THE A-INFINITY ALGEBRA OF

A CURVE AND THE

j-INVARIANT

by

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DISSERTATION ABSTRACT

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We choose a generator G of the derived category of coherent sheaves on a smooth curve X of genus g which corresponds to a choice of g distinguished points $P_1, \dots, P_g$ on X. We compute the Hochschild cohomology of the algebra $B = \mathrm{Ext}^*(G, G)$ in certain internal degrees relevant to extending the associative algebra structure on B to an A_∞-structure, which demonstrates that A_∞-structures on B are finitely determined for curves of arbitrary genus.

When the curve is taken over $\mathbb{C}$ and $g = 1$, we amend an explicit A_∞-structure on B computed by Polishchuk so that the higher products m_6 and m_8 become Hochschild cocycles. We use the cohomology classes of m_6 and m_8 to recover the j-invariant of the curve. When $g \geq 2$, we use Massey products in $D^b(X)$ to show that in the A_∞-structure on B, m_3 is homotopic to 0 if and only if X is hyperelliptic and $P_1, \dots, P_g$ are chosen to be Weierstrass points.

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CHAPTER I

INTRODUCTION

A-infinity (A_∞-) algebras were invented in the sixties by Stasheff ([24]), and occupy a central role in modern problems related to homological mirror symmetry. We consider a graded vector space A over a field $\mathbb{K}$. An A_∞-structure on A is a certain generalization of an associative algebra structure, where we relax the associativity condition in the presence of a differential, requiring that multiplication be associative only up to homotopy.

More specifically, an A_∞-algebra is the space A together with a set $\{m_n\}_{n=1}^{\infty}$ of operations $m_n : A^{\otimes n} \to A$ of homogeneous degree $2 - n$ satisfying certain compatibility relations. For example, m_1 is a differential and a derivation with respect to m_2; m_2 is a homotopy associative multiplication with homotopy given by m_3. The cohomology H^*A with respect to m_1 is an associative algebra, but also inherits some higher-order operations (to compensate for the loss of chain information) which make it also an A_∞-algebra. Section II.3. gives a brief overview of A_∞-algebras and their morphisms. The reader is referred to [10] for a more detailed introduction.

Despite the evident motivation to forget associativity, a particular case of interest is when A is a differential graded (dg-) algebra; considered as an A_∞-algebra, m_1 is taken as the differential, m_2 is the usual (associative) multiplication, and $m_n = 0$ for $n \geq 3$. There is no sense in which A and H^*A are equivalent (in general) if we restrict to the context of usual algebras. A shadow of the power of the generalization to A_∞-algebras appears in the equivalence as triangulated categories of some derived A_∞-module categories over A and $H^*(A)$, respectively. Such a result implies that by living in the context of A_∞-algebras we do not lose so much information by passing to cohomology.

An A_∞-category is a natural generalization of an A_∞-algebra to a categorical setting, where we recover the notion of an A_∞-algebra by considering an A_∞-category with one object. The Fukaya category $\mathrm{Fuk}(M)$ of a symplectic manifold (M, ω) and the bounded derived category of coherent sheaves $D^b(X)$ on an algebraic variety X are interesting examples of A_∞-categories which appear naturally in practice. Indeed, these two examples are the focus of much present interest in A_∞-structures, due primarily to the influential paper [11] of Kontsevich from 1994.

In that paper, Kontsevich conjectured that string theoretic mirror symmetry between two Calabi-Yau manifolds X and Y should be understood mathematically as *homological mirror symmetry* (HMS), or equivalences of A_∞-categories,

$$\mathrm{Fuk}(X) \simeq D^b(Y) \text{ and } D^b(X) \simeq \mathrm{Fuk}(Y).$$

This gives, roughly, an exchanging of the symplectic and complex structures by passing from X to Y. Polishchuk and Zaslow proved the conjecture in this form for elliptic curves in [16], [18]; Seidel proved it for the quartic surface in [23]; and Abouzaid and Smith have treated abelian surfaces in [2]. In [9], Katzarkov proposed a generalization of mirror symmetry for varieties of general type which are not Calabi-Yau; that version of the conjecture was proved in one direction for curves of genus two by Seidel [22] and for curves of higher genus by Efimov [4]. These works on curves of higher genus exhibit the equivalence of the symplectic structure on the curve with the complex structure on its proposed mirror dual, without considering the other direction.

In proving the homological mirror conjecture, little explicit knowledge of the A_∞-structures involved is needed. As some evidence for this claim, the full proof of HMS for elliptic curves (see [18]) only required specific calculation of m_2 and m_3, while not until [17] was a complete A_∞-structure relevant to this problem computed. In this paper we seek to refine and expand the explicit knowledge of the A_∞-structures for curves on the complex side of this problem.

Let X be a variety over an algebraically closed field $\mathbb{K}$ and let $G \in \mathrm{Coh}(X)$, where $\mathrm{Coh}(X)$ is the category of coherent sheaves on X. Let $G \to I^\bullet$ be an injective resolution of G in $\mathrm{Coh}(X)$; then $A = \mathrm{Hom}(I^\bullet, I^\bullet)$ has the structure of a dg-algebra whose cohomology is

$B = \mathrm{Ext}^*(G, G)$. By a theorem of Kadeishvili [8], there is a minimal ($m_1 = 0$) A_∞-structure on B (unique up to A_∞-equivalence) such that A and B are quasi-isomorphic as A_∞-algebras.

Let $\mathscr{C}$ be a triangulated category, and $T \in \mathrm{Ob}(\mathscr{C})$. We let $\mathbf{tria}(T) \subset \mathscr{C}$ be the smallest triangulated full subcategory containing T which is closed under passage to direct summands; in other words, $\mathbf{tria}(T)$ is the closure of T under shifts, extensions, and taking direct summands. We say that T is a *generator* of $\mathscr{C}$ if $\mathbf{tria}(T) = \mathscr{C}$.

Let Mod-B be the category of A_∞-modules over B, $D^b(\text{Mod-}B)$ its bounded derived category. Then $\mathbf{tria}(B) \subset D^b(\text{Mod-}B)$ is the triangulated subcategory generated by the free B-module of rank one. This is the derived category of *perfect* B-modules, and is sometimes denoted $\mathbf{perf}(B)$. If we suppose further that G is a generator of $D^b(X)$, then a theorem in the thesis of Lefévre-Hasegawa ([12], 7.6) implies that there is an equivalence of triangulated categories, $D^b(X) \simeq \mathbf{perf}(B)$.

When X is a smooth curve, it is known that we do not lose any information by passing from X to $D^b(X)$; that is, $X_1 \cong X_2$ if and only if $D^b(X_1) \simeq D^b(X_2)$. This follows that the Bondal-Orlov reconstruction theorem ([3], 2.5) for curve of genus $g > 1$, and is proved for elliptic curves by Hille and Van den Bergh ([7], 5.1). It follows that when X is a smooth complex curve and G is a generator of $D^b(X)$, the A_∞-structure on B is sufficient to recover X up to isomorphism. Therefore it is of value to investigate this structure in some detail.

Let E be a complex elliptic curve with structure sheaf $\mathcal{O}_E$ and let L be a line bundle of degree 1. In [17], Polishchuk studies the case where $G = \mathcal{O}_E \oplus L$, and computes explicitly an A_∞-structure on $B(E) = \mathrm{Ext}^*(G, G)$ in terms of the Eisenstein series of E. The A_∞-algebra $B(E)$ recovers E up to isomorphism, but the associative algebra $B = (B(E), m_2)$ is independent of E. In this way we get a family of non-equivalent A_∞-structures extending the associative algebra structure of B. That is, if we denote by $\mathcal{M}_\infty^B$ the moduli space of A_∞-structures on B up to equivalence, and let $\mathcal{M}_{g,n}$ be the usual moduli space of smooth curves of genus g with n marked points, there is a map,

$$\mathcal{M}_{1,1} \to \mathcal{M}_\infty^B, \quad E \mapsto B(E).$$

This leads naturally to several questions.

First, can we describe the equivalence classes of A_∞-structures extending the multiplication on B? It is well-known that such extensions are governed by certain components of the Hochschild cohomology of B. The specifics of this relationship are recalled in detail in Section II.5.. In Chapter III, we compute relevant components of $HH^*(B)$. There is no particular need to restrict this calculation to the complex numbers. The main result is Theorem III.4.1, and it applies whenever $\operatorname{char}\mathbb{K} \neq 2, 3$. Prior to this writing, Perutz and Lekili published an independent calculation of this cohomology ([13], Thm.4); nonetheless, the author feels the calculation here is of value both for its method and for its usefulness in extending to the case of $g \geq 2$.

Let $HH^n_{(m)}(B)$ be the Hochschild cohomology of B in dimension n for maps of homogeneous degree m. Since $HH^n_{(2-n)}(B)$ vanishes for $n > 8$ (by Theorem III.4.1), A_∞-structures on B are determined up to equivalence by the set $\{m_n\}_{n=3}^8$. (In general, when an A_∞-structure is determined up to equivalence by a finite number of operations, we say that it is *finitely determined*.) This also implies that $HH^*_{(2-n)}(B)$ is a finite-dimensional vector space, so $\mathcal{M}_\infty^B$ can be realized as a quotient of an affine scheme of finite type.

Second, the appearance of the Eisenstein series of E in the higher operations on $B(E)$ suggests that we might find some other interesting functions lurking there. In particular, can we recover the j-invariant $j(E)$ from this A_∞-structure? The A_∞-structure (m) in [17] has the property that for $k \in \mathbb{Z}_{\geq 1}$, $m_{2k-1} = 0$ and $m_{2k} \neq 0$. Theorem III.4.1 implies the existence of an equivalent structure (m') (Proposition IV.1.2) such that $m'_{2k-1} = 0$, $m'_4 = 0$, and m'_6, m'_8 are Hochschild cocycles. We compute (m') explicitly in Chapter IV, and recover $j(E)$ in Theorem IV.2.3 as the value of a rational function on $V = HH^6_{(-4)}(B) \oplus HH^8_{(-6)}(B) \cong \mathbb{C}^2$ evaluated at the point $(m'_6(E), m'_8(E))$ (with suitably chosen coordinates on V).

In Chapters V and VI we consider curves X of genus $g \geq 2$. We choose a generator G of $D^b(X)$, which amounts to choosing g distinguished points on X. When we choose these points across all genus g curves to satisfy a certain open condition on $\mathcal{M}_{g,g}$, we get a family of A_∞-algebras $B^g(X) = \operatorname{Ext}^*(G, G)$ which restrict to the same associative algebra, $B^g = (B^g(X), m_2)$. That is, there is a map,

$$\mathcal{M}_{g,g} \to \mathcal{M}_\infty^{B^g}, \quad X \mapsto B^g(X).$$

Then we can ask, is the A_∞-structure on $B^g(X)$ finitely determined? In Chapter V we calculate the relevant components of Hochschild cohomology for the associative algebra B^g. The result in Theorem V.4.10 shows that $HH^n_{(2-n)}(B^g)$ vanishes for $n > 6$, so $B^g(X)$ is determined up to equivalence knowing only up to m_6.

The question of finite determination is important for the HMS problem for curves of higher genus. If we are trying to determine an equivalence between two A_∞-structures, it is useful to know that we need only force their equality up to some finite level. Our choice of the generator G is not unique, and there is no reason to expect that an arbitrary such choice gives a finitely determined A_∞-structure. At present, there is no general program for finding a generator with the finite determination property; therefore these positive results in the simple cases of curves might also be useful for suggesting some patterns for finding such generators of arbitrary varieties.

Finally, what additional information can be determined about the A_∞-structure on $B^g(X)$? A result in [19] implies that m_3 can be chosen to represent Massey products in $D^b(X)$. In Chapter VI we compute one of these Massey products in order to make some comments about if and when m_3 is trivial (homotopic to 0). Theorem VI.2.1 shows that m_3 is trivial only if X is hyperelliptic, and the g distinguished points are Weierstrass points. It would be interesting to find an explicit homotopy of this structure which takes m_3 to 0, thus making m_4 and m_5 Hochschild cocycles. One can then hope to recover analogs of the j-invariant from this A_∞-structure in the case of hyperelliptic curves.

CHAPTER II

PRELIMINARIES

This chapter compiles many of the results, definitions and notation used throughout this paper. The reader who prefers may skip this chapter and return to these sections as needed when they are referenced later. While there are no new results in this chapter, the proof of Lemma II.5.1 is new, as a suitable reference could not be found.

II.1. Reduced Hochschild cohomology

Let $\mathbb{K}$ be a field, B a (unital) $\mathbb{K}$-algebra. Let M be a B-bimodule. We define the full Hochschild cochain complex, $C^\bullet(B, M)$, where,

$$C^n(B, M) := \mathrm{Hom}_{\mathbb{K}}(B^{\otimes_{\mathbb{K}} n}, M),\ n \geq 0.$$

For a cochain $\phi \in C^n(B, M)$, we define the cochain $\delta\phi \in C^{n+1}(B, M)$ as,

$$(\delta\phi)(a_0, \ldots, a_n) = (-1)^{|a_0||\phi|} a_0\phi(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i \phi(a_0, \ldots, a_{i-1}a_i, \ldots, a_n) \\ + (-1)^{n+1}\phi(a_0, \ldots, a_{n-1})a_n.$$

The cohomology of this complex is the Hochschild cohomology of B with coefficients in M, denoted $HH^\bullet(B, M)$.

Remark II.1.1. *The sign on the first term applies to the case when B is $\mathbb{Z}$-graded, in which case $|\phi|$ is the homogeneous degree of the map ϕ. By convention, if B is not graded, then $|b| = 0$ for all $b \in B$.*

With $\mathbb{K}$, B as above, let $R \subset B$ be a semi-simple subring such that $B \cong R \oplus B/R$ as R-modules. Let B_+ be the R-submodule of B isomorphic to B/R. We say in this case that B is *augmented* over R.

We define the reduced cochain complex $C^\bullet_R(B, M)$ by replacing $\otimes_{\mathbb{K}}$ and $\mathrm{Hom}_{\mathbb{K}}$ with $\otimes_R$ and Hom_R and replacing B with B_+ in the full complex. That is,

$$C^n_R(B, M) := \mathrm{Hom}_R(B_+^{\otimes_R n}, M),\ n \geq 0,$$

with the differential defined the same as above. The cohomology of this complex we call the reduced Hochschild cohomology of B over R with coefficients in M, denoted $HH^\bullet_R(B, M)$.

In all applications, we will use a reduced Hochschild cochain complex. The two complexes are known to be quasi-isomorphic for unital algebras ([21] (20d)). We will therefore usually write $C^\bullet(M)$ and $HH^\bullet(M)$ when B and R are understood, and it is understood that everything is reduced over R.

When B is a ($\mathbb{Z}$-)graded $\mathbb{K}$-algebra, it follows that B_+ is a graded R-module. Then the reduced tensor algebra,

$$T(B_+) = \bigoplus_{i=1}^{\infty} B_+^{\otimes_R i},$$

is bigraded. The cohomological grading $\deg_{\mathrm{coh}}$ gives the length of a tensor, and the *internal grading* $\deg_{\mathrm{int}}$ is inherited from the grading on B_+; that is,

$$\deg_{\mathrm{coh}}(x_1 \otimes \cdots \otimes x_n) = n, \quad \deg_{\mathrm{int}}(x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} \deg(x_i).$$

When R is concentrated in degree 0 and M is a graded B-bimodule, the complex $C^\bullet(M)$ is bigraded. Recall that if $f : N \to M$ is a map of graded B-bimodules, we say that f is *homogeneous of internal degree* m if $\deg(f(x)) = \deg(x) + m$ for all x in N.

We let $C^n_{(m)}(M) \subset C^n(M)$ be the R-submodule of homogeneous maps of internal degree m. It is easy to check that δ preserves m, so $C^\bullet(M)$ is a direct sum of complexes with fixed internal degree,

$$C^\bullet(M) = \bigoplus_{m \in \mathbb{Z}} C^\bullet_{(m)}(M), \quad HH^\bullet(M) = \bigoplus_{m \in \mathbb{Z}} HH^\bullet_{(m)}(M).$$

In cases when we have fixed the internal degree m and it does not cause ambiguity in later sections, we will suppress m also in the notation and write simply $C^\bullet(M)$ and $HH^\bullet(M)$.

II.2. The normalized bar complex and duality

Let $\mathbb{K}$, B, R, B_+ be as in Section II.2.. We define the chain complex $\mathrm{Bar}^R_\bullet(B)$ as

$$\mathrm{Bar}^R_n(B) := B_+^{\otimes_R n},\ n \geq 0,$$

with differential defined by

$$d(x_1 \otimes \cdots x_n) = \sum_{i=1}^{n-1} x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n.$$

This is the normalized bar complex of B over R, which we denote $HB^R_\bullet(B)$. When R is understood we will write $\mathrm{Bar}_\bullet(B)$ and $HB_\bullet(B)$.

The complex $\mathrm{Bar}_\bullet(B)$ has the bigrading of $T(B_+)$ described in Section II.1.. We let $\mathrm{Bar}^{(m)}_n(B)$ be the space of tensors of internal degree m.

Remark II.2.1. *In general, when V is a chain (or cochain) space with an internal grading, we will write $V^{(m)}$ (or $V_{(m)}$, respectively) to denote the subspace of chains with internal degree m.*

The differential d preserves the internal grading, so $\mathrm{Bar}_\bullet(B)$ is a direct sum of complexes,

$$\mathrm{Bar}_\bullet(B) = \bigoplus_{m \in \mathbb{Z}} \mathrm{Bar}^{(m)}_\bullet(B).$$

When the internal degree is fixed and it does not cause ambiguity, we will suppress m in the notation and write simply $\mathrm{Bar}_\bullet(B)$ and $HB_\bullet(B)$.

Let B be graded, M a graded B bimodule, and $C^\bullet(M)$ the Hochschild complex reduced over R. When the first and last terms of the Hochschild differential vanish, there is an isomorphism of complexes

$$[\mathrm{Bar}_\bullet(B)]^* \otimes M \xrightarrow{\sim} C^\bullet(B, M),$$

since in this case $\delta f = f \circ d$ for $f \in C^n(M)$. When M is concentrated in a single degree k, we have more specifically

$$[\mathrm{Bar}^{(m)}_{\bullet}(B)]^* \otimes M \xrightarrow{\sim} C^{\bullet}_{(k-m)}(B, M).$$

II.3. A_∞-algebras

We use [10] as a reference. Let $\mathbb{K}$ be a field, V a graded $\mathbb{K}$-vector space. An A_∞-*structure* on V is a collection of maps $m_n \in \mathrm{Hom}_{\mathbb{K}}(V^{\otimes_{\mathbb{K}} n}, V)$ of internal degree $2-n$ for each $n \geq 1$, which satisfy the compatibility relations:

$$\sum_{n=r+s+t} (-1)^{r+st} m_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0, \tag{II.1}$$

for each $n \geq 1$, where $u = r+1+t$. We call the relation in which $r+s+t = k$ the A_∞-relation of order k. The space V endowed with such maps is called an A_∞-*algebra*. We denote this space by (V, m), where $(m) = \{m_n\}_{n=1}^{\infty}$ is an A_∞-structure on V.

Remark II.3.1. *If $|g|$ is the internal degree of a graded map g and $|x|$ is the internal degree of a tensor x, we use the sign convention that $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$.*

An A_∞-structure is called *minimal* if $m_1 = 0$. The A_∞-relation of order 1 implies that m_1 has internal degree 1 and $m_1^2 = 0$, so (V, m_1) is a complex. We denote its cohomology by H^*V. The algebra V with multiplication m_2 (denoted (V, m_2)) is not associative in general, but the A_∞-relation of order 3 implies that if either $m_1 = 0$ or $m_3 = 0$, then m_2 is associative. In particular, m_2 induces a multiplication on H^*V such that (H^*V, m_2) is an associative algebra.

Let (V, m) and (V', m') be two A_∞-algebras. A *morphism* of A_∞-algebras is a collection of maps

$f_n \in \mathrm{Hom}_{\mathbb{K}}(V^{\otimes_{\mathbb{K}} n}, V')$ of internal degree $1-n$ for each $n \geq 1$ satisfying the compatibility axioms:

$$\sum_{n=r+s+t} (-1)^{r+st} f_u(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) \quad = \quad \sum (-1)^s m'_r(f_{i_1} \otimes \cdots \otimes f_{i_r}), \tag{II.2}$$

where $u = r+1+t$, and the sum on the right is over all $1 \leq r \leq n$ and all decompositions

$n = i_1 + \cdots + i_r$, where we define

$$s = \sum_{j=1}^{r} (r-j)(i_j - 1).$$

We denote by $(f) : (V, m) \to (V, m')$ the A_∞-morphism with component maps f_n.

As a reference for strict A_∞-isomorphisms, we use [20], Section 2.1. Let (m), (m') be two A_∞-structures on V. Then we call a morphism $(f) : (V, m) \to (V, m')$ a *strict A_∞-isomorphism* provided $f_1 = \mathrm{id}_V$.

Proposition II.3.2. *([18], Lemma 1.1) Let (V, m) be an A_∞-algebra, and $(f) = \{f_n\}_{n=1}^{\infty}$ a collection of $\mathbb{K}$-linear maps $f_n : V^{\otimes n} \to V$, homogeneous of internal degree $1-n$, with $f_1 = id_V$. Then there is a unique A_∞-structure (m') on V such that $(f) : (V, m) \to (V, m')$ is a strict A_∞-isomorphism.*

In particular, this means that the strict A_∞-isomorphisms act on A_∞-structures on V in an appropriate sense. In the situation of the proposition, we write $m' = f * m$.

([12] 1.2.1.7) Let $f, g : (V, m) \to (V', m')$ be two A_∞-morphisms. A *homotopy* between f and g is a collection of maps

$$h_n : V^{\otimes n} \to V', \ n \geq 1,$$

homogeneous of internal degree $-i$ satisfying for each n the equation,

$$\begin{aligned} f_n - g_n = & \sum (-1)^s m_{r+1+t}(f_{1_i} \otimes \cdots \otimes f_{i_r} \otimes h_k \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \\ & + \sum (-1)^{jk+l} h_i(\mathbf{1}^{\otimes j} \otimes m_k \otimes \mathbf{1}^{\otimes l}, \end{aligned}$$

where

$$j + k + l = i_1 + \cdots + i_r + j_1 + \cdots j_t + k = n,$$

and s determines a sign. We denote such a collection by (h). If such an (h) exists for f and g, we say that f and g are *homotopic*. We will not use these relations explicitly (justifying ignorance of the sign s), but note: 1) homotopy between morphisms is an equivalence relation and 2) a morphism f and any collection (h) of such maps determines a unique morphism g such that (h) is a homotopy from f to g ([20], Lemma 2.1). In this case we say that $h * f = g$.

We now return to the situation of Section II.1. where B is a graded $\mathbb{K}$-algebra, $R \subset B$ is a semi-simple subring concentrated in degree 0 such that $B = R \oplus B_+$ as an R-bimodule. A minimal A_∞-structure on B *reduced over* R is an A_∞-structure on B such that $m_1 = 0$, m_2 is the usual multiplication in B, and $m_n \in C^n_R(B, B)$ for $n \geq 3$. For $r \in R$ and $a_i \in B$, we require for $n \geq 3$ that

$$m_n(a_1, \ldots, r, \ldots, a_n) = 0.$$

We define morphisms and strict A_∞-isomorphisms of such structures by replacing $\otimes_{\mathbb{K}}$ and $\mathrm{Hom}_{\mathbb{K}}$ and $\otimes_R$ and Hom_R in the above definitions.

