INFINITE DIMENSIONAL VERSIONS OF THE SCHUR-HORN THEOREM

by

JOHN JASPER

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Student: John D. Jasper

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Marcin Bownik	Chair
N. Christopher Phillips	Member
Yuan Xu	Member
David Levin	Member
Dietrich Belitz	Outside Member

and

Richard Linton	Vice President for Research and Graduate Studies/Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

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We characterize the diagonals of four classes of self-adjoint operators on infinite dimensional Hilbert spaces. These results are motivated by the classical Schur-Horn theorem, which characterizes the diagonals of self-adjoint matrices on finite dimensional Hilbert spaces.

In Chapters II and III we present some known results. First, we generalize the Schur-Horn theorem to finite rank operators. Next, we state Kadison's theorem, which gives a simple necessary and sufficient condition for a sequence to be the diagonal of a projection. We present a new constructive proof of the sufficiency direction of Kadison's theorem, which is referred to as the Carpenter's Theorem.

Our first original Schur-Horn type theorem is presented in Chapter IV. We look at operators with three points in the spectrum and obtain a characterization of the diagonals analogous to Kadison's result.

In the final two chapters we investigate a Schur-Horn type problem motivated by a problem in frame theory. In Chapter V we look at the connection between frames and diagonals of locally invertible operators. Finally, in Chapter VI we give a characterization of the diagonals of locally invertible operators, which

in turn gives a characterization of the sequences which arise as the norms of frames with specified frame bounds.

This dissertation includes previously published co-authored material.

CURRICULUM VITAE

NAME OF AUTHOR: John D. Jasper

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2011, Univeristy of Oregon
Master of Science, Mathematics, 2007, University of Oregon
Bachelor of Science, Mathematics, 2003, Western Washington University

AREAS OF SPECIAL INTEREST:

Functional Analysis, Operator Theory, Frame Theory

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, Mathematics, 2005 to present

GRANTS, AWARDS, AND HONORS:

Johnson Research Fellowship, University of Oregon Department of Mathematics, 2009

Borsting Graduate Student Award, University of Oregon Department of Mathematics, 2010

PUBLICATIONS:

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CHAPTER I

INTRODUCTION

The classical Schur-Horn Theorem gives a necessary and sufficient condition in terms of linear inequalities for a sequence $\{d_i\}_{i=1}^N$ to be the diagonal of a self-adjoint matrix with eigenvalues $\{\lambda_i\}_{i=1}^N$ counting multiplicity. More precisely, if we let $\mathcal{U}$ be the set of unitary operators on $\mathcal{H}$ and let $\{e_i\}_{i=1}^N$ be a fixed orthonormal basis, then the Schur-Horn Theorem gives a characterization of the set of diagonals of the unitary orbit of E, which is denoted

(I.0.1) $$\mathcal{D}(E) := \{\{\langle UEU^*e_i, e_i\rangle\}_{i=1}^N : U \in \mathcal{U}\}.$$

In this dissertation we will give a complete characterization of the set $\mathcal{D}(E)$ for four distinct classes of operators.

Chapters II and III present previously known results. In Chapter II we present a proof of the Schur-Horn Theorem for finite rank operators, including a proof of the classical finite dimensional case. Though the proofs in this section are new, the results are contained in work of Gohberg and Markus [13] and Arveson and Kadison [5].

Chapter III deals with projections on infinite dimensional, not necessarily separable spaces. In 2002 Kadison [16, 17] gave a complete characterization of the set of diagonals of projections. This amazing theorem gives a simple condition on a sequence that is necessary and sufficient for it to be the diagonal of a projection. In the original paper Kadison refers to the necessity of this theorem as the Pythagorean theorem and the sufficiency as the Carpenter's Theorem. Chapter III is devoted to a complete proof of the Carpenter's Theorem. Our proof uses new techniques and yields the real case, which the original does not.

In Chapter IV we give an analogous characterization of the diagonals of self-adjoint operators on separable Hilbert spaces with three eigenvalues. By first scaling and shifting, Kadison's theorem gives a complete characterization of self-adjoint operators with at most two eigenvalues. Thus, it is a natural next step to consider the case of three eigenvalues. The full characterization is quite complex, requiring eight distinct cases depending on the configuration of the multiplicities of the eigenvalues. However, we also present a condensed version where the multiplicities of the eigenvalues are not specified. At the end of Chapter IV we give some examples to demonstrate the use of our characterization. We consider several examples of fixed diagonal sequences and find all possible sets of three points that arise as the spectrum of a self-adjoint operator with the given diagonal.

Several researchers have been motivated by problems in frame theory to look at the Schur-Horn Theorem and problems related to it. In Chapter V we give an example of the connection between frames and diagonals of self-adjoint operators. Specifically, given a frame $\{f_i\}_{i\in I}$ with optimal frame bounds A and B, we consider the problem of characterizing the sequence $\{\|f_i\|\}_{i\in I}$. The purpose of this chapter is to reformulate this problem into the problem of characterizing the diagonals of positive locally invertible operators, that is, self-adjoint operators E such that $\{A,B\}\subseteq\sigma(E)\subseteq\{0\}\cup[A,B]$. Chapter VI gives the characterization of the diagonals of such operators, which in turn gives an answer to the problem in frame theory. Distinct from the previous cases, in this chapter we do not characterize $\mathcal{D}(E)$ for each locally invertible operator E. Instead, we give a characterization of $\mathcal{D}(E)$ for all self-adjoint operators E with $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ for fixed A and B.

Chapters III, V and VI contain co-authored material, which has been accepted for publication and will appear in *Journal für die reine und angewandte Mathematik*.

CHAPTER II

FINITE RANK OPERATORS

II.1. Finite dimensional Hilbert spaces

The classical Schur-Horn Theorem gives a characterization of the set of diagonals of self-adjoint operators with prescribed eigenvalues and multiplicities. This characterization comes in the form of a set of linear inequalities (II.1.1) which are known as majorization.

Theorem II.1.1 (Schur-Horn Theorem)**.** *Let $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^N$ be real sequences with nonincreasing order. If*

$$\begin{aligned} &\sum_{i=1}^{n} d_i \leq \sum_{i=1}^{n} \lambda_i \quad \textit{for } n = 1, \ldots, N, \\ &\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} d_i, \end{aligned} \tag{II.1.1}$$

then there is a self-adjoint operator $E : \mathbb{R}^N \to \mathbb{R}^N$ with eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$.

