NC-ALGEBROID THICKENINGS OF MODULI SPACES AND BIMODULE EXTENSIONS OF VECTOR BUNDLES OVER NC-SMOOTH SCHEMES

by

BEN DYER

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

December 2017

DISSERTATION APPROVAL PAGE

Student: Ben Dyer

Title: NC-Algebroid Thickenings of Moduli Spaces and Bimodule Extensions of Vector Bundles over NC-Smooth Schemes

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Nicholas Proudfoot	Chair
Alexander Polishchuk	Core Member
Arkady Berenstein	Core Member
Boris Botvinnik	Core Member
Jens Nöckel	Institutional Representative

and

Sara D. Hodges	Interim Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded December 2017

DISSERTATION ABSTRACT

Ben Dyer

Doctor of Philosophy

Department of Mathematics

December 2017

Title: NC-Algebroid Thickenings of Moduli Spaces and Bimodule Extensions of Vector Bundles over NC-Smooth Schemes

We begin by reviewing the theory of NC-schemes and NC-smoothness, as introduced by Kapranov in [11] and developed further by Polishchuk and Tu in [20].

For a smooth algebraic variety X with a torsion-free connection ∇, we study modules over the NC-smooth thickening $\widetilde{\mathcal{O}}_X$ of X constructed in [20] via NC-connections. In particular we show that the NC-vector bundle $\widetilde{E}_{\bar{\nabla}}$ constructed via mNC-connections in [20] from a vector bundle $(E, \bar{\nabla})$ with connection additionally admits a bimodule extension at least to nilpotency degree 3.

Next, in joint work with A. Polishchuk [7], we show that the gap, as first noticed in [20], in the proof from [11] that certain functors are representable by NC-smooth thickenings of moduli spaces of vector bundles is unfixable. Although the functors do not represent NC-smooth thickenings, they lead to a weaker structure of *NC-algebroid thickening*, which we define. We also consider a similar construction for families of quiver representations, in particular upgrading some of the quasi-NC-structures of [23] to NC-smooth algebroid thickenings.

This thesis includes unpublished co-authored material.

CURRICULUM VITAE

NAME OF AUTHOR: Ben Dyer

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
The Evergreen State College, Olympia, WA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2017, University of Oregon
Masters of Science, Mathematics, 2017, University of Oregon
Bachelors of Science, Mathematics and Biochemistry, 2013, The Evergreen State College

AREAS OF SPECIAL INTEREST:

Noncommutative Geometry
Algebraic Geometry
Deformation Quantization
Quantum Algebra

PROFESSIONAL EXPERIENCE:

Faculty, The Evergreen State College, September 2017

Graduate Teaching Fellow, University of Oregon, September 2013—September 2017

PUBLICATIONS:

B. Dyer and A. Polishchuk. NC-smooth algebroid thickenings for families of vector bundles and quiver representations. `arXiv:1710.00243`

ACKNOWLEDGEMENTS

Thanks to Sasha for sharing this project with me, and for being an exceptional davisor. Thanks to Nick Proudfoot and Jessica Simoes, whose assistance was invaluable in the graduation process. Thanks also to Sherilyn Schwartz, Jens Nöckel, Arkady Berenstein, and Boris Botvinnik.

Thanks to Ryan Takahashi and Keegan Boyle for numerous discussions over the last four years. Thanks to Jeff Musyt, too :0). Thanks to the older Oregonian algebraic geometers for showing me the ropes, Bronson Lim, Nick Howell, Max Kutler and Justin Hilburn, and to all my friends from UO.

Thanks to Kate for being the best sister in the world. Thanks to my mother for her love. Thanks to Brian for all the tennis matches, that kept me sane while writing this thesis. Thanks to Ellie, who entered my life at just the right moment.

Thanks to everyone who helped me find my way here: Krishna Chowdary for suggesting I relearn calculus; Allen Mauney for teaching it; Lydia McKinstry and Ben Simon for believing in me early on; Ari Herman for the encouragement and inspiration; Brian Walter, Rachel Hastings and my classmates in Math Systems 2011—12 for creating a superb learning environment; my earliest mathematical collaborators, Roger Royset and Marena Shear; my teachers Richard Weiss and David McAvity, who made time for many independent learning contracts; Jason Mock at TESC and Richard Dahlen at SPSCC, where I wrote much of this thesis. My profound gratitude to Vauhn Foster-Grahler and The QuaSR Center for many invaluable lessons. <Thanks to you too IG.>

To my father.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In this chapter we attempt locate the present work in the larger context of noncommutative geometry, give some intuition for the idea of NC-schemes, and indicate the main results.

1.1 Noncommutative Algebraic Geometry

In the case of noncommutative differential geometry (cf. [4]) one may take a noncommutative C*-algebra as the basic object of study, analogy with Gelfand duality. However, the similar duality in algebraic geometry is only between commutative rings and *affine* schemes, with apparently no equally obvious notion of a noncommutative scheme. The problem with the naive approach is the following: there are natural noncommutative analogs either for just the affine schemes, or for all locally ringed spaces, but there is no obvious relation between them. In particular we lack an embedding of (noncommutative) rings into locally (noncommutative) ringed spaces.

One approach to noncommutative algebraic geometry then is to try to associate a space $\operatorname{Spec} R$ to a noncommutative ring R. As shown in [21], it is not possible to do this faithfully functorially in a way which extends classical algebraic geometry (although see [24]), e.g. any such theory defines $\operatorname{Spec} M_3(\mathbb{C}) = \varnothing$.

Instead one way choose to settle for only a subcategory of all noncommutative rings, for which the theory of algebraic geometry will have satisfying properties. This is the approach taken in [25], which defines $\operatorname{Spec} R$ for rings with "enough"

Ore sets. Another option is that of [16], where one defines noncommutative spaces "virtually" in terms of their categories of sheaves.

Kapranov's theory [11] of NC-schemes, the subject of this thesis, avoids the difficulty of producing any new topological spaces or additive categories by taking a formal approach to noncommutative algebraic geometry.

1.2 NC-Schemes as Formal Noncommutative Neighborhoods

Recall that to a closed embedding of schemes $X \subset Y$ there is a *formal scheme* $\hat{X}_Y \to Y$, called the formal neighborhood of X in Y. This is a locally ringed space whose underlying space is that of X, but whose sheaf of formal functions $\widehat{\mathcal{O}}_X = \varprojlim \mathcal{O}_Y / \mathcal{I}_X^n$ carries infinitesimal information about the embedding.

In the noncommutative set up, given any ring R one has a natural surjection $\pi_{ab} : R \to R_{ab}$, with kernel the two-sided ideal generated by commutators of elements in R. One may imagine this as being dual to some hypothetical closed immersion of noncommutative spaces:

$$\operatorname{Spec} R_{ab} \hookrightarrow \text{“}\operatorname{Spec} R\text{”}$$

Although we don't know how to define the latter space, all we need in order to study the formal neighborhood is a new structure sheaf on $\operatorname{Spec} R_{ab}$, which remembers "infinitesimal noncommutative" information about R.

The formal noncommutative neighborhood of this embedding is modeled by the algebra $R_{[\![ab]\!]}$, called the *NC-completion* of R, which is the completion with respect to a natural filtration $\mathcal{I}^d R$. Unlike the commutative formal neighborhoods, this filtration is not simply the powers of $\mathcal{I}^1 R = R[R,R]$; it is an important aspect of the theory that one imposes convergence not only of higher products of commutators $[x_1, y_1][x_2, y_2]\cdots[x_n, y_n]$, but also of higher nestings of commutators $[x_1, [x_2, [\cdots, x_n]\cdots]]$. This reduces the R-linear structure on $\mathrm{gr}^{\bullet}_{\mathcal{I}}(R)$ to a single R_{ab}-module (similar to the commutative case), causes many natural localizations to be of Ore type, and ultimately enables the construction of a locally ringed space $X = \operatorname{Spec} R$ with $\Gamma(X, \mathcal{O}_X) = R$.

In particular, Kapranov's NC-nilpotent algebras are so-called *schematic algebras*, and the theory of NC-schemes fits into the larger picture of noncommutative algebraic geometry described in [25].

1.2.1 NC-manifolds and quantization

One of the interesting features of Kapranov's theory is the existence of a unique NC-smooth algebra thickening a fixed smooth commutative algebra. Because this structure is not at all canonical, it is an interesting question when a non-affine variety admits an NC-smooth thickening (e.g. when local NC-smooth thickenings can glue together).

It is pointed out in [17] that the affine NC-smooth thickenings defined by Kapranov had already been discovered as certain microlocalizations (cf. [25]). In some sense the DG-resolutions of NC-smooth thickenings defined in [20] are analogs of Fedosov's construction of deformation quantization (cf. [1, 8] and [19]). In the present work, we show that algebroids, which also first came up in the study of deformation quantization (cf. [15]) fit naturally into this theory as well.

In particular we find NC-algebroid thickenings of certain moduli spaces.

1.3 Summary of Results

In Chapter V we consider a question of [20, Rem. 3.3.10] on bimodule extendability of NC-vector bundles. It was suggested that for an NC-vector bundle coming from an mNC-connection, perhaps extendability to a 2-nilpotent bimodule would imply flatness of the connection. However, we show that an NC-vector bundle coming from an mNC-connection always admits a 2- and even a 3-nilpotent bimodule extension. Chapter IV and Sections 2.1,2.2 are necessary for these computations.

In VI we define a notion of *almost NC-schemes*, modeled on the category $a\mathcal{N}$ of NC-nilpotent algebras up to inner automorphism, and observe that a functor which factors through $a\mathcal{N}$ cannot represent an NC-smooth scheme. However, we introduce the weakened notion of an *NC-smooth algebroid thickening* and prove that a formally smooth functor which is locally representable in $a\mathcal{N}$ determines an NC-smooth algebroid thickening.

It follows from VI that the natural functors defined by Kapranov [11] and Toda [23] are not representable by NC-smooth schemes. In VII we define certain moduli spaces which we call *excellent families* of vector bundles (correcting the definition of [11]) and of quiver representations (which have some overlap with [23]), and construct NC-smooth algebroid thickenings of each using the results of VI.

Chapters VI and VII are unpublished joint work with A. Polishchuk.

CHAPTER II

PRELIMINARIES

The purpose of this chapter is to collect some facts an notation which will be useful for the later chapters. The first two sections are in preparation for the computations on bimodule extensions of NC-vector bundles, while the sections on non-abelian (hyper)cohomology are for the NC-algebroid thickenings of moduli spaces.

2.1 Algebraic de Rham Complex

For affine space $\mathbb{A}^n$, the operation of contraction with the Euler vector field $E = x^i \dfrac{\partial}{\partial x^i}$ defines an explicit homotopy equivalence of chain complexes $\Omega^\bullet_{\mathbb{A}^n} \simeq \mathbb{C}$ between the algebraic de Rham complex and its cohomology. Moreover, the construction is GL_n-equivariant, hence leads to similar contraction for the relative DGAs of any vector bundle.

2.1.1 Algebraic de Rham complex of affine space $\mathbb{A}^n$

Definition 2.1.1. Let $x_1, \ldots, x_n$ be coordinates for $\mathbb{A}^n$. The *Euler vector field* is

$$E = x_1 \cdot \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n},$$

and for a differential k-form $\omega(v_1, \ldots, v_k)$, the operation of *contraction with Euler vector field* is denoted ι_E and is defined by

$$\iota_E(\omega)(w_1, \ldots, w_{k-1}) = \omega(E, w_1, \ldots, w_{k-1}).$$

What follows are some elementary properties of the operation ι_E.

Lemma 2.1.2. *The following properties of the contraction ι_E hold:*

(a.) For any $v, w \in \Omega^\bullet$, then $\iota_E(v \wedge w) = \iota_E(v) \wedge w + (-1)^{|v|}\iota_E(w)$

(b.) If $\omega \in \Lambda^k(V)$ then $d \circ \iota_E(\omega) = k \cdot \omega$

(c.) If $\omega \in \Lambda^i(V) \otimes S^j(V)$ then $\iota_E \circ d(\omega) = j \cdot \omega$

Proof. (a) is clear. For (b) $k = 0, 1$ are clear, and the rest follow inductively using (a) as follows. Consider $v \in \Lambda^1(V), w \in \Lambda^k(V)$, then

$$\begin{aligned} d\iota_E(v \wedge w) &= d\big(\iota_E(v) \wedge w - v \wedge \iota_E(w)\big) \\ &= d\iota_E(v) \wedge w + \iota_E(v) \wedge dw - dv \wedge \iota_E(w) + v \wedge d\iota_E(w) \\ &= d\iota_E(v) \wedge w + v \wedge d\iota_E(w) \\ &= (k+1) \cdot v \wedge w \end{aligned}$$

For (c) just note that

$$\iota_E(\sum_i \partial_i f dx_i \wedge \omega) = \partial_i f \cdot x_i \cdot \omega = (\sum_i \deg_i(f)) \cdot f \cdot \omega = \deg(f) \cdot f \cdot \omega$$

where $\deg_i$ denotes the degree of f with respect to x_i. and $\deg f = \sum_i \deg_i$ is the total degree. □

Definition 2.1.3. The homotopy operator $h_E : \Omega^\bullet_{\mathbb{A}^n} \to \Omega^{\bullet-1}_{\mathbb{A}^n}$ is defined to be $h_E = \bigoplus h_{i,j}$ where $h_{i,j} : \Lambda^i(V) \otimes S^j(V) \to \Lambda^{i-1}(V) \otimes S^{j+1}(V)$ is given by $h_{i,j} = \frac{1}{i+j}\iota_E$.

Definition 2.1.4. In this thesis, a *retraction* of a complex $B^\bullet$ onto a subcomplex $A^\bullet \overset{\iota}{\hookrightarrow} B^\bullet$ is the data of:

(i) a map $r : B^\bullet \to A^\bullet$ such that $r|_{A^\bullet} = \mathrm{id}_{A^\bullet}$;

(ii) a homotopy h from r to id_B, i.e. such that $d_B h + h d_B = \mathrm{id}_B - r$.

which also satisfy the side conditions, $h|_{A^\bullet} = 0$ and $rh = h^2 = 0$.

Proposition 2.1.5. *The projection $\Omega^\bullet_{\mathbb{A}^n} \to \mathbb{C}$ is a retraction with homotopy operator h_E.*

Proof. Let $\omega \in \Lambda^i(V) \otimes S^j(V)$ such that $i + j \geq 1$. Then we have:

$$\begin{aligned}(h_E d + d h_E)(\omega) &= \frac{\iota_E(d\omega)}{(i+1)+(j-1)} + \frac{d(\iota_E \omega)}{i+j} \\ &= \frac{j \cdot \omega + i \cdot \omega}{i+j} \\ &= \omega.\end{aligned}$$

□

Similarly, for any variety X there is a relative Euler vector field on $\mathbb{A}^n \times X$, parallel to $\mathbb{A}^n$, giving a homotopy equivalence of the relative de Rham complex

$$\Omega^\bullet_{(\mathbb{A}^n \times X)/X} \xrightarrow{\sim} \mathcal{O}_X.$$

Furthermore, the Euler vector field is GL_n-invariant, so this works for any vector bundle $\mathcal{V}$ on X to obtain a retraction

$$\Omega^\bullet_{\mathcal{V}/X} \xrightarrow{\sim} \mathcal{O}_X.$$

2.1.2 Relative algebraic de Rham complex of a vector bundle

The following fact is obvious.

Lemma 2.1.6. *There is a natural identification* $\Omega^1_{\mathcal{V}/X} = p^*\mathcal{V}^*$

Proof. We consider the affine case. Let $B = S_A(P^*)$ for a projective module P.

$$\begin{aligned}\mathrm{Hom}_B(\Omega^1_{B/A}, M) &= \mathrm{Der}_A(B, M)\\ &= \mathrm{Hom}_A(P^*, \mathrm{Hom}_B(B, M))\\ &= \mathrm{Hom}_B(P^* \otimes_A B, M)\end{aligned}$$

Thus $\Omega^1_{B/A} = P^* \otimes_A B$ by the Yoneda lemma. □

In particular $\Omega^1_{TX/X} = \Omega^1_X \otimes_{\mathcal{O}_X} S(\Omega^1_X)$.

Proposition 2.1.7. *For any vector bundle* $p : \mathcal{V} \to X$*, the projection to degree 0*

$$\Omega^\bullet_{\mathcal{V}/X} \tilde{\to} \mathcal{O}_X$$

is a retraction with homotopy h_E*.*

2.2 Free Lie Algebras

In this section we fix notation regarding free Lie algebras used later in the section on DG-resolutions.

For a vector space V, we denote by TV the tensor algebra, SV the symmetric algebra, and $\mathcal{L}V$ the free Lie algebra. The derived subalgebra of $\mathcal{L}V$ is denoted $\mathcal{L}_+V = [\mathcal{L}V, \mathcal{L}V]$. The universal enveloping algebra of a Lie algebra L is denoted UL. When L is a graded Lie algebra, as is the case for $\mathcal{L}V$ and $\mathcal{L}_+V$, the grading extends uniquely over the inclusion $L \subset UL$ to UL, and we denote $U^dL := (UL)^d$, and $U^+L = \bigoplus_{d>0} U^dL$.

Example 2.2.1. Under the identification $TV = U\mathcal{L}V$, $U^d\mathcal{L}V$ corresponds to T^dV.

In [11, §3], it is observed that given an ordered basis $x_1, \dots, x_n$ for V, if one considers the subspace $S_{ord}V \subset TV$ of ordered polynomials, then restriction of the multiplication $TV \otimes TV \to TV$,

$$S_{ord}V \otimes U\mathcal{L}_+V \xrightarrow{\sim} TV$$

is a bijection. The inverse is a rewriting process $f = \sum_\lambda [\![f_\lambda(x_1, \dots x_n)]\!] \cdot M_\lambda$, where $\{M_\lambda\}_{\lambda \in \Lambda}$ is a basis for $U\mathcal{L}_+V$, $f_\lambda \in SV$ and $[\![f_\lambda]\!]$ is the corresponding ordered polynomial in TV.

Although this decomposition was convenient in [11] for describing the multiplication rule via the Feynman-Maslov operator calculus, it is inconvenient for this thesis as $S_{ord}V \subset TV$ is not $GL(V)$-invariant, hence it doesn't lead to a similar decomposition for vector bundles.

Instead we use a different rewriting process involving $SV \subset TV$ viewed as the symmetric polynomials.

Proposition 2.2.2. *The restriction of the multiplication $TV \otimes TV \to TV$,*

$$\mu : SV \otimes U\mathcal{L}_+V \longrightarrow TV,$$

is a right $U\mathcal{L}_+V$-linear isomorphism of graded $GL(V)$-representations.

Proof. Follows easily by comparing with the identification of $TV \cong SV \otimes U\mathcal{L}_+V$ via ordered monomials. □

Example 2.2.3. In Kapranov's set up $x_2x_1 \in \mathbb{C}\langle x_1, x_2\rangle$ gets rewritten as $x_2x_1 = x_1x_2 - [x_1, x_2]$. Instead, we rewrite this as $x_2x_1 = \frac{1}{2}(x_1x_2 + x_2x_1) - \frac{1}{2}[x_1, x_2]$.

The following projections are useful later in computations of chapter V.

Corollary 2.2.4. *There is a natural map of graded $GL(V)$-representations*

$$\Pi : TV \longrightarrow U\mathcal{L}_+V$$

with kernel $\mu(S^+V \otimes U\mathcal{L}_+V)$, and corresponding projections $\Pi_d : T^dV \to U^d\mathcal{L}_+V$.

2.3 Nonabelian Cohomology

In this section we review nonabelian cohomology (cf. [9, Sec. 3.3-3.4], [18, Sec. 2.6.8]).

Definition 2.3.1. Consider a sheaf of groups $\mathcal{G}$ on a topological space X and an open covering $\mathcal{U} = (U_i)$ of X.

(i) The set of 1-cocycles $Z^1(\mathcal{U}, \mathcal{G})$ consists of $g_{ij} \in \mathcal{G}(U_{ij})$, such that $g_{ii} = 1$, $g_{ij}g_{ji} = 1$, and $g_{ij}|_{U_{ijk}} \cdot g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}}$.

(ii) Two such 1-cocycles (g_{ij}) and (g'_{ij}) are *cohomologous* if for some $h_i \in \mathcal{G}(U_i)$,

$$g'_{ij} = h_i|_{U_{ij}} g_{ij} h_j^{-1}|_{U_{ij}}.$$

for some $h_i \in \mathcal{G}(U_i)$.

(iii) The pointed set of equivalence classes in $Z^1(\mathcal{U}, \mathcal{G})$ is denoted $H^1(\mathcal{U}, \mathcal{G})$

Completely analogously to abelian Čech cohomology we define $H^1(X, \mathcal{G})$ as the limit over all covers $\mathcal{U}$, i.e. $H^1(X, \mathcal{G}) = \varprojlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{G})$.

Now assume we are given an *abelian extension* of sheaves of groups

$$0 \to \mathcal{A} \to \mathcal{G}' \xrightarrow{p} \mathcal{G} \to 1,$$

i.e. $\mathcal{A}$ is a sheaf of abelian normal subgroups of $\mathcal{G}'$. Then we have a natural connecting map

$$\delta_0 : H^0(X, \mathcal{G}) \to H^1(X, \mathcal{A})$$

such that $\delta_0(g) = 0$ if and only if g lifts to a global section of $\mathcal{G}'$.

Definition 2.3.2. The connecting map δ_0 is defined in terms of cocycles by lifting $g_i := g|_{U_i}$ locally to $\tilde{g}_i \in \mathcal{G}'(U_i)$ and forming the element:

$$\delta_0(g) = (\tilde{g}_i)^{-1}\tilde{g}_j \tag{2.1}$$

(Note that this differs from the choice made in [18] that $\delta(g) = \tilde{g}_j \cdot \tilde{g}_i^{-1}$.)

Note that δ_0 is *not a homomorphism* in general. Rather, it satisfies

$$\delta_0(g_1 g_2) = g_2^{-1}(\delta_0(g_1)) + \delta_0(g_2), \tag{2.2}$$

where we write the group structure in $H^1(X, \mathcal{A})$ additively and use the natural action of $H^0(X, \mathcal{G})$ on $H^1(X, \mathcal{A})$ induced by the adjoint action of $\mathcal{G}$ on $\mathcal{A}$. (This means that $g \mapsto \delta_0(g^{-1})$ is a *crossed homomorphism.*) An equivalent restatement of (2.2) is that there is a *twisted action* of $H^0(X, \mathcal{G})$ on $H^1(X, \mathcal{A})$ given by

$$g \times a = g(a) + \delta_0(g^{-1}), \quad \text{where } g \in H^0(X, \mathcal{G}), a \in H^1(X, \mathcal{A}). \tag{2.3}$$

Explicitly, the usual action of $g \in H^0(X, \mathcal{G})$ on a class of a Cech 1-cocycle (a_{ij}) with values in $\mathcal{A}$ is given by $g_i' a_{ij} (g_i')^{-1}$, where $g_i' \in \mathcal{G}'(U_i)$ are liftings of g. On the other hand, the twisted action of g on a_{ij} is given by $g_i' a_{ij} (g_j')^{-1} = \tilde{g}_i a_{ij} \tilde{g}_i^{-1} (\tilde{g}_i \tilde{g}_j^{-1})$.

Next, starting from a class $g \in H^1(X, \mathcal{G})$ we can construct a class

$$\delta_1(g) \in H^2(X, \mathcal{A}^g)$$

such that $\delta_1(g) = 0$ if and only if g is in the image of the map $H^1(X, \mathcal{G}') \to H^1(X, \mathcal{G})$. Here $\mathcal{A}^g$ is the sheaf obtained from $\mathcal{A}$ by twisting with g. Namely, if g is represented by a Cech 1-cocycle $g_{ij} \in \mathcal{G}(U_i)$ then we have isomorphisms $\psi_i : \mathcal{A}|_{U_i} \to \mathcal{A}^g|_{U_i}$ such that $\psi_j = \psi_i \circ g_{ij}$ over U_{ij}. To construct $\delta_1(g)$, for some covering (U_i), we can choose liftings $g'_{ij} \in \mathcal{G}'(U_{ij})$ for a 1-cocycle (g_{ij}) representing g (such that $g'_{ij} g'_{ji} = 1$ and $g'_{ii} = 1$). Then $\delta_1(g)$ is the class of the 2-cocycle $(\psi_i(g'_{ij} g'_{jk} g'_{ki}))$ with values in $\mathcal{A}^g$.

Finally, for a given class $g \in H^1(X, \mathcal{G})$ we need the following description of the fiber of the map

$$H^1(X, \mathcal{G}') \stackrel{H^1(p)}{\to} H^1(X, \mathcal{G})$$

over g. Assume that this fiber is nonempty and let us choose an element $g' \in H^1(X, \mathcal{G}')$ projecting to g. Then we have an exact sequence of twisted groups

$$1 \to \mathcal{A}^g \to (\mathcal{G}')^{g'} \to \mathcal{G}^g \to 1.$$

Thus, as before we have two actions of the group $H^0(X, \mathcal{G}^g)$ on $H^1(X, \mathcal{A}^g)$. Now we can construct a surjective map

$$H^1(X, \mathcal{A}^g) \to H^1(p)^{-1}(g), \tag{2.4}$$

such that the fibers of this map are the orbits of the twisted action of $H^0(X, \mathcal{G}^g)$ on $H^1(X, \mathcal{A}^g)$ (see (2.3)). Namely, let (g'_{ij}) be a Cech 1-cocycle representing g', and let

$a_{ij} \in \mathcal{A}(U_{ij})$ be the g-twisted 1-cocycle, so that $\psi_i(a_{ij})$ is a 1-cocycle with values in $\mathcal{A}^g$. This means that over U_{ijk} one has

$$a_{ij} Ad(g_{ij})(a_{jk}) = a_{ik}.$$

Then our map (2.4) sends (a_{ij}) to the class of $(a_{ij} g'_{ij})$.

