_k \longrightarrow f(\mathbf{0} \oplus \mathbf{W}_k)$.

Since $r_k(\mathbf{N} \oplus \mathbf{W}_k) \geq 3$ and $\mathbf{W}_k \cong C_k \oplus C_k$, it follows from Corollary 4.3.4 that $r_k(\mathbf{M}) \geq 1$. From this, it follows that $L(\mathbf{M})_k$ is non-empty. This concludes the proof of the proposition. □

*Proof of Theorem 4.3.2.* We proceed by induction on $g$. The base case of the induction, which is the case of the theorem where $g = 4$ and $\bar{r}(\mathbf{M}) \geq 4$, follows immediately from Proposition 4.3.5. Now suppose that the theorem holds for the $g-1$ case. Let $\mathbf{M}$ be a linking form with $\bar{r}_k(\mathbf{M}) \geq g$ and $g \geq 4$. By Proposition 4.3.5 there exists a morphism $f : \mathbf{W}_k \longrightarrow \mathbf{M}$ and by Corollary 4.3.1 it follows that $\bar{r}_k(f(\mathbf{W}_k)^{\perp}) \geq g-1$. Let $\mathbf{M}'$ denote the orthogonal compliment $f(\mathbf{W}_k)^{\perp}$ and consider the subgroup $\mathbf{M}' \perp \langle f(\sigma) \rangle \leq \mathbf{M}$, where $\sigma$ is one of the standard generators of $\mathbf{W}_k$ ($\mathbf{M}' \perp \langle f(\sigma) \rangle$ indicates an orthogonal direct sum). The chain of inclusions $\mathbf{M}' \hookrightarrow \mathbf{M}' \perp \langle f(\sigma) \rangle \hookrightarrow \mathbf{M}$ induces a chain of embeddings of sub-simplicial-complexes

$$L(\mathbf{M}')_k \xrightarrow{\quad i_1 \quad} L(\mathbf{M}' \perp \langle f(\sigma) \rangle)_k \xrightarrow{\quad i_2 \quad} L(\mathbf{M})_k. \tag{4.4}$$

The composition is null-homotopic since the vertex in $L(\mathbf{M})_k$ determined by the morphism $f : \mathbf{W}_k \longrightarrow \mathbf{M}$, is adjacent to every simplex in the subcomplex $L(\mathbf{M}')_k \leq L(\mathbf{M})_k$. To prove that $L(\mathbf{M})_k$ is $\frac{1}{2}(g-4)$-connected, we apply Proposition 2.1.5 to the maps $i_1$ and $i_2$ with $n := \frac{1}{2}(g-4)$. Since $L(\mathbf{M}')_k$ is $(n-1)$-connected by the induction assumption (recall that $\bar{r}(\mathbf{M}') \geq g-1$), Proposition 2.1.5 together with the fact that $i_2 \circ i_1$ is null-homotopic will imply that $L(\mathbf{M})_k$ is $\frac{1}{2}(g-4)$-connected.

Let $\xi$ be a $p$-simplex of $L(\mathbf{M}' \perp \langle f(\sigma) \rangle)_k$. The linking form on the subgroup $f(\sigma) \leq \mathbf{M}'$ is trivial and thus it follows that the projection homomorphism, $\pi : \mathbf{M}' \perp \langle f(\sigma) \rangle \longrightarrow \mathbf{M}'$ preserves the linking form structure. Thus, there is an induced

simplicial map

$$\bar{\pi} : L(\mathbf{M}' \perp \langle f(\sigma)\rangle)_k \longrightarrow L(\mathbf{M}')_k,$$

and it follows easily that $i_1$ is a section of $\bar{\pi}$. It follows from (4.3) that there is an equality of simplicial complexes,

$$[\mathrm{lk}_{L(\mathbf{M}' \perp \langle f(\sigma)\rangle)_k}(\xi)] \cap L(\mathbf{M}')_k \;=\; \mathrm{lk}_{L(\mathbf{M}')_k}(\bar{\pi}(\xi)).$$

Since $\bar{r}_k(\mathbf{M}') \geq g-1$, the induction assumption (which is that $lCM(L(\mathbf{M}')_k) \geq \frac{1}{2}(g-2)$) implies that the above complex is

$$\tfrac{1}{2}(g-2) - p - 2 = (n-p-1) - \text{connected},$$

where recall, $n = \frac{1}{2}(g-4)$. Proposition 2.1.5 then implies that the map $i_1$ is $n$-connected.

We now focus on the map $i_2$. Since $b(\sigma,\sigma) = 0$, it follows that the subgroup

$$\mathbf{M}' \perp \langle f(\sigma)\rangle \leq \mathbf{M}$$

is precisely the orthogonal compliment of $\langle f(\sigma)\rangle$ in $\mathbf{M}$. Let $\zeta := \{f_0, \ldots, f_p\} \leq L(\mathbf{M})_k$ be a $p$-simplex, and denote $\mathbf{M}'' := [\sum(f_i(\mathbf{W}_k))]^{\perp} \leq \mathbf{M}$. We have,

$$L(\mathbf{M}' \perp \langle f(\sigma)\rangle)_k \cap \mathrm{lk}_{L(\mathbf{M})}(\zeta) \;=\; L(\mathbf{M}'' \cap \langle f(\sigma)\rangle^{\perp})_k. \tag{4.5}$$

Corollary 4.3.1 implies that $\bar{r}_k(\mathbf{M}'') \geq g-p-1$. Passing to the kernel of the homomorphism

$$b(\_, f(\sigma))|_{\mathbf{M}''} : \mathbf{M}'' \longrightarrow C_k,$$

reduces the stable $k$-rank by 1, and so we have $\bar{r}_k(\mathbf{M}'' \cap \langle f(\sigma)\rangle^{\perp}) \geq g-p-2$. By the induction assumption, it follows that the complex $L(\mathbf{M}'' \cap \langle f(\sigma)\rangle^{\perp})_k$ is at least

$$\tfrac{1}{2}(g-p-2-4) \geq (n-p-1)-\text{connected}.$$

By Proposition 2.1.5 it follows that the inclusion $i_2$ is $n$-connected. Combining with the previous paragraph implies that $i_2 \circ i_1$ is $n$-connected. It then follows that $L(\mathbf{M})_k$ is $n = \frac{1}{2}(g-4)$-connected since $i_2 \circ i_1$ is null-homotopic.

The fact that $lCM(L(\mathbf{M})_k) \geq \frac{1}{2}(g-1)$ is proven as follows. Let $\zeta = \{f_0, \ldots, f_p\} \leq L(\mathbf{M})_k$ be a $p$-simplex and let $\mathbf{V}$ denote the orthogonal compliment $[\sum f_i(\mathbf{W})]^{\perp}$. We have $\bar{r}_k(\mathbf{V}) \geq g-p-1$. By (4.3) we have $\mathrm{lk}_{L(\mathbf{M})_k}(\zeta) \cong L(\mathbf{V})_k$ and so by the induction assumption it follows that $|\mathrm{lk}_{L(\mathbf{M})_k}(\zeta)|$ is $\frac{1}{2}(g-p-1-4)$-connected. The inequality

$$\tfrac{1}{2}(g-p-1-4) = \tfrac{1}{2}(g-p-1)-2 \geq \tfrac{1}{2}(g-1)-p-2$$

implies that $|\mathrm{lk}_{L(\mathbf{M})_k}(\zeta)|$ is $(\frac{1}{2}(g-1)-p-2)$-connected. This proves that $lCM(L(\mathbf{M})_k) \geq \frac{1}{2}(g-1)$ and concludes the proof of the Theorem. $\square$

# CHAPTER V

# $\mathbb{Z}/K$-MANIFOLDS

## 5.1. Basic Definitions

One of the main tools we will use to study the diffeomorphism groups of odd dimensional manifolds will be manifolds with certain types of *Baas-Sullivan* singularities, namely $\mathbb{Z}/k$-manifolds (which in this paper we refer to as $\langle k \rangle$-manifolds). We will use these manifolds to construct a geometric model for the linking form. Here we give an overview of the definition and basic properties of such manifolds. For further reference on $\mathbb{Z}/k$-manifolds or manifolds with general Baas-Sullivan singularities, see (3), (5), and (33).

**Notational Convention 5.1.1.** For a positive integer $k$, we let $\langle k \rangle$ denote the set consisting of $k$-elements, $\{1, \ldots, k\}$. We will consider this set to be a zero-dimensional manifold.

**Definition 5.1.1.** Let $k$ be a positive integer. Let $P$ be a $p$-dimensional smooth manifold equipped with the following extra structure:

i. The boundary of $P$ has the decomposition, $\partial P = \partial_0 P \cup \partial_1 P$ where $\partial_0 P$ and $\partial_1 P$ are $(p-1)$-dimensional manifolds with boundary and

$$\partial_{0,1} P := (\partial_0 P) \cap (\partial_1 P) = \partial(\partial_0 P) = \partial(\partial_1 P)$$

is a $(d-2)$-dimensional closed manifold.

ii. There is a manifold $\beta P$ and diffeomorphism $\Phi : \partial_1 P \xrightarrow{\cong} \beta P \times \langle k \rangle$.

With $P$, $\beta P$, and $\Phi$ as above, the pair $(P, \Phi)$ is said to be a $\langle k \rangle$-manifold. The diffeomorphism $\Phi$ is referred to as the *structure-map* and the manifold $\beta P$ is called the *Bockstein*.

**Notational Convention 5.1.2.** We will usually drop the structure-map from the notation and denote $P := (P, \Phi)$. We will always denote the structure-map associated to a $\langle k \rangle$-manifold by the same capital greek letter $\Phi$. If another $\langle k \rangle$-manifold is present, say $Q$, we will decorate the structure map with the subscript $Q$, i.e. $\Phi_Q$.

Any smooth manifold $M$ is automatically a $\langle k \rangle$-manifold by setting $\partial_0 M = \partial M$, $\partial_1 M = \emptyset$, and $\beta M = \emptyset$. Such a $\langle k \rangle$-manifold $M$ with $\partial_1 M = \emptyset$, $\beta M = \emptyset$ is said to be *non-singular*.

Now, let $P$ be a $\langle k \rangle$-manifold as in the above definition. Notice that the diffeomorphism $\Phi$ maps the submanifold $\partial_{0,1} P \subset \partial_1 P$ diffeomorphically onto $\partial(\beta P)$. In this way, if we set

$$\partial_0(\partial_0 P) := \emptyset, \quad \partial_1(\partial_0 P) := (\partial_0 P) \cap (\partial_1 P) = \partial_{0,1} P, \quad \text{and} \quad \beta(\partial_0 P) = \partial(\beta P),$$

the pair $\partial_0 P := (\partial_0 P, \ \Phi|_{\partial_{0,1} P})$ is a $\langle k \rangle$-manifold. We will refer to $\partial_0 P$ as the *boundary* of $P$. If $\partial_0 P = \emptyset$, then $P$ is said to be a *closed* $\langle k \rangle$-manifold.

Given a $\langle k \rangle$-manifold $P$, one can construct a manifold with *cone-type singularities* in a natural way as follows.

**Definition 5.1.2.** Let $P$ be a $\langle k \rangle$- manifold. Let $\bar{\Phi} : \partial_1 P \longrightarrow \beta P$ be the map given by the composition $\partial_1 P \xrightarrow[\cong]{\Phi} \beta P \times \langle k \rangle \xrightarrow{\text{proj}_{\beta P}} \beta P$. We define $\widehat{P}$ to be the quotient space obtained from $P$ by identifying points $x, y \in \partial_1 P$ if and only if $\bar{\Phi}(x) = \bar{\Phi}(y)$.

We will need to consider maps from $\langle k \rangle$-manifolds to non-singular manifolds.

**Definition 5.1.3.** Let $P$ be a $\langle k \rangle$-manifold and let $X$ be a topological space. A map $f : P \longrightarrow X$ is said to be a $\langle k \rangle$-map if there exists a map $f_\beta : \beta P \to X$ such that the restriction of $f$ to $\partial_1 P$ has the factorization $\partial_1 P \xrightarrow{\bar{\Phi}} \beta P \xrightarrow{f_\beta} X$, where $\bar{\Phi} : \partial_1 P \longrightarrow \beta P$ is the map from Definition 5.1.2. Clearly the map $f_\beta$ is uniquely determined by $f$.

We denote by $\mathrm{Maps}_{\langle k \rangle}(P, X)$ the space of $\langle k \rangle$-maps $P \to M$, topologized as a subspace of $\mathrm{Maps}(P, X)$ with the compact-open topology. It is immediate that any $\langle k \rangle$-map $f : P \to X$ induces a unique map $\widehat{f} : \widehat{P} \longrightarrow X$ and that the correspondence, $f \mapsto \widehat{f}$ induces a homeomorphism, $\mathrm{Maps}_{\langle k \rangle}(P, X) \cong \mathrm{Maps}(\widehat{P}, X)$. Throughout the paper we will denote by $\hat{f} : \widehat{P} \longrightarrow Y$, the map induced by the $\langle k \rangle$-map $f$. In the case that $X$ is a smooth manifold, $f$ is said to be a smooth $\langle k \rangle$-map if both $f$ and $f_\beta$ are both smooth.

## 5.2. Bordism of $\langle k \rangle$-Manifolds

We will need to consider the oriented bordism groups of $\langle k \rangle$-manifolds. For a space $X$ and non-negative integer $j$, we denote by $\Omega_j^{SO}(X)_{\langle k \rangle}$ the bordism group of $j$-dimensional, oriented $\langle k \rangle$-manifolds associated to $X$. We refer the reader to (5) and (33) for precise details of the definitions. We have the following Theorem from (5).

**Theorem 5.2.1.** *For any space $X$ and integer $k \geq 2$, there is a long exact sequence:*

$$\cdots \longrightarrow \Omega_j^{SO}(X) \xrightarrow{\times k} \Omega_j^{SO}(X) \xrightarrow{j_k} \Omega_j^{SO}(X)_{\langle k \rangle} \xrightarrow{\beta} \Omega_{j-1}^{SO}(X) \longrightarrow \cdots \tag{5.1}$$

*where $\times k$ denotes multiplication by the integer $k$, $j_k$ is induced by inclusion (since an oriented smooth manifold is an oriented $\langle k\rangle$-manifold), and $\beta$ is the map induced by $P \mapsto \beta P$.*

It is immediate from the above long exact sequence that for all integers $k \geq 2$, there are isomorphisms

$$\Omega_0^{SO}(\text{pt.})_{\langle k\rangle} \cong \mathbb{Z}/k \quad \text{and} \quad \Omega_1^{SO}(\text{pt.})_{\langle k\rangle} \cong 0. \tag{5.2}$$

## 5.3. $\mathbb{Z}/k$-Homotopy Groups

For integers $k, n \geq 2$, let $M(\mathbb{Z}/k, n)$ denote the $n$-th $\mathbb{Z}/k$-Moore-space. Recall that $M(\mathbb{Z}/k, n)$ is uniquely determined up to homotopy by the calculation,

$$H_j(M(\mathbb{Z}/k, n);\ \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k & \text{if } j = n, \\ \mathbb{Z} & \text{if } j = 0, \\ 0 & \text{else.} \end{cases}$$

For a space $X$, we denote by $\pi_n(X;\ \mathbb{Z}/k)$ the set of based homotopy classes of maps $M(\mathbb{Z}/k, n) \longrightarrow X$. Since $M(\mathbb{Z}/k, n)$ is a suspension when $n \geq 2$, the set $\pi_n(X;\ \mathbb{Z}/k)$ has the structure of a group, which is abelian when $n \geq 3$.

For integers $n, k \geq 2$, we define a $\langle k\rangle$-manifold which will play the role of the sphere in the category of $\langle k\rangle$-manifolds.

**Construction 5.3.1.** Choose an embedding $\Phi' : D^n \times \langle k\rangle \longrightarrow S^n$. Let $V_k^n$ denote the manifold obtained from $S^n$ by removing the interior of $\Phi'(D^n \times \langle k\rangle)$ from $S^n$. The inverse of the restriction of the map $\Phi'$ to $\partial D^n \times \langle k\rangle$ induces a diffeomorphism,

$\Phi : \partial V_k^n \xrightarrow{\cong} S^{n-1} \times \langle k \rangle$. By setting $\beta V_k^n = S^{n-1}$, the above diffeomorphism $\Phi$ gives $V_k^n$ the structure of a closed $\langle k \rangle$-manifold.

Let $\widehat{V}_k^n$ denote the singular space obtained from $V_k^n$ as in Definition 5.1.2. An elementary calculation shows that,

$$H_j(\widehat{V}_k^n) \cong \begin{cases} \mathbb{Z}/k & \text{if } j = n-1 \text{ or } 0, \\ \mathbb{Z}^{\oplus(k-1)} & \text{if } j = 1, \end{cases} \qquad \text{and} \qquad \pi_1(\widehat{V}_k^n) \cong \mathbb{Z}^{\star(k-1)}, \tag{5.3}$$

where $\mathbb{Z}^{\star(k-1)}$ denotes the free group on $(k-1)$-generators. It follows that the Moore-space $M(\mathbb{Z}/k, n-1)$ can be constructed from $\widehat{V}_k^n$ by attaching $(k-1)$-many 2-cells, one for each generator of the fundamental group. This yields the following result.

**Lemma 5.3.1.** *Let $X$ be a 2-connected space and let $k \geq 2$ and $n \geq 3$ be integers. The inclusion map $\widehat{V}_k^n \hookrightarrow M(\mathbb{Z}/k, n-1)$ induces a bijection of sets, $\pi_0(\mathrm{Maps}_{\langle k \rangle}(V_k^n, X)) \xrightarrow{\cong} \pi_{n-1}(X; \mathbb{Z}/k)$.*

*Proof.* Since $X$ is simply connected, any map $\widehat{V}_k^n \longrightarrow X$ extends to a map $M(\mathbb{Z}/k, n-1) \longrightarrow X$ and since $X$ is 2-connected, it follows that any such extension is unique up to homotopy. This proves that the inclusion $\widehat{V}_k^n \hookrightarrow M(\mathbb{Z}/k, n-1)$ induces a bijection $\pi_0(\mathrm{Maps}(\widehat{V}_k^n, X)) \cong \pi_{n-1}(X; \mathbb{Z}/k)$. The lemma then follows from composing this bijection with the natural bijection, $\pi_0(\mathrm{Maps}_{\langle k \rangle}(V_k^n, X)) \xrightarrow{\cong} \pi_0(\mathrm{Maps}(\widehat{V}_k^n, X))$. □

**Corollary 5.3.2.** *Let $X$ be a 2-connected space and let $k \geq 2$ and $n \geq 3$ be integers. Let $x \in \pi_{n-1}(X)$ be an element of order $k$. Then there exists a $\langle k \rangle$-map $f : V_k^n \longrightarrow X$ such that the associated map $f_\beta : S^{n-1} \longrightarrow X$ is a representative of $x$.*

*Proof.* The cofibre sequence $S^j \xrightarrow{\times k} S^j \longrightarrow M(\mathbb{Z}/k, j)$ induces a long exact sequence,

$$\cdots \longrightarrow \pi_n(X) \xrightarrow{\times k} \pi_n(X) \longrightarrow \pi_{n-1}(X; \mathbb{Z}/k) \xrightarrow{\partial} \pi_{n-1}(X) \xrightarrow{\times k} \pi_{n-1}(X) \longrightarrow \cdots$$

It follows that if $x \in \pi_{n-1}(X)$ is of order $k$, then there is an element $y \in \pi_{n-1}(X; \mathbb{Z}/k)$ such that $\partial y = x$. Let $r_\beta : \pi_0(\mathrm{Maps}_{\langle k \rangle}(V_k^n, X)) \longrightarrow \pi_{n-1}(X)$ denote the map induced by, $f \mapsto f_\beta$. It follows from the construction of the the map $\partial$ in the above long exact sequence that the diagram,

$$\begin{array}{ccc} \pi_0(\mathrm{Maps}_{\langle k \rangle}(V_k^n, X)) & \xrightarrow{\cong} & \pi_{n-1}(X; \mathbb{Z}/k) \\ & \searrow^{r_\beta} & \downarrow \partial \\ & & \pi_{n-1}(X) \end{array}$$

commutes, where the upper horizontal map is the bijection from Lemma 5.3.1. The result then follows from commutativity of this diagram. □

## 5.4. Immersions and Embeddings of $\langle k \rangle$-Manifolds

We will need to consider immersions and embeddings of a $\langle k \rangle$-manifold into a smooth manifold. For what follows, let $P$ be a $\langle k \rangle$-manifold and let $M$ be a manifold.

**Definition 5.4.1.** A $\langle k \rangle$-map $f : P \longrightarrow M$ is said to be a $\langle k \rangle$-*immersion* if it is an immersion when considering $P$ as a smooth manifold with boundary. Two $\langle k \rangle$-immersions $f, g : P \longrightarrow M$ are said to be regularly homotopic if there exists a homotopy $F_t : P \longrightarrow M$ with $F_0 = f$ and $F_1 = g$ such that $F_t$ is a $\langle k \rangle$-immersion for all $t \in [0, 1]$.

In addition to immersions we will mainly need to deal with embeddings of $\langle k \rangle$-manifolds.

**Definition 5.4.2.** A $\langle k \rangle$-immersion $f : P \longrightarrow M$ is said to be a $\langle k \rangle$-embedding if the induced map $\widehat{f} : \widehat{P} \longrightarrow M$ is an embedding.