Conversely, if $E : \mathbb{C}^N \to \mathbb{C}^N$ is a self-adjoint operator with eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$, then (II.1.1) *holds.*

The theorem states that a sequence is the diagonal of a self-adjoint matrix if and only if it is majorized by the eigenvalue sequence. The necessity of (II.1.1) was shown by Schur in 1923 [21], while sufficiency was shown by Horn in 1955 [15].

The following is a new proof of Horn's direction of the finite dimensional Schur-Horn Theorem.

Theorem II.1.2 (Finite Horn's Theorem)**.** *Let* $\{\lambda_i\}_{i=1}^N$ *and* $\{d_i\}_{i=1}^N$ *be real sequences in nonincreasing order. If* (II.1.1) *holds then there is a self-adjoint operator* $S : \mathbb{R}^N \to \mathbb{R}^N$ *with eigenvalues* $\{\lambda_i\}$ *and diagonal* $\{d_i\}$*.*

Proof. We will show by induction on N that S exists. The case of $N = 1$ is trivial, so assume the claim is proved for all integers less than N. Define the number

$$\delta = \min\left\{ \sum_{i=1}^{n} \lambda_i - \sum_{i=1}^{n} d_i : n < N \right\}.$$

If $\delta = 0$, then there is some $m < N$ such that (II.1.1) holds for the sequences $\{d_i\}_{i=1}^m$ and $\{\lambda_i\}_{i=1}^m$. Clearly, (II.1.1) also holds for $\{d_i\}_{i=m+1}^N$ and $\{\lambda_i\}_{i=m+1}^N$. We use the inductive assumption to find self-adjoint operators $S_1 : \mathbb{R}^m \to \mathbb{R}^m$ with eigenvalues $\{\lambda_i\}_{i=1}^m$ and diagonal $\{d_i\}_{i=1}^m$, and $S_2 : \mathbb{R}^{N-m} \to \mathbb{R}^{N-m}$ with eigenvalues $\{\lambda_i\}_{i=m+1}^N$ and diagonal $\{d_i\}_{i=m+1}^N$. Then $S = S_1 \oplus S_2$ is the desired operator.

If $\delta > 0$ then there is some $m < N$ such that

$$\sum_{i=1}^{m} \lambda_i - \sum_{i=1}^{m} d_i = \delta.$$

Note that

$$\delta \leq \sum_{i=1}^{N-1} \lambda_i - \sum_{i=1}^{N-1} d_i = d_N - \lambda_N,$$

and

$$\delta \leq \lambda_1 - d_1.$$

Now, define the sequence

$$\widetilde{d}_i = \begin{cases} d_1 + \delta & i = 1 \\ d_i & i = 2, \ldots, N-1 \\ d_N - \delta & i = N. \end{cases}$$

By the minimality of δ the sequences $\{\widetilde{d}_i\}_{i=1}^N$ and $\{\lambda_i\}_{i=1}^N$ still satisfy (II.1.1). Now, the sequences $\{\widetilde{d}_i\}_{i=1}^m$ and $\{\lambda_i\}_{i=1}^m$ satisfy (II.1.1). Similarly the sequences $\{\widetilde{d}_i\}_{i=m+1}^N$ and $\{\lambda_i\}_{i=m+1}^N$ still satisfy (II.1.1). Since these pairs of sequences have length less than N we can apply the inductive assumption to obtain $S_1 : \mathbb{R}^m \to \mathbb{R}^m$ with eigenvalues $\{\lambda_i\}_{i=1}^m$ and diagonal $\{\widetilde{d}_i\}_{i=1}^m$, and $S_2 : \mathbb{R}^{N-m} \to \mathbb{R}^{N-m}$ with eigenvalues $\{\lambda_i\}_{i=m+1}^N$ and diagonal $\{\widetilde{d}_i\}_{i=m+1}^N$. As before, we consider the operator $S = S_1 \oplus S_2$. Let $\alpha \in [0,1]$ be such that that $\alpha(d_1+\delta)+(1-\alpha)(d_N-\delta) = d_1$. Define the unitary operator U on the standard orthonormal basis $\{e_i\}_{i=1}^N$ by

$$U(e_i) = \begin{cases} \sqrt{\alpha}e_1 - \sqrt{1-\alpha}e_N & i = 1 \\ \sqrt{1-\alpha}e_i + \sqrt{\alpha}e_N & i = N \\ e_i & \text{otherwise} \end{cases}$$

It is a simple calculation to see that U^*SU has the desired diagonal by noting that $\langle Se_1, e_N\rangle = 0$. □

Next, we present a simple proof of Schur's direction of the Schur-Horn Theorem. The proof also covers the case of positive compact operators.

Theorem II.1.3 (Schur)**.** *If $S : \mathcal{H} \to \mathcal{H}$ is a positive compact operator with eigenvalue list $\{\lambda_i\}_{i=1}^\infty$ in nonincreasing order, then for any orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\mathcal{H}$ we have*

(II.1.2) $$\sum_{i=1}^n \langle Se_i, e_i\rangle \le \sum_{i=1}^n \lambda_i \quad \textit{for all } n \in \mathbb{N}$$

Proof. Let $\{f_i\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$ of eigenvectors with $Sf_i = \lambda_i f_i$ and let $\{e_i\}_{i=1}^\infty$ be any orthonormal basis. Now we calculate

$$\begin{aligned}\sum_{i=1}^{n}\langle Se_i, e_i\rangle &= \sum_{i=1}^{n}\left\langle S\left(\sum_{j=1}^{\infty}\langle e_i, f_j\rangle f_j\right), e_i\right\rangle = \sum_{i=1}^{n}\left\langle \sum_{j=1}^{\infty}\langle e_i, f_j\rangle Sf_j, e_i\right\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{\infty}\lambda_j\langle e_i, f_j\rangle\langle f_j, e_i\rangle = \sum_{j=1}^{\infty}\lambda_j\sum_{i=1}^{n}|\langle e_i, f_j\rangle|^2 = \sum_{j=1}^{\infty}\lambda_j a_j,\end{aligned} \tag{II.1.3}$$

where $a_j = \sum_{i=1}^n |\langle e_i, f_j\rangle|^2$. Note that $\sum_{j=1}^\infty a_j = n$ and $0 \le a_j \le 1$ for all j, independent of choice of basis $\{e_i\}_{i=1}^\infty$. Since $\{\lambda_i\}_{i=1}^\infty$ is in nonincreasing order, the choice of sequence $\{a_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty a_j = n$ and $0 \le a_j \le 1$ for all i, that maximizes the last quantity of (II.1.3) consists of n ones followed by zeros. This yields (II.1.2). □

II.2. Positive finite rank operators

The analogue of the Schur-Horn Theorem for trace class operators was proved by Arveson and Kadison in [5]. It was further generalized to compact operators by Kaftal and Weiss in [18]. The following is a special case of Arveson-Kadison theorem [5, Theorem 4.1] for finite rank operators. Theorem II.2.1 can also be deduced from the Kaftal and Weiss infinite dimensional extension of the Schur-Horn Theorem [18, Theorem 6.1].

