ON THE SOLVABILITY OF BETA-ENSEMBLES WHEN BETA IS A SQUARE INTEGER

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2019

DISSERTATION APPROVAL PAGE

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Title: On the Solvability of Beta-Ensembles when Beta is a Square Integer

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Degree awarded June 2019

DISSERTATION ABSTRACT

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We use combinatorial identities in the shuffle and exterior algebra to present hyperpfaffian formulations of partition functions for β-ensembles with arbitrary probability measure when β is a square integer. This is an analogue of the de Bruijn integral identities for the $\beta = 1$ and $\beta = 4$ ensembles. We also generalize several classic algebraic identities for determinants and Pfaffians to identities for Hyperpfaffians, extending the fermionic and bosonic Wick formulas which frequently arise in Quantum Field Theory.

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R. A. Gordon and J. M. Wells, *On the perimeter of integral triangles*, International Journal of Pure and Applied Mathematics **64** (2010).

ACKNOWLEDGEMENTS

I am greatly indebted to my doctoral adviser, Christopher Sinclair, for his assistance in preparation of this manuscript, for his guidance and mentorship during my long tenure as a graduate student, for his ever-insightful discourse and instruction, and for introducing me to the wonderful world of random matrix theory.

I also thank the many scholars who provided helpful conversation, comment, and critique during the creation of this document, including David Levin, Benjamin Young, Jonathan Brundan, Elisha Wolff, Joe Webster, Doeke Buursma, Andrew Wray, Eric Hogle, and Matthew Ruppel.

I owe much gratitude to my undergraduate mentor at Whitman College, Barry Balof, for finding a wayward mathematician and guiding him back home, and to my undergraduate adviser, Russel Gordon, for instilling in me the virtues of excellence, precision, and tenacity.

I am also grateful to the many mathematicians at the University of Oregon and Whitman College with whom I have shared innumerable mathematical discussions, on topics ranging from the trivial to the profound.

Finally, I must express extreme thanks to my family: to my parents Russ and Connie, whose unconditional love allowed me to choose the path that led me here, to my brother Andrew, who ceaselessly motivates me to achieve higher and higher goals, to my wife Kayla, whose unfailing support keeps me standing even in darkest days, and to my son Oliver, who I know will one day read this dissertation and remark on how obvious it all is.

For my father, the best teacher I have ever had.

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CHAPTER I

INTRODUCTION

Introduction

Random matrix theory is the study of the eigenvalue statistics of ensembles of matrices, which are a collection of (typically) square matrices along with a probability measure defined on this set. Often, the probability measure is presented as the distribution of individual entries of the matrix, along with a specification for inter-entry dependence. This probability measure then induces a probability measure on the eigenvalues of the matrix. Although a rich field of study on its own, random matrix theory also enjoys rather abundant application throughout mathematics and physics, since the eigenvalue statistics of many matrix ensembles can be used to model a wide variety of phenomena. The statistics of discrete energy levels in atomic spectra bear many of the same features as the eigenvalues of Hermitian matrices. Additionally, these same Hermitian ensembles can be used to describe the statics of a multi-particle system in one dimension interacting via a repulsive force and subject to a fixed potential. Alternatively, the *Wishart* ensemble of matrices can be used in the estimation of the covariance matrix for a population vector, given a large sample. The widespread applicability of random matrices evinces a universal paradigm—a collection of theorems akin to the classical Central Limit Theorem. But the utility of such theorems depends on an available supply of *solvable* ensembles in each universality class—collections of matrices for which the densities of eigenvalues can be expressed in terms of 'known' functions whose properties and asymptotics are well-studied.

The β-ensembles are one such collection, and are composed of random matrices whose eigenvalue densities take a common form, indexed by a non-negative, real parameter β. The *classic* β-ensembles ($\beta = 1, 2, 4$) correspond to Hermitian matrices with real, complex, or quaternionic Gaussian entries, and were first studied in the 1920s by John Wishart in multivariate statistics [27] and the 1950s by Eugene Wigner in nuclear physics [26]. In the subsequent decade, Freeman Dyson and Madan Mehta[12] unified a previously disparate collection of random matrix models by demonstrating that the three classic β-ensembles are each variations of a single action on random Hermitian matrices (representing the three associative division algebras over $\mathbb{R}$). More recently, the development [10] of matrix models representing arbitrary, non-negative values of β, as well as the discovery and expansion of Central Limit Theory-like results [20] lead to a renewed focus on these ensembles. In addition to their historical role in the development of random matrix theory, the classic β-ensembles remain essential to the current study of random matrices due to their membership in the class of *integrable* probability models—a somewhat nebulously-defined collection of objects which are enriched by some essential, overarching algebraic structure. The $\beta = 2$ ensemble is an example of a *determinantal point process*, while the $\beta = 1, 4$ ensembles are examples of *Pfaffian point processes.*

Background

In the sequel, suppose μ is a finite measure on $\mathbb{R}$ (historically, $d\mu(x) = \exp(-x^2/2)dx$). For each $\beta \in \mathbb{R}_+$, consider the N-point process specified by the joint probability density

$$\rho_N(x_1, \ldots, x_N) = \frac{1}{Z_N(\beta) N!} \prod_{i<j} |x_j - x_i|^\beta$$

where $Z_N(\beta)$ denotes the *partition function* of β, and is the normalizing constant required for ρ_N to be a probability density function. For each $1 \leq n \leq N$, define the nth correlation function by

$$R_n(x_1, \ldots, x_n) = \frac{1}{Z_N(\beta)(N-n)!} \int_{\mathbb{R}^{N-n}} \rho_N(x_1, \ldots, x_n, y_1, \ldots, y_{N-n})\, d\mu^{N-n}(\mathbf{y})$$

where $d\mu^{N-n}(\mathbf{y})$ denotes the $(N - n)$-fold product measure on $\mathbb{R}^{N-n}$. For β-ensembles, the correlation functions are nothing more than rescaled marginal density functions, and are completely determined by the joint density function. However, the study of the local statistics of the ensemble is made simpler by using the correlation functions in place of the marginal densities.

When $\beta = 1$, 2, or 4, these correlation functions can be rewritten in a particularly nice form. For $\beta = 2$, elementary matrix operations and Fubini's Theorem can be used to show that

$$R_n(x_1, \ldots, x_n) = \frac{1}{Z_N(2)} \det \left(K(x_i, x_j)_{1 \leq i,j \leq n} \right)$$

where the *kernel* $K(x, y)$ is a certain square integrable function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that can most easily be expressed in terms of the orthogonal polynomials for the measure μ. The details of this derivation are given in [20].

And for $\beta = 1$ or 4,

$$R_n(x_1, \ldots, x_n) = \frac{1}{Z_N(\beta)} \mathrm{Pf}\left(K(x_i, x_j)_{1 \leq i,j \leq n}\right)$$

where $\mathrm{Pf}(\mathbf{A})$ denotes the Pfaffian of an antisymmetric matrix $\mathbf{A}$ (with $\mathrm{Pf}(\mathbf{A}) := \sqrt{\det(\mathbf{A})}$) and where $K(x, y)$ is an antisymmetric matrix kernel. This result was first shown for Hermitian ensembles by Mehta in [19], and then for general weights by Mehta and Mahoux in [18] (except for the case $\beta = 1$ and N odd). Finally, the last remaining case was given by Adler, Forrester and Nagao in [1]

Of fundamental concern in the theory of random matrices is the behavior of eigenvalue statistics of matrix ensembles as $N \to \infty$. The immediate advantage of these determinantal and Pfaffian formulations for the correlation functions is that the fundamental characteristics of the eigenvalues are encoded in the kernel function, which does not increase in complexity as N grows large, considerably simplifying the asymptotic analysis of the eigenvalue statistics in β-ensembles.

The Method of Tracy and Widom

Derivations of the determinantal/Pfaffian forms of the correlation functions have been presented in numerous guises over the past several decades. However, of particular note is the method of Tracy and Widom [24], which evokes the underlying algebraic structure of the ensemble, and proceeds by first establishing that the partition function $Z_N(\beta)$ takes a determinant/Pfaffian form, and then uses the Sylvester Determinant Identity

$$\det(\mathbf{I}_m + \mathbf{BA}) = \det(\mathbf{I}_n + \mathbf{AB}) \qquad \mathbf{A} \in M_{n\times m}(\mathbb{R}),\ \mathbf{B} \in M_{m\times n}(\mathbb{R})$$

(where I_m and I_n are the identity matrices of rank m and n, respectively) to show that the correlation functions must have the same form as well. For $\beta = 2$, the result follows almost immediately from the application of the Sylvester Identity, while the proof in the case when β is 1 or 4 required considerably more ingenuity.

This complication arises chiefly from the fact that, for most of the history of random matrix theory, results involving Pfaffians were often stated in terms of a *quaterion* determinant of a matrix, and calculations involving the Pfaffian were performed by first transforming the expressions into corresponding ones involving the determinant. However, the Pfaffian can also be viewed in an independent light as a particular evaluation in the exterior algebra of a vector space, and doing so allows for further generalization of the Tracy-Widom method.

The Exterior Algebra

For a vector space of even dimension $K = 2N$, the kth exterior power $\bigwedge^k V$ of V consists of all antisymmetric k-tensors of elements in V, and the exterior algebra $\bigwedge V$ is formed from the direct sum of all the exterior powers of V, with multiplication given by the antisymmetric tensor operation. Of key importance is the exterior square $\bigwedge^2 V$, whose elements bijectively correspond to $K \times K$ antisymmetric matrices. Under this correspondence, the Pfaffian of an antisymmetric matrix $\mathbf{A}$ can be obtained by taking the Nth power of the associated antisymmetric tensor. That is, if $\{\vec{e}_1, \ldots, \vec{e}_K\}$ is a basis for V and $\omega \in \bigwedge^2 V$ is the antisymmetric tensor associated with the matrix A, then

$$\frac{\omega^{\wedge N}}{N!} = \{\mathrm{Pf}(\mathbf{A})\}\ \vec{e}_1 \wedge \cdots \wedge \vec{e}_K$$

But from this perspective, Pfaffian identities naturally arise as structural properties of the exterior algebra. In particular, the Sylvester Determinant Identity has a Pfaffian analogue

$$\frac{\mathrm{Pf}(\mathbf{Z}^{-1}+\mathbf{B}^T\mathbf{A}\mathbf{B})}{\mathrm{Pf}\mathbf{Z}^{-1}} = \frac{\mathrm{Pf}(\mathbf{A}^{-1}+\mathbf{B}\mathbf{Z}\mathbf{B}^T)}{\mathrm{Pf}\mathbf{A}^{-1}}$$

where $\mathbf{A} \in M_n(\mathbb{R})$ and $\mathbf{Z} \in M_m(\mathbb{R})$ are invertible antisymmetric matrices, and $\mathbf{B} \in M_{n\times m}(\mathbb{R})$ is arbitrary, which then can be used to give a nearly identical proof of the Pfaffian form for the $\beta = 1, 4$ correlation functions as was used to prove the determinantal form for the $\beta = 2$ correlation functions.

Partition Functions as Hyperpfaffians

For the classical β-ensembles, the first step for rewriting the correlation functions as determinants/Pfaffians is to observe that the partition function $Z_N(\beta)$ is itself a determinant/Pfaffian of a matrix of integrals of appropriately chosen orthogonal polynomials. One way to do so is to apply the Andreief determinant identity [2] to the partition function $Z_N(\beta)$. The result follows immediately when $\beta = 2$, and by viewing the Pfaffian as the square root of a determinant, the result can also be shown when $\beta = 1, 4$ with some additional finesse. But when the Pfaffian is viewed from the context of the exterior algebra, the Andreief determinant identity can be extended to analogous Pfaffian identities (referred to in the literature as the de Bruijn identities [8]). In fact, adopting this perspective illuminates an underlying algebraic structure, allowing the identity to be further generalized in the case when β is an arbitrary square integer.

Recalling the exterior algebra definition of the Pfaffian of an antisymmetric 2-tensor, we can define the *Hyperpfaffian*, $\mathrm{PF}(\omega)$, of an antisymmetric L-tensor

$\omega \in \bigwedge^L V$ by

$$\frac{\omega^{\wedge N}}{N!} = \{\mathrm{PF}(\omega)\}\, \vec{e}_1 \wedge \cdots \wedge \vec{e}_K,$$

provided that V has dimension $K = NL$.

In 2002, Jean-Gabriel Luque and Jean-Ives Thibon [17] used techniques in the shuffle algebra to show that when $\beta = L^2$ is an even square integer, the partition function $Z_N(\beta)$ can be written as a Hyperpfaffian of an L form whose coefficients are integrals of Wronskians of suitable orthogonal polynomials. Then in 2011, Chris Sinclair [23] used combinatorial methods to show that the result also holds when β is an odd square integer.

In the sequel, we show that the shuffle algebra techniques first implemented by Thibon and Luque can, with some modification, be adopted to give a universal proof that $Z_N(\beta)$ can be written as a Hyperpfaffian when β is a square integer, regardless of whether β is even or odd.

Outline

The Shuffle Algebra and de Bruijn's Identities

Given a set X and a commutative ring with unity R, the shuffle algebra can be obtained by endowing the free R-algebra on X, $R\langle X\rangle$, with an additional product $\sqcup\!\sqcup$, which is first defined on basis elements and then extended linearly. For words $v = u_1 \ldots u_k$ and $w = u_{k+1} \ldots u_{k+n}$ in $R\langle A\rangle$, let

$$v \sqcup\!\sqcup\, w := \sum_{\sigma \in \mathrm{Sh}(k,n)} u_{\sigma^{-1}(1)} \ldots u_{\sigma^{-1}(n+k)}$$

where $\mathrm{Sh}(k,n)$ is the subset of the symmetric group on $n+k$ letters consisting of permutations satisfying

$$\sigma(1) < \cdots < \sigma(k) \quad \sigma(k+1) < \cdots < \sigma(n).$$

That is, given two words v and w of length k and n, the product $v \sqcup\!\sqcup\, w$ is the sum of all $\binom{n+k}{k}$ words formed by interlacing the letters in v and w.

While the shuffle algebra is of great interest in its own right in representation theory and combinatorics (If V is a free R-module and V^* is its algebraic dual space, then $R\langle V^*\rangle_{\sqcup\!\sqcup}$ is isomorphic as a Hopf algebra to the graded dual of the tensor algebra $T(V)$ [21]), of more immediate relevance, it also appears to be the correct setting for performing the iterated integrals that appear in calculations of the partition function, $Z_N(\beta)$.

In particular, since the joint density function $\rho_N(x)$ for the β-ensemble is completely symmetric in the N variables, it can be restricted to the N-simplex $\Delta^N = \{x_1 < \cdots < x_N\}$ at the cost of a combinatorial factor $N!$. That is,

$$Z_N(\beta) = \frac{1}{N!}\int_{\mathbb{R}^N} \prod_{i<j} |x_j - x_i|^\beta \, d\mu^N(\mathbf{x}) = \int_{\Delta^N} \prod_{i<j} (x_j - x_i)^\beta d\mu^N(\mathbf{x}).$$

This has the added benefit of eliminating a pesky absolute value sign in the case when β is an odd integer.

Now, take H to be a vector space of suitably integrable functions with basis set X, (say, $H = L^2(\mathbb{R}, \mu)$, where $d\mu(x) = e^{-x^2/2}\, dx$) and define a linear functional $\langle - \rangle$ on $\mathbb{R}\langle X\rangle_{\sqcup\!\sqcup}$ by

$$\langle f_1 \cdots f_n \rangle = \int_{\Delta^N} f_1(x_1) \cdots f_n(x_n)\, d\mu(x) \qquad \text{where } f_1 \cdots f_n \text{ is a word in } \mathbb{R}\langle X\rangle_{\sqcup\!\sqcup}\,.$$

Then $\langle - \rangle$ is, in fact, an algebra homomorphism, in the sense that

$$\Big\langle (f_1 \cdots f_k) \sqcup (f_{k+1} \cdots f_{k+n}) \Big\rangle = \Big\langle f_1 \cdots f_k \Big\rangle \cdot \Big\langle f_{k+1} \cdots f_{k+n} \Big\rangle.$$

This result is equivalent to the celebrated lemma first proved by K.T. Chen in the context of cohomology of loop space [6].

With this lemma in hand, the main result (an extension of the de Bruijn identities to the case when $\beta = L^2$) can be obtained. Proof of part (1) is first due to J. Thibon and J. Luque in [17], while the first proofs of parts (2) and (3) appear in [23] by C. Sinclair. In the sequel, we present a new proof of the result covering all three cases.

Theorem (Wells)**.** *Suppose N and L are positive integers, that $\mathbf{p} = \{p_1, p_2, \ldots, p_N; x\}$ is a complete family of monic orthogonal polynomials in the variable x, that $\mathfrak{t} = \{\mathfrak{t}_i\}$ and $\mathfrak{s} = \{\mathfrak{s}_i\}$ are subsets of $\{1, \ldots, NL\}$ of size L, that $\vec{e}_{\mathfrak{t}} = \vec{e}_{\mathfrak{t}_1} \wedge \cdots \wedge \vec{e}_{\mathfrak{t}_L}$, and that $\mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}; x)$ is the Wronskian of a subset of polynomials $\mathbf{p}_{\mathfrak{t}} = \{p_{\mathfrak{t}_1}, \ldots, p_{\mathfrak{t}_L}\}$ in the variable x. Then $Z_N(\beta)$ is given by the following expressions:*

1. *if $\beta = L^2$ is even,*

$$\mathrm{PF}\left(\sum_{\mathfrak{t}} \Big\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}; x) \Big\rangle \vec{e}_{\mathfrak{t}} \right)$$

2. *if $\beta = L^2$ is odd and N is even,*

$$\mathrm{PF}\left(\left[\sum_{\mathfrak{t}} \Big\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}; x) \Big\rangle \vec{e}_{\mathfrak{t}} \right] \wedge \left[\sum_{\mathfrak{s}} \Big\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{s}}; y) \Big\rangle \vec{e}_{\mathfrak{s}} \right] \right)$$

3. if $\beta = L^2$ is odd and N is odd,

$$\mathrm{PF}\left(\left[\sum_{\mathfrak{t}} \left\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}; x) \right\rangle \vec{e}_{\mathfrak{t}}\right] \wedge \left[\sum_{\mathfrak{s}} \left\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{s}}; y) \right\rangle \vec{e}_{\mathfrak{s}}\right] + \left[\sum_{\mathfrak{t}} \left\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}; x) \right\rangle \vec{e}_{\mathfrak{t}} \wedge \vec{e}_{\mathfrak{i}'}\right] \right)$$

where $\vec{e}_{\mathfrak{i}'} = \vec{e}_{NL+1} \wedge \cdots \wedge \vec{e}_{(N+1)L}$.

Identities in the Exterior Algebra

Several algebraic identities of the determinant (the Laplace Expansion, the Cauchy-Binet Formula, and the Jacobi Minor Inverse Formula, for example) have been well-studied since the days of Jacobi, Cayley, and Sylvester. Given that the Pfaffian is the square root of the determinant, it is not terribly surprising that many of these identities have a Pfaffian analogue (a few of which appear in other guises elsewhere in the literature as the famed Wick Formulas [4],[3]). However, by representing an antisymmetric matrix as an antisymmetric 2-tensor, many of these Pfaffian identities can be realized simply as structural properties of the exterior algebra. And moreover, by adopting this perspective, it becomes clear that there is little unique about the Pfaffian of an antisymmetric 2-tensor—many of the same identities also hold for the Hyperpfaffian of an antisymmetric L-tensor.

Concise statements of the Laplace Expansion, the Cauchy-Binet formula, the Jacobi Minor Inverse formula, and the Sylvester identity are facilitated by introducing two auxillary transformations on the exterior algebra as follows:

For $\omega \in \bigwedge^L V$, define $\exp(\omega) \in \bigwedge V$ by

$$\exp(\omega) = \sum_{k=1} \frac{\omega^{\wedge k}}{k!}$$

and note that $\exp(\omega)$ is actually a finite sum, since $\omega^{\wedge k} = 0$ for all k with $kL > \dim V$.

Suppose $\langle \cdot \,|\, \cdot \rangle$ is an inner product on V, which is extended to an inner product on $\bigwedge^k V$ by

$$\langle f_1 f_2 \cdots f_k | g_1 g_2 \cdots g_k \rangle = \det \left(\langle f_i | g_j \rangle \right)_{i,j=1}^{k} \quad \text{for } f_1, \ldots, f_k, g_1, \ldots, g_\ell \in \textstyle\bigwedge^1(\eta).$$

The *Hodge dual* operator $*$ on $\bigoplus_k \bigwedge^{2k} V$ is defined for $\alpha \in \bigwedge^{2k} V$ by $*(\alpha) = \beta$, where β is the unique antisymmetric $(n-2k)$-tensor so that

$$\alpha \wedge \beta = \langle \alpha | \alpha \rangle \vec{e}_1 \wedge \cdots \wedge \vec{e}_n.$$

It turns out that in some cases, an antisymmetric $2k$-tensor α has an inverse antisymmetric $2k$-tensor α' with the property that

$$\exp(\alpha) = \frac{1}{\mathrm{PF}(\alpha)} * [\exp(\alpha')],$$

in which case the coefficients of α contain essentially the same information as those of α'. Remarkably, if α is an antisymmetric 2-tensor with $\mathrm{Pf}(\alpha) \neq 0$ (corresponding to an invertible antisymmetric matrix A), then the inverse form α' always exists and corresponds to the matrix inverse of A.

Now, the Cauchy-Binet formula for Hyperpfaffians is equivalent to the observation that the k-homogeneous component of $\exp(\omega)$ is given by

$$\frac{\omega^{\wedge k}}{k!} = \sum_I \mathrm{PF}(\omega_I)$$

where the sum is taken over all k-element subsets $I \subset \{1, 2, \ldots, N\}$, and where ω_I is called the I-minor of ω and denotes the L-formed obtained by setting the basis vectors $\vec{e}_i$ equal to 0, when $i \notin I$.

Meanwhile, the Laplace Expansion for Hyperpfaffians arises by combining the Cauchy-Binet Formula with observation that if $\omega \in \bigwedge^L V$ and $\alpha \in \bigwedge^K V$ with $\omega = \alpha^{\wedge k}$, then $\mathrm{PF}(\omega) = \mathrm{PF}(\alpha)$.

By combining the equation $\exp(\alpha) = \frac{1}{\mathrm{PF}(\alpha)} * [\exp(\alpha')]$ with the Cauchy-Binet formula, the Hyperpfaffians of minors of α can be expressed in terms of the Hyperpfaffians of minors of α'. In particular, when α is an antisymmetric 2-tensor, this result is equivalent to the Jacobi Minor Inverse Formula for Pfaffians.

In the language of the exterior algebra, the Pfaffian Sylvester Identity can be restated as

$$\frac{\mathrm{PF}(\zeta' + \mathbf{B} \cdot \alpha)}{\mathrm{PF}(\zeta')} = \frac{\mathrm{PF}(\alpha' + \mathbf{B}^T \cdot \zeta)}{\mathrm{PF}(\alpha')} \qquad \text{for } \alpha, \zeta \in \textstyle\bigwedge^2 V \text{ and } \mathbf{B} \in \mathrm{M}_n(\mathbb{R})$$

where $\mathbf{B} \cdot \alpha$ denotes the antisymmetric 2-tensor given by

$$B \cdot \alpha = \sum_{i<j} \alpha_{ij} \, (\mathbf{B} v_i) \wedge (\mathbf{B} v_j) \qquad \text{when } \alpha = \sum_{i<j} \alpha_{ij} \, \vec{e}_i \wedge \vec{e}_j.$$

But more generally, by using the Hyperpfaffian in place of the Pfaffian, the same identity also holds for antisymmetric L-tensors α and ζ, provided that the inverse forms for α and ζ exist.

