CATEGORICAL ACTIONS ON SUPERCATEGORY $\mathcal{O}$

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DISSERTATION ABSTRACT

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This dissertation uses techniques from the theory of categorical actions of Kac-Moody algebras to study the analog of the BGG category $\mathcal{O}$ for the queer Lie superalgebra. Chen recently reduced many questions about this category to its so-called types A, B, and C blocks. The type A blocks were completely described in joint work with Brundan in terms of the general linear Lie superalgebra. This dissertation proves that the type C blocks admit the structure of a tensor product categorification of the n-fold tensor power of the natural $\mathfrak{sp}_\infty(\mathbb{C})$-module. Using this result, we relate the combinatorics for these blocks to Webster's orthodox bases for the quantum group of type C_∞, verifying the truth of a recent conjecture of Cheng-Kwon-Wang. This dissertation contains coauthored material.

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CHAPTER I

INTRODUCTION

This chapter contains excerpts from the introduction of coauthored material in [BD2]. J. Brundan and I worked closely in the writing of that introduction.

1.1. Categorical actions: a broad overview

In the early 1990s, Lusztig constructed canonical bases in certain integrable representations of quantum groups. These bases possess amazing integrality and positivity properties. The study of categorical actions of Kac-Moody algebras has its roots in the idea that these bases must be the shadows cast by some higher structures. Some of the first examples studied include:

- The representation theory of symmetric groups and their Hecke algebras [LLT, A, G].
- Rational representations of the general linear group [BK1].
- The BGG category $\mathcal{O}$ for the general linear Lie (super)algebra [BFK, B1].

Inspired by common elements in these examples, Chuang and Rouquier [CR] unified them under the axiomatic framework of $\mathfrak{sl}_2$-*categorifications*. Subsequently, Rouquier [R] extended the ideas to categorical actions of arbitrary Kac-Moody algebras. Independently, motivated instead by the low-dimensional topology problem of categorifying Reshetikhin-Turaev invariants, Khovanov and Lauda [KL1, KL2] came up with equivalent formulations of the same notions.

Let us give a brief overview of the idea of categorical actions. Let $\mathfrak{s}$ be any any complex, symmetrizable Kac-Moody algebra with simple roots $\{\alpha_i \mid i \in I\}$, weight lattice P, Chevalley generators $\{e_i, f_i \mid i \in I\}$, and coroots $h_i := [e_i, f_i]$. In classical representation theory, one is often interested in integrable linear representations of $\mathfrak{s}$. Roughly speaking, this is the data of:

– A complex vector space V with weight decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$.

– Locally nilpotent maps $e_i : V_\lambda \to V_{\lambda+\alpha_i}$, and $f_i : V_{\lambda+\alpha_i} \to V_\lambda$ for each $i \in I$ and $\lambda \in P$.

The linear maps are required to satisfy certain relations. For example, the commutator $e_i f_i - f_i e_i$ must act on V_λ as the scalar $\lambda(h_i)$. Put into fancier language, classical representation theory studies representations of $\mathfrak{s}$ in the category $\mathcal{V}ec$, whose objects are vector spaces, and whose morphisms are linear maps.

In higher representation theory, one replaces the category $\mathcal{V}ec$ with the 2-category $\mathfrak{Cat}$ of categories, functors, and natural transformations. A *weak categorical representation* of $\mathfrak{s}$ is the data of:

– A (suitably finite, additive, linear...) category $\mathcal{C}$, equipped with a decomposition $\mathcal{C} = \bigoplus_{\lambda \in P} \mathcal{C}_\lambda$.

– Biadjoint functors $F_i : \mathcal{C}_\lambda \to \mathcal{C}_{\lambda-\alpha_i}$ and $E_i : \mathcal{C}_{\lambda-\alpha_i} \to \mathcal{C}_\lambda$ for each $i \in I$ and $\lambda \in P$.

These functors are necessarily exact, so they induce linear operators $e_i := [E_i]$ and $f_i := [F_i]$ on the split Grothendieck group $[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$. We require that these operators make $[\mathcal{C}]$ into an integrable linear representation of $\mathfrak{s}$, with λ-weight space $[\mathcal{C}_\lambda]$.

The key idea of higher representation theory is that, rather than specifying relations between functors on the Grothendieck group, one should look for natural transformations between the functors which induce the required Grothendieck group relations. This line of thinking leads to the notion of a *strong categorical action*, first introduced by Chuang-Rouquier [CR] in the case $\mathfrak{s} = \mathfrak{sl}_2$, and then for general $\mathfrak{s}$ in [R, KL2]. By exploiting the "higher structure" afforded by the natural transformations, Chuang and Rouquier proved Broué's Abelian defect conjecture for the symmetric group. This was the first example demonstrating that the techniques of higher representation theory can provide more information than classical techniques alone.

We will not give the full definition of strong categorical actions here; it may be found in Chapter 2. To illustrate the general idea, the axioms assert amongst other things that for every weight $\lambda \in P$ and each $i \in I$ for which $\lambda(h_i)$ is a positive integer, the functors E_iF_i and $F_iE_i : \mathcal{C}_\lambda \to \mathcal{C}_\lambda$ should admit a distinguished natural isomorphism:

$$\rho_{i,\lambda} : E_iF_i \Rightarrow F_iE_i \oplus \mathrm{Id}_{\mathcal{C}_\lambda}^{\lambda(h_i)} . \tag{1.1}$$

On the level of Grothendieck groups, this isomorphism of functors induces the relation that $[e_i, f_i]$ acts on the λ-weight space $[\mathcal{C}_\lambda]$ as multiplication by the scalar $\lambda(h_i)$. We remark that the data of a strong categorical action is equivalent to a strict 2-functor $\mathfrak{U}(\mathfrak{s}) \to \mathfrak{Cat}$, where $\mathfrak{U}(\mathfrak{s})$ is the Kac-Moody 2-category of Khovanov-Lauda and Rouquier. This 2-category is discussed at length in [BD1].

1.2. Super category $\mathcal{O}$

In this dissertation, we use the rich structure arising from categorical actions to study the analog of the BGG category $\mathcal{O}$ associated to the queer Lie superalgebra

$\mathfrak{q}_n(\Bbbk)$. Chen [C] reduced most questions about this category to the study of the so-called types A, B, C blocks. Already in the early 2000s, Brundan [B2] had investigated the type B blocks (which correspond to integer weights) and formulated a version of the Kazhdan-Lusztig conjecture for characters of irreducibles in those blocks in terms of certain canonical bases for the quantum group of type B_∞. Recently, Cheng, Kwon and Wang [CKW] noted the type A blocks (defined below) and type C blocks (which correspond to half-integer weights) mirror combinatorics of the quantum groups of type A_∞ and C_∞, respectively. This led to analogs of Brundan's Kazhdan-Lusztig conjecture for the type A and type C blocks.

In joint work with Brundan [BD2], we have proved the truth of the Cheng-Kwon-Wang conjecture for type A blocks ([CKW, Conjecture 5.13]). In fact, we establish an equivalence of categories between the type A blocks for $\mathfrak{q}_n(\Bbbk)$ and the integral blocks of category $\mathcal{O}$ for a general linear Lie superalgebra. This reduces the Cheng-Kwon-Wang conjecture for type A blocks to the Kazhdan-Lusztig conjecture of [B1], which was proved already in [CLW, BLW].

While the type A conjecture has been verified, the type B conjecture from [B2] and the original type C conjecture [CKW, Conjecture 5.9] appear to be incorrect. Tsuchioka discovered in 2010 that the type B canonical bases considered in [B2] fail to satisfy appropriate positivity properties, so that the conjecture is certainly false. After [CKW] appeared, Tsuchioka also pointed out similar issues with the type C canonical bases studied in [CKW], so that conjecture is likely incorrect, as well.

Despite the fact that there are problems with the original type C conjecture, many of the arguments used to prove the type A conjecture can be modified to study the type C blocks. Chapters 3 and 4 of this dissertation develop these ideas. In particular, we prove a modified version of the Cheng-Kwon-Wang conjecture for type C blocks, where one replaces Lusztig's canonical basis with Webster's "orthodox basis" arising from the indecomposable projective modules of the tensor product algebras [W1, §4]. This modified conjecture was proposed independently by Cheng, Kwon and Wang in a revision of their article ([CKW, Conjecture 5.10]). It is not as satisfactory as the situation for type A blocks, however, since there is no elementary algorithm to compute Webster's basis explicitly (unlike the canonical basis).

We will review our results for type A blocks in more detail in Section 1.4.1 below. In Section 1.4.2 we will discuss our new results for the type C blocks, and Section 1.4.3 will say a little more about the type B blocks.

1.3. Super background

To formulate our results in more detail, we need to briefly recall some basic notions of supercategories and superalgebra.

1.3.1. Supercategories

Let $\Bbbk$ be a ground field which is algebraically closed of characteristic zero, and fix for eternity a choice of $\sqrt{-1} \in \Bbbk$. We adopt the language of [BE1, Definition 1.1]:

- A *supercategory* is a category enriched in the monoidal category of vector superspaces (i.e., $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces over $\Bbbk$ with morphisms that are parity-preserving linear maps).

- Any morphism in a supercategory decomposes uniquely into an even and an odd morphism as $f = f_{\bar{0}} + f_{\bar{1}}$. If f is homogeneous we write $|f| \in \mathbb{Z}/2\mathbb{Z}$ for its parity. A *superfunctor* between supercategories means a $\Bbbk$-linear functor which preserves the parities of morphisms.

– Given superfunctors $F, G : \mathcal{C} \to \mathcal{D}$, a *supernatural transformation* $\eta : F \Rightarrow G$ is a family of morphisms $\eta_M = \eta_{M,\bar{0}} + \eta_{M,\bar{1}} : FM \to GM$ for each $M \in \operatorname{ob}\mathcal{C}$, such that $\eta_{N,p} \circ Ff = (-1)^{|f|p} Gf \circ \eta_{M,p}$ for every homogeneous morphism $f : M \to N$ in $\mathcal{C}$ and each $p \in \mathbb{Z}/2\mathbb{Z}$. It is *even* (resp. *odd*) if $\eta_M = \eta_{M,\bar{0}}$ (resp. $\eta_M = \eta_{M,\bar{1}}$) for all M.

For example, suppose that A is a locally unital superalgebra, i.e. an associative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ equipped with a distinguished collection $\{1_x \mid x \in X\}$ of mutually orthogonal even idempotents such that $A = \bigoplus_{x,y \in X} 1_x A 1_y$. Then there is a supercategory A-smod consisting of finite dimensional left A-supermodules M which are locally unital in the sense that $M = \bigoplus_{x \in X} 1_x M$. Even morphisms in A-smod are parity-preserving linear maps such that $f(av) = af(v)$ for all $a \in A, v \in M$; odd morphisms are parity-reversing linear maps such that $f(av) = (-1)^{|a|} af(v)$ for homogeneous a.

For any supercategory $\mathcal{C}$, the *Clifford twist* $\mathcal{C}^{\mathtt{CT}}$ is the supercategory whose objects are pairs (X, ϕ) where $X \in \operatorname{ob}\mathcal{C}$ and $\phi \in \operatorname{End}_{\mathcal{C}}(X)$ is an odd involution. A morphism $f : (X, \phi) \to (X', \phi')$ in $\mathcal{C}^{\mathtt{CT}}$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $f_p \circ \phi = (-1)^p \phi' \circ f_p$ for each $p \in \mathbb{Z}/2\mathbb{Z}$. One can also take Clifford twists of superfunctors and supernatural transformations, so that $\mathtt{CT}$ is actually a 2-superfunctor from the 2-supercategory of supercategories to itself in the sense of [BE1, Definition 2.2]. The following lemma is a variation on [KKT, Lemma 2.3].

Lemma 1.3.1. *Suppose $\mathcal{C}$ is an additive supercategory in which all even idempotents split. Also assume that for every $X \in \operatorname{ob}\mathcal{C}$, there exists another object $\Pi X \in \operatorname{ob}\mathcal{C}$, and an odd isomorphism $\zeta_X : \Pi X \xrightarrow{\sim} X$. Then, the supercategories $\mathcal{C}$ and $(\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$ are superequivalent.*

Proof. Note that objects in the supercategory $(\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$ consist of triples (X, ϕ, ψ) for $X \in \mathrm{ob}\,\mathcal{C}$ and $\phi, \psi \in \mathrm{End}_{\mathcal{C}}(X)_{\bar{1}}$ such that $\phi^2 = \psi^2 = \mathrm{id}$ and $\phi \circ \psi = -\psi \circ \phi$. Morphisms $f : (X, \phi, \psi) \to (X', \phi', \psi')$ in $(\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$ are morphisms $f : X \to X'$ in $\mathcal{C}$ such that $f_p \circ \phi = (-1)^p \phi' \circ f_p$, and $f_p \circ \psi = (-1)^p \psi' \circ f_p$ for each $p \in \mathbb{Z}/2$.

Define a superfunctor $F : \mathcal{C} \to (\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$ as follows. On an object $X \in \mathrm{ob}\,\mathcal{C}$, let $FX := (X \oplus \Pi X, \phi, \psi)$ where

$$\phi = \begin{pmatrix} 0 & \zeta_X \\ \zeta_X^{-1} & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} 0 & -\sqrt{-1}\zeta_X \\ \sqrt{-1}\zeta_X^{-1} & 0 \end{pmatrix}.$$

On a homogeneous morphism $f : X \to X'$, we let $Ff : FX \to FX'$ be the morphism defined by the matrix $\begin{pmatrix} f & 0 \\ 0 & \Pi f \end{pmatrix}$, where $\Pi f : \Pi X \to \Pi X'$ denotes $(-1)^{|f|}\zeta_{X'}^{-1} \circ f \circ \zeta_X$. We show that F is a superequivalence by checking that it is full, faithful and evenly dense (see [BE1]). It is obviously faithful. To see that it is full, take an arbitrary homogeneous morphism $f : FX \to FX'$ in $(\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$. Viewing f as a 2×2 matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ of morphisms in $\mathcal{C}$, we need to show that $f_{12} = f_{21} = 0$ and $f_{22} = \Pi f_{11}$. This follows easily on considering the matrix entries in the identities $\phi' \circ f = (-1)^{|f|} f \circ \phi$ and $\psi' \circ f = (-1)^{|f|} f \circ \psi$.

Finally, to check that F is evenly dense, we take any object $(X, \phi, \psi) \in \mathrm{ob}(\mathcal{C}^{\mathtt{CT}})^{\mathtt{CT}}$, and must show that it is evenly isomorphic to an object in the image of F. Let

$$e_1 := \frac{1 - \sqrt{-1}\phi \circ \psi}{2}, \qquad e_2 := \frac{1 + \sqrt{-1}\phi \circ \psi}{2}.$$

These are mutually orthogonal idempotents summing to the identity in $\mathrm{End}_{\mathcal{C}}(X)_{\bar{0}}$. Hence, we may decompose X as $X = X_1 \oplus X_2$ with X_i being the image of e_i. We then have that $\phi = e_2 \circ \phi \circ e_1 + e_1 \circ \phi \circ e_2$, and similarly for ψ. Now we observe that

$$e_2 \circ \psi \circ e_1 = e_2 \circ \psi = \frac{\psi + \sqrt{-1}\phi}{2} = \sqrt{-1}\,\phi \circ e_1 = \sqrt{-1}\,e_2 \circ \phi \circ e_1.$$

Similarly, $e_1 \circ \psi \circ e_2 = -\sqrt{-1}\,e_1 \circ \phi \circ e_2$. The map $e_2 \circ \phi \circ e_1 \circ \zeta_{X_1}$ is an even isomorphism $\Pi X_1 \xrightarrow{\sim} X_2$, hence, $X = X_1 \oplus X_2 \cong X_1 \oplus \Pi X_1 = FX_1$. Under this isomorphism, $\phi = e_2 \circ \phi \circ e_1 + e_1 \circ \phi \circ e_2$ corresponds to the matrix $\begin{pmatrix} 0 & \zeta_{X_1} \\ \zeta_{X_1}^{-1} & 0 \end{pmatrix}$. Similarly, $\psi = \sqrt{-1}\,e_2 \circ \phi \circ e_1 - \sqrt{-1}\,e_1 \circ \phi \circ e_2$ corresponds to $\begin{pmatrix} 0 & -\sqrt{-1}\zeta_{X_1} \\ \sqrt{-1}\zeta_{X_1}^{-1} & 0 \end{pmatrix}$. This verifies that (X, ϕ, ψ) is evenly isomorphic to FX_1. $\square$

For example, if A is a locally unital superalgebra, then there is an obvious isomorphism between the Clifford twist $A\text{-smod}^{\mathtt{CT}}$ of this supercategory and the supercategory $A \otimes C_1\text{-smod}$, where C_1 denotes the rank one Clifford superalgebra generated by an odd involution c, and $A \otimes C_1$ is the usual braided tensor product of superalgebras. Hence, $(A\text{-smod}^{\mathtt{CT}})^{\mathtt{CT}}$ is isomorphic to $A \otimes C_2\text{-smod}$ where $C_2 := C_1 \otimes C_1$ is the rank two Clifford superalgebra generated by $c_1 := c \otimes 1$ and $c_2 := 1 \otimes c$. In this situation, the above lemma is obvious as $A \otimes C_2$ is isomorphic to the matrix superalgebra $M_{1|1}(A)$, which is Morita superequivalent to A.

For another construction of a supercategory, suppose that $\mathcal{C}$ is any $\Bbbk$-linear category. Then we let $\mathcal{C} \oplus \Pi\mathcal{C}$ be the supercategory whose objects are formal direct sums $V_1 \oplus \Pi V_2$ for $V_1, V_2 \in \operatorname{ob}\mathcal{C}$, with morphisms $V_1 \oplus \Pi V_2 \to W_1 \oplus \Pi W_2$ being matrices of the form $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ for $f_{ij} \in \operatorname{Hom}_{\mathcal{C}}(V_j, W_i)$. The $\mathbb{Z}/2\mathbb{Z}$-grading is defined so that $f_{\bar{0}} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$ and $f_{\bar{1}} = \begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix}$.

For example, if B is any locally unital algebra (no super!) and $\mathcal{C} = B$-mod is the category of finite-dimensional locally unital B-modules, then the supercategory $\mathcal{C} \oplus \Pi\mathcal{C}$ may be identified with the category B-smod, where we view B as a purely even superalgebra.

1.3.2. Lie superalgebras

A *Lie superalgebra* $\mathfrak{g}$ is a superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, equipped with an even linear map $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies the following conditions for every homogeneous $x, y, z \in \mathfrak{g}$:

- Super skew-symmetry: $[x, y] = -(-1)^{|x||y|}[y, x]$.
- Super Jacobi identity: $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

These axioms imply that the restriction of the bracket makes the subspace $\mathfrak{g}_{\bar{0}}$ into a Lie algebra in the ordinary sense.

The basic example is the *general linear Lie superalgebra* $\mathfrak{gl}_{m|n}(\Bbbk)$. The elements of this Lie algebra are $(m+n) \times (m+n)$ matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{1.2}$$

where A is an $m \times m$ matrix, D is an $n \times n$ matrix, etc. The grading is given by declaring that the even subspace $\mathfrak{gl}_{m|n}(\Bbbk)_{\bar{0}}$ consists of all such matrices with B and C being zero, while the odd subspace consists of those matrices for which A and D are zero. The bracket on $\mathfrak{gl}_{m|n}(\Bbbk)$ is given by the matrix supercommutator:

$$[x, y] := xy - (-1)^{|x||y|}yx.$$

While this equation only makes sense when x and y are homogeneous, it may be extended to non-homogeneous elements of $\mathfrak{gl}_{m|n}(\Bbbk)$ by linearity.

1.4. Statement of results

Now fix $n \geq 1$ and let $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ be the *queer Lie superalgebra* $\mathfrak{q}_n(\Bbbk)$. This is the subalgebra of $\mathfrak{gl}_{n|n}(\Bbbk)$ consisting of all matrices of block form

$$\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right). \tag{1.3}$$

Let $\mathfrak{b}$ (resp. $\mathfrak{h}$) denote the standard Borel (resp. Cartan) subalgebra of $\mathfrak{q}$ consisting matrices of the form (1.3) in which A and B are upper triangular (resp. diagonal). Let $\mathfrak{t} := \mathfrak{h}_{\bar{0}}$. We let $\delta_1, \ldots, \delta_n$ denote the basis for $\mathfrak{t}^*$ such that δ_i picks out the ith diagonal entry of the matrix A.

Given $\lambda \in \mathfrak{t}^*$, write $\lambda = \sum_{r=1}^{n} \lambda_r \delta_r$. It will be convenient to define certain subsets of $\mathfrak{t}^*$ as follows:

- Given some $z \in \Bbbk$ with $2z \notin \mathbb{Z}$, and some *sign sequence* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ with each $\sigma_r = \pm$, we let $\Lambda_{\boldsymbol{\sigma} z}$ denote the collection of all $\lambda \in \mathfrak{t}^*$ such that each λ_r is in the set $\sigma_r(z + \mathbb{Z})$.

- Let Λ_0 denote the collection of all $\lambda \in \mathfrak{t}^*$ where each λ_r is an integer. These are the *integral weights*.

- Let $\Lambda_{\frac{1}{2}}$ denote the collection of all $\lambda \in \mathfrak{t}^*$ with each $\lambda_r \in \frac{1}{2} + \mathbb{Z}$. These are the *half-integer weights*.

Fixing a choice of symbols $X \in \{\boldsymbol{\sigma} z, 0, \frac{1}{2}\}$, we let $s\mathcal{O}_X$ denote the supercategory of all $\mathfrak{q}$-supermodules M such that:

– M is finitely generated as a $\mathfrak{q}$-supermodule;

– M is locally finite-dimensional over $\mathfrak{b}$;

– M is semisimple over $\mathfrak{t}$ with weights in Λ_X.

Morphisms in $s\mathcal{O}_X$ are arbitrary (not necessarily even) $\mathfrak{q}$-supermodule homomorphisms, so $s\mathcal{O}_X$ is indeed a supercategory.

1.4.1. Type A blocks

Letting $\boldsymbol{\sigma}$ and z vary, the blocks of the categories $s\mathcal{O}_{\boldsymbol{\sigma}z}$ are the *type A blocks*. This dissertation does not contain a detailed study of these blocks, but we give an overview here.

Recent work by Cheng-Kwon-Wang [CKW] observed that the combinatorics of the type A blocks can be described in terms of the Kac-Moody algebra $\mathfrak{sl}_\infty$ associated to the Dynkin diagram A_∞, and its quantization $U_q(\mathfrak{sl}_\infty)$. Building off their observations, joint work with Brundan [BD2] proved the following:

Main Theorem (Type A). *When n is even (resp. odd), the supercategory $s\mathcal{O}_{\boldsymbol{\sigma}z}$ (resp. $s\mathcal{O}^{\mathtt{CT}}_{\boldsymbol{\sigma}z}$) splits a direct sum $\mathcal{O} \oplus \Pi\mathcal{O}$, for some $\mathbb{k}$-linear category $\mathcal{O}$. Moreover, the category $\mathcal{O}$ admits the structure of a tensor product categorification of $V^{\otimes\boldsymbol{\sigma}} := V^{\sigma_1} \otimes \cdots V^{\sigma_n}$, where V^+ is the natural module associated to the Lie algebra $\mathfrak{sl}_\infty$-module, and V^- is its dual.*

When n is odd, the theorem actually describes the Clifford twists of the type A blocks. We can recover the type A blocks by Clifford twisting again, since $(s\mathcal{O}^{\mathtt{CT}}_{\boldsymbol{\sigma}z})^{\mathtt{CT}}$ is equivalent to $s\mathcal{O}_{\boldsymbol{\sigma}z}$, by Lemma 1.3.1.

The detailed definition of tensor product categorifications will be given in Chapter 2. The assertion that $\mathcal{O}$ is a tensor product categorification of $V^{\otimes \boldsymbol{\sigma}}$ roughly means that:

- The category $\mathcal{O}$ is a highest weight category in the sense of Cline, Parshall, and Scott [CPS], with irreducible objects $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ labeled by the set $\mathbf{B} = \mathbb{Z}^n$, which is partially ordered using the *Bruhat ordering.* This is a certain canonically defined partial ordering implicit in Lusztig's work on based modules [Lu]. For $\boldsymbol{b} \in \mathbf{B}$, we let $P(\boldsymbol{b})$ denote a fixed projective cover of $L(\boldsymbol{b})$, and $M(\boldsymbol{b})$ the corresponding standard object. The category $\mathcal{O}^{\Delta}$ denotes the exact subcategory of objects with a filtration by standard modules.

- There are biadjoint functors $F_i, E_i : \mathcal{O} \to \mathcal{O}$ for every $i \in \mathbb{Z}$ which preserve $\mathcal{O}^{\Delta}$, and induce operators on the Grothendieck group such that he linear isomorphism $[\mathcal{O}^{\Delta}] \to V^{\otimes \boldsymbol{\sigma}}$ given by $[M(\boldsymbol{b})] \mapsto v_{\boldsymbol{b}}$ is an isomorphism of $\mathfrak{sl}_\infty$-modules. In particular, $[\mathcal{O}^{\Delta}]$ is an integrable representation of $\mathfrak{sl}_\infty$, so there is a weak categorical action of $\mathfrak{sl}_\infty$ on $\mathcal{O}$. Here $\{v_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$ is the *monomial basis* for $V^{\otimes \boldsymbol{\sigma}}$.

- The compositions of the functors F_i and E_i admit certain natural transformations which upgrade the weak categorical action of $\mathfrak{sl}_\infty$ on $\mathcal{O}$ to a strong categorical action.

Combining the above theorem with a powerful result for the uniqueness of tensor product categorifications of $V^{\otimes \boldsymbol{\sigma}}$ [BLW, Theorem 2.12], we see that $\mathcal{O}$ is equivalent to the sum of the integral blocks in the BGG category $\mathcal{O}$ associated to a general linear Lie (super)algebra. Results from [CLW, BLW] about general linear Lie superalgebras further imply that

$$(P(\boldsymbol{a}) : M(\boldsymbol{b})) = [M(\boldsymbol{b}) : L(\boldsymbol{a})] = d_{\boldsymbol{a},\boldsymbol{b}}(1)$$

where the polynomials $d_{\boldsymbol{a},\boldsymbol{b}}(q)$ are the entries of the transition matrix used to express Lusztig's canonical basis for the corresponding quantum deformation of $V^{\otimes \boldsymbol{\sigma}}$ in terms of the monomial basis. An algorithm to compute the canonical bases, hence the polynomials $d_{\boldsymbol{a},\boldsymbol{b}}(q)$, can be found in [BD2, §8]. This demonstrates the combinatorics in the type A blocks are governed by combinatorics associated to the quantum group $U_q(\mathfrak{sl}_\infty)$, which is completely explicit.

1.4.2. Type C blocks

This dissertation is devoted to the study of the *type C blocks* which are the blocks in the category $s\mathcal{O}_{\frac{1}{2}}$. Let $\mathfrak{sp}_\infty$ denote the Kac-Moody algebra associated to the Dynkin diagram C_∞ with V its natural module. The main result of this thesis can be summarized as follows:

Main Theorem (Type C). *When n is even (resp. odd) the supercategory $s\mathcal{O}_{\frac{1}{2}}$ (resp. $s\mathcal{O}_{\frac{1}{2}}^{\mathtt{CT}}$) splits as a direct sum $\mathcal{O} \oplus \Pi\mathcal{O}$, where $\mathcal{O}$ is a $\Bbbk$-linear category. Moreover, $\mathcal{O}$ admits the structure of a tensor product categorification of $V^{\otimes n}$, where V is the natural $\mathfrak{sp}_\infty$-module.*

As in the type A situation above, this theorem actually describes the Clifford twists of the type C blocks when n is odd. We may recover the type C blocks by Clifford twisting a second time and applying Lemma 1.3.1.