In the particular case when the (usual) action of $H^0(X, \mathcal{G}^g)$ on $H^1(X, \mathcal{A}^g)$ is trivial, the corresponding connecting map

$$\delta_0 : H^0(X, \mathcal{G}^g) \to H^1(X, \mathcal{A}^g)$$

is a group homomorphism, and the map (2.4) induces an identification of the cokernel of this homomorphism with $H^1(p)^{-1}(g)$. Equivalently, in this case the map (2.4) corresponds to a transitive action of $H^1(X, \mathcal{A}^g)$ on $H^1(p)^{-1}(g)$, such that the stabilizer of any element is the image of δ_0.

Remark 2.3.3. Later we will consider cases in which $\mathcal{A} \subset \mathcal{G}$ is central, so that the action is indeed trivial.

2.4 Nonabelian Hypercohomology

We will use below the following simple generalization of nonabelian H^1. Let $\mathcal{G}$ be a sheaf of groups over a topological space X, and let $\mathcal{E}$ be a sheaf of sets, equipped with a $\mathcal{G}$-action. We view a pair $\mathcal{G} \curvearrowright \mathcal{E}$ as a generalization of a length 2 complex.

Definition 2.4.1. For an open covering $\mathcal{U} = (U_i)_{i \in I}$ of X, we define

(i) The *set of* 1-*cocycles* $Z^1(\mathcal{U}, \mathcal{G} \curvearrowright \mathcal{E})$ over $\mathcal{U}$ for the pair $\mathcal{G} \curvearrowright \mathcal{E}$ to be the pointed set of (g_{ij}, e_i) where $g_{ij} \in Z^1(\mathcal{U}, \mathcal{G})$ and $e_i \in \mathcal{E}(U_i)$ such that $e_i = g_{ij}(e_j)$.

(ii) Two 1-cocycles over $\mathcal{U}$, (g_{ij}, e_i) and $(\widetilde{g}_{ij}, \widetilde{e}_i)$ are called *cohomologous* if for some collection $h_i \in \mathcal{G}(U_i)$ we have

$$\tilde{g}_{ij} = h_i g_{ij} h_j^{-1}, \quad \widetilde{e}_i = h_i(e_i).$$

(iii) The *nonabelian hypercohomology* $\mathbb{H}^1(\mathcal{U}, \mathcal{G} \curvearrowright \mathcal{E})$ with respect to $\mathcal{U}$ is the pointed set of equivalence classes.

Again, passing to the limit over all open coverings $\mathcal{U}$, we get the *nonabelian hypercohomology*, $\mathbb{H}^1(X, \mathcal{G} \curvearrowright \mathcal{E})$.

This construction is natural: if we have a homomorphism of sheaves of groups $\mathcal{G}_1 \to \mathcal{G}_2$ and the compatible map of sheaves of set $\mathcal{E}_1 \to \mathcal{E}_2$, then we get the induced map

$$\mathbb{H}^1(X, \mathcal{G}_1 \curvearrowright \mathcal{E}_1) \to \mathbb{H}^1(X, \mathcal{G}_2 \curvearrowright \mathcal{E}_2).$$

Also, sending (g_{ij}, e_i) to g_{ij} defines a projection to the usual nonabelian H^1,

$$\mathbb{H}^1(X, \mathcal{G} \curvearrowright \mathcal{E}) \to H^1(X, \mathcal{G}).$$

Remark 2.4.2. While $H^1(X, \mathcal{G})$ classifies $\mathcal{G}$-torsors, $\mathbb{H}^1(X, \mathcal{G} \curvearrowright \mathcal{E})$ can be identified with the isomorphism classes of pairs (P, e), where P is a $\mathcal{G}$-torsor, and e is a global section of the twisted sheaf $\mathcal{E}_P = P \times_{\mathcal{G}} \mathcal{E}$.

Next, we have the following analog of the connecting homomorphism $H^1 \to H^2$. Assume that we have an abelian extension of sheaves of groups

$$1 \to \mathcal{A}_0 \to \mathcal{G}' \xrightarrow{p} \mathcal{G} \to 1$$

over X, and sheaves of sets $\mathcal{E}'$ and $\mathcal{E}$, where $\mathcal{G}'$ (resp., $\mathcal{G}$) acts on $\mathcal{E}'$ (resp., $\mathcal{E}$). Further, assume that we have a sheaf of abelian groups $\mathcal{A}_1$ acting freely on $\mathcal{E}'$, and an identifcation $\mathcal{E} = \mathcal{E}'/\mathcal{A}_1$. We denote this action as $a_1 + e'$, where $a_1 \in \mathcal{A}_1$, $e' \in \mathcal{E}'$. We require the following compatibilities between these data. First, the projections $p : \mathcal{E}' \to \mathcal{E}$ and $p : \mathcal{G}' \to \mathcal{G}$ should be compatible with the actions (of $\mathcal{G}'$ on $\mathcal{E}'$ and of $\mathcal{G}$ on $\mathcal{E}$). Note that this implies that there is an action of $\mathcal{G}'$ on $\mathcal{A}_1$, compatible with the group structure on $\mathcal{A}_1$, such that

$$g'(a_1 + e') = g'(a_1) + g'(e').$$

Secondly, we require that the subgroup $\mathcal{A}_0 \subset \mathcal{G}'$ acts trivially on $\mathcal{A}_1$, so that there is an induced action of $\mathcal{G}$ on $\mathcal{A}_1$, such that the above formula becomes

$$g'(a_1 + e') = p(g)(a_1) + g'(e').$$

In particular, for $g' = a_0 \in \mathcal{A}_0$, we get

$$a_0(a_1 + e') = a_1 + a_0(e'). \tag{2.5}$$

For $e' \in \mathcal{E}'$ and $a_0 \in \mathcal{A}_0$, let us define $d_{e'}(a_0) \in \mathcal{A}_1$ from the equation

$$a_0(e') = d_{e'}(a_0) + e'$$

(this is possible since a_0 acts trivially on $\mathcal{E}$). Furthermore, (2.5) easily implies that $d_{a_1+e'}(a_0) = d_{e'}(a_0)$, so we have a well defined map of sheaves

$$\mathcal{E} \times \mathcal{A}_0 \to \mathcal{A}_1 : (e, a_0) \mapsto d_e(a_0),$$

compatible with the group structures in $\mathcal{A}_0$ and $\mathcal{A}_1$, such that

$$a_0(e') = d_{p(e')}(a_0) + e'.$$

In particular, for every section e of $\mathcal{E}$ over an open subset $U \subset X$ we have a complex of abelian groups over U, $(\mathcal{A}_\bullet, d_e)$. Note that $\mathcal{G}$ acts on $\mathcal{A}_0$ (via adjoint action $Ad(g)$), $\mathcal{A}_1$ and $\mathcal{E}$, and we have

$$g(d_e(a_0)) = d_{g(e)}(Ad(g)a_0). \tag{2.6}$$

Now assume we have a class $c \in \mathbb{H}^1(X, \mathcal{G} \curvearrowright \mathcal{E})$ represented by a Cech 1-cocycle (g_{ij}, e_i). Let $g = (g_{ij})$ be the induced class in $H^1(X, \mathcal{G})$. We have the corresponding twisted sheaves $\mathcal{A}_0^g$, $\mathcal{A}_1^g$, and (2.6) implies that the d_{e_i}'s glue into a global differential

$$d_e : \mathcal{A}_0^g \to \mathcal{A}_1^g.$$

We are going to define an obstruction class $\delta_1(c)$ with values in

$$\mathbb{H}^2(X, (\mathcal{A}_\bullet^g, d_e)),$$

such that it vanishes if and only if (g_{ij}, e_i) can be lifted to a class in $\mathbb{H}^1(X, \mathcal{G}' \curvearrowright \mathcal{E}')$. Namely, by making the covering small enough, we can assume that

$$g_{ij} = p(g'_{ij}),\ g'_{ij} \in \mathcal{G}'(U_{ij}),\quad e_i = p(e'_i),\ e'_i \in \mathcal{E}'(U_i).$$

Then we have well defined elements $a_{0,ijk} \in \mathcal{A}_0(U_{ijk})$ and $a_{1,ij} \in \mathcal{A}_1(U_{ij})$, such that

$$g'_{ij} g'_{jk} = a_{0,ijk} g'_{ik},$$

$$g'_{ij}(e'_j) = a_{1,ij} + e'_i.$$

It is easy to check that $(a_{0,ijk}, a_{1,ij})$ satisfy the equations

$$a_{0,ijk} + a_{0,ikl} = Ad(g_{ij}) a_{0,jkl} + a_{0,ijl},\quad a_{1,ij} + g_{ij}(a_{1,jk}) = d_{e_i}(a_{0,ijk}) + a_{1,ik},$$

which means that we get a 2-cocycle $\delta_1(g_{ij}, e_i)$ with values in $(\mathcal{A}^g_\bullet, d_e)$.

One can check that this construction gives a well defined element $\delta_1(c) \in \mathbb{H}^2(X, (\mathcal{A}^g_\bullet, d_e))$. Namely, a different choice of liftings $g'_{ij} \mapsto a_{0,ij} g'_{ij}$, $e'_i \mapsto a_{1,i} + e'_i$ would lead to adding the coboundary of $(a_{0,ij}, a_{1,i})$ to the twisted 2-cocycle $(a_{0,ijk}, a_{1,ij})$. On the other hand, changing (g_{ij}, e_i) to $(h_i g_{ij} h_j^{-1}, h_i(e_i))$ would lead to a different presentation of the twisted sheaves $\mathcal{A}^g_\bullet$, so that the action of h_i glues into isomorphism between two presentations. Our 2-cocycles $\delta_1(g_{ij}, e_i)$ and $\delta_1(h_i g_{ij} h_j^{-1}, h_i(e_i))$ correspond to each other under this isomorphism.

Next, let us assume that a class $c \in \mathbb{H}^1(X, \mathcal{G} \curvearrowright \mathcal{E})$ is lifted to a class $c' \in \mathbb{H}^1(X, \mathcal{G}' \curvearrowright \mathcal{E}')$. (More precisely, we need to fix the corresponding pair (P', e') where P' is $\mathcal{G}'$-torsor and e' is a global section of $\mathcal{E}'_{P'}$.) Let $g \in H^1(X, \mathcal{G})$ be the image of c. We define the following subgroup in $H^0(X,\mathcal{G}^g)$:

$$\mathbb{H}^0(X,\mathcal{G},c) := \{(\alpha_i \in \mathcal{G}(U_i)) \mid \alpha_i = g_{ij}\alpha_j g_{ij}^{-1},\ \alpha_i(e_i) = e_i\},$$

where (g_{ij}, e_i) is a Cech representative of c. We have a natural connecting map (depending on a choice of c')

$$\delta_0 : \mathbb{H}^0(X,\mathcal{G},c) \to \mathbb{H}^1(X,(\mathcal{A}_\bullet^g, d_e)),$$

defined as follows. We can assume (g_{ij}, e_i) comes from a Cech representative (g'_{ij}, e'_i) for c'. Let $\alpha = (\alpha_i)$ be an element in $\mathbb{H}^0(X,\mathcal{G},c)$. We can assume that each α_i can be lifted to $\alpha'_i \in \mathcal{G}'(U_i)$. Then we have

$$\alpha'_i \cdot a_{0,ij} = g'_{ij}\alpha'_j(g'_{ij})^{-1}, \quad \alpha'_i(a_{1,i} + e'_i) = e'_i,$$

for uniquely defined $a_{0,ij} \in \mathcal{A}_0(U_{ij})$, $a_{1,i} \in \mathcal{A}_0(U_i)$. It is easy to check that the following equations are satisfied:

$$a_{0,ij} + Ad(g_{ij})(a_{0,jk}) = a_{0,ik}, \quad d_{e_i}(a_{0,ij}) = a_{1,i} - g_{ij}(a_{1,j}), \tag{2.7}$$

which mean that $(a_{0,ij}, a_{1,i})$ define a 1-cocycle with values in $(\mathcal{A}_\bullet^g, d_e)$. We set $\delta_0(\alpha_i)$ to be the class of this 1-cocycle. As in Sec. 2.3, one can check that $\alpha \mapsto \delta_0(\alpha^{-1})$ is a crossed homomorphism, i.e., equation (2.2) is satisfied.

Next, we have a natural surjective map (depending on c')

$$\mathbb{H}^1(X,(\mathcal{A}_\bullet^g, d_e)) \to L_c, \tag{2.8}$$

where $L_c \subset \mathbb{H}^1(X, \mathcal{G}' \curvearrowright \mathcal{E}')$ is the set of liftings of c. Namely, given a twisted Cech 1-cocycle with values in $(\mathcal{A}^g_\bullet, d_e)$, $(a_{0,ij}, a_{1,i})$, so that equations (2.7) are satisfied, and a representative (g'_{ij}, e'_i) of c' we get a new lifting $(a_{0,ij}g'_{ij}, a_{1,i} + e'_i)$. Furthermore, as in Sec. 2.3, we can identify the fibers of (2.8) with the orbits of the twisted action of $\mathbb{H}^0(X, \mathcal{G}, c)$ on $\mathbb{H}^1(X, (\mathcal{A}^g_\bullet, d_e))$, which is defined similarly to (2.3). In particular, in the case when the usual action of $\mathbb{H}^0(X, \mathcal{G}, c)$ on $\mathbb{H}^1(X, (\mathcal{A}^g_\bullet, d_e))$ is trivial (or equivalently, δ_0 is a group homomorphism), these orbits are simply the cosets for the image of δ_0.

CHAPTER III

BASIC THEORY OF NC-SCHEMES

3.1 NC-Nilpotent and NC-Complete Algebras

We recall the category $\mathcal{N}$ of NC-nilpotent algebras. They are naturally described as those algebras for which any expression involving sufficiently many commutator brackets vanishes. It is convenient to associate to a given algebra A a *degree of NC-nilpotency*, a measure of how many brackets a non-zero expression in A may have, in other words to define subcategories $\mathcal{N}_d$,

$$\mathcal{C}om = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots \subset \bigcup \mathcal{N}_d = \mathcal{N}$$

consisting of algebras which are *NC-nilpotent of degree d*, or *d-nilpotent algebras*. The notion of degree of NC-nilpotency depends on a choice of NC-filtration. Originally in [11] the commutator filtration F^dR was used for this purpose, and it has many pleasant features, but the filtration $\mathcal{I}^dR$ introduced in [20] turns out to be more convenient for the study of NC-smoothness via DG-resolutions. In this thesis, we will always work with the filtration $\mathcal{I}^dR$ of [20], which we call *the NC-filtration*.

3.1.1 The NC-filtration & NC-nilpotent algebras

For a Lie algebra L the lower central series L_i is the decreasing filtration

$$L \supset [L, L] \supset [L, [L, L]] \supset \cdots,$$

the convention for its indexing being to start with $L_1 = L$ and then $L_n = [L, L_{n-1}]$ for $n \geq 2$. Any ring R may also be considered as a Lie algebra R^{Lie} equipped with its commutator bracket $[a,b] = ab - ba$, and there is a corresponding associative lower central series $R_{(i)}$ where $R_{(i)} = R \cdot R_i^{\text{Lie}}$ is the two-sided ideal generated by the lower central series of R^{Lie}.

Definition 3.1.1. For any ring R, define the *NC-filtration* $\mathcal{I}^\bullet R$ to be the smallest decreasing filtration of R by two-sided ideals

$$\mathcal{I}^0 R \supset \mathcal{I}^1 R \supset \mathcal{I}^2 R \supset \cdots$$

having $\mathcal{I}^0 R = R$ and $\mathcal{I}^1 R = R[R,R]$, such that $R_d^{\text{Lie}} \subset \mathcal{I}^d R$ for $d \geq 2$, and which is an *algebra filtration*, i.e. $(\mathcal{I}^m R) \cdot (\mathcal{I}^n R) \subset \mathcal{I}^{m+n} R$ for $m, n \geq 0$.

This means that for $d \geq 2$ we have $\mathcal{I}^d R = R \cdot R_d^{\text{Lie}} + \sum_{i+j=d} (\mathcal{I}^i R) \cdot (\mathcal{I}^j R)$, so the first few terms of the NC-filtration are:

$$\begin{aligned}
&\mathcal{I}^0 R = R \\
&\mathcal{I}^1 R = \mathcal{I}^2 R = R[R,R] \\
&\mathcal{I}^3 R = R[R,[R,R]] + R[R,R]^2 \\
&\mathcal{I}^4 R = R[R,R]^2 + R[R,[R,[R,R]]] \\
&\quad \vdots
\end{aligned}$$

This is the same as the definition in [20]:

$$\mathcal{I}^d R = \sum_{i_1 \geq 2, \ldots, i_m \geq 2, i_1 + \cdots + i_m \geq d} R \cdot R_{i_1}^{\text{Lie}} \cdot R \cdots R \cdot R_{i_m}^{\text{Lie}} \cdot R.$$

Example 3.1.2. From the decomposition $TV = SV \otimes U\mathcal{L}_+V$, it is obvious that $\mathcal{I}^d TV = SV \otimes U^{\geq d}\mathcal{L}_+V$ and $\mathrm{gr}^d_{\mathcal{I}}(TV) = U^d\mathcal{L}_+V$.

Remark 3.1.3. The commutator filtration F^dR of [11] is defined analogously to $\mathcal{I}^dR$ but with a shift: that is, one takes $R \cdot R^{\mathrm{Lie}}_{d+1} \subset F^dR$ instead of $R \cdot R^{\mathrm{Lie}}_{d} \subset \mathcal{I}^dR$ for $d \geq 2$.

Definition 3.1.4. An algebra R is called *NC-nilpotent* if $\mathcal{I}^nR = 0$ for some n, and *NC-nilpotent of degree d*, or *d-nilpotent*, if $\mathcal{I}^{d+2}R = 0$. The category of NC-nilpotent algebras (resp. of degree $\leq d$) is denoted $\mathcal{N}$ (resp. $\mathcal{N}_d$).

In particular note that $\mathcal{N}_0 = \mathcal{C}om$, and $\mathcal{N}_1$ consists of the *central extensions* of commutative algebras, which we define in the next section. One of the most pleasant aspects of NC-nilpotent rings is their behavior with respect to localization:

Proposition 3.1.5 ([11, (2.1.5)])**.** *Let R be NC-nilpotent, and $S \subset R_{ab}$ any proper multiplicative set with preimage $\widetilde{S} = \pi_{ab}^{-1}(S)$. Then $\widetilde{S}$ is a (two-sided) Ore set. In particular there exists a localized ring $\widetilde{S}^{-1}R$, flat over R, satisfying the universal property.*

3.1.2 Central extensions

Definition 3.1.6. An R-bimodule M is called *central* if $rm = mr$ for all $r \in R, m \in M$.

It is easy to see a central R-bimodule M is equivalent data to an R_{ab}-module:

$$(r_1r_2).m = r_1.(r_2.m) = r_1.(m.r_2) = (m.r_2).r_1 = m.(r_2r_1) = (r_2r_1).m$$

Definition 3.1.7. A *central extension* R' of R by I is an exact sequence of algebras $I \to R' \to R$ such that $I^2 = 0$ and I is a central bimodule.

An algebra extension $R' \in \mathrm{Exal}(R, I) = H^2(R, I)$ includes I in the center $Z(R')$ if and only if I is a central R-bimodule.

Central extensions enter the story in the following way:

Example 3.1.8. Let $R' \in \mathcal{N}_{d+1}$ and $R \in \mathcal{N}_d$ be the truncation $R = R'/\mathcal{I}^{d+2}R'$. Then $\mathcal{I}^{d+2}R' \to R' \to R$ is a central extension.

Proposition 3.1.9. *The associated graded algebra $gr^{\bullet}_{\mathcal{I}}(R)$ is a central R-bimodule.*

Proof. That $[R, \mathcal{I}^d R] \subset \mathcal{I}^{d+1}R$ follows easily by induction using that $\mathcal{I}^d R$ is an algebra filtration, so that $[R, (\mathcal{I}^i R) \cdot (\mathcal{I}^j R)] = [R, \mathcal{I}^i R] + [R, \mathcal{I}^j R]$. □

Corollary 3.1.10. *Any $(d+1)$-nilpotent algebra R is a central extension of a d-nilpotent algebra. The category $\mathcal{N}$ of NC-nilpotent algebras is the same as the iterated central extensions of commutative algebras.*

We now record some useful facts about central extensions. The following proposition is a rephrasing of [11, 1.2.5(a), 1.2.6, and 1.2.7]

Proposition 3.1.11. *Let $I \xrightarrow{\iota} R' \xrightarrow{p} R$ be a central extension, and $f : S \to R$ be a homomorphism.*

(a.) *The set of homomorphisms $f' : S \to R'$ lifting f (such that $pf' = f$) is a pseudo-torsor for* $\mathrm{Der}(S, I) = \mathrm{Der}(S_{ab}, I)$.

(b.) *The set of endomorphisms ψ of R' such that $p\psi = p$ and $\psi|_I = \mathrm{id}_I$ is a group under composition, naturally isomorphic to* $\mathrm{Der}(R, I) = \mathrm{Der}(R_{ab}, I)$.

(c.) If $R'_{ab} = R_{ab}$ then any endomorphism ψ of R' for which $p\psi = p$ also satisfies $\psi|_I = \mathrm{id}_I$. In particular it is an automorphism.

Proof. (a) is familiar from classical deformation theory (see e.g. [**?** , II.6.2(a)]) and only requires $I^2 = 0$, not centrality of I. Given two lifts f', f'' then their difference $\delta : S \to I$ is a derivation:

$$\begin{aligned}
\delta(s_1 s_2) &= f'(s_1 s_2) - f''(s_1 s_2) \\
&= f'(s_1)f'(s_2) + \big(f'(s_1)f''(s_2) - f'(s_1)f''(s_2)\big) - f''(s_1)f''(s_2) \\
&= f'(s_1) \cdot \delta(s_2) + \delta(s_1) \cdot f''(s_2) \\
&= f(s_1).\delta(s_2) + \delta(s_1).f(s_2).
\end{aligned}$$

Note that $\mathrm{Der}(S, I) = \mathrm{Der}(S_{ab}, I)$ whenever I is a central S-bimodule because of $\delta([s_1, s_2]) = [\delta s_1, s_2] + [s_1, \delta s_2]$.

For (b), denote by $\mathrm{End}(p)$ the set of endomorphisms, and $\mathrm{End}(p, \iota)$ the endomorphisms restricting to identity on I. Although $\mathrm{End}(p)$ is only a monoid, $\mathrm{End}(p, \iota)$ is a group by the 5-lemma. There is a commutative diagram with horizontal bijections:

$$\begin{array}{ccc}
\mathrm{Der}(R, I) & \longrightarrow & \mathrm{End}(p, \iota) \\
\big\downarrow & & \big\downarrow \\
\mathrm{Der}(R', I) & \xrightarrow{\ !\ } & \mathrm{End}(p)
\end{array}$$

However, the lower horizontal arrow "!" is not structure-preserving. The extra condition in $\mathrm{End}(p, \iota)$ insures the horizontal map is a homomorphism as it implies $\delta_1 \circ \delta_2 = 0$, so that $(1 + \delta_1) \circ (1 + \delta_2) = 1 + (\delta_1 + \delta_2)$.

For (c) simply note that in this case we have

$$\mathrm{Der}(R', I) = \mathrm{Der}(R'_{ab}, I) = \mathrm{Der}(R_{ab}, I) = \mathrm{Der}(R, I)$$

so that the diagram from (ii) identifies $\mathrm{End}(p,\iota) = \mathrm{End}(p)$. □

The following is also familiar from deformation theory (cf. [22, (2.16)]) and is crucial for representability theorems later.

Proposition 3.1.12. *[11, 1.2.5(b)] Let $I \to R' \to R$ be a central extension. There is an isomorphism of rings,*

$$R' \times_R R' \cong R' \times_{R_{ab}} R_{ab}[I]$$

sending $(x,y) \mapsto (x, y_{ab} + (y-x))$.

3.1.3 NC-complete algebras

Definition 3.1.13. For any algebra R, the *NC-completion* $R_{[\![ab]\!]}$ is the limit

$$R_{[\![ab]\!]} \longrightarrow \big(\cdots\cdots \longrightarrow R/\mathcal{I}^3R \longrightarrow R/\mathcal{I}^2R \longrightarrow R_{ab}\big),$$

i.e. the completion with respect to the NC-filtration, $R_{[\![ab]\!]} = \varprojlim R/\mathcal{I}^dR$.

If the natural map $R \longrightarrow R_{[\![ab]\!]}$ is an isomorphism then R is called *NC-complete*, but in general this map is neither injective nor surjective.

Example 3.1.14. Let L be a Lie algebra. Then $U(L)_{[\![ab]\!]}$ is just the completion with respect to the PBW filtration. If L is nilpotent, then $U(L) = U(L)_{[\![ab]\!]}$ is NC-nilpotent.

Example 3.1.15. Here are some elementary examples that show how in general the NC-completion may be quite un-interesting.

1. Let L be the non-abelian 2-dimensional Lie algebra with basis x,y such that $[x,y] = x$. Then $x \in \mathcal{I}^dU(L)$ for all $d \geq 1$ so $U(L)_{[\![ab]\!]} = U(L)_{ab} = \mathbb{C}[y]$.

2. Let $R = M_n(\mathbb{C})$ for $n \geq 2$ (or consider $R = M_n(R')$ for any ring R'). Then $[R,R] = \mathfrak{sl}_n(\mathbb{C})$ contains a unit (such as a permutation matrix), so $\mathcal{I}^1 R = R$. This implies $R_{[\![ab]\!]} = R_{ab} = 0$.