CHAPTER II

PRELIMINARY DEFINITIONS AND LEMMAS

Conventions

Unless otherwise stated, we assume that every ring R is a commutative ring with unity, and moreover, that every non-negative integer is a unit in R. Of course, while several of the properties discussed in the sequel can indeed be extended to arbitrary commutative rings with unity, this restriction is sufficient to cover all cases relevant to the main results in the following chapters, with the benefit of obviating the need for distinguishing between results which hold for general rings, and those which only hold for the restricted class of rings.

To simplify matters when writing multi-indices and permutations, we make use of the following conventions:

- For any positive integer n, let $\underline{n}$ denote the set of the first n integers $\{1, \dots, n\}$.
- For any multi-index $(i_1, \dots, i_k)$ where each i_j is an integer in $\underline{n}$, we define a function $\mathfrak{t} : \underline{k} \to \underline{n}$ by $\mathfrak{t}(1) = i_1, \dots, \mathfrak{t}(k) = i_k$ and write $Q_{\mathfrak{t}}$ for $Q_{i_1,\dots,i_k}$.
- Frequently, we will need to permute the order of indices in a multi-indexed symbol, and so viewing a permutation σ of the first k integers as a bijection $\sigma : \underline{k} \to \underline{k}$, we write the symbol $Q_{i_{\sigma(1)},\dots,i_{\sigma(k)}}$ as $Q_{\mathfrak{t}\circ\sigma}$.
- It will often be useful to write $\mathfrak{t} : \underline{k} \nearrow \underline{n}$ for a strictly increasing function from $\underline{k}$ to $\underline{n}$. In particular, note that any such $\mathfrak{t}$ simply represents a choice of k integers from among the first n integers. Moreover, in light of the previous

convention, observe that every injective function $\mathfrak{s} : \underline{k} \nearrow \underline{n}$ can be written uniquely as the composition $\mathfrak{s} = \mathfrak{t} \circ \tau$ of a strictly increasing function $\mathfrak{t} : \underline{k} \nearrow \underline{n}$ and a permutation $\tau \in S_k$.

– Given an increasing function $\mathfrak{t} : \underline{k} \nearrow \underline{n}$, let $\mathfrak{t}'$ denote the complimentary increasing function $\mathfrak{t}' : \underline{n-k} \nearrow \underline{n}$ with $\mathfrak{t}(\underline{k}) \sqcup \mathfrak{t}'(\underline{n-k}) = \underline{n}$. Let $\mathrm{sgn}(\mathfrak{t})$ denote the signature of the permutation $\sigma \in S_n$ given by

$$\sigma(i) = \begin{cases} \mathfrak{t}(i), & \text{if } i \in \underline{k} \\ \mathfrak{t}'(i-k), & \text{if } i \in \underline{n} \setminus \underline{k} \end{cases}$$

– In the sequel, we will need to consider the restriction of permutations and increasing functions to subsets of their domain. If $\mathfrak{t} : \underline{k} \to \underline{n}$ is a function and $k = NL$ for positive integers N and L, we write $\mathfrak{t} = (\mathfrak{t}_1 | \dots | \mathfrak{t}_N)$, where each $\mathfrak{t}_n$ is the restriction of $\mathfrak{t}$ to the set $\underline{nL} \setminus \underline{nL - L}$.

– Finally, although a slight abuse of notation, given functions $\mathfrak{t}_1, \dots, \mathfrak{t}_N : \underline{L} \to \underline{n}$, we let $\mathfrak{t} = (\mathfrak{t}_1 | \dots | \mathfrak{t}_N)$ denote the function $\mathfrak{t} : \underline{NL} \to \underline{n}$ given by

$$\mathfrak{t}(m) = \begin{cases} \mathfrak{t}_1(m), & \text{if } 1 \leq m \leq L, \\ \mathfrak{t}_2(m-L), & \text{if } L+1 \leq m \leq 2L, \\ \dots, & \\ \mathfrak{t}_N(m-(N-1)L), & \text{if } (N-1)L+1 \leq m \leq NL \end{cases}$$

In this way, we will often equate the restriction $\mathfrak{t}_i$ of $\mathfrak{t} = (\mathfrak{t}_1 | \dots | \mathfrak{t}_N)$ with the function $\mathfrak{t}_i : \underline{k} \to \underline{n}$, although strictly speaking, the two functions have different domains. In many cases, the distinction will not matter, and when it does, the domain will be made clear.

The Symmetric Group

The symmetric group S_N of degree N consists of all set automorphisms of the set $X = \{1, \ldots, N\}$, with multiplication given by function composition. For a standard reference on the symmetric group, see [22].

Given any subset $Y = \{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$, let $S_{i_1,\ldots,i_k}$ denote the set of permutations of Y, given by $\sigma(j) = j$ unless $j \in Y$. We can identify $S_{i_1,\ldots,i_k}$ with S_k in the obvious way.

A partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of a non-negative integer n is a sequence of non-negative integers so that $\sum \lambda_i = n$. If $n = NL$ for non-negative integers N, L, let Λ^N denote the partition of n into N equal parts, each of size L. That is, $\Lambda^N = (L, L, \ldots, L)$.

For each partition λ of n, we define the following subgroups and subsets of the symmetric group:

1. The Young subgroup H_λ of S_n is the internal direct product

$$H_\lambda = S_{1,\ldots,\lambda_1} \times S_{\lambda_1+1,\ldots,\lambda_1+\lambda_2} \times \cdots \times S_{n-\lambda_k+1,\ldots,n},$$

where $S_{m+1,\ldots,m+\lambda_i} = \{\sigma \in S_n \,|\, \sigma(j) = j, \text{ unless } m+1 \leq j \leq m+\lambda_i\}$. Demonstrably,

$$|H_\lambda| = \lambda_1!\lambda_2! \cdots \lambda_k!.$$

2. The subset of block permutations $\mathrm{Bl}(\lambda) \subset S_n$ consists of those permutations satisfying

$$\sigma(j) + 1 = \sigma(j+1) \qquad \text{for all } j \text{ except possibly for } j = \lambda_1 + \cdots + \lambda_i.$$

Block permutations preserve the contiguity of elements in the 'blocks'

$$\{1, \ldots, \lambda_1\}, \quad \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \quad \ldots \quad \{n - \lambda_k + 1, \ldots, n\}$$

while permuting the blocks themselves. As block permutations are completely determined by the values of $\sigma(\lambda_1 + \cdots + \lambda_i + 1)$ for $1 \leq i \leq k$, then

$$|\mathrm{Bl}(\lambda)| = k!.$$

For any block permutation $\theta \in \mathrm{Bl}(\lambda)$, let β_θ denote the corresponding permutation in S_k.

3. The subset of shuffle permutations $\mathrm{Sh}(\lambda) \subset S_n$ consists of those permutations satisfying

$$\sigma(j_1) < \sigma(j_2) \qquad \text{whenever } \lambda_1 + \cdots + \lambda_i \leq j_1 < j_2 \leq \lambda_1 + \cdots + \lambda_{i+1}$$

Shuffle permutations in $\mathrm{Sh}(\lambda)$ are so called because they represent iterative riffle shuffles on a set of n cards sorted into k stacks of $\lambda_1, \ldots, \lambda_k$ ordered cards each. A straightforward induction argument on k can be used to show that

$$|\mathrm{Sh}(\lambda)| = \binom{n}{\lambda_1, \ldots, \lambda_k} = \frac{n!}{\lambda_1! \ldots \lambda_k!}.$$

4. The subset of λ-ordered permutations $O(\lambda) \subset S_n$ consists of those permutations which satisfy

$$\sigma(1) < \sigma(\lambda_1 + 1) < \sigma(\lambda_1 + \lambda_2 + 1) < \cdots < \sigma(n - \lambda_k + 1).$$

   Since each permutation in $O(\lambda)$ can produce $k!$ distinct permutations in S_n by composing with elements of $\mathrm{Bl}(\lambda)$, then

$$|O(\lambda)| = \frac{n!}{k!}$$

5. Let $\mathrm{Sh}(\lambda)/\mathrm{Bl}(\lambda)$ denote the collection of equivalence classes of elements of $\mathrm{Sh}(\lambda)$ under multiplication by $\mathrm{Bl}(\lambda)$. Observe that if $\sigma \in \mathrm{Sh}(\lambda)$ and $\theta \in \mathrm{Bl}(\lambda)$, then $\sigma \circ \theta \in \mathrm{Sh}(\lambda^\theta)$, where λ^θ is the partition $(\lambda_{\beta_\theta(1)}, \ldots \lambda_{\beta_\theta(k)})$. Let $\mathrm{Sh}^o(\lambda)$ denote the subset of permutations of the form $\sigma \circ \theta \in O(\lambda^\theta) \cap \mathrm{Sh}(\lambda^\theta)$, where $\sigma \in \mathrm{Sh}(\lambda)$ and $\theta \in \mathrm{Bl}(\lambda)$.

   Of course, in the case when $\lambda = \Lambda^N$ (that is, $\lambda_i = \lambda_j$ for all i, j), much of the preceding obfuscatory notation can be avoided, since for any $\theta \in \mathrm{Bl}(\lambda)$, we have $\lambda^\theta = \lambda$, and so each equivalence class in $\mathrm{Sh}(\lambda)/\mathrm{Bl}(\lambda)$ is actually a subset of $\mathrm{Sh}(\lambda)$, in which case $\mathrm{Sh}^o(\lambda) = O(\lambda) \cap \mathrm{Sh}(\lambda)$.

Properties of the Symmetric Group

Lemma 1. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n, and let H_λ be the Young subgroup of this partition. Then for each coset $K \in S_n/H_\lambda$, there is exactly one permutation $\sigma \in K$ with $\sigma \in \mathrm{Sh}(\lambda)$.*

Proof. Suppose $\sigma \in S_n$. For each $1 \leq i \leq k$ and for each $1 \leq j \leq \lambda_i$, let $t^i_j = \sigma(\lambda_1 + \cdots + \lambda_{i-1} + j)$. Define an ordering permutation $\pi \in S_{\lambda_i}$ by

$$t^i_{\pi^{-1}(1)} < \cdots < t^i_{\pi^{-1}(\lambda_i)}$$

Then the permutation $\pi = (\pi_1 | \ldots | \pi_k)$ is an element of the Young subgroup H_λ, and

$$\sigma \circ \pi(j_1) < \sigma \circ \pi(j_2) \qquad \text{whenever } \lambda_1 + \cdots + \lambda_i \leq j_1 < j_2 \leq \lambda_1 + \cdots + \lambda_{i+1},$$

which shows that every coset $K \in S_n / H_\lambda$ contains at least one shuffle permutation. But $|S_n\ H_\lambda| = n!/(\lambda_1! \ldots \lambda_k!) = |\mathrm{Sh}(\lambda)|$, so these shuffle permutations must necessarily be unique. $\square$

Corollary 1. *Given any $\sigma \in S_{NL}$, there exist unique permutations π, τ so that $\sigma = \tau \circ \pi$, where π is an element of the Young subgroup $H_{\Lambda^N} \subset S_{NL}$, and where $\tau \in \mathrm{Sh}(\Lambda^N)$.*

Lemma 2. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n, let $\mathrm{Sh}(\lambda)$ be the set of shuffle permutations for this partition, and let $\mathrm{Bl}(\lambda)$ be the set of block permutations for this partition. Then for each equivalence class $K \in \mathrm{Sh}(\lambda)/\mathrm{Bl}(\lambda)$, there exists exactly one permutations $\sigma \in K$ with $\sigma \in \mathrm{Sh}^o(\lambda)$.*

Proof. Suppose $\sigma \in \mathrm{Sh}(\lambda)$, and for each $1 \leq i \leq k$, let $s_i = \sigma(\lambda_1 + \cdots + \lambda_{i-1} + 1)$. Define an ordering permutation $\beta \in S_k$ by

$$s_{\beta^{-1}(1)} < \cdots < s_{\beta^{-1}(k)}$$

and define a new partition $\mu = \{\mu_1, \ldots, \mu_k\} = \{\lambda_{\beta^{-1}(1)}, \lambda_{\beta^{-1}(2)}, \ldots, \lambda_{\beta^{-1}(k)}\}$. Let $\theta \in \mathrm{Bl}(\mu)$ be defined by

$$\theta(\mu_1 + \cdots + \mu_{i-1} + j) = \lambda_1 + \cdots + \lambda_{\beta^{-1}(i-1)} + j \qquad \text{for } 1 \leq j \leq \mu_i$$

Then by construction, $\sigma \circ \theta \in O(\mu) \cap \mathrm{Sh}(\mu) = \mathrm{Sh}^o(\lambda)$, as desired. Moreover, since

$$|O(\mu) \cap \mathrm{Sh}(\mu)| = \frac{n!}{\lambda_{\beta^{-1}(1)}! \cdots \lambda_{\beta^{-1}(k)}!} = \frac{n!}{\lambda_1! \cdots \lambda_k!} = |\mathrm{Sh}(\lambda)/\mathrm{Bl}(\lambda)|$$

then each such ordered shuffle permutation is indeed unique.

$\square$

The Tensor Algebra

The following discussion of the tensor algebra mirrors the treatment in [11]. Let R be a commutative ring with unity, and let V be an R-module. The tensor product $T^2(V) = V \otimes V$ is formed by taking the quotient of the free abelian group on $V \times V$ by the ideal $\mathcal{I}$ generated by elements of the form

$$(v_1+v_2, w_1)-(v_1, w_1)-(v_2, w_1), \;\; (v_1, w_1+w_2)-(v_1, w_1)-(v_1, w_2), \;\; (rv_1, w_1)-(v_1, rw_1)$$

for $r \in R, v_i, w_i \in V$.

For each integer $k \geq 1$, define the kth tensor power of V by

$$T^k(V) = V \otimes V \otimes \cdots \otimes V \quad (k \text{ factors})$$

and let $T^0(V) = R$. We will call elements of $T^k(V)$ k-tensors.

Theorem 1. *If V is a rank d free R-module with basis $X = \{\vec{e}_1, \ldots \vec{e}_d\}$, then $T^k(V)$ has a basis*

$$\{\vec{e}_{i_1} \otimes \cdots \otimes \vec{e}_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq d\}.$$

Define the Tensor Algebra $T(V)$ by

$$T(V) = \bigoplus_{k=0}^{\infty} T^k(V).$$

and observe that $T(V)$ is indeed an R-algebra with multiplication

$$(v_1 \otimes \cdots \otimes v_k) \otimes (w_1 \otimes \cdots \otimes w_\ell) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_\ell$$

For a set X and a ring R, let $R\langle X\rangle$ denote the free unital algebra on X over R, with multiplicative unit e. We may (and often will) identify the tensor algebra $T(V)$ of a free R-module V with the free R-algebra $R\langle X\rangle$, where X is an R-basis for V. The identification is given by $v \otimes w = vw$, for $v, w \in X$.

The Shuffle Algebra

The shuffle product was first introduced by Eilenberg and Mac Lane in [13]. Define an operation $\sqcup\!\sqcup$ on $R\langle X\rangle$ inductively as follows:

1. $e \sqcup\!\sqcup e = e$

2. For all $a \in X$,

$$a \sqcup\!\sqcup e = e \sqcup\!\sqcup a = a$$

3. For all $w, u \in S\langle X\rangle$ and $a, b \in X$,

$$ua \sqcup\!\sqcup\, wb = (u \sqcup\!\sqcup\, wb)a + (ua \sqcup\!\sqcup\, w)b$$

The algebra $R\langle X\rangle$ with $\sqcup\!\sqcup$ is called the Shuffle Algebra on X, and denoted $R\langle X\rangle_{\sqcup\!\sqcup}$. By construction, this algebra is commutative and associative.

An explicit presentation of the operation $\sqcup\!\sqcup$ is also possible. For $1 \leq k \leq n$, let $\mathrm{Sh}(k, n-k)$ denote the subset of the permutation group S_n of permutations σ satisfying

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(n).$$

By convention, let $\mathrm{Sh}(0, n) = \{\mathrm{id}\}$. Thus, if $u = u_1 \ldots u_k$ and $u' = u_{k+1} \ldots u_n$ are words in A of length k and $n-k$, respectively, then

$$u \sqcup\!\sqcup\, u' = \sum_{\sigma \in \mathrm{Sh}(k,n-k)} u_{\sigma^{-1}(1)} \ldots u_{\sigma^{-1}(n)}.$$

That is, given two words w and u of length k and $n-k$, the product $w \sqcup\!\sqcup\, u$ is the sum of all $\binom{n}{k}$ words formed by interlacing the letters in u and w.

Demonstrating that the two definitions of $\sqcup\!\sqcup$ coincide is a straightforward induction exercise.

Example. Let $w = abc$ and $u = xy$. Then

$$w \sqcup\!\sqcup\, u = abcxy + abxcy + axbcy + xabcy + abxyc + axbyc + xabyc + axybc + xaybc + xyabc$$

The Exterior Algebra

Suppose V is a real vector space of dimension d, with basis $X = \{\vec{e}_1, \ldots, \vec{e}_d\}$, and let $T(V)$ be the tensor algebra of V over $\mathbb{R}$. The *exterior algebra* of V is obtained by taking quotients of $T(V)$ by the ideal $\mathcal{I}$ generated by elements of the form $v \otimes v$, for $v \in V$. The exterior algebra $T(V)/\mathcal{I}$ is denoted by $\bigwedge_{\mathbb{R}} V$ and the image of $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ in $\bigwedge_{\mathbb{R}} V$ is denoted by $v_1 \wedge v_2 \wedge \cdots \wedge v_k$.

Note that $\mathcal{I}$ is generated by homogeneous elements, and so is a graded ideal. Hence, $\bigwedge_{\mathbb{R}} V$ is a graded algebra, and the kth homogeneous component $\bigwedge^k_{\mathbb{R}} V = T^k(V)/\mathcal{I}_k$ is called the kth exterior power of V. Elements of $\bigwedge^k_{\mathbb{R}} V$ are called antisymmetric k-tensors, or k-forms. The multiplication

$$(v_1 \wedge \cdots \wedge v_k) \wedge (w_1 \wedge \cdots \wedge w_\ell) = v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_\ell$$

in the exterior algebra is called the *wedge* or exterior product. Multiplication is anticommutiative, since for all $v, w \in V$,

$$0 = (v + w) \wedge (w + v) = v \wedge v + w \wedge w + v \wedge w + w \wedge v = v \wedge w + w \wedge v$$

Theorem 2. *For any non-negative integer k, the kth exterior power $\bigwedge^k_{\mathbb{R}} V$ has a basis*

$$\{\vec{e}_{i_1} \wedge \cdots \wedge \vec{e}_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq d, \}.$$

In particular, $\bigwedge^k_{\mathbb{R}} V$ has dimension $\binom{d}{k}$.

As with multi-indexed symbols above, we write $e_{\mathfrak{t}}$ for $e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $\mathfrak{t} : \underline{k} \nearrow \underline{n}$ is an increasing function with $\mathfrak{t}(j) = i_j$. We will write $\vec{e}_{\mathrm{vol}}$ for the 'volume form' element $\vec{e}_1 \wedge \cdots \wedge \vec{e}_d$.

More generally, for any commutative ring with unity R and finite-rank free R-module V, we may define the exterior algebra $\bigwedge_R V$ just as above by taking a suitable quotient of the tensor algebra $T(V)$. All of the aforementioned properties of the exterior algebra still hold, where subspace and dimension are replaced with submodule and rank, as appropriate.

It will often be convenient to extend the linear functional $\langle\,\cdot\,\rangle : T(H) \to \mathbb{R}$ defined previously to an $\mathbb{R}$-linear map $\langle\,\cdot\,\rangle : \bigwedge_R V \to \bigwedge V$, where $R = T(H)_{\sqcup\!\sqcup}$ and V is a free R-module of rank N. In this case, we first define $\langle\,\cdot\,\rangle$ on a basis vector $\vec{e}_{i_1} \wedge \cdots \wedge \vec{e}_{i_k}$ for $\bigwedge_R V$ by

$$\langle f\, \vec{e}_{i_1} \wedge \cdots \wedge \vec{e}_{i_k} \rangle = \langle f \rangle\, \vec{e}_{i_1} \wedge \cdots \wedge \vec{e}_{i_k} \qquad \text{where } f \in T(H)$$

and then extend $\mathbb{R}$-linearly to all of $\bigwedge_R V$.

Properties of the Exterior Algebra

The following properties of the exterior algebra are well-known, but we include proofs due to their importance in the sequel.

Theorem 3. *If V is a free R-module of rank N, and $\alpha \in \bigwedge^k V$ and $\beta \in \bigwedge^\ell V$ with*

$$\alpha = \sum_{\mathfrak{t}:\underline{k}\nearrow\underline{N}} a_{\mathfrak{t}}\, \vec{e}_{\mathfrak{t}} \qquad \beta = \sum_{\mathfrak{t}:\underline{\ell}\nearrow\underline{N}} b_{\mathfrak{t}}\, \vec{e}_{\mathfrak{t}}$$

then

$$\alpha \wedge \beta = \sum_{\mathfrak{t}:\underline{k+\ell}\nearrow\underline{N}} c_{\mathfrak{t}}\, \vec{e}_{\mathfrak{t}}$$

where

$$c_{\mathfrak{t}} = \sum_{\sigma \in \mathrm{Sh}(k,\ell)} \mathrm{sgn}(\sigma) a_{\mathfrak{t}\circ\sigma_1} b_{\mathfrak{t}\circ\sigma_2}\, \vec{e}_{\mathfrak{t}}.$$

Proof. This follows from Lemma 1. $\square$

Theorem 4 (Normal form for antisymmetric 2-forms)**.** *Suppose V is a free R-module of rank N and $\alpha \in \bigwedge^2 V$. Then there exists a basis $\{\vec{v}_1, \dots, \vec{v}_N\}$ for V in which*

$$\alpha = \sum_{i=1}^{k} \vec{v}_{2i-1} \wedge \vec{v}_{2i}$$

for some k with $2k \leq N$.