Theorem II.2.1 (Kadison, Arveson)**.** *Let $\{\lambda_i\}_{i=1}^N$ be strictly positive and nonincreasing and let $\{d_i\}_{i=1}^\infty$ be nonnegative nonincreasing. There is a positive rank N operator with positive eigenvalues $\{\lambda_i\}$ and diagonal $\{d_i\}$ if and only if*

$$\begin{aligned}&\sum_{i=1}^{n} d_i \le \sum_{i=1}^{n}\lambda_i \quad \textit{for all } n \le N \\ &\sum_{i=1}^{\infty} d_i = \sum_{i=1}^{N}\lambda_i.\end{aligned} \tag{II.2.4}$$

We need only prove that (II.2.4) is sufficient, since Theorem II.1.3 implies that (II.2.4) is necessary. First, we will handle the case of rank one operators with the following lemma.

Lemma II.2.2. *If* $\{d_i\}_{i=1}^{\infty}$ *is a nonnegative sequence with*

$$\sum_{i=1}^{\infty} d_i = \lambda < \infty$$

then there is a positive rank 1 *(or rank* 0 *is* $\lambda = 0$*) operator* S *with eigenvalue* λ *and diagonal* $\{d_i\}$*.*

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Set

$$v = \sum_{i=1}^{\infty} \sqrt{d_i} e_i,$$

and define $S : \mathcal{H} \to \mathcal{H}$ by $Sf = \langle f, v\rangle v$ for each $f \in \mathcal{H}$. Clearly S is rank 1, and since $\|v\|^2 = \lambda$ the vector v is an eigenvector with eigenvalue λ. Finally, it is simple to check that S has the desired diagonal. □

Theorem II.2.3 (Finite Rank Horn's Theorem)**.** *Let* $\{\lambda_i\}_{i=1}^{N}$ *be a strictly positive nonincreasing sequence and let* $\{d_i\}_{i=1}^{\infty}$ *be a nonnegative nonincreasing sequences. If*

$$\begin{aligned} &\sum_{i=1}^{n} d_i \leq \sum_{i=1}^{n} \lambda_i \quad \textit{for all } n \leq N \\ &\sum_{i=1}^{\infty} d_i = \sum_{i=1}^{N} \lambda_i \end{aligned} \tag{II.2.5}$$

Then there is a positive rank N *operator* S *on a real Hilbert space* $\mathcal{H}$ *with eigenvalues* $\{\lambda_i\}_{i=1}^{N}$ *and diagonal* $\{d_i\}_{i=1}^{\infty}$*.*

Proof. Define

$$m_0 = \max\left\{ m : \sum_{i=m}^{\infty} d_i \geq \lambda_N \right\}$$

and

$$\delta = \left(\sum_{i=m_0}^{\infty} d_i \right) - \lambda_N.$$

Note that $m_0 \geq N$ and define $\{\widetilde{\lambda}_i\}_{i=1}^{m_0-1}$ by

$$\widetilde{\lambda}_i = \begin{cases} \lambda_i & i = 1, 2, \ldots, N-1 \\ 0 & i = N, \ldots, m_0 - 1. \end{cases}$$

Note that $d_{m_0} > \delta$, and define the sequence $\{\widetilde{d}_i\}_{i=m_0}^{\infty}$ by

$$\widetilde{d}_i = \begin{cases} d_{m_0} - \delta & i = m_0 \\ d_i & i > m_0. \end{cases}$$

Since

$$\sum_{i=m_0}^{\infty} \widetilde{d}_i = \lambda_N,$$

we can apply Lemma II.2.2 to get a positive, rank 1 operator $\widetilde{S}_2$ with eigenvalue λ_N and diagonal $\{\widetilde{d}_i\}_{i=m_0}^{\infty}$. Now define $\{\widetilde{d}_i\}_{i=1}^{m_0-1}$ by

$$\widetilde{d}_i = \begin{cases} d_i & i < m_0 - 1 \\ d_{m_0-1} + \delta & i = m_0 - 1. \end{cases}$$

Note that

$$\sum_{i=1}^{m_0-1} \widetilde{d}_i = \sum_{i=1}^{m_0-1} \widetilde{\lambda}_i$$

and clearly we have

$$\sum_{i=1}^{n} \widetilde{d}_i \leq \sum_{i=1}^{n} \widetilde{\lambda}_i$$

for all $n = 1, \ldots, m_0 - 2$. Thus, by Theorem II.1.2 there is a positive operator $\widetilde{S}_1$ with diagonal $\{\widetilde{d}_i\}_{i=1}^{m_0-1}$ and eigenvalues $\{\widetilde{\lambda}_i\}_{i=1}^{m_0-1}$. Now, the operator $\widetilde{S} = \widetilde{S}_1 \oplus \widetilde{S}_2$ has the desired eigenvalues, but diagonal $\{\widetilde{d}_i\}_{i=1}^{\infty}$. However, $\{\widetilde{d}_i\}$ only differs from $\{d_i\}$ at $i = m_0 - 1$ and m_0. Let $\alpha \in [0, 1]$ such that $\alpha(d_{m_0-1} + \delta) + (1 - \alpha)(d_{m_0} - \delta) = d_{m_0-1}$.

Define the unitary operator U on the standard orthonormal basis $\{e_i\}_{i=1}^{\infty}$ by

$$U(e_i) = \begin{cases} \sqrt{\alpha}e_{m_0-1} - \sqrt{1-\alpha}e_{m_0} & i = m_0 - 1 \\ \sqrt{1-\alpha}e_{m_0-1} + \sqrt{\alpha}e_{m_0} & i = m_0 \\ e_i & \text{otherwise.} \end{cases}$$

It is a simple calculation to see that $U^*\widetilde{S}U$ has the desired diagonal. This gives an operator with the desired diagonal. $\square$

CHAPTER III

PROJECTIONS

III.1. Statement of Kadison's theorem

In [16] and [17] Kadison gave a complete characterization of the diagonals of projections which are self-adjoint operators with only 0 and 1 as eigenvalues. The goal of this chapter is to prove one direction of Kadison's theorem, namely that the condition (III.1.1) is sufficient. This direction is known as the Carpenter's Theorem. While the theorem is known, our proof is new and has several advantages over the original. First, the original proof does not yield the real case, which ours does. Second, our proof is not existential in that it gives a concrete process for finding the desired projection. The material in Section III.2 is contained in a paper co-authored with Marcin Bownik [7] which has been accepted for publication in *Journal für die reine und angewandte Mathematik*.