Example 3.1.16. For a free algebra we have $TV = U\mathcal{L}V$. By the PBW theorem there is an isomorphism of vector spaces $U\mathcal{L}V \cong SV \otimes U\mathcal{L}_+V$, and hence $\hat{T}V \cong \hat{S}V \hat{\otimes} \hat{U}\mathcal{L}_+V$. The NC-completion is the subalgebra $TV_{[\![ab]\!]} \subset \hat{T}V$,

$$TV_{[\![ab]\!]} \cong SV \hat{\otimes} \hat{U}\mathcal{L}_+V.$$

(The right hand side means the vector space $\varprojlim(SV \otimes U^{\leq d}\mathcal{L}_+V)$.)

Localization doesn't work as well for general NC-complete algebras (in particular, localization does not commute with completion).

Definition 3.1.17 ([11, Def. (2.1.8)])**.** Let R be an NC-complete algebra and let $T \in R_{ab}$ be a multiplicative subset. We set

$$R[\![T^{-1}]\!] := \varprojlim(R/\mathcal{I}^d R)[T_d^{-1}],$$

where $T_d \subset R/\mathcal{I}^d R$ is the preimage of T. In the case when $T = \{f^n \mid n \geq 0\}$, for some element $f \in R_{ab}$, we denote the above algebra simply as $R[\![f^{-1}]\!]$.

Proposition 3.1.18. *[11, 2.1.1] For a central extension $R' \to R$, the natural map $GL_n(R') \to GL_n(R)$ is surjective.*

In particular the case $n = 1$ ensures that the NC-schemes defined in the next section are locally ringed spaces:

Corollary 3.1.19. *[11, 2.1.2] A central extension of a local ring is again a local ring.*

3.2 NC-Schemes

3.2.1 The spectrum of an NC-nilpotent ring

We are now going to define affine NC-schemes by constructing for any NC-complete ring R a locally ringed space $X = \operatorname{Spec} R$ whose underlying space is simply $\operatorname{Spec} R_{ab}$.

Proposition 3.2.1. *[11, 2.2.1] For an NC-nilpotent algebra R and $f \in R_{ab}$, let $S = \{f, f^2, f^3, \dots\}$ and $\widetilde{S} = \pi_{ab}^{-1}(S)$. There is a unique structure $\mathcal{O}_X$ of locally ringed space on $\operatorname{Spec} R_{ab}$ such that*

$$\Gamma(D(f), \mathcal{O}_X) = \widetilde{S}^{-1} R$$

for all $f \in R_{ab}$ and such that the maps are the corresponding localizations.

There is also a locally ringed space associated to NC-complete algebras.

Definition 3.2.2. The spectrum $\operatorname{Spec} R$ of an NC-complete algebra R is the locally ringed space

$$\operatorname{Spec} R = \varprojlim \left(\operatorname{Spec} R/\mathcal{I}^d R \right).$$

3.2.2 NC-schemes

In this section we define general (non-affine) NC-schemes.

Definition 3.2.3. An *affine NC-scheme* is a locally ringed space isomorphic to $\operatorname{Spec} \Lambda$ for an NC-complete ring Λ. An *NC-scheme* is a locally ringed space $(X, \mathcal{O}_X)$ with a covering by open sets U_i such that $(U_i, \mathcal{O}_X|_{U_i})$ are affine NC-schemes.

Definition 3.2.4. An NC-scheme X is of *finite type* if X_{ab} is of finite type and $\mathrm{gr}^d_{\mathcal{I}}(\mathcal{O}_X)$ is a coherent sheaf on X_{ab} for all d.

Definition 3.2.5. For NC-schemes X, Y we denote $X \hat{\times} Y$ the categorical product in NC-schemes, whose structure sheaf is $\mathcal{O}_X \hat{*} \mathcal{O}_Y = (\mathcal{O}_X * \mathcal{O}_Y)_{[\![ab]\!]}$. Denote by $X \times Y$ the NC-scheme for which $\mathcal{O}_{X \times Y} = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_Y$.

Example 3.2.6. Let X be any scheme, there is an NC-scheme $\widetilde{X}^{(1)}$ with structure sheaf $\widetilde{\mathcal{O}}_X = \mathcal{O}_X \oplus \Omega^2_X$ and product structure given by $(f_1, \omega_1)(f_2, \omega_2) = (f_1 f_2, f_1\omega_2 + f_2\omega_1 + df_1 \wedge df_2)$. Then $\widetilde{X}^{(1)}$ is an NC-scheme which is NC-nilpotent of degree 1, called the *standard 1-smooth thickening* of X.

3.3 NC-Smooth Algebras

3.3.1 NC-smooth algebras

Definition 3.3.1. An NC-nilpotent algebra of degree d (resp. NC-complete algebra) is called *d-smooth* (resp. *NC-smooth*) if for any square-zero extension $I \to \Lambda' \to \Lambda$ in $\mathcal{N}_d$ (resp. $\mathcal{N}$) and any morphism $A \to \Lambda$, there exists a lift as in the diagram:

$$\begin{array}{ccc} & \overset{f'}{\nearrow} & \Lambda' \\ & & \downarrow \\ A & \xrightarrow{f} & \Lambda \end{array}$$

Example 3.3.2. The NC-completion $TV_{[\![ab]\!]}$ of a free algebra is NC-smooth (it's free as an NC-complete algebra).

Example 3.3.3. More generally, it is easy to see if A is quasi-free (cf. [6]), then $A_{[\![ab]\!]}$ is NC-smooth. Besides free algebras, these include path algebras of quivers and coordinate rings of curves, both of which have commutative NC-completions.

Note that if R is d-smooth, then for any $k \leq d$, the truncation $R/\mathcal{I}^{k+1}R$ is k-smooth. In particular if R is NC-smooth then R_{ab} is formally smooth. The most important fact about NC-smooth algebras is the existence and uniqueness (up to non-canonical isomorphism) for any formally smooth commutative algebra R of an NC-smooth algebra R' such that $R'_{ab} \cong R$.

Theorem 3.3.4 ([11], [20])**.** *There is a unique (up to noncanonical isomorphism) d-smooth thickening of any $(d-1)$-smooth algebra.*

Uniqueness follows easily from Proposition 2.1.8(iii), whereas the proof of existence is constructive.

Remark 3.3.5. Working with the commutator filtration, Kapranov constructed the NC-smooth thickening R of a formally smooth algebra R_{ab} as the limit $R = \varprojlim R_d$, where $R_0 = R_{ab}$ and R_{d+1} is a universal central extension of R_d by $H_2(R_d, R_{ab})$. This may then be truncated to obtain d-smooth thickenings with respect to our NC-filtration. However, this is not very explicit as the relevant Hochschild cohomology groups have not been computed. Originally in [11] there was a proposed solution of this problem in terms of certain polynomial functors Q^d on the category of R_{ab}-modules such that $H_2(R_d, R_{ab}) = Q^d(\Omega^1_{R_{ab}})$, however a gap was noticed in [5] and has not been resolved except in the case of R_{ab} a local ring.

The purpose of the NC-filtration $\mathcal{I}^d R$ introduced in [20] is that the corresponding polynomial functors are easily identified. In [20], starting from the initial data of a torsion-free connection on Ω^1_A, d-smooth thickenings of all orders are constructed as certain subalgebras of $T^{\leq d}_{\mathcal{O}}(\Omega^1_A)$.

3.4 NC-Smooth Schemes

3.4.1 NC-smooth schemes

Definition 3.4.1. An NC-scheme X is called *NC-smooth* if for any central extension $\Lambda' \to \Lambda$ of NC-nilpotent algebras, and any map $f : \operatorname{Spec}\Lambda \to X$, there exists a lift f' as in the diagram:

$$\begin{array}{ccc} \operatorname{Spec}\Lambda & \xrightarrow{f} & X \\ \big\downarrow & \nearrow_{f'} & \\ \operatorname{Spec}\Lambda' & & \end{array}$$

Equivalently, the natural map $\operatorname{Hom}(\operatorname{Spec}\Lambda', X) \longrightarrow \operatorname{Hom}(\operatorname{Spec}\Lambda, X)$ is surjective.

Any NC-scheme X has an abelianization X_{ab}, which is an ordinary scheme whose structure sheaf is $\mathcal{O}_{X_{ab}} = (\mathcal{O}_X)_{ab}$. In this case we call X an *NC-thickening* of X_{ab}, and if X is NC-smooth, an *NC-smooth thickening*.

NC-smoothness is a local condition; the proof of following proposition illustrates the usefulness of central extensions in this theory.

Proposition 3.4.2. *[20, 2.1.4] An NC-scheme is NC-smooth if and only if it has a cover by open NC-subschemes which are NC-smooth.*

Example 3.4.3. Recall the transition functions for $\mathbb{P}^n$ in distinguished charts with coordinate algebra $\mathbb{C}[x_0^{(i)}, \ldots, x_n^{(i)}]$ and glued on localizations by the relations

$$x_\alpha^{(i)} = x_\alpha^{(j)} \cdot (x_i^{(j)})^{-1}$$

which in particular for $\alpha = j$ says that $x_j^{(i)} = (x_i^{(j)})^{-1}$. That these glue compatibly is a cocycle relation:

$$\begin{aligned} x_\alpha^{(i)} &= x_\alpha^{(j)} \cdot (x_i^{(j)})^{-1} \\ &= \left(x_\alpha^{(k)} \cdot (x_j^{(k)})^{-1} \right) \cdot (x_i^{(j)})^{-1} \\ &= x_\alpha^{(k)} \cdot \left(x_k^{(j)} \cdot (x_i^{(j)})^{-1} \right) \\ &= x_\alpha^{(k)} \cdot \left(x_i^{(j)} \cdot (x_k^{(j)})^{-1} \right)^{-1} \\ &= x_\alpha^{(k)} \cdot (x_i^{(k)})^{-1} \end{aligned}$$

Note that this just uses the one relation above, and *not* commutativity of the variables. This means that one gets a cocycle relation for free algebras $\mathbb{C}\langle x_0^{(i)}, \ldots, x_n^{(i)} \rangle$, and passing to the NC-completions determines an NC-smooth thickening of $\mathbb{P}^n$. If one instead writes all the inverses on the left, as in $x_\alpha^{(i)} = \cdot (x_i^{(j)})^{-1} \cdot x_\alpha^{(j)}$, one similarly obtains an NC-smooth thickening. These are in fact not isomorphic as NC-schemes, even at the 1-nilpotent level [20].

More generally any time one has a variety, locally isomorphic to $\mathbb{A}^n$, such that checking the cocycle condition doesn't necessitate commuting any of the variables, one obtains an NC-smooth thickening. The following example is new to this thesis.

Example 3.4.4. Let $\mathcal{H}_q = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(q))$ be a Hirzebruch surface. Then it has a cover by four open sets $V_i \cong \mathbb{A}^2$ glued together as implicit by the choice of coordinates: $V_1 = \mathbb{C}[z_1, z_2]$, $V_2 = \mathbb{C}[z_1^{-1}, z_2]$, $V_3 = \mathbb{C}[z_2^{-q} z_1^{-1}, z_2^{-1}]$, $V_4 = \mathbb{C}[z_1 z_2^q, z_2^{-1}]$, see e.g. [2, Ex. 3.8]. It is easy to see the cocycle condition does not require the commutativity of the variables, so this lifts to an NC-smooth thickening of $\mathcal{H}_q$.

One of the interesting questions in this area is:

Question 3.4.5. Which varieties admit NC-smooth thickenings?

The question is trivial in dimension 1 — smooth curves are already NC-smooth (they are quasi-free). The present state of knowledge is that they exist for curves, flag varieties, affine varieties, and abelian varieties, and products of these (if X and Y are NC-smooth, then so is $X \hat{\times} Y$). All of these examples can be found in [11] or [20].

Remark 3.4.6. On the other hand, nothing is known about which varieties *don't* admit NC-smooth thickenings, although an obstruction theory is outlined in [11, §4]. For example, it is possible to show that the standard 1-smooth thickening of a K3 surface does not extend to a 2-smooth thickening. However, the space of all 1-smooth thickenings is 20-dimensional (identified with $H^1(X, T \otimes \Omega^2) \cong H^1(X, \Omega^1)$) and the obstruction depends on this class. Furthermore, later in this thesis a weaker (but perhaps more natural) structure of *NC-smooth algebroid thickening* is introduced, and perhaps the question 3.4.5 should be modified accordingly.

3.5 NC-Functor of Points

For any NC-scheme X there is the corresponding representable functor $h^X : \mathcal{N}^{\mathrm{op}} \to Sets$ sending $\Lambda \mapsto \mathrm{Hom}(\operatorname{Spec}\Lambda, X)$. In the case that $X = \operatorname{Spec} A$ this is the same as the functor $h_A : \mathcal{N} \to Sets$ sending $\Lambda \mapsto \mathrm{Hom}(A, \Lambda)$.

Proposition 3.5.1 ([11])**.** *The category $\mathcal{N}^{op}$ is equivalent to the affine nilpotent NC-schemes. NC-schemes is a full subcategory of Fun$(\mathcal{N}^{op}, Sets)$.*

This point of view is useful in the study of NC-smooth thickenings of smooth variety M, because of the following representability criterion of [11]. We consider pairs of central extensions, which we denote for $i = 1, 2$

$$I_i \longrightarrow \Lambda_i \xrightarrow{p_i} \Lambda \tag{3.1}$$

and the natural map

$$j : h(\Lambda_1 \times_\Lambda \Lambda_2) \longrightarrow h(\Lambda_1) \times_{h(\Lambda)} h(\Lambda_2). \tag{3.2}$$

Proposition 3.5.2. *([11, 2.3.5]) Let M be a smooth algebraic variety and $h : \mathcal{N}_d \to$ Sets a formally smooth functor such that $h|_{\mathcal{C}om} = h_M$. Then $h|_{\mathcal{N}_d}$ is representable by a d-smooth NC-scheme if and only if for any pair of central extensions in $\mathcal{N}_d$ the natural map*

$$h(\Lambda_1 \times_\Lambda \Lambda_2) \to h(\Lambda_1) \times_{h(\Lambda)} h(\Lambda_2)$$

is an isomorphism. Moreover, it suffices to check the cases when

(a) Λ *is commutative and* $\Lambda_1 = \Lambda \oplus I_1$,

(b) $\Lambda_1 = \Lambda_2$ *and* $p_1 = p_2$.

The NC-smooth thickenings of $\mathbb{P}^n$ described above in 3.4.3 have nice descriptions in terms of their NC-functors of points.

Example 3.5.3. For $\Lambda \in \mathcal{N}$ define $h(\Lambda)$ to be the set of rank 1 projective left submodules of Λ^n. Clearly $h|_{\mathcal{C}om} = h_{\mathbb{P}^n}$, and formal smoothness comes from lifting idempotents. To see j is a bijection, one constructs an inverse sending $P_i \to P$ for $i = 1, 2$ to $P_1 \times_P P_2$.

This example more and more generally leads to NC-smooth thickenings of the Grassmannians, and of flag varieties, by considering (flags of) left projective submodules in Λ^n of the appropriate rank(s).

3.6 Associated Graded & Center of an NC-Smooth Thickening

We end this section with the observation that any NC-smooth thickening $\widetilde{\mathcal{O}}_X$ of an ordinary scheme X has a pre-determined associated graded algebra, functorial in X, as shown in [20]. Additionally, if $\widetilde{\mathcal{O}}_X$ is properly noncommutative (i.e. if $\dim X \geq 2$) then $\widetilde{\mathcal{O}}_X$ has trivial center.

Both of these results use the theory of DG-resolutions, which we don't discuss until the next chapter. The NC-smooth thickening R^{NC} of a smooth commutative algebra R is embedded as $R^{NC} \hookrightarrow \hat{T}_R(\Omega^1_R)$ in such a way that if $\tilde{f} \in R^{NC}$ lifts $f \in R$, then $\tilde{f} = f - df$ modulo $T^{\geq 2}$.

Theorem 3.6.1. *[20, 2.1.6, 2.3.15] For any NC-thickening $\widetilde{\mathcal{O}}$ of a smooth variety X, there is a natural surjective homomorphism of graded algebras,*

$$\xi : U\mathcal{L}_{+}\Omega^1_X \twoheadrightarrow gr^{\bullet}_{\mathcal{I}}(\widetilde{\mathcal{O}})$$

determined by $\xi([df, \omega]) = [\tilde{f}, \xi(\omega)]$ where $\tilde{f}_{ab} = f$. It is an isomorphism if and only if $\widetilde{\mathcal{O}}$ is NC-smooth.

Proposition 3.6.2. *Let R be a d-smooth algebra for some $d \geq 1$, such that $\dim R_{ab} \geq 2$ and R_{ab} is connected. Then the center of R is $\mathbb{C} + \mathcal{I}^{d+1}R$.*

Proof. (i) In the case $d = 1$, R is the standard 1-smooth thickening. If $\tilde{f}, \tilde{g} \in R$ then $[\tilde{f}, \tilde{g}] = df \wedge dg$ where $f = \tilde{f}_{ab}$. The map $\Omega^1 \to T \otimes \Omega^2$ sending $df \mapsto [dg \mapsto df \wedge dg]$ is injective for $\dim X \geq 2$, so if $\tilde{f}$ is central then $df = 0$ and $f \in \mathbb{C}$.

Now let $d \geq 2$, and let $R' = R/\mathcal{I}^{d+1}R$ denote the truncation of R. If $Z(R') = \mathbb{C} + \mathcal{I}^d R'$, then since an element $\tilde{f} \in Z(R)$ also has central truncation $\tilde{f}' \in Z(R')$, we know already that $Z(R) \subset \mathbb{C} + \mathcal{I}^d R$. By d-nilpotency $[\mathcal{I}^d R, \mathcal{I}^1 R] \subset \mathcal{I}^{d+2}R = 0$, so whether or not $\tilde{f} \in \mathcal{I}^d R$ is central is determined completely by the map $\mathcal{I}^d R \times R_{ab} \to \mathcal{I}^{d+1}R$ sending $(\tilde{f}, g) \mapsto [\tilde{f}, \tilde{g}] = [\tilde{f}, g - dg] = [\tilde{f}, -dg]$.

This is equivalent to the commutator pairing $U^d \mathcal{L}_+ \Omega^1_{R_{ab}} \times \Omega^1_{R_{ab}} \to U^{d+1}\mathcal{L}_+ \Omega^1_{R_{ab}}$ restricted from $T(\Omega^1_{R_{ab}}) \times T(\Omega^1_{R_{ab}}) \to T(\Omega^1_{R_{ab}})$. Since the center of $T(\Omega_{R_{ab}})^1$ is trivial, the reuslt follows. □

This easily implies:

Corollary 3.6.3. *Let $\mathcal{O}_X^{NC}$ be an NC-smooth thickening of a smooth scheme X, where $\dim X \geq 2$. Then the center of $\mathcal{O}_X^{NC}$ is the constant sheaf $\mathbb{C}_X$.*

CHAPTER IV

NC-SMOOTHNESS VIA DG-RESOLUTIONS

In this chapter we review the construction of DG-resolutions of NC-smooth thickenings via algebraic NC-connections developed in [20], which we use in the following chapter to study bimodule extensions.

4.1 Relative NC-de Rham Complex

Definition 4.1.1. For a smooth variety X define the *relative NC-de Rham complex of* X to be the DG-algebra $(\bar{\mathcal{A}}^\bullet_X, \tau)$ with $\bar{\mathcal{A}}^\bullet_X = \Omega^\bullet_X \otimes_{\mathcal{O}} \hat{T}_{\mathcal{O}}(\Omega^1_X)$, graded by the de Rham degree in $\Omega^\bullet_X$, and with graded differential τ determined by the rule $\tau(1 \otimes \alpha) = \alpha \otimes 1$ for $\alpha \in T^1\Omega^1_X$.

It follows from $\tau(1 \otimes \alpha) = \alpha \otimes 1$ and $\tau^2 = 0$ that $\tau|_{\Omega^\bullet_X} = 0$.

Remark 4.1.2. Geometrically, this is a noncommutative version of the relative de Rham complex of the projection $p : TX \to X$, which is identified with $\Omega^\bullet_{TX/X} = (\Omega^\bullet_X \otimes S(\Omega^1_X), d_r)$. Or perhaps even more appropriately, the projection $p^{(\infty)} : X^{(\infty)}_{TX} \to X$ from the formal neighborhood of the zero section, functions on which are $\hat{S}(\Omega^1_X)$.)

Proposition 4.1.3. *There exist right $\hat{U}\mathcal{L}_+(\Omega^1)$-linear homotopy operators*

$$h : \Omega^i_X \otimes_{\mathcal{O}} T^j(\Omega^1) \longrightarrow \Omega^{i-1}_X \otimes_{\mathcal{O}} T^{j+1}(\Omega^1)$$

such that $h\tau + \tau h = \mathrm{id}$ *for* $i \geq 1$, *and* $h^2 = 0$.

Proof. As a right $U\mathcal{L}_+\Omega^1$-module we may identify $\Omega_X^\bullet \otimes T(\Omega^1) = \Omega_X^\bullet \otimes S(\Omega^1) \otimes U\mathcal{L}_+\Omega^1$. It is easy to see that τ vanishes on $U\mathcal{L}_+\Omega^1$ (see [20]). So we can identify $\tau = d_{rel} \otimes 1$ where d_{rel} is the relative de Rham differential on $\Omega^\bullet_{TX/X}$. Thus we set $h = h_E \otimes 1$. □

This allows us to compute the cohomology of $\bar{\mathcal{A}}_X^\bullet$.

Corollary 4.1.4 ([20]). *The projection* $\pi : \bar{\mathcal{A}}_X^\bullet \twoheadrightarrow \hat{U}\mathcal{L}_+\Omega_X^1$ *is a retraction. In particular the cohomology of* $\bar{\mathcal{A}}_X^\bullet$ *is* $\hat{U}\mathcal{L}_+\Omega_X^1$ *(in degree 0).*

It is significant that $\bar{\mathcal{A}}_X^\bullet$ is a DG-resolution of $\hat{U}\mathcal{L}_+\Omega_X^1$ as this has associated graded $\mathrm{gr}^\bullet_{\mathcal{F}_{\mathrm{tot}}}(\hat{U}\mathcal{L}_+\Omega_X^1) = U\mathcal{L}_+\Omega_X^1$. For an NC-complete algebra $\widetilde{\mathcal{O}}_X^{NC}$ the condition $\mathrm{gr}^\bullet_{\mathcal{I}}(\widetilde{\mathcal{O}}^{NC}) = U\mathcal{L}_+\Omega_X^1$ is equivalent to NC-smoothness.

The main idea of [20, §2] is to consider another dga $(\mathcal{A}_X^\bullet, D)$ with the same underlying graded algebra as $\bar{\mathcal{A}}_X^\bullet$, but with higher terms added to perturb the differential $D = \tau + D_1 + D_2 + \cdots$, until $\mathcal{I}^\bullet(\ker(D)) = \mathcal{F}^\bullet_{\mathrm{tot}}(\ker(D))$. Since adding higher terms does not change the associated graded with respect to $\mathcal{F}^\bullet_{\mathrm{tot}}$, in this way we obtain an NC-smooth thickening.

Before moving on, it is convenient to also introduce a few filtrations on $\bar{\mathcal{A}}_X^\bullet$.

Definition 4.1.5. $\bar{\mathcal{A}}_X^\bullet$ has filtrations $\mathcal{F}_T^d, \mathcal{F}_{\mathrm{tot}}^d, \mathcal{I}_T^d$, given by:

$$\mathcal{F}_T^d(\bar{\mathcal{A}}_X^\bullet) = \Omega_X^\bullet \otimes T^{\geq d}\Omega_X^1, \qquad \mathcal{F}_{\mathrm{tot}}^d(\bar{\mathcal{A}}_X^\bullet) = \sum_i \Omega_X^{\geq d-i} \otimes T^{\geq i}\Omega_X^1, \qquad \mathcal{I}_T^d(\bar{\mathcal{A}}_X^\bullet) = \Omega_X^\bullet \otimes \mathcal{I}^d(\hat{T}\Omega_X^1)$$

Intersecting with $\bar{\mathcal{A}}_X^0 = \hat{T}(\Omega_X^1)$ one obtains two filtrations — the filtration by degree $\mathcal{F}^d\hat{T}(\Omega_X^1) = \hat{T}^{\geq d}\Omega_X^1$ and the NC-filtration $\mathcal{I}^d\hat{T}(\Omega_X^1)$. Note that $\mathcal{I}^d\mathcal{A}_X^0 \subset \mathcal{F}_{\mathrm{tot}}^d\mathcal{A}_X^0$.

4.2 Algebraic NC-Connections

Since τ preserves the total degree, any derivation on $\bar{\mathcal{A}}^\bullet_X$ obtained from τ by adding higher terms in $\hat{T}(\Omega^1_X)$ will preserve $\mathcal{F}^\bullet_{\mathrm{tot}}$. Any derivation D of $\bar{\mathcal{A}}^\bullet_X$ which preserves the filtration by total degree $\mathcal{F}_{\mathrm{tot}}$ can be written as a formal sum $D = D_0 + D_1 + D_2 + \cdots$, where D_k is the term which raises the total degree by k.

The following notion was introduced in [20, Def. 1.2.1]:

Definition 4.2.1 ([20]). Let X be a smooth variety. An *(algebraic) NC-connection* on X is a degree one graded derivation D of the graded algebra $\mathcal{A}^\bullet_X = \Omega^\bullet_X \otimes_{\mathcal{O}} \hat{T}_{\mathcal{O}}(\Omega^1_X)$ extending the de Rham differential on $\Omega^\bullet_X$, such that $D^2 = 0$ and $D_0 = \tau$.

Note that each D_i is determined just by its value on $\alpha \in T^1(\Omega^1)$, so we denote by $\nabla_i : \Omega^1 \to \Omega^1 \otimes_{\mathcal{O}} T^i(\Omega^1)$ the restriction $D_i|_{T^1\Omega^1}$. For $f \in \mathcal{O}$ and $s \in \Omega^1$ we have the equation

$$D(1 \otimes fs) = D(f \otimes s) = df \otimes s + f \cdot D(1 \otimes s)$$

thus ∇_1 is a usual algebraic connection, whereas for $i = 0, 2, 3, \ldots$ the ∇_i are $\mathcal{O}$-linear.