Proof. We proceed by inducting on N. When $N = 1$, the result is vacuously true, since $\bigwedge^2 V$ is rank 0. When $N = 2$, the exterior square $\bigwedge^2 V$ is rank 1, and so every non-degenerate element $\alpha \in \bigwedge^2 V$ is of the form $\alpha = a\, \vec{e}_1 \wedge \vec{e}_2$. Define $\vec{v}_i = \sqrt{a}\, \vec{e}_i$, and observe that $\alpha = \vec{v}_1 \wedge \vec{v}_2$. Suppose now that the result holds for all $N = 1, \dots, n$. Let V be a rank $n+1$ free R-module, and suppose $\alpha \in \bigwedge^2 V$ is a non-degenerate element with $\alpha = \sum_{i<j} a_{ij}\, \vec{e}_i \wedge \vec{e}_j$. By relabeling indices and scaling both $\vec{e}_n$ and $\vec{e}_{n+1}$ by $\sqrt{a_{n,n+1}}$, assume $a_{n,n+1} = 1$. Let

$$\vec{v}_n = \vec{e}_n + \sum_{i=1}^{n-1} a_{i,n+1} \vec{e}_i \qquad \vec{v}_n + 1 = \vec{e}_{n+1} - \sum_{i=1}^{N-1} a_{i,n} \vec{e}_i$$

and observe that

$$\alpha = \vec{v}_n \wedge \vec{v}_{n+1} + \beta$$

for some $\beta \in \bigwedge^2 V$ with

$$\beta = \sum_{1 \leq i < j \leq N-1} b_{ij}\, \vec{e}_i \wedge \vec{e}_j.$$

But by the induction hypothesis, there exist $\{\vec{v}_1, \dots, \vec{v}_{2k}\}$ with $2k \leq N - 1$ so that

$$\beta = \sum_{i=1}^{k} \vec{v}_{2i-1} \wedge \vec{v}_{2i}$$

and hence,

$$\alpha = \vec{v}_n \wedge \vec{v}_{n+1} + \sum_{i=1}^{k} \vec{v}_{2i-1} \wedge \vec{v}_{2i},$$

as desired. $\square$

Pfaffians and Hyperpfaffians

Throughout this section, we assume that R is a commutative ring with unity such that every positive integer k is a unit, and that V is a free R-module of rank $2N$.

Pfaffians

Let $\mathbf{A}$ be a $2N \times 2N$ antisymmetric matrix. Define the Pfaffian of $\mathbf{A}$, $\mathrm{Pf}(\mathbf{A})$, by

$$\mathrm{Pf}(\mathbf{A}) = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma(2i-1),\sigma(2i)}$$

To each antisymmetric matrix A, associate a 2-form $\omega \in \bigwedge^2 V$ with V by

$$\omega = \sum_{i<j} \mathbf{A}_{ij} \vec{e}_i \wedge \vec{e}_j.$$

Similarly, to each two form $\omega \in \bigwedge^2 V$ with $\omega = \sum_{i<j} a_{ij} \vec{e}_i \wedge \vec{e}_j$, associate the antisymmetric matrix $\mathbf{A}$ by

$$\mathbf{A}_{ij} = \begin{cases} a_{ij}, & \text{if } i < j, \\ -a_{ij}, & \text{if } i > j, \\ 0, & \text{if } i = j. \end{cases}$$

This gives a bijective correspondence between 2-forms and antisymmetric matrices. We define the Pfaffian, $\mathrm{Pf}(\omega)$, of a 2-form ω to be the Pfaffian of the associated antisymmetric matrix.

Theorem 5. *The Pfaffian of a $2N \times 2N$ antisymmetric matrix $\mathbf{A}$ is given by*

$$\mathrm{Pf}(\mathbf{A}) = \frac{1}{N!} \sum_{\sigma \in \mathrm{Sh}(\Lambda^N)} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma(2i-1),\sigma(2i)}.$$

Proof. Since $\mathbf{A}$ is antisymmetric, the expression

$$\mathrm{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma(2i-1),\sigma(2i)}$$

is constant across each coset in S_{2N}/H_{Λ^N}. But $|H_{\Lambda^N}| = 2^N$, and so

$$\frac{1}{N!} \sum_{\sigma \in \mathrm{Sh}(\Lambda^N)} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma(2i-1),\sigma(2i)} = \frac{1}{2^N N!} \sum_{\sigma \in S_{2N}} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma(2i-1),\sigma(2i)}$$

as desired. $\square$

Corollary 2. *The Pfaffian of a $2N \times 2N$ antisymmetric matrix $\mathbf{A}$ with associated 2-form ω is given by*

$$\frac{1}{N!}\omega^{\wedge N} = \mathrm{Pf}(\mathbf{A})\, \vec{e}_{\mathrm{vol}}.$$

Proof. This follows by combining the previous theorem with Theorem 3. $\square$

Theorem 6. *The Pfaffian of a $2N \times 2N$ antisymmetric matrix $\mathbf{A}$ is given by*

$$\mathrm{Pf}(\mathbf{A}) = \sum_{\sigma \in \mathrm{Sh}^o(\Lambda^N)} \mathrm{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma(2i-1),\sigma(2i)}.$$

Proof. Since the entries of $\mathbf{A}$ are elements of a commutative ring R, then the expression

$$\operatorname{sgn}(\sigma)\prod_{i=1}^{N}\mathbf{A}_{\sigma(2i-1),\sigma(2i)}$$

is constant across each coset in $\mathrm{Sh}(\Lambda^N)/\mathrm{Bl}(\Lambda^N)$. But $|\mathrm{Bl}(\Lambda^N)| = N!$, and so

$$\sum_{\sigma\in\mathrm{Sh}^o(\Lambda^N)}\operatorname{sgn}(\sigma)\prod_{i=1}^{N}\mathbf{A}_{\sigma(2i-1),\sigma(2i)} = \frac{1}{N!}\sum_{\sigma\in\mathrm{Sh}(\Lambda^N)}\operatorname{sgn}(\sigma)\prod_{i=1}^{N}\mathbf{A}_{\sigma(2i-1),\sigma(2i)},$$

as desired $\square$

Corollary 3. *If* $\omega\in\bigwedge^2 V$ *with* $\omega=\sum_{i<j}\mathbf{A}_{ij}\,\vec{e}_i\wedge\vec{e}_j$, *then*

$$\mathrm{Pf}(\omega)\,\vec{e}_{\mathrm{vol}} = \bigwedge_{i=1}^{2N-1}\left(\sum_{j>i}\mathbf{A}_{ij}\,\vec{e}_i\wedge\vec{e}_j\right).$$

Proof. This follows from the previous theorem, along with Lemma 2. $\square$

Hyperpfaffians

Throughout this section, we assume that R is a commutative ring and V is a free R-module of rank NL.

Let $\mathbf{A}=\{A_{\mathfrak{t}}\,|\,\mathfrak{t}:\underline{L}\to\underline{NL}\}$ be an array of values in R with the property that for each $\sigma\in S_L$,

$$\operatorname{sgn}(\sigma)\mathbf{A}_{\mathfrak{t}\circ\sigma}=\mathbf{A}_{\mathfrak{t}}.$$

By way of analogy with antisymmetric matrices, we will call such collections *antisymmetric* $(NL)^L$*-arrays.* Define the Hyperpfaffian of $\mathbf{A}$, $\mathrm{PF}(\mathbf{A})$, by

$$\mathrm{Pf}(\mathbf{A}) = \frac{1}{(L!)^N N!}\sum_{\sigma\in S_{NL}}\operatorname{sgn}(\sigma)\prod_{i=1}^{N}\mathbf{A}_{\sigma_i}$$

Now, to each antisymmetric $(NL)^L$-array $\mathbf{A}$, associate an L-form $\omega \in \bigwedge^L V$ by

$$\omega = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} \mathbf{A}_{\mathfrak{t}} \vec{e}_{\mathfrak{t}}.$$

Similarly, to each L-form $\omega \in \bigwedge^L V$ with $\omega = \sum_{\mathfrak{t}} a_{\mathfrak{t}} \vec{e}_{\mathfrak{t}}$, associate the antisymmetric $(NL)^L$ array $\mathbf{A}$ by

$$\mathbf{A}_{\mathfrak{t}\circ\sigma} = \operatorname{sgn}(\sigma) a_{\mathfrak{t}} \qquad \text{for } \mathfrak{t} : \underline{L} \nearrow \underline{NL} \text{ and } \sigma \in S_L$$

This gives a bijective correspondence between L-forms and antisymmetric $(NL)^L$ arrays. We define the Hyperpfaffian $\mathrm{PF}(\omega)$ of an L-form to be the Hyperpfaffian of the associated array.

Proofs for the following results follow from nearly identical arguments to those of the corresponding Pfaffian properties.

Theorem 7. *The Hyperpfaffian of an antisymmetric $(NL)^L$-array $\mathbf{A}$ is given by*

$$\mathrm{PF}(\mathbf{A}) = \frac{1}{N!} \sum_{\sigma \in \mathrm{Sh}(\Lambda^N)} \operatorname{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma_i}.$$

Corollary 4. *The Hyperpfaffian of an antisymmetric $(NL)^L$-array $\mathbf{A}$ with associated L-form ω is given by*

$$\frac{1}{N!}\omega^{\wedge N} = \mathrm{PF}(\mathbf{A})\, \vec{e}_{\mathrm{vol}}.$$

Theorem 8. *The Hyperpfaffian of an antisymmetric $(NL)^L$-array A is given by*

$$\mathrm{PF}(\mathbf{A}) = \sum_{\sigma \in \mathrm{Sh}^o(\Lambda^N)} \operatorname{sgn}(\sigma) \prod_{i=1}^{N} \mathbf{A}_{\sigma_i}$$

Corollary 5. *If* $\omega \in \bigwedge^L V$ *with* $\omega = \sum_{\mathfrak{t}} \mathbf{A}_{\mathfrak{t}}\, \vec{e}_{\mathfrak{t}}$*, then*

$$\mathrm{PF}(\omega)\, \vec{e}_{\mathrm{vol}} = \bigwedge_{i=1}^{NL-L+1} \left(\sum_{\substack{\mathfrak{t}:\underline{L} \nearrow \underline{NL} \\ \mathfrak{t}(1)=i}} \mathbf{A}_{\mathfrak{t}}\, \vec{e}_{\mathfrak{t}} \right).$$

CHAPTER III

ANDREIEF AND DE BRUIJN IDENTITIES

A Few Important Polynomial Definitions and Identities

The Wronskian

A *complete* k-family of monic polynomials is a collection $\{p_i(x)\}_{i=0}^k$ such that each p_i is monic of degree $i-1$. Given an increasing function $\mathfrak{t} : \underline{k} \nearrow \underline{n}$, let $\mathbf{p}_{\mathfrak{t}}(x)$ denote the subset $\{p_{\mathfrak{t}(i)}(x)\}_{i=1}^k$.

To simplify the appearance of several formula, we define the ℓth modified differential operator D^ℓ for any non-negative integer ℓ, by

$$D^\ell f(x) = \frac{1}{\ell!}\frac{d^\ell f}{dx^\ell}$$

In particular, this allows us to define the Wronskian $\mathrm{Wr}(\mathbf{f})$ of a family $\mathbf{f} = \{f_i(x)\}_{i=1}^n$ of n many differentiable functions as

$$\mathrm{Wr}(\mathbf{f}) = \det\left[D^{\ell-1} f_i\right]_{i,\ell=1}^n$$

Observe that our Wronskian differs by a combinatorial factor from the typical Wronskian which appears in the study of differential equations.

Now, for each complete NL-family of monic polynomials $\mathbf{p}$, we define an L-form $\omega_{\mathbf{p}} \in \bigwedge_R^L V$, where $R = T(H)_{\sqcup\!\sqcup}$ and V is a free R-module of rank N, by

$$\omega_{\mathbf{p}} = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}})\, \vec{e}_{\mathfrak{t}}. \tag{3.1}$$

Observe that the coefficients of $\omega_{\mathbf{p}}$ are in fact polynomials of a single variable. To obtain an L-form with real coefficients, $\langle\omega_{\mathbf{p}}\rangle$, we integrate term-by-term with respect to $d\mu$. That is,

$$\langle\omega_{\mathbf{p}}\rangle = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} \big\langle \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}(x))\big\rangle\, \vec{e}_{\mathfrak{t}} = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} \left\{\int \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}(x))\, d\mu(x)\right\} \vec{e}_{\mathfrak{t}}.$$

When the complete monic family of polynomials $\mathbf{p}$ is clear from context, we will omit the subscript and simply write ω. In several places, it will be convenient to instead view ω as a collection of forms $\omega(x) \in \bigwedge_{\mathbb{R}}^{L} V$ indexed by $x \in \mathbb{R}$. Of course, $\omega(x)$ is simply the evaluation of ω at $x \in \mathbb{R}$.

The Vandermonde and Confluent Vandermonde

While the following is a standard result, due to its importance to the work that follows, its proof deserves a moment of reflection.

Theorem 9. *Let $\{p_1(x), \ldots, p_N(x)\}$ be a collection monic polynomials such that each $p_k(x)$ is of degree k. Then*

$$\prod_{i<j}(x_j - x_i) = \det\Big[p_i(x_j)\Big]_{i,j=1}^{N}$$

Proof. Let $\mathbf{V}(\mathbf{x}) = \Big[p_i(x_j)\Big]$, and observe that $\det \mathbf{V}(\mathbf{x}) = 0$ whenever $x_i = x_j$ with $i \neq j$. Therefore, $(x_i - x_j)$ must be a factor of $\det \mathbf{V}(\mathbf{x})$ for each $i \neq j$. Moreover, by the Liebniz formula for the determinant, $\det \mathbf{V}(\mathbf{x})$ must be a homogeneous polynomial of degree $\sum_{i=0}^{N-1} i = \frac{N(N-1)}{2} = \binom{N}{2}$. Of course, $\prod_{i<j}(x_j - x_i)$ is also a homogeneous polynomial of the same degree with the same irreducible factors, and so $\det \mathbf{V}(\mathbf{x}) = C\prod_{i<j}(x_j - x_i)$ for some constant C.

Now, as each p_k is monic of degree k, then $p_k(x_k) = x^k + q_k(x_k)$ where the degree of q_k is less than k. Appealing again to the Leibniz formula for the determinant, observe that the only term of the form $\alpha \cdot x_{N-1}^{N-1} x_{N-2}^{N-2} \cdots x_1$ is obtained by expanding along the main diagonal, and so the coefficient on $x_{N-1}^{N-1} x_{N-2}^{N-2} \cdots x_1$ in $\det \mathbf{V}(\mathbf{x})$ is 1. On the other hand, the only way to obtain a term of the form $\alpha \cdot x_{N-1}^{N-1} x_{N-2}^{N-2} \cdots x_1$ in the expansion of $\prod_{i<j}(x_j - x_i)$ is by taking the product of the the first term x_i in each factor $(x_j - x_i)$, which shows that the coefficient on this monomial is also 1. The result then follows. $\square$

For any complete NL-family of monic polynomials $\mathbf{p}$, we define the Lth confluent $NL \times NL$ Vandermonde matrix $\mathbf{V}_{\mathbf{p}}^{L}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^N$ by first defining the $NL \times L$ matrix

$$\mathbf{V}_{\mathbf{p}}^{L}(x) = \left[D^{\ell-1} p_n(x)\right]_{n,\ell=1}^{NL,L}$$

where $\mathbf{p}$ is a complete NL-family of monic polynomials. Then take

$$\mathbf{V}_{\mathbf{p}}^{L}(\mathbf{x}) = \begin{bmatrix} \mathbf{V}_{\mathbf{p}}^{L}(x_1) & \mathbf{V}_{\mathbf{p}}^{L}(x_2) & \ldots & \mathbf{V}_{\mathbf{p}}^{L}(x_N) \end{bmatrix} \qquad \text{for } \mathbf{x} = (x_1, x_2, \ldots, x_N)$$

Observe that the confluent Vandermonde specializes to the ordinary Vandermonde matrix when $L = 1$.

The confluent Vandermonde matrix satisfies the following identity which shows that the determinant of the confluent Vandermonde doesn't depend on the choice of family of monic polynomials.

Theorem 10. *For any complete NL-family of monic polynomials* $\mathbf{p}$,

$$\det \mathbf{V}_{\mathbf{p}}^{L}(\mathbf{x}) = \prod_{i<j} (x_j - x_i)^{L^2}$$

An efficient proof of this result proceeds by expressing each higher derivative $p_k^{(n)}$ in the confluent Vandermonde matrix as a higher-order difference quotient of the polynomial p_k. Doing so makes use of the following lemma, which while well-known in numerical approximation circles, is proved in full here for completeness.

Lemma 3. *Suppose f is a smooth function, and let $\Delta_h^n[f](x)$ be the n-step finite forward difference formula for f at x defined by*

$$\Delta_h^n[f](x) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f\big(x+(n-k)h\big).$$

Then

$$\lim_{h\to 0} \frac{\Delta_h^n[f](x)}{h^n} = f^{(n)}(x)$$

Proof. We proceed by induction. When $n=1$, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\Delta_h^1[f](x)}{h},$$

by definition of the derivative. Suppose the formula holds for $N=1,\dots,n$. Then

$$\begin{aligned}
\Delta_h^n[f](x+h) - \Delta_h^n[x] =& \sum_{k=0}^{n}(-1)^k \binom{n}{k} f\big(x+h+(n-k)h\big) \\
& - \sum_{k=0}^{n}(-1)^k \binom{n}{k} f\big(x+(n-k)h\big) \\
=& f(x+(n+1)h) + (-1)^{n+1} f(x) \\
& + \sum_{k=0}^{n-1}(-1)^{k+1} f\big(x+(n-k)h\big)\left[\binom{n}{k+1}+\binom{n}{k}\right] \\
=& \sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} f\big(x+((n+1)-k)h\big) \\
=& \Delta_h^{n+1}[f](x),
\end{aligned}$$

where the 3rd equality follows from the *Vandermonde identity*, $\binom{n+1}{k} = \binom{N}{k+1} + \binom{n}{k}$. Thus, since f is smooth, then we may freely interchange the following limits, and

$$\begin{aligned} f^{(n+1)}(x) &= \lim_{r \to 0} \frac{f^{(n)}(x+r) - f^{(n)}(x)}{r} \\ &= \lim_{r \to 0} \frac{1}{r} \left[\lim_{h \to 0} \frac{\Delta_h^n[f](x+r) - \Delta_h^n[f](x)}{h^n} \right] \\ &= \lim_{h \to 0} \frac{\Delta_h^n[f](x+h) - \Delta_h^n[f](x)}{h^{n+1}} \\ &= \lim_{h \to 0} \frac{\Delta_h^{n+1}[f](x)}{h^{n+1}}, \end{aligned}$$

as desired. □

We now proceed to a proof of Theorem 10.

Proof. Suppose now that $\mathbf{x} \in \mathbb{R}^N$ and $h \in \mathbb{R}$. To improve readability, we use the convention that for any single variable function f, the expression $f(\mathbf{x})$ denotes the vector $(f(x_1), f(x_2), \ldots, f(x_N)) \in \mathbb{R}^N$.

For each $0 \leq k \leq L-1$, let $f_k(x) = x + kh$, and define $\mathbf{y}^k = f_k(\mathbf{x})$, and let $\mathbf{z}$ denote the NL-tuple $\mathbf{z} = (\mathbf{y}^0, \ldots, \mathbf{y}^{L-1})$. Observe that the n-step finite forward difference formula $\Delta_h^k[f](x)$ (as defined in the preceding Lemma) is a linear combination of $\{f(x+kh) \,|\, 0 \leq k \leq n-1\}$. Hence, using matrix column operations, the Vandermonde determinant, $\det \mathbf{V_p}(\mathbf{z})$, may be expressed as

$$\det \mathbf{V_p}(\mathbf{z}) = \det \begin{bmatrix} p_1(\mathbf{y}^0) & p_1(\mathbf{y}^1) & p_1(\mathbf{y}^2) & \dots & p_1(\mathbf{y}^{L-1}) \\ \vdots & & & & \vdots \\ p_{NL}(\mathbf{y}^0) & \dots & & \dots & p_{NL}(\mathbf{y}^{L-1}) \end{bmatrix}$$

$$= \det \begin{bmatrix} p_1(\mathbf{x}) & \Delta_h^1[p_1](\mathbf{x}) & \Delta_h^2[p_1](\mathbf{x}) & \dots & \Delta_h^{L-1}[p_1](\mathbf{x}) \\ \vdots & & & & \vdots \\ p_{NL}(\mathbf{x}) & \dots & & \dots & \Delta_h^{L-1}[p_{NL}](\mathbf{x}) \end{bmatrix}$$

By factoring out common powers of h from each column, we obtain

$$\det \mathbf{V_p}(\mathbf{z}) = C(h) \cdot \det \begin{bmatrix} p_1(\mathbf{x}) & \frac{\Delta_h^1[p_1](\mathbf{x})}{h} & \frac{\Delta_h^2[p_1](\mathbf{x})}{h^2 2!} & \dots & \frac{\Delta_h^{L-1}[p_1](\mathbf{x})}{h^{L-1}(L-1)!} \\ \vdots & & & & \vdots \\ p_{NL}(\mathbf{x}) & \dots & & \dots & \frac{\Delta_h^{L-1}[p_{NL}](\mathbf{x})}{h^{L-1}(L-1)!} \end{bmatrix} \tag{3.2}$$

where

$$C(h) = h^{\frac{NL(L-1)}{2}} \left[\prod_{k=1}^{L-1} k! \right]^N$$

Let $\mathbf{M}_h$ denote the matrix which appears in line 3.2 above, and observe that by Lemma 3,

$$\det \mathbf{V_p^L}(\mathbf{x}) = \lim_{h \to 0} \det \mathbf{M}_h.$$

On the other hand, by expanding the Vandermonde determinant according to Theorem 9 and grouping products depending on whether pairwise difference are taken between entries in the same vector $\mathbf{y}^k$ or between entries in different vectors

$\mathbf{y}^m$ and $\mathbf{y}^n$, then

$$\begin{aligned}
\mathbf{V}_{\mathbf{p}}(\mathbf{z}) &= \left[\prod_{k=0}^{L-1}\prod_{i<j}\big((x_j+kh)-(x_i+kh)\big)\right]\cdot\left[\prod_{m<n}\prod_{i,j=1}^{N}\big((x_j+nh)-(x_i+mh)\big)\right] \\
&= \left[\prod_{i<j}\big(x_j-x_i\big)^L\right]\cdot\left[\prod_{m<n}\prod_{i,j=1}^{N}\big((x_j-x_i)+(n-m)h\big)\right] \\
&= \left[\prod_{i<j}\big(x_j-x_i\big)^L\right]\cdot\left[\prod_{m<n}\prod_{i\neq j}^{N}\big((x_j-x_i)+(n-m)h\big)\right] \\
&\quad\cdot\left[h^{\frac{NL(L-1)}{2}}\prod_{m<n}(n-m)^N\right] \\
&=C(h)\cdot\left[\prod_{i<j}\big(x_j-x_i\big)^L\right]\cdot\left[\prod_{m<n}\prod_{i\neq j}^{N}\big((x_j-x_i)+(n-m)h\big)\right].
\end{aligned}$$

Thus,

$$\left[\prod_{i<j}\big(x_j-x_i\big)^L\right]\cdot\left[\prod_{m<n}\prod_{i\neq j}^{N}\big((x_j-x_i)+(n-m)h\big)\right]=\frac{\mathbf{V}_{\mathbf{p}}(\mathbf{z})}{C(h)}=\det\mathbf{M}_h$$

Taking limits as $h\to 0$ of the left and right expressions above, we obtain

$$\prod_{i<j}(x_j-x_i)^{L^2}=\det\mathbf{V}_{\mathbf{p}}^L(\mathbf{x}),$$

as desired. $\square$

Expectation Operator and Chen's Lemma

The Shuffle and Function Algebras

Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R},\mu)$ of square integrable functions with respect to a finite measure μ on $\mathbb{R}$, and suppose H is a finite-dimensional subspace of $\mathcal{H}$ with basis X (We assume H is large enough to contain all functions of interest to the current program). For each k, view the kth tensor space $T^k(H)$ as a subset of $L^2(\mathbb{R}^k, \mu)$ by defining

$$(f_1 \otimes \cdots \otimes f_k)(x_1, \ldots, x_k) = f_1(x_1) \cdots f_k(x_1)$$

By identifying the tensor algebra $T(H)$ with the free algebra $\mathbb{R}\langle X\rangle$ (via the map $f_1 \otimes \cdots \otimes f_k \mapsto f_1 \cdots f_k$), we may define the shuffle of two functions $f = f_1 \otimes \cdots \otimes f_k$ and $g = f_{k+1} \otimes \cdots \otimes f_n$ by

$$f \sqcup\!\sqcup\, g = \sum_{\sigma \in \mathrm{Sh}(k,n)} f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)}$$

Now, for each non-negative integer k, we define a linear functional $\langle \cdot \rangle_k$ on $T^k(V)$ by

$$\langle f_1 \otimes \cdots \otimes f_k \rangle_k = \int_{-\infty}^{\infty} \int_{-\infty}^{x_k} \cdots \int_{-\infty}^{x_1} f_1(x_1) \cdots f_k(x_k)\, dx_1 \cdots dx_k$$

Moreover, this collection $\{\langle \cdot \rangle_k | k \in \mathbb{Z}_{\geq 0}\}$ induces a functional $\langle \cdot \rangle$ on $T(V)$, whereby $\langle \cdot \rangle$ acts as $\langle \cdot \rangle_k$ on the kth graded component of a non-homogeneous tensor.