Combining this theorem with the uniqueness of tensor product categorifications, we also demonstrate that the combinatorics in the type C blocks can be expressed in terms of Webster's *orthodox bases* associated to the quantum group $U_q(\mathfrak{sp}_\infty)$. This was independently conjectured by Cheng, Kwon, and Wang in the updated version of their paper ([CKW, Conjecture 5.11]). Although this does not provide as complete a description of the combinatorics as we have for the type A blocks, it still has significant consequences for our example. For example, we will also use our type C theorem to determine the associated crystal underlying the type C blocks, and then classify the indecomposable projective-injective (or *prinjective*) objects of $\mathcal{O}$.

1.4.3. Type B blocks

The type B blocks are the blocks of the category $s\mathcal{O}_0$. These are the most interesting blocks of all, but due to time constraints in the preparation of this dissertation, a study of these blocks will not be included. However, it should be mentioned that, from the standpoint of higher representation theory, the behavior of the type B blocks differs substantially from the types A and C blocks. Indeed, in the type A and C cases, the supercategory $s\mathcal{O}$ splits as a direct sum $\mathcal{O} \oplus \Pi\mathcal{O}$, for a $\Bbbk$-linear category $\mathcal{O}$, i.e., we can "de-superize" the theory, thereby fitting the types A and C cases into the existing framework of categorical actions. In contrast, the behavior exhibited in the type B blocks is genuinely "super." Because of this, the detailed study of these blocks requires a notion of *supercategorical actions*. This theory is still under-developed, but recent work by Brundan and Ellis [BE2] on super Kac-Moody 2-categories has laid the foundations for the subject.

We let $s\mathcal{O}_0^{\mathtt{Q}}$ denote the Serre subcategory of $s\mathcal{O}_0$ generated by the type $\mathtt{Q}$ irreducibles, i.e., those which admit an odd involution. Similarly, we let $s\mathcal{O}_0^{\mathtt{M}}$ denote the Serre subcategory of $s\mathcal{O}_0$ generated by the type $\mathtt{M}$ irreducibles, i.e., those which have no odd involution. Despite the fact that the axiomatic definitions have not been fixed, some version of the following conjecture appears to be true:

Main Conjecture (Type B). *The supercategory* $s\mathcal{O}_0^{\mathtt{Q}} \oplus (s\mathcal{O}_0^{\mathtt{M}})^{\mathtt{CT}}$ *admits the structure of a tensor product supercategorification of* $V^{\otimes n}$*, where* V *is the natural module for the Lie algebra* $\mathfrak{so}_\infty$ *associated to the Dynkin diagram* B_∞*, where all long simple roots are* even *but the short simple root is* odd.

In future work, I plan to make these definitions more explicit and prove this conjecture. Just as Chuang-Rouquier [CR] used categorical actions to prove Broué's Abelian defect conjecture for the symmetric group, studying these supercategorical actions may provide insights towards a proof of Broué's conjecture for the spin symmetric group.

1.5. Organization

This dissertation is organized as follows:

- Chapter 2 will provide the necessary background material on categorical actions and tensor product categorifications. It will recall the proof of the well-known result that the integral blocks in the BGG category $\mathcal{O}$ for $\mathfrak{gl}_n(\mathbb{k})$ admits the structure of a tensor product categorification of $(V^+)^{\otimes n}$, where V^+ is as above. Having recalled the required definitions, the chapter concludes with a brief discussion of the type A blocks.

- Chapter 3 is dedicated to the proof of the Main Theorem for type C blocks which we described in the previous section. Many of results in this section have proofs which are nearly identical to analogous results proved in [BD2]. Because of this, the chapter contains many excerpts from coauthored material.

– Chapter 4 will apply the results of Chapter 3 to study the structure of the type C blocks. In particular, it will relate the combinatorics of the block with Webster's type C orthodox bases from [W1]. It will also give a concrete description of the canonically defined *associated crystal* for the type C blocks, and use it to classify the prinjective objects in these blocks.

CHAPTER II

CATEGORICAL ACTIONS

This chapter gives an overview of the theory of categorical actions of Kac-Moody algebras. We will use these categorical actions in chapters 3 and 4 to prove results about the type C blocks in the BGG category $\mathcal{O}$ associated to the Lie superalgebra $\mathfrak{q}_n(\mathbb{k})$.

2.1. Schurian and highest weight categories

Before giving the formal definition of categorical actions of Kac-Moody algebras, we must review some basic notions.

2.1.1. Schurian categories

We begin by recalling the notion of a Schurian category from [BLW, §2]:

Definition 2.1.1. A *Schurian category* is a $\mathbb{k}$-linear, Abelian category $\mathcal{C}$ such that:

- $\mathcal{C}$ has enough projectives and injectives.
- Every object of $\mathcal{C}$ has finite length.
- The endomorphism ring of any irreducible object is one-dimensional.

We can view Schurian categories as generalization of the categories of finite dimensional modules over a finite dimensional algebra. To make this more precise, suppose that A is a locally unital algebra, i.e., A is equipped with a system of orthogonal idempotents $\{1_x \mid x \in X\}$ such that $A = \bigoplus_{x,y,\in X} 1_x A 1_y$, and let mod-$A$ denote the category of finite dimensional, right A-modules M for which

$M = \bigoplus_{x \in X} M1_x$. We have the following characterization of Schurian categories, the proof of which is outlined in [BLW, §2.1].

Proposition 2.1.2. *A category $\mathcal{C}$ is Schurian if and only if it is equivalent to* mod-A*, where A is a locally unital algebra for which the one-sided ideals $1_x A$ and $A1_x$ are finite dimensional for every $x \in X$.*

If $\mathcal{C}$ is a Schurian category, we let p$\mathcal{C}$ denote the additive category of projective objects of $\mathcal{C}$, and write $[\mathcal{C}]$ for the complexified split Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathrm{p}\mathcal{C})$.

2.1.2. Highest weight categories

We also need the notion of a highest weight category, which was first introduced by Cline, Parshall, and Scott in [CPS].

Definition 2.1.3. A *highest weight category* $(\mathcal{C}, \mathbf{B}, \preceq)$ is the data of:

- A Schurian category $\mathcal{C}$.
- A set $\mathbf{B}$ which indexes a complete set of non-isomorphic irreducible objects $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ in $\mathcal{C}$.
- A partial order $\preceq$ on $\mathbf{B}$.

For each $\boldsymbol{b} \in \mathbf{B}$, fix a projective cover $P(\boldsymbol{b})$ of $L(\boldsymbol{b})$. Define the *standard object* $\Delta(\boldsymbol{b})$ as the maximal quotient of $P(\boldsymbol{b})$ whose composition multiplicities satisfy the properties

$$[\Delta(\boldsymbol{b}) : L(\boldsymbol{b})] = 1 \quad \text{and} \quad [\Delta(\boldsymbol{b}) : L(\boldsymbol{c})] = 0 \text{ unless } \boldsymbol{c} \preceq \boldsymbol{b}.$$

In order for $(\mathcal{C}, \mathbf{B}, \preceq)$ to be a highest weight category, we require that each projective $P(\boldsymbol{b})$ has a finite filtration

$$0 = P_0 \subset P_1 \subset \cdots \subset P_d = P(\boldsymbol{b})$$

such that $P_d/P_{d-1} \cong \Delta(\boldsymbol{b})$, and for every $1 \leq r < d$, there is some $\boldsymbol{c}_r \succ \boldsymbol{b}$ such that $P_r/P_{r-1} \cong \Delta(\boldsymbol{c}_r)$.

If $\mathcal{C}$ is a highest weight category, any filtration of an object of $\mathcal{C}$ with subquotients isomorphic to standard objects is called a Δ-*flag*. We let $\mathcal{C}^{\Delta}$ denote the exact subcategory of all objects of $\mathcal{C}$ with a Δ-flag, and let $[\mathcal{C}^{\Delta}]$ denote its corresponding complexified Grothendieck group. The classes $\{[\Delta(\boldsymbol{b})] \mid \boldsymbol{b} \in \mathbf{B}\}$ form a basis for $[\mathcal{C}^{\Delta}]$, and our condition on the Δ-flags of projectives in $\mathcal{C}$ implies that there is an inclusion $[\mathcal{C}] \hookrightarrow [\mathcal{C}^{\Delta}]$. This is an isomorphism when $\mathbf{B}$ is finite, or, more generally, when every $\boldsymbol{b} \in \mathbf{B}$ is comparable to only finitely many $\boldsymbol{c} \in \mathbf{B}$.

2.1.3. Serre subcategories and Serre quotients

We next recall the well-known definitions of Serre subcategories and Serre quotients. These will play an important role in chapter 4.

Assume that $\mathcal{C}$ is a Schurian category with its irreducible objects labeled by a set $\mathbf{B}$, and suppose $\mathbf{B}' \subset \mathbf{B}$. Let $\mathcal{C}'$ denote the *Serre subcategory* of $\mathcal{C}$ generated by the irreducible objects

$$\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}'\}.$$

This means that $\mathcal{C}'$ is the full subcategory of $\mathcal{C}$ whose objects are those $M \in \operatorname{ob}\mathcal{C}$ satisfying the condition that whenever $[M : L(\boldsymbol{b})] \neq 0$, then $\boldsymbol{b} \in \mathbf{B}'$. Note that $\mathcal{C}'$ is also Schurian.

Let $\mathcal{C}'' = \mathcal{C}/\mathcal{C}'$ denote the corresponding *Serre quotient category*. This is a Schurian category, too. By definition, the objects of the category $\mathcal{C}''$ are the same as the objects of $\mathcal{C}$. The morphisms in $\mathcal{C}''$ are constructed as follows. Given objects $M, N \in \operatorname{ob} \mathcal{C}'' = \operatorname{ob} \mathcal{C}$, let $\Omega(M, N)$ denote the collection of all pairs (M', N'), where $M' \subset M$ and $N' \subset N$ satisfy $M/M', N' \in \operatorname{ob} \mathcal{C}'$. The set $\Omega(M, N)$ is partially ordered by $(M', N') \leq (M'', N'')$ if and only if $M'' \subset M'$ and $N' \subset N''$. When this happens, composing with the inclusion $M'' \hookrightarrow M'$ and the quotient map $N/N' \twoheadrightarrow N/N''$ induces a linear map $\operatorname{Hom}_{\mathcal{C}}(M', N/N') \to \operatorname{Hom}_{\mathcal{C}}(M'', N/N'')$. Define

$$\operatorname{Hom}_{\mathcal{C}''}(M, N) := \varinjlim \operatorname{Hom}_{\mathcal{C}}(M', N/N')$$

where the colimit is taken over all pairs $(M', N') \in \Omega(M, N)$.

Let $\pi : \mathcal{C} \to \mathcal{C}''$ be the obvious exact quotient functor. The definition of morphisms in $\mathcal{C}''$ imply that whenever M is an object of $\mathcal{C}'$, then $\pi M \cong 0$ in $\mathcal{C}''$. In fact, the category $\mathcal{C}''$ and the functor π are universal with respect to this property: whenever there is a Schurian category $\mathcal{E}$ and an exact functor $G : \mathcal{C} \to \mathcal{E}$ for which $G(M) \cong 0$ for every $M \in \operatorname{ob} \mathcal{C}'$, then there exists a unique exact functor $\bar{G} : \mathcal{C}'' \to \mathcal{D}$ such that $\bar{G} \circ \pi = G$.

Next, suppose that $\mathcal{C}$ is a highest weight category, so $\mathbf{B}$ is equipped with a partial order $\preceq$. An *ideal* or *lower set* in $\mathbf{B}$ is a subset $\mathbf{B}' \subset \mathbf{B}$ which satisfies the property that whenever $\boldsymbol{b} \in \mathbf{B}'$ and $\boldsymbol{a} \preceq \boldsymbol{b}$, then $\boldsymbol{a} \in \mathbf{B}'$, too. Assume that $\mathbf{B}'$ is an ideal, and set $\mathbf{B}'' = \mathbf{B} \setminus \mathbf{B}'$.

The assumption that $\mathbf{B}'$ is an ideal implies that that the subcategory $\mathcal{C}'$ defined above inherits a highest weight structure from $\mathcal{C}$, with poset $(\mathbf{B}', \preceq)$. The irreducibles and standards in $\mathcal{C}'$ are precisely those $L(\boldsymbol{b})$ and $\Delta(\boldsymbol{b})$ with $\boldsymbol{b} \in \mathbf{B}'$. The projective cover of $L(\boldsymbol{b})$ is the maximal quotient $P'(\boldsymbol{b})$ of $P(\boldsymbol{b})$ which lies in $\mathcal{C}'$. The category

$\mathcal{C}''$ inherits a highest weights structure, with poset $(\mathbf{B}'', \preceq)$. The irreducibles, projective indecomposables, and standard objects in $\mathcal{C}''$ are given by applying π to the corresponding objects in $\mathcal{C}$.

We record the following lemma for future use:

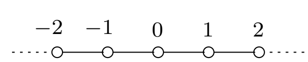
Lemma 2.1.4 ([BD1, Lemma 2.13]). *Assume $M, N \in \operatorname{ob}\mathcal{C}$ are such that all irreducible constituents of the head of V and the the socle of W are of the form $L(\boldsymbol{b})$ for $\boldsymbol{b} \in \mathbf{B}''$. The quotient functor π induces an isomorphism*

$$\operatorname{Hom}_{\mathcal{C}}(V, W) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}''}(\pi V, \pi W)$$

2.2. Type A and C combinatorics

The types A and C blocks for $\mathfrak{q}_n(\Bbbk)$ described in the introduction give rise to highest weight categories, where the labeling set $\mathbf{B} = \mathbb{Z}^n$ is partially ordered by the respective types A and C *Bruhat order*. These are particular instances of the "inverse dominance ordering" of [LW, Definition 3.2], which appears implicitly in Lusztig's work on tensor products of based modules in [Lu, §27.3]. The goal of this section is to introduce the necessary combinatorics to define these orders.

Let A_∞ denote the Dynkin diagram



which has nodes indexed by $I = \mathbb{Z}$. We denote the associated Kac-Moody algebra by $\mathfrak{sl}_\infty$, which we identify with the Lie algebra of finitely-supported complex matrices whose rows and columns are indexed by I. It is generated by the matrix units $f_i := e_{i+1,i}$ and $e_i := e_{i,i+1}$ for $i \in I$. The *natural representation* V^+ of $\mathfrak{sl}_\infty$ is the module of column vectors with standard basis $\{v_i^+ \mid i \in I\}$. We also need the *dual natural representation* V^- with basis $\{v_i^- \mid i \in I\}$. The action of the Chevalley generators on these bases is given by

$$e_i v_j^+ = \delta_{i+1,j} v_i^+, \qquad e_i v_j^- = \delta_{i,j} v_{i+1}^-, \tag{2.1}$$

$$f_i v_j^+ = \delta_{i,j} v_{i+1}^+, \qquad f_i v_j^- = \delta_{i+1,j} v_i^-. \tag{2.2}$$

Fix a sign sequence $\boldsymbol{\sigma} \in \{\pm\}^n$. We stress that the subsequent notation depends implicitly on this choice! Define

$$V^{\otimes \boldsymbol{\sigma}} := V^{\sigma_1} \otimes \cdots \otimes V^{\sigma_n}$$

which has monomial basis $\{v_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$ defined from $v_{\boldsymbol{b}} := v_{b_1}^{\sigma_1} \otimes \cdots \otimes v_{b_n}^{\sigma_n}$. For any $i \in I$, we define the *i-signature* of $\boldsymbol{b} \in \mathbf{B}$ by $i\text{-sig}(\boldsymbol{b}) = (i\text{-sig}(\boldsymbol{b})_1, \ldots, i\text{-sig}(\boldsymbol{b})_n) \in \{\mathtt{f}, \mathtt{e}, \bullet\}^n$ where

$$i\text{-sig}(\boldsymbol{b})_t := \begin{cases} \mathtt{f} & \text{if either } \sigma_t = + \text{ and } b_t = i, \text{ or } \sigma_t = - \text{ and } b_t = i+1, \\ \mathtt{e} & \text{if either } \sigma_t = + \text{ and } b_t = i+1, \text{ or } \sigma_t = - \text{ and } b_t = i, \\ \bullet & \text{otherwise.} \end{cases} \tag{2.3}$$

For $1 \leq t \leq n$, let $\boldsymbol{d}_t$ denote the element of $\mathbf{B}$ with $\sigma_t 1$ in the t-th entry, and zero everywhere else. Then, the Chevalley generators act on the monomial basis of $V^{\otimes \boldsymbol{\sigma}}$ by

$$f_i v_{\boldsymbol{b}} = \sum_{\substack{1 \leq t \leq n \\ i\text{-sig}(\boldsymbol{b})_t = \mathtt{f}}} v_{\boldsymbol{b}+\boldsymbol{d}_t}, \qquad e_i v_{\boldsymbol{b}} = \sum_{\substack{1 \leq t \leq n \\ i\text{-sig}(\boldsymbol{b})_t = \mathtt{e}}} v_{\boldsymbol{b}-\boldsymbol{d}_t}. \tag{2.4}$$

The root system of $\mathfrak{sl}_\infty$ has *weight lattice* $P := \bigoplus_{i \in I} \mathbb{Z}\,\omega_i$ where ω_i is the ith fundamental weight. For $i \in I$, we set

$$\varepsilon_i := \omega_i - \omega_{i-1}, \qquad \alpha_i := \varepsilon_i - \varepsilon_{i+1}.$$

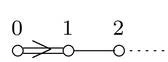
We identify ε_i with the weight of the vector v_i^+ in the $\mathfrak{sl}_\infty$-module V^+. Then, $v_i^- \in V^-$ is of weight $-\varepsilon_i$, and $v_{\boldsymbol{b}} \in V^{\otimes \boldsymbol{\sigma}}$ has weight

$$\mathrm{wt}(\boldsymbol{b}) := \sum_{r=1}^{n} \sigma_r \varepsilon_{b_r} \in P. \tag{2.5}$$

Let $\trianglelefteq$ denote the *dominance order* on P defined from $\beta \trianglelefteq \gamma$ if and only if $\gamma - \beta \in \bigoplus_{i \in I} \mathbb{N}\,\alpha_i$. For $1 \leq s \leq n$, it will also be convenient to define $\mathrm{wt}_s(\boldsymbol{b}) := \sum_{1 \leq r \leq s} \sigma_r \varepsilon_{b_r}$, so that $\mathrm{wt}_n(\boldsymbol{b}) = \mathrm{wt}(\boldsymbol{b})$.

Definition 2.2.1. The type A *Bruhat order* $\preceq$ on $\mathbf{B}$ associated to $\boldsymbol{\sigma}$ is the partial order defined by declaring that $\boldsymbol{b} \preceq \boldsymbol{a}$ if and only if $\mathrm{wt}_s(\boldsymbol{b}) \trianglerighteq \mathrm{wt}_s(\boldsymbol{a})$ for all $s = 1, \ldots, n$, with equality when $s = n$. In particular, $\boldsymbol{b}$ and $\boldsymbol{a}$ are comparable only when $\mathrm{wt}(\boldsymbol{b}) = \mathrm{wt}(\boldsymbol{a})$.

Next, we mimic the same constructions, replacing $\mathfrak{sl}_\infty$ with the Kac-Moody algebra $\mathfrak{sp}_\infty$ associated to the Dynkin diagram C_∞



with nodes indexed by $I = \mathbb{N}$.

Denote the Chevalley generators of $\mathfrak{sp}_\infty$ by $\{e_i, f_i \mid i \in I\}$. The natural $\mathfrak{sp}_\infty$-module V has basis $\{v_j \mid j \in \mathbb{Z}\}$ and action defined from

$$f_i v_j = \begin{cases} v_{j+1} & \text{if } j = \pm i \\ 0 & \text{otherwise} \end{cases}, \qquad e_i v_j = \begin{cases} v_{j-1} & \text{if } j = 1 \pm i \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\mathfrak{sp}_\infty$ preserves the non-degenerate symplectic form

$$(\cdot,\cdot) : V \otimes V \to \Bbbk \qquad (v_j, v_k) = \operatorname{sgn}(j-k)\delta_{j,1-k},$$

where sgn is the sign function. Hence, V is isomorphic to its dual representation, in contrast to the type A setting.

We redefine the i-signature $i\text{-sig}(\boldsymbol{b})$ for the type C case as

$$i\text{-sig}(\boldsymbol{b})_t := \begin{cases} \mathtt{f} & \text{if } \boldsymbol{b} = \pm i \\ \mathtt{e} & \text{if } \boldsymbol{b} = 1 \pm i \\ \bullet & \text{otherwise} \end{cases} \tag{2.6}$$

Also, let $\boldsymbol{d}_t$ denote the element of $\mathbf{B}$ with a 1 in the t-th entry, and zero everywhere else. The nth tensor power $V^{\otimes n}$ of the natural module has basis $\{v_{\boldsymbol{b}} := v_{b_1} \otimes \cdots \otimes v_{b_n} \mid \boldsymbol{b} \in \mathbf{B}\}$ and the action of $\mathfrak{sp}_\infty$ on this basis is given by

$$f_i v_{\boldsymbol{b}} = \sum_{\substack{1 \le t \le n \\ i\text{-sig}(\boldsymbol{b})_t = \mathtt{f}}} v_{\boldsymbol{b}+\boldsymbol{d}_t}, \qquad e_i v_{\boldsymbol{b}} = \sum_{\substack{1 \le t \le n \\ i\text{-sig}(\boldsymbol{b})_t = \mathtt{e}}} v_{\boldsymbol{b}-\boldsymbol{d}_t}. \tag{2.7}$$

In the *weight lattice* $P := \bigoplus_{i \in I} \mathbb{Z}\varepsilon_i$, we have the *simple roots* $\alpha_0 := -2\varepsilon_0$ and $\alpha_i := \varepsilon_{i-1} - \varepsilon_i$ for $i > 0$, where v_i is of weight ε_i. We denote the corresponding *dominance* *order* on P by $\trianglelefteq$. For $1 \leq s \leq n$, define

$$\mathrm{wt}_s(\boldsymbol{b}) := \sum_{\substack{1 \leq r \leq s \\ b_r \geq 1}} \varepsilon_{b_r - 1} - \sum_{\substack{1 \leq r \leq s \\ b_r \leq 0}} \varepsilon_{-b_r}.$$

The vector $v_{\boldsymbol{b}} \in V^{\otimes n}$ is of weight $\mathrm{wt}(\boldsymbol{b}) := \mathrm{wt}_n(\boldsymbol{b})$.

Definition 2.2.2. The *type C Bruhat order* $\preceq$ on $\mathbf{B}$ by is given by $\boldsymbol{b} \preceq \boldsymbol{a}$ if and only if $\mathrm{wt}_s(\boldsymbol{b}) \trianglerighteq \mathrm{wt}_s(\boldsymbol{a})$ for every s in $1, \ldots, n$, with equality when $s = n$.

2.3. Quiver Hecke categories

In the the next three sections, we are going to introduce quiver Hecke categories and categorical actions associated to $\mathfrak{sl}_\infty$ and $\mathfrak{sp}_\infty$. In order to unify our treatement of the two cases, we set $\mathfrak{s} = \mathfrak{sl}_\infty$ or $\mathfrak{sp}_\infty$, and make the following assumptions:

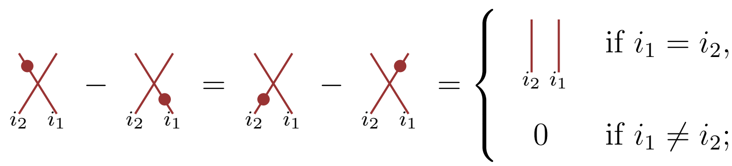
- If $\mathfrak{s} = \mathfrak{sl}_\infty$ let $I = \mathbb{Z}$ index its simple roots, let P denote its weight lattice. Given our fixed choice of $\boldsymbol{\sigma}$ as above, let T denote the mixed tensor space $T = V^{\otimes \boldsymbol{\sigma}}$. We make $\mathbf{B} = \mathbb{Z}^n$ into a poset using the type A Bruhat order $\preceq$ associated to $\boldsymbol{\sigma}$.

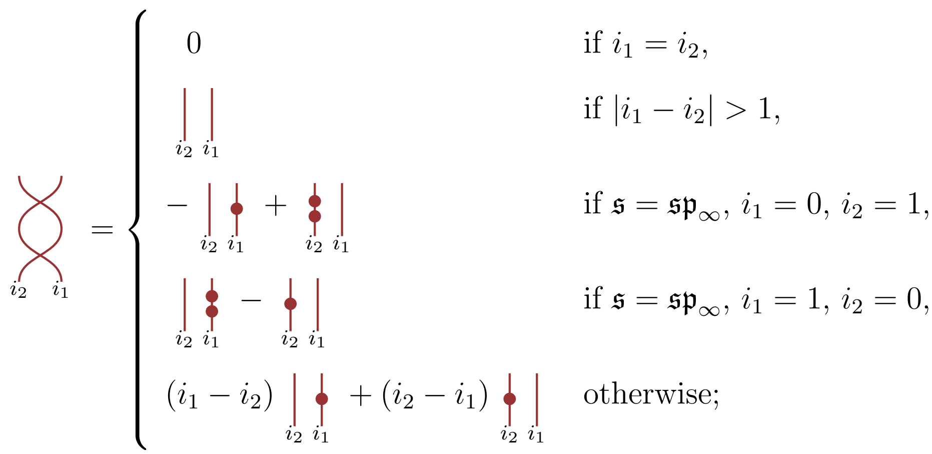
- If $\mathfrak{s} = \mathfrak{sp}_\infty$, let $I = \mathbb{N}$ index its simple roots, let P denote its weight lattice, and let T denote the tensor space $T = V^{\otimes n}$. We partially order $\mathbf{B} = \mathbb{Z}^n$ with the type C Bruhat order $\preceq$.

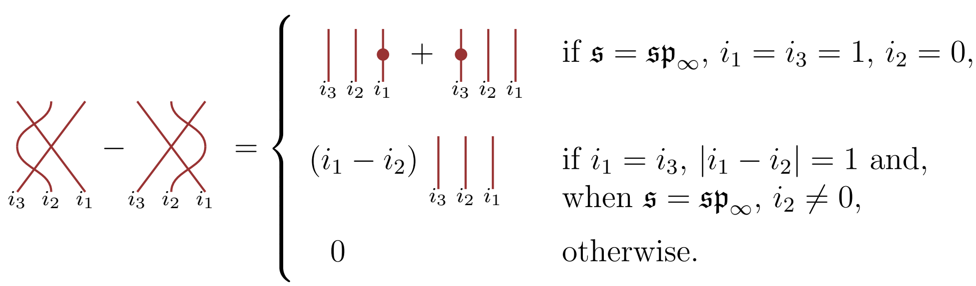
Next, we use the string calculus of [KL1] to define the *quiver Hecke category* $\mathcal{QH}$ associated to $\mathfrak{s}$. An expository account of the string calculus can be found in [BD1, §3].