Lemma 4.2.2 ([20, Cor. 2.3.9]). *For any NC-connection D, there is a $\mathbb{C}$-linear isomorphism $(\mathcal{A}^\bullet_X, D) \xrightarrow{\Psi_D} (\bar{\mathcal{A}}^\bullet_X, \tau)$ of complexes given by $\Psi_D = (1 + hD_{\geq 1})$.*

Thus there is a corresponding homotopy operator h_D for $\mathcal{A}^\bullet_X$, given by $h_D = \Psi_D^{-1} \circ h \circ \Psi_D$, such that $h_D D + D h_D = 1$ for $\bullet > 0$. Note that since our choice of homotopy satisfies $h^2 = 0$ then we have the simplification $h_D = \Psi_D^{-1} h$.

It immediately follows that the associated graded of the cohomology is the same as before. In fact we have:

Theorem 4.2.3 ([20]). *For any NC-connection D, the two filtrations coincide:*

$$\mathcal{I}^d(\ker^0(D)) = \mathcal{F}^d(\ker^0(D)).$$

In particular, $\ker^0(D)$ *is NC-complete and has* $gr^\bullet_{\mathcal{I}}(\ker^0(D)) = U\mathcal{L}_+\Omega^1_X$, *so* $\widetilde{\mathcal{O}}_X = \ker^0(D)$ *is an NC-smooth thickening of* X.

It is useful in the computations that follow to be able to lift elements of truncated NC-thickenings $\widetilde{\mathcal{O}}^{\le d}$ to higher NC-thickenings $\widetilde{\mathcal{O}}^{\le d+k}$. We have the following:

Proposition 4.2.4. *There is a retraction* $\Sigma : \mathcal{A}^\bullet_X \to \ker(D)$ *given by* $\Sigma = (1 - h_D D)$.

Proof. Let $x \in \mathcal{A}^\bullet_X$. Then $Dx \in \mathcal{A}^{\ge 1}_X$ so that we may use $h_D D + D h_D = 1$ and $D^2 = 0$ to get $D(x - h_D Dx) = Dx - (1 - h_D D)Dx = h_D D^2 x = 0$. Hence $\Sigma(x) \in \ker(D) = \widetilde{\mathcal{O}}_X$. □

Since we have a natural $\mathcal{O}$-linear inclusion $\iota_d : \widetilde{\mathcal{O}}^{\le d}_X \subset \mathcal{A}^\bullet_X$ we obtain a section

$$\sigma : \widetilde{\mathcal{O}}^{\le d} \to \widetilde{\mathcal{O}}^{\le d+k} \tag{4.1}$$

by including $\widetilde{\mathcal{O}}^{\le d} \subset T^{\le d}(\Omega^1)$ into $\mathcal{A}^0_X$ as terms of degree $\le d$, then applying Σ, then projecting to terms of degree $\le d + k$. We omit d, k from the notation for brevity — it should always be clear from context the meaning. In particular we will use the formula for $k = 1$, denoting by $(x_0, \dots, x_n)$ a local section of $\widetilde{\mathcal{O}}^{\le n}_X \subset T^{\le n}(\Omega^1_X)$,

$$\sigma(x_0, \dots, x_n) = (x_0, \dots, x_n, -hD_1 x_n - hD_2 x_{n-1} - \dots - hD_n x_0). \tag{4.2}$$

The existence of an NC-connection D on X is equivalent to the existence of a usual torsion-free connection ∇ (on the cotangent bundle).

Proposition 4.2.5 ([20])**.** *For any torsion-free connection* ∇ *on* X *there exists an NC-connection* D *such that* $D_1 = \nabla$.

Moreover, the isomorphism type of the NC-smooth thickening obtained as $\ker(D)$ is independent of the choice of connection.

Proposition 4.2.6 ([20])**.** *Let* D *and* D' *be two NC-connections on* X*. There exists an algebra automorphism* Ψ_0 *of* $\hat{T}(\Omega^1_X)$ *such that* $\Psi = \mathrm{id} \otimes \Psi_0$ *is isomorphism* $(\mathcal{A}^\bullet_X, D) \xrightarrow{\Psi} (\mathcal{A}^\bullet_X, D')$ *of chain complexes.*

Remark 4.2.7. The DG-resolutions considered here fit into the general picture of homotopy perturbation theory (cf. [26, Sec. 2]).

CHAPTER V

BIMODULE EXTENSIONS OF NC-VECTOR BUNDLES

In this chapter we make frequent use of the section σ from (4.1).

5.1 NC-vector bundles

5.1.1 NC-Vector Bundles

Definition 5.1.1. Let $\widetilde{X}$ be a d-smooth or NC-smooth thickening of X. An *NC-vector bundle* on $\widetilde{X}$ is a locally free right $\widetilde{\mathcal{O}}_X$-module $\widetilde{E}$. We say that $\widetilde{E}$ *extends* an ordinary vector bundle E on X_{ab} if $\widetilde{E}_{ab} \cong E$.

Right modules have endomorphisms given by matrices with coefficients in $\widetilde{\mathcal{O}}_X$, so that an NC-vector bundle of rank r on $\widetilde{X}$ is equivalent to the data of a 1-cocycle $\tilde{g}_{ij} \in \check{H}^1(X, GL_r\widetilde{\mathcal{O}}_X)$.

In [20, Sec. 3] it shown how to construct via mNC-connections (similar to the construction of NC-connections) an NC-vector bundle extending an ordinary vector bundle $E, \bar{\nabla}$ with connection, on a smooth thickening coming from a connection.

Proposition 5.1.2 ([20])**.** *Let $\widetilde{X}$ be an NC-thickening of X from a connection ∇. Let $(E, \bar{\nabla})$ be a vector bundle with connection on X. Then there is a natural extension of E an NC-vector bundle $\widetilde{E}_{\bar{\nabla}}$ on $\widetilde{X}$.*

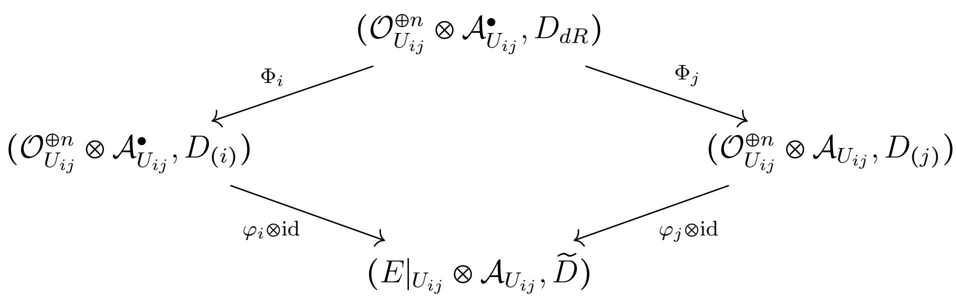
5.1.2 Cocycle description of $\widetilde{E}_{\bar{\nabla}}$

We compute a formula for the cocycle representing the NC-vector bundle coming from an mNC-connection.

We pick trivializations $\varphi_i : \mathcal{O}^{\oplus n}|_{U_i} \to E|_{U_i}$, with transition functions $g_{ij} = \varphi_i^{-1}\varphi_j$. We have matrices of 1-forms $B_i = \varphi_i^{-1} \circ \bar{\nabla} \circ \varphi_i - d$ on $\mathcal{O}^{\oplus n}|_{U_i}$.

Proposition 5.1.3. *The cocycle $\tilde{g}_{ij} \in \check{H}^1(X, GL_r\widetilde{\mathcal{O}}_X)$ given by $\tilde{g}_{ij} = \Phi_i^{-1} \circ (g_{ij} \otimes 1) \circ \Phi_j$ represents the NC-vector bundle $\widetilde{E}_{\bar{\nabla}}$.*

Proof. One uses [20, Thm. 3.1.1] to construct maps as in the diagram:



Here $D_{(i)}$ is the mNC-connection on the trivial bundle extending the connection $d + B_i$, D_{dR} is the mNC-connection extending d, and $\widetilde{D}$ the mNC-connection extending $\bar{\nabla}$. □

We use this to compute the first few terms of the cocycle.

Proposition 5.1.4. *The truncated cocycle $\tilde{g}_{ij}^{\le 3}$ (up to degree 3) given below represents the NC-vector bundle $(\widetilde{E}_{\bar{\nabla}})^{\le 3}$.*

$$\begin{aligned}\tilde{g}_{ij}^{\le 3} = g_{ij} &+ \left[g_{ij}\theta_1^j - \theta_1^i g_{ij}\right] + \left[g_{ij}\theta_2^j + ((\theta_1^i)^2 - \theta_2^i)g_{ij} - \theta_1^i g_{ij}\theta_1^j\right] \\ &+ \left[(\theta_2^i\theta_1^i - (\theta_1^i)^3 - \theta_3^i)g_{ij} + g_{ij}\theta_3^j + ((\theta_1^i)^2 - \theta_2^i)g_{ij}\theta_1^j - \theta_1^i g_{ij}\theta_2^j\right]\end{aligned} \tag{5.1}$$

Proof. The maps Φ_j and Φ_i^{-1} are determined step by step according to the algorithm in [20]:

$$\begin{aligned}
\Phi_j &= (1+\theta_1^j)(1+\theta_2^j)(1+\theta_3^j)\cdots \\
&= 1+\theta_1^j+\theta_2^j+(\theta_1^j\theta_2^j+\theta_3^j)+\cdots \\
\Phi_i^{-1} &= \cdots(1+\theta_3^j)^{-1}(1+\theta_2^j)^{-1}(1+\theta_1^i)^{-1} \\
&= \cdots(1-\theta_3^i+\cdots)(1-\theta_2^i+\cdots)(1-\theta_1^i+(\theta_1^i)^2-(\theta_1^i)^3+\cdots) \\
&= 1-\theta_1^i+\left[(\theta_1^i)^2-\theta_2^i\right]+\left[\theta_2^i\theta_1^i-(\theta_1^i)^3-\theta_3^i\right]+\cdots
\end{aligned}$$

Now collect terms of degree ≤ 3. □

Remark 5.1.5. As a check, note that according to this formula we should have

$$-dg_{ij} = [\tilde{g}_{ij}]_1 = g_{ij}\theta_1^j - \theta_1^i g_{ij}. \tag{5.2}$$

This is true since $\theta_1^i = h(D_{dR} - D_{(i)}) = h(d-(d+B_i)) = -B_i$ and because

$$-dg_{ij} = \varphi_i^{-1}\alpha_{ij}\varphi_j = B_i g_{ij} - g_{ij}B_j.$$

5.2 Bimodule Extendability of NC-Vector Bundles

5.2.1 NC-bimodule extensions

The following notion is introduced in [20, Sec. 3].

Definition 5.2.1. A *bimodule extension* of an NC-vector bundle $\widetilde{E}$ is the structure of a left module given by a homomorphism $\widetilde{\mathcal{O}} \to \mathcal{E}nd(\widetilde{E})$ whose abelianization is the diagonal embedding $\mathcal{O} \to \mathcal{E}nd(E)$.

The latter condition ensures that if we abelianize $\widetilde{E}$ either as a left or right module we get the same E. Concretely in terms of trivializations, a bimodule structure is determined by homomorphisms $A_i : \widetilde{\mathcal{O}}|_{U_i} \to M_r\widetilde{\mathcal{O}}|_{U_i}$ such that on U_{ij},

$$A_i \cdot \tilde{g}_{ij} = \tilde{g}_{ij} \cdot A_j. \tag{5.3}$$

We also have the completely analogous notion for a d-nilpotent NC-vector bundle $\widetilde{E}^{\leq d}$ of a *d-nilpotent bimodule extension*, given by $A_i^{\leq d}$ such that $A_i^{\leq d}\tilde{g}_{ij}^{\leq d} = \tilde{g}_{ij}^{\leq d}A_j^{\leq d}$.

The maps $A_i^{\leq d}$ automatically send the center $Z(\widetilde{\mathcal{O}}^{\leq d}) = \mathbb{C}_X \oplus U^d\mathcal{L}_+\Omega^1_X$ to diagonal matrices:

Lemma 5.2.2. *Let $A^{\leq d} : \widetilde{\mathcal{O}}^{\leq d} \to M_r\widetilde{\mathcal{O}}_{\leq d}$ be a bimodule extension. Then for $\tilde{f} \in \widetilde{\mathcal{O}}^{\leq d}$,*

$$A^{\leq d}(\tilde{f}_{\leq d}) = A^{\leq d}(\sigma\tilde{f}_{\leq d-1}) + (\tilde{f}_{\leq d} - \sigma\tilde{f}_{\leq d-1})I.$$

In other words $A^{\leq d+1}|_{\mathcal{I}^{d+1}}$ is the diagonal embedding.

Proof. By 3.6.1 there is a natural identification $\mathcal{I}^d\widetilde{\mathcal{O}}^d = U^d\mathcal{L}_+\Omega^1_X$. Thus we may assume $z \in \mathcal{I}^d\widetilde{\mathcal{O}}^{\leq d} \subset T^d\Omega^1_X$ is a sum of products $F_{n_1}\cdots F_{n_k}$ of elements of the form $F_n = f_0[df_1[df_2[\cdots, df_n]\cdots]]$ such that $d = \sum_{i=1}^k n_i$. It is easy to see that modulo $\mathcal{I}^{d+1}$, $F_{n_1}\cdots F_{n_k} = \widetilde{F}_{n_1}\cdots\widetilde{F}_{n_k}$ where $\widetilde{F}_n = \sigma(f_0)[\sigma(f_1)[\sigma(f_2)[\cdots, \sigma(f_n)]\cdots]]$. Indeed, since these expressions have arity d, and because degree 0 is central, working modulo $T^{\geq d+1}$ it suffices to replace $\sigma(f_i)$ by $-df_i$ for $i > 0$.

Then since $\widetilde{F}_i$ are commutators of elements in $\widetilde{\mathcal{O}}_X^{\leq d}$, it follows that $A_i(\widetilde{F}_i) = f_0[A_i(\sigma(f_1))[A_i(\sigma(f_2))[\cdots, A_i(\sigma(f_n))], \cdots]] = f_0[-df_1I[-df_2I[\cdots, -df_nI]\cdots]]$. (This is all analogous to "by considering commutators" in the proof of [20, 3.3.3(ii)].) □

Because of the previous lemma, and because $A^{\le 1}(\tilde{f}) = f - df$, it makes sense to define the following:

Definition 5.2.3. Define $\bar{A}_i^{\le d} : \widetilde{\mathcal{O}}^{\le d-1} \to M_r \widetilde{\mathcal{O}}^{\le d}$ by the equation

$$A_i^{\le d}(\tilde{f}_{\le d}) = \tilde{f}_{\le d} + \bar{A}_i^{\le d}(\tilde{f}_{\le d-1}).$$

Next we study when there exists a bimodule extension $A_i^{\le d+1}$ extending $A_i^{\le d}$.

Definition 5.2.4. For a bimodule extension $A_i^{\le d+1}$ extending $A_i^{\le d}$. Define the maps $\eta_i^{(d+1)} : \widetilde{\mathcal{O}}^{\le d} \to M_r(U^{d+1}\mathcal{L}_+\Omega^1)$ by

$$A_i^{\le d+1}(\sigma \tilde{f}_{\le d}) = \sigma A_i^{\le d}(\tilde{f}_{\le d}) + \eta_i^{(d+1)}(\tilde{f}_{\le d}) \tag{5.4}$$

and $\bar{\eta}_i^{(d+1)} : \widetilde{\mathcal{O}}^{\le d} \to M_r T^{\le d+1}(\Omega^1)$ by

$$A_i^{\le d+1}(\tilde{f}_{\le d}) = A_i^{\le d}(\tilde{f}_{\le d-1}) + \bar{\eta}_i^{(d+1)}(\tilde{f}_{\le d}) \tag{5.5}$$

Note that

$$\bar{A}_i^{\le d+1}(\tilde{f}_{\le d}) = \sigma \bar{A}_i^{\le d}(\tilde{f}_{\le d-1}) + \eta_i^{(d+1)}(\tilde{f}_{\le d}) \tag{5.6}$$

because $\sigma \tilde{f}_{\le d} + \bar{A}_i^{\le d+1}(\tilde{f}_{\le d}) = A_i^{\le d+1}(\sigma \tilde{f}_{\le d}) = \sigma(\tilde{f}_{\le d} + \bar{A}_i^{\le d}(\tilde{f}_{\le d})) + \eta_i^{(d+1)}(\tilde{f}_{\le d})$.

The data of such $A^{\le d+1}$ is equivalent to $\eta^{(d+1)}$ satisfying certain conditions.

Proposition 5.2.5. $A_i^{\le d+1}(\tilde{f}_{\le d+1}) = \tilde{f}_{\le d+1} + \sigma_d \bar{A}_i^{\le d}(\tilde{f}_{\le d-1}) + \eta_i^{(d+1)}(\tilde{f}_{\le d})$ *defines a homomorphism if and only if*

$$\delta(\eta_i^{(d+1)}) = -\delta(\sigma \bar{A}_i^{\le d}) = -\Pi_{d+1}\left(\tilde{f} \cdot \sigma \bar{A}_i^{\le d}(\tilde{g}) + \sigma \bar{A}_i^{\le d}(\tilde{f}) \cdot \tilde{g}\right) \tag{5.7}$$

Proof. First note that $A_i^{\le d+1}$ is a homomorphism if and only if $\delta(\bar{A}_i^{\le d+1}) = 0$:

$$
\begin{aligned}
A_i^{\le d+1}(\tilde{f})A_i^{\le d+1}(\tilde{g}) - A_i^{\le d+1}(\tilde{f}\tilde{g}) &= \big(\tilde{f} + \bar{A}_i^{\le d+1}(\tilde{f})\big)\big(\tilde{g} + \bar{A}_i^{\le d+1}(\tilde{g})\big) - \big(\tilde{f}\tilde{g} + \bar{A}_i^{\le d+1}(\tilde{f}\tilde{g})\big) \\
&= \tilde{f} \cdot \bar{A}_i^{\le d+1}(\tilde{g}) + \bar{A}_i^{\le d+1}(\tilde{f}) \cdot \tilde{g} - \bar{A}_i^{\le d+1}(\tilde{f}\tilde{g}) \\
&= \delta(\bar{A}_i^{\le d+1})(\tilde{f}, \tilde{g}).
\end{aligned}
$$

Because of 5.6, $\delta(\bar{A}_i^{\le d+1})(\tilde{f}, \tilde{g}) = 0$ is equivalent to $\delta(\eta_i^{(d+1)}) = -\delta(\sigma\bar{A}_i^{\le d})$.

The expression $A_i^{\le d+1}(\tilde{f})A_i^{\le d+1}(\tilde{g}) - A_i^{\le d+1}(\tilde{f}\tilde{g})$ takes values in $M_r(U^{d+1}\mathcal{L}_+\Omega^1)$ because its truncation $A_i^{\le d}$ we assume satisfies 5.3. So both $\delta(\bar{A}_i^{\le d+1})$ and $\delta(\eta_i^{(d+1)})$ have coefficients in $U^{d+1}\mathcal{L}_+\Omega^1$, hence so does $\delta(\sigma\bar{A}_i^{\le d})$. This means that

$$
\begin{aligned}
\delta(\sigma\bar{A}_i^{\le d})(\tilde{f}, \tilde{g}) &= \Pi_{d+1}\Big(\delta(\sigma\bar{A}_i^{\le d})(\tilde{f}, \tilde{g})\Big) \\
&= \Pi_{d+1}\Big(\tilde{f} \cdot \sigma\bar{A}_i^{\le d}(\tilde{g}) + \sigma\bar{A}_i^{\le d}(\tilde{f}) \cdot \tilde{g} - \sigma\bar{A}_i^{\le d}(\tilde{f}\tilde{g})\Big) \\
&= \Pi_{d+1}\Big(\tilde{f} \cdot \sigma\bar{A}_i^{\le d}(\tilde{g}) + \sigma\bar{A}_i^{\le d}(\tilde{f}) \cdot \tilde{g}\Big).
\end{aligned}
$$

□

Remark 5.2.6. Note that for $d = 1$ we have $\bar{A}_i^{\le 1} = 0$, hence the condition is that $\delta(\eta_i^{(2)}) = 0$, e.g. $\eta_i^{(2)}$ is a derivation, as stated in [20, 3.3.3(ii)]. However, for $d \ge 2$ the solutions are only a (pseudo)torsor over derivations.

Proposition 5.2.7. *If $A_i^{\le d}$ is a degree d bimodule structure, then there is an extension to a degree $(d+1)$-bimodule $A_i^{\le d+1}$ if and only if there exist $\mathbb{C}$-linear maps $\eta_i^{(d+1)} : \widetilde{\mathcal{O}}^{\le d} \to M_r(U^{d+1}\mathcal{L}_+\Omega_X^1)$ such that for all $\tilde{f} \in \widetilde{\mathcal{O}}^{\le d}$,*

$$
A_i^{\le d}(\tilde{f}_{\le d}) \cdot \tilde{g}_{ij}^{\le d} \quad - \quad \tilde{g}_{ij}^{\le d} \cdot A_j^{\le d}(\tilde{f}_{\le d}) \quad = \quad g_{ij}\bar{\eta}_j^{(d+1)}(\tilde{f}_{\le d}) \quad - \quad \bar{\eta}_i^{(d+1)}(\tilde{f}_{\le d})g_{ij} \tag{5.8}
$$

or

$$\Pi_{d+1}\left(A_i^{\leq d}(\tilde{f}_{\leq d})\cdot\tilde{g}_{ij}^{\leq d} \quad - \quad \tilde{g}_{ij}^{\leq d}\cdot A_j^{\leq d}(\tilde{f}_{\leq d})\right) \quad = \quad g_{ij}\eta_j^{(d+1)}(\tilde{f}_{\leq d}) \quad - \quad \eta_i^{(d+1)}(\tilde{f}_{\leq d})g_{ij} \tag{5.9}$$

where $\eta_i^{(d+1)} + (\sigma-1)A_i^{\leq d} = \bar{\eta}_i^{(d+1)}$*, and which have the Hochschild coboundary from above.*

Proof. Extendability to a $(d+1)$-nilpotent bimodule is determined by the equation $A_i^{\leq d+1}(\tilde{f}_{\leq d+1})\tilde{g}_{ij}^{\leq d+1} = \tilde{g}_{ij}^{\leq d+1}A_j^{\leq d+1}(\tilde{f}_{\leq d+1})$ in $M_r(T^{\leq d+1}\Omega^1)$. Because $A_i^0(\tilde{f}) = fI$ is diagonal with central coefficients, this reduces to $A_i^{\leq d+1}(\tilde{f}_{\leq d})\tilde{g}_{ij}^{\leq d} = \tilde{g}_{ij}^{\leq d}A_j^{\leq d+1}(\tilde{f}_{\leq d})$. From 5.2.2, we also know $A_i^{\leq d+1}|_{\mathcal{I}^{d+1}}$ is diagonal with central coefficients, thus we only need to consider 5.3 for $\tilde{f}_{\leq d+1} = \sigma\tilde{f}_{\leq d}$. In this case we may use 5.5 to write $A_i^{\leq d+1}(\sigma\tilde{f}_{\leq d}) = A_i^{\leq d}(\tilde{f}_{\leq d}) + \bar{\eta}_i^{(d+1)}(\tilde{f}_{\leq d})$, thus reducing to 5.8. This is equivalent to 5.9 because $(\sigma-1)A_i^{\leq d}$ is in the kernel of Π_{d+1}.

□

5.2.2 Bimodule extendability to degree 2

First we recall from [20, Prop. 3.3.3(iii)] that any vector bundle E with a connection admits a 1-nilpotent bimodule extension. We reproduce the proof here in our current notation.

Proposition 5.2.8. *The* $\mathbb{C}$*-linear map* $\eta_i^{(2)} : \mathcal{O} \to M_r\widetilde{\mathcal{O}}^{\leq 2}$ *given by*

$$\eta_i^{(2)}(f) = [\theta_1^i, df] \tag{5.10}$$

determines a central bimodule structure $A_i^{\leq 2}(\tilde{f}) = \tilde{f}\cdot I + \eta_i^{(2)}(f)$*.*

Proof. From 5.8, finding suitable homomorphisms $A_i^{\leq 2}$ is equivalent to finding $\mathbb{C}$-linear maps $\bar{\eta}_i^{(2)}$ satisfying:

$$\left[A_i^{\leq 1}(\tilde{f})\tilde{g}_{ij}^{\leq 1} - \tilde{g}_{ij}^{\leq 1}A_j^{\leq 1}(\tilde{f})\right]_2 = g_{ij}\bar{\eta}_j^{(2)} - \bar{\eta}_i^{(2)}g_{ij}$$

Since $A_i^{\leq 1}(\tilde{f}) = f - df$ and $-dg_{ij} = g_{ij}\theta_1^j - \theta_1^i g_{ij}$, we have

$$\begin{aligned}\left[A_i^{\leq 1}\tilde{g}_{ij}^{\leq 1} - \tilde{g}_{ij}^{\leq 1}A_j^{\leq 1}\right]_2 &= [df, dg_{ij}] \\ &= [df, \theta_1^i g_{ij} - g_{ij}\theta_1^j] \\ &= [df, \theta_1^i]g_{ij} - g_{ij}[df, \theta_1^j]\end{aligned}$$

In the last line we have used that $\theta_1^i[df, g_{ij}] = 0$ because the coefficients of g_{ij} are central and dfI is diagonal. Thus we arrive at a solution $\eta_i^{(2)}(f) = \bar{\eta}_i^{(2)}(f) = [\theta_1^i, df]$.

By 5.2.5, the condition that $A_i^{\leq 2}$ is a homomorphism is just that $\eta_i^{(2)}$ is a derivation.

□

Remark 5.2.9. By 5.2.5, any other 1-nilpotent bimodule extension differs from this one by a global derivation $\eta_\Gamma^{(2)}$ such that $[\eta_\Gamma^{(2)}, g_{ij}] = 0$.