We can think of $\langle \cdot \rangle$ as an *expectation operation* on the space of functions $\mathcal{H}$ which are measurable with respect to a random variable with distribution μ.

The following lemma, due to Chen in [6], asserts that this expectation operator $\langle \cdot \rangle$ is actually an algebra homomorphism from $T(V)_{\sqcup\!\sqcup}$ to $\mathbb{R}$:

Lemma 4 (Chen)**.** *If $f, g \in T(V)$, then*

$$\langle f \sqcup\!\sqcup\, g \rangle = \langle f \rangle \, \langle g \rangle.$$

Using Fubuni's Theorem, each iterated integral that appears in $\langle f\rangle_n$ can also be realized as an n-fold integral across the n-simplex $\{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_1 < x_2 < \cdots < x_n\}$, and thus, $\langle f\rangle_k\langle g\rangle_{n-k}$ represents an integral across the Cartesian product of two such regions. Chen's Lemma, in essence, asserts that this product can be written as a (nearly) disjoint union of several n-simplices, indexed by shuffle permutations.

Let X be the set of points in $\mathbb{R}^n$ whose coordinates are all distinct, and observe that $\mathbb{R}^n\setminus X$ has Lebesgue measure 0. Each point $(x_1,\dots,x_n)\in X$ can be associated with an ordering permutation $\sigma\in S_n$, where $\sigma(i)=j$ when x_i is the jth smallest coordinate. Then for each $\sigma\in S_n$, let Δ^n_σ be the set of all points in A with ordering permutation σ, that is,

$$\Delta^n_\sigma = \{(x_1,\dots,x_n)\in X \mid x_{\sigma^{-1}(1)} < \cdots < x_{\sigma^{-1}(n)}\}.$$

In particular, observe that the collection $\{\Delta^n_\sigma\}$ partitions X into mutually disjoint subsets, and that for $(x_1,\dots,x_n)\in\Delta^n_\sigma$, $x_i<x_j$ if and only if $\sigma(i)<\sigma(j)$.

Now, the key observation for Chen's Lemma can be stated as

$$\Delta^k_{\mathrm{id}}\times\Delta^{n-k}_{\mathrm{id}} = Y \sqcup \bigsqcup_{\sigma\in\mathrm{Sh}(k,n-k)} \Delta^n_\sigma. \tag{3.3}$$

where $Y=(\mathrm{R}^n\setminus X)\cap(\Delta^k_{\mathrm{id}}\times\Delta^{n-k}_{\mathrm{id}})$. Note that as $Y\subset\mathrm{R}^n\setminus X$, then Y has Lebesgue measure 0.

For one inclusion, suppose $(x_1,\dots,x_n)\in\Delta^k_{\mathrm{id}}\times\Delta^{n-k}_{\mathrm{id}}$ with ordering permutation σ. Since $(x_1,\dots,x_k)\in\Delta^k_{\mathrm{id}}$, then $x_1<\cdots<x_k$ and so $\sigma(1)<\cdots<$

$\sigma(k)$. Similarly, since $(x_{k+1}, \ldots, x_n) \in \Delta_\sigma^{n-k}$, then $\sigma(k+1) < \cdots < \sigma(n)$. Hence, $\sigma \in \mathrm{Sh}(k, n-k)$. On the other hand, if $(x_1, \ldots, x_n) \in \Delta_\sigma^n$ for some $\sigma \in \mathrm{Sh}(k, n-k)$, then $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$, and so $x_1 < \cdots < x_k$ and $x_{k+1} < \cdots < x_n$.

To prove Chen's Lemma, suppose $f = f_1 \otimes \cdots \otimes f_k$ and $g = f_{k+1} \otimes \cdots \otimes f_n$ are functions in $T(V)$. Then

$$
\begin{aligned}
\langle f \rangle \langle g \rangle &= \left(\int_{\Delta_{\mathrm{id}}^k} f_1(x_1) \ldots f_k(x_k)\, dx_1 \cdots dx_k \right) \\
&\quad \cdot \left(\int_{\Delta_{\mathrm{id}}^{n-k}} f_{k+1}(x_{k+1}) \cdots f_n(x_n)\, dx_{k+1} \ldots dx_n \right) \\
&= \int_{\Delta_{\mathrm{id}}^k \times \Delta_{\mathrm{id}}^{n-k}} f_1(x_1) \cdots f_n(x_n)\, dx_1 \ldots dx_n \\
&= \int_{\bigsqcup_{\sigma \in \mathrm{Sh}(k,n)} \Delta_\sigma^n} f_1(x_1) \cdots f_n(x_n)\, dx_1 \ldots dx_n \\
&= \sum_{\sigma \in \mathrm{Sh}(k,n)} \int_{\Delta_\sigma^n} f_1(x_1) \cdots f_n(x_n)\, dx_1 \ldots dx_n \\
&= \sum_{\sigma \in \mathrm{Sh}(k,n)} \int_{\Delta_{\mathrm{id}}^n} f_1(y_{\sigma(1)}) \cdots f_n(y_{\sigma(n)})\, dy_1 \ldots dy_n \\
&= \sum_{\sigma \in \mathrm{Sh}(k,n)} \int_{\Delta_{\mathrm{id}}^n} f_{\sigma^{-1}(1)}(y_1) \cdots f_{\sigma^{-1}(n)}(y_n)\, dy_1 \ldots dy_n \\
&= \langle f \sqcup g \rangle
\end{aligned}
$$

where the second equality follows from Fubini's Theorem, where the third follows from 3.3 and the observation that $Y \subset \mathrm{R}^n \setminus X$ has Lebesgue measure 0, and where the fourth follows from the relabeling of variables $x_i = y_{\sigma(i)}$.

A Key Lemma

Let $\{Q_{\mathfrak{t}} \,|\, \mathfrak{t} : \underline{L} \to \underline{NL}\}$ be a subset of an alphabet I, let F be a field of characteristic 0, and for each $\mathfrak{t} : \underline{L} \to \underline{NL}$, define $A_{\mathfrak{t}} \in F\langle I\rangle$ by

$$A_{\mathfrak{t}} = \sum_{\tau \in S_L} \operatorname{sgn}(\tau) Q_{\mathfrak{t}\circ\tau}.$$

For example, when $L = 2$, we can view $\{Q_{\mathfrak{t}\circ\tau}\} = \{Q_{i,j} \,|\, 1 \le i, j \le N\}$ as in an $N \times N$ matrix with entries in I, in which case $A_{i,j} = Q_{i,j} - Q_{j,i}$ is the (i,j) entry in the matrix $A = Q - Q^T$.

Lemma 5. *Let L be an even integer, let $\{Q_{\mathfrak{t}}\}$ and $\{A_{\mathfrak{t}}\}$ be as above, let F be a field of characteristic 0, let V be a rank NL free module over $R = F\langle I\rangle_{\sqcup\!\sqcup}$, and consider $\left(\sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} A_{\mathfrak{t}}\,\vec{e}_{\mathfrak{t}}\right)$ as an element of $\bigwedge_R V$. Then*

$$\sum_{\sigma \in S_{NL}} \operatorname{sgn}(\sigma) Q_{\sigma_1} Q_{\sigma_2} \cdots Q_{\sigma_N} = \mathrm{PF}_R\left(\sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} A_{\mathfrak{t}}\,\vec{e}_{\mathfrak{t}}\right)$$

Proof. For each $\sigma \in H_{NL}$, write $\sigma = (\sigma_1|\ldots|\sigma_N)$. Then

$$\begin{aligned}
\mathrm{PF}_R\left(\sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} A_{\mathfrak{t}}\,\vec{e}_{\mathfrak{t}}\right) &= \sum_{\sigma \in H_{NL}} \operatorname{sgn}(\sigma) A_{\sigma_1} \sqcup\!\sqcup A_{\sigma_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup A_{\sigma_N} \\
&= \sum_{\sigma \in H_{NL}} \operatorname{sgn}(\sigma)\Big(\sum_{\tau \in S_L} \operatorname{sgn}(\tau) Q_{\sigma_1\circ\tau}\Big) \sqcup\!\sqcup \cdots \sqcup\!\sqcup \Big(\sum_{\tau \in S_L} \operatorname{sgn}(\tau) Q_{\sigma_N\circ\tau}\Big) \\
&= \sum_{\sigma \in H_{NL}} \operatorname{sgn}(\sigma) \sum_{\tau_1,\ldots,\tau_N \in S_L} \operatorname{sgn}(\tau_1 \cdots \tau_N) Q_{\sigma_1\circ\tau} \sqcup\!\sqcup \cdots \sqcup\!\sqcup Q_{\sigma_N\circ\tau} \\
&= \sum_{\sigma \in H_{NL}} \operatorname{sgn}(\sigma) \sum_{\tau_1,\ldots,\tau_N \in S_L} \operatorname{sgn}(\tau_1 \cdots \tau_N) \sum_{\pi \in S_N} Q_{\sigma_{\pi(1)}\circ\tau} \cdots Q_{\sigma_{\pi(N)}\circ\tau} \\
&= \sum_{\alpha \in S_{NL}} \operatorname{sgn}(\alpha) Q_{\alpha_1} \cdots Q_{\alpha_N}.
\end{aligned}$$

Here, the equalities in lines (1) through (4) follow from appropriate application of definition and elementary properties of the shuffle operation. To obtain the equality in line (5), observe that every permutation $\alpha \in S_{NL}$ can be transformed into a unique element $\sigma \in H_{NL}$ via a collection of ordering permutations $(\pi, \tau_1, \ldots, \tau_N) \in S_N \times S_L \times \cdots \times S_L$, where π is specified by

$$\alpha\big(\pi(0)L+1\big) < \alpha\big(\pi(1)L+1\big) < \cdots < \alpha\big(\pi(N-1)L+1\big),$$

where each τ_n is specified by

$$\alpha\big((n-1)L+\tau_n(1)\big) < \alpha\big((n-1)L+\tau_n(2)\big) < \cdots < \alpha\big((n-1)L+\tau_n(L)\big),$$

and where

$$\sigma\big((n-1)L+\ell\big) = \alpha\big(\pi(n-1)+\tau_n(\ell)\big) \qquad \text{for } 1 \leq n \leq N, 1 \leq \ell \leq L.$$

Then observe that as L is even, $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\tau_1 \cdots \tau_N)\operatorname{sgn}(\sigma)$. $\square$

Statement of the Main Result

Theorem 11. *Suppose L and N are positive integers, and let ω be the L-form given in 3.1 for any complete NL-family of monic polynomials. That is,*

$$\omega(x) = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}}; x)\, \vec{e}_{\mathfrak{t}} \in \textstyle\bigwedge_R^L V$$

where $R = T(H)_{\sqcup\!\sqcup}$ and V is a free R-module of rank N. Then the partition function $Z_N(\beta)$ is given by the following expressions:

1. *if* $\beta = L^2$ *is even,*

$$\mathrm{PF}\left(\int \omega(x)\, d\mu(x)\right).$$

2. *if* $\beta = L^2$ *is odd and* N *is even,*

$$\mathrm{PF}\left(\iint_{x<y} \omega(x) \wedge \omega(y)\, d\mu(x) d\mu(y)\right).$$

3. *if* $\beta = L^2$ *is odd and* N *is odd,*

$$\mathrm{PF}\left(\iint_{x<y} \omega(x) \wedge \omega(y)\, d\mu(x) d\mu(y) + \int \omega(x) \wedge \vec{e}_{\mathfrak{i}'}\, d\mu(x)\right)$$

where $\vec{e}_{\mathfrak{i}'} = \vec{e}_{NL+1} \wedge \cdots \wedge \vec{e}_{(N+1)L}$.

Method of Proof

For each of the three cases, the proof proceeds along similar lines. We first use the Laplace expansion of the determinant to express the confluent Vandermonde as a sum of products of polynomials in several variables, which is then interpreted as an element of $T(H)$. After this, we apply Lemma 5 to express the confluent Vandermonde as the hyperpfaffian of a form in $\bigwedge_R V$, where $R = T(H)_{\sqcup\!\sqcup}$. Finally, we apply the linear functional $\langle \cdot \rangle$ and use Chen's Lemma to represent the partition function Z_N as the hyperpfaffian of a particular form in $\bigwedge_{\mathbb{R}} V$.

In order to exhibit the method of proof that will be used in the general case, we first present proofs of the result for when $\beta = 1$ and $\beta = 4$.

Using the Laplace expansion to write $\det \mathbf{V}_{\mathbf{p}}^2$ as a sum of N-fold products of polynomials,

$$\begin{aligned}\prod_{i<j}(x_j - x_i)^4 =& \mathbf{V}_{\mathbf{p}}^2(\mathbf{x}) \\ =& \sum_{\sigma\in S_{2N}} \epsilon(\sigma) p_{\sigma(1)}(x_1)p'_{\sigma(2)}(x_1)\cdots p_{\sigma(2n-1)}(x_N)p'_{\sigma(2N)}(x_N). \qquad (3.4)\end{aligned}$$

Then by grouping terms in the same variable, we define a collection of functions $\{Q_{ij}\}_{i,j=1}^{2N}$ by $Q_{ij}(x) = p_i(x)p'_j(x)$. Recalling that in the tensor algebra of function $T(V)$ the tensor$(f \otimes g)$ can be viewed as a function of two variables, $(f \otimes g)(x, y) = f(x)g(y)$, we can express the sum on the right of 3.4 as the function F of N variables defined by

$$F = \sum_{\sigma\in S_{2N}} \epsilon(\sigma) Q_{\sigma(1)\sigma(2)} \otimes \cdots \otimes Q_{\sigma(2N-1)\sigma(2N)}.$$

And by Lemma 5, we have

$$F = \mathrm{Pf}_R\left(Q_{ij} - Q_{ji}\right),$$

and so

$$\prod_{i<j}(x_j - x_i)^4 = \left[\mathrm{Pf}_R\left(Q_{ij} - Q_{ji}\right)\right](x_1, \ldots, x_N).$$

Now, by applying the linear functional $\langle\cdot\rangle$ to both sides above, and using Chen's Lemma to pass from the Pfaffian over the shuffle algebra $R = T(H)_{\sqcup\!\sqcup}$ to a Pfaffian over $\mathbb{R}$, we have

$$Z_N(4) = \left\langle \prod_{i<j} (x_j - x_i)^4 \right\rangle = \langle \mathrm{Pf}_R(Q_{ij} - Q_{ji}) \rangle$$
$$= \mathrm{Pf}\left(\langle Q_{ij} - Q_{ji} \rangle \right)$$
$$= \mathrm{Pf}\left(\int p_i(x) p_j'(x) - p_j(x) p_i'(x) \, d\mu(x) \right),$$

as desired.

The Case when $\beta = 1$ and N is Even

The $\beta = 1$ case proceeds almost analogously to the $\beta = 4$ case. However, because Lemma 5 required that the length L of the multi-index on Q be even, we instead group polynomial factors together two variables at a time. Ultimately, this will be the reason for the appearance of double integrals in the Pfaffian formulation for the partition function for $\beta = 1$ (and more generally, $\beta = 2k + 1$).

Suppose $N = 2M$. As above, use the Laplace expansion to express $\det \mathbf{V}_{\mathbf{p}}^1$ as a sum:

$$\prod_{i<j} (x_j - x_i) = \det \mathbf{V}_{\mathbf{p}}^1(\mathbf{x})$$
$$= \sum_{\sigma \in S_{2M}} \epsilon(\sigma) p_{\sigma(1)}(x_1) p_{\sigma(2)}(x_2) \cdots p_{\sigma(2M-1)}(x_{2M-1}) p_{\sigma(2M)}(x_{2M}).$$

Define the collection $\{Q_{ij}\}_{i,j=1}^{2M}$ of 2-variable functions by $Q_{ij}(x, y) = p_i(x) p_j(y)$, so that the right side above can be expressed as a function F of N variables

$$F = \sum_{\sigma \in S_{2M}} \epsilon(\sigma) Q_{\sigma(1)\sigma(2)} \otimes \cdots \otimes Q_{\sigma(2M-1)\sigma(2M)}.$$

The next step is to apply Lemma 5 to $F \sqcup \mathbb{1}$ in order to obtain a Pfaffian of an anti-symmetric 2-form. However, since each of $\widetilde{Q}_{ij}$ is a function of two variables, we must work over the ring $R = T(H_2)$, where H_2 is a finite-dimensional subspace of the Hilbert Space $\mathcal{H}^{\otimes}2 = L^2(\mathbb{R}^2, \mu \times \mu)$ of square-integrable functions in two variables. But now, by Lemma 5,

$$F = \mathrm{Pf}_R\left(Q_{ij} - Q_{ji}\right)$$

which shows that

$$\prod_{i<j}(x_j - x_i) = \left[\mathrm{Pf}_R\left(Q_{ij} - Q_{ji}\right)\right](x_1, \ldots, x_N)$$

Hence, upon applying the linear functional $\langle \cdot \rangle$ to both sides above, appealing to Chen's Lemma, and recalling that each Q_{ij} is a function in two variables,

$$\begin{aligned} Z_N(1) = \left\langle \prod_{i<j}(x_j - x_i) \right\rangle &= \langle \mathrm{Pf}_R(Q_{ij} - Q_{ji}) \rangle \\ &= \mathrm{Pf}\left(\langle Q_{ij} - Q_{ji} \rangle\right) \\ &= \mathrm{Pf}\left(\iint_{x<y} p_i(x)p_j(y) - p_j(x)p_i(y)\, d\mu(x)d\mu(y)\right) \end{aligned}$$

The Case when $\beta = 1$ and N is Odd

When $\beta = 1$ and N is odd, we cannot use the same trick we did previously to group polynomial factors two variables at a time (since we have an odd number of polynomial factors). Instead, we will introduce an additional auxillary variable which ultimately won't change the value of the partition function. Suppose $N = 2M + 1$, and let $\widetilde{N} = N + 1$. As before, expand the Vandermonde determinant using the Laplace expansion to obtain

$$\prod_{i<j}(x_j - x_i) = \det \mathbf{V}^1_{\mathbf{p}}(\mathbf{x}) = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) p_{\sigma(1)}(x_1) \cdots p_{\sigma(N)}(x_N).$$

Define a collection of single-variable functions $\{Q_i\}$ by $Q_i(x) = p_i(x)$ (Although redundant here, the need for such Q_i will become apparent in the general case). Then the right side of the equation above can be expressed as a function F of N variables as

$$F = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) Q_{\sigma(1)} \otimes \cdots \otimes Q_{\sigma(N)}.$$

Let $\mathbb{1}$ denote the constant function $\mathbb{1}(x) = 1$. Then

$$\begin{aligned} F \sqcup\!\sqcup\, \mathbb{1} = & \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \mathbb{1} \otimes Q_{\sigma(1)} \otimes \cdots \otimes Q_{\sigma(N)} \\ & + \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) Q_{\sigma(1)} \otimes \mathbb{1} \otimes \cdots \otimes Q_{\sigma(N)} \\ & + \dots \\ & + \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) Q_{\sigma(1)} \otimes \cdots \otimes Q_{\sigma(N)} \otimes \mathbb{1} \end{aligned}$$

We wish to represent $F \sqcup\!\sqcup\, \mathbb{1}$ in terms of a set of two-variable symbols $\{\widetilde{Q}_{ij}\}$, defined as follows:

$$\widetilde{Q}_{ij} = \begin{cases} Q_i \otimes Q_j, & \text{if } 1 \le i, j \le N, \\ Q_i \otimes \mathbb{1}, & \text{if } 1 \le i \le N, j = \widetilde{N}, \\ -1 \otimes Q_j, & \text{if } i = \widetilde{N}, 1 \le j \le N, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$
\begin{aligned}
F \sqcup\!\sqcup \mathbb{1} = &\sum_{\sigma \in S_N} \operatorname{sgn}(\sigma)(-\widetilde{Q}_{\widetilde{N},\sigma(1)}) \otimes \widetilde{Q}_{\sigma(2),\sigma(3)} \otimes \cdots \otimes Q_{\sigma(N-1),\sigma(N)} \\
&+ \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma)\widetilde{Q}_{\sigma(1),\widetilde{N}} \otimes \widetilde{Q}_{\sigma(2),\sigma(3)} \otimes \cdots \otimes \widetilde{Q}_{\sigma(N-1),\sigma(N)} \\
&+ \dots \\
&+ \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma)\widetilde{Q}_{\sigma(1),\sigma(2)} \otimes \cdots \otimes \widetilde{Q}_{\sigma(N-2),\sigma(N-1)} \otimes \widetilde{Q}_{\sigma(N),\widetilde{N}} \\
= &\sum_{\tau \in S_{\widetilde{N}}} \operatorname{sgn}(\tau)\widetilde{Q}_{\tau(1),\tau(2)} \otimes \cdots \otimes \widetilde{Q}_{\tau(N),\tau(\widetilde{N})},
\end{aligned}
$$

where the final equality was obtained by noting that each permutation $\tau \in S_{\widetilde{N}}$ corresponds to a choice of $1 \leq k \leq \widetilde{N}$ so that $\tau(k) = \widetilde{N}$, along with a permutation $\sigma \in S_N$ so that for $\ell < k$, $\tau(\ell) = \sigma(\ell)$, for $\ell > k$, $\tau(\ell) = \sigma(\ell - 1)$, and so that $\operatorname{sgn}(\tau) = (-1)^k \operatorname{sgn}(\sigma)$.