II.4. The Lie superalgebra of superderivations

Let V be a graded $\mathbb{K}$-vector space, $f : T(V) \to T(V)$ a $\mathbb{K}$-linear map, homogeneous with respect to the internal grading. We call f a *superderivation* if for $v_1, v_2 \in T(V)$, we have,

$$f(v_1 \otimes v_2) = f(v_1) \otimes v_2 + (-1)^{|v_1||f|} v_1 \otimes f(v_2).$$

We denote by $\mathrm{Der}_l\, T(V)$ the vector space of superderivations of internal degree l. We define,

$$\mathrm{Der}\, T(V) := \bigoplus_{l \in \mathbb{Z}} \mathrm{Der}_l\, T(V).$$

There is a bracket defined on $\mathrm{Der}\, T(V)$ which gives it the structure of a Lie superalgebra. For $d_1 \in \mathrm{Der}_i\, T(V)$, $d_2 \in \mathrm{Der}_j\, T(V)$, and $d_3 \in \mathrm{Der}_k\, T(V)$, the bracket is defined by,

$$[d_1, d_2] = d_1 d_2 - (-1)^{ij} d_2 d_1,$$

and the Jacobi identity states that

$$(-1)^{ik}[d_1, [d_2, d_3]] + (-1)^{ij}[d_2, [d_3, d_1]] + (-1)^{jk}[d_3, [d_1, d_2] = 0.$$

II.5. Hochschild cohomology and A_∞-structures

We restrict to the case where $\operatorname{char}\mathbb{K} \neq 2, 3$. Let $n \geq 2$. A *minimal A_n-structure* on a graded $\mathbb{K}$-vector space V is a collection of maps $m_i \in \operatorname{Hom}_{\mathbb{K}}(V^{\otimes i}, V)$ for $1 \leq i \leq n$, each of internal degree $2 - i$, satisfying the A_∞-relations of order k for all $k \leq n+1$ (ref. Equation II.1), and such that $m_1 = 0$. Note that the relation of order $n+1$ would typically include the terms,

$$\pm m_1(m_{n+1}) \pm m_{n+1}\left(\sum_{r+1+t=n+1} \mathbf{1}^{\otimes r} \otimes m_1 \otimes \mathbf{1}^t\right),$$

but that these terms vanish when $m_1 = 0$. In particular, note that a minimal A_∞-structure on V restricts to a minimal A_n-structure for all $n \geq 2$. In this section we recall the relationship between Hochschild cohomology of an associative algebra B and minimal A_n-structures on B. (See [1], [20])

Since $m_1 = 0$, the A_∞-relation of order $k+1 \geq 3$ can be rewritten in the form,

$$m_2(1 \otimes m_k \pm m_k \otimes 1) - m_k\left(\sum_{r+2+t=k+1} (-1)^{r+1}\mathbf{1}^{\otimes r} \otimes m_2 \otimes \mathbf{1}^{\otimes s}\right) = \Phi_k(m_3, \ldots, m_{k-1}),$$

where Φ_k is a quadratic expression. Since m_2 is an associative multiplication, the left side of this equation is exactly δm_k, where δ is the Hochschild differential; that is, we have the equation

$$\delta m_k = \Phi_k(m_3, \ldots, m_{k-1}).$$

Lemma II.5.1. *([1], Lemma 2.3) Assume the maps $\{m_i | 1 \leq i \leq k-1\}$ determine an A_{k-1}-structure on V. Then with Φ_k as defined above,*

1. *$\Phi_k(m_3, \ldots, m_{k-1}) : V^{\otimes k+1} \to V$ is homogeneous of internal degree $2-k$; and*
2. *$\delta(\Phi_k(m_3, \ldots, m_{k-1})) = 0$.*

That is, the A_{k-1}-structure provides a particular cocycle, $\Phi_k(m_3, \ldots, m_{k-1})$. We can find m_k to extend this to an A_k-structure if the relation of order $k+1$ can be solved for m_k, i.e., if this cocycle is a coboundary. Therefore this result implies that if $HH^{k+1}_{(2-k)}(V)$ vanishes, any A_{k-1}-structure can be extended to an A_k-structure. Statement (1) is a trivial check, but we include a proof of statement (2).

Proof. Each given m_i, $1 \le i \le k-1$, determines a map $\hat{m}_i : T(V) \to V$, homogeneous of internal degree $2-i$, equal to m_i on $V^{\otimes i}$ and zero otherwise. Let $S : V \to V$ be the grading shift of degree -1, i.e., $(SV)_j = V_{j+1}$. Following [15] §4, we let $W = (SV)^*$. Then each $\hat{m}_i$ has a dual map $\hat{d}_i : W \to T(W)$ of internal degree 1, which can be uniquely extended to a superderivation $d_i : T(W) \to T(W)$ of internal degree 1.

The A_∞-constraints arise from certain relations among the brackets of the superderivations $d_i \in \mathrm{Der}_1\, T(W)$. The Lie bracket $[-,-]_D$ on superderivations (see Section II.4.) corresponds to the Gerstenhaber bracket ([5]) $[-,-]_G$ on Hochschild cochains in the dual picture. We suppress G and D when no confusion will arise. If δ is the Hochschild differential, then on cochains we have $\delta f = [m_2, f]_G$ and,

$$\Phi_k(m_3,\ldots,m_{k-1}) = \begin{cases} -[m_3,m_{k-1}] - [m_4,m_{k-2}] + \\ \quad \cdots - [m_{(k+1)/2}, m_{(k+3)/2}] & \text{if } k \text{ is odd,} \\ \\ -[m_3,m_{k-1}] - [m_4,m_{k-2}] - \\ \quad \cdots - \dfrac{1}{2}[m_{(k+2)/2}, m_{(k+2)/2}] & \text{if } k \text{ is even.} \end{cases} \tag{II.3}$$

(Here we need that 2 is invertible in $\mathbb{K}$.) For compactness, we will write Φ_k in place of $\Phi_k(m_3,\ldots,m_{k-1})$. The condition that $[m_2,\Phi_k]_G = 0$ is dual to the condition that $[d_2,\Phi_k^*]_D = 0$. Since Φ_k can be written as a sum of brackets of the m_i, dualizing presents Φ_k^* as a sum of brackets of derivations, making Φ_k^* itself a derivation. We show that $[d_2,\Phi_k^*]_D = 0$ by induction in k.

The base case $k = 3$ is obvious since $\Phi_3 = \Phi_3^* = 0$. Now suppose that superderivations $d_2,\ldots,d_{k-1}$ are defined such that $\deg_{\mathrm{int}} d_j = 1$ and $[d_2,d_j] = \Phi_j^*$ for all $j < k$ with k even. Since each $d_j \in \mathrm{Der}_1\, T(W)$, the bracket and Jacobi identity reduce to,

$$[d_1,d_2] = [d_2,d_1] = d_1d_2 + d_2d_1, \quad [d_1,[d_2,d_3]] + [d_2,[d_3,d_1]] + [d_3,[d_1,d_2]] = 0.$$

Then

$$\begin{aligned}
[d_2, \Phi_k^*] = & -[d_2, [d_3, d_{k-1}] + [d_4, d_{k-2}] + \cdots + \frac{1}{2}[d_{(k+2)/2}, d_{(k+2)/2}]] \\
&= [d_3, [d_2, d_{k-1}] + [d_{k-1}, [d_2, d_3]] + [d_4, [d_2, d_{k-2}]] + [d_{k-2}, [d_2, d_4]] + \\
&\quad \cdots + [d_{(k+2)/2}, [d_2, d_{(k+2)/2}]] \\
&= \sum_{i=3}^{k-1} [d_i, \Phi_{k+2-i}^*]
\end{aligned}$$

When Φ_{k+2-i}^* are expanded as in equation II.3, the sum on the right will have terms:

1. $-[d_i, [d_j, d_t]]$, $-[d_j, [d_i, d_t]]$, $-[d_t, [d_i, d_j]]$ where $i + j + t = k + 4$ and i, j, t are all distinct. Each term appears once, in the expansion of $[d_i, \Phi_{k+2-i}^*]$, $[d_j, \Phi_{k+2-j}^*]$ and $[d_k, \Phi_{k+2-t}^*]$ respectively, and their sum vanishes by the Jacobi identity.

2. $-[d_i, [d_i, d_j]]$, $-\frac{1}{2}[d_j, [d_i, d_i]]$ where $2i + j = k + 4$ and $i \neq j$. Each term appears once, in $[d_i, \Phi_{k+2-i}^*]$ and $[d_j, \Phi_{k+2-j}^*]$, respectively, and their sum vanishes by the Jacobi identity. By the Jacobi identity we have

$$-\frac{1}{2}[d_j, [d_i, d_i]] = [d_i, [d_i, d_j]],$$

so these terms cancel.

3. $-[d_i, [d_i, d_i]]$ where $3i = k + 4$. By the Jacobi identity we have that $3[d_i, [d_i, d_i]] = 0$, so $[d_i, [d_i, d_i]] = 0$ since $\operatorname{char} \mathbb{K} \neq 3$.

This completes the step for k even. When k is odd we have

$$\begin{aligned}
[d_2, \Phi_k^*] &= -[d_2, [d_3, d_{k-1}] + \cdots + [d_{(k+1)/2}, d_{(k+3)/2}]] \\
&= \sum_{i=3}^{k-1} [d_i, [d_2, d_{k+2-i}]] \\
&= \sum_{i=3}^{k-1} [d_i, \Phi_{k+2-i}^*],
\end{aligned}$$

with the same result as the k even case. □

Lemma II.5.2. *([20], Lemma 2.2) Let* (m) *and* (m') *be two minimal* A_∞*-structures on* V *whose restriction to* A_{k-1}*-structures are equal for some* $k \geq 3$*. Then,*

1. $m_k - m'_k$ *is a Hochschild cocycle; and*

2. $m_k - m'_k$ *is a Hochschild coboundary if and only if there exists a strict* A_∞*-isomorphism* $(f) : V \to V$ *such that* $f * m_i = m'_i$ *for all* $i \leq k$*.*

Therefore if $HH^k_{(2-k)}(V) = 0$, all extensions of a particular A_{k-1}-structure on V to A_k-structures are equivalent in this precise sense. The details of the proof of this lemma also show that we should take define f by $f_1 = \mathrm{id}_V$; f_{k-1} such that $\delta f_{k-1} = m_k - m'_k$; and $f_n = 0$ otherwise.

Lemma II.5.3. *([20], Lemma 2.3) Let* (m), (m') *be two minimal* A_∞*-structures on* V *and let* (f), (f') *be strict* A_∞*-isomorphisms such that 1)* $f * m = f' * m = m'$ *and 2)* $f_i = f'_i$ *for* $1 \leq i < k$*, where* $k \geq 2$*. Then,*

1. $f_k - f'_k$ *is a Hochschild cocycle; and*

2. $f_k - f'_k$ *is a Hochschild coboundary if and only if there is a homotopy* h *such that* $h * f_i = f'_i$ *for all* $i \leq k$*.*

Therefore if $HH^k_{(1-k)}(V)$ vanishes and $f_1, \ldots, f_{k-1}$ (the start of an A_∞-morphism) are given, the choice of f_k which satisfies 1) and 2) in the statement of the lemma is unique up to homotopy.

II.6. The spectral sequence of a filtration

Let $(C_\bullet, d)$ be a chain complex ($d : C_n \to C_{n-1}$) with $\{F_i C_\bullet\}_{i \in \mathbb{Z}}$ an ordered family of subcomplexes,

$$\cdots \supset F_i C \supset F_{i+1} C \supset F_{i+2} C \supset \cdots$$

Such a family is called a *decreasing filtration* of $C_\bullet$. The filtration is called *exhaustive* if $C_\bullet = \bigcup_{i \in \mathbb{Z}} F_i C_\bullet$; *bounded below* if for each n there exists an integer s such that $F_i C_n = 0$ for all $i \geq s$; *bounded above* if for each n there exists an integer t such that and $F_iC_n = C_n$ for all $i \leq t$; and *bounded* if it is bounded above and bounded below.

Theorem II.6.1. *([25], Thm. 5.4.1) A filtration F of a chain complex C naturally determines a spectral sequence starting with $E^0_{pq} = F_pC_{p+q}/F_{p+1}C_{p+q}$ and $E^1_{pq} = H_{p+q}(E^0_{p*})$.*

The differential d_0^{pq} in the spectral sequence is induced by the differential d in C, and is easy to understand. Let $x \in F_iC_n/F_{i+1}C_n$ (on the zero page of the sequence). Let $\tilde{x}$ be a lifting of x in F_iC_n. Then $d_0^{pq}(x)$ is the class of $d(\tilde{x})$ in $F_iC_{n-1}/F_{i+1}C_{n-1}$, which is well-defined since $F_{i+1}C$ is a complex. Then $d_0^{pq}(x) = 0$ (that is, x represents a class in E^1_{pq}) if and only if $d(\tilde{x}) \in F_{i+1}C_{n-1}$. Then we can consider $d_1^{pq}(x)$ similarly, with an appropriately defined target space.

Theorem II.6.2. *([25]), Thm. 5.5.1) Suppose that the filtration on C is either a) bounded or b) bounded below and exhaustive. Then the spectral sequence from Theorem II.6.1 converges to $H_*(C)$.*

II.7. Eisenstein series

We use [17] as the main reference. Let T be the space of all oriented bases of $\mathbb{C}$ as an $\mathbb{R}$-vector space, and let $k \in \mathbb{Z}$. A C^∞-function $F : T \to \mathbb{C}$ is called *modular* if 1) it is invariant under the action of $\mathrm{SL}(2, \mathbb{Z})$ on T and 2) $F(1, \tau) = f(e^{2\pi i \tau})$, where $f(q)$ is meromorphic at $q = 0$. We say that F has weight k provided,

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^k F(\omega_1, \omega_2)$$

for all $(\omega_1, \omega_2) \in T$.

For $(\omega_1, \omega_2) \in T$, let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. The Eisenstein series e_{2k} for $k \geq 2$ is defined as,

$$e_{2k}(\omega_1, \omega_2) = \sum_{\omega \in \Lambda/\{0\}} \frac{1}{\omega^{2k}}.$$

The function e_{2k} is modular of weight $2k$. We consider the analogous series for $k = 1$ defined as

$$e_2(\omega_1, \omega_2) = \sum_m \sum_{n; n \neq 0 \text{ if } m=0} \frac{1}{(m\omega_2 + n\omega_1)^2}.$$

Unfortunately e_2 is not modular, but the correction

$$e_2^*(\omega_1, \omega_2) = e_2(\omega_1, \omega_2) - \frac{\pi}{a(\Lambda)} \cdot \frac{\bar{\omega_1}}{\omega_1},$$

where $a(\Lambda) = \mathrm{Im}(\bar{\omega_1}\omega_2)$ is the area of $\mathbb{C}/\Lambda$, is $\mathrm{SL}(2, \mathbb{Z})$-invariant of weight 2. For convenience we set $e_{2k}^* = e_{2k}$ for $k \geq 2$.

For integers m, n of the same parity, we set

$$f_{m,n}(\Lambda) = \left(\frac{\pi}{a(\Lambda)}\right)^m \sum_{\omega \in \Lambda/\{0\}} \frac{\bar{\omega}^m}{\omega^n} \exp\left(-\frac{\pi}{a(\Lambda)}|\omega|^2\right).$$

Then for integers $a, b \geq 0$ of different parity we set

$$g_{a,b}(\Lambda) = \sum_{k \geq 0} k! \left(\binom{a}{k} + \binom{b}{k}\right) f_{a+b-k,k+1}(\Lambda).$$

When m, n are of different parity or a, b the same parity, then $f_{m,n}(\Lambda) = g_{a,b}(\Lambda) = 0$. It is shown in [17] that $g_{a,b}$ is a polynomial in $e_2^*, e_4, \ldots, e_{a+b+1}$ with rational coefficients. A few of these polynomial relations we will use later. When the lattice is understood, we will write e_{2k} in place of $e_{2k}(\Lambda)$, $g_{a,b}$ in place of $g_{a,b}(\Lambda)$, and so on.

Lemma II.7.1.

1. $g_{3,0} = 6e_4$,
2. $g_{2,1} = -[e_2^*]^2 + 5e_4$,
3. $g_{5,0} = 120e_6$,
4. $g_{4,1} = -5g_{3,0}g_{1,0} + \frac{7}{10}g_{5,0}$,
5. $g_{3,2} = -2g_{2,1}g_{1,0} + \frac{5}{6}g_{4,1}$,

Proof. These follow immediately from [17] Prop. 2.6.1. $\square$

When $C = \mathbb{C}/\langle \mathbb{Z} \oplus \tau\mathbb{Z}\rangle$ is a complex elliptic curve, we set $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ for the purpose of computing the Eisenstein series of the curve. We set $t = \dfrac{\operatorname{Im}\tau}{\pi}$, and for non-negative integers a, b, c, d define

$$M(a,b,c,d) := (-1)^{\binom{a+b+c+d+1}{2}} \frac{1}{a!b!c!d!} \cdot t^{a+b+c+d+1} \cdot g_{a+c,b+d}(\Lambda).$$

Note that the expression $M(a,b,c,d)$ is invariant under transpositions of a and c, and of b and d.

II.8. Massey products in a triangulated category

We use [19] as the reference. Let $\mathcal{D}$ be a triangulated category, with X, Y, Z, T objects in $\mathcal{D}$, $f \in \operatorname{Hom}(X,Y)$, $g \in \operatorname{Hom}(Y, Z[1])$, $h \in \operatorname{Hom}(Z,T)$ be morphisms such that $g \circ f = 0$. We define the Massey product

$$\operatorname{MP}(f,g,h) \in \operatorname{coker}(\operatorname{Hom}(X,Z) \oplus \operatorname{Hom}(Y,T)) \xrightarrow{(h,f)} \operatorname{Hom}(X,T).$$

Let

$$Z \xrightarrow{\alpha} C \xrightarrow{\beta} Y \xrightarrow{g} Z[1] \to \cdots$$

be a distinguished triangle in $\mathcal{D}$. Then by the axioms of the triangulated category there exist liftings $\tilde{f} \in \operatorname{Hom}(X,C)$ and $\tilde{h} \in \operatorname{Hom}(C,T)$ such that

$$\beta \circ \tilde{f} = f, \;\; \tilde{h} \circ \alpha = h.$$

Then we define

$$\operatorname{MP}(f,g,h) = [\tilde{h} \circ \tilde{f}].$$

CHAPTER III

HOCHSCHILD COHOMOLOGY AND THE ELLIPTIC CURVE

Let E be an elliptic curve with structure sheaf $\mathcal{O}$ over an algebraically closed field $\mathbb{K}$ with $\operatorname{char}\mathbb{K} \neq 2, 3$. Let $D^b(E)$ be the bounded derived category of coherent sheaves on E. Let P be a closed point on E, $\mathbb{K}(P) = \mathcal{O}(P)/\mathcal{O}$ the skyscraper sheaf at P. Let $G = \mathcal{O} \oplus \mathbb{K}(P)$, and $B(E) := \operatorname{Ext}^*(G, G)$.

We realize $B(E)$ as the cohomology of a differential graded (dg-) algebra as follows. We construct an injective resolution of G, $G \to I_\bullet$. Then $A = \operatorname{Hom}(I_\bullet, I_\bullet)$ is a dg-algebra whose cohomology if $B(E)$. We consider A as an A_∞-algebra with $m_n = 0$ for $n \geq 3$. Kadeishvili's paper [8] then implies that $B(E) = H^*A$ has a minimal A_∞-structure (m) such that m_2 is induced by the multiplication in A and $(B(E), m)$ is A_∞-quasi-isomorphic to A. (An A_∞-quasi-isomorphism is an A_∞-morphism such that f_1 is a quasi-isomorphism of complexes.) Moreover, such (m) is unique up to (non-unique) strict equivalence.

Thus $B(E)$ inherits an A_∞-structure. By comments in the Introduction, it is clear that the equivalence class of this A_∞-structure depends on E. Since $m_1 = 0$, the remarks in Section II.3. imply that $(B(E), m_2)$ is an associative algebra, and it is not hard to see that the isomorphism class of this associative algebra is independent of E. The discussion in Section II.5. implies that extensions of an associative algebra to an A_∞-algebra are governed by Hochschild cohomology in cohomological degree n and internal degrees $1 - n$, $2 - n$, and $3 - n$.

The goal of this chapter is calculate the Hochschild cohomology of $(B(E), m_2)$ in these internal degrees. The main result is Theorem III.4.1.

III.1. The associative algebra B

With notation as above, let $B = \mathrm{Ext}^*(G(E), G(E))$ considered as an associative algebra. Then B is a direct sum (as a $\mathbb{K}$-vector space) of components:

(i) $\mathrm{Hom}(\mathcal{O}, \mathcal{O})$ and $\mathrm{Hom}(\mathbb{K}(P), \mathbb{K}(P))$, both one-dimensional generated by the identity maps $\mathrm{id}_{\mathcal{O}}$, id_P;

(ii) $\mathrm{Hom}(\mathcal{O}, \mathbb{K}(P))$, a one-dimensional space, generated by a function θ;

(iii) $\mathrm{Ext}^1(\mathbb{K}(P), \mathcal{O})$, a one-dimensional space generated by a function η;

(iv) $\mathrm{Ext}^1(\mathbb{K}(P), \mathbb{K}(P))$ and $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O})$, both isomorphic to the one-dimensional space $H^1(\mathcal{O})$.

By Serre duality the products $\theta\eta = \xi \in \mathrm{Ext}^1(\mathcal{O}, \mathcal{O})$ and $\eta\theta = \psi \in \mathrm{Ext}^1(\mathbb{K}(P), \mathbb{K}(P))$ are nonzero, so we take ξ and ψ as generators of these spaces. For degree reasons all other products (except those involving the identities) are zero. Figure 1 gives a diagrammatic representation of B.

$$\begin{tikzcd}
\bullet \arrow[loop left, "{\mathrm{id}_{\mathcal{O}}, \xi[1]}"] \arrow[r, bend left, "\theta"] & \bullet \arrow[l, bend left, "{\eta[1]}"] \arrow[loop right, "{\mathrm{id}_P, \psi[1]}"]
\end{tikzcd}$$

Figure 1: Arrow diagram for B

B is a graded $\mathbb{K}$-algebra, $B = B_0 \oplus B_1$, where

$$B_0 = \langle \mathrm{id}_P, \mathrm{id}_{\mathcal{O}}, \theta \rangle, \ B_1 = \langle \eta, \xi, \psi \rangle.$$

Let $R = \langle \mathrm{id}_P, \mathrm{id}_{\mathcal{O}} \rangle \subset B$. Then $R \cong \mathbb{K} \times \mathbb{K}$ is a semi-simple subring of B, and we consider B as an R-algebra. As R-bimodules,

$$B \cong R \oplus B/R, \quad B/R \cong B_+ = \langle \theta, \eta, \xi, \psi \rangle,$$

so B is augmented over R in the sense of Section II.1..

III.2. A filtration of the reduced Hochschild cochain complex

Let $C^\bullet_R(B,B)$ be the Hochschild cochain complex of B with coefficients in B, reduced over R. The reader may refer to Section II.1. for definitions, conventions, and notation concerning Hochschild cohomology.

We consider the decreasing filtration on B as a B-bimodule,

$$F_0 = B \supset F_1 = B_+ \supset F_2 = \langle \xi, \psi \rangle \supset F_3 = 0.$$

For any fixed internal degree m, this provides a decreasing filtration of the Hochschild cochain complex, $C^\bullet_{(m)}(B)$, by Hochschild complexes of the sub-bimodules F_i,

$$C^\bullet_{(m)}(F_0) \supset C^\bullet_{(m)}(F_1) \supset C^\bullet_{(m)}(F_2) \supset 0.$$

We will use the spectral sequence on Hochschild cohomology associated to this filtration (see Section II.6.). Since the filtration is bounded, the sequence converges to $HH^\bullet_{(m)}(B)$. From here we suppress m in the notation when it will cause no confusion. Isomorphisms

$$C^\bullet(F_i)/C^\bullet(F_{i+1}) \cong C^\bullet(F_i/F_{i+1})$$

imply that on the zero page of this sequence we will compute $HH^n(F_i/F_{i+1})$ for $i = 0, 1, 2$.

III.2.1. $HH^\bullet(F_2)$ and $HH^\bullet(F_0/F_1)$

First, we reduce to a calculation on a subcomplex of the bar complex (see Section II.2.). Since $F_2 = \langle \xi \rangle \oplus \langle \psi \rangle$ as a B-bimodule,

$$C^\bullet(F_2) = C^\bullet(\langle \psi \rangle) \oplus C^\bullet(\langle \xi \rangle)$$

as complexes. For the graded B-bimodule $\langle \xi \rangle$ concentrated in degree 1, the concluding remark in Section II.2. implies that,

$$C^\bullet_{(m)}(\langle \xi \rangle) \cong [\mathrm{Bar}^{(1-m)}_\bullet(B)]^* \otimes \xi \cong [\mathrm{id}_{\mathcal{O}} \otimes \mathrm{Bar}^{(1-m)}_\bullet(B) \otimes \mathrm{id}_{\mathcal{O}}]^* \otimes \xi.$$

So we compute homology of the complex,

$$\mathrm{Bar}_\bullet(\mathcal{O}) := \mathrm{id}_{\mathcal{O}} \otimes_R \mathrm{Bar}_\bullet(B) \otimes_R \mathrm{id}_{\mathcal{O}}.$$

We interpret this complex as tensors of composable paths in Figure 1 starting and ending at the left vertex.