5.3 Bimodule Extendability in Degree 3

Now we extend the $\widetilde{\mathcal{O}}_X^{\leq 2}$-bimodule structure on $(\widetilde{E}_{\bar{\nabla}})^{\leq 2}$ of the previous section to an $\widetilde{\mathcal{O}}_X^{\leq 3}$-bimodule structure on $(\widetilde{E}_{\bar{\nabla}})^{\leq 3}$. In this section $\tilde{f}$ denotes an element of $\widetilde{\mathcal{O}}_X^{\leq 3}$ and $f = \tilde{f}_0 = \tilde{f}_{ab}$.

Proposition 5.3.1. *The* $\mathbb{C}$*-linear map* $\eta_i^{(3)} : \widetilde{\mathcal{O}}^{\leq 2} \to M_r(U^3\mathcal{L}_+\Omega_X^1)$ *given by*

$$\eta_i^{(3)}(\tilde{f}_{\leq 2}) = [\tilde{f}_2, \theta_1^i] - [df, \theta_2^i] + \theta_1^i[df, \theta_1^i] + (\sigma - 1)A_i^{\leq 2}(\tilde{f}) \tag{5.11}$$

or

$$\eta_i^{(3)}(\tilde{f}_{\leq 2}) = \Pi_3\big([\tilde{f}_2, \theta_1^i] - [df, \theta_2^i]\big) \tag{5.12}$$

along with the previously defined $\eta_i^{(2)}$ *determine a homomorphism*

$$A_i^{\leq 3}(\tilde{f}) = \tilde{f}I + \sigma\eta_i^{(2)}(f) + \eta_i^{(3)}(f) \tag{5.13}$$

which satisfies 5.3 extending that of $\eta_i^{(2)}$*. Thus* $A_i^{\leq 3}$ *is a 2-nilpotent bimodule extension.*

Proof. To obtain the formula 5.11, we consider the equation 5.8 for $d = 2$:

$$A_i^{\leq 2}(\tilde{f}_{\leq 2})\tilde{g}_{ij}^{\leq 2} - \tilde{g}_{ij}^{\leq 2}A_j^{\leq 2}(\tilde{f}_{\leq 2}) = g_{ij}\bar{\eta}_j^{(3)} - \bar{\eta}_i^{(3)}g_{ij}.$$

Recall the formulas $A_i^{\leq 2}(\tilde{f}_{\leq 2}) = \tilde{f}_{\leq 2} I + \eta_i^{(2)}(f)$, and $\tilde{g}_{ij}^{(2)} = (g_{ij}\theta_2^j - \theta_2^i g_{ij}) + \theta_1^i \cdot dg_{ij}$ from 5.1 and separate terms as follows:

$$\begin{aligned}
A_i^{\leq 2}(\tilde{f}_{\leq 2})\tilde{g}_{ij}^{\leq 2} - \tilde{g}_{ij}^{\leq 2} A_j^{\leq 2}(\tilde{f}_{\leq 2}) &= -[df I, \tilde{g}_{ij}^{(2)}] + [\tilde{f}_2, dg_{ij}] + dg_{ij} \cdot \eta_j^{(2)}(f) - \eta_i^{(2)}(f) \cdot dg_{ij} \\
&= \Big([\tilde{f}_2, dg_{ij}] - [df, (g_{ij}\theta_2^j - \theta_2^i g_{ij})] \Big) \\
&\qquad - [df, \theta_1^i \cdot dg_{ij}] + dg_{ij}[\theta_1^j, df] - [\theta_1^i, df] dg_{ij} \\
&= \Big([\tilde{f}_2, dg_{ij}] - [df, (g_{ij}\theta_2^j - \theta_2^i g_{ij})] \Big) - \theta_1^i [df, dg_{ij}] - dg_{ij}[df, \theta_1^j] \\
&= \Big([\tilde{f}_2, dg_{ij}] - [df, (g_{ij}\theta_2^j - \theta_2^i g_{ij})] \Big) - \theta_1^i [df, \theta_1^i] g_{ij} + g_{ij}\theta_1^j [df, \theta_1^j] \\
&= g_{ij} \Big([\tilde{f}_2, \theta_1^j] - [df, \theta_2^j] + \theta_1^j [df, \theta_1^j] \Big) \\
&\qquad - \Big([\tilde{f}_2, \theta_1^i] - [df, \theta_2^i] + \theta_1^i [df, \theta_1^i] \Big) g_{ij}
\end{aligned}$$

Thus we may set $\bar{\eta}_i^{(3)} = [\tilde{f}_2, \theta_1^i] - [df, \theta_2^i] + \theta_1^i [df, \theta_1^i]$. This means $\eta_i^{(3)} = [\tilde{f}_2, \theta_1^i] - [df, \theta_2^i] + \theta_1^i [df, \theta_1^i] + hD_1(\eta_i^{(2)}(f))$. We also find the following useful formula:

$$\begin{aligned}
\eta_i^{(3)}(\tilde{f}) = \Pi_3\big(\eta_i^{(3)}(\tilde{f})\big) &= \Pi_3\big(\bar{\eta}_i^{(3)}(\tilde{f})\big) \\
&= \Pi_3\big([\tilde{f}_2, \theta_1^i] + \theta_1^i [df, \theta_1^i] - [df, \theta_2^i]\big) \\
&= \Pi_3\big([\tilde{f}_2, \theta_1^i] - [df, \theta_2^i]\big).
\end{aligned}$$

It remains to check that $A_i^{\leq 3}$ is a homomorphism, using 5.7. This says that $\delta(\eta_i^{(3)})$ must be equal to

$$
\begin{aligned}
-\Pi_3\big(\bar{A}_i^{\leq 2}(\tilde{f})\cdot\tilde{g}+\tilde{f}\cdot\bar{A}_i^{\leq 2}(\tilde{g})\big) &= -\Pi_3\big(-df\cdot\eta_i^{(2)}(g)-\eta_i^{(2)}(f)\cdot dg\big)\\
&= \Pi_3(\eta_i^{(2)}(f)\cdot dg)\\
&= [\eta_i^{(2)}(f),dg].
\end{aligned}
$$

Using the formula $\eta_i^{(3)}(\tilde{f}) = \Pi_3\big([\tilde{f}_2,\theta_1^i]-[df,\theta_2^i]\big)$, the term $\Pi_3([df,\theta_2^i]))$ is a derivation, hence does not contribute to the Hochschild differential. The remaining terms come from $f\tilde{g}_2+g\tilde{f}_2-(\tilde{f}\tilde{g})_2=\tilde{f}_1\tilde{g}_1$,

$$
\begin{aligned}
\delta(\eta_i^{(3)})(\tilde{f},\tilde{g}) &= -\Pi_3[\tilde{f}_1\tilde{g}_1,\theta_1^i]\\
&= -\Pi_3(df[dg,\theta_1^i]+[df,\theta_1^i]dg)\\
&= [dg[df,\theta_1^i]].
\end{aligned}
$$

Since $\eta_i^{(2)}(f)=[\theta_1^i,df]$, this agrees with the above. □

Remark 5.3.2. Again, another bimodule extension differs from this by a global derivation $\eta_\Gamma^{(3)}$ satisfying $[g_{ij},\eta_\Gamma^{(3)}]=0$. If we consider extendability of another 1-nilpotent bimodule $\eta_i^{(2)}+\eta_\Gamma^{(2)}$, this will just add the term $[\eta_\Gamma^{(2)},-dg_{ij}]$ to the left hand side of 5.8, which is easily rewritten using 5.2 as $g_{ij}[\eta_\Gamma^{(2)},\theta_1^j]-[\eta_\Gamma^{(2)},\theta_1^j]g_{ij}$, and $\eta_i^{(3)}$ may be modified accordingly. Moreover, because $[\eta_\Gamma^{(2)},\theta_1^i]$ is a derivation, it doesn't contribute to the Hochschild differential.

5.4 Bimodule Extendability in Degree 4

Proposition 5.4.1. *The* $\mathbb{C}$*-linear map* $\eta_i^{(4)} : \widetilde{\mathcal{O}}^{\leq 3} \to M_r(U^4\mathcal{L}_+\Omega_X^1)$,

$$\begin{aligned}\eta_i^{(4)}(\tilde{f}) = [\theta_2^i, [df, \theta_1^i]] - \theta_1^i[\theta_1^i df + \tilde{f}_2, \theta_1^i] - [\tilde{f}_3, \theta_1^i] + [\tilde{f}_2, \theta_2^i] \\ - hD_1\big(- hD_1\eta_i^{(2)}(f) + \eta_i^{(3)}(\tilde{f})\big) - hD_2\big(\eta_i^{(2)}(f)\big)\end{aligned} \tag{5.14}$$

or,

$$\eta_i^{(4)}(\tilde{f}) = \Pi_4\Bigg([\theta_2^i, [df, \theta_1^i]] - \theta_1^i[\theta_1^i df + \tilde{f}_2, \theta_1^i] - [\tilde{f}_3, \theta_1^i] + [\tilde{f}_2, \theta_2^i]\Bigg) \tag{5.15}$$

determines a 3-nilpotent bimodule extension,

$$A_i^{\leq 4}(\tilde{f}_{\leq 4}) = \tilde{f}_{\leq 4} + \sigma\bar{A}_i^{\leq 3}(\tilde{f}_{\leq 2}) + \eta_i^{(4)}(\tilde{f}_{\leq 3}) \tag{5.16}$$

extending the 2-nilpotent bimodule extension $A_i^{\leq 3}$ *defined in 5.13.*

Proof. We will need formulas (5.1), (5.2), (5.10) and (5.11), which we collect below:

$$\tilde{g}_{ij} = g_{ij} - dg_{ij} + (g_{ij}\theta_2^j - \theta_2^i g_{ij} + \theta_1^i \cdot dg_{ij}) + \cdots,$$

$$dg_{ij} = \theta_1^i g_{ij} - g_{ij}\theta_1^j,$$

$$\eta_i^{(2)}(f) = [\theta_1^i, df],$$

$$\eta_i^{(3)}(\tilde{f}) = [\tilde{f}_2, \theta_1^i] - [df, \theta_2^i] + \theta_1^i[df, \theta_1^i] + hD_1(\eta_i^{(2)}(f)).$$

Calculate $\bar{\eta}_i^{(4)}$ as in 5.8 for $d = 3$.

$$\begin{aligned}
A_i^{\le 3}(\tilde{f}_{\le 3})\tilde{g}_{ij}^{\le 3} - \tilde{g}_{ij}^{\le 3}A_j^{\le 3}(\tilde{f}_{\le 3}) = {} & [-df, \theta_2^i dg_{ij} - dg_{ij}\theta_2^j - (\theta_1^i)^2 dg_{ij}] \\
& + [\tilde{f}_2, g_{ij}\theta_2^j - \theta_2^i g_{ij}] + [\tilde{f}_2, \theta_1^i dg_{ij}] + [\tilde{f}_3, -dg_{ij}] \\
& - [df, \theta_1^i](g_{ij}\theta_2^j - \theta_2^i g_{ij} + \theta_1^i dg_{ij}) \\
& - (g_{ij}\theta_2^j - \theta_2^i g_{ij} + \theta_1^i dg_{ij})(-[df, \theta_1^j]) \\
& + hD_1(-[df, \theta_1^i])dg_{ij} - dg_{ij}hD_1(-[df, \theta_1^j]) \\
& + \big([\tilde{f}_2, \theta_1^i] - [df, \theta_2^i] + \theta_1^i[df, \theta_1^i] + hD_1\eta_i^{(2)}(f)\big)(-dg_{ij}) \\
& - (-dg_{ij})\big([\tilde{f}_2, \theta_1^j] - [df, \theta_2^j] + \theta_1^j[df, \theta_1^j] + hD_1\eta_j^{(2)}(f)\big)
\end{aligned}$$

Row by row, there are 3, 4, 3, 3, 2, 4, and 4 terms. We number these terms in order from 1 to 23.

Combine terms 2, 21, 8 and 9:

$$\begin{aligned}
& [df, dg_{ij}\theta_2^j] - dg_{ij}[df, \theta_2^j] - [df, \theta_1^i](g_{ij}\theta_2^j - \theta_2^i g_{ij}) \\
&= [df, dg_{ij}]\theta_2^j - [df, \theta_1^i](g_{ij}\theta_2^j - \theta_2^i g_{ij}) \\
&= [df, \theta_1^i g_{ij} - g_{ij}\theta_1^j]\theta_2^j - [df, \theta_1^i](g_{ij}\theta_2^j - \theta_2^i g_{ij}) \\
&= [df, \theta_1^i]\theta_2^i g_{ij} - g_{ij}[df, \theta_1^j]\theta_2^j.
\end{aligned}$$

Combine terms 1, 17, 11 and 12:

$$
\begin{aligned}
&-[df,\theta_2^i dg_{ij}] + [df,\theta_2^i]dg_{ij} + (g_{ij}\theta_2^j - \theta_2^i g_{ij})[df,\theta_1^j] \\
&= \big(-\theta_2^i[df,dg_{ij}] - [df,\theta_2^i]dg_{ij}\big) + [df,\theta_2^i]dg_{ij} + (g_{ij}\theta_2^j - \theta_2^i g_{ij})[df,\theta_1^j] \\
&= -\theta_2^i[df,\theta_1^i g_{ij} - g_{ij}\theta_1^j] + (g_{ij}\theta_2^j - \theta_2^i g_{ij})[df,\theta_1^j] \\
&= g_{ij}\theta_2^j[df,\theta_1^j] - \theta_2^i[df,\theta_1^i]g_{ij}.
\end{aligned}
$$

Combine terms 3, 10, 18, 13 and 22:

$$
\begin{aligned}
&\Big([df,(\theta_1^i)^2 dg_{ij}] - [df,\theta_1^i]\theta_1^i dg_{ij} - \theta_1^i[df,\theta_1^i]dg_{ij}\Big) + \theta_1^i dg_{ij}[df,\theta_1^j] + dg_{ij}\theta_1^j[df,\theta_1^j] \\
&= (\theta_1^i)^2[df,dg_{ij}] + \theta_1^i dg_{ij}[df,\theta_1^j] + dg_{ij}\theta_1^j[df,\theta_1^j] \\
&= (\theta_1^i)^2[df,(\theta_1^i g_{ij} - g_{ij}\theta_1^j)] + \theta_1^i(\theta_1^i g_{ij} - g_{ij}\theta_1^j)[df,\theta_1^j] + (\theta_1^i g_{ij} - g_{ij}\theta_1^j)\theta_1^j[df,\theta_1^j] \\
&= (\theta_1^i)^2[df,\theta_1^i]g_{ij} - g_{ij}(\theta_1^j)^2[df,\theta_1^j].
\end{aligned}
$$

Combine terms 6, 16, and 20:

$$
\begin{aligned}
&[\tilde{f}_2,\theta_1^i dg_{ij}] - [\tilde{f}_2,\theta_1^i]dg_{ij} + dg_{ij}[\tilde{f}_2,\theta_1^j] \\
&= \theta_1^i[\tilde{f}_2,dg_{ij}] + dg_{ij}[\tilde{f}_2,\theta_1^j] \\
&= \theta_1^i[\tilde{f}_2,(\theta_1^i g_{ij} - g_{ij}\theta_1^j)] + (\theta_1^i g_{ij} - g_{ij}\theta_1^j)[\tilde{f}_2,\theta_1^j] \\
&= \theta_1^i[\tilde{f}_2,\theta_1^i]g_{ij} - g_{ij}\theta_1^j[\tilde{f}_2,\theta_1^j]
\end{aligned}
$$

Terms 4, 5, and 7 are easy:

$$
\begin{aligned}
&[\tilde{f}_2, g_{ij}\theta_2^j - \theta_2^i g_{ij}] + [\tilde{f}_3, dg_{ij}] \\
&= ([\tilde{f}_3,\theta_1^i] - [\tilde{f}_2,\theta_2^i])g_{ij} - g_{ij}([\tilde{f}_3,\theta_1^j] - [\tilde{f}_2,\theta_2^i])
\end{aligned}
$$

Terms 14 and 19, and terms 15 and 23 each cancel directly.

Collect all of this:

$$\begin{aligned}\bar{\eta}_i^{(4)}(\tilde{f}) &= -[df,\theta_1^i]\theta_2^i + \theta_2^i[df,\theta_1^i] - (\theta_1^i)^2[df,\theta_1^i] - \theta_1^i[\tilde{f}_2,\theta_1^i] - [\tilde{f}_3,\theta_1^i] \\ &= [\theta_2^i,[df,\theta_1^i]] - \theta_1^i[\theta_1^i df + \tilde{f}_2,\theta_1^i] - [\tilde{f}_3,\theta_1^i] + [\tilde{f}_2,\theta_2^i].\end{aligned}$$

We may set $\eta_i^{(4)} = \Pi_4(\bar{\eta}_i^{(4)})$:

$$\eta_i^{(4)}(\tilde{f}) = \Pi_4\Big[[\theta_2^i,[df,\theta_1^i]] - \theta_1^i[\theta_1^i df + \tilde{f}_2,\theta_1^i] - [\tilde{f}_3,\theta_1^i] + [\tilde{f}_2,\theta_2^i]\Big].$$

Next we must check the map $A_i^{\leq 4}$ of 5.16 is a homomorphism, using 5.2.5. The terms involving only "*df*" are derivations, and dissapear in $\delta(\eta_i^{(4)})$. We have:

$$\begin{aligned}\delta(\eta_i^{(4)})(\tilde{f},\tilde{g}) &= \Pi_4(-\theta_1^i[df\,dg,\theta_1^i] + [df\,dg,\theta_2^i]) + \Pi_4([df\,\tilde{g}_2 + \tilde{f}_2 dg,\theta_1^i]) \\ &= \Pi_4\big(-\theta_1^i df[dg,\theta_1^i] - \theta_1^i[df,\theta_1^i]dg\big) \\ &\qquad + \Pi_4\big(df[\tilde{g}_2,\theta_1^i] + [df,\theta_1^i]\tilde{g}_2 + \tilde{f}_2[dg,\theta_1^i] + [\tilde{f}_2,\theta_1^i]dg\big) \\ &= -[\theta_1^i,df][dg,\theta_1^i] - [\theta_1^i,dg][df,\theta_1^i] \\ &\qquad + \Pi_4\big(df[\tilde{g}_2,\theta_1^i] + [df,\theta_1^i]\tilde{g}_2 + \tilde{f}_2[dg,\theta_1^i] + [\tilde{f}_2,\theta_1^i]dg\big).\end{aligned}$$

Compare this to $\Pi_4(\delta \bar{A}_i^{\leq 3})$, using that $\bar{A}_i^{\leq 3} = \sigma \eta_i^{(2)} + \eta_i^{(3)}$. Then:

$$\begin{aligned}
\Pi_4\Big(\delta(\bar{A}_i^{\leq 3})(\tilde{f},\tilde{g})\Big) &= \Pi_4\Big(-df\big(\eta_i^{(3)}(\tilde{g}) - hD_1\eta_i^{(2)}(\tilde{g})\big) + \tilde{f}_2\cdot\eta_i^{(2)}(\tilde{g})\Big) \\
&\quad + \Pi_4\Big(\big(\eta_i^{(3)}(\tilde{f}) - hD_1\eta_i^{(2)}(\tilde{f})\big)(-dg) + \eta_i^{(2)}(\tilde{f})\cdot\tilde{g}_2\Big) \\
&= \Pi_4\Big(-df\big([\tilde{g}_2,\theta_1^i] - [dg,\theta_2^i] + \theta_1^i[dg,\theta_1^i]\big) + \tilde{f}_2\cdot\eta_i^{(2)}(\tilde{g})\Big) \\
&\quad + \Pi_4\Big(\big([\tilde{f}_2,\theta_1^i] - [df,\theta_2^i] + \theta_1^i[df,\theta_1^i]\big)(-dg) + \eta_i^{(2)}(\tilde{f})\cdot\tilde{g}_2\Big) \\
&= \Pi_4\Big(-df\theta_1^i[dg,\theta_1^i] + \theta_1^i[df,\theta_1^i](-dg)\Big) \\
&\quad + \Pi_4\Big(-df[\tilde{g}_2,\theta_1^i] + \tilde{f}_2\eta_i^{(2)}(\tilde{g}) + [\tilde{f}_2,\theta_1^i](-dg) + \eta_i^{(2)}(\tilde{f})\tilde{g}_2\Big) \\
&\quad + \Pi_4\Big([df\,dg,\theta_2^i])\Big)
\end{aligned}$$

as desired. □

CHAPTER VI

ALMOST NC-SCHEMES AND NC-ALGEBROIDS

The results of this chapter are all in collaboration with A. Polishchuk, and appear in our co-written pre-print [7].

In this chapter we introduce weaker notions than NC-smooth thickenings, which we call *almost NC-schemes*, or aNC-schemes for short, which sometimes lead to global objects called *NC-algebroids*.

The inspiration for this chapter was certain functors studied in [11] and [23] extending representable functors on $\mathcal{C}om$, but which fail to be representable by an NC-scheme due to the existence of inner automorphisms in $\mathcal{N}$. We begin by considering the general situation in which functors can fail to be representable due to inner automorphisms.

6.1 Almost NC-Schemes

Definition 6.1.1. The category $a\mathcal{N}$ has the same objects as $\mathcal{N}$, while the morphisms in $a\mathcal{N}$ are equivalence classes of homomorphisms $A \to B$, where $f_1, f_2 : A \to B$ are equivalent if there exists $b \in B^*$ such that $f_2 = bf_1b^{-1}$. We denote $a\mathcal{N}_d \subset a\mathcal{N}$ the full subcategory of NC-nilpotent algebras of degree d.

Proposition 6.1.2. *Let $h : \mathcal{N} \to Sets$ be a formally smooth functor such that $h|_{\mathcal{C}om} = h_M$ for a smooth variety M of dimension at least 1, and which factors through $a\mathcal{N}$. Then $h|_{\mathcal{N}_d}$ is not representable by an NC-nilpotent scheme of degree d.*

Proof. Let $U = \operatorname{Spec} A \subset X$ be an affine NC-subscheme corresponding to an open affine subscheme of B of dimension ≥ 1. Then A is 1-smooth, as is $A' = A \hat{*} \mathbb{C}[z, z^{-1}]$. Since z abelianizes to a non-scalar, then z is not central in A', hence it does not commute with some element of $A \subset A'$. □

6.1.1 The category of affine almost NC schemes

$f^{ab} : A^{ab} \to B^{ab}$ factors through $A^{ab}[\overline{S}^{-1}]$, where $\overline{S} \subset A^{ab}$ is the image of S. It follows that we have a cartesian square of sets

$$\begin{array}{ccc} h_{A[S^{-1}]}(B) & \longrightarrow & h_A(B) \\ \downarrow & & \downarrow \\ h_{A^{ab}[\overline{S}^{-1}]}(B^{ab}) & \longrightarrow & h_{A^{ab}}(B^{ab}) \end{array}$$

Definition 6.1.3. For an NC-complete algebra R we denote by $\overline{h}_R$ the corresponding functor on $a\mathcal{N}$: $\overline{h}_R(B)$ is the set of conjugacy classes of algebra homomorphisms $R \to B$.

Since the images of both horizontal arrows in the above cartesian square are stable under the action of inner automorphisms of B, we deduce that the similar square

$$\begin{array}{ccc} \overline{h}_{R[\![T^{-1}]\!]}(B) & \longrightarrow & \overline{h}_R(B) \\ \downarrow & & \downarrow \\ h_{R^{ab}[T^{-1}]}(B^{ab}) & \longrightarrow & h_{R^{ab}}(B^{ab}) \end{array} \tag{6.1}$$

is still Cartesian for any $B \in \mathcal{N}$.

Let $a\mathcal{NC}$ denote the category of NC-complete algebras with morphisms given by algebra homomorphisms viewed up to conjugation, i.e., up to post-composing with an inner automorphism. We denote by $a\mathcal{NCS}_{is}$ the subcategory in $a\mathcal{NC}$, whose objects are *NC-smooth* algebras, with *isomorphisms* in $a\mathcal{NC}$ as morphisms.

Lemma 6.1.4. *The functor*

$$a\mathcal{NCS}_{is}^{op} \to Fun_{is}(a\mathcal{N}, Sets) : R \mapsto \overline{h}_R$$

is fully faithful, where Fun_{is} *is the category of functors and natural isomorphisms between them.*

Proof. Note that for any $d \geq 0$, the restriction $\overline{h}_R|_{a\mathcal{N}_d}$ is naturally isomorphic to the representable functor $\overline{h}_{R/\mathcal{I}^{d+2}R}$. Thus, for NC-complete algebras R and R', we have a natural identification

$$Isom(\overline{h}_{R'}, \overline{h}_R) \simeq \varprojlim_d Isom_{a\mathcal{N}}(R/\mathcal{I}^{d+2}R, R'/\mathcal{I}^{d+2}R'),$$

where $Isom_{a\mathcal{N}}$ denotes the set of isomorphisms in the category $a\mathcal{N}$. Thus, it suffices to prove that if R and R' are NC-smooth then the natural map

$$Isom_{a\mathcal{NC}}(R, R') \to \varprojlim_d Isom_{a\mathcal{N}}(R/\mathcal{I}^{d+2}R, R'/\mathcal{I}^{d+2}R') \tag{6.2}$$

is a bijection. To check surjectivity, assume we are given a collection of algebra homomorphisms

$$f_d : R/\mathcal{I}^{d+2}R \to R'/\mathcal{I}^{d+2}R',$$

which are compatible up to conjugation, i.e., the homomorphism $f_{d+1,d} : R/\mathcal{I}^{d+2}R \to R'/\mathcal{I}^{d+2}R'$ induced by f_{d+1} is equal to $\theta_{u_d} f_d$, where θ_{u_d} is the inned automorphism associated with a unit $u_d \in R'/\mathcal{I}^{d+2}R'$. Now, starting from $d = 0$, we can recursively correct f_{d+1} by an inner automorphism of $R'/\mathcal{I}^{d+3}R'$, so that the homomorphisms (f_d) become compatible on the nose (not up to an inner automorphism). Since R' is NC-complete, this defines a unique homomorphism $f : R \to R'$ inducing (f_d).

