THE GEOMETRY OF QUASI-SASAKI MANIFOLDS

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ADAM P. WELLY

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Student: Adam P. Welly

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Dr. Weiyong He	Chair
Dr. Peter Gilkey	Core Member
Dr. Peng Lu	Core Member
Dr. Christopher Sinclair	Core Member
Dr. Dietrich Belitz	Institutional Representative

and

Scott Pratt	Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Adam P. Welly

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Let (M, g) be a quasi-Sasaki manifold with Reeb vector field ξ. Our goal is to understand the structure of M when g is an Einstein metric. Assuming that the S^1 action induced by ξ is locally free or assuming a certain non-negativity condition on the transverse curvature, we prove some rigidity results on the structure of (M, g).

Naturally associated to a quasi-Sasaki metric g is a transverse Kähler metric g^T. The *transverse Kähler-Ricci flow* of g^T is the normalized Ricci flow of the transverse metric. Exploiting the transverse Kähler geometry of (M, g), we can extend results in Kähler-Ricci flow to our transverse version. In particular, we show that a deep and beautiful theorem due to Perleman has its counterpart in the quasi-Sasaki setting.

We also consider evolving a Sasaki metric g by Ricci flow. Unfortunately, if $g(0)$ is Sasaki then $g(t)$ is not Sasaki for $t > 0$. However, in some instances $g(t)$ is quasi-Sasaki. We examine this and give some qualitative results and examples in the special case that the initial metric is η-Einstein.

CURRICULUM VITAE

NAME OF AUTHOR: Adam P. Welly

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Western Washington University, Bellingham, WA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2016, University of Oregon
Bachelor of Science, Mathematics, 2010, Western Washington University

AREAS OF SPECIAL INTEREST:

Differential Geometry
Geometric Analysis

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, Department of Mathematics, University of Oregon, Eugene OR, 2010–2016

Peer Tutor, Department of Mathematics, Western Washington University, Bellingham Wa, 2008–2010

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Dedicated to my wife Allison and my son Owen.

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CHAPTER I

INTRODUCTION

1.1. Overview

This dissertation is an exploration into the geometry of quasi-Sasaki manifolds. As the name suggests, a quasi-Sasaki structure is similar to a Sasaki one, but is more flexible than the latter. Indeed, Sasaki manifolds form a subset of the quasi-Sasaki manifolds, but in general the two can have quite different features. For example, cosymplectic manifolds are also quasi-Sasaki. A cosymplectic manifold has an integrable contact bundle whereas the contact bundle of a Sasaki manifold is as far from being integrable as possible. But like a Sasaki manifold, a quasi-Sasaki manifold (M, g) comes equipped with a one-dimensional Riemannian foliation, $\mathcal{F}_\xi$, determined by a nowhere vanishing vector field ξ. The space transverse to this foliation carries a Kähler structure. Dual to the vector field ξ (via the metric g) is a 1-form η. The kernel of η is called the contact bundle. Perhaps the most significant difference between a quasi-Sasaki and a true Sasaki structure has to do with the rank of the 2-form $d\eta$. For a Sasaki structure, $\eta \wedge (d\eta)^n$ is a volume form and the transverse Kähler metric $\omega = d\eta$. If a quasi-Sasaki manifold has $(d\eta)^n = 0$, then it lacks a contact structure and, even when $(d\eta)^n \neq 0$, there may be no clear relationship between the transverse Kähler metric and the 2-form $d\eta$. In the next sections we will recall all of the necessary definitions and elementary facts concerning quasi-Sasaki manifolds that will be used throughout the remainder of this dissertation.

The pioneering work on quasi-Sasaki manifolds was iniated by Blair in [3] during the late 1960's. However, his work was not without errors. These errors were later recognized by Tanno who then gave complete and correct revisions of Blair's results in [37]. The main results of these papers describe a splitting of quasi-Sasaki manifolds by assuming the integrability of a certain almost product structure $TM = P \oplus Q$ and positive definiteness of $d\eta(\cdot, \Phi\cdot)$ when restricted to P. These assumptions are somewhat unnatural and unmotivated and we do not make them here. Rather, we want to prove a splitting result like their's, but under the assumption that the metric g is Einstein. In fact, one of our motivating reasons for studying quasi-Sasaki manifolds is the search for "new" Einstein metrics. A Riemannian metric g is called Einstein if its Ricci tensor is proportional to g. That is, if there is a constant $\lambda \in \mathbb{R}$ such that $\mathrm{Ric}_g = \lambda g$. Differential geometers are interested in Einstein metrics for a wide variety of reasons. Entire books have been written on the subject; the book by Besse is a classic. Often times Einstein metrics are somehow the "best" or most desirable metrics. They appear as critical points of certain geometrically defined functionals. Physicists are interested in Einstein metrics for their role in the theory of general relativity. Thus we are curious about the rigidity of quasi-Sasaki-Einstein metrics.

For example, we can create quasi-Sasaki-Einstein metrics by taking products of Sasaki-Einstein and Kähler-Einstein metrics, but we don't consider these to be "new" Einstein metrics. So the question is, are all quasi-Sasaki-Einstein metrics of this form or are there more "interesting" examples possible? We investigate this in chapter II. By assuming some regularity on the orbits of the Reeb vector field ξ, we can give a precise description of the possible Einstein metrics. We define a transverse version of non-negative quadratic orthogonal bisectional curvature (NQOBC) and show that a quasi-Sasaki-Einstein metric satisfying this curvature conditions is rigid. We show that M is locally the product of a Kähler-Einstein manifold M_0 and a *full rank* quasi-Sasaki-Einstein manifold M_1. Furthermore, the transverse Kähler metric of M_1

is a product of Kähler-Einstein metrics. If it happens that the second *basic Betti number* $b_2^B(M_1) = 1$, then the metric on M_1 is simply a scaling of a Sasaki-Einstein metric. Note that Sasaki metrics are not preserved by homotheties, but quasi-Sasaki metrics are. Under the stronger assumption of positive or non-negative transverse holomorphic bisectional curvature, we conjecture that something like the Frankel and the generalized Frankel conjectures (which are actually no longer conjectures) in Sasaki geometry ought to be true. See [20] and [21].

As we will see later in this chapter, a quasi-Sasaki manifold (M, g) has naturally associated to it a transverse Kähler metric g^T. It is known by the work of Cao in [7] that if the first Chern class of a Kähler manifold is negative or null, then the Kähler-Ricci flow will converge to a Kähler-Einstein metric. This gives us some motivation to consider the *transverse Kähler-Ricci flow* (TKRF). By exploiting the transverse Kähler geometry of (M, g), we can extend deep results in Kähler-Ricci flow to our transverse version. In particular, if the *basic first Chern class* is negative or null, we can reproduce the results of Cao in [7] in our transverse setting. If M is compact and the basic first Chern class is positive, then defining functionals which are monotonic along the TKRF, we can mimic the work of Perleman in [34] and prove uniform bounds on the diameter and scalar curvature along the flow. We also show that, analogous to [2] and [29], positivity of transverse holomorphic bisectional curvature is preserved along the TKRF. All of this and more is carried out in chapter III.

In chapter IV we start to explore Hamilton's Ricci flow on a Sasaki manifold (M, g). That is, we let the Sasaki metric g itself evolve by the Ricci flow:

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_g.$$

It is well-known that evolving a Kähler metric by Ricci flow preserves the Kähler condition, but the Ricci flow does not preserve the Sasaki condition. However, we will show that in the special case where the initial Sasaki metric is η-Einstein, that there is a quasi-Sasaki structure associated to the evolving metric. The geometry of an η-Einstein manifold is special enough that the Ricci flow equation can be reduced to a system of two ordinary differential equations. The qualitative behavior of the flow is discussed in each of the three cases determined by the transverse Kähler-Einstein structure. We then exhibit concrete examples of η-Einstein metrics where the transverse metric g^T satisfies $\mathrm{Ric}^T_g = \lambda g^T$ for $\lambda = -1, 0, 4$. We mention in passing the classification theorem of three-dimensional Sasaki manifolds due to Geiges and the uniformization theorem due to Belgun and how our examples tie in. We raise some questions about the general case of evolving an arbitrary Sasaki metric by the Ricci flow.

1.2. Preliminaries

In this section we will review the (Riemannian) basics of quasi-Sasaki geometry. Throughout, $M = M^{2n+1}$ will denote a smooth, oriented, connected, Riemannian manifold of real dimension $2n+1$ for $n \geq 1$. The metric will be denoted by g and ∇ will denote the Levi-Civita connection of g. We begin by defining a quasi-Sasaki structure as a certain type of almost contact structure. This is the definition that Blair gave in [3] and the one that we shall work with here.

Definition 1.2.1. An *almost contact* structure on M^{2n+1} is a triple (ξ, η, Φ) where ξ is a nowhere vanishing vector field (called the *Reeb* vector field), η is a 1-form with $\eta(\xi) = 1$ and Φ is a (1,1)-tensor satisfying $\Phi^2 = -I + \xi \otimes \eta$.

We let $L_\xi \subset TM$ denote the line bundle generated by ξ. The *contact bundle* $\mathcal{D} \subset TM$ is defined to be the kernel of η. We have the direct sum decomposition

$$TM = L_\xi \oplus \mathcal{D}.$$

The following theorem will be used frequently.

Theorem 1.2.2. *Suppose M^{2n+1} has an almost contact structure (ξ, η, Φ). Then $\Phi\xi = 0$, the endomorphism Φ has rank $2n$ and $\eta \circ \Phi = 0$.*

Proof. Since $\Phi^2 = -I$ on $\mathcal{D}$, we see that the rank of Φ is at least $2n$. Now $\Phi^2\xi = 0$ implies that $\Phi\xi \notin \mathcal{D} \setminus \{0\}$. Thus $\Phi\xi = f\xi$ for some $f \in C^\infty(M)$. But we have that $0 = \Phi^2\xi = f\Phi\xi = f^2\xi$. Since ξ is nowhere vanishing, $f = 0$. Thus $\Phi\xi = 0$ and this implies that the rank of Φ is exactly $2n$. For any vector field $X \in TM$, we now have

$$\begin{aligned}
\eta(\Phi X)\xi &= \Phi^2\Phi X + \Phi X \\
&= \Phi(\Phi^2 X + X) \\
&= \eta(X)\Phi\xi \\
&= 0.
\end{aligned}$$

Therefore $\eta \circ \Phi = 0$. $\square$

Definition 1.2.3. An *almost contact metric* structure on M^{2n+1} is a quadruple (ξ, η, Φ, g) where (ξ, η, Φ) is an almost contact structure and g is a Riemannian metric on M that is compatible with Φ in the sense that $\Phi^* g = g - \eta \otimes \eta$.

If (ξ, η, Φ, g) is an almost contact metric structure, the metric compatibility with Φ and $\Phi\xi = 0$ implies that $\eta = g(\xi, \cdot)$. Hence η and ξ are dual to each other via the metric g. Given an almost contact structure, a compatible metric always exists.

Proposition 1.2.4. *Given an almost contact structure* (ξ, η, Φ) *on* M*, there is a Riemannian metric* g *such that* (ξ, η, Φ, g) *is an almost contact metric structure.*

Proof. Let $\tilde{g}$ be any Riemannian metric on M. First define a metric g' by

$$g'(X,Y) := \tilde{g}\left(\Phi^2 X, \Phi^2 Y\right) + \eta(X)\eta(Y).$$

Then we have $g'(X,\xi) = \eta(X)$ for all $X \in TM$. Now define a metric g by

$$g(X,Y) := \frac{1}{2}\left(g'(X,Y) + g'(\Phi X, \Phi Y) + \eta(X)\eta(Y)\right)$$

and check that $g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all $X, Y \in TM$. $\square$

Let M^{2n+1} be equipped with an almost contact structure. Consider the manifold $M \times \mathbb{R}$ with the almost complex structure $J : T(M \times \mathbb{R}) \to T(M \times \mathbb{R})$, defined by

$$J\Big(f_1 X + f_2 \frac{d}{dt}\Big) = f_1 \Phi X - f_2 \xi + f_1 \eta(X) \frac{d}{dt}. \tag{1.1}$$

Here $X \in TM$, $f_1, f_2 \in C^\infty(M \times \mathbb{R})$ and $\dfrac{d}{dt}$ spans the tangent space to $\mathbb{R}$. It is straightforward to verify that $J^2 = -I$.

Definition 1.2.5. An almost contact structure on M is called *normal* if the almost complex structure J, defined by (1.1), is integrable.

Recall that an almost complex structure is called integrable if it is induced by an actual complex structure. The celebrated theorem of Newlander and Nirenberg asserts that an almost complex structure J is integrable if and only if the Nijenhuis torsion tensor $N_J \equiv 0$. For a (1,1)-tensor T, the Nijenhuis torsion tensor N_T is the

(1,2)-tensor defined by

$$N_T(X,Y) := T^2[X,Y] + [TX,TY] - T[TX,Y] - T[X,TY]. \tag{1.2}$$

So if our almost product structure is normal, then $N_J \equiv 0$ and this implies that for any vector fields $X, Y \in TM$,

$$\begin{aligned} N^{(1)}(X,Y) &:= N_\Phi(X,Y) + 2d\eta(X,Y)\xi = 0, \\ N^{(2)}(X,Y) &:= (\mathcal{L}_{\Phi X}\eta)(Y) - (\mathcal{L}_{\Phi Y}\eta)(X) = 0, \\ N^{(3)}(X) &:= (\mathcal{L}_\xi \Phi)(X) = 0, \\ N^{(4)}(X) &:= (\mathcal{L}_\xi \eta)(X) = 0. \end{aligned}$$

Conversely, the vanishing of these four tensor fields implies that $N_J \equiv 0$. In fact, the vanishing of N_J is equivalent to the vanishing of $N^{(1)}$. A proof of these facts and the following proposition can be found in chapter 6 of [4].

Proposition 1.2.6. *If $N^{(1)}$ vanishes then so do $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$. Therefore, an almost contact structure is normal if and only if $N^{(1)} \equiv 0$.*

An equation that we will use frequently, and one that holds for any 1-form η is

$$2d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]). \tag{1.3}$$

The normality condition has some interesting consequences. We record a few of these in the following lemma.

Lemma 1.2.7. *Let (ξ, η, Φ) be a normal, almost contact structure on M. Then $d\eta(\xi, \cdot) = 0$ and $\Phi^* d\eta = d\eta$. Moreover, if g is a compatible metric then the integral curves of ξ are geodesics.*

Proof. Since $N^{(4)}(X) = (\mathcal{L}_\xi \eta)(X) = 0$ for any $X \in TM$, we have

$$\begin{aligned} 2d\eta(\xi, X) &= \xi(\eta(X)) - X(\eta(\xi)) - \eta([\xi, X]) \\ &= \xi(\eta(X)) - \eta([\xi, X]) \\ &= (\mathcal{L}_\xi \eta)(X) = 0. \end{aligned}$$

Thus $d\eta(\xi, \cdot) = 0$. Cartan's magic formula says that $\mathcal{L}_{\Phi X}\eta = d\iota_{\Phi X}\eta + \iota_{\Phi X} d\eta$. Since $\eta \circ \Phi = 0$, we get

$$(\mathcal{L}_{\Phi X}\eta)(Y) = d\eta(\Phi X, Y).$$

From $N^{(2)}(X, Y) = 0$ and the above, we have $d\eta(\Phi X, Y) = d\eta(\Phi Y, X)$. Now replacing Y with ΦY and using $d\eta(\xi, \cdot) = 0$, we see that $d\eta(\Phi X, \Phi Y) = -d\eta(Y, X) = d\eta(X, Y)$. For the last claim, notice that $g(\nabla_X \xi, \xi) = \frac{1}{2} X g(\xi, \xi) = 0$. Again using $N^{(4)}(X) = 0$, we compute that for any $X \in TM$

$$\begin{aligned} g(X, \nabla_\xi \xi) &= \xi g(X, \xi) - g(\nabla_\xi X, \xi) + g(\nabla_X \xi, \xi) \\ &= \xi(\eta(X)) - \eta([\xi, X]) \\ &= (\mathcal{L}_\xi \eta)(X) = 0. \end{aligned}$$

Thus $\nabla_\xi \xi = 0$. Hence the integral curves of ξ are geodesics. □

The *fundamental 2-form* associated to an almost contact metric structure (ξ, η, Φ, g) is the 2-form ω defined for all $X, Y \in TM$ by

$$\omega(X, Y) := g(X, \Phi Y). \tag{1.4}$$

From $\Phi^* g = g - \eta \otimes \eta$ and $\eta \circ \Phi = 0$, it follows that $\Phi^* \omega = \omega$. Note also that $\omega(\xi, \cdot) = 0$. Now we are ready to give the definition of a quasi-Sasaki structure.

Definition 1.2.8. A *quasi-Sasaki structure* is a normal, almost contact metric structure whose fundamental 2-form is closed (i.e. $d\omega = 0$).

A smooth manifold equipped with a quasi-Sasaki structure is called a *quasi-Sasaki manifold*. A fundamental difference between a Sasaki and a quasi-Sasaki structure has to do with whether or not $d\eta$ is non-degenerate on $\mathcal{D}$. We can categorize quasi-Sasaki structures according to their rank, which indicates how degenerate $d\eta$ is. The following definition was introduced in [3].

Definition 1.2.9. The *rank* of a quasi-Sasaki structure is $2p$ when $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$. The rank is $2p + 1$ when $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$.

A quasi-Sasaki structure of rank 1 is commonly called a *cosymplectic* structure. In this case we have $d\eta = 0$. A quasi-Sasaki structure of rank $2n + 1$ with $\omega = d\eta$ is Sasaki. The next proposition follows easily from lemma 1.2.7.

Proposition 1.2.10. *There are no quasi-Sasaki structures of even rank.*

Proof. Since $d\eta(\xi, \cdot) = 0$, if $(d\eta)^p \neq 0$ there are vector fields $X_1, \ldots, X_{2p} \in \mathcal{D}$ such that $(d\eta)^p(X_1, \ldots, X_{2p}) \neq 0$. Thus $\eta \wedge (d\eta)^p(\xi, X_1, \ldots, X_{2p}) = (d\eta)^p(X_1, \ldots, X_{2p}) \neq 0$. $\square$

Some facts about the geometry of a quasi-Sasaki manifold are contained in the next lemma. Important is the fact that ξ is a Killing vector field (i.e. $\mathcal{L}_\xi g = 0$). This means that moving along the integral curves of ξ is an isometry of the metric.

Lemma 1.2.11. *Let (ξ, η, Φ, g) be a quasi-Sasaki structure on M. Then $\mathcal{L}_\xi g = 0$, $g(\nabla_X \xi, Y) = d\eta(X, Y)$ and $\nabla_\xi \Phi = 0$.*

Proof. Since $N^{(3)}(Y) = (\mathcal{L}_\xi \Phi)(Y) = 0$, we have $\Phi[\xi, Y] = [\xi, \Phi Y]$. Thus we see that

$$(\mathcal{L}_\xi \omega)(X, Y) = (\mathcal{L}_\xi g)(X, \Phi Y).$$

The Lie derivative $\mathcal{L}_\xi \omega = d\iota_\xi \omega + \iota_\xi d\omega = 0$. Hence we have $(\mathcal{L}_\xi g)(X, \Phi Y) = 0$. Next, using $N^{(4)}(X) = (\mathcal{L}_\xi \eta)(X) = 0$ and $\nabla_\xi \xi = 0$, one computes that

$$(\mathcal{L}_\xi g)(X, \eta(Y)\xi) = 0.$$

The above shows that $(\mathcal{L}_\xi g)(X, (\Phi + \xi \otimes \eta)Y) = 0$. Since the map $\Phi + \xi \otimes \eta$ is non-singular, it follows that $\mathcal{L}_\xi g = 0$. Therefore, ξ is a Klling vector field.

For a vector field Z, let θ_Z be the 1-form defined by $\theta_Z(Y) := g(Y, Z)$. It is then straightforward to verify that

$$(\mathcal{L}_Z g)(X, Y) + 2d\theta_Z(X, Y) = 2g(\nabla_X Z, Y).$$

Taking $Z = \xi$ and using that ξ is a Killing vector field gives $g(\nabla_X \xi, Y) = d\eta(X, Y)$.

With this and the vanishing of tensors $N^{(2)}$ and $N^{(3)}$, a straightforward computation, using that ∇ is Riemannian and torsion free, reveals that $\nabla_\xi \Phi = 0$. $\square$

Remark 1.2.12. The condition that ξ is a Killing vector field is equivalent to the condition $g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$ for all $X, Y \in TM$. Also, one can actually show more than just $\nabla_\xi \Phi = 0$. In particular, lemma 6.1 of [4] shows that

$$g((\nabla_X \Phi)Y, Z) = d\eta(\Phi Y, X)\eta(Z) - d\eta(\Phi Z, X)\eta(Y). \tag{1.5}$$

We end this section with some results which characterize cosymplectic manifolds (those with $d\eta = 0$) among quasi-Sasak manifolds.

Lemma 1.2.13. *A quasi-Sasaki manifold is cosymplectic if and only if $\nabla\Phi = 0$ if and only if $\nabla\omega = 0$.*

Proof. If $\nabla\Phi = 0$ then $N_\Phi \equiv 0$. By normality, $2d\eta(X,Y)\xi = -N_\Phi(X,Y) = 0$ for all $X, Y \in TM$. Hence $d\eta = 0$. Conversely, if $d\eta = 0$ then from equation (1.5) we conclude that $\nabla\Phi = 0$. The second if and only if statement follows easily from $\omega = g(\cdot, \Phi\cdot)$ and $\nabla g = 0$. □

Lemma 1.2.14. *A quasi-Sasaki manifold is cosymplectic if and only if $\nabla\eta = 0$.*

Proof. Since ξ is a Killing vector field and $\eta = g(\xi, \cdot)$, we compute that

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -g(\nabla_Y \xi, X) = -(\nabla_Y \eta)(X).$$

We can write equation (1.3) as

$$2d\eta(X,Y) = (\nabla_X \eta)(Y) - (\nabla_Y \eta)(X).$$

Combining these equations we have $d\eta(X,Y) = (\nabla_X \eta)(Y)$ and the result follows. □

Remark 1.2.15. The cosymplectic case is the only one where the contact subbundle $\mathcal{D}$ is integrable. Indeed, it follows from equation (1.3) that a subbundle defined as the kernel of a closed 1-form is involutive, hence integrable (see definition 5.0.2 and theorem 5.0.3). Furthermore, from $\nabla\eta = 0$ we see that $\mathcal{D}$ is a parallel distribution. As $TM = L_\xi \oplus \mathcal{D}$ and both L_ξ and $\mathcal{D}$ are integrable in this case, it follows that M is locally the product of a 1-dimensional manifold (line or circle) and a Kähler manifold.

1.3. Transverse Kähler Geometry

In this section we discuss the basics of Riemannian foliations and the associated transverse geometry, as it applies to our study of quasi-Sasaki manifolds. For a more thorough treatment of foliations, the interested reader should see [30] and/or [39].

Let (M, ξ, η, Φ, g) be a quasi-Sasaki manifold. Recall that L_ξ is the line bundle generated by ξ and $\mathcal{D} = \ker(\eta)$. From $\Phi^* g = g - \eta \otimes \eta$ and $\Phi\xi = 0$ we see that the decomposition $TM = L_\xi \oplus \mathcal{D}$ is orthogonal with respect to g. Hence $\mathcal{D} = L_\xi^\perp$. The integral submanifolds of L_ξ, which are the integral curves of ξ, define the *leaves* of a 1-dimensional foliation of M, which we denote by $\mathcal{F}_\xi$. We let $\nu(\mathcal{F}_\xi) := TM/L_\xi$ denote the normal bundle of the foliation. Then we have the short exact sequence

$$0 \to L_\xi \to TM \to \nu(\mathcal{F}_\xi) \to 0.$$

The metric provides a splitting $\sigma : \nu(\mathcal{F}_\xi) \to \mathcal{D}$ of this short exact sequence. Explicitly, for $\bar{X} \in \nu(\mathcal{F}_\xi)$ we have $\bar{X} = X + L_\xi$ for some $X \in TM$. Then $\sigma(\bar{X}) = X - g(X, \xi)\xi$. Clearly $\sigma(\bar{X}) \in \mathcal{D}$ and it is straightforward to check that σ is a well-defined bijection. Thus we may identify $\mathcal{D}$ and $\nu(\mathcal{F}_\xi)$. A *transverse metric* is a Riemannian metric on the quotient bundle $\nu(\mathcal{F}_\xi)$. Since $\mathcal{D} \simeq \nu(\mathcal{F}_\xi)$, we define the transverse metric g^T by

$$g^T := g|_{\mathcal{D}}.$$

The orthogonal projection $\pi_{\mathcal{D}} : TM \to \mathcal{D}$ is given by $\pi_{\mathcal{D}}(X) = X - \eta(X)\xi$. Observe:

$$\begin{aligned} g^T(X,Y) &= g(\pi_{\mathcal{D}}(X), \pi_{\mathcal{D}}(Y)) \\ &= g(X,Y) - \eta(X)\eta(Y) \\ &= \Phi^* g(X,Y). \end{aligned}$$

Thus the transverse metric is related to the quasi-Sasaki metric via $g = \eta \otimes \eta + g^T$. We want to know, under what conditions does the transverse metric reflect the geometry of the *leaf space* $M/\mathcal{F}_\xi$? To answer this question, we need the next definition.

Definition 1.3.1. A vector field $X \in TM$ is called *foliate* with respect to $\mathcal{F}_\xi$ if $[X, \xi] \in L_\xi$. A Riemannian metric g is called *bundle-like* with respect to $\mathcal{F}_\xi$ if for any foliate vector fields X and Y which are orthogonal to L_ξ, we have $\xi g(X,Y) = 0$.

Bundle-like metrics are important in the theory of foliations because they provide globally defined transverse metrics which are locally the pull-back of a metric on the local Riemannian quotient. The following is proposition 2.5.7 in [5].

Proposition 1.3.2. *If g is bundle-like with respect to $\mathcal{F}_\xi$, then $\mathcal{F}_\xi$ is a Riemannian foliation. Conversely, if $\mathcal{F}_\xi$ is a Riemannian foliation with transverse metric g^T, then there is a bundle-like metric g whose associated transverse metric is g^T.*

Thus we see that the transverse metric is locally the pull-back of a metric on the local Riemannian quotient if and only if g is bundle-like. In this way, bundle-like metrics reflect the geometry of the leaf space $M/\mathcal{F}_\xi$. The next lemma is immediate.

Lemma 1.3.3. *Quasi-Sasaki metrics are bundle-like with respect to $\mathcal{F}_\xi$.*

Proof. This follows easily from definition 1.3.1 and the fact that $\mathcal{L}_\xi g = 0$. $\square$

The following definition is used to describe the behavior of the orbits of ξ.

Definition 1.3.4. The foliation $\mathcal{F}_\xi$ is *quasi-regular* if each $x \in M$ has a neighborhood U_x such that any leaf of $\mathcal{F}_\xi$ passes through U_x at most $m_x \in \mathbb{Z}^+$ times. If $m_x = 1$ for all $x \in M$, then $\mathcal{F}_\xi$ is called *regular*. If for some point, no such m exists, then $\mathcal{F}_\xi$ is called *irregular*.

If all of the orbits of ξ are compact then $\mathcal{F}_\xi$ is quasi-regular. In this case, ξ generates an S^1 action on M. If this action is free, then $\mathcal{F}_\xi$ is regular. If the S^1 action is only locally free, then $\mathcal{F}_\xi$ is strictly quasi-regular. Assuming M is compact, quasi-regularity implies that the orbits of ξ are all compact. Thus, if ξ has a non-compact orbit, then $\mathcal{F}_\xi$ is irregular. We will often say that M is regular (quasi-regular, irregular) to mean that $\mathcal{F}_\xi$ is regular (quasi-regular, irregular).

Example 1.3.5. A good example to keep in mind is the *weighted Sasaki structure* on $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. Pick real numbers $a_0, a_1 > 0$ and consider the vector field

$$\xi = \sqrt{-1} \sum_j a_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

Restricting to S^3, ξ is everywhere tangent to S^3 and nowhere vanishing. The 1-form

$$\eta = -\frac{\sqrt{-1}}{2} \sum_j \frac{1}{a_j} \left(\bar{z}_j dz^j - z_j d\bar{z}_j \right)$$

is a contact form with $\eta(\xi) = 1$. On $\mathcal{D} \subset TS^3$, we let Φ be the restriction of the standard complex structure on $\mathbb{C}^2$ and extend it trivially in the ξ direction. Then $g = \eta \otimes \eta + d\eta(\Phi \cdot, \cdot)$ defines a Sasaki-metric on S^3. The integral curve of ξ through the point $(z_0, z_1) \in S^3$ is given by

$$(z_0, z_1) \mapsto (e^{2\pi i a_0 t} z_0, e^{2\pi i a_1 t} z_1).$$

Choosing $a_0 = a_1 = 1$ gives the standard Sasaki structure on S^3. If a_0 and a_1 are chosen to be relatively prime integers, then $\mathcal{F}_\xi$ is regular. If a_0 and a_1 are not coprime or $a_0, a_1 \in \mathbb{Q} \setminus \mathbb{Z}$, then $\mathcal{F}_\xi$ is quasi-regular. Suppose that $\frac{a_0}{a_1}$ is irrational. Orbits through points of the form $(z_0, 0)$ are circles as they return to $(z_0, 0)$ at time $t = a_0^{-1}$. Similarly, orbits through $(0, z_1)$ are circles that return in time $t = a_1^{-1}$. However, the orbits through points (z_0, z_1) where $z_0 z_1 \neq 0$ are not compact. Since $\frac{a_0}{a_1}$ is irrational, there is no time t such that both $a_0 t$ and $a_1 t$ are integers. Thus the integral curve never returns to (z_0, z_1). The closure of these orbits is the torus T^2. This example easily generalizes to higher odd dimensional spheres.

Adapting theorem 6.3.8 of [5] to the special case of a quasi-Sasaki manifold, we get the following structure theorem. This important theorem will be used frequently in the sequel. See chapter 4.3 of [5] for the definition of the orbifold cohomology. If the orbifold is in fact a manifold, then the orbifold cohomology is the usual cohomology.