Proposition III.2.1. *Let $HB_\bullet^{(m)}(\mathcal{O})$ be the homology of the complex $\mathrm{Bar}_\bullet^{(m)}(\mathcal{O})$. Then,*

1. $\dim HB_n^{(n)}(\mathcal{O}) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$

2. $\dim HB_n^{(n-1)}(\mathcal{O}) = \begin{cases} 1 & \text{if } n = 3, 4, \\ 0 & \text{otherwise.} \end{cases}$

3. $\dim HB_n^{(n-2)}(\mathcal{O}) = \begin{cases} 1 & \text{if } n = 7, 8, \\ 0 & \text{otherwise.} \end{cases}$

4. $\dim HB_n^{(n-3)}(\mathcal{O}) = \begin{cases} 1 & \text{if } n = 11, 12, \\ 0 & \text{otherwise.} \end{cases}$

Remark III.2.2. *We use juxtaposition in place of $\otimes$ to represent tensors in $\mathrm{Bar}_\bullet(B)$.*

For $m \geq 0$, we consider the decreasing filtration on the complex $\mathrm{Bar}_\bullet^{(m)}(\mathcal{O})$,

$$F^0 \,\mathrm{Bar}_\bullet^{(m)}(\mathcal{O}) \supset F^1 \,\mathrm{Bar}_\bullet^{(m)}(\mathcal{O}) \supset \cdots,$$

where,

$$F^i \,\mathrm{Bar}_n^{(m)}(\mathcal{O}) = \langle \xi^{k_1}\theta\psi^{c_1}\eta\xi^{k_2}\cdots\eta\xi^{k_{n-m+1}} | \sum k_j \geq i \rangle.$$

For fixed n, m, the space $F^i \,\mathrm{Bar}_n^{(m)}(\mathcal{O}) = 0$ for $i \gg 0$; so the filtration is bounded and the spectral sequence of the filtration therefore converges to the homology of the complex. On the zero page of this sequence, there are complexes $(\mathrm{gr}_i\mathcal{O})_\bullet^{(m)}$ for each $i \geq 0$, where

$$(\mathrm{gr}_i\mathcal{O})_n^{(m)} = F^i \,\mathrm{Bar}_n^{(m)}(\mathcal{O}) / F^{i+1} \,\mathrm{Bar}_n^{(m)}(\mathcal{O}) \cong \langle \xi^{k_1}\theta\psi^{c_1}\eta\xi^{k_2}\cdots\theta\xi^{k_{n-m+1}} | \sum k_j = i \rangle.$$

We need the following lemma.

Lemma III.2.3. *Let $HB_\bullet^{(m)}(gr_i\mathcal{O})$ be the bar homology of the complex $(gr_i\mathcal{O})_\bullet^{(m)}$. Then,*

1. $HB_{n-1}^{(n-1)}(gr_{n-1}\mathcal{O}) = \langle \xi^{n-1} \rangle$;
2. $HB_{n}^{(n-1)}(gr_{n-2}\mathcal{O}) = \langle \xi^a\theta\eta\xi^b | a+b = n-2 \rangle$;
3. $HB_{n+1}^{(n-1)}(gr_{n-3}\mathcal{O}) = \langle \xi^a\theta\eta\xi^b\theta\eta\xi^c | a+b+c = n-3,\ b \neq 0 \rangle$;
4. $HB_{n+2}^{(n-1)}(gr_{n-4}\mathcal{O}) = \langle \xi^a\theta\eta\xi^b\theta\eta\xi^c\theta\eta\xi^d | a+b+c+d = n-4,\ b,c \neq 0 \rangle$;
5. $HB_{n+3}^{(n-1)}(gr_{n-5}\mathcal{O}) = \langle \xi^a\theta\eta\xi^b\theta\eta\xi^c\theta\eta\xi^d\theta\eta\xi^e | a+b+c+d+e = n-5,\ b,c,d \neq 0 \rangle$;
6. *These are the only nonzero spaces on page one in internal degree $n-1$.*

Proof. By Lemma A.1,

$$(\mathrm{gr}_i\mathcal{O})_\bullet^{(n-1)} = X_\bullet^{(n-1)} \oplus Y_\bullet^{(n-1)},$$

and the only homology of this complex is the space $X_l^{(n-1)}$. We therefore find the values of l and i such that $X_l^{(n-1)}$ is nonzero.

Suppose $\xi^{a_0}\theta\eta\xi^{a_1}\cdots\theta\eta\xi^{a_j} \in (\mathrm{gr}_i\mathcal{O})_l^{(n-1)}$. Then there are j factors of θ in this string, so the internal degree of this tensor is $l - j = n-1$; thus $j = l-n+1$. The sum of powers of ξ must be $i = l-2j$, so

$$i = l - 2(l-n+1) = 2n-l-2.$$

Substituting $l = n-1,\ n,\ n+1,\ n+2,\ n+3$ gives the values of i in the lemma. □

Proof of Proposition III.2.1. Lemma III.2.3 implies that on the first page of the spectral sequence there is one nontrivial complex,

$$\cdots \to HB_{n+3}^{(n-1)}(\mathrm{gr}_{n-5}\mathcal{O}) \to HB_{n+2}^{(n-1)}(\mathrm{gr}_{n-4}\mathcal{O}) \to HB_{n+1}^{(n-1)}(\mathrm{gr}_{n-3}\mathcal{O}) \to \tag{III.1}$$

$$HB_{n}^{(n-1)}(\mathrm{gr}_{n-2}\mathcal{O}) \to HB_{n-1}^{(n-1)}(\mathrm{gr}_{n-1}\mathcal{O}) \to 0. \tag{III.2}$$

For $n = 1$, only the last space in III.1 is nonzero, $HB_0^{(0)}(\mathrm{gr}_{n-1}\mathcal{O}) = \langle 1 \rangle$, so this space survives in the limit, proving part (1) for $n = 0$. From here we assume $n > 1$. Let d_1 be the differential in III.1. Since,

$$d_1(\xi^{k_1}\theta\eta\xi^{k_2}\cdots\eta\xi^{k_m}) = \sum_{i=1}^{m-1} \pm\xi^{k_1}\theta\cdots\xi^{k_{i-1}}\theta\eta\xi^{k_i+k_{i+1}+1}\cdots\eta\xi^{k_j},$$

we can represent the differential as,

$$(k_1,\ldots,k_j) \mapsto (k_1+k_2+1,k_3,\ldots,k_j) \pm\cdots\pm (k_1,\ldots,k_{j-1}+k_j+1).$$

We map the complex III.1 to a subcomplex of the simplex as in the proof of Lemma A.1. We make the change of variable $k_i' = k_i$, then map,

$$(k_1,\cdots,k_j) \mapsto (k_1',\cdots,k_j') \mapsto (k_1',k_1'+k_2',\cdots,\sum_{i=1}^{j-1} k_i').$$

In our case, $\sum k_i = n-j$; thus $\sum_{i=1}^{j-1} k_i' \leq n-1$, so consider this a map to the $(n-1)$-simplex. Since we assume $k_i \geq 1$ for $2 \leq i \leq j-1$, it follows that the corresponding $k_i' \geq 2$; thus the image of the map consists of the subcomplex of the $(n-1)$-simplex in which we require that the difference between adjacent vertices be at least 2. So our complex maps isomorphically to the dimension 3, 2, 1, 0, -1 part of the simplicial complex $\Delta[n-1]$ from Appendix B.

From Proposition B.1, the resulting simplicial complex has no homology in dimension -1, so III.1 has none in external degree $n-1$ when $n > 1$. The simplicial complex has one-dimensional homology: (reduced) in dimension 0 for $n-1 = 2,3$, so III.1 has one-dimensional homology in dimension n for $n = 3,4$; in dimension 1 for $n-1 = 5,6$, so III.1 has one-dimensional homology in dimension $n+1$ for $n+1 = 7,8$; and in dimension 2 for $n-1 = 8,9$, so III.1 has one-dimensional homology in dimension $n+2$ for $n+2 = 11,12$. This completes the claim. □

The correspondence with the simplicial complex allows us to find explicit representatives of all classes. In principle, the class representatives in Appendix B only correspond to homology classes on page one of the sequence and have no direct connection to the classes in $HB_\bullet(\mathcal{O})$. However, the proof of the proposition shows that the complex III.1 can be viewed directly as a subcomplex of $\mathrm{Bar}_\bullet(\mathcal{O})$. Thus we extract directly from the simplicial correspondence the representatives listed in the second column of Table 1.

Space	Simplicial representative	Simplified representative
$HB_3^{(2)}(\mathcal{O})$	$\sigma_3^{(2)} := \theta\eta\xi + \xi\theta\eta$	$\theta\psi\eta$
$HB_4^{(3)}(\mathcal{O})$	$\sigma_3^{(2)} \otimes \xi$	$\theta\psi\eta\xi \sim \xi\theta\psi\eta$
$HB_7^{(5)}(\mathcal{O})$	$\sigma_7^{(5)} := \theta\eta\xi^2\theta\eta\xi + \theta\eta\xi^3\theta\eta + \xi\theta\eta\xi\theta\eta\xi + \xi\theta\eta\xi^2\theta\eta$	$\theta\psi\eta\xi\theta\psi\eta$
$HB_8^{(6)}(\mathcal{O})$	$\sigma_7^{(5)} \otimes \xi$	$\theta\psi\eta\xi\theta\psi\eta\xi \sim \xi\theta\psi\eta\xi\theta\psi\eta$
$HB_{11}^{(8)}(\mathcal{O})$	$\sigma_{11}^{(8)} := \theta\eta\xi^3\theta\eta\xi\theta\eta\xi + \theta\eta\xi^2\theta\eta\xi^2\theta\eta\xi + \xi\theta\eta\xi^2\theta\eta\xi\theta\eta\xi +$ $\xi\theta\eta\xi\theta\eta\xi^2\theta\eta\xi + \theta\eta\xi^3\theta\eta\xi^2\theta\eta + \theta\eta\xi^2\theta\eta\xi^3\theta\eta +$ $\xi\theta\eta\xi^2\theta\eta\xi^2\theta\eta + \xi\theta\eta\xi\theta\eta\xi^3\theta\eta$	$\theta\psi\eta\xi\theta\psi\eta\xi\theta\psi\eta$
$HB_{12}^{(9)}(\mathcal{O})$	$\sigma_{11}^{(8)} \otimes \xi$	$\theta\psi\eta\xi\theta\psi\eta\xi\theta\psi\eta\xi \sim$ $\xi\theta\psi\eta\xi\theta\psi\eta\xi\theta\psi\eta$

Table 1: Representatives of $HB_\bullet(\mathcal{O})$

The simplified representative is a more algebraically manageable homolog of the representative taken directly from the simplicial correspondence. We show the calculation that $\sigma_7^{(5)}$ is homologous to $\theta\psi\eta\xi\theta\psi\eta$ as an example of the simplification procedure. Let d be the bar differential. Then

$$\begin{aligned}
x_1 = d(\theta\eta\theta\eta\xi^2\theta\eta) &= \xi\theta\eta\xi^2\theta\eta - \theta\psi\eta\xi^2\theta\eta + \theta\eta\xi^3\theta\eta + \theta\eta\theta\eta\xi^3, \\
x_2 = d(\theta\eta\theta\eta\xi\theta\eta\xi) &= \xi\theta\eta\xi\theta\eta\xi - \theta\psi\eta\xi\theta\eta\xi + \theta\eta\xi^2\theta\eta\xi - \theta\eta\theta\eta\xi^3, \\
x_3 = d(\theta\psi\eta\xi\theta\eta\theta\eta) &= \theta\psi\eta\xi^2\theta\eta - \theta\psi\eta\xi\theta\psi\eta + \theta\psi\eta\xi\theta\eta\xi, \\
\theta\psi\eta\xi\theta\psi\eta &= \sigma_7^{(5)} - x_1 - x_2 - x_3.
\end{aligned}$$

Let

$$\mathrm{Bar}_\bullet(P) := \mathrm{id}_P \otimes_R \mathrm{Bar}_\bullet(B) \otimes_R \mathrm{id}_P.$$

We interpret this complex as tensors of composable paths in Figure 1 starting and ending at the right vertex.

Corollary III.2.4. *Let $HB^{(m)}_\bullet(P)$ be the homology of the complex* $\mathrm{Bar}^{(m)}_\bullet(P)$*. Then,*

1. $\dim HB^{(n)}_n(P) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$

2. $\dim HB^{(n-1)}_n(P) = \begin{cases} 1 & \text{if } n = 3, 4, \\ 0 & \text{otherwise.} \end{cases}$

3. $\dim HB^{(n-2)}_n(P) = \begin{cases} 1 & \text{if } n = 7, 8, \\ 0 & \text{otherwise.} \end{cases}$

4. $\dim HB^{(n-3)}_n(P) = \begin{cases} 1 & \text{if } n = 11, 12, \\ 0 & \text{otherwise.} \end{cases}$

Proof. There is an isomorphism of complexes $\mathrm{Bar}^{(n-1)}_\bullet(\mathcal{O}) \to \mathrm{Bar}^{(n-1)}_\bullet(P)$ induced by the following map on B_+:

$$\eta \mapsto \theta,\ \theta \mapsto \eta,\ \xi \mapsto \psi,\ \psi \mapsto \xi.$$

$\square$

This map of complexes provides the representatives of classes listed in Table 2.

Space	Simplified representative
$HB^{(2)}_3(P)$	$\eta\xi\theta$
$HB^{(3)}_4(P)$	$\eta\xi\theta\psi \sim \psi\eta\xi\theta$
$HB^{(5)}_7(P)$	$\eta\xi\theta\psi\eta\xi\theta$
$HB^{(6)}_8(P)$	$\eta\xi\theta\psi\eta\xi\theta\psi \sim \psi\eta\xi\theta\psi\eta\xi\theta$
$HB^{(8)}_{11}(P)$	$\eta\xi\theta\psi\eta\xi\theta\psi\eta\xi\theta$
$HB^{(9)}_{12}(P)$	$\eta\xi\theta\psi\eta\xi\theta\psi\eta\xi\theta\psi \sim \psi\eta\xi\theta\psi\eta\xi\theta\psi\eta\xi\theta$

Table 2: Representatives of $HB_\bullet(P)$.

Corollary III.2.5. *1.* $\dim HH^n_{(1-n)}(F_2) = \dim HH^n_{(-n)}(F_0/F_1) = \begin{cases} 2 & \textit{if } n = 0 \\ 0 & \textit{otherwise,} \end{cases}$

2. $\dim HH^n_{(2-n)}(F_2) = \dim HH^n_{(1-n)}(F_0/F_1) = \begin{cases} 2 & \textit{if } n = 3,4 \\ 0 & \textit{otherwise,} \end{cases}$

3. $\dim HH^n_{(3-n)}(F_2) = \dim HH^n_{(2-n)}(F_0/F_1) = \begin{cases} 2 & \textit{if } n = 7,8 \\ 0 & \textit{otherwise,} \end{cases}$

4. $\dim HH^n_{(4-n)}(F_2) = \dim HH^n_{(3-n)}(F_0/F_1) = \begin{cases} 2 & \textit{if } n = 11,12 \\ 0 & \textit{otherwise.} \end{cases}$

Proof. There are isomorphisms $C^\bullet_{(1-m)}(\langle\psi\rangle) \cong [\mathrm{Bar}^{(m)}_\bullet(P)]^*$ and $C^\bullet_{(1-m)}(\langle\xi\rangle) \cong [\mathrm{Bar}^{(m)}_\bullet(\mathcal{O})]^*$, according to the duality of Section II.2.. Since $F_2 = \langle\psi\rangle \oplus \langle\xi\rangle$, this is the claim for F_2. There is an isomorphism of B-bimodules $F_0/F_1 \cong F_2[1]$, so the result for F_0/F_1 differs by 1 in the internal degree. ☐

III.2.2. $HH^\bullet(F_1/F_2)$

Again we reduce to a calculation on the bar complex since the first and last terms of the cohomology differential in $C^\bullet(F_1/F_2)$ vanish. As B-bimodules, there is an isomorphism

$$F_1/F_2 \cong (F_2 + \langle\theta\rangle)/F_2 \oplus (F_2 + \langle\eta\rangle)/F_2 := V_\theta \oplus V_\eta.$$

We begin by calculating $HH^\bullet_{(-m)}(V_\theta)$, which is dual to,

$$\mathrm{Bar}^{(m)}_\bullet(\theta) := \mathrm{id}_{\mathcal{O}} \otimes_R \mathrm{Bar}^{(m)}_\bullet(B) \otimes_R \mathrm{id}_P \subset \mathrm{Bar}^{(m)}_n(B).$$

We interpret tensors in this complex as paths in Figure 1 starting at the left vertex and ending at the right.

Proposition III.2.6. *Let $HB^{(m)}_\bullet(\theta)$ be the homology of the complex $\mathrm{Bar}^{(m)}_\bullet(\theta)$. Then,*

1. $\dim HB^{(n-1)}_n(\theta) = \begin{cases} 1 & \text{if } n = 1, 2 \\ 0 & \text{otherwise,} \end{cases}$

2. $\dim HB^{(n-2)}_n(\theta) = \begin{cases} 1 & \text{if } n = 5, 6 \\ 0 & \text{otherwise,} \end{cases}$

3. $\dim HB^{(n-3)}_n(\theta) = \begin{cases} 1 & \text{if } n = 9, 10 \\ 0 & \text{otherwise.} \end{cases}$

We mimic the procedure from Section III.2.1.. For $m, i \geq 0$, we consider the decreasing filtration such that,

$$F^i \,\mathrm{Bar}^{(m)}_n(\theta) = \langle \xi^{k_1}\theta\psi^{c_1}\cdots\xi^{k_{n-m}}\theta\psi^{c_{n-m}} | \sum k_j \geq i\rangle,$$
$$(\mathrm{gr}_i\theta)^{(m)}_n = F^i \,\mathrm{Bar}^{(m)}_n(\theta)/F^{i+1}\,\mathrm{Bar}^{(m)}_n(\theta) = \langle \xi^{k_1}\theta\psi^{c_1}\cdots\xi^{k_{n-m}}\theta\psi^{c_{n-m}} | \sum k_j = i\rangle.$$

Lemma III.2.7. *Let $HB^{(m)}_\bullet(gr_i\theta)$ be the bar homology of the complex $(gr_i\theta)^{(m)}_\bullet$. Let $n \geq 2$. Then,*

1. $HB^{(n-2)}_{n-1}(gr_{n-2}\theta) = \langle \xi^{n-2}\theta\rangle$;
2. $HB^{(n-2)}_{n}(gr_{n-3}\theta) = \langle \xi^{a}\theta\eta\xi^b\theta | a+b = n-3,\ b \neq 0\rangle$;
3. $HB^{(n-2)}_{n+1}(gr_{n-4}\theta) = \langle \xi^{a}\theta\eta\xi^b\theta\eta\xi^c\theta | a+b+c = n-4,\ b, c \neq 0\rangle$;
4. $HB^{(n-2)}_{n+2}(gr_{n-5}\theta) = \langle \xi^{a}\theta\eta\xi^b\theta\eta\xi^c\theta\eta\xi^d\theta | a+b+c+d = n-5,\ b, c, d \neq 0\rangle$;
5. $HB^{(n-2)}_{k}(gr_i\theta) = 0$ *for* $k = n-1, n, n+1, n+2$ *and all other values of* i.

Proof. This follows as in Lemma III.2.3. □

Proof of Proposition III.2.6. Lemma III.2.7 implies that on the first page of the spectral sequence of the given filtration we have one nontrivial complex,

$$\cdots \to HB^{(n-2)}_{n+2}(\mathrm{gr}_{n-5}\theta) \to HB^{(n-2)}_{n+1}(\mathrm{gr}_{n-4}\theta) \to HB^{(n-2)}_{n}(\mathrm{gr}_{n-3}\theta) \to HB^{(n-2)}_{n-1}(\mathrm{gr}_{n-2}\theta) \to 0.$$

For $n = 2, 3$ we have only $HB_{n-1}^{(n-2)} \neq 0$ in the above sequence, so these spaces survive when we pass to the limit. This gives the result in (1) after a change of variable in n.

For $n \geq 4$, we map to the simplex as in the proof of Proposition III.2.1 to recover the simplicial complex from Appendix B; the image is the complex $\Delta[n-3]$ ($n-3$ since $\sum_{i=1}^{j} k'_i = n-1$ and $k'_j \geq 2$.) According to Proposition B.1, the simplicial complex has one-dimensional homology: (reduced) in dimension 0 for $n-3 = 2, 3$, so the original complex has one-dimensional homology in dimension n for $n = 5, 6$; in dimension 1 for $n-3 = 5, 6$, so the original complex has one-dimensional homology in dimension $n+1$ for $n+1 = 9, 10$. This gives the result in (2), (3) after a change of variable in n. $\square$

The simplicial correspondence also provides representatives of classes, listed in Table 3.

Space	Simplified representative
$HB_1^{(0)}(\theta)$	θ
$HB_2^{(1)}(\theta)$	$\xi\theta \sim \theta\psi$
$HB_5^{(3)}(\theta)$	$\theta\psi\eta\xi\theta$
$HB_6^{(4)}(\theta)$	$\xi\theta\psi\eta\xi\theta \sim \theta\psi\eta\xi\theta\psi$
$HB_9^{(6)}(\theta)$	$\theta\psi\eta\xi\theta\psi\eta\xi\theta$
$HB_{10}^{(7)}(\theta)$	$\xi\theta\psi\eta\xi\theta\psi\eta\xi\theta \sim \theta\psi\eta\xi\theta\psi\eta\xi\theta\psi$

Table 3: Representatives of $HB_\bullet(\theta)$

Corollary III.2.8. *1.* $\dim HH^n_{(1-n)}(V_\theta) = \begin{cases} 1 & \text{if } n = 1, 2 \\ 0 & \text{otherwise,} \end{cases}$

2. $\dim HH^n_{(2-n)}(V_\theta) = \begin{cases} 1 & \text{if } n = 5, 6 \\ 0 & \text{otherwise,} \end{cases}$

3. $\dim HH^n_{(3-n)}(V_\theta) = \begin{cases} 1 & \text{if } n = 9, 10 \\ 0 & \text{otherwise.} \end{cases}$

Proof. There is an isomorphism $C^\bullet_{(-m)}(V_\theta) \cong [\mathrm{Bar}_\bullet^{(m)}(\theta)]^*$ according to the duality of Section II.2.. $\square$

Finally define,

$$\mathrm{Bar}_\bullet^{(m)}(\eta) := \mathrm{id}_P \otimes \mathrm{Bar}_\bullet^{(m)}(B) \otimes \mathrm{id}_{\mathcal{O}}.$$

We interpret tensors in this complex as paths in Figure 1 starting at the left vertex and ending at the right.

Corollary III.2.9. *Let* $HB_\bullet^{(m)}(\eta)$ *be the homology of* $\mathrm{Bar}_\bullet^{(m)}(\eta)$. *Then,*

1. $\dim HB_n^{(n)}(\eta) = \dim HH^n_{(1-n)}(V_\eta) = \begin{cases} 1 & \text{if } n = 1,2 \\ 0 & \text{otherwise,} \end{cases}$

2. $\dim HB_n^{(n-1)}(\eta) = \dim HH^n_{(2-n)}(V_\eta) = \begin{cases} 1 & \text{if } n = 5,6 \\ 0 & \text{otherwise,} \end{cases}$

3. $\dim HB_n^{(n-2)}(\eta) = \dim HH^n_{(3-n)}(V_\eta) = \begin{cases} 1 & \text{if } n = 9,10 \\ 0 & \text{otherwise.} \end{cases}$

Proof. There is an isomorphism of complexes $\mathrm{Bar}_\bullet^{(m-1)}(\theta) \cong \mathrm{Bar}_\bullet^{(m)}(\eta)$ induced by the following map on B_+:

$$\eta \mapsto \theta,\ \theta \mapsto \eta,\ \psi \mapsto \xi,\ \xi \mapsto \psi.$$

Finally there is an isomorphism $C^\bullet_{(1-m)}(V_\eta) \cong [\mathrm{Bar}_\bullet^{(m)}(\eta)]^*$. □

There are homology representatives listed in Table 4.