Furthermore, since R is NC-complete, we see that f is an isomorphism if and only if all f_d are isomorphisms.

It remains to check that (6.2) is injective. Thus, given two isomorphisms $f, f' : R \to R'$ such that the induced isomorphisms f_d and f'_d are conjugate for each d, we have to check that f and f' are conjugate. By considering $f^{-1}f'$, we reduce the problem to checking that if we have an automorphism $f : R \to R$ such that f_d is an inner automorphism of $R/\mathcal{I}^{d+2}R$ for each d, then f is inner. For any algebra A, let us denote by $Inn(A)$ the group of inner automorphisms of A. Note that we have an exact sequence of groups

$$1 \to Z(A)^* \to A^* \to Inn(A) \to 1.$$

Applying this to each algebra $R/\mathcal{I}^{d+2}R$, and passing to projective limits, we have an exact sequence

$$1 \to \varprojlim_d Z(R/\mathcal{I}^{d+2}R)^* \to \varprojlim_d (R/\mathcal{I}^{d+2}R)^* \xrightarrow{\rho} \varprojlim_d Inn(R/\mathcal{I}^{d+2}R).$$

We claim that the arrow ρ in this sequence is surjective. Indeed, it is enough to check that the inverse system $(Z(R/\mathcal{I}^{d+2}R)^*)$ satisfies the Mittag-Leffler condition. But by Lemma 3.6.2(i), for $d \geq 1$, the image of the projection

$$Z(R/\mathcal{I}^{d+2}R)^* \to Z(R/\mathcal{I}^{d+1}R)^*$$

is equal to $\mathbb{C}^*$, which implies the required stabilization. Thus, the map ρ is surjective. Note that the source of this map can be identified with R^*. Thus, we deduce the surjectivity of the natural map

$$R^* \to \varprojlim_d Inn(R/\mathcal{I}^{d+2}R).$$

It follows that in we can compose f with an inner automorphism θ_u of R, such that $f' = \theta_u f$ induces the identity automorphism of $R/\mathcal{I}^{d+2}R$ for each d. It follows $f' = \mathrm{id}$, i.e., f is inner. □

6.2 Local Representability Criterion for Almost NC-Schemes

In this section we prove a local analog of Kapranov's representability criterion 3.5.2 for aNC-schemes. As in the case of NC-schemes the main idea is to study fibers of the map $h(p) : h(\Lambda') \to h(\Lambda)$ for a central extension

$$0 \to I \to \Lambda' \overset{p}{\to} \Lambda \to 0 \tag{6.3}$$

For $d \geq 1$, let $h : a\mathcal{N}_d \to Sets$ be a functor such that $h|_{a\mathcal{N}_{d-1}}$ is representable by $A \in a\mathcal{N}_{d-1}$. The key new ingredient we have to use is the following. Given a central extension (6.3) with $\Lambda' \in \mathcal{N}_d$, $\Lambda \in \mathcal{N}_{d-1}$, and a homomorphism $f : A \to \Lambda$, we set

$$U(f) := \{u \in \Lambda^* \mid uf(a)u^{-1} = f(a) \forall a \in A\}.$$

Then we have a natural map

$$\Delta_f : U(f) \to \mathrm{Der}(A, I) = \mathrm{Der}(A^{ab}, I).$$

where

$$\Delta_f(u) : A \to I : a \mapsto [u, f(a)]_{\Lambda'} u^{-1}, \tag{6.4}$$

where for $l_1, l_2 \in \Lambda$, we define $[l_1, l_2]_{\Lambda'} \in \Lambda'$ by

$$[l_1, l_2]_{\Lambda'} := [\widetilde{l_1}, \widetilde{l_2}], \tag{6.5}$$

where $\widetilde{l_i}$ is a lifting of l_i to Λ'. Note that $[u, f(a)]_{\Lambda'} \in I$.

Furthermore, one can check that the image of Δ_f depends only on the image of f in $\mathrm{Hom}_{a\mathcal{N}}(A, \Lambda) = h(\Lambda)$. Also, using the fact that I is central we immediately check that Δ_f is a group homomorphism. The next result shows that in the case when h itself is representable, the cokernel of Δ_f maps bijectively to $h(p)^{-1}(f)$.

Lemma 6.2.1. *Let A' be an NC-nilpotent algebra of degree d such that $A = A'/I_{d+1}A'$. Then for any central extension* (6.3)*, with $\Lambda' \in \mathcal{N}_d$ and $\Lambda \in \mathcal{N}_{d-1}$, and any algebra homomorphism $f : A' \to \Lambda$ there exists a natural transitive action of the group* $\mathrm{Der}(A, I)$ *on the fiber $h_{A'}(p)^{-1}(f)$ of the map $h_{A'}(p) : h_{A'}(\Lambda') \to h_{A'}(\Lambda)$, such that the action of* $\mathrm{Der}(A, I)$ *on any element of this fiber induces a bijection*

$$coker(\Delta_f) \tilde{\to} h_{A'}(p)^{-1}(f).$$

Proof. It is well known that the difference between two homomorphisms $A' \to \Lambda'$ lifting $f : A' \to \Lambda$ is a derivation $A' \to I$, and that this induces a simply transitive action of $\mathrm{Der}(A', I) = \mathrm{Der}(A, I)$ on the set of such liftings. Now assume that we have two homomorphisms $f_1', f_2' : A \to \Lambda'$, such that both $p \circ f_1'$ and $p \circ f_2'$ are conjugate to f. Then replacing f_1' and f_2' by conjugate homomorphisms we can assume that $p \circ f_1' = p \circ f_2' = f$. Now It is easy to see that if f_2' and f_1' are conjugate by $u \in (\Lambda')^*$ then $u \in U(f)$, and the difference $f_2' - f_1'$ is the derivation $a \mapsto [u, f(a)]_{\Lambda'} u^{-1}$. This establishes the required bijection. □

Next, we return to the situation when only $h|_{a\mathcal{N}_{d-1}}$ is representable. Recall from 3.1.12 that for any central extension (6.3) there is a natural isomorphism

$$\Lambda' \times_\Lambda \Lambda' \tilde{\to} \Lambda' \times_{\Lambda^{ab}} (\Lambda^{ab} \oplus I) : (x, y) \mapsto (x, (x_{ab}, y - x)), \tag{6.6}$$

Let us assume in addition that h commutes with pull-backs by commutative nilpotent extension, so that

$$h(\Lambda' \times_{\Lambda^{ab}} (\Lambda^{ab} \oplus I)) \simeq h(\Lambda') \times_{h(\Lambda^{ab})} h(\Lambda^{ab} \oplus I).$$

Combining this with the above isomorphism we get a natural map

$$h(\Lambda') \times_{h(\Lambda^{ab})} h(\Lambda^{ab} \oplus I) \simeq h(\Lambda' \times_\Lambda \Lambda') \to h(\Lambda') \times_{h(\Lambda)} h(\Lambda'). \tag{6.7}$$

Now assume $\Lambda' \in \mathcal{N}_d$, $\Lambda \in \mathcal{N}_{d-1}$ and we are given an element $f' \in h(\Lambda')$ lifting $f \in h(\Lambda)$. Since $h|_{a\mathcal{N}_{d-1}} \simeq h_A$ we have a natural identification of the fiber of $h(\Lambda^{ab} \oplus I) \to h(\Lambda^{ab}) = \mathrm{Hom}_{alg}(A, \Lambda^{ab})$ over f^{ab} with $\mathrm{Der}(A, I)$. Thus, for any $D \in \mathrm{Der}(A, I)$ we can consider a pair $(f', f^{ab} + D)$ in the left-hand side of (6.7). Let us define $f' + D \in h(p)^{-1}(f)$, so that $(f', f' + D)$ is the image of $(f', f^{ab} + D)$ under (6.7). In this way we get a map

$$\delta_{f'} : \mathrm{Der}(A, I) \to h(p)^{-1}(f) : D \mapsto f' + D. \tag{6.8}$$

It is easy to see (by considering $\Lambda' \times_\Lambda \Lambda' \times_\Lambda \Lambda'$) that in this way we get an action of the group $\mathrm{Der}(A, I)$ on $h(p)^{-1}(f)$. Note that in the case when h is representable by some $A' \in \mathcal{N}_d$, this operation is exactly the operation of adding a derivation $A' \to A \to I$ to a homomorphism $A' \to \Lambda'$.

Now we can prove the following local aNC version of Proposition 3.5.2.

Proposition 6.2.2. *Let A be a $(d-1)$-smooth algebra in $a\mathcal{N}_{d-1}$, and let $h : a\mathcal{N}_d \to Sets$, be a formally smooth functor such that $h|_{a\mathcal{N}_{d-1}} \simeq h_A$. Then h is representable by a d-smooth algebra in $a\mathcal{N}_d$ if and only if the following two conditions hold.*

(i) For any nilpotent extension $\Lambda' \to \Lambda$ with $\Lambda' \in a\mathcal{N}_d$ and $\Lambda \in \mathcal{C}om$, and any commutative nilpotent extension $\Lambda'' \to \Lambda$, the natural map

$$h(\Lambda' \times_\Lambda \Lambda'') \to h(\Lambda') \times_{h(\Lambda)} h(\Lambda'')$$

is a bijection.

(ii) For every central extension (6.3), for any $f' \in h(\Lambda')$ extending $f \in h(\Lambda)$, the map $\delta_{f'}$, which is well defined due to condition (i), induces a bijection

$$coker(\Delta_f) \tilde{\to} h(p)^{-1}(f).$$

Proof. Assume first that h is representable by $A' \in a\mathcal{N}_d$. To check condition (i) for $h = h_{A'}$ we first note that since Λ and Λ'' are commutative, the set $h(\Lambda') \times_{h(\Lambda)} h(\Lambda'')$ can be described as pairs of homomorphisms $f' : A \to \Lambda'$ and $f'' : A \to \Lambda''$ lifting the same homomorphism $f : A \to \Lambda$, up to the equivalence replacing f' by a conjugate homomorphism. Clearly, this is the same as giving a homomorphism $A' \to \Lambda' \times_\Lambda \Lambda''$

up to conjugacy. On the other hand, condition (ii) for $h_{A'}$ follows from Lemma 6.2.1.

Now assume that conditions (i) and (ii) hold, and let $A' \to A$ be a d-smooth thickening of A (it exists by [11, Prop. (1.6.2)]). Let $e \in h(A)$ be the family corresponding to the isomorphism $h|_{a\mathcal{N}_{d-1}} \simeq h_A$. Since h is formally smooth, there exists an element $e' \in h(A')$ lifting e. Let $h_{A'} \to h$ be the induced morphism of functors. We already know that it is an isomorphism on $a\mathcal{N}_{d-1}$, and we claim that it is an isomorphism on $a\mathcal{N}_d$. The argument is similar to that of Proposition 7.1.3. Given $\Lambda' \in \mathcal{N}_d$, we can fit it into a central extension (6.3) with $\Lambda \in \mathcal{N}_{d-1}$. Then we consider the commutative square

$$\begin{array}{ccc} h_{A'}(\Lambda') & \longrightarrow & h_{A'}(\Lambda) \\ \downarrow & & \downarrow \\ h(\Lambda') & \longrightarrow & h(\Lambda) \end{array}$$

Since $h_{A'}(\Lambda) \simeq h_A(\Lambda) \simeq h(\Lambda)$, we know that the right vertical arrow is an isomorphism. Also, both horizontal arrows are surjective. Let us fix a homomorphism $f \in h_A(\Lambda)$, and its lifting $f' \in h_{A'}(\Lambda')$. As we have seen in Lemma 6.2.1, the fiber of the top horizontal arrow over f is identified with $\operatorname{coker}(\Delta_f)$. The same is true for the fiber of the bottom horizontal arrow over f, by condition (ii). It remains to observe that both isomorphism are induced by the operation (6.8) of adding a derivation in $\operatorname{Der}(A, I)$, which is compatible with morphisms of functors on $a\mathcal{N}_d$, extending h_A on $a\mathcal{N}_{d-1}$. Thus, the left vertical arrow induces an isomorphism between the fibers of the horizontal arrows over f. Since f was arbitrary, we deduce that the left vertical arrow is an isomorphism. □

Remark 6.2.3. All fiber products are in $\mathcal{N}_d$. Fiber products of central extensions usually do not exist in $a\mathcal{N}_d$ unless one factor is commutative.

6.3 NC-Algebroids

Definition 6.3.1. A $\mathbb{C}$-*algebroid* is a $\mathbb{C}$-linear stack on space X, locally non-empty, and such that any two objects over an open set U are locally isomorphic.

See [15] and [13] for references on algebroids, and [12] for stacks.

Definition 6.3.2. Let X be a smooth scheme. An *NC-smooth algebroid thickening* of X is a $\mathbb{C}$-algebroid $\mathcal{A}$ over X such that for every object $\sigma \in \mathcal{A}(U)$ over an open subset $U \subset X$ the sheaf of algebras $\mathcal{E}nd_{\mathcal{A}}(\sigma)$ is an NC-smooth thickening of U.

For a functor h on $a\mathcal{N}$ such that $h_{\mathcal{C}om} = h_X$ and an open subset $U \subset X$ we define the subfunctor $h_{/U} \subset h$ by

$$h_{/U}(\Lambda) = h(\Lambda) \times_{h_X(\Lambda^{ab})} h_U(\Lambda^{ab}),$$

where we use the identification $h(\Lambda^{ab}) \simeq h_X(\Lambda^{ab})$.

Lemma 6.3.3. *Let $h = \overline{h}_R$, where R is an NC-complete algebra. Then for any distinguished affine $D(f) \subset Spec(A^{ab})$ we have an equality of subfunctors $h_{/D(f)} = \overline{h}_{A[\![f^{-1}]\!]}$.*

Proof. This follows immediately from the cartesian square (6.1) with $T = \{f^n \mid n \geq 0\}$. □

Lemma 6.3.4. *Let h be a functor on $a\mathcal{N}$ such that $h|_{\mathcal{C}om} = h_X$ for some scheme X. Assume that (U_i) is an affine covering of X, such that for every i we have an isomorphism $h_{/U_i} \simeq \overline{h}_{A_i}$ for some $A_i \in \mathcal{N}$. Let us denote also by A_i the corresponding sheaf of algebras over U_i. Then for every open subset $V \subset U_i \cap U_j$, which is* *distingushed in both U_i and U_j, we have an isomorphism*

$$\alpha_{ij,V} : A_i|_V \simeq A_j|_V$$

compatible with the isomorphisms $\overline{h}_{A_i(V)} \simeq h_{/V} \simeq \overline{h}_{A_j(V)}$. Furthermore, for another such open $V' \subset U_i \cap U_j$ the isomorphisms $\alpha_{ij,V}|_{V\cap V'}$ and $\alpha_{ij,V'}|_{V\cap V'}$ differ by an inner automorphism. Also, for any open $V \subset U_i \cap U_j \cap U_k$, distinguished in U_i, U_j and U_k, we have

$$\alpha_{jk}|_V \circ \alpha_{ij}|_V = \alpha_{ik}|_V \circ Ad(u_{ijk})$$

for some $u_{ijk} \in A_i(V)^*$.

Proof. Let us fix an isomorphism $h_{/U_i} \simeq \overline{h}_{A_i}$ for each i. Suppose $V \subset U_i \cap U_j$ is a distinguished affine open in both U_i and U_j. Then

$$\overline{h}_{A_i,/V} \simeq h_{/V} \simeq \overline{h}_{A_j,V}.$$

Thus, by Lemmas 6.3.3 and 6.1.4, we have an isomorphism between the corresponding localizations of A_i and A_j in $a\mathcal{N}$, and hence, an isomorphism $\alpha_{ij} : A_i|_V \simeq A_j|_V$, defined uniquely up to an inner automorphism. For $V \subset U_i \cap U_j \cap U_k$ the compatibility between α_{ij}, α_{jk} and α_{ik}, up to an inner automorphism, follows from the compatibility of all of these isomorphisms with the isomorphisms of $\overline{h}_{A_i,/V}$, $\overline{h}_{A_j,/V}$ and $\overline{h}_{A_k,/V}$ with $h_{/V}$. □

Lemma 6.3.5. *(i) Let $\mathcal{A}$ and $\mathcal{A}'$ be a pair of NC-smooth algebroids over a smooth scheme X, and $F, G : \mathcal{A} \to \mathcal{A}'$ is a pair of equivalences. Assume that for an open covering (U_i) of X we have an isomorphism $F|_{U_i} \simeq G|_{U_i}$. Then there exists an isomorphism $F \simeq G$.*

(ii) Let $\mathcal{A}$ and $\mathcal{A}'$ be a pair of NC-smooth algebroids over a smooth scheme X. Assume that for an open covering (U_i) of X we have an equivalence

$$F_i : \mathcal{A}|_{U_i} \to \mathcal{A}'|_{U_i}$$

and that for each pair i, j, we have an isomorphism

$$F_i|_{U_{ij}} \simeq F_j|_{U_{ij}}.$$

Then there exists an equivalence $F : \mathcal{A} \to \mathcal{A}'$ such that $F|_{U_i} \simeq F_i$.

(iii) Let U_i be an open covering of a smooth scheme X, and for each i let $\mathcal{A}_i$ be an NC-smooth algebroid over U_i. Assume that for every i, j, we have an equivalence

$$F_{ij} : \mathcal{A}_i|_{U_{ij}} \to \mathcal{A}_j|_{U_{ij}},$$

such that for every i, j, k, there is an isomorphism

$$F_{jk}|_{U_{ijk}} \circ F_{ij}|_{U_{ijk}} \simeq F_{ik}|_{U_{ijk}},$$

where $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$. Then there exists an NC-smooth algebroid $\mathcal{A}$ over X and equivalences $F_i : \mathcal{A}|_{U_i} \to \mathcal{A}_i$, such that for every i, j, there is an isomorphism

$$F_{ij} \circ F_i|_{U_{ij}} \simeq F_j|_{U_{ij}}.$$

Proof. Without loss of generality we can assume that X is connected.

(i) Let us choose for each i an isomorphism $\phi_i : F|_{U_i} \to G|_{U_i}$. Then for each i, j, we have

$$\phi_j|_{U_{ij}} = \phi_i|_{U_{ij}} \circ c_{ij},$$

where c_{ij} is an autoequivalence of $F_i|_{U_{ij}}$. Since F_i is an equivalence, we have $\underline{Aut}(F) \simeq \underline{Aut}(\mathrm{id}_{\mathcal{A}})$. Locally, the sheaf $\underline{Aut}(\mathrm{id}_{\mathcal{A}})$ is given by the center of $\mathrm{End}_{\mathcal{A}}(\sigma)$, where σ is an object of $\mathcal{A}$. Hence, by Lemma 3.6.2, the natural morphism of sheaves $\mathbb{C}^*_X \to \underline{Aut}(\mathrm{id}_{\mathcal{A}})$ is an isomorphism. Thus, c_{ij} is a Cech 1-cocycle with values in $\mathbb{C}^*_X$. Since X is irreducible, the corresponding Cech cohomology is trivial, so we can multiply ϕ_i by appropriate constants in $\mathbb{C}^*$, to make them compatible on double intersections. The corrected isomorphisms glue into a global isomorphism $F \to G$.

(ii) Let us choose for each i, j an isomorphism $\phi_{ij} : F_i|_{U_{ij}} \to F_j|_{U_{ij}}$. Then for each i, j, k, the composition $c_{ijk} = \phi_{ki}\phi_{jk}\phi_{ij}$ is an autoequivalence of $F_i|_{U_{ijk}}$, where c_{ijk} is a Cech 2-cocycle with values in $\mathbb{C}^*_X$. As above, choosing representation of c_{ijk} as a coboundary allows to correct ϕ_{ij} by constants in $\mathbb{C}^*$, so that the isomorphisms ϕ_{ij} are compatible on triple intersections. Hence, we can glue (F_i) into the required global equivalence $F : \mathcal{A} \to \mathcal{A}'$.

(iii) For every i, j, k, let us choose an isomorphism

$$g_{ijk} : F_{jk}|_{U_{ijk}} \circ F_{ij}|_{U_{ijk}} \to F_{ik}|_{U_{ijk}}.$$

Then for every i, j, k, l, we have over U_{ijkl},

$$g_{ikl}(F_{kl} * g_{ijk}) = c_{ijkl} g_{ijl}(g_{jkl} * F_{ij})$$

for some $c_{ijkl} \in \underline{Aut}(F_{il})(U_{ijkl}) = \mathbb{C}^*$. Furthermore, (c_{ijkl}) is a Cech 3-cocycle with values in $\mathbb{C}^*_X$. Hence, we can mulitply g_{ijk} with appropriate constants to make them compatible on quadruple intersections. This allows to glue $(\mathcal{A}_i)$ into a global $\mathbb{C}$-algebroid over X (see [13, Prop. 2.1.13]). □

Theorem 6.3.6. *Let h be a formally smooth functor on $a\mathcal{N}$ such that $h|_{\mathcal{C}om} = h_X$ and h is locally representable, i.e., there exists an open affine covering (U_i) of X, and isomorphisms*

$$h_{/U_i} \simeq \overline{h}_{A_i},$$

where A_i is an NC-smooth thickening of U_i. Then there exists an NC-smooth algebroid $\mathcal{A}$ over X and equivalences of algebroids

$$F_i : \mathcal{A}|_{U_i} \to A_i,$$

such that for every open subset $V \subset U_i \cap U_j$, distinguished in both U_i and U_j, there is an isomorphism

$$g_{ij} \circ F_i|_V \simeq F_j|_V,$$

where $g_{ij} : A_i|_V \to A_j|_V$ is a representative (up to conjugation) of the isomorphism $\overline{h}_{A_i|_V} \simeq h_{/V} \simeq \overline{h}_{A_j|_V}$.

Proof. First, we apply Lemma 6.3.4 and obtain isomorphisms

$$\alpha_{ij,V} : A_i|_V \to A_j|_V$$

for every open $V \subset U_i \cap U_j$, distinguished in both U_i and U_j, such that these isomorphisms for V and V' and for $V \subset U_i \cap U_j \cap U_k$, are compatible up to an inner automorphism. Let $\mathcal{A}_i$ denote an NC-smooth algebroid over U_i associated with A_i. Note that $U_i \cap U_j$ can be covered by open subsets V, which are distinguished in both U_i and U_j. Each isomorphism $\alpha_{ij,V}$ gives an equivalence

$$F_{ij,V} : \mathcal{A}_i|_V \to \mathcal{A}_j|_V.$$

Since the local autoequivalence of $\mathcal{A}_i$ associated with an inner automorphism of A_i is isomorphic to the identity, we get that $F_{ij,V}$ and $F_{ij,V'}$ induce isomorphic equivalences over $V \cap V'$. By Lemma 6.3.5(ii), we obtain an equivalence defined over U_{ij},

$$F_{ij} : \mathcal{A}_i|_{U_{ij}} \to \mathcal{A}_j|_{U_{ij}},$$

such that for every $V \subset U_{ij}$, distinguished in both U_i and U_j, one has $F_{ij}|_V \simeq \alpha_{ij,V}$.

Furthermore, we claim that over U_{ijk} there is an equivalence

$$F_{jk}|_{U_{ijk}} \circ F_{ij}|_{U_{ijk}} \simeq F_{ik}|_{U_{ijk}}. \tag{6.9}$$

Indeed, by Lemma 6.3.4, we have a similar equivalence over V, for every $V \subset U_{ijk}$. Thus, our claim follows immediately from Lemma 6.3.5(i), applied to the equivalences in both sides of (6.9). Finally, we can apply Lemma 6.3.5(iii) to conclude the existence of the required NC-smooth algebroid $\mathcal{A}$ over X. □

CHAPTER VII

NC-ALGEBROID THICKENINGS OF MODULI SPACES

The results of this chapter are all in collaboration with A. Polishchuk, and appear in our co-written pre-print [7].

In this section we consider two functors, which are NC-thickenings of of certain families of either vector bundles or quiver representations. The first is the same functor as defined in [11] and claimed to be representable, although a gap was discovered in [20]. The second is similar to those considered by Toda in [23].

In each case the functor factors through $a\mathcal{N}$, so is not representable by an NC-scheme, but we show it is locally representable in $a\mathcal{N}$. It follows from the results of the previous chapter that these functors lead to natural NC-smooth algebroid thickenings of the parameter spaces they thicken.

7.1 Excellent Families of Vector Bundles

Let Z be a projective algebraic variety, B a smooth variety, and let $\mathcal{E}^{ab}$ be a vector bundle over B. We denote by $\rho : B \times Z \to B$ the natural projection.

Definition 7.1.1. We say that $\mathcal{E}^{ab}$ is an *excellent family* of bundles on Z if

(a) $\mathcal{O}_B \to \rho_* \mathcal{E}nd(\mathcal{E}^{ab})$ is an isomorphism,

(b) the Kodaira-Spencer map $\kappa : T_B \to R^1\rho_* \mathcal{E}nd(\mathcal{E}^{ab})$ is an isomorphism,

(c) $R^2\rho_* \mathcal{E}nd(\mathcal{E}^{ab}) = 0$,

(d) $R^d\rho_* \mathcal{E}nd(\mathcal{E}^{ab})$ is locally free for $d \geq 3$.

Note that our definition is slightly stronger than [11, Def. (5.4.1)] in that we add condition (d), which is used crucially in the base change calculations. Note also that condition (a) is satisfied whenever $\mathcal{E}$ is a family of stable bundles on Z, cf. [10] Lemma 4.6.3. For the definition of the Kodaira-Spencer map, refer to [10] section 10.

Following [11] we consider the natural functor on $\mathcal{N}$ of noncommutative families of vector bundles extending $\mathcal{E}$.