We begin with the statement of Kadison's Theorem.

Theorem III.1.1 (Kadison)**.** *Let* $\{d_i\}_{i\in I}$ *be a sequence in* $[0,1]$ *and* $\alpha \in (0,1)$. *Define*

$$a = \sum_{d_i<\alpha} d_i \quad \textit{and} \quad b = \sum_{d_i\geq\alpha}(1-d_i).$$

There is a projection with diagonal $\{d_i\}_{i\in I}$ *if and only if*

$$a - b \in \mathbb{Z} \cup \{\pm\infty\}, \tag{III.1.1}$$

with the convention that $\infty - \infty = 0$.

Remark III.1.2. Observe that in Theorem III.1.1, if there exists a partition $I = I_1 \cup I_2$ such that

$$\sum_{i\in I_1} d_i, \sum_{i\in I_2}(1-d_i) < \infty \quad \text{and} \quad \sum_{i\in I_1} d_i - \sum_{i\in I_2}(1-d_i) \in \mathbb{Z},$$

then we have $a - b \in \mathbb{Z}$ for all $\alpha \in (0,1)$. Thus, the existence of such a partition is also a sufficient condition for a sequence to the be the diagonal of a projection. We will find use for these more general partitions in the sequel.

Remark III.1.3. Note that the indexing set I is not assumed to be countable. In [16, 17] the possibility that I is an uncountable set is addressed in all but the most difficult case where $\{d_i\}$ and $\{B - d_i\}$ are nonsummable [17, Theorem 15]. However, the case where I is uncountable is a simple extension of the countable case, as we will now explain.

Proof of reduction of Theorem III.1.1 to countable case. First, we consider a projection P with diagonal $\{d_i\}_{i\in I}$ with respect to some orthonormal basis $\{e_i\}$. If a or b is infinite then there is nothing to show, so we may assume $a, b < \infty$. Set $J = \{i \in I : d_i = 0\} \cup \{i \in I : d_i = 1\}$, and let P' be the operator P acting on $\overline{\text{span}}\{e_i\}_{i\in I\setminus J}$. Since e_i is an eigenvector for each $i \in J$, P' is a projection with diagonal $\{d_i\}_{i\in I\setminus J}$. The assumption that $a, b < \infty$ implies $I \setminus J$ is at most countable. Thus, the countable case of Theorem VI.3.3 applied to the operator P' yields $a - b \in \mathbb{Z}$. This shows that (III.1.1) is necessary.

To show that (III.1.1) is sufficient, we claim that it is enough to assume that all of the d_i are in $(0,1)$. If we can find a projection P with only these d_i, then we take I

to be the identity and $\mathbf{0}$ the zero operator on Hilbert spaces with dimensions chosen so that $P \oplus I \oplus \mathbf{0}$ has diagonal $\{d_i\}$. Since a and b do not change when we restrict to $(0,1)$, we may assume that $\{d_i\}_{i\in I}$ has uncountably many terms and is contained in $(0,1)$. There is some $n \in \mathbb{N}$ such that $J = \{i \in I : 1/n < d_i < 1-1/n\}$ has the same cardinality as I. Thus, we can partition I into a collection of countable infinite sets $\{I_k\}_{k\in K}$ such that $I_k \cap J$ is infinite for each $k \in K$. Each sequence $\{d_i\}_{i\in I_k}$ contains infinitely many terms bounded away from 0 and 1, thus (III.1.1) holds with a or b infinite. Again, by the countable case of Theorem VI.3.3, for each $k \in K$ there is a projection P_k with diagonal $\{d_i\}_{i\in I_k}$. Thus, $\bigoplus_{k\in K} P_k$ is a projection with diagonal $\{d_i\}_{i\in I}$. □

Theorem III.1.1 gives a characterization of the set of diagonals of all projections without reference to the multiplicities of the eigenvalues. However, given the diagonal $\{d_i\}$ of a projection P, we can recover the multiplicities from the following formulas:

$$\dim\ker P = \sum(1-d_i) \qquad \dim\operatorname{ran} P = \sum d_i.$$

Thus, Kadison's theorem gives a complete characterization of the set of diagonals of the unitary orbit of any single orthogonal projection, and can be considered the Schur-Horn Theorem for operators with two points in the spectrum.

In [16, 17] Kadison refers to the necessity of (III.1.1) as the Pythagorean theorem and the sufficiency as the Carpenter's Theorem. Thus, in the case of projections Horn's theorem is called the Carpenter's Theorem and Schur's theorem is called the Pythagorean theorem. We will adopt this terminology.

The goal of the rest of this chapter is give a new proof of the Carpenter's Theorem.

Theorem III.1.4 (The Carpenter's Theorem)**.** *Let* $\{d_i\}_{i\in I}$ *be a sequence in* $[0,1]$ *and* $\alpha \in (0,1)$. *Define*

$$a = \sum_{d_i<\alpha} d_i \quad \textit{and} \quad b = \sum_{d_i\geq\alpha} (1-d_i).$$

If one of the following holds

(i) $a = \infty$,

(ii) $b = \infty$,

(iii) $a, b < \infty$ *and* $a - b \in \mathbb{Z}$,

then there is a projection P *with diagonal* $\{d_i\}$.

Though there are many cases to consider, in each case we give an explicit construction of a projection with the desired diagonal. This is distinct from Kadison's original proof, which is more existential. Note that our proof also yields the real case, which the original proof does not.

III.2. The $0-1$ Lemma

The following lemma, along with Theorem III.1.1, is the main tool we will use to construct operators with the desired diagonal in this and the remaining chapters. This lemma first appeared in [7].