Definition 2.3.1. The *quiver Hecke category* $\mathcal{QH}$ associated to $\mathfrak{s}$ is the strict $\Bbbk$-linear monoidal category generated by objects I and morphisms  $: i \to i$ and

 $: i_2 \otimes i_1 \to i_1 \otimes i_2$ subject to the following relations:







Let I^d denote the set of words $\boldsymbol{i} = i_d \cdots i_1$ of length d in the alphabet I, and identify $\boldsymbol{i} \in I^d$ with the object $i_d \otimes \cdots \otimes i_1 \in \mathrm{ob}\, \mathcal{QH}$. Then, the locally unital algebra

$$QH_d := \bigoplus_{\boldsymbol{i}, \boldsymbol{i}' \in I^d} \mathrm{Hom}_{\mathcal{QH}}(\boldsymbol{i}, \boldsymbol{i}') \tag{2.8}$$

is the *quiver Hecke algebra* associated to $\mathfrak{s}$. These algebras were originally defined by Khovanov and Lauda [KL1] and Rouquier [R].

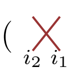
2.4. Categorical actions

In this section, we give axioms for (integrable) categorical actions of the Kac-Moody algebra $\mathfrak{s}$. While we are specializing to the case where $\mathfrak{s} = \mathfrak{sl}_\infty$ or $\mathfrak{sp}_\infty$, the definition has obvious analogs for other Kac-Moody algebras. Recall that a linear representation M of $\mathfrak{s}$ is *integrable* if it decomposes into weight spaces $M = \bigoplus_{\lambda \in P} M_\lambda$, and the Chevalley generators e_i and f_i act locally nilpotently on M.

Definition 2.4.1. A categorical action of $\mathfrak{s}$ on a Schurian category $\mathcal{C}$ is the data of:

(D1) A *weight decomposition* $\mathcal{C} = \bigoplus_{\lambda \in P} \mathcal{C}_\lambda$.

(D2) A strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{C})$, where $\mathcal{E}nd(\mathcal{C})$ is the strict monoidal category whose objects are $\Bbbk$-linear functors and whose morphisms are natural transformations. For $i, i_1, i_2 \in I$, let F_i denote the functor $\Phi(i)$, and define natural transformations

$\xi_i := \Phi($  $) : F_i \Rightarrow F_i \quad \tau_{i_2,i_1} := \Phi($  $) : F_{i_2}F_{i_1} \Rightarrow F_{i_1}F_{i_2}.$

(D3) A functor E_i for every $i \in I$, plus natural transformations $\eta_i : \mathrm{Id}_\mathcal{C} \Rightarrow E_iF_i$ and $\varepsilon_i : F_iE_i \Rightarrow \mathrm{Id}_\mathcal{C}$ making (F_i, E_i) into an adjoint pair.

This data must satisfy the following conditions:

(D4) The natural transformation ξ_i is locally nilpotent, i.e., for every object $M \in \operatorname{ob} \mathcal{C}$, the induced endomorphism $\xi_{i,M} : F_iM \to F_iM$ is nilpotent.

(D5) The functor E_i is isomorphic to a left adjoint of F_i.

(D6) For every $\lambda \in P$, the restriction of F_i sends $\mathcal{C}_\lambda$ into $\mathcal{C}_{\lambda-\alpha_i}$.

(D7) The induced operators $f_i := [F_i]$ and $e_i := [E_i]$ make the complexified Grothendieck group $[\mathcal{C}]$ into an integrable linear representation of $\mathfrak{s}$, where $[\mathcal{C}_\lambda]$ is the λ-weight space.

If a category $\mathcal{C}$ is equipped with a categorical action of $\mathfrak{s}$, we will call $\mathcal{C}$ a *categorical representation of* $\mathfrak{s}$.

Remark 2.4.2. This is one of several equivalent definitions of categorical actions of $\mathfrak{s}$ found in the literature. Theorem 5.30 in [R] shows that this definition is equivalent to an integrable 2-representation of Rouquier's Kac-Moody 2-category $\mathfrak{U}(\mathfrak{s})$. Khovanov and Lauda originally gave a different definition of $\mathfrak{U}(\mathfrak{s})$, but recent work in [B5] shows that the 2-categories are isomorphic. My expository paper with J. Brundan [BD1] contains a detailed account of $\mathfrak{U}(\mathfrak{s})$.

Morphisms between categorical representations are *strongly equivariant functors*:

Definition 2.4.3. Suppose that $\mathcal{C}$ and $\mathcal{C}'$ are categorical representations of $\mathfrak{s}$. For clarity, let Φ', F_i', E_i', etc., denote the data associated with the categorical action on $\mathcal{C}'$. We say that a functor $G : \mathcal{C} \to \mathcal{C}'$ is *strongly equivariant* if its restriction to $\mathcal{C}_\lambda$ has its image in $\mathcal{C}'_\lambda$, and if there exists natural isomorphisms $\zeta_i : F_i'G \Rightarrow GF_i$ such that:

(E1) The natural transformation

$$E_i'G\varepsilon_i \circ E_i'\zeta_i E_i \circ \eta_i GE_i : GE_i \to E_i'G$$

is invertible.

(E2) We have the equality of natural transformations

$$G\xi_i \circ \zeta_i = \zeta_i \circ \xi_i' G : F_i' G \Rightarrow GF_i$$

(E3) We have the equality

$$G\tau_{i_2 i_1} \circ \zeta_{i_2} F_{i_1} \circ F_{i_2}' \zeta_{i_1} = \zeta_{i_1} F_{i_2} \circ F_{i_1}' \zeta_{i_2} \circ \tau_{i_2,i_1}' G,$$

where these are viewed as natural transformations $F_{i_2}' F_{i_1}' G \Rightarrow GF_{i_2} F_{i_2}$

2.5. Tensor product categorifications

Given categorical representations $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathfrak{s}$, we would like a notion of a category $\mathcal{C}_1 \otimes \mathcal{C}_2$ which serves as the "tensor product" of the categorical representations $\mathcal{C}_1$ and $\mathcal{C}_2$. At present, there is no known method to construct a general tensor product of categorical representations, although it it hoped that a construction will eventually emerge. In the meantime, Losev and Webster have introduced a notion of *tensor product categorification* in [LW], which provides a list of properties allowing one to recognize a given categorical representation $\mathcal{C}$ as a tensor product. In that definition, they assume that the Grothendieck group of $\mathcal{C}$ is a tensor product of highest weight modules. In our setting, the underlying $\mathfrak{s}$-module T is not a tensor product of integrable highest weight modules, so we need to extend the Losev-Webster definition. For the case where $\mathfrak{s} = \mathfrak{sl}_\infty$, the following definition is equivalent to the one found in [BLW, Definition 2.9]. The $\mathfrak{s} = \mathfrak{sp}_\infty$ case is the obvious reformulation of the $\mathfrak{sl}_\infty$ definition.

Definition 2.5.1. A *tensor product categorification* of T is a highest weight category $(\mathcal{C}, \mathbf{B}, \preceq)$ ($\mathbf{B}$ and $\preceq$ as above) equipped with the data (D2) and (D3), satisfying the conditions (D5) and (D4). We also require that:

(TPC1) The functors E_i and F_i preserve the category $\mathcal{C}^\Delta$.

(TPC2) The linear isomorphism $[\mathcal{C}^\Delta] \to T$ defined by $[\Delta(\boldsymbol{b})] \mapsto v_{\boldsymbol{b}}$ intertwines the actions of the induced operators $[F_i]$ and $[E_i]$ on $[\mathcal{C}^\Delta]$ with the action of the Chevalley generators e_i and f_i of $\mathfrak{s}$ on T.

If $\mathcal{C}$ is a tensor product categorification of T, it is automatic that $\mathcal{C}$ decomposes as a direct sum $\mathcal{C} = \bigoplus_{\lambda \in P} \mathcal{C}_\lambda$, where $\mathcal{C}_\lambda$ is the Serre subcategory of $\mathcal{C}$ generated by the irreducibles of the form $L(\boldsymbol{b})$, where $\mathrm{wt}(\boldsymbol{b}) = \lambda$. This shows that the prescribed highest weight structure on $\mathcal{C}$ induces a decomposition as in (D1). In addition, because $[\mathcal{C}]$ embeds into $[\mathcal{C}^\Delta] \cong T$, we see that $[\mathcal{C}]$ is itself an integrable $\mathfrak{s}$-module. This implies that tensor product categorifications are also categorical representations of $\mathfrak{s}$, in the sense of Definition 2.4.1.

In their paper [LW], Losev and Webster prove a powerful uniqueness result for tensor product categorifications associated to tensor products of integrable highest weight modules. In our infinite rank setting, the tensor space T is not a tensor product of highest weight modules, so the uniqueness theorem from [LW] does not immediately apply. Instead we need the following theorem, which extends the uniqueness in the type A setting.

Theorem 2.5.2 ([BLW, Theorem 2.11])**.** *For any $\boldsymbol{\sigma} \in \{\pm\}^n$, there exists a tensor product categorification $\mathcal{C}$ of $V^{\otimes \boldsymbol{\sigma}}$. Moreover, if $\mathcal{C}'$ is any other tensor product categorification of $V^{\otimes \boldsymbol{\sigma}}$, there is a equivalence of categories $G : \mathcal{C} \to \mathcal{C}'$ which is strongly equivariant, and satisfies $G(L(\boldsymbol{b})) \cong L'(\boldsymbol{b})$ for every $\boldsymbol{b} \in \mathbf{B}$.*

We remark that the result actually proved in [BLW] is slightly more general, as it includes tensor products of exterior powers of the modules V^+ and V^-. The techniques used to prove uniqueness in this theorem should also yield a proof that tensor product categorifications of the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$ are essentially unique. To our knowledge, the type C blocks of Chapter 3 are the only known categorification of $V^{\otimes n}$, so any uniqueness theorem along these lines has no application at this point, and we do not prove uniqueness in this dissertation.

Because machinery to construct general tensor product categorifications is currently unavailable, the existence statement in Theorem 2.5.2 relies on an explicit but ad hoc construction using categories of modules associated to the general linear Lie (super)algebra.

2.6. An example

To illustrate the theory in this chapter, we recall the construction of the tensor product categorification of the $\mathfrak{sl}_\infty$-module $V^{\otimes \boldsymbol{\sigma}}$ found in [BLW] for the special case where each $\sigma_r = +$. In fact, this construction is the basic model for the approach we are going to follow in Chapter 3 to construct tensor product categorifications of the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$.

For this section only, we will denote V^+ by V, so we write $V^{\otimes n}$ for $V^{\otimes \boldsymbol{\sigma}}$. Set $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{k})$. Let $\mathfrak{b}$ denote its standard Borel subalgebra of upper triangular matrices, and $\mathfrak{t}$ its Cartan subalgebra of diagonal matrices. Let $\delta_1, ..., \delta_n$ denote the usual coordinate functions in $\mathfrak{t}^*$. Define the *weight lattice* $P = \bigoplus_{i \in I} \mathbb{Z}\, \delta_i$, and a *weight dictionary* $\mathbf{B} \to P$ given by $\boldsymbol{b} \mapsto \lambda_{\boldsymbol{b}}$, where $\lambda_{\boldsymbol{b}} := \sum_{r=1}^n \lambda_{\boldsymbol{b},r} \delta_r$ where $\lambda_{\boldsymbol{b},r} = b_r - r + 1$. The presence of the $-r+1$ in the definition of $\lambda_{\boldsymbol{b},r}$ is a sort of "ρ-shift." We let $\mathcal{O}$

denote the category of finitely generated $\mathfrak{g}$-modules which are locally finite over $\mathfrak{b}$ and semisimple over $\mathfrak{t}$, with weights in the set P.

We stress that all of this notation applies only to this section. The notation $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}, \mathcal{O}$, etc. has different meaning outside of this section. The category $\mathcal{O}$ which we define here is the subcategory of the usual BGG category $\mathcal{O}$ associated to $\mathfrak{g}$ corresponding to integral weights. The fact that $\mathcal{O}$ categorifies $V^{\otimes n}$ was well-known before [BLW], for example, see [CR, §7.4]. In fact, this was a motivating examples for Losev and Webster's definition of tensor product categorification.

2.6.1. Highest weight structure

To construct the irreducible objects in $\mathcal{O}$, first define the Verma module $M(\boldsymbol{b})$ by

$$M(\boldsymbol{b}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b})$$

where $V(\boldsymbol{b})$ is the one-dimensional $U(\mathfrak{b})$ module associated to the $\mathfrak{t}$-weight $\lambda_{\boldsymbol{b}}$. By standard arguments, $M(\boldsymbol{b})$ has irreducible head $L(\boldsymbol{b})$, and every irreducible in $\mathcal{O}$ is isomorphic to some $L(\boldsymbol{b})$.

It is well-known that the category $\mathcal{O}$ is a highest weight category with poset $(\mathbf{B}, \preceq)$, where the standard modules correspond to the Vermas. This fact can be extracted from Chapters 1, 3, and 5 of [H].

2.6.2. Functors F_i and E_i

Let U denote the natural $\mathfrak{g}$-module of column vectors, with standard basis $u_1, ..., u_n$, and U^* its dual representation, with dual basis $\phi_1, ..., \phi_n$.

If $M \in \operatorname{ob} \mathcal{O}$, then $U \otimes M$ and $U^* \otimes M$ are also objects of $\mathcal{O}$, so we have functors $F := U \otimes -$ and $E := U^* \otimes - : \mathcal{O} \to \mathcal{O}$. There are canonical $\mathfrak{g}$-module homomorphisms $U \otimes U^* \to \Bbbk$ and $\Bbbk \to U^* \otimes U$ which induce natural transformations

$$\varepsilon : FE \Rightarrow \mathrm{Id}_{\mathcal{O}} \text{ and } \eta : \mathrm{Id}_{\mathcal{O}} \Rightarrow EF$$

making the (F, E) into an adjoint pair. The symmetric braiding $U \otimes U^* \to U^* \otimes U$ induces a natural isomorphism $FE \Rightarrow EF$, so there are also natural transformations

$$\varepsilon' : EF \Rightarrow \mathrm{Id}_{\mathcal{O}} \text{ and } \eta' : \mathrm{Id}_{\mathcal{O}} \Rightarrow FE$$

making making (E, F) into an adjoint pair.

For $1 \leq r, s, \leq n$, let $e_{r,s}$ denote the corresponding matrix unit in $\mathfrak{g}$. The trace form

$$\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \quad e_{r,s} \otimes e_{s',r'} \mapsto \delta_{r,r'}\delta_{s,s'}$$

defines an invariant element of the dual module $(\mathfrak{g} \otimes \mathfrak{g})^* = \mathfrak{g}^* \otimes \mathfrak{g}^*$. Using the trace form to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, we see that κ becomes identified with the *Casimir tensor*

$$\Omega = \sum_{r,s} e_{r,s} \otimes e_{s,r}.$$

Hence, Ω is invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$. Working in the associative algebra $U(\mathfrak{g}) \otimes U(\mathfrak{g})$, it follows that Ω commutes with the image of the coproduct $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Define a natural transformation $x : F \Rightarrow F$ given by letting $x_M : FM \to FM$ be endomorphism $FM = U \otimes M$ induced by multiplication by Ω. This is a $\mathfrak{g}$-module homomorphism because Ω commutes with all coproducts. Similarly, define a natural transformation $x^* : E \Rightarrow E$ given by letting x^*_M be the endomorphism of EM induced by multiplication by $-\Omega$. The following theorem can be extracted from [CR, §7.4.3].

Theorem 2.6.1. *1. For every $\boldsymbol{b} \in \mathbf{B}$, the object $M := FM(\boldsymbol{b})$ has a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where $M_t/M_{t-1} \cong M(\boldsymbol{b} + \boldsymbol{d}_t)$ for $1 \leq t \leq n$. Moreover, the endomorphism x_M preserves this filtration, and acts on the subquotient M_t/M_{t-1} as multiplication by the scalar b_t.

2. For every $\boldsymbol{b} \in \mathbf{B}$, the object $M = EM(\boldsymbol{b})$ has a filtration

$$0 = M^n \subset M^{n-1} \subset \cdots M^0 = M$$

where $M^{t-1}/M^t \cong M(\boldsymbol{b} - \boldsymbol{d}_t)$ for $1 \leq t \leq n$. The endomorphism x_M^* preserves this filtration, and acts on the subquotient M^{t-1}/M^t as multiplication by the scalar $b_t - 1$.

Proof. (1) The existence of the filtration uses the tensor identity:

$$FM = U \otimes_{\Bbbk} \big(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b})\big) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U \otimes_{\Bbbk} V(\lambda_{\boldsymbol{b}}))\,.$$

As a $\mathfrak{b}$-module, U has an obvious filtration $0 = U_0 \subset \cdots \subset U_n$, where U_t is the span of the vectors $u_1, ..., u_t \in U$. It follows that U_t/U_{t-1} is the one-dimensional $\mathfrak{b}$-module of weight δ_t. Because $\lambda_{\boldsymbol{b}} + \delta_t = \lambda_{\boldsymbol{b}+\boldsymbol{d}_t}$, the $\mathfrak{b}$-module $U \otimes V(\boldsymbol{b})$ has a filtration with subquotients of the form $V(\boldsymbol{b} + \boldsymbol{d}_t)$. The exactness of the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} - : U(\mathfrak{b})\text{-mod} \to U(\mathfrak{g})\text{-mod}$ implies that $M = U \otimes M(\boldsymbol{b})$ has the desired filtration.

The fact that x_M preserves the filtration comes from the observation that

$$\Omega = \frac{1}{2}(\Delta(C) - 1 \otimes C - C \otimes 1) \tag{2.9}$$

where $C = \sum_{r,s} e_{r,s} e_{s,r}$ is the central Casimir element of $U(\mathfrak{g})$, and Δ is the usual coproduct. The eigenvalues for the induced operator on M_t/M_{t-1} can be calculated using (2.9), along with the fact that C acts on any highest weight vector of weight $\lambda = \sum_{r=1}^{n} \lambda_r \delta_r$ as multiplication by the scalar $c_\lambda := \sum_{r=1}^{n} \lambda_r^2 + \sum_{1 \le r \le s \le n} (\lambda_r - \lambda_s)$. Hence, x_M acts on any subquotient of M isomorphic to $M(\boldsymbol{b} + \boldsymbol{d}_t)$ as multiplication by the scalar $\frac{1}{2}(c_{\lambda_{\boldsymbol{b}} + \boldsymbol{d}_t} - c_{\boldsymbol{d}_t} - c_{\lambda_{\boldsymbol{b}}})$. This magically simplifies to b_t.

The proof of (2) is similar. □

Corollary 2.6.2. *For any $M \in \mathrm{ob}\,\mathcal{O}$, the object FM (resp. EM) is a finite direct sum of generalized eigenspaces with respect to the operator x_M (resp. x_M^*). The eigenvalues of x_M and x_M^* lie in the set I.*

Proof. The theorem implies this immediately when M is a Verma module. Using exactness of E and F, the corollary also holds for every irreducible M, and hence for any M. □

Using the corollary, the functor F decomposes as $F = \bigoplus_{i \in I} F_i$, where $F_i M$ is defined to be the i-generalized eigenspace for $x_M : FM \to FM$. We have a similar decomposition of the functor $E = \bigoplus_{i \in I} E_i$, where the decomposition is done by taking eigenvalues with respect to the natural transformation x^*.

The natural transformations η and ε making (F, E) into an adjoint pair induce and adjunction making (F_i, E_i) into an adjoint pair for any $i \in I$. Indeed, a straightforward check shows that $x^* : E \Rightarrow E$ is the *right mate* to $x : F \Rightarrow F$, with respect to the adjunction (F, E). In other words, the composition

$$E \overset{\eta E}{\Rightarrow} EFE \overset{ExE}{\Rightarrow} EFE \overset{E\varepsilon}{\Rightarrow} E$$

is equal to $x^* : E \Rightarrow E$. Elementary facts about adjunctions imply that ε and η induce an adjunction making (F_i, E_i) and adjoint pair for every $i \in I$. Similarly, x^* is the *left mate* for x with respect to the adjunction (E, F), so (E_i, F_i) is also an adjoint pair.

2.6.3. Grothendieck group relations

Given any $\boldsymbol{b} \in \mathbf{B}$ and any $i \in I$, Theorem 2.6.1 implies that the object $F_i M(\boldsymbol{b})$ has a filtration with subquotients of the form $M(\boldsymbol{b} + \boldsymbol{d}_t)$, where t ranges over all indices such that $b_t = i$. Similarly, $E_i M(\boldsymbol{b})$ has a filtration with subquotients of the form $M(\boldsymbol{b} - \boldsymbol{d}_t)$, where now t ranges over all indices with $b_t = i+1$. Because they are exact, it follows that these functors preserve the category $\mathcal{O}^\Delta$ of objects of $\mathcal{O}$ with a Verma flag, and (TPC1) is satisfied.

Recall the definition of the i-signature from (2.3) for the special case where each $\sigma_r = +$. The observations from the previous paragraph demonstrate that

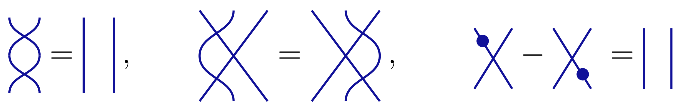
$$[F_i M(\boldsymbol{b})] = \sum_{i\text{-sig}(\boldsymbol{b})_t = \mathtt{f}} [M(\boldsymbol{b} + \boldsymbol{d}_t)], \quad [E_i M(\boldsymbol{b})] = \sum_{i\text{-sig}(\boldsymbol{b})_t = \mathtt{e}} [M(\boldsymbol{b} - \boldsymbol{d}_t)].$$

Comparing this with (2.4), the linear isomorphism $[\mathcal{O}^\Delta] \overset{\sim}{\to} (V^+)^{\otimes n}$ given by $[M(\boldsymbol{b})] \mapsto v_{\boldsymbol{b}}$ can be upgraded to an $\mathfrak{sl}_\infty$-module isomorphism, where the Chevalley generators e_i, f_i of $\mathfrak{sl}_\infty$ act on $[\mathcal{O}^\Delta]$ as $[E_i]$ and $[F_i]$, respectively. Hence, we have also checked (TPC2).

2.6.4. Affine Hecke Category

The rest of this section will be devoted to the construction of a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{End}(\mathcal{C})$, sending $i \in \operatorname{ob} \mathcal{QH}$ to F_i, for which the natural transformation $\xi_i : F_i \Rightarrow F_i$ is locally nilpotent, as required by (D2) and (D4). The construction of this functor is quite subtle. To define it, we need to pass through an intermediate category.

Definition 2.6.3. The (degenerate) *affine Hecke category* $\mathcal{AH}$ is the strict monoidal category with generating object 1, and generating morphisms  $: 1 \to 1$ and  $: 1 \otimes 1 \to 1 \otimes 1$, satisfying the relations:



Define $AH_d := \operatorname{Hom}_{\mathcal{AH}}(1^{\otimes d}, 1^{\otimes d})$. This is is the well-known (degenerate) *affine Hecke algebra.*

In contrast to the category $\mathcal{QH}$, the following theorem demonstrates that the affine Hecke category $\mathcal{AH}$ appears easily in our setting. It was first observed by Arakawa-Suzuki in Theorem 2.2.2 of [AS]. Its proof is an elementary verification of relations.

Theorem 2.6.4. *There is a strict monoidal functor $\Psi : \mathcal{AH} \to \mathcal{End}(\mathcal{O})$ given by*

$\Psi(1) = F$ $\quad$ $\Psi($  $) = x$ $\quad$ $\Psi($$) = t.$

Here $t : F^2 \Rightarrow F^2$ is the natural endomorphism of $F^2 = U \otimes U \otimes -$ given by swapping the two factors of U.

2.6.5. Cyclotomic quotients

In order to pass from the affine Hecke category $\mathcal{AH}$ to the quiver Hecke category $\mathcal{QH}$, we will need to exploit an isomophism between *cyclotomic quotients* of the algebras AH_d and QH_d. As this is analogous to the situation in Section 3.5 below, we review the details here.

To define the quotients, we label strings of diagrams from right to left. Given any $1 \leq r \leq d$, let x_r denote the diagram in AH_d with a dot on the rth string. Similarly, given any $\boldsymbol{i} \in I^d$, let $\xi_r 1_{\boldsymbol{i}}$ denote the diagram in $1_{\boldsymbol{i}} QH_d 1_{\boldsymbol{i}} = \mathrm{Hom}_{\mathcal{QH}}(\boldsymbol{i}, \boldsymbol{i})$ with a dot on the r-th string.

Fix $\mu = \sum_{i \in I} \mu_i \varepsilon_i \in P$, and define the *cyclotomic quotient* $AH_d(\mu)$ (resp. $QH_d(\mu)$) to be the quotient of AH_d (resp. QH_d) by the two-sided ideal generated by the the polynomial $\prod_{i \in I}(x_1 - i)^{\mu_i}$ (resp. the elements $\{\xi_1^{\mu_i} 1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^d\}$). By abuse, we let x_r and $\xi_r 1_{\boldsymbol{i}}$ denote the image of these elements in $AH_d(\mu)$ and $QH_d(\mu)$, respectively.

The cyclotomic quotients are finite-dimensional algebras. By [K, Lemma 7.1.4], the minimum polynomial of each x_r (calculated in $AH_d(\mu)$) has its roots in I. Therefore, the commutative subalgebra of $AH_d(\mu)$ generated by $x_1, ..., x_d$ contains a set of mutually orthogonal idempotents $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^d\}$ projecting any $AH_d(\mu)$-module onto its $\boldsymbol{i}$-word space:

$$1_{\boldsymbol{i}} M = \{m \in M | (x_r - i_r)^N m = 0 \text{ for } N \gg 0\}$$

A striking theorem of Brundan and Kleshchev [BK3], which was also noted by Rouquier [R, Proposition 3.15], shows that there is an isomorphism of locally unital algebras $QH_d(\mu) \xrightarrow{\sim} AH_d(\mu)$ given by

$$1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}} \text{ and } \xi_r 1_{\boldsymbol{i}} \mapsto (x_r - i_r)1_{\boldsymbol{i}}.$$

There is also an explicit formula for crossings which is a bit more complicated. We will not need the explicit formula here!

We remark that whenever $\mu, \mu' \in P$ satisfy $\mu_i \leq \mu'_i$ for each i, we have surjections $QH_d(\mu') \twoheadrightarrow QH_d(\mu)$ and $AH_d(\mu') \twoheadrightarrow AH_d(\mu)$. Hence, the sets of cyclotomic quotients $\{QH_d(\mu) \mid \mu \in P\}$ and $\{AH_d(\mu) \mid \mu \in P\}$ each form an inverse system of locally unital algebras with idempotents indexed by I^d. Taking the inverse limit of each system, we obtain the completions

$$\widehat{QH}_d := \lim_{\leftarrow} QH_d(\mu), \quad \widehat{AH}_d = \lim_{\leftarrow} \widehat{AH}_d.$$

As noted by Webster [W3], the isomorphisms of cyclotomic quotients $QH_d(\mu) \xrightarrow{\sim} AH_d(\mu)$ induce an isomorphism of completions $\widehat{QH}_d \xrightarrow{\sim} \widehat{AH}_d$.