The main result about $\langle k \rangle$-embeddings that we will use is the following. The proof is given in Section 11.6, using the techniques developed throughout all of Section XI.

**Theorem 5.4.1.** *Let $n \geq 2$ be an integer and let $k > 2$ be an odd integer. Let $M$ be a 2-connected, oriented manifold of dimension $4n + 1$. Then any $\langle k \rangle$-map $f : V_k^{2n+1} \longrightarrow M$ is homotopic through $\langle k \rangle$-maps to a $\langle k \rangle$-embedding.*

The following corollary follows immediately by combining Theorem 5.4.1 with Corollary 5.3.2.

**Corollary 5.4.2.** *Let $n \geq 2$ be an integer and let $k > 2$ be an odd integer. Let $M$ be a 2-connected, oriented manifold of dimension $4n + 1$. Let $x \in \pi_{2n}(M)$ be a class of order $k$. Then there exists a $\langle k \rangle$-embedding $f : V_k^{2n+1} \longrightarrow M$ such that the embedding $f_\beta : \beta V_k^{2n+1} = S^{2n} \longrightarrow M$ is a representative of the class $x$.*

# CHAPTER VI

# $\langle K, L\rangle$-MANIFOLDS

## 6.1. Basic definitions

We will have to consider certain spaces with more complicated singularity structure than that of the $\langle k\rangle$-manifolds encountered in the previous section.

**Definition 6.1.1.** Let $k$ and $l$ be positive integers. Let $N$ be a smooth $d$-dimensional manifold equipped with the following extra structure:

i. The boundary $\partial N$ has the decomposition,

$$\partial N \ = \partial_0 N \cup \partial_1 N \cup \partial_2 N$$

such that $\partial_0 N$, $\partial_1 N$ and $\partial_2 N$ are $(d-1)$-dimensional manifolds, the intersections

$$\partial_{0,1} N := \partial_{0,1} N, \quad \partial_{0,2} N := \partial_0 N \cap \partial_2 N, \quad \partial_{1,2} N := \partial_1 N \cap \partial_2 N$$

are $(d-2)$-dimensional manifolds, and

$$\partial_{0,1,2} N := \partial_0 N \cap \partial_1 N \cap \partial_2 N$$

is a $(d-3)$-dimensional closed manifold.

ii. There exist manifolds $\beta_1 N$, $\beta_2 N$, and $\beta_{1,2} N$, and diffeomorphisms

$$\begin{aligned}\Phi_1 : \partial_1 N &\xrightarrow{\cong} \beta_1 N \times \langle k \rangle,\\ \Phi_2 : \partial_2 N &\xrightarrow{\cong} \beta_2 N \times \langle l \rangle,\\ \Phi_{1,2} : \partial_{1,2} N &\xrightarrow{\cong} \beta_{1,2} N \times \langle k \rangle \times \langle l \rangle,\end{aligned}$$

such that the maps

$$\begin{aligned}\Phi_1 \circ \Phi_{1,2}^{-1} : \beta_{1,2} N \times \langle k \rangle \times \langle l \rangle &\longrightarrow \beta_1 N \times \langle k \rangle,\\ \Phi_2 \circ \Phi_{1,2}^{-1} : \beta_{1,2} N \times \langle k \rangle \times \langle l \rangle &\longrightarrow \beta_1 N \times \langle l \rangle,\end{aligned}$$

are identical on the direct factors of $\langle k \rangle$ and $\langle l \rangle$ respectively.

With the above conditions satisfied, the 4-tuple $N := (N, \Phi_1, \Phi_2, \Phi_{1,2})$ is said to be a $\langle k, l \rangle$-manifold of dimension $d$.

**Remark 6.1.1.** The above definition is a specialization of $\Sigma$-manifold from (5, Definition 1.1.1) and a generalization of the definition of $\langle k \rangle$-manifold. In fact, any $\langle k \rangle$-manifold $P$ is a $\langle k, l \rangle$-manifold with $\beta_2 P = \emptyset$.

As for the case with $\langle k \rangle$- manifolds, we will drop the structure maps $\Phi_1, \Phi_2, \Phi_{1,2}$ from the notation and denote $N := (N, \Phi_1, \Phi_2, \Phi_{1,2})$. The manifold $\partial_0 W$ is referred to as the boundary of the $\langle k, l \rangle$-manifold and is a $\langle k, l \rangle$-manifold in its own right. A $\langle k, l \rangle$-manifold $N$ is said to be *closed* if $\partial_0 N = \emptyset$.

From a $\langle k, l \rangle$-manifold $N$, one obtains a manifold with cone-type singularities in the following way.

**Definition 6.1.2.** Let $N$ be a $\langle k, l \rangle$-manifold. Let $\bar{\Phi}_1 : \partial_1 N \longrightarrow \beta_1 N$ be the map defined by the composition $\partial_1 N \xrightarrow[\cong]{\Phi_1} \beta_1 N \times \langle k \rangle \xrightarrow{\mathrm{proj}_{\beta_1 N}} \beta_1 N$. Define $\bar{\Phi}_2 :$

$\partial_2 N \longrightarrow \beta_2 N$ similarly. We define $\widehat{N}$ to be the quotient space obtained from $N$ by identifying two points $x, y$ if and only if for $i = 1$ or 2, both $x$ and $y$ are in $\partial_i W$ and $\bar{\Phi}_i(x) = \bar{\Phi}_i(y)$.

## 6.2. Oriented $\langle k, l\rangle$-Bordism

We will need to make use of the oriented bordism groups of $\langle k, l\rangle$-manifolds. For any space $X$ and non-negative integer $j$, we denote by $\Omega_j^{SO}(X)_{\langle k,l\rangle}$ the $j$-th $\langle k, l\rangle$-bordism group associated to the space $X$. We refer the reader to (5) for details on the definition. There are maps

$$\beta_1 : \Omega_j^{SO}(X)_{\langle k,l\rangle} \longrightarrow \Omega_{j-1}^{SO}(X)_{\langle l\rangle}, \quad \beta_2 : \Omega_j^{SO}(X)_{\langle k,l\rangle} \longrightarrow \Omega_{j-1}^{SO}(X)_{\langle k\rangle}$$

defined by sending a $\langle k, l\rangle$-manifold $N$ to $\beta_1 N$ and $\beta_2 N$ respectively. We also have maps

$$j_1 : \Omega_j^{SO}(X)_{\langle k\rangle} \longrightarrow \Omega_j^{SO}(X)_{\langle k,l\rangle}, \quad j_2 : \Omega_j^{SO}(X)_{\langle l\rangle} \longrightarrow \Omega_j^{SO}(X)_{\langle k,l\rangle}$$

defined by considering a $\langle k\rangle$-manifold or an $\langle l\rangle$-manifold as a $\langle k, l\rangle$-manifold. We have the following theorem from (5).

**Theorem 6.2.1.** *The following sequences are exact,*

$$\cdots \to \Omega_j^{SO}(X)_{\langle l\rangle} \xrightarrow{\times l} \Omega_j^{SO}(X)_{\langle l\rangle} \xrightarrow{j_1} \Omega_j^{SO}(X)_{\langle k,l\rangle} \xrightarrow{\beta_1} \Omega_{j-1}^{SO}(X)_{\langle l\rangle} \to \cdots$$

$$\cdots \to \Omega_j^{SO}(X)_{\langle k\rangle} \xrightarrow{\times k} \Omega_j^{SO}(X)_{\langle k\rangle} \xrightarrow{j_2} \Omega_j^{SO}(X)_{\langle k,l\rangle} \xrightarrow{\beta_2} \Omega_{j-1}^{SO}(X)_{\langle k\rangle} \to \cdots$$

Using the isomorphisms $\Omega_0^{SO}(\text{pt.})_{\langle k\rangle} \cong \mathbb{Z}/k$ and $\Omega_1^{SO}(\text{pt.})_{\langle k\rangle} = 0$, we obtain the following basic calculations using the above exact sequence.

**Corollary 6.2.2.** *For any two integers $k, l \geq 2$ we have the following isomorphisms,*

$$\Omega_0^{SO}(pt.)_{\langle k,l\rangle} \cong \mathbb{Z}/\gcd(k,l) \quad \textit{and} \quad \Omega_1^{SO}(pt.)_{\langle k,l\rangle} \cong \mathbb{Z}/\gcd(k,l).$$

*In particular we have,*

$$\Omega_0^{SO}(pt.)_{\langle k,k\rangle} \cong \mathbb{Z}/k \quad \textit{and} \quad \Omega_1^{SO}(pt.)_{\langle k,k\rangle} \cong \mathbb{Z}/k.$$

## 6.3. 1-dimensional, Closed, Oriented, $\langle k, k\rangle$-Manifolds

We will need to consider 1-dimensional $\langle k, k\rangle$-manifolds. They will arise for us as the intersections of $(n+1)$-dimensional $\langle k\rangle$-manifolds immersed in a $(2n+1)$-dimensional manifold. Denote by $A_k$ the space $[0,1] \times \langle k\rangle$. By setting

$$\partial_1 A_k = \{0\} \times \langle k\rangle \quad \text{and} \quad \partial_2 A_k = \{1\} \times \langle k\rangle,$$

$A_k$ naturally has the structure of a closed $\langle k, k\rangle$-manifold with, $\beta_1 A_k = \langle 1\rangle = \beta_2 A_k$ (the single point space). We denote by $+A_k$ the oriented $\langle k, k\rangle$-manifold with orientation induced by the standard orientation on $[0,1]$. We denote by $-A_k$ the $\langle k, k\rangle$-manifold equipped with the opposite orientation. It follows that

$$\beta_1(\pm A_k) = \pm\langle 1\rangle \quad \text{and} \quad \beta_2(\pm A_k) = \mp\langle 1\rangle. \tag{6.1}$$

Using the fact that the map $\beta_i : \Omega_1^{SO}(\text{pt.})_{\langle k,k\rangle} \longrightarrow \Omega_0^{SO}(\text{pt.})_{\langle k\rangle}$ for $i = 1, 2$ is an isomorphism (this follows from Corollary 6.2.2 and the exact sequence in Theorem 6.2.1), we have the following proposition.

**Proposition 6.3.1.** *The oriented, closed, $\langle k,k\rangle$ manifold $+A_k$ represents a generator for $\Omega_1^{SO}(pt.)_{\langle k,k\rangle}$. Furthermore, any oriented, closed, 1-dimensional $\langle k,k\rangle$-manifold that represents a generator of $\Omega_1^{SO}(pt.)_{\langle k,k\rangle}$, is of the form*

$$(+A_k \times \langle r\rangle) \sqcup (-A_k \times \langle s\rangle) \sqcup X,$$

*where $r,s \in \mathbb{N}$ are such that $r-s$ is relatively prime to $k$, and where $X$ is some null-bordant $\langle k,k\rangle$-manifold such that $\beta_1 X = \emptyset$ or $\beta_2 X = \emptyset$ (in other words, $X$ has the structure of $\langle k\rangle$-manifold).*

Throughout, we will consider the element of $\Omega_1^{SO}(\text{pt.})_{\langle k,k\rangle}$ determined by the oriented $\langle k,k\rangle$-manifold $+A_k$ to be the *standard generator*.

# CHAPTER VII

# INTERSECTION THEORY OF $\mathbb{Z}/K$-MANIFOLDS

In this section and the next two sections after, we will discuss the intersections of embeddings of $\langle k \rangle$-manifolds.

## 7.1. Preliminaries

Here we review some of the basics about intersections of embedded smooth manifolds and introduce some terminology and notation.

For what follows, let $M$, $X$, and $Y$ be oriented smooth manifolds of dimension $m$, $r$, and $s$ respectively and let $t$ denote the integer $r + s - m$. Let

$$\varphi : (X, \partial X) \longrightarrow (M, \partial M) \quad \text{and} \quad \psi : (Y, \partial Y) \longrightarrow (M, \partial M) \tag{7.1}$$

be smooth, transversal maps such that $\varphi(\partial X) \cap \psi(\partial Y) = \emptyset$ (for these two maps to be transversal, we mean that the product map $\varphi \times \psi : X \times Y \longrightarrow M \times M$ is transverse to the diagonal submanifold $\triangle_M \subset M \times M$). We let $\varphi \pitchfork \psi$ denote the *transverse pull-back* $(\varphi \times \psi)^{-1}(\triangle_M)$, which is a closed submanifold of $X \times Y$ of dimension $t$. The orientations on $X$, $Y$, and $M$ induce an orientation on $\varphi \pitchfork \psi$ and thus $\varphi \pitchfork \psi$ determines a bordism class in $\Omega_t^{SO}(\text{pt.})$ which we denote by $\Lambda^t(\varphi, \psi; M)$. It follows easily that, $\Lambda^t(\varphi, \psi; M) = (-1)^{(m-s)\cdot(m-r)}\Lambda^t(\psi, \varphi; M)$.

## 7.2. Intersections of $\langle k \rangle$-Manifolds

We now proceed to consider intersections of $\langle k \rangle$-manifolds. Let $M$ be an oriented manifold of dimension $m$, let $X$ be an oriented manifold of dimension

$r$, and let $P$ be an oriented $\langle k \rangle$-manifold of dimension $p$. Let $t$ denote the integer $r + p - m$. Let

$$\varphi : (X, \partial X) \longrightarrow (M, \partial M) \quad \text{and} \quad f : (P, \partial_0 P) \longrightarrow (M, \partial M)$$

be a smooth map and a smooth $\langle k \rangle$-map respectively. Suppose that $f$ and $\varphi$ are transversal and that $f(\partial_0 P) \cap \varphi(\partial X) = \emptyset$ (when we say that $f$ and $\varphi$ are transversal, we mean that both $f$ and $f_\beta$ are transverse to $\varphi$ as smooth maps). The pull-back,

$$f \pitchfork \varphi = (f \times \varphi)^{-1}(\triangle_M) \subset P \times X$$

has the structure of a closed $\langle k \rangle$-manifold as follows. We denote,

$$\partial_1(f \pitchfork \varphi) := f|_{\partial_1 P} \pitchfork \varphi \quad \text{and} \quad \beta(f \pitchfork \varphi) := f_\beta \pitchfork \varphi.$$

The factorization, $\partial_1 P \xrightarrow{\bar{\Phi}} \beta P \xrightarrow{f_\beta} M$ of the restriction map $f|_{\partial_1 P}$ implies that the diffeomorphism,

$$\Phi \times Id_X : \partial_1 P \times X \xrightarrow{\cong} (\beta P \times \langle k \rangle) \times X$$

maps $\partial_1(f \pitchfork X)$ diffeomorphically onto $\beta(f \pitchfork X) \times \langle k \rangle$. It follows that $f \pitchfork \varphi$ has the structure of a $\langle k \rangle$-manifold of dimension $t = p + r - m$. Furthermore, $f \pitchfork \varphi$ inherits an orientation from the orientations of $X$, $P$ and $M$.

**Definition 7.2.1.** Let $f : (P, \partial_0 P) \longrightarrow (M, \partial M)$ and $\varphi : (X, \partial X) \longrightarrow (M, \partial M)$ be exactly as above. We define $\Lambda_k^t(f, \varphi; M) \in \Omega_t^{SO}(\text{pt.})_{\langle k \rangle}$ to be the oriented bordism class determined by the pull-back $f \pitchfork \varphi$ and its induced orientation.

Recall from Section V the Bockstein homomorphism, $\beta : \Omega_t^{SO}(\text{pt.})_{\langle k \rangle} \longrightarrow \Omega_{t-1}^{SO}(\text{pt.})$. We have the following proposition.

**Proposition 7.2.1.** *Let $f : (P, \partial_0 P) \longrightarrow (M, \partial M)$ and $\varphi : (X, \partial X) \longrightarrow (M, \partial M)$ be exactly as above. Then*

$$\beta(\Lambda_k^t(f, \varphi; M)) = \Lambda^{t-1}(f_\beta, \varphi; M),$$

*where $\Lambda^{t-1}(f_\beta, \varphi; M) \in \Omega_{t-1}^{SO}(pt.)$ is the bordism class defined in Section 7.1.*

## 7.3. $\langle k, l \rangle$-Manifolds and Intersections

We now consider the intersection of a $\langle k \rangle$-manifold with an $\langle l \rangle$-manifold. For what follows, let $P$ be an oriented $\langle k \rangle$-manifold of dimension $p$, let $Q$ be an oriented $\langle l \rangle$-manifold of dimension $q$, and let $M$ be an oriented manifold of dimension $m$. Let

$$f : (P, \partial_0 P) \longrightarrow (M, \partial M) \quad \text{and} \quad g : (Q, \partial_0 Q) \longrightarrow (M, \partial M)$$

be a smooth $\langle k \rangle$-map and a smooth $\langle l \rangle$-map respectively. Suppose that $f$ and $g$ are transversal and that $f(\partial_0 P) \cap g(\partial_0 Q) = \emptyset$ (when we say that $f$ and $g$ are transversal, we mean that $f$ and $f_\beta$ are each transverse to both $g$ and $g_\beta$ as smooth maps). Let $t$ denote the integer $p + q - m$. We will analyze the $t$-dimensional submanifold

$$f \pitchfork g = (f \times g)^{-1}(\triangle_M) \subset P \times Q.$$

The transversality condition on $f$ and $g$ implies that the space $f \pitchfork g$, and the subspaces

$$\begin{array}{ccc} f|_{\partial P} \pitchfork g \subset \partial P \times Q, & f \pitchfork g|_{\partial Q} \subset P \times \partial Q, & f|_{\partial P} \pitchfork g|_{\partial Q} \subset \partial P \times \partial Q, \\ f_\beta \pitchfork g \subset \beta P \times Q, & f \pitchfork g_\beta \subset P \times \beta Q, & f_\beta \pitchfork g_\beta \subset \beta P \times \beta Q, \end{array}$$

are all smooth submanifolds. We define

$$\begin{array}{ccc} \partial_1(f \pitchfork g) := f|_{\partial P} \pitchfork g, & \partial_2(f \pitchfork g) := f \pitchfork g|_{\partial Q}, & \partial_{1,2}(f \pitchfork g) := f|_{\partial P} \pitchfork g|_{\partial Q}, \\ \beta_1(f \pitchfork g) := f_\beta \pitchfork g, & \beta_2(f \pitchfork g) := f \pitchfork g_\beta, & \beta_{1,2}(f \pitchfork g) := f_\beta \pitchfork g_\beta. \end{array}$$

The structure maps, $\Phi_P : \partial P \xrightarrow{\cong} \beta P \times \langle k \rangle$ and $\Phi_Q : \partial Q \xrightarrow{\cong} \beta Q \times \langle l \rangle$ induce diffeomorphisms,

$$\begin{array}{r} \Phi_P \times Id : \partial P \times Q \xrightarrow{\quad\cong\quad} \beta P \times \langle k \rangle \times Q, \\ Id \times \Phi_Q : P \times \partial Q \xrightarrow{\quad\cong\quad} P \times \beta Q \times \langle l \rangle, \\ \Phi_P \times \Phi_Q : \partial P \times \partial Q \xrightarrow{\quad\cong\quad} \beta P \times \langle k \rangle \times \beta Q \times \langle l \rangle. \end{array} \tag{7.2}$$

The factorizations,

$$\begin{array}{l} \partial P \xrightarrow{\bar{\Phi}_P} \beta P \xrightarrow{f_\beta} M, \\ \partial Q \xrightarrow{\bar{\Phi}_Q} \beta Q \xrightarrow{g_\beta} M, \end{array}$$

of the restriction maps $f|_{\partial P}$ and $g|_{\partial Q}$ imply that the diffeomorphisms from (7.2) map the submanifolds

$$\partial_1(f \pitchfork g) \subset \partial P \times Q, \quad \partial_2(f \pitchfork g) \subset P \times \partial Q, \quad \text{and} \quad \partial_{1,2}(f \pitchfork g) \subset \partial P \times \partial Q$$

diffeomorphically onto

$$\beta_1(f \pitchfork g) \times \langle k \rangle, \quad \beta_2(f \pitchfork g) \times \langle l \rangle, \quad \text{and} \quad \beta_{1,2}(f \pitchfork g) \times \langle k \rangle \times \langle l \rangle$$

respectively. It follows that $f \pitchfork g$ has the structure of an oriented $\langle k, l\rangle$-manifold of dimension $t = p + q - m$.

**Definition 7.3.1.** Let $f : (P, \partial_0 P) \longrightarrow (M, \partial M)$ and $g : (Q, \partial_0 Q) \longrightarrow (M, \partial M)$ be exactly as above. We denote by $\Lambda^1_{k,l}(f, g; M) \in \Omega^{SO}_t(\text{pt.})_{\langle k,l\rangle}$ the bordism class determined by the pull-back $f \pitchfork g$.

For the following proposition, recall from Section 5.2 the Bockstein homomorphisms,

$$\beta_1 : \Omega^{SO}_t(\text{pt.})_{\langle k,l\rangle} \longrightarrow \Omega^{SO}_{t-1}(\text{pt.})_{\langle l\rangle} \quad \text{and} \quad \beta_2 : \Omega^{SO}_t(\text{pt.})_{\langle k,l\rangle} \longrightarrow \Omega^{SO}_{t-1}(\text{pt.})_{\langle k\rangle}.$$

**Proposition 7.3.1.** *The bordism class $\Lambda^t_{k,l}(f, g; M) \in \Omega^{SO}_t(pt.)_{\langle k,l\rangle}$ satisfies the following equations*

*i.* $\Lambda^t_{k,l}(f, g; M) = (-1)^{(m-p)\cdot(m-q)} \cdot \Lambda^t_{l,k}(g, f; M)$,

*ii.* $\beta_1(\Lambda^t_{k,l}(f, g; M)) = \Lambda^{t-1}_l(f_\beta, g; M)$,

*iii.* $\beta_2(\Lambda^t_{k,l}(f, g; M)) = \Lambda^{t-1}_k(f, g_\beta; M)$.