Space	Representative
$HB_1^{(1)}(\eta)$	η
$HB_2^{(2)}(\eta)$	$\psi\eta \sim \eta\xi$
$HB_5^{(4)}(\eta)$	$\eta\xi\theta\psi\eta$
$HB_6^{(5)}(\eta)$	$\psi\eta\xi\theta\psi\eta \sim \eta\xi\theta\psi\eta\xi$
$HB_9^{(7)}(\eta)$	$\eta\xi\theta\psi\eta\xi\theta\psi\eta$
$HB_{10}^{(8)}(\eta)$	$\psi\eta\xi\theta\psi\eta\xi\theta\psi\eta \sim \eta\xi\theta\psi\eta\xi\theta\psi\eta\xi$

Table 4: Representatives of $HB_\bullet(\eta)$

Since $F_1/F_2 \cong V_\theta \oplus V_\eta$ as R-bimodules, it follows that $HH^n_{(m)}(F_1/F_2) = HH^n_{(m)}(V_\theta) \oplus HH^n_{(m)}(V_\eta)$.

III.3. Calculating differentials in the spectral sequence

We piece together the spectral sequence on Hochschild cohomology associated with the filtration of B given in the last section. We can understand all maps on the first page. Let $\delta_1^{i,n-i} : HH^n(F_i/F_{i+1}) \to HH^{n+1}(F_{i+1}/F_{i+2})$ be the differential on page one. We will write simply δ_1 if superscript information is understood.

Lemma III.3.1. *(1) $\delta_1(HH^n(F_2)) = 0$ for all n;*

(2) $\delta_1^{0,4j} : HH^{4j}_{(-3j)}(F_0/F_1) \to HH^{4j+1}_{(-3j)}(F_1/F_2)$ has rank 1 for $j = 0, 2$ and rank 2 for $j = 1, 3$;

(3) $\delta_1(HH^{4j+1}_{(-3j)}(F_1/F_2)) = 0$ for $j = 0, 1, 2$;

(4) $\delta_1^{1,4j+1} : HH^{4j+2}_{(-3j-1)}(F_1/F_2) \to HH^{4j+3}_{(-3j-1)}(F_2)$ has rank 1 for $j = 1$ and rank 2 for $j = 0, 2$;

(5) $\delta_1(HH^{4j+3}_{(-3j-3)}(F_0/F_1)) = 0$ for $j = 0, 1, 2$.

Proof. (1) This is a trivial observation about the spectral sequence since $F_3 = 0$.

(2) Note that here δ_1 is a map between two-dimensional spaces. Let x, x' be the representatives of classes in $HB^{(3j)}_{4j}(\mathcal{O})$ from Table 1, respectively; let y, y' be the representatives of classes in $HB^{(3j)}_{4j}(P)$ from Table 2. Then

$$\alpha := [x]^* \otimes \mathrm{id}_{\mathcal{O}} = [x']^* \otimes \mathrm{id}_{\mathcal{O}},\ \beta := [y]^* \otimes \mathrm{id}_P = [y']^* \otimes \mathrm{id}_P$$

generate $HH^{4j}_{(-3j)}(F_0/F_1)$. From Table 3, the class of $\omega_1 := x \otimes \theta = \theta \otimes y'$ generates $HB^{(3j)}_{4j+1}(\theta)$ and the class of $\omega_2 = \eta \otimes x' = y \otimes \eta$ generates $HB^{(3j+1)}_{4j+1}(\eta)$. Thus $[\omega_1]^* \otimes \theta$

and $[\omega_2]^* \otimes \eta$ generate $HH^{4j+1}_{(-3j)}(F_1/F_2)$. Then we calculate,

$$\begin{aligned}
\delta_1(\alpha)(\omega_1) &= \delta([x]^* \otimes \mathrm{id}_{\mathcal{O}})([x \otimes \theta]) \\
&= -[x]^*([x]) \cdot \mathrm{id}_{\mathcal{O}} \cdot \theta = -\theta, \\
\delta_1(\alpha)(\omega_2) &= \delta([x']^* \otimes \mathrm{id}_{\mathcal{O}})(\eta \otimes x') \\
&= (-1)^{-3j}\eta \cdot [x']^*([x']) \cdot \mathrm{id}_{\mathcal{O}} = (-1)^j \eta, \\
\delta_1(\beta)(\omega_1) &= \delta([y']^* \otimes \mathrm{id}_L)([\theta \otimes y']) \\
&= \theta \cdot [y']^*([y']) \cdot \mathrm{id}_L = \theta, \\
\delta_1(\beta)(\omega_2) &= \delta([y]^* \otimes \mathrm{id}_L)([y \otimes \eta]) \\
&= -[y]^*([y]) \cdot \mathrm{id}_L \cdot \eta = -\eta.
\end{aligned}$$

We conclude that $\delta_1(\alpha + \beta) = 0$ when j is even, so $\delta_1^{0,4j}$ has rank 1 in that case. When j is odd, we see that there is no kernel.

(3) For $\phi \in HH^{4j+1}_{(-3j)}(F_0/F_1)$, we have $\delta_1\phi \in HH^{4j+1}_{(-3j)}(F_2)$. But $HH^{4j+1}_{(-3j)}(F_2) = 0$ by Lemma III.2.5.

(4) Let x, x' be the representatives of the class in $HB^{(3j+1)}_{4j+2}(\theta)$, and y, y' be the representatives of the class in $HB^{(3j+2)}_{4j+2}(\eta)$ from Tables 3 and 4. Then the representative of the class in $HB^{(3j+2)}_{4j+3}(\mathcal{O})$ is $\theta \otimes y = x' \otimes \eta$ and the representative of the class in $HB^{(3j+2)}_{4j+3}(P)$ is $y' \otimes \theta = \eta \otimes x$. So it follows as in part (2) that $\delta_1^{1,4j+1}$ has rank 1 when $3j+1$ is even (i.e., j is odd) and rank 2 otherwise.

(5) This follows since $HH^{4j+4}_{(-3j-3)}(F_1/F_2) = 0$.

□

On page two, the only nonzero maps are $\delta_2^{0,n} : \ker \delta_1^{0,n} \to \operatorname{coker} \delta_1^{1,n-1}$. We see from Lemma III.3.1 that $\ker \delta_1^{0,n}$ is one-dimensional for $n = 4j$ in internal degree $-3j$, and two-dimensional for $n = 4j+3$ in internal degree $-3j-3$.

Lemma III.3.2. *(1)* $\delta_2^{0,4j} = 0$

(2) $\operatorname{rank} \delta_2^{0,4j+3} = 2$.

Proof. (1) Since $HH^{4j+1}_{(-3j)}(F_2)=0$, and thus $\operatorname{coker}\delta_1^{1,4j-1}=0$, so this is trivial.

(2) From Lemma III.3.1 we know that $\ker\delta_1^{0,4j+3}=HH^{4j+3}_{(-3j-3)}(F_0/F_1)$ and $\operatorname{coker}\delta_1^{1,4j+2}=HH^{4j+4}_{(-3j-3)}$. Let x be the representative of $HB^{(3j+3)}_{4j+3}(\mathcal{O})$ from Table 1, and let y be the representative of $HB^{(3j+3)}_{4j+3}(P)$ from Table 2. So $[x]^*\otimes \mathrm{id}_{\mathcal{O}}, [y]^*\otimes \mathrm{id}_P$ generate $HH^{4j+3}_{(-3j-3)}(F_0/F_1)$. Then $\omega_1=[\xi\otimes x]=[x\otimes\xi]$ generates $HB^{(3j+4)}_{4j+4}(\mathcal{O})$ and $\omega_2=[\psi\otimes y]=[y\otimes\psi]$ generates $HB^{(3j+4)}_{4j+4}(P)$. So we calculate,

$$\delta_2([x]^*\otimes\mathrm{id}_{\mathcal{O}})(\omega_1)=\pm\xi\cdot[x]^*([x])\cdot\mathrm{id}_{\mathcal{O}}=\pm\xi,$$
$$\delta_2([x]^*\otimes\mathrm{id}_{\mathcal{O}})(\omega_2)=0,$$
$$\delta_2([y]^*\otimes\mathrm{id}_{P})(\omega_1)=0,$$
$$\delta_2([y]^*\otimes\mathrm{id}_{P})(\omega_2)=\pm\psi\cdot[y]^*([y])\cdot\mathrm{id}_{P}=\pm\psi.$$

Thus $\delta_2^{0,4j+3}$ has rank 2.

∎

III.4. Hochschild cohomology of B

Theorem III.4.1. *The Hochschild cohomology of B for maps in internal degrees $1-n$, $2-n$, and $3-n$ is,*

1. $\dim HH^n_{(1-n)}(B)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{otherwise,}\end{cases}$
2. $\dim HH^n_{(2-n)}(B)=\begin{cases}1 & \text{if } n=6,8\\ 0 & \text{otherwise,}\end{cases}$
3. $\dim HH^n_{(3-n)}(B)=\begin{cases}1 & \text{if } n=7,9\\ 0 & \text{otherwise.}\end{cases}$

Proof. • First we fix internal degree $-3j$, for $j = 1, 2, 3$. We know,

$$\dim HH^{4j}_{(-3j)}(F_0/F_1) = \dim HH^{4j+1}_{(-3j)}(F_1/F_2) = 2,$$
$$HH^{4j+2}_{(-3j)}(F_i/F_{i+1}) = HH^{4j-1}_{(-3j)}(F_i/F_{i+1}) = 0,$$

for all i from Corollaries III.2.5, III.2.8, and III.2.9. Now $\operatorname{rank} \delta_1^{0,4j} = 1$ when j is even from Lemma III.3.1 part (2), and the sequence degenerates at this point since $HH^{4j+1}_{(-3j)}(F_2) = 0$. It follows that $\dim HH^{4j}_{(-3j)}(B) = \dim HH^{4j+1}_{(-3j)}(B) = 1$ when j is even. When j is odd, $\delta_1^{0,4j}$ is an isomorphism and the sequence also degenerates here; so in that case there is no cohomology.

- We fix internal degree $-3j - 1$ for $j = 0, 1, 2$. The same corollaries show

$$\dim HH^{4j+2}_{(-3j-1)}(F_1/F_2) = \dim HH^{4j+3}_{(-3j-1)}(F_2) = 2,$$
$$HH^{4j+1}_{(-3j-1)}(F_i/F_{i+1}) = HH^{4j+4}_{(-3j-1)}(F_i/F_{i+1}) = 0.$$

Lemma III.3.1 part (4) shows that $delta_1^{1,4j+1}$ has rank 1 when j is odd and is an isomorphism when j is even. The sequence degenerates at this point since $HH^{4j+2}_{(-3j-1)}(F_0/F_1) = 0$. So $\dim HH^{4j+2}_{(-3j-1)}(B) = \dim HH^{4j+3}_{(-3j-3)}(B) = 1$ when j is odd and that all cohomology vanishes when j is even.

- We fix internal degree $-3j - 2$ for $j = 0, 1, 2$. Then Lemma III.3.1 part (5) shows that the nontrivial map on page two is a map between two-dimensional spaces; Lemma III.3.2 shows that map has rank 2. Therefore in the limit there is no cohomology. □

CHAPTER IV

THE A_∞-ALGEBRA OF A COMPLEX ELLIPTIC CURVE AND THE J-INVARIANT

Let $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be a complex elliptic curve (in particular $\operatorname{Im}\tau > 0$); P a closed point on E_τ; $L = \mathcal{O}(P)$ the line bundle of degree 1; and $A(E_\tau) = \operatorname{Ext}^*(\mathcal{O} \oplus L, \mathcal{O} \oplus L)$. Let B be the associative algebra from Chapter III, $B(E_\tau) = \operatorname{Ext}^*(\mathcal{O} \oplus \mathbb{K}(P), \mathcal{O} \oplus \mathbb{K}(P))$.

In [17], Polishchuk calculates the A_∞-structure on $A(E_\tau)$ arising from using the Dolbeault complex to compute Ext. An obvious isomorphism of algebras $B \cong (A(E_\tau), m_2)$ naturally gives an A_∞-structure $(m(\tau)) = \{m_n(\tau)\}_{n=1}^{\infty}$ on $B(E_\tau)$. We suppress τ when no confusion will arise.

In this chapter we construct a family of strict A_∞-equivalences

$$(f_\tau) : B(E_\tau) \to B(E_\tau),$$

indexed by τ. The (f_τ) are constructed so that if

$$(m'(\tau)) = f * (m(\tau)),$$

we have that $m'_6(\tau)$ and $m'_8(\tau)$ are Hochschild cocycles (Proposition IV.1.2). We then construct an explicit $\mathbb{C}$-linear isomorphism,

$$HH^6_{(-4)}(B) \oplus HH^8_{(-6)}(B) \xrightarrow{\sim} \mathbb{C}^2,$$

which allows us to recover the j-invariant of E_τ from the point $(m'_6(\tau), m'_8(\tau))$ (Theorem IV.2.3).

IV.1. An A_∞-structure on B

Fix a curve E. Recall definitions and results concerning Eisenstein series from Section II.7..

Theorem IV.1.1. *([17], 2.5.1) The only non-trivial higher products m_n on $B(E)$ are of the form,*

$$m_n(\xi^a\theta\psi^b\eta\xi^c\theta\psi^d) = M(a,b,c,d)\cdot\theta,$$
$$m_n(\psi^a\eta\xi^b\theta\psi^c\eta\xi^d) = M(a,b,c,d)\cdot\eta,$$
$$m_n(\xi^a\theta\psi^b\eta\xi^c\theta\psi^d\eta\xi^e) = M(a+e+1,b,c,d)\cdot id_{\mathcal{O}},$$
$$m_n(\psi^a\eta\xi^b\theta\psi^c\eta\xi^d\theta\psi^e) = M(a+e+1,b,c,d)\cdot id_P.$$

All products m_n with odd n vanish.

Since $m_3 = 0$, the A_∞-relation of order 5 implies that m_4 is a Hochschild cocycle. In Theorem III.4.1 we conclude that $HH^4_{(-2)}(B) = 0$; so m_4 is a coboundary and therefore by Lemma II.5.2, all choices of m_4 are related by some strict A_∞-equivalence. Thus even though $m_4 \neq 0$ in Theorem IV.1.1, there must exist some strict equivalence $f : B \to B$ such that $f * m_4 = 0$. Indeed, we take $f_1 = \mathrm{id}_B$, f_3 to be such that $\delta f_3 = m_4$, and $f_n = 0$ otherwise. Moreover since $HH^3_{(-2)}(B) = 0$, the choice of f_3 in this equivalence is unique up to homotopy. We take as f_3,

$$\begin{aligned} f_3 =& M(1,0,0,0)[([\eta\xi^2]^* - [\psi^2\eta]^* - [\psi\eta\xi]^*)\otimes\eta + ([\theta\psi^2]^* - [\xi^2\theta]^* - [\xi\theta\psi]^*)\otimes\theta + \\ & ([\xi\theta\eta]^* + [\theta\psi\eta]^* - [\theta\eta\xi]^*)\otimes\mathrm{id}_{\mathcal{O}} + ([\psi\eta\theta]^* + [\eta\xi\theta]^* - [\eta\theta\psi]^*)\otimes\mathrm{id}_P].\end{aligned}$$

Proposition IV.1.2. *Let $(m') = f * (m)$. Then,*

1. *$m'_k = 0$ for k odd and $m'_2 = m_2$;*
2. *$m'_4 = 0$; and*
3. *m'_6 and m'_8 are Hochschild cocycles.*

Proof. 1. The relation in Equation II.2 of order 1 shows immediately that $m'_1 = 0$, after which the relations of order 2 and 3 show that $m'_2 = m_2$ and $m'_3 = 0$. We proceed by induction in k to show that $m'_{2k+1} = 0$.

Assume $m_{2j+1} = 0$ for $j \leq k$, and consider the relation of order $2k+3$. This relation reduces to the equation,

$$f_1 m_{2k+3} + f_3(m_{2k+1} \otimes \mathbf{1}^{\otimes 2} + \mathbf{1} \otimes m_{2k+1} \otimes \mathbf{1} + \mathbf{1}^{\otimes 2} \otimes m_{2k+1}) = m'_{2k+3}.$$

Other terms on the left vanish since only f_1 and f_3 are nonzero. Other terms on the right vanish by the induction assumption: in order for $i_1 + \cdots + i_r = 2k+3$ where all i_j are 1 or 3, we must have r odd, and $m'_r = 0$ for r odd and $r < 2k+3$. By Theorem IV.1.1 we know that $m_{2k+1} = m_{2k+3} = 0$, so the left side vanishes completely, implying $m'_{2k+3} = 0$.

2. The morphism relation of order 4 reduces to

$$\delta f_3 = m_4 - m'_4.$$

The claim that $\delta f_3 = m_4$ can be verified by direct calculation, which implies that $m'_4 = 0$.

3. The A_∞-relations for (m'_n) give

$$\delta m'_6 = \Phi_6(m'_3, m'_4, m'_5), \tag{IV.1}$$

$$\delta m'_8 = \Phi_8(m'_3, m'_4, m'_5, m'_6, m'_7), \tag{IV.2}$$

where Φ_k is a quadratic expression. The right side of Equation IV.1 is zero since $m'_3 = m'_4 = m'_5 = 0$. The right side of Equation IV.2 is zero since only $m'_6 \neq 0$, and m'_6 is paired with m'_4 in this quadratic expression.

□

We determine from the morphism relations of order 6 and 8 that,

$$m_6' = m_6 + f_3\left(\sum_{r+s=2} \mathbf{1}^{\otimes r} \otimes m_4 \otimes \mathbf{1}^{\otimes s}\right) - m_2(f_3 \otimes f_3), \tag{IV.3}$$

$$m_8' = m_8 + f_3\left(\sum_{r+s=2} \mathbf{1}^{\otimes r} \otimes m_6 \otimes \mathbf{1}^{\otimes s}\right) - m_6'\left(\sum_{r+s=5} \mathbf{1}^{\otimes r} \otimes f_3 \otimes \mathbf{1}^{\otimes s}\right). \tag{IV.4}$$

IV.2. Recovering the j-invariant

We construct an explicit isomorphism,

$$HH^6_{(-4)}(B) \oplus HH^8_{(-6)}(B) \xrightarrow{\sim} \mathbb{C}^2$$

from which we extract the j-invariant of E. Let S be our standard basis of B as a $\mathbb{K}$-algebra. For every $x \in B_+^{\otimes n}$ and $y \in S$, we define an R-linear function $\mathrm{ev}_x^y : \mathrm{Hom}_R(B_+^{\otimes n}, B) \to \mathbb{C}$, where $\mathrm{ev}_x^y(\phi)$ is the coefficient on y in the expansion of $\phi(x)$ in the basis S. Let (f') be another strict equivalence with $f_i' = 0$ for $i \neq 1, 3$ and $\delta f_3' = m_4$.

Proposition IV.2.1. *Let*

$$\begin{aligned} x =& -\eta\theta\eta\xi\theta\psi - \psi\eta\xi\theta\eta\theta + \psi\eta\theta\eta\theta\psi + \eta\theta\eta\theta\psi^2 - \eta\theta\psi^2\eta\theta + \psi^2\eta\theta\eta\theta - \\ &\eta\xi\theta\eta\theta\psi - \psi\eta\theta\eta\xi\theta, \\ x' =& \theta\psi\eta\xi\theta\psi. \end{aligned}$$

1. *If $\hat{\phi} \in Hom_R^{(-4)}(B_+^{\otimes 6}, B)$ is a Hochschild cocycle, then*

$$\hat{\beta}(\hat{\phi}) := (\mathrm{ev}_x^{id_P} - \mathrm{ev}_{x'}^{\theta})(\hat{\phi}) = 0$$

   if and only if $\hat{\phi}$ is a coboundary; thus $\hat{\beta}$ detects cohomology classes.

2. *Let $c \in HH^6_{(-4)}(B)$ such that $\hat{\beta}(c) = -15$. Then the $\mathbb{C}$-linear function $\beta : HH^6_{(-4)}(B) \to \mathbb{C}$ defined by $c \mapsto 1$ is an isomorphism such that $\beta(m_6') = t^4 e_4$.*

3. *$m_6' - f' * m_6$ is a Hochschild coboundary; thus the class of m_6' is independent of the choice of f_3.*

Proof. 1. Let $\phi \in \operatorname{Hom}_R^{(-4)}(B_+^{\otimes 5}, B)$. Then the first and last terms of $(\delta\phi)(x)$ vanish for degree reasons and,

$$\begin{aligned}
(\delta\phi)(x) &= \phi(-[-\psi\eta\xi\theta\psi + \psi\eta\theta\psi^2 - \psi^3\eta\theta] + [-\eta\xi^2\theta\psi + \psi^2\eta\theta\psi + \eta\xi\theta\psi^2 - \psi^2\eta\xi\theta]- \\
&\quad [\psi\eta\xi\theta\psi + \eta\theta\psi^3 + \psi^3\eta\theta - \eta\xi^2\theta\psi - \psi\eta\xi^2\theta] + [-\psi\eta\xi^2\theta + \psi\eta\theta\psi^2 + \psi^2\eta\xi\theta - \eta\xi\theta\psi^2]- \\
&\quad [-\psi\eta\xi\theta\psi - \eta\theta\psi^3 + \psi^2\eta\theta\psi]) \\
&= \phi(\psi\eta\xi\theta\psi) \\
&= \operatorname{ev}_{\psi\eta\xi\theta\psi}^{\mathrm{id}_P}(\phi) \cdot \mathrm{id}_P, \\
(\delta\phi)(x') &= \theta \cdot \phi(\psi\eta\xi\theta\psi) \\
&= \theta \cdot \operatorname{ev}_{\psi\eta\xi\theta\psi}^{\mathrm{id}_P}(\phi) \cdot \mathrm{id}_P \\
&= \operatorname{ev}_{\psi\eta\xi\theta\psi}^{\mathrm{id}_P}(\phi) \cdot \theta,
\end{aligned}$$

Since β takes the difference of the coefficients, this proves that β vanishes on coboundaries. The other direction will follow from (2).

2. Since m_6' is a cocycle and Theorem III.4.1 showed that $HH^6_{(-4)}(B)$ is one-dimensional, calculating that $\beta(m_6') = -15t^4e_4$ will prove (2) as well as the remaining direction in (1). We use the relations given in Proposition II.7.1. Let

$$z = m_4 \otimes \mathbf{1}^2 - \mathbf{1} \otimes m_4 \otimes \mathbf{1} + \mathbf{1}^2 \otimes m_4.$$

Recall the definition of m_6' from Equation IV.3. We have,

$$\begin{aligned}
m_6(x) &= (-M(2,0,0,1) - M(2,1,0,0) + M(3,0,0,0) + M(3,0,0,0) - \\
&\quad M(1,0,2,0) + M(3,0,0,0) - M(2,1,0,0) - M(2,0,0,1)) \cdot \mathrm{id}_P \\
&= (-4M(2,1,0,0) + 3M(3,0,0,0) - M(1,0,2,0)) \cdot \mathrm{id}_P \\
&= -2t^4 g_{2,1} \cdot \mathrm{id}_P, \\
f_3(z(x)) &= M(1,0,0,0) g_{1,0} f_3(-\eta\theta\psi + \eta\theta\psi + \psi\eta\theta - \psi\eta\theta + \eta\theta\psi + \psi\eta\theta - \\
&\quad \eta\theta\psi - \psi\eta\theta - \eta\theta\psi + \eta\theta\psi - \eta\xi\theta - \eta\xi\theta + \psi\eta\theta - \psi\eta\theta) \\
&= M(1,0,0,0) f_3(-2\eta\xi\theta) \\
&= -2[M(1,0,0,0)]^2 \cdot \mathrm{id}_P, \\
&= -2t^4 [g_{1,0}]^2 \cdot \mathrm{id}_P \\
-m_2(f_3 \otimes f_3)(x) &= -f_3(\psi\eta\theta) \cdot f_3(\eta\theta\psi) + f_3(\eta\theta\psi) \cdot f_3(\psi\eta\theta) + f_3(\eta\xi\theta) \cdot f_3(\eta\theta\psi) + \\
&\quad f_3(\psi\eta\theta) \cdot f_3(\eta\xi\theta) \\
&= 0; \text{ therefore} \\
\mathrm{ev}_x^{\mathrm{id}_P}(m_6') &= -2t^4 g_{2,1} - 2t^4 [g_{1,0}]^2 \\
&= 2t^4([e_2^*]^2 - 5e_4 - [e_2^*]^2) = -10t^4 e_4.
\end{aligned}$$

Also,

$$\begin{aligned}
m_6'(x') &= M(0,1,1,1) \cdot \theta - f_3(\theta\psi\eta) \cdot f_3(\xi\theta\psi) \\
&= t^4(g_{2,1} + g_{1,0}^2) \cdot \theta = 5t^4 e_4 \cdot \theta; \text{ therefore} \\
\mathrm{ev}_{x'}^{\theta}(m_6') &= 5t^4 e_4, \\
\beta(m_6') &= (\mathrm{ev}_x^{\mathrm{id}_P} - \mathrm{ev}_{x'}^{\theta})(m_6') = -15t^4 e_4.
\end{aligned}$$

3. By Lemma II.5.3, $f_3 - f_3'$ is a cocycle; since $HH^3_{(-2)}(B) = 0$ it is also a coboundary. Let

$h \in C^2_{(-2)}(B)$ such that $\delta h = f_3 - f_3'$. Then,

$$\begin{aligned} m_6' - f' * m_6 &= \delta h\left(\sum_{r+s=2} \mathbf{1}^{\otimes r} \otimes m_4 \otimes \mathbf{1}^{\otimes s}\right) - m_2(f_3 \otimes f_3) \cdot m_2(f_3' \otimes f_3') \\ &= \delta h\left(\sum_{r+s=2} \mathbf{1}^{\otimes r} \otimes m_4 \otimes \mathbf{1}^{\otimes s}\right) - m_2(f_3 \otimes \delta h + \delta h \otimes f_3 - \delta h \otimes \delta h). \end{aligned}$$

From this last expression, it is a straightforward check that $\beta(m_6' - f' * m_6) = 0$.