Theorem 1.3.6. *Let (M, ξ, η, Φ, g) be a quasi-Sasaki manifold of rank $2p+1$ such that the leaves of $\mathcal{F}_\xi$ are all compact. Then*

1. *The leaf space $M/\mathcal{F}_\xi$ is a Kähler orbifold such that the canonical projection $\pi : M \to M/\mathcal{F}_\xi$ is an orbifold Riemannian submersion. The fibers are totally geodesic submanifolds of M diffeomorphic to S^1.*

2. *M is the total space of a principal S^1-orbibundle over $M/\mathcal{F}_\xi$ with connection 1-form η. The curvature form $d\eta = \pi^*\Omega$ where Ω is a closed (1,1)-form of rank p on $M/\mathcal{F}_\xi$ such that $\frac{1}{2\pi}[\Omega] \in H^2_{orb}(M/\mathcal{F}_\xi, \mathbb{Z})$.*

3. *If $\mathcal{F}_\xi$ is regular, then M is the total space of a principal S^1-bundle over the Kähler manifold $M/\mathcal{F}_\xi$.*

Now we will show that the transverse geometry of a quasi-Sasaki manifold has a Kähler structure. From $\Phi^2 = -I + \xi \otimes \eta$, we have $(\Phi|_{\mathcal{D}})^2 = -I$. Hence $\Phi|_{\mathcal{D}}$ is an almost complex structure on $\mathcal{D}$. We consider the complexified contact bundle $\mathcal{D}_{\mathbb{C}} := \mathcal{D} \otimes_{\mathbb{R}} \mathbb{C}$. Then $\Phi|_{\mathcal{D}}$ extends $\mathbb{C}$-linearly to an almost complex structure on $\mathcal{D}_{\mathbb{C}}$, which we continue to call Φ. Let $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$ denote the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of Φ. Then $\mathcal{D}_{\mathbb{C}} = \mathcal{D}^{1,0} \oplus \mathcal{D}^{0,1}$. Complex conjugation is an $\mathbb{R}$-linear isomorphism between $\mathcal{D}^{1,0}$ and $\mathcal{D}^{0,1}$.

As the transverse metric is Φ-invariant (i.e. $\Phi^* g^T = g^T$), it defines a Hermitian metric on $\mathcal{D}^{1,0}$. Explicitly, we $\mathbb{C}$-linearly extend g^T in both entries to a symmetric, complex bilinear form on $\mathcal{D}_{\mathbb{C}}$, which we continue to call g^T. Then for any $U, V \in \mathcal{D}^{1,0}$, we define the Hermitian metric h by

$$h(U,V) := g^T(U, \bar{V}).$$

The map $\sigma : \mathcal{D} \to \mathcal{D}^{1,0}$ that sends $X \mapsto \frac{1}{2}\left(X - \sqrt{-1}\Phi X\right)$ is an isomorphism of complex vector bundles $(\mathcal{D}, \Phi) \simeq (\mathcal{D}^{1,0}, \sqrt{-1})$. Under this isomorphism, a straightforward computation reveals that

$$\sigma^* h = \frac{1}{2}\left(g^T + \sqrt{-1}\omega\right).$$

It is well-known that the holomorphic tangent bundle of an almost complex manifold is involutive if and only if the almost complex structure is integrable. In the almost contact setting, we have a similar theorem.

Theorem 1.3.7. *An almost contact structure is normal if and only if* $[\mathcal{D}^{1,0}, L_\xi] \subset \mathcal{D}^{1,0}$ *and* $[\mathcal{D}^{1,0}, \mathcal{D}^{1,0}] \subset \mathcal{D}^{1,0}$.

Proof. By proposition 1.2.6, normality is equivalent to $N_\Phi + 2d\eta \otimes \xi \equiv 0$. Now let $X, Y \in \mathcal{D}^{1,0}$. Using equation (1.3) and $\Phi^2 = -I + \xi \otimes \eta$, we compute that

$$
\begin{aligned}
N_\Phi(X,Y) + 2d\eta(X,Y)\xi &= -2[X,Y] - 2\sqrt{-1}\Phi[X,Y], \\
N_\Phi(X,\bar{Y}) + 2d\eta(X,\bar{Y})\xi &= 0, \\
N_\Phi(X,\xi) + 2d\eta(X,\xi)\xi &= -[X,\xi] - \sqrt{-1}\Phi[X,\xi].
\end{aligned}
$$

So $N_\Phi + 2d\eta \otimes \xi \equiv 0$ if and only if $\Phi[X,Y] = \sqrt{-1}[X,Y]$ and $\Phi[X,\xi] = \sqrt{-1}[X,\xi]$. □

Corollary 1.3.8. *An almost contact structure is normal if and only if the Nijenhuis torsion tensor N_Φ vanishes on $\mathcal{D}^{1,0} \oplus L_\xi$.*

Proof. Since $\Phi^* d\eta = d\eta$, we have $d\eta(X,Y) = 0$ for $X, Y \in \mathcal{D}^{1,0}$. By lemma 1.2.7, $d\eta(X,\xi) = 0$. The corollary now follows from the computation in theorem 1.3.7. □

So for a quasi-Sasaki structure, the contact bundle $\mathcal{D}$ is endowed with a complex structure Φ and a Hermitian metric whose Kähler form is closed. Thus the data $(\mathcal{D}, \Phi, \omega)$ gives M a transverse Kähler structure. We define a connection ∇^T on $\mathcal{D}$ by

$$
\begin{aligned}
\nabla^T_X Y &:= \pi_{\mathcal{D}}\left(\nabla_X Y\right), \ X \in \mathcal{D}, \\
\nabla^T_\xi Y &:= \pi_{\mathcal{D}}\left[\xi, Y\right].
\end{aligned}
$$

One can check that ∇^T is the unique, torsion-free connection on $\mathcal{D}$ with $\nabla^T g^T = 0$. For this reason we call ∇^T the *transverse Levi-Civita connection.* We define the *transverse curvature operator* for $X, Y \in \mathcal{D}$ by

$$
R^T(X,Y) := \nabla^T_X \nabla^T_Y - \nabla^T_Y \nabla^T_X - \nabla^T_{[X,Y]}.
$$

With this we can then define the transverse Riemannian curvature tensor, the transverse Ricci curvature and the transverse scalar curvature in the obvious way. When computing transverse curvatures, we have all of the familiar formulas from Kähler geometry. Next we briefly mention the transverse de Rham cohomology and its associated Hodge theory. For a detailed discussion, see [15] and [24].

Definition 1.3.9. A function $f \in C^\infty(M)$ is called basic if $df(\xi) = 0$. A p-form $\theta \in \Omega^p(M)$ is called *basic* if $\iota_\xi\theta = 0$ and $\mathcal{L}_\xi\theta = 0$.

We let $\Lambda^p_B(\mathcal{F}_\xi)$ be the sheaf of germs of basic p-forms and $\Omega^p_B(\mathcal{F}_\xi)$ the set of global sections of $\Lambda^p_B(\mathcal{F}_\xi)$. Observe that if θ is basic then so is $d\theta$. This follows from $\mathcal{L}_\xi\theta = \iota_\xi d\theta + d\iota_\xi\theta$ and $\mathcal{L}_\xi d\theta = d\mathcal{L}_\xi\theta$. We set $d_B := d|_{\Omega^p_B}$. Then $d_B : \Omega^p_B \to \Omega^{p+1}_B$ is well-defined, $d_B \circ d_B = 0$ and we have a subcomplex of the de Rham complex:

$$0 \to C^\infty_B \to \Omega^1_B \to \cdots \to \Omega^{2n}_B \to 0.$$

We denote the cohomology ring by $H^*_B(\mathcal{F}_\xi)$ and call it the *basic cohomology ring.* There is a transverse Hodge theory for the basic cohomology ring and in many ways it resembles the de Rham-Hodge theory of a Kähler manifold. The transverse Hodge star, $*_B$, is defined in terms of the usual Hodge star by $*_B\alpha := *(\eta \wedge \alpha)$. The adjoint operator $\delta_B : \Omega^p_B \to \Omega^{p-1}_B$ of d_B is defined by

$$\delta_B := - *_B \circ d_B \circ *_B.$$

We define the *basic Hodge Laplacian* (with respect to d_B) by $\triangle_B := d_B\delta_B + \delta_B d_B$. We say that $\theta \in \Omega^p_B$ is harmonic if $\triangle_B\theta = 0$. The *transverse Hodge theorem* of [14] states that every basic cohomology class has a unique harmonic representative. We define the *transverse tensor Laplacian* by $\Delta^T := \nabla^T\nabla^T$. For a basic function f, one can check that

$$\triangle_B f = -\Delta^T f = -g^{i\bar{j}}\partial_i\partial_{\bar{j}} f.$$

The transverse complex structure Φ induces a natural decomposition of the basic p-forms into basic (i,j)-forms where $i+j=p$:

$$\Omega_B^p \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{i+j=p}\Omega_B^{i,j}$$

and the exterior derivative splits as $d_B = \partial_B + \bar{\partial}_B$ where

$$\partial_B : \Omega_B^{i,j} \to \Omega_B^{i+1,j} \quad \text{and} \quad \bar{\partial}_B : \Omega_B^{i,j} \to \Omega_B^{i,j+1}.$$

From $d_B^2 = 0$ we get $\partial_B^2 = \bar{\partial}_B^2 = 0$ and $\partial_B\bar{\partial}_B + \bar{\partial}_B\partial_B = 0$. As we did above, we can define the basic Hodge Laplacians with respect to ∂_B and $\bar{\partial}_B$. We can also define a transverse Lefschetz operator $L : \theta \mapsto \theta \wedge \omega$. One can then recover the expected transverse Kähler identities. We set

$$d_B^c := \frac{\sqrt{-1}}{2}(\bar{\partial}_B - \partial_B).$$

Then we have $d_B d_B^c = \sqrt{-1}\partial_B\bar{\partial}_B$ and $d_B^c \circ d_B^c = 0$. The *transverse Ricci form* ρ^T is the basic 2-form defined by

$$\rho^T := \mathrm{Ric}^T(\cdot, \Phi\cdot).$$

Just as in the Kähler setting, one can show that

$$\rho^T = -\sqrt{-1}\partial_B\bar{\partial}_B \log(\det(g^T)).$$

Thus ρ^T is a d_B closed (1,1)-form and so defines a basic cohomology class. The class $[\rho^T] \in H^2_B(\mathcal{F}_\xi)$ is independent of the choice of transverse Kähler metric and, as in the Kähler setting, $2\pi c^1_B = [\rho^T]$, where c^1_B is the *basic first Chern class.*

1.4. Transverse Distance and Preferred Coordinates

First we introduce the transverse distance function that will be important for us in chapter III. Let (M, g) be quasi-Sasaki and assume that M is compact and quasi-regular. For a point $x \in M$, let γ_x denote the orbit of ξ through x. Since γ_x is compact for all $x \in M$, we can make the following definition:

Definition 1.4.1. The *transverse distance* function d^T is defined on M by

$$d^T(x, y) := d(\gamma_x, \gamma_y)$$

where d is the geodesic distance function on M, determined by g. The *transverse ball* of radius $r > 0$ about a point $x \in M$ is defined as

$$B^T(x, r) := \{y \in M : d^T(x, y) < r\}.$$

The *transverse diameter* of M is the number Θ^T defined as

$$\Theta^T := \max_{x,y \in M} d^T(x, y).$$

The transverse distance function defines a metric on the leaf space $M/\mathcal{F}_\xi$. It turns out to be the same as the geodesic distance function on $M/\mathcal{F}_\xi$, determined by g^T. In order to prove that d^T is a metric, we need to show that it satisfies the triangle inequality. To do so, we make use of the following proposition:

Proposition 1.4.2. *For $x, y \in M$ and $p \in \gamma_x$, there is $q \in \gamma_y$ such that $d^T(x, y) = d(p, q)$.*

Proof. Since the curves γ_x and γ_y are compact, there are points $x_1 \in \gamma_x$ and $y_1 \in \gamma_y$ such that $d^T(x, y) = d(x_1, y_1)$. Now let $\sigma : [0, 1] \to M$ be the length minimizing geodesic with $\sigma(0) = x_1$ and $\sigma(1) = y_1$. Flowing along γ_x is an isometry of M. As γ_x is compact, it is diffeomorphic to S^1. Thus there is $\zeta \in S^1$ such that $\zeta(x_1) = p$. Let $q := \zeta(y_1)$ and $\tilde{\sigma} = \zeta \circ \sigma$. Then $\tilde{\sigma}$ is a geodesic connecting p and q whose length is the same as σ. Therefore, $d^T(x, y) = d(x_1, y_1) = d(p, q)$. □

Corollary 1.4.3. *The transverse distance function satisfies the triangle inequality. For fixed $x_0 \in M$, $r(x) := d^T(x, x_0)$ is a basic Lipschitz function.*

Proof. Pick points $x, y, z \in M$. By the previous proposition, there are points $p \in \gamma_x$, $q \in \gamma_y$ and $s \in \gamma_z$ such that $d^T(x, y) = d(p, q)$ and $d^T(y, z) = d(q, s)$. Thus

$$d^T(x, z) \leq d(p, s) \leq d(p, q) + d(q, s) = d^T(x, y) + d^T(y, z).$$

Let r be as above. By the triangle inequality, $|r(x) - r(y)| \leq d^T(x, y) \leq d(x, y)$. Thus r is Lipschitz continuous with Lipschitz constant 1. Hence r is differentiable almost everywhere and $|\nabla r| \leq 1$ at any point of differentiability. Since r is constant along the orbits of ξ, $dr(\xi) = 0$. Therefore, r is basic. □

We conclude this chapter by introducing the local coordinates and local frame that we will use in our computations in chapter II, III and IV. By normality, the almost complex structure J, defined by (1.1), on $M \times \mathbb{R}$ is integrable; hence it is induced by a complex structure. This means that there are (real) local coordinates $(x_0, y_0, \ldots, x_n, y_n)$ on $M \times \mathbb{R}$ such that $J\partial_{x_j} = \partial_{y_j}$ and $J\partial_{y_j} = -\partial_{x_j}$.

We will identify M as a submanifold of $M \times \mathbb{R}$ via $M \simeq M \times \{0\}$. Let t be the coordinate on $\mathbb{R}$. Then the tangent bundle $T(M\times\mathbb{R}) = TM \oplus \mathbb{R}[\frac{d}{dt}] = L_\xi \oplus \mathcal{D} \oplus \mathbb{R}[\frac{d}{dt}]$. Observe that $J\xi = \frac{d}{dt}$ and $J\frac{d}{dt} = -\xi$. Thus, $L_\xi \oplus \mathbb{R}[\frac{d}{dt}]$ forms an involutive, complex subbundle of $T(M \times \mathbb{R})$. By the complex version of the Frobenius theorem, there are local coordinates as above with $\partial_{x_0} = \xi$ and $\partial_{y_0} = \frac{d}{dt}$. Thus $y_0 = t$.

Now consider the complexification $T(M \times \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C}$-linearly extend J. For $j = 0, \ldots, n$, put $z_j = \frac{1}{2}(x_j - \sqrt{-1}y_j)$. Then $(z_0, \ldots, z_n, \bar{z}_0, \ldots, \bar{z}_n)$ are local coordinates on $M \times \mathbb{R}$ with $J\partial_{z_j} = \sqrt{-1}\partial_{z_j}$ and $J\partial_{\bar{z}_j} = -\sqrt{-1}\partial_{\bar{z}_j}$. We know that $\partial_{z_0} = \frac{1}{2}(\xi - \sqrt{-1}\frac{d}{dt})$. For $j \geq 1$, we can write $\partial_{z_j} = Z_j + f_j\frac{d}{dt}$ for some vector field $Z_j \in TM \otimes_{\mathbb{R}} \mathbb{C}$ and some complex-valued function f_j. By the definition of J,

$$J\partial_{z_j} = \Phi Z_j - f_j\xi + \eta(Z_j)\frac{d}{dt}.$$

But from $J\partial_{z_j} = \sqrt{-1}\partial_{z_j}$, we have

$$J\partial_{z_j} = \sqrt{-1}\left(Z_j + f_j\frac{d}{dt}\right).$$

Equating the two expression for $J\partial_{z_j}$, we find that $f_j = -\sqrt{-1}\eta(Z_j)$ and

$$\Phi Z_j = \sqrt{-1}(Z_j - \eta(Z_j)\xi).$$

Setting $t = 0$, the local coordinates $(z_0, \ldots, z_n, \bar{z}_0, \ldots, \bar{z}_n)$ on $M \times \mathbb{R}$ induce local coordinates on M. Since $t = y_0$, we get the local coordinates $(x, z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ where $x = x_0$. The coordinate vectors satisfy $\partial_x = \xi$ and $\Phi\partial_{z_j} = \sqrt{-1}(\partial_{z_j} - \eta(\partial_{z_j})\xi)$. We call these *preferred local coordinates*.

Since η is a real 1-form and $d\eta$ is a basic (1,1)-form, in a small coordinate neighborhood U we can write

$$\eta = dx - \sqrt{-1}(h_j dz^j - h_{\bar{j}} d\bar{z}^j)$$

where $h : U \to \mathbb{R}$ is a (local) real basic function (i.e. $\xi h = \partial_x h = 0$). We use the notation $h_{\bar{j}} = \partial_{\bar{z}_j} h = \frac{\partial h}{\partial \bar{z}_j} = \overline{\frac{\partial h}{\partial z_j}} = \overline{\partial_{z_j} h} = \overline{h_j}$. Then, defining $d\eta_{i\bar{j}} := h_{i\bar{j}}$, we have

$$d\eta = \sqrt{-1} d\eta_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Now, for $j = 1, \dots, n$, define vector fields

$$X_j := \partial_{z_j} + \sqrt{-1} h_j \xi \quad \text{and} \quad X_{\bar{j}} := \bar{X}_j = \partial_{\bar{z}_j} - \sqrt{-1} h_{\bar{j}} \xi.$$

Notice that $\eta(X_j) = \eta(\bar{X}_j) = 0$, while $\Phi X_j = \sqrt{-1} X_j$ and $\Phi \bar{X}_j = -\sqrt{-1} \bar{X}_j$. Thus, $\mathcal{D}^{1,0} = \text{span}\{X_j\}$ and $\mathcal{D}^{0,1} = \text{span}\{\bar{X}_j\}$. We call the frame $(\xi, X_1, \dots, X_n, \bar{X}_1, \dots, \bar{X}_n)$ a *preferred local frame*. Since $d\eta(\cdot, \xi) = 0$, we see that

$$d\eta(X_i, \bar{X}_j) = d\eta(\partial_{z_i}, \partial_{\bar{z}_j}) = \sqrt{-1} d\eta_{i\bar{j}}.$$

Since the h_j are basic functions, we compute that

$$[X_i, X_j] = [\bar{X}_i, \bar{X}_j] = [\xi, X_i] = [\xi, \bar{X}_j] = 0$$

and

$$[X_i, \bar{X}_j] = -2\sqrt{-1} d\eta_{i\bar{j}} \xi.$$

From the compatibility of Φ with g, in preferred local coordinates we can write

$$g = \eta \otimes \eta + g^T_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

where $g^T_{i\bar{j}} = \overline{g^T_{\bar{j}i}} = g^T_{\bar{j}i}$. Since $\mathcal{L}_\xi g = \mathcal{L}_\xi \eta = 0$, the functions $g^T_{i\bar{j}}$ are basic. The transverse metric and Kähler form are, respectively,

$$g^T = g^T_{i\bar{j}} dz^i \otimes d\bar{z}^j \quad \text{and} \quad \omega = \sqrt{-1} g^T_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Hence

$$g^T(X_i, \bar{X}_j) = g^T(\partial_{z_i}, \partial_{\bar{z}_j}) = g^T_{i\bar{j}}$$

and

$$\omega(X_i, \bar{X}_j) = \omega(\partial_{z_i}, \partial_{\bar{z}_j}) = \sqrt{-1} g^T_{i\bar{j}}.$$

The transverse metric defines a pointwise norm on basic forms. If $\alpha = \alpha_i dz^i$ and $\beta = \beta_{\bar{j}} d\bar{z}^j$ are basic (1,0) and (0,1)-forms, respectively, then we define $|\alpha|^2_{g^T} = g^{i\bar{j}} \alpha_i \overline{\alpha_j}$ and $|\beta|^2_{g^T} = g^{i\bar{j}} \beta_{\bar{j}} \overline{\beta_{\bar{i}}}$. We can then extend this to basic forms of type (p, q). For example, the norm of a $(1,1)$-form $\theta = \theta_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is given by $|\theta|^2_{g^T} = g^{i\bar{j}} g^{k\bar{l}} \theta_{i\bar{l}} \overline{\theta_{j\bar{k}}}$. At a point where $g_{i\bar{j}} = \delta_{ij}$, we see that $|\theta|^2_{g^T} = \sum_{i,k} \theta_{i\bar{k}} \overline{\theta_{i\bar{k}}} = \sum_{i,k} |\theta_{i\bar{k}}|^2$, as we would expect. For any basic form θ, the norm with respect to the quasi-Sasaki metric g is twice that of the norm with respect to the transverse metric g^T; $|\theta|^2_g = 2|\theta|^2_{g^T}$.

CHAPTER II

RIGIDITY OF QUASI-SASAKI-EINSTEIN METRICS

2.1. The Ricci tensor and Einstein equation

A Sasaki metric g and $d\eta$ are intimately related by $g = \eta \otimes \eta + d\eta(\Phi\cdot, \cdot)$. It is not so clear, a priori, how a quasi-Sasaki metric and $d\eta$ are related. In this section we compute the Ricci tensor of a quasi-Sasaki metric. To better understand the relationship between the metric and $d\eta$, we want to know how the Ricci tensor depends on $d\eta$. Given that a quasi-Sasaki structure has a Riemannian foliation associated to it, one could compute the Ricci tensor of g using curvature formulas involving the O'Neill tensors A and T (see theorems 2.5.16 and 2.5.18 of [5]). However, these curvature formulas will not give us what we want directly and to use them requires a sufficiently large amount of computation anyway. Thus, we will present a barehanded computation of the Ricci tensor and we will see the role that $d\eta$ plays in the curvature. We shall compute the Ricci tensor with respect to a preferred local frame $\{\xi, X_i, \bar{X}_i\}$. Recall that $\{X_i\}$ is a local frame for $\mathcal{D}^{1,0}$, $g^T(X_i, \bar{X}_j) = g^T_{i\bar{j}}$ and $[\xi, X_i] = [\xi, \bar{X}_i] = 0$.

In lemma 1.2.7 we showed that $\nabla_\xi \xi = 0$. Now we observe that

$$\nabla_{X_i}\xi = \sqrt{-1} g^{k\bar{j}} d\eta_{i\bar{j}} X_k \quad \text{and} \quad \nabla_{\bar{X}_i}\xi = \overline{\nabla_{X_i}\xi} = -\sqrt{-1} g^{j\bar{k}} d\eta_{j\bar{i}} \bar{X}_k. \tag{2.1}$$

To see this, write $\nabla_{X_i}\xi = \alpha\xi + \beta^j X_j + \gamma^k \bar{X}_k$ and use $g(\nabla_X \xi, Y) = d\eta(X, Y)$, keeping in mind that $d\eta$ is a basic (1,1)-form. Let $\pi_{L_\xi} : TM \to L_\xi$ be the orthogonal projection onto L_ξ. Proposition 2.5.14 in [5] asserts that for $X, Y \in \mathcal{D}$,

$$\pi_{L_\xi}\left(\nabla_X Y\right) = \frac{1}{2}\pi_{L_\xi}\left([X, Y]\right). \tag{2.2}$$

Writing $[X, Y] = \alpha\xi + \beta^i X_i + \gamma^i \bar{X}_i$ and applying η to both sides yields $\alpha = \eta([X, Y])$. Hence $\pi_{L_\xi}([X, Y]) = \eta([X, Y])\xi$. By equation (1.3), for $X, Y \in \mathcal{D}$, we have $\eta([X, Y]) = -2d\eta(X, Y)$. Therefore, by (2.2) and the above,

$$\pi_{L_\xi}(\nabla_X Y) = -d\eta(X, Y)\xi. \tag{2.3}$$

Now, as $\nabla_X Y = \nabla^T_X Y + \pi_{L_\xi}(\nabla_X Y)$ for $X, Y \in \mathcal{D}$, equation (2.3) yields

$$\nabla_X Y = \nabla^T_X Y - d\eta(X, Y)\xi. \tag{2.4}$$

Since the transverse metric is Kähler and $d\eta$ is a (1,1)-form, it follows from (2.4) that

$$\nabla_{X_i} X_j = \nabla^T_{X_i} X_j \quad \text{and} \quad \nabla_{X_i} \bar{X}_j = -\sqrt{-1}\, d\eta_{i\bar{j}}\xi. \tag{2.5}$$

Equations (2.1) and (2.5) along with $\nabla_\xi \xi = 0$ allow us to compute the curvature tensor of a quasi-Sasaki structure with respect to our preferred frame. For notational convenience, we will drop the superscript T from the transverse metric g^T.

First we compute

$$\begin{aligned} R(X_i, \xi)\xi &= \nabla_{X_i}\nabla_\xi \xi - \nabla_\xi \nabla_{X_i}\xi - \nabla_{[X_i,\xi]}\xi \\ &= 0 - \sqrt{-1} g^{k\bar{j}} d\eta_{i\bar{j}} \nabla_\xi X_k - 0 \\ &= g^{k\bar{j}} g^{m\bar{p}} d\eta_{i\bar{j}} d\eta_{k\bar{p}} X_m. \end{aligned}$$

Hence we have $g(R(X_i, \xi)\xi, X_{\bar{l}}) = g^{k\bar{j}} d\eta_{i\bar{j}} d\eta_{k\bar{l}} = g^{k\bar{j}} d\eta_{i\bar{j}} \overline{d\eta_{l\bar{k}}}$ and therefore,

$$\mathrm{Ric}(\xi, \xi) = 2g^{i\bar{l}} g(R(X_i, \xi)\xi, X_{\bar{l}}) = 2g^{i\bar{l}} g^{k\bar{j}} d\eta_{i\bar{j}} \overline{d\eta_{l\bar{k}}} = |d\eta|^2_g \geq 0.$$

Next we will compute $\mathrm{Ric}(X_j, \xi)$. We start with

$$\begin{aligned}
R(\bar{X}_i, \xi)X_j &= \nabla_{\bar{X}_i}\nabla_\xi X_j - \nabla_\xi \nabla_{\bar{X}_i} X_j - \nabla_{[\bar{X}_i,\xi]}X_j \\
&= \sqrt{-1}g^{k\bar{p}}\nabla_{\bar{X}_i} d\eta_{j\bar{p}}X_k - \sqrt{-1}\nabla_\xi(d\eta_{j\bar{i}}\xi) - 0 \\
&= \sqrt{-1}g^{k\bar{p}}\left(\bar{X}_i d\eta_{j\bar{p}}X_k + d\eta_{j\bar{p}}\nabla_{\bar{X}_i}X_k\right) - 0 \\
&= \sqrt{-1}g^{k\bar{p}}\left(\bar{X}_i d\eta_{j\bar{p}}X_k + \sqrt{-1}d\eta_{j\bar{p}}d\eta_{k\bar{i}}\xi\right).
\end{aligned}$$

From equation (2.5) we have $[X_i, X_{\bar{j}}] = -2\sqrt{-1}d\eta_{i\bar{j}}\xi$. Using this we compute that

$$\begin{aligned}
R(\bar{X}_i, X_j)\xi &= \nabla_{\bar{X}_i}\nabla_{X_j}\xi - \nabla_{X_j}\nabla_{\bar{X}_i}\xi - \nabla_{[\bar{X}_i,X_j]}\xi \\
&= \sqrt{-1}g^{k\bar{p}}\left(\nabla_{\bar{X}_i} d\eta_{j\bar{p}}X_k\right) + \sqrt{-1}g^{m\bar{q}}\left(\nabla_{X_j} d\eta_{m\bar{i}}X_{\bar{q}}\right) - 2\sqrt{-1}d\eta_{j\bar{i}}\nabla_\xi\xi \\
&= \sqrt{-1}g^{k\bar{p}}\left(\bar{X}_i d\eta_{j\bar{p}}X_k + d\eta_{j\bar{p}}\nabla_{\bar{X}_i}X_k\right) + \sqrt{-1}g^{m\bar{q}}\left(X_j d\eta_{m\bar{i}}X_{\bar{q}} + d\eta_{m\bar{i}}\nabla_{X_j}X_{\bar{q}}\right) - 0 \\
&= \sqrt{-1}g^{k\bar{p}}\left(\bar{X}_i d\eta_{j\bar{p}}X_k + \sqrt{-1}d\eta_{j\bar{p}}d\eta_{k\bar{i}}\xi\right) + \sqrt{-1}g^{m\bar{q}}\left(X_j d\eta_{m\bar{i}}X_{\bar{q}} - \sqrt{-1}d\eta_{m\bar{i}}d\eta_{j\bar{q}}\xi\right).
\end{aligned}$$

Now we can compute the Ricci curvature

$$\begin{aligned}
\mathrm{Ric}(X_j, \xi) &= g^{l\bar{i}}\left(g(R(X_l, X_j)\xi, \bar{X}_i) + g(R(\bar{X}_i, X_j)\xi, X_l)\right) \\
&= g^{l\bar{i}}\left(g(R(\bar{X}_i, \xi)X_j, X_l) + g(R(\bar{X}_i, X_j)\xi, X_l)\right) \\
&= 0 + \sqrt{-1}g^{l\bar{i}}(\nabla_{X_j} g^{m\bar{q}} d\eta_{m\bar{i}})g_{l\bar{q}} \\
&= \sqrt{-1}\nabla_{X_j} g^{m\bar{i}} d\eta_{m\bar{i}} \\
&= \nabla_{X_j}\mathrm{tr}_{g^T}(d\eta)
\end{aligned}$$

where tr_{g^T} denotes the trace by the transverse metric g^T. For basic forms, $\mathrm{tr}_{g^T} = \frac{1}{2}\mathrm{tr}_g$.