## 7.4. Main Disjunction Theorem

We now discuss the main result that we will need to use regarding the intersections of $k$-manifolds. We will need the following terminology.

**Definition 7.4.1.** Let $M$ be a manifold. We will call a smooth, one parameter family of diffeomorphisms $\Psi_t : M \longrightarrow M$ with $t \in [0, 1]$ and $\Psi_0 = Id_M$ a *diffeotopy*. For a subspace $N \subset M$, we say that $\Psi_t$ is a *diffeotopy relative* $N$, and we write $\Psi_t : M \longrightarrow M$ rel $N$, if in addition, $\Psi_t|_N = Id_N$ for all $t \in [0, 1]$.

The main case of intersections of $\langle k \rangle$ and $\langle l \rangle$-manifolds that we will need to consider is the case when

$$k = l \quad \text{and} \quad \dim(P) + \dim(Q) - \dim(M) = 1.$$

For $n \geq 2$, let $M$ be an oriented manifold of dimension $4n+1$ and and let $P$ and $Q$ be oriented $k$-manifolds of dimension $2n+1$. Let

$$f : (P, \partial_0 P) \longrightarrow (M, \partial M) \quad \textit{and} \quad g : (Q, \partial_0 Q) \longrightarrow (M, \partial M) \tag{7.3}$$

be transversal $\langle k \rangle$-embeddings such that $f(\partial_0 P) \cap g(\partial_0 Q) = \emptyset$. Suppose further that $M$, $P$, and $Q$ are 2-connected.

**Theorem 7.4.1.** *With $f$ and $g$ the $\langle k \rangle$-embeddings given above, suppose that $\Lambda^1_{k,k}(f, g; M) = 0$. If the integer $k$ is odd, then there exists a diffeotopy*

$$\Psi_t : M \longrightarrow M \ \operatorname{rel} \partial M$$

*such that $\Psi_1(f(P)) \cap g(Q) = \emptyset$.*

We also have:

**Corollary 7.4.2.** *Suppose that the class $\Lambda^1_{k,k}(f, g; M) \in \Omega^{SO}_1(pt.)_{\langle k,k \rangle}$ is equal to the class represented by the closed 1-dimensional $\langle k, k \rangle$-manifold $+A_k$. If $k$ is odd, there exists a diffeotopy $\Psi_t : M \longrightarrow M \ \operatorname{rel} \partial M$ such that the $\langle k, k \rangle$-manifold given by the transverse pull-back $(\Psi_1 \circ f) \pitchfork g$, is diffeomorphic to $A_k$.*

**Remark 7.4.1.** Both of these results are proven in Section **??** (see Theorem 10.5.1 and Corollary 10.5.4). These above results are crucial in the proof of our main

homological stability theorem. The key place (only place) that they are used is in the proof of Lemma 8.1.2.

## 7.5. Connection to the Linking Form

In practice we will need to consider intersections of $\langle k \rangle$ embeddings $f, g : V_k^{2n+1} \longrightarrow M$. We will need to relate $\Lambda^1_{k,k}(f, g; M)$ to the homotopical linking form $b : \pi^\tau_{2n}(M) \otimes \pi^\tau_{2n}(M) \longrightarrow \mathbb{Q}/\mathbb{Z}$. Let

$$T_k : \Omega_1^{SO}(\text{pt.})_{\langle k,k \rangle} \longrightarrow \mathbb{Q}/\mathbb{Z}$$

be the homomorphism given by the composition

$$\Omega_1^{SO}(\text{pt.})_{\langle k,k \rangle} \xrightarrow{+A_k \mapsto 1} \mathbb{Z}/k \xrightarrow{1 \mapsto 1/k} \mathbb{Q}/\mathbb{Z}.$$

The following proposition follows easily from the definition of the homological linking form (3.5).

**Proposition 7.5.1.** *Let $M$ be a $(4n+1)$-dimensional, oriented manifold. Let*

$$f, g : V^{2n+1} \longrightarrow M$$

*be $k$-embeddings. Consider the homotopy classes $[f_\beta], [g_\beta] \in \pi^\tau_{2n}(M)$, which both have order $k$. Then*

$$b([f_\beta], [g_\beta]) \;=\; T_k(\Lambda^1_{k,k}(f, g; M)).$$

Combining this with Theorem 7.4.1 and Corollary 10.5.4 yields the following.

**Corollary 7.5.2.** *Let*

$$f, g : V^{2n+1} \longrightarrow M$$

*be $k$-embeddings and suppose that $b([f_\beta], [g_\beta]) = 0$. Then there exists a diffeotopy $\Psi_t : M \longrightarrow M$ such that $\Psi_1(f(V_k^{2n+1}) \cap g(V_k^{2n+1}) = \emptyset$. Suppose now that $b([f_\beta], [g_\beta]) = \frac{1}{k} \mod 1$. Then there exists a diffeotopy $\Psi_t : M \longrightarrow M$ such that there is a diffeomorphism $(\Psi_1 \circ f) \pitchfork g \cong +A_k$.*

# CHAPTER VIII

# HIGH CONNECTIVITY OF $|X_\bullet(M,A)_K|$

Let $n \geq 2$ be an integer. Let $M$ be a 2-connected, $(4n+1)$-dimensional manifold with non-empty boundary. Fix an embedding $a : [0,\infty) \times \mathbb{R}^{4n} \longrightarrow M$ with $a^{-1}(\partial M) = \{0\} \times \mathbb{R}^{4n}$. Recall from Definition 4.1.1 the topological flag complex $X_\bullet(M,a)_k$. In this chapter we prove Theorem 4.1.3 which we restate again below for the convenience of the reader. Recall from (1.9) the *k-rank* $r_k(M)$, of a $(4n+1)$-dimensional manifold $M$.

**Theorem 8.0.3.** *Let $n, k \geq 2$ be integers with $k$ odd. Let $M$ be a 2-connected, $(4n+1)$-dimensional manifold with non-empty boundary. Let $g \in \mathbb{N}$ be an integer such that $r_k(M) \geq g$. Then the geometric realization $|X_\bullet(M,a)_k|$ is $\frac{1}{2}(g-4)$-connected.*

The proof of this theorem will require several intermediate constructions.

## 8.1. The Complex of $\langle k \rangle$-Embeddings

Fix integers $n, k \geq 2$. Let $M$ be a manifold of dimension $(4n+1)$ with non-empty boundary. Consider transversal $\langle k \rangle$-embeddings

$$\varphi^0, \varphi^1 : V_k^{2n+1} \longrightarrow M$$

such that the transverse pull-back $\varphi^0 \pitchfork \varphi^1$ is diffeomorphic to $A_k$ as a $\langle k,k \rangle$-manifold. It follows that $\varphi^0(V_k^{2n+1}) \cap \varphi^1(V_k^{2n+1}) \cong \widehat{A}_k$, where $\widehat{A}_k$ is the singular space obtained from $A_k$ as in Definition 6.1.2. It will be useful to have an abstract model for the space given by the union, $\varphi^0(V_k^{2n+1}) \cup \varphi^1(V_k^{2n+1})$.

**Construction 8.1.1.** To begin the construction, fix a point $y \in \mathrm{Int}(V_k^{2n+1})$. For $i = 1, \ldots, k$, let $\partial_1^i V_k^{2n+1}$ denote the component of the boundary given by $\Phi^{-1}(\beta V_k^{2n+1} \times \{i\})$, where $\langle k \rangle = \{1, \ldots, k\}$. Let $\bar{\Phi} : \partial_1 V_k^{2n+1} \longrightarrow \beta V_k^{2n+1}$ be the map used in Definition 5.1.2.

i. For $i = 1, \ldots, k$, fix points $x_i \in \partial_1^i V_k^{2n+1}$ such that, $\bar{\Phi}(x_1) = \cdots = \bar{\Phi}(x_k)$.

ii. For $i = 1, \ldots, k$, choose embeddings $\gamma_i : [0,1] \longrightarrow V_k^{2n+1}$ such that

$$\gamma_i(0) = x_i, \qquad \gamma_i^{-1}(\partial_1 V_k^{2n+1}) = \{0\}, \qquad \text{and} \qquad \gamma_i(1) = y.$$

Then for each $i$, let $\bar{\gamma}_i : [0,1] \longrightarrow V_k^{2n+1}$ be the embedding given by the formula

$$\bar{\gamma}_i(t) = \gamma(1-t).$$

iii. Recall that $A_k = [0,1] \times \langle k \rangle = \sqcup_{i=1}^k [0,1]$. The maps

$$\sqcup_{i=1}^k \gamma_i : A_k \longrightarrow V_k^{2n+1} \quad \text{and} \quad \sqcup_{i=1}^k \bar{\gamma}_i : A_k \longrightarrow V_k^{2n+1},$$

yield embeddings

$$\Gamma : \widehat{A}_k \longrightarrow \widehat{V}_k^{2n+1} \quad \text{and} \quad \bar{\Gamma} : \widehat{A}_k \longrightarrow \widehat{V}_k^{2n+1}.$$

iv. We define $Y_k^{2n+1}$ to be the space obtained by forming the push-out of the diagram,

$$\begin{array}{ccc} & \widehat{A}_k & \\ {\scriptstyle \Gamma}\swarrow & & \searrow{\scriptstyle \bar{\Gamma}} \\ \widehat{V}_k^{2n+1} & & \widehat{V}_k^{2n+1} \end{array} \tag{8.1}$$

v. By applying the *Mayer-Vietoris sequence* and *Van Kampen's theorem* we compute,

$$H_s(Y_k^{2n+1};\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k \oplus \mathbb{Z}/k & \text{if } s = 2n, \\ \mathbb{Z}^{\oplus(k-1)} & \text{if } s = 1, \\ \mathbb{Z} & \textit{if } s = 0, \end{cases} \qquad \pi_1(Y_k) \cong \mathbb{Z}^{\star(k-1)},$$

where $\mathbb{Z}^{\star(k-1)}$ denotes the free-group on $(k-1)$-generators.

The next proposition follows easily by inspection.

**Proposition 8.1.1.** *Let $\varphi^0, \varphi^1 : V_k^{2n+1} \longrightarrow M$ be transversal $\langle k \rangle$-embeddings such that the pull-back is diffeomorphic to $A_k$ as a $\langle k, k \rangle$-manifold. Then the union $\varphi^0(V_k^{2n+1}) \cup \varphi^1(V_k^{2n+1})$ is homeomorphic to the space $Y_k^{2n+1}$.*

**Notation 8.1.1.** *Let $\varphi = (\varphi^0, \varphi^1)$ be a pair of $\langle k \rangle$-embeddings $\varphi^0, \varphi^1 : V_k^{2n+1} \longrightarrow M$ such that the transverse pull-back is diffeomorphic to $A_k$ as a $\langle k, k \rangle$-manifold. We will denote by $Y_k(\varphi^0, \varphi^1)$ the subspace of $M$ given by the union $\varphi^0(V_k^{2n+1}) \cup \varphi^1(V_k^{2n+1})$.*

We now define a simplicial complex based on pairs of $\langle k \rangle$-embeddings, $V_k^{2n+1} \to M$ as above.

**Definition 8.1.1.** Let $M$ and $k$ be as above. Let $K(M)_k$ be the simplicial complex with vertex set given by the set of all pairs $(\varphi^0, \varphi^1)$ of transverse $\langle k \rangle$-embeddings

$$\varphi^0, \varphi^1 : V_k^{2n+1} \longrightarrow M$$

such that the transverse pull-back is diffeomorphic to $A_k$ as a $\langle k, k\rangle$-manifold. A set

$$\{(\varphi_0^0, \varphi_0^1) \ldots, (\varphi_p^0, \varphi_p^1)\}$$

of vertices forms a $p$-simplex if $Y_k(\varphi_i^0, \varphi_i^1) \cap Y_k(\varphi_j^0, \varphi_j^1) = \emptyset$ whenever $i \neq j$.

Now, recall from Section 4.2, the simplicial complex $L(\mathbf{M})_k$ associated to an object $\mathbf{M}$ of the category $\mathcal{L}_{-1}$ of anti-symmetric linking forms. We will need to compare the simplical complex $K(M)_k$ to the simplicial complex $L(\pi_{2n}^\tau(M))_k$, where $(\pi_{2n}^\tau(M),\ b)$ is the homotopical linking form associated to $M$, see (3.6). We construct a simplicial map

$$F : K(M)_k \longrightarrow L(\pi_{2n}^\tau(M))_k \tag{8.2}$$

as follows. For a vertex $\varphi = (\varphi^0, \varphi^1) \in K(M)_k$, let $\langle [\varphi_\beta^0], [\varphi_\beta^1]\rangle \leq \pi_{2n}^\tau(M)$ denote the subgroup generated by the homotopy classes determined by the embeddings $\varphi_\beta^\nu : S^{2n} \to M$ for $\nu = 0, 1$. The classes $[\varphi_\beta^\nu]$, $\nu = 0, 1$ each have order $k$ and $b([\varphi_\beta^0], [\varphi_\beta^1]) = \frac{1}{k}$. It follows that the sub-linking form given by $\langle [\varphi_\beta^0], [\varphi_\beta^1]\rangle \leq \pi_{2n}^\tau(M)$ is isomorphic to the standard non-singular linking form $\mathbf{W}_k$ from Definition 3.1.1. The map $F$ from (8.2), is then defined by sending a vertex $\varphi$ to the morphism of linking forms $\mathbf{W}_k \to \pi_{2n}^\tau(M)$ determined by

$$\rho \mapsto [\varphi_\beta^0], \quad \sigma \mapsto [\varphi_\beta^1],$$

where $\rho$ and $\sigma$ are the standard generators of $\mathbf{W}_k$. The disjointness condition from condition ii. of Definition 8.1.1, implies that this formula preserves all adjacencies and thus yields a well defined simplicial map. It follows easily that for any $(4n+1)$-

dimensional manifold $M$ and integer $k \geq 2$ that

$$r_k(\pi^\tau_{2n}(M)) \geq r_k(M) \tag{8.3}$$

where recall, $r_k(\pi^\tau_{2n}(M))$ is the $k$-rank of the linking form $(\pi^\tau_{2n}(M), b)$ as defined in Definition 4.3.1, and $r_k(M)$ is the $k$-rank of the manifold $M$ as defined in the introduction.

**Lemma 8.1.2.** *Let $n, k \geq 2$ be integers with $k$ odd. Let $M$ be a 2-connected manifold of dimension $4n+1$. Then the geometric realization $|K(M)_k|$ is $\frac{1}{2}(r_k(M)-4)$-connected and $lCM(K(M)_k) \geq \frac{1}{2}(r_k(M)-1)$.*

*Proof.* Let $r_k(M) \geq g$. Since $L(\pi^\tau_{2n}(M))_k$ is $\frac{1}{2}(g-4)$-connected and $lCM(L(\pi^\tau_{2n}(M))_k) \geq \frac{1}{2}(g-1)$, the proof of the lemma will follow directly from Corollary 2.1.4 once we verify two things:

i. the map $F$ has the link lifting property (see Definition 2.1.3),

ii. $F(\mathrm{lk}_{K(M)_k}(\zeta)) \leq \mathrm{lk}_{L(\pi^\tau_{2n}(M))_k}(F(\zeta))$ for any simplex $\zeta \in K(M)_k$.

Property i. is proven by applying Corollary 5.4.2, Corollary 10.5.4, and Theorem 7.4.1 as follows. Let $f : \mathbf{W}_k \longrightarrow \pi^\tau_{2n}(M)$ be a morphism of linking forms (which determines a vertex in $L(\pi^\tau_{2n}(M))_k$). Let $\rho, \sigma \in \mathbf{W}_k$ denote the standard generators as defined in Section **??**. The elements $f(\rho), f(\sigma) \in \pi^\tau_{2n}(M)$ have order $k$ and thus by Corollary 5.4.2 we may choose $\langle k \rangle$-embeddings $\varphi^0, \varphi^1 : V^{2n+1}_k \longrightarrow M$ such that $[\varphi^0_\beta] = f(\rho)$ and $[\varphi^1_\beta] = f(\sigma)$. Furthermore, since $b(f(\rho), f(\sigma)) = \frac{1}{k} \mod 1$, it follows from Proposition 7.5.1 that,

$$\Lambda^1_{k,k}([\varphi^0], [\varphi^1]) = [+A_k] \in \Omega^{SO}_1(\mathrm{pt.})_{\langle k,k \rangle}.$$

We then may apply Corollary 10.5.4 (or Corollary 7.4.2) so as to obtain an isotopy of $\varphi^0$ through $\langle k\rangle$-embeddings to a $\langle k\rangle$-embedding $\bar{\varphi}^0$, so that $\bar{\varphi}^0 \pitchfork \varphi^1 \cong A_k$. The pair $(\bar{\varphi}^0, \varphi^1)$ determines a vertex in $K(M)_k$ and clearly $F((\bar{\varphi}^0, \varphi^1)) = f$. This shows how to lift any vertex $v \in L(\pi^\tau_{2n}(M))_k$ to a vertex $\hat{v} \in K(M)_k$ such that $F(\hat{v}) = v$.

Now let $f : \mathbf{W}_k \to \pi^\tau_{2n}(M)$ be a morphism representing a vertex of $L(\pi^\tau_{2n}(M))_k$, let $f_1, \ldots, f_m$ an arbitrary set of vertices in $L(\pi^\tau_{2n}(M))_k$ that are adjacent to $f$, and let

$$(\varphi^0_1, \varphi^1_1), \ldots, (\varphi^0_m, \varphi^1_m)$$

be vertices in $K(M)_k$ with $F((\varphi^0_i, \varphi^1_i)) = f_i$, for $i = 1, \ldots, m$. To show that $F$ has the link lifting property, it will suffice to construct a vertex $(\bar{\varphi}^0, \bar{\varphi}^1) \in K(M)_k$ mapping to $f$, such that $(\bar{\varphi}^0, \bar{\varphi}^1)$ is adjacent to $(\varphi^0_i, \varphi^1_i)$ for $i = 1, \ldots, m$.

For each $i$ and $\nu = 0, 1$, let $\varphi^\nu_{\beta,i} : S^{2n} = \beta V^{2n+1}_k \longrightarrow M$ denote the map associated to $\varphi^\nu_i$ and let $[\varphi^\nu_{\beta,i}]$ denote the associated class in $\pi_{2n}(M)$. For $i = 1, \ldots, m$ we have:

$$b(f(\rho),\ [\varphi^\nu_{\beta,i}]) = b(f(\sigma),\ [\varphi^\nu_{\beta,i}]) = 0 \quad \text{for } \nu = 0, 1. \tag{8.4}$$

By Corollary 5.4.2, we may choose $\langle k\rangle$-embeddings $\varphi^0, \varphi^1 : V^{2n+1}_k \longrightarrow M$ with $[\varphi^0_\beta] = f(\sigma)$ and $[\varphi^1_\beta] = f(\rho)$. By (8.4) we may inductively apply Corollary 7.5.2 (or Theorem 7.4.1) to find isotopies of $\varphi^0$ and $\varphi^1$ (through $\langle k\rangle$-embeddings) to new $\langle k\rangle$-embeddings

$$\bar{\varphi}^0, \bar{\varphi}^1 : V^{2n+1}_k \longrightarrow M$$

such that:

(a) $Y_k(\varphi_i^0, \varphi_i^1) \cap Y_k(\bar{\varphi}^0, \bar{\varphi}^1) = \emptyset$ for $i = 1, \ldots m$,

(b) $\bar{\varphi}^0 \pitchfork \bar{\varphi}^1 \cong A_k$.

This proves that $F$ has the link lifting property.

The fact that $F(\mathrm{lk}_{K(M)_k}(\zeta)) \leq \mathrm{lk}_{L(\pi^\tau_{2n}(M))_k}(F(\zeta))$ for any simplex $\zeta \in K(M)_k$ follows immediately from the fact that if $\phi, \psi : V_k^{2n+1} \longrightarrow M$ are disjoint $\langle k \rangle$-embeddings, then $b([\phi_\beta], [\psi_\beta]) = 0$. This establishes property ii. and completes the proof of the Lemma. □

## 8.2. A Modification of $K(M)_k$

Let $(\varphi^0, \varphi^1)$ be a vertex of $K(M)_k$ and consider the subspace $Y_k(\varphi^0, \varphi^1) \subset M$. We will need to make a further modification of $Y_k(\varphi^0, \varphi^1)$ as follows.

**Construction 8.2.1.** Let $(\varphi^0, \varphi^1)$ be as above. Since $2 < \dim(M)/2$, we may choose an embedding

$$G : (\sqcup_{i=1}^{k-1} D_i^2,\ \sqcup_{i=1}^{k-1} S_i^1) \longrightarrow (M,\ Y_k(\varphi^0, \varphi^1)) \tag{8.5}$$

which satisfies the following conditions:

(a)
$$G(\sqcup_{i=1}^{k-1} \mathrm{Int}(D_i^2)) \bigcap Y_k(\varphi^0, \varphi^1) \ = \ \emptyset.$$

(b) The maps
$$G|_{S_i^1} : S^1 \longrightarrow Y_k(\varphi^0, \varphi^1) \qquad \text{for } i = 1, \ldots, k-1,$$
represent a minimal set of generators for $\pi_1(Y_k(\varphi^0, \varphi^1))$, which by Proposition 8.1.1 is the free group on $k-1$ generators.