Lemma III.2.1. *Let* $\{a_i\}_{i=1}^N$ *and* $\{b_i\}_{i=1}^M$ *be sequences in* $[0,B]$ *with* $\max\{a_i\} \leq \min\{b_i\}$. *Let* $\eta_0 \geq 0$ *and*

$$\eta_0 \leq \min\left\{ \sum_{i=1}^N a_i, \sum_{i=1}^M (B - b_i) \right\}.$$

(i) There exist sequences $\{\widetilde{a}_i\}_{i=1}^N$ *and* $\{\widetilde{b}_i\}_{i=1}^M$ *in* $[0, B]$ *satisfying*

$$\text{(III.2.2)} \qquad \widetilde{a}_i \le a_i \quad i = 1, \ldots, N, \quad \textit{and} \quad b_i \le \widetilde{b}_i \quad i = 1, \ldots, M,$$

$$\text{(III.2.3)} \qquad \eta_0 + \sum_{i=1}^N \widetilde{a}_i = \sum_{i=1}^N a_i \quad \textit{and} \quad \eta_0 + \sum_{i=1}^M (B - \widetilde{b}_i) = \sum_{i=1}^M (B - b_i),$$

(ii) Given any sequences $\{\widetilde{a}_i\}$ *and* $\{\widetilde{b}_i\}$ *as in (i) and any finite or infinite bounded sequence of real numbers* $\{c_i\}$*, if there is a self-adjoint operator* $\widetilde{E}$ *on* $\mathcal{H}$ *with diagonal*

$$\{\widetilde{a}_1, \ldots, \widetilde{a}_N, \widetilde{b}_1, \ldots, \widetilde{b}_M, c_1, c_2, \ldots\},$$

there exists an operator E *on* $\mathcal{H}$ *unitarily equivalent to* $\widetilde{E}$ *with diagonal*

$$\{a_1, \ldots, a_N, b_1, \ldots, b_M, c_1, c_2, \ldots\}.$$

Proof. By scaling the sequences, we can reduce Lemma III.2.1 to the case $B = 1$. Set

$$\{a_i^{(0)}\}_{i=1}^N = \{a_i\}_{i=1}^N \quad \text{and} \quad \{b_i^{(0)}\}_{i=1}^M = \{b_i\}_{i=1}^M.$$

Define a series of new sequences by applying the following algorithm:

Step i: If $\eta_{i-1} = 0$ then we are done. Otherwise set

$$a_{n_i}^{(i-1)} = \max\{a_n^{(i-1)}\} \quad \text{and} \quad b_{m_i}^{(i-1)} = \min\{b_m^{(i-1)}\}.$$

Then define

$$\delta_i = \min\{a_{n_i}^{(i-1)}, 1 - b_{m_i}^{(i-1)}, \eta_{i-1}\}.$$

Now define the sequences $\{a_n^{(i)}\}$ and $\{b_m^{(i)}\}$ by

$$a_n^{(i)} = \begin{cases} a_{n_i}^{(i-1)} - \delta_i & n = n_i \\ a_n^{(i-1)} & \text{otherwise} \end{cases}, \quad b_m^{(i)} = \begin{cases} b_{m_i}^{(i-1)} + \delta_i & m = m_i \\ b_m^{(i-1)} & \text{otherwise} \end{cases}$$

Define

$$\eta_i = \eta_{i-1} - \delta_i$$

and proceed to step $i+1$.

We claim that the above algorithm will stop after $K \le N+M-1$ steps. Notice that if $\delta_i = \eta_{i-1}$, then $\eta_i = 0$ and the algorithm stops. So, assume that for each i either $\delta_i = a_{n_i}$ or $\delta_i = 1 - b_{m_i}$. If $\delta_i = a_{n_i}$, then the sequence $\{a_n^{(i)}\}$ will have one more zero than $\{a_n^{(i-1)}\}$. If $\delta_i = 1 - b_{m_i}$, then the sequence $\{b_m^{(i)}\}$ will have one more 1 than $\{b_m^{(i-1)}\}$. If $\{a_n^{(i)}\}$ is a sequence of zeros then the algorithm must have stopped, since $\eta_i \le \sum_{n=1}^{N} a_n^{(i)}$. Similarly, if $\{b_m^{(i)}\}$ is a sequence of ones, then the algorithm must have stopped, since $\eta_i \le \sum_{m=1}^{M}(1 - b_m^{(i)})$. Thus, the algorithm can continue for at most $N+M-1$ steps. Finally, set $\widetilde{a}_i = a_i^{(K)}$ and $\widetilde{b}_j = b_j^{(K)}$ for all i and j. This completes the proof of (i).

Let $\{e_i\}$ be the orthonormal basis with respect to which $\widetilde{E}$ has diagonal

$$\{\widetilde{b}_1, \dots, \widetilde{b}_M, \widetilde{a}_1, \dots, \widetilde{a}_N, c_1, c_2, \dots\}.$$

We may assume $\{\widetilde{b}_1, \dots, \widetilde{b}_M, \widetilde{a}_1, \dots, \widetilde{a}_N\}$ is written in nonincreasing order. Let P be the orthogonal projection onto the finite dimensional Hilbert space $\mathcal{H}_0 = \operatorname{span}\{e_i\}_{i=1}^{N+M}$, and let $\widetilde{E}_0 : \mathcal{H}_0 \to \mathcal{H}_0$ be the operator $P\widetilde{E}$ restricted to $\mathcal{H}_0$. In other words, $\widetilde{E}_0$ is the $(N+M)\times(N+M)$ corner of $\widetilde{E}$ with diagonal $\{\widetilde{b}_1, \dots, \widetilde{b}_M, \widetilde{a}_1, \dots, \widetilde{a}_N\}$.

Let $\{\lambda_i\}_{i=1}^{N+M}$ be the eigenvalues of $\widetilde{E}_0$, written in nonincreasing order. By Theorem II.1.2 we have the majorization property (II.1.1) for the diagonal of $\widetilde{E}_0$ and $\{\lambda_i\}$. Using (III.2.2) and (III.2.3) yields

$$\sum_{i=1}^{k} b_i \le \sum_{i=1}^{k} \widetilde{b}_i \qquad \text{for } k = 1, \dots, M,$$

$$\sum_{i=1}^{M} b_i + \sum_{i=1}^{k} a_i = \sum_{i=1}^{M} \widetilde{b}_i - \eta_0 + \sum_{i=1}^{k} a_i \le \sum_{i=1}^{M} \widetilde{b}_i + \sum_{i=1}^{k} \widetilde{a}_i \quad \text{for } k = 1, \dots, N.$$

This shows that the majorization property also holds for $\{b_1, \ldots, b_M, a_1, \ldots, a_N\}$ and $\{\lambda_i\}$. By Theorem II.1.2 there is an operator $E_0 : \mathcal{H}_0 \to \mathcal{H}_0$ with diagonal $\{b_1, \ldots, b_M, a_1, \ldots, a_N\}$ and eigenvalues $\{\lambda_i\}$, and thus there is a unitary $U_0 : \mathcal{H}_0 \to \mathcal{H}_0$ such that $E_0 = U_0^* \widetilde{E}_0 U_0$.