□

Proposition IV.2.2. *Let $x = \eta\xi\theta\psi^2\eta\xi\theta$.*

1. *If $\hat{\phi} \in Hom_R(B_+^{\otimes 8}, B)$ is a cocycle, then $\mathrm{ev}_x^{id_P}(\hat{\phi}) = 0$ if and only if $\hat{\phi}$ is a coboundary.*
2. *Let $c \in HH^8_{(-6)}(B)$ such that $\mathrm{ev}_x^{id_P}(c) = -35$. Then the $\mathbb{C}$-linear function*

   $$\gamma : HH^8_{(-6)}(B) \to \mathbb{C}$$

   defined by $c \mapsto 1$ is an isomorphism such that $\gamma(m_8') = t^6 e_6$.
3. *$m_8' - f' * m_8$ is a Hochschild coboundary; thus the class of m_8' is independent of the choice of f_3.*

Proof. 1. For $\phi \in \mathrm{Hom}_R^{(-6)}(B_+^{\otimes 7}, B)$ we have,

$$(\delta\phi)(x) = \eta \cdot \phi(\xi\theta\psi^2\eta\xi\theta) + \phi(\eta\xi\theta\psi^2\eta\xi) \cdot \theta,$$

which is 0 for degree reasons. The other direction will follow from (2).

2. Since $HH^8_{(-6)}(B)$ is one-dimensional and m_8' is a cocycle, we need only check that $\gamma(m_8') = -35t^6 e_6 \cdot \mathrm{id}_P$ to prove (2). Let

$$\begin{aligned} \kappa_1 &= f_3(m_6 \otimes \mathbf{1}^{\otimes 2} - \mathbf{1} \otimes m_6 \otimes \mathbf{1} + \mathbf{1}^{\otimes 2} \otimes m_6), \\ \kappa_2 &= -\, m_6'\left(\sum_{r+s=5} \mathbf{1}^{\otimes r} \otimes f_3 \otimes \mathbf{1}^{\otimes s}\right), \end{aligned}$$

so $m_8' = m_8 + \kappa_1 + \kappa_2$. Then,

$$\begin{aligned}
m_8(x) &= M(1,1,2,1)\cdot \mathrm{id}_P \\
&= -\frac{1}{2}t^6 g_{3,2}\cdot \mathrm{id}_P, \\
\kappa_1(x) &= t^4\cdot f_3(M(0,1,2,0)\cdot \eta\xi\theta + M(0,2,1,0)\cdot\eta\xi\theta) \\
&= 2t^6\cdot M(2,1,0,0)\cdot M(1,0,0,0)\cdot \mathrm{id}_P \\
&= -t^6 g_{2,1}g_{1,0}\cdot \mathrm{id}_P, \\
\kappa_2(x) &= -M(1,0,0,0)\cdot m_6'(\mathrm{id}_P\psi^2\eta\xi\theta - \eta\theta\psi\eta\xi\theta + \eta\xi\theta\eta\xi\theta - \eta\xi\theta\eta\xi\theta - \eta\xi\theta\psi\eta\theta + \eta\xi\theta\psi^2\mathrm{id}_P) \\
&= M(1,0,0,0)\cdot m_6'(\eta\theta\psi\eta\xi\theta + \eta\xi\theta\psi\eta\theta) \\
&= M(1,0,0,0)\cdot [M(1,0,1,1)\cdot \mathrm{id}_P + M(1,1,1,0)\cdot \mathrm{id}_P + M(1,0,0,0)\cdot f_3(\eta\xi\theta+\eta\xi\theta) \\
&\quad - f_3(\eta\theta\psi)\cdot f_3(\eta\xi\theta) - f_3(\eta\xi\theta)\cdot f_3(\psi\eta\theta)] \\
&= -t^6(2g_{2,1}g_{1,0} + 2[g_{1,0}]^3)\cdot \mathrm{id}_P,
\end{aligned}$$

so

$$\begin{aligned}
\mathrm{ev}_x^{\mathrm{id}_P}(m_8') &= t^6\left(-\frac{1}{2}g_{3,2} - g_{2,1}g_{1,0} - 2g_{2,1}g_{1,0} - 2[g_{1,0}]^3\right) \\
&= t^6\cdot\left(-\frac{1}{2}g_{3,2} - 3g_{2,1}g_{1,0} - 2[g_{1,0}]^3\right) \\
&= -35t^6 e_6.
\end{aligned}$$

3. Let $m_6' - f_3' * m_6 = \delta g$, $f_3 - f_3' = \delta h$. Then,

$$\begin{aligned}
m_8' - f' * m_8 &= \delta h\left(\sum_{r+s=2}\mathbf{1}^{\otimes r}\otimes m_6\otimes \mathbf{1}^{\otimes s}\right) - m_6'\left(\sum_{r+s=5}\mathbf{1}^{\otimes r}\otimes f_3\otimes\mathbf{1}^{\otimes s}\right) + \\
&= (f' * m_6)\left(\sum_{r+s=5}\mathbf{1}^{\otimes r}\otimes (f_3-\delta h)\otimes \mathbf{1}^{\otimes s}\right) \\
&= \delta h\left(\sum_{r+s=2}\mathbf{1}^{\otimes r}\otimes m_6\otimes\mathbf{1}^{\otimes s}\right) - \delta g\left(\sum_{r+s=5}\mathbf{1}^{\otimes r}\otimes f_3\otimes\mathbf{1}^{\otimes s}\right).
\end{aligned}$$

Using this last expression, it is a straightforward check that $\gamma(m_8' - f' * m_6) = 0$.

$\square$

Theorem IV.2.3. *Let α, γ be as defined in Propositions IV.2.1 and IV.2.2. Then the j-invariant of E is*

$$j(E) = 1728 \cdot \frac{[\beta(m_6')]^3}{[\beta(m_6')]^3 - 27[\gamma(m_8')]^2}.$$

Proof. Since,

$$j(E) = 1728 \cdot \frac{[e_4]^3}{[e_4]^3 - 27[e_6]^2},$$

this follows immediately from the previous propositions.

□

CHAPTER V

HOCHSCHILD COHOMOLOGY AND THE CURVE OF GENUS $g \geq 2$

Let X be a smooth curve of genus $g \geq 1$ over an algebraically closed field $\mathbb{K}$ with $\operatorname{char} \mathbb{K} \neq 2, 3$, with $\mathcal{O}_X$ the structure sheaf on X. Let P be a point on X. Then there is a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(P) \to \mathbb{K}(P) \to 0$$

which gives rise to a long exact sequence on Ext,

$$0 \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X(P)) \to \operatorname{Hom}(\mathcal{O}_X, \mathbb{K}(P)) \xrightarrow{\phi_P} \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \to \cdots .$$

Let θ_P generate the one-dimensional space $\operatorname{Hom}(\mathcal{O}_X, \mathbb{K}(P))$. Since,

$$\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X(P))$$

is an isomorphism, ϕ_P is an injection; so $\phi_P(\theta_P) \neq 0$. Note that

$$\dim \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X) = g.$$

Let $P_1, \dots, P_g$ be distinct points on X, with $\mathbb{K}(P_i) = \mathcal{O}_X(P_i)/\mathcal{O}_X$ the skyscraper sheaf at P_i, such that the classes $\{\phi_{P_i}(\theta_{P_i}\}_{i=1}^{g}$ generate $\operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$. Then $G(X) = \mathcal{O}_X \oplus \bigoplus_{i=1}^{g} \mathbb{K}(P_i)$ generates $D^b(X)$. Since $D^b(X)$ determines X uniquely, the minimal A_∞-algebra

$$B^g(X) = \operatorname{Ext}^*(G(X), G(X))$$

also recovers X. (Note that $B^1 = B$ from Chapter III.) Since the restriction of $B^g(X)$ to an associative algebra is independent of X when g is fixed, it is useful to study A_∞-extensions of the algebra (B^g, m_2). In this chapter we calculate $HH^n_{(m)}$ for $g \geq 2$ and $m = 1-n,\ 2-n,\ 3-n$. The main result is Theorem V.4.10.

V.1. The associative algebra B^g

In the remainder of the chapter we assume $g \geq 2$ unless otherwise noted. With notation as above, let B^g be the algebra $\mathrm{Ext}^*(G(X), G(X))$ considered as an associative algebra. Then B^g is a direct sum (as a $\mathbb{K}$-vector space) of components,

(i) $\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$, a one-dimensional space generated by the identity map, $\mathrm{id}_{\mathcal{O}_X}$;

(ii) $\mathrm{Hom}(\mathbb{K}(P_i), \mathbb{K}(P_i))$, each one-dimensional generated by the identity map id_{P_i};

(iii) $\mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(P_i))$, each one-dimensional generated by a function θ_i;

(iv) $\mathrm{Ext}^1(\mathbb{K}(P_i), \mathcal{O}_X)$, each one-dimensional generated by a function η_i;

(v) $\mathrm{Ext}^1(\mathbb{K}(P_i), \mathbb{K}(P_i))$, each one-dimensional generated by $\eta_i\theta_i = \psi_i$;

(vi) $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$, a g-dimensional space generated by the set of functions $\theta_i\eta_i = \xi_i$.

This also gives us a standard basis S of B^g over $\mathbb{K}$. Figure 2 gives a diagrammatic representation of B^g.

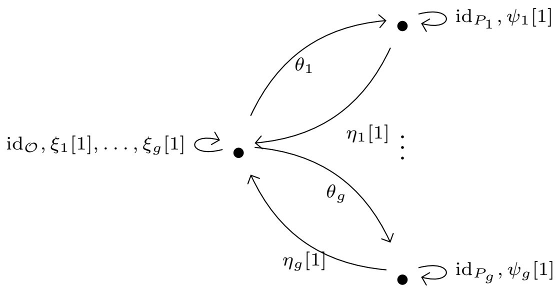

Figure 2: Arrow diagram for B^g

Then B^g is a graded $\mathbb{K}$-algebra with,

$$B^g_0 = \langle \mathrm{id}_{\mathcal{O}_X}, \{\mathrm{id}_{P_i}, \theta_i\}_{i=1}^g \rangle,\ B^g_1 = \langle \{\eta_i, \xi_i, \psi_i\}_{i=1}^g \rangle.$$

We also consider B^g as an algebra over the semi-simple subring,

$$R^g = \langle \mathrm{id}_{\mathcal{O}_X}, \{\mathrm{id}_{P_i}\}_{i=1}^g \rangle,$$

over which B^g splits as a direct sum, $B^g = R^g \oplus B^g_+$, with

$$B^g_+ = \langle \{\theta_i, \eta_i, \psi_i, \xi_i\}_{i=1}^g \rangle.$$

Thus B^g is augmented over R^g in the sense of Section II.1.. We write B is place of B^g and R for R^g when this will cause no confusion.

V.2. A filtration of the Hochschild complex

Our goal is to compute $HH^n_R(B)$ in internal degrees $1-n$, $2-n$, and $3-n$ for all n. By $\otimes$ we will mean $\otimes_R$ and by Hom we mean Hom_R. We consider the decreasing filtration on B as a B-bimodule,

$$F_0 = B \supset F_1 = B_+ \supset F_2 = \langle \{\xi_i, \psi_i\}_{i=1}^g \rangle \supset F_3 = 0.$$

The decreasing filtration on B gives rise to a decreasing filtration on the reduced Hochschild complex with coefficients in B,

$$C^\bullet(F_0) \supset C^\bullet(F_1) \supset C^\bullet(F_2) \supset 0.$$

We consider the spectral sequence on Hochschild cohomology associated with this filtration. Since the filtration is finite (and therefore bounded) the spectral sequence converges to the cohomology of the complex, $HH^\bullet(B)$. On the zero page we have isomorphisms (fitting together into isomorphisms of complexes),

$$C^\bullet(F_i)/C^\bullet(F_{i+1}) \cong C^\bullet(F_i/F_{i+1}).$$

So we should compute $HH^n(F_i/F_{i+1})$ to fill out page one of the spectral sequence. We define:

1. $\mathrm{Bar}_\bullet(\mathcal{O}_X, B^g) := \mathrm{id}_{\mathcal{O}_X} \otimes_R \mathrm{Bar}_\bullet(B^g) \otimes_R \mathrm{id}_{\mathcal{O}_X}$, with homology $HB_\bullet(\mathcal{O}_X, B^g)$;
2. $\mathrm{Bar}_\bullet(P_i, B^g) := \mathrm{id}_{P_i} \otimes_R \mathrm{Bar}_\bullet(B^g) \otimes_R \mathrm{id}_{P_i}$, with homology $HB_\bullet(P_i, B^g)$;
3. $\mathrm{Bar}_\bullet(\theta_i, B^g) := \mathrm{id}_{\mathcal{O}_X} \otimes_R \mathrm{Bar}_\bullet(B^g) \otimes_R \mathrm{id}_{P_i}$, with homology $HB_\bullet(\theta_i, B^g)$;
4. $\mathrm{Bar}_\bullet(\eta_i, B^g) := \mathrm{id}_{P_i} \otimes_R \mathrm{Bar}_\bullet(B^g) \otimes_R \mathrm{id}_{\mathcal{O}_X}$, with homology $HB_\bullet(\eta_i, B^g)$.

We interpret these subcomplexes as tensors corresponding to paths in Figure 2 where we specify the source and target. We write $\mathrm{Bar}_\bullet(\mathcal{O}_X)$, $HB_\bullet(\mathcal{O}_X)$, $\mathrm{Bar}_\bullet(P_i)$, etc. when it causes no confusion.

Lemma V.2.1. *There are isomorphisms,*

1. $HH^n_{(-k)}(F_0/F_1) \cong [HB^{(k)}_n(\mathcal{O}_X)]^* \otimes id_{\mathcal{O}_X} \oplus \bigoplus_{i=1}^g [HB^{(k)}_n(P_i)]^* \otimes id_{P_i}$;
2. $HH^n_{(-k)}(F_1/F_2) \cong \bigoplus_{i=1}^g [HB^{(k)}_n(\theta_i)]^* \otimes \theta_i \oplus \bigoplus_{i=1}^g [HB^{(k+1)}_n(\eta_i)]^* \otimes \eta_i$;
3. $HH^n_{(-k)}(F_2) \cong \bigoplus_{i=1}^g [HB^{(k+1)}_n(\mathcal{O}_X)]^* \otimes \xi_i \oplus \bigoplus_{i=1}^g [HB^{(k+1)}_n(P_i)]^* \otimes \psi_i$.

Proof. There are isomorphisms of B-bimodules,

$$\begin{aligned} F_0/F_1 &\cong (F_1 + \langle \mathrm{id}_{\mathcal{O}_X}\rangle)/F_1 \oplus \bigoplus_{i=1}^g (F_1 + \langle \mathrm{id}_{P_i}\rangle)/F_1, \\ F_1/F_2 &\cong \bigoplus_{i=1}^g (F_2 + \langle \theta_i\rangle)/F_2 \oplus \bigoplus_{i=1}^g (F_2 + \langle \eta_i\rangle)/F_2, \\ F_2 &\cong \bigoplus_{i=1}^g (\xi_i) \oplus \bigoplus_{i=1}^g (\psi_i), \end{aligned}$$

so these results would follow from isomorphisms for each corresponding summand. Those follow from the duality of Section II.2. since the first and last terms in the cohomology differential are zero in each quotient. ☐

Remark V.2.2. *We define*

$$V_\theta = \bigoplus_{i=1}^g (F_2 + \langle \theta_i\rangle)/F_2, \ V_\eta = \bigoplus_{i=1}^g (F_2 + \langle \eta_i\rangle)/F_2.$$

The second equation in this proof claims $F_1/F_2 \cong V_\theta \oplus V_\eta$.

Let $B^g(i) = \langle \mathrm{id}_{\mathcal{O}_X}, \mathrm{id}_{P_i}, \theta_i, \eta_i, \xi_i, \psi_i \rangle \subset B^g$. There is an obvious isomorphism $B^g(i) \cong B^1$ as $\mathbb{K}$-algebras. Let $B^g_+(i)$ be the B^g-submodule corresponding to B^1_+ under this isomorphism. Then there is an isomorphism of B^g-bimodules, $B^g_+ \cong \bigoplus_{i=1}^g B^g_+(i)$. We define,

(1) $\widetilde{HB}(\mathcal{O}_X, B^g(i)) : \bigoplus_{n=0}^{\infty} \bigoplus_{m=n-3}^{n} HB_n^{(m)}(\mathcal{O}_X, B^1)$, summarized in Proposition III.2.1;

(2) $\widetilde{HB}(P_i) := \bigoplus_{n=0}^{\infty} \bigoplus_{m=n-3}^{n} HB_n^{(m)}(P_i, B^1)$, summarized in Corollary III.2.4;

(3) $\widetilde{HB}(\theta_i) := \bigoplus_{n=0}^{\infty} \bigoplus_{m=n-3}^{n-1} HB_n^{(m)}(\theta_i, B^1)$, summarized in Proposition III.2.6;

(4) $\widetilde{HB}(\eta_i) := \bigoplus_{n=0}^{\infty} \bigoplus_{m=n-2}^{n} HB_n^{(m)}(\eta_i, B^1)$, summarized in Corollary III.2.9;

with representatives of all classes in cohomological degree n taken in $B^g_+(i)^{\otimes n}$. We rephrase the referenced results for our purposes here.

Lemma V.2.3. *1. $\widetilde{HB}(\mathcal{O}_X, B^g(i))$ and $\widetilde{HB}(P_i)$ are nonzero only for $n = 0, 3, 4, 7, 8, 11, 12$.*

2. $\widetilde{HB}(\theta_i)$ and $\widetilde{HB}(\eta_i)$ are nonzero only for $n = 1, 2, 5, 6, 9, 10$.

For $m > 0$,

$$S_m := \{(i_1, \ldots, i_m) \in \mathbb{Z}^m | 1 \leq i_j \leq g \text{ and } i_j \neq i_{j+1} \text{ for all } j\},$$
$$W(\mathcal{O}_X) := \bigoplus_{m=1}^{\infty} \bigoplus_{\sigma \in S_m} \widetilde{HB}(\mathcal{O}_X, B^g(i_1)) \otimes_R \otimes \cdots \otimes_R \widetilde{HB}(\mathcal{O}_X, B^g(i_m)).$$

Remark V.2.4. *We interpret $W(\mathcal{O}_X)$ as sequences of loops in B^g starting and ending at $\mathcal{O}_X$ built by bar homology classes from B^1.*

Lemma V.2.5. *1. $HB_n(\mathcal{O}_X)$ is the subspace of classes of cohomological degree n in $W(\mathcal{O}_X)$;*

2. $HB_n(P_i)$ is the subspace of classes of cohomological degree n in $\widetilde{HB}(P_i) \oplus [\widetilde{HB}(\eta_i) \otimes_R W(\mathcal{O}_X) \otimes_R \widetilde{HB}(\theta_i)]$;

3. $\widetilde{HB}_n(\theta_i)$ is the subspace of classes of cohomological degree n in $W(\mathcal{O}_X) \otimes_R \widetilde{HB}(\theta_i)$;

4. $\widetilde{HB}_n(\eta_i)$ is the subspace of classes of external degree n in $\widetilde{HB}(\eta_i) \otimes_R W(\mathcal{O}_X)$.

Proof. We have

$$(B^g_+)^{\otimes n} = \left(\bigoplus_{i=1}^{g} B^g_+(i) \right)^{\otimes n} = \bigoplus_{m=1}^{\infty} \bigoplus_{\sigma \in S_m} \left(\bigoplus_{\sum n_j = n, n_j \neq 0} B^g_+(i_1)^{\otimes n_1} \otimes \cdots \otimes B^g_+(i_m)^{\otimes n_m} \right).$$

For each σ, the corresponding summands form the n-chains of the complex $T(B^g_+(i_1)) \otimes \cdots \otimes T(B^g_+(i_m))$, which is a tensor product of bar complexes. Since $\otimes$ is $\otimes_R$, tensors represent paths in B^g; since $i_j \neq i_{j+1}$ for all j, it follows that we can only attach paths at the vertex corresponding to $\mathcal{O}_X$ in Figure 2, i.e.,

$$\mathrm{Bar}_\bullet(\mathcal{O}_X) = \bigoplus_{m=1}^{\infty} \bigoplus_{\sigma \in S_m} \mathrm{Bar}_\bullet(\mathcal{O}_X, B^g(i_1)) \otimes \cdots \otimes \mathrm{Bar}_\bullet(\mathcal{O}_X, B^g(i_m)).$$

Thus the homology of this complex is the tensor product of homologies, which is the first result. The others follow similarly. □

V.3. Differentials in the spectral sequence

This section compiles several technical lemmas concerning the behavior of the differentials in the spectral sequence associated with the filtration from Section V.2..

Lemmas V.2.1 and V.2.5 explain how to build each space $E_1^{i,n-i} = HH^n(F_i/F_{i+1})$ by considering certain sequences n_1-n_2-$\cdots$-n_m of positive integers such that $\sum_{j=1}^m n_j = n$.

Remark V.3.1. *These are the numbers n_j from the proof of Lemma V.2.5, with possible values of these numbers given in Lemma V.2.3. That is, the n_j are the lengths of classes in $\widetilde{HB}(\mathcal{O}_X, B^g(i_j))$,, $\widetilde{HB}(P_i)$, $\widetilde{HB}(\theta_i)$, or $\widetilde{HB}(\eta_i)$ which are tensored to give a class in $HB_n(\mathcal{O}_X)$, $HB_n(P_i)$, $HB_n(\theta_i)$, or $HB_n(\eta_i)$, according to Lemma V.2.5.*

Since we are only concerned with internal degrees $1-n$, $2-n$, and $3-n$, this places a limit on the lengths of admissible such sequences. In particular, since the θ_i are the only basis elements of B_+ in graded degree 0, bar homology classes of internal degree $n-k$ are those which have exactly k factors of some (possibly different) θ_i. Each factor of θ_i corresponds either to an entire loop in B_+ (in the sense of Remark V.2.4 and Figure 2), or to a path along θ_i.

The following comments describe the classes we will call *admissible*. A sequence is admissible if it corresponds to an admissible class. In $HB_n(\mathcal{O}_X)$ and $HB_n(P_i)$ we take classes of internal degree at least $n-3$ for each n (i.e., having no more than 3 loops in B_+); in $HB_n(\theta_i)$ we take classes of internal degree at most $n-3$ (i.e., having no more than 2 loops in B_+ and an additional path along θ_i); in $HB_n(\eta_i)$ we take classes of internal degree at most $n-2$ (i.e., having no more than 3 loops in B_+ and an additional path along η_i).