The next step is to apply Lemma 5 to $F \sqcup\!\sqcup \mathbb{1}$ in order to obtain a Pfaffian of an anti-symmetric 2-form. As before, take $R = T(H_2)$ in Lemma 5, which shows

$$
F \sqcup\!\sqcup \mathbb{1} = \operatorname{Pf}_R(\widetilde{Q}_{ij} - \widetilde{Q}_{ji}).
$$

Therefore, applying the linear functional $\langle \cdot \rangle$ to both sides above and using Chen's Lemma, along with the observation that as μ is a probability measure, $\langle \mathbb{1} \rangle = 1$, then

$$\begin{aligned}\left\langle \prod_{i<j}(x_j - x_i) \right\rangle =& \langle F \rangle \\ =& \langle F \sqcup\!\sqcup\, \mathbb{1} \rangle \\ =& \langle \mathrm{Pf}_R(\widetilde{Q}_{ij} - \widetilde{Q}_{ji}) \rangle \\ =& \mathrm{Pf}\left(\langle \widetilde{Q}_{ij} - \widetilde{Q}_{ji} \rangle \right).\end{aligned}$$

In particular, we have

$$\langle \widetilde{Q}_{ij} - \widetilde{Q}_{ji} \rangle = \begin{cases} \displaystyle\iint_{x<y} p_i(x)p_j(y) - p_j(x)p_i(y)\, d\mu(x)d\mu(y), & \text{if } 1 \leq i < j \leq N, \\ \displaystyle\iint_{x<y} p_i(x) + p_i(y)\, d\mu(x)d\mu(y), & \text{if } 1 \leq i \leq N, j = \widetilde{N}. \end{cases}$$

But by Chen's Lemma, since $p_i(x) + p_i(y) = p_i \sqcup\!\sqcup\, \mathbb{1}$ in $T(H)$, then

$$\langle p_i(x) + p_i(y) \rangle = \langle p_i \sqcup\!\sqcup\, \mathbb{1} \rangle = \langle p_i \rangle$$

and so

$$\langle \widetilde{Q}_{ij} - \widetilde{Q}_{ji} \rangle = \begin{cases} \displaystyle\iint_{x<y} p_i(x)p_j(y) - p_j(x)p_i(y)\, d\mu(x)d\mu(y), & \text{if } 1 \leq i < j \leq N, \\ \displaystyle\int p_i(x)\, d\mu(x), & \text{if } 1 \leq i \leq N, j = \widetilde{N}, \end{cases}$$

as desired.

The Case when $\beta = L^2$ is Even

We now proceed to the main results. For each $\sigma \in S_{NL}$, write $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_N$. We begin by again using the Laplace expansion to express the confluent Vandermonde $\mathbf{V}_{\mathbf{p}}^L(\mathbf{x})$ as a sum of products

$$\prod_{i<j}(x_j - x_i)^\beta = \det \mathbf{V}_{\mathbf{p}}^L(\mathbf{x}) = \sum_{\sigma \in S_{LN}} \operatorname{sgn}(\sigma) \prod_{n=1}^{N} \left[\prod_{\ell=1}^{L} D^{\ell-1} p_{\sigma((n-1)L+\ell)}(x_n) \right].$$

Now, we define a collection of functions $\{Q_{\mathfrak{t}} \,|\, \mathfrak{t} : \underline{L} \to \underline{NL}\}$ by $Q_{\mathfrak{t}}(x) = \prod_{\ell=1}^{L} D^{\ell-1} p_{\mathfrak{t}(\ell)}(x)$ and use properties of the tensor algebra of functions to write the right hand side as the function F of N variables

$$F = \sum_{\sigma \in S_{LN}} \operatorname{sgn}(\sigma) \left[\bigotimes_{n=1}^{N} Q_{\sigma_n} \right].$$

Then, for each increasing function $\mathfrak{t} : \underline{L} \nearrow \underline{NL}$ we define a function $A_{\mathfrak{t}}$ as in the proof of Lemma 5 by

$$A_{\mathfrak{t}} = \sum_{\tau \in S_L} \operatorname{sgn}(\tau) Q_{\mathfrak{t} \circ \tau}.$$

Thus, by Lemma 5,

$$F = \operatorname{PF}_R \left(\sum_{\mathfrak{t}:\underline{L} \nearrow \underline{NL}} A_{\mathfrak{t}} \, \vec{e}_{\mathfrak{t}} \right),$$

But observe that $A_{\mathfrak{t}}(x) = \operatorname{Wr}(\mathbf{p}_{\mathfrak{t}}(x))$, and so

$$\sum_{\mathfrak{t}:\underline{L} \nearrow \underline{NL}} A_{\mathfrak{t}}(x) \, \vec{e}_{\mathfrak{t}} = \omega(x). \tag{3.5}$$

Therefore,

$$\prod_{i<j}(x_j - x_i)^\beta = \left[\mathrm{PF}_R\left(\sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} A_\mathfrak{t}\,\vec{e}_\mathfrak{t}\right)\right](x_1,\ldots,x_N) = \left[\mathrm{PF}_R\,(\omega)\,\right](x_1,\ldots,x_N).$$

Now, by applying the functional $\langle\cdot\rangle$ to both sides above, and using Chen's Lemma,

$$\begin{aligned} Z_N(\beta) = \left\langle \prod_{i<j}(x_j - x_i)^\beta \right\rangle =& \langle \mathrm{PF}_R(\omega)\rangle \\ =& \mathrm{PF}\,(\langle\omega\rangle) \\ =& \mathrm{PF}\left(\sum_{\mathfrak{t}:\underline{L}\nearrow\underline{NL}} \left\{\int \mathrm{Wr}(p_\mathfrak{t}(x))\,d\mu(x)\right\}\,\vec{e}_\mathfrak{t}\right). \end{aligned}$$

The Case when $\beta = L^2$ is Odd and N is Even

Suppose $N = 2M$. Following as an example the $\beta = 1$ case, we expand the confluent Vandermonde determinant as a sum of products, and group factors together two variables at a time:

$$\begin{aligned} \prod_{i<j}(x_j - x_i)^\beta =& \det \mathbf{V}_\mathbf{p}^L(\mathbf{x}) \\ =& \sum_{\sigma\in S_{LN}} \mathrm{sgn}(\sigma) \prod_{m=1}^{M}\left[\prod_{\ell=1}^{L} D^{\ell-1}p_{\sigma(2(m-1)L+\ell)}(x_{2m})\right] \\ & \cdot \left[\prod_{\ell=1}^{L} D^{\ell-1}p_{\sigma((2(m-1)+1)L+\ell)}(x_{2m+1})\right] \end{aligned}$$

Define a collection of two-variable functions $\{Q_\mathfrak{t}\,|\,\mathfrak{t} : \underline{2L} \to \underline{NL}\}$ by

$$Q_\mathfrak{t}(x,y) = \prod_{\ell=1}^{L} D^{\ell-1}p_{\mathfrak{t}(\ell)}(x)D^{\ell-1}p_{\mathfrak{t}(L+\ell)}(y)$$

and observe that the right side of the Vandermonde expansion above can be expressed as an N-variable function F by

$$F = \sum_{\sigma \in S_{LN}} \operatorname{sgn}(\sigma) \left[\bigotimes_{m=1}^{M} Q_{\sigma_m} \right].$$

As in the β even case, we define functions $A_{\mathfrak{t}}$ for each increasing function $\mathfrak{t} : \underline{2L} \nearrow \underline{NL}$ by

$$A_{\mathfrak{t}} = \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) Q_{\mathfrak{t} \circ \tau}.$$

However, since each $A_{\mathfrak{t}}$ is a function of two variables, in order to apply Lemma 5, we must work over the ring $R = T(H_2)$, where H_2 is a finite-dimensional subspace of the Hilbert space $\mathcal{H}_2 = L^2(\mathbb{R}^2, \mu \times \mu)$ of square integrable functions in two variables. With this modification, Lemma 5 gives

$$F = \mathrm{PF}_R \left(\sum_{\mathfrak{t}:\underline{2L} \nearrow \underline{NL}} A_{\mathfrak{t}} \, \vec{e}_{\mathfrak{t}} \right).$$

But the L-form above can also be expressed in another way as

$$\sum_{\mathfrak{t}:\underline{2L} \nearrow \underline{NL}} A_{\mathfrak{t}}(x, y) \, \vec{e}_{\mathfrak{t}} = \omega(x) \wedge \omega(y). \tag{3.6}$$

where $\omega(x) \wedge \omega(y)$ is calculated pointwise in $\bigwedge_{\mathbb{R}}^{L} V$ for each $x, y \in \mathbb{R}$ (rather than as $\omega \wedge \omega$ in $\bigwedge_{R}^{L} V$).

To see this, first observe that for each $x, y \in \mathbb{R}$,

$$\omega(x) \wedge \omega(y) = \sum_{\mathfrak{s}:\underline{L} \nearrow \underline{NL}} \sum_{\mathfrak{u}:\underline{L} \nearrow \underline{NL}} \mathrm{Wr}(p_{\mathfrak{s}}(x)) \mathrm{Wr}(p_{\mathfrak{u}}(y)) \, \vec{e}_{\mathfrak{s}} \wedge \vec{e}_{\mathfrak{u}}$$

and use the Laplace expansion of the determinant to write

$$\mathrm{Wr}(p_{\mathfrak{s}}(x))\mathrm{Wr}(p_{\mathfrak{u}}(y)) = \sum_{\sigma\in S_L}\sum_{\tau\in S_L} \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\prod_{\ell=1}^{L} D^{\ell-1}p_{\mathfrak{s}\sigma(\ell)}(x)D^{\ell-1}p_{\mathfrak{u}\tau(\ell)}(y).$$

On the other hand,

$$A_{\mathfrak{t}} = \sum_{\pi\in S_{2L}} \mathrm{sgn}(\pi)\prod_{\ell=1}^{L} D^{\ell-1}p_{\mathfrak{t}\pi(\ell)}(x)D^{\ell-1}p_{\mathfrak{t}\pi(L+\ell)}(y)$$

Hence, it suffices to show that for any $\mathfrak{t} : \underline{2L} \nearrow \underline{NL}$ and $\pi \in S_{2L}$, there exists a unique quartet $(\mathfrak{s}, \mathfrak{u}, \sigma, \tau)$ with $\mathfrak{s}, \mathfrak{u} : \underline{L} \nearrow \underline{NL}$ and with $\sigma, \tau \in S_L$ so that $\vec{e}_{\mathfrak{t}} = \vec{e}_{\mathfrak{s}} \wedge \vec{e}_{\mathfrak{u}}$ and

$$\mathfrak{t}\pi(k) = \begin{cases} \mathfrak{s}\sigma(k), & \text{if } 1 \leq k \leq L, \\ \mathfrak{u}\tau(k-L), & \text{if } L+1 \leq k \leq 2L. \end{cases}$$

But this now follows immediately from Lemma 1. Therefore,

$$\prod_{i<j}(x_j - x_i)^{\beta} = \left[\mathrm{PF}_R\left(\sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{NL}} A_{\mathfrak{t}}\,\vec{e}_{\mathfrak{t}}\right)\right](x_1, \ldots, x_N) = \mathrm{PF}_R(\omega(x) \wedge \omega(y))$$

Again, note that the multiplication in $\omega(x) \wedge \omega(y)$ is performed pointwise in $\bigwedge_{\mathbb{R}} V$, while the multiplication used to obtained the hyperpfaffian of the resulting form is performed in $\bigwedge_R V$.

Now, by applying the functional $\langle \cdot \rangle$ to both sides above and appealing to Chen's Lemma,

$$\begin{aligned} Z_N(\beta) &= \left\langle \prod_{i<j} (x_j - x_i)^\beta \right\rangle \\ &= \big\langle \mathrm{PF}_R(\omega(x) \wedge \omega(y)) \big\rangle \\ &= \mathrm{PF}(\langle \omega(x) \wedge \omega(y) \rangle) \\ &= \mathrm{PF}\left(\sum_{\mathfrak{s}:\underline{L}\nearrow\underline{NL}} \sum_{\mathfrak{u}:\underline{L}\nearrow\underline{NL}} \left\{ \iint_{x<y} \mathrm{Wr}(\mathbf{p}_{\mathfrak{s}}(x)) \mathrm{Wr}(\mathbf{p}_{\mathfrak{u}}(y))\, d\mu(x) d\mu(y) \right\} \vec{e}_{\mathfrak{s}} \wedge \vec{e}_{\mathfrak{u}} \right) \end{aligned}$$

The Case when $\beta = L^2$ and N are Odd

With some finesse, the final case can be handled much like the others. Suppose $N = 2M + 1$, and let $\widetilde{N} = N + 1$. As before, expand the confluent Vandermonde determinant as a sum of products,

$$\prod_{i<j} (x_j - x_i)^\beta = \det \mathbf{V}_{\mathbf{p}}^L(\mathbf{x}) = \sum_{\sigma \in S_{LN}} \mathrm{sgn}(\sigma) \prod_{n=1}^{N} \left[\prod_{\ell=1}^{L} D^{\ell-1} p_{\sigma((n-1)L+\ell)}(x_n) \right].$$

Define a collection of single-variable functions $\{Q_{\mathfrak{t}} \,|\, , \mathfrak{t} : \underline{L} \to \underline{NL}\}$ by $Q_{\mathfrak{t}}(x) = \prod_{\ell=1}^{L} D^{\ell-1} p_{\mathfrak{t}(\ell)}(x)$, and use properties of the tensor algebra to express the right side of the equation above as a function F of N variables,

$$F = \sum_{\sigma \in S_{LN}} \mathrm{sgn}(\sigma) \left[\bigotimes_{n=1}^{N} Q_{\sigma_n} \right].$$

Unfortunately, since L is not even, we cannot immediately apply Lemma 5. Moreover, since N also is not even, we cannot apply the same trick we used previously to group functions two variables at a time. Instead, we will add an auxiliary variable x_{N+1} to the mix in a way that ultimately does not change the fundamental integral in the partition function, allowing us to express the confluent Vandermonde in a form similar to the one that appears in the case when L is odd and N is even.

Let $\mathbb{1}$ denote the constant function $\mathbb{1}(x) = 1$ and consider the $\widetilde{N}$-variable function, $F \sqcup \mathbb{1}$. Importantly, since μ is assumed to be a probability measure, then $\langle \mathbb{1} \rangle = 1$ and so by Chen's Lemma,

$$\langle F \rangle = \langle F \rangle \langle \mathbb{1} \rangle = \langle F \sqcup \mathbb{1} \rangle. \tag{3.7}$$

We wish to extend the collection $\{Q_{\mathfrak{t}}\}$ to a collection of two-variable functions $\{\widetilde{Q}_{\mathfrak{t}} \,|\, \mathfrak{t} : \underline{2L} \to \underline{\tilde{N}L}\}$ in order to express $F \sqcup \mathbb{1}$ in terms of $\{\widetilde{Q}_{\mathfrak{t}}\}$. For each $\mathfrak{t} : \underline{2L} \to \underline{\tilde{N}L}$, let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ denote the restriction of $\mathfrak{t}$ to $\underline{L}$ and $\underline{2L} \setminus \underline{L}$, respectively. We define $\{\widetilde{Q}_{\mathfrak{t}}\}$ as follows:

$$\widetilde{Q}_{\mathfrak{t}} = \begin{cases} Q_{\mathfrak{t}_1} \otimes Q_{\mathfrak{t}_2}, & \text{if } \mathfrak{t} \subset \underline{NL}, \\ Q_{\mathfrak{t}_1} \otimes \mathbb{1}, & \text{if } \mathfrak{t}_1 \subset \underline{NL}, \mathfrak{t}_2 = \mathfrak{i}', \\ -\mathbb{1} \otimes Q_{\mathfrak{t}_2}, & \text{if } \mathfrak{t}_1 = \mathfrak{i}', \mathfrak{t}_2 \subset \underline{NL}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathfrak{i}'$ denotes the increasing function on L integers whose image is $\underline{\tilde{N}L} \setminus \underline{NL}$. We claim that

$$F \sqcup \mathbb{1} = \sum_{\sigma \in S_{\tilde{N}L}} \operatorname{sgn}(\sigma) \bigotimes_{n=1}^{\tilde{N}/2} \widetilde{Q}_{\sigma_n}, \tag{3.8}$$

To verify 3.8, observe that for each permutation $\sigma \in S_{NL}$ with $\sigma = (\sigma_1|\ldots|\sigma_N)$ and each integer $1 \leq k < N$, let

$$T_\sigma^k = \mathrm{sgn}(\sigma)\left[\bigotimes_{n=1}^{k} Q_{\sigma_n}\right] \otimes \mathbb{1} \otimes \left[\bigotimes_{n=k+1}^{N} Q_{\sigma_n}\right]$$

and let

$$T_\sigma^0 = \mathrm{sgn}(\sigma)\mathbb{1} \otimes \left[\bigotimes_{n=1}^{N} Q_{\sigma_n}\right] \qquad \text{and} \qquad T_\sigma^N = \mathrm{sgn}(\sigma)\left[\bigotimes_{n=1}^{N} Q_{\sigma_n}\right] \otimes \mathbb{1}.$$

Then by the definition of the shuffle product, we have

$$F \sqcup\!\sqcup \mathbb{1} = \sum_{\sigma \in S_{NL}} \sum_{k=0}^{N} T_\sigma^k.$$

Now, if k is even, then

$$T_\sigma^k = \widetilde{Q}_{(\sigma_1|\sigma_2)} \otimes \cdots \otimes \widetilde{Q}_{(\sigma_{k-1}|\sigma_k)} \otimes -\widetilde{Q}_{(\mathbb{i}'|\sigma_{k+1})} \otimes \cdots \otimes \widetilde{Q}_{(\sigma_{N-1}|\sigma_N)},$$

while if k is odd, then

$$T_\sigma^k = \widetilde{Q}_{(\sigma_1|\sigma_2)} \otimes \cdots \otimes \widetilde{Q}_{(\sigma_{k-2}|\sigma_{k-1})} \otimes \widetilde{Q}_{(\sigma_k|\mathbb{i}')} \otimes \cdots \otimes \widetilde{Q}_{(\sigma_{N-1}|\sigma_N)},$$

For each $0 \leq k \leq N$, let $\pi^k \in S_{\widetilde{N}L}$ with $\pi^k = (\sigma_1|\ldots|\sigma_k|\mathbb{i}'|\sigma_{k+1}|\ldots|\sigma_N)$ and note that as L is odd and that as each σ_n is a restriction to L integers, then $\mathrm{sgn}(\pi^k) = (-1)^{N-k}\mathrm{sgn}(\sigma)$. Hence, writing $\pi^k = (\pi_1|\ldots|\pi_{\widetilde{N}/2})$, then

$$T_\sigma^k = \mathrm{sgn}(\pi^k)\widetilde{Q}_{\pi_1^k} \otimes \cdots \otimes \widetilde{Q}_{\pi_{\widetilde{N}/2}^k},$$

and so

$$F \sqcup \mathbb{1} = \sum_{\sigma \in S_{NL}} \sum_{k=0}^{N} \operatorname{sgn}(\pi^k) \widetilde{Q}_{\pi_1^k} \otimes \cdots \otimes \widetilde{Q}_{\pi^k_{\widetilde{N}/2}}.$$

On the other hand, if $\pi \in S_{\widetilde{N}L}$ is not of the form $(\sigma_1|\ldots|\sigma_k|\mathfrak{i}'|\sigma_{k+1}|\ldots|\sigma_N)$ for some $\sigma \in S_{NL}$ and $0 \le k \le N$, then $\widetilde{Q}_{\pi_n} = 0$ for some $1 \le n \le \widetilde{N}/2$. It follows that

$$F \sqcup \mathbb{1} = \sum_{\sigma \in S_{\widetilde{N}L}} \operatorname{sgn}(\sigma) \bigotimes_{n=1}^{\widetilde{N}/2} \widetilde{Q}_{\sigma_n},$$

as desired.

Now, for each increasing $\mathfrak{t} : \underline{2L} \nearrow \underline{\widetilde{N}L}$, define $A_{\mathfrak{t}}$ by

$$A_{\mathfrak{t}} = \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \widetilde{Q}_{\mathfrak{t} \circ \tau}.$$

Let $\vec{e}_{\mathfrak{i}'} = \vec{e}_{NL+1} \wedge \cdots \wedge \vec{e}_{\widetilde{N}L} \in \bigwedge_R^L (V \oplus R^L)$ and for each $x, y \in \mathbb{R}$, define $\widetilde{\omega}_1(x), \widetilde{\omega}_2(y) \in \bigwedge_{\mathbb{R}}^L (V \oplus R^L)$ by

$$\widetilde{\omega}_1(x) = \omega(x) - \vec{e}_{\mathfrak{i}'} \qquad \widetilde{\omega}_2(y) = \omega(y) + \vec{e}_{\mathfrak{i}'}.$$

We claim

$$\sum_{\mathfrak{t}:\underline{2L} \nearrow \underline{\widetilde{N}L}} A_{\mathfrak{t}} \, \vec{e}_{\mathfrak{t}}.$$

To verify this, first observe that $A_{\mathfrak{t}} = 0$, unless either $\mathfrak{t} \subset \underline{NL}$ or $\mathfrak{t} = (\mathfrak{t}_1|\mathfrak{i}')$ with $\mathfrak{t}_1 \subset \underline{NL}$.

If $\mathfrak{t} \subset \underline{NL}$, then for each $\tau \in S_{2L}$,

$$\widetilde{Q}_{\mathfrak{t} \circ \tau} = Q_{\mathfrak{t} \circ \tau_1} \otimes Q_{\mathfrak{t} \circ \tau_2}$$

and so

$$\sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{NL}} A_{\mathfrak{t}}\,\vec{e}_{\mathfrak{t}} = \omega(x)\wedge\omega(y) \tag{3.9}$$

by equation 3.6.

On the other hand, suppose $\mathfrak{t} = (\mathfrak{t}_1|\mathfrak{i}')$ with $\mathfrak{t}_1 \subset \underline{NL}$. If $\tau \in S_{2L}$ with $\tau = (\tau_1|\tau_2)$, then $Q_{\mathfrak{t}\circ\tau} = 0$ unless either τ_1 is an increasing function with image $\underline{2L}\setminus\underline{L}$ or τ_2 is the identity function with image $\underline{2L}\setminus\underline{L}$. In the former case,

$$\widetilde{Q}_{\mathfrak{t}\circ\tau} = -\mathbb{1}\otimes Q_{\mathfrak{t}_2\circ\tau_2},$$

while in the latter case,

$$\widetilde{Q}_{\mathfrak{t}\circ\tau} = Q_{\mathfrak{t}_1\circ\tau_1}\otimes\mathbb{1}.$$

Thus, by equation 3.5,

$$A_{(\mathfrak{t}_1|\mathfrak{i}')} = \mathrm{Wr}(\mathbf{p}_{\mathfrak{t}_1})\otimes\mathbb{1} + \mathbb{1}\otimes\mathrm{Wr}(\mathbf{p}_{\mathfrak{t}_1}). \tag{3.10}$$

Therefore, combining equations 3.9 and 3.10 above, we have

$$\widetilde{\omega}_1(x)\wedge\widetilde{\omega}_2(y) = \omega(x)\wedge\omega(y) + [\omega(x)+\omega(y)]\wedge\vec{e}_{\mathfrak{i}'} = \sum_{\mathfrak{t}:\underline{2L}\nearrow\underline{\widetilde{N}L}} A_{\mathfrak{t}}\,\vec{e}_{\mathfrak{t}}. \tag{3.11}$$

As in the previous case when L was odd and N was even, we work over the ring $R = T(H_2)$ of tensors of two-variable functions. By combining equations 3.8

and 3.11 with Lemma 5,

$$F \sqcup \mathbb{1} = \mathrm{PF}_R \left(\sum_{\mathfrak{t}:\underline{2L} \nearrow \underline{NL}} A_{\mathfrak{t}}\, \vec{e}_{\mathfrak{t}} \right) = \mathrm{PF}_R\big(\widetilde{\omega}_1(x) \wedge \widetilde{\omega}_2(y)\big).$$

Applying the linear functional $\langle \cdot \rangle$ to both sides above and appealing to Chen's Lemma and equation 3.7, we have

$$\begin{aligned} Z_N(\beta) = & \left\langle \prod_{i<j} (x_j - x_i)^\beta \right\rangle \\ =& \langle F \rangle \\ =& \langle F \sqcup \mathbb{1} \rangle \\ =& \Big\langle \mathrm{PF}_R\big(\widetilde{\omega}_1(x) \wedge \widetilde{\omega}_2(y)\big) \Big\rangle \\ =& \mathrm{PF}\left(\langle \widetilde{\omega}_1(x) \wedge \widetilde{\omega}_2(y) \rangle \right). \end{aligned}$$

Finally, since

$$\omega(x) + \omega(y) = \omega \sqcup \mathbb{1}$$

then by Chen's Lemma,

$$\langle [\omega(x) + \omega(y)] \wedge \vec{e}_{\mathfrak{i}'} \rangle = \langle (\omega \sqcup \mathbb{1}) \wedge \vec{e}_{\mathfrak{i}'} \rangle = \langle \omega \wedge \vec{e}_{\mathfrak{i}'} \rangle,$$

which shows

$$
\begin{aligned}
Z_N(\beta) =& \mathrm{PF}\left(\langle \widetilde{\omega}_1(x) \wedge \widetilde{\omega}_2(y) \rangle\right) \\
=& \mathrm{PF}\left(\langle \omega(x) \wedge \omega(y) \rangle + \langle [\omega(x) + \omega(y)] \wedge \vec{e}_{\mathrm{i}'} \rangle\right) \\
=& \mathrm{PF}\left(\langle \omega(x) \wedge \omega(y) \rangle + \langle \omega \wedge \vec{e}_{\mathrm{i}'} \rangle\right) \\
=& \mathrm{PF}\left(\iint_{x<y} \omega(x) \wedge \omega(y)\, d\mu(x) d\mu(y) + \int \omega(x) \wedge \vec{e}_{\mathrm{i}'}\, d\mu(x)\right),
\end{aligned}
$$

as desired.