Define the unitary $U = U_0 \oplus I$, where I is the identity operator on $\overline{\operatorname{span}}\{e_i\}_{i>N+M}$. Hence, the operator $E = U^* \widetilde{E} U$ has diagonal

$$\{a_1, \ldots, a_N, b_1, \ldots, b_M, c_1, c_2, \ldots\}.$$

□

III.3. The Carpenter's Theorem part i

The goal of this section is to give a proof of the case of the Carpenter's Theorem where $a < \infty$, $b < \infty$ and $a - b \in \mathbb{Z}$.

As a corollary of Theorem II.2.3 we have the summable versions of the Pythagorean and Carpenter's Theorem.

Theorem III.3.1. *Let $M \in \mathbb{N} \cup \{\infty\}$ and $\{d_i\}_{i=1}^{M}$ a summable sequence in $[0, 1]$. There is a projection P with diagonal $\{d_i\}$ if and only if $\sum_{i=1}^{M} d_i \in \mathbb{N}$.*

Proof. First, assume $\{d_i\}$ is the diagonal of a projection P. We know that

$$\dim \operatorname{ran} P = \sum_{i=1}^{M} d_i.$$

Since $\{d_i\}$ is summable, this implies that P has finite dimensional range, and thus $\sum d_i \in \mathbb{N}$.

Next, assume $\sum d_i = N \in \mathbb{N}$. Define the sequence $\{\lambda_i\}_{i=1}^{M}$ by

$$\lambda_i = \begin{cases} 1 & i = 1, \ldots, N \\ 0 & i > N. \end{cases}$$

Since $\{d_i\}$ is summable, by reindexing we may assume $\{d_i\}_{i=1}^N$ consists of the N largest terms of $\{d_i\}$ in nonincreasing order. Since $d_i \le 1$ for all i we have

$$\sum_{i=1}^n d_i \le \sum_{i=1}^n \lambda_i \qquad \text{for } n = 1, 2, \ldots, N. \tag{III.3.4}$$

We also have

$$\sum_{i=1}^M d_i = N = \sum_{i=1}^M \lambda_i,$$

which implies (III.3.4) holds for all $n \in \mathbb{N}$. By Theorem II.2.3 (or Theorem II.1.2 if $M < \infty$) there is a self-adjoint operator P with eigenvalues $\{\lambda_i\}_{i=1}^M$ and diagonal $\{d_i\}_{i=1}^M$. Since $\lambda_i = 0$ or 1 the operator P is a projection. □

Corollary III.3.2. *Let $M \in \mathbb{N} \cup \{\infty\}$ and $\{d_i\}_{i=1}^M$ be a sequence in $[0, 1]$ such that $\{1 - d_i\}$ is summable. There is a projection P with diagonal $\{d_i\}$ if and only if $\sum_{i=1}^M (1 - d_i) \in \mathbb{N}$.*

Proof. This follows immediately from the observation that a projection P has diagonal $\{d_i\}$ if and only if $I - P$ is a projection with diagonal $\{1 - d_i\}$. □

Next, we handle the case that

$$a = \sum_{d_i < 1/2} d_i, b = \sum_{d_i \ge 1/2} (1 - d_i) < \infty.$$

Proposition III.3.3. *Let $\{d_i\}_{i \in I}$ be a sequence in $[0, 1]$. If $a, b < \infty$ and*

$$\sum_{d_i < 1/2} d_i - \sum_{d_i \ge 1/2} (1 - d_i) = k \in \mathbb{Z} \tag{III.3.5}$$

then there exists a projection P with diagonal $\{d_i\}$.

Proof. First, note that if $\{d_i\}$ or $\{1 - d_i\}$ summable then from (III.3.5) we see that the sum is in $\mathbb{N}$ and thus we can appeal to Theorem III.3.1 or Corollary III.3.2 to see that the desired projection exists. Thus, we may assume both 0 and 1 are limit points of the sequence $\{d_i\}$.

Next, we claim that it is enough to prove the theorem under the assumption that $d_i \in (0,1)$ for all i. Indeed, if P is a projection with diagonal $\{d_i\}_{d_i \in (0,1)}$, I is the identity operator on a space of dimension $|\{i\colon d_i = 1\}|$ and $\mathbf{0}$ is the zero operator on a space of dimension $|\{i\colon d_i = 0\}|$ then $P \oplus I \oplus \mathbf{0}$ is a projection with diagonal $\{d_i\}$.

Define $I_1 = \{i \in I : d_i < 1/2\}$ and $I_2 = \{i \in I : d_i \geq 1/2\}$. Choose $i_2 \in I_2$ such that $d_{i_2} \leq d_i$ for all $i \in I_2$. Choose $J_1 \subseteq I_1$ such that $I_1 \setminus J_1$ is finite and

$$\sum_{i \in J_1} d_i < 1 - d_{i_2}.$$

Let $i_1 \in I_2$ be such that $d_{i_1} > d_{i_2}$ and

$$d_{i_1} + \sum_{i \in J_1} d_i \geq 1.$$

Set

$$\eta_0 = d_{i_1} + \sum_{i \in J_1} d_i - 1 < \sum_{i \in J_1} d_i.$$

Let $F_1 \subset J_1$ be a finite set such that

$$\sum_{i \in F_1} d_i > \eta_0.$$

Also, note that $1 - d_{i_2} > \eta_0$, so that we may apply Lemma III.2.1 to the sequences $\{d_i\}_{i \in F_1}$ and $\{d_i\}_{i = i_2}$ to obtain sequences $\{\widetilde{d}_i\}_{i \in F_1}$ and $\{\widetilde{d}_i\}_{i = i_2}$ such that

$$\sum_{i \in F_1} \widetilde{d}_i = \sum_{i \in F_1} d_i - \eta_0 \qquad \text{and} \qquad 1 - \widetilde{d}_{i_2} = 1 - d_{i_2} - \eta_0.$$

Set $\widetilde{d}_i = d_i$ for $i \in I \setminus (F_1 \cup \{i_2\})$. Note that

$$\sum_{i\in J_1\cup\{i_1\}} \widetilde{d}_i = d_{i_1} + \sum_{i\in J_1\setminus F_1} d_i + \sum_{i\in F_1} \widetilde{d}_i = d_{i_1} + \sum_{i\in J_1\setminus F_1} d_i + \sum_{i\in F_1} d_i - \eta_0 = 1.$$

By Theorem III.3.1 there is a projection P_1 with diagonal $\{\widetilde{d}_i\}_{i\in J_1\cup\{i_1\}}$.