Given such an embedding $G$ as in (8.5), we denote

$$Y_k^G(\varphi^0, \varphi^1) \ := \ Y_k(\varphi^0, \varphi^1) \bigcup G(\sqcup_{i=1}^{k-1} D_i^2). \tag{8.6}$$

It follows from conditions i. and ii. above that $Y_k^G(\varphi^0, \varphi^1)$ is simply connected and that

$$H_s(Y_k^G(\varphi^0, \varphi^1);\ \mathbb{Z}) = \begin{cases} \mathbb{Z}/k \oplus \mathbb{Z}/k & \text{if } s = 2n, \\ \mathbb{Z} & \text{if } s = 0, \\ 0 & \text{else.} \end{cases}$$

It follows that $Y_k^G(\varphi^0, \varphi^1)$ has the homotopy type of the Moore-space $M(\mathbb{Z}/k \oplus \mathbb{Z}/k, 2n)$ and hence is homotopy equivalent to the manifold $W_k'$. We will think of $Y_k^G(\varphi^0, \varphi^1) \hookrightarrow M$ as being a choice of embedding of the $(2n+1)$-skeleton of $W_k'$ into $M$.

Using the construction given above, we define a modification of the simplicial complex $K(M)_k$. Let $M$ be a $(4n + 1)$-dimensional manifold with non-empty boundary. Let

$$a : [0, \infty) \times \mathbb{R}^{4n} \longrightarrow M$$

be an embedding with $a^{-1}(\partial M) = \{0\} \times \mathbb{R}^{4n}$.

**Definition 8.2.1.** Let $\bar{K}(M, a)_k$ be the simplicial complex whose vertices are given by 4-tuples $(\varphi, G, \gamma, t)$ which satisfy the following conditions:

i. $\varphi = (\varphi^0, \varphi^1)$ is a vertex in $K(M)_k$.

ii. $G : (\sqcup_{i=1}^{k-1} D_i^2,\ \sqcup_{i=1}^{k-1} S_i^1) \longrightarrow (M,\ Y_k(\varphi^0, \varphi^1))$ is an embedding as in Construction (8.2.1).

iii. $t$ is a real number.

iv. $\gamma : [0,1] \longrightarrow M$ is an embedded path which satisfies:

(a) $\gamma^{-1}(Y_k^G(\varphi^0, \varphi^1)) = \{1\}$,

(b) there exists $\epsilon > 0$ such that for $s \in [0, \epsilon)$, the equality

$$\gamma(s) = a(s, te_1) \in [0,1] \times \mathbb{R}^{4n}$$

is satisfied, where $e_1 \in \mathbb{R}^{4n}$ denotes the first basis vector.

A set of vertices $\{(\varphi_0, G_0, \gamma_0, t_0), \ldots, (\varphi_p, G_p, \gamma_p, t_p)\}$ forms a $p$-simplex if and only if

$$\left(\gamma_i([0,1]) \cup Y_k^{G_i}(\varphi_i^0, \varphi_i^1)\right) \bigcap \left(\gamma_j([0,1]) \cup Y_k^{G_j}(\varphi_j^0, \varphi_j^1)\right) = \emptyset \quad \text{whenever } i \neq j.$$

There is a simplicial map

$$\bar{F} : \bar{K}(M,a)_k \longrightarrow K(M)_k, \quad (\varphi, G, \gamma, t) \mapsto \varphi. \tag{8.7}$$

**Proposition 8.2.1.** *Let $n, k \geq 2$ be integers with $k$ odd. Let $M$ be a 2-connected, manifold of dimension $4n+1$ and let $g \in \mathbb{N}$ be such that $r_k(M) \geq g$. Then the geometric realization $|\bar{K}(M)_k|$ is $\frac{1}{2}(g-4)$-connected and $lCM(\bar{K}(M)_k) \geq \frac{1}{2}(g-1)$.*

*Proof.* The proof of this proposition is proven by the same method as Lemma 8.1.2. It is proven by verifying that the map $\bar{F}$ from (8.7) has the link lifting property (Definition 2.1.3) and that it preserves links. Since $|K(M)_k|$ is $\frac{1}{2}(g-4)$-connected and $lCM(K(M)_k) \geq \frac{1}{2}(g-1)$, we then may apply Corollary 2.1.4 to deduce the claim of the proposition.

Let $\varphi = (\varphi^0, \varphi^1)$ be a vertex in $K(M)_k$. Let $(\varphi_1, G_1, \gamma_1, t_1), \ldots, (\varphi_m, G_m, \gamma_m, t_m)$ be vertices in $\bar{K}(M,a)_k$ such that $\varphi_i$ is adjacent to $\varphi$ for $i = 1, \ldots, m$. Since $\dim(M)/2 > 2$, there is no obstruction to choosing an embedding $G$ as in Construction 8.2.1 (with respect to $\varphi$ so as to construct $Y_k^G(\varphi^0, \varphi^1)$) so that the image of $G$ is disjoint from the images of $G_i$ and $\gamma_i$ for all $i$. Furthermore, with $G$ chosen, we may then choose an embedded path $\gamma : [0,1] \longrightarrow M$, connecting $Y_k^G(\varphi^0, \varphi^1)$ to $\partial M$ so as to yield a vertex $(\varphi, G, \gamma, t) \in \bar{K}(M,a)_k$, which maps to $\varphi$ under $\bar{F}$ and is adjacent to $(\varphi_i, G_i, \gamma_i, t_i)$ for all $i$. This proves the fact that $\bar{F}$ has the link lifting property. The fact that $\bar{F}$ preserves links is immediate from the definition of $\bar{F}$. This concludes the proof of the proposition. □

## 8.3. Reconstructing Embeddings

Let $(\varphi, G, \gamma, t)$ be a vertex in $\bar{K}(M,a)_k$. We will need to consider smooth regular neighborhoods of the subspace $Y_k^G(\varphi^0, \varphi^1) \cup \gamma([0,1]) \subset M$. The following lemma identifies the diffeomorphism type of such a regular neighborhood. Recall from Section 4.1 the manifold $\bar{W}_k = W_k' \cup_\alpha ([0,1] \times D^{4n})$ used in the definition of $X_\bullet(M,a)_k$.

**Lemma 8.3.1.** *Let $(\varphi, G, \gamma, t)$ be a vertex in $\bar{K}(M,a)_k$. If $k$ is odd then any closed regular neighborhood $U$ of the subspace $Y_k^G(\varphi^0, \varphi^1) \cup \gamma([0,1]) \subset M$, is diffeomorphic to the manifold $\bar{W}_k = W_k' \cup_\alpha ([0,1] \times D^{4n})$.*

*Proof.* By definition of regular neighborhood, the inclusion map $Y_k^G(\varphi^0, \varphi^1) \hookrightarrow U$ is a homotopy equivalence ($U$ collapses to $Y_k^G(\varphi^0, \varphi^1)$, see (17)). The maps $\varphi_\beta^0, \varphi_\beta^1 : S^{2n} \longrightarrow U$ represent generators for $\pi_{2n}(U)$ and since $\varphi^0 \pitchfork \varphi^1 \cong A_k$, it follows that

$$b([\varphi_\beta^0],\ [\varphi_\beta^1]) = \tfrac{1}{k} \mod 1$$

and hence, the linking form $(\pi^{\tau}_{2n}(U), b)$ is isomorphic to $\mathbf{W}_k$. It follows from Constructions 8.1.1 and 8.2.1 that the regular neighborhood $U$ is $(2n-1)$-connected. Now, $U$ is homotopy equivalent to the Moore-space $M(\mathbb{Z}/k \oplus \mathbb{Z}/k, 2n)$, and so the set of isomorphism classes of $(4n+1)$-dimensional vector bundles over $U$ is in bijective correspondence with the set $[M(\mathbb{Z}/k \oplus \mathbb{Z}/k, 2n),\ \ BSO]$. Since $\pi_{2n}(BSO; \mathbb{Z}/k) = 0$ whenever $k$ is odd, it follows that the tangent bundle $TU \to U$ is trivial and thus $U \in \mathcal{W}^S_{4n+1}$ (i.e. $U$ is stably parallelizable). We will show that the boundary $\partial U$ is diffeomorphic to $S^{4n}$. Once this is demonstrated, it will follow from the classification theorem, Theorem 3.3.2 (and Remark 3.3.1), that $U$ is diffeomorphic to the manifold $W'_k$.

Since $U$ is parallelizable, by (21, Theorem 5.1) it will be enough to show that $\partial U$ is homotopy equivalent to $S^{4n}$. From Constructions 8.1.1, 8.2.1 and the *Universal Coefficient Theorem*, we have

$$H^s(U; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k \oplus \mathbb{Z}/k & \text{if } s = 2n+1, \\ \mathbb{Z} & \text{if } s = 0, \\ 0 & \text{else.} \end{cases}$$

Using *Lefschetz Duality* it then follows that

$$H_s(U, \partial U; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/k \oplus \mathbb{Z}/k & \text{if } s = 2n, \\ 0 & \text{else.} \end{cases}$$

Consider the long exact sequence on homology associated to $(U, \partial U)$. It follows immediately that $\partial U$ is $(2n-2)$-connected and that the long exact sequence reduces

to

$$0 \longrightarrow H_{2n}(\partial U;\mathbb{Z}) \longrightarrow H_{2n}(U;\mathbb{Z}) \longrightarrow H_{2n}(U,\partial U;\mathbb{Z}) \longrightarrow H_{2n-1}(\partial U;\mathbb{Z}) \longrightarrow 0. \tag{8.8}$$

We claim that the map $H_{2n}(U,\mathbb{Z}) \to H_{2n}(U,\partial U;\mathbb{Z})$ is an isomorphism. To see this, consider the commutative diagram

$$\begin{array}{ccc} H_{2n}(U;\mathbb{Z}) & \xrightarrow[\cong]{x\mapsto b(x,\_)} & H^{2n}(U;\mathbb{Q}/\mathbb{Z}) \\ \downarrow & & \downarrow\cong \\ H_{2n}(U,\partial U;\mathbb{Z}) & \xrightarrow{\cong} & H^{2n+1}(U;\mathbb{Z}). \end{array}$$

In the above diagram the bottom-horizontal map is the Leftshetz duality isomorphism, the right vertical map is the boundary homomorphism in the Bockstein exact sequence (which in this case is an isomorphism), and the top-horizontal map $x \mapsto b(x,\_)$ is an isomorphism since the homological linking form $(H_{2n}(U),\ b)$ is non-singular. It follows that the map $H_{2n}(U;\mathbb{Z}) \to H_{2n}(U,\partial U;\mathbb{Z})$ is indeed an isomorphism and it then follows from the exact sequence of (8.8) that $\partial U$ has the same homology type of $S^{4n}$.

To prove that $\partial U$ has the same homotopy type of $S^{4n}$, we must show that $\partial U$ is simply connected. To to this it will suffice to show that $\pi_i(U,\partial U) = 0$ for $i = 1, 2$. For $i = 1, 2$, let $f : (D^i, \partial D^i) \longrightarrow (U, \partial U)$ be a map. Since

$$\dim(U) - \dim(Y_k^G(\varphi^0,\varphi^1)) \geq 3,$$

we may deform $f$ so that its image is disjoint from $Y_k^G(\varphi^0,\varphi^1)$. We then may then find another (strictly smaller) regular neighborhood $U'$ of $Y_k^G(\varphi^0,\varphi^1)$ such that

$U' \subsetneq U$ and $f(D^i) \subset U \setminus U'$. The class $[f] \in \pi_i(U, \partial U)$ is in the image of the map

$$\pi_i(U \setminus \mathrm{Int}(U'),\ \partial U) \longrightarrow \pi_i(U,\ \partial U)$$

induced by inclusion. Using the uniqueness theorem for smooth regular neighborhoods (see (17)), it follows that the manifold $U \setminus \mathrm{Int}(U')$ is an $H$-cobordism from $\partial U$ to $\partial U'$ and so it follows that$\pi_i(U \setminus \mathrm{Int}(U'), \partial U) = 0$. This proves that $[f] = 0$ and thus $\pi_i(U, \partial U) = 0$ since $f$ was arbitrary. It follows by considering the exact sequence on homotopy groups associated to the pair $(U, \partial U)$ that $\partial U$ is simply connected.

Since $\partial U$ is simply connected and has the homology type of a sphere, it follows that $\partial U$ is a homotopy sphere. It then follows from (21, Theorem 5.1) that $\partial U$ is diffeomorphic to $S^{4n}$ since $\partial U$ bounds a parallelizable manifold, namely $U$. This concludes the proof of the lemma. □

We now define a new simplicial complex.

**Definition 8.3.1.** Let $\widehat{K}(M, a)_k$ be the simplicial complex whose vertices are given by triples $(\bar{\varphi}, \Psi, s)$ which satisfy the following conditions:

i. The 4-tuple $\bar{\varphi} = (\varphi, G, \gamma, t)$ is a vertex in $\bar{K}(M, a)_k$.

ii. $s$ is a real number.

ii. $\Psi : \bar{W}_k \times [s, \infty) \longrightarrow M$ is a smooth family of embeddings $\bar{W}_k \hookrightarrow M$ that satisfies the following:

   (a) for each $t \in [s, \infty)$, the embedding $\Psi(\_,\ t) : \bar{W}_k \longrightarrow M$ is an element of $X_0(M, a)_k$,

(b) $Y^G(\varphi^0, \varphi^1) \cup \gamma([0,1]) \subset \Psi(\bar{W}_k, t)$ for all $t \in [s, \infty)$,

(c) for any neighborhood $U$ of $Y^G(\varphi^0, \varphi^1) \cup \gamma([0,1])$, there is $t_U \in [s, \infty)$ such that $\Psi(\bar{W}_k, t) \subset U$ when $t \geq t_U$.

A set of vertices $\{(\bar{\varphi}_0, \Psi_0, s_0), \ldots, (\bar{\varphi}_p, \Psi_p, s_p)\}$ forms a $p$-simplex if the associated set $\{\bar{\varphi}_0, \ldots, \bar{\varphi}_p\}$ is a $p$-simplex in the complex $\bar{K}(M,a)_k$ (no extra pairwise condition on the $\Psi_i$ and $s_i$ are required).

By construction of $\widehat{K}(M,a)_k$, there is a simplicial map,

$$\widehat{F} : \widehat{K}(M,a)_k \longrightarrow \bar{K}(M,a)_k, \quad (\bar{\varphi}, \Psi, s) \mapsto \bar{\varphi}. \tag{8.9}$$

**Proposition 8.3.2.** *Let $n, k \geq 2$ be integers with $k$ odd. Let $M$ be a compact, 2-connected, manifold of dimension $4n+1$. Let $g \in \mathbb{N}$ be such that $r_k(M) \geq g$. Then the geometric realization $|\widehat{K}(M)_k|$ is $\frac{1}{2}(g-4)$-connected and $lCM(\bar{K}(M)_k) \geq \frac{1}{2}(g-1)$.*

*Proof.* The proof of this proposition again follows the same strategy as Lemma 8.1.2. We check that the map $\widehat{F}$ has the cone lifting property, that it preserves links, and then we apply Corollary 2.1.4.

The fact that $\widehat{F}$ preserves links follows immediately from the definition. We will verify the link lifting property. Let $\bar{\varphi} = (\varphi, G, \gamma, t)$ be a vertex of $\bar{K}(M)_k$. Let

$$(\bar{\varphi}_1, \Psi_1, s_1), \ldots, (\bar{\varphi}_m, \Psi_m, s_m)$$

be a collection of vertices in $\widehat{K}(M)_k$ such that $\bar{\varphi}$ is adjacent to $\bar{\varphi}_i$ in $\bar{K}(M)_k$ for $i = 1, \ldots, m$. We will denote

$$Y_k(\bar{\varphi}) := Y_k^G(\varphi^0, \varphi^1) \cup \gamma([0,1]). \tag{8.10}$$

Let $U \subset M$ be a regular neighborhood of $Y_k(\bar{\varphi})$. Since $U$ collapses to $Y_k(\bar{\varphi})$ (by definition of regular neighborhood), we may choose a one-parameter family of embeddings:

$$\rho : U \times [s, \infty) \longrightarrow U \tag{8.11}$$

which satisfies the following:

i. For all $t \in [s, \infty)$, the embedding $\rho_t = \rho|_{U \times \{t\}} : U \to U$ is the identity on $Y_k(\bar{\varphi})$.

ii. Given any neighborhood $U' \subset U$ of $Y_k(\bar{\varphi})$, there exists $t' > s$ such that $\rho_t(U) \subset U'$ for all $t \geq t'$.

We call such an isotopy a *compression isotopy* of $U$ to $Y_k(\bar{\varphi})$. By Lemma 8.3.1, there exists a diffeomorphism $\Psi : \bar{W}_k \xrightarrow{\cong} U$ such that the composition $\bar{W}_k \xrightarrow{\Psi} U \hookrightarrow M$ satisfies the conditions of Definition 8.2.1. It then follows that the triple $(\bar{\varphi},\ \Psi \circ \rho,\ s)$ is a vertex of $\widehat{K}(M, a)_k$ that maps to $\bar{\varphi}$ under $\widehat{F}$. It follows from the definition of $\widehat{K}(M)_k$ that $(\bar{\varphi},\ \Psi \circ \rho,\ s)$ is automatically adjacent to $(\bar{\varphi}_i, \Psi_i, s_i)$ for $i = 1, \ldots, m$. This proves that $\widehat{F}$ has the link lifting property. This completes the proof of the proposition. $\square$

## 8.4. Comparison with $X_\bullet(M,a)_k$

We are now in a position to finally prove Theorem 8.0.3 by comparing $|X_\bullet(M,a)_k|$ to $|\widehat{K}(M,a)_k|$. We will need to construct an auxiliary semi-simplicial space related to the simplicial complex $\widehat{K}(M,a)_k$. Let $M$ be a $(4n+1)$-dimensional manifold with non-empty boundary and let $a : [0,\infty)\times\mathbb{R}^{4n} \longrightarrow M$ be an embedding as used in Definition 8.2.1. We define two semi simplicial spaces $\widehat{K}_\bullet(M,a)_k$ and $\widehat{K}'_\bullet(M,a)_k$.

**Definition 8.4.1.** The space of $p$-simplices $\widehat{K}_p(M,a)_k$ is defined as follows:

i. The space of 0-simplices $\widehat{K}_0(M,a)_k$ is defined to have the same underlying set as the set of vertices of the simplicial complex $\widehat{K}(M,a)_k$.

ii. The space of $p$-simplices $\widehat{K}_p(M,a)_k \subset (\widehat{K}_0(M,a)_k)^{\times(p+1)}$ consists of the ordered $(p+1)$-tuples $((\bar{\varphi}_0,\Psi_0,s_0),\cdots,(\bar{\varphi}_p,\Psi_p,s_p))$ such that the associated unordered set

$$\{(\bar{\varphi}_0,\Psi_0,s_0),\cdots,(\bar{\varphi}_p,\Psi_p,s_p)\}$$

is a $p$-simplex in the simplicial complex $\widehat{K}(M,a)_k$.

The spaces $\widehat{K}_p(M,a)_k$ are topologized using the $C^\infty$-topology on the spaces of embeddings. The assignments $[p] \mapsto \widehat{K}_p(M,a)_k$ define a semi-simplicial space which we denote by $\widehat{K}_\bullet(M,a)_k$.

Finally, $\widehat{K}'_\bullet(M,a)_k \subset \widehat{K}_\bullet(M,a)_k$ is defined to be the sub-semi-simplicial space consisting of all $(p+1)$-tuples $((\bar{\varphi}_0,\Psi_0,s_0),\cdots,(\bar{\varphi}_p,\Psi_p,s_p))$ such that $\Psi_i(\bar{W}_k)\cap\Psi_j(\bar{W}_k)=\emptyset$ whenever $i\neq j$.

It is easily verified that both $\widehat{K}_\bullet(M,a)_k$ and $\widehat{K}'_\bullet(M,a)_k$ are topological flag complexes.

**Proposition 8.4.1.** *Let $k, n \geq 2$ be integers with $k$ odd. Let $M$ be a 2-connected $(4n+1)$-dimensional manifold and let $g \geq 0$ be such that $r_k(M) \geq g$. Then the geometric realization $|\widehat{K}_\bullet(M,a)_k|$ is $\frac{1}{2}(g-4)$-connected.*

*Proof.* Let $\widehat{K}_\bullet(M,a)^\delta_k$ denote the the *discretization* of $\widehat{K}_\bullet(M,a)_k$ as defined in Definition 2.2.3. Consider the map

$$|\widehat{K}_\bullet(M,a)^\delta_k| \longrightarrow |\widehat{K}(M,a)_k| \tag{8.12}$$

induced by sending an ordered list $((\bar{\varphi}_0, \Psi_0, s_0), \cdots, (\bar{\varphi}_p, \Psi_p, s_p))$ to its associated underlying set. For any such set $\{(\bar{\varphi}_0, \Psi_0, s_0), \cdots, (\bar{\varphi}_p, \Psi_p, s_p)\}$ which forms a $p$-simplex in $\widehat{K}(M,a)^\delta_k$, there is only one possible ordering on it which yields an element of $\widehat{K}_\bullet(M,a)^\delta_k$. Thus the map (8.12) is a homeomorphism. By Proposition 8.3.2, it follows that $\widehat{K}_\bullet(M,a)^\delta_k$ (which is clearly a topological flag-complex) is weakly Cohen-Macaulay of dimension $\frac{1}{2}(g-2)$, as defined in Definition 2.2.2. It then follows from Theorem 2.2.1 that $|\widehat{K}_\bullet(M,a)_k|$ is $\frac{1}{2}(g-4)$-connected. □

We now consider the inclusion map $\widehat{K}'_\bullet(M,a)_k \longrightarrow \widehat{K}_\bullet(M,a)_k$.