2.6.6. From $\mathcal{AH}$ to $\mathcal{QH}$

Recall the monoidal functor $\Psi : \mathcal{AH} \to \mathcal{E}nd(\mathcal{O})$ defined above. Given any $d > 0$, Ψ induces an algebra homomorphism $\psi_d : AH_d \to NT_d$, the algebra of natural transformations $F^d \to F^d$. For any $M \in \operatorname{ob} \mathcal{O}$, there is an induced algebra homomorphism $\psi_{d,M} : AH_d \to \operatorname{End}_{\mathcal{O}}(F^d M)$. Corollary 2.6.2 shows that $\psi_{d,M}(x_1) = F^{d-1}x$ acts locally finitely with eigenvalues in I. It follows that there is some $\mu \in P$ such that $\psi_{d,M}$ factors through the cyclotomic quotient $AH_d(\mu)$, where the idempotent $1_{\boldsymbol{i}}$ projects F^d onto the summand $F_{\boldsymbol{i}}M := F_{i_d} \cdots F_{i_1} M$. Because we have such a cyclotomic quotient for any M, it follows that there is a locally unital algebra homomorphism

$$\hat{\psi}_d : \widehat{AH}_d \to NT_d(F)$$

where the algebra of natural transformations NT_d is made into a locally unital by equipping it with the distinguished idempotents $1_{\boldsymbol{i}}$ which project F^d onto the summand $F_{\boldsymbol{i}}$.

Pulling back along the maps $QH_d \hookrightarrow \widehat{QH}_d \xrightarrow{\sim} \widehat{AH}_d$, we obtain a locally unital algebra homomorphism $\varphi_d : QH_d \to NT_d$. The algebra homorphism φ_d is the data of a map between between morphisms spaces in $\mathcal{QH}$ and $\mathcal{End}(\mathcal{O})$. It is compatible with the monoidal structure in these categories, so we have a monoidal functor $\Phi : \mathcal{QH} \to \mathcal{End}(\mathcal{O})$ defined on the objects of $\mathcal{QH}$ by $\Phi(\boldsymbol{i}) = F_{\boldsymbol{i}}$. On a general morphism $f \in QH_d$, we define $\Phi(f) := \varphi_d(f)$. Hence, we have constructed the data of (D2). The fact that dotted strings act locally nilpotently is a consequence of our formula for the image of dotted strings under the isomorphism of cyclotomic quotients. This completes the proof that $\mathcal{O}$ is a tensor product categorification of $V^{\otimes n}$.

Remark 2.6.5. In early definitions of categorical actions of type A Kac-Moody algebras (e.g. [CR]), the data of (D2) satisfying (D4) was instead replaced by the data of:

(D2*) A strict monoidal functor $\Psi : \mathcal{AH} \to \mathcal{End}(\mathcal{C})$ with

$F := \Psi(1)$ $\quad x := \Psi($  $) : F \Rightarrow F$ $\quad t := \Psi($$) : F^2 \Rightarrow F^2.$

which satisfies:

(D4*) The functor F decomposes as $F = \bigoplus_{i \in I} F_i$, where F_i is defined so that F_iM is the i-generalized eigenspace for $x_M : F_iM \to F_iM$.

Hence, given the functor $\Psi : \mathcal{AH} \to \mathcal{E}nd(\mathcal{O})$ from theorem, we have satisfied the axioms for a type A categorical action in the sense of Chuang-Rouquier, without needing the additional complexity of passing to $\mathcal{QH}$. While it might be more convenient to reformulate our definitions in terms of the affine Hecke category, the approach using the category $\mathcal{AH}$ and the algebras AH_d has two main disadvantages:

- There is no obvious graded structure, which is essential when studying categorical actions of quantum groups.
- There is no adaptation of $\mathcal{AH}$ to Kac-Moody algebras outside of type A. In particular, we could not define categorical action of C_∞ using $\mathcal{AH}$.

In contrast, the category $\mathcal{QH}$ and its associated algebras QH_d do not possess these deficiencies.

2.7. Type A blocks revisited

Having introduced the required definitions, we revisit the classification of type A blocks in [BD2]. Fix some $\boldsymbol{\sigma} \in \{\pm\}^n$, and $z \in \Bbbk$ with $2z \notin \mathbb{Z}$. Recall the main type A theorem of Section 1.4.1, which says that the supercategory

$$s\mathcal{O} = \begin{cases} s\mathcal{O}_{\boldsymbol{\sigma} z} & \text{if } n \text{ is even} \\ s\mathcal{O}^{\mathtt{CT}}_{\boldsymbol{\sigma} z} & \text{if } n \text{ is odd} \end{cases}$$

decomposes as $\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$ where $\mathcal{O}$ is a $\Bbbk$-linear category admitting the structure of a tensor product categorification of $V^{\otimes \boldsymbol{\sigma}}$.

To construct a second tensor product categorification of $V^{\otimes \boldsymbol{\sigma}}$, set $p = \#\{r \mid \sigma_r = +\}$ and $q = n - p$. Let $s\mathcal{O}'$ denote the subcategory of the BGG supercategory $\mathcal{O}$ associated to $\mathfrak{gl}_{p|q}(\Bbbk)$ whose objects have integer weights. Arguments in [BLW] demonstrate that the supercategory $s\mathcal{O}'$ splits as $\mathcal{O}' \oplus \Pi\mathcal{O}'$, where $\mathcal{O}'$ is a $\Bbbk$-linear category. Using techniques similar to those employed in Section 2.6 above, Section 3 in [BLW] demonstrates that the category $\mathcal{O}'$ admits the structure of a tensor product categorification of $V^{\otimes n}$. Actually, in the case where each $\sigma_r = +$, the category $\mathcal{O}'$ is precisely the category of $\mathfrak{gl}_{n|0}(\Bbbk) = \mathfrak{gl}_n(\Bbbk)$ modules studied in Section 2.6.

Applying Theorem 2.5.2, there is strongly equivariant equivalence $\mathcal{O} \xrightarrow{\sim} \mathcal{O}'$. Hence, the categories $s\mathcal{O}$ and $s\mathcal{O}'$ are superequivalent. In the case where n is even, this demonstrates that every type A block is equivalent to a block in the category $s\mathcal{O}'$. In the case where n is odd, we Clifford twist and apply Lemma 1.3.1 to show that every type A block is equivalent to the Clifford twist of a block in the category $s\mathcal{O}'$.

Because the blocks in the category $s\mathcal{O}'$ have been studied extensively (e.g. [B1, CLW, BLW]) this result has strong implications for the type A blocks. As mentioned in Section 1.4.1, it shows that the combinatorics of the composition multiplicities of Verma modules is controlled by the computable combinatorics of canonical bases associated to the quantum group $U_q(\mathfrak{sl}_\infty)$.

We also remark that, while the number z was important in defining $s\mathcal{O}_{\boldsymbol{\sigma} z}$, it is irrelevant for defining the category $s\mathcal{O}'$. Therefore, for any $z, z' \in \Bbbk$ for which $2z, 2z' \notin \mathbb{Z}$, we have a superequivalence $s\mathcal{O}_{\boldsymbol{\sigma} z} \xrightarrow{\sim} s\mathcal{O}_{\boldsymbol{\sigma} z'}$.

Example 2.7.1. Suppose $n = 2$ and fix some $\boldsymbol{\sigma} \in \{\pm\}^n$. The set $\mathbf{B}$ labels the irreducible objects in the category $\mathcal{O} \subset s\mathcal{O}_{\boldsymbol{\sigma} z}$, where the irreducible $L(\boldsymbol{b})$ has highest weight

$$\lambda_{\boldsymbol{b}} := \sigma_1(b_1 + z)\delta_1 + \sigma_2(b_2 + z)\delta_2 \in \mathfrak{t}^*.$$

Here $\delta_1, \delta_2 \in \mathfrak{t}^*$ are the coordinate functions on the diagonal matrices in $\mathfrak{q}_2(\Bbbk)$, see Section 1.4. Using standard facts about the BGG category $\mathcal{O}$ for $\mathfrak{gl}_2(\Bbbk)$ (when $\sigma_1 =$

σ_2) and $\mathfrak{gl}_{1|1}(\Bbbk)$ (when $\sigma_1 \neq \sigma_2$), we can completely describe the type A blocks as follows. For any type A block $\mathcal{A}$, one of the following three cases must apply.

Case 1. Suppose that either:

- $\sigma_1 = \sigma_2$, and $\mathcal{A}$ contains an irreducible of the form $L(\boldsymbol{b})$, where $b_1 = b_2$.
- $\sigma_1 \neq \sigma_2$, and $\mathcal{A}$ contains an irreducible of the form $L(\boldsymbol{b})$, where $b_1 + b_2 \neq 0$.

Then, $\mathcal{A}$ is equivalent to the category of finite-dimensional vector spaces over $\Bbbk$. In particular $\mathcal{A}$ is semisimple and contains one irreducible object.

Case 2. Suppose that $\sigma_1 = \sigma_2$, and $\mathcal{A}$ contains an irreducible of the form $L(\boldsymbol{b})$, where $b_1 \neq b_2$. Then, $\mathcal{A}$ is equivalent to a regular block in the BGG category $\mathcal{O}$ for $\mathfrak{gl}_2(\Bbbk)$. Hence, $\mathcal{A}$ also contains the irreducible $L(\boldsymbol{b}')$, where $\boldsymbol{b}' = w\boldsymbol{b}$ is the tuple obtained from $\boldsymbol{b}$ by interchanging its entries. Without loss of generality, assume $b_1 > b_2$. Then, we have equalities

$$P(\boldsymbol{b}) = \Delta(\boldsymbol{b}) \text{ and } \Delta(\boldsymbol{b}') = L(\boldsymbol{b}').$$

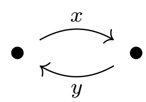
Moreover, the multiplicities of standard objects in projectives in $\mathcal{A}$ satisfies:

$$\begin{aligned} (P(\boldsymbol{b}) : \Delta(\boldsymbol{b})) = 1 \quad (P(\boldsymbol{b}) : \Delta(\boldsymbol{b}')) = 0 \\ (P(\boldsymbol{b}') : \Delta(\boldsymbol{b})) = 1 \quad (P(\boldsymbol{b}') : \Delta(\boldsymbol{b}')) = 1 \end{aligned}$$

From these numbers, we may calculate the composition multiplicities of standard modules using BGG reciprocity. We have the following relations in the Grothendieck group, which we identify with $V^{\otimes \boldsymbol{\sigma}}$ by $[\Delta(\boldsymbol{b})] \leftrightarrow v_{\boldsymbol{b}}$:

$$[P(\boldsymbol{b}) = v_{b_1} \otimes v_{b_2} \qquad \text{and} \qquad [P(\boldsymbol{b}')] = v_{b_2} \otimes v_{b_1} + v_{b_1} \otimes v_{b_2}.$$

These define elements of the canonical basis of $V^{\otimes\boldsymbol{\sigma}}$. In addition the block $\mathcal{A}$ is equivalent to the category mod-P of finite dimensional modules P, the path algebra of the quiver



modulo the relation $xy = 0$.

Case 3. Suppose that $\sigma_1 \neq \sigma_2$, and $\mathcal{A}$ contains an irreducible of the form $L(\boldsymbol{b})$, where $b_1 + b_2 = 0$. This is perhaps the most interesting case of all. In this case, $\mathcal{A}$ contains *all* irreducibles of the form $L(\boldsymbol{b})$, where $b_1 = -b_2$. The non-zero multiplicities of standard objects inside the projectives are given by

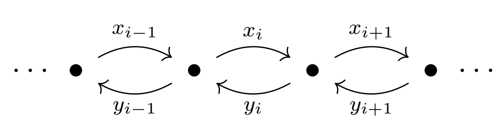
$$[P(b,-b) : \Delta(b,-b)] = 1 \qquad [P(b) : \Delta(b+1, -(b+1))] = 1$$

Again, identifying the the Grothendieck group with $V^{\otimes\boldsymbol{\sigma}}$, this means that

$$[P(b,-b)] = v_b \otimes v_{-b} + v_{b+1} \otimes v_{-(b+1)}$$

Again, these are elements of the canonical basis.

The block $\mathcal{A}$ is equivalent to the category mod-Q of finite dimensional modules over Q, the path algebra of the quiver



with vertex set $\mathbb{Z}$, modulo the relations $x_i y_i = -y_{i+1}x_{i+1}$ and $x_{i+1}x_i = y_i y_{i+1} = 0$ for all $i \in \mathbb{Z}$.

CHAPTER III

TYPE C BLOCKS

Recall from the Chapter 1 that the type C blocks for $\mathfrak{q}_n(\Bbbk)$ are the blocks of the category $s\mathcal{O}_{\frac{1}{2}}$ of $\mathfrak{q}_n(\Bbbk)$-supermodules whose weights are in the set $\Lambda_{\frac{1}{2}}$ of half-integer weights. The goal of this chapter is to demonstrate that sum of these blocks admits the structure of a tensor product categorification of the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$. Because the results in this chapter are analogous to results for type A blocks from [BD2], the structure of this chapter closely follows that paper, and many excerpts are taken directly from the coauthored material. Because J. Brundan and I worked closely, it would be impossible to separate our contributions to that paper. The results in this chapter will be formulated into the coauthored paper [BD3].

3.1. The supercategory $s\mathcal{O}$

3.1.1. Choice of square roots

Recall from Chapter 1 that $\Bbbk$ is an algebraically closed field of characteristic 0 with a fixed choice of $\sqrt{-1} \in \Bbbk$. Here we also fix a choice of $\sqrt{i+\frac{1}{2}} \in \Bbbk$, for each non-negative integer i. Next, define $\sqrt{-i-\frac{1}{2}} := (-1)^i\sqrt{-1}\sqrt{i+\frac{1}{2}}$. Thus, we have a fixed choice of a square root of every element of $\mathbb{Z}+\frac{1}{2}$, and our choices satisfy the equation

$$\sqrt{i+\frac{1}{2}}\cdot\sqrt{i-\frac{1}{2}} = \sqrt{-i-\frac{1}{2}}\cdot\sqrt{-i+\frac{1}{2}} \tag{3.1}$$

for every $i \in \mathbb{Z}$. We write write

$$I := \mathbb{N}, \qquad J := \left\{\pm\sqrt{i+\frac{1}{2}}\sqrt{i-\frac{1}{2}}\;\middle|\; i \in I\right\}. \tag{3.2}$$

We follow the convention that $0 \in \mathbb{N}$.

3.1.2. General linear superalgebras revisited

Fix some $n \geq 1$, and set $m := \lceil n/2 \rceil$, so that $n = 2m$ or $2m-1$. Let $\widehat{\mathfrak{g}}$ denote the Lie superalgebra $\mathfrak{gl}_{2m|2m}(\Bbbk)$ of $4m \times 4m$ matrices. Its natural representation of column vectors $\widehat{U}$ has basis $u_1, ..., u_{2m}$ of $\widehat{U}_{\bar{0}}$ and $u_{2m+1}, ..., u_{4m}$ of $\widehat{U}_{\bar{1}}$. Write $x_{r,s}$ for the rs-matrix unit in $\widehat{\mathfrak{g}}$, so $x_{r,s}u_t = \delta_{s,t}u_r$. For $1 \leq r, s \leq 2m$, we define

$$e_{r,s} := x_{r,s} + x_{2m+r,2m+s}, \qquad e'_{r,s} := x_{2m+r,s} + x_{r,2m+s}, \tag{3.3}$$

$$f_{r,s} := x_{r,s} - x_{2m+r,2m+s}, \qquad f'_{r,s} := x_{2m+r,s} - x_{r,2m+s}. \tag{3.4}$$

Also let

$$h_r := e_{r,r}, \qquad h'_r := e'_{r,r}. \tag{3.5}$$

Note that the elements $e_{r,s}, f_{r,s}$ and h_r define even elements of $\widehat{\mathfrak{g}}$, while the elements $e'_{r,s}, f'_{r,s}$, and h'_r are odd.

We record how these distinguished elements of $\widehat{\mathfrak{g}}$ act on the natural module $\widehat{U}$. For the sake of simplicity, write u'_r for u_{2m+r}, so that $\widehat{U}_{\bar{1}}$ has basis $u'_1, ..., u'_{2m}$. We have that

$$e_{r,s}u_t = \delta_{s,t}u_r, \qquad e'_{r,s}u_t = \delta_{s,t}u'_r, \qquad e_{r,s}u'_t = \delta_{s,t}u'_r, \qquad e'_{r,s}u'_t = \delta_{s,t}u_r, \tag{3.6}$$

$$f_{r,s}u_t = \delta_{s,t}u_r, \qquad f'_{r,s}u_t = \delta_{s,t}u'_r, \qquad f_{r,s}u'_t = -\delta_{s,t}u'_r, \qquad f'_{r,s}u'_t = -\delta_{s,t}u_r. \tag{3.7}$$

Finally let $\widehat{U}^*$ be the dual supermodule to $\widehat{U}$, with basis $\phi_1, \ldots, \phi_{2m}, \phi'_1, \ldots, \phi'_{2m}$ that is dual to the basis $u_1, \ldots, u_{2m}, u'_1, \ldots, u'_{2m}$. The action of the distinguished elements of $\widehat{\mathfrak{g}}$ is given by

$$e_{r,s}\phi_t = -\delta_{r,t}\phi_s, \quad e_{r,s}\phi'_t = -\delta_{r,t}\phi'_s, \quad e'_{r,s}\phi_t = -\delta_{r,t}\phi'_s, \quad e'_{r,s}\phi'_t = \delta_{r,t}\phi_s, \tag{3.8}$$

$$f_{r,s}\phi_t = -\delta_{r,t}\phi_s, \quad f_{r,s}\phi'_t = \delta_{r,t}\phi'_s, \quad f'_{r,s}\phi_t = \delta_{r,t}\phi'_s, \quad f'_{r,s}\phi'_t = \delta_{r,t}\phi_s. \tag{3.9}$$

3.1.3. Updated definitions

As implied by the statement of the main type C theorem in section 1.4.2, the type C blocks behave differently depending on whether n is even or odd. We define a new Lie superalgebra $\mathfrak{g}$ to unify the two cases. When $n = 2m$ is even, we set $\mathfrak{g} = \mathfrak{q}_n(\mathbb{k})$, with standard Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$ as defined in the Chapter 1. In particular, $\mathfrak{g}$ is the subalgebra of $\widehat{\mathfrak{g}}$ spanned by $\{e_{r,s}, e'_{r,s} \mid 1 \le r, s \le n\}$, while $\mathfrak{h}$ has basis $\{h_r, h'_r \mid 1 \le r \le 2m\}$.

When $n = 2m - 1$ is odd, we let $\mathfrak{g}$ denotes the Lie superalgebra $\mathfrak{q}_n(\mathbb{k}) \oplus \mathfrak{q}_1(\mathbb{k})$, which we identify with the subalgebra of $\widehat{\mathfrak{g}}$ spanned by $\{e_{r,s}, e'_{r,s} \mid 1 \le r, s \le n\} \sqcup \{h_{2m}, h'_{2m}\}$. In this case, we must change our definitions of $\mathfrak{b}$ and $\mathfrak{h}$ from Chapter 1 to the following:

- $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$ spanned by $\{e_{r,s}, e'_{r,s} \mid 1 \le r \le s \le n\} \sqcup \{h_{2m}, h'_{2m}\}$;
- $\mathfrak{h}$ is the Cartan subalgebra spanned by $\{h_r, h'_r \mid 1 \le r \le 2m\}$.

In both the even and the odd cases, the subspaces $U \subseteq \widehat{U}$ and $U^* \subseteq \widehat{U}^*$ spanned by $u_1, \dots, u_n, u'_1, \dots, u'_n$ and $\phi_1, \dots, \phi_n, \phi'_1, \dots, \phi'_n$, respectively, may be viewed as $\mathfrak{g}$-supermodules. Also set $\mathfrak{t} := \mathfrak{h}_{\bar{0}}$ and let $\delta_1, \dots, \delta_{2m}$ be the basis for $\mathfrak{t}^*$ that is dual to the basis $h_1, \dots, h_{2m}$ for $\mathfrak{t}$. Again, when n is odd, these definitions differs from the ones given in Chapter 1.

It will be convenient to index the subset of $\mathfrak{t}^*$ corresponding to half-integer weights with n-tuples of integers. Set $\mathbf{B} = \mathbb{Z}^n$ as in Chapter 2. Given $\boldsymbol{b} \in \mathbf{B}$ and $1 \leq r \leq n$, define $\lambda_{\boldsymbol{b}} \in \mathfrak{t}^*$ by

$$\lambda_{\boldsymbol{b}} := \begin{cases} \sum_{r=1}^{n} \lambda_{\boldsymbol{b},r}\delta_r & \text{if } n = 2m \text{ is even} \\ \\ \sum_{r=1}^{n} \lambda_{\boldsymbol{b},r}\delta_r + \delta_{2m} & \text{if } n = 2m-1 \text{ is odd} \end{cases} \tag{3.10}$$

where we write $\lambda_{\boldsymbol{b},r} = b_r - \frac{1}{2}$. Recall also from the type C combinatorics in Chapter 2 that $\boldsymbol{d}_r$ denotes the element of $\mathbf{B}$ with a 1 in the r-th entry, and zero everywhere else, and note that

$$\lambda_{\boldsymbol{b}\pm\boldsymbol{d}_r} = \lambda_{\boldsymbol{b}} \pm \delta_r. \tag{3.11}$$

Let Λ denote the collection of elements of $\mathfrak{t}^*$ of the form $\{\lambda_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}\}$. When n is even, Λ is the set $\Lambda_{\frac{1}{2}}$ from Chapter 1. In the odd case this is a new definition.

We let $s\mathcal{O}$ denote the category of $\mathfrak{g}$-supermodules which are:

- Finitely generated over $\mathfrak{g}$.

- Locally finite over $\mathfrak{b}$.

- Semisimple $\mathfrak{t}$-modules, with weights in the set $\Lambda \subset \mathfrak{t}^*$.

Hence, when n is even, $s\mathcal{O}$ is the category $s\mathcal{O}_{\frac{1}{2}}$ defined in Chapter 1. When n is odd, $s\mathcal{O}$ is equivalent to the supercategory $(s\mathcal{O}_{\frac{1}{2}})^{\mathtt{CT}}$. Indeed, if M is a supermodule in $s\mathcal{O}$, the restriction of M to the subalgebra $\mathfrak{q}_n(\mathbb{k})$, equipped with the odd involution defined by the action of h'_{2m}, gives an object of the Clifford twist of $s\mathcal{O}_{\frac{1}{2}}$.

3.1.4. Construction of irreducibles

In contrast to the situation for reductive Lie algebras, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *supercommutative* rather than commutative. Hence, the irreducible modules over the Cartan are not necessarily one-dimensional. We proceed to define some irreducible $\mathfrak{h}$-supermodules $\{V(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$. Let C_2 be the rank 2 Clifford superalgebra with odd generators c_1, c_2 subject to the relations $c_1^2 = c_2^2 = 1, c_1c_2 = -c_2c_1$. Let V be the irreducible C_2-supermodule on basis v, v' with v even and v' odd, and action defined by

$$c_1 v = v', \quad c_1 v' = v, \quad c_2 v = \sqrt{-1}v', \quad c_2 v' = -\sqrt{-1}v.$$

Then, for $\boldsymbol{b} \in \mathbf{B}$, we set $V(\boldsymbol{b}) := V^{\otimes m}$. For $1 \le r \le n$, we let h_r act by the scalar $\lambda_{\boldsymbol{b},r}$ and h'_r act by left multiplication by $\sqrt{\lambda_{\boldsymbol{b},r}}\ \mathrm{id}^{\otimes(s-1)} \otimes c_{r+1-2s} \otimes \mathrm{id}^{(m-s)}$ where $s := \lfloor r/2 \rfloor$ (and we are using the usual superalgebra sign rules). In the odd case, we also need to define the actions of h_{2m} and h'_{2m}: these act as the identity and the odd involution $\mathrm{id}^{\otimes(m-1)} \otimes c_2$, respectively. In all cases, $V(\boldsymbol{b})$ is an irreducible $\mathfrak{h}$-supermodule of type $\mathtt{M}$, and its $\mathfrak{t}$-weight is $\lambda_{\boldsymbol{b}}$. Moreover, by construction, $h'_1 \cdots h'_{2m}$ acts on any even (resp. odd) vector in $V(\boldsymbol{b})$ as $c_{\boldsymbol{b}}$ (resp. $-c_{\boldsymbol{b}}$), where

$$c_{\boldsymbol{b}} := (\sqrt{-1})^m \sqrt{\lambda_{\boldsymbol{b},1}} \cdots \sqrt{\lambda_{\boldsymbol{b},n}}. \tag{3.12}$$

The sign here distinguishes $V(\boldsymbol{b})$ from its parity flip.

Lemma 3.1.1. *For $\boldsymbol{b} \in \mathbf{B}$, any $\mathfrak{h}$-supermodule that is semisimple of weight $\lambda_{\boldsymbol{b}}$ over $\mathfrak{t}$ decomposes as a direct sum of copies of the supermodules $V(\boldsymbol{b})$ and $\Pi V(\boldsymbol{b})$.*

Proof. We can identify $\mathfrak{h}$-supermodules that are semisimple of weight $\lambda_{\boldsymbol{b}}$ over $\mathfrak{t}$ with supermodules over the Clifford superalgebra $C_{2m} := C_2^{\otimes m}$, so that h'_r $(r = 1, \dots, n)$

acts in the same way as $\sqrt{\lambda_{\boldsymbol{b},r}}\ \mathrm{id}^{\otimes(s-1)} \otimes c_{r+1-2s} \otimes \mathrm{id}^{(m-s)}$ where $s := \lfloor r/2 \rfloor$, and in the odd case h'_{2m} acts as $\mathrm{id}^{\otimes(m-1)} \otimes c_2$. The lemma then follows since C_{2m} is simple, indeed, it is isomorphic to the matrix superalgebra $M_{2^{n-1}|2^{n-1}}(\Bbbk)$. $\square$

Let $\underline{s\mathcal{O}}$ denote the *underlying category* consisting of the same objects as $s\mathcal{O}$ but only the even morphisms. This is obviously an Abelian category. In order to parametrize its irreducible objects explicitly, we introduce the *Verma supermodule* $M(\boldsymbol{b})$ for $\boldsymbol{b} \in \mathbf{B}$ by setting

$$M(\boldsymbol{b}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b}), \tag{3.13}$$

where we are viewing $V(\boldsymbol{b})$ as a $\mathfrak{b}$-supermodule by inflating along the surjection $\mathfrak{b} \twoheadrightarrow \mathfrak{h}$. The weight $\lambda_{\boldsymbol{b}}$ is the highest weight of $M(\boldsymbol{b})$ in the usual *dominance order* on $\mathfrak{t}^*$, i.e. $\lambda \leq \mu$ if and only if $\mu - \lambda \in \bigoplus_{r=1}^{n-1} \mathbb{N}(\delta_r - \delta_{r+1})$. Note also that we can distinguish $M(\boldsymbol{b})$ from its parity flip in the same way as for $V(\boldsymbol{b})$: the element $h'_1 \cdots h'_{2m}$ acts on any even (resp. odd) vector in the highest weight space $M(\boldsymbol{b})_{\lambda_{\boldsymbol{b}}}$ as the scalar $c_{\boldsymbol{b}}$ (resp. $-c_{\boldsymbol{b}}$).