Table 5 in rows 0-12 summarizes admissible sequences. We include admissible sequences for θ_i, P_i for $n = 13$ only because they appear as boundaries of admissible sequences for $\mathcal{O}_X, P_i$ for $n = 12$. The column labeled $\mathcal{O}_X$ in row n lists sequences corresponding to admissible classes in $HB_n(\mathcal{O}_X)$, and similarly for P_i and θ_i. Since the admissible sequences for η_i are the reverse of those admissible for θ_i, only the sequences for θ_i are listed.

n	$\mathcal{O}_X$	P_i	θ_i (η_i)
0	0	0	
1			1
2			2
3	3	3	
4	4	4	3-1
5		1-3-1	5, 4-1, 3-2
6	3-3	1-4-1, 2-3-1, 1-3-2	
7	7, 3-4, 4-3	7, 2-4-1, 1-4-2, 2-3-2	3-3-1
8	8, 4-4	8, 2-4-2	7-1, 3-5, 4-3-1, 3-4-1, 3-3-2
9	3-3-3	1-7-1, 2-3-3-1, 1-3-3-2, 1-4-3-1, 1-3-4-1	9, 8-1, 7-2, 4-5, 3-6, 4-4-1, 4-3-2, 3-4-2
10	3-7, 7-3, 3-3-4, 3-4-3, 4-3-3	1-7-2, 2-7-1, 1-8-1, 2-3-3-2, 1-4-4-1, 1-4-3-2, 2-4-3-1, 1-3-4-2, 2-3-4-1	10, 8-2, 4-4-2, 3-3-3-1
11	11, 3-8, 8-3, 4-7, 7-4, 3-4-4, 4-3-4, 4-4-3	11, 1-8-2, 2-8-1, 2-7-2, 1-4-4-2, 2-4-4-1, 2-4-3-2, 2-3-4-2	3-3-3-2, 7-3-1, 3-7-1, 4-3-3-1, 3-4-3-1, 3-3-4-1
12	12, 4-8, 8-4, 4-4-4	12, 2-8-2, 2-4-4-2	11-1, 3-8-1, 8-3-1, 4-7-1, 7-4-1, 3-7-2, 7-3-2, 3-4-4-1, 4-3-4-1, 4-4-3-1, 3-3-4-2, 3-4-3-2, 4-3-3-2
13			13, 12-1, 11-2, 4-8-1, 8-4-1, 3-8-2, 8-3-2, 4-7-2, 7-4-2, 3-4-4-2, 4-3-4-2, 4-4-3-2, 4-4-4-1

Table 5: Admissible sequences

We decorate a sequence n_1-$\cdots$-n_k with an element $\sigma = (i_1, \ldots, i_m) \in S_m$ (defined after Lemma V.2.3 by writing $n_1(i_1)$-$\cdots$-$n_m(i_m)$ to refer to the sequence where the class of length n_j is represented in $B(i_j)$. We will calculate differentials on page one of the spectral sequence by playing a combinatorial game with these decorated sequences.

To describe this game, we assume for simplicity that we have a sequence n_1-n_2. It will be obvious that these comments extend to arbitrary sequences. Let α be the bar cycle corresponding to the sequence $n_1(i_1)$-$n_2(i_2)$, and ω be the bar cycle corresponding to the sequence $m_1(j_1)$-$\cdots$-$m_k(j_k)$. Then the only way for $(\delta[\alpha]^*)(\omega)$ to be nonzero is if truncating the first or last factor in the tensor ω gives α (see calculations in the proof of Lemma III.3.1 part (2)); that is, the sequence for ω must be one of the following:

$$
\begin{aligned}
s_1(j) &= 1(\mathrm{j})\text{-}n_1(i_1)\text{-}n_2(i_2),\ j \neq i_1 \\
s_2(j) &= n_1(i_1)\text{-}n_2(i_2)\text{-}1(\mathrm{j}),\ j \neq i_2 \\
s_3 &= (n_1+1)(i_1)\text{-}n_2(i_2),\ \text{or} \\
s_4 &= n_1(i_1)\text{-}(n_2+1)(i_2).
\end{aligned}
$$

Therefore $\delta([\alpha]^*)$ is some linear combination of cocycles corresponding to the sequences $s_1(j)$, $s_2(j)$, s_3, and s_4. Now we consider specific cases of how to compute δ using sequences:

1. Consider a sequence $m_1(j_1)$-$\cdots$-$m_k(j_k)$-$n(i)$ admissible for θ_i, corresponding to some loops in $B_+(j_1), \ldots, B_+(j_k)$ (ref. Figure 2) and a path along θ_i. The dual of this path is a function,

$$[m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}n(i)]^* \otimes \theta_i \in HH^n(F_1/F_2).$$

Since $\delta_1(HH^n(F_1/F_2)) \subset HH^{n+1}(F_2)$, we look for sequences $s_1(j)$, $s_2(j)$, s_3, and s_4 admissible for $\mathcal{O}_X$ or P_i. To build such sequences:

(a) We can precede our sequence by a path along η_i whenever $j_1 \neq i$, corresponding to the sequence $1(i)$-$m_1(j_1)$-$\cdots$-$m_k(j_k)$-$n(i)$ and the cohomology class,

$$[1(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}n(i)]^* \otimes \psi_i;$$

(b) We can precede our sequence by a path along η_i whenever $j_1 = i$ and m_1 is even, corresponding to the sequence $(m_1+1)(i)$-$\cdots$-$m_k(j_k)$-$n(i)$ and the cohomology class,

$$[(m_1+1)(i)\text{-}\cdots\text{-}m_k(j_k)\text{-}n(i)]^* \otimes \psi_i;$$

(c) We can follow our sequence by η_i whenever n is even, corresponding to the sequence $m_1(j_1)$-$\cdots$-$m_k(j_k)$-$(n+1)(i)$, and the cohomology class,

$$[m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}(n+1)(i)]^* \otimes \xi_i.$$

2. By analogy, the boundary of a class $[n(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)]^* \otimes \eta_i$ admissible for η_i is a linear combination of the classes:

   (a) $[n(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}1(i)]^* \otimes \psi_i$, if $j_k \neq i$;

   (b) $[n(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}(m_k+1)(i)]^* \otimes \psi_i$, if $j_k = i$ and m_k is even;

   (c) $[(n+1)(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)]^* \otimes \xi_i$, if n is even.

3. Consider a sequence $m_1(j_1)$-$\cdots$-$m_k(j_k)$ admissible for $\mathcal{O}_X$, corresponding to some loops in $B_+(j_1), \ldots, B_+(j_k)$. These sequences correspond to cohomology classes in $HH^n(F_2)$ and $HH^n(F_0/F_1)$. Of course $\delta_1(HH^n(F_2)) = 0$, so our only interest in in computing,

$$\delta_1([m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)]^* \otimes \mathrm{id}_{\mathcal{O}_X} \in HH^{n+1}(F_1/F_2).$$

   Therefore we look for sequences $s_1(j)$, $s_2(j)$, s_3, and s_4 admissible for θ_i or η_i for some i.

   (a) We can precede this sequence by a path along η_i, which gives the classes,

$$[1(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)]^* \otimes \eta_i \quad \text{if } i \neq j_1, \text{ or}$$
$$[(m_1+1)(j_1)\text{-}\cdots\text{-}m_k(j_k)]^* \otimes \eta_{j_1} \quad \text{if } i = j_1 \text{ and } m_1 \text{ even.}$$

   (b) We can follow this sequence with a path along θ_i, which gives the classes,

$$[m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}1(i)]^* \otimes \theta_i \quad \text{if } i \neq j_k, \text{ or}$$
$$[m_1(j_1)\text{-}\cdots\text{-}(m_k+1)(j_k)]^* \otimes \theta_{j_k} \quad \text{if } i = j_k \text{ and } m_k \text{ is even.}$$

4. Finally, consider sequences $n_1(i)$-$m_1(j_1)$-$\cdots$-$m_k(j_k)$-$n_2(i)$ or $n(i)$ admissible for P_i. These correspond to: a path along η_i, some loops in different $B_+(j_l)$, and a path along θ_i; and some loops in $B_+(i)$ starting and ending at the vertex corresponding to $\mathbb{K}(P_i)$, respectively. Again we look for sequences allowable for θ_i or η_i.

   (a) We can precede this sequence by a path along θ_i if n_1 is even, giving the class,

   $$[(n_1+1)(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}n_2(i)]^* \otimes \theta_i;$$

   (b) We can follow it by a path along η_i if n_2 is even, giving the class,

   $$[n_1(i)\text{-}m_1(j_1)\text{-}\cdots\text{-}m_k(j_k)\text{-}(n_2+1)(i)]^* \otimes \eta_i.$$

Remark V.3.2. *The calculations here apply only to sequences of positive length. In particular, the boundaries of* $[0]^* \otimes id_{\mathcal{O}_X}$ *and* $[0]^* \otimes id_{P_i}$ *are special cases since these sequences therefore cannot be decorated with some* $\sigma \in S_m$ *for* $m > 0$.

These rules allow us to calculate $\delta_1^{0,n}$ and $\delta_1^{1,n-1}$. The reader may also reference Table 5 for admissible sequence information.

Lemma V.3.3. *The following form a basis for* $\ker \delta_1^{1,n-1}$ *for* $n > 0$*:*

1. $[n(i)]^* \otimes \theta_i$ *and* $[n(i)]^* \otimes \eta_i$ *with* n *odd;*
2. $[n(i)]^* \otimes \theta_i + [n(i)]^* \otimes \eta_i$ *with* $n \equiv 6 \mod 8$*;*
3. $[m(j)\text{-}1(i)]^* \otimes \theta_i \pm [1(i)\text{-}m(j)]^* \otimes \eta_i$*; and*
   $[m_1(j_1)\text{-}m_2(j_2)\text{-}1(i)]^* \otimes \theta_i \pm [1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_i$*;*
4. $[m(j)\text{-}5(i)]^* \otimes \theta_i \pm [1(i)\text{-}m(j)\text{-}4(i)]^* \otimes \eta_i$*, and*
   $[5(i)\text{-}m(j)]^* \otimes \eta_i \pm [4(i)\text{-}m(j)\text{-}1(i)]^* \otimes \theta_i$*;*
5. $[3(i)\text{-}m(j)\text{-}2(i)]^* \otimes \theta_i \pm [2(i)\text{-}m(j)\text{-}3(i)]^* \otimes \eta_i$*;*
6. $[3(i)\text{-}m(j)\text{-}1(i)]^* \otimes \theta_i$ *and* $[1(i)\text{-}m(j)\text{-}3(i)]^* \otimes \eta_i$

Proof. We first prove that the sequences described give classes in $\ker \delta$.

1. When n is odd, $n+1$, 1-n, and n-1 are not admissible sequences for $\mathcal{O}_X$ or P_i.

2. When n is even, $n+1$ is an admissible sequence for both $\mathcal{O}_X$ and P_i, so,

$$\delta_1([n(i)]^* \otimes \theta_i) = (-1)^{(n+2)/4}[(n+1)(i)]^* \otimes \psi_i - [(n+1)(i)]^* \otimes \xi_i,$$
$$\delta_1([n(i)]^* \otimes \eta_i) = [(n+1)(i)]^* \otimes \xi_i - [(n+1)(i)]^* \otimes \psi_i.$$

Then the sum of these is therefore in $\ker \delta_1$ if and only if $(n+2)/4$ is even.

3. We have,

$$\delta_1([1(\mathrm{i})\text{-}m(j)]^* \otimes \eta_i) = \pm [1(\mathrm{i})\text{-}m(j)\text{-}1(\mathrm{i})] \otimes \psi_i,$$
$$\delta_1([m(j)\text{-}1(\mathrm{i})]^* \otimes \theta_i = \pm [1(\mathrm{i})\text{-}m(j)\text{-}1(\mathrm{i})]^* \otimes \xi_i.$$

So a sum or difference of these is in $\ker \delta_1$. The same calculation gives this result for 1-m_1-m_2 and m_1-m_2-1 by substituting m_1-m_2 for m.

4. We have,

$$\delta_1([1(i)\text{-}m(j)\text{-}4(i)]^* \otimes \eta_i) = \pm [1(i)\text{-}m(j)\text{-}5(i)]^* \otimes \psi_i,$$
$$\delta_1([m(j)\text{-}5(i)]^* \otimes \theta_i) = \pm[1(i)\text{-}m(j)\text{-}5(i)]^* \otimes \psi_i.$$

This shows that a sum or difference of these is in $\ker \delta_1$, and the calculation for the reverse sequences is analogous.

5. We have,

$$\delta_1([2(i)\text{-}m(j)\text{-}3(i)]^* \otimes \eta_i) = [3(i)\text{-}m(j)\text{-}3(i)]^* \otimes \xi_i,$$
$$\delta_1([3(i)\text{-}m(j)\text{-}2(i)]^* \otimes \theta_i = \pm [3(i)\text{-}m(j)\text{-}3(i)]^* \otimes \psi_i.$$

So a sum or difference of these is in $\ker \delta_1$.

6. These follow as in (1), since 4-m-1, 3-m-2 are not admissible for $\mathcal{O}_X$ or P_i.

It remains to show that this is the entire kernel. Let $\Omega(F_1/F_2)$ be the sequence basis for $HH^n(F_1/F_2)$ defined by all decorated admissible sequences. Let $V_1 \subset HH^n(F_1/F_2)$ be the space with basis consisting of the set $\Omega_1(F_1/F_2)$ of elements of $\Omega(F_1/F_2)$ not yet considered;

i.e.,

$$V_1 = \langle\{[m(j)\text{-}k(i)]^* \otimes \theta_i, [k(i)\text{-}m(j)]^* \otimes \eta_i, |k \text{ is even}, \},$$
$$\{[m_1(j_1)\text{-}m_2(j_2)\text{-}2(i)]^* \otimes \theta_i, [2(i)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_i\}\rangle.$$

Let V_2 be the space with basis $\Omega(F_1/F_2)/\Omega_1(F_1/F_2)$, so that $HH^n(F_1/F_2) = V_1 \oplus V_2$.

We will construct a decomposition $HH^{n+1}(F_2) = W_1 \oplus W_2$ such that 1) $\delta_1(V_2) \subset W_2$, and 2) the projection of $\delta(V_1)$ onto W_1 is an isomorphism. This will complete the claim.

We begin by calculating $\delta_1(V_1)$.

$$\begin{aligned}
\delta_1([m(j)\text{-}k(i)]^* \otimes \theta_i) =& \pm [1(\mathrm{i})\text{-}m(j)\text{-}k(i)]^* \otimes \psi_i \pm [m(j)\text{-}(k+1)(\mathrm{i})]^* \otimes \xi_i,\\
\delta_1([k(i)\text{-}m(j)]^* \otimes \eta_i) =& \pm [(k+1)(\mathrm{i})\text{-}m(j)]^* \otimes \xi_i \pm [k(i)\text{-}m(j)\text{-}1(\mathrm{i})]^* \otimes \psi_i,\\
\delta_1([2(\mathrm{i})\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_i) =& [3(\mathrm{i})\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \xi_i \pm\\
& [2(\mathrm{i})\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}1(\mathrm{i})]^* \otimes \psi_i,\\
\delta_1([m_1(j_1)\text{-}m_2(j_2)\text{-}2(\mathrm{i})]^* \otimes \theta_i =& \pm [1(\mathrm{i})\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}2(\mathrm{i})]^* \otimes \psi_i \pm\\
& [m_1(j_1)\text{-}m_2(j_2)\text{-}3(\mathrm{i})]^* \otimes \xi_i.
\end{aligned}$$

Let $W_1 \subset HH^{n+1}(F_2)$ have basis consisting of all

$$[m(j)\text{-}(k+1)(i)]^* \otimes \xi_i,\ \ [(k+1)(\mathrm{i})\text{-}m(j)]^* \otimes \xi_i,$$

$$[3(\mathrm{i})\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \xi_i, [m_1(j_1)\text{-}m_2(j_2)\text{-}3(\mathrm{i})]^* \otimes \xi_i,$$

and let W_2 be the complement of W_1 with respect to the sequence basis. The calculations in 1-6 and in the equations above show that this decomposition satifies the required properties. In particular, each basis vector v of V_1 has exactly one basis vector w of W_1 in the expansion of $\delta_1(v)$, and this gives a one-to-one correspondence between bases. So $\ker \delta_1 \subset V_2$, and this completes the proof. $\square$

Lemma V.3.4. *The following form a basis of* $\ker \delta_1^{0,n}$ *for* $n > 0$*:*

1. $[n(i)]^* \otimes id_{P_i}$, *where* n *is odd;*

2. $[n_1(i)\text{-}m(j)\text{-}n_2(i)]^* \otimes id_{P_i}$, *where* n_1, n_2 *are odd;*

3. $[n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}n_2(i)]^* \otimes id_{P_i}$ *where* n_1, n_2 *are odd;*

4. $[m_1(j_1)\text{-}m_2(j_2)]^* \otimes id_{\mathcal{O}_X} \pm [(m_1-1)(j_1)\text{-}m_2(j_2)\text{-}1(j_1)]^* \otimes id_{P_i} \pm$ $[1(j_2)\text{-}m_1(j_1)\text{-}(m_2-1)(j_2)]^* \otimes id_{P_i}$, *where* m_1,m_2 *are odd, in genus 2 only.*

Proof. We first show that the vectors described in 1-4 are in $\ker \delta_1$. Note that we restrict to $n > 0$ due to Remark V.3.2.

1. When n odd is admissible for P_i, $n+1$ is not admissible for θ_i, η_i.

2,3. In both cases, adding 1 to the outside sequences n_1 and n_2 do not produce admissible sequences for $\mathcal{O}_X$ or P_i since n_1, n_2 are odd.

4. Since m_1, m_2 are odd, neither of the sequences $(m_1+1)\text{-}m_2$ or $m_1\text{-}(m_2+1)$ are admissible. Since we assume $g = 2$,

$$\begin{aligned}
\delta_1([m_1(j_1)\text{-}m_2(j_2)]^* \otimes \mathrm{id}_{\mathcal{O}_X}) = {} & \pm [1(j_2)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_{j_2} - \\
& [m_1(j_1)\text{-}m_2(j_2)\text{-}1(j_1)]^* \otimes \theta_{j_1}, \\
\delta_1([(m_1-1)(j_1)\text{-}m_2(j_2)\text{-}1(j_1)]^* \otimes \mathrm{id}_{P_i}) = {} & \pm [m_1(j_1)\text{-}m_2(j_2)\text{-}1(j_1)]^* \otimes \theta_{j_1}, \\
\delta_1(1(j_2)\text{-}m_1(j_1)\text{-}(m_2-1)(j_2)]^* \otimes \mathrm{id}_{P_i}) = {} & - [1(j_2)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_{j_2}.
\end{aligned}$$

It follows that a linear combination of these is in $\ker \delta_1$.

We will prove that this is the entire kernel as follows. Let $\Omega(F_0/F_1)$ be the sequence basis for $HH^n(F_0/F_1)$, and $\Omega(F_1/F_2)$ be the sequence basis for $HH^{n+1}(F_1/F_2)$.

Let $V_1 \subset HH^n(F_0/F_1)$ be the subspace with basis $\Omega_1(F_0/F_1) \subset \Omega(F_0/F_1)$ consisting of those basis vectors considered in 1-4 above. The complement of $\Omega_1(F_0/F_1)$ in $\Omega(F_0/F_1)$ can be partitioned into two classes: those classes of the form $[s]^* \otimes \mathrm{id}_{P_i}$ for some i, and those of the form $[s]^* \otimes \mathrm{id}_{\mathcal{O}_X}$. Let V_2 be the space spanned by the former classes with sequence basis $\Omega_2(F_0/F_1)$, and V_3 be the space spanned by the latter with sequence basis $\Omega_3(F_0/F_1)$, so $HH^n(F_0/F_1) = V_1 \oplus V_2 \oplus V_3$.

We will provide a decomposition $HH^{n+1}(F_1/F_2) = W_1 \oplus W_2 \oplus W_3$ together with bases $\Omega_1(F_1/F_2)$, $\Omega_2(F_1/F_2)$, and $\Omega_3(F_1/F_2)$ having the properties that:

(1) $\delta(V_1) \subset W_1$;

(2) the projection of $\delta(V_2)$ onto W_2 gives a one-to-one correspondence between basis vectors, so that in some choice of basis orderings the matrix of this transformation is the identity;

(3) $\delta(V_1 \oplus V_2) \subset W_1 \oplus W_2$; and

(4) the projection of $\delta(V_3)$ onto W_3 gives a one-to-one correspondence between basis vectors, so that in some choice of basis orderings the matrix of this transformation is the identity.

Thus in particular $\delta(V_1) \subset W_1$ and the matrix of the projection of $\delta(V_2 \oplus V_3)$ onto $W_2 \oplus W_3$ can be chosen to be upper triangular with ones on the diagonal; this would complete the proof.

The spaces W_2 and W_3 are defined in Appendix C. We then define W_1 as the complement of W_2 and W_3 in $HH^{n+1}(F_1/F_2)$ with respect to the basis $\Omega(F_1/F_2)$. One then checks based on the calculations in 1-4 and Appendix C that this decomposition satisfies these properties. $\square$

There is only one nontrivial map on page two of this spectral sequence,

$$\delta_2^{0,n} : \ker \delta_1^{0,n} \to \operatorname{coker} \delta_1^{1,n-2}.$$

We calculate δ_2 using sequences from Table 5 similar to the computation of δ_1. To calculate δ_2 on $\ker \delta_1^{0,n}$, we take admissible sequences of type s_3 and s_4, defined after Table 5.

Lemma V.3.5. *$\delta_2^{0,n}$ is injective for all $n > 0$.*

Proof. We restrict to $n > 0$ due to Remark V.3.2. Lemma V.3.4 calculates a basis for $\ker \delta_1^{0,n}$. We prove that $\delta_2^{0,n}$ is injective by describing a decomposition $HH^{n+1}(F_2) = W_1 \oplus W_2$ together with bases of W_1 and W_2 such that: 1) $\delta_1^{1,n-1}(HH^n(F_1/F_2)) \subset W_1$; and 2) the projection of $\delta_2^{0,n}(\ker \delta_1^{0,n})$ onto W_2 provides a one-to-one correspondence of basis vectors, so that under some ordering of bases the matrix of this transformation is the identity. This would complete the proof.

Let $\Omega(F_2)$ be the sequence basis for $HH^{n+1}(F_2)$. We start by calculating values of $\delta_2^{0,n}$ on the basis from Lemma V.3.4, letting $x(j_1, j_2)$ represent the term from (4). We take the underlined terms on the right below as the basis $\Omega_2(F_2)$ of W_2.

$$\begin{aligned}
\delta_2([n(i)]^* \otimes \mathrm{id}_{P_i}) = &\pm \underline{[(n+1)(i)]^* \otimes \psi_i},\\
\delta_2([n_1(i)\text{-}m(j)\text{-}n_2(i)]^* \otimes \mathrm{id}_{P_i}) = &\pm \underline{[(n_1+1)(i)\text{-}m(j)\text{-}n_2(i)]^* \otimes \psi_i}+\\
&(-1)^m[n_1(i)\text{-}m(j)\text{-}(n_2+1)(i)]^* \otimes \psi_i,\\
\delta_2([n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}n_2(i)]^* \otimes \mathrm{id}_{P_i}) = &\pm \underline{[(n_1+1)(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}n_2(i)]^* \otimes \psi_i}+\\
&(-1)^{m_1+m_2}[n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}(n_2+1)(i)]^* \otimes \psi_i,\\
\delta_2(x(j_1,j_2)) = &\pm \underline{[(m_1+1)(j_1)\text{-}m_2(j_2)]^* \otimes \xi_{j_1}}\pm\\
&[m_1(j_1)\text{-}(m_2+1)(j_2)]^* \otimes \xi_{j_2}\pm\\
&[(m_1-1)(j_1)\text{-}m_2(j_2)\text{-}2(j_1)]^* \otimes \psi_{j_1}\pm\\
&[2(j_2)\text{-}m_1(j_1)\text{-}(m_2-1)(j_2)]^* \otimes \psi_{j_2}.
\end{aligned}$$

We define W_1 to be the complement of W_2 with respect to the basis $\Omega(F_2)$. We compare these results to those in Lemma V.3.3 to verify that this decomposition satisfies the given conditions. □

Lemmas V.3.5 and V.3.3 give an easy way to count the dimension of $HH^n(B)$.

Lemma V.3.6. *In nonzero internal degrees,*

$$\dim HH^n(B) = \dim HH^n(F_2) - \operatorname{rank} \delta_1^{1,n-2} + \dim \ker \delta_1^{1,n-1} - \dim HH^{n-1}(F_0/F_1).$$

Proof. We restrict to nonzero internal degrees due to Remark V.3.2. In cohomological degree n on page two of the spectral sequence are the spaces,

$$\operatorname{coker} \delta_1^{1,n-2},\ \ker \delta_1^{1,n-1}/\operatorname{image} \delta_1^{0,n-1},\ \text{and}\ \ker \delta_1^{0,n}.$$

Taking cohomology on this page leaves the spaces

$$\ker \delta_1^{1,n-1}/\operatorname{image} \delta_1^{0,n-1}\ \text{and}\ \operatorname{coker} \delta_2^{0,n-1} = \operatorname{coker} \delta_1^{1,n-2}/\ker \delta_1^{0,n-1}.$$

Counting dimensions we have,

$$\dim(\ker \delta_1^{1,n-1} / \operatorname{image} \delta_1^{0,n-1}) = \dim\ker \delta_1^{1,n-1} - \operatorname{rank} \delta_1^{0,n-1},$$
$$\dim(\operatorname{coker} \delta_1^{1,n-2} / \ker \delta_1^{0,n-1}) = \dim HH^n(F_2) - \operatorname{rank} \delta_1^{1,n-2} - \dim\ker \delta_1^{0,n-1}.$$

Adding these and using the rank-nullity identity for $\delta_1^{0,n-1}$ gives the result. □

V.4. Hochschild cohomology of B^g

We now proceed to the main computation. Recall from Remark V.2.2 that

$$F_1/F_2 = V_\theta \oplus V_\eta, \text{ so } HH^n(F_1/F_2) = HH^n(V_\theta) \oplus HH^n(V_\eta).$$

Lemmas V.4.1-V.4.8 have tables that contain the nonzero spaces on page one of the spectral sequence in some fixed internal degree; the sequence types from Table 5 which correspond to classes in that space; and the dimension of the space spanned by classes of that sequence type.