**Proposition 8.4.2.** *For any $(4n+1)$-dimensional manifold $M$ with non-empty boundary, the map $|\widehat{K}'_\bullet(M,a)_k| \longrightarrow |\widehat{K}_\bullet(M,a)_k|$ induced by inclusion is a weak homotopy equivalence.*

*Proof.* For $p \geq 0$, let

$$x \mapsto ((\bar{\varphi}^x_0, \Psi^x_0, s^x_0), \cdots, (\bar{\varphi}^x_p, \Psi^x_p, s^x_p)) \quad \text{for } x \in D^j \tag{8.13}$$

represent an element of the relative homotopy group

$$\pi_j\left(\widehat{K}_p(M,a)_k,\ \widehat{K}'_p(M,a)_k\right) = 0. \tag{8.14}$$

For each $x$, $Y_k(\bar{\varphi}_i^x) \bigcap Y_k(\bar{\varphi}_j^x) = \emptyset$ whenever $i \neq j$. Using condition (c) in Definition 8.3.1, since $D^j$ is compact we may choose a real number $s \geq \max\{s_i^x \mid i = 0, \ldots, p, \text{ and } x \in D^j\}$, such that for any $x \in D^j$,

$$\Psi_i^x(\bar{W}_k,\ t) \cap \Psi_j^x(\bar{W}_k,\ t) = \emptyset \quad \text{whenever } t \geq s \text{ and } i \neq j.$$

For each $x \in D^j$, $t \in [0,1]$, and $i = 0, \ldots, p$, let $s_i^x(t)$ denote the real number given by the sum

$$(1-t) \cdot s_i^x + t \cdot s$$

and let $\Psi_i^x(t)$ denote the restriction of $\Psi_i^x$ to $\bar{W}_k \times [s_i^x(t),\ \infty)$. The formula,

$$(x,t) \ \mapsto\ ((\bar{\varphi}_0^x,\ \Psi_0^x(t),\ s_0^x(t)),\ \cdots,\ (\bar{\varphi}_p^x,\ \Psi_p^x(t),\ s_p^x(t))) \qquad \text{for } t \in [0,1]$$

yields a homotopy from the map defined in (8.13) to a map which represents the trivial element in the relative homotopy group (8.14). This implies that for all $p, j \geq 0$, the relative homotopy group (8.14) is trivial and thus the inclusion $\widehat{K}'_p(M,a)_k \longrightarrow \widehat{K}_p(M,a)_k$ is a weak homotopy equivalence for all $p$. It follows that the induced map $|\widehat{K}'_\bullet(M,a)_k| \longrightarrow |\widehat{K}_\bullet(M,a)_k|$ is a weak homotopy equivalence. $\square$

Finally, we consider the map

$$\widehat{K}'_\bullet(M,a)_k \longrightarrow X_\bullet(M,a)_k, \quad (\bar{\varphi}, \Psi, s) \mapsto \Psi_s = \Psi|_{\bar{W}_k \times \{s\}}. \tag{8.15}$$

The following proposition implies Theorem 8.0.3.

**Proposition 8.4.3.** *Let $n \geq 2$ and suppose that $k > 2$ is an odd integer. Then for any $(4n+1)$-dimensional manifold $M$ with non-empty boundary, the degree of connectivity of $|X_\bullet(M,a)_k|$ is bounded below by the degree of connectivity of $|\widehat{K}'_\bullet(M,a)_k|$.*

*Proof.* To prove the proposition it will suffice to construct a section of the map (8.15). The existence of such a section implies that the map on homotopy groups induced by (8.15) is a surjection. The result then follows. Let $x, y \in \pi^\tau_{2n}(\bar{W}_k)$ be two generators such that $b(x,y) = \frac{1}{k} \mod 1$. By combining Corollary 5.4.2 and Corollary 10.5.4, we may choose $\langle k \rangle$-embeddings $\varphi^0, \varphi^1 : V_k^{2n+1} \longrightarrow M$ such that

$$[\varphi^0_\beta] = x, \quad [\varphi^1_\beta] = y, \quad \text{and} \quad \varphi^0 \pitchfork \varphi^1 \cong A_k.$$

We then may apply Construction 8.2.1 to obtain a vertex $\bar{\varphi} = (\varphi, G, \gamma, t) \in \bar{K}(\bar{W}_k, a)_k$. Now, the whole manifold $\bar{W}_k$ is a regular neighborhood for $Y_k(\bar{\varphi})$. We may choose a compression isotopy $\rho : \bar{W}_k \times [0,\infty) \longrightarrow \bar{W}_k$ of $\bar{W}_k$ to $Y_k(\bar{\varphi})$ as in (8.11) and which satisfies the same conditions associated to the isotopy (8.11). It follows that $(\bar{\varphi}, \rho, 0)$ is an element of $\widehat{K}'_0(\bar{W}_k, a)_k$. Using $\bar{\varphi}$ and the compression isotopy $\rho$, we then define a simplicial map

$$X_\bullet(M,a)_k \longrightarrow \widehat{K}'_\bullet(M,a)_k, \quad \Psi \mapsto (\Psi \circ \bar{\varphi},\ \Psi \circ \rho,\ 0), \tag{8.16}$$

where $\Psi \circ \bar{\varphi}$ is the vertex in $\bar{K}(M,a)_k$ given by the 4-tuple, $((\Psi \circ \varphi^0,\ \Psi \circ \varphi^1),\ \Psi \circ G,\ \Psi \circ \gamma,\ t)$. It follows that this map is a section of (8.15). □

# CHAPTER IX

# HOMOLOGICAL STABILITY

With our main technical result Theorem 8.0.3 established, in this section we show how Theorem 8.0.3 implies the main result of the paper which is Theorem 1.3.2.

## 9.1. A Model for $\mathrm{BDiff}^{\partial}(M)$

Let $M$ be a compact manifold of dimension $m$ with non-empty boundary. We now construct a concrete model for $\mathrm{BDiff}^{\partial}(M)$. Fix a collar embedding,

$$h : [0, \infty) \times \partial M \longrightarrow M$$

with $h^{-1}(\partial M) = \{0\} \times \partial M$. Fix once and for all an embedding, $\theta : \partial M \longrightarrow \mathbb{R}^{\infty}$ and let $S$ denote the submanifold $\theta(\partial M) \subset \mathbb{R}^{\infty}$.

**Definition 9.1.1.** We define $\mathcal{M}(M)$ to be the set of compact $m$-dimensional submanifolds $M' \subset [0, \infty) \times \mathbb{R}^{\infty}$ that satisfy:

i. $M' \cap (\{0\} \times \mathbb{R}^{\infty}) = S$ and $M'$ contains $[0, \epsilon) \times S$ for some $\epsilon > 0$.

ii. The boundary of $M'$ is precisely $\{0\} \times S$.

iii. $M'$ is diffeomorphic to $M$ relative to $S$.

Denote by $\mathcal{E}(M)$ the space of embeddings $\psi : M \to [0, \infty) \times \mathbb{R}^{\infty}$ for which there exists $\epsilon > 0$ such that $\psi \circ h(t, x) = (t, \theta(x))$ for all $(t, x) \in [0, \epsilon) \times \partial M$. The space $\mathcal{M}(M)$ is topologized as a quotient of the space $\mathcal{E}(M)$ where two embeddings are identified if they have the same image.

It follows from Definition 9.1.1 that $\mathcal{M}(M)$ is equal to the orbit space, $\mathcal{E}(M)/\operatorname{Diff}^{\partial}(M)$. By the main result of (4), the quotient map, $\mathcal{E}(M) \longrightarrow \mathcal{E}(M)/\operatorname{Diff}^{\partial}(M) = \mathcal{M}(M)$ is a locally trivial fibre-bundle. This together with the fact that $\mathcal{E}(M)$ is weakly contractible implies that there is a weak-homotopy equivalence, $\mathcal{M}(M) \simeq \operatorname{BDiff}^{\partial}(M)$.

Now suppose that $m = 4n + 1$ with $n \geq 2$. Let $k \geq 2$ be an integer. Recall from Section I the manifold $\widetilde{W}_k$, given by forming the connected sum of $[0,1] \times \partial M$ with $W_k$. Choose a collared embedding $\alpha : \widetilde{W}_k \longrightarrow [0,1] \times \mathbb{R}^{\infty}$ such that for $(i,x) \in \{0,1\} \times \partial M \subset \widetilde{W}_k$, the equation $\alpha(i,x) = (i,\theta(x))$ is satisfied. For any submanifold $M' \subset [0,\infty) \times \mathbb{R}^{\infty}$, denote by $M' + e_1 \subset [1,\infty) \times \mathbb{R}^{\infty}$ the submanifold obtained by linearly translating $M'$ over 1-unit in the first coordinate. Then for $M' \in \mathcal{M}(M)$, the submanifold $\alpha(\widetilde{W}_k) \cup (M' \cup e_1) \subset [0,\infty) \times \mathbb{R}^{\infty}$ is an element of $\mathcal{M}(M \cup_{\partial M} \widetilde{W}_k)$. Thus, we have a continuous map,

$$s_k : \mathcal{M}(M) \longrightarrow \mathcal{M}(M \cup_{\partial M} \widetilde{W}_k); \quad V \mapsto \alpha(\widetilde{W}_k) \cup (V + e_1). \tag{9.1}$$

As in the introduction, we will refer to this map as the *kth-stabilization map.*

**Remark 9.1.1.** The construction of the stabilization map $s_k$ depends on the choice of embedding $\alpha : \tilde{W}_k \to [0,1] \times \mathbb{R}^{\infty}$. However, any two such embeddings are isotopic (the space of all such embeddings is weakly contractible). It follows that the homotopy class of $s_k$ does not depend on any of the choices made. In this way, the manifold $\tilde{W}_k$ determines a unique homotopy class of maps $\operatorname{BDiff}^{\partial}(M) \longrightarrow \operatorname{BDiff}^{\partial}(M \cup_{\partial M} \tilde{W}_k)$ which is in the same homotopy class as the map (1.8) used in the statement of Theorem 1.3.2.

## 9.2. A Semi-Simplicial Resolution

Let $M$ be as in Section 9.1. For each positive integer $K$, we construct a semi-simplicial space $Z_\bullet(M)_k$, equipped with an augmentation $\epsilon_k : Z_\bullet(M)_k \longrightarrow \mathcal{M}(M)$ such that the induced map $|Z_\bullet(M)_k| \longrightarrow \mathcal{M}(M)$ is highly connected. Such an augmented semi-simplicial space is called a *semi-simplicial resolution*.

Let $\theta : \partial M \hookrightarrow \mathbb{R}^\infty$ be the embedding used in the construction of $\mathcal{M}(M)$. Pick once and for all a coordinate patch $c_0 : \mathbb{R}^{m-1} \longrightarrow S = \theta(\partial M)$. This choice of coordinate patch induces for any $M' \in \mathcal{M}(M)$, a germ of an embedding $[0,1) \times \mathbb{R}^{m-1} \longrightarrow M'$ as used in the construction of the semi-simplicial space $\bar{K}_\bullet(M')_k$ from Definition 4.1.1.

**Definition 9.2.1.** For each non-negative integer $l$, let $Z_l(M)_k$ be the set of pairs $(M', \bar{\phi})$ where $M' \in \mathcal{M}(M)$ and $\bar{\phi} \in Z_l(M')_k$, where $X_l(M')_k$ is defined using the embedding germ

$$[0,1) \times \mathbb{R}^{m-1} \longrightarrow M'$$

induced by the chosen coordinate patch $c_0 : \mathbb{R}^{m-1} \longrightarrow S$. The space $Z_l(M)_k$ is topologized as the quotient, $Z_l(M)_k = (\mathcal{E}(M) \times X_l(M)_k)/\operatorname{Diff}^\partial(M)$. The assignments $[l] \mapsto Z_l(M)_k$ make $Z_\bullet(M)_k$ into a semi-simplicial space where the face maps are induced by the face maps in $X_\bullet(M)_k$.

The projection maps $Z_l(M)_k \longrightarrow \mathcal{M}(M)$ given by $(V, \bar{\phi}) \mapsto V$ yield an augmentation map $\epsilon_k : Z_l(M)_k \longrightarrow \mathcal{M}(M)$. We denote by $Z_{-1}(M)_k$ the space $\mathcal{M}(M)$.

By construction, the projection maps $Z_l(M)_k \to \mathcal{M}(M)$ are locally trivial fibre-bundles with standard fibre given by $X_l(M)_k$. From this we have:

**Corollary 9.2.1.** *The map* $|\epsilon_k| : |Z_l(M)_k| \longrightarrow \mathcal{M}(M)$ *induced by the augmentation is* $\frac{1}{2}(r_k(M) - 2)$*-connected.*

*Proof.* It follows from (31, Lemma 2.1) that there is a homotopy-fibre sequence $|X_l(M)_k| \to |Z_l(M)_k| \to \mathcal{M}(M)$. The result follows from the long-exact sequence on homotopy groups. □

## 9.3. Proof of the Main Theorem

We show how to use the *semi-simplicial resolution* $\epsilon_k : Z_\bullet(M)_k \to \mathcal{M}(M)$ to complete the proof of Theorem 1.3.2. First, we fix some new notation which will make the steps of the proof easier to state.

Let $M$ be a compact $(4n+1)$-dimensional manifold with non-empty boundary. Fix an odd integer $k > 2$ (the integer $k$ will be same throughout the entire section). For each $g \in \mathbb{N}$ we denote by $M_{g,k}$ the manifold obtained by forming the connected-sum of $M$ with $W_k^{\#g}$. Notice that $\partial M = \partial M_{g,k}$ for all $g \in \mathbb{N}$. We consider the spaces $\mathcal{M}(M_{g,k})$. For each $g \in \mathbb{N}$, the stabilization map from (9.1) yields a map,

$$s_k : \mathcal{M}(M_{g,k}) \longrightarrow \mathcal{M}(M_{g+1,k}), \quad M' \mapsto \widetilde{W}_k \cup (M' + e_1).$$

Using the weak equivalence $\mathcal{M}(M_{g,k}) \simeq \mathrm{BDiff}^\partial(M_{g,k})$, Theorem 1.3.2 translates to the following:

**Theorem 9.3.1.** *The induced map* $(s_k)_* : H_l(\mathcal{M}(M_{g,k})) \longrightarrow H_l(\mathcal{M}(M_{g+1,k}))$ *is an isomorphism when* $l \leq \frac{1}{2}(g-3)$ *and is an epimorphism when* $l \leq \frac{1}{2}(g-1)$*.*

We will need to consider the augmented semi-simplicial space $Z_\bullet(M_{g,k})_k \longrightarrow \mathcal{M}(M_{g,k})$ that was constructed in the previous section.

**Notational Convention 9.3.1.** In order to prevent overcrowding in our notation, thought the rest of this section we will drop $k$ from the notation and denote

$$M_g := M_{g,k} \quad \text{and} \quad Z_\bullet(M_g) := Z_\bullet(M_{g,k})_k.$$

Since $k$ is fixed throughout this section, this is not an issue.

Since $r_k(M_g) \geq g$ for $g \in \mathbb{N}$, it follows from Corollary 9.2.1 that the map

$$|\epsilon_k| : |Z_\bullet(M_g)| \longrightarrow Z_{-1}(M_g) := \mathcal{M}(M_g).$$

is $\frac{1}{2}(g-2)$-connected. With this established, the proof of Theorem 9.3.1 proceeds in exactly the same way as in (9, Section 5). We provide an outline for how to complete the proof and refer the reader to (9, Section 5) for details.

For what follows we fix $g \in \mathbb{N}$. For each non-negative integer $l \leq g$ there is a map

$$F_l : \mathcal{M}(M_{g-l-1}) \longrightarrow Z_l(M_g) \tag{9.2}$$

which is defined in exactly the same way as the map from (9, Proposition 5.3). From (9, Proposition 5.3, 5.4 and 5.5) we have the following.

**Proposition 9.3.2.** *Let $g \geq 4$.*

*i. The map $F_l : \mathcal{M}(M_{g-l-1}) \longrightarrow Z_l(M_g)$ is a weak homotopy equivalence.*

*ii. The following diagram is commutative,*

$$\begin{array}{ccc} \mathcal{M}(M_{g-l-1}) & \xrightarrow{s_k} & \mathcal{M}(M_{g-l}) \\ \downarrow{\scriptstyle F_l} & & \downarrow{\scriptstyle F_l} \\ Z_l(M_g) & \xrightarrow{d_l} & Z_{l-1}(M_g). \end{array}$$

*iii. The face maps $d_i : Z_l(M_g) \longrightarrow Z_{l-1}(M_g)$ are weakly homotopic.*

**Remark 9.3.1.** The proof of Proposition 9.3.2 proceeds in the same way as the proofs of (9, Proposition 5.3, 5.4 and 5.5). The key ingredients of this proof are Propositions 4.2.1 and 4.1.2.

We consider the spectral sequence $E^*_{*,*}$ associated to the skeletal filtration of the augmented semi-simplicial space

$$Z_\bullet(M_g) \longrightarrow \mathcal{M}(M_g).$$

This spectral sequence has the following properties:

- The $E^1$-term given by $E^1_{p,q} = H_q(Z_p(M_g))$ for $l \geq -1$ and $j \geq 0$.
- The differential is given by $d^1 = \sum(-1)^i(d_i)_*$, where $(d_i)_*$ is the map on homology induced by the $i$th face map in $Z_\bullet(M_g)$.
- The group $E^\infty_{p,q}$ is a subquotient of the relative homology group $H_{p+q+1}(Z_{-1}(M_g),\ |Z_\bullet(M_g)|)$.

Proposition 9.3.2 together with Corollary 9.2.1 imply the following:

(a) For $g \geq 4 + p$, there are isomorphisms $E^1_{p,q} \cong H_q(\mathcal{M}(M_{g-p-1}))$.

(b) When $p$ is even, the differential $d^1 : E^1_{p,q} \longrightarrow E^1_{p-1,l}$ is equal to $(s_k)_*$ (the map on homology induced by the $k$-th stabilization map) when $p$ is even. It is equal to zero when $p$ is odd.

(c) The term $E^\infty_{p,q}$ is equal to 0 when $p + q \leq \frac{1}{2}(g - 4)$.

We now complete the proof of Theorem 9.3.1 by applying the spectral sequence from (9, Proof of Theorem 1.2). We repeat the argument below for the convenience of the reader.

*Proof of Theorem 9.3.1.* Let $a$ denote the integer $\frac{1}{2}(g-4)$. We will use the above spectral sequence to prove that $H_q(\mathcal{M}_{g-1}) \longrightarrow H_q(\mathcal{M}_g)$ is an isomorphism for $q \leq a$, assuming that we know inductively that for $j > 0$, the stabilization maps $H_q(\mathcal{M}_{g-2j-1}) \longrightarrow H_q(\mathcal{M}_{g-2j})$ are isomorphisms for $q \leq a - j$.

This inductive assumption implies that the differential

$$d^1 : E^1_{2j,q} \longrightarrow E^1_{2j-1,q}$$

is an isomorphism for $0 < j \leq a - q$. Hence, it follows that $E^2_{p,q} = 0$ whenever $0 < p \leq 2(a-q)$. In particular the term $E^2_{p,q}$ vanishes whenever

$$p \geq 1, \quad q \leq a-1, \quad \text{and} \quad p+q \leq a+1.$$

Thus for $r \geq 2$ and $q \leq a$, it follows that the differentials

$$d^r : E^r_{r-1,q-r+1} \longrightarrow E^r_{-1,q} \quad \text{and} \quad d^r : E^r_{r,q-r+1} \longrightarrow E^r_{0,q}$$

both vanish. It follows that for $q \leq a$ we have

$$E^\infty_{0,q} = E^2_{0,q} = \mathrm{Ker}(H_q(\mathcal{M}(M_{g-1})) \to H_q(\mathcal{M}(M_g)),$$

$$E^\infty_{-1,q} = E^2_{-1,q} = \mathrm{Coker}(H_q(\mathcal{M}(M_{g-1})) \to H_q(\mathcal{M}(M_g)).$$

Since the group $E^\infty_{p,q}$ vanishes for $p+q \leq a$ we see that the stabilization map

$$H_q(\mathcal{M}(M_{g-1})) \longrightarrow H_q(\mathcal{M}(M_g))$$

has vanishing kernel and cockerel for $q \leq a$, establishing the induction step. The statement of the theorem is vacuous for $g = 1$ and $g = 2$, which establishes the base case of the induction. A similar argument proves that the stabilization map $H_q(\mathcal{M}(M_{g-1})) \longrightarrow H_q(\mathcal{M}(M_g))$ is an epimorphism when $q = a + 1$. This completes the proof of the theorem. $\square$

# CHAPTER X

# EMBEDDINGS AND DISJUNCTION FOR $\mathbb{Z}/K$-MANIFOLDS

We now develop a technique for modifying the intersections of embedded $\langle k \rangle$-manifolds that will allow us to prove Theorem 7.4.1 stated in Section VII. Recall from Section 7.4 the definition of *diffeotopy.*

**Definition 10.0.1.** Let $M$ be a manifold. We will call a smooth, one parameter family of diffeomorphisms $\Psi_t : M \longrightarrow M$ with $t \in [0,1]$ and $\Psi_0 = Id_M$ a *diffeotopy.* For a subspace $N \subset M$, we say that $\Psi_t$ is a *diffeotopy relative* $N$, and we write $\Psi_t : M \longrightarrow M$ rel $N$, if in addition, $\Psi_t|_N = Id_N$ for all $t \in [0,1]$.