Next, we note that

$$\sum_{i\in I_1\setminus J_1}(1-\widetilde{d}_i) + \sum_{i\in I_2\setminus\{i_1\}}(1-\widetilde{d}_i) = |I_1\setminus J_1| - \sum_{i\in I_1\setminus J_1} d_i + \sum_{i\in I_2\setminus\{i_1\}}(1-d_i) - \eta_0$$

$$= |I_1\setminus J_1| - \sum_{i\in I_1} d_i + \sum_{i\in I_2}(1-d_i) = |I_1\setminus J_1| - k \in \mathbb{N}.$$

By Corollary III.3.2 there is a projection P_2 with diagonal $\{\widetilde{d}_i\}_{i\in I\setminus(J_1\cup\{i_1\})}$.

The projection $P_1 \oplus P_2$ has diagonal $\{\widetilde{d}_i\}_{i\in\mathbb{N}}$. By Lemma III.2.1 part (ii) there is an operator P with diagonal $\{d_i\}$ which is unitarily equivalent to $P_1 \oplus P_2$ and is thus a projection. □

III.4. The algorithm

In this section we introduce a new technique for finding a projection with prescribed diagonal. The main result of this section (Theorem III.4.3) may be thought of as a generalization of Lemma II.2.2. As in that lemma, given a sequence $\{d_i\}$ we produce an orthonormal set $\{v_i\}$ such that the projection onto $\overline{\text{span}}\{v_i\}$ is an orthogonal projection with the desired diagonal.

In order to find the vectors $\{v_i\}$ we must make a technical assumption on the sequence $\{d_i\}$ involving the order of the terms of the sequence. Not every sequence will satisfy this condition. However, in Lemma III.4.2 we show that some subsequence will. Fortunately, we can arrange it so that the subsequence contains a prescribed term of the original sequence. In the next section we apply Theorem III.4.3 countably many times to obtain a projection with diagonal consisting of the full sequence $\{d_i\}$.

Lemma III.4.1. *Let $\sigma, d_1, d_2 \in [0,1]$. If $\max\{d_1, d_2\} \le \sigma$ and $\sigma \le d_1 + d_2$ then there exists a number $a \in [0,1]$ such that the matrix*

$$\begin{pmatrix} a & \sigma - a \\ d_1 - a & d_2 - \sigma + a \end{pmatrix} \tag{III.4.6}$$

has entries in $[0,1]$ and

$$a(d_1 - a) = (\sigma - a)(d_2 - \sigma + a). \tag{III.4.7}$$

Moreover, if $d_1 + d_2 < 2\sigma$ then a is unique and given by

$$a = \frac{\sigma(\sigma - d_2)}{2\sigma - d_1 - d_2}. \tag{III.4.8}$$

Proof. First, assume $\max\{d_1, d_2\} \le \sigma$ and $\sigma \le d_1 + d_2$. If $d_1 = d_2 = \sigma$ then any $a \in [0, \sigma]$ will satisfy (III.4.7) and the matrix (III.4.6) will have entries in [0,1]. Thus, we may additionally assume $d_1 + d_2 < 2\sigma$. Since the quadratic terms in (III.4.7) cancel out, the equation is linear and the unique solution is given by (III.4.8). It remains to show that the entries of the matrix in (III.4.6) are in $[0,1]$. It is clear that $a \ge 0$. Next, we calculate

$$\sigma - a = \sigma\left(1 - \frac{\sigma - d_2}{2\sigma - d_1 - d_2}\right) = \frac{\sigma(\sigma - d_1)}{2\sigma - d_1 - d_2}, \tag{III.4.9}$$

which implies that $\sigma - a \ge 0$ or $\sigma \ge a$. Since $\sigma \le 1$ we clearly have $a, \sigma - a \in [0,1]$. Since $d_1 + d_2 \in [\sigma, 2\sigma)$ we have

$$(d_1 - a) + (d_2 - \sigma + a) = d_1 + d_2 - \sigma \in [0, \sigma).$$

If $d_1 - a = d_2 - \sigma + a = 0$ then the proof is complete. If one of $d_1 - a$ and $d_2 - \sigma + a$ is negative then the other must be strictly positive. From (III.4.7) we see that $a = \sigma - a = 0$ and thus $\sigma = 0$. Since $2\sigma > d_1 + d_2 \geq 0$ it is clear that $\sigma \neq 0$. Thus, neither of $d_1 - a$ and $d_2 - \sigma + a$ is negative. $\square$

Lemma III.4.2. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a sequence such that $d_1 \in [0,1)$, $d_i \in [0, \frac{1}{2}]$ for $i \geq 2$ and $\sum_{i=1}^{\infty} d_i = \infty$. There is an injection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(1) = 1$ and for each $n \in \mathbb{N}$ we have*

$$d_{\pi(k_n - 1)} \geq d_{\pi(k_n)} \quad \textit{where} \quad k_n = \min\left\{k : \sum_{i=1}^{k} d_{\pi(i)} \geq n\right\}. \tag{III.4.10}$$

Proof. First, we will define a sequence of bijections $\pi_n : \mathbb{N} \to \mathbb{N}$.

Set

$$m_1 = \min\left\{k : \sum_{j=1}^{k} d_j \geq 1\right\} \tag{III.4.11}$$

and let $\pi_1 : \{1, \ldots, m_1\} \to \{1, \ldots, m_1\}$ be a bijection such that $\pi_1(1) = 1$ and $\{d_{\pi(i)}\}_{i=2}^{m_1}$ is in nonincreasing order. Extend π_1 to a bijection of $\mathbb{N}$ by defining $\pi_1(i) = i$ for $i \notin \{1, \ldots, m_1\}$.