As usual, the Verma supermodule $M(\boldsymbol{b})$ has a unique irreducible quotient denoted $L(\boldsymbol{b})$. Thus, $L(\boldsymbol{b})$ is an irreducible $\mathfrak{g}$-supermodule of highest weight $\lambda_{\boldsymbol{b}}$, and the action of $h'_1 \cdots h'_{2m}$ on its highest weight space distinguishes it from its parity flip. The irreducible supermodules $\{L(\boldsymbol{b}), \Pi L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$ give a complete set of pairwise inequivalent irreducible supermodules in $\underline{s\mathcal{O}}$. The endomorphism algebras of these objects are all one-dimensional, so they are irreducibles of type M. Moreover, by a standard argument involving restricting to the underlying even Lie algebra as in [B3, Lemma 7.3], we get that $\underline{s\mathcal{O}}$ is a Schurian category in the sense of Definition 2.1.1.

3.1.5. Duality on $s\mathcal{O}$

Let x^T denote the usual transpose of a matrix $x \in \widehat{\mathfrak{g}}$. This induces an antiautomorphism of $\mathfrak{g}$, i.e., we have that $[x,y]^T = [y^T, x^T]$. Given any $M \in \operatorname{ob} s\mathcal{O}$, the definition of $s\mathcal{O}$ implies that M has finite dimensional weight spaces. We view the direct sum $\bigoplus_{\boldsymbol{b}\in\mathbf{B}} M^*_{\lambda_{\boldsymbol{b}}}$ of the linear duals of the weight spaces of M as a $\mathfrak{g}$-supermodule with action defined by $(xf)(v) := f(x^T v)$. Let $M^\star$ be the object of $s\mathcal{O}$ obtained from this by applying also the parity switching functor Π^m. Making the obvious definition on morphisms, this gives us a contravariant superequivalence $\star : s\mathcal{O} \to s\mathcal{O}$. We have incorporated the parity flip into this definition in order to get the following lemma.

Lemma 3.1.2. *For $\boldsymbol{b} \in \mathbf{B}$, we have that $L(\boldsymbol{b})^\star \cong L(\boldsymbol{b})$ via an even isomorphism.*

Proof. By weight considerations, we either have that $L(\boldsymbol{b})^\star$ is evenly isomorphic to $L(\boldsymbol{b})$ or to $\Pi L(\boldsymbol{b})$. To show that the former holds, take an even highest weight vector $f \in L(\boldsymbol{b})^\star$. We must show that $h'_1 \cdots h'_{2m} f = c_{\boldsymbol{b}} f$ (rather than $-c_{\boldsymbol{b}} f$). Remembering the twist by Π^m in our definition of $\star$, there is a highest weight vector $v \in L(\boldsymbol{b})$ of parity $m \pmod 2$ such that $f(v) = 1$. Then we get that

$$(h'_1 \cdots h'_{2m} f)(v) = f(h'_{2m} \cdots h'_1 v) = (-1)^m f(h'_1 \cdots h'_{2m} v) = c_{\boldsymbol{b}} f(v).$$

Hence, $h'_1 \cdots h'_{2m} f = c_{\boldsymbol{b}} f$. □

Let $P(\boldsymbol{b})$ be a projective cover of $L(\boldsymbol{b})$ in $\underline{s\mathcal{O}}$. There are even epimorphisms $P(\boldsymbol{b}) \twoheadrightarrow M(\boldsymbol{b}) \twoheadrightarrow L(\boldsymbol{b})$. Applying $\star$, we deduce that there are even monomorphisms $L(\boldsymbol{b}) \hookrightarrow M(\boldsymbol{b})^\star \hookrightarrow P(\boldsymbol{b})^\star$. The supermodule $P(\boldsymbol{b})^\star$ is an injective hull of $L(\boldsymbol{b})$, while $M(\boldsymbol{b})^\star$ is the *dual Verma supermodule.*

3.1.6. Verma flags

The following lemma is well known; it follows from central character considerations (e.g. see [CW, Theorem 2.48]) plus the universal property of Verma supermodules.

Lemma 3.1.3. *Suppose that $\lambda_{\boldsymbol{b}}$ is dominant and typical, i.e., whenever $1 \leq r < s \leq n$, we have $\lambda_{b,r} > \lambda_{b,s}$ and $\lambda_{b,r} + \lambda_{b,s} \neq 0$. Then $P(\boldsymbol{b}) = M(\boldsymbol{b})$.*

Remark 3.1.4. The condition $\lambda_{b,r} > \lambda_{b,s}$ is equivalent to $b_r > b_s$, while the condition $\lambda_{b,r} + \lambda_{b,s} \neq 0$ is equivalent to $b_r + b_s \neq 1$.

Let $s\mathcal{O}^{\Delta}$ be the full subcategory of $s\mathcal{O}$ consisting of all supermodules possessing a Verma flag, i.e., for which there is a finite filtration $0 = M_0 \subset \cdots \subset M_l = M$ with subquotients M_k/M_{k-1} evenly isomorphic to $M(\boldsymbol{b})$'s or $\Pi M(\boldsymbol{b})$'s for $\boldsymbol{b} \in \mathbf{B}$. Since the classes of all $M(\boldsymbol{b})$ and $\Pi M(\boldsymbol{b})$ are linearly independent in the Grothendieck group of the underlying category $\underline{s\mathcal{O}}$, the multiplicities $(M : M(\boldsymbol{b}))$ and $(M : \Pi M(\boldsymbol{b}))$ of $M(\boldsymbol{b})$ and $\Pi M(\boldsymbol{b})$ in a Verma flag of M are independent of the particular choice of flag. The following lemma is quite standard.

Lemma 3.1.5. *For $M \in \operatorname{ob} s\mathcal{O}^{\Delta}$ and $\boldsymbol{b} \in \mathbf{B}$, we have that*

$$(M : M(\boldsymbol{b})) = \dim \operatorname{Hom}_{s\mathcal{O}}(M, M(\boldsymbol{b})^{\star})_{\bar{0}},$$

$$(M : \Pi M(\boldsymbol{b})) = \dim \operatorname{Hom}_{s\mathcal{O}}(M, M(\boldsymbol{b})^{\star})_{\bar{1}}.$$

Also, any direct summand of $M \in \operatorname{ob} s\mathcal{O}^{\Delta}$ possesses a Verma flag.

Proof. The first part of the lemma follows by induction on the length of the Verma flag, using the following two observations: for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$ we have that

– $\mathrm{Hom}_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{b})^\star)$ is zero if $\boldsymbol{a} \neq \boldsymbol{b}$, and it is one-dimensional of even parity if $\boldsymbol{a} = \boldsymbol{b}$;

– $\mathrm{Ext}^1_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{b})^\star) = 0$.

To check these, for the first one, we use the universal property of $M(\boldsymbol{a})$ to see that $\mathrm{Hom}_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{b})^\star)$ is zero unless $\lambda_{\boldsymbol{a}} \leq \lambda_{\boldsymbol{b}}$. Similarly, on applying $\star$, it is zero unless $\lambda_{\boldsymbol{b}} \leq \lambda_{\boldsymbol{a}}$. Hence, we may assume that $\boldsymbol{a} = \boldsymbol{b}$. Finally, due to weight considerations, any non-zero homomorphism $M(\boldsymbol{a}) \to M(\boldsymbol{a})^\star$ must send the head to the socle, so $\mathrm{Hom}_{s\mathcal{O}}(M(\boldsymbol{a}), M(\boldsymbol{a})^\star)$ is evenly isomorphic to $\mathrm{Hom}_{s\mathcal{O}}(L(\boldsymbol{a}), L(\boldsymbol{a})^\star)$, which is one-dimensional and even thanks to Lemma 3.1.2. For the second property, we must show that all short exact sequences in $\underline{s\mathcal{O}}$ of the form

$$0 \to M(\boldsymbol{a})^\star \to M \to M(\boldsymbol{b}) \to 0 \qquad \text{or} \qquad 0 \to \Pi M(\boldsymbol{a})^\star \to M \to M(\boldsymbol{b}) \to 0$$

split. Either $\lambda_{\boldsymbol{a}}$ or $\lambda_{\boldsymbol{b}}$ is a maximal weight of M. In the latter case, using also Lemma 3.1.1, we can use the universal property of $M(\boldsymbol{b})$ to construct a splitting of $M \twoheadrightarrow M(\boldsymbol{b})$. In the former case, we apply $\star$, the resulting short exact sequence splits as before, and then we dualize again.

The final statement of the lemma may be proved by mimicking the argument for semisimple Lie algebras from [H, §3.2]. □

3.2. Special projective superfunctors

Next, we investigate the superfunctors $U \otimes -$ and $U^* \otimes -$ defined by tensoring with the $\mathfrak{g}$-supermodules U and U^* introduced in the previous section. They clearly preserve the properties of being finitely generated over $\mathfrak{g}$, locally finite-dimensional over $\mathfrak{b}$, and semisimple over $\mathfrak{t}$. Since the $\mathfrak{t}$-weights of U and U^* are $\delta_1, \dots, \delta_n$ and

$-\delta_1, \dots, -\delta_n$, respectively, and using (3.11), we get for each $M \in \operatorname{ob} s\mathcal{O}$ that all weights of $U \otimes M$ and $U^* \otimes M$ are of the form $\lambda_{\boldsymbol{b}}$ for $\boldsymbol{b} \in \mathbf{B}$. Hence, these superfunctors send objects of $s\mathcal{O}$ to objects of $s\mathcal{O}$, i.e. we have defined

$$sF := U \otimes - : s\mathcal{O} \to s\mathcal{O}, \qquad sE := U^* \otimes - : s\mathcal{O} \to s\mathcal{O}. \tag{3.14}$$

Let

$$\omega := \sum_{r,s=1}^{n} \left(f_{r,s} \otimes e_{s,r} - f'_{r,s} \otimes e'_{s,r} \right) \in U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g}). \tag{3.15}$$

Left multiplication by ω (resp. by $-\omega$) defines a linear map $x_M : U \otimes M \to U \otimes M$ (resp. $x^*_M : U^* \otimes M \to U^* \otimes M$) for each $\mathfrak{g}$-supermodule M. In view of the next lemma, these maps define a pair of even supernatural transformations

$$x : sF \Rightarrow sF, \qquad x^* : sE \Rightarrow sE. \tag{3.16}$$

Lemma 3.2.1. *The linear maps x_M and x^*_M just defined are even $\mathfrak{g}$-supermodule homomorphisms.*

Proof. This is straightforward to verify directly, but we give a more conceptual argument which better explains the origin of these maps. The odd element

$$f' := \sum_{t=1}^{n} f'_{t,t} \in U(\widehat{\mathfrak{g}}) \tag{3.17}$$

supercommutes with the elements of $U(\mathfrak{g})$. Hence, $f' \otimes 1 \in U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g})$ supercommutes with the image of the comultiplication $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \subset$

$U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g})$. The odd Casimir tensor

$$\Omega' := \sum_{r,s=1}^{n} \left(e_{r,s} \otimes e'_{s,r} - e'_{r,s} \otimes e_{s,r}\right) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

also supercommutes with the image of Δ. Hence, the even tensor

$$\Omega := \Omega'(f' \otimes 1) = -\sum_{r,s,t=1}^{n} \left(e_{r,s} f'_{t,t} \otimes e'_{s,r} + e'_{r,s} f'_{t,t} \otimes e_{s,r}\right) \in U(\widehat{\mathfrak{g}}) \otimes U(\mathfrak{g})$$

commutes with the image of Δ. Consequently, left multiplication by Ω defines even $\mathfrak{g}$-supermodule endomorphisms $x_M : U \otimes M \to U \otimes M$ and $x_M^* : U^* \otimes M \to U^* \otimes M$. It remains to observe that these endomorphisms agree with the linear maps defined by left multiplication by ω and $-\omega$, respectively. Indeed, by a calculation using (3.6)–(3.9), the elements $e_{r,s} f'_{t,t}$ and $e'_{r,s} f'_{t,t}$ of $U(\widehat{\mathfrak{g}})$ act on vectors in U (resp. U^*) in the same way as $\delta_{s,t} f'_{r,s}$ and $-\delta_{s,t} f_{r,s}$ (resp. $-\delta_{r,t} f'_{r,s}$ and $\delta_{r,t} f_{r,s}$), respectively. □

Lemma 3.2.2. *Suppose that $\boldsymbol{b} \in \mathbf{B}$ and let $M := M(\boldsymbol{b})$.*

1. *There is a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = U \otimes M$$

with $M_t/M_{t-1} \cong M(\boldsymbol{b} + \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} + \boldsymbol{d}_t)$ for each $t = 1, \ldots, n$. The endomorphism x_M preserves this filtration, and the induced endomorphism of the subquotient M_t/M_{t-1} is diagonalizable with exactly two eigenvalues $\pm\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$. Its $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace is evenly isomorphic to $M(\boldsymbol{b}+\boldsymbol{d}_t)$, while the other eigenspace is evenly isomorphic to $\Pi M(\boldsymbol{b} + \boldsymbol{d}_t)$.

2. There is a filtration

$$0 = M^n \subset \cdots \subset M^1 \subset M^0 = U^* \otimes M$$

with $M^{t-1}/M^t \cong M(\boldsymbol{b} - \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} - \boldsymbol{d}_t)$ *for each* $t = 1, \ldots, n$. *The endomorphism* x_M^* *preserves this filtration, and the induced endomorphism of the subquotient* M^{t-1}/M^t *is diagonalizable with exactly two eigenvalues* $\pm\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t} - 1}$. *Its* $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t} - 1}$*-eigenspace is evenly isomorphic to* $M(\boldsymbol{b} - \boldsymbol{d}_t)$*, while the other eigenspace is evenly isomorphic to* $\Pi M(\boldsymbol{b} - \boldsymbol{d}_t)$.

Proof. (1) The filtration is constructed in [B2, Lemma 4.3.7], as follows. By the tensor identity

$$U \otimes M = U \otimes (U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V(\boldsymbol{b})) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U \otimes V(\boldsymbol{b})).$$

As a $\mathfrak{b}$-supermodule, U has a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$ in which the section U_t/U_{t-1} is spanned by the images of u_t and u_t'. Let M_t be the submodule of $U \otimes M$ that maps to $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U_t \otimes V(\lambda))$ under this isomorphism.

Now fix $t \in \{1, \ldots, n\}$. Let $v_1, \ldots, v_k$ be a basis for the even highest weight space $M(\boldsymbol{b})_{\lambda_{\boldsymbol{b}},\bar{0}}$, so that $h_t'v_1, \ldots, h_t'v_k$ is a basis for $M(\boldsymbol{b})_{\lambda_{\boldsymbol{b}},\bar{1}}$. The subquotient $M_t/M_{t-1} \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (U_t/U_{t-1} \otimes V(\boldsymbol{b}))$ is generated by the images of the vectors $\{u_t \otimes v_i, u_t' \otimes v_i, u_t \otimes h_t'v_i, u_t' \otimes h_t'v_i \mid i = 1, \ldots, k\}$, which by weight considerations span a $\mathfrak{b}$-supermodule isomorphic to $V(\boldsymbol{b} + \boldsymbol{d}_t) \oplus \Pi V(\boldsymbol{b} + \boldsymbol{d}_t)$. Hence,

$$M_t/M_{t-1} \cong M(\boldsymbol{b} + \boldsymbol{d}_t) \oplus \Pi M(\boldsymbol{b} + \boldsymbol{d}_t).$$

The action of $f_{r,s}\otimes e_{s,r}-f'_{r,s}\otimes e'_{s,r}$ on any of $u_t\otimes v_i, u'_t\otimes v_i, u_t\otimes h'_tv_i$ or $u'_t\otimes h'_tv_i$ is zero unless $r\leq s=t$, and if $r<s=t$ then it sends these vectors into M_{t-1}. Therefore, x_M preserves the filtration. Moreover, this argument shows that it acts on the highest weight space of the quotient M_t/M_{t-1} in the same way as $x_t := f_{t,t}\otimes h_t - f'_{t,t}\otimes h'_t$.

Now consider the purely even subspace $S_{i,t}$ of M_t/M_{t-1} with basis given by the images of $u_t\otimes v_i, u'_t\otimes h'_tv_i$. Recalling that h_t acts on v_i and on h'_tv_i by $\lambda_{\boldsymbol{b},t}$, and that $(h'_t)^2=h_t$, it is straightforward to check that the matrix of the endomorphism x_t of $S_{i,t}$ in the given basis is equal to

$$A := \begin{pmatrix} \lambda_{\boldsymbol{b},t} & \lambda_{\boldsymbol{b},t} \\ 1 & -\lambda_{\boldsymbol{b},t} \end{pmatrix}.$$

Also recall from our construction of $V(\boldsymbol{b})$ that $h'_1\cdots h'_{2m}$ acts on v_i as the scalar $c_{\boldsymbol{b}}$ from (3.12), and it acts on h'_tv_i as $-c_{\boldsymbol{b}}$. Using this, another calculation shows that $h'_1\cdots h'_{2m}$ acts on $S_{i,t}$ as the matrix $\frac{c_{\boldsymbol{b}}}{\lambda_{\boldsymbol{b},t}}A$. Similarly, on the purely odd subspace $S'_{i,t}$ with basis given by the images of $u'_t\otimes v_i, u_t\otimes h'_tv_i$, x_t has matrix $-A$ and $h'_1\cdots h'_{2m}$ has matrix $-\frac{c_{\boldsymbol{b}}}{\lambda_{\boldsymbol{b},t}}A$.

Since the matrix A has eigenvalues $\pm\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$, the calculation made in the previous paragraph implies that x_t is diagonalizable on M_t/M_{t-1} with exactly these eigenvalues. Moreover on any even highest weight vector in its $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace, we get that $h'_1\cdots h'_{2m}$ acts as

$$\frac{c_{\boldsymbol{b}}}{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}=c_{\boldsymbol{b}+\boldsymbol{d}_t}.$$

This implies that the $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace is evenly isomorphic to $M(\boldsymbol{b}+\boldsymbol{d}_t)$. Similarly, the $-\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1}$-eigenspace is evenly isomorphic to $\Pi M(\boldsymbol{b}+\boldsymbol{d}_t)$.

(2) Similar. $\square$

Corollary 3.2.3. *For $M \in \operatorname{ob} s\mathcal{O}$, all roots of the minimal polynomials of x_M and x_M^* (computed in the finite dimensional superalgebras $\operatorname{End}_{s\mathcal{O}}(sF\,M)$ and $\operatorname{End}_{s\mathcal{O}}(sE\,M)$) belong to the set J from (3.2).*

Proof. This is immediate from the theorem in case M is a Verma supermodule. We may then deduce that it is true for all irreducibles, hence, for any $M \in \operatorname{ob} s\mathcal{O}$. $\square$

Corollary 3.2.3 implies that we can decompose

$$sF = \bigoplus_{j \in J} sF_j, \qquad sE = \bigoplus_{j \in J} sE_j, \tag{3.18}$$

where sF_j (resp. sE_j) is the subfunctor of sF (resp. sE) defined by letting $sF_j\,M$ (resp. $sE_j\,M$) be the generalized j-eigenspace of x_M (resp. x_M^*) for each $M \in \operatorname{ob} s\mathcal{O}$. For $i \in I$, recall the definition of i-sig$(\boldsymbol{b})$ from (2.6). The following theorem relates the combinatorics for the Verma subquotients of $sF_jM(\boldsymbol{b})$ with the summands of $f_iv_{\boldsymbol{b}} \in V^{\otimes n}$, see (2.7).

Theorem 3.2.4. *Given $\boldsymbol{b} \in \mathbf{B}$ and $i \in I$, let $j := \sqrt{i - \frac{1}{2}}\sqrt{i + \frac{1}{2}}$. Then:*

(1) $sF_j\,M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the Verma supermodules

$$\{M(\boldsymbol{b} + \boldsymbol{d}_t) \mid \textit{for } 1 \leq t \leq n \textit{ such that } i\text{-sig}(\boldsymbol{b})_t = \texttt{f}\}.$$

(2) $sE_j\,M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the Verma supermodules

$$\{M(\boldsymbol{b} - \boldsymbol{d}_t) \mid \textit{for } 1 \leq t \leq n \textit{ such that } i\text{-sig}(\boldsymbol{b})_t = \texttt{e}\}.$$

Proof. (1) It is immediate from Lemma 3.2.2 that $sF_j\,M(\boldsymbol{b})$ has a multiplicity-free filtration with sections that are evenly isomorphic to the supermodules $M(\boldsymbol{b}+\boldsymbol{d}_t)$ for $t = 1, \dots, n$ such that $\sqrt{\lambda_{\boldsymbol{b},t}}\sqrt{\lambda_{\boldsymbol{b},t}+1} = j$. Squaring both sides, this equation implies that $(\lambda_{\boldsymbol{b},t}+\frac{1}{2})^2 = i^2$. Hence,

$$\lambda_{\boldsymbol{b},t} = b_t - \frac{1}{2} = -\frac{1}{2} \pm i$$

We deduce that $b_t = \pm i$. Since we squared our original equation, it remains to check that we do indeed get solutions to that in both cases. This follows from 3.1.

(2) Similar. $\square$

The superfunctors sF and sE are both left and right adjoint to each other via some canonical (even) adjunctions. The adjunction making (sE, sF) into an adjoint pair is induced by the linear maps

$$\varepsilon : U^* \otimes U \to \Bbbk,\ \phi \otimes u \mapsto \phi(u), \qquad \eta : \Bbbk \to U \otimes U^*,\ 1 \mapsto \sum_{r=1}^{n} (u_r \otimes \phi_r + u'_r \otimes \phi'_r).$$

Thus, the unit of adunction $c : 1 \Rightarrow sF\,sE$ is defined on supermodule M by the map $c_M : M \xrightarrow{\text{can}} \Bbbk \otimes M \xrightarrow{\eta\otimes\text{id}} U \otimes U^* \otimes M$, and the counit of adjunction $d : sE\,sF \Rightarrow 1$ is defined by $d_M : U^* \otimes U \otimes M \xrightarrow{\varepsilon\otimes\text{id}} \Bbbk \otimes M \xrightarrow{\text{can}} M$. Similarly, the adjunction making (sF, sE) into an adjoint pair is induced by the linear maps

$$U \otimes U^* \to \Bbbk,\ u \otimes \phi \mapsto (-1)^{|\phi||u|}\phi(u), \quad \Bbbk \to U^* \otimes U,\ 1 \mapsto \sum_{r=1}^{n} (\phi_r \otimes u_r - \phi'_r \otimes u'_r).$$

The following lemma implies that these adjunctions restrict to adjunctions making (sF_j, sE_j) and (sE_j, sF_j) into adjoint pairs for each $j \in J$. It follows that all of these superfunctors send projectives to projectives, and they are all exact, i.e. they preserve short exact sequences in $\underline{s\mathcal{O}}$.

Lemma 3.2.5. *The supernatural transformation $x^* : sE \Rightarrow sE$ is both the left and right mate of $x : sF \Rightarrow sF$ with respect to the canonical adjunctions defined above.*

Proof. We just explain how to check that x^* is the left mate of x with respect to the adjunction (sE, sF); the argument for right mate is similar. We need to show for each $M \in \operatorname{ob} s\mathcal{O}$ that the composition

$$U^* \otimes M \stackrel{\mathrm{id} \otimes c_M}{\longrightarrow} U^* \otimes U \otimes U^* \otimes M \stackrel{\mathrm{id} \otimes x_{U^* \otimes M}}{\longrightarrow} U^* \otimes U \otimes U^* \otimes M \stackrel{d_{U^* \otimes M}}{\longrightarrow} U^* \otimes M$$

is equal to $x^*_M : U^* \otimes M \to U^* \otimes M$. Recall for this that x^*_M is defined by left multiplication by $\sum_{r,s=1}^n \left(f'_{r,s} \otimes e'_{s,r} - f_{r,s} \otimes e_{s,r}\right)$, while $x_{U^* \otimes M}$ is defined by left multiplication by $\sum_{r,s=1}^n (f_{r,s} \otimes e_{s,r} \otimes 1 + f_{r,s} \otimes 1 \otimes e_{s,r} - f'_{r,s} \otimes e'_{s,r} \otimes 1 - f'_{r,s} \otimes 1 \otimes e'_{s,r})$. Now one computes the effect of both maps on homogeneous vectors of the form $\phi_t \otimes v$ and $\phi'_t \otimes v$ using (3.6)–(3.9). $\square$

3.3. Bruhat order revisited

In this section we describe the irreducible subquotients of $M(\boldsymbol{b})$ in terms of the Bruhat order. Recall the type C combinatorics from Section 2.2. In particular, for the remainder of this dissertation, we let P denote the weight lattice for $\mathfrak{sp}_\infty$, containing elements $\{\varepsilon_i \mid i \in I\}$ and simple roots $\alpha_0 = -2\varepsilon_0$ and $\alpha_i = \varepsilon_{i-1} - \varepsilon_i$ for $i > 0$. We also have the *dominance ordering* $\trianglelefteq$ on P and the type C *Bruhat ordering* $\preceq$ on the set $\mathbf{B}$.

We begin with several technical lemmas to make the ordering more concrete. Using these lemmas, we can prove in particular that whenever $\boldsymbol{a} \preceq \boldsymbol{b}$, it is also true that $\lambda_{\boldsymbol{a}} \leq \lambda_{\boldsymbol{b}}$ in the usual dominance ordering on $\mathfrak{t}^*$.

Lemma 3.3.1. *Let $\beta = \sum_{i \in I} \beta_i \varepsilon_i$ and $\gamma = \sum_{i \in I} \gamma_i \varepsilon_i$ be elements of P. Then, $\beta \trianglelefteq \gamma$ if and only if the sum $\sum_{i \geq j} (\beta_i - \gamma_i)$ is positive for every $j \in I$, and in addition, when $j = 0$, the sum is even.*

Proof. First suppose that $\beta \trianglelefteq \gamma$, so that $\gamma - \beta = \sum_{i \in I} (\gamma_i - \beta_i)\varepsilon_i$ is a (finite) sum $\sum_{i \in I} h_i \alpha_i$ where each $h_i \geq 0$. Writing each α_i in terms of the ε_j's, we see that

$$\gamma_i - \beta_i = \begin{cases} h_1 - 2h_0 & \text{if } i = 0 \\ h_{i+1} - h_i & \text{if } i > 0 \end{cases}, \qquad \text{hence} \qquad \sum_{i \geq j} (\beta_i - \gamma_i) = \begin{cases} 2h_0 & \text{if } j = 0 \\ h_j & \text{if } j > 0 \end{cases}.$$

This proves the sums are positive for any $j \in I$, and even when $j = 0$. The other implication is proved by reversing these steps. $\square$

Lemma 3.3.2. *For $\boldsymbol{a} \in \mathbf{B}, i \in I$ and $1 \leq s \leq n$, define*

$$N_{[1,s]}(\boldsymbol{a}, i) := \#\{1 \leq r \leq s \mid a_r > i\} - \#\{1 \leq r \leq s \mid a_r \leq -i\}.$$

Then $\boldsymbol{a} \succeq \boldsymbol{b}$ if and only if

- *$N_{[1,n]}(\boldsymbol{a}, i) = N_{[1,n]}(\boldsymbol{b}, i)$ for all $i \in I$;*
- *$N_{[1,s]}(\boldsymbol{a}, 0) \equiv N_{[1,s]}(\boldsymbol{b}, 0) \pmod 2$ for each $s = 1, \ldots, n-1$;*
- *$N_{[1,s]}(\boldsymbol{a}, i) \geq N_{[1,s]}(\boldsymbol{b}, i)$ for all $i \in I$ and $s = 1, \ldots, n-1$.*

Proof. Given $\boldsymbol{a} \in \mathbf{B}$ and $1 \leq s \leq n$, write $\mathrm{wt}_s(\boldsymbol{a}) = \sum_{i \in I} \beta_i \varepsilon_i$. For $j \in I$, the number $N_{[1,s]}(\boldsymbol{a}, j)$ is precisely the sum $\sum_{j \geq i} \beta_i$. The lemma then follows from the definition of $\preceq$, along with Lemma 3.3.1. $\square$

Recall from (2.7) the Chevalley generators of $\mathfrak{sp}_\infty$ act on the monomial basis of $V^{\otimes n}$ by

$$f_i v_{\boldsymbol{b}} = \sum_{i\text{-sig}(\boldsymbol{b})_t = \,\mathtt{f}} v_{\boldsymbol{b}+\boldsymbol{d}_t}.$$

The following lemmas explores the interaction between the action of f_i and the Bruhat ordering.