Lastly we want to compute $\mathrm{Ric}(\bar{X}_j, X_k)$. We start by computing $R(X_i, \bar{X}_j)X_k$. Since we will eventually pair this with $X_{\bar{l}}$, we will neglect any terms in the ξ direction.

$$\begin{aligned}
R(X_i, \bar{X}_j)X_k &= \nabla_{X_i}\nabla_{\bar{X}_j}X_k - \nabla_{\bar{X}_j}\nabla_{X_i}X_k - \nabla_{[X_i,\bar{X}_j]}X_k \\
&= \nabla_{X_i}(\sqrt{-1}d\eta_{k\bar{j}}\xi + \nabla^T_{\bar{X}_j}X_k) - \nabla_{\bar{X}_j}(\nabla^T_{X_i}X_k) + 2\sqrt{-1}d\eta_{i\bar{j}}\nabla_\xi X_k \\
&= \sqrt{-1}d\eta_{k\bar{j}}\nabla_{X_i}\xi + \nabla_{X_i}\nabla^T_{\bar{X}_j}X_k - \nabla_{\bar{X}_j}\nabla^T_{X_i}X_k - 2g^{m\bar{p}}d\eta_{i\bar{j}}d\eta_{k\bar{p}}X_m \\
&= -g^{m\bar{p}}d\eta_{k\bar{j}}d\eta_{i\bar{p}}X_m + \nabla^T_{X_i}\nabla^T_{\bar{X}_j}X_k - \nabla^T_{\bar{X}_j}\nabla^T_{X_i}X_k - 2g^{m\bar{p}}d\eta_{i\bar{j}}d\eta_{k\bar{p}}X_m.
\end{aligned}$$

As $[\xi, X_k] = 0$, $\nabla^T_\xi X_k = \pi_{\mathcal{D}}([\xi, X_k]) = 0$. Since $[X_i, \bar{X}_j] = -2\sqrt{-1}d\eta_{i\bar{j}}\xi$, it follows that $\nabla^T_{[X_i,\bar{X}_j]}X_k = 0$. Thus the above can be written as

$$R(X_i, \bar{X}_j)X_k = R^T(X_i, \bar{X}_j)X_k - g^{m\bar{p}}\left(d\eta_{k\bar{j}}d\eta_{i\bar{p}} + 2d\eta_{i\bar{j}}d\eta_{k\bar{p}}\right)X_m.$$

By a computation similar to the one above, modulo the terms in the ξ direction,

$$R(\bar{X}_l, \bar{X}_j)X_k = R^T(\bar{X}_l, \bar{X}_j)X_k + g^{m\bar{p}}\left(d\eta_{k\bar{j}}d\eta_{m\bar{l}} - d\eta_{k\bar{l}}d\eta_{m\bar{j}}\right)\bar{X}_p.$$

Now we are ready to compute the Ricci curvature. Using the above we find

$$\begin{aligned}
\mathrm{Ric}(\bar{X}_j, X_k) &= g^{i\bar{l}}\left(g(R(X_i, \bar{X}_j)X_k, X_{\bar{l}}) + g(R(\bar{X}_l, \bar{X}_j)X_k, X_i)\right) + g(R(\xi, \bar{X}_j)X_k, \xi) \\
&= \mathrm{Ric}^T(\bar{X}_j, X_k) - 2g^{i\bar{l}}d\eta_{i\bar{j}}d\eta_{k\bar{l}}.
\end{aligned}$$

The above computations prove our next proposition.

Proposition 2.1.1. *With respect to a preferred frame, the Ricci tensor of a quasi-Sasaki structure* (ξ, η, Φ, g) *satisfies*

$$\mathrm{Ric}(\xi,\xi) = 2g^{i\bar{l}}g^{k\bar{j}}d\eta_{i\bar{j}}d\eta_{k\bar{l}} = |d\eta|_g^2, \tag{2.6}$$

$$\mathrm{Ric}(X_k,\xi) = \sqrt{-1}\nabla_{X_k}g^{i\bar{j}}d\eta_{i\bar{j}} = \nabla_{X_k}\mathrm{tr}_{g^T}(d\eta), \tag{2.7}$$

$$\mathrm{Ric}(X_i,\bar{X}_l) = \mathrm{Ric}^T(X_i,\bar{X}_l) - 2g^{k\bar{j}}d\eta_{i\bar{j}}d\eta_{k\bar{l}}. \tag{2.8}$$

If our quasi-Sasaki structure is actually Sasaki, then the above equations should yield the well-known formulae for the Ricci tensor of a Sasaki metric. In the Sasaki setting, one has $d\eta_{i\bar{j}} = g_{i\bar{j}}$. Substituting this into the above equations we get

$$\begin{aligned}\mathrm{Ric}(\xi,\xi) &= 2n,\\ \mathrm{Ric}(X_k,\xi) &= 0,\\ \mathrm{Ric}(X_i,\bar{X}_l) &= \mathrm{Ric}^T(X_i,\bar{X}_l) - 2g(X_i,\bar{X}_l),\end{aligned}$$

which is indeed correct.

Recall that an Einstein metric g is one where $\mathrm{Ric}_g = \lambda g$ for some constant $\lambda \in \mathbb{R}$. Proposition 2.1.1 gives us more information about the nature of a quasi-Sasaki-Einstein metric. Since ξ has unit length, $L_\xi \perp \mathcal{D}$ and $g(X_i,\bar{X}_j) = g^T(X_i,\bar{X}_j)$, proposition 2.1.1 gives us our next result.

Proposition 2.1.2. *A quasi-Sasaki metric is Einstein with* $\mathrm{Ric}_g = \lambda g$ *if and only if*

$$|d\eta|_g^2 = \lambda, \tag{2.9}$$

$$\nabla_{X_k}\mathrm{tr}_{g^T}(d\eta) = 0, \tag{2.10}$$

$$\mathrm{Ric}^T(X_i,\bar{X}_l) - 2g^{k\bar{j}}d\eta_{i\bar{j}}d\eta_{k\bar{l}} = \lambda g^T(X_i,\bar{X}_l). \tag{2.11}$$

Remark 2.1.3. The above proposition lends us some useful information. First we see that $\lambda \geq 0$ and that the norm of $d\eta$ is constant over M. If $\lambda > 0$ (i.e. $d\eta \neq 0$), then by Myers's theorem M is compact. Since $d\eta$ is basic, the second equation tells us that the trace of $d\eta$ is constant. Since $d\eta$ is an exact (1,1)-form, this implies that $\delta(d\eta) = 0$ and therefore $d\eta$ is harmonic. Finally, the trace of the third equation by the transverse metric yields $R^T = \lambda(n+1) \geq 0$. Thus the transverse scalar curvature is constant. This implies that the transverse Ricci form ρ^T is harmonic.

Proposition 2.1.4. *If a Kähler metric has constant scalar curvature then the Ricci form is harmonic.*

Proof. Let $g = g_{\alpha\bar{\beta}} dz^\alpha \otimes dz^{\bar{\beta}}$ be a Kähler metric and $\rho = \sqrt{-1} R_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$ the Ricci form. Recall that ρ is a closed (1,1)-form and $R_{\alpha\bar{\beta}} = -\partial_\alpha \partial_{\bar{\beta}} \log\det(g)$. Using $\delta = -g^{kl} \iota_{\partial_k} \nabla_{\partial_l}$ we compute that

$$\begin{aligned}
\delta\rho &= \sqrt{-1} g^{\lambda\bar{\mu}} \left(R_{\alpha\bar{\mu}/\lambda} dz^\alpha - R_{\lambda\bar{\beta}/\bar{\mu}} dz^{\bar{\beta}} \right) \\
&= \sqrt{-1} g^{\lambda\bar{\mu}} \left(R_{\lambda\bar{\mu}/\alpha} dz^\alpha - R_{\lambda\bar{\mu}/\bar{\beta}} dz^{\bar{\beta}} \right) \\
&= \sqrt{-1} \left(\frac{\partial R}{\partial z_\alpha} dz^\alpha - \frac{\partial R}{\partial z_{\bar{\beta}}} dz^{\bar{\beta}} \right).
\end{aligned}$$

So if the scalar curvature R is constant, then $\delta\rho = d\rho = 0$. Hence ρ is harmonic. $\square$

Remark 2.1.5. If the manifold is compact, then the converse to the above is true. That is, a compact Kähler metric with harmonic Ricci form has constant scalar curvature. To see this, let Λ denote the dual Lefschetz operator and recall that $\Lambda\rho = R$ and Λ commutes with the Hodge Laplacian $\triangle = d\delta + \delta d$. So if ρ is harmonic then so is R. Harmonic functions on compact manifolds are constant. Hence R is constant.

2.2. The Regular Case

Let (M, ξ, η, Φ, g) be a quasi-Sasaki manifold. In this section we assume that the orbits of ξ are compact and the induced S^1 action is free. Hence $\mathcal{F}_\xi$ is regular. By theorem 1.3.6, M is the total space of a principal circle bundle over the Kähler manifold $B := M/\mathcal{F}_\xi$. The transverse geometry of M is the Kähler geometry of (B, g^T). For a manifold B (not necessarily Kähler), we let $\mathcal{P}(B, S^1)$ denote the collection of all principal circle bundles $\pi : P \to B$. We will quickly review some facts about $\mathcal{P}(B, S^1)$. For a reference to these facts, see chapter two of [4].

Theorem 2.2.1. *There is a binary operation on $\mathcal{P}(B, S^1)$ giving it the structure of a group isomorphic to $H^2(B, \mathbb{Z})$.*

For any integer $m \neq 0$, the cyclic group $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ is a subgroup of S^1. Thus, for any $P \in \mathcal{P}(B, S^1)$ there is a $\mathbb{Z}_m$ action on P and $P/\mathbb{Z}_m$ is a principal $S^1/\mathbb{Z}_m$ bundle over B. Since $S^1/\mathbb{Z}_m \simeq S^1$, we can consider $P/\mathbb{Z}_m \in \mathcal{P}(B, S^1)$.

Theorem 2.2.2. *For $P \in \mathcal{P}(B, S^1)$ and an integer $m \neq 0$, $P/\mathbb{Z}_m \simeq mP$. This implies that if P is simply connected and $|m| > 1$, then $P \neq mP'$ for any $P' \in \mathcal{P}(B, S^1)$.*

We let $V := \ker(\pi_*) \subset TP$ and call it the *vertical* distribution. For $\theta \in S^1$, we define $\hat{\theta} : P \to P$ by $p \mapsto \theta \cdot p$. A *connection* on $P \in \mathcal{P}(B, S^1)$ is a smooth distribution $H \subset TP$ such that $TP = H \oplus V$ and $\hat{\theta}_*(H) = H$ for all $\theta \in S^1$. We call H the *horizontal* distribution. A point $p \in P$ induces a map $\hat{p} : S^1 \to P$ by $\theta \mapsto \theta \cdot p$. The image of $\hat{p}$ is the orbit through p. It is diffeomorphic to $\pi^{-1}(\pi(p)) \simeq S^1$. At $1 \in S^1$, the differential of $\hat{p}$ is a map $\hat{p}_* : \mathbb{R} \to T_pP$. The image of $\hat{p}_*$ is V_p. The *connection form* of H is the S^1 invariant 1-form η on P such that for $X \in T_pP$, $\eta(X) = t$ where $\hat{p}_*(t)$ is the vertical part of X. Note that $H = \ker(\eta)$. For a principal S^1 bundle, the *curvature form* of the connection H is $d\eta$.

Lemma 2.2.3. *Given a connection form η on $P \in \mathcal{P}(B, S^1)$, there exists a unique, closed 2-form Ω on B such that $\frac{1}{2\pi}[\Omega] \in H^2(B, \mathbb{Z})$ and $d\eta = \pi^*\Omega$. The cohomology class of Ω is independent of the choice of connection form.*

The cohomology class $\frac{1}{2\pi}[\Omega]$ is called the *characteristic class* of the principal circle bundle P. Theorem 2.2.1 shows that $\mathcal{P}(B, S^1)$ is in bijection with $H^2(B, \mathbb{Z})$. The above lemma and the following theorem due to Kobayashi in [28] show that specifying a connection form η on P corresponds to choosing a representative of the characteristic class. Principal circle bundles are classified by their characteristic class.

Theorem 2.2.4. *Let Ω be a closed 2-form on B with $\frac{1}{2\pi}[\Omega] \in H^2(B, \mathbb{Z})$. Then there is $P \in \mathcal{P}(B, S^1)$ and a connection form η on P such that $d\eta = \pi^*\Omega$.*

If g is Einstein with $\mathrm{Ric}_g = \lambda g$, the first equation in proposition 2.1.2 shows that $\lambda \geq 0$. If $\lambda = 0$ then $d\eta = 0$ and by (2.11), $\mathrm{Ric}_{g^T} = 0$. Therefore, M is cosymplectic and B is Calabi-Yau. We know already that M is locally the product of a circle and a Kähler manifold (see remark 1.2.15), but in this case we can say more. Since $d\eta = 0$, the characteristic class $\frac{1}{2\pi}[\Omega] = 0$. This means that M is the trivial circle bundle over B. Therefore, $M = S^1 \times B$. This proves our next theorem.

Theorem 2.2.5. *If (M, g) is a quasi-Sasaki manifold with $\mathrm{Ric}_g = 0$ and $\mathcal{F}_\xi$ is regular with compact leaves, then $M = S^1 \times B$ where B is a Calabi-Yau manifold and g is a product metric.*

Now we assume that g is Einstein with $\mathrm{Ric}_g = \lambda g$ for $\lambda > 0$. Then by Myers's theorem M is compact, in which case the regularity of $\mathcal{F}_\xi$ is equivalent to the assumption that the orbits of ξ are compact and the S^1 action is free. The (1,1)-tensor θ associated to $d\eta$ is defined implicitly by

$$g(\theta(X), Y) = d\eta(X, Y).$$

If we write $\theta(X_i) = \theta_i^k X_k$, then $\theta_i^k = \sqrt{-1} g^{k\bar{j}} d\eta_{i\bar{j}}$. With this and proposition 2.1.2, the condition that g is Einstein with $\mathrm{Ric}_g = \lambda g$ is equivalent to

$$
\begin{aligned}
|d\eta|_g^2 &= \lambda, \\
\triangle_B d\eta &= 0, \\
\mathrm{Ric}_{g^T}(X,Y) &= \lambda g^T(X,Y) + 2g^T(\theta(X), \theta(Y)).
\end{aligned}
$$

From the third equation above we see that $\mathrm{Ric}_{g^T} > 0$. By theorem 1 of [27], B is simply connected. Up to a factor of 2π, the Ricci form of g^T is a representative of the first Chern class of B. Thus $c^1(B) > 0$. Before we give the main theorem of this section, we will prove the existence of quasi-Sasaki-Einstein metrics on certain principal circle bundles. These examples illustrate the general case.

Let $(B_i^{n_i}, g_i)$, $i = 1, \dots, k$, be Kähler-Einstein manifolds with complex dimension n_i and $c^1(B_i) > 0$. We can write $c^1(B_i) = q_i \alpha_i$ where $q_i \in \mathbb{Z}^+$ and $\alpha_i \in H^2(B_i, \mathbb{Z})$ is an indivisible class. By scaling the metrics appropriately, we can assume that $\mathrm{Ric}_{g_i} = q_i g_i$. Then we have $2\pi c^1(B_i) = [\rho_i] = q_i[\omega_i]$ and, hence, $\frac{1}{2\pi}[\omega_i] = \alpha_i$ is an indivisible, integral class. Let $B = B_1 \times \cdots \times B_k$ and $\pi_i : B \to B_i$ the projections. Pick integers b_i, not all zero, and set

$$\Omega = b_1 \pi_1^* \omega_1 + \cdots + b_k \pi_k^* \omega_k.$$

Then $0 \neq \frac{1}{2\pi}[\Omega] \in H^2(B, \mathbb{Z})$, so by theorem 2.2.4 there is a nontrivial principal circle bundle $\pi : M \to B$ and a connection form η on M such that $d\eta = \pi^*\Omega$.

Theorem 2.2.6. *There exists a quasi-Sasaki-Einstein metric of positive scalar curvature on the principal circle bundle described above.*

The proof of the theorem is mostly the same as the proof of theorem 1.4 in [41]. We use the construction due to Wang and Ziller in [41] to produce the Einstein metrics and then we show that they are in fact quasi-Sasaki. For the sake of completeness, we include the construction here.

Proof. We consider a metric on M of the form $g = a^2\eta \otimes \eta + \pi^* h$ where

$$h = a_1\pi_1^* g_1 + \cdots + a_k \pi_k^* g_k$$

for constants $a \neq 0$ and $a_i > 0$. The choice of a is inconsequential. We will show that we can choose the a_i so that g is Einstein, but first let us verify that this indeed defines a quasi-Sasaki metric.

The 1-form $\eta_a = a\eta$ determines the contact bundle $\mathcal{D} = \ker(\eta_a) = \ker(\eta)$. Since it is the horizontal distribution of the connection, it is invariant under the S^1 action and $TM = \mathcal{D} \oplus L_\xi$ where $L_\xi = \ker(\pi_*)$ is the vertical distribution and ξ is a vector field on M with $\eta(\xi) = 1$. Let $\xi_a = a^{-1}\xi$. Let J be the complex structure on B. Define an endomorphism $\Phi : TM \to TM$ by $\Phi := \tilde{\pi} \circ J \circ \pi_*$, where $\tilde{\pi}$ denotes the unique horizontal (with respect to η) lift of a vector field on B. Then $\Phi\xi = 0$ and for horizontal vector fields $X \in \mathcal{D}$, we have $\Phi^2 X = -X$. Therefore $\Phi^2 = -I + \xi_a \otimes \eta_a$. As h is invariant under J, it is easy to show that $\pi^* h(\Phi X, \Phi Y) = \pi^* h(X, Y)$. Hence

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta_a(X)\eta_a(Y).$$

Thus (ξ_a, η_a, Φ, g) is an almost contact metric structure on M. Since the base manifold B is complex, a computation shows that the structure is normal. The fundamental 2-form $\omega := g(\cdot, \Phi\cdot) = \pi^* h(\cdot, \Phi\cdot) = \pi^*(a_1\pi_1^*\omega_1 + \cdots + a_k\pi_k^*\omega_k)$, where ω_i is the Kähler form of g_i. Since each ω_i is closed, it follows that $d\omega = 0$. Therefore, (ξ_a, η_a, Φ, g) is quasi-Sasaki. Note that it is Sasaki precisely when $a_i = ab_i > 0$ for all $i = 1, \ldots, k$.

If the metric g is to satisfy $\mathrm{Ric}_g = \lambda g$, then by proposition 2.1.2, we must have

$$\lambda = |d\eta|_g^2 = 2a^2 \sum_{i=1}^{k} n_i \left(\frac{b_i}{a_i}\right)^2 > 0. \tag{2.12}$$

Since the Kähler forms ω_i are harmonic, we have that $d\eta$ is harmonic. Thus equation (2.10) is satisfied. The contact bundle has a natural splitting $\mathcal{D} = \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_k$ where $\pi_* \mathcal{D}_i = TB_i$. On $\mathcal{D}_i$ we have $d\eta = b_i \pi_i^* \omega_1$. So to satisfy equation (2.11), for $i = 1, \ldots, k$, we need

$$\frac{q_i}{a_i} = \lambda + 2a^2 \left(\frac{b_i}{a_i}\right)^2. \tag{2.13}$$

Combining equations (2.12) and (2.13), for each $j = 1, \ldots, k$, we have

$$\left(\sum_{i=1}^{k} n_i \left(\frac{b_i}{a_i}\right)^2\right) \left(\frac{q_j}{a_j} - \lambda\right) = \lambda \left(\frac{b_j}{a_j}\right)^2. \tag{2.14}$$

Introducing the new variables $s_j := \dfrac{q_j}{\lambda a_j}$, equation (2.14) becomes

$$\left(\sum_{i=1}^{k} n_i \left(\frac{b_i}{q_i}\right)^2 s_i^2\right) (s_j - 1) = \left(\frac{b_j}{q_j}\right)^2 s_j^2. \tag{2.15}$$

Since not all $b_j = 0$, this shows that $s_j \geq 1$ with equality if and only if $b_j = 0$. We can rearrange (2.15) to get

$$\left(1 + \frac{1}{n_j} - s_j\right) \sum_{i=1}^{k} n_i \left(\frac{b_i}{q_i}\right)^2 s_i^2 = \frac{1}{n_j} \sum_{i \neq j} n_i \left(\frac{b_i}{q_i}\right)^2 s_i^2. \tag{2.16}$$

This shows that $s_j \leq 1 + \frac{1}{n_j}$ with equality if and only if $b_i = 0$ for all $i \neq j$. Let us assume that all $b_j \neq 0$; we will comment on the general case later. Then $1 < s_j < 1 + \frac{1}{n_j} \leq 2$. If we multiply equation (2.15) by n_j and then sum on j, we get

$$\sum_{j=1}^{k} n_j(s_j - 1) = 1. \tag{2.17}$$

Thus, solving the system (2.15) is equivalent to solving

$$\left(\frac{b_j}{q_j}\right)^2 \frac{s_j^2}{s_j - 1} = c, \tag{2.18}$$

for each $j = 1, \ldots, k$ and some constant $c > 0$, subject to the constraint (2.17).

On the interval $(1, 2]$, the function $f(s) = \frac{s^2}{s-1}$ decreases monotonically with $f(s) \to \infty$ as $s \to 1^+$ and $f(2) = 4$. So we start by choosing c large enough that $c > 4\left(\frac{b_j}{q_j}\right)^2$ for all j. Then there is a unique solution to (2.18) with $1 < s_j < 2$. If c is extremely large, then all s_j must be very close to 1. But then we will have $\sum_{j=1}^{k} n_j(s_j - 1) < 1$. As c decreases, the s_j increase monotonically. So we decrease c until the largest $s_j = 2$. Then we will have $\sum_{j=1}^{k} n_j(s_j - 1) > 1$ since $n_j \geq 1$. Thus there is a unique value of c for which (2.18) has a solution with $1 < s_j < 2$ and $\sum_{j=1}^{k} n_j(s_j - 1) = 1$. With the values of the s_j in hand, we return to equation (2.12) and solve for λ. Once we know s_j and λ, we have a_j and we are done. $\square$

Remark 2.2.7. If one of the b_i, say $b_1 = 0$, then on $\mathcal{D}_1$ we have $d\eta = 0$ and the bundle over B_1 is trivial so M splits as $M' \times B_1$. Here $\pi' : M' \to B_2 \times \cdots \times B_k$ is a principal S^1 bundle with connection form η' such that $d\eta' = \pi'^*\Omega$. See section 2.5. We should also note that if any of the $b_i = 0$, then the quasi-Sasaki structure has rank $< 2n+1$.

Example 2.2.8. Suppose the base $B = B_1^n$ consists of a single Kähler-Einstein manifold and consider the principal circle bundle $\pi : M \to B$ with $d\eta = b\pi^*\omega_1$, $b \neq 0$. By theorem 2.2.6,

$$g = \eta \otimes \eta + \frac{2b^2(n+1)}{q}\pi^* g_1$$

is quasi-Sasaki-Einstein with

$$\mathrm{Ric}_g = \frac{nq^2}{2b^2(n+1)^2} g.$$

The quasi-Sasaki structure has full rank $2n+1$. If we set $a = \frac{q}{2b(n+1)}$ and $\eta_a = a\eta$, then $a^2 g = \eta_a \otimes \eta_a + ab\pi_1^* g$. Since $d\eta_a = ab\pi^*\omega_1$, we see that $a^2 g$ is Sasaki-Einstein.

Example 2.2.9. Recall that $\mathbb{C}P^n$ with the Fubini-Study metric ω_{FS} is Kähler-Einstein with $\rho = (n+1)\omega_{FS}$. It is well known that $H^2(\mathbb{C}P^n, \mathbb{Z}) \simeq \mathbb{Z}$ and that $\alpha = \frac{1}{2\pi}[\omega_{FS}]$ is a generator. Thus $c^1(\mathbb{C}P^n) = (n+1)\alpha$ and every principal circle bundle $\pi : P \to \mathbb{C}P^n$ has characteristic class $\frac{1}{2\pi}[\Omega] = b\alpha$ for some integer b. A classic example is the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$. Since the total space S^{2n+1} is simply connected, by theorem 2.2.2, $b = \pm 1$. We choose an orientation so that $b = 1$.

Let $B_i = \mathbb{C}P^{n_i}$, $i = 1, \ldots, k$, and $B = B_1 \times \cdots \times B_k$. Then $H^2(B, \mathbb{Z}) \simeq \oplus_{i=1}^k \mathbb{Z}$ with generators $\alpha_i \in H^2(B_i, \mathbb{Z})$. Thus every principal circle bundle $\pi : P \to B$ has characteristic class $\frac{1}{2\pi}[\Omega] = \sum_{i=1}^k b_i \alpha_i$ for some integers b_i. The total space P is simply connected if and only if the characteristic class is indivisible if and only if $\gcd(b_1, \ldots, b_k) = 1$.

For a concrete example, take $n_i = 1$ for $i = 1, 2, 3$, $b_1 = 1$, $b_2 = -1$ and $b_3 = 0$. Then $q_i = 2$ and it is not too hard to solve the equations from theorem 2.2.6. Working through them, we find that $a_1 = a_2 = 3$ and $a_3 = \frac{9}{2}$. Thus the fundamental 2-form

$\omega = \pi^*(3\pi_1^*\omega_{FS} + 3\pi_2^*\omega_{FS} + \frac{9}{2}\pi_3^*\omega_{FS})$ and the metric $g = \eta \otimes \eta + \omega(\Phi\cdot, \cdot)$ is quasi-Sasaki-Einstein with $\mathrm{Ric}_g = \frac{4}{9}g$.

This example illustrates some interesting features of quasi-Sasaki-Einstein metrics. First, the rank of this quasi-Sasaki structure is 5, whereas full rank is 7. Thus this quasi-Sasaki structure is not a deformation of a Sasaki structure. However, since $b_3 = 0$, the total space P splits as $P' \times B_3$ where $\pi' : P' \to B_1 \times B_2$ is a principal circle bundle with connection form η' such that $d\eta' = \pi'^*\Omega$. Thus $g' = \eta \otimes \eta + \omega'(\Phi\cdot, \cdot)$, where $\omega' = \pi'^*(3\pi_1^*\omega_{FS} + 3\pi_2^*\omega_{FS})$, is a quasi-Sasaki-Einstein metric on P' with $\mathrm{Ric}'_g = \frac{4}{9}g'$. This quasi-Sasaki structure does indeed have full rank, but still it cannot be a deformation of a Sasaki structure because $d\eta'(\Phi\cdot, \cdot)$ is not positive definite; it is negative definite over B_2.

Now we will present the main theorem of this section. It is a consequence of theorem 1 in [40].

Theorem 2.2.10. *If (M, ξ, η, Φ, g) is a regular, quasi-Sasaki-Einstein manifold with positive scalar curvature, then M is a principal circle bundle over a compact Kähler manifold B, where B is a product of Kähler-Einstein manifolds and the metric g is among those constructed in theorem 2.2.6.*

Proof. Since g is Einstein with positive scalar curvature, M is compact. Then the regularity of $\mathcal{F}_\xi$ implies that the orbits of ξ are compact, so by theorem 1.3.6 M is a principal circle bundle over a compact Kähler manifold B with η as the connection form. By lemma 1.2.7, $d\eta(\Phi\cdot, \Phi\cdot) = d\eta$. Thus the curvature form of the circle bundle is of type (1,1). Now by theorem 1 in [40], B is a product of Kähler-Einstein manifolds and (M, g) is among those constructed in theorem 2.2.6. $\square$

Theorems 2.2.5 and 2.2.10 give us a clear picture of the possible quasi-Sasaki-Einstein metrics in the regular case. As we have seen, they are very rigid and do not offer us "new" Einstein metrics that we didn't already know about.

2.3. The quasi-Regular Case

In this section we assume, more generally, that the orbits of ξ are compact and the induced S^1 action is locally free. Then $\mathcal{F}_\xi$ is quasi-regular. By theorem 1.3.6, M is the total space of a principal S^1-orbibundle over the Kähler orbifold $B := M/\mathcal{F}_\xi$ with connection form η and curvature form $d\eta = \pi^*\Omega$. Here Ω is a closed (1,1)-form with $\frac{1}{2\pi}[\Omega] \in H^2_{orb}(B, \mathbb{Z})$. The treatment here closely follows that of the previous section. We will not dive too deeply into the general theory of orbifolds and orbibundles, but only reference pertinent information. For definitions and fundamental results, see [5] and [26] and the references therein.

Many facts about principal circle bundles have an orbibundle counterpart. For instance, theorem 4.3.15 of [5] says that the (isomorphism classes of) principal S^1-orbibundles over an orbifold B are in one-to-one correspondence with the elements of $H^2_{orb}(B, \mathbb{Z})$ and the bijection is given by the orbifold first Chern class. In particular, the trivial orbibundle corresponds to the trivial orbifold cohomology class. Also, it is not hard to produce orbibundle versions of lemma 2.2.3 and theorem 2.2.4. The same argument as in the previous section gives us the orbifold version of theorem 2.2.5.

Theorem 2.3.1. *If (M, g) is a quasi-Sasaki manifold with $\mathrm{Ric}_g = 0$ and $\mathcal{F}_\xi$ has compact leaves, then $M = S^1 \times |B|$ where $|B|$ is the underlying topological space of a Calabi-Yau orbifold B.*

Remark 2.3.2. Since M is a manifold, $|B|$ must be a manifold too.

The orbifolds given by theorem 1.3.6 (i.e. those coming from the leaf space of a quasi-regular quasi-Sasaki structure) are *locally cyclic*, meaning that all of the local uniformizing groups are cyclic groups. Furthermore, given an orbifold B, theorem 4.3.15 of [5] also states that the total space of an S^1-orbibundle over B is a smooth manifold when all of the local uniformizing groups of B inject into S^1. For these reasons, we will only concern ourselves with locally cyclic orbifolds. As a sort of converse to theorem 1.3.6, we have the following:

Proposition 2.3.3. *Given a locally cyclic Kähler orbifold B with $\frac{1}{2\pi}[\Omega] \in H^2_{orb}(B, \mathbb{Z})$, there is a quasi-Sasaki structure on the principal S^1-orbibundle $\pi : M \to B$ corresponding to $\frac{1}{2\pi}[\Omega]$ with connection form η such that $d\eta = \pi^*\Omega$.*

Proof. Let ξ be the unit vector field along the fibers and consider the metric on M given by $g = \eta \otimes \eta + \pi^* g_B$ where g_B is a Kähler metric on B. Define $\Phi = \tilde{\pi} \circ J \circ \pi_*$ where J is the complex structure on B and $\tilde{\pi}$ is the horizontal lift with respect to η. Then $\Phi^2 = -I + \xi \otimes \eta$ and $\Phi^* g = g - \eta \otimes \eta$. It follows that M is quasi-Sasaki. $\square$

We also have the orbi-analogue of theorem 2.2.6. Since the equations we must solve are the same, the proof follows mutatis mutandis, and so we omit it.

Theorem 2.3.4. *Let $(B_i^{n_i}, \omega_i)$, $i = 1, \ldots, k$, be locally cyclic Kähler-Einstein orbifolds with $c^1_{orb}(B_i) > 0$. We can assume that the metrics have been normalized so that $\frac{1}{2\pi}[\omega_i]$ is an indivisible, integral class and $c^1_{orb}(B_i) = q_i \frac{1}{2\pi}[\omega_i]$ for some $q_i \in \mathbb{Z}^+$. Let $B = B_1 \times \cdots \times B_k$ and $\pi_i : B \to B_i$ the projections. Pick integers b_i, not all zero, and set*

$$\Omega = b_1 \pi_1^* \omega_1 + \cdots + b_k \pi_k^* \omega_k.$$

Then there is a quasi-Sasaki-Einstein metric with positive scalar curvature on the principal S^1-orbibundle $\pi : M \to B$ corresponding to $\frac{1}{2\pi}[\Omega]$ with connection form η such that $d\eta = \pi^*\Omega$.

We conclude this section with the orbi-analogue of the rigidity theorem 2.2.10. We will show that theorem 1 in [40] can be carried over to the orbifold setting and, as in the previous section, it implies our theorem.