CHAPTER IV

PFAFFIAN AND HYPERPFAFFIAN IDENTITIES

Bra-Ket Notation

In order to facilitate several calculations using matrices and operators, we will adopt the following notation inspired by the "Bra-Ket" notation from quantum physics, first introduction by P. Dirac in [9].

For present purposes, we use the Ket, $|v\rangle$, to denote a column vector labeled by v, and the Bra, $\langle w|$, to denote a row vector labeled by w. When we need to discuss coordinates of row or column vectors, we use the indices themselves for the labels of the vector. For example,

$$\langle v| = \begin{pmatrix} v_1 & \dots & v_N \end{pmatrix} = v_1\langle 1| + \cdots + v_N\langle N|$$

We may also multiply a bra and a ket together, and doing so corresponds to computing the inner product of two vectors:

$$\langle v|\,|w\rangle = \langle v|w\rangle = \sum_{i=1}^{N} v_i w_i.$$

The conjugate transpose of a bra is a ket, and vice versa:

$$\langle v|^T = |v\rangle, \qquad |v\rangle^T = \langle v|.$$

Finally, we may also use bra-kets with linear operators. If $\mathbf{A}$ is an $N \times M$ matrix, $\langle v|$ is an N-dimensional row vector, and $|w\rangle$ is an M-dimensional column vector, then $\mathbf{A}|w\rangle$ denotes the matrix product of $\mathbf{A}$ and $|w\rangle$, and $\langle v|\mathbf{A}$ denotes the matrix product of $\langle v|$ and $\mathbf{A}$, or

$$\mathbf{A}|w\rangle = \sum_{i=1}^{N}\left(\sum_{j=1}^{N}\mathbf{A}_{ij}w_j\right)|i\rangle \qquad \langle v|\mathbf{A} = \sum_{j=1}^{M}\left(\sum_{i=1}^{N}v_i\mathbf{A}_{ij}\right)\langle j|.$$

In this way, $\langle n|\mathbf{A}|m\rangle$ denotes the (n,m)-entry of the matrix $\mathbf{A}$, while if v, w are labels of vectors, then $\langle v|\mathbf{A}|w\rangle$ denotes the pairing of v and w with respect to the bilinear form given by the matrix $\mathbf{A}$.

Observe, in particular, that if $\mathbf{A}$ is an $N\times L$ matrix and $\mathbf{B}$ is an $L\times N$ matrix, then

$$\langle n|\mathbf{AB}|m\rangle = \sum_{k=1}^{L}\langle n|\mathbf{A}|k\rangle\langle k|\mathbf{B}|m\rangle$$

which, if nothing else, evinces the beautiful efficiency of bra-ket notation.

Properties of Pfaffians

The Relationship between Pfaffians and Determinants

The following property can be used to give a coordinate-free definition of the determinant of a linear transformation.

Theorem 12. *Let $\mathbf{A}$ be a linear transformation $\mathbf{A} : V \to V$ of a rank N free R-module V, and let $\{\vec{e}_1, \ldots, \vec{e}_N\}$ be a basis for V. Then*

$$\mathbf{A}\vec{e}_1 \wedge \cdots \wedge \mathbf{A}\vec{e}_N = \det(\mathbf{A})\,\vec{e}_1 \wedge \cdots \wedge \vec{e}_N.$$

Suppose $\mathfrak{t} : \underline{2k} \nearrow \underline{2N}$ and $\mathfrak{u} : \underline{2\ell} \nearrow \underline{2N}$ are increasing functions. Let $\mathfrak{t}\mathbf{A}\mathfrak{u}$ denote the $2k \times 2\ell$ minor of $\mathbf{A}$ so that

$$\langle m|\mathfrak{t}\mathbf{A}\mathfrak{u}|n\rangle = \langle \mathfrak{t}(m)|\mathbf{A}|\mathfrak{u}(n)\rangle.$$

The following theorem generalizes the previous result, and is connected to the *Grassmann embedding* in the study of algebraic geometry, as detailed in [15].

Theorem 13. *Let V be a rank N free R-module with basis $\{\vec{v}_1, \dots, \vec{v}_N\}$, let W be a rank L free R-module with basis $\{\vec{w}_1, \dots, \vec{w}_L\}$, and let* $\mathbf{C}$ *be the matrix of a linear transformation $V \to W$. Then for any $J \leq N$, and increasing function $\mathfrak{u} : \underline{J} \nearrow \underline{N}$,*

$$\mathbf{C}\vec{v}_{\mathfrak{u}(1)} \wedge \cdots \wedge \mathbf{C}\vec{v}_{\mathfrak{u}(J)} = \sum_{\mathfrak{t}:\underline{J}\nearrow\underline{L}} \det(\mathfrak{t}\mathbf{C}\mathfrak{u})\, \vec{w}_{\mathfrak{t}(1)} \wedge \cdots \wedge \vec{w}_{\mathfrak{t}(J)}.$$

Proof. We prove the case when $\mathfrak{u} : \underline{J} \nearrow \underline{L}$ is the increasing function $\mathfrak{u}(j) = j$ for $1 \leq j \leq J$. The general case follows from an analogous argument. Write

$$\mathbf{C}\vec{v}_i = \sum_{\ell=1}^{L} \langle \ell | \mathbf{C} | k \rangle\, \vec{w}_\ell$$

and observe

$$\begin{aligned}\mathbf{C}\vec{v}_1 \wedge \cdots \wedge \mathbf{C}\vec{v}_J &= \sum_{\ell_1=1}^{L} \cdots \sum_{\ell_J=1}^{L} \langle \ell_1 | \mathbf{C} | 1 \rangle \cdots \langle \ell_J | \mathbf{C} | J \rangle\, \vec{w}_1 \wedge \cdots \wedge \vec{w}_{\ell_J} \\ &= \sum_{\mathfrak{j}:\underline{J}\to\underline{L}} \prod_{j=1}^{J} \langle \mathfrak{j}(j) | \mathbf{C} | j \rangle\, \vec{w}_{\mathfrak{j}(1)} \wedge \cdots \wedge \vec{w}_{\mathfrak{j}(J)}.\end{aligned}$$

But by Lemma 1, the above sum can be expressed as

$$\sum_{\mathfrak{t}:\underline{J}\nearrow\underline{L}} \sum_{\sigma\in S_N} \operatorname{sgn}(\sigma) \prod_{j=1}^{J} \langle \mathfrak{t} \circ \sigma(j) | \mathbf{C} | j \rangle\, \vec{w}_{\mathfrak{t}(1)} \wedge \cdots \wedge \vec{w}_{\mathfrak{t}(J)} = \sum_{\mathfrak{t}:\underline{J}\nearrow\underline{L}} \det(\mathfrak{t}\mathbf{C}\mathfrak{u})\, \vec{w}_{\mathfrak{t}(1)} \wedge \cdots \wedge \vec{w}_{\mathfrak{t}(J)},$$

giving the desired result. □

Theorem 14. *Let* $\mathbf{A}$ *be a $2N \times 2N$ antisymmetric matrix and* $\mathbf{B}$ *be a $2N \times 2N$ matrix. Then*

1. $\mathrm{Pf}(\mathbf{B}\mathbf{A}\mathbf{B}^T) = \mathrm{Pf}(\mathbf{A})\det(\mathbf{B})$.

2. $\mathrm{Pf}(\mathbf{A})^2 = \det(\mathbf{A})$.

Proof. For the first identity, let $\omega, \nu \in \bigwedge^2 V$ be given by

$$\omega = \sum_{i<j} \langle i|\mathbf{B}\mathbf{A}\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j$$

and

$$\nu = \sum_{i<j} \langle i|\mathbf{A}|j\rangle \mathbf{B}\vec{e}_i \wedge \mathbf{B}\vec{e}_j$$

Observe that

$$\begin{aligned}\langle i|\mathbf{A}|j\rangle \mathbf{B}\vec{e}_i \wedge \mathbf{B}\vec{e}_j =& \langle i|\mathbf{A}|j\rangle \left(\sum_k \langle k|\mathbf{B}|i\rangle \vec{e}_k\right) \wedge \left(\sum_\ell \langle \ell|\mathbf{B}|j\rangle \vec{e}_\ell\right) \\ =& \sum_{k\neq\ell} \langle k|\mathbf{B}|i\rangle \langle i|\mathbf{A}|j\rangle \langle j|\mathbf{B}^T|\ell\rangle \vec{e}_k \wedge \vec{e}_\ell.\end{aligned}$$

Then the $\vec{e}_i \wedge \vec{e}_j$ term in ν is

$$\begin{aligned}&\sum_{a<b} \langle i|\mathbf{B}|a\rangle \langle a|\mathbf{A}|b\rangle \langle b|\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j + \sum_{a<b} \langle j|\mathbf{B}|a\rangle \langle a|\mathbf{A}|b\rangle \langle b|\mathbf{B}^T|i\rangle \vec{e}_j \wedge \vec{e}_i \\ &= \sum_{a<b} \langle i|\mathbf{B}|a\rangle \langle a|\mathbf{A}|b\rangle \langle b|\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j - \sum_{a<b} \langle a|\mathbf{B}^T|j\rangle \langle b|\mathbf{A}^T|a\rangle \langle i|\mathbf{B}|b\rangle \vec{e}_i \wedge \vec{e}_j \\ &= \sum_{a<b} \langle i|\mathbf{B}|a\rangle \langle a|\mathbf{A}|b\rangle \langle b|\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j + \sum_{a<b} \langle i|\mathbf{B}|b\rangle \langle b|\mathbf{A}|a\rangle \langle a|\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j \\ &= \sum_{a\neq b} \langle i|\mathbf{B}|a\rangle \langle a|\mathbf{A}|b\rangle \langle b|\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j\end{aligned}$$

On the other hand,

$$\langle i|\mathbf{B}\mathbf{A}\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j = \sum_{k\neq\ell} \langle i|\mathbf{B}|k\rangle \langle k|\mathbf{A}|\ell\rangle \langle \ell|\mathbf{B}^T|j\rangle \vec{e}_i \wedge \vec{e}_j.$$

Hence, $\omega = \nu$. Taking the N^{th} exterior power of each two form,

$$N!\mathrm{Pf}(\mathbf{BAB}^T)\vec{e}_{\text{vol}} = \omega^{\wedge N} = \nu^{\wedge N} = N!\mathrm{Pf}(\mathbf{A})\det(\mathbf{B})\vec{e}_{\text{vol}},$$

as desired.

Now, by Theorem 4, there exists an $2N \times 2N$ matrix $\mathbf{B}$ so that the 2-form associated to $\mathbf{BAB}^T$ is

$$\omega = \sum_{i=1}^{n} \lambda_i \vec{e}_{2i-1} \wedge \vec{e}_{2i}.$$

Hence,

$$\mathrm{Pf}(\mathbf{A})\det(\mathbf{B}) = \mathrm{Pf}(\mathbf{BAB}^T) = \prod_i \lambda_i.$$

Squaring both sides,

$$\mathrm{Pf}(\mathbf{A})^2 \det(\mathbf{BB}^T) = \prod_i \lambda_i^2.$$

On the other hand, since $\mathbf{BAB}^T$ is in block diagonal form, then $\det(\mathbf{BAB}^T)$ can be readily computed, and

$$\prod_i \lambda_i^2 = \det(\mathbf{BAB}^T) = \det(\mathbf{A})\det(\mathbf{BB}^T).$$

It follows that $\mathrm{Pf}(\mathbf{A})^2 = \det(\mathbf{A})$. $\square$

As part of the preceding proof, we also proved the following result:

Corollary 6. *Let $\mathbf{A}$ be an $N \times N$ antisymmetric matrix, let $\mathbf{C}$ be an $L \times N$ matrix, let $\{\vec{v}_1, \ldots, \vec{v}_N\}$ be a basis for a rank N free R-module V, and let $\{\vec{w}_1, \ldots, \vec{w}_L\}$ be a basis for a rank L free R-module W. If $\gamma \in \bigwedge^2 W$ with*

$$\gamma = \sum_{i<j} \langle i|\mathbf{CAC}^T|j\rangle \, \vec{w}_i \wedge \vec{w}_j$$

then

$$\gamma = \sum_{i<j} \langle i|\mathbf{A}|j\rangle \, (\mathbf{C}\vec{v}_i) \wedge (\mathbf{C}\vec{v}_j).$$

The Laplace Formulas

Theorem 15. *For any* $1 \leq k \leq N$*,*

$$\frac{\omega^{\wedge k}}{k!} = \sum_{\mathfrak{t}:\underline{2k}\nearrow\underline{2N}} \mathrm{Pf}(\mathbf{t}\mathbf{A}\mathbf{t})\vec{e}_{\mathfrak{t}}$$

Proof. Relabel indices according to $\mathbf{t}$ and apply Corollary 2. $\square$

For any $\alpha \in \bigwedge^L V$ for $L \geq 1$, let $\exp(\alpha) \in \bigwedge V$ be given by

$$\exp(\alpha) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$$

Since V is of finite rank, then α is nilpotent, and so $\exp(\alpha)$ is actually a finite sum.

Corollary 7. *Suppose* $\omega \in \bigwedge^2 V$ *and let* $\mathbf{A}$ *be the* $2N \times 2N$ *antisymmetric matrix associated with* ω*. Then*

$$\exp(\omega) = \sum_{k=0}^{N} \sum_{\mathfrak{t}:\underline{2k}\nearrow\underline{2N}} \mathrm{Pf}(\mathbf{t}\mathbf{A}\mathbf{t})\vec{e}_{\mathfrak{t}}$$

Theorem 16 (Laplace Formula)**.** *Suppose* $\mathbf{A}$ *is a* $2N \times 2N$ *antisymmetric matrix. For any* $1 \leq m < n \leq 2N$*, let* $\tau_{m,n} : \underline{2} \nearrow \underline{2N}$ *denote the increasing function*

$1 \mapsto m, 2 \mapsto n$. *Then for any* $1 \leq n \leq 2N$,

$$\begin{aligned}\mathrm{Pf}(\mathbf{A}) = &\sum_{m=1}^{n-1} \mathrm{sgn}(\tau_{m,n})\langle m|\mathbf{A}|n\rangle \mathrm{Pf}(\tau'_{m,n}\mathbf{A}\tau'_{m,n}) \\ &+ \sum_{m=n+1}^{2N} \mathrm{sgn}(\tau_{n,m})\langle n|\mathbf{A}|m\rangle \mathrm{Pf}(\tau'_{n,m}\mathbf{A}\tau'_{n,m})\end{aligned}$$

In the special case when $n = 1$*, the Laplace expansion reduces to the following simpler expression:*

$$\mathrm{Pf}(\mathbf{A}) = \sum_{m=2}^{2N} (-1)^m \langle 1|\mathbf{A}|m\rangle \mathrm{Pf}(\tau'_{1,m}\mathbf{A}\tau'_{1,m})$$

Proof. Let ω be the 2-form corresponding to $\mathbf{A}$. Consider $\frac{\omega^N}{N!} = \frac{\omega}{N} \wedge \frac{\omega^{N-1}}{(N-1)!}$ and apply the previous theorem. $\square$

Theorem 17 (Generalized Laplace Formula)**.** *Suppose* $\mathbf{A}$ *is an antisymmetric* $2N \times 2N$ *matrix, and that* $K < N$*. Then*

$$\mathrm{Pf}(\mathbf{A}) = \frac{1}{\binom{N}{K}} \sum_{\mathfrak{t}:\underline{2K} \nearrow \underline{2N}} \mathrm{sgn}(\mathfrak{t})\mathrm{Pf}(\mathfrak{t}\mathbf{A}\mathfrak{t})\mathrm{Pf}(\mathfrak{t}'\mathbf{A}\mathfrak{t}')$$

Proof. Use the Laplace formula and induct on k. $\square$

The Cauchy-Binet Formula

The following theorem (first due to M. Ishikawa and M. Wakayama in [16]) is reminiscent of the Cauchy-Binet Formula for determinants, and sometimes goes under the name of *Minor Summation Formula.*

Theorem 18 (The Pfaffian Cauchy-Binet Formula)**.** *Suppose* $L \leq N$ *and that* $L = 2K$ *and* $N = 2M$*. Let* A *be an* $N \times N$ *antisymmetric matrix and let* $\mathbf{C}$ *be an*

$L \times N$ matrix. Then

$$\mathrm{Pf}(\mathbf{C}\mathbf{A}\mathbf{C}^T) = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \mathrm{Pf}(\mathfrak{t}\mathbf{A}\mathfrak{t})\det(\mathbf{C}\mathfrak{t}).$$

Proof. Suppose V is a rank N free R-module with basis $\{\vec{v}_1, \ldots, \vec{v}_N\}$, and suppose W is a rank L free R-module with basis $\{\vec{w}_1, \ldots, \vec{w}_L\}$. Let $\alpha \in \bigwedge^2 V$ be the antisymmetric form associated with $\mathbf{A}$, and let $\beta \in \bigwedge^2 W$ be the antisymmetric form associated with $\mathbf{C}\mathbf{A}\mathbf{C}^T$. By Corollary 6,

$$\gamma = \sum_{i<j} \langle i|\mathbf{C}\mathbf{A}\mathbf{C}^T|j\rangle\, \vec{w}_i \wedge \vec{w}_j = \sum_{i<j} \langle i|\mathbf{A}|j\rangle\, (\mathbf{C}\vec{v}_i) \wedge (\mathbf{C}\vec{v}_j),$$

and so by the Theorem 15, along with Theorem 13,

$$\frac{\gamma^{\wedge K}}{K!} = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \mathrm{Pf}(\mathfrak{t}\mathbf{A}\mathfrak{t})\, (\mathbf{C}\vec{v}_{\mathfrak{t}(1)}) \wedge \cdots \wedge (\mathbf{C}\vec{v}_{\mathfrak{t}(L)}) = \sum_{\mathfrak{t}:\underline{L}\nearrow\underline{N}} \mathrm{Pf}(\mathfrak{t}\mathbf{A}\mathfrak{t})\det(\mathbf{C}\mathfrak{t})\, \vec{w}_1 \wedge \cdots \wedge \vec{w}_L$$

□

The exterior algebra gives a particularly convenient expression for the Pfaffian of the sum of two antisymmetric matrices.

Theorem 19 (Pfaffian Summation Formula)**.** *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are antisymmetric* $2N \times 2N$ *matrices. Then*

$$\mathrm{Pf}(\mathbf{A} + \mathbf{B}) = \sum_{k=0}^{N} \sum_{\mathfrak{t}:\underline{2k}\nearrow\underline{2N}} \mathrm{sgn}(\mathfrak{t})\mathrm{Pf}(\mathfrak{t}\mathbf{A}\mathfrak{t})\mathrm{Pf}(\mathfrak{t}'\mathbf{B}\mathfrak{t}')$$

Proof. Let α and β be the 2-forms corresponding to $\mathbf{A}$ and $\mathbf{B}$, respectively. By Theorem 15,

$$\begin{aligned}
\frac{1}{N!}(\alpha+\beta)^N &= \frac{1}{N!}\sum_{k=0}^{N}\binom{N}{k}\alpha^{\wedge k}\wedge\beta^{\wedge(N-k)} \\
&= \sum_{k=0}^{N}\left\{\frac{1}{k!}\alpha^{\wedge k}\right\}\wedge\left\{\frac{1}{(N-k)!}\beta^{\wedge(N-k)}\right\} \\
&= \sum_{k=0}^{N}\left\{\sum_{\mathrm{t}:\underline{2k}\nearrow\underline{2N}}\mathrm{Pf}(\mathbf{tAt})\vec{e}_{\mathrm{t}}\right\}\wedge\left\{\sum_{\mathrm{t}:\underline{2N-2k}\nearrow\underline{2N}}\mathrm{Pf}(\mathbf{tBt})\vec{e}_{\mathrm{t}}\right\} \\
&= \sum_{k=0}^{N}\sum_{\mathrm{t}:\underline{2k}\nearrow\underline{2N}}\mathrm{sgn}(\mathbf{t})\mathrm{Pf}(\mathbf{tAt})\mathrm{Pf}(\mathbf{t'Bt'})\vec{e}_{\mathrm{vol}}
\end{aligned}$$

as desired. $\square$

The Jacobi Minor Inverse Formula

For general matrices, the Jacobi Minor Inverse formula expresses a relationship between the determinant of a minor of the matrix, and the determinant of the complementary minor of the inverse transpose of the matrix. For invertible antisymmetric matrices, a similar identity can be obtained involving the Pfaffians of minors of the matrix.

For any $1 \le i < j \le 2N$, let $\tau_{ij} : \underline{2} \nearrow \underline{2N}$ denote the increasing function $1 \mapsto i, 2 \mapsto j$, and let τ'_{ij} denote the complementary index function. We define the *Pfaffian cofactor matrix* of an antisymmetric matrix $\mathbf{A}$ by

$$\langle m|\overline{\mathbf{A}}|n\rangle = \begin{cases} -\mathrm{sgn}(\tau_{m,n})\mathrm{Pf}(\tau'_{m,n}\mathbf{A}\tau'_{m,n}), & \text{if } m < n \\ \mathrm{sgn}(\tau_{n,m})\mathrm{Pf}(\tau'_{n,m}\mathbf{A}\tau'_{n,m}), & \text{if } m > n, \\ 0, & \text{if } m = n \end{cases}$$

In particular, the cofactor matrix is the antisymmetric matrix whose (m, n)-entry is given as the signed Pfaffian of the $(2N-2)\times(2N-2)$ minor formed by deleting the mth and nth rows and columns.