Lemma 3.3.3. *Suppose that* $\boldsymbol{a} \succeq \boldsymbol{b}$ *and* $i\text{-sig}(\boldsymbol{a})_r = i\text{-sig}(\boldsymbol{b})_n = \mathtt{f}$ *for some* $i \in I$ *and* $1 \leq r \leq n$. *Then* $\boldsymbol{a} + \boldsymbol{d}_r \succeq \boldsymbol{b} + \boldsymbol{d}_n$, *with equality if and only if* $\boldsymbol{a} = \boldsymbol{b}$ *and* $r = n$.

Proof. We use the conditions from Lemma 3.3.2. For *either* $j \neq i$ and $1 \leq s \leq n$, *or* $j = i$ and $1 \leq s < r$, we have that

$$N_{[1,s]}(\boldsymbol{a} + \boldsymbol{d}_r, j) = N_{[1,s]}(\boldsymbol{a}, j) \geq N_{[1,s]}(\boldsymbol{b}, j) = N_{[1,s]}(\boldsymbol{b} + \boldsymbol{d}_n, j).$$

When $j = i$ and $r \leq s < n$, we have that:

$$\begin{aligned} N_{[1,s]}(\boldsymbol{a} + \boldsymbol{d}_r, i) = N_{[1,s]}(\boldsymbol{a}, i) + 2^{\delta_{i,0}} \geq N_{[1,s]}(\boldsymbol{b}, i) + 2^{\delta_{i,0}} &> N_{[1,s]}(\boldsymbol{b}, i) \\ &= N_{[1,s]}(\boldsymbol{b} + \boldsymbol{d}_n, i). \end{aligned}$$

Finally,

$$\begin{aligned} N_{[1,n]}(\boldsymbol{a} + \boldsymbol{d}_r, i) = N_{[1,n]}(\boldsymbol{a}, i) + 1 &= N_{[1,n]}(\boldsymbol{b}, i) + 1 \\ &= N_{[1,n]}(\boldsymbol{b} + \boldsymbol{d}_n, i). \end{aligned}$$

□

Lemma 3.3.4. *Suppose we are given $\boldsymbol{b} \in \mathbf{B}$. Define $\boldsymbol{a} \in \mathbf{B}$ by setting $a_1 := b_1$, and then inductively define each a_s for $s = 2, \ldots, n$ to be the greatest integer such that $a_s \leq b_s$ and the following hold for all $1 \leq r < s$:*

- *The entry $a_s < a_r$;*
- *The entry $a_s \leq -b_r$.*

Define a monomial $X = X_n \cdots X_2$ in the Chevalley generators $\{f_i \mid i \in I\}$ by setting

$$X_r := \begin{cases} f_{b_r-1} f_{b_r-2} \cdots f_{a_r+1} f_{a_r} & \text{if } a_r \geq 0, \\ f_{1-b_r} f_{2-b_r} \cdots f_{-a_r} & \text{if } b_r \leq 0, \\ f_{b_r-1} \cdots f_1 f_0 f_1 \cdots f_{-a_r} & \text{if } b_r > 0 > a_r, \end{cases}$$

for each $r = 1, \ldots, n$. Then, working in the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$, we have that

$$X v_{\boldsymbol{a}} = v_{\boldsymbol{b}} + (\text{a sum of } v_{\boldsymbol{c}}\text{'s for } \boldsymbol{c} \succ \boldsymbol{b}),$$

Proof. We proceed by induction on n, where the base case $n = 1$ is trivial. When $n > 1$, define $\bar{\boldsymbol{b}} = (b_1, ..., b_{n-1}), \bar{\boldsymbol{a}} = (a_1, ..., a_{n-1}), \bar{X} = X_{n-1} \cdots X_2$. Applying the induction hypothesis in the $\mathfrak{sp}_\infty$-module $V^{\otimes n-1}$, we can write

$$\bar{X} v_{\bar{\boldsymbol{a}}} = v_{\bar{\boldsymbol{b}}} + (\text{a sum of } v_{\bar{\boldsymbol{c}}}\text{'s for } \bar{\boldsymbol{c}} \succ \bar{\boldsymbol{b}})$$

We observe that whenever f_i appears as a factor of some X_r with $r < n$, then $f_i v_{a_n} = 0$. To prove this, we need to demonstrate that $a_n \neq \pm i$. This follows from the definition of the monomials X_r:

- When $a_r \geq 0$, we have $a_r \leq i \leq b_r - 1$, or equivalently, $1 - b_r \leq -i \leq a_r \leq 0$. Because $a_n \leq -b_r$, it follows that $a_n < -i$.

– When, $b_r \leq 0$, we have $1 - b_r \leq i \leq -a_r$, or equivalently, $a_r \leq -i \leq b_r - 1$. Because $a_n < a_r$, it follows that $a_n < -i$.

– When $a_r < 0 < b_r$, we have $i \leq \max\{-a_r, b_r - 1\}$, or equivalently, $-i \geq \min\{a_r, -b_r + 1\}$. Because $a_n < a_r$ and $-b_r + 1$, it follows that $a_n < -i$.

Therefore, if $\tilde{\boldsymbol{b}} = (b_1, ..., b_{n-1}, a_n)$, then

$$\bar{X} v_{\boldsymbol{a}} = v_{\tilde{\boldsymbol{b}}} + (\text{a sum of } v_{\boldsymbol{c}} \text{ where } \boldsymbol{c} \succeq \tilde{\boldsymbol{b}})$$

Lastly, we act with X_n, which sends v_{a_n} to v_{b_n}, and apply Lemma 3.3.3. □

Example 3.3.5. If $\boldsymbol{b} = (2, 0, -1, 0, 2, 0, -1)$ then $\boldsymbol{a} = (2, -2, -3, -4, -5, -6, -7)$ and $X = (f_2 f_3 f_4 f_5 f_6 f_7)(f_1 f_2 f_3 f_4 f_5 f_6)(f_1 f_0 f_1 f_2 f_3 f_4 f_5)(f_1 f_2 f_3 f_4)(f_2 f_3)(f_1 f_2)$.

Theorem 3.3.6. *For every $\boldsymbol{b} \in \mathbf{B}$, the indecomposable projective supermodule $P(\boldsymbol{b})$ has a Verma flag with top section evenly isomorphic to $M(\boldsymbol{b})$ and other sections evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \in \mathbf{B}$ with $\boldsymbol{c} \succ \boldsymbol{b}$.*

Proof. Let notation be as in the statement of Lemma 3.3.4. Let $i_1, \ldots, i_l \in I$ be defined so that X is the monomial $f_{i_l} \cdots f_{i_2} f_{i_1}$. Let $j_k := \sqrt{i_k - \frac{1}{2}}\sqrt{i_k + \frac{1}{2}}$ for each k and consider the supermodule

$$P := sF_{j_l} \cdots sF_{j_2} sF_{j_1} M(\boldsymbol{a}).$$

For each $1 \leq r < s \leq n$, we have that $a_r > a_s$. In addition, $a_s \leq -b_r \leq -a_r$, so $a_s + a_r < 1$. This implies that the weight $\lambda_{\boldsymbol{a}}$ is typical and dominant, hence $M(\boldsymbol{a})$ is projective by Lemma 3.1.3. Since each sF_j is left adjoint to the exact functor sE_j, it sends projectives to projectives, and we deduce that P is projective. The combinatorics for how the Chevalley generators f_i act on the elements $v_{\boldsymbol{b}}$ from (2.7) matches that of Theorem 3.2.4, we can reinterpret Lemma 3.3.4 as saying that P has a Verma flag with $M(\boldsymbol{b})$ as a subquotient, and all other subquotients evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \succ \boldsymbol{b}$. Actually, the order of induction from Lemma 3.3.4 constructs $M(\boldsymbol{b})$ as a quotient of P, hence $P(\boldsymbol{b})$ is evenly isomorphic to a summand of P, and it just remains to apply Lemma 3.1.5. $\square$

Corollary 3.3.7. *For $\boldsymbol{b} \in \mathbf{B}$, we have that $[M(\boldsymbol{b}) : L(\boldsymbol{b})] = 1$. All other composition factors of $M(\boldsymbol{b})$ are evenly isomorphic to $L(\boldsymbol{c})$'s for $\boldsymbol{c} \prec \boldsymbol{b}$.*

Proof. This follows immediately from Theorem 3.3.6 and the following analog of *BGG reciprocity*:

$$[M(\boldsymbol{b}) : L(\boldsymbol{c})] = \mathrm{Hom}_{s\mathcal{O}}(P(\boldsymbol{c}), M(\boldsymbol{b})^{\star})_{\bar{0}} = (P(\boldsymbol{c}) : M(\boldsymbol{b}))$$

$$[M(\boldsymbol{b}) : \Pi L(\boldsymbol{c})] = \mathrm{Hom}_{s\mathcal{O}}(P(\boldsymbol{c}), M(\boldsymbol{b})^{\star})_{\bar{1}} = (P(\boldsymbol{c}) : \Pi M(\boldsymbol{c}))$$

These equalities are given by Lemma 3.1.2 and 3.1.5. $\square$

Corollary 3.3.8. *For any $\boldsymbol{b} \in \mathbf{B}$, every irreducible subquotient of the indecomposable projective $P(\boldsymbol{b})$ is evenly isomorphic to $L(\boldsymbol{a})$ for $\boldsymbol{a} \in \mathbf{B}$ with $\mathrm{wt}(\boldsymbol{a}) = \mathrm{wt}(\boldsymbol{b})$.*

Proof. By Theorem 3.3.6, $P(\boldsymbol{b})$ has a Verma flag with sections $M(\boldsymbol{c})$ for $\boldsymbol{c} \succeq \boldsymbol{b}$. By Corollary 3.3.7, the composition factors of $M(\boldsymbol{c})$ are $L(\boldsymbol{a})$'s for $\boldsymbol{a} \preceq \boldsymbol{c}$. Hence, every irreducible subquotient of $P(\boldsymbol{b})$ is evenly isomorphic to $L(\boldsymbol{a})$ for $\boldsymbol{a} \in \mathbf{B}$ such that $\boldsymbol{a} \preceq \boldsymbol{c} \succeq \boldsymbol{b}$ for some $\boldsymbol{c}$. This condition implies that $\mathrm{wt}(\boldsymbol{a}) = \mathrm{wt}(\boldsymbol{b})$. $\square$

3.4. Weak categorical action

Let $\mathcal{O}$ be the Serre subcategory of $s\mathcal{O}$ generated by the supermodules

$$\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\},$$

i.e. it is the full subcategory of $s\mathcal{O}$ consisting of all supermodules whose composition factors are evenly isomorphic to $L(\boldsymbol{b})$'s for $\boldsymbol{b} \in \mathbf{B}$ (and not $\Pi L(\boldsymbol{b})$'s). Since each $L(\boldsymbol{b})$ is of type $\mathtt{M}$, there are no non-zero odd morphisms between objects of $\mathcal{O}$. Because of this, we forget the super and view $\mathcal{O}$ as a $\Bbbk$-linear category.

Theorem 3.4.1. *We have that* $s\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$ *in the sense of Chapter 1.*

Proof. Let $\Pi\mathcal{O}$ be the Serre subcategory of $s\mathcal{O}$ generated by the supermodules $\{\Pi L(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbf{B}\}$. By Corollary 3.3.8, all even extensions between $\Pi L(\boldsymbol{a})$ and $L(\boldsymbol{b})$ are split. Hence, every supermodule in $s\mathcal{O}$ decomposes uniquely as a direct sum of an object of $\mathcal{O}$ and an object of $\Pi\mathcal{O}$. The result follows. □

Theorem 3.4.2. *The category $\mathcal{O}$ is a highest weight category in the sense of Definition 2.1.3, with weight poset* $(\mathbf{B}, \preceq)$. *Its standard objects are the Verma supermodules* $\{M(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$.

Proof. It is clear that $\mathcal{O}$ is a Schurian category with isomorphism classes of irreducible objects represented by $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}\}$. By Theorem 3.3.6, $P(\boldsymbol{b})$ has a Verma flag with $M(\boldsymbol{b})$ at the top and other sections that are evenly isomorphic to $M(\boldsymbol{c})$'s for $\boldsymbol{c} \succ \boldsymbol{b}$. It just remains to observe that the Verma supermodules $M(\boldsymbol{b})$ coincide with the standard objects $\Delta(\boldsymbol{b})$. This follows using the filtration just described plus Corollary 3.3.7. □

Remark 3.4.3. By Lemma 3.1.2, the duality $\star$ on $s\mathcal{O}$ restricts to a duality $\star : \mathcal{O} \to \mathcal{O}$ fixing isomorphism classes of irreducible objects.

Next, take $i \in I$ and set $j := \sqrt{i - \frac{1}{2}} \cdot \sqrt{i + \frac{1}{2}}$. Theorem 3.2.4 implies that the exact functors sF_j and sE_j send the standard objects in $\mathcal{O}$ to objects of $\mathcal{O}$ with a Δ-flag. Using this, we deduce that they send irreducibles in $\mathcal{O}$ to irreducibles in $\mathcal{O}$, and then that they send arbitrary object of $\mathcal{O}$ to objects in $\mathcal{O}$. Thus, their restrictions define exact endofunctors

$$F_i := sF_j|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}, \quad E_i := sE_j|_{\mathcal{O}} : \mathcal{O} \to \mathcal{O}. \tag{3.19}$$

Again, these functors are biadjoint. Let $\mathcal{O}^\Delta$ be the full subcategory of $\mathcal{O}$ consisting of all objects possessing a Verma flag. This is an exact subcategory of $\mathcal{O}$. Its complexified Grothendieck group $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}^\Delta)$ has basis $\{[M(\boldsymbol{b})] \mid \boldsymbol{b} \in \mathbf{B}\}$.

Theorem 3.4.4. *For each $i \in I$, the functors F_i and E_i are exact endofunctors of $\mathcal{O}^\Delta$. Moreover, if we identify $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{O}^\Delta)$ with $V^{\otimes n}$ so that $[M(\boldsymbol{b})] \leftrightarrow v_{\boldsymbol{b}}$ for each $\boldsymbol{b} \in \mathbf{B}$, then the induced endomorphisms $[F_i]$ and $[E_i]$ of the Grothendieck group act in the same way as the Chevalley generators f_i and e_i of $\mathfrak{sp}_\infty$.*

Proof. Compare Theorem 3.2.4 with (2.7). $\square$

Thus, we have constructed a highest weight category $\mathcal{O}$ with weight poset $\preceq$, and equipped it with biadjoint endofunctors E_i and F_i for every $i \in I$, which induce an action of $\mathfrak{sp}_\infty$ on the Grothendieck group. This gives us the data of a weak categorical action of $\mathfrak{sp}_\infty$ in the sense of [CR, R].

3.5. Strong categorical action

In this section, we upgrade the weak categorical action of $\mathfrak{sp}_\infty$ on $\mathcal{O}$ constructed so far to a strong categorical action. Recalling (D2) and (D4) from the definition of categorical actions (Definition 2.4.1), we must prove the following:

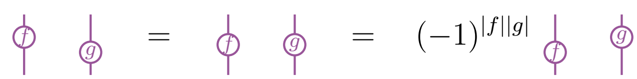
Theorem 3.5.1. *There is a strict monoidal functor* $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ *sending the generating objects* $i \in I$ *to the endofunctors* F_i *from (3.19). Moreover, for all* $M \in \operatorname{ob}\mathcal{O}$ *and* $i \in I$*, the endomorphism* $F_iM \to F_iM$ *defined by the natural transformation* $\Phi\Big(\underset{i}{\bullet}\Big)$ *is nilpotent.*

Here, the construction of Φ is similar to the construction of the monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ in Subsection (2.6.4)–(2.6.6). In those subsections $\mathcal{O}$ was a category of $\mathfrak{gl}_n(\mathbb{k})$-modules, and Φ was constructed by first defining a "obvious" monoidal functor $\Psi : \mathcal{AH} \to \mathcal{E}nd(\mathcal{O}')$ and then exploiting some isomorphism theorems to pass to $\mathcal{QH}$.

In our present situation, the construction of $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ is analogous, but even more subtle. We begin by producing an easily-defined *monoidal superfunctor* $\Psi : \mathcal{AHC} \to \mathcal{E}nd(s\mathcal{O})$, where $\mathcal{AHC}$ is the *affine Hecke-Clifford supercategory* (defined below), and $\mathcal{E}nd(s\mathcal{O})$ is the monoidal supercategory whose objects are superfunctors $s\mathcal{O} \to s\mathcal{O}$, and whose morphisms are supernatural transformations. From there, we exploit a remarkable isomorphism theorem of [KKT] to produce a monoidal superfunctor $\mathcal{QHC} \to \mathcal{E}nd(s\mathcal{O})$, where $\mathcal{QHC}$ is the *quiver Hecke-Clifford supercategory* (also defined below). Lastly, we realize $\mathcal{QH}$ as a full subcategory of $\mathcal{QHC}$ with only even morphisms to obtain the functor Φ required by Theorem 3.5.1.

3.5.1. Intermediate categories

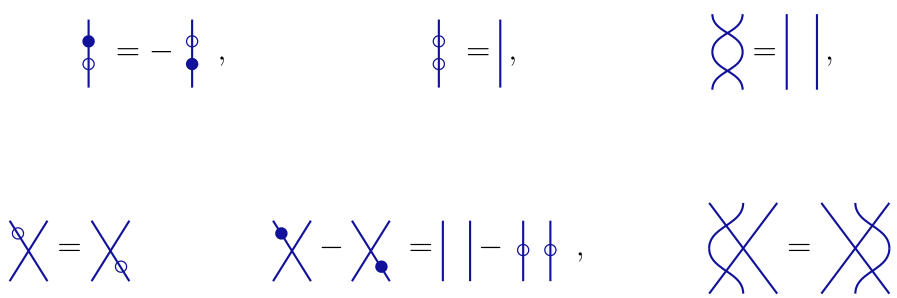
Both $\mathcal{AHC}$ and $\mathcal{QHC}$ are examples of (strict) *monoidal supercategories*, meaning that they are supercategories equipped with a monoidal product in an appropriate enriched sense. We refer the reader to the introduction of [BE1] for the precise definition, just recalling that morphisms in a monoidal supercategory satisfy the *super interchange law* rather than the usual interchange law of a monoidal category: in terms of the string calculus as in [BE1] we have that



(3.20)

for homogeneous morphisms f and g of parities $|f|$ and $|g|$, respectively.

Definition 3.5.2. The (degenerate) *affine Hecke-Clifford supercategory* $\mathcal{AHC}$ is the strict monoidal supercategory with a single generating object 1, even generating morphisms  $: 1 \to 1$ and  $: 1 \otimes 1 \to 1 \otimes 1$, and an odd generating morphism  $: 1 \to 1$. These are subject to the following relations:



Denoting the object $1^{\otimes d} \in \mathrm{ob}\,\mathcal{AHC}$ simply by d, the (degenerate) *affine Hecke-Clifford superalgebra* is the superalgebra

$$AHC_d := \mathrm{End}_{\mathcal{AHC}}(d). \tag{3.21}$$

This was introduced originally by Jones and Nazarov [JN].

For a supercategory $\mathcal{C}$, we write $\mathcal{E}nd(\mathcal{C})$ for the strict monoidal supercategory consisting of superfunctors and supernatural transformations.

Theorem 3.5.3. *There is a strict monoidal superfunctor* $\Psi : \mathcal{AHC} \to \mathcal{E}nd(s\mathcal{O})$ *sending the generating object* 1 *to the endofunctor* $sF = U \otimes -$ *from (3.14), and the generating morphisms*  ,  *and*  *to the supernatural transformations* x, c *and* t *which are defined on* $M \in \mathrm{ob}\, s\mathcal{O}$ *as follows:*

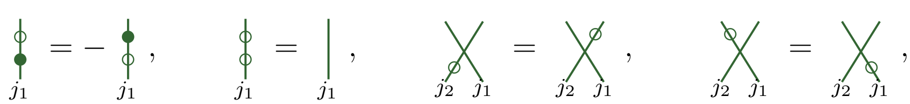
– $x_M : U \otimes M \to U \otimes M$ *is left multiplication by the tensor* ω *from (3.15);*

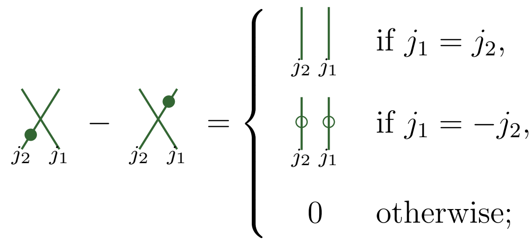
– $c_M : U \otimes M \to U \otimes M$ *is left multiplication by* $\sqrt{-1}\, f' \otimes 1$ *for* f' *as in (3.17);*

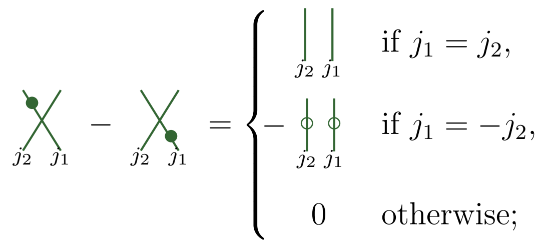
– $t_M : U \otimes U \otimes M \to U \otimes U \otimes M$ *sends* $u \otimes v \otimes m \mapsto (-1)^{|u||v|} v \otimes u \otimes m$.

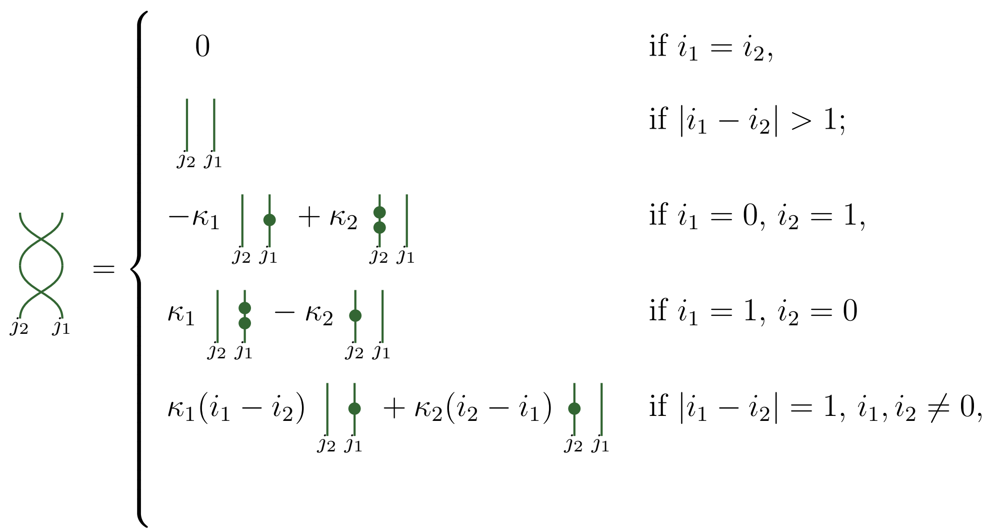
Proof. This an elementary check of relations, similar to the one made in the proof of [HKS, Theorem 7.4.1]. □

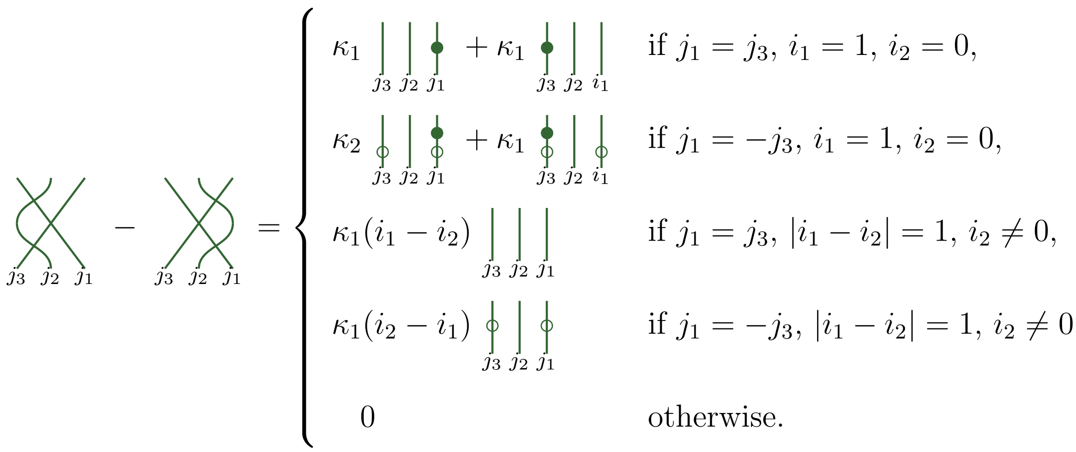
Definition 3.5.4. The *quiver Hecke-Clifford supercategory* of type $\mathfrak{sp}_\infty$ is the monoidal supercategory $\mathcal{QHC}$ with objects generated by the set J, even generating morphisms  $: j_1 \to j_1$ and  $: j_2 \otimes j_1 \to j_1 \otimes j_2$, and odd generating morphisms  $: j_1 \to -j_1$, for all $j_1, j_2 \in J$. These are subject to the following relations:




j_2 j_1
if $j_1 = j_2$,
if $j_1 = -j_2$,
0 otherwise;


j_2 j_1
if $j_1 = j_2$,
if $j_1 = -j_2$,
0 otherwise;


j_2 j_1
0 if $i_1 = i_2$,
if $|i_1 - i_2| > 1$;
$-\kappa_1$ $+ \kappa_2$ if $i_1 = 0$, $i_2 = 1$,
κ_1 $- \kappa_2$ if $i_1 = 1$, $i_2 = 0$
$\kappa_1(i_1 - i_2)$ $+ \kappa_2(i_2 - i_1)$ if $|i_1 - i_2| = 1$, $i_1, i_2 \neq 0$,



In the above, we have adopted the convention given $j_r \in J$ that $i_r \in I$ and $\kappa_r \in \{\pm 1\}$ are defined from $j_r = \kappa_r \sqrt{i_r - \frac{1}{2}}\sqrt{i_r + \frac{1}{2}}$. Identifying the word $\boldsymbol{j} = j_d \cdots j_1 \in J^d$ with $j_d \otimes \cdots \otimes j_1 \in \operatorname{ob} \mathcal{QHC}$, the *quiver Hecke-Clifford superalgebra* is the locally unital algebra

$$QHC_d := \bigoplus_{\boldsymbol{j}, \boldsymbol{j}' \in J^d} \operatorname{Hom}_{\mathcal{QHC}}(\boldsymbol{j}, \boldsymbol{j}'). \tag{3.22}$$

This is exactly as in [KKT, Definition 3.5] in the special case of the $\mathfrak{sp}_\infty$-quiver.