Theorem 2.3.5. *If (M, g) is a quasi-regular, quasi-Sasaki-Einstein manifold with positive scalar curvature, then M is a principal circle orbibundle over a compact Kähler orbifold B, where B is a product of Kähler-Einstein orbifolds and the metric g is among those constructed in theorem 2.3.4.*

Proof. Since (M, g) is compact, quasi-regularity of $\mathcal{F}_\xi$ implies that the orbits of ξ are compact. By the structure theorem 1.3.6, M is a principal S^1-orbibundle over a compact Kähler orbifold (B, g^T) with connection form η. By lemma 1.2.7, the curvature form $d\eta$ is of type (1,1). The conditions on $d\eta$ and ρ^T imposed by the Einstein equations are the same as the conditions on the curvature form and Ricci form in [40]. Proposition 1 in [40] is proved by local computations, exploiting these conditions. Thus we would have the same proposition in orbifold setting.

Proposition 2.3.6. *Under the assumptions of the theorem, there is a global, orthogonal decomposition $TB = E_1 \oplus \cdots \oplus E_r$ where the E_i are the eigenspaces of the Ricci curvature. The eigenvalues are constant and any sum of the eigenspaces is integrable and invariant under the complex structure.*

The leaves of the foliation $TB = E_1 \oplus \cdots \oplus E_r$ have an induced Kähler structure and the Einstein equations for g show that they have positive Ricci curvature. Thus the leaves are compact and simply connected by Myers's theorem and Kobayashi's theorem in orbifold setting, respectively. By the same argument as in step 3 of proposition 2 of [40] (using the local stability theorem of foliations with compact leaves), we can find a local frame for E_1 consisting of vector fields which are foliate with respect to $E_1^\perp$. Then we can perform the same local computation as in [40] to show that both E_1 and $E_1^\perp$ are totally geodesic. Then it follows that E_1 and $E_1^\perp$ are parallel. Iterating this argument we find that each E_i is parallel. Then, since B is compact (hence complete) and simply connected, by the de Rham decomposition theorem for orbifolds (see [26]) B splits isometrically as a product of Kähler orbifolds $B = B_1 \times \cdots \times B_r$ according to the decomposition of TB. Hence B is a product of Kähler-Einstein orbifolds with $c^1_{orb}(B_i) > 0$. Then since the curvature form is of type (1,1), it follows that the characteristic class of the S^1-orbibundle must be of the form in theorem 2.3.4. $\square$

2.4. Transverse Curvature Conditions

In the previous two sections we assumed that all of the orbits of ξ were compact. We want to make no assumptions on the orbits of the Reeb field, but the irregular case, where M is compact and ξ has noncompact orbits, is inherently more difficult to deal with. The difficulty comes from the potential unruliness of the leaf space $M/\mathcal{F}_\xi$. It may not have a tractable structure; for instance, it may not even be a Hausdorff space! Since we are unable to work with the irregular case directly, to get a handle on it we will assume that the transverse curvature satisfies a certain non-negativity condition. In this section, we introduce the curvature conditions that we will work with and mention a few of their topological implications. We first consider a curvature condition known as *non-negative quadratic orthogonal bisectional curvature* (NQOBC).

Definition 2.4.1. A Kähler manifold of complex dimension n has NQOBC at a point $x \in M$ if for any unitary frame $\{e_1, \ldots, e_n\}$ of $T_x^{1,0}M$ and any real numbers $\alpha_1, \ldots, \alpha_n$,

$$\sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}}(\alpha_i - \alpha_j)^2 \geq 0.$$

We say that the manifold has NQOBC if it does so at every point $x \in M$.

For more information about Kähler manifolds with NQOBC, see the paper [8]. The primary reason why we are interested in this curvature condition is for the following theorem:

Theorem 2.4.2. *If a closed (compact with no boundary) Kähler manifold M has NQOBC, then every harmonic (1,1)-form is parallel.*

Proof. The Bochner formula for (1,1)-forms on a Kähler manifold is

$$\triangle\alpha_{i\bar{j}} = -\Delta\alpha_{i\bar{j}} - 2R_{i\bar{j}k\bar{l}}\alpha_{l\bar{k}} + R_{i\bar{k}}\alpha_{k\bar{j}} + R_{k\bar{j}}\alpha_{i\bar{k}}.$$

Here $\triangle = d\delta + \delta d$ is the Hodge laplacian and $\Delta = \mathrm{tr}(\nabla^2)$ is the tensor laplacian. If $\triangle\alpha = 0$, then the Bochner formula yields

$$\alpha_{i\bar{j}}\Delta\alpha_{i\bar{j}} = -2R_{i\bar{j}k\bar{l}}\alpha_{l\bar{k}}\alpha_{i\bar{j}} + R_{i\bar{k}}\alpha_{k\bar{j}}\alpha_{i\bar{j}} + R_{k\bar{j}}\alpha_{i\bar{k}}\alpha_{i\bar{j}}. \tag{2.19}$$

Since α is a (1,1)-form, $A := \alpha(J\cdot, \cdot)$ is a symmetric bilinear form that is invariant under the complex structure J. Hence A defines a Hermitian form and thus there is a local unitary frame $\{e_i, \overline{e_i}\}$ that diagonalizes A. So with respect to this frame,

$\alpha_{i\bar{i}} := \alpha(e_i, \overline{e_i})$ are the only nonzero components of α and equation (2.19) becomes

$$\begin{aligned}\alpha_{i\bar{i}}\Delta\alpha_{i\bar{i}} &= -2R_{i\bar{i}k\bar{k}}\alpha_{k\bar{k}}\alpha_{i\bar{i}} + R_{i\bar{i}}\alpha_{i\bar{i}}^2 + R_{k\bar{k}}\alpha_{k\bar{k}}^2 \\ &= R_{i\bar{i}k\bar{k}}(\alpha_{i\bar{i}} - \alpha_{k\bar{k}})^2. \end{aligned} \tag{2.20}$$

Integrating equation (2.20) and using integration by parts, we get

$$\int_M -|\nabla\alpha|^2 \, dV = \int_M R_{i\bar{i}k\bar{k}}(\alpha_{i\bar{i}} - \alpha_{k\bar{k}})^2 \, dV.$$

From this, the assumption of NQOBC implies that $\nabla\alpha = 0$. $\square$

If we assume that the transverse Kähler metric of a quasi-Sasaki structure has non-negative quadratic orthogonal transverse bisectional curvature, then the following corollary is immediate.

Corollary 2.4.3. *If the transverse Kähler metric of a closed quasi-Sasaki manifold has NQOBC, then every basic, harmonic (1,1)-form is transversely parallel. That is, if α is a basic (1,1)-form and $\triangle_B\alpha = 0$, then $\nabla^T\alpha = 0$.*

The other curvature condition that we will consider is *transverse holomorphic bisectional curvature* (THBC). This is just the holomorphic bisectional curvature of the transverse Kähler metric.

Definition 2.4.4. Let σ be a plane in $\mathcal{D}_p \subset T_pM$. We say that σ is Φ-*invariant* if $\Phi\sigma = \sigma$. It follows that $\sigma = \operatorname{span}\{X_p, \Phi X_p\}$ for some $0 \neq X_p \in \mathcal{D}_p$.

Definition 2.4.5. Let σ_1 and σ_2 be Φ-invariant planes in $\mathcal{D}_p \subset T_pM$. The *transverse holomorphic bisectional curvature* (THBC) $H^T(\sigma_1, \sigma_2)$ is defined by

$$H^T(\sigma_1, \sigma_2) := \langle R^T(X, \Phi X)\Phi Y, Y\rangle,$$

where $X \in \sigma_1$, $Y \in \sigma_2$ and $|X| = |Y| = 1$.

It is a straightforward exercise to check that $H^T(\sigma_1, \sigma_2)$ is well-defined. That is, $H^T(\sigma_1, \sigma_2)$ depends only on σ_1 and σ_2, not on the choice of X and Y. We say that $H^T \geq 0$ at a point $p \in M$ if $H^T(\sigma_1, \sigma_2) \geq 0$ for any two Φ-invariant planes $\sigma_1, \sigma_2 \subset \mathcal{D}_p$. We say that M has non-negative transverse holomorphic bisectional curvature if $H^T \geq 0$ for all $p \in M$.

Let $\sigma_u = \text{span}\{u, \Phi u\}$ and $\sigma_v = \text{span}\{v, \Phi v\}$ for some unit vectors $u, v \in \mathcal{D}$. Writing $u = U + \bar{U}$ and $v = V + \bar{V}$ where $U = \frac{1}{2}(u - \sqrt{-1}\Phi u)$ and $V = \frac{1}{2}(v - \sqrt{-1}\Phi v)$, we have $H^T(\sigma_u, \sigma_v) = 4\langle R^T(U, \bar{U})V, \bar{V}\rangle$. So $H^T \geq 0$ if and only if $\langle R^T(U, \bar{U})V, \bar{V}\rangle \geq 0$ for all $U, V \in \mathcal{D}^{1,0}$.

In the Kähler setting, positive and non-negative holomorphic bisectional curvature has been studied extensively. For example, see any of [2], [18], [22], [29], [35] or [42]. Positive and non-negative THBC has been studied in the Sasaki setting in [20] and [21]. The following is a corollary of theorem 4 in [18]. Since the proof would follow in exactly the same way, we shall omit it.

Corollary 2.4.6. *If M is a compact, connected quasi-Sasaki manifold with positive transverse bisectional curvature, then the second basic Betti number $b_B^2 = 1$.*

Note that NQOBC is a strictly weaker condition than non-negative holomorphic bisectional curvature. Clearly NHBC implies NQOBC and indeed there are examples of Kähler manifolds with NQOBC that do not admit any Kähler metrics with NHBC.

2.5. Splitting the Contact Bundle

In this section we show that the presence of transversely parallel (1,1)-forms induces a splitting of the contact bundle of a quasi-Sasaki manifold. We begin with a definition.

Definition 2.5.1. A subbundle $\mathcal{D}_1 \subset \mathcal{D}$ is called *invariant* or *transversely parallel* if for all $Y \in \mathcal{D}_1$, $\nabla^T_X Y \in \mathcal{D}_1$ for any $X \in TM$. The contact bundle $\mathcal{D}$ is *reducible* (with respect to g^T) if there are invariant subbundles $\mathcal{D}_1$ and $\mathcal{D}_2$ and an orthogonal decomposition $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$. We say that g^T is (irreducible) reducible if $\mathcal{D}$ is (not) reducible with respect to g^T.

Let (M, g) be a quasi-Sasaki manifold. The fundamental 2-form ω (i.e. the Kähler form of the transverse metric g^T) is transversely parallel (i.e. $\nabla^T \omega = 0$). Suppose that α is another basic, transversely parallel (1,1)-form. Then for $X, Y \in \mathcal{D}$, define the (1,1)-tensor S by the equation

$$\alpha(X, Y) = \omega(S(X), Y). \tag{2.21}$$

Notice that S is self-adjoint with respect to ω. Indeed,

$$\omega(S(X), Y) = \alpha(X, Y) = -\alpha(Y, X) = -\omega(S(Y), X) = \omega(X, S(Y)).$$

Hence S is a diagonalizable linear map on each contact space $\mathcal{D}_p \subset T_pM$. Since both α and ω are transversely parallel, it follows from (2.21) that S is transversely parallel. Therefore, the linear map $S_p : \mathcal{D}_p \to \mathcal{D}_p$ has the same set of eigenvalues for every point $p \in M$. Let $\{c_0, \ldots, c_r\}$ be the distinct eigenvalues of S. As S is self-adjoint, the eigenvalues $c_i \in \mathbb{R}$. If α is not a multiple of ω, then S is not a multiple of the identity and thus $r \geq 1$.

Let $\mathcal{D}_i \subset \mathcal{D}$ be the eigenbundle corresponding to the eigenvalue c_i of S. Then

$$\mathcal{D} = \mathcal{D}_0 \oplus \cdots \oplus \mathcal{D}_r \tag{2.22}$$

and the distribution $p \mapsto \mathcal{D}_i(p)$ is transversely parallel. Indeed, let $Y \in \mathcal{D}_i(p)$ and $X \in T_pM$. Since $\nabla^T S = 0$, we have $S(\nabla^T_X Y) = \nabla^T_X S(Y) = c_i \nabla^T_X Y$. Therefore $\nabla^T_X Y \in \mathcal{D}_i(p)$ and $\mathcal{D}_i$ is invariant in the sense of definition 2.5.1. The splitting (2.22) is orthogonal with respect to ω. Indeed, if $X \in \mathcal{D}_i$ and $Y \in \mathcal{D}_j$ and $i \neq j$, then

$$c_i\omega(X, Y) = \omega(S(X), Y) = \omega(X, S(Y)) = c_j\omega(X, Y).$$

Since $c_i \neq c_j$, this shows that $\omega(X, Y) = 0$. Since c_i is real, $\mathcal{D}_i$ is closed under complex conjugation (i.e. $\overline{\mathcal{D}_i} = \mathcal{D}_i$). Since both α and ω are (1,1)-forms, they are Φ-invariant. As S is self-adjoint, we get that

$$\begin{aligned}\omega(X, S(\Phi Y)) = \omega(S(X), \Phi Y) &= \alpha(X, \Phi Y)\\ &= -\alpha(\Phi X, Y)\\ &= -\omega(S(\Phi X), Y) = -\omega(\Phi X, S(Y)) = \omega(X, \Phi S(Y)).\end{aligned}$$

Since ω is non-degenerate on $\mathcal{D}$, this implies that $S \circ \Phi = \Phi \circ S$. Hence, $\mathcal{D}_i$ is invariant under Φ (i.e. $\Phi\mathcal{D}_i = \mathcal{D}_i$). With this we see that the splitting of $\mathcal{D}$ in (2.22) is orthogonal with respect to g^T and therefore, if α and ω are not proportional to each other, $\mathcal{D}$ is reducible. Corresponding to this splitting, we write $\omega = \omega_0 \oplus \cdots \oplus \omega_r$, $\alpha = \alpha_0 \oplus \cdots \oplus \alpha_r$ and $\Phi = \Phi_0 \oplus \cdots \oplus \Phi_r$, where

$$\Phi_i(X) = \begin{cases} \Phi(X) & \text{if } X \in \mathcal{D}_i, \\ 0 & \text{otherwise} \end{cases}.$$

On $\mathcal{D}_i$ we have $\alpha_i = c_i\omega_i$. Since $\mathcal{D}_i$ has a Käher structure, $\mathcal{D}_i$ has even real dimension. Therefore, $\mathcal{D}_i$ has a local frame $\{X_1, \ldots, X_{p_i}, \bar{X}_1, \ldots, \bar{X}_{p_i}\}$ where $X_j \in \mathcal{D}^{1,0}$.

Lemma 2.5.2. *Let M be a compact quasi-Sasaki manifold with transverse NQOBC. Then $\mathcal{D}$ is reducible if and only if $b_B^{1,1} > 1$.*

Proof. If $b_B^{1,1} > 1$ then by the transverse Hodge theorem, we have at least two distinct, basic, harmonic (1,1)-forms. With the assumption of transverse NQOBC, these harmonic forms are transversely parallel (corollary 2.4.3) and as we just saw above, this implies that $\mathcal{D}$ is reducible.

Conversely, suppose $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$ and the $\mathcal{D}_i$ are invariant with respect to g^T. Let $\omega_i = \omega|_{\mathcal{D}_i}$. Then $\omega = \omega_1 \oplus \omega_2$ and $0 \neq [\omega_i]_B \in H_B^{1,1}(M, \mathcal{F}_\xi)$. Furthermore, since $\omega^n = (\omega_1 \oplus \omega_2)^n \neq 0$, $[\omega_1]_B$ and $[\omega_2]_B$ are independent. Thus $b_B^{1,1} > 1$. $\square$

2.6. The General Case with a Curvature Condition

In this section we assume that the quasi-Sasaki manifold (M, g) is Einstein with $\mathrm{Ric}_g = \lambda g$. Then $d\eta$ is harmonic and, as we saw in remark 2.1.3, the transverse scalar curvature $R^T = \lambda(n+1)$ is constant. Proposition 2.1.4 then implies that the transverse Ricci form ρ^T is harmonic. We also assume further that (M, g) satisfies a transverse curvature condition at least as strong as NQOBC. Then by corollary 2.4.3, $d\eta$ and ρ^T are transversely parallel. Thus we can split the contact bundle as in the previous section.

Let's split the contact bundle with respect to $d\eta$. Let $c_0, \ldots, c_r$, be the distinct eigenvalues of S (the (1,1)-tensor form of $d\eta$). Then we have an orthogonal splitting of $\mathcal{D}$ into transversely parallel distributions, $\mathcal{D} = \mathcal{D}_0 \oplus \cdots \oplus \mathcal{D}_r$. On each $\mathcal{D}_i$ we have $d\eta_i = c_i \omega_i$. If we write $\rho^T = \oplus \rho_i^T$ according to this decomposition, then equation (2.11) shows that $\rho_i^T = (\lambda + 2c_i^2)\omega_i \geq 0$. This implies that the splitting of $\mathcal{D}$ with respect to ρ^T is the same as the splitting of $\mathcal{D}$ with respect to $d\eta$.

If our quasi-Sasaki structure has rank $2p+1$ for $p < n$, then $c_i = 0$ for some i. Reindexing if neccessary, we may assume that $c_0 = 0$ and hence $d\eta_0 = 0$. That $\mathcal{D}_i$ is transversely parallel implies that for any $X, Y \in \mathcal{D}_i$ we have $\pi_{\mathcal{D}}([X,Y]) \in \mathcal{D}_i$. Since $\pi_{L_\xi}([X,Y]) = -2d\eta(X,Y)\xi$, we see that $\mathcal{D}_0$ and $L_\xi \oplus \mathcal{D}_i$ are involutive, hence integrable. Again using that the $\mathcal{D}_i$ are transversely parallel, it is not hard to see that $L_\xi \oplus \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_r$ is also integrable and thus M is locally a product manifold. In fact, (M,g) is locally a Riemannian product, as we will see next.

Since g^T and $d\eta$ are both transversely parallel and we have the relations $c_i g^T(X, \Phi_i Y) = d\eta_i(X,Y)$, it follows that $\nabla^T \Phi_i = 0$. We will show that Φ_0, in particular, is parallel with respect to the metric g. That is, $\nabla\Phi_0 = 0$. Note that this need not be the case for the other Φ_i. Remember that by lemma 1.2.13, $\nabla\Phi = 0$ if and only if M is cosymplectic. First we prove the following useful proposition.

Proposition 2.6.1. *If $\mathcal{D} = \mathcal{D}_0 \oplus \cdots \oplus \mathcal{D}_r$ admits a splitting such that $d\eta_i = c_i\omega_i$, then for $Y \in \mathcal{D}$, $\nabla_Y \xi = -\sum_i c_i \Phi_i Y$. In particular, if $Y \in \mathcal{D}_i$ then $\nabla_Y \xi = -c_i \Phi Y$.*

Proof. By lemma 1.2.11 we have $d\eta(X,Y) = -g(X, \nabla_Y \xi)$. Since $d\eta = \sum_i d\eta_i$ and $d\eta_i(X,Y) = c_i g(X, \Phi_i Y)$, the above equation gives

$$\sum_i g(X, c_i \Phi_i Y) = -g(X, \nabla_Y \xi).$$

Since X was arbitrary, the result follows. □

Now we will show that $\nabla\Phi_0 = 0$. Recall that the transverse Levi-Civita connection ∇^T is defined by

$$\nabla^T_X Y := \pi_{\mathcal{D}}(\nabla_X Y), \ X \in \mathcal{D},$$

$$\nabla^T_\xi Y := \pi_{\mathcal{D}}[\xi, Y].$$

Lemma 2.6.2. *The (1,1)-tensor Φ_0 is parallel with respect to the metric g.*

Proof. As $\Phi : TM \to \mathcal{D}$ and $\Phi\xi = 0$, for $X, Y \in \mathcal{D}$ we have

$$0 = (\nabla^T_X \Phi_0)(Y) = \pi_{\mathcal{D}} \left((\nabla_X \Phi_0)(Y) \right) = \pi_{\mathcal{D}}(\nabla_X \Phi_0(Y) - \Phi_0(\nabla_X Y)).$$

This says that $(\nabla_X \Phi_0)(Y)$ has no part in $\mathcal{D}$. Does it have any part in L_ξ? Well, $\Phi_0 : \mathcal{D}_0 \to \mathcal{D}_0$ and the term $\nabla_X \Phi_0(Y) \in \mathcal{D}$ because the ξ part is $d\eta(X, \Phi_0(Y))\xi = 0$ since $d\eta = 0$ on $\mathcal{D}_0$. Thus, for $X, Y \in \mathcal{D}$, we have $(\nabla_X \Phi_0)(Y) = 0$. By proposition 2.6.1 and $\Phi_i \circ \Phi_j = 0$ for $i \neq j$ we compute that

$$\begin{aligned}
(\nabla_X \Phi_0)(\xi) &= \nabla_X \Phi_0(\xi) - \Phi_0(\nabla_X \xi) \\
&= \sum_i c_i \Phi_0(\Phi_i X) \\
&= 0.
\end{aligned}$$

Thus $\nabla_X \Phi_0 = 0$ for $X \in \mathcal{D}$. By lemma 1.2.11, $\nabla_\xi \Phi = 0$ which implies that $\nabla_\xi \Phi_0 = 0$ (we could also show this by direct computation). Therefore $\nabla \Phi_0 = 0$. $\square$

Now let $\tilde{\Phi} := \sum_{i=1}^r \Phi_i$ and $\tilde{\mathcal{D}} := \sum_{i=1}^r \mathcal{D}_i$. The map $P := -\tilde{\Phi}^2 + \xi \otimes \eta$ is projection onto $L_\xi \oplus \tilde{\mathcal{D}}$ and $Q := -\Phi_0^2$ is projection onto $\mathcal{D}_0$. Thus $P^2 = P$, $Q^2 = Q$ and $PQ = QP = 0$. As in section **??**, $F := P - Q$ defines an almost product structure. Since $\Phi = \Phi_0 + \tilde{\Phi}$ and $\Phi^2 = -I + \xi \otimes \eta$, we have $F = I + 2\Phi_0^2$. By proposition 2.6.2, $\nabla \Phi_0 = 0$ and this clearly implies that $\nabla \Phi_0^2 = 0$. Hence $\nabla F = 0$. So by proposition 5.0.7, (M, g) is a locally decomposable Riemannian manifold. Thus, locally, (M, g) is the Riemannian product of manifolds (M_0, g_0) and (M_1, g_1) whose tangent spaces are (isomorphic to) $\mathcal{D}_0$ and $L_\xi \oplus \tilde{\mathcal{D}}$ respectively. We will show that (M_0, g_0) is Kähler-Einstein and (M_1, g_1) is quasi-Sasaki-Einstein.

Pick a point $x \in M$ and let M_0 be a maximal integral submanifold of $\mathcal{D}_0$ containing the point x. Then $\dim_{\mathbb{R}}(M_0) = 2(n-p)$. Define 2-forms ω_0 and $\tilde{\omega}$ by $\omega_0(X,Y) = g(X, \Phi_0 Y)$ and $\tilde{\omega}(X,Y) = g(X, \tilde{\Phi} Y)$. Then $\omega_0 = \omega - \tilde{\omega}$. Since ω is closed and $\tilde{\omega} = \sum_{i=1}^{r} c_i^{-1} d\eta_i$ is proportional to $d\eta$, we have $d\omega_0 = 0$. Because Φ_0 has rank $2(n-p)$, $\omega_0|_{\mathcal{D}_0}$ is non-degenerate. We have $(\Phi_0|_{\mathcal{D}_0})^2 = -I$, $\nabla\Phi_0 = 0$ and $g|_{\mathcal{D}_0}$ is compatible with Φ_0. Therefore, $(M_0, \Phi_0|_{\mathcal{D}_0}, g|_{\mathcal{D}_0})$ is Kähler. To see that the metric on M_0 is Einstein we refer back to equations (2.8) and (2.11). Since $d\eta = 0$ on $\mathcal{D}_0$, these equations say precisely that $g|_{\mathcal{D}_0}$ is an Einstein metric on M_0

Let M_1 be a maximal integral submanifold of $L_\xi \oplus \tilde{\mathcal{D}}$ containing x. Then $\dim_{\mathbb{R}}(M_1) = 2p+1$. Since $\eta \wedge (d\eta)^p \neq 0$ on $L_\xi \oplus \tilde{\mathcal{D}}$, restricting η to $L_\xi \oplus \tilde{\mathcal{D}}$ gives M_1 a contact structure. As Φ and $\tilde{\Phi}$ agree on $\tilde{\mathcal{D}}$ and both are zero on L_ξ, it is easily verified that the restriction of $(\xi, \eta, \tilde{\Phi}, g)$ to $L_\xi \oplus \tilde{\mathcal{D}}$ gives M_1 an almost contact metric structure. Now we check the normality condition.

Recall that $N_\Phi + \xi \otimes 2d\eta = 0$ and since $\nabla\Phi_0 = 0$, we have $N_{\Phi_0} = 0$. Take $X, Y \in L_\xi \oplus \tilde{\mathcal{D}}$. Since $\tilde{\Phi} = \Phi - \Phi_0$ and $\Phi_0 X = \Phi_0 Y = 0$, we compute that

$$N_{\tilde{\Phi}}(X,Y) + 2d\eta(X,Y)\xi = \Phi_0[X, \Phi Y] + \Phi_0[\Phi X, Y] - \Phi\Phi_0[X,Y] - \Phi_0\Phi[X,Y]. \quad (2.23)$$

As the distribution $L_\xi \oplus \tilde{\mathcal{D}}$ is integrable, it is involutive. Hence, each of the bracket terms in (2.23) are in $L_\xi \oplus \tilde{\mathcal{D}}$. Since Φ_0 annihilates $L_\xi \oplus \tilde{\mathcal{D}}$, it follows that $N_{\tilde{\Phi}}(X,Y) + 2d\eta(X,Y)\xi = 0$. Therefore, M_1 has a normal almost contact metric structure. Clearly, the fundamental 2-form $\tilde{\omega}$ associated to this almost contact metric structure is closed. Hence $(\xi, \eta, \tilde{\Phi}, g)$ restricted to $L_\xi \oplus \tilde{\mathcal{D}}$ gives M_1 a full rank quasi-Sasaki structure. Moreover, the metric is Einstein. Summarizing, we have proved:

Theorem 2.6.3. *Let M^{2n+1} be a quasi-Sasaki-Einstein manifold of rank $2p+1$ for $p < n$ with transverse NQOBC. Then (M, g) is locally the Riemannian product of a Kähler-Einstein manifold (M_0, g_0) with $\dim_{\mathbb{R}}(M_0) = 2(n-p)$ and quasi-Sasaki-Einstein manifold (M_1, g_1) of rank $2p+1$. Moreover, the transverse Kähler metric of g_1 is a product of transverse Kähler-Einstein metrics.*

Remark 2.6.4. That M is locally a product manifold does not require the transverse curvature assumption. The Einstein equations alone are enough to get this and the proof can go along the same lines as the proof of proposition 1 in [40]. However, to get a local Riemannian product, the almost product structure must be parallel (proposition 5.0.7). At this time, we don't know if this can be proved solely on the strength of the Einstein equations in the irregular case.

For the remainder of this section we may assume that our quasi-Sasaki structures have full rank and if $\mathcal{D}$ is reducible, then none of the $\mathcal{D}_i$ are integrable on their own. If $\mathcal{D}$ is irreducible, then the transversely parallel forms ω, $d\eta$ and ρ^T must all be proportional to each other. That is, $d\eta = c\omega$ and $\rho^T = \mu\omega$ for some constants c and μ. Since $c \neq 0$, our quasi-Sasaki structure is simply a scaling of a Sasaki-Einstein structure. By replacing η with $c\eta$, replacing ξ with $c^{-1}\xi$ and scaling the transverse metric by c^2 we get a Sasaki-Einstein structure.

Combining corollaries 2.4.3, 2.4.6 and lemma 2.5.2 we get:

Theorem 2.6.5. *A quasi-Sasaki-Einstein manifold with positive transverse holomorphic bisectional curvature is, up to homothey, a Sasaki-Einstein manifold.*

Now by [21] we get that

Corollary 2.6.6. *A quasi-Sasaki-Einstein manifold with positive transverse holomorphic bisectional curvature is diffeomorphic to S^{2n+1} and the metric can be deformed to the round metric.*

Lemma 2.6.7. *A reducible quasi-Sasaki manifold with positive Ricci curvature is quasi-regular.*

Proof. Suppose that M is quasi-Sasaki with positive Ricci curvature and $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$ is reducible. Then the $\mathcal{D}_i$ are transversely parallel and this implies that $L_\xi \oplus \mathcal{D}_i$ is an integrable subbundle of TM. Since the $\mathcal{D}_i$ are Φ-invariant, using proposition 2.6.1 and equation (2.4), it is not hard to show that the leaves of $L_\xi \oplus \mathcal{D}_i$ are totally geodesic submanifolds of M. Hence the induced metric on them is complete. By assumption, $\mathcal{D}_1$ and $\mathcal{D}_2$ have positive Ricci curvature. So by Myers's theorem, the leaves of $L_\xi \oplus \mathcal{D}_i$ are compact. An integral curve of ξ is a transversal intersection of a leaf from $L_\xi \oplus \mathcal{D}_1$ and a leaf from $L_\xi \oplus \mathcal{D}_2$. Thus the ξ orbits are compact and therefore $\mathcal{F}_\xi$ is quasi-regular. □

From this lemma we see that an irregular quasi-Sasaki-Einstein manifold with positive scalar curvature is irreducible. Hence we have the following corollary:

Corollary 2.6.8. *An irregular quasi-Sasaki-Einstein manifold with positive scalar curvature and transverse NQOBC is a scaling of a Sasaki-Einstein manifold.*

CHAPTER III

TRANSVERSE KÄHLER-RICCI FLOW

The aim of this chapter is to extend results in Kähler-Ricci flow to quasi-Sasaki manifolds. The transverse Kähler-Ricci flow is the normalized Kähler-Ricci flow of the quasi-Sasaki manifold's transverse Kähler metric. We give a proof of its short-time existence. The long-time existence follows from the work in [36]. In comparison with [7], we can show that when the *basic* first Chern class $c_B^1 \leq 0$, the flow converges to a transverse Kähler-Einstein metric. This can be done just as in [36], but we choose not to pursue it here. Rather, we are interested in the case that $c_B^1 > 0$. In [31], Perelman introduced his $\mathcal{W}$ functional. The $\mathcal{W}$ functional is monotone increasing along the Ricci flow and this property has many important applications. In particular, for a Kähler manifold with positive first Chern class, the monotonicity of the $\mathcal{W}$ functional can be used to prove uniform bounds on the scalar curvature and the diameter of the manifold along the flow. This was first presented in [34].