Lemma V.4.1. $\dim HH^1_{(0)}(B) = g$, *and* $HH^2_{(0)}(B) = HH^3_{(0)}(B) = 0.$

Proof. It follows from Table 5 that the spaces on page one in internal degree 0 have the dimensions listed in Table 6.

Space	sequence(s)	dimension
$HH^0_{(0)}(F_0/F_1)$	0	$g+1$
$HH^1_{(0)}(V_\eta)$	1	g
$HH^1_{(0)}(V_\theta)$	1	g

Table 6: Dimensions in internal degree 0

Since $HH^1_{(0)}(F_2) = 0$, $\dim\ker \delta_1^{1,-1} = 2g$. Using the representatives of $HH^0_{(0)}(F_0/F_1)$, we calculate,

$$\delta_1([1]^* \otimes \mathrm{id}_{\mathcal{O}_X}) = \sum_{i=1}^{g}([\eta_i]^* \otimes \eta_i + [\theta_i]^* \otimes \theta_i),$$
$$\delta_1([1]^* \otimes \mathrm{id}_{P_i}) = [\eta_i]^* \otimes \eta_i + [\theta_i]^* \otimes \theta_i.$$

So there is a 1-dimensional kernel, and $\operatorname{rank} \delta_1^{0,0} = g$. The result follows. $\square$

Lemma V.4.2. $\dim HH^2_{(-1)}(B) = 0$, $\dim HH^3_{(-1)}(B) = g^2 - g$, *and* $HH^4_{(-1)}(B) = 0$.

Proof. Table 7 summarizes the spaces in degree -1 on page one.

Space	sequence	dimension
$HH^2_{(-1)}(V_\eta)$	2	g
$HH^2_{(-1)}(V_\theta)$	2	g
$HH^3_{(-1)}(F_2)$	3	$g^2 + g$

Table 7: Dimensions in internal degree -1

By Lemma V.3.3 part (2), $\delta_1^{1,1}$ has no kernel; so $\operatorname{rank} \delta_1^{1,1} = 2g$. The result follows from Lemma V.3.6. $\square$

Lemma V.4.3. $HH^3_{(-2)}(B) = 0$, $\dim HH^4_{(-2)}(B) = 2g^2 - 2g$, *and* $HH^5_{(-2)}(B) = 0$.

Proof. Table 8 summarizes the spaces on page one in degree -2.

Space	sequence	dimension
$HH^3_{(-2)}(F_0/F_1)$	3	$2g$
$HH^4_{(-2)}(F_2)$	4	$g^2 + g$
$HH^4_{(-2)}(V_\eta)$	1-3	$g(g-1)$
$HH^4_{(-2)}(V_\theta)$	3-1	$g(g-1)$
$HH^5_{(-2)}(F_2)$	1-3-1	$g(g-1)$

Table 8: Dimensions in internal degree -2

By Lemma V.3.3 part (3) and V.3.5, $\ker \delta_1^{1,3}$ has basis,

$$\{[3(i)\text{-}1(j)]^* \otimes \theta_j - [1(j)\text{-}3(i)]^* \otimes \eta_j | j \neq i\} \quad g(g-1).$$

So $\dim \ker \delta_1^{1,3} = g^2 - g$ and $\operatorname{rank} \delta_1^{1,3} = g^2 - g$.

$\square$

Lemma V.4.4. *$HH^4_{(-3)}(B) = 0$, $\dim HH^5_{(-3)}(B) = g^2 - g$, and $\dim HH^6_{(-3)}(B) = g^3 - 2g^2 + g$.*

Proof. Table 9 summarizes the spaces on page one in degree -3.

Space	sequence(s)	dimension
$HH^4_{(-3)}(F_0/F_1)$	4	$2g$
$HH^5_{(-3)}(V_\eta)$	5	g
	1-4	$g(g-1)$
	2-3	$g(g-1)$
$HH^5_{(-3)}(V_\theta)$	5, 4-1, 3-2	$g + 2g(g-1)$
$HH^5_{(-3)}(F_0/F_1)$	1-3-1	$g(g-1)$
$HH^6_{(-3)}(F_2)$	3-3	$g^2(g-1)$
	1-4-1, 2-3-1, 1-3-2	$3g(g-1)$

Table 9: Dimensions in internal degree -3

Lemma V.3.3 implies that $\ker \delta_1^{1,4}$ has basis

$$\{[5(i)]^* \otimes \theta_i, [5(i)]^* \otimes \eta_i | i = 1, \dots, g\} \quad 2g,$$

$$\{[4(i)\text{-}1(j)]^* \otimes \theta_j - [1(j)\text{-}4(i)]^* \otimes \eta_j\} \quad g(g-1).$$

So $\ker \delta_1^{1,4}$ has dimension $g^2 + g$; $\operatorname{rank} \delta_1^{1,4} = 3g^2 - 3g$. □

Lemma V.4.5. *$\dim HH^6_{(-4)}(B) = g$ and $\dim HH^7_{(-4)}(B) = 3g^3 - 5g^2 + 3g$.*

Proof. Table 10 summarizes the spaces on page one in degree -4.

Space	sequence(s)	dimension
$HH^6_{(-4)}(V_\eta)$	6	g
	2-4	$g(g-1)$
$HH^6_{(-4)}(V_\theta)$	6, 4-2	$g + g(g-1)$
$HH^6_{(-4)}(F_0/F_1)$	3-3	$g(g-1)$
	1-4-1, 2-3-1, 1-3-2	$3g(g-1)$
$HH^7_{(-4)}(F_2)$	7	$g^2 + g$
	4-3, 3-4	$2g^2(g-1)$
	2-4-1, 1-4-2, 2-3-2	$3g(g-1)$
$HH^7_{(-4)}(V_\eta)$	1-3-3	$g(g-1)^2$
$HH^7_{(-4)}(V_\theta)$	3-3-1	$g(g-1)^2$

Table 10: Dimensions in internal degree -4

By Lemma V.3.3, $\ker \delta_1^{1,5}$ has basis

$$\{[6(i)]^* \otimes \theta_i - [6(i)]^* \otimes \eta_i | i = 1, \ldots, g\},$$

and so has dimension g; thus $\operatorname{rank} \delta_1^{1,5} = 2g^2 - g$.

By Lemma V.3.5 and V.3.4, $\ker \delta_1^{1,6}$ has basis,

$$\begin{aligned} &\{[3(i)\text{-}3(j)\text{-}1(i)]^* \otimes \theta_i, [1(i)\text{-}3(j)\text{-}3(i)]^* \otimes \eta_i\} \quad 2g(g-1), \\ \{[3(j_1)\text{-}3(j_2)\text{-}1(i)]^* &\otimes \theta_i + [1(i)\text{-}3(j_1)\text{-}3(j_2)]^* \otimes \eta_i\} \quad g(g-1)(g-2). \end{aligned}$$

So $\dim \ker \delta_1^{1,6} = g^3 - g^2$.

□

Lemma V.4.6. *$HH^7_{(-5)}(B) = 0$ and $\dim HH^8_{(-5)}(B) = 3g^3 - 4g^2 + g$.*

Proof. Table 11 summarizes the spaces in degree -5 on page one.

Space	sequence(s)	dimension
$HH^7_{(-5)}(F_0/F_1)$	7	$2g$
	3-4, 4-3	$2g(g-1)$
	2-4-1, 1-4-2, 2-3-2	$3g(g-1)$
$HH^8_{(-5)}(F_2)$	8	g^2+g
	4-4	$g^2(g-1)$
	2-4-2	$g(g-1)$
$HH^8_{(-5)}(V_\eta)$	5-3, 1-7	$2g(g-1)$
	1-4-3, 1-3-4	$2g(g-1)^2$
	2-3-3	$g(g-1)^2$
$HH^8_{(-5)}(V_\theta)$	3-5, 7-1	$2g(g-1)$
	3-4-1, 4-3-1, 3-3-2	$3g(g-1)^2$

Table 11: Dimensions in internal degree -5

By Lemma V.3.3, $\ker \delta_1^{1,7}$ has basis:

$$\begin{aligned}
\{[7(i)\text{-}1(j)]^* \otimes \theta_j + [1(j)\text{-}7(i)]^* \otimes \eta_j\} &\quad g(g-1),\\
\{[3(j_1)\text{-}4(j_2)\text{-}1(i)]^* \otimes \theta_i + [1(i)\text{-}3(j_1)\text{-}4(j_2)]^* \otimes \eta_i\} &\quad g(g-1)(g-2),\\
\{[4(j_1)\text{-}3(j-2)\text{-}1(i)]^* \otimes \theta_i + [1(i)\text{-}4(j_1)\text{-}3(j_2)]^* \otimes \eta_i\} &\quad g(g-1)(g-2),\\
\{[1(i)\text{-}3(j)\text{-}4(i)]^* \otimes \eta_i + [3(j)\text{-}5(i)]^* \otimes \theta_i\} &\quad g(g-1),\\
\{[4(i)\text{-}3(j)\text{-}1(i)]^* \otimes \theta_i + [5(i)\text{-}3(j)]^* \otimes \eta_i\} &\quad g(g-1),\\
\{[3(i)\text{-}3(j)\text{-}2(i)]^* \otimes \theta_i + [2(i)\text{-}3(j)\text{-}3(i)]^* \otimes \eta_i\} &\quad g(g-1),\\
\{[1(i)\text{-}4(j)\text{-}3(i)]^* \otimes \eta_i, [3(i)\text{-}4(j)\text{-}1(i)]^* \otimes \theta_i\} &\quad 2g(g-1).
\end{aligned}$$

So $\dim \ker \delta_1^{1,7} = 2g^3 - 2g$. □

Lemma V.4.7. *$HH^8_{(-6)}(B) = 0$ and $\dim HH^9_{(-6)}(B) = g^3 - g^2$.*

Proof. Table 12 summarizes the spaces in degree -6 on page one.

Space	sequence(s)	dimension
$HH^8_{(-6)}(F_0/F_1)$	8	$2g$
	4-4	$g(g-1)$
	2-4-2	$g(g-1)$
$HH^9_{(-6)}(V_\eta)$	9	g
	1-8, 2-7, 5-4, 6-3	$4g(g-1)$
	1-4-4, 2-3-4, 2-4-3	$3g(g-1)^2$
$HH^9_{(-6)}(V_\theta)$	9	g
	8-1, 7-2, 4-5, 3-6	$4g(g-1)$
	4-4-1, 4-3-2, 3-4-2	$3g(g-1)^2$

Table 12: Dimensions in internal degree -6

By Lemma V.3.3 $\ker \delta_1^{1,8}$ has basis:

$$\{[9(i)]^* \otimes \theta_i, [9(i)]^* \otimes \eta_i\} \quad 2g$$

$$\{[8(i)\text{-}1(j)]^* \otimes \theta_j + [1(j)\text{-}8(i)]^* \otimes \eta_j\} \quad g(g-1),$$

$$\{[4(j_1)\text{-}4(j_2)\text{-}1(i)]^* \otimes \theta_i + [1(i)\text{-}4(j_1)\text{-}4(j_2)]^* \otimes \eta_i\} \quad g(g-1)(g-2),$$

$$\{[4(i)\text{-}4(j)\text{-}1(i)]^* \otimes \theta_i + [5(i)\text{-}4(j)]^* \otimes \eta_i\} \quad g(g-1),$$

$$\{[1(i)\text{-}4(j)\text{-}4(i)]^* \otimes \eta_i + [4(j)\text{-}5(i)]^* \otimes \theta_i\} \quad g(g-1),$$

$$\{[2(i)\text{-}4(j)\text{-}3(i)]^* \otimes \eta_i + [3(i)\text{-}4(j)\text{-}2(i)]^* \otimes \theta_i\} \quad g(g-1).$$

So $\dim \ker \delta_1^{1,8} = g^3 + g^2$.

□

Lemma V.4.8. $HH^{10}_{(-7)}(B) = 0$.

Proof. Table 13 summarizes the spaces in degree -7 on page one.

Space	sequence(s)	dimension
$HH^{10}_{(-7)}(V_\eta)$	10	g
	8-2	$g(g-1)$
	4-4-2	$g(g-1)^2$
$HH^{10}_{(-7)}(V_\theta)$	10	g
	2-8	$g(g-1)$
	2-4-4	$g(g-1)^2$

Table 13: Dimensions in internal degree -7

From Lemma V.3.3, $\delta_1^{1,9}$ has no kernel. □

Lemma V.4.9. *$HH^n(B) = 0$ in internal degrees $1-n$, $2-n$, and $3-n$ for $n > 10$.*

Proof. The positive terms in the result in Lemma V.3.6 depend on $HH^n(F_2)$ and $HH^n(F_1/F_2)$ being nonzero. For $n > 10$, these spaces are 0 in the specified internal degrees. □

In summary we have the following result.

Theorem V.4.10. *Let X be a curve of genus $g \geq 2$. Then,*

1. $\dim HH^n_{(1-n)}(B^g) = \begin{cases} g & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$

2. $\dim HH^n_{(2-n)}(B^g) = \begin{cases} g^2 - g & \text{if } n = 3 \\ 2g^2 - 2g & \text{if } n = 4 \\ g^2 - g & \text{if } n = 5 \\ g & \text{if } n = 6 \\ 0 & \text{otherwise.} \end{cases}$

3. $\dim HH^n_{(3-n)}(B^g) = \begin{cases} g^3 - 2g^2 + g & \text{if } n = 6 \\ 3g^3 - 5g^2 + 3g & \text{if } n = 7 \\ 3g^3 - 4g^2 + g & \text{if } n = 8 \\ g^3 - g^2 & \text{if } n = 9 \\ 0 & \text{otherwise.} \end{cases}$

V.5. An explicit $\mathbb{K}$-linear isomorphism $HH^3_{(-1)}(B^g) \cong \mathbb{K}^{g^2-g}$

This isomorphism will be used explicitly in Chapter VI. We define,

$$\alpha : HH^3_{(-1)}(B) \to \mathrm{Mat}_g(\mathbb{K}) \cong \mathbb{K}^{g^2},\ f \mapsto (\alpha_{ij}(f)),$$

where $\alpha_{ij}(f)$ is defined by the equation,

$$f(\theta_i \psi_i \eta_i) = \lambda \cdot \xi_i + \sum_{j \neq i} \alpha_{ij}(f) \cdot \xi_j.$$

Proposition V.5.1. *1. α is well-defined;*

2. α induces an isomorphism of $HH^3_{(-1)}(B)$ with the subspace consisting of those matrices (α_{ij}) with $\alpha_{ii} = 0$ for all i.

Proof. 1. We first show that α is well-defined by showing that it vanishes on boundaries. Let

$h \in C^2_{(-1)}(B^g)$. Then,

$$\begin{aligned}(\delta h)(\theta_i\psi_i\eta_i) &= \theta_i \cdot h(\psi_i\eta_i) - h(\theta_i\psi_i) \cdot \eta_i \\ &= \lambda \cdot \xi_i,\end{aligned}$$

since $h(\psi_i\eta_i)$ is proportional to η_i for degree reasons and R-linearity of h, and $h(\theta_i\psi_i)$ is proportional to θ_i similarly. So α is well-defined.

2. Since $\dim HH^3_{(-1)}(B) = g^2 - g$, it is enough to show that the image of α contains the subspace described, which clearly has dimension $g^2 - g$. For $i \neq j$, let

$$f_{ij} = (\theta_i\psi_i\eta_i)^* \otimes \xi_j + (\xi_i\theta_i\eta_i)^* \otimes \xi_j \in C^3_{(-1)}(F_2) \subset C^3_{(-1)}(B).$$

The first and last terms of the Hochschild differential vanish since f_{ij} maps to F_2, so for $(x, y, z, w) \in B_+^{\otimes 4}$,

$$(\delta f_{ij})(x, y, z, w) = -f_{ij}(xy, z, w) + f_{ij}(x, yz, w) - f_{ij}(x, y, zw).$$

In order for one of these three terms not to vanish, we must have either $(xy, z, w) = \xi_i\theta_i\eta_i$ or $(x, yz, w) = \theta_i\psi_i\eta_i$. So we must have $(x, y, z, w) = \theta_i\eta_i\theta_i\eta_i$; but in this case the nonzero terms cancel, so we conclude that f_{ij} is a cocycle. Furthermore $\alpha(f_{ij}) = E_{ij}$, where E_{ij} is the ij-matrix unit. This completes the claim.

□

CHAPTER VI

THE A_∞-ALGEBRA OF A CURVE OF GENUS $g \geq 2$

We continue using the notation of Chapter V, with X a smooth complex curve of genus $g \geq 2$. Let $P \in X$ be a closed point, with

$$\theta \in \mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(P)),\ \ \eta \in \mathrm{Ext}^1(\mathbb{K}(P), \mathcal{O}_X)$$

generators of these one-dimensional spaces, with

$$\xi = \theta\eta \in \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X),\ \ \psi = \eta\theta \in \mathrm{Ext}^1(\mathbb{K}(P), \mathbb{K}(P)).$$

The reader may recall from Section II.8. the definitions of the Massey product in a triangulated category.

VI.1. A Massey product in $D^b(X)$

We consider the sequence in $D^b(X)$,

$$\mathcal{O}_X \xrightarrow{\theta} \mathbb{K}(P) \xrightarrow{\psi} \mathbb{K}(P)[1] \xrightarrow{\eta} \mathcal{O}_X[1].$$

Since $\psi \circ \theta = \eta \circ \psi = 0$, the Massey product,

$$\mathrm{MP}(\theta, \psi, \eta) \in \mathrm{coker}(\mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(P)) \oplus \mathrm{Ext}^1(\mathbb{K}(P), \mathcal{O}_X) \xrightarrow{(\eta,\theta)} \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)),$$

is well-defined. This target space is simply $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)/\langle\xi\rangle$.

Let $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X, with

$$\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3 \in \mathrm{Ob}(\mathrm{Coh}(X)).$$

We recall the correspondence between elements $f \in \mathrm{Ext}^1(\mathscr{F}_1, \mathscr{F}_2)$ and pairs $(\mathscr{F}, g)$ with $\mathscr{F} \in \mathrm{Ob}(\mathrm{Coh}(X))$, $g \in \mathrm{Hom}_{\mathrm{Coh}(X)}(\mathscr{F}_2, \mathscr{F})$, such that there is an exact sequence

$$0 \to \mathscr{F}_2 \xrightarrow{g} \mathscr{F} \to \mathscr{F}_1 \to 0.$$

When f corresponds to $(\mathscr{F}, g)$ this way, we will say that $(f, \mathscr{F}, g)$ is an *extension triple.*

If $(f, \mathscr{F}, g)$ is an extension triple and $h \in \mathrm{Hom}_{(\mathrm{Coh}(X)}(\mathscr{F}_3, \mathscr{F}_1)$, then

$$f \circ h \in \mathrm{Ext}^1(\mathscr{F}_3, \mathscr{F}_2).$$

We understand this composition as the extension triple $(f \circ h, \widetilde{\mathscr{F}}, \tilde{g})$ where $\widetilde{\mathscr{F}}$ is a pullback in the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{F}_2 \arrow[r, "g"] \arrow[dr, dotted, "\tilde{g}"'] & \mathscr{F} \arrow[r] & \mathscr{F}_1 \arrow[r] & 0 \\
& & \widetilde{\mathscr{F}} \arrow[u, "\pi_1"] \arrow[r, "\pi_2"] & \mathscr{F}_3 \arrow[u, "h"'] \arrow[r] & 0
\end{tikzcd}$$

We now consider the maps θ, η, ξ, ψ in more detail. Let t be a local parameter in $\mathcal{O}_{X,P}$. Recall that we identify $\mathbb{K}(P)$ with $\mathcal{O}_X(P)/\mathcal{O}_X$.

1. The generator $\theta \in \mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(P))$ corresponds to the composition,

$$\mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_X/\mathcal{O}_X(P) \xrightarrow{\cdot t^{-1}} \mathbb{K}(P).$$

2. The generator $\eta \in \mathrm{Ext}^1(\mathbb{K}(P), \mathcal{O}_X)$ is in the extension triple $(\eta, \mathcal{O}_X(P), i)$; i.e., it corresponds to the sequence,

$$0 \to \mathcal{O}_X \xrightarrow{i} \mathcal{O}_X(P) \to \mathbb{K}(P) \to 0.$$

3. Let $\mathbb{K}(2P) = \mathcal{O}_X(2P)/\mathcal{O}_X$. The generator $\psi \in \mathrm{Ext}^1(\mathbb{K}(P), \mathbb{K}(P))$ is in the extension triple $(\psi, \mathbb{K}(2P), i)$; i.e., it corresponds to the sequence,

$$0 \to \mathbb{K}(P) \xrightarrow{i} \mathbb{K}(2P) \to \mathbb{K}(P) \to 0.$$

4. The map $\xi \in \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$ is in the extension triple $(\xi, Z, \tilde{i})$ which corresponds to the bottom row in the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i"] \arrow[dr, dotted, "\tilde{i}"'] & \mathcal{O}_X(P) \arrow[r] & \mathbb{K}(P) \arrow[r] & 0 \\
& & Z \arrow[u, "\pi_1"] \arrow[r, "\pi_2"] & \mathcal{O}_X \arrow[u, "\theta"'] \arrow[r] & 0
\end{tikzcd}$$

Since

$$\mathbb{K}(P) \xrightarrow{i} \mathbb{K}(2P) \xrightarrow{\pi} \mathbb{K}(P) \xrightarrow{\psi} \mathbb{K}(P)[1]$$

is a distinguished triangle in $D^b(X)$, there exist liftings $\tilde{\theta} \in \mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(2P))$ and $\tilde{\eta} \in \mathrm{Ext}^1(\mathbb{K}(2P), \mathcal{O}_X)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\theta"] \arrow[dr, dotted, "\tilde{\theta}"'] & \mathbb{K}(P) \arrow[r, "\psi"] & \mathbb{K}(P) \arrow[r, "\eta"] \arrow[dl, "i"'] & \mathcal{O}_X \\
& \mathbb{K}(2P) \arrow[u, "\pi"] \arrow[urr, dotted, "\tilde{\eta}"'] & &
\end{tikzcd}$$

Then $\mathrm{MP}(\theta, \psi, \eta) = [\tilde{\eta} \circ \tilde{\theta}] \in \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)/\langle \xi \rangle$.