Theorem 20 (The Pfaffian Jacobi Minor Inverse Formula)**.** *Let* $\overline{\mathbf{A}}$ *be the Pfaffian cofactor matrix of a* $2N \times 2N$ *antisymmetric matrix* $\mathbf{A}$*. Then*

$$\mathbf{A}\overline{\mathbf{A}} = \mathrm{Pf}(\mathbf{A})I_{2N}.$$

Moreover, if $\mathrm{Pf}(\mathbf{A}) \neq 0$*, then*

$$\mathbf{A}^{-1} = \frac{1}{\mathrm{Pf}(\mathbf{A})}\overline{\mathbf{A}}.$$

Proof. By the Laplace Formula, for any $1 \le n \le 2N$,

$$\begin{aligned}
\mathrm{Pf}(\mathbf{A}) &= \sum_{m=1}^{n-1} \mathrm{sgn}(\tau_{m,n})\langle m|\mathbf{A}|n\rangle \mathrm{Pf}(\tau'_{m,n}\mathbf{A}\tau'_{m,n}) \\
&\quad + \sum_{m=n+1}^{n} \mathrm{sgn}(\tau_{n,m})\langle n|\mathbf{A}|m\rangle \mathrm{Pf}(\tau'_{n,m}\mathbf{A}\tau'_{n,m}) \\
&= \sum_{m=1}^{n-1} -\langle m|\mathbf{A}|n\rangle\langle m|\overline{\mathbf{A}}|n\rangle + \sum_{m=n+1}^{n} \langle n|\mathbf{A}|m\rangle\langle m|\overline{\mathbf{A}}|n\rangle \\
&= \sum_{m=1}^{2N} \langle n|\mathbf{A}|m\rangle\langle m|\overline{\mathbf{A}}|n\rangle \\
&= \sum_{m=1}^{2N} \langle n|\mathbf{A}\overline{\mathbf{A}}|n\rangle
\end{aligned}$$

Hence, the diagonal elements of $\mathbf{A}\overline{\mathbf{A}}$ are all equal to $\mathrm{Pf}(\mathbf{A})$. All that remains is to show that the off-diagonal elements vanish.

To do so, consider the matrix $\mathbf{B}$ obtained by replacing the nth row of $\mathbf{A}$ with the ℓth row of $\mathbf{A}$, and then by replacing the nth column of the resulting matrix with the ℓth column of that matrix. Since $\mathbf{B}$ has two identical columns, then $\det(\mathbf{B}) = 0$, and hence, $\mathrm{Pf}(\mathbf{B}) = 0$.

Note that $\langle n|\mathbf{B}|\ell\rangle = 0$, and that $\langle n|\mathbf{B}|m\rangle = \langle \ell|\mathbf{B}|m\rangle$ and $\langle m|\overline{\mathbf{B}}|n\rangle = \langle m|\overline{\mathbf{A}}|n\rangle$ for any $m \neq \ell, n$. Thus, by the preceding argument,

$$0 = \mathrm{Pf}(\mathbf{B}) = \sum_{m=1}^{2N} \langle n|\mathbf{B}|m\rangle\langle m|\overline{\mathbf{B}}|n\rangle = \sum_{m\neq \ell,n} \langle n|\mathbf{B}|m\rangle\langle m|\overline{\mathbf{B}}|n\rangle = \sum_{m\neq \ell,n} \langle \ell|\mathbf{A}|m\rangle\langle m|\overline{\mathbf{A}}|n\rangle$$

as desired. $\square$

The Sylvester Identities

The following identities are sometimes referred to as the *Schur Formulas for partitioned matrices.*

Theorem 21 (The Determinant Sylvester Identity)**.** *Suppose* $\mathbf{B}$ *is an* $N \times L$ *matrix and* $\mathbf{C}$ *is an* $L \times N$ *matrix. Then*

$$\det(\mathbf{I}_N + \mathbf{BC}) = \det(\mathbf{I}_L + \mathbf{CB})$$

Proof. Consider the $(N + L) \times (N + L)$ block matrix

$$\begin{bmatrix} \mathbf{I}_L & 0 \\ \mathbf{B} & \mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{I}_L & -\mathbf{C} \\ 0 & \mathbf{I}_N + \mathbf{BC} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_L & -\mathbf{C} \\ \mathbf{B} & \mathbf{I}_N \end{bmatrix} = \begin{bmatrix} \mathbf{I}_L + \mathbf{CB} & -\mathbf{C} \\ 0 & \mathbf{I}_N \end{bmatrix} \begin{bmatrix} \mathbf{I}_L & 0 \\ \mathbf{B} & \mathbf{I}_N \end{bmatrix}.$$

Taking determinants of the left and right side above, and noting that the determinant of a block triangular matrix is the product of the determinants of the diagonal blocks, gives the desired result. $\square$

Theorem 22 (The Pfaffian Sylvester Identity)**.** *Suppose* $\mathbf{A}$ *is a* $2N \times 2N$ *antisymmetric matrix, that* $\mathbf{Z}$ *is a* $2K \times 2K$ *antisymmetric matrix, and that* $\mathbf{B}$ *is a* $2N \times 2K$ *matrix. If* $\mathbf{A}$ *and* $\mathbf{Z}$ *are invertible, then*

$$\frac{\mathrm{Pf}(\mathbf{Z}^{-1} + \mathbf{B}^T\mathbf{A}\mathbf{B})}{\mathrm{Pf}(\mathbf{Z}^{-1})} = \frac{\mathrm{Pf}(\mathbf{A}^{-1} + \mathbf{B}\mathbf{Z}\mathbf{B}^T)}{\mathrm{Pf}(\mathbf{A}^{-1})}.$$

Proof. Consider the block matrix products

$$\begin{bmatrix} \mathbf{I}_{2N} & 0 \\ \mathbf{B}^T\mathbf{A} & \mathbf{I}_{2K} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{Z}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{2N} & \mathbf{A}^T\mathbf{B} \\ 0 & \mathbf{I}_{2K} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1} & 0 \\ 0 & \mathbf{Z}^{-1} + \mathbf{B}^T\mathbf{A}\mathbf{B} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathbf{I}_{2N} & -\mathbf{B}\mathbf{Z} \\ 0 & \mathbf{I}_{2K} \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{B} \\ -\mathbf{B}^T & \mathbf{Z}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{2N} & 0 \\ \mathbf{Z}\mathbf{B}^T & \mathbf{I}_{2K} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{B}\mathbf{Z}\mathbf{B}^T & 0 \\ 0 & \mathbf{Z}^{-1} \end{bmatrix}$$

By Theorem 14, the Pfaffian of the left sides of both equations above are equal, and so the Pfaffians of the right sides of both equations above must be equal as well. By the Laplace formula,

$$\mathrm{Pf}(\mathbf{A}^{-1})\mathrm{Pf}(\mathbf{Z}^{-1} + \mathbf{B}^T\mathbf{A}\mathbf{B}) = \mathrm{Pf}(\mathbf{A}^{-1} + \mathbf{B}\mathbf{Z}\mathbf{B}^T)\mathrm{Pf}(\mathbf{Z}^{-1})$$

Dividing both sides above by $\mathrm{Pf}(\mathbf{A}^{-1})$ and $\mathrm{Pf}(\mathbf{Z}^{-1})$ gives the desired result. $\square$

The Wick Formulas

The Wick Formulas (which are also known as the Isserlis Formulas in probability literature) provides a combinatorial method for computing higher order derivatives of multivariate exponential functions. A detailed treatment of the origin and application of the Wick formulas can be found in [25] and [14]. In the study of probability and statistics, the Wick formulas are used to calculate the mixed moments of multivariate Gaussian distributions. The Wick formulas also find frequent application in particle physics and quantum field theory, where they are used to express correlations in many-body problems as a sequence of pairwise interactions. Historically, the Wick formulas take one of two forms (either *bosonic* or *fermionic*), depending on whether the particles in the underling physical system obey the Pauli exclusion principle, and hence, whether the relevant equations contain commuting variables, in the bosonic case, or anticommuting variables, in the fermionic case.

The Bosonic Wick Formula

The Bosonic Wick Formula expresses the moments of a multivariate Gaussian variable in terms of its covariance matrix. Let $\mathbf{X}$ be a centered, multivariate Gaussian N-vector, with density $d\mu^N(\mathbf{x}) = C \exp\left(-\frac{1}{2}\mathbf{x}^T \mathbf{\Sigma}_X^{-1}\mathbf{x}\right) d^N\mathbf{x}$ on $\mathbb{R}^N$ (where $\mathbf{\Sigma}_X$ is the covariance matrix for $\mathbf{X}$). For $n \in \mathbb{N}$, let $f_1, \ldots, f_n$ be linear forms in $x_1, \ldots, x_N$ (so that $f_1 \ldots f_n$ is a homogeneous polynomial of degree n). Define $\langle f \rangle = \int f d\mu(\mathbf{x})$. Then

$$\langle f_1 \cdots f_n \rangle = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \operatorname{Haf}\left(\langle f_i f_j \rangle\right), & \text{if } n \text{ is even.} \end{cases}$$

Here, the *Hafnian*, $\mathrm{Haf}(\mathbf{S})$, of a symmetric $n \times n$ matrix $\mathbf{S}$ of even order $n = 2N$ is

$$\mathrm{Haf}(\mathbf{S}) = \sum_{\sigma \in \mathrm{Sh}^o(\Lambda^N)} \mathbf{S}_{\sigma(1)\sigma(2)} \cdots \mathbf{S}_{\sigma(2N-1),\sigma(2N)}.$$

Of course, there is obvious similarity between the Hafnian of a symmetric matrix and the Pfaffian of an antisymmetric matrix, where the Hafnian is simply the Pfaffian with the signature of the permutation omitted. Remarkably, this omission makes the Hafnian, in practice, significantly more difficult to compute!

To see why the Bosonic Wick formula holds, define a random n-vector $\mathbf{Y}$ by $Y_i = f_i(\mathbf{X})$ and let $\mathbf{A}$ be the matrix so that $\mathbf{Y} = \mathbf{AX}$. Then $\langle f_1 \cdots f_n \rangle = \mathbb{E}[Y_1 \cdots Y_n]$. Since $\mathbf{X}$ is multivariate Gaussian with covariance matrix $\mathbf{K}_X$, then $\mathbf{Y}$ is also multivariate Gaussian, with covariance matrix $\mathbf{K}_Y = \mathbf{A}\mathbf{K}_X\mathbf{A}^T$. Thus, the moment-generating function $m_Y(\mathbf{t})$ of $\mathbf{Y}$ is given by

$$\mathbb{E}\left[\exp\left(\mathbf{t} \cdot \mathbf{Y}\right)\right] = \exp\left(\frac{1}{2}\mathbf{t}'\mathbf{\Sigma}_Y\mathbf{t}\right).$$

Assuming $n = 2N$, expand the nth order terms of both sides:

$$\mathbb{E}\left[\frac{1}{n!}\left(\mathbf{t} \cdot \mathbf{Y}\right)^n\right] = \frac{1}{2^n n!}\left(\mathbf{t}'\mathbf{\Sigma}_Y\mathbf{t}\right)^n.$$

Matching the $t_1 \ldots t_n$ terms gives

$$\mathbb{E}\left[Y_1 \ldots Y_n\right] t_1 \ldots t_n = \left(\sum_{\sigma \in \mathrm{Sh}^o(\Lambda^N)} \mathbb{E}[Y_{\sigma(1)}Y_{\sigma(2)}] \cdots \mathbb{E}[Y_{\sigma(n-1}Y_{\sigma(n)}]\right) t_1 \ldots t_n,$$

as desired

The Berezin Integral

In the following section, it is best to view the exterior algebra as an anticommuting analogue of a polynomial algebra $R[x_1, \ldots, x_n]$, rather than as an extension of an R-module. To emphasize this change in perspective, we adopt the following notation. Let $\eta = \{\eta_1, \ldots, \eta_{NL}\}$ be an ordered set of NL anticommuting variables, let R be a commutative ring with unity, and let $\bigwedge(\eta)$ be the R-algebra generated by these variables. We can, of course, identify $\bigwedge(\eta)$ with $\bigwedge_R V$, where V is a free R-module of rank NL with basis $\{\eta_1, \ldots, \eta_{NL}\}$.

For each i, define $\partial/\partial\eta_i$ to be the left anti-derivation given by

1. $\dfrac{\partial}{\partial\eta_i}\eta_k = \delta_{ik}$.
2. $\dfrac{\partial}{\partial\eta_i}1 = 0$
3. $\dfrac{\partial}{\partial\eta_i}(\alpha\beta) = \dfrac{\partial\alpha}{\partial\eta_i}\beta + (-1)^p\alpha\dfrac{\partial\beta}{\partial\eta_i}$, where $\alpha \in \bigwedge_K^p(\eta)$ and $\beta \in \bigwedge_K^{n-p}(\eta)$.

Alternatively, $\partial/\partial\eta_i$ acts on a word in the η_j by moving η_i to the leftmost position (with sign) and deleting it.

The Berezin (or Grassmann) integral of a linear function f of one anticommuting variable is defined by

$$\int f(\theta)d\theta = \frac{\partial}{\partial\theta}f(\theta)$$

And the Berezin integral of a polynomial f in n anticommuting variables $\eta_1, \ldots, \eta_n$ is defined by

$$\int f(\eta_1, \ldots, \eta_n)d\eta_1 \ldots d\eta_n = \frac{\partial}{\partial\eta_n} \cdots \frac{\partial}{\partial\eta_1}f(\eta_1, \ldots, \eta_n)$$

Let η^T denote the row vector $[\, \eta_1 \cdots \eta_n \,]$. For any anti-symmetric matrix $\mathbf{A}$, the associated 2-form ω is given by

$$\omega = \frac{1}{2}\eta^T \mathbf{A}\eta.$$

And conversely, given any alternating 2-form ω, the associated antisymmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A}_{ij} = \int \omega \, d\eta_i \, d\eta_j.$$

Like the classic integral, there is a change-of-variables formula for the Berezin integral. Let $\mathbf{B}$ be an invertible $N \times N$ matrix and define anticommuting variables $\theta = (\theta_1, \ldots, \theta_N)$ by $\theta = \mathbf{B}\eta$. Then for any polynomial f in n anti-commuting variables θ,

$$\int f(\theta)\, d\theta = \det(\mathbf{B}) \int f(\mathbf{B}(\eta))\, d\eta$$

which follows from the observation that $\theta_i = \mathbf{B}^T \eta_i$ and

$$\theta_1 \cdots \theta_N = (\mathbf{B}^T \eta_1) \cdots (\mathbf{B}^T \eta_N) = \det(\mathbf{B})\eta_1 \cdots \eta_N.$$

The Fermionic Wick Formula

Now, suppose $\omega \in \bigwedge^2 V$, and let $\mathbf{A}$ be the associated antisymmetric matrix. If $\mathbf{A}$ is invertible, we may define the "fermionic Gaussian probability measure" $d\mu(\eta)$ by

$$d\mu(\eta) = C \exp\left(\frac{1}{2}\eta^T \mathbf{A}^{-1}\eta\right) d\eta, \qquad \text{where } C \text{ is such that } \int d\mu(\eta) = 1.$$

For $n \in \mathbb{N}$, let $f_1, \ldots, f_n$ be linear forms in $\eta_1, \ldots, \eta_N$ (so that $f_1 \ldots f_n$ is a homogeneous polynomial of degree n). Note that if $n > N$, then $f_1 \cdots f_n \equiv 0$.

To emphasize the connection between the Fermionic and Bosonic Wick Formulas, define the anticommuting 'expectation' operator $\langle f \rangle$ by

$$\langle f \rangle = \int f(\eta)\, d\mu(\eta).$$

The fermionic Wick formula can then be stated as

$$\langle f_1 \dots f_n \rangle = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \operatorname{Pf}\left(\langle f_i f_j \rangle\right), & \text{if } n \text{ is even.} \end{cases}$$

In the special case when $f_i = \eta_{\mathfrak{t}(i)}$ for $1 \le i \le 2k$ and an increasing function $\mathfrak{t} : \underline{2k} \nearrow \underline{2N}$, then the Wick formula can be stated as

$$\langle f_1 \cdots f_{2k} \rangle = \frac{1}{\operatorname{Pf}(\mathbf{A})} \int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta^T \mathbf{A}^{-1} \eta\right) d\eta = \frac{1}{\operatorname{Pf}(\mathbf{A})} \int \eta_{\mathfrak{t}} \left(\frac{1}{2}\eta^T \mathbf{A}^{-1} \eta\right)^{N-k} d\eta.$$

But this, it turns out, is simply an equivalent formulation for the Jacobi Minor Inverse Formula in Theorem 20. In fact, each of the other Pfaffian identities in this section can be expressed using the Berezin integral. The following theorem summarizes this correspondence, with proofs adapted from those which appear in [5].

Theorem 23 (Berezin Integral Formula for Pfaffian Identities)**.** *Let R be a commutative ring with unity and let $\mathbf{A}$ be a $2N \times 2N$ antisymmetric matrix over R. Then*

1. *(The Gaussian Integral)*

$$\int \exp\left(\frac{1}{2}\eta^T \mathbf{A} \eta\right) d\eta = \operatorname{Pf}\mathbf{A}.$$

2. *(Pfaffian Summation Formula) If* $\mathbf{A}$ *is an invertible* $2N \times 2N$ *antisymmetric matrix and* $\lambda = \left[\begin{smallmatrix} \lambda_1 & \cdots & \lambda_{2N} \end{smallmatrix}\right]$ *are anticommuting variables not contained in* $\bigwedge(\eta)$*, then*

$$\int \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta + \lambda^T \eta\right) d\eta = (\mathrm{Pf}\mathbf{A}) \exp\left(\frac{1}{2}\lambda^T \mathbf{A}^{-1}\lambda\right).$$

3. *(The Laplace Formula) For any* $\mathfrak{t} : \underline{k} \nearrow \underline{2N}$ *we have*

$$\int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta = \begin{cases} 0, & \textit{if } k \textit{ is odd,} \\ \operatorname{sgn}(\mathfrak{t})\mathrm{Pf}(\mathfrak{t}'\mathbf{A}\mathfrak{t}'), & \textit{if } k \textit{ is even.} \end{cases}$$

4. *(The Jacobi Minor Inverse Formula) For any* $\mathfrak{t} : \underline{k} \nearrow \underline{2N}$*, if* $\mathbf{A}$ *is invertible, then*

$$\int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta = \begin{cases} 0, & \textit{if } k \textit{ is odd,} \\ (\mathrm{Pf}\mathbf{A})\mathrm{Pf}\left(\mathfrak{t}\mathbf{A}^{-T}\mathfrak{t}\right), & \textit{if } k \textit{ is even.} \end{cases}$$

5. *(The Cauchy-Binet Formula) More generally, for any* $k \times 2N$ *matrix* $\mathbf{C}$ *over* R*,*

$$\begin{aligned} &\int (\mathbf{C}\eta)_1 \ldots (\mathbf{C}\eta)_k \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta \\ &\qquad = \begin{cases} 0, & \textit{if } k \textit{ is odd,} \\ \displaystyle\sum_{\mathfrak{t}:\underline{k}\nearrow\underline{2N}} \operatorname{sgn}(\mathfrak{t}) \det(\mathfrak{t}\mathbf{C}\mathfrak{t})\mathrm{Pf}\left(\mathfrak{t}'\mathbf{A}\mathfrak{t}'\right), & \textit{if } k \textit{ is even.} \end{cases} \end{aligned}$$

and if $\mathbf{A}$ *is invertible, then*

$$\int (\mathbf{C}\eta)_1 \dots (\mathbf{C}\eta)_k \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta = \begin{cases} 0, & \textit{if } k \textit{ is odd,} \\ (\mathrm{Pf}\mathbf{A})\,\mathrm{Pf}\left(\mathbf{C}\mathbf{A}^{-T}\mathbf{C}^T\right), & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. For part (1), observe

$$\begin{aligned} \int \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta =& \frac{\partial}{\partial \eta_{2N}} \cdots \frac{\partial}{\partial \eta_1} \sum_{k=1}^{N} \frac{1}{k!} \left(\frac{1}{2}\eta^T \mathbf{A}\eta\right)^k \\ =& \frac{1}{N!} \left(\frac{1}{2}\eta^T \mathbf{A}\eta\right)^N \\ =& \mathrm{Pf}(\mathbf{A}) \end{aligned}$$

For part (2), we make the substitution $\eta = \theta + \mathbf{A}^{-1}\lambda$ and note that as $\lambda^T\theta = -\theta^T\lambda$ and $\mathbf{A}^{-T} = -\mathbf{A}^{-1}$, then

$$\begin{aligned} \eta^T \mathbf{A}\eta =& (\theta^T + \lambda^T \mathbf{A}^{-T})(\mathbf{A}\theta + \lambda) \\ =& \theta^T \mathbf{A}\theta + \chi^T\lambda + \lambda^T \mathbf{A}^{-T}\mathbf{A}\theta + \lambda^T \mathbf{A}^{-1}\lambda \\ =& \theta^T \mathbf{A}\theta - 2\lambda^T\theta - \lambda^T \mathbf{A}^{-1}\lambda. \end{aligned}$$

Therefore, since the Berezin integral is translation invariant,

$$\begin{aligned} \int \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta + \lambda^T\eta\right) d\eta =& \int \exp\Big(\frac{1}{2}\left[\theta^T \mathbf{A}\chi - 2\lambda^T\theta - \lambda^T \mathbf{A}^{-1}\lambda\right] \\ & + \left[\lambda^T\theta + \lambda^T \mathbf{A}^{-1}\lambda\right]\Big) d\theta \\ =& \int \exp\left(\frac{1}{2}\theta^T A\theta\right) \exp\left(\frac{1}{2}\lambda^T \mathbf{A}^{-1}\lambda\right) d\theta \\ =& \mathrm{Pf}(\mathbf{A}) \exp\left(\frac{1}{2}\lambda^T \mathbf{A}^{-1}\lambda\right). \end{aligned}$$

For part (3), compute

$$\begin{aligned}
\int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta = &\int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta_{\mathfrak{t}'}^T(\mathfrak{t}'\mathbf{A}\mathfrak{t}')\eta_{\mathfrak{t}'}\right) d\eta \\
=&\operatorname{sgn}(\mathfrak{t}) \int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta_{\mathfrak{t}'}^T(\mathfrak{t}'\mathbf{A}\mathfrak{t}')\eta_{\mathfrak{t}'}\right) d\eta_{\mathfrak{t}}\, d\eta_{\mathfrak{t}'} \\
=&\operatorname{sgn}(\mathfrak{t})(-1)^{k(2r-k)} \left(\int \eta_{\mathfrak{t}}\, d\eta_{\mathfrak{t}}\right) \left(\int \exp\left(\frac{1}{2}\eta_{\mathfrak{t}'}^T(\mathfrak{t}'\mathbf{A}\mathfrak{t}')\eta_{\mathfrak{t}'}\right) d\eta_{\mathfrak{t}'}\right) \\
=&\operatorname{sgn}(\mathfrak{t})\operatorname{Pf}(\mathfrak{t}'\mathbf{A}\mathfrak{t}').
\end{aligned}$$

For part (4), we first differentiate the result of part (b) with respect to $\frac{\partial}{\partial\lambda_{i_r}} \cdots \frac{\partial}{\partial\lambda_{i_1}}$ and then set $\lambda = 0$.