We have two primary goals in this chapter. The first is to establish uniform bounds on the transverse scalar curvature and the transverse diameter along the transverse Kähler-Ricci flow when $c_B^1 > 0$. This will imply uniform bounds on the scalar curvature and the diameter of the manifold along the flow. We first define the $\mathcal{W}^T$ functional, which is the analogue of the $\mathcal{W}$ functional for the quasi-Sasaki setting. We show that it is monotonic along the flow and then we proceed in a way similar to [34]. A subtle difference in our situation is that the $\mathcal{W}^T$ functional is only involved with *basic* functions. That is, functions which are constant in the ξ direction. This poses some problems as the distance function on the manifold is not basic. To deal with this, we utilize the *transverse distance function* from chapter I.

This approach was taken in [12] and [19], where these results were, independently, established for Sasaki manifolds. The essential ingredient is really the transverse Kähler structure of the manifold. The additional contact structure afforded by Sasaki manifolds is not required. Thus we can prove these results in the quasi-Sasaki setting as well. On a compact Kähler manifold, it is known that if the initial metric has nonnegative bisectional curvature, then this property is preserved along the Kähler-Ricci flow. The second goal of this chapter is to show that nonnegativity of the transverse (holomorphic) bisectional curvature, as defined in 2.4.5, is preserved along the transverse Kähler-Ricci flow.

This chapter is organized as follows: First we define the transverse Kähler-Ricci flow and prove its short-time existence. Next we give an evolving quasi-Sasaki structure, compatible with the evolving transverse metric, and provide some commentary on how the *rank* of this structure may change along the flow. Next we define the $\mathcal{W}^T$ and μ^T functionals and prove their monotonicity. After that we show that the transverse scalar curvature and Ricci potential are bounded along the flow by the diameter of the manifold. Then we prove a uniform upper bound on the *transverse diameter* of the manifold along the flow. This implies a uniform bound on the diameter of the manifold, and hence on the scalar curvature and Ricci potential. The presentation in sections 3 and 4 is much like that in [34], but here we provide more details for the arguments and computations. The last section is devoted to studying the transverse Kähler-Ricci flow when the initial metric has non-negative (positive) transverse bisectional curvature. Some of the technical lemmas that get used in this chapter are collected in the appendix.

3.1. Transverse Kähler-Ricci Flow

Following the introduction of Hamilton's Ricci flow, similar evolution equations were considered on Kähler manifolds. Since the 1980's, Kähler-Ricci flow (KRF) has been a very fruitful area of mathematics. Many deep and beautiful results are known and KRF is still an active area of research today. It is well-known in Kähler geometry via the Chern-Weil theory, that the first Chern class, c^1, of a Kähler manifold is represented by a multiple of the Ricci form and that the de Rham cohomology class of the Ricci form is independent of the Kähler metric. Thus a necessary condition for the existence of a Kähler-Einstein metric ω is that $c^1 = \lambda[\omega]$. Hence c^1 must be positive definite, negative definite or null depending on whether λ is positive, negative or zero respectively. The existence of Kähler-Einstein metrics in the cases $c^1 < 0$ and $c^1 = 0$ was proved independently by Aubin in [1] and Yau in [43]. In [7], Cao showed that the normalized KRF converges to a Kähler-Einstein metric when $c^1 \leq 0$. When $c^1 > 0$, things are more complicated and there are obstructions to the existence of Kähler-Einstein metrics, see [16]. However, in certain cases, it is known that the KRF converges to a Kähler-Ricci soliton when $c^1 > 0$. See [33] and [38].

In [36], Smoczyk, Wang and Zhang defined the Sasaki-Ricci flow (SRF) as the KRF of the underlying transverse Kähler metric associated with a Sasaki structure. They proved results analogous to those in [7]. The work in [36] on the SRF does not rely heavily on the contact structure of a Sasaki manifold and many of their results can be carried over to the quasi-Sasaki setting. Collins in [12] and He in [19], independently, extended a remarkable result of Perelman in KRF with $c^1 > 0$ (see [34]) to the SRF with $c^1_B > 0$. In this chapter, we will do the same in the quasi-Sasaki setting. In this section we define the flow and then prove its short time existence.

Definition 3.1.1. Let $(\xi, \eta_0, \Phi_0, g_0)$ be a quasi-Sasaki structure with $2\pi c_B^1 = \kappa[\omega_0]$ for some constant κ. The *transverse Kähler-Ricci flow* (TKRF) is given by

$$\frac{\partial}{\partial t} g^T = \kappa g^T - \mathrm{Ric}_g^T, \quad g^T(0) = g_0^T. \tag{3.1}$$

Let $\rho^T := \mathrm{Ric}^T(\cdot, \Phi\cdot)$ be the transverse Ricci form. Recall that ρ^T is independent of the transverse Kähler metric and that the class $[\rho^T] = 2\pi c_B^1$. As $\omega = g^T(\cdot, \Phi\cdot)$, equation (3.1) is equivalent to

$$\frac{\partial}{\partial t}\omega = \kappa\omega - \rho^T, \quad \omega(0) = \omega_0. \tag{3.2}$$

A *transverse homothety*, also called a $\mathcal{D}$-homothety, is a deformation of a quasi-Sasaki structure that scales the transverse metric. Given a quasi-Sasaki structure (ξ, η, Φ, g) and $\lambda > 0$, we set

$$\xi_\lambda := \lambda^{-1}\xi, \quad \eta_\lambda := \lambda\eta \ \text{ and } \ g_\lambda := \lambda(\lambda - 1)\eta \otimes \eta + \lambda g.$$

Then $(\xi_\lambda, \eta_\lambda, \Phi, g_\lambda)$ is a quasi-Sasaki structure and the transverse metric $g_\lambda^T = \lambda g^T$. So if $2\pi c_B^1 = \kappa[\omega]$, then by a $\mathcal{D}$-homothety we may assume that $\kappa = -1, 0$ or 1, depending on the sign of κ.

3.11. Short-Time Existence

To prove the short-time existence of a solution to equation (3.2), we reduce it to a parabolic Monge-Ampére equation. Having done this, the *standard theory* of parabolic equations guarantees the existence of a solution for at least a short time. We will prove the case $c_B^1 > 0$ since we shall assume this in the sequel. The cases

$c_B^1 = 0$ and $c_B^1 < 0$ can be proved similarly. We also note, although we don't prove it here, that equation (3.1) has a short-time solution even with no assumption on c_B^1.

We assume that $[\omega_0] = 2\pi c_B^1$ and we know that $2\pi c_B^1 = [\rho_0^T]$. Thus, by the transverse $\partial\bar{\partial}$-lemma of [15], there is a basic function f such that $\omega_0 = \rho_0^T + \sqrt{-1}\partial\bar{\partial}f$. Recall that $\rho_0^T = -\sqrt{-1}\partial\bar{\partial}\log(\det(\omega_0))$ Thus

$$\begin{aligned}\omega_0 &= -\sqrt{-1}\partial\bar{\partial}\log(\det(\omega_0)) + \sqrt{-1}\partial\bar{\partial}f \\ &= -\sqrt{-1}\partial\bar{\partial}\log(e^{-f}\det(\omega_0)) \\ &= -\sqrt{-1}\partial\bar{\partial}\log(\det(e^{-f/n}\omega_0)).\end{aligned}$$

We let $\Omega := \det(e^{-f/n}\omega_0)$ so that we can write $\omega_0 = -\sqrt{-1}\partial\bar{\partial}\log(\Omega)$.

Next we consider the equation

$$\frac{\partial\phi}{\partial t} = \log\frac{\det(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)}{\Omega} + \phi + c(t),\ \phi(0) \equiv 1, \tag{3.3}$$

defined for all $t \geq 0$ such that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0$. Here $c(t)$ is a time-dependent constant. If ϕ solves (3.3), then it is easy to check that the metric $\omega := \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ solves (3.2). Conversely, given a solution to (3.2), its basic cohomology class must evolve as

$$\frac{\partial}{\partial t}[\omega] = [\omega] - 2\pi c_B^1,\ [\omega](0) = [\omega_0].$$

Solving this ODE, we find that $[\omega] = 2\pi c_B^1 + e^t([\omega_0] - 2\pi c_B^1)$. Since we assume that $[\omega_0] = 2\pi c_B^1$, we have $[\omega(t)] = [\omega_0]$. By the transverse $\partial\bar{\partial}$-lemma, there is $\phi \in C_B^\infty(M)$ such that

$$\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi. \tag{3.4}$$

Since $\omega(0) = \omega_0$, we have that $\partial\bar{\partial}\phi(0) = 0$. Differentiating the above, we get

$$\omega - \rho^T = \sqrt{-1}\partial\bar{\partial}\frac{\partial\phi}{\partial t}. \tag{3.5}$$

Putting (3.4) into (3.5) and writing $\rho^T = -\sqrt{-1}\partial\bar{\partial}\log(\det(\omega))$, we then have

$$\sqrt{-1}\partial\bar{\partial}\frac{\partial\phi}{\partial t} = \sqrt{-1}\partial\bar{\partial}\left(\log\frac{\det(\omega)}{\Omega} + \phi\right). \tag{3.6}$$

Thus there is a time-dependent constant $c(t)$ such that

$$\frac{\partial\phi}{\partial t} = \log\frac{\det(\omega)}{\Omega} + \phi + c(t).$$

Hence ϕ solves (3.3). Therefore, equations (3.2) and (3.3) are equivalent.

Now consider the parabolic Monge-Ampére equation, given by

$$\frac{\partial\phi}{\partial t} = \log\frac{\det(\omega_{0_{i\bar{j}}} + \phi_{,i\bar{j}})}{\Omega} + \xi^2\phi + \phi. \tag{3.7}$$

Here ϕ is not assumed to be basic and the indices after the comma denote covariant derivatives.

Lemma 3.1.2. *Equation (3.7) preserves the property $\xi\phi = 0$.*

Proof. The proof of this lemma is by a maximum principal argument and is the same as the proof of Lemma 5.2 in [36]. See [36] for the details. □

With this we can prove the short time existence of our flow.

Proposition 3.1.3. *For some $T > 0$, equation (3.2) has a solution for $t \in [0, T)$.*

Proof. Equation (3.7) is parabolic whenever we have

$$\det(\omega_{0_{i\bar{j}}} + \phi_{,i\bar{j}}) > 0. \tag{3.8}$$

By the standard parabolic theory, for any initial function $\phi(0)$ satisfying (3.8), there is $T > 0$ and a solution, $\phi : M \times [0, T) \to \mathbb{R}$, to (3.7) with $\det(\omega_{0_{i\bar{j}}} + \phi_{,i\bar{j}}) > 0$ for all $t \in [0, T)$.

Take $\phi(0) \equiv 1$. Then (3.8) is satisfied and since $\xi\phi(0) = 0$, the solution ϕ is a basic function by lemma 3.1.2. Recall that

$$\phi_{,i\bar{j}} := \nabla^2\phi(X_i, X_{\bar{j}}) = \nabla d\phi(X_i, X_{\bar{j}}) = X_i X_{\bar{j}}\phi - d\phi(\nabla_{X_i} X_{\bar{j}}).$$

By (2.5) we have $\nabla_{X_i} X_{\bar{j}} = -d\eta(X_i, X_{\bar{j}})\xi$. So when ϕ is basic we have $\phi_{,i\bar{j}} = \phi_{i\bar{j}}$. Thus our solution to (3.7) also solves (3.3) (with $c(t) = 0$). Hence, we have a solution to (3.2) on $[0, T)$. $\square$

3.12. Long-Time Existence

The main theorem in [7] implies that for a compact Kähler manifold (M, g_0) with $c^1 = \kappa[\omega_0]$, the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g = \kappa g - \mathrm{Ric}_g, \quad g(0) = g_0,$$

has a unique solution defined for all $t \geq 0$ and when $\kappa \leq 0$ the flow $g(t)$ converges to the unique Kähler-Einstein metric in c^1. Using this result, it is shown in [36] that for a compact Sasaki manifold (M, η_0, g_0) with with $c_B^1 = \kappa[d\eta_0]$, the Sasaki-Ricci flow

$$\frac{\partial}{\partial t} g^T = \kappa g^T - \mathrm{Ric}_g^T, \quad g^T(0) = g_0^T,$$

has a unique solution for $t \in [0, \infty)$ and when $\kappa \leq 0$ the flow $g^T(t)$ converges to a transverse Kähler-Einstein metric.

In the Sasaki setting, $d\eta$ is non-degenerate on $\mathcal{D}$ and it is the transverse Kähler form of the Sasaki metric. So the Sasaki-Ricci flow is really just our TKRF where the initial structure is Sasaki (rank $2n+1$ quasi-Sasaki with $\omega = d\eta$). To prove the long-time existence of the flow, one needs uniform C^k-estimates for the metric potential function ϕ. The estimates obtained in [36] do not rely on the fact that the initial structure is Sasaki and thus can be carried over to the setting of our TKRF. Therefore, the long-time existence of the TKRF follows from the work in [36], which, in turn, follows from the work in [7]. We record this as a corollary.

Corollary 3.1.4. *Let $(M, \xi, \eta_0, \Phi, g_0)$ be a compact quasi-Sasaki manifold with $c_B^1 = \kappa[\omega_0]$. Then the transverse Kähler-Ricci flow*

$$\frac{\partial}{\partial t} g^T = \kappa g^T - \mathrm{Ric}_g^T, \quad g^T(0) = g_0^T,$$

has a unique solution that exists for all $t \geq 0$. When $\kappa \leq 0$, the flow $g^T(t)$ converges to a transverse Kähler-Einstein metric.

3.13. The Evolving quasi-Sasaki Structure

Now that we have an evolving transverse Kähler metric $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi(t)$ that satisfies (3.2), we need an associated evolving quasi-Sasaki structure. We define the evolving structure $(\xi, \eta_t, \Phi_t, g_t)$ by

$$\eta_t := \eta_0 + d^c_B \phi,$$

$$\Phi_t := \Phi_0 - \xi \otimes d^c_B \phi \circ \Phi_0,$$

$$g_t := \eta_t \otimes \eta_t + \omega_t(\Phi_t \cdot, \cdot).$$

As $d^c_B\phi$ is basic, it is clear that $\eta_t(\xi) = \eta_0(\xi) = 1$. It is not hard to check that the relations $\Phi_t^2 = -I + \xi \otimes \eta_t$ and $g_t(\Phi_t \cdot, \Phi_t \cdot) = g_t - \eta_t \otimes \eta_t$ hold for this evolving structure. Hence, for each $t \geq 0$, we have a quasi-Sasaki structure $(\xi, \eta_t, \Phi_t, g_t)$ where the transverse metric g_t^T evolves by TKRF.

It is natural to ask what happens to the rank of the quasi-Sasaki structure as it evolves under the TKRF. If the initial structure is Sasaki (rank $2n+1$), then $d\eta_t = \omega_t$ and thus the evolving structure remains Sasaki. However, the rank need not be constant along the flow. If the initial structure is co-symplectic (rank 1), then $d\eta_0 = 0$. If the rank stays constant then $d\eta_t = \sqrt{-1}\partial\bar{\partial}\phi = 0$ for all t. This implies that ω_0 is a constant solution to (3.2). Hence ω_0 is a transverse Kähler-Einstein metric. So if the initial structure is rank 1 and ω_0 is not Einstein, then the rank cannot remain constant. To construct such a manifold, simply take a compact Kähler manifold $(\tilde{M}, J, \tilde{g})$ with $c^1 > 0$ which is not Einstein (these exist) and let $M = S^1 \times \tilde{M}$ and $g = d\theta \otimes d\theta + \tilde{g}$. Then $\{M, \partial_\theta, d\theta, J, g\}$ is co-symplectic and ω is not transverse Kähler-Einstein.

If the initial structure has rank $2p+1$ with $p < n$, we would like to know what happens to the rank of the evolving structure. If the initial structure is rank 1, then the rank can only increase along the flow, but is this what happens in general?

We end this section with a few words on the volume form naturally associated to a quasi-Sasaki manifold. Given a quasi-Sasaki structure on M, the top form $\eta \wedge (\omega)^n$ defines a volume form. If $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ evolves by transverse Kähler-Ricci flow, then as we noted above, $\eta_t = \eta_0 + d_B^c\phi$ is the evolving (almost) contact form. Since $d_B^c\phi$ is basic,

$$d_B^c\phi \wedge (\omega_t)^n = 0,$$

because it is a basic form of degree $2n+1$. In a preferred coordinate chart we have

$$g_t = \eta_t \otimes \eta_t + (g^T_{i\bar{j}} + \phi_{i\bar{j}})dz^i \otimes d\bar{z}^j.$$

It follows that

$$(\omega_t)^n = c(n)\det(g^T_{i\bar{j}} + \phi_{i\bar{j}})dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n,$$

where the constant

$$c(n) = n!\frac{(\sqrt{-1})^n}{2^n}.$$

Therefore, the evolving volume form

$$\begin{aligned}
\eta_t \wedge (\omega_t)^n &= \eta_0 \wedge (\omega_t)^n \\
&= c(n)\det(g^T_{i\bar{j}} + \phi_{i\bar{j}})dx \wedge dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \\
&= n! dV_{g_t}.
\end{aligned}$$

where dV_{g_t} is the Riemmanian volume form associated to g_t.

3.2. Perelman's entropy functional on quasi-Sasaki manifolds

In this section we develop the analog of Perelman's $\mathcal{W}$ functional in the quasi-Sasaki setting. Our *transverse entropy* functional $\mathcal{W}^T$ can be viewed as the entropy functional $\mathcal{W}$ for the transverse Kähler structure. We will show that $\mathcal{W}^T$ is monotonic along the transverse Kähler-Ricci flow. This will allow us, in the next section, to prove a non-collapsing result which can then be used to prove bounds on the scalar curvature and the diameter of the manifold along the TKRF, as in [34].

3.21. The transverse entropy functional

Let M be a quasi-Sasaki manifold. The $\mathcal{W}^T$ functional is defined in terms of the quasi-Sasaki metric g, but it really only depends on the induced transverse metric. We make the following definition:

Definition 3.2.1. For a quasi-Sasaki manifold (M, ξ, η, Φ, g), a basic function $f \in C^\infty_B(M)$ and a real number $\tau > 0$, the transverse entropy functional $\mathcal{W}^T$ is defined by

$$\mathcal{W}^T(g, f, \tau) := \int_M (\tau(R^T + |\nabla f|^2) + f)e^{-f}\tau^{-n}\, dV, \tag{3.9}$$

where $dV = \eta \wedge (\omega)^n$.

Even though the $\mathcal{W}^T$ functional is defined in terms of the metric g, it really only depends on the induced transverse metric g^T. For this reason we may write $\mathcal{W}^T(g^T, \cdot, \cdot)$ to mean $\mathcal{W}^T(\bar{g}, \cdot, \cdot)$ where $\bar{g}$ is any bundle-like metric inducing g^T.

Since the function f in (3.9) is basic, the integrand of the $\mathcal{W}^T$ functional only involves the transverse Kähler structure. Thus, when we compute the first variation of $\mathcal{W}^T$ we are essentially just computing as we would in the Kähler setting, with all of the familiar integration by parts formulas from Kähler geometry at our disposal.

Proposition 5.0.10 and corollary 5.0.11 help make this precise. As noted in [17], corollary 5.0.11 implies a general principle that will aid us greatly in our computations. The principle is that if a result for a compact Kähler manifold can be proved using only Stokes theorem and integration by parts, then the same result holds for compact quasi-Sasaki manifolds and the proof can go as in the Kähler setting by inserting a "$\wedge\eta$" in each line of the proof. See also proposition 4.6 in [19].

We consider a variation of the quasi-Sasaki structure that fixes ξ and the transverse holomorphic structure. Let $v_{i\bar{j}} = \delta g^T_{i\bar{j}}$, $v = (g^T)^{i\bar{j}} v_{i\bar{j}}$, $\delta f = h \in C^\infty_B(M)$ and $\delta\tau = \sigma$. We assume that, at least locally, there is a basic function ψ such that $v_{i\bar{j}} = \partial_i \partial_{\bar{j}} \psi$. Then we have $v_{i\bar{j},i} = v_{,i}$.

Proposition 3.2.2. *The first variation of $\mathcal{W}^T$ on compact quasi-Sasaki manifolds is given by*

$$\begin{aligned}\delta\mathcal{W}^T(v_{i\bar{j}}, h, \sigma) = &\int_M \left(\sigma(R^T + \Delta^T f) - \tau\langle v_{i\bar{j}}, R^T_{i\bar{j}} + f_{i\bar{j}}\rangle + h\right)\tau^{-n} e^{-f}\ dV \\ &+ \int_M (v - h - n\sigma\tau^{-1})(\tau(2\Delta^T f - |\nabla^T f|^2 + R^T) + f)\tau^{-n} e^{-f}\ dV.\end{aligned}$$

Before we prove the proposition, a few comments and observations are in order. First, the Christoffel symbols of the transverse Levi-Civita connection in the ξ direction, as well as those with mixed barred and unbarred indices, are all zero. The others are given by the usual formula from Kähler geometry,

$$\Gamma^k_{ij} = (g^T)^{k\bar{l}} \frac{\partial g^T_{i\bar{l}}}{\partial z_j} \quad \text{and} \quad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \overline{\Gamma^k_{ij}}.$$

For a function f on M, we will use the notation $f_i := \frac{\partial f}{\partial z_i} = \partial_i f$ and $f_{i\bar{j}} := \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} = \partial_i \partial_{\bar{j}} f$. Next, if f is a basic function, then

$$|\nabla f|^2 = |\nabla^T f|^2 = (g^T)^{i\bar{j}} f_i f_{\bar{j}}$$

and

$$\Delta f = \Delta^T f = (g^T)^{i\bar{j}} \nabla_i \nabla_{\bar{j}} f = (g^T)^{i\bar{j}} f_{i\bar{j}}.$$

Finally, for symmetric (0,2)-tensors $S_{i\bar{j}}$ and $T_{i\bar{j}}$, we have an inner product given by

$$\langle S_{i\bar{j}}, T_{i\bar{j}} \rangle := (g^T)^{i\bar{l}} (g^T)^{k\bar{j}} S_{i\bar{j}} T_{k\bar{l}},$$

and we write $|S_{i\bar{j}}|^2 = \langle S_{i\bar{j}}, S_{i\bar{j}} \rangle$. To ease notation, we will drop the superscript T from the transverse connection, Laplacian and metric in what follows. As all of the functions involved are basic, this should not cause any confusion. The presence of one barred index and one unbarred index in the metric indicates that we are considering the transverse Kähler metric. Now for the proof:

Proof. From the relation $g^{i\bar{l}} g_{k\bar{l}} = \delta_i^k$, we find that $\delta g^{i\bar{j}} = -g^{i\bar{l}} v_{k\bar{l}} g^{k\bar{j}}$. Now the variation of the transverse scalar curvature is

$$\begin{aligned} \delta R^T &= \delta(g^{i\bar{j}} R^T_{i\bar{j}}) \\ &= -\delta(g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log(\det(g^T_{i\bar{j}}))) \\ &= -\langle v_{i\bar{j}}, R^T_{i\bar{j}} \rangle - \Delta v. \end{aligned}$$

Next we compute

$$\begin{aligned}\delta|\nabla f|^2 &= \delta(g^{i\bar{j}} f_i f_{\bar{j}}) \\ &= -\langle v_{i\bar{j}}, f_i f_{\bar{j}}\rangle + g^{i\bar{j}}(h_i f_{\bar{j}} + f_i h_{\bar{j}}) \\ &= -\langle v_{i\bar{j}}, f_i f_{\bar{j}}\rangle + 2g^T(\nabla f, \nabla h).\end{aligned}$$

Recall that $dV = \eta \wedge (\omega)^n = c(n)\det(g^T_{i\bar{j}})dx \wedge dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n$. Hence $\delta dV = v dV$ and it follows that

$$\delta(\tau^{-n} e^{-f} dV) = (v - h - n\sigma\tau^{-1})\tau^{-n} e^{-f} dV.$$

Putting these all together, we have

$$\begin{aligned}\delta\mathcal{W}^T(v_{ij}, h, \sigma) = &\int_M (\sigma(R^T + |\nabla f|^2) + h)\tau^{-n} e^{-f} dV \\ &+ \int_M \tau(-\langle v_{i\bar{j}}, R^T_{i\bar{j}} + f_i f_{\bar{j}}\rangle - \Delta v + 2g^T(\nabla f, \nabla h))\tau^{-n} e^{-f} dV \\ &+ \int_M (\tau(R^T + |\nabla f|^2) + f)(v - h - n\sigma\tau^{-1})\tau^{-n} e^{-f} dV.\end{aligned}$$

Observe that $\Delta e^{-f} = (|\nabla f|^2 - \Delta f)e^{-f}$. So, since M is closed, by Stokes' theorem we have

$$\int_M (|\nabla f|^2 - \Delta f)e^{-f} dV = 0. \tag{3.10}$$

Since $f, h, v \in C^\infty_B(M)$, we have the following integration by parts formulas:

$$\int_M e^{-f}\Delta v \, dV = \int_M v(|\nabla f|^2 - \Delta f)e^{-f} \, dV, \tag{3.11}$$

$$\int_M \langle v_{i\bar{j}}, f_i f_{\bar{j}}\rangle e^{-f} dV = \int_M (\langle v_{i\bar{j}}, f_{i\bar{j}}\rangle - v(\Delta f - |\nabla f|^2))e^{-f} \, dV, \tag{3.12}$$

$$\int_M g^T(\nabla f, \nabla h)e^{-f} dV = \int_M h(|\nabla f|^2 - \Delta f)e^{-f} \, dV. \tag{3.13}$$

Equations (3.11) and (3.13) are straightforward to verify. For a proof of (3.12), see proposition 4.3 in [19]. Now, using equations (3.10)-(3.13) in our above expression for $\delta\mathcal{W}^T$ we get

$$\begin{aligned}\delta\mathcal{W}^T(v_{i\bar{j}}, h, \sigma) = &\int_M \left(\sigma(R^T + \Delta f) - \tau\langle v_{i\bar{j}}, R^T_{i\bar{j}} + f_{i\bar{j}}\rangle + h\right)\tau^{-n}e^{-f} \, dV \\ &+ \int_M (v - h - n\sigma\tau^{-1})(\tau(2\Delta f - |\nabla f|^2 + R^T) + f)\tau^{-n}e^{-f} \, dV.\end{aligned}$$

$\square$

If we choose $h = v - n\sigma\tau^{-1}$, then the second integral in the expression for $\delta\mathcal{W}^T$ vanishes. Now let $v_{i\bar{j}} = g_{i\bar{j}} - R^T_{i\bar{j}} - f_{i\bar{j}}$. Then $v = n - R^T - \Delta f$ and $h = n - R^T - \Delta f - n\sigma\tau^{-1}$. Putting these choices of $v_{i\bar{j}}$ and h into the above and noting that $\langle g_{i\bar{j}}, R^T_{i\bar{j}} + f_{i\bar{j}}\rangle = R^T + \Delta f$, we find that

$$\delta\mathcal{W}^T(v_{i\bar{j}}, h, \sigma) = \int_M \left(\tau|R_{i\bar{j}} + f_{i\bar{j}}|^2 - (\tau - \sigma + 1)(R + \Delta f) + n - n\sigma\tau^{-1}\right)\tau^{-n}e^{-f} \, dV.$$

Now setting $\sigma = \tau - 1$, this becomes

$$\begin{aligned}\delta\mathcal{W}^T(v_{i\bar{j}}, h, \sigma) &= \int_M \left(\tau|R_{i\bar{j}} + f_{i\bar{j}}|^2 - 2(R + \Delta f) + n\tau^{-1}\right)\tau^{-n}e^{-f}\ dV \\ &= \int_M |R_{i\bar{j}} + f_{i\bar{j}} - \tau^{-1}g_{i\bar{j}}|^2\tau^{-n+1}e^{-f}\ dV \geq 0. \qquad (3.14)\end{aligned}$$

The next goal is to show that $\mathcal{W}^T$ is monotonically increasing along the TKRF. Let us outline one way to accomplish this.

(i) First show that for a diffeomorphism $\varphi : M \to M$ we have

$$\mathcal{W}^T(g, f, \tau) = \mathcal{W}^T(\varphi^* g, \varphi^* f, \tau). \qquad (3.15)$$

This is relatively straightforward.

(ii) Next find a smooth basic function $f = f(t)$ satisfying

$$\frac{\partial f}{\partial t} = n\tau^{-1} - R^T - \Delta f + |\nabla f|^2.$$

This is a backward heat equation and for any time $T > 0$, if $f(T) \in C_B^\infty(M)$, then there is a unique basic solution on $[0, T]$. To prove this, one can argue as follows:

Proof (sketch). Pick $T > 0$ and consider the equation

$$\frac{\partial w}{\partial s} = \Delta w - (R^T - n\tau^{-1})w. \qquad (3.16)$$

This is a linear parabolic equation as long as g^T remains positive definite. Since g^T is positive definite for all time along the TKRF, by the standard parabolic theory, given an initial condition, a unique solution exists for $s \in [0,T]$. If $w(0)$ is basic, then the solution remains basic. Indeed, as R^T is basic, ξw satisfies

$$\frac{\partial}{\partial s}(\xi w) = \xi \frac{\partial w}{\partial s} = \xi \Delta w - \xi (R^T + n\tau^{-1}) w = \Delta(\xi w) - (R^T + n\tau^{-1})(\xi w).$$

So if $\xi w(0) = 0$, then by the uniqueness of the solution, we must have $\xi w \equiv 0$.

If $w(0) \geq 0$, then $w(s) \geq 0$. Furthermore, if $w(0)$ is positive at some point $p \in M$, then $w(s) > 0$. By the standard regularity theory for linear parabolic equations, if we have $w(0) \in W^{1,2}(M)$ then $w(s) \in C^\infty(M)$. See proposition 4.9 in [19].

We can solve equation (3.16) with $w(0) > 0$ a basic function. Then we can write $w(s) = e^{-f(s)}$ for some smooth function f. As w is basic for all s, so is f. Let $t = T - s$. Then for $t \in [0,T]$, it's not hard to check that $f(t)$ solves the backward heat equation

$$\frac{\partial f}{\partial t} = n\tau^{-1} - R^T - \Delta_B f + |\nabla f|^2.$$

$\square$

(iii) Now consider the time-dependent diffeomorphisms $\varphi(t,\cdot) : M \to M$ having

$$\frac{\partial}{\partial t}\varphi(t,p) = -\nabla f(t)|_{\varphi(t,p)}, \ \varphi(0,\cdot) = I.$$

That these diffeomorphisms exist is somewhat standard and a proof can be found, for example, in lemma 3.15 of [10]. Next, direct computation shows that if g evolves by TKRF, then $\tilde{g} := \varphi^* g$ induces a transverse metric which evolves by

$$\frac{\partial}{\partial t}\tilde{g}^T_{i\bar{j}} = \tilde{g}_{i\bar{j}} - \tilde{R}^T_{i\bar{j}} - \tilde{f}_{i\bar{j}},$$

where $\tilde{R}^T_{i\bar{j}}$ is the transverse Ricci tensor of $\tilde{g}$ and $\tilde{f} = \varphi^* f$ is a basic function satisfing

$$\frac{\partial \tilde{f}}{\partial t} = n\tau^{-1} - \tilde{R}^T - \Delta \tilde{f}.$$

(iv) Finally, letting τ evolve by

$$\frac{\partial \tau}{\partial t} = \tau - 1,$$

the computations in (3.14) and (3.15) then imply that $\frac{\partial \mathcal{W}^T}{\partial t} \geq 0$ along the TKRF.