7.1.1 The functor of NC-families extending an excellent family

Definition 7.1.2. For an excellent family $\mathcal{E}$ over a smooth (commutative) base B, we define the functor $h_B^{NC} : \mathcal{N} \to Sets$ sending $\Lambda \in \mathcal{N}$ to the isomorphism classes of objects in the following category $\mathcal{C}_\Lambda$. Consider NC-schemes $X = \operatorname{Spec}(\Lambda)$ and $X \times Z$. Let us denote by $X_{ab}^0 = \operatorname{Spec}(\Lambda_0^{ab})$ the reduced scheme associated with the abelianization of X. Then the objects of $\mathcal{C}_\Lambda$ are the triples (f, E_Λ, ϕ) consisting of

(i) a morphism $f : X_{ab}^0 \to B$ of schemes,

(ii) a locally free sheaf of right $\mathcal{O}_{X\times Z}$-modules E_Λ,

(iii) an isomorphism $\phi : \mathcal{O}_{X_{ab}^0 \times Z} \otimes E_\Lambda \tilde{\to} (f \times \mathrm{id})^* \mathcal{E}$.

A morphism $(f_1, E_1, \phi_1) \to (f_2, E_2, \phi_2)$ exists only if $f_1 = f_2$ and is given by an isomorphism $E_1 \to E_2$ commuting with the ϕ_i. On morphisms h_B^{NC} is the usual pullback.

The following result is stated in [11] (see [11, Prop. (5.4.3)(a)(b)]). However, we believe our stronger assumptions on the family $\mathcal{E}$, including condition (d), are needed for it to hold.

Proposition 7.1.3. *The functor h_B^{NC} is formally smooth and the natural morphism of functors $h_B \to h_B^{NC}|_{\mathcal{C}om}$ is an isomorphism.*

Lemma 7.1.4. *For any commutative algebra Λ and any $(f, E_\Lambda, \phi) \in h_B^{NC}(\Lambda)$ the natural map*

$$\Lambda \to End(E_\Lambda)$$

is an isomorphism.

Proof. We prove this by the degree of nilpotency of the nilradical of Λ. Assume first that Λ is reduced. Then we have $E_\Lambda = (f \times \mathrm{id})^* \mathcal{E}$. Hence, by the base change theorem,

$$\begin{aligned} H^0(X \times Z, (f \times \mathrm{id})^* \mathcal{E}nd(\mathcal{E})) &\simeq H^0(X, Rp_{X,*}(f \times \mathrm{id})^* \mathcal{E}nd(\mathcal{E})) \\ &\simeq H^0(X, \mathcal{H}^0(Lf^* R\rho_* \mathcal{E}nd(\mathcal{E}))), \end{aligned}$$

where $X = \mathrm{Spec}\,(\Lambda)$. Since $R^i \rho_* \mathcal{E}nd(\mathcal{E})$ are locally free for $i \geq 1$, we have

$$\mathcal{H}^0(Lf^* R\rho_* \mathcal{E}nd(\mathcal{E})) \simeq f^* \rho_* \mathcal{E}nd(\mathcal{E}) \simeq \mathcal{O}_X,$$

where in the last isomorphism we used assumption (a). This shows that our assertion holds for such Λ.

Next, assume we have a central extension $0 \to I \to \Lambda' \to \Lambda \to 0$ of commutative algebras, such that I is a module over Λ_0, the quotient of Λ by its nilradical. Assume that $\Lambda \to \mathrm{End}(E_\Lambda)$ is an isomorphism for any $(f, E_\Lambda, \phi) \in h_B^{NC}(\Lambda)$ and let us prove a similar statement over Λ'. Given $(f, E_{\Lambda'}, \phi') \in h_B^{NC}(\Lambda')$, let E_Λ be the induced locally free sheaf over $\mathrm{Spec}\,(\Lambda) \times Z$. Then we have an exact sequence of coherent sheaves on $\operatorname{Spec}(\Lambda') \times Z$,

$$0 \to \mathcal{E}_{\Lambda_0} \otimes p_1^* \mathcal{I} \to \mathcal{E}_{\Lambda'} \to \mathcal{E}_{\Lambda} \to 0,$$

where $\mathcal{I}$ is the ideal sheaf on $\operatorname{Spec}(\Lambda')$ corresponding to I. Taking sheaves of homomorphisms from $\mathcal{E}_{\Lambda'}$ we get an exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}_{\Lambda_0}) \otimes p_1^* \mathcal{I} \to \mathcal{E}nd(\mathcal{E}_{\Lambda'}) \to \mathcal{E}nd(\mathcal{E}_{\Lambda}) \to 0$$

Passing to global sections we obtain a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \Lambda' & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(\mathcal{E}nd(\mathcal{E}_{\Lambda_0}) \otimes p_1^* \mathcal{I}) & \longrightarrow & \operatorname{End}(E_{\Lambda'}) & \longrightarrow & \operatorname{End}(E_{\Lambda}) & \longrightarrow & 0 \end{array} \tag{7.1}$$

Note that $\mathcal{E}_{\Lambda_0} \simeq (f \times \mathrm{id})^* \mathcal{E}$, so as before we get

$$\begin{aligned} H^0(X^0 \times Z, \mathcal{E}nd(\mathcal{E}_{\Lambda_0}) \otimes p_1^* \mathcal{I}) &\simeq H^0(X^0, \mathcal{I} \otimes \mathcal{H}^0(Lf^* R\rho_* \mathcal{E}nd(\mathcal{E}))) \\ &\simeq H^0(X^0, \mathcal{I} \otimes f^* \rho_* \mathcal{E}nd(\mathcal{E})) \\ &\simeq I, \end{aligned}$$

where $X^0 = \operatorname{Spec}(\Lambda_0)$. Thus, in the above morphism of exact sequences the leftmost and the rightmost vertical arrows are isomorphisms. Hence, the middle vertical arrow is also an isomorphism. □

Proof of Proposition 7.1.3. Assume we are given a central extension

$$0 \to I \to \Lambda' \to \Lambda \to 0 \tag{7.2}$$

in $\mathcal{N}$ and an element $(f, E_\Lambda, \phi) \in h_B^{NC}(\Lambda)$, so that E_Λ is a locally free sheaf of right $\mathcal{O}_{X\times Z}$-modules of rank r, where $X = \mathrm{Spec}\,(\Lambda)$. We have to check that it lifts to a locally free sheaf of right $\mathcal{O}_{X'\times Z}$-modules, where $X' = \mathrm{Spec}\,(\Lambda')$. Furthermore, it is enough to consider central extensions as above, where the nilradical of Λ^{ab} acts trivially on I, so that I is a Λ_0^{ab}-module.

We have a natural abelian extension of sheaves of groups on $X_{ab} \times Z$,

$$1 \to M_r(\mathcal{O}_{X_{ab}\times Z}) \otimes p_1^*\mathcal{I} \to GL_r(\mathcal{O}_{X'\times Z}) \to GL_r(\mathcal{O}_{X\times Z}) \to 1 \tag{7.3}$$

where $\mathcal{I}$ is the coherent sheaf on X_{ab} corrresponding to I. The isomorphism class of E_Λ corresponds to an element of the nonabelian cohomology $H^1(X_{ab} \times Z, GL_r(\mathcal{O}_{X\times Z}))$. By the standard formalism (see Sec. 2.3) the obstruction to lifting this class to a class in $H^1(X_{ab}\times Z, GL_r(\mathcal{O}_{X'\times Z}))$ lies in $H^2(X_{ab}\times Z, \mathcal{E}nd(E_{\Lambda_0^{ab}})\otimes p_1^*\mathcal{I})$, where $E_{\Lambda_0^{ab}}$ is induced by E_Λ. We claim that this group H^2 vanishes. Indeed, we have $E_{\Lambda_0^{ab}} \simeq (f \times \mathrm{id})^*\mathcal{E}$. Applying the base change theorem we get an isomorphism

$$R\Gamma(X_{ab}^0 \times Z, (f \times \mathrm{id})^*\mathcal{E}nd(\mathcal{E}) \otimes p_1^*\mathcal{I}) \simeq R\Gamma(X_{ab}^0, \mathcal{I} \otimes Lf^*R\rho_*\mathcal{E}nd(\mathcal{E})).$$

It remains to observe that by our assumptions (c) and (d), the complex of sheaves $Lf^*R\rho_*\mathcal{E}nd(\mathcal{E})$ has no cohomology in degrees ≥ 2.

To prove that second assertion we argue by induction on the degree of nilpotency of the nilradical of a test algebra Λ. Thus, we consider a square zero extension (7.2) of commutative algebras, where I is a Λ_0^{ab}-module, and study the corresponding commutative square

$$\begin{array}{ccc} h_B(\Lambda') & \longrightarrow & h_B(\Lambda) \\ \downarrow & & \downarrow \\ h_B^{NC}(\Lambda') & \longrightarrow & h_B^{NC}(\Lambda) \end{array} \tag{7.4}$$

We assume that the right vertical arrow is an isomorphism and we would like to prove the same about the left vertical arrow. We know that both horizontal arrows are surjective. Furthermore, using the interpretation in terms of nonabelian H^1 and the exact sequence (7.3) we can get a description of the preimage of an element $E_\Lambda \in h_B^{NC}(\Lambda)$ under the bottom arrow. Namely, the corresponding sequence of twisted sheaves is

$$0 \to \mathcal{E}nd(E^{ab}) \otimes p_1^*\mathcal{I} \to \underline{Aut}(E_{\Lambda'}) \to \underline{Aut}(E_\Lambda) \to 1. \tag{7.5}$$

By Lemma 7.1.4, we have $Aut(E_\Lambda) = \Lambda^*$, and it is easy to see that this group acts trivially on $H^1(X_{ab} \times Z, \mathcal{E}nd(E_{\Lambda^{ab}}) \otimes p_1^*\mathcal{I})$ (since Λ' is in the center of $\underline{Aut}(E_{\Lambda'})$). It follows that the preimage of E_Λ in $h_B^{NC}(\Lambda')$ is the principal homogeneous space for the abelian group

$$\text{coker}\,(Aut(E_\Lambda) \xrightarrow{\delta_0} H^1(X_{ab} \times Z, \mathcal{E}nd(E_{\Lambda^{ab}}) \otimes p_1^*\mathcal{I})),$$

where δ_0 is the connecting homomorphism associated with (7.5). However, by Lemma 7.1.4, fixing a lifting $E_{\Lambda'} \in h_B^{NC}(\Lambda')$, we get that the previous map in the long exact sequence, $Aut(E_{\Lambda'}) \to Aut(E_\Lambda)$ is just the projection $(\Lambda')^* \to \Lambda^*$, so it is surjective. This implies that the preimage of E_Λ is the principal homogeneous space for

$$H^1(X_{ab} \times Z, \mathcal{E}nd(E_{\Lambda_0^{ab}}) \otimes p_1^* \mathcal{I}) \simeq H^0(X_{ab}^0, \mathcal{I} \otimes \mathcal{H}^1(Lf^* R\rho_* \mathcal{E}nd(\mathcal{E}))).$$

By our assumptions (c) and (d), we have

$$\mathcal{H}^1(Lf^* R\rho_* \mathcal{E}nd(\mathcal{E})) \simeq f^* R^1 \rho_* \mathcal{E}nd(\mathcal{E}),$$

thus, the above group is $H^0(X_{ab}^0, I \otimes f^* R^1 \rho_* \mathcal{E}nd(\mathcal{E}))$.

On the other hand, different extensions of $\mathrm{Spec}\,(\Lambda) \to B$ to $\mathrm{Spec}\,(\Lambda') \to B$ correspond to $H^0(B, f_* I \otimes \mathcal{T}_B)$. It is easy to check that the map $h_B(\Lambda') \to h_B^{NC}(\Lambda')$ is compatible with the Kodaira-Spencer map

$$H^0(B, f_* \mathcal{I} \otimes \mathcal{T}_B) \simeq H^0(X_{ab}^0, \mathcal{I} \otimes f^* \mathcal{T}_B) \to H^0(X_{ab}^0, \mathcal{I} \otimes f^* R^1 \rho_* \mathcal{E}nd(\mathcal{E})),$$

which is an isomorphism by assumption (b). It follows that the map $h_B(\Lambda') \to h_B^{NC}(\Lambda')$ is an isomorphism. □

We have the following simple observation.

Proposition 7.1.5. *The functor $h_B^{NC} : \mathcal{N} \to Sets$ factors through $a\mathcal{N}$.*

Proof. Suppose we have two homomorphims $f_1, f_2 : \Lambda' \to \Lambda$ in $\mathcal{N}$ such that they are conjugate, i.e., $f_2 = \theta f_1$, where $\theta = \theta_u$ is an inner automorphism of Λ: $\theta_u(x) = uxu^{-1}$ for some unit u in Λ. We have to check that f_1 and f_2 induce the same map $h(\Lambda') \to h(\Lambda)$. Equivalently, we have to check that the map $h(\theta) : h(\Lambda) \to h(\Lambda)$ is equal to the identity. Note that θ_u induces an automorphism of the NC-scheme $X = \mathrm{Spec}\,(\Lambda)$, which we still denote by θ, and the map $h(\theta)$ sends a right $\mathcal{O}_{X \times Z}$-module E_Λ to $(\theta \times \mathrm{id}_Z)^* E_\Lambda$. Now we observe that the automorphism $\theta \times \mathrm{id}$ of $X \times Z$ acts trivially on the underlying topological space and is given by the inner automorphism θ_u of the structure sheaf $\mathcal{O} = \mathcal{O}_{X\times Z}$, associated with u which we view as a global section of $\mathcal{O}^*$. Thus, the operation $(\theta\times\mathrm{id}_Z)^*$ is given by tensoring on the right with $\mathcal{O}-\mathcal{O}$ bimodule $\underset{\theta_u}{\mathcal{O}}$ (which is the structure sheaf with the left $\mathcal{O}$-action twisted by θ_u).

Now we use the general fact that twisting by an inner automorphism does not change an isomorphism class of a bimodule. Namely, if M is an $R-S$-bimodule and θ_u is the inner automorphism of R associated with $u \in R^*$, then we have an isomorphism of (R,S)-bimodules,

$$M \tilde{\to} {}_{\theta_u}M : m \mapsto um.$$

This construction also works for bimodules over sheaves of rings and an inner automorphism associated with a global unit. This implies that in our situation the functor $(\theta\times\mathrm{id}_Z)^*$ is isomorphic to identity, and our claim follows □

Remark 7.1.6. In fact, our proof of Proposition 7.1.5 shows a little more. We can enhance h_B^{NC} to a functor with values in groupoids, by considering the category of the data as in Definition 7.1.2 and isomorphisms between them. On the other hand, we can consider a 2-category of algebras in $\mathcal{N}$ with the usual 1-morphisms and with 2-morphisms between $f_1, f_2 : \Lambda' \to \Lambda$ given by $u \in \Lambda^*$ such that $f_2 = \theta_u f_1$. Then the functor h_B^{NC} lifts to a 2-functor from this 2-category to the 2-category of groupoids.

7.1.2 Local representability in $a\mathcal{N}$

By Proposition 7.1.5, we can view h_B^{NC} as a functor on the category $a\mathcal{N}$, our main goal is to prove the local representability of the corresponding functor $h_B^{NC}|_{a\mathcal{N}_d}$ by a d-smooth NC-algebra.

Theorem 7.1.7. *Assume that the base B of an excellent family is affine. Then for every $d \geq 0$ the functor $h_B^{NC}|_{a\mathcal{N}_d}$ is representable in $a\mathcal{N}_d$ by a d-smooth thickening of B. Hence the functor h_B^{NC} is representable in $a\mathcal{N}$ by a NC-smooth thickening of B.*

The proof will proceed by induction on d. We need two technical lemmas (the second of which is a noncommutative extension of Lemma 7.1.4).

Lemma 7.1.8. *Assume that $h_B^{NC}|_{a\mathcal{N}_{d-1}}$ is representable by $A \in \mathcal{N}_{d-1}$. Then for any central extension (6.3) with $\Lambda \in a\mathcal{N}_{d-1}$, $\Lambda' \in a\mathcal{N}_d$, and any homomorphism $f : A \to \Lambda$, there is a commutative square*

$$\begin{array}{ccc} U(f) & \xrightarrow{\quad\Delta_f\quad} & \mathrm{Der}(A^{ab}, I) \\ \downarrow & & \downarrow{\scriptstyle -KS} \\ Aut(E_\Lambda) & \xrightarrow{\delta_0} & H^1(Spec\,(\Lambda^{ab}) \times Z, End(E^{ab}) \otimes I) \end{array} \tag{7.6}$$

Here Δ_f is given by (6.4); $E_\Lambda = E_f$ is the family in $h_B^{NC}(\Lambda)$ induced by f; the map KS is induced by the Kodaira-Spencer map; and the homomorphism $U(f) \to Aut(E_f)$ associates with $u \in \Lambda^*$ an automorphism of E_f induced by the left multiplication by u on Λ. The map δ_0 is the connecting map associated with the exact sequence of sheaves (7.5), where $E_{\Lambda'}$ is a vector bundle over $Spec\,(\Lambda') \times Z$ lifting E_Λ. In particular, in this situation δ_0 is a group homomorphism.

Proof. We are going to compute the maps in the square (7.6) using local trivializations. Let us denote by $\mathcal{E}^{ab}$ the original family over $B \times Z$, and let $\mathcal{E}$ be the family over $\mathrm{Spec}\,(A) \times Z$ corresponding to the element $\mathrm{id}_A \in h_A(A) \simeq h_B^{NC}(A)$. We denote by f^{ab} the homomorphism $A^{ab} \to \Lambda^{ab}$ induced by f and the corresponding morphism of affine schemes $\mathrm{Spec}\,(\Lambda^{ab}) \to \mathrm{Spec}\,(A^{ab}) = B$. Note that by Proposition 7.1.3, we have an isomorphism $E^{ab} = (f^{ab} \times \mathrm{id})^* \mathcal{E}^{ab}$.

Step 1. Computation of $\delta_0 : Aut(E_f) \to H^1(\mathrm{Spec}\,(\Lambda^{ab}) \times Z, \mathcal{E}nd(E^{ab}) \otimes p_1^* \mathcal{I})$.

Let us fix an open affine covering (U_i) of $\mathrm{Spec}\,(\Lambda^{ab})\times Z$ such that $E_{f'}$ is trivial over U_i. Then, given an automorphism $\alpha \in Aut(E_f)$, over U_i we can lift α to an automorphism α_i of $E_{\Lambda'}$. Now over $U_i \cap U_j$ the endomorphism $\alpha_i^{-1}\alpha_j - \mathrm{id}$ of $E_{\Lambda'}$ factors through the kernel of the projection $E_{f'} \to E_f$, i.e., $E^{ab} \otimes p_1^*\mathcal{I}$. This gives the Cech 1-cocycle with values in $\mathcal{E}nd(E^{ab}) \otimes p_1^*\mathcal{I}$, representing the class $\delta_0(\alpha)$.

Step 2. Computation of the KS-map

$$\mathrm{Der}(A^{ab}, I) \to H^1(\mathrm{Spec}\,(\Lambda^{ab}) \times Z, \mathcal{E}nd(E^{ab}) \otimes p_1^*\mathcal{I}). \tag{7.7}$$

Note that we have an identification

$$\mathrm{Der}(A^{ab}, I) \simeq H^0(B, \mathcal{T}_B \otimes f_*^{ab}\mathcal{I}).$$

Let us fix trivializations $\varphi_i^{ab} : \mathcal{O}^n \to \mathcal{E}^{ab}$ over an affine open covering (U_i) of $B \times Z$, and let $g_{ij}^{ab} = (\varphi_i^{ab})^{-1}\varphi_j^{ab} \in M_n(\mathcal{O}(U_i \cap U_j))$ be the corresponding transition functions. Then to a vector field v on B with values in $f_*^{ab}\mathcal{I}$ the KS-map associates the Cech 1-cocycle $\varphi_i^{ab} v(g_{ij}^{ab})(g_{ij}^{ab})^{-1}(\varphi_i^{ab})^{-1}$ on $B \times Z$ with values in $\mathcal{E}nd(\mathcal{E}^{ab}) \otimes p_1^* f_*^{ab}\mathcal{I}$.

We also need to calculate the image of this class under the isomorphism induced by the projection formula

$$H^1(B \times Z, \mathcal{E}nd(\mathcal{E}^{ab}) \otimes p_1^* f_*\mathcal{I}) \tilde{\to} H^1(B \times Z, (f \times \mathrm{id})_*((f \times \mathrm{id})^*\mathcal{E}nd(\mathcal{E}^{ab}) \otimes p_1^*\mathcal{I})) \simeq$$
$$H^1(\mathrm{Spec}\,(\Lambda^{ab}) \times Z, \mathcal{E}nd(E^{ab}) \otimes p_1^*\mathcal{I}).$$

To this end we note that the morphism $f^{ab} \times \mathrm{id} : \mathrm{Spec}\,(\Lambda^{ab}) \times Z \to B \times Z$ is affine, and so $\widetilde{U}_i := (f^{ab} \times \mathrm{id})^{-1}(U_i)$ is an affine open covering of $\mathrm{Spec}\,(\Lambda^{ab}) \times Z$, over which we have the induced trivializations of $E^{ab} = (f^{ab} \times \mathrm{id})^*\mathcal{E}^{ab}$, which we still denote by

φ_i^{ab}. Now it is easy to see that the corresponding Cech 1-cocycle on $\operatorname{Spec}(\Lambda^{ab}) \times Z$ with values in $\mathcal{E}nd(E^{ab}) \otimes I$ is given by

$$\varphi_i^{ab} v(g_{ij}^{ab}) f^{ab}(g_{ij}^{ab})^{-1} (\varphi_i^{ab})^{-1},$$

where we denote still by $f^{ab} : \mathcal{O}(U_i \cap U_j) \to \mathcal{O}(\widetilde{U}_i \cap \widetilde{U}_i)$ the homomorphism induced by f^{ab}, and also extend v to a derivation $\mathcal{O}(U_i \cap U_j) \to p_1^* \mathcal{I}(\widetilde{U}_i \cap \widetilde{U}_j)$.

Step 3. Now we can check the commutativity of the square (7.6)

We start by choosing an affine open covering (U_i) of $B \times Z$ and trivializations of $\mathcal{E}^{ab}$ over U_i. Then we can lift these trivializations to some trivializations φ_i : $\mathcal{O}^n_{\operatorname{Spec}(A)\times Z}|_{U_i} \to \mathcal{E}$. We denote by g_{ij} the corresponding transition functions in $GL_n(\mathcal{O}_{\operatorname{Spec}(A)\times Z}(U_i \cap U_j))$.

By definition, $\Delta_f(u)$ is the derivation

$$v(a) = [u, f(a)]_{\Lambda'} u^{-1} = [\tilde{u}, \widetilde{f(a)}]\tilde{u}^{-1},$$

where $\tilde{u}, \widetilde{f(a)} \in \Lambda'$ are some lifts of u and $f(a)$ (note that $\operatorname{Der}(A, I) = \operatorname{Der}(A^{ab}, I)$). Hence, $KS(\Delta_f(u))$ is represented by the 1-cocycle

$$\varphi_i [\tilde{u}, \widetilde{f(g_{ij})}]_{\Lambda'} \tilde{u}^{-1} \widetilde{f(g_{ij})}^{-1} \varphi_i^{-1} = \varphi_i (\tilde{u} \widetilde{f(g_{ij})} \tilde{u}^{-1} \widetilde{f(g_{ij})}^{-1} - \mathrm{id}) \varphi_i^{-1}. \tag{7.8}$$

As in Step 2, we have the induced affine open covering $\widetilde{U}_i$ of $\operatorname{Spec}(\Lambda^{ab}) \times Z$, and the induced trivializations ψ_i of E_f over $\widetilde{U}_i$. Let us choose a lifting $E_{\Lambda'}$ of E_f to a vector bundle over $\operatorname{Spec}(\Lambda') \times Z$ (it exists by formal smoothness of h_B^{NC}), and liftings ψ_i' of ψ_i to trivializations of $E_{\Lambda'}$ over $\widetilde{U}_i$. Note that we have $\psi_i^{-1}\psi_j = f(g_{ij})$, and hence $(\psi_i')^{-1}\psi_j'$ provide liftings $\widetilde{f(g_{ij})} \in \Lambda'$ of $f(g_{ij})$. The image of $u \in$

$U(f)$ in $Aut(E_f)$ can be represented over $\widetilde{U}_i$ as $\psi_i u \psi_i^{-1}$, where we view u as the corresponding operator of the left multiplication by u (note that these operators are compatible on intersections because $u \cdot f(g_{ij}) = f(g_{ij}) \cdot u$, due to the inclusion $u \in U(f)$). Using the lifting $\tilde{u} \in \Lambda'$ of u we get local automorphisms of $E_{f'}$ over $\widetilde{U}_i$, $\alpha_i = \psi_i' \tilde{u} (\psi_i')^{-1}$. Then

$$\delta_0(\alpha) = \alpha_i^{-1}\alpha_j - \mathrm{id} = (\psi_i'\tilde{u}^{-1}(\psi_i')^{-1})(\psi_j'\tilde{u}\tilde{\psi}_j^{-1}) - \mathrm{id} = \psi_i'(\tilde{u}^{-1}\widetilde{f(g_{ij})}\tilde{u}\widetilde{f(g_{ij})}^{-1} - \mathrm{id})(\psi_i')^{-1}.$$

Comparing this with (7.8) we see that

$$\delta_0(\alpha) = KS(\Delta_f(u^{-1})) = KS(-\Delta_f(u)) = -KS(\Delta_f(u)).$$

□

Lemma 7.1.9. *Assume that $h_B^{NC}|_{a\mathcal{N}_d}$ is representable by $A \in a\mathcal{N}_d$, so $h_B^{NC}|_{a\mathcal{N}_d} \simeq h_A$. Then for every d-nilpotent algebra Λ and every homomorphism $f : A \to \Lambda$, the induced homomorphism $U(f) \to Aut(E_f)$ is an isomorphism. Here E_f represents the family in $h_B^{NC}(\Lambda)$ induced by f.*

Proof. We will prove the assertion by induction on $d' \leq d$ such that Λ is d'-nilpotent. For $d' = 0$, i.e., when Λ is commutative, we have $U(f) = \Lambda^*$ and the assertion follows from Lemma 7.1.4.