## 10.1. A $\mathbb{Z}/k$ Version of the Whitney Trick

We now discus a certain version of the Whitney trick for $\langle k \rangle$-manifolds. Let $M$ be an oriented manifold of dimension $m$, let $X$ be an oriented manifold of dimension $r$, and let $P$ be an oriented $\langle k \rangle$-manifold of dimension $p$. Suppose that:

- both $P$ and $X$ are path-connected,
- $m \geq 6$,
- $p + r = m$,
- $p, r \geq 2$.

Let

$$\varphi : (X, \partial X) \longrightarrow (M, \partial M) \quad \text{and} \quad f : (P, \partial_0 P) \longrightarrow (M, \partial M) \tag{10.1}$$

be a smooth embedding and a $\langle k \rangle$-embedding respectively, such that

$$\varphi(\partial X) \cap f(\partial_0 P) = \emptyset.$$

We will need to consider the invariant $\Lambda_k^0(f, \varphi; M)$ defined in Section VII. Using the standard identification

$$\Omega_0^{SO}(\text{pt.})_{\langle k \rangle} = \mathbb{Z}/k,$$

the element $\Lambda_k^0(f, \varphi; M)$ is equal to the modulo $k$ reduction of the oriented, algebraic intersection number associated to the intersection of $f(\text{Int}(P))$ and $\varphi(X)$. The following theorem is a version of the classical *Whitney trick* for $\langle k \rangle$-manifolds.

**Theorem 10.1.1.** *Let $f$ and $\varphi$ be exactly as in (10.1) above. Using the identification $\Omega_0^{SO}(pt.)_{\langle k \rangle} = \mathbb{Z}/k$, suppose that*

$$\Lambda_k^0(f, \varphi; M) \ = \ j \quad \mod k.$$

*Then there exists a diffeotopy $\Psi_t : M \longrightarrow M$ $\text{rel}\, \partial M$ such that,*

$$\Psi_1(\varphi(X)) \cap f(\text{Int}(P)) \cong +\langle j \rangle.$$

To prove the above theorem we will need to use the next lemma.

**Lemma 10.1.2.** *Let $P$ be a $\langle k \rangle$-manifold of dimension $p \geq 2$, let $M$ be a smooth manifold of dimension $m \geq 6$, and let $f : (P, \partial_0 P) \longrightarrow (M, \partial M)$ be a $\langle k \rangle$-embedding. Let $r$ denote the integer $m - p$. Given any any positive integer $n$, there exists an embedding*

$$g : S^r \longrightarrow \text{Int}(M \setminus f_{\beta P}(\beta P))$$

*that satisfies the following:*

*i.* $g(S^r) \cap f(\mathrm{Int}(P)) \cong \pm\langle n \cdot k \rangle$,

*ii. the composition* $S^r \xrightarrow{g} \mathrm{Int}(M \setminus f_{\beta P}(\beta P)) \hookrightarrow \mathrm{Int}(M)$ *extends to an embedding*

$$D^{r+1} \hookrightarrow \mathrm{Int}(M).$$

*Proof.* We first prove this explicitly for the case that $n = 1$. So, suppose that $n$ is equal to 1. We construct the embedding $g : S^r \longrightarrow \mathrm{Int}(M)$ in stages as follows.

**Construction 10.1.1.** Let $\bar{\Phi} : \partial_1 P \longrightarrow \beta P$ be the map given by the composition

$$\partial_1 P \xrightarrow[\cong]{\Phi} \beta P \times \langle k \rangle \xrightarrow{\text{proj.}} \beta P.$$

For $i = 1, \ldots, k$, let $\partial_1^i P$ denote the submanifold given by $\Phi^{-1}(\partial_1 P \times \{i\})$.

i. Choose a collar embedding $h : \partial_1 P \times [0, \infty) \longrightarrow P$ such that $h^{-1}(\partial_1 P) = \partial_1 P \times \{0\}$.

ii. Choose a point $y \in \beta P$. For $i = 1, \ldots, k$, let $y_i \in \partial_1^i P$ be the point such that $\bar{\Phi}(y_i) = y$. Then define embeddings

$$\gamma_i : [0, 1] \longrightarrow f(P), \quad \gamma_i(t) = f(h(y_i, t)).$$

It is clear that $\gamma_i(0) = f(y)$ for all $i$. We then denote

$$x_i := \gamma_i(1) \quad \text{for } i = 1, \ldots, k.$$

iii. Choose an embedding $\alpha : D^2 \longrightarrow M$ that satisfies the following conditions:

(a) $\alpha(D^2) \cap f(P) = \sqcup_{i=1}^{k} \gamma_i([0,1])$,

(b) $\alpha(\partial D^2) \cap f(P) = \{x_1, \ldots, x_k\}$,

(c) $\alpha(D^2)$ intersects $f(P)$ orthogonally (with resect to some metric on $M$),

(d) $f(\beta P) \cap \alpha(D^2) \subset \alpha(\mathrm{Int}(D^2))$.

Since $2 < m/2$, there is no obstruction to choosing such an embedding.

iv. Let $r$ denote the integer $m - p$. Choose a $(r-1)$-frame of orthogonal vector fields $(v_1, \ldots, v_{r-1})$ over the embedded disk $\alpha(D^2) \subset M$ with the property that $v_i$ is orthogonal to $\alpha(D^2)$ and orthogonal to $f(P)$ over the intersection $\alpha(D^2) \cap f(P)$, for $i = 1, \ldots, r-1$. Since the disk is contractable, there is no obstruction to the existence of such a frame.

The orthogonal $(r-1)$-frame chosen in step iv. induces an embedding

$$\bar{g} : D^{r+1} \longrightarrow M.$$

The orthogonality condition (condition (c)) in Step iii. of the above construction, together with the orthogonality condition on the frame chosen in step iv., implies that $\bar{g}(D^{r+1})$ is transverse to $f(P)$. Furthermore, condition (b) from step iii. of the above construction implies that

$$g(\partial D^{r+1}) \cap f(\mathrm{Int}(P)) = \{x_1, \ldots, x_k\},$$

and all points on the right-hand side of the equality have the same orientation. We then set the map $g : S^r \longrightarrow M$ equal to the embedding obtained by restricting $\bar{g}$ to the boundary of $D^{r+1}$. This proves the lemma in the case that $n = 1$. To prove

the lemma for general $n$ one simply iterates $n$-times the exact construction given above. $\square$

*Proof of Theorem 10.1.1.* It will suffice to prove the following: suppose that $f(\mathrm{Int}(P)) \cap X$ consists of exactly $k$ points, all of which are positively oriented. Then there exists a diffeotopy $\Psi_t : M \longrightarrow M \ \text{ rel } \ \partial M$ such that $\Psi_0 = Id_M$ and $\Psi_1(X) \cap f(P) = \emptyset$. So, suppose that $f(\mathrm{Int}(P)) \cap X$ consists of exactly $k$ points, all of which are positively oriented. By the previous lemma, there exists and embedding

$$g : S^r \longrightarrow \mathrm{Int}(M \setminus X \cup f_\beta(\beta P))$$

such that $g(S^r) \cap f(\mathrm{Int}(P))$ consists of exactly $k$ points, all of which are negatively oriented. Furthermore, the embedding $g$ can be chosen so that it admits an extension to an embedding

$$\bar{g} : D^{r+1} \longrightarrow \mathrm{Int}(M \setminus X).$$

Let $\tilde{X} \subset M$ be the submanifold obtained by forming the connected sum of $g(S^r) \subset M$ with $X$ along some embedded arc in $M$ that is disjoint from $f(P)$. It follows easily from the fact that $g$ extends to an embedding of a disk, that $\tilde{X}$ is ambient isotopic to $X$. By construction, it follows that we have

$$f(\mathrm{Int}(P)) \cap \tilde{X} \cong +\langle k \rangle \sqcup -\langle k \rangle.$$

Since both $f(\mathrm{Int}(P))$ and $\tilde{X}$ are path connected and $M$ is simply connected by assumption, we may then apply the *Whitney trick* to obtain a diffeotopy

$$\Psi_t : M \longrightarrow M \,\mathrm{rel}\,\partial M$$

with $\Psi_1(X) \cap f(P) = \emptyset$. This concludes the proof of the theorem. □

## 10.2. A Higher Dimensional Intersection Invariant

We recall now a certain construction developed by Hatcher and Quinn in (16). Let $M$, $X$, and $Y$ be smooth manifolds of dimension $m$, $r$, and $s$ respectively. Let $t = r + s - m$. Let

$$\varphi : (X, \partial X) \longrightarrow (M, \partial M) \quad \text{and} \quad \psi : (Y, \partial Y) \longrightarrow (M, \partial M)$$

be smooth maps. Let $E(\varphi, \psi)$ denote the *homotopy pull-back* of $\varphi$ and $\psi$. Specifically, this is given by

$$E(\varphi, \psi) = \{(x, y, \gamma) \in X \times Y \times \mathrm{Path}(M) \mid \varphi(x) = \gamma(0),\ \psi(y) = \gamma(1)\ \}.$$

Consider the diagram,

$$\begin{array}{ccc} E(\varphi,\psi) & \xrightarrow{\pi_X} & X \\ \downarrow{\scriptstyle \pi_Y} & \searrow^{\pi_M} & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{\psi} & M \end{array} \tag{10.2}$$

where $\pi_X$ and $\pi_Y$ are projection maps and $\pi_M$ is given by the formula $(x, y, \gamma) \mapsto \gamma(1/2)$. It is easily verified that this diagram commutes up to homotopy. Let $\nu_X$ and $\nu_Y$ denote the stable normal bundles associated to $X$ and $Y$. We will need to

consider the stable vector bundle over $E(\varphi, \psi)$ given by the Whitney sum

$$\pi_X^*(\nu_X) \oplus \pi_Y^*(\nu_Y) \oplus \pi_M^*(TM).$$

We will denote this stable bundle by $\widehat{\nu}(\varphi, \psi)$. We will need to consider the *normal bordism group*

$$\Omega_t^{\mathrm{fr}\cdot}(E(\varphi, \psi),\ \widehat{\nu}(\varphi, \psi)).$$

Elements of this bordism group are represented by triples $(N, f, F)$, where $N$ is a $t$-dimensional closed manifold, $f : N \longrightarrow E(\varphi, \psi)$ is a map, and $F : \nu_N \longrightarrow \widehat{\nu}(\varphi, \psi)$ is an isomorphism of stable vector bundles covering the map $f$.

Now, suppose that the maps $\varphi$ and $\psi$ are transversal and that

$$\varphi(\partial X) \cap \psi(\partial Y) = \emptyset.$$

It follows that the pullback $\varphi \pitchfork \psi \subset X \times Y$ is a closed submanifold of dimension $t$. There is a natural map

$$\iota_{\varphi,\psi} : \varphi \pitchfork \psi \longrightarrow E(\varphi, \psi), \quad (x, y) \mapsto (x, y, c_{\varphi(x)}),$$

where $c_{\varphi(x)}$ is the constant path at point $\varphi(x)$. Let $\nu_{\varphi \pitchfork \psi}$ denote the stable normal bundle associated to the pull-back $\varphi \pitchfork \psi$. The following is given in (16, Proposition 2.1) (see also the discussion on Pages 331-332).

**Proposition 10.2.1.** *There is a natural bundle isomorphism* $\hat{\iota}_{\varphi,\psi} : \nu_{\varphi \pitchfork \psi} \xrightarrow{\cong} \nu(\varphi, \psi)$*, determined uniquely by the homotopy classes of* $\varphi$ *and* $\psi$*, that covers the map* $\iota_{\varphi,\psi}$*. In this way, the triple* $(\varphi \pitchfork \psi,\ \iota_{\varphi,\psi},\ \hat{\iota}_{\varphi,\psi})$ *determines a bordism class in* $\Omega_t^{fr\cdot}(E(\varphi, \psi),\ \widehat{\nu}(\varphi, \psi))$.

The bordism group $\Omega_t^{\mathrm{fr}\cdot}(E(\varphi,\psi),\ \widehat{\nu}(\varphi,\psi))$ can be quite difficult to compute in general. However, in the case that the manifolds $X$, $Y$, and $M$ are highly connected, the group $\Omega_t^{\mathrm{fr}\cdot}(E(\varphi,\psi),\ \widehat{\nu}(\varphi,\psi))$ reduces to something much more simple. The following proposition is proven in (16, Section 3).

**Proposition 10.2.2.** *Suppose that $X$, $Y$, and $M$ are $(t+1)$-connected (recall that $t = \dim(X) + \dim(Y) - \dim(M) = r + s - m$). Then the homomorphism*

$$\Omega_t^{fr\cdot}(pt.) \to \Omega_t^{fr\cdot}(E(\varphi,\psi),\ \widehat{\nu}(\varphi,\psi))$$

*induced by the inclusion of any point into $E(\varphi,\psi)$, is an isomorphism.*

**Definition 10.2.1.** In the case that $X$, $Y$, and $M$ are $(t+1)$-connected, we will denote by

$$\alpha_t(\varphi,\psi;M) \in \Omega_t^{\mathrm{fr}\cdot}(\mathrm{pt.}) \tag{10.3}$$

the image of the bordism class in $\Omega_t^{\mathrm{fr}\cdot}(E(\varphi,\psi),\ \widehat{\nu}(\varphi,\psi))$ associated to $\varphi \pitchfork \psi$ under the isomorphism of the previous proposition.

**Remark 10.2.1.** We emphasize that it is not necessary for both $\varphi$ and $\psi$ to be embeddings in order for the class $\alpha_t(\varphi,\psi;M)$ to be defined. It is only necessary that $\varphi$ and $\psi$ be transversal as smooth maps. Furthermore its is easy to see that $\alpha_t(\varphi,\psi,M)$ is an invariant of the homotopy class of $\varphi$ and $\psi$. However, the next theorem (Theorem 10.2.3) does require that $\varphi$ and $\psi$ be embeddings.

The following is proven in (16, Theorem 2.2) (and in (36)).

**Theorem 10.2.3.** *Let*

$$\varphi : (X,\partial X) \longrightarrow (M,\partial M) \quad \textit{and} \quad \psi : (Y,\partial Y) \longrightarrow (M,\partial M)$$

*be embeddings such that* $\varphi(\partial X) \cap \psi(\partial Y) = \emptyset$. *Suppose that* $m > r + \frac{s}{2} + 1$ *and* $m > s + \frac{r}{2} + 1$, *and that* $X$, $Y$, *and* $M$ *are* $(t+1)$*-connected. Then if* $\alpha_t(\varphi, \psi; M) = 0$, *there exists a diffeotopy*

$$\Psi_t : M \longrightarrow M \ \text{rel}\, \partial M$$

*such that* $\Psi_1(\varphi(X)) \cap \psi(Y) = \emptyset$.

**Remark 10.2.2.** In (16) the above theorem is only explicitly proven in the case when $X$ and $Y$ are closed manifolds, though their proof can easily by modified to yield the version stated above. In (41), a proof of the relative version stated exactly as above is given.

The next lemma, which we will use latter, is a restatement of (16, Theorem 1.1).

**Lemma 10.2.4.** *Let*

$$\varphi : (X, \partial X) \longrightarrow (M, \partial M) \quad \textit{and} \quad \psi : (Y, \partial Y) \longrightarrow (M, \partial M)$$

*be embeddings with* $\varphi(\partial X) \cap \psi(\partial Y) = \emptyset$. *Suppose that* $\varphi$ *is homotopic relative* $\partial X$, *to a map* $\varphi'$ *such that* $\varphi'(X) \cap \psi(Y) = \emptyset$. *If* $m > r + s/2 + 1$, *then there exists a diffeotopy*

$$\Psi_t : M \longrightarrow M \, \text{rel}\, \partial M$$

*such that* $(\Psi_1 \circ \varphi(X)) \cap \psi(Y) = \emptyset$.

**Remark 10.2.3.** The main dimensional case when will use Theorem 10.2.3 and Lemma 10.2.4 is when $\dim(M) = 2n + 1$, $\dim(X) = \dim(Y) = n + 1$, and $n \geq 4$.

## 10.3. Creating Intersections

There is a particular application of the above theorem that we will need to use. Let $M$ and $Y$ be oriented, connected manifolds of dimension $m$ and $s$ respectively and let

$$\psi : (Y, \partial Y) \longrightarrow (M, \partial M)$$

be an embedding. Let $r = m - s$ and let $\varphi : S^r \longrightarrow \mathrm{Int}(M)$ be a smooth map transverse to $\psi(Y) \subset M$. Let $j \geq 0$ be an integer strictly less than $r$ and let $\gamma : S^{r+j} \longrightarrow S^r$ be a smooth map. Denote by

$$\mathcal{P}_j : \pi_{r+j}(S^r) \xrightarrow{\cong} \Omega_j^{\mathrm{fr}\cdot}(\mathrm{pt.}) \tag{10.4}$$

the *Pontryagin-Thom* isomorphism (see (23)). The following lemma shows how to compute

$$\alpha_j(\varphi \circ \gamma,\ \psi;\ M)$$

in terms of $\alpha_0(\varphi, \psi, M)$ and the element $\mathcal{P}_j([\gamma]) \in \Omega_j^{\mathrm{fr}\cdot}(\mathrm{pt.})$.

**Lemma 10.3.1.** *Let $\psi$, $\varphi$ and $\gamma : S^{r+j} \to S^r$ be exactly as above. Then*

$$\alpha_j(\varphi \circ \gamma,\ \psi;\ M) \ =\ \alpha_0(\varphi, \psi; M) \cdot \mathcal{P}_j([\gamma]),$$

*where the product on the right-hand side is the product in the graded bordism ring $\Omega_*^{fr\cdot}(pt.)$.*

*Proof.* Let $s \in \mathbb{Z}$ denote the oriented, algebraic intersection number associated to the intersection of $\varphi(S^r)$ and $\psi(Y)$. By application of the Whitney trick, we may

deform $\varphi$ so that

$$\varphi(S^r) \cap \psi(Y) = \{x_1, \ldots, x_\ell\}, \tag{10.5}$$

where the points $x_i$ for $i = 1, \ldots, \ell$ all have the same sign. It follows that

$$(\varphi \circ \gamma)^{-1}(\psi(Y)) = \coprod_{i=1}^{\ell} \gamma^{-1}(x_i).$$

For each $i \in \{1, \ldots, \ell\}$, the framing at $x_i$ (induced by the orientations of $\gamma(S^r)$, $\psi(Y)$ and $M$) induces a framing on $\gamma^{-1}(x_i)$. We denote the element of $\Omega_1^{\mathrm{fr.}}(\mathrm{pt.})$ given by $\gamma^{-1}(x_i)$ with this induced framing by $[\gamma^{-1}(x_i)]$. By definition of the Pontryagon-Thom map $\mathcal{P}_j$ (see (23, Section 7)), the element $[\gamma^{-1}(x_i)]$ is equal to $\mathcal{P}_j([\gamma])$ for $i = 1, \ldots, \ell$. Using the equality (10.5), it follows that

$$\alpha_j(\varphi \circ \gamma,\ \psi;\ M) = \ell \cdot \mathcal{P}_j([\gamma]).$$

The proof then follows from the fact that $\alpha_0(\varphi, \psi, M)$ is identified with the algebraic intersection number associated to $\varphi(S^r)$ and $\psi(Y)$. □

## 10.4. A Technical Lemma

Before we proceed further, we develop a technical result that will play an important role in the proof of Theorem 10.5.1. For $n \geq 4$, let $M$ be a 2-connected, oriented $(2n+1)$-dimensional manifold and let $P$ be a 2-connected, oriented, $\langle k \rangle$-manifold of dimension $n+1$. Let $f : (P, \partial_0 P) \longrightarrow (M, \partial M)$ be a $\langle k \rangle$-embedding. Let $U$ be a tubular neighborhood of $f_\beta(\beta P) \subset M$ whose boundary intersects $f(\mathrm{Int}(P))$ transversally. Denote,

$$Z := M \setminus \mathrm{Int}(U), \qquad P' := f^{-1}(Z), \qquad f' := f|_{P'}. \tag{10.6}$$

It follows from the fact that $\partial U$ intersects $\mathrm{Int}(f(P))$ transversally that $P'$ is a smooth manifold with boundary (after smoothing corners) and that $f'$ maps $\partial P'$ into $\partial M$. Let $\xi$ denote the generator of the framed bordism group $\Omega_1^{\mathrm{fr}\cdot}(\mathrm{pt.})$, which is isomorphic to $\mathbb{Z}/2$.