This outline is similar to the approach in [25] for the Ricci flow and the one taken in [12] for the Sasaki-Ricci flow. We can also make this outline work for us, but there are some technical difficulties involved. One difficulty encountered with this approach is that the diffeomorphisms φ may not preserve the transverse holomorphic structure. The pulled back metrics $\varphi^* g$ may not be quasi-Sasaki as the induced transverse metric may not even by Hermitian. This is not a terrible difficulty to overcome though. One must simply define $\mathcal{W}^T$ on a larger class of metrics. In particular, the class $\mathcal{M}$ of Riemannian metrics on M for which ξ is a unit-length Killing vector field will suffice. The metrics in $\mathcal{M}$ are bundle-like with respect to $\mathcal{F}_\xi$ and thus induce transverse metrics. But now one must check that $\mathcal{W}^T$ has the same variational formula when the metrics vary in $\mathcal{M}$ rather than as quasi-Sasaki metrics. This can be done and the proof is the same as in the Riemannian setting. Then one must show that if $g \in \mathcal{M}$ then $\varphi^* g \in \mathcal{M}$, that $\varphi_* \xi = \xi$ and that if f is basic then $\varphi^* f$ is basic too. All of this can be done but we will choose not to pursue it any further.

Another approach is to show by direct computation that $\frac{\partial \mathcal{W}^T}{\partial t} \geq 0$ under the *coupled transverse Kähler-Ricci flow*

$$\begin{cases} \dfrac{\partial g^T}{\partial t} = g^T - \mathrm{Ric}_g^T \\ \\ \dfrac{\partial f}{\partial t} = n\tau^{-1} - R^T - \Delta f + |\nabla f|^2 \\ \\ \dfrac{\partial \tau}{\partial t} = \tau - 1 \end{cases} \tag{3.17}$$

This approach takes advantage of the transverse Kähler structure and we can remain in the category of quasi-Sasaki metrics. We can prove the following proposition, which is the same as proposition 4.4 in [19]:

Proposition 3.2.3. *Under the coupled transverse Kähler-Ricci flow (3.17), the $\mathcal{W}^T$ functional on quasi-Sasaki manifolds satisfies*

$$\frac{\partial \mathcal{W}^T}{\partial t} = \int_M |R_{i\bar{j}} + f_{i\bar{j}} - \tau^{-1} g_{i\bar{j}}|^2 \tau^{-n+1} e^{-f}\ dV + \int_M |f_{ij}|^2 \tau^{-n+1} e^{-f}\ dV \geq 0. \tag{3.18}$$

Moreover, $\mathcal{W}^T$ is strictly increasing unless $R^T_{i\bar{j}} + f_{i\bar{j}} - \tau^{-1} g^T_{i\bar{j}} = 0$ and $f_{ij} = 0$. In this case, ∇f is a real holomorphic vector field and g^T is a transverse (gradient-shrinking) Kähler-Ricci soliton.

The proof involves a careful computation with several uses of integration by parts. Some of the integration by parts formulas are not obvious. As the proof is the same as in [19], we do not reproduce it here. Next we discuss the analog of Perelman's μ functional.

3.22. The transverse μ-functional

In the last subsection, we saw that if g evolves as $\partial_t g_{i\bar{j}} = g_{i\bar{j}} - R^T_{i\bar{j}} - f_{i\bar{j}}$, f evolves as $\partial_t f = n\tau^{-1} - R^T - \Delta_B f$ and τ evolves as $\partial_t \tau = \tau - 1$, then $\mathcal{W}^T(g, f, \tau)$ is non-decreasing with t and $g^{i\bar{j}}\partial_t g_{i\bar{j}} - \partial_t f - \frac{n}{\tau}\partial_t \tau = 0$. Hence

$$\frac{\partial}{\partial t}(\tau^{-n} e^{-f} dV) = 0.$$

This implies that $\int_M \tau^{-n} e^{-f} dV$ is constant and motivates our next definition.

Definition 3.2.4. The transverse μ-functional, μ^T, is defined by

$$\mu^T(g, \tau) := \inf_{f \in C^\infty_B(M)} \{\mathcal{W}^T(g, f, \tau) : \int_M \tau^{-n} e^{-f} dV = 1\}.$$

As it is the case with the $\mathcal{W}^T$-functional, it is not surprising that the μ^T-functional is also monotonically non-decreasing along the TKRF when $\partial_t \tau = \tau - 1$. We prove this next.

Proposition 3.2.5. *If $g(t)$ evolves by TKRF and τ satisfies $\partial_t \tau = \tau - 1$, then $\mu^T(g(t), \tau(t))$ is monotone non-decreasing in t. In particular, $\mu^T(g(t), 1)$ is non-decreasing along the transverse Kähler-Ricci flow.*

Proof. Fix a time $t_1 > 0$ and let $f_{t_1} \in C^\infty_B(M)$ be normalized by $\int_M \tau^{-n} e^{-f_{t_1}} dV = 1$. Then solve the backward heat equation

$$\frac{\partial f}{\partial t} = n\tau^{-1} - R^T - \Delta_B f + |\nabla f|^2$$

for $t \in [0, t_1]$ with $f(t_1) = f_{t_1}$. We know that the solution $f(t)$ remains basic. When g evolves by TKRF and $\partial_t \tau = \tau - 1$, we can compute that

$$\frac{\partial}{\partial t}(\tau^{-n} e^{-f} dV) = (\Delta f - |\nabla f|^2)\tau^{-n} e^{-f} dV = -\tau^{-n} \Delta e^{-f} dV.$$

Thus,

$$\frac{\partial}{\partial t} \int_M \tau^{-n} e^{-f} \, dV = - \int_M \tau^{-n} \Delta e^{-f} \, dV = 0.$$

This implies that $\int_M \tau^{-n} e^{-f(t)} \, dV = 1$ for all $t \in [0, t_1]$. Now chose $0 \leq t_0 < t_1$. By the definition of μ^T and the monotonicity of $\mathcal{W}^T$, we get

$$\mu^T(g(t_0), \tau(t_0)) \leq \mathcal{W}^T(g(t_0), f(t_0), \tau(t_0)) \leq \mathcal{W}^T(g(t_1), f(t_1), \tau(t_1)).$$

Since $f_{t_1} = f(t_1)$ was chosen arbitrarily, we conclude that

$$\mu^T(g(t_0), \tau(t_0)) \leq \inf_{f(t_1)} \mathcal{W}^T(g(t_1), f(t_1), \tau(t_1)) = \mu^T(g(t_1), \tau(t_1)).$$

□

3.3. Bounds on Scalar Curvature and the Transverse Ricci Potential

We assume there is a transverse Kähler metric ω such that $[\omega] = 2\pi c_B^1$. Then by the transverse $\partial\bar{\partial}$-lemma, there is smooth function $u(x, t)$ such that $g_{i\bar{j}}^T - R_{i\bar{j}}^T = \partial_i \partial_{\bar{j}} u$. We call u a *transverse Ricci potential.* We may assume that u is normalized so that $\int_M e^{-u} \, dV = 1$. If ϕ is a solution to (3.3), then

$$\partial_i \partial_{\bar{j}} \frac{\partial \phi}{\partial t} = \frac{\partial g_{i\bar{j}}^T}{\partial t} = g_{i\bar{j}}^T - R_{i\bar{j}}^T = \partial_i \partial_{\bar{j}} u.$$

So we can take $u = \partial_t \phi$. As $R^T_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log(\det(g^T))$, we compute that

$$\begin{aligned}
\partial_i \partial_{\bar{j}} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}(g^T_{i\bar{j}} - R^T_{i\bar{j}}) \\
&= g^T_{i\bar{j}} - R^T_{i\bar{j}} + \partial_i \partial_{\bar{j}} g^{i\bar{j}} \frac{\partial g^T_{i\bar{j}}}{\partial t} \\
&= \partial_i \partial_{\bar{j}}(u + g^{i\bar{j}} \partial_i \partial_{\bar{j}} u) \\
&= \partial_i \partial_{\bar{j}}(u + \Delta^T u).
\end{aligned}$$

Thus, there is time-dependent constant $A(t)$ such that

$$\frac{\partial u}{\partial t} = u + \Delta^T u + A(t). \tag{3.19}$$

As $\partial_t dV = \Delta^T u \; dV$, differentiating $\int_M e^{-u} \; dV = 1$ with respect to t, we find that

$$A(t) = -\int_M u e^{-u} \; dV.$$

We can use the monotonicity of the μ^T-functional to show that A is uniformly bounded. Recall from proposition 3.2.5 that $\mu^T(g(t), 1)$ is non-decreasing in t.

Lemma 3.3.1. *The function $A(t)$ is uniformly bounded.*

Proof. It is a routine calculus exercise to show that $f(t) = te^{-t}$ is bounded above by e^{-1}. From $\partial_t dV = \Delta^T u \; dV$, we see that the volume of the manifold is constant along the TKRF. For simplicity, let us assume that $\mathrm{Vol}(M) = \int_M dV = 1$. Then

$$A(t) = -\int_M u e^{-u} \; dV \geq -e^{-1} \int_M dV = -e^{-1}.$$

Tracing $g^T_{i\bar{j}} - R^T_{i\bar{j}} = \partial_i \partial_{\bar{j}} u$, we have $\Delta^T u = n - R^T$. Thus

$$\begin{aligned}\mathcal{W}^T(g(t), u(t), 1) &= \int_M (n - \Delta^T u + |\nabla u|^2 + u)e^{-u}\ dV \\ &= \int_M \Delta^T e^{-u}\ dV + \int_M (n+u)e^{-u}\ dV = n - A(t).\end{aligned}$$

Therefore,

$$A(t) = n - \mathcal{W}^T(g(t), u(t), 1) \leq n - \mu^T(g(t), 1) \leq n - \mu^T(g(0), 1).$$

$\square$

Next we will see how the transverse scalar curvature evolves along the TKRF, but first we recall a useful lemma. The proof is a standard calculus argument.

Lemma 3.3.2. *If $f = f(x,t)$ is a smooth function on $M \times [0,T]$ for some $T > 0$ and f achieves its minimum (maximum) at (x_0, t_0), then either $t_0 = 0$ or at the point (x_0, t_0)*

$$\frac{\partial f}{\partial t} \leq 0\ (\geq 0),\ \ \nabla f = 0 \textit{ and } \Delta f \geq 0\ (\leq 0).$$

Lemma 3.3.3. *Along the TKRF, the transverse scalar curvature evolves by*

$$\frac{\partial R^T}{\partial t} = -R^T + |\mathrm{Ric}^T|^2 + \Delta^T R^T$$

and is uniformly bounded from below.

Proof. Because of the transverse Kähler structure, the Ricci tensor $R^T_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log(\det(g_{i\bar{j}}))$. Thus we can write the scalar curvature as

$$R^T = -g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log(\det(g_{i\bar{j}})).$$

From $g^{i\bar{l}}g_{k\bar{l}} = \delta^i_k$, we compute $\partial_t g^{i\bar{j}} = -g^{i\bar{l}}(\partial_t g_{k\bar{l}})g^{k\bar{j}}$. Therefore,

$$\begin{aligned}
\frac{\partial R^T}{\partial t} &= g^{i\bar{l}}(\partial_t g_{k\bar{l}})g^{k\bar{j}}\partial_i\partial_{\bar{j}}\log(\det(g_{i\bar{j}})) - g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{i\bar{j}}\partial_t g_{i\bar{j}} \\
&= -g^{i\bar{l}}(g_{k\bar{l}} - R^T_{k\bar{l}})g^{k\bar{j}}R^T_{i\bar{j}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{i\bar{j}}(g_{i\bar{j}} - R^T_{i\bar{j}}) \\
&= -\delta^i_k g^{k\bar{j}}R^T_{i\bar{j}} + g^{i\bar{l}}g^{k\bar{j}}R^T_{k\bar{l}}R^T_{i\bar{j}} - g^{i\bar{j}}\partial_i\partial_{\bar{j}}(n) + g^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{i\bar{j}}R^T_{i\bar{j}} \\
&= -R^T + |\mathrm{Ric}^T|^2 + \Delta^T R^T.
\end{aligned}$$

Now fix a time $T > 0$ and suppose that $R^T(x,t)$ on $M \times [0,T]$ takes its minimum at (x_0,t_0). If $t_0 = 0$, then we have

$$\inf_{M\times[0,T]} R^T(x,t) \geq \inf_M R^T(x,0).$$

Since R^T is basic, $\Delta^T R^T = \Delta R^T$. So if $t_0 \neq 0$, then by the above lemma, at (x_0,t_0),

$$0 \geq \frac{\partial R^T}{\partial t} - \Delta^T R^T = -R^T + |\mathrm{Ric}^T|^2 \geq -R^T.$$

So either way, as $T > 0$ was arbitrary, we have $R^T \geq \min\{0, \inf_M R^T(x,0)\}$. $\square$

Lemma 3.3.4. *The function $u(x,t)$ is uniformly bounded from below.*

Proof. Fix $T > 0$. By lemmas 3.3.1 and 3.3.3, $A(t) - R^T$ is uniformly bounded above. Set

$$C = \max\{0, \sup_{x\in M, t\in[0,T]} n - R^T + A(t)\} < \infty.$$

Suppose that $u(x,t)$ is not uniformly bounded below. Then there is a point (x_0,t_0) where $u(x_0,t_0) + C < 0$. Since u is smooth in x, there is a neighborhood U of x_0 such that

$$u(x,t_0) + C < 0$$

for all $x \in U$. Recall that $\partial_t u = u + \Delta^T u + A(t)$ and $\Delta^T u = n - R^T$. Thus, for $x \in U$,

$$\frac{\partial u}{\partial t}|_{(x,t_0)} = u(x,t_0) + n - R^T(x,t_0) + A(t_0) < 0.$$

Since u is smooth in t, $u(x,t) \leq u(x,t_0)$ for $x \in U$ and $t \geq t_0$. From $\partial_t u < u + C$ we get

$$u(x,t) < (C + u(x,t_0))e^{t-t_0} \leq -C_1 e^t \tag{3.20}$$

for $x \in U$, $t \geq t_0$ and some constant $C_1 > 0$ that depends on t_0.

Recall that $\partial_t \phi = u$. Integrating equation (3.20) from t_0 to t gives

$$\phi(x,t) \leq \phi(x,t_0) - C_1(e^t - e^{t_0}) \leq -C_2 e^t \tag{3.21}$$

for $x \in U$, $t \geq t_0$ sufficiently large and $C_2 > 0$. As u is normalized so that $\int_M e^{-u}\, dV = 1$, $-u$ cannot be very large. Thus there is a constant $C_3 > 0$ such that

$$u(x_t,t) := \max_{x \in M} u(x,t) \geq -C_3.$$

From $\partial_t(u - \phi) = n - R^T + A(t) \leq C$ we get

$$u(x_t,t) - \phi(x_t,t) \leq \max_{x \in M}(u(x,0) - \phi(x,0)) + Ct.$$

This implies there is a constant C_4 so that

$$\phi(x'_t,t) := \max_{x \in M} \phi(x,t) \geq -C_4 - Ct. \tag{3.22}$$

Tracing $g^T_{i\bar{j}}(t) = g_{i\bar{j}}(0) + \partial_i\partial_{\bar{j}}\phi$, with the metric $g(0)$ gives $n + \Delta_{g(0)}\phi = g^{i\bar{j}}(0)g^T_{i\bar{j}}(t) > 0$. Recall that the volume of the manifold is constant along the TKRF and we are assuming that $\mathrm{vol}(M) = 1$. Let G_0 be the Green's function for $\Delta_{g(0)}$. The function $\phi - \int_M \phi\, dV_0$ integrates to zero with respect to dV_0, so we may apply Green's formula to write

$$\phi(x'_t, t) = \int_M \phi(y,t)\, dV_0 - \int_M \Delta_{g(0)}\phi(y,t)G_0(x'_t, y)\, dV_0. \tag{3.23}$$

By (3.21), $\phi(x,t) \leq -C_2e^t$ on $U \times [t_0, \infty)$. Since $-\Delta_{g(0)}\phi < n$ and $G_0(x,y)$ is integrable, from (3.23) we get that for $t \geq t_0$

$$\phi(x'_t, t) \leq \mathrm{vol}(M \setminus U)\phi(x'_t, t) - \mathrm{vol}(U)C_2e^t + \tilde{C}.$$

Note that $\mathrm{vol}(M \setminus U) < 1$. So for large enough $t \geq t_0$,

$$\phi(x'_t, t) \leq -C_5e^t + C_6. \tag{3.24}$$

Combining (3.22) and (3.24), we have

$$-C_4 - Ct \leq \phi(x'_t, t) \leq -C_5e^t + C_6.$$

This is a contradiction for sufficiently large t as all of the constants involved are independent of $t > t_0$. Therefore, $u(x,t)$ is uniformly bounded from below. □

Lemma 3.3.5. *There is a uniform constant $C > 0$ such that $|\nabla u|^2 \leq C(u + C)$.*

Before we begin the proof, we need one new bit of notation. For a Kähler metric g, we define ∇f to be the vector field dual to $\bar{\partial} f$. We define $\bar{\nabla} f$ to be the vector field dual to ∂f. In local coordinates, $\nabla f = g^{i\bar{j}} f_{\bar{j}} dz^i$, $\bar{\nabla} f = g^{i\bar{j}} f_i d\bar{z}^j$ and $g(\nabla f, \bar{\nabla} f) = g^{i\bar{j}} f_i f_{\bar{j}} = |\nabla f|^2$. We make the analogous definition for a basic function and the transverse Kähler metric of a quasi-Sasaki structure.

Proof. Define the parabolic operator $\Box := \partial_t - \Delta$ and denote $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$. One computes

$$\Box\left(\frac{f}{h}\right) = \frac{\Box f}{h} - \frac{f\Box h}{h^2} + \frac{2\Re\langle \nabla f, \bar{\nabla} h\rangle}{h^2} - \frac{2f|\nabla h|^2}{h^3},$$

where $\Re$ denotes the real part of a complex number. From (3.19) we have

$$\Box u = u + A(t).$$

A local coordinate computation gives

$$\partial_t |\nabla u|^2 = |\nabla u|^2 + \langle R_{i\bar{j}}, u_i u_{\bar{j}}\rangle + 2\Re\langle \nabla u, \bar{\nabla}\Delta u\rangle,$$

and the Bochner-Kodaira formula gives

$$\Delta|\nabla u|^2 = |\nabla\nabla u|^2 + |\nabla\bar{\nabla} u|^2 + \langle R_{i\bar{j}}, u_i u_{\bar{j}}\rangle + 2\Re\langle \nabla u, \bar{\nabla}\Delta u\rangle.$$

Therefore

$$\Box|\nabla u|^2 = |\nabla u|^2 - |\nabla\nabla u|^2 - |\nabla\bar{\nabla} u|^2.$$

By lemma 3.3.4, there is a constant $B > 0$ such that $u(x,t) + B > 0$ for all $x \in M$ and all $t \in [0,\infty)$. Now let $H := |\nabla u|^2/(u+2B)$. Then

$$\begin{aligned}
\Box H &= \frac{\Box|\nabla u|^2}{u+2B} - \frac{|\nabla u|^2\Box u}{(u+2B)^2} + \frac{2\Re\langle\nabla|\nabla u|^2, \bar{\nabla} u\rangle}{(u+2B)^2} - \frac{2|\nabla u|^2|\nabla u|^2}{(u+2B)^3} \\
&= \frac{|\nabla u|^2 - |\nabla\nabla u|^2 - |\nabla\bar{\nabla} u|^2}{u+2B} - \frac{|\nabla u|^2(u+A)}{(u+2B)^2} + \frac{2\Re\langle\nabla|\nabla u|^2, \bar{\nabla} u\rangle}{(u+2B)^2} - \frac{2|\nabla u|^4}{(u+2B)^3} \\
&= \frac{-|\nabla\nabla u|^2 - |\nabla\bar{\nabla} u|^2}{u+2B} + \frac{|\nabla u|^2(2B-A)}{(u+2B)^2} + \frac{2\Re\langle\nabla|\nabla u|^2, \bar{\nabla} u\rangle}{(u+2B)^2} - \frac{2|\nabla u|^4}{(u+2B)^3}.
\end{aligned} \tag{3.25}$$

By the definition of H,

$$\nabla H = \frac{\nabla|\nabla u|^2}{u+2B} - \frac{|\nabla u|^2\nabla u}{(u+2B)^2}.$$

Hence

$$\frac{\Re\langle\nabla H, \bar{\nabla} u\rangle}{u+2B} = \frac{\Re\langle\nabla|\nabla u|^2, \bar{\nabla} u\rangle}{(u+2B)^2} - \frac{|\nabla u|^4}{(u+2B)^3}. \tag{3.26}$$

Next we observe that

$$\begin{aligned}
\frac{|\langle\nabla|\nabla u|^2, \bar{\nabla} u\rangle|}{(u+2B)^2} &\le \frac{|\nabla u|^2(|\nabla\nabla u| + |\nabla\bar{\nabla} u|)}{(u+2B)^{3/2}(u+2B)^{1/2}} \\
&\le \frac{|\nabla u|^4}{2(u+2B)^3} + \frac{|\nabla\nabla u|^2 + |\nabla\bar{\nabla} u|^2}{u+2B}.
\end{aligned} \tag{3.27}$$

Choosing $0 < \varepsilon < 1/4$ and combining (3.25), (3.26) and (3.27), yields

$$\Box H \le \frac{|\nabla u|^2(2B-A)}{(u+2B)^2} + \frac{(2-\varepsilon)\Re\langle\nabla H, \bar{\nabla} u\rangle}{u+2B} - \frac{\varepsilon|\nabla u|^4}{2(u+2B)^3}. \tag{3.28}$$

If H is unbounded from above, then there is a time t_0 such that

$$\max_{M\times[0,t_0]} H > \frac{2B-A}{\varepsilon/2}.$$

At the point where H achieves its maximum, $\nabla H = 0$ and $\Box H \geq 0$. At this point, (3.28) gives

$$0 \leq \Box H \leq \frac{|\nabla u|^2}{(u+2B)^2}\Big(2B - A - \frac{\varepsilon}{2}H\Big) < 0.$$

This is a contradiction and therefore H is bounded from above. Thus there is a constant $C > 0$ such that $|\nabla u|^2 \leq C(u+C)$. $\square$

Lemma 3.3.6. *There is a uniform constant $C > 0$ such that $R^T \leq C(u+C)$.*

Proof. From $\partial_i\partial_{\bar{j}} u = g^T_{i\bar{j}} - R^T_{i\bar{j}}$, $\Delta u = n - R^T$ and $\partial_t R^T = -R^T + |\mathrm{Ric}^T|^2 + \Delta^T R^T$, we compute

$$\begin{aligned}
\Box(\Delta u) &= -\Box R^T \\
&= R^T - |\mathrm{Ric}^T|^2 \\
&= n - \Delta u - \langle g^T_{i\bar{j}} - u_{i\bar{j}}, g^T_{i\bar{j}} - u_{i\bar{j}}\rangle \\
&= n - \Delta u - (n - 2\Delta u + |u_{i\bar{j}}|^2) \\
&= \Delta u - |u_{i\bar{j}}|^2.
\end{aligned}$$

Let $K := -\Delta u/(u+2B)$ where $B > 0$ is as in the previous lemma. We compute

$$\begin{aligned}
\Box K &= \frac{-\Box \Delta u}{u+2B} + \frac{\Delta u \Box u}{(u+2B)^2} - \frac{2\Re\langle \nabla\Delta u, \bar{\nabla} u\rangle}{(u+2B)^2} + \frac{2\Delta u|\nabla u|^2}{(u+2B)^3} \\
&= \frac{-\Delta u + |u_{i\bar{j}}|^2}{u+2B} + \frac{\Delta u(u+A)}{(u+2B)^2} - \frac{2\Re\langle \nabla\Delta u, \bar{\nabla} u\rangle}{(u+2B)^2} + \frac{2\Delta u|\nabla u|^2}{(u+2B)^3} \\
&= \frac{|u_{i\bar{j}}|^2}{u+2B} + \frac{\Delta u(A-2B)}{(u+2B)^2} - \frac{2\Re\langle \nabla\Delta u, \bar{\nabla} u\rangle}{(u+2B)^2} + \frac{2\Delta u|\nabla u|^2}{(u+2B)^3}. \qquad (3.29)
\end{aligned}$$

By the definition of K,

$$\nabla K = \frac{-\nabla\Delta u}{u+2B} + \frac{\Delta u \nabla u}{(u+2B)^2}.$$

Hence

$$\frac{\Re\langle \nabla K, \bar\nabla u\rangle}{u+2B} = -\frac{\Re\langle \nabla\Delta u, \bar\nabla u\rangle}{(u+2B)^2} + \frac{\Delta u|\nabla u|^2}{(u+2B)^3}.$$

Combining this with (3.29) yields

$$\Box K = \frac{|u_{i\bar j}|^2}{u+2B} + \frac{\Delta u(A-2B)}{(u+2B)^2} + \frac{2\Re\langle \nabla K, \bar\nabla u\rangle}{u+2B}. \tag{3.30}$$

Let H be as in the previous lemma. There we computed that

$$\Box H = \frac{-|u_{ij}|^2 - |u_{i\bar j}|^2}{u+2B} + \frac{|\nabla u|^2(2B-A)}{(u+2B)^2} + \frac{2\Re\langle \nabla H, \bar\nabla u\rangle}{u+2B}.$$

Now let $G := 2H + K$. Then by (3.30) and the above, we have

$$\Box G = \frac{-2|u_{ij}|^2 - |u_{i\bar j}|^2}{u+2B} + \frac{(2|\nabla u|^2 - \Delta u)(2B-A)}{(u+2B)^2} + \frac{2\Re\langle \nabla G, \bar\nabla u\rangle}{u+2B}. \tag{3.31}$$

At the point where G achieves its maximum, $\nabla G = 0$ and $\Box G \geq 0$. So at this point, equation (3.31) yields

$$0 \leq \Box G \leq \frac{-|u_{i\bar j}|^2}{u+2B} + \frac{2|\nabla u|^2(2B-A)}{(u+2B)^2} - \frac{\Delta u(2B-A)}{(u+2B)^2}. \tag{3.32}$$

In normal coordinates at the point where G is at a maximum, by Cauchy-Schwarz

$$(\Delta u)^2 = \Big(\sum_i u_{i\bar i}\Big)^2 \leq n \sum_i u_{i\bar i}^2 = n|u_{i\bar j}|^2.$$

So by (3.32) we get

$$0 \le \Box G \le \frac{2|\nabla u|^2(2B-A)}{(u+2B)^2} - \frac{\Delta u}{u+2B}\left(\frac{\Delta u}{n} + \frac{2B-A}{u+2B}\right). \tag{3.33}$$

Since $\inf u + 2B > 0$ and $A(t)$ is uniformly bounded, $(2B-A)/(u+2B)$ is uniformly bounded from above. By the bound on $|\nabla u|^2$ from the previous lemma, the first term on the right hand side of (3.33) is uniformly bounded from above.

So if $-\Delta u/(u+2B)$ becomes arbitrarily large, then $\Delta u \ll -(u+2B) < 0$ and the right hand side of (3.33) goes negative. This is a contradiction. Thus $-\Delta u/(u+2B)$ is bounded from above. Hence there is a constant $C > 0$ such that $-\Delta u \le C(u+C)$. Since $R^T = n - \Delta u$, the proof is complete. □

Proposition 3.3.7. *Let $x_t \in M$ be such that $u(x_t, t) = \min_{y \in M} u(y,t)$. Then there is a uniform constant $C > 0$ such that*

$$u(y,t) \le C(d_t^2(x_t, y) + 1),$$

$$|\nabla u| \le C(d_t(x_t, y) + 1),$$

$$R^T(y,t) \le C(d_t^2(x_t, y) + 1),$$

where d_t is the distance function for the metric $g(t)$.

We prove the bound on u. The bounds on $|\nabla u|$ and R^T then follow from lemmas 3.3.5 and 3.3.6 respectively. By lemma 3.3.4, u is uniformly bounded below. Thus we may assume that $u > 0$. If not, then we just consider the function $\tilde{u} := u + C$ where $C > 0$ is a constant. Then $\tilde{u}$ is a transverse Ricci potential and choosing C large enough makes $\tilde{u} > 0$.

Proof. The bound on $|\nabla u|^2$ from lemma 3.3.5 implies that $|\nabla\sqrt{u}| \le C_0$. So all of the first derivatives of $\sqrt{u}$ are uniformly bounded (recall that u is basic). Thus $\sqrt{u}$ is uniformly Lipschitz. Hence

$$|\sqrt{u(y,t)} - \sqrt{u(z,t)}| \le C_0 d_t(y,z).$$

Now if $u(x_t,t) \ge C(t)$ and $C(t) \to \infty$ as $t \to \infty$, then by the normalization of u and the fact that the volume of the manifold is constant along the flow, we get

$$1 = \int_M e^{-u}\, dV \le \mathrm{vol}(M) e^{-C(t)} \to 0.$$

Contradiction. Thus $u(x_t,t) \le C_1$ for some constant $C_1 > 0$. Therefore,

$$\begin{aligned}\sqrt{u(y,t)} &\le C_0 d_t(y,x_t) + \sqrt{u(x_t,t)}\\ &\le C_0 d_t(y,x_t) + \sqrt{C_1}.\end{aligned}$$

This implies that $u(y,t) \le C d_t^2(y,x_t) + C$ for some constant $C > 0$. $\square$

We see from this proposition that if we can bound the diameter of the manifold along the TKRF, then we will have uniform bounds on the transverse Ricci potential u and the transverse scalar curvature R^T. We take this up in the next section.

3.4. Upper Bound on Diameter

In this section we will prove a uniform upper bound on the diameter of a quasi-Sasaki manifold along the TKRF. This along with proposition 3.3.7 will provide uniform bounds on u, ∇u and R^T. We take an approach analogous to that of Sesum and Tian in [34], but we must make some adjustments. The W^T functional only involves basic functions, but the distance function of a quasi-Sasaki manifold is not basic. To deal with this, we work with the transverse distance function from definition 1.4.1.

We begin with a non-collapsing theorem. Since R^T is bounded from below along the TKRF (lemma 3.3.3), there is $r_0 > 0$ small enough so that $R^T(x,t) \geq -r_0^{-2}$ for all x and t.

Theorem 3.4.1. *Let M be a quasi-Sasaki manifold whose ξ orbits are compact. Let $g(t)$ be a solution to the TKRF. Then there is a constant $C > 0$ such that for every $x \in M$, if $R^T \leq Cr^{-2}$ on $B^T_{g(t)}(x,r)$ for $r \in (0, r_0]$, then* $\mathrm{Vol}(B^T_{g(t)}(x,r)) \geq Cr^{2n}$.