3.5.2. Isomorphisms of completions

As stated at the beginning of the section, we are going to exploit a remarkable isomorphism theorem between certain completions $\widehat{AHC}_d$ and $\widehat{QHC}_d$ of the superalgebras superalgebras AHC_d and QHC_d from (3.21) and (3.22), which was constructed in [KKT]. To define these, we need some further notation.

Numbering strands of a diagram by $1, \ldots, d$ from right to left, AHC_d is generated by its elements $x_r, c_r (1 \leq r \leq d)$ and $t_r (1 \leq r < d)$ corresponding to the closed dot on the rth strand, the open dot on the rth strand, and the crossing of the rth and $(r+1)$th strands, respectively. Let $HC_d := S_d \ltimes C_d$ be the *Sergeev superalgebra*, that is, the smash product of the symmetric group S_d with basic transpositions $t_1, \ldots, t_{d-1}$ acting on the Clifford superalgebra C_d on generators $c_1, \dots, c_d$. Let A_d denote the purely even polynomial superalgebra $\Bbbk[x_1, \dots, x_d]$. Then the natural multiplication map gives a superspace isomorphism $HC_d \otimes A_d \xrightarrow{\sim} AHC_d$. Transporting the multiplication on AHC_d to $HC_d \otimes A_d$ via this isomorphism, the following describe how to commute a polynomial $f \in A_d$ past the generators of HC_d:

$$(1 \otimes f)(c_r \otimes 1) = c_r \otimes c_r(f), \tag{3.23}$$

$$(1 \otimes f)(t_r \otimes 1) = t_r \otimes t_r(f) + 1 \otimes \partial_r(f) + c_r c_{r+1} \otimes \tilde{\partial}_r(f), \tag{3.24}$$

for operators $c_r, t_r, \partial_r, \tilde{\partial}_r : A_d \to A_d$ such that

- t_r is the automorphism that interchanges x_r and x_{r+1} and fixes all other generators;
- c_r is the automorphism that sends $x_r \mapsto -x_r$ and fixes all other generators;
- ∂_r is the Demazure operator $\partial_r(f) := \frac{t_r(f) - f}{x_r - x_{r+1}}$;
- $\tilde{\partial}_r$ is the twisted Demazure operator $c_{r+1} \circ \partial_r \circ c_r$, so $\tilde{\partial}_r(f) = \frac{t_r(f) - c_{r+1}(c_r(f))}{x_r + x_{r+1}}$.

Given a tuple $\mu = (\mu_i)_{i \in I}$ of non-negative integers all but finitely many of which are zero, the quotient superalgebra

$$AHC_d(\mu) := AHC_d \Big/ \Big\langle \prod_{i \in I} \left(x_1^2 - i^2 + \frac{1}{4} \right)^{\mu_i} \Big\rangle \tag{3.25}$$

is a (degenerate) *cyclotomic Hecke-Clifford superalgebra* in the sense of [BK2, §3.e]. It is finite dimensional. Moreover, all roots of the minimal polynomials of all $x_r \in AHC_d(\mu)$ belong to the set J. It follows for each $\boldsymbol{j} = j_d \cdots j_1$ in the set J^d of words of length d in letters J that there is an idempotent $1_{\boldsymbol{j}} \in AHC_d(\mu)$ defined by the projection onto the simultaneous generalized eigenspaces for $x_1, \ldots, x_d$ with eigenvalues $j_1, \ldots, j_d$, respectively. Moreover, we have that

$$AHC_d(\mu) = \bigoplus_{\boldsymbol{j},\boldsymbol{j}' \in J^d} 1_{\boldsymbol{j}'} AHC_d(\mu) 1_{\boldsymbol{j}}.$$

If $\mu \leq \mu'$, i.e. $\mu_i \leq \mu'_i$ for all i, there is a canonical surjection $AHC_d(\mu') \twoheadrightarrow AHC_d(\mu)$ sending $x_r, c_r, t_r, 1_{\boldsymbol{j}} \in AHC_d(\mu')$ to the elements of $AHC_d(\mu)$ with the same names. Let

$$\widehat{AHC}_d := \varprojlim_{\mu} AHC_d(\mu) \tag{3.26}$$

be the inverse limit of this system of superalgebras taken in the category of locally unital superalgebras with distinguished idempotents indexed by J^d. Using the basis theorem for the cyclotomic quotients $AHC_d(\mu)$ from [BK2, §3-e], one can identify $\widehat{AHC}_d$ with the completion defined in [KKT, Definition 5.3][1]. In particular, letting

$$\widehat{A}_d := \bigoplus_{\boldsymbol{j} \in J^d} \mathbb{k}[[x_1 - j_1, \ldots, x_d - j_d]] 1_{\boldsymbol{j}},$$

there is a superspace isomorphism $HC_d \otimes \widehat{A}_d \xrightarrow{\sim} \widehat{AHC}_d$ induced by the obvious multiplication maps $HC_d \otimes \widehat{A}_d \twoheadrightarrow AHC_d(\mu)$ for all μ. The multiplication on $HC_d \otimes \widehat{A}_d$ corresponding to the one on $\widehat{AHC}_d$ via this isomorphism has the following properties

[1]Note there is a sign error in [KKT, (5.5)]: it should read $-C_a C_{a+1} \ldots$.

for all $f \in \widehat{A}_d$:

$$(1 \otimes f1_{\boldsymbol{j}})(c_r \otimes 1_{\boldsymbol{j}'}) = c_r \otimes c_r(f)1_{c_r(\boldsymbol{j})}1_{\boldsymbol{j}'}, \tag{3.27}$$

$$(1 \otimes f1_{\boldsymbol{j}})(t_r \otimes 1_{\boldsymbol{j}'}) = t_r \otimes t_r(f)1_{t_r(\boldsymbol{j})}1'_{\boldsymbol{j}} + 1 \otimes \frac{t_r(f)1_{t_r(\boldsymbol{j})} - f1_{\boldsymbol{j}}}{x_r - x_{r+1}}1_{\boldsymbol{j}'}$$
$$+ c_r c_{r+1} \otimes \frac{t_r(f)1_{t_r(\boldsymbol{j})} - c_{r+1}(c_r(f))1_{c_{r+1}(c_r(\boldsymbol{j}))}}{x_r + x_{r+1}}1_{\boldsymbol{j}'}. \tag{3.28}$$

The fractions on the right hand side of (3.28) make sense: in the first, $(x_r - x_{r+1})1_{\boldsymbol{j}'}$ is invertible unless $j'_r = j'_{r+1}$, in which case the expression equals $\partial_r(f)1_{\boldsymbol{j}}1_{\boldsymbol{j}'}$; the second is fine when $j'_r \neq -j'_{r+1}$ as then $(x_r + x_{r+1})1_{\boldsymbol{j}'}$ is invertible, while if $j'_r = -j'_{r+1}$ it equals $\tilde{\partial}_r(f)1_{t_r(\boldsymbol{j})}1_{\boldsymbol{j}'}$.

Similarly, there is a completion $\widehat{QHC}_d$ of QHC_d. To introduce this, we denote the elements of $QHC_d 1_{\boldsymbol{j}}$ defined by an open dot on the rth strand, a closed dot on the rth strand and a crossing of the rth and $(r+1)$th strands by $\gamma_r 1_{\boldsymbol{j}}, \xi_r 1_{\boldsymbol{j}}$ and $\tau_r 1_{\boldsymbol{j}}$, respectively. For $\mu = (\mu_i)_{i \in I}$ as above, we define the *cyclotomic quiver Hecke-Clifford superalgebra*

$$QHC_d(\mu) := QHC_d \Big/ \left\langle \xi_1^{2\mu_i} 1_{\boldsymbol{j}} \,\middle|\, \boldsymbol{j} \in J^d, i \in I \text{ with } j_1^2 = (z+i)(z+i+1) \right\rangle. \tag{3.29}$$

Using the relations, it is easy to see that the images of all $\xi_r 1_{\boldsymbol{j}}$ are nilpotent in $QHC_d(\mu)$. Then we set

$$\widehat{QHC}_d := \varprojlim_{\mu} QHC_d(\mu), \tag{3.30}$$

taking the inverse limit once again in the category of locally unital superalgebras with distinguished idempotents indexed by J^d. The obvious locally unital homomorphisms $QHC_d \otimes_{\Bbbk[\xi_1,\dots,\xi_d]} \Bbbk[[\xi_1,\dots,\xi_d]] \twoheadrightarrow QHC_d(\mu)$ for each μ induce a surjective homomorphism

$$QHC_d \otimes_{\Bbbk[\xi_1,\dots,\xi_d]} \Bbbk[[\xi_1,\dots,\xi_d]] \to \widehat{QHC}_d.$$

This map is actually an isomorphism, as may be deduced using the basis theorem for QHC_d from [KKT, Corollary 3.9] plus the observation that the image of any non-zero element $u \in QHC_d$ is non-zero in $QHC_d(\mu)$ for sufficiently large μ; the latter assertion follows by elementary considerations involving the natural $\mathbb{Z}$-grading on QHC_d. Consequently, $\widehat{QHC}_d$ is isomorphic to the completion introduced in a slightly different way in [KKT, Definition 3.16]. Moreover, there is a locally unital embedding $QHC_d \hookrightarrow \widehat{QHC}_d$.

At last, we are ready to state the crucial theorem from [KKT]. We need this only in the special situation of [KKT, §5.2(i)(c)], but emphasize that the results obtained in [KKT] are substantially more general. In particular, for us, all elements of the set I are even in the sense of [KKT, §3.5], so that we do not need the more general quiver Hecke *super*algebras of [KKT].

Theorem 3.5.5. *There is a superalgebra isomorphism* $\widehat{QHC}_d \xrightarrow{\sim} \widehat{AHC}_d$ *such that*

$$1_{\boldsymbol{j}} \mapsto 1_{\boldsymbol{j}}, \quad \gamma_r 1_{\boldsymbol{j}} \mapsto c_r 1_{\boldsymbol{j}}, \quad \xi_r 1_{\boldsymbol{j}} \mapsto y_r 1_{\boldsymbol{j}}, \quad \tau_r 1_{\boldsymbol{j}} \mapsto t_r g_r 1_{\boldsymbol{j}} + f_r 1_{\boldsymbol{j}} + c_r c_{r+1} \tilde{f}_r 1_{\boldsymbol{j}},$$

for all $\boldsymbol{j} \in J^d$ and r. Here, $y_r \in \Bbbk[[x_r - j_r]]$ and $g_r, f_r, \tilde{f}_r \in \Bbbk[[x_r - j_r, x_{r+1} - j_{r+1}]]$ are the power series determined uniquely by the following:

$$
\begin{aligned}
j_r &= \kappa_r \sqrt{i - \frac{1}{2}}\sqrt{i + \frac{1}{2}} \text{ for } i_r \in I \text{ and } \kappa_r \in \{\pm\}, \\
y_r &= \begin{cases} \kappa_r \left(\sqrt{i_r^2 + 2j_r(x_r - j_r) + (x_r - j_r)^2} - i_r\right) \in (x_r - j_r) & \text{if } i_r \neq 0 \\ \kappa_r \left((x_r - j_r)^2 + 2j_r(x_r - j_r)\right) & \text{if } i_r = 0 \end{cases} \\
p_r &= \frac{(x_r^2 - x_{r+1}^2)^2}{2(x_r^2 + x_{r+1}^2) - (x_r^2 - x_{r+1}^2)^2}, \\
g_r &= \begin{cases} -1 & \text{if } i_r < i_{r+1}, \\ p_r\left(\kappa_r y_r - \kappa_{r+1} y_{r+1}\right) & \text{if } i_r = i_{r+1} + 1, \\ p_r & \text{if } i_r > i_{r+1} + 1, \\ \frac{\sqrt{p_r}}{y_r - y_{r+1}} \in \frac{x_r - x_{r+1}}{y_r - y_{r+1}} + (x_r - x_{r+1}) & \text{if } j_r = j_{r+1}, \\ \frac{\sqrt{p_r}}{y_r + y_{r+1}} \in \frac{x_r + x_{r+1}}{y_r + y_{r+1}} + (x_r + x_{r+1}) & \text{if } j_r = -j_{r+1}; \end{cases} \\
f_r &= \frac{g_r}{x_r - x_{r+1}} - \frac{\delta_{j_r, j_{r+1}}}{y_r - y_{r+1}}, \qquad \tilde{f}_r = \frac{g_r}{x_r + x_{r+1}} - \frac{\delta_{j_r, -j_{r+1}}}{y_r + y_{r+1}}.
\end{aligned}
$$

(All of this notation depends implicitly on $\boldsymbol{j}$.)

Proof. This is a special case of [KKT, Theorem 5.4]. To help the reader to translate between our notation and that of [KKT], we note that the set J in [KKT] is the same as our set J, but the set I there is $J^2 = \{j^2 \mid j \in J\}$. We have made various other choices as stipulated in [KKT] in order to produce concrete formulae:

- We have taken the functions $\varepsilon : J \to \{0, 1\}$ and $h : J^2 \to \Bbbk$ from [KKT, (5.7)] so that $\varepsilon(j) = (1 - \kappa)/2$ and $\lambda(j) := h(j^2) = i$ for $j = \kappa\sqrt{i - \frac{1}{2}}\sqrt{i + \frac{1}{2}}$;
- For [KKT, (5.11)] we took $G_{j_r, j_{r+1}}$ (our g_r) to be -1 when $i_r < i_{r+1}$.

We remark that the existence of well-defined fractions g_r, f_r, and $\tilde{f}_r \in \Bbbk[[x_r - j_r, x_{r+1} - j_{r+1}]]$ is justified by [KKT, Lemma 5.5]. In the case where $j_r = \pm j_{r+1}$, the choice of $\sqrt{p_r}$ used in g_r is uniquely determined by the specified containment. A different choice of square root would have forced g_r to be an element of the set $-\frac{x_r-x_{r+1}}{y_r-y_{r+1}} + (x_r - x_{r+1})$ or $-\frac{x_r+x_{r+1}}{y_r+y_{r+1}} + (x_r + x_{r+1})$, respectively. Similarly, the square root defining y_r when $i_r \neq 0$ is uniquely determined by the specified containment $y_r \in (x_r - j_r)$, as a different choice of square root would have a constant term in the power series expansion formula for y_r in terms of $x_r - j_r$.

The formula for y_r can be extracted from Subsection 5.3.2 of [KKT] as follows. In that section, $\lambda_r \in \Bbbk[[x_r - j_r]]$ satisfies

$$\lambda_r^2 = x_r^2 + \frac{1}{4} = (x_r - j_r)^2 - 2j_r(x_r - j_r) + j_r^2.$$

In our C_∞ situation, there are two cases depending on i_r, which is the same as their $\lambda(j_r)$:

- When $i_r = 0$, $y_r = \lambda_r^2$.
- When $i_r > 0$, $y_r = \lambda_r - i_r$.

In each of these cases, there is a unique choice of square root allowing us to write y_r as a power series in the ideal generated by $(x_r - j_r)$.

The formula defining the image of $\tau_r 1_{\boldsymbol{j}}$ is a bit more complicated. First, as in [KKT, (5.12)], define $\tilde{s}_r 1_{\boldsymbol{j}} \in \widehat{AHC}_d$ by

$$\tilde{s}_r 1_{\boldsymbol{j}} := \varphi_r g_r 1_{\boldsymbol{j}} = t_r g_r 1_{\boldsymbol{j}} + \frac{g_r}{x_1 - x_2} 1_{\boldsymbol{j}} + c_1 c_2 \frac{g_r}{x_1 + x_2} 1_{\boldsymbol{j}},$$

where $\varphi_r = t_r 1_{\boldsymbol{j}} + \frac{1}{x_1 - x_2} 1_{\boldsymbol{j}} + c_1 c_2 \frac{1}{x_1 + x_2} 1_{\boldsymbol{j}}$ is the *intertwiner* for $\mathcal{AHC}_d$ given in [KKT, (5.1)]. Next, using [KKT, Theorem 3.8], define

$$\sigma_r 1_{\boldsymbol{j}} = \begin{cases} \tilde{s}_r 1_{\boldsymbol{j}} & \text{if } j_r \neq \pm j_{r+1} \\ \tilde{s}_r 1_{\boldsymbol{j}} - \frac{1}{y_r - y_{r+1}} 1_{\boldsymbol{j}} & \text{if } j_r = j_{r+1} \\ \tilde{s}_r 1_{\boldsymbol{j}} - \frac{1}{y_r + y_{r+1}} 1_{\boldsymbol{j}} & \text{if } j_r = -j_{r+1} \end{cases}$$

Then, $\sigma_r 1_{\boldsymbol{j}}$ is precisely the element $t_r g_r 1_{\boldsymbol{j}} + f_r 1_{\boldsymbol{j}} + c_r c_{r+1} \tilde{f}_r 1_{\boldsymbol{j}}$ from the statement of the theorem.

□

3.5.3. Construction of Φ

We are ready to prove Theorem 3.5.1 by constructing $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$:

Proof of Theorem 3.5.1. For $\boldsymbol{i} = i_d \cdots i_1 \in I^d$, let $F_{\boldsymbol{i}} := F_{i_d} \cdots F_{i_1} : \mathcal{O} \to \mathcal{O}$. The usual composition of natural transformations makes the vector space

$$NT_d := \bigoplus_{\boldsymbol{i}, \boldsymbol{i}' \in I^d} \operatorname{Hom}(F_{\boldsymbol{i}}, F_{\boldsymbol{i}'})$$

into a locally unital algebra with distinguished idempotents $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^d\}$ arising from the identity endomorphisms of each $F_{\boldsymbol{i}}$. Also horizontal composition of natural transformations defines homomorphisms $a_{d_2,d_1} : NT_{d_2} \otimes NT_{d_1} \to NT_{d_2+d_1}$ for all $d_1, d_2 \geq 0$. Recalling (2.8), the data of a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ sending i to F_i is just the same as a family of locally unital algebra homomorphisms $\Phi_d : QH_d \to NT_d$ for all $d \geq 0$, such that $1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}}$ for each $\boldsymbol{i} \in I^d$ and

$$a_{d_2,d_1} \circ \Phi_{d_2} \otimes \Phi_{d_1} = \Phi_{d_2+d_1} \circ b_{d_2,d_1} \tag{3.31}$$

for all $d_1, d_2 \geq 0$, where $b_{d_2,d_1} : QH_{d_2} \otimes QH_{d_1} \to QH_{d_2+d_1}$ is the obvious embedding defined by horizontal concatenation of diagrams.

To construct Φ_d, we start from the monoidal superfunctor Ψ from Theorem 3.5.3. This induces superalgebra homomorphisms $\Psi_d : AHC_d \to \mathrm{End}(sF^d)$ for all $d \geq 0$, where $\mathrm{End}(sF^d)$ denotes supernatural endomorphisms of $sF^d : s\mathcal{O} \to s\mathcal{O}$. For each $M \in \mathrm{ob}\, s\mathcal{O}$, Corollary 3.2.3 implies that $\mathrm{ev}_M \circ \Psi_d : AHC_d \to \mathrm{End}_{s\mathcal{O}}(sF^d M)$ factors through all sufficiently large cyclotomic quotients $AHC_d(\mu)$. Hence, Ψ_d extends uniquely to a locally unital superalgebra homomorphism $\widehat{\Psi}_d : \widehat{AHC}_d \to SNT_d$, where

$$SNT_d := \bigoplus_{\boldsymbol{j},\boldsymbol{j}' \in J^d} \mathrm{Hom}(sF_{\boldsymbol{j}}, sF_{\boldsymbol{j}'}) \subset \mathrm{End}(sF^d)$$

and $sF_{\boldsymbol{j}} := sF_{j_d} \cdots sF_{j_1}$. Composing $\widehat{\Psi}_d$ with the isomorphism from Theorem 3.5.5 and the inclusion $QHC_d \hookrightarrow \widehat{QHC}_d$, we obtain a locally unital superalgebra homomorphism $\Theta_d : QHC_d \to SNT_d$. It is obvious from Definitions 2.3.1 and 3.5.4 that there is a locally unital algebra homomorphism $\mathrm{in} : QH_d \to (QHC_d)_{\bar{0}}$ sending the idempotent $1_{\boldsymbol{i}}$ to $1_{\boldsymbol{j}}$ for $\boldsymbol{j}$ with $j_r := \sqrt{i_r - \frac{1}{2}}\sqrt{i_r + \frac{1}{2}}$, and taking the elements of $QH_d 1_{\boldsymbol{i}}$ defined by the dot on the rth strand and the crossing of the rth and $(r+1)$th strands to $\xi_r 1_{\boldsymbol{j}}$ and $\tau_r 1_{\boldsymbol{j}}$, respectively. Also, recalling (3.19), restriction from $s\mathcal{O}$ to $\mathcal{O}$ defines a homomorphism

$$\mathrm{pr} : \bigoplus_{\boldsymbol{j},\boldsymbol{j}' \in J_+^d} 1_{\boldsymbol{j}'} (SNT_d)_{\bar{0}} 1_{\boldsymbol{j}} \to NT_d$$

where $J_+ := \left\{ \sqrt{i - \frac{1}{2}}\sqrt{i + \frac{1}{2}} \;\middle|\; i \in I \right\} \subset J$. Then the composition $\mathrm{pr} \circ \Theta_d \circ \mathrm{in}$ gives us the desired locally unital homomorphism $\Phi_d : QH_d \to NT_d$ sending $1_{\boldsymbol{i}} \mapsto 1_{\boldsymbol{i}}$ for each $\boldsymbol{i} \in I^d$. It just remains to observe that the property (3.31) is satisfied, and that

$\Phi_d(x_r 1_{\boldsymbol{i}})_M$ is nilpotent for each r, $\boldsymbol{i} \in I^d$ and $M \in \operatorname{ob}\mathcal{O}$. These things follow from the explicit formulae in Theorems 3.5.3 and 3.5.5 plus Corollary 3.2.3 once again. $\square$

3.6. Proof of main theorem (type C)

Recall the following theorem from Chapter 1:

Main Theorem (Type C). *When n is even, the supercategory $s\mathcal{O}_{\frac{1}{2}}$ splits as a direct sum $\mathcal{O}_{\frac{1}{2}} \oplus \Pi\mathcal{O}_{\frac{1}{2}}$, where $\mathcal{O}_{\frac{1}{2}}$ is a $\Bbbk$-linear category. Moreover, $\mathcal{O}_{\frac{1}{2}}$ admits the structure of a tensor product categorification of $V^{\otimes n}$, where V is the natural $\mathfrak{sp}_\infty$-module. When n is odd, analogous results hold where the supercategory $s\mathcal{O}_{\frac{1}{2}}$ is replaced by $s\mathcal{O}^{\mathtt{CT}}_{\frac{1}{2}}$.*

Proof. Theorem 3.4.1 shows that there is a decomposition $s\mathcal{O} = \mathcal{O} \oplus \Pi\mathcal{O}$, where $s\mathcal{O}$ denotes the category $s\mathcal{O}_{\frac{1}{2}}$ from Chapter 1 when n is even, and the category $s\mathcal{O}^{\mathtt{CT}}_{\frac{1}{2}}$ when n is odd. We have also checked the following:

- Theorem 3.4.2: The category $\mathcal{O}$ is highest weight, with $\mathbf{B}$ labeling its irreducible objects, partially ordered by $\preceq$.

- Theorem 3.5.1: There is a strict monoidal functor $\Phi : \mathcal{QH} \to \operatorname{End}(\mathcal{O})$, such that $\Phi(i) = F_i$. The natural transformation $\Phi\Big(\underset{i}{\bullet}\Big) : F_i \Rightarrow F_i$ is locally nilpotent. This gives us the data of (D2) satisfying (D4).

- The functor E_i is both left and right adjoint to F_i, as implied by Lemma 3.2.5. This gives us the data of (D3) satisfying axiom (D5).

Theorem 3.4.4 shows that the exact functors F_i and E_i preserve $\mathcal{O}^\Delta$, and induced linear maps give an action of $\mathfrak{sp}_\infty$ on $[\mathcal{O}^\Delta]$. Further, the map $[\mathcal{O}^\Delta] \to V^{\otimes n}$ given by $[M(\boldsymbol{b})] \mapsto v_{\boldsymbol{b}}$ induces an isomorphism of $\mathfrak{sp}_\infty$-modules, so axioms (TPC1) and

(TPC2) hold. Thus, we have verified that $\mathcal{O}$ admits the structure of a tensor product categorification of $V^{\otimes n}$. □

CHAPTER IV

APPLICATIONS

In this chapter we continue with all notation set up in Chapter 3, including the category $\mathcal{O}$ defined in Section 3.4, functors $E_i, F_i : \mathcal{O} \to \mathcal{O}$, etc.

In their paper [LW], Losev and Webster prove several theorems about tensor product categorifications for tensor products of highest weight modules, including a uniqueness theorem [LW, Theorem 6.1] and an explicit description of underlying crystals [LW, Theorem 7.2]. Because the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$ is not a tensor product of highest weight modules, these theorems cannot be directly applied to our category $\mathcal{O}$ from Chapter 3. To invoke them, for $k > 0$, we pass to the Serre subquotient $\mathcal{O}_k$ of $\mathcal{O}$ defined below. This category admits the structure of a tensor product categorification of $V_k^{\otimes n}$, where V_k is the natural $\mathfrak{sp}_k$-module. In particular, V_k is a highest weight module. By applying Losev and Webster's theorems to the category $\mathcal{O}_k$ and then transporting these results back to $\mathcal{O}$, we achieve the following:

- We prove that the combinatorics of the type C blocks are determined by Webster's *orthodox basis*.
- We calculate the underlying crystal for the category $\mathcal{O}$.
- We classify all projective-injective (or *prinjective*) indecomposable objects of $\mathcal{O}$.

4.1. The root system $\mathbf{C}_k$

Recall that the nodes of the Dynkin diagram C_∞ are labeled by the index set $I = \mathbb{N}$, where $0 \in I$ indexes the long root in the corresponding root system. Given $k \geq 0$, let $I_k \subset I$ denote the set of all natural numbers between 0 and $k-1$, and let

C_k denote the subdiagram of C_∞ with nodes indexed by I_k. Denote its associated Kac-Moody algebra by $\mathfrak{sp}_k$, which we realize as the subalgebra of $\mathfrak{sp}_\infty$ with Chevalley generators $\{e_i, f_i \mid i \in I_k\}$. Note that $\mathfrak{sp}_k$ inherits many notions from $\mathfrak{sp}_\infty$, including a weight lattice $\bigoplus_{i \in I_k} \mathbb{Z}\,\varepsilon_i$, set of simple roots $\{\alpha_i \mid i \in I_k\}$, etc.