Lemma VI.1.1. *For one choice of $\tilde{\eta}$, we have an extension triple $(\tilde{\eta}, \mathcal{O}_X(2P), i)$.*

Proof. Let $(\beta, \mathcal{O}_X(2P), i)$ be an extension triple, corresponding to the sequence,

$$0 \to \mathcal{O}_X \xrightarrow{i} \mathcal{O}_X(2P) \to \mathbb{K}(2P) \to 0.$$

It is sufficient to show that $\beta \circ i = \eta$. Composing $\beta \circ i$ gives the extension Z' in the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i"] \arrow[dr] & \mathcal{O}_X(2P) \arrow[r, "\pi"] & \mathbb{K}(2P) \arrow[r] & 0 \\
& & Z' \arrow[u] \arrow[r] & \mathbb{K}(P) \arrow[u, "i"'] \arrow[r] & 0
\end{tikzcd}$$

The square,

$$\begin{tikzcd}
\mathcal{O}_X(2P) \arrow[r, "\pi"] & \mathbb{K}(2P) \\
\mathcal{O}_X(P) \arrow[u, "i"] \arrow[r, "\pi"] & \mathbb{K}(P) \arrow[u, "i"']
\end{tikzcd}$$

commutes, so by the universal property of the pullback, i and π from $\mathcal{O}_{X,P}$ factor through Z' by a map $(i, \pi) : \mathcal{O}_X(P) \to Z'$.

$$\begin{tikzcd}
& \mathcal{O}_X(2P) \arrow[dr] & \\
\mathcal{O}_X(P) \arrow[ur] \arrow[r, dotted, "(i,\pi)"] \arrow[dr] & Z' \arrow[u] \arrow[d] & \mathbb{K}(2P) \\
& \mathbb{K}(P) \arrow[ur] &
\end{tikzcd}$$

We claim that all squares in the following diagram commute, giving an equivalence of extensions $\mathcal{O}_X(P) \sim Z'$.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "="'] & \mathcal{O}_X(P) \arrow[r] \arrow[d, "(i,\pi)"'] & \mathbb{K}(P) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & Z' \arrow[r] & \mathbb{K}(P) \arrow[r] & 0
\end{tikzcd}$$

The second square commutes trivially by the diagram above, and the first square commutes since all of these maps are inclusions. $\square$

Let $K(\mathcal{O}_{X,P})$ be the fraction field of $\mathcal{O}_{X,P}$, which has $\mathbb{K}$-basis $\{\ldots, t^{-2}, t^{-1}, 1, t, t^2, \ldots\}$. For $f \in K(\mathcal{O}_{X,P})$, let $\mathrm{pol}(f)$ be the projection of f onto $V = \mathrm{Span}\{t^{-1}, t^{-2}, \ldots\}$, i.e., $\mathrm{pol}(f)$ is the polar part of f at P.

Lemma VI.1.2. *One choice of $\tilde{\theta}$ is the composition*

$$\mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_X(-2P) \xrightarrow{\cdot t^{-2}} \mathbb{K}(2P).$$

Proof. Since the target is a skyscraper at P, it is sufficient to prove that this choice works on stalks at P. Let $f \in \mathcal{O}_{X,P}$. By the discussion preceding these lemmas we know that $\theta_P(f) = \mathrm{pol}(t^{-1} \cdot f)$. The projection $\pi : \mathbb{K}(2P) \to \mathbb{K}(P)$ is given by $\pi_P(f) = \mathrm{pol}(t \cdot f)$. The composition in the statement of the lemma maps $f \mapsto t^{-2} \cdot f$. So with this composition at $\tilde{\theta}$ we have $\pi_P \circ \tilde{\theta}_P = \theta_P$ as needed. $\square$

We compose $\tilde{\eta} \circ \tilde{\theta}$ to get W in the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i"] \arrow[dr, "\tilde{i}"] & \mathcal{O}_X(2P) \arrow[r] & \mathbb{K}(2P) \arrow[r] & 0 \\
 & & W \arrow[u] \arrow[r] & \mathcal{O}_X \arrow[u, "\tilde{\theta}"'] \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(2P))$ is two-dimensional, we also consider the morphism $t \cdot \tilde{\theta}$, which is the map on stalks $f \mapsto \mathrm{pol}(t^{-1} f)$, corresponding to the extension W' in the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r, "i"] \arrow[dr, "\tilde{i}"] & \mathcal{O}_X(2P) \arrow[r] & \mathbb{K}(2P) \arrow[r] & 0 \\
 & & W' \arrow[u] \arrow[r] & \mathcal{O}_X \arrow[u, "t \cdot \tilde{\theta}"']
\end{tikzcd}$$

Lemma VI.1.3. $W' \cong Z$

Proof. We have a diagram,

$$\begin{tikzcd}
Z \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{O}_X(P) \arrow[r, "i"] & \mathcal{O}_X(2P) \arrow[d, "\pi"] \\
 & \mathcal{O}_X \arrow[r, "t \cdot \tilde{\theta}"'] & \mathbb{K}(2P)
\end{tikzcd}$$

For this diagram to commute it is enough to check on stalks at P, since the target is a skyscraper at P. We can describe Z explicitly,

$$Z = \ker(\mathcal{O}_X(P) \oplus \mathcal{O}_X \xrightarrow{(\pi,-\theta)} \mathbb{K}(P)),$$

so

$$Z_P = \{(f_1, f_2) \in \mathcal{O}_X(P)_P \oplus \mathcal{O}_{X,P} | \operatorname{pol}(f_1) = \operatorname{pol}(t^{-1} f_2).$$

Then,

$$(\pi \circ i \circ \pi_1)_P(f_1, f_2) = \pi_P(f_1) = \operatorname{pol}(f_1),$$
$$(t \cdot \tilde{\theta} \circ \pi_2)_P(f_1, f_2) = t \cdot \tilde{\theta}_P(f_2) = \operatorname{pol}(t^{-1} f_2).$$

Since the diagram commutes, the maps to $\mathcal{O}_X(2P)$ and $\mathcal{O}_X$ factor through W',

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr, dotted, "\alpha"] \arrow[ddr] & \mathcal{O}_X(P) \arrow[r] & \mathcal{O}_X(2P) \arrow[d] \\
& W' \arrow[ur] \arrow[d] & \mathbb{K}(2P) \\
& \mathcal{O}_X \arrow[ur] &
\end{tikzcd}$$

The map α fits into the diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "="'] & Z \arrow[r] \arrow[d, "\alpha"'] & \mathcal{O}_X \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & W' \arrow[r] & \mathcal{O}_X \arrow[r] & 0
\end{tikzcd}$$

The top and bottom maps in the left square are inclusion in the first factor and α is the identity on the first factor, so this square commutes; the right square is the bottom left triangle on the diagram defining α. Therefore this diagram commutes, and α is an isomorphism by the Five Lemma. $\square$

Proposition VI.1.4. $\mathrm{MP}(\theta, \psi, \eta) = 0$ *if and only if X is hyperelliptic and P is a Weierstrass point on X.*

Proof. The coset $\mathrm{MP}(\theta, \psi, \eta) \in \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)/\langle \xi \rangle$ is represented by W, while the coset 0 is represented by Z. It follows that $\mathrm{MP}(\theta, \psi, \eta) = 0$ if and only if W and Z are proportional in $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$.

The short exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(2P) \to \mathbb{K}(2P) \to 0$$

gives rise to a long exact sequence on Ext,

$$0 \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(2P)) \to \mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(2P)) \xrightarrow{\gamma} \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \to \cdots .$$

The map γ is injective if and only if

$$\dim \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(2P)) = \dim \Gamma(\mathcal{O}_X(2P)) = 1,$$

i.e., P is not a Weierstrass point.

Thus when P is not a Weierstrass point, W and W' are linearly independent; since $W' \cong Z$ by Lemma VI.1.3, it must be that W and Z linearly independent. When P is Weierstrass, γ is not injective, so W and W' map to the same line in $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$, therefore in that case the Massey product vanishes. Non-hyperelliptic curves have no Weierstrass points, so this forces X to be hyperelliptic. ☐

VI.2. Homotopy class of triple products on a curve

An immediate consequence of [19] Proposition 1.1 is that since the A_∞-structure on B^g is minimal:

1. $\mathrm{MP}(x, y, z) = [m_3(x, y, z)]$, for $x, y, z \in B_+$ such that $y \circ x = z \circ y = 0$ and $[m_3(x, y, z)]$ denotes the coset of the element $m_3(x, y, z)$ in the space where $\mathrm{MP}(x, y, z)$ is defined; and

2. if $(f) : B^g \to B^g$ is a strict equivalence such that $m'_3 = f * m_3$, then $[m'_3(x,y,z)] = \mathrm{MP}(x,y,z)$ as well.

In this way we may think of the Massey product as a strict equivalence-invariant version of m_3. Together with Proposition VI.1.4 this gives the following result.

Theorem VI.2.1. *Let X be a smooth curve of genus $g \geq 2$ over $\mathbb{C}$. Then m_3 is homotopic to 0 if and only if X is hyperelliptic and $P_1, \ldots, P_g$ are Weierstrass points.*

Proof. Let α be the map in Proposition V.5.1. Then Proposition VI.1.4 implies that $\alpha_{ij}(m_3) = 0$ (i.e., m_3 is a coboundary in Hochschild cohomology) for $i \neq j$ if and only if each point is a Weierstrass point. Since there are $2g+2$ of them for any particular curve, there are sufficiently many to choose g.

It remains to show that g Weierstrass points satisfy the generation condition, that $\xi_1, \ldots, \xi_g$ generate $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$. Let $P_1, \ldots, P_g$ be points on X hyperelliptic, with $D = \sum_{i=1}^{g} P_i$. Let $f : X \to \mathbb{P}^1$ be the morphism of degree 2. We claim that $P_1, \ldots, P_g$ do not satisfy the generation condition if and only if $f(P_i) = f(P_j)$ for some i, j; that is, $P_i + P_j$ is in the hyperelliptic system on X.

The short exact sequence,

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \bigoplus_{i=1}^{g} \mathbb{K}(P_i) \to 0$$

gives rise to a long exact sequence on Ext,

$$0 \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(D)) \to \bigoplus_{i=1}^{g} \mathrm{Hom}(\mathcal{O}_X, \mathbb{K}(P_i)) \to$$

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(D)) \to 0.$$

Thus the generation condition is equivalent to the statement that

$$\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(D)) \cong H^1(X, \mathcal{O}_X(D)) = 0,$$

i.e., D is non-special. By Serre duality $[H^1(X, \mathcal{O}_X(D))]* \cong \mathrm{Hom}(\mathcal{O}_X(D), \omega_X)$ where ω_X is the canonical sheaf on X; this space is nonzero if and only if D is a subdivisor of an effective canonical divisor on X.

By [6] Prop. IV.5.3, every effective canonical divisor on X is of the form $K = D_1 + \cdots + D_{g-1}$ where each D_i is in the hyperelliptic system. Since D has degree g, $D \subset K$ if and only if D contains some D_i. This proves the claim.

The Weierstrass points are exactly the ramification points of f. It follows that if the P_i are distinct Weierstrass points, the divisor D is non-special. This completes the claim. $\square$

APPENDIX A

A MAP TO A PRODUCT OF SIMPLICES

This lemma will apply directly to Lemma III.2.3, and uses the same notation. Let $\mathscr{K}(l)$ be the reduced simplicial complex of the simplicial l-cell; that is, the full simplicial complex with l vertices.

Lemma A.1. *1.* $(gr_i\mathcal{O})^{(m)}_\bullet = X^{(m)}_\bullet \oplus Y^{(m)}_\bullet$ *where*

$$X^{(m)}_n = \langle\{\xi^{a_0}\theta\eta\xi^{a_1}\cdots\theta\eta\xi^{a_{n-m}}|a_k \neq 0 \textit{ for } 1 \leq k \leq n-m-1\}\rangle,$$
$$Y^{(m)}_n = \langle\{\xi^{a_0}\theta\psi^{b_1}\eta\xi^{a_1}\cdots\eta\xi^{a_{n-m}}|b_k \neq 0 \textit{ for some } k \textit{ or } a_k = 0 \textit{ for some } 1 \leq k \leq n-m-1\}\rangle.$$

2. $HB_n(X^{(m)}_\bullet) = X^{(m)}_n$ *and* $HB_n(Y^{(m)}_\bullet) = 0$ *for all* n.

Proof. 1. Each standard basis tensor in $(\mathrm{gr}_i\mathcal{O})^{(n-m)}_n$ is obviously in either X_n or Y_n (we suppress m since it is arbitrary). In X_n, d increases the sum of powers of ξ by 1 in each term, so $d|_{X_n} = 0$. On Y_n, any tensor that satisfies one of the two properties listed will have a boundary each of whose terms has a nonzero power on some b_k, which will therefore be in Y_{n-1}.

2. Since $d|_{X_n} = 0$, the first result is clear.

For every $j \geq 0$, there is a subcomplex $Y^j_\bullet \subset Y_\bullet$ where

$$Y^j_n = \{\xi^{a_0}\theta\psi^{b_1}\eta\xi^{a_1}\cdots\eta\xi^{a_{n-m}}|\text{exactly } j \text{ of } a_1,\ldots,a_{n-m-1} \text{ are nonzero}\},$$

such that

$$Y_\bullet = \bigoplus_{j=1}^{m-1} Y^j_\bullet,$$

so it is enough to show that $HB_n(Y^j_\bullet) = 0$ for all j. We will use induction in j.

For the base case, we map $Y^0_\bullet$ to the simplicial complex as follows. A standard basis tensor in Y^0_n is determined by the $(n-m)$-tuple $(b_1, \ldots, b_{n-m})$ where $\sum_{k=1}^{n-m} b_k = 2m - n - i$. The differential acts by

$$d(b_1, \ldots, b_m) = \sum_{k=1}^{n-m-1} \pm(b_1, \ldots, b_k + b_{k+1} + 1, \ldots, b_{n-m}).$$

After the change of variable $b'_k = b_k + 1$, we have $\sum_{k=1}^{n-m} b'_k = m - i$ (which is constant since m, i are fixed), $b'_k \geq 1$ for all k, and

$$d(b'_1, \ldots, b_{n-m}) = \sum_{k=1}^{n-m-1} \pm(b'_1, \ldots, b'_k + b'_{k+1}, \ldots, b'_{n-m}).$$

Finally we map

$$(b'_1, \ldots, b'_{n-m}) \mapsto \left(b'_1, b'_1 + b'_2, \ldots, \sum_{k=1}^{n-m-1} b'_k \right) \in \mathscr{K}(m - i - 1).$$

This is a map of complexes which surjects onto $\mathscr{K}(m-i-1)$. Since $\mathscr{K}(m-i-1)$ has no homology, it follows that $HB_n(Y^0_\bullet) = 0$ for all n.

Now suppose that $HB_n(Y^{j,(m)}_\bullet) = 0$ for all n, all $j \leq l$, and all m. A standard basis tensor y in Y^{l+1}_n can be written as $y_1 \otimes y_2$ where $y_1 \in Y^{l,(m_1)}_s$ (for some internal degree m_1) and $y_2 \in Y^{0,(m_2)}_{n-s}$ (for some internal degree m_2), by splicing y immediately before the $(l+1)$-st nonzero internal power of ξ. This gives an isomorphism

$$Y^{l+1,(m)}_\bullet \cong \bigoplus_{m_1+m_2=m} Y^{l,(m_1)}_\bullet \otimes Y^{0,(m_2)}_\bullet.$$

It follows from the induction hypothesis and the Kunneth formula that the homology of the complex on the right is 0, therefore $HB_n(Y^{l+1}_\bullet) = 0$ for all n.

□

APPENDIX B

HOMOLOGY OF A SIMPLICIAL COMPLEX

Let $[n] = \{1, 2, \ldots, n\}$. We define a simplicial complex $\Delta[n] \subset P([n])$ such that

$$\Delta_0 = \{\{i\} | i \in [n]\},\ \Delta_1 = \{\{i, j\} | j - i \geq 2\},\ \Delta_2 = \{\{i, j, k\} | j - i \geq 2, k - j \geq 2\},$$

and in general

$$\Delta_m = \{\{i_1, i_2, \ldots, i_m\} | i_{j+1} - i_j \geq 2,\ j = 1, \ldots, m-1\}.$$

Proposition B.1. *For all $k \in \mathbb{N}$,*

$$\Delta[3k+1] \cong \textit{point},$$

$$\Delta[3k+2] \cong S^k,$$

$$\Delta[3k+3] \cong S^k.$$

Proof. We proceed by induction on k, starting at $k = 0$. The complex $\Delta[1]$ is a point, $\Delta[2]$ is two points and no edges, and $\Delta[3]$ is three points and the edge $\{1, 3\}$, so this establishes the base case.

Suppose the result for k. Then

$$\Delta[3k+4] = A \cup B$$

where

$$A = \{1, 2, 3, \ldots, 3k+3\} \cap \Delta[3k+4],$$
$$B = \{\text{all simplices containing the vertex } 3k+4\}.$$

Then B is contractible so $B \cong D^k$ and $A \cong \Delta[3k+3] \cong S^k$. Their intersection is

$$A \cap B = \Delta[3k+3] \cap \{1, 2, 3, \ldots, 3k+2\} \cong \Delta[3k+2] \cong S^k.$$

So $\Delta[3k+4] = D^k \cup S^k$ with $D^k \cap S^k = S^k$, which is contractible. We proceed similarly in the other cases. Now $\Delta[3k+5] = A \cup B$ where

$$A = \{1, 2, \ldots, 3k+4\} \cap \Delta[3k+5] \cong \Delta[3k+4] \cong D^{k+1},$$
$$B = \{\text{all simplices containing 3k+5}\} \cong D^{k+1},$$
$$A \cap B \cong \Delta[3k+3] \cong S^k.$$

So now we have two disks intersecting in S^k, which gives S^{k+1}. Finally $\Delta[3k+6] = A \cup B$ where

$$A = \{1, 2, \ldots, 3k+5\} \cap \Delta[3k+6] \cong \Delta[3k+5] \cong S^{k+1},$$
$$B = \{\text{all simplices containing 3k+6}\} \cong D^{k+1},$$
$$A \cap B \cong \Delta[3k+4] \cong D^{k+1}.$$

So we have an S^{k+1} and a disk intersecting in a disk, which gives S^{k+1}.

□

For $k = 0, 1, 2$, it will be helpful to have explicit representatives of the resulting homology class in $\Delta[3k+2]$ and $\Delta[3k+3]$. For $\Delta[2]$ and $\Delta[3]$ we use $\{1\} - \{2\}$.

The loop in $\Delta[5]$ is constructed from gluing the contractible complex $\Delta[5] \cap \{1, 2, 3, 4\}$ with the contractible complex of those simplices touching $\{5\}$. The intersection is the S^0 in $\{1, 2, 3\}$. The easiest way to realize this class is by taking the cone over $\{1\} \cup \{2\}$ to $\{4\}$ and another cone to $\{5\}$. Thus the resulting loop is $\{1,5\} \pm \{1,4\} \pm \{2,5\} \pm \{2,4\}$. We also use this class in $\Delta[6]$. (The choice between $+$ and $-$ not relevant for our application of this calculation, so we do not make it.)

The class in $\Delta[8]$ we realize similarly. The intersection of the two contractible parts is $\Delta[6]$, which we consider as the loop above. We make a cone over this loop to the points $\{7\}$ and $\{8\}$ to get the class representative

$$\{1,5,7\} \pm \{1,4,7\} \pm \{2,5,7\} \pm \{2,4,7\} \pm \{1,5,8\} \pm \{1,4,8\} \pm \{2,5,8\} \pm \{2,4,8\}.$$

This class will also represent the loop in $\Delta[9]$.

APPENDIX C

W_2 AND W_3 IN THE PROOF OF LEMMA V.3.4

We first define W_2. The table below collects the basis vectors of V_2 and their boundaries, with the underlined term in each boundary designated to be in $\Omega_2(F_1/F_2)$. We also define for integers a, b,

$$\kappa(a,b) = \begin{cases} 1 & \text{if } a \equiv b \mod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Element of $\Omega_2(F_0/F_1)$	$\Omega_2(F_1/F_2)$ expansion of boundary
$[n(i)]^* \otimes \mathrm{id}_{P_i}$ n even	$\underline{[(n+1)(i)]^* \otimes \theta_i} \pm [(n+1)(i)]^* \otimes \eta_i$
$[1(i)\text{-}m(j)\text{-}k(i)]^* \otimes \mathrm{id}_{P_i}$ k and m even; or k even, m odd and $g > 2$	$\pm\underline{[1(i)\text{-}m(j)\text{-}(k+1)(i)]^* \otimes \eta_i}$
$[k(i)\text{-}m(j)\text{-}1(i)]^* \otimes \mathrm{id}_{P_i}$ k and m even; or k even, m odd and $g > 2$	$\pm\underline{[(k+1)(i)\text{-}m(j)\text{-}1(i)]^* \otimes \theta_i}$
$[n_1(i)\text{-}m(j)\text{-}n_2(i)]^* \otimes \mathrm{id}_{P_i}$ n_1 even and $n_1, n_2 > 1$	$\underline{[(n_1+1)(i)\text{-}m(j)\text{-}n_2(i)]^* \otimes \theta_i}$ $\pm\kappa(n_2, 0)[n_1(i)\text{-}m(j)\text{-}(n_2+1)(i)]^* \otimes \eta_i$
$[n_1(i)\text{-}m(j)\text{-}n_2(i)]^* \otimes \mathrm{id}_{P_i}$ n_1 odd and n_2 even	$\pm\underline{[n_1(i)\text{-}m(j)\text{-}(n_2+1)(i)]^* \otimes \eta_i}$
$[n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}n_2(i)]^* \otimes \mathrm{id}_{P_i}$ n_1 even	$\underline{[(n_1+1)(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}n_2(i)]^* \otimes \theta_i}$ $\pm\kappa(n_2, 0)[n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}(n_2+1)(i)]^* \otimes \eta_i$
$[n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}n_2(i)]^* \otimes \mathrm{id}_{P_i}$ n_1 odd and n_2 even	$\pm\underline{[n_1(i)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}(n_2+1)(i)]^* \otimes \eta_i}$

Table 14: Definition of $\Omega_2(F_1/F_2)$

We define W_3 similarly in the table below.

Element of $\Omega_3(F_0/F_1)$	$\Omega_3(F_1/F_2)$ expansion of boundary
$[n(i)]^* \otimes \mathrm{id}_{\mathcal{O}_X}$	$\pm\underline{[n(i)\text{-}1(j)]^* \otimes \theta_j}$ (for some $j \neq i$) $\pm\kappa(n,0)[(n+1)(i)]^* \otimes \theta_i \pm \kappa(n,0)[(n+1)(i)]^* \otimes \eta_i$ $\pm\sum_{k\neq i,j}[n(i)\text{-}1(k)]^* \otimes \theta_k \pm \sum_{k\neq i}[1(k)\text{-}n(i)]^* \otimes \eta_k$
$[m_1(j_1)\text{-}m_2(j_2)]^* \otimes \mathrm{id}_{\mathcal{O}_X}$ m_1, m_2 odd and $g > 2$	$\pm\underline{[m_1(j_1)\text{-}m_2(j_2)\text{-}1(j)]^* \otimes \theta_j}$ (for some $j \neq j_1$ and $j \neq j_2$) $\pm\sum_{k\neq j_2,j}[m_1(j_1)\text{-}m_2(j_2)\text{-}1(k)]^* \otimes \theta_k \pm$ $\sum_{k\neq j_1}[1(k)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_k$
$[m_1(j_1)\text{-}m_2(j_2)]^* \otimes \mathrm{id}_{\mathcal{O}_X}$ m_1 even	$\underline{[(m_1+1)(j_1)\text{-}m_2(j_2)]^* \otimes \eta_{j_1}}$ $\pm\kappa(m_2,0)[m_1(j_1)\text{-}(m_2+1)(j_2)]^* \otimes \theta_{j_2} \pm$ $\sum_{k\neq j_2}[m_1(j_1)\text{-}m_2(j_2)\text{-}1(k)]^* \otimes \theta_k$ $\pm\sum_{k\neq j_1}[1(k)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_k$
$[m_1(j_1)\text{-}m_2(j_2)]^* \otimes \mathrm{id}_{\mathcal{O}_X}$ m_1 odd and m_2 even	$\pm\underline{[m_1(j_1)\text{-}(m_2+1)(j_2)]^* \otimes \theta_{j_2}}$ $\pm\sum_{k\neq j_2}[m_1(j_1)\text{-}m_2(j_2)\text{-}1(k)]^* \otimes \theta_k \pm$ $\sum_{k\neq j_1}[1(k)\text{-}m_1(j_1)\text{-}m_2(j_2)]^* \otimes \eta_k$
$[m_1(j_1)\text{-}m_2(j_2)\text{-}m_3(j_3)]^* \otimes \mathrm{id}_{\mathcal{O}_X}$	$\pm\underline{[m_1(j_1)\text{-}m_2(j_2)\text{-}m_3(j_3)\text{-}1(j_2)]^* \otimes \theta_{j_2}}$ $\pm\kappa(m_1,0)[(m_1+1)(j_1)\text{-}m_2(j_2)\text{-}m_3(j_3)]^* \otimes \eta_{j_1} \pm$ $\kappa(m_3,0)[m_1(j_1)\text{-}m_2(j_2)\text{-}(m_3+1)(j_3)]^* \otimes \theta_{j_3}$ $\pm\sum_{k\neq j_2,j_3}[m_1(j_1)\text{-}m_2(j_2)\text{-}m_3(j_3)\text{-}1(k)]^* \otimes \theta_k \pm$ $\sum_{k\neq j_1}[1(k)\text{-}m_1(j_1)\text{-}m_2(j_2)\text{-}m_3(j_3)]^* \otimes \eta_k$

Table 15: Definition of $\Omega_3(F_1/F_2)$

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