Proof. We give a proof by contradiction. Suppose that the theorem is false. Then there is a sequence of points and times $(p_k, t_k) \in M \times [0,\infty)$ with $t_k \to \infty$ such that $R^T \leq Cr_k^{-2}$ on $B_k := B^T_{g(t_k)}(p_k, r_k)$ but $r_k^{-2n}\mathrm{Vol}(B_k) \to 0$ as $k \to \infty$.

Let $\psi : [0,\infty) \to \mathbb{R}$ be a *plateau* function such that $\psi = 1$ on $[0, \frac{1}{2}]$, is decreasing on $[\frac{1}{2}, 1]$ with bounded derivative and $\psi = 0$ on $[1,\infty)$. Let $\tau_k := r_k^2$ and define

$$w_k(x) := e^{C_k}\psi\left(r_k^{-1} d^T_{g(t_k)}(p_k, x)\right) =: e^{C_k}\psi_k(x),$$

where the constant C_k is chosen so that $\int_M w_k^2 \tau_k^{-n}\, dV_k = 1$. Here $dV_k = \eta_{t_k} \wedge (\omega_{t_k})^n$ is the volume form at time t_k. Note that the support of w_k is contained in B_k. Thus

$$1 = \int_M w_k^2 \tau_k^{-n}\, dV_k = e^{2C_k} r_k^{-2n} \int_M \psi^2\left(r_k^{-1} d_k^T(p_k, x)\right) dV_k \leq e^{2C_k} r_k^{-2n}\mathrm{vol}(B_k).$$

Since $1 \leq e^{2C_k} r_k^{-2n}\mathrm{Vol}(B_k)$ and $r_k^{-2n}\mathrm{Vol}(B_k) \to 0$, it must be that $C_k \to \infty$ as $k \to \infty$. Recall, $\mathcal{W}^T(g,f,\tau) = \tau^{-n}\int_M \tau(R^T w^2 + 4|\nabla w|^2) - w^2\log(w^2)\, dV$, where $w = e^{-f/2}$.

Now

$$\begin{aligned}\mathcal{W}^T(g(t_k), w_k, \tau_k) &= \tau_k^{-n} \int_M r_k^2(R^T w_k^2 + 4|\nabla w_k|^2) - w_k^2 \log(w_k^2)\ dV_k \\ &\leq e^{2C_k}\tau_k^{-n} \int_M 4|\psi_k'|^2 - \psi_k^2 \log(\psi_k^2)\ dV_k + r_k^2 \max_{B_k} R^T - 2C_k,\end{aligned}$$

Here we have used that $d^T_{g(t_k)}$ is Lipschitz continuous, hence differentiable almost everywhere, with Lipschitz constant 1. Let $V_k(r) := \mathrm{Vol}(B_k(r))$.

Notice that for $t \in \mathbb{R}$, $\lim_{t\to 0} t^2 \log(t^2) = 0$ and for $t \in (0, 1]$, $t^2 \log(t^2) \geq -1$. Since ψ_k takes its values in $[0, 1]$ and ψ_k' is bounded with support in $[\frac{1}{2}, 1]$, there is some constant $C' > 0$ such that

$$\int_M 4|\psi_k'|^2 - \psi_k^2 \log(\psi_k^2)\ dV_k \leq C'(V_k(r_k) - V_k(r_k/2)).$$

By lemma 5.0.13, we may assume that $V_k(r_k) \leq 5^n V_k(r_k/2)$. Hence,

$$\begin{aligned}\int_M 4|\psi_k'|^2 - \psi_k^2 \log(\psi_k^2)\ dV g_k &\leq \tilde{C} V_k(r_k/2) \\ &\leq \tilde{C} \int_M \psi_k^2\ dV_k.\end{aligned}$$

Combining this with the above, we find that

$$\begin{aligned}\mu^T(g(t_k), \tau_k) &\leq \mathcal{W}^T(g(t_k), w_k, \tau_k) \\ &\leq \tilde{C} \int_M w_k^2 \tau_k^{-n}\ dV_k + r_k^2 \max_{B_k} R^T - 2C_k \\ &\leq \tilde{C} + C - 2C_k \to -\infty.\end{aligned}$$

By proposition 3.2.5, $\mu(g(t), 1 + (\tau_0 - 1)e^t)$ is non-decreasing in t. Now for each k, set $\tau_0^k := 1 - (1 - \tau_k)e^{-t_k}$. Then $\mu(g(0), \tau_0^k) \leq \mu(g(t_k), \tau_k) \to -\infty$ as $k \to -\infty$. This is a contradiction because $\mu(g(0), \tau)$ is a continuous function of τ and $\tau_0^k \to 1$ as $k \to \infty$. $\square$

Let u be a transverse Ricci potential. Recall that u is a basic function. By lemma 3.3.4, u is uniformly bounded from below. Let $x_t \in M$ be such that $u(x_t, t) = \min_{x \in M} u(x, t)$ and let $d_t^T(y) = d_{g(t)}^T(x_t, y)$. For $k_1 \leq k_2$, we define the *transverse annulus*

$$B_\xi(k_1, k_2) := \{y \in M : 2^{k_1} \leq d_t^T(y) \leq 2^{k_2}\}.$$

Now we can prove the diameter bound.

Theorem 3.4.2. *Let M be a quasi-Sasaki manifold whose ξ orbits are compact. Let $g(t)$ be a solution to the TKRF. Then there is a uniform constant C such that the transverse diameter $\Theta_{g(t)}^T < C$. This implies the diameter of M is uniformly bounded along the flow.*

Proof. Suppose that the transverse diameters are not uniformly bounded. Then there is a sequence of times $t_i \to \infty$ such that $\Theta_{g(t_i)}^T \to \infty$. Let $\varepsilon_i > 0$ be a sequence such that $\varepsilon_i \to 0$. By lemma 5.0.14, there are sequences k_1^i, k_2^i, r_1^i, r_2^i and a uniform constant C such that

1. $V_i(k_1^i, k_2^i) := \mathrm{Vol}_{g(t_i)}(B_\xi(k_1^i, k_2^i)) < \varepsilon_i$

2. $V_i(k_1^i, k_2^i) \leq 2^{10n} V_i(k_1^i + 2, k_2^i - 2)$

3. $r_1^i \in [k_1^i, k_1^i + 1]$, $r_2^i \in [k_2^i - 1, k_2^i]$ and $\int_{B_\xi(r_1^i, r_2^i)} R^T \, dV_{t_i} \leq C V_i(k_1^i, k_2^i)$.

Now let ψ_i be a cut-off function such that $\psi_i(t) = 1$ for $t \in [2^{k_1^i+2}, 2^{k_2^i-2}]$, $\psi_i(t) = 0$ for $t \in (-\infty, 2^{r_1^i}] \cup [2^{r_2^i}, \infty)$ and $|\psi'(t)| < 2$. Define functions $w_i(y) := e^{C_i} \psi_i(d_{t_i}^T(y))$

where the constant C_i is chosen so that $\int_M w_i^2 \; dV_{t_i} = 1$. Then we have

$$1 = e^{2C_i} \int_M \psi_i^2 \; dV_{t_i} \leq e^{2C_i} V_1(k_1^i, k_2^i) < e^{2C_i} \varepsilon_i.$$

Since $\varepsilon_i \to 0$, it must be that $C_i \to \infty$.

Now we compute

$$\begin{aligned}
\mathcal{W}^T(g(t_i), w_i, 1) &= \int_M R^T w_i^2 + 4|\nabla w_i|^2 - w_i^2 \log(w_i^2) \; dV_{t_i} \\
&= e^{2C_i} \int_M R^T \psi_i^2 + 4|\psi_i'|^2 - \psi_i^2 \log(\psi_i)^2 - 2C_i \psi_i^2 \; dV_{t_i} \\
&= e^{2C_i} \int_M R^T \psi_i^2 + 4|\psi_i'|^2 - \psi_i^2 \log(\psi_i^2) \; dV_{t_i} - 2C_i. \qquad (3.34)
\end{aligned}$$

By lemma 5.0.14, we can estimate the first term in the integrand of (3.34) as

$$\begin{aligned}
e^{2C_i} \int_M R^T \psi_i^2 &\leq e^{2C_i} \int_{B_\xi(r_1^i, r_2^i)} R^T \; dV_{t_i} \\
&\leq e^{2C_i} C V_i(k_1^i, k_2^i) \\
&\leq e^{2C_i} C 2^{10n} V_i(k_1^i + 2, k_2^i - 2) \\
&\leq C 2^{10n} \int_m w_i^2 \; dV_{t_i} \\
&= C 2^{10n}
\end{aligned}$$

Since $0 \leq \psi_i \leq 1$ and ψ_i' is uniformly bounded, there is a uniform constant C such that $4|\nabla \psi_i|^2 - \psi_i^2 \log(\psi_i^2) \leq C$. Using lemma 5.0.14 again we can estimate the remaining terms in (3.34). Note that $B_\xi(r_1^i, r_2^i) \subseteq B_\xi(k_1^i, k_2^i)$.

$$
\begin{aligned}
e^{2C_i}\int_M 4|\psi_i'|^2 - \psi_i^2 \log(\psi_i^2)\ dV_{g(t_i)} &\leq e^{2C_i} C V_i(k_1^i, k_2^i) \\
&\leq e^{2C_i} C 2^{10n} V_i(k_1^i + 2, k_2^i - 2) \\
&\leq C 2^{10n} \int_M w_i^2\ dV_{t_i} \\
&= C 2^{10n}
\end{aligned}
$$

Therefore

$$
\mu^T(g(0), 1) \leq \mu^T(g(t_i), 1) \leq \mathcal{W}^T(g(t_i), w_i, 1) \leq C2^{10n} - 2C_i \to -\infty.
$$

This contradicts lemma 5.0.12. Thus the transverse diameter is uniformly bounded along the TKRF. $\square$

3.5. Transverse Holomorphic Bisectional Curvature

Recall the transverse holomorphic bisectional curvature defined in section 2.4. In this section we will prove

Theorem 3.5.1. *Let $g^T(t)$ be a solution to the TKRF. If the initial metric $g^T(0)$ has nonnegative transverse holomorphic bisectional curvature then so does $g^T(t)$ for $t \geq 0$. If the initial metric has positive transverse holomorphic bisectional curvature at some point $x \in M$, then $g^T(t)$ has positive transverse holomorphic bisectional curvature for $t > 0$.*

To prove theorem 3.5.1, we need to compute the evolution of the transverse holomorphic bisectional curvature along the TKRF. By standard computations in

Kähler-Ricci flow,

$$\frac{\partial}{\partial t} R_{i\bar{i}j\bar{j}} = \Delta R_{i\bar{i}j\bar{j}} + F(Rm)_{i\bar{i}j\bar{j}} + R_{i\bar{i}j\bar{j}},$$

where, for a tensor S with the same type as the curvature tensor,

$$F(S)_{i\bar{i}j\bar{j}} = \sum_{k,l} \left(|S_{i\bar{k}l\bar{j}}|^2 - |S_{i\bar{k}j\bar{l}}|^2 + S_{i\bar{i}l\bar{k}} S_{k\bar{l}j\bar{j}} \right) - \sum_{k} \Re \left(S_{i\bar{k}} S_{k\bar{i}j\bar{j}} + S_{j\bar{k}} S_{i\bar{i}k\bar{j}} \right).$$

See proposition 2.79 and corollary 2.82 in [9] for the computations in the Kähler setting. We can easily mimic these computations and derive a similar formula in the quasi-Sasaki setting. We let $H^T = H^T_{ij} := R^T_{i\bar{i}j\bar{j}}$. Then we have

$$\frac{\partial}{\partial t} H^T = \Delta^T H^T + F(H^T) + H^T.$$

Using Hamilton's maximum principle for tensors, Bando ($n \leq 3$) in [2] and Mok ($n \geq 4$) in [29] proved the analog of theorem 3.5.1 for the Kähler-Ricci flow. To apply this maximum principle, one must show that F has the *null vector property.* That is, if there are nonzero $X, Y \in T_pM$ such that $S_p(X, \bar{X}, Y, \bar{Y}) = 0$, then $F(S)_p(X, \bar{X}, Y, \bar{Y}) \geq 0$. We have the following:

Proposition 3.5.2. *If $S \geq 0$ and there exists $X, Y \in \mathcal{D}^{1,0}$, both non-zero, such that $S_p(X, \bar{X}, Y, \bar{Y}) = 0$, then $F(S)_p(X, \bar{X}, Y, \bar{Y}) \geq 0$.*

Proof. This is a local problem, so we choose a coordinate patch $\pi : U \subset M \to V \subset \mathbb{C}^n$ with $p \in U$. Then the transverse metric g^T on U induces a Kähler metric g_V on V which is evolving by Kähler-Ricci flow. So $F(R^T)$ restricted to V is the same as in the Kähler case. The result now follows from the analogous result in the Kähler setting. See [2], [29] or page 107 of [9]. □

Now we can prove theorem 3.5.1:

Proof. It suffices to prove the theorem for short time, say for $t \in [0, t_0]$ for some $t_0 > 0$. Then all metrics involved have bounded geometry. We define an auxiliary tensor S in local coordinates by

$$S_{i\bar{j}k\bar{l}} := \frac{1}{2}\left(g^T_{i\bar{j}}g^T_{k\bar{l}} + g^T_{i\bar{l}}g^T_{k\bar{j}}\right).$$

Note that S has the same type as the curvature tensor, $S > 0$ at all points of M and S is parallel, hence $\Delta S = 0$. Then there is a constant $C_1 > 0$ such that

$$-C_1 S \le \frac{\partial S}{\partial t} \le C_1 S.$$

Since F is smooth and $S > 0$, there is a constant $C_2 > 0$ such that

$$F(H^T) - F(H^T + fS) \ge -C_2|f|S$$

for $f \in \mathbb{R}$ with $|f| \le 1$. When f is a basic function, fS has the same type as the transverse curvature tensor. Let $f = f(t,x)$ be a basic function for all t and choose $\varepsilon > 0$ small enough so that $\varepsilon|f(t,x)| \le 1$ for all (t,x). Then we compute

$$\begin{aligned}
\frac{\partial}{\partial t}(H^T + \varepsilon f S) &= \Delta H^T + F(H^T) + H^T + \varepsilon\frac{\partial f}{\partial t}S + \varepsilon f\frac{\partial S}{\partial t} \\
&= F(H^T) - F(H^T + \varepsilon f S) + F(H^T + \varepsilon f S) + H^T + \varepsilon f S \\
&+ \Delta(H^T + \varepsilon f S) + \varepsilon S\left(\frac{\partial f}{\partial t} - \Delta f\right) + \varepsilon f\left(\frac{\partial S}{\partial t} - S\right) \\
&\ge F(H^T + \varepsilon f S) + \Delta(H^T + \varepsilon f S) + H^T + \varepsilon f S \\
&+ \varepsilon S\left(\frac{\partial f}{\partial t} - \Delta f - (C_1 + 1 + C_2)|f|\right).
\end{aligned}$$

Now let f satisfy

$$\frac{\partial f}{\partial t} - \Delta f - (C_1 + 1 + C_2) f = 1$$

with initial condition $f(\cdot, 0) \equiv 1$. In local coordinates, $\Delta f = \xi^2 f + g^{i\bar{j}} \partial_i \partial_{\bar{j}} f$. Since $\xi g^{i\bar{j}} = 0$, we have $\xi(\Delta f) = \Delta(\xi f)$. Hence

$$\frac{\partial (\xi f)}{\partial t} = \Delta(\xi f) + (C_1 + 1 + C_2) \xi f.$$

Since $\xi f(0) = 0$, it follows from the uniqueness of the solution that $\xi f \equiv 0$. Thus $f(\cdot, t)$ is basic. By the maximum principle, $f > 0$ for $t > 0$. Therefore

$$\frac{\partial}{\partial t}(H^T + \varepsilon f S) \geq F(H^T + \varepsilon f S) + \Delta(H^T + \varepsilon f S) + H^T + \varepsilon f S + \varepsilon S. \tag{3.35}$$

Since $H^T \geq 0$ at $t = 0$, we have $H^T + \varepsilon f S > 0$ at $t = 0$. In fact, $H^T + \varepsilon f S > 0$ for all t. We give a proof by contradiction.

If not, then there is a first time $t_0 > 0$, a point $p \in M$ and unit vectors $X, Y \in \mathcal{D}_p^{1,0}$ such that $(H^T + \varepsilon f S)(X, \bar{X}, Y, \bar{Y}) = 0$ at (t_0, p). By proposition 3.5.2, at (t_0, p),

$$F(H^T + \varepsilon f S)(X, \bar{X}, Y, \bar{Y}) \geq 0.$$

Now choose a coordinate patch U about $p \in M$. Then $\pi : U \to V \subset \mathbb{C}^n$ is a Riemannian submersion. Let $X_V = \pi_* X$, $Y_V = \pi_* Y$. Extend X_V and Y_V to a normal neighborhood of $(\pi(p), t_0)$ by parallel translation along radial geodesics such that $\nabla X_V = \nabla Y_V = 0$ at $(\pi(p), t_0)$. Then extend X_V and Y_V around t_0 such that

$\partial_t X_V = \partial_t Y_V = 0$ at $(\pi(p), t_0)$. Using (3.35) at (p, t_0) we have

$$\begin{aligned}
0 &\geq \frac{\partial}{\partial t}\left(H^T + \varepsilon f S\right)(X_V, \bar{X_V}, Y_V, \bar{Y_V}) \\
&= \left(\frac{\partial}{\partial t}(H^T + \varepsilon f S)\right)(X_V, \bar{X_V}, Y_V, \bar{Y_V}) \\
&\geq \left(\Delta(H^T + \varepsilon f S)\right)(X_V, \bar{X_V}, Y_V, \bar{Y_V}) + \varepsilon S(X_V, \bar{X_V}, Y_V, \bar{Y_V}) \\
&= \Delta\left((H^T + \varepsilon f S)(X_V, \bar{X_V}, Y_V, \bar{Y_V})\right) + \varepsilon S(X_V, \bar{X_V}, Y_V, \bar{Y_V}) \\
&> 0.
\end{aligned}$$

Contradiction. Thus $H^T + \varepsilon f S > 0$ for all t. Letting $\varepsilon \to 0$, we have $H^T \geq 0$ for all t. Now suppose that $H^T > 0$ at some point in M when $t = 0$. Define $f_0(p) := \min H^T_p(X, \bar{X}, Y, \bar{Y})$ for $X, Y \in \mathcal{D}^{1,0}_p$ with $|X| = |Y| = 1$. Then $f_0 \geq 0$ everywhere and $f_0 > 0$ somewhere. As ξ acts on g isometrically, f_0 is constant along the orbits of ξ. Hence $\xi f_0 = 0$. As before, we compute that

$$\frac{\partial}{\partial t}(H^T - fS) \geq F(H^T - fS) + \Delta(H^T - fS) + S\left(-\frac{\partial f}{\partial t} + \Delta f - (C_1 + C_2)|f|\right).$$

Now let f satisfy

$$\frac{\partial f}{\partial t} = \Delta f - (C_1 + C_2) f$$

with initial condition $f(\cdot, 0) = f_0$. Again, since $\xi f_0 = 0$ we have $\xi f \equiv 0$. If we let $\tilde{f} = e^{(C_1+C_2)t} f$, then $\tilde{f}$ solves the heat equation

$$\frac{\partial \tilde{f}}{\partial t} = \Delta \tilde{f}.$$

It is well known that a function $\tilde{f}$ satisfying the heat equation with nonnegative initial condition is positive for $t > 0$, see [32]. Thus $f > 0$ for $t > 0$ and

$$\frac{\partial}{\partial t}(H^T - fS) \geq F(H^T - fS) + \Delta(H^T - fS).$$

By the definition of f_0, we have $H^T - fS \geq 0$ when $t = 0$. Arguing as above, we can show that $H^T - fS \geq 0$ for $t > 0$. Since $S > 0$ and $f > 0$, it follows that $H^T > 0$. $\square$

CHAPTER IV

THE RICCI FLOW OF A SASAKI METRIC

In this chapter we want to investigate the behavior of the Ricci flow when the initial metric is Sasaki. We are not talking about the Sasaki-Ricci flow as in [12], [19] or [36], which is just the transverse Kähler-Ricci flow of a Sasaki manifold, but rather the Ricci flow of the actual Sasaki metric. The Ricci flow of a Kähler metric preserves the Kähler condition and this has been an extremely fruitful area of mathematical research since the 1980s. Since Sasaki geometry is closely related to Kähler geometry, it seems reasonable to suspect that the Ricci flow of a Sasaki metric should be a tractable subject, but one quickly runs into trouble; the Ricci flow need not preserve the Sasaki condition. For example, consider a Sasaki-Einstein metric g_0 with Einstein constant λ. The Ricci flow $\dfrac{\partial g}{\partial t} = -2\mathrm{Ric}_g$ with $g(0) = g_0$ is given by $g(t) = (1-2\lambda t)g_0$. Since Sasaki metrics are not preserved by homothety, $g(t)$ is not Sasaki for $t > 0$. However, a scaling of a Sasaki metric is a quasi-Sasaki metric. This gives us hope that the Ricci flow does preserve some sort of structure related to the Sasaki geometry.

We begin with the following useful observation: Given a Sasaki structure (ξ, η, Φ, g), we can create a two-parameter family of quasi-Sasaki structures. Given $A, B > 0$, let $\xi_B = \frac{1}{\sqrt{B}}\xi$, $\eta_B = \sqrt{B}\eta$ and $g_{A,B} = \eta_B \otimes \eta_B + Ag^T$. Using the same (1,1)-tensor Φ, it is easy to see that the (almost) contact structure is preserved. A straighforward computation shows that the metric compatibility and normality conditions are also preserved. However, $g_{A,B}(X, \Phi Y) \neq d\eta_B(X, Y)$. Rather we have $g_{A,B}(X, \Phi Y) = \frac{A}{\sqrt{B}} d\eta_B(X, Y)$. Since Kähler metrics are preserved by homothety, the transverse metric $g^T_{A,B} = Ag^T$ is Kähler and thus $(\xi_B, \eta_B, \Phi, g_{A,B})$ defines a quasi-Sasaki structure.

Definition 4.0.3. A quasi-Sasaki structure obtained via the above construction will be called an (A, B)*-deformation* of a Sasaki structure.

By choosing $A = \sqrt{B}$, the deformed structure remains Sasaki. These type of deformations are known in the literature as transverse or $\mathcal{D}$-homotheties. Note also that an (A, A)-deformation results in an honest scaling of the metric, but does not preserve the Sasaki structure unless $A = 1$. From proposition 2.1.1 we get the following corollary.

Corollary 4.0.4. *If* (ξ, η, Φ, g) *is an* (A, B)*-deformation of a Sasaki structure, then for* $X, Y \in \mathcal{D}$ *the Ricci tensor of* g *satisfies*

$$\begin{aligned}
\operatorname{Ric}(\xi, \xi) &= 2n\frac{B}{A^2} \\
\operatorname{Ric}(X, \xi) &= 0 \\
\operatorname{Ric}(X, Y) &= \operatorname{Ric}^T(X, Y) - 2\frac{B}{A^2}g^T(X, Y).
\end{aligned}$$

4.1. η-Einstein metrics

It is difficult (as far as the author can tell) to say much about the Ricci flow of an arbitrary Sasaki metric. However, for a special class of Sasaki metrics known as η*-Einstein*, we can actually say quite a lot. It is these metrics whose Ricci flow we shall investigate.

Definition 4.1.1. A Sasaki metric g is η-Einstein if there are real numbers μ and ν such that Ricci tensor of g satisfies

$$\operatorname{Ric}_g = \mu g + \nu \eta \otimes \eta. \tag{4.1}$$

Recall that for a Sasaki structure (ξ, η, Φ, g), $\eta \wedge (d\eta)^n$ is a volume form, $g(X, \Phi Y) = d\eta(X, Y)$ for all $X, Y \in TM$ and for $X, Y \in \mathcal{D}$ the Ricci tensor satisfies

$$\mathrm{Ric}(\xi, \xi) = 2n, \tag{4.2}$$

$$\mathrm{Ric}(X, \xi) = 0, \tag{4.3}$$

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}^T(X, Y) - 2g^T(X, Y), \tag{4.4}$$

From equations (4.1) and (4.2) we see that $\mu + \nu = 2n$. From equations (4.3) and (4.4) we see that a Sasaki manifold is η-Einstein if and only if the transverse metric is Kähler-Einstein. Indeed, if $\mathrm{Ric}^T = \lambda g^T$ then $\mathrm{Ric} = (\lambda - 2)g + (2n + 2 - \lambda)\eta \otimes \eta$. Conversely, if $\mathrm{Ric} = \mu g + \nu \eta \otimes \eta$ then $\mathrm{Ric}^T = (\mu + 2)g^T$. We should also note that an η-Einstein manifold has constant scalar curvature. Tracing equation (4.1) we see that the scalar curvature $R = (2n + 1)\mu + \nu = 2n(\mu + 1)$. An excellent reference for the geometry of η-Einstein manifolds is the paper [6].

Let $(\xi_0, \eta_0, \Phi, g_0)$ be η-Einstein. Then we can write $g_0 = \eta_0 \otimes \eta_0 + g_0^T$ where g_0^T is transverse Kähler-Einstein with $\mathrm{Ric}_0^T = \lambda g_0^T$. Now consider an (A, B)-deformation of this Sasaki structure. By corollary 4.0.4, for $X, Y \in \mathcal{D}$, the Ricci tensor of the deformed metric satisfies

$$\begin{aligned}
\mathrm{Ric}(\xi_0, \xi_0) &= 2n\frac{B^2}{A^2}, \\
\mathrm{Ric}(X, \xi_0) &= 0, \\
\mathrm{Ric}(X, Y) &= \left(\lambda - 2\frac{B}{A}\right) g_0^T(X, Y).
\end{aligned}$$

Therefore, by the above equations, a metric of the form

$$g(t) := A(t)g_0^T + B(t)\eta_0 \otimes \eta_0 \tag{4.5}$$

is the solution to the Ricci flow equation

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_g, \quad g(0) = g_0, \tag{4.6}$$

with g_0 an η-Einstein metric, if and only if $A(t), B(t) > 0$ satisfy

$$\frac{dA}{dt} = 4\frac{B}{A} - 2\lambda, \quad A(0) = 1, \tag{4.7}$$

$$\frac{dB}{dt} = -4n\frac{B^2}{A^2}, \quad B(0) = 1. \tag{4.8}$$

By standard existence and uniqueness theory for ODEs (e.g. the Picard-Lindelöf theorem), the system (4.7), (4.8) has a unique solution that exists for at least a short time. Note that if we try to impose the additional restriction that $A = \sqrt{B}$, then the system has no solution. Hence the metric cannot evolve by Sasaki metrics in this way. But, as we mentioned before, the metrics in (4.5) are quasi-Sasaki.

For our qualitative analysis of these differential equations, we will make use of the conserved quantity

$$\Lambda := \left(\frac{\lambda}{2(n+1)} - \frac{B}{A}\right) B^{-\frac{n+1}{n}}. \tag{4.9}$$

Indeed, a straightforward computation reveals that $\frac{d}{dt}\Lambda = 0$. From the requirement that $A(0) = B(0) = 1$, we find that $\Lambda = \frac{\lambda}{2(n+1)} - 1$.

Since a Sasaki metric is η-Einstein if and only if the transverse metric is Kähler-Einstein, we naturally have three cases to consider. In the following sections we investigate the Ricci flow equations (4.7) and (4.8) when the transverse Einstein constant λ is zero, positive and negative, respectively. We want to understand the behavior of the functions A and B in each case. Knowing their behavior, we can determine how the curvature is changing along the Ricci flow. Computing as in chapter II, the sectional curvature of a plane spanned by ξ and $X \in \mathcal{D}$ is

$$\kappa(X,\xi) = \frac{B}{A^2}\kappa_0(X,\xi), \tag{4.10}$$

where κ_0 denotes the sectional curvature with respect to the initial metric g_0. The sectional curvature of a plane spanned by $X, Y \in \mathcal{D}$ is

$$\kappa(X,Y) = \frac{1}{A}\kappa_0^T(X,Y) + \mathcal{O}\left(\frac{B}{A^2}\right). \tag{4.11}$$

4.2. Transverse Calabi-Yau Structure

When $\lambda = 0$ we have $c_B^1 = 0$ and the transverse structure is Calabi-Yau. In this case we can solve the Ricci flow equations explicitly. We compute

$$\frac{d}{dt}\left(\frac{B}{A^2}\right) = -4(n+2)\left(\frac{B}{A^2}\right)^2. \tag{4.12}$$

We solve this to get

$$\frac{B}{A^2} = \frac{1}{1+4(n+2)t}. \tag{4.13}$$

Now we have

$$\frac{dB}{dt} = -4n\left(\frac{B}{A^2}\right)B = \frac{-4n}{1+4(n+2)t}B.$$

This yields

$$B(t) = (1 + 4(n+2)t)^{\frac{-n}{n+2}}.$$

Then from equation (4.13) we get

$$A(t) = (1 + 4(n+2)t)^{\frac{1}{n+2}}.$$

Therefore, the flow exists for all $t \in [0, \infty)$ and $A(t) \to \infty$ while $B(t) \to 0$ as $t \to \infty$. In the limit, all sectional curvatures tend to zero and the metric degenerates to a flat metric on the transverse space.

Example 4.2.1. For an example of an η-Einstein metric with transverse Calabi-Yau structure, we consider the Heisenberg group $H = H(3, \mathbb{R})$. It is a nilpotent Lie group that can be identified algebraically as

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

We identify this Lie group topologically with $\mathbb{R}^3$ in the obvious way. We consider the group action to be from the right. Hence, the action of right multiplication by (a, b, c) is $(x, y, z) \mapsto (x + a, y + b, z + c + bx)$. The Lie algebra is generated by the right invariant vector fields $\xi = 2\frac{\partial}{\partial z}$, $Y = 2\frac{\partial}{\partial y}$ and $X = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})$ which satisfy the bracket relations $[\xi, Y] = [\xi, X] = 0$ and $[Y, X] = 2\xi$.

Define a 1-form $\eta := \frac{1}{2}(dz - ydx)$. Note that η is invariant under the group action. As $\eta \wedge d\eta = \frac{1}{4}dx \wedge dy \wedge dz$, we see that η is a contact form. We have $\mathcal{D} := \ker(\eta) = \{X, Y\}$. The complex structure $J : \mathcal{D} \to \mathcal{D}$ defined by $JX = Y$ and

$JY = -X$ gives $(\mathcal{D}, d\eta, J)$ a Kähler structure. This is just the standard, flat, Kähler structure on $\mathbb{R}^2 = \mathbb{C}$.