**Lemma 10.4.1.** *Let $f : (P, \partial_0 P) \longrightarrow (M, \partial M)$ be as above and let $i_Z : Z \hookrightarrow M$ denote the inclusion map. There exists an embedding $\varphi : S^{n+1} \longrightarrow Z$ which satisfies:*

*i.* $\alpha_1(f', \varphi; Z) = k \cdot \xi \in \Omega_1^{fr\cdot}(pt.)$,

*ii. the composition $i_Z \circ \varphi : S^{n+1} \longrightarrow M$ is null-homotopic.*

*Proof.* By Lemma 10.1.2, we may choose an embedding $\phi : S^n \longrightarrow M \setminus f_\beta(\beta P)$ an embedding that satisfies:

- $\phi(S^n)$ intersects $f(\mathrm{Int}(P))$ transversally,
- $\phi(S^n) \cap f(\mathrm{Int}(P)) \cong +\langle k \rangle$,
- $i_Z \circ \phi : S^n \to M$ extends to an embedding $D^{n+1} \hookrightarrow M$.

By shrinking the tubular neighborhood $U$ of $f_\beta(\beta P)$ if necessary, we may assume that

$$\phi(S^n) \subset Z = M \setminus \mathrm{Int}(U).$$

Denote by $\widehat{\phi} : S^n \longrightarrow Z$ the map obtained by restricting the codomain of $\phi$. Let

$$\gamma : S^{n+1} \longrightarrow S^n$$

represent the generator of $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$. By Lemma 10.3.1 it follows that,

$$\alpha_1(\widehat{\phi} \circ \gamma,\ f'; Z) = \alpha_0(\widehat{\phi},\ f'; Z) \cdot \mathcal{P}_1([\gamma]) = k \cdot \mathcal{P}_1([\gamma]) = k \cdot \xi,$$

where $\mathcal{P}_1 : \pi_{n+1}(S^n) \longrightarrow \Omega_1^{\text{fr.}}(\text{pt.})$ is the Pontryagin-Thom map for framed bordism. Since $Z$ is 2-connected and $n \geq 4$, we may apply (37, Proposition 1) (or the main theorem of (12)), and find a homotopy of the map $\widehat{\phi} \circ \gamma$, to an embedding $\varphi : S^{n+1} \longrightarrow Z$. Since the map $i_Z \circ \phi : S^n \longrightarrow M$ is null-homotopic, it follows that $i_Z \circ \varphi : S^{n+1} \to M$ is null-homotopic as well. This completes the proof of the lemma. □

## 10.5. Modifying Intersections

We now state the main result of this section (which is a restatement of Theorem 7.4.1 from Section 7.4). Fix an integer $n \geq 4$, let $M$ be an oriented, 2-connected manifold of dimension $2n + 1$. Let $P$ and $Q$ be oriented, 2-connected, $\langle k \rangle$-manifolds of dimension $n + 1$. We will use the same $M$, $P$, and $Q$ throughout this section.

**Theorem 10.5.1.** *With $M$, $P$, and $Q$ as above and let*

$$f : (P, \partial_0 P) \longrightarrow (M, \partial M) \quad \textit{and} \quad g : (Q, \partial_0 Q) \longrightarrow (M, \partial M)$$

*be transversal $\langle k \rangle$-embeddings such that $f(\partial_0 P) \cap g(\partial_0 Q) = \emptyset$. Suppose that $\Lambda^1_{k,k}(f, g; M) = 0$. If the integer $k$ is odd, then there exists a diffeotopy $\Psi_t : M \longrightarrow M$ rel $\partial M$ such that $\Psi_1(f(P)) \cap g(Q) = \emptyset$.*

The proof of the above theorem is proven in stages via several intermediate propositions.

**Proposition 10.5.2.** *Let*

$$f : (P, \partial_0 P) \longrightarrow (M, \partial M) \quad \textit{and} \quad g : (Q, \partial_0 Q) \longrightarrow (M, \partial M)$$

*be $\langle k \rangle$-embeddings as above and suppose that*

$$\beta_1(\Lambda^1_{k,k}(f, g; M)) \;=\; \Lambda^0_k(f_\beta, g; M) \;=\; 0.$$

*Then there exists a diffeotopy $\Psi_t : M \longrightarrow M \operatorname{rel} \partial M$ such that $\Psi_1(f_\beta(\beta P)) \cap g(Q) = \emptyset$.*

*Proof.* Since $0 \;=\; \beta_1(\Lambda^1_{k,k}(f, g; M)) \;=\; \Lambda^0_k(g, f_\beta; M)$, it follows that the algebraic intersection number associated to $f_\beta(\beta P)$ and $g(\operatorname{Int} Q)$ is a multiple of $k$. The desired diffeotopy exists by Theorem 10.1.1. □

**Proposition 10.5.3.** *Let $g : (Q, \partial_0 Q) \longrightarrow (M, \partial M)$ be a $\langle k \rangle$-embedding as above. Let $X$ be a smooth manifold of dimension $n + 1$ and let $\varphi : (X, \partial X) \longrightarrow (M, \partial M)$ be a smooth embedding such that*

$$\varphi(\partial X) \cap g(\partial_0 Q) = \emptyset.$$

*If the integer $k$ is odd, then there exists a diffeotopy, $\Psi_t : M \to M \;\operatorname{rel} \partial M$ such that,*

$$\Psi_1(\varphi(X)) \cap g(Q) \;=\; \emptyset.$$

*Proof.* By Proposition 7.2.1, we have

$$\beta(\Lambda^1_k(g,\varphi;M)) \;=\; \Lambda^0(g_\beta,\varphi;M) \in \Omega^{SO}_0(\text{pt.})$$

where

$$\beta : \Omega^{SO}_1(\text{pt.})_{\langle k\rangle} \longrightarrow \Omega^{SO}_0(\text{pt.}), \quad [V] \mapsto [\beta V]$$

is the Bockstein homomorphism. By (5.2), this Bockstein homomorphism is the zero map for all $k$ (the group $\Omega^{SO}_1(\text{pt.})_{\langle k\rangle}$ is equal to zero). It follows that $\Lambda^0(g_\beta,\varphi;M) \in \Omega^{SO}_0(\text{pt.})$ is the zero element and thus the oriented, algebraic intersection number associated to $g_\beta(\beta Q)\cap X$ is equal to zero. By application of the *Whitney trick* (24, Theorem 6.6), we may find a diffeotopy of $M$, relative $\partial M$, which pushes $X$ off of the submanifold $g_\beta(\beta Q) \subset M$. Using this, we may now assume that $\varphi(X)\cap g(\partial_1 Q) = \emptyset$.

Let $U \subset M$ be a closed tubular neighborhood of $f_\beta(\beta P)$, disjoint from $X$, such that the boundary of $U$ intersects $f(P)$ transversely. As in (10.6), we denote

$$Z := M \setminus \operatorname{Int} U, \qquad P' := f^{-1}(Z), \qquad f' := f|_{P'}.$$

Notice that $P'$ is a manifold with boundary and that $f'$ is an embedding which maps $(P',\partial P')$ into $(Z,\partial Z)$. Furthermore, $\varphi$ maps $(X,\partial X)$ into $(Z,\partial Z)$. To prove the corollary it will suffice to construct a diffeotopy $\Psi'_t : Z \longrightarrow Z \;\operatorname{rel}\partial Z$ such that $\Psi'_1(X)\cap P' = \emptyset$. By Theorem 10.2.3, the obstruction to the existence of such a diffeotopy is the class $\alpha_1(f',\varphi;Z) \in \Omega^{\text{fr}}_1(\text{pt.})$. If $\alpha_1(f',\varphi;Z)$ is equal to zero, we are done. So suppose that $\alpha_1(f',\varphi;Z) = \xi$ where $\xi$ is the non-trivial element in

$\Omega_1^{\mathrm{fr}}(\mathrm{pt.}) \cong \mathbb{Z}/2$. Denote by $i_Z : Z \hookrightarrow M$ the inclusion map. By Lemma 10.4.1 there exits an embedding $\phi : S^{n+1} \longrightarrow Z$ such that:

- $\alpha_1(f', \phi; Z) = k \cdot \xi$ where $\xi \in \Omega_1^{\mathrm{fr}}(\mathrm{pt.}) \cong \mathbb{Z}/2$ is the standard generator,
- the embedding $i_Z \circ \phi : S^{n+1} \longrightarrow M$ is null-homotopic.

Since $k$ is odd, we have $\alpha_1(f', \phi; Z) = \xi$. We denote by $\widehat{\varphi} : X \longrightarrow M$ the embedding obtained by forming the connected sum of $\varphi(X)$ with $i_Z \circ \varphi(S^{n+1})$ along the thickening of an embedded arc that is disjoint from $f(P)$, $U$, and $X$. Since $i_Z \circ \varphi : S^{n+1} \longrightarrow M$ is null-homotopic, it follows that $\widehat{\varphi}$ is homotopic, relative to $\partial X$, to the original embedding $\varphi$. We have

$$\alpha_1(f', \widehat{\varphi}; Z) = \alpha_1(f', \varphi; Z) + \alpha_1(f', \phi; Z) = \xi + \xi = 0,$$

and so there exists a diffeotopy $\Psi'_t : Z \to Z \ \mathrm{rel}\, \partial Z$ such that $\Psi'_1(\widehat{\varphi}(X)) \cap f'(P') = \emptyset$. We then extend $\Psi'_t$ identically over $M \setminus Z$ to obtain a diffeotopy

$$\widehat{\Psi}_t : M \longrightarrow M \ \mathrm{rel}\, \partial M$$

such that $\widehat{\Psi}_1(\widehat{\varphi}(X)) \cap f(P) = \emptyset$. Now, since $\varphi$ is homotopic relative $\partial X$ to the embedding $\widehat{\Psi}_1 \circ \widehat{\varphi}$ and $\widehat{\Psi}_1(\widehat{\varphi}(X)) \cap f(P) = \emptyset$, we may apply Lemma 10.2.4 to obtain a diffeotopy

$$\Psi_t : M \longrightarrow M \, \mathrm{rel}\, \partial M$$

such that $(\Psi_1 \circ \varphi(X)) \cap f(P) = \emptyset$. This concludes the proof of the proposition. $\square$

We can now complete the proof of Theorem 10.5.1.

*Proof of Theorem 10.5.1.* By hypothesis we have $\Lambda^1_{k,k}(f,g;M) = 0$, and thus $\Lambda^0_k(f_\beta,g;M) = 0$, and so by Proposition 10.5.2 we may assume that $f_\beta(\beta P) \cap g(Q) = \emptyset$. Choose a closed tubular neighborhood $U \subset M$ of $f_\beta(\beta P)$, disjoint from $g(Q)$, with boundary transverse to $f(P)$. As in (10.6) we denote,

$$Z := M \setminus \operatorname{Int} U, \quad P' := f^{-1}(Z), \quad \text{and} \quad f' := f|_{P'}. \tag{10.7}$$

With these definitions, $P'$ is an oriented manifold with boundary and

$$f' : (P', \partial P') \longrightarrow (Z, \partial Z)$$

is an embedding. Furthermore, since $U$ was chosen to be disjoint from $g(Q)$, we have $g(Q) \subset Z$. Let $g' : (Q, \partial_0) \longrightarrow (Z, \partial Z)$ denote the $\langle k \rangle$-embedding obtained by restricting the codomain of $g$. To finish the proof, we then apply Proposition 10.5.3 to the embedding $f' : (P', \partial P') \longrightarrow (Z, \partial Z)$ and $\langle k \rangle$-embedding $g' : (Q, \partial_0 Q) \longrightarrow (Z, \partial Z)$, to obtain a diffeotopy of $Z$ (relative $\partial Z$) that pushes $f'(P')$ off of $g'(Q)$. This completes the proof of the theorem. □

We now come to an important corollary. Recall from Section 6.3 the $\langle k,k \rangle$-manifold $A_k$.

**Corollary 10.5.4.** *Let $f$ and $g$ be exactly as in the statement of Theorem 10.5.1. Suppose that the class $\Lambda^1_{k,k}(f,g;M)$ is equal to the class represented by the closed 1-dimensional $\langle k,k \rangle$-manifold $+A_k$. If $k$ is odd then there exists a diffeotopy $\Psi_t : M \longrightarrow M \operatorname{rel} \partial M$ such that the transverse pull-back $(\Psi_1 \circ f) \pitchfork g$ is diffeomorphic to $A_k$.*

*Proof.* Since $\Lambda^1_{k,k}(f,g;M)$ is equal to the class represented by $+A_k$ in $\Omega^{SO}_1(\text{pt.})_{\langle k,k\rangle}$, it follows that $f \pitchfork g$ is diffeomorphic (as an oriented $\langle k,k\rangle$-manifold) to the disjoint union of precisely one copy of $+A_k$ together with some other oriented $\langle k,k\rangle$-manifold, that represents the zero element in $\Omega^{SO}_1(\text{pt.})_{\langle k,k\rangle}$. We may write

$$f(P) \cap g(Q) \;=\; A \sqcup Y, \tag{10.8}$$

where

$$A \cong +A_k \quad \text{and} \quad [Y] = 0 \text{ in } \Omega^{SO}_1(\text{pt.})_{\langle k,k\rangle}.$$

Let $U \subset M$ be a closed neighborhood of $f_\beta(\beta P) \cup A$, disjoint from $Y$, with boundary transverse to both $f(P)$ and $g(Q)$. We then denote

$$Z := M \setminus \text{Int}(U), \qquad P' := f^{-1}(Z), \qquad Q' := g^{-1}(Z). \tag{10.9}$$

Notice that both $P'$ and $Q'$ are $\langle k\rangle$-manifolds with

$$\begin{aligned} &\partial_0 P' = f^{-1}(\partial Z), \qquad \partial_1 P' = (f|_{\partial_1 P})^{-1}(Z), \qquad \beta P' = f_\beta^{-1}(Z), \\ &\partial_0 Q' = g^{-1}(\partial Z), \qquad \partial_1 Q' = (g|_{\partial_1 Q})^{-1}(Z), \qquad \beta Q' = g_\beta^{-1}(Z). \end{aligned}$$

We denote by

$$f' : (P', \partial_0 P') \longrightarrow (Z, \partial Z) \quad \text{and} \quad g' : (Q', \partial_0 Q') \longrightarrow (Z, \partial Z)$$

the $\langle k\rangle$-embeddings given by restricting $f$ and $g$. By construction, the pull-back $f' \pitchfork g'$ is diffeomorphic as an oriented $\langle k,k\rangle$-manifold to $Y$, which represents the zero element in $\Omega^{SO}_1(\text{pt.})_{\langle k,k\rangle}$. It follows that $\Lambda^1_{k,k}(f',g';Z) = 0$. By Theorem 10.5.1

we obtain a diffeotopy $\Psi_t : Z \longrightarrow Z \ \text{rel}\,\partial Z$, such that $\Psi_1(f^{'}(P')) \cap g'(Q') = \emptyset$. This concludes the proof. $\square$

# CHAPTER XI

# IMMERSIONS AND EMBEDDINGS OF $\mathbb{Z}/K$-MANIFOLDS

In this section we determine the conditions for when a $\langle k \rangle$-map can be deformed to a $\langle k \rangle$-immersion or a $\langle k \rangle$-embedding. The techniques of this section enable us to prove Theorem 5.4.1 (which is restated again in this section as Theorem 11.6.1).

## 11.1. A Recollection of Smale-Hirsch Theory

Let $N$ and $M$ be smooth manifolds of dimensions $n$ and $m$ respectively. Denote by $\mathrm{Imm}(N, M)$ the space of immersions $N \to M$, topologized in the $C^\infty$-topology. Let $\mathrm{Imm}^f(N, M)$ denote the space of bundle maps $TN \longrightarrow TM$ which are fibre-wise injective. Elements of the space $\mathrm{Imm}^f(N, M)$ are called *formal immersions*. There is a map $\mathcal{D} : \mathrm{Imm}(N, M) \longrightarrow \mathrm{Imm}^f(N, M)$ defined by sending an immersion $\phi : N \longrightarrow M$ to the bundle injection given by its differential $D\phi : TN \longrightarrow TM$. The following theorem is proven in (1, Chapter III, Section 9) and is originally due to Hirsch and Smale.

**Theorem 11.1.1.** *The if* $\dim(N) < \dim(M)$*, then the map* $\mathcal{D} : \mathrm{Imm}(N, M) \longrightarrow \mathrm{Imm}^f(N, M)$ *is a weak homotopy equivalence. In the case that* $\dim(N) = \dim(M)$*, then* $\mathcal{D}$ *is a weak homotopy equivalence if* $N$ *is an open manifold.*

Let $\widehat{\mathrm{Imm}}(N, M)$ denote the space of pairs $(\phi, \mathbf{v}) \in \mathrm{Imm}(N, M) \times \mathrm{Maps}(N, TM)$ that satisfy:

i. $\pi(\mathbf{v}(x)) = \phi(x)$ for all $x \in N$, where $\pi : TM \to M$ is the bundle projection,

ii. for each $x \in N$, the vector $\mathbf{v}(x)$ is transverse to the vector subspace

$$D\phi(T_xN) \subset T_{\phi(x)}M,$$

where $D\phi$ is the differential of $\phi$.

Similarly, we define $\widehat{\mathrm{Imm}}^f(N, M)$ to be the space of pairs $(\psi, \mathbf{v}) \in \mathrm{Imm}^f(N, M) \times \mathrm{Maps}(N, TM)$ which satisfy:

i. $\pi(\mathbf{v}(x)) = \pi(\psi(x))$ for all $x \in N$, where $\pi : TM \to M$ is the bundle projection,

ii. for all $x \in N$, the vector $\mathbf{v}(x)$ is transverse to the vector subspace

$$\psi(T_xN) \subset T_{\pi(\psi(x))}M.$$

There is a map

$$\widehat{\mathcal{D}} : \widehat{\mathrm{Imm}}(N, M) \longrightarrow \widehat{\mathrm{Imm}}^f(N, M), \quad (\phi, \mathbf{v}) \mapsto (D\phi, \mathbf{v}). \tag{11.1}$$

The following is an easy corollary of Theorem 11.1.1.

**Corollary 11.1.2.** *Suppose that* $\dim(N) < \dim(M)$. *Then the map* $\widehat{\mathcal{D}}$ *from (11.1) is a weak homotopy equivalence.*

## 11.2. The Space of $\langle k \rangle$-Immersions

We now proceed to prove a version of Theorem 11.1.2 for immersions of $\langle k \rangle$-manifolds. For what follows, let $M$ be a manifold of dimension $m$ and let $P$ be

a $\langle k \rangle$-manifold of dimension $p$. We will need to construct a suitable space of $\langle k \rangle$-immersions and formal $\langle k \rangle$-immersions.

Choose a collar embedding $h : \partial_1 P \times [0, \infty) \longrightarrow P$, with $h^{-1}(\partial_1 P) = \partial_1 P \times \{0\}$. Denote by $\mathbf{v}_h \in \Gamma_{\partial_1 P}(TP)$ the inward pointing vector field along $\partial_1 P$ determined by the differential of the collar embedding $h$. Using $\mathbf{v}_h$ we have maps,

$$\begin{aligned} R &: \mathrm{Imm}(P, M) \longrightarrow \widehat{\mathrm{Imm}}(\partial_1 P, M), \qquad \phi \mapsto (\phi|_{\partial P},\ D\phi \circ \mathbf{v}_h), \\ R^f &: \mathrm{Imm}^f(P, M) \longrightarrow \widehat{\mathrm{Imm}}^f(\partial_1 P, M), \qquad \psi \mapsto (\psi|_{\partial P},\ \psi \circ \mathbf{v}_h). \end{aligned} \tag{11.2}$$

The next lemma follows from the basic results of (1, Chapter III: Section 9).

**Lemma 11.2.1.** *The map $R^f$ is a Serre-fibration in the case that* $\dim(P) \leq \dim(M)$. *The map $R$ is a Serre-fibration in the case that* $\dim(P) < \dim(M)$.

Let $\bar{\Phi} : \partial_1 P \longrightarrow \beta P$ be the map given by the composition $\partial_1 P \xrightarrow[\cong]{\Phi} \beta P \times \langle k \rangle \xrightarrow{\mathrm{proj}_{\beta P}} \beta P$. Using $\bar{\Phi}$ we have a map

$$T_k : \widehat{\mathrm{Imm}}(\beta P, M) \longrightarrow \widehat{\mathrm{Imm}}(\partial_1 P, M), \quad (\phi, \mathbf{v}) \mapsto (\phi \circ \bar{\Phi},\ \mathbf{v} \circ \bar{\Phi}). \tag{11.3}$$

Similarly, by using the differential $D\bar{\Phi}$ of $\bar{\Phi}$, we define a map

$$T_k^f : \widehat{\mathrm{Imm}}^f(\beta P, M) \longrightarrow \widehat{\mathrm{Imm}}^f(\partial_1 P, M), \quad (\psi, \mathbf{v}) \mapsto (\psi \circ D\bar{\Phi},\ \mathbf{v} \circ \bar{\Phi}). \tag{11.4}$$

**Definition 11.2.1.** We define $\mathrm{Imm}_{\langle k \rangle}(P, M)$ to be the space of pairs

$$(\phi, (\phi', \mathbf{v})) \in \mathrm{Imm}(P, M) \times \widehat{\mathrm{Imm}}(\beta P, M)$$

such that $T_k(\phi', \mathbf{v}) = R(\phi)$. Similarly we define $\mathrm{Imm}^f_{\langle k\rangle}(P, M)$ to be the space of pairs

$$(\psi, (\psi', \mathbf{v})) \in \mathrm{Imm}^f(P, M) \times \widehat{\mathrm{Imm}}^f(\beta P, M)$$

such that $T^f_k(\psi', \mathbf{v}) = R^f(\psi)$.