$$\begin{aligned}
&\frac{\partial}{\partial\lambda_{i_r}} \cdots \frac{\partial}{\partial\lambda_{i_1}} \operatorname{Pf}(\mathbf{A}) \exp\left(\frac{1}{2}\lambda^T \mathbf{A}^{-1}\lambda\right)\Big|_{\lambda=0} \\
&\quad = \frac{\partial}{\partial\lambda_{i_r}} \cdots \frac{\partial}{\partial\lambda_{i_1}} \int \exp\left(\frac{1}{2}\eta^T\mathbf{A}\eta + \lambda^T\eta\right) d\eta\,\Big|_{\lambda=0} \\
&\quad = \int \left(\int \exp\left(\frac{1}{2}\eta^T\mathbf{A}\eta + \lambda^T\eta\right) d\eta\right) d\lambda_{\mathfrak{t}}\,\Big|_{\lambda=0} \\
&\quad = \int \left(\int \exp\left(\lambda^T\eta\right) d\lambda_{\mathfrak{t}}\Big|_{\lambda=0}\right) \exp\left(\frac{1}{2}\eta^T\mathbf{A}\eta\right) d\eta \\
&\quad = \int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta^T\mathbf{A}\eta\right) d\eta
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\frac{\partial}{\partial \lambda_{i_r}} \cdots \frac{\partial}{\partial \lambda_{i_1}} \mathrm{Pf}(\mathbf{A}) \exp\left(\frac{1}{2}\lambda^T \mathbf{A}^{-1}\lambda\right)\Big|_{\lambda=0} \\
&\qquad = (\mathrm{Pf}\mathbf{A}) \int \exp\left(\frac{1}{2}\lambda^T \mathbf{A}^{-1}\lambda\right) d\lambda_{\mathfrak{t}}\Big|_{\lambda=0} \\
&\qquad = (\mathrm{Pf}\mathbf{A}) \int \exp\left(\frac{1}{2}\lambda_{\mathfrak{t}}^T (\mathfrak{t}\mathbf{A}\mathfrak{t})^{-1}\lambda_{\mathfrak{t}}\right) d\lambda_{\mathfrak{t}} \\
&\qquad = (\mathrm{Pf}\mathbf{A})\, (-1)^{k(k-1)/2} \int \exp\left(\frac{1}{2}\lambda_{\mathfrak{t}}^T (\mathfrak{t}\mathbf{A}\mathfrak{t})^{-1}\lambda_{\mathfrak{t}}\right) d\lambda_{\mathfrak{t}} \\
&\qquad = (\mathrm{Pf}\mathbf{A})\, (-1)^{k(k-1)/2} \mathrm{Pf}\left(\mathfrak{t}\mathbf{A}^{-1}\mathfrak{t}\right) \\
&\qquad = (\mathrm{Pf}\mathbf{A})\, (-1)^{k(k-1)/2} (-1)^{k/2} \mathrm{Pf}\left(\mathfrak{t}\mathbf{A}^{-T}\mathfrak{t}\right) \\
&\qquad = (\mathrm{Pf}\mathbf{A})\, \mathrm{Pf}\left(\mathfrak{t}\mathbf{A}^{-T}\mathfrak{t}\right)
\end{aligned}
$$

Finally, for part (5), observe

$$
\begin{aligned}
\int (\mathbf{C}\eta)_1 \ldots (\mathbf{C}\eta)_k \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta &= \sum_{\mathfrak{t}:\underline{k}\nearrow \underline{2N}} \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma) C_{1,\mathfrak{t}\circ\sigma(1)} \cdots C_{k,\mathfrak{t}\circ\sigma(k)} \\
&\quad \cdot \int \eta_{\mathfrak{t}(1)} \cdots \eta_{\mathfrak{t}(k)} \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta \\
&= \sum_{\mathfrak{t}:\underline{k}\nearrow \underline{2N}} \mathrm{sgn}(\mathfrak{t}) \det(\mathfrak{t}\mathbf{C}\mathfrak{t}) \mathrm{Pf}(\mathfrak{t}'\mathbf{A}\mathfrak{t}')
\end{aligned}
$$

And when $\mathbf{A}$ is invertible,

$$
\begin{aligned}
\int (\mathbf{C}\eta)_1 \ldots (\mathbf{C}\eta)_k \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta =& \frac{\partial}{\partial \lambda_{i_1}} \cdots \frac{\partial}{\partial \lambda_{i_k}} \int \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta + \lambda^T \mathbf{C}\eta\right)\Big|_{\lambda=0} \\
=& \frac{\partial}{\partial \lambda_{i_1}} \cdots \frac{\partial}{\partial \lambda_{i_k}} \mathrm{Pf}(\mathbf{A}) \exp\left(\frac{1}{2}\lambda^T \mathbf{C}\mathbf{A}^{-1}\mathbf{C}^T\lambda\right)\Big|_{\lambda=0} \\
=& \mathrm{Pf}(\mathbf{A})\mathrm{Pf}(\mathbf{C}\mathbf{A}^{-T}\mathbf{C}^T)
\end{aligned}
$$

$\square$

Hyperpfaffian Identities

Before presenting the Hyperpfaffian analogues for the fermionic Wick formulas, it will be useful to define a few important operations on the exterior algebra, which act as extensions of familiar matrix operations.

Even and Odd Components of $\bigwedge(\eta)$

Let $\bigwedge^{\text{even}}(\eta)$ denote the subalgebra of $\bigwedge(\eta)$ generated by all homogeneous forms of even degree, let $\bigwedge^{\text{odd}}(\eta)$ denote the subalgebra of $\bigwedge(\eta)$ generated by all homogeneous forms of odd degree, and let $\bigwedge^{\text{even}}_{>0}(\eta)$ denote the subalgebra of $\bigwedge^{\text{even}}(\eta)$ generated by all non-constant homogeneous forms of even degree. Evidently,

$$\bigwedge(\eta) = \bigwedge^{\text{even}}(\eta) \bigoplus \bigwedge^{\text{odd}}(\eta) = R \bigoplus \bigwedge^{\text{even}}_{>0}(\eta) \bigoplus \bigwedge^{\text{odd}}(\eta).$$

Minors of L-Forms

The first task is to extend the notion of matrix minors to arbitrary forms. For each increasing function $\mathfrak{t} : \underline{k} \nearrow \underline{NL}$ with $0 \le k \le NL$, and for each $\omega \in \bigwedge^L(\eta)$ with $\omega = \sum_{\mathfrak{t}} A_{\mathfrak{t}}\, \eta_{\mathfrak{t}}$, define $\omega_{\mathfrak{t}} \in \bigwedge^L(\eta)$ by

$$\omega_{\mathfrak{t}} = \sum_{\mathfrak{s}:\underline{L} \nearrow \mathfrak{t}(\underline{k})} A_{\mathfrak{s}}\, \eta_{\mathfrak{s}}.$$

That is, $\omega_{\mathfrak{t}}$ is the L-form obtained by evaluating ω at $\eta_i = 0$ for all i outside the range of $\mathfrak{t}$. Note that in the special case when $L = 2$, if $\omega \in \bigwedge^2(\eta)$ is a 2-form with associated antisymmetric matrix $\mathbf{A}$, then ω_{t} is the 2-form associated to the antisymmetric matrix $\mathbf{tAt}$, as defined previously.

The Exponentiation Map

As before, for any $\alpha \in \bigwedge^L(\eta)$ with $L \geq 1$, we let $\exp(\alpha) \in \bigwedge(\eta)$ be given by

$$\exp(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha^k}{k!}$$

Now, if $\alpha, \beta \in \bigwedge^L(\eta)$ with $L \geq 1$ even, then $\alpha\beta = \beta\alpha$, and so we may expand $(\alpha + \beta)^n$ using the binomial theorem:

$$(\alpha + \beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k}.$$

Hence,

$$\exp(\alpha + \beta) = \exp(\alpha)\exp(\beta). \tag{4.1}$$

Change-of-Basis

Suppose $\alpha \in \bigwedge^L(\eta)$ with

$$\alpha = \sum_{\mathrm{t}:\underline{L}\nearrow\underline{NL}} A_{\mathrm{t}}\, \eta_{\mathrm{t}}$$

and that $\mathbf{C}$ is a $K \times NL$ matrix. We let $\mathbf{C} \cdot \alpha$ denote the L-form given by

$$\mathbf{C} \cdot \alpha = \sum_{\mathrm{t}:\underline{L}\nearrow\underline{NL}} A_{\mathrm{t}}\, \mathbf{C}\eta_{\mathrm{t}(1)} \cdots \mathbf{C}\eta_{\mathrm{t}(L)}$$

By Theorem 13, if $K \geq L$, then

$$\mathbf{C} \cdot \alpha = \sum_{\mathfrak{t}:\underline{L} \nearrow \underline{NL}} \sum_{\mathfrak{u}:\underline{L} \nearrow \underline{K}} A_{\mathfrak{t}} \det(\mathfrak{u}\mathbf{C}\mathfrak{t})\eta_{\mathfrak{u}}. \tag{4.2}$$

Observe that if $\alpha \in \bigwedge^2(\eta)$ with associated antisymmetric matrix $\mathbf{A}$, then $\mathbf{C} \cdot \alpha$ is the 2-form associated to the antisymmetric matrix $\mathbf{CAC}^T$, as shown in Corollary 6.

More generally, if $\lambda \in \bigwedge(\eta)$ is not homogeneous, we may define $\mathbf{C} \cdot \lambda$ first on the homogeneous components of λ as above, and then extend linearly. We observe now that for $\lambda \in \bigwedge_{>0}(\eta)$, then

$$\exp(\mathbf{C} \cdot \lambda) = \mathbf{C} \cdot \exp(\lambda). \tag{4.3}$$

The Hodge Dual

The discussion of the Hodge dual operator here follows R. W. R. Darling's treatment of the topic in [7]. Let $\langle \cdot | \cdot \rangle$ denote the symmetric inner product on V which is orthonormal with respect to the basis η, and for each $1 \leq k \leq NL$, extend $\langle \cdot | \cdot \rangle$ to a symmetric inner product on $\bigwedge(\eta)$ by

$$\langle f_1 f_2 \cdots f_k | g_1 g_2 \cdots g_\ell \rangle = \delta_{k,\ell} \det \left(\langle f_i | g_j \rangle \right)_{i,j=1}^{k} \quad \text{for } f_1, \ldots, f_k, g_1, \ldots, g_\ell \in \bigwedge^1(\eta).$$

Let $\mathcal{E} = \{\eta_{\mathfrak{t}} \,|\, \mathfrak{t} : \underline{k} \nearrow \underline{NL}, 0 \leq k \leq k \leq NL\}$ denote the basis for $\bigwedge(\eta)$. The *Hodge dual* is the operator first defined on $\mathcal{E}$ by

$$*\eta_{\mathfrak{t}} = \operatorname{sgn}(\mathfrak{t})\eta_{\mathfrak{t}'} \qquad \text{where } \mathfrak{t} : \underline{k} \nearrow \underline{NL} \tag{4.4}$$

and then extended linearly to all of $\bigwedge(\eta)$. Observe that for $\eta_{\mathfrak{t}} \in \mathcal{E}$ with $\mathfrak{t} : \underline{k} \nearrow \underline{NL}$, then

$$**\eta_{\mathfrak{t}} = (-1)^{k(NL-k)}\eta_{\mathfrak{t}}. \tag{4.5}$$

In particular, this implies that the Hodge star operator is an involution on the subalgebra $\bigwedge^{\text{even}}(\eta)$.

Now, using the formula for multiplication in the exterior algebra provided by Theorem 3, if $\alpha, \beta \in \bigwedge^k(\eta)$, we see that the Hodge dual operator satisfies

$$\alpha \wedge (*\beta) = \langle \alpha|\beta\rangle\, \eta_{\text{vol}}. \tag{4.6}$$

In particular,

$$\alpha \wedge (*\alpha) = |\alpha|^2\eta_{\text{vol}}.$$

Suppose $\alpha \in \bigwedge^k(\eta)$ and $\beta \in \bigwedge^{NL-k}(\eta)$. Then we have

$$\alpha \wedge \beta = (-1)^{k(NL-k)}\beta \wedge \alpha.$$

Combining equations (4.5) and (4.6) with the observation that $\langle\cdot\,|\,\cdot\rangle$ is symmetric,

$$\begin{aligned}
\langle \alpha| *\beta\rangle\eta_{\text{vol}} =& \alpha \wedge **\beta \\
=& (-1)^{k(NL-k)}\alpha \wedge \beta \\
=& \beta \wedge \alpha \\
=& (-1)^{k(NL-k)}\langle \beta| *\alpha\rangle\eta_{\text{vol}} \\
=& (-1)^{k(NL-k)}\langle *\alpha|\beta\rangle\eta_{\text{vol}}.
\end{aligned}$$

In particular, this shows that the Hodge star operator is self-adjoint when restricted to the subalgebra $\bigwedge^{\text{even}}(\eta)$. That is,

$$(*\alpha) \wedge \beta = \alpha \wedge (*\beta) \tag{4.7}$$

for $\alpha, \beta \in \bigwedge^{\text{even}}(\eta)$.

Inverse Forms

For an invertible $2N \times 2N$ antisymmetric matrix $\mathbf{A}$, the Jacobi Minor Inverse Formula from Theorem 20 states that

$$\mathbf{A} = \frac{1}{\mathrm{Pf}(\mathbf{A}^{-1})} \overline{\mathbf{A}^{-1}},$$

which shows that the entries of $\mathbf{A}$ can be expressed in terms of Pfaffians of minors of $\mathbf{A}^{-1}$. Theorem 23 restates this result in terms of antisymmetric forms:

$$\int \eta_{\mathfrak{t}} \exp\left(\frac{1}{2}\eta^T \mathbf{A}\eta\right) d\eta = \mathrm{sgn}(\mathfrak{t})\mathrm{Pf}(\mathfrak{t}'\mathbf{A}\mathfrak{t}') = (\mathrm{Pf}\mathbf{A})\mathrm{Pf}\left(\mathfrak{t}\mathbf{A}^{-T}\mathfrak{t}\right) \quad \mathfrak{t} : \underline{2k} \nearrow \underline{2N}.$$

Alternatively, if ω is the 2-form associated with $\mathbf{A}$ and ω' is the 2-form associated with $\mathbf{A}^{-1}$, then

$$\int \eta_{\mathfrak{t}} \exp(\omega)\, d\eta = \mathrm{sgn}(\mathfrak{t})\mathrm{Pf}(\omega_{\mathfrak{t}'}) = \mathrm{Pf}(\omega)\mathrm{Pf}(\omega'_{\mathfrak{t}})$$

In particular, combining Theorem 23 with the definition of the Hodge star operator in Equation 4.4 shows that

$$\frac{1}{\mathrm{Pf}(\omega)} \exp(\omega) = * \exp(\omega').$$

We now extend this notion of an inverse to forms of higher degree. Suppose $\alpha, \alpha' \in \bigwedge^L(\eta)$ with $\mathrm{PF}(\alpha) \neq 0$. We say that α' is an *inverse form* to α provided that

$$\frac{1}{\mathrm{PF}(\alpha)} \exp(\alpha) = *[\exp(\alpha')]. \tag{4.8}$$

Just as the minors of the inverse to a matrix can be used to generate the entries of the original matrix, the powers of the inverse form can be used to generate the coefficients of the original form. But unlike the case when $L = 2$, when $L > 2$, the condition $\mathrm{PF}(\alpha) \neq 0$ is not sufficient to guarantee the existence of an inverse form, as the following example shows.

Example. Suppose $|\eta| = 16$, and define $\alpha, \beta, \gamma \in \bigwedge^4(\eta)$ by

$$\alpha = \eta_1\eta_2\eta_3\eta_4 + \eta_5\eta_6\eta_7\eta_8 + \eta_9\eta_{10}\eta_{11}\eta_{12} + \eta_{13}\eta_{14}\eta_{15}\eta_{16}$$

and

$$\beta = \eta_1\eta_5\eta_9\eta_{13} \qquad \gamma = \alpha + \beta$$

Observe that as α, β are even forms, then $\alpha\beta = \beta\alpha$. Moreover, as β shares a variable with each term in the expansion of α, then $\alpha\beta = 0$. Finally, as β is a pure form, then $\beta^2 = 0$. Hence, for any $n > 1$, the binomial expansion of $\gamma^n = (\alpha + \beta)^n$ is given by

$$\gamma^n = (\alpha + \beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^k \beta^{n-k} = \alpha^n,$$

and so

$$\exp(\gamma) = \exp(\alpha) + \beta.$$

But note that $\mathrm{PF}(\alpha) = \mathrm{PF}(\gamma) = 1$ and that α satisfies

$$\exp(\alpha) = *\exp(\alpha).$$

Thus, if $\gamma' \in \bigwedge^4(\eta)$ satisfies

$$\exp(\gamma) = *\exp(\gamma'),$$

then in particular,

$$\alpha' = \frac{1}{3!} * (\alpha^3) = \frac{1}{3!} * (\gamma^3) = \gamma'$$

which would imply that

$$\alpha + \beta = \gamma = \frac{1}{3!} * (\gamma')^3 = \frac{1}{3!} * (\alpha')^3 = \alpha,$$

a contradiction.

Hyperpfaffian Formulas

With the preceding conventions in hand, we now present Hyperpfaffian analogues for the Laplace, Jacobi Minor Inverse, and Cauchy-Binet Formulas, as well as the Sylvester Identity.

Theorem 24. *Let R be a commutative ring with unity, let $\eta = \{\eta_1, \cdots, \eta_{NL}\}$ be a collection of NL anticommuting variables, and let $\alpha, \zeta \in \bigwedge^L(\eta)$.*

1. *(The Laplace Formula)*

$$\exp(\alpha) = \sum_{k=1}^{N} \sum_{\mathfrak{t}:\underline{kL}\nearrow\underline{NL}} \mathrm{PF}(\alpha_{\mathfrak{t}})\eta_{\mathfrak{t}}$$

2. *(The Jacobi Minor Inverse Formula) If $\alpha' \in \bigwedge^L(\eta)$ is an inverse form for α, then*

$$\exp(\alpha) = \frac{1}{\mathrm{PF}(\alpha)} \sum_{k=1}^{N} \sum_{\mathfrak{t}:\underline{kL}\nearrow\underline{NL}} \mathrm{sgn}(\mathfrak{t})\mathrm{PF}(\alpha'_{\mathfrak{t}'})\eta_{\mathfrak{t}}$$

3. *(The Cauchy-Binet Formula) For any $ML \times NL$ matrix $\mathbf{C}$ over R with $M \leq N$,*

$$\exp(\mathbf{C} \cdot \alpha) = \sum_{k=1}^{M} \sum_{\mathfrak{t}:\underline{kL}\nearrow\underline{NL}} \sum_{\mathfrak{u}:\underline{kL}\nearrow\underline{ML}} \mathrm{PF}(\alpha_{\mathfrak{t}}) \det(\mathfrak{u}\mathbf{C}\mathfrak{t})\eta_{\mathfrak{u}}$$

4. *(The Sylvester Identity) If L is even, if $\alpha', \zeta' \in \bigwedge^L(\eta)$ are inverse forms for α, ζ, and if $\mathbf{C}$ is a $NL \times NL$ matrix, then*

$$\frac{\mathrm{PF}(\zeta' + \mathbf{C} \cdot \alpha)}{\mathrm{PF}(\zeta')} = \frac{\mathrm{PF}(\alpha' + \mathbf{C}^T \cdot \zeta)}{\mathrm{PF}(\alpha')}.$$

Proof. Part (1) follows by relabeling indices according to $\mathfrak{t}$ and applying Corollary 4. Part (2) then follows by combining part (1) with Equations 4.4 and 4.8, while part (3) follows by combining part (1) with Equations 4.2 and 4.3.

For part (4), observe

$$\begin{aligned}
\exp(\zeta' + \mathbf{C} \cdot \alpha) &= \exp(\zeta') \exp(\mathbf{C} \cdot \alpha) \\
&= \exp(\mathbf{C}^T \cdot \zeta') \exp(\alpha) \\
&= \frac{1}{\mathrm{PF}(\zeta)} [* \exp(\mathbf{C}^T \cdot \zeta)] \exp(\alpha) \\
&= \frac{1}{\mathrm{PF}(\zeta)} \exp(\mathbf{C}^T \cdot \zeta) [* \exp(\alpha)] \\
&= \frac{\mathrm{PF}(\alpha)}{\mathrm{PF}(\zeta)} \exp(\mathbf{C}^T \cdot \zeta) \exp(\alpha') \\
&= \frac{\mathrm{PF}(\alpha)}{\mathrm{PF}(\zeta)} \exp(\alpha' + \mathbf{C}^T \cdot \zeta),
\end{aligned}$$

where the first and sixth equalities follow from 4.1, the second from part (3) of this theorem, the third and fifth from 4.8, and the fourth from 4.7. The result then follows by using part (1) of this theorem to express the coefficients on term in the exponential series as hyperpfaffians.

□

Hyperpfaffian Wick Formulas

The Berezin integral may also be used to restate the preceding identities in the guise of Wick Formulas, analogous to Theorem 23.

Theorem 25 (Berezin Integral Formula for Hyperpfaffian Identities)**.** *Let R be a commutative ring with unity, let $\eta^T = [\, \eta_1 \cdots \eta_{NL} \,]$ be a vector of NL anticommuting variables, and let $\alpha \in \bigwedge^L(\eta)$. Then*

1. *(The Gaussian Integral)*

$$\int \exp(\alpha)\ d\eta = \mathrm{PF}(\alpha).$$

2. *(The Laplace Formula) For any $\mathfrak{t} : \underline{k} \nearrow \underline{NL}$ we have*

$$\int \eta_{\mathfrak{t}} \exp(\alpha)\ d\eta = \begin{cases} 0, & \text{if } L \nmid k, \\ \mathrm{sgn}(\mathfrak{t})\mathrm{PF}(\alpha_{\mathfrak{t}'}), & \text{if } L \mid k. \end{cases}$$

3. *(The Jacobi Minor Inverse Formula) For any* $\mathfrak{t} : \underline{k} \nearrow \underline{NL}$, *if* α *has an inverse form* α', *then*

$$\int \eta_{\mathfrak{t}} \exp(\alpha)\ d\eta = \begin{cases} 0, & \text{if } L \nmid k, \\ \mathrm{PF}(\alpha)\mathrm{PF}(\alpha'_{\mathfrak{t}}), & \text{if } L \mid k. \end{cases}$$

4. *(The Cauchy-Binet Formula) More generally, for any* $k \times NL$ *matrix* $\mathbf{C}$ *over* R,

$$\int (\mathbf{C}\eta)_1 \dots (\mathbf{C}\eta)_k \exp(\alpha)\ d\eta = \begin{cases} 0, & \text{if } L \nmid k, \\ \sum_{\mathfrak{t}:\underline{k} \nearrow \underline{NL}} \mathrm{sgn}(\mathfrak{t}) \det(\mathfrak{t}\mathbf{C}\mathfrak{t}) \mathrm{PF}(\alpha_{\mathfrak{t}'}), & \text{if } L \mid k. \end{cases}$$

and if α *has a inverse form* α', *then*

$$\int (\mathbf{C}\eta)_1 \dots (\mathbf{C}\eta)_k \exp(\alpha)\ d\eta = \begin{cases} 0, & \text{if } L \nmid k, \\ \mathrm{PF}(\alpha)\,\mathrm{PF}(\mathbf{C} \cdot \alpha'), & \text{if } L \mid k. \end{cases}$$

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