Next, we have to see that both groups fit into the same exact sequences, when Λ' is a central extension of Λ by I. Namely, if $f' : A \to \Lambda'$ is a homomorphism lifting f, then by Lemma 7.1.8, we have a morphism of exact sequences

$$\begin{array}{ccccccc} 1+I & \longrightarrow & U(f') & \longrightarrow & U(f) & \xrightarrow{\Delta_f} & \mathrm{Der}(A^{ab}, I) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle -KS} \\ 1+I & \longrightarrow & Aut(E_{f'}) & \longrightarrow & Aut(E_f) & \xrightarrow{\delta_0} & H^1(\mathcal{E}nd(E^{ab}) \otimes p_1^*\mathcal{I}) \end{array} \tag{7.9}$$

Note that the map KS is an isomorphism. Since the map $U(f) \to Aut(E_f)$ is an isomorphism by the induction assumption, we deduce that $U(f') \to Aut(E_{f'})$ is also an isomorphism. □

Proof of Theorem 7.1.7. By Proposition 7.1.3, we know that the assertion is true for $d = 0$. Now, assuming that the functor $h_B^{NC}|_{a\mathcal{N}_{d-1}}$ is representable, we will apply Proposition 6.2.2 to prove that $h_B^{NC}|_{a\mathcal{N}_d}$ is representable. It suffices to check conditions (i) and (ii) of this Proposition. To prove condition (i) assume that $\Lambda' \to \Lambda$ and $\Lambda'' \to \Lambda$ and nilpotent extensions with $\Lambda, \Lambda'' \in \mathcal{C}om$. To see that the map

$$h(\Lambda' \times_\Lambda \Lambda'') \to h(\Lambda') \times_{h(\Lambda)} h(\Lambda'')$$

is a bijection, we construct (as in [11, Lem. (5.4.4)]) the inverse map as follows. Starting with families $\mathcal{E}_{\Lambda'}$ and $\mathcal{E}_{\Lambda''}$ over Λ' and Λ'', and choosing an arbitrary isomorphism of the induced families over Λ, we define the family over $\Lambda' \times_\Lambda \Lambda''$ as the fibered product $\mathcal{E}_{\Lambda'} \times_{\mathcal{E}_\Lambda} \mathcal{E}_{\Lambda''}$. One has to check that the result does not depend on a choice of isomorphism of families over Λ (this may fail in general, but works for commutative Λ''). Note that different choices differ by an automorphism of $\mathcal{E}_\Lambda$, so it is enough to see that any such automorphism can be lifted to an automorphism of $\mathcal{E}_{\Lambda''}$. But this follows immediately from Lemma 7.1.4.

Next, let us check condition (ii). Given a central extension (6.3) with $\Lambda' \in \mathcal{N}_d$, $\Lambda \in \mathcal{N}_{d-1}$, and a family $(f^{ab}, E_\Lambda, \phi)$ in $h_B^{NC}(\Lambda)$, then choosing a lifting $E_{\Lambda'}$ to a family over Λ', from the corresponding exact sequence of sheaves of groups (7.5) we get a connecting map

$$\delta_0 : Aut(E_\Lambda) \to H^1(X_{ab} \times Z, \mathcal{E}nd(E_{\Lambda^{ab}}) \otimes p_1^* \mathcal{I}).$$

Furthermore, by Lemma 7.1.8, δ_0 is actually a group homomorphism (and the source of this map acts trivially on the target). Thus, from the formalism of nonabelian cohomology applied to the abelian extension of sheaves of groups (7.3) we get that different liftings of E_Λ to a family over Λ' form a principal homogeneous space over $\operatorname{coker}(\delta_0)$ (see Sec.2.3). Note that by Lemma (7.1.9), we have an isomorphism $U(f) \simeq Aut(E_\Lambda)$, where $f : A \to \Lambda$ is the homomorphism giving E_Λ. Thus, by Lemma 7.1.8, we can identify $\operatorname{coker}(\delta_0)$ with $\operatorname{coker}(\Delta_f)$. Thus, to prove condition (ii), it remains to check that the two actions of $\operatorname{Der}(A, I)$ on the set of liftings of E_Λ are the same (the one coming from the formalism of non-abelian cohomology, and the other one given by the map (6.8)).

To this end we use the computation of the Kodaira-Spencer map (7.7) using local trivializations. Namely, we choose trivializations of the universal bundle $\mathcal{E}$ over an open covering of $\operatorname{Spec}(A) \times Z$, and denote by g_{ij} the corresponding transition functions, so that $f(g_{ij})$ are the transition functions for E_Λ. Then, in the notation of Lemma 7.1.8, a derivation $v \in \operatorname{Der}(A, I) = \operatorname{Der}(A^{ab}, I)$ gives rise to the Cech 1-cocycle

$$\varphi_i v(g_{ij}) f(g_{ij})^{-1} \varphi_i^{-1}$$

on $\operatorname{Spec}(\Lambda^{ab}) \times Z$ with values in $\mathcal{E}nd(E^{ab}) \otimes p_1^* \mathcal{I}$. The corresponding $f(g_{ij})$-twisted 1-cocycle with values in $M_r(\mathcal{O}) \otimes p_1^* \mathcal{I}$ is $(v(g_{ij}) f(g_{ij})^{-1})$. Now by definition, the action of v on the set of liftings of $f(g_{ij})$ to a 1-cocycle with values in $GL_r(\mathcal{O}_{\operatorname{Spec}(\Lambda') \times Z})$

sends $(\tilde{g}_{ij})$ to

$$((1+v(g_{ij})f(g_{ij})^{-1})\cdot\widetilde{g}_{ij} = (\widetilde{g}_{ij}+v(g_{ij})). \tag{7.10}$$

On the other hand, from v we get a homomorphism $f^{ab}+v : A \to \Lambda^{ab}\oplus I$, and hence, the 1-cocycle $(f^{ab}+v)(g_{ij})$ with values in $GL_r(\mathcal{O}_{\mathrm{Spec}\,(\Lambda^{ab}\oplus I)\times Z})$ lifting $f^{ab}(g_{ij})$. Hence, a lifting $\widetilde{g}_{ij}$ of $f(g_{ij})$ together with v defines a 1-cocycle

$$(\widetilde{g}_{ij},(f^{ab}+v)(g_{ij}))$$

with values in $GL_r(\mathcal{O}_{\mathrm{Spec}\,(\Lambda'\times_{\Lambda^{ab}}(\Lambda^{ab}\oplus I))\times Z})$. It remains to observe that under the isomorphism (6.6) it corresponds to the 1-cocycle

$$(\widetilde{g}_{ij},\widetilde{g}_{ij}+v(g_{ij}))$$

with values in $GL_r(\mathcal{O}_{\mathrm{Spec}\,(\Lambda'\times_{\Lambda}\Lambda')\times Z})$, which has (7.10) as the same second component. □

7.2 Excellent Families of Quiver Representations

Let Q be a finite quiver with the set of vertices Q_0 and the set of arrows Q_1. We denote by $h,t : Q_1 \to Q_0$ the maps associating with an arrow its head and tail.

As in [23], we can consider representations of Q over an NC-scheme X.

Definition 7.2.1. A representation of Q over an NC-scheme X is a collection of vector bundles $(\mathcal{V}_v)_{v\in Q_0}$ over X, and a collection of morphisms $e_a : \mathcal{V}_t(a) \to \mathcal{V}_{h(a)}$, for each $a\in Q_1$.

In order to impose conditions on a family of quiver representation analogous to Kapranov's functor for vector bundles, we first need the analog of the Kodaira-Spencer map.

7.2.1 Kodaira-Spencer map for quiver representations

With a collection $\mathcal{V} = (\mathcal{V}_v)_{v\in Q_0}$ of vector bundles over X we associate a triple of sheaves of groups on the underlying topological space of X,

$$\mathcal{G}(\mathcal{V}) := \prod_v \underline{Aut}(\mathcal{V}_v), \quad \mathcal{E}_0(\mathcal{V}) := \prod_v \mathcal{E}nd(\mathcal{V}_v), \quad \mathcal{E}_1(\mathcal{V}) := \prod_a \mathcal{H}om(\mathcal{V}_{t(a)}, \mathcal{V}_{h(a)}).$$

Note that there is a natural action of $\mathcal{G}(\mathcal{V})$ on $\mathcal{E}_1(\mathcal{V})$ given by

$$(g_v)\cdot(\phi_a) = (g_{h(a)}\phi_a g_{t(a)}^{-1}).$$

In the case of trivial bundles $\mathcal{V}_v = \mathcal{O}^{n_v}$, for a dimension vector $n_\bullet$, we denote these sheaves as $\mathcal{G}(n_\bullet)$, $\mathcal{E}_0(n_\bullet)$ and $\mathcal{E}_1(n_\bullet)$. When we want to stress the dependence on the NC-scheme X we write $\mathcal{G}(n_\bullet, X)$, etc.

A structure of a representation of Q on $\mathcal{V}$ is given by a global section $e = (e_a)$ of $\mathcal{E}_1(\mathcal{V})$. For such a structure e we can build a 2-term complex

$$\mathcal{E}_\bullet(\mathcal{V}, e) : \mathcal{E}_0(\mathcal{V}) \overset{d_f}{\to} \mathcal{E}_1(\mathcal{V}),$$

where the differential is given by $d_e(\phi_v) = \phi_{h(a)}e_a - e_a\phi_{t(a)}$. Note that $\mathcal{H}^0\mathcal{E}_\bullet(\mathcal{V}, e)$ is precisely the sheaf of endomorphisms of $(\mathcal{V}, e)$ as a representation of Q.

Let $(\mathcal{V}, e)$ be a representation of Q over X. Over some open affine covering $\mathcal{U} = (U_i)$ of X we can choose a trivialization $\varphi_i = (\varphi_{v,i}) : \bigoplus_v \mathcal{O}_{U_i}^{n_v} \to \bigoplus_v \mathcal{V}_v|_{U_i}$. Then over each U_i we have morphisms

$$e_{a,i} := \varphi_{h(a),i}^{-1} e_a \varphi_{t(a),i} \in M_{n_{t(a)} \times n_{h(a)}}(\mathcal{O}(U_i)) = \mathcal{E}_1(n_\bullet)(U_i),$$

and over intersections $U_i \cap U_j$ we have transition functions

$$g_{ij} = (g_{v,ij}) = \varphi_i^{-1}\varphi_j \in \prod_v GL_{n_v}(\mathcal{O}(U_i \cap U_j)) = \mathcal{G}(n_\bullet)(U_i \cap U_j).$$

One immediately checks that $(g_{ij}, e_{a,i})$ defines a Cech 1-cocycle with values in the pair $\mathcal{G}(n_\bullet) \curvearrowright \mathcal{E}_1(n_\bullet)$ (see Sec. 2.4). Furthermore, a different choice of trivializations (φ_i) leads to a cohomologous cocycle, so we have a well defined element of $\mathbb{H}^1(X, \mathcal{G}(n_\bullet) \curvearrowright \mathcal{E}_1(n_\bullet))$. One can easily check that in this way we get a bijection between the latter nonabelian hypercohomology group and the set of isomorphism classes of representations $(\mathcal{V}, e)$ of Q, such that the underlying vector bundle has dimension vector $n_\bullet$.

For a central extension (6.3) we have an abelian extension of sheaves of groups

$$1 \to \mathcal{E}_0(n_\bullet, \mathcal{O}_{X_{ab}}) \otimes \mathcal{I} \to \mathcal{G}(n_\bullet, \mathcal{O}_{X'}) \to \mathcal{G}(n_\bullet, \mathcal{O}_X) \to 1 \tag{7.11}$$

where $X = \mathrm{Spec}\,(\Lambda)$, $X' = \mathrm{Spec}\,(\Lambda')$, $\mathcal{I} \subset \mathcal{O}_{X'}$ is the ideal sheaf associated with I, and an exact sequence of abelian groups

$$0 \to \mathcal{E}_1(n_\bullet) \otimes \mathcal{I} \to \mathcal{E}_1(n_\bullet, X') \to \mathcal{E}_1(n_\bullet, X) \to 0,$$

compatible with the actions of the groups from (7.11). From Sec. 2.4 we get that the obstacle to lifting a representation $(\mathcal{V}, e)$ of Q over $\mathrm{Spec}\,(\Lambda)$ to a representation of Q over $\mathrm{Spec}\,(\Lambda')$ is an element of the hypercohomology $\mathbb{H}^2(X^{ab}, \mathcal{E}_\bullet(\mathcal{V}, e) \otimes \mathcal{I})$.

But the latter group $\mathbb{H}^2$ fits into the exact sequence

$$\ldots \to H^1(X^{ab}, \mathcal{E}_1(\mathcal{V}) \otimes \mathcal{I}) \to \mathbb{H}^2 \to H^2(X^{ab}, \mathcal{E}_0(\mathcal{V}) \otimes \mathcal{I}) \to \ldots$$

Since X^{ab} is an affine scheme, we deduce that our $\mathbb{H}^2$ vanishes. Thus, the functor of families of Q-representations on $\mathcal{N}$ is formally smooth.

Definition 7.2.2. With a representation $(\mathcal{V}, e)$ of Q over a commutative scheme B we associate the KS-map, which is a morphism of coherent sheaves on B,

$$KS : \mathcal{T}_B \to \mathcal{H}^1\mathcal{E}_\bullet(\mathcal{V}, e), \tag{7.12}$$

defined as follows. Locally we can choose trivializations $\varphi : \bigoplus_v \mathcal{O}^{n_v} \to \bigoplus_v \mathcal{V}_v$ and set for a local derivation v of $\mathcal{O}_B$,

$$KS(v) := \varphi v(\varphi^{-1} e_a \varphi)\varphi^{-1} \mod \operatorname{im}(d_e) \in \mathcal{E}_1(\mathcal{V}, f))/\operatorname{im}(d_e).$$

It is easy to check that a change of a local trivialization leads to an addition of a term in $\operatorname{im}(d_e)$, so the map KS is well defined.

Remark 7.2.3. This definition is motivated by the fact that in the case when $B =$ Spec (k) is the point and (V, e) is a Q-representation over k, the space $H^1\mathcal{E}_\bullet(V, e)$ is isomorphic to $\operatorname{Ext}^1((V, e), (V, e))$ (see [3, Cor. 1.4.2]), which is the tangent space to deformations of (V, e) as a Q-representation.

7.2.2 Excellent families of quiver representations

Now let us fix a family $(\mathcal{V}^{ab}, e^{ab})$ of representations of Q over a smooth commutative base scheme B. We have the following analog of Definition 7.1.1.

Definition 7.2.4. We say that $(\mathcal{V}^{ab}, e^{ab})$ is an *excellent family* of representations of Q if

(a) the natural map $\mathcal{O}_B \to \mathcal{E}nd(\mathcal{V}^{ab}, e^{ab}) = \mathcal{H}^0\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab})$ is an isomorphism;

(b) the Kodaira-Spencer map $KS : \mathcal{T}_B \to \mathcal{H}^1\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab})$ is an isomorphism.

For example, these conditions are satisfied for the moduli spaces of stable quiver representations corresponding to an indivisible dimension vector (see [14, 5.3]).

Let us point out some consequences of the assumptions (a) and (b). Given $f : S \to B$ (where S is a commutative scheme), for $(V, e) = (f^*\mathcal{V}^{ab}, f^*e)$ we have

$$\mathcal{E}nd(V,e) = \mathcal{H}^0\mathcal{E}_\bullet(V,e) = \mathcal{H}^0 Lf^*\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \simeq f^*\mathcal{H}^0\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \simeq f^*\mathcal{O}_B \simeq \mathcal{O}_S,$$

where we used the fact that $\mathcal{H}^1\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \simeq \mathcal{T}_B$ is locally free. Also, if S is affine, then for any coherent sheaf $\mathcal{F}$ on S we have

$$\mathcal{H}^1(\mathcal{E}_\bullet(V,e) \otimes \mathcal{F}) \simeq \mathcal{H}^1\mathcal{E}_\bullet(V,e) \otimes \mathcal{F} \simeq f^*\mathcal{T}_B \otimes \mathcal{F}.$$

Now we consider the following analog of Definition 7.1.2 for quiver representations.

7.2.3 Functor of NC-families extending an excellent family of representations

Definition 7.2.5. For an excellent family $(\mathcal{V}^{ab}, e^{ab})$ of representations of Q over a smooth (commutative) base B, we define the functor $h_B^{NC} : \mathcal{N} \to \mathit{Sets}$ by letting h_B^{NC} to be the set of isomorphism classes of the following data (f, V_Λ, ϕ). Let $X = \operatorname{Spec}(\Lambda)$ and let X_{ab}^0 be the reduced scheme of the abelianization of X.

Then $f : X^0_{ab} \to B$ is a morphism, (V_Λ, e_Λ) is a representation of Q over X, and $\phi : (E_\Lambda, e_\Lambda)|_{X^0_{ab}} \simeq (f^*\mathcal{V}^{ab}, f^*e^{ab})$ is an isomorphism of representations of Q.

We have the following analog of Theorem 7.1.7 (and Propositions 7.1.5).

Theorem 7.2.6. *The functor h^{NC}_B is formally smooth and factors through the category $a\mathcal{N}$. If the base B is affine then for every $d \geq 0$ the functor $h^{NC}_B|_{a\mathcal{N}_d}$ is representable by a d-smooth thickening of B.*

Proof. The proof follows the same steps as in the case of families of vector bundles. We already shown before that h^{NC}_B is formally smooth. The fact that h^{NC}_B factors through $a\mathcal{N}$ is proved similarly to Proposition 7.1.5.

The key technical computation is the analog of Lemma 7.1.8, which in our case claims commutativity of the diagram

$$\begin{array}{ccc} U(f) & \xrightarrow{\quad \Delta_f \quad} & \mathrm{Der}(A^{ab}, I) \\ \downarrow & & \downarrow{\scriptstyle -KS} \\ Aut(\mathcal{V}_\Lambda, e_\Lambda) & \xrightarrow{\delta_0} & H^0(X^{ab}, \mathcal{H}^1\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \otimes \mathcal{I}) \end{array} \tag{7.13}$$

associated with a central extension (6.3) and a representation $(\mathcal{V}_{\Lambda'}, e_{\Lambda'})$ of Q over $X' = \mathrm{Spec}\,(\Lambda')$. Here we assume that $h^{NC}_B|_{a\mathcal{N}_{d-1}}$ is represented by $A \in \mathcal{N}_{d-1}$, and that $\Lambda \in a\mathcal{N}_{d-1}$ and (V_Λ, e_Λ) is a Q-representation over $X = \mathrm{Spec}\,(\Lambda)$ corresponding to a homomorphism $f : A \to \Lambda$. Also, $(\mathcal{V}_{\Lambda'}, e_{\Lambda'})$ is a Q-representation over X', extending $(\mathcal{V}_\Lambda, e_\Lambda)$. The right vertical arrow in (7.13) is induced by the KS-map (7.12), and the bottom arrow is the connecting map defined in Sec. 2.4. More precisely, we use here the identification for any quiver representation $(\mathcal{V}, e)$ over X of the automorphism group $Aut(\mathcal{V}, e)$ with the group $\mathbb{H}^0(X, \mathcal{G}(n_\bullet), c)$, where $c \in \mathbb{H}^1(X, \mathcal{G}(n_\bullet) \curvearrowright \mathcal{E}_1(n_\bullet))$ is the class of $(\mathcal{V}, e)$. Also, we use the natural isomorphism

$$\mathbb{H}^1(X, \mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \otimes \mathcal{I}) \tilde{\to} H^0(X, \mathcal{H}^1\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \otimes \mathcal{I}) \tag{7.14}$$

induced by the projection $\mathcal{E}_1(\mathcal{V}^{ab}) \to \mathbb{H}^1\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab})$.

We assume that there is an open covering (U_i) of B and trivializations φ_i^{ab} of $\mathcal{V}^{ab}|_{U_i}$ and the compatible trivializations ψ_i of $\mathcal{V}_\Lambda$ and $V_{\Lambda'}$ over the covering $\widetilde{U}_i = q^{-1}U_i$. Let (g_{ij}, e_i) be the Cech 1-cocycle corresponding to the universal family over Spec(A), so that the corresponding cocycle for $(\mathcal{V}_\Lambda, e_\Lambda)$ is $(f(g_{ij}), f(e_i))$.

By definition of δ_0 (see Sec. 2.4), starting from an automorphism α of $Aut(\mathcal{V}_\Lambda, e_\Lambda)$ we can lift it over $\widetilde{U}_i$ to an automorphism α'_i of $(\mathcal{V}_{\Lambda'}, e_{\Lambda'})$ and then define $\delta_0(\alpha)$ is the class of the Cech 1-cocycle with values in $\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \otimes I$, given by

$$a_{0,ij} = (\alpha'_i)^{-1}\alpha'_j - \mathrm{id}, \quad a_{1,i} = (\alpha'_i)^{-1}e_{i,\Lambda'} - e_{i,\Lambda'}.$$

Calculating as in the proof of Lemma 7.1.8, and recalling that the action of $\mathcal{G}_0(n_\bullet)$ on $\mathcal{E}_1(n_\bullet)$ is given by conjugation, we get

$$a_{0,ij} = \psi_i([\widetilde{u}^{-1}, \widetilde{f(g_{ij})}] - \mathrm{id})\psi_i^{-1} = \psi_i\Delta_f(u^{-1})(f(g_{ij}))f(g_{ij})^{-1}\psi_i^{-1},$$

$$a_{1,i} = \psi_i(\widetilde{u}^{-1}e_{i,\Lambda'}\widetilde{u} - e_{\Lambda'})\psi_i^{-1} = \psi_i\Delta_f(u^{-1})(e_i)\psi_i^{-1},$$

where we extend the derivation $\Delta_f : A \to I$ to matrices with entries in A. Now we note that the image of the class of this Cech 1-cocycle under the isomorphism (7.14) is simply the global section of $\mathcal{H}^1\mathcal{E}_\bullet(\mathcal{V}^{ab}, e^{ab}) \otimes \mathcal{I}$ given by

$$(a_{1,i} \mod \mathrm{im}(d_e)) = KS(\Delta_f(u^{-1})) = -KS(\Delta_f(u)).$$

REFERENCES CITED

[1] Bezrukavnikov, R. and Kaledin, D. (2004). Fedosov quantization in algebraic context. *Mosc. Math. J.*, 4:559–592.

[2] Brasselet, J.-P. (2008). Introduction to toric varieties. *Publicações Matemáticas, IMPA*.

[3] Brion, M. (2012). Representations of quivers. *in Geometric methods in representation theory. I*, pages 103–104.

[4] Connes, A. (1994). *Noncommutative Geometry.* Academic Press.

[5] Cortiñas, G. (2004). The structure of smooth algebras in Kapranov's framework for noncommutative geometry. *J. of Algebra*, 281:679–694.

[6] Cuntz, J. and Quillen, D. (1995). Algebra extensions and nonsingularity. *J. Amer. Math. Soc.*, 200:33–88.

[7] Dyer, B. and Polishchuk, A. (2017). NC-smooth algebroid thickenings for families of vector bundles and quiver representations, arxiv:1710.00243.

[8] Fedosov, B. V. (1994). A simple geometrical construction of deformation quantization. *J. of Diff. Geom.*, 40:213–238.

[9] Giraud, J. (1971). *Cohomologie non abélienne*. Springer-Verlag.

[10] Huybrechts, D. and Lehn, M. (2010). *The Geometry of Moduli Spaces of Sheaves*. Cambridge University Press.

[11] Kapranov, M. (1998). Noncommutative geometry based on commutator expansions. *J. Reine Angew. Math*, 505:73–118.

[12] Kashiwara, M. and Schapira, P. (2006). *Categories and Sheaves*. Springer-Verlag.

[13] Kashiwara, M. and Schapira, P. (2012). Deformation quantization modules. *Astérisque*, 345:xii+147.

[14] King, A. (1994). Moduli of representations of finite dimensional algebras. *Quarterly J. of Math*, 45:515–530.

[15] Kontsevich, M. (2001). Deformation quantization of algebraic varieties. *Lett. Math. Phys*, 56:271294.

[16] Kontsevich, M. and Rosenberg, A. (2000). Noncommutative smooth spaces. *The Gelfand Mathematical Seminars 1996-1999*, pages 85–108.

[17] Le Bruyn, L. and Van de Weyer, G. (2002). Formal structures and representation spaces. *J. of Alg.*, 247:616–635.

[18] Manin, Y. (1991). *Gauge Field Theory and Complex Geometry*. Springer-Verlag.

[19] Orem, H. (2014). Formal geometry for noncommutative manifolds, arxiv:1408.0830.

[20] Polishchuk, A. and Tu, J. (2014). DG-resolutions of NC-smooth thickenings and NC-Fourier-Mukai transforms. *Mathematische Annalen*, 360:79–156.

[21] Reyes, M. (2012). Obstructing extensions of the functor Spec to noncommutative rings. *Israel J. Math.*, 192:667–698.

[22] Schlessinger, M. (1968). Functors of Artin rings. *Trans. AMS*, 130:208–222.

[23] Toda, Y. (2017). Non-commutative thickening of moduli spaces of stable sheaves. *Compositio Math.*, 153:1153–1195.

[24] Vale, R. (2008). On the opposite category of rings, arxiv:0806.1476.

[25] Van Oystaeyen, F. (2000). *Algebraic Geometry for Associative Algebras*. CRC Press.

[26] Yu, S. (2015). Todd class via homotopy perturbation theory, arXiv:1510.07936.