Restricting the natural $\mathfrak{sp}_\infty$-module V to $\mathfrak{sp}_k$, let V_k denote the *natural* $\mathfrak{sp}_k$-*module* spanned by the vectors $\{v_b \mid -k < b \leq k\} \subset V$. This is a $2k$-dimensional irreducible module of highest weight $-\varepsilon_{k-1}$. The tensor space $V_k^{\otimes n} \subset V^{\otimes n}$ has basis $\{v_{\boldsymbol{b}} \mid \boldsymbol{b} \in \mathbf{B}_k\}$, where

$$\mathbf{B}_k = \{\boldsymbol{b} \in \mathbf{B} \mid -k < b_s \leq k \text{ for all } 1 \leq s \leq n\}.$$

Also, the type C Bruhat ordering $\preceq$ on $\mathbf{B}$ restricts to a partial ordering on $\mathbf{B}_k$.

Because $V_k^{\otimes n}$ is a tensor product of highest weight modules, [LW, Definition 3.2] applies. To rephrase their definition in terms similar to Definition 2.5.1, we replace $\mathbf{B}$ in Definition 2.5.1 with $\mathbf{B}_k$, I with I_k, and the monoidal category $\mathcal{QH}$ with the full subcategory $\mathcal{QH}_k$ generated as a monoidal category by the objects $i \in I_k$.

4.2. The category $\mathcal{O}_k$

Let $\mathbf{B}_{\leq k}$ denote the set of all $\boldsymbol{b} \in \mathbf{B}$ such that $N_{[1,s]}(\boldsymbol{b}, k) \leq 0$ for all $s = 1, \ldots, n$. Next, define $\mathbf{B}_{<k}$ as the set of all $\boldsymbol{b} \in \mathbf{B}_{\leq k}$ such that $N_{[1,s]}(\boldsymbol{b}, k) < 0$ for at least one s. Lemma 3.3.2 implies immediately that $\mathbf{B}_{\leq k}$ and $\mathbf{B}_{<k}$ are ideals in the sense of Section 2.1.3.

Observe that the set $\mathbf{B}_k$ is the difference $\mathbf{B}_{\leq k} \setminus \mathbf{B}_{<k}$. Indeed, the elements $\boldsymbol{b}$ of $B_{\leq k} \setminus \mathbf{B}_{<k}$ satisfy $N_{[1,s]}(\boldsymbol{b}, k) = 0$ for every $1 \leq s \leq n$. A straight-forward induction starting with $s = 1$ then shows that $-k < b_s \leq k$ for every s.

We are precisely in the situation described in Section 2.1.3. Let $\mathcal{O}_{\leq k}$ denote the Serre subcategory of $\mathcal{O}$ generated by the irreducible supermodules $\{L(\boldsymbol{b}) \mid \boldsymbol{b} \in \mathbf{B}_{\leq k}\}$, and similarly define $\mathcal{O}_{<k}$. Because $\mathbf{B}_{\leq k}$ and $\mathbf{B}_{<k}$ are ideals, the categories $\mathcal{O}_{\leq k}$ and $\mathcal{O}_{<k}$ inherit a highest weight structure from $\mathcal{O}$. Hence, the quotient category $\mathcal{O}_k := \mathcal{O}_{\leq k}/\mathcal{O}_{<k}$ has an induced highest weight structure with poset $(\mathbf{B}_k, \preceq)$.

Let $\pi : \mathcal{O}_{\leq k} \to \mathcal{O}_k$ be the quotient functor. For $\boldsymbol{b} \in \mathbf{B}_k$, define $L_k(\boldsymbol{b}) = \pi L(\boldsymbol{b})$ and $M_k(\boldsymbol{b}) = \pi M(\boldsymbol{b})$. The object $L_k(\boldsymbol{b})$ is irreducible in $\mathcal{O}_k$, and $M_k(\boldsymbol{b})$ is a standard module in the highest weight structure.

Lemma 4.2.1. *Given any $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}$, let k be large enough so that $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{B}_k$. Then,*

$$[M(\boldsymbol{b}) : L(\boldsymbol{a})] = [M_k(\boldsymbol{b}) : L_k(\boldsymbol{a})].$$

Proof. This is immediate from the exactness of the quotient functor π. □

4.3. Categorical actions on $\mathcal{O}_k$

Lemma 4.3.1. *For every $i \in I_k$, the functors F_i and $E_i : \mathcal{O} \to \mathcal{O}$ preserve the subcategories $\mathcal{O}_{\leq k}$ and $\mathcal{O}_{<k}$.*

Proof. We will check this for F_i. The proof for E_i is similar. As in [BLW, Lemma 2.18], it suffices to check that $F_i M(\boldsymbol{b})$ is an object of $\mathcal{O}_{\leq k}$ (resp. $\mathcal{O}_{\leq k}$) whenever $\boldsymbol{b} \in \mathbf{B}_{\leq k}$ (resp. $\mathbf{B}_{<k}$).

Both cases follow from the observation that if $i \in I_k$, and $i\text{-sig}(\boldsymbol{b})_t = \mathtt{f}$,

$$N_{[1,s]}(\boldsymbol{b}, k) = N_{[1,s]}(\boldsymbol{b} + \boldsymbol{d}_t, k). \tag{4.1}$$

Indeed, if $i\text{-sig}(\boldsymbol{b})_t = \mathtt{f}$, then $b_t = \pm i$. From the restriction $i \in I_k$, it follows that $-k+1 \leq b_t \leq k-1$, so $-k+2 \leq b_t + 1 \leq k$. Hence,

$$N_{[1,s]}(\boldsymbol{b}, k) = N_{[1,s]}(\boldsymbol{b} + \boldsymbol{d}_t, k)$$

for any $1 \leq s \leq n$, because the t-th entry does not contribute to either sum in 4.1.

Because $F_i M(\boldsymbol{b})$ is filtered by Vermas of the form $M(\boldsymbol{b}+\boldsymbol{d}_t)$ where $i\text{-sig}(\boldsymbol{b})_t = \mathtt{f}$, this observation proves that $F_i M(\boldsymbol{b})$ is in $\mathcal{O}_{\leq k}$ when $\boldsymbol{b} \in \mathbf{B}_{\leq k}$, and similarly for $\mathcal{O}_{<k}$ and $\mathbf{B}_{<k}$.

□

Theorem 4.3.2. *The category $\mathcal{O}_k$ admits the structure of a tensor product categorification of $V_k^{\otimes n}$.*

Proof. Using the lemma, we see that F_i and E_i restrict to biadjoint, exact endofunctors of $\mathcal{O}_{\leq k}$ and $\mathcal{O}_{<k}$. Hence, they induce biadjoint, exact endofunctors $\bar{F}_i, \bar{E}_i$ of the quotient category $\mathcal{O}_k$ which satisfy $\pi F_i = \bar{F}_i \pi$, and $\pi E_i = \bar{E}_i \pi$.

Recall from Theorem 3.5.1 that there is a strict monoidal functor $\Phi : \mathcal{QH} \to \mathcal{E}nd(\mathcal{O})$ sending the generating object $i \in I \subset \text{ob}\, \mathcal{QH}$ to F_i. Restricting to the subcategory $\mathcal{QH}_k$ of $\mathcal{QH}$ generated by $i \in I_k$, and restricting the functors F_i to $\mathcal{O}_{\leq k}$ and $\mathcal{O}_{<k}$, we see that there are well-defined monoidal functors:

$$\mathcal{QH}_k \to \mathcal{E}nd(\mathcal{O}_{\leq k}) \text{ and } \mathcal{QH}_k \to \mathcal{E}nd(\mathcal{O}_{<k})$$

sending the generating object $i \in I_k$ to F_i. Hence, there is an induced monoidal functor $\Phi_k : \mathcal{QH}_k \to \mathcal{E}nd(\mathcal{O}_k)$. The fact that $\Phi_k\Big($  $\Big)$ is locally nilpotent follows from the fact that the corresponding natural transformation is locally nilpotent in $\mathcal{O}_{\leq k}$.

Hence, we have equipped the highest weight category $\mathcal{O}_k$ with the $\mathfrak{sp}_k$-analogs of (D2) and (D3), satisfying (D4) and (D5). Whenever $i \in I_k$, the fact that the functors $\bar{F}_i$ and $\bar{E}_i$ satisfy the $\mathfrak{sp}_k$ versions of the axioms (TPC1) and (TPC2) follows immediately from the corresponding statements for the functors F_i and E_i on $\mathcal{O}$. □

Webster's general theory from [W1] gives another construction of an $\mathfrak{sp}_k$-tensor product categorification of $V_k^{\otimes n}$ in terms of the category T_k^n-mod of finite dimensional modules over the tensor product algebra T_k^n of [W1, §4] associated to the $\mathfrak{sp}_k$-module $V_k^{\otimes n}$. See also [LW, Proposition 3.11].

Theorem 4.3.3. *The category $\mathcal{O}_k$ is strongly equivariantly equivalent to T_k^n-mod.*

Proof. We are now in finite rank, so this follows from the uniqueness of tensor product categorifications exactly as established in [LW, Theorem 6.1]. □

As another application, by choosing k sufficiently large, we combine Theorem 4.3.3 with Lemma 4.2.1 to see that the composition multiplicities of the Verma supermodules in $\mathcal{O}$ are the same as the corresponding composition multiplicities of the standard objects in T_k^n-mod. This was conjectured independently in [CKW, Conjecture 5.11].

Along similar lines, we have isomorphisms of Grothendieck groups

$$[T_k^n\text{-mod}^\Delta] \cong [\mathcal{O}_k^\Delta] \cong V_k^{\otimes n}$$

given by identifying the classes of standard objects in T_n^k-mod and $\mathcal{O}_k$ with the monomial basis in $V_k^{\otimes n}$. In [W1, W2], Webster defines the *orthodox basis* of $V_k^{\otimes n}$ to be the basis arising from the classes of indecomposable projectives in T_k^n-mod. Taking k sufficiently large so that $P(\boldsymbol{b})$ is an object in $\mathcal{O}_{\leq k}$ and hence $\mathcal{O}_k$, we see that the class $[P(\boldsymbol{b})]$ corresponds to an orthodox basis element of $V_k^{\otimes n}$.

Webster's general theory constructs tensor product categorifications for any tensor product of highest weight modules over a symmetrizable Kac-Moody algebra. In the case where the Kac-Moody algebra is of finite type A, D, E, the orthodox bases described here are precisely Lusztig's canonical basis for the tensor space. Outside of those types, in particular, for the type C_k case described here, there is no known elementary algorithm to compute these bases, and hence, the combinatorics underlying the type C blocks.

This contrasts with the situation in the type A blocks, where the composition multiplicities of Verma supermodules can be computed in terms of certain canonical bases associated to the quantum group of type A_∞, which can can be constructed using elementary methods. See Section 1 of [BD2].

4.4. Crystals

As another application of the results of this dissertation, we identify the associated crystal for $\mathcal{O}$ with the crystal underlying the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$. Crystals were originally developed by Kashiwara [Ka]. We recall the definition for $\mathfrak{sp}_\infty$ here, although the definition adapts easily to any other Kac-Moody algebra.

Definition 4.4.1. A *normal* $\mathfrak{sp}_\infty$ *crystal* is a set $\mathbf{B}$ with a decomposition $\mathbf{B} = \bigsqcup_{\lambda \in P} \mathbf{B}_\lambda$, plus *crystal operators* $\tilde{e}_i, \tilde{f}_i : \mathbf{B} \to \mathbf{B} \sqcup \{0\}$ for each $i \in I$ satisfying the following axioms:

(C1) for every $\lambda \in P$, the crystal operator $\tilde{e}_i$ restricts to a map $\mathbf{B}_\lambda \to \mathbf{B}_{\lambda+\alpha_i} \sqcup \{0\}$;

(C2) for $b \in \mathbf{B}$, we have that $\tilde{e}_i(b) = b' \neq 0$ if and only if $\tilde{f}_i(b') = b \neq 0$;

(C3) for every $b \in \mathbf{B}$, there is an $r \in \mathbb{N}$ such that $\tilde{e}_i^r(b) = \tilde{f}_i^r(b) = 0$.

For each i, define functions $\varepsilon_i, \varphi_i : \mathbf{B} \to \mathbb{N}$ by

$$\varepsilon_i(b) = \max\{r \in \mathbb{N} \mid \tilde{e}_i^r(b) \neq 0\}, \qquad \varphi_i(b) = \max\{r \in \mathbb{N} \mid \tilde{f}_i^r(b) \neq 0\}.$$

Then we also require that

(C4) $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \lambda \rangle$ for each $b \in \mathbf{B}_\lambda$ and $i \in I$.

There is an $\mathfrak{sp}_\infty$-crystal structure on $\mathbf{B}$ defined using the combinatorics underlying the $\mathfrak{sp}_\infty$-module $V^{\otimes n}$. The decomposition $\mathbf{B} = \bigsqcup_{\lambda \in P} \mathbf{B}_\lambda$ is given by setting $\mathbf{B}_\lambda = \{\boldsymbol{b} \in \mathbf{B} \mid \mathrm{wt}(\boldsymbol{b}) = \lambda\}$. The crystal operators are defined using *Kashiwara's tensor product rule,* described as follows. Pick any $\boldsymbol{b} \in \mathbf{B}$. Starting from the *$i$-signature* i-sig$(\boldsymbol{b})$ from (2.6), we inductively define the *reduced i-signature* by replacing pairs of entries of the form `ef` (possibly separated only by $\bullet$'s) with $\bullet$'s. This process iterates until all `e` entries appear to the right of the entries `f`. Then define $\tilde{f}_i\boldsymbol{b}$ to be $\boldsymbol{b} + \boldsymbol{d}_r$ if the rightmost `f` in the reduced i-signature appears in position r, or 0 if there are no `f`'s remaining in the reduced i-signature. Similarly, define $\tilde{e}_i\boldsymbol{b}$ to be $\boldsymbol{b} - \boldsymbol{d}_s$ if the leftmost `e` in the reduced i-signature appears in position s, or 0 if there are no `e`'s present. This defines our crystal operators $\tilde{e}_i$ and $\tilde{f}_i$. A routine check shows that this data makes $\mathbf{B}$ into a normal $\mathfrak{sp}_\infty$ crystal. By restricting to the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for $i \in I_k$, we can make the set $\mathbf{B}_k \subset \mathbf{B}$ into an $\mathfrak{sp}_k$ crystal.

Example 4.4.2. Take $\boldsymbol{b} = (1, 2, -1, 4, -2, -2, 3, 1, -1)$. The 2-signature of $\boldsymbol{b}$ is the tuple $(\bullet, \mathtt{f}, \mathtt{e}, \bullet, \mathtt{f}, \mathtt{f}, \mathtt{e}, \bullet, \mathtt{e})$. Reducing the 2-signature replaces all `ef` pairs (possibly separated by $\bullet$'s) with $\bullet$'s, so the reduced 2-signature is $(\bullet, \mathtt{f}, \bullet, \bullet, \bullet, \mathtt{f}, \mathtt{e}, \bullet, \mathtt{e})$. Because the rightmost `f` occurs in the sixth entry, we have

$$\tilde{f}_2\boldsymbol{b} = \boldsymbol{b} + \boldsymbol{d}_6 = (1, 2, -1, 4, -2, -1, 3, 1, -1).$$

Similarly, the leftmost e occurs in the seventh entry, and it follows that

$$\tilde{e}_2 \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{d}_7 = (1, 2, -1, 4, -2, -2, 2, -1).$$

The following theorem demonstrates that the crystal structure on $\mathbf{B}$ can be induced by applying the functors F_i and E_i to irreducibles in $\mathcal{O}$. In this way, the irreducibles objects in $\mathcal{O}$ categorify the crystal $\mathbf{B}$.

Theorem 4.4.3. *If $\tilde{f}_i \boldsymbol{b} \neq 0$, then both the the head and socle of $F_i L(\boldsymbol{b})$ are isomorphic to $L(\tilde{f}_i \boldsymbol{b})$; otherwise, $F_i L(\boldsymbol{b}) = 0$. A similar statement holds with F_i and $\tilde{f}_i$ replaced by E_i and $\tilde{e}_i$.*

Proof. Let $k > i$ be sufficiently large so that $\boldsymbol{b} \in \mathbf{B}_k$, and all of the composition factors $F_i L(\boldsymbol{b})$ are labeled by elements of $\mathbf{B}_k$. Arguing in $\mathcal{O}_k$, Theorem 7.2 from [LW] demonstrate that $\bar{F}_i L_k(\boldsymbol{b})$ is non-zero if and only if $\tilde{f}_i \boldsymbol{b} \neq 0$, in which case the head and socle of $\bar{F}_i L_k(\boldsymbol{b})$ is isomorphic to $L_k(\tilde{f}_i \boldsymbol{b})$.

Lemma 2.1.4 and the fact that $\pi F_i \cong \bar{F}_i \pi$ imply that $\mathrm{Hom}_{\mathcal{O}_k}(L_k(\boldsymbol{c}), \bar{F}_i L_k(\boldsymbol{b})) \cong \mathrm{Hom}_{\mathcal{O}_{\leq k}}(L(\boldsymbol{c}), F_i L(\boldsymbol{b}))$. Hence, $L_k(\boldsymbol{c})$ is in the socle of $\bar{F}_i L_k(\boldsymbol{b})$ if and only if $L(\boldsymbol{c})$ is in the socle of $F_i L(\boldsymbol{b})$, and similarly for the heads. This proves the version of the theorem involving F_i. The proof for E_i is similar. □

4.5. Classification of Prinjectives

An object P in a category $\mathcal{C}$ is said to be *prinjective* if it is both projective and injective. In this section, we use the expicit description of the associated crystal for $\mathcal{O}$ from Theorem 4.4.3 to classify the prinjectives in $\mathcal{O}$. We prove the following analog of Theorem 2.24 in [BLW]:

Theorem 4.5.1. *Given $\boldsymbol{b} \in \mathbf{B}$, the following are equivalent:*

1. *The label* $\boldsymbol{b}$ *is antidominant, i.e.,* $b_1 \leq b_2 \leq \cdots \leq b_n$.

2. *The projective indecomposable* $P(\boldsymbol{b})$ *is prinjective.*

3. *We have an isomorphism* $P(\boldsymbol{b}) \cong P(\boldsymbol{b})^\star$.

4. *There exists some* $\boldsymbol{a} \in \mathbf{B}$ *for which* $L(\boldsymbol{b})$ *is in the socle of* $M(\boldsymbol{a})$.

We can easily prove a weaker version of this theorem in the case where the entries of $\boldsymbol{b}$ are constant:

Lemma 4.5.2. *Suppose that* $\boldsymbol{b} \in \mathbf{B}$ *satisfies* $b_1 = \cdots = b_n$. *Then,* $L(\boldsymbol{b}) = M(\boldsymbol{b}) = P(\boldsymbol{b})$. *Because* $L(\boldsymbol{b}) \cong L(\boldsymbol{b})^\star$, *it follows that* $P(\boldsymbol{b}) \cong P(\boldsymbol{b})^\star$ *is prinjective.*

Proof. Let b denote the common value of the entries b_t. Then,

$$\mathrm{wt}(\boldsymbol{b}) = \begin{cases} n\varepsilon_{b-1} & \text{if } b > 0, \\ -n\varepsilon_{-b} & \text{if } b \leq 0, \end{cases}$$

From this, it is clear that there is no other $\boldsymbol{c} \in \mathbf{B}$ with $\mathrm{wt}(\boldsymbol{c}) = \mathrm{wt}(\boldsymbol{b})$, so $\boldsymbol{b}$ is not Bruhat-comparable to any other element of $\mathbf{B}$. The lemma then follows from the observation that $P(\boldsymbol{b})$ can have no Verma subquotients other than $M(\boldsymbol{b})$, and $M(\boldsymbol{b})$ can have no composition factors other than $L(\boldsymbol{b})$. □

Let $\mathbf{B}^\circ$ denote the collection all elements of $\mathbf{B}$ which can be obtained from $\mathbf{z} = (0, \ldots, 0) \in \mathbf{B}$ by applying a sequence of crystal operators. In other words, $\mathbf{B}^\circ$ is the *connected component* of $\mathbf{B}$ containing $\mathbf{z}$.

Lemma 4.5.3. *We have* $\boldsymbol{b} \in \mathbf{B}^\circ$ *if and only if* $\boldsymbol{b}$ *is antidominant.*

Proof. For the first implication, we show that any $\boldsymbol{b}$ which can be obtained from the antidominant $\mathbf{z} \in \mathbf{B}$ using crystal operators is also antidominant. Hence, it suffices to show that whenever $\boldsymbol{a} \in \mathbf{B}$ is antidominant, then so are $\tilde{f}_i\boldsymbol{a}$ and $\tilde{e}_i\boldsymbol{a}$.

To check that the entries in $\tilde{f}_i\boldsymbol{a}$ are in increasing order, we recall that $\tilde{f}_i\boldsymbol{a} = \boldsymbol{a}+\boldsymbol{d}_s$ where s is the maximal index for which the reduced i-signature of $\boldsymbol{a}$ contains an $\mathtt{f}$. Then, $\tilde{f}_i\boldsymbol{a}$ is antidominant only if $b_s < b_{s+1}$. On the contrary, if they were equal, then we would have $i\text{-sig}(\boldsymbol{a})_s = i\text{-sig}(\boldsymbol{a})_{s+1} = \mathtt{f}$. Because we cancel $\mathtt{ef}$ pairs (and not $\mathtt{fe}$!) it would then follow that the reduced i-signature of $\boldsymbol{a}$ contains a $\mathtt{f}$ in its $(s+1)$ entry, which contradicts our assumption about s. The proof of the antidominance of $\tilde{e}_i\boldsymbol{a}$ is similar, and it follows that the elements of $\mathbf{B}^\circ$ are antidominant.

To prove the reverse implication, suppose that $b_1 \le \cdots \le b_n$. For every index s, define a monomial

$$\widetilde{X}_s := \begin{cases} \tilde{f}_{b_s-1}\tilde{f}_{b_s-2}\cdots\tilde{f}_0 & \text{if } b_s \ge 0 \\ \tilde{e}_{-b_s}\tilde{e}_{-b_s+1}\cdots\tilde{e}_1 & \text{if } b_s < 0 \end{cases}$$

Note that, by definition $\widetilde{X}_s$ is an empty monomial if $b_s = 0$. Let t denote the maximal index for which $b_t < 0$, and define

$$\widetilde{X} := \widetilde{X}_t\widetilde{X}_{t-1}\cdots\widetilde{X}_1\widetilde{X}_{t+1}\widetilde{X}_{t+2}\cdots\widetilde{X}_n.$$

A straightforward calculation shows that $X\mathbf{z} = \boldsymbol{b}$, so $\boldsymbol{b} \in \mathbf{B}^\circ$. We emphasize that this calculation relies heavily on the fact that the b_t are in increasing order. □

Proof of Theorem 4.5.1. The proof is similar to that of [BLW, Theorem 2.24].

(1) $\implies$ (2). Fix a antidominant $\boldsymbol{b} \in \mathbf{B}$. Let $\widetilde{X}$ be the corresponding monomial in the crystal operators $\{\tilde{e}_i, \tilde{f}_i \mid i \in I\}$ defined in the proof of Lemma 4.5.3. By replacing each $\tilde{e}_i$ with E_i and each $\tilde{f}_i$ with F_i, we obtain a monomial X in the functors $\{E_i, F_i \mid i \in I\}$. Using Lemma 4.5.2, the irreducible $L(\mathbf{z})$ is prinjective, and because the functors F_i and E_i are right and left adjoint to exact functors, it follows that $T := XL(\mathbf{z})$ is prinjective, too. Using exactness of the E_i and F_i and iterating

Theorem 4.4.3 demonstrates that $L(\boldsymbol{b})$ is in the head and socle of T. Therefore, the projective indecomposable $P(\boldsymbol{b})$ is a summand of T. Because $P(\boldsymbol{b})$ is a summand of an injective object, it follows that $P(\boldsymbol{b})$ is injective, too.

(2) $\implies$ (3). If $P(\boldsymbol{b})$ is prinjective, then it must be isomorphic to an indecomposable injective $P(\boldsymbol{c})^\star$ for some $\boldsymbol{c} \in \mathbf{B}$. We need to verify that $\boldsymbol{c} = \boldsymbol{b}$. Because $\star$ is exact and preserves irreducibles, we see that the indecomposable projectives $P(\boldsymbol{b})$ and $P(\boldsymbol{c})$ have the same composition multiplicities. Because the classes of indecompsable projectives are linearly independent in the Grothendieck group of $\mathcal{O}$, it follows that $\boldsymbol{b} = \boldsymbol{c}$.

(3) $\implies$ (4). If $P(\boldsymbol{b}) \cong P(\boldsymbol{b})^\star$, then the socle of $P(\boldsymbol{b})$ is isomorphic to $L(\boldsymbol{b})$. Using the filtration of $P(\boldsymbol{b})$ by Vermas, there is some $\boldsymbol{a} \succeq \boldsymbol{b}$ with $L(\boldsymbol{b}) \hookrightarrow M(\boldsymbol{a})$.

(4) $\implies$ (1). Suppose that $L(\boldsymbol{b}) \hookrightarrow M(\boldsymbol{a})$. Pick k large enough that all composition factors of $\Delta(\boldsymbol{a})$ are labeled by elements of $\mathbf{B}_k$. Passing to $\mathcal{O}_k$, the irreducible $L_k(\boldsymbol{b})$ is in the socle of $M_k(\boldsymbol{a})$. Because $\boldsymbol{\kappa} = (1-k, ..., 1-k) \in \mathbf{B}_k$ labels the vector in the maximal weight space of $V_k^{\otimes n}$, the weight $\mathrm{wt}(\boldsymbol{\kappa}) = -n\varepsilon_{k-1}$ is maximal among the weights of elements of $\mathbf{B}_k$. In particular,

$$\mathrm{wt}(\boldsymbol{\kappa}) - \mathrm{wt}(\boldsymbol{b}) = \sum_{i \in I_k} h_i \alpha_i \quad (h_i \geq 0)$$

is a sum of simple roots for $\mathfrak{sp}_k$. We let $h := \sum_{i \in I_k} h_i$ denote the height of $\mathrm{wt}(\boldsymbol{\kappa}) - \mathrm{wt}(\boldsymbol{b})$. We prove that $\boldsymbol{b}$ is antidominant by induction on h.

When $h = 0$, we have $\boldsymbol{b} = \boldsymbol{\kappa}$, so $\boldsymbol{b}$ is weakly decreasing. When $h > 0$, we apply Proposition 5.2 of [LW] in the quotient category $\mathcal{O}_k$ to deduce that there is some $i \in I_k$ with $\bar{E}_i L_k(\boldsymbol{b})$ non-zero. Theorem 7.2 from [LW] shows that $L_k(\tilde{e}_i \boldsymbol{b}) \hookrightarrow \bar{F}_i L_k(\boldsymbol{b}) \hookrightarrow \bar{F}_i M_k(\boldsymbol{a})$. Because $\bar{F}_i M_k(\boldsymbol{a})$ is filtered by standard objects, it follows that

$L_k(\tilde{e}_i\boldsymbol{b})$ embeds into a standard object in $\mathcal{O}_k$. The height of $\mathrm{wt}(\boldsymbol{\kappa}) - \mathrm{wt}(\tilde{e}_i\boldsymbol{b})$ is $h-1$, so the induction shows that $\tilde{e}_i\boldsymbol{b}$ is antidominant. Because $\tilde{f}_i\tilde{e}_i\boldsymbol{b} = \boldsymbol{b}$, arguments in the proof of Lemma 4.5.3 shows that $\boldsymbol{b}$ is antidominant. $\square$

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