Define a metric $g_0 := \eta \otimes \eta + \frac{1}{4}(dx \otimes dx + dy \otimes dy)$. Then g_0 is invariant under the group action, $\{\xi, Y, X\}$ is an orthonormal frame and (ξ, η, Φ, g_0) gives H a Sasaki structure. Here Φ is just the extension of J such that $\Phi\xi = 0$. Explicitly we have $\Phi = \frac{\partial}{\partial y} \otimes dx - (\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}) \otimes dy$. The nonzero components of the Ricci tensor of g_0 are $\mathrm{Ric}(\xi, \xi) = 2$ and $\mathrm{Ric}(Y, Y) = \mathrm{Ric}(X, X) = -2$. Therefore $\mathrm{Ric}_{g_0} = -2g_0 + 4\eta \otimes \eta$. Hence g_0 is an η-Einstein metric. Under the Ricci flow, the curvature tends to zero and the metric degenerates to a flat metric on $\mathbb{R}^2$.

Now define the compact quotient manifold $N := H(3, \mathbb{R})/H(3, \mathbb{Z})$ where $H(3, \mathbb{Z})$ denotes matrices of the above form with $x, y, z \in \mathbb{Z}$. Then N is a non-trivial circle bundle over a complex torus and the Sasaki structure on H descends to a Sasaki structure on N. The S^1 fiber corresponds to the integral curves of the Reeb vector field $\xi = 2\frac{\partial}{\partial z}$. The Ricci flow on N behaves the same as the Ricci flow on H. Under the flow, the S^1 fiber shrinks to a point and we have convergence to a flat torus.

Remark 4.2.2. The behavior of the Ricci flow seen in this example is precisely the behavior of the Ricci flow on any Nil 3-manifold with an invariant metric. See, for example, chapter 1.6 of [11].

4.3. Transverse Fano Structure

When $\lambda > 0$ we have $c_B^1 > 0$ and the transverse structure is Fano. In general, the flow only exists for finite time $t \in [0, T)$, $T < \infty$, and both $A(t), B(t) \to 0$ as $t \to T$. We will consider three cases:

1. If $\lambda > 2(n+1)$ then, from equation (4.9), $\Lambda > 0$. In this case we have

$$\frac{B}{A} = \frac{\lambda}{2(n+1)} - \Lambda B^{\frac{n+1}{n}} < \frac{\lambda}{2(n+1)}.$$

   Then

$$\frac{dA}{dt} = 4\frac{B}{A} - 2\lambda < \frac{2\lambda}{n+1} - 2\lambda < 0.$$

   This shows that $A(t) \to 0$ in finite time. Since $0 < B < \frac{\lambda}{2(n+1)}A$, it follows that $B(t) \to 0$ in finite time as well.

2. If $\lambda = 2(n+1)$, then $\Lambda = 0$ and the solution is given by $A(t) = B(t) = 1 - 4nt$, as is easily checked. In this case the initial metric is Sasaki-Einstein. Notice that the flow only exists for $t \in \left[0, \frac{1}{4n}\right)$ and both $A(t), B(t) \to 0$ as $t \to \frac{1}{4n}$.

3. If $0 < \lambda < 2(n+1)$, then $\Lambda < 0$. In this case we have

$$\frac{B}{A} = \frac{\lambda}{2(n+1)} - \Lambda B^{\frac{n+1}{n}} > \frac{\lambda}{2(n+1)}.$$

   Then

$$\frac{dB}{dt} = -4n\left(\frac{B}{A}\right)^2 < \frac{-n\lambda^2}{(n+1)^2} < 0.$$

   This implies that $B(t) \to 0$ in finite time. Since $0 < A < \frac{2(n+1)}{\lambda}B$, it follows that $A(t) \to 0$ in finite time as well.

In this transverse Fano case, we see that all the sectionals curvatures tend to infinity as we approach the terminal time. This is to be expected (at least in one direction) of a Ricci flow with a finite-time singularity. It would be interesting to determine the terminal time explicitly in terms of λ and n in cases 1 and 3 above.

Example 4.3.1. The canonical example of an η-Einstein manifold with transverse Fano structure is the classical Hopf fibration. This gives S^3 the structure of a principle circle bundle over $\mathbb{C}P^1$ and the standard Sasaki structure on S^3 (compare with example 1.3.5). Writing $S^3 = \{(w,z) \in \mathbb{C}^2 : |w|^2 + |z|^2 = 1\}$, the Hopf fibration $S^1 \hookrightarrow S^3 \to \mathbb{C}P^1$ is induced by the map $\pi : S^3 \to \mathbb{C}P^1$, where $(w,z) \mapsto [w : z]$. Here $[w : z]$ denotes the homogenous coordinates on $\mathbb{C}P^1$. As $[w : z] = [\lambda w : \lambda z]$ for $\lambda \in \mathbb{C}^*$, it is clear that the fibers of π are circles $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

As we saw in chapter II, the collection of principal circle bundles over $\mathbb{C}P^1$ has a group structure isomorphic to $H^2(\mathbb{C}P^1, \mathbb{Z}) \simeq \mathbb{Z}$ with generator $\frac{1}{2\pi}[\omega_{FS}]$. The Fubini-Study metric has constant holomorphic sectional curvature equal to 4. As in the proof of theorem 2.2.6, there is a contact form η on S^3 such that $\frac{1}{2}d\eta = \pi^*\omega_{FS}$ and

$$g := \eta \otimes \eta + \pi^*\omega_{FS}(J\cdot, \cdot)$$

defines a Sasaki metric. The metric g is in fact the round metric of constant sectional curvature 1. So g is an Einstein (hence η-Einstein) metric with $\mathrm{Ric} = 2g$. The Ricci flow on S^3 with initial metric g is $g(t) := (1-4t)g$. The flow exists for $t \in \left[0, \frac{1}{4}\right)$ and S^3 shrinks to a point at the terminal time $T = \frac{1}{4}$. As $t \to T$ the sectional curvatures blows up.

Now fix $\varepsilon \in (0,1)$ and define a Sasaki metric g_ε via a $\mathcal{D}$-homothetic deformation of g. That is, $g_\varepsilon := \varepsilon g + (\varepsilon^2 - \varepsilon)\eta \otimes \eta$. Then $\bar{g} := \varepsilon^{-1} g_\varepsilon = \varepsilon\eta \otimes \eta + \pi^*\omega_{FS}(J\cdot, \cdot)$ and $(S^3, \bar{g})$ is the classical ε-collapsed Berger sphere. Let $\bar{g}(t)$ be the Ricci flow with initial metric $\bar{g}$. This flow is well-understood (see chapter 1.5 of [11]). The flow exists on a maximal finite time interval $[0, T)$ and the metric becomes asymptotically round as $t \to T$. Thus, $g_\varepsilon(\varepsilon^{-1}t) = \varepsilon\bar{g}(\varepsilon^{-1}t)$ is the Ricci flow on $[0, \varepsilon T)$ with intitial metric g_ε and it behaves the same as the flow $\bar{g}(t)$.

4.4. Transverse Canonical Structure

When $\lambda < 0$ we have $c_B^1 < 0$ and we say that the transverse structure is canonical. In this case the flow exists for all $t \geq 0$, $A(t) \to \infty$ as $t \to \infty$ and $B(t)$ tends to a constant $c(n, \lambda) \in (0, 1)$ as $t \to \infty$.

Since $\frac{dB}{dt} = -4n\frac{B^2}{A^2} < 0$ and $B(0) = 1$, we have $0 < B(t) \leq 1$. From (4.8) and (4.9),

$$\frac{dB}{dt} = -4n\left(\frac{\lambda}{2(n+1)} - \Lambda B^{\frac{n+1}{n}}\right)^2. \tag{4.14}$$

Thus $\frac{dB}{dt}$ is uniformly bounded. Hence, the right hand side of (4.14) and its first derivative are uniformly bounded. By a standard result in ODEs (see [23]), this implies that (4.14) with the initial condition $B(0) = 1$ has a unique solution defined for all $t \geq 0$. Then it follows from (4.9) that equation (4.7) has a unique solution defined for all $t \geq 0$.

Since we must have $A, B > 0$, right away we see that $\frac{dA}{dt} = 4\frac{B}{A} - 2\lambda \geq -2\lambda > 0$. Thus

$$A(t) \geq 1 - 2\lambda t.$$

Hence $A(t) \to \infty$ as $t \to \infty$. Now, from equation (4.9) we have

$$\frac{B}{A} = \frac{\lambda}{2(n+1)} - \Lambda B^{\frac{n+1}{n}},$$

and it follows that

$$\lim_{t\to\infty} B(t) = \left(\frac{\lambda}{2\Lambda(n+1)}\right)^{\frac{n}{n+1}} = \left(\frac{\lambda}{\lambda - 2(n+1)}\right)^{\frac{n}{n+1}}.$$

As in the $\lambda = 0$ case, as $t \to \infty$ all of the sectional curvatures tend to zero and the metric degenerates to a flat metric on the transverse space.

Example 4.4.1. For an example of an η-Einstein metric with transvere canonical structure, we consider $M = B \times \mathbb{R}$ where $B = \{z \in \mathbb{C} : |z| < 1\}$. Let ω be the Bergman metric on B. We take

$$\omega = 2i\partial\bar{\partial}\log(1-|z|^2) = 2i\frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}.$$

Then (B, ω, J) is Kähler with constant, negative, holomorphic sectional curvature. Here J is the usual complex structure on $\mathbb{C}$. Since ω is a real, closed 2-form on B, there is some real 1-form α such that $\omega = d\alpha$. In particular,

$$\alpha = i\frac{zd\bar{z} - \bar{z}dz}{(1-|z|^2)}.$$

Let $\pi : M \to B$ be the projection. Let t be the coordinate on $\mathbb{R}$ and define $\frac{1}{2}\eta := \pi^*\alpha + dt$. Then $\frac{1}{2}d\eta = \pi^*\omega$ and we see that η is a contact form. The Reeb vector field $\xi = \frac{1}{2}\frac{\partial}{\partial t}$. We take $\Phi = \tilde{\pi} \circ J \circ \pi_*$. Now define a metric

$$g_0 := \pi^*\omega(\cdot, J\cdot) + \eta \otimes \eta = \frac{1}{2}d\eta(\cdot, \Phi\cdot) + \eta \otimes \eta.$$

Then (ξ, η, Φ, g_0) is a Sasaki structure on M.

In real coordinates (x, y), we have

$$\alpha = 2\frac{xdy - ydx}{1-x^2-y^2} \quad \text{and} \quad \omega = 4\frac{dx \wedge dy}{(1-x^2-y^2)^2}.$$

Define vector fields

$$X := \frac{1-x^2-y^2}{2}\left(\frac{\partial}{\partial x} - \eta\left(\frac{\partial}{\partial x}\right)\xi\right) \quad \text{and} \quad Y := \frac{1-x^2-y^2}{2}\left(\frac{\partial}{\partial y} - \eta\left(\frac{\partial}{\partial y}\right)\xi\right).$$

Then $\mathcal{D} = \text{span}\{X, Y\}$ and $\{\xi, X, Y\}$ is an orthonormal frame for g. Working with this frame, we compute the nonzero components of the Ricci tensor to be $\text{Ric}(\xi, \xi) = 2$ and $\text{Ric}(X, X) = \text{Ric}(Y, Y) = -3$. Thus $\text{Ric} = -3g + 5\eta \otimes \eta$. Hence, g is η-Einstein.

If we consider a metric of the form $g = A\pi^*\omega(\cdot, J\cdot) + B\eta \otimes \eta$, then we compute that, with respect to the frame $\{\xi, X, Y\}$, the nonzero components of the Ricci tensor of g are $\text{Ric}(\xi, \xi) = 2\frac{B^2}{A^2}$ and $\text{Ric}(X, X) = \text{Ric}(Y, Y) = -1 - 2\frac{B}{A}$. Thus, we will have the Ricci flow with inital metric g_0 if we can solve

$$\begin{aligned} \frac{dA}{dt} &= 2 + 4\frac{B}{A}, \quad A(0) = 1, \\ \frac{dB}{dt} &= -4\frac{B^2}{A^2}, \quad B(0) = 1. \end{aligned}$$

From the above we know that the solution exists for all $t \geq 0$ and as $t \to \infty$, $A(t) \to \infty$, $B(t) \to \frac{\sqrt{5}}{5}$ and the sectional curvatures all tend to zero.

Remark 4.4.2. There is a classification of 3-dimensional Sasaki manifolds due to Geiges. The three examples we have presented in this chapter are the universal covers of the geometries given in the classification. Our examples easily generalize to higher dimensions, but in higher dimensions there is no such classification. In dimension 3, the basic first Chern class is always signed or null. As one might expect, there is a correspondence between the type of Sasaki structure (positive, negative or null c_B^1) and the three geometries given in Geiges' classification. A well-known theorem of Belgun makes this precise. See [6] for more information about this.

4.5. Rigidity

To have a Ricci flow of the form (4.5), the initial Sasaki metric g_0 must have very special geometry. Indeed, we can prove the following:

Proposition 4.5.1. *Let (ξ, η_0, Φ, g_0) be a Sasaki structure. Suppose that the Ricci flow with initial metric g_0 is given by $g(t) = A(t)g_0^T + B(t)\eta_0 \otimes \eta_0$. Then g_0 is an η-Einstein metric.*

Proof. As we saw before, the metric $g(t)$ provides a quasi-Sasaki structure and we have $\mathrm{Ric}_t(\xi, \xi) = 2n\frac{B^2}{A^2}$, $\mathrm{Ric}_t|_{\mathcal{D}} = \mathrm{Ric}_t^T - 2\frac{B}{A}g_0^T$ and $\mathrm{Ric}_t(X, \xi) = 0$ for all $X \in \mathcal{D}$. Since $g^T(t) = A(t)g_0^T$ is just a scaling of g_0^T, we have $\mathrm{Ric}_t^T = \mathrm{Ric}_0^T$. Hence, A satisfies

$$\frac{dA}{dt} g_0^T(X, Y) = 4\frac{B}{A} g_0^T(X, Y) - 2\mathrm{Ric}_0^T(X, Y)$$

for all $X, Y \in \mathcal{D}$. Rewrite the above as

$$2\mathrm{Ric}_0^T(X, Y) = \left(4\frac{B}{A} - \frac{dA}{dt}\right) g_0^T(X, Y).$$

Since the left-hand side is independent of time, $4\frac{B}{A} - \frac{dA}{dt}$ must be a constant, call it 2δ. Then $\mathrm{Ric}_0^T = \delta g_0^T$. Thus, the initial transverse metric g_0^T is Kähler-Einstein. Therefore, the initial Sasaki metric is η-Einstein. □

4.6. Questions and Directions for Future Research

We have merely scratched the surface of the investigation into the Ricci flow of a Sasaki metric and there are many questions still to be answered. We list just a few of them below.

1. The Ricci flow with an initial Sasaki metric is not Sasaki for $t > 0$, but is it quasi-Sasaki in general?

2. The author can show that, in general, the length of ξ with respect to the evolving metric is decreasing. If the Ricci flow has a finite time singularity, does the length of ξ going to zero characterize the singular time?

3. Is there a relationship between the Ricci flow of a Sasaki metric and the transverse Kähler-Ricci flow of a Sasaki metric?

CHAPTER V

APPENDIX

In chapter II we proved that under certain conditions a quasi-Sasaki-Einstein manifold has a local Riemannian product structure. The proof of theorem 2.6.3 made use of an integrable, in fact parallel, almost product structure. We decided to include this short section here to review the necessary terminology and some criteria for integrability.

Definition 5.0.1. An **almost product structure** on a smooth manifold M is a (1,1)-tensor F (i.e. an endomorphism of the tangent bundle) such that $F^2 = I$.

If we let $P = \frac{1}{2}(I + F)$ and $Q = \frac{1}{2}(I - F)$, then

$$P^2 = P,\ Q^2 = Q \text{ and } PQ = QP = 0. \tag{5.1}$$

Conversely, given (1,1)-tensors P and Q satisfying (5.1), setting $F = P - Q$ gives an almost product structure. The images of P and Q define complementary distributions $\mathcal{P}$ and $\mathcal{Q}$ of TM. We will assume that P and Q have constant rank p and q, respectively. That is, the dimension of the corresponding distribution is constant for all $x \in M$. On the other hand, given two distributions $\mathcal{P}$ and $\mathcal{Q}$ such that $TM = \mathcal{P} \oplus \mathcal{Q}$, setting P and Q to be their respective projections, we see that P and Q satisfy (5.1).

Definition 5.0.2. Let D be a k-dimensional distribution on a smooth m-dimensional manifold M. We say that D is **involutive** if for any $X, Y \in D$, the Lie bracket $[X, Y] \in D$.

We say that D is **integrable** if for each point $x \in M$ there is a k-dimensional submanifold N containing x such that $T_xN = D_x$. We call N an integral submanifold of D (through x).

We say that D is **completely integrable** if for each point $x \in M$, there is a coordinate neighborhood $(U, x_1, \ldots, x_m)$ of x such that $\{\partial_{x_1}, \ldots, \partial_{x_k}\}$ is a local frame for D. Setting $x_i = c_i$ for $i = k+1, \ldots, m$, and some constants c_i gives an integral submanifold of D.

The famous theorem of Frobenius says that these three notions are equivalent.

Theorem 5.0.3 (Frobenius)**.** *A distribution is involutive if and only if it is integrable if and only if it is completely integrable.*

Definition 5.0.4. An almost product structure F on M is **integrable** if the distributions induced by P and Q are integrable in the sense of Frobenius. In this case we say that M is a locally product manifold.

So if $F = P - Q$ is an integrable almost product structure on M, then by the Frobenius theorem, for each point $x \in M$, there is a coordinate neighborhood $(U, x_1, \ldots, x_m)$ such that $\{\partial_{x_1}, \ldots, \partial_{x_p}\}$ is a local frame for $\mathcal{P}$ and $\{\partial_{x_{p+1}}, \ldots, \partial_{x_m}\}$ is a local frame for $\mathcal{Q}$. Setting $x_{p+1}, \ldots, x_m$ constant gives an integral submanifold of $\mathcal{P}$ and setting $x_1, \ldots, x_p$ constant gives an integral submanifold of $\mathcal{Q}$. Thus M is locally the product of two manifolds whose tangent spaces are (isomorphic to) $\mathcal{P}$ and $\mathcal{Q}$.

By proposition 3.1.4 in [13], the integrability of the almost product structure F is equivalent to the vanishing of the Nijenhuis tensor N_F defined by (1.2). Furthermore, since $N_I = 0$, one can check that $N_F = 2N_P = -2N_Q$. Therefore:

Proposition 5.0.5. *An almost product structure $F = P - Q$ is integrable if and only if $N_F = 0$ if and only if $N_P = 0$ if and only if $N_Q = 0$.*

Now let g be a Riemannian metric and ∇ the Levi-Civita connection of g. Recall that $\nabla g = 0$ and $\nabla_X Y - \nabla_Y X = [X, Y]$. If T is a parallel (1,1)-tensor (i.e. $\nabla T = 0$), then it is straightforward to check that the Nijenhuis tensor $N_T = 0$. It is also easy to check that the image of a parallel (1,1)-tensor is an involutive distribution. Since $\nabla F = 2\nabla P = -2\nabla Q$, we see that if any of F, P or Q is parallel then the almost product structure is integrable, but in this case even more is true.

Definition 5.0.6. Let (M, g) be a locally product manifold with $TM = \mathcal{P} \oplus \mathcal{Q}$ an orthogonal decomposition. We say that (M, g) is a **locally decomposable Riemannian manifold** if for any $U, V \in \mathcal{P}$ and $X, Y \in \mathcal{Q}$ we have $Xg(U, V) = 0$ and $Ug(X, Y) = 0$.

In this case the metric splits as an honest product metric corresponding to the locally product structure. It is known by work of Yano that a necessary and sufficient condition for a locally product manifold to be a locally decomposable Riemannian manifold is $\nabla F = 0$. Therefore we have the following:

Proposition 5.0.7. *Let $F = P - Q$ be an almost product structure on (M, g). Then (M, g) is a locally decomposable Riemannian manifold if and only if $\nabla F = 0$ if and only if $\nabla P = 0$ if and only if $\nabla Q = 0$ if and only if $\mathcal{P}$ and $\mathcal{Q}$ are parallel distributions.*

Remark 5.0.8. We remark that if F is an integrable almost product structure, then there exists a torsion-free, affine connection (not necessarily the Levi-Civita connection) with respect to which F is parallel. See the book [13] for reference.

In this section we have collected the technical lemmas and propositions used in chapter III. We begin with

Lemma 5.0.9. *Let $x \in M$ and $V(r) := \mathrm{Vol}(B^T(x,r))$. If $\mathcal{F}_\xi$ is quasi-regular, then*

$$\lim_{r\to 0}\frac{V(r)}{V(r/2)} = 4^n.$$

Proof. Recall that $\pi : M \to Z$ is a principle S^1-orbibundle over the orbifold $Z = M/\mathcal{F}_\xi$. Pick a point $x \in M$ and choose $r > 0$ small enough so that $B^T(x,r)$ is a trivial S^1 bundle over the geodesic ball $\pi(B^T(x,r)) \subset Z$. Since the set of orbifold singularities in Z has measure zero, it does not contribute anything when we compute volumes. Thus we may assume that $\pi(B^T(x,r))$ contains only smooth points. For smooth points in Z, all S^1 fibers have the same length, call it ℓ. On $B^T(x,r)$ the metric and volume form can be written as

$$g = \eta\otimes\eta + \pi^* h,$$
$$dV = \eta\wedge(\pi^*\omega_h)^n,$$

where h is a Kähler metric on $\pi(B^T(x,r))$ and ω_h is the Kähler form. Hence

$$V(r) = \int_{B_g^T(x,r)} dV = \int_{S^1\times\pi(B_g^T(x,r))}\eta\wedge(\pi^*\omega_h)^n = \ell\cdot\mathrm{Vol}(\pi(B_g^T(x,r)).$$

The result follows from this and the fact that for any geodesic ball $B(r) \subset Z$ of radius r,

$$\lim_{r\to 0}\frac{\mathrm{Vol}(B(r))}{\mathrm{Vol}(B(r/2))} = 2^{2n} = 4^n.$$

□

Proposition 5.0.10. *Let M^{2n+1} be a compact quasi-Sasaki manifold without boundary. Let α and β be basic forms with $\deg(\alpha) + \deg(\beta) = 2n - 1$. Then*

$$\int_M d\alpha \wedge \beta \wedge \eta + (-1)^{\deg(\alpha)} \int_M \alpha \wedge d\beta \wedge \eta = 0.$$

Proof. We compute that

$$d(\alpha \wedge \beta \wedge \eta) = d\alpha \wedge \beta \wedge \eta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta \wedge \eta - \alpha \wedge \beta \wedge d\eta.$$

The form $\alpha \wedge \beta \wedge d\eta$ is basic and of degree $2n + 1$, thus $\alpha \wedge \beta \wedge d\eta = 0$. The result now follows from Stokes' theorem. □

Corollary 5.0.11. *Let α be a basic $(2n - 1)$-form on a closed quasi-Sasaki manifold M^{2n+1}. Then*

$$\int_M d_B \alpha \wedge \eta = 0.$$

Proof. Take $\beta = 1$ in the above proposition. □

Lemma 5.0.12. *For any bundle-like metric g and $\tau > 0$, we have $\mu^T(g, \tau) > -\infty$.*

Proof. Letting $w := \tau^{-n/2} e^{-f/2}$, we can write the $\mathcal{W}^T$-functional as

$$\mathcal{W}^T(g, f, \tau) = \int_M \tau(R^T w^2 + 4|\nabla w|^2) - 2(\log(w) + n \log(\tau)) w^2 \, dV.$$

For a constant $c > 0$, it is not hard to see that $\mathcal{W}^T(cg^T, f, c\tau) = \mathcal{W}^T(g^T, f, \tau)$. Hence we may assume without loss of generality that $\tau = 1$. Thus we need to show that there is a constant $C = C(g)$ such that

$$\int_M R^T w^2 + 4|\nabla w|^2 - 2w^2 \log(w) \, dV \geq C.$$

Since $\inf_{x\in M} R^T > -\infty$ and $\int_M w^2\, dV = 1$, we only have to show that

$$\int_M 4|\nabla w|^2 - 2w^2 \log(w)\, dV \geq C.$$

This follows from the log Sobolev inequality (see, for example, lemma 6.36 in [9]). $\square$

Lemma 5.0.13. *Suppose there is a sequence $r_k > 0$ and a sequence of points and times $(p_k, t_k) \in M \times [0, \infty)$ with $t_k \to \infty$ such that $r_k^{-2n} V_k(r_k) \to 0$ as $k \to \infty$, where*

$$V_k(r_k) = \mathrm{Vol}(B^T_{g(t_k)}(p_k, r_k)).$$

Then we may assume that $V_k(r_k) \leq 5^n V_k(r_k/2)$

Proof. If it is not the case that $V_k(r_k) \leq 5^n V_k(r_k/2)$, then let $r_k^i = 2^{-i} r_k$ for integers $i > 0$. By lemma 5.0.9, we can choose i_0 to be the least integer such that $V_k(r_k^{i_0}) \leq 5^n V_k(r_k^{i_0}/2)$. So for $i < i_0$ we have $V_k(r_k^i) > 5^n V_k(r_k^i/2) = 5^n V_k(r_k^{i+1})$. Iterating this inequality, we get

$$V_k(r_k^{i_0}) < 5^{-i_0 n} V_k(r_k).$$

Therefore, as $k \to \infty$,

$$(r_k^{i_0})^{-2n} V_k(r_k^{i_0}) < \left(\frac{4}{5}\right)^{i_0 n} r_k^{-2n} V_k(r_k) \to 0.$$

Thus we can replace r_k with $r_k^{i_0}$ and we have $V_k(r_k^{i_0}) \leq 5^n V_k(r_k^{i_0}/2)$. $\square$

Lemma 5.0.14. *For any $\varepsilon > 0$ there are $k_1 < k_2$ such that if the transverse diameter is sufficiently large then*

1. $\mathrm{Vol}(B_\xi(k_1, k_2)) < \varepsilon$

2. $\mathrm{Vol}(B_\xi(k_1, k_2)) \leq 2^{10n} \mathrm{Vol}(B_\xi(k_1 + 2, k_2 - 2))$

3. *There exists* $r_1 \in [k_1, k_1 + 1]$, $r_2 \in [k_2 - 1, k_2]$ *and a uniform constant* C *such that*

$$\int_{B_\xi(r_1,r_2)} R^T \, dV \leq C\mathrm{Vol}(B_\xi(k_1, k_2)).$$

Proof. Let $\varepsilon > 0$ be given. As the manifold M is closed and $\partial_t dV_t = \Delta^T u \; dV_t = \Delta u \; dV_t$, the volume of M is constant along the flow. So if the transverse diameter becomes sufficiently large, then there is K big enough so that for all $k_2 > k_1 \geq K$ we have $\mathrm{Vol}(B_\xi(k_1, k_2)) < \varepsilon$.

Take $k_2 > k_1 \geq K$. If (2) holds then we are done. Suppose that (2) does not hold. We then consider the annulus $B_\xi(k_1 + 2, k_2 - 2)$ and check to see if (2) holds now. If it does then we are done, otherwise we repeat this step. Suppose that after repeating this step m times, we are still unable to find k_1 and k_2 so that (1) and (2) hold. Then we would have

$$\mathrm{Vol}(B_\xi(k_1, k_2)) > 2^{10nm}\mathrm{Vol}(B_\xi(k_1 + 2m, k_2 - 2m)).$$

We can assume that $k_2 - k_1 - 4m$ is very close to 1. Then $2m$ is close to $\frac{1}{2}(k_2 - k_1 - 1)$. If we choose $k \gg 1$ and set $k_1 = k/2$ and $k_2 = 3k/2$, then m is close to $k/4$, $k_1 + 2m$ is close to k and $k_2 - 2m$ is close to $k + 1$.

The length of ξ is constant along the flow. Hence, the lengths of the integral curves of ξ (which are closed geodesics) are constant along the flow. Thus they are uniformly bounded above by some $\ell > 0$. So for any $p, q \in M$ we have $d_{g(t)}(p, q) \leq d^T_{g(t)}(p, q) + \ell$. This, along with proposition 3.3.7, gives $R^T \leq C2^{2k}$ on $B_\xi(k, k + 1)$ for some uniform constant C. The annulus $B_\xi(k, k + 1)$ contains at least 2^{2k} disjoint transverse balls of radius 2^{-k}. Then by theorem 3.4.1 we have

$$\mathrm{Vol}(B_\xi(k,k+1)) \geq \sum_{i=1}^{2^{2k}} \mathrm{Vol}(B^T_{g(t)}(2^{-k})) \geq C2^{2k}2^{-2kn}.$$

Combining this with the above we find that

$$\varepsilon > \mathrm{Vol}(B_\xi(k_1,k_2)) > 2^{10nm}\mathrm{Vol}(B_\xi(k,k+1)) \geq C2^{10nm}2^{2k}2^{-2kn}.$$

If k is large enough, then $10m > 2k$ and the above is a contradiction. Therefore (1) and (2) hold.

To prove (3) we first define the transverse sphere $S^T_g(x,r) := \{y \in M : d^T_g(x,y) = r\}$. Then $\frac{d}{dr}\mathrm{Vol}(B^T_g(x,r)) = \mathrm{Vol}(S^T_g(x,r))$. Given $k_1 \ll k_2$ such that (1) and (2) hold, there is $r_1 \in [k_1, k_1+1]$ such that

$$\mathrm{Vol}(S^T_{g(t)}(x,2^{r_1})) \leq \frac{\mathrm{Vol}(B_\xi(k_1,k_2))}{2^{k_1-1}}.$$

If not, then we would have

$$\mathrm{Vol}(B_\xi(k_1,k_1+1)) = \int_{2^{k_1}}^{2^{k_1+1}} \mathrm{Vol}(S^T(x,r))\ dr > 2\mathrm{Vol}(B_\xi(k_1,k_2)).$$

This is a contradiction since $k_1 \ll k_2$. Similarly we can prove there is $r_2 \in [k_2-1,k_2]$ such that

$$\mathrm{Vol}(S^T_{g(t)}(x,2^{r_2})) \leq \frac{\mathrm{Vol}(B_\xi(k_1,k_2))}{2^{k_2-1}}.$$

Using these volume estimates for the spheres and proposition 3.3.7, we have

$$\begin{aligned}
\int_{B_\xi(r_1,r_2)} R^T - n \; dV &= \int_{B_\xi(r_1,r_2)} -\Delta u \; dV \\
&= \int_{S^T(x,2^{r_1})} |\nabla u| \; dV + \int_{S^T(x,2^{r_2})} |\nabla u| \; dV \\
&\leq \frac{\mathrm{Vol}(B_\xi(k_1,k_2))}{2^{k_1-1}} C 2^{k_1+1} + \frac{\mathrm{Vol}(B_\xi(k_1,k_2))}{2^{k_2-1}} C 2^{k_2} \\
&= C\mathrm{Vol}(B_\xi(k_1,k_2)).
\end{aligned}$$

$\square$

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