**Remark 11.2.1.** Let $(\phi, (\phi', \mathbf{v})) \in \mathrm{Imm}_{\langle k\rangle}(P, M)$. By construction, the immersion $\phi : P \longrightarrow M$ is a $\langle k\rangle$-immersion and $\phi' = \phi_\beta$. The pair $(\phi', \mathbf{v})$ is completely determined by the $\langle k\rangle$-immersion $\phi$ and so, the space $\mathrm{Imm}_{\langle k\rangle}(P, M)$ is homeomorphic to the subspace of $\mathrm{Maps}_{\langle k\rangle}(P, M)$ consisting of all $\langle k\rangle$-immersions $P \to M$.

**Lemma 11.2.2.** *The following two commutative diagrams*

$$\begin{array}{ccc} \mathrm{Imm}_{\langle k\rangle}(P, M) & \longrightarrow & \mathrm{Imm}(P, M) \\ \downarrow & & \downarrow R \\ \widehat{\mathrm{Imm}}(\beta P, M) & \xrightarrow{T_k} & \widehat{\mathrm{Imm}}(\partial_1 P, M), \end{array} \qquad \begin{array}{ccc} \mathrm{Imm}^f_{\langle k\rangle}(P, M) & \longrightarrow & \mathrm{Imm}^f(P, M) \\ \downarrow & & \downarrow R^f \\ \widehat{\mathrm{Imm}}^f(\beta P, M) & \xrightarrow{T^f_k} & \widehat{\mathrm{Imm}}^f(\partial_1 P, M), \end{array}$$

*are homotopy cartesian.*

*Proof.* This follows immediately from Lemma 11.2.1 and the fact that both of the diagrams are pull-backs. □

Finally we may consider the map

$$\mathcal{D}_k : \widehat{\mathrm{Imm}}_{\langle k\rangle}(P, M) \longrightarrow \widehat{\mathrm{Imm}}^f_{\langle k\rangle}(P, M), \qquad (\phi,\ (\phi', \mathbf{v})) \mapsto (D\phi,\ (D\phi', \mathbf{v})). \quad (11.5)$$

We have the following theorem.

**Theorem 11.2.3.** *Suppose that* $\dim(P) < \dim(M)$. *Then the map* $\mathcal{D}_k$ *of (11.5) is a weak homotopy equivalence.*

*Proof.* The map from (11.5) induces a map between the two commutative squares in Lemma 11.2.2. The maps between the entries on the bottom row and the entries on the upper-right are weak homotopy equivalences by Theorem 11.1.1 and Corollary 11.1.2. It then follows from Lemma 11.2.2 that the upper-left map (which is (11.5)) is a weak homotopy equivalence. □

## 11.3. Representing Classes of $\langle k \rangle$-Maps by $\langle k \rangle$-Immersions

Let $P$ be a $\langle k \rangle$-manifold of dimension $p$ and let $h : \partial_1 \times [0, \infty) \longrightarrow P$ be a collar embedding with $h^{-1}(\partial_1 P) = \partial_1 P \times \{0\}$. We have a bundle map

$$\Phi^* : TP|_{\partial_1 P} \longrightarrow T(\beta P) \oplus \epsilon^1 \tag{11.6}$$

given by the composition, $TP|_{\partial_1 P} \xrightarrow{\cong} T(\partial_1 P) \oplus \epsilon^1 \xrightarrow{D\bar{\Phi} \oplus Id_{\epsilon^1}} T(\beta P) \oplus \epsilon^1$, where the first map is the bundle isomorphism induced by the collar embedding $h$. Using this bundle isomorphism $\Phi^*$, we define a new space $T\widehat{P}$ as a quotient of $TP$ by identifying two points $v, v' \in TP|_{\partial_1 P} \subset TP$ if and only if $\Phi^* v = \Phi^* v'$. With this definition, there is a natural projection $\widehat{\pi} : T\widehat{P} \longrightarrow \widehat{P}$ which makes the diagram

$$\begin{array}{ccc} TP & \longrightarrow & T\widehat{P} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \widehat{\pi}} \\ P & \longrightarrow & \widehat{P} \end{array} \tag{11.7}$$

commute. It is easy to verify that the projection map $\widehat{\pi} : T\widehat{P} \longrightarrow \widehat{P}$ is a vector bundle and that the upper-horizontal map in the above diagram is a bundle map that is an isomorphism on each fibre.

**Definition 11.3.1.** The $\langle k \rangle$-manifold $P$ is said to be *parallelizable* if the induced vector bundle $\widehat{\pi} : T\widehat{P} \to \widehat{P}$ is trivial.

**Corollary 11.3.1.** *Let $P$ be a parallelizable $\langle k\rangle$-manifold and let $M$ be a manifold of dimension greater than* $\dim(P)$*. Let $f : P \longrightarrow M$ be a $\langle k\rangle$-map and consider the induced map $\widehat{f} : \widehat{P} \longrightarrow M$. Suppose that the pull-back bundle $\widehat{f}^*(TM) \longrightarrow \widehat{P}$ is trivial. Then $f$ is homotopic through $\langle k\rangle$-maps to a $\langle k\rangle$-immersion.*

*Proof.* Since both $T\widehat{P} \to \widehat{P}$ and $\widehat{f}^*(TM) \to \widehat{P}$ are trivial vector bundles and $\dim(M) > \dim(P)$, we may choose a bundle injection $T\widehat{P} \to \widehat{f}^*(TM)$ covering the identity on $\widehat{P}$, and hence a fibrewise injective bundle map $\widehat{\psi} : T\widehat{P} \longrightarrow TM$ that covers the map $\widehat{f}$. Using the quotient construction from (11.7), the bundle map $\widehat{\psi}$ induces a unique formal $\langle k\rangle$-immersion $\psi \in \mathrm{Imm}^f_{\langle k\rangle}(P, M)$ whose underlying $\langle k\rangle$-map is $f$. It then follows from Theorem 11.2.3 that there exists a $\langle k\rangle$-immersion $\phi \in \mathrm{Imm}_{\langle k\rangle}(P, M)$ such that $\mathcal{D}(\phi)$ is on the same path component as $\psi$. It then follows that $\phi$ is homotopic through $\langle k\rangle$-maps to the map that underlies $\psi$, which is $f$. This completes the proof of the corollary. □

## 11.4. The Self-Intersections of a $\langle k\rangle$-Immersion

For what follows let $M$ be a manifold of dimension $m$ and let $P$ be a $\langle k\rangle$-manifold of dimension $p$. We will need to analyze the self-intersections of $\langle k\rangle$-immersions $P \to M$.

**Definition 11.4.1.** For $M$ a manifold and $P$ a $\langle k\rangle$-manifold, a $\langle k\rangle$-immersion $f : P \longrightarrow M$ is said to be in *general position* if the following conditions are met:

i. The immersion $f_\beta : \beta P \to M$ is self-transverse.

ii. The restriction map $f|_{\mathrm{Int}(P)} : \mathrm{Int}(P) \longrightarrow M$ is a self-transverse immersion and is transverse to the immersed submanifold $f_\beta(\beta P) \subset M$.

Let $f : P \longrightarrow M$ be a $\langle k \rangle$-immersion that is in general position. Let $\hat{q} : P \longrightarrow \widehat{P}$ denote the quotient projection and let $\widehat{\triangle}_P \subset P \times P$ be the subspace defined by setting

$$\widehat{\triangle}_P = (\hat{q} \times \hat{q})^{-1}(\triangle_{\widehat{P}}),$$

where $\triangle_{\widehat{P}} \subset \widehat{P} \times \widehat{P}$ is the diagonal subspace. It follows from Definition 11.4.1 that the map

$$(f \times f)|_{(P\times P)\setminus\widehat{\triangle}_P} : (P \times P) \setminus \widehat{\triangle}_P \longrightarrow M \times M$$

is transverse to the diagonal submanifold $\triangle_M \subset M \times M$. We denote by $\Sigma_f \subset (P \times P) \setminus \widehat{\triangle}_P$ the submanifold given by

$$\Sigma_f := \left((f \times f)|_{(P\times P)\setminus\widehat{\triangle}_P}\right)^{-1}(\triangle_M). \tag{11.8}$$

By the techniques of Section 7.3, $\Sigma_f$ has the structure of a $\langle k, k \rangle$-manifold with

$$\begin{array}{ccc} \partial_1\Sigma_f = f|_{\partial_1 P} \pitchfork f, & \partial_2\Sigma_f = f \pitchfork f|_{\partial_1 P}, & \partial_{1,2}\Sigma_f = f|_{\partial_1 P} \pitchfork f|_{\partial_1 P}, \\ \beta_1\Sigma_f = f_\beta \pitchfork f, & \beta_2\Sigma_f = f \pitchfork f_\beta, & \beta_{1,2}\Sigma_f = f_\beta \pitchfork f_\beta. \end{array}$$

The involution

$$P \times P \setminus \widehat{\triangle}_P \longrightarrow P \times P \setminus \widehat{\triangle}_P, \quad (x, y) \mapsto (y, x)$$

restricts to an involution on $\Sigma_f \subset P \times P \setminus \widehat{\triangle}_P$ which we denote by

$$T_{\Sigma_f} : \Sigma_f \longrightarrow \Sigma_f. \tag{11.9}$$

It is clear that the involution $T_{\Sigma_f}$ has no fixed-points. Since

$$\partial_1\Sigma_f \subset (\partial_1 P) \times P \quad \text{and} \quad \partial_2\Sigma_f \subset P \times (\partial_1 P),$$

it follows that

$$T_{\Sigma_f}(\partial_1\Sigma_f) \subset \partial_2\Sigma_f \quad \text{and} \quad T_{\Sigma_f}(\partial_2\Sigma_f) \subset \partial_1\Sigma_f.$$

If both $M$ and $P$ are oriented, then $\Sigma_f$ obtains a unique orientation induced from orientations on $P$ and $M$ in the standard way. Furthermore, $T_{\Sigma_f}$ preserves orientation if $m - p$ is even and reverses orientation if $m - p$ is odd. We sum up the observations made above into the following proposition.

**Proposition 11.4.1.** *Let $P$ be an oriented $\langle k \rangle$-manifold of dimension $p$ and let $M$ be an oriented manifold of dimension $m$. Let $f : P \longrightarrow M$ be a $\langle k \rangle$-immersion which is in general position. Then the double-point set $\Sigma_f$ has the structure of an oriented $\langle k, k \rangle$-manifold of dimension $2p - m$, equipped with a free involution $T_{\Sigma_f} : \Sigma_f \longrightarrow \Sigma_f$ such that*

$$T_{\Sigma_f}(\partial_1\Sigma_f) \subset \partial_2\Sigma_f \quad \textit{and} \quad T_{\Sigma_f}(\partial_2\Sigma_f) \subset \partial_1\Sigma_f.$$

*The involution $T_{\Sigma_f}$ preserves orientation if $m - p$ is even and reverses orientation if $m - p$ is odd.*

## 11.5. Modifying Self-Intersections

In this section, we develop a technique for eliminating the self-intersections of a $\langle k \rangle$-immersion $P \to M$ by deforming the $\langle k \rangle$-immersion to a $\langle k \rangle$-embedding via a homotopy through $\langle k \rangle$-maps. We will solve this problem in the special case

that $P$ is a 2-connected, oriented, $(2n+1)$-dimensional $\langle k \rangle$-manifold and $M$ is a 2-connected, oriented, $(4n+1)$-dimensional manifold and $n \geq 2$.

By Proposition 11.4.1, if $f : P \longrightarrow M$ is such a $\langle k \rangle$-immersion in general position, then the double-point set $\Sigma_f$ is a 1-dimensional $\langle k, k \rangle$-manifold with an orientation preserving, involution $T : \Sigma_f \longrightarrow \Sigma_f$ with no fixed points, such that

$$T(\partial_1 \Sigma_f) = \partial_2 \Sigma_f \quad \text{and} \quad T(\partial_2 \Sigma_f) = \partial_1 \Sigma_f.$$

We will need the following general result about such 1-dimensional, $\langle k, k \rangle$-manifolds equipped with such an involution as above.

**Lemma 11.5.1.** *Let $N$ be a 1-dimensional, closed, oriented, $\langle k, k \rangle$-manifold. Suppose that $N$ is equipped with an orientation preserving, involution $T : N \longrightarrow N$ with no fixed points, such that*

$$T(\partial_1 N) = \partial_2 N \quad \textit{and} \quad T(\partial_2 N) = \partial_1 N.$$

*Then,*

$$\beta_1 N = \beta_2 N = +\langle j \rangle \sqcup -\langle j \rangle$$

*for some integer $j$.*

*Proof.* We prove this by contradiction. Suppose that $\beta_1 N = +\langle j \rangle \sqcup -\langle l \rangle$ where $j \neq l$. Since $T$ preserves orientation and $T(\partial_1 N) = \partial_2 N$ and $T(\partial_2 N) = \partial_1 N$, it follows that $\beta_2 N = +\langle j \rangle \sqcup -\langle l \rangle$ as well.

If we forget the $\langle k, k\rangle$-structure on $N$, then $N$ is just an oriented, 1-dimensional manifold with boundary equal to

$$\partial_1 N \sqcup \partial_2 N = [(+\langle j\rangle \sqcup -\langle l\rangle) \times \langle k\rangle] \bigcup [(+\langle j\rangle \sqcup -\langle l\rangle) \times \langle k\rangle]. \tag{11.10}$$

By reorganizing the above union, we see that the zero-dimensional manifold in (11.10) is equal to $+\langle 2 \cdot k \cdot j\rangle \sqcup -\langle 2 \cdot k \cdot l\rangle$.

However since $j \neq l$, there is no oriented, one dimensional manifold with boundary equal to $+\langle 2 \cdot k \cdot j\rangle \sqcup -\langle 2 \cdot k \cdot l\rangle$. This yields a contradiction. This proves the lemma. □

**Proposition 11.5.2.** *Let $P$ be a closed $\langle k\rangle$-manifold of dimension $2n+1$, let $M$ be a manifold of dimension $4n+1$ and let $f : P \longrightarrow M$ be a $\langle k\rangle$-immersion. Then there is a regular homotopy (through $\langle k\rangle$-immersions) of $f$ to a $\langle k\rangle$-immersion $f' : P \longrightarrow M$, such that*

$$\beta_1 \Sigma_{f'} \;=\; \beta_2 \Sigma_{f'} \;=\; f'_\beta(\beta P) \cap f'(\mathrm{Int}(P)) \;=\; \emptyset.$$

*Proof.* First, by choosing a small, regular homotopy, we may assume that $f$ is in general position. Since $\beta P$ is a closed $2n$-dimensional manifold and $2n < \frac{4n+1}{2}$, the fact that $f$ is in general position implies that $f_\beta : \beta P \longrightarrow M$ is an embedding.

Furthermore, we may assume that $f_\beta(\beta P)$ is disjoint from the image of the double point set of the immersion $f|_{\mathrm{Int}(P)} : \mathrm{Int}(P) \longrightarrow M$.

Consider the intersection $f_\beta(\beta P) \cap f(\mathrm{Int}(P))$. We choose a closed, disk neighborhood $U \subset \mathrm{Int}(P)$ that contains $f|^{-1}_{\mathrm{Int}(P)}(f_\beta(\beta P))$, such that the restriction $f|_U : U \longrightarrow M$ is an embedding (we may choose $U$ so that $f|_U$ is an embedding because $f_\beta(\beta P)$ is disjoint from the image of the double point set of $f|_{\mathrm{Int}(P)}$). By

Lemma 11.5.1 it follows that there is a diffeomorphism

$$f|_U^{-1}(f_\beta(\beta P)) \;\cong\; \beta_1\Sigma_f \;\cong\; +\langle j\rangle \sqcup -\langle j\rangle$$

for some integer $j$, and so, the oriented, algebraic intersection number associated to the intersection $f(U) \cap f_\beta(\beta P)$ is equal to zero. By the Whitney trick, we may find an isotopy through embeddings $\phi_t : U \longrightarrow M$ with

$$\phi_0 = f|_U \quad \text{and} \quad \phi_t|_{\partial U} = f|_{\partial U} \quad \text{for all } t \in [0,1]$$

such that $\phi_1(U) \cap f_\beta(\beta P)$.

We then may extend this isotopy over the rest of $P$ by setting it equal to $f$ for all $t \in [0,1]$ on the compliment of $U \subset P$. This concludes the proof of the lemma. □

**Corollary 11.5.3.** *Let $P$ be a 2-connected, closed, oriented $\langle k\rangle$-manifold of dimension $2n+1$. Let $M$ be a 2-connected, oriented, manifold of dimension $4n+1$, and let $f : P \longrightarrow M$ be a $\langle k\rangle$-immersion. Then $f$ is homotopic through $\langle k\rangle$-maps to a $\langle k\rangle$-embedding.*

**Remark 11.5.1.** In the statement of the above corollary, we are not asserting that any $\langle k\rangle$-immersion $f : P \longrightarrow M$ is regularly homotopic to a $\langle k\rangle$-embedding. The homotopy through $\langle k\rangle$-maps constructed in the proof of this result may very well not be a homotopy through $\langle k\rangle$-immersions.

*Proof.* Assume that $f$ is in general position. By the previous proposition we may assume that $f_\beta : \beta P \longrightarrow M$ is an embedding and that $\beta_1\Sigma_f = \emptyset$. We may choose a

collar embedding

$$h : \partial_1 P \times [0, \infty) \longrightarrow P \quad \text{with} \quad h^{-1}(\partial_1 P) = \partial P_1 \times \{0\},$$

such that for each $i \in \langle k \rangle$, the restriction map

$$f|_{h(\partial_1^i P \times [0,\infty))} : h(\partial_1^i P \times [0, \infty)) \longrightarrow M$$

is an embedding, where $\partial_1^i P = \Phi^{-1}(\beta P \times \{i\})$. Now let $U \subset M$ be a closed tubular neighborhood of $f_{\beta P}(\beta P) \subset M$, disjoint from the image $f(P \setminus h(\partial_1 P \times [0, \infty)))$, such that the boundary $\partial U$ is transverse to $f(P)$. We define,

$$Z := M \setminus \mathrm{Int}(U), \qquad P' := f^{-1}(Z), \qquad f' := f|_{P'}. \tag{11.11}$$

By construction, $P'$ and $Z$ are a manifolds with boundary, $f'$ maps $\partial P'$ into $\partial Z$, and $f'(\mathrm{Int}(P')) \subset \mathrm{Int}(Z)$. The corollary will be proven if we can find a homotopy of $f'$, relative $\partial P'$, to a map

$$f'' : (P', \partial P') \longrightarrow (Z, \partial Z)$$

which is an embedding. Using the 2-connectivity of both $P'$ and $Z$ (and the dimensional conditions on $P'$ and $Z$), the existence of such a homotopy follows from (12, Theorem 4.1) (or from (19, Theorem 1.1)). □

## 11.6. Representing $\langle k \rangle$-Maps by $\langle k \rangle$-Embeddings

We are now in a position to prove Theorem 5.4.1 from Section 5.4. It follows as a corollary of the results developed throughout this section. Here is the theorem restated again for the convenience of the reader.

**Theorem 11.6.1.** *Let $n \geq 2$ be an integer and let $k > 2$ be an odd integer. Let $M$ be a 2-connected, oriented manifold of dimension $4n+1$. Then any $\langle k \rangle$-map $f : V_k^{2n+1} \longrightarrow M$ is homotopic through $\langle k \rangle$-maps to a $\langle k \rangle$-embedding.*

*Proof.* Since $M$ is 2-connected, it follows that the map $\widehat{f} : \widehat{V}_k^{2n+1} \longrightarrow M$ (which is the map induced by the $\langle k \rangle$-map $f$), extends to a map $M(\mathbb{Z}/k, 2n) \longrightarrow M$, where $M(\mathbb{Z}/k, 2n)$ is a $\mathbb{Z}/k$-Moore-space (see Lemma 5.3.1). It then follows that the vector bundle $\widehat{f}^*(TM) \longrightarrow \widehat{V}_k^{2n+1}$ is classified by a map $\widehat{V}_k^{2n+1} \longrightarrow BSO$ that factors through a map $M(\mathbb{Z}/k, 2n) \longrightarrow BSO$. When $k$ is odd, the $\mathbb{Z}/k$-homotopy group $\pi_{2n}(BSO; \mathbb{Z}/k)$ is trivial. It follows that the bundle $\widehat{f}^*(TM) \longrightarrow \widehat{P}$ is trivial. Now, it is easy to verify that the $\langle k \rangle$-manifold $V_k^{2n+1}$ is parallelizable as a $\langle k \rangle$-manifold (see Definition 11.3.1). It then follows from Corollary 11.3.1 that the map $f$ is homotopic through $k$-maps to a $\langle k \rangle$-immersion, which we denote by $f' : V_k^{2n+1} \longrightarrow M$. The proof of the theorem then follows by applying Corollary 11.5.3 to the $\langle k \rangle$-immersion $f'$. □

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