Now, define

$$d_i^{(1)} = d_{\pi_1(i)} \text{ for all } i \in \mathbb{N}$$

and

$$k_1 = \min\left\{k : \sum_{i=1}^{k} d_i^{(1)} \geq 1\right\}.$$

For $n \geq 2$ define the bijection π_n as follows. Set

$$m_n = \min\left\{k : \sum_{j=1}^{k} d_j^{(n-1)} \geq n\right\}. \tag{III.4.12}$$

Let $\pi_n : \{k_{n-1}+1, \ldots, m_n\} \to \{k_{n-1}+1, \ldots, m_n\}$ be a bijection such that $\{d_i^{(n-1)}\}_{i=k_{n-1}+1}^{m_n}$ is in nonincreasing order. Extend π_n to all of $\mathbb{N}$ by defining $\pi_n(i) = i$ for $i \notin \{k_{n-1}+1, \ldots, m_n\}$.

Set

$$d_i^{(n)} = d_{\pi_n(i)}^{(n-1)} \text{ for all } i \in \mathbb{N}$$

and

$$k_n = \min\left\{k : \sum_{i=1}^{k} d_i^{(n)} \geq n\right\}. \tag{III.4.13}$$

Set $m_0 = k_0 = 0$. From (III.4.11) we see $m_1 \geq 2$ since $d_1 < 1$. Using (III.4.12), that $d_i^{(n)} = d_i^{(n-1)}$ for $i \leq k_{n-1}$, and the assumption that $d_i \leq 1/2$ for $i \geq 2$ we see that

$$\sum_{i=1}^{k_{n-1}+1} d_i^{(n)} = \sum_{i=1}^{k_{n-1}-1} d_i^{(n-1)} + d_{k_{n-1}}^{(n)} + d_{k_{n-1}+1}^{(n)} < n-1+d_{k_{n-1}}^{(n)} + d_{k_{n-1}+1}^{(n)} \leq n-1+\frac{1}{2}+\frac{1}{2} = n,$$

which implies that $k_n \geq k_{n-1}+2$.

For $i = k_{n-1}+1, \ldots, k_n$ we define

$$\pi(i) = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_n(i).$$

We claim that π is an injection. Assume $\pi(i) = \pi(j)$. Without loss of generality we have $n, m \in \mathbb{N}$ with $m \geq n$ such that $k_{n-1}+1 \leq i \leq k_n$ and $k_{m-1}+1 \leq j \leq k_m$, and thus

$$\pi_1 \circ \pi_2 \circ \cdots \circ \pi_n(i) = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_m(j).$$

For each $\ell \in \mathbb{N}$ the map π_ℓ is a bijection. This implies that

$$i = \pi_{n+1} \circ \cdots \circ \pi_m(j). \tag{III.4.14}$$

Since $i \leq k_n$, for any $\ell > n$ we have $\pi_\ell(i) = i = \pi_\ell^{-1}(i)$. Applying π_ℓ^{-1} to both sides of (III.4.14) for $\ell = n+1, \ldots, m$ we obtain $i = j$. This shows that π is an injection. It is clear that $\pi(1) = 1$ since $\pi_1(1) = 1$.

Next, note that for $j = k_{n-1}+1, \ldots, k_n$ we have

$$d_j^{(n)} = d_{\pi_n(j)}^{(n-1)} = d_{\pi_{n-1}(\pi_n(j))}^{(n-2)} = \cdots = d_{\pi(j)}.$$

Thus, for $n \geq 2$ the sequences $\{d_{\pi(i)}\}_{i=k_{n-1}+1}^{k_n}$ are nonincreasing. For $n = 1$ we notice that if $k_1 = 2$ then $d_1 > 1/2$, which implies $d_{\pi(k_1-1)} \geq d_{\pi(k_1)}$. If $k_1 \geq 3$ then we also have $d_{\pi(k_1-1)} \geq d_{\pi(k_1)}$ since $\{d_{\pi(i)}\}_{i=2}^{k_1}$ is nonincreasing. Thus, for all n we have

(III.4.15) $$d_{\pi(k_n-1)} \geq d_{\pi(k_n)}.$$

□

Theorem III.4.3. *Let $i_0 \in I$ and let $\{d_i\}_{i\in I}$ be a sequence such that $d_{i_0} \in [0,1)$, $d_i \in [0, \frac{1}{2}]$ for $i \neq i_0$ and $\sum_{i\in I} d_j = \infty$. There exists a subset $J \subset I$ with $i_0 \in J$ and an orthogonal projection P with diagonal $\{d_i\}_{i\in J}$.*

Proof. Since I is a countable set and $\sum d_j = \infty$ we may assume without loss of generality that $I = \mathbb{N}$ and $i_0 = 1$. By Lemma III.4.2 there is an injection $\pi : \mathbb{N} \to \mathbb{N}$ such that $\pi(1) = 1$ and (III.4.10) holds.

For each n set

(III.4.16) $$\sigma_n = n - \sum_{i=1}^{k_n-2} d_{\pi(i)}.$$

From the definition of k_n we see that

(III.4.17) $$\sigma_n = n - \sum_{i=1}^{k_n} d_{\pi(i)} + d_{\pi(k_n-1)} + d_{\pi(k_n)} \leq d_{\pi(k_n-1)} + d_{\pi(k_n)}.$$

From the minimality of k_n and (III.4.10) we see that

$$\sigma_n = n - \sum_{i=1}^{k_n-1} d_{\pi(i)} + d_{\pi(k_n-1)} \geq d_{\pi(k_n-1)} \geq d_{\pi(k_n)},$$

which implies that

$$\sigma_n \geq \max\{d_{\pi(k_n-1)}, d_{\pi(k_n)}\}. \tag{III.4.18}$$

From Lemma III.4.1 for each n there exists $a_n \in [0,1]$ such that the matrix

$$\begin{pmatrix} a_n & \sigma_n - a_n \\ d_{\pi(k_n-1)} - a_n & d_{\pi(k_n)} - \sigma_n + a_n \end{pmatrix}$$

has non-negative entries and

$$a_n(d_{\pi(k_n-1)} - a_n) = (\sigma_n - a_n)(d_{\pi(k_n)} - \sigma_n + a_n). \tag{III.4.19}$$

Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Set

$$v_1 = \sum_{i=1}^{k_1-2} d_{\pi(i)}^{1/2} e_i + a_1^{1/2} e_{k_1-1} - (\sigma_1 - a_1)^{1/2} e_{k_1},$$

and for $n \geq 2$ define

$$\begin{aligned} v_n = {} & (d_{\pi(k_{n-1}-1)} - a_{n-1})^{1/2} e_{k_{n-1}-1} + (d_{\pi(k_{n-1})} - \sigma_{n-1} + a_{n-1})^{1/2} e_{k_{n-1}} \\ & + \sum_{i=k_{n-1}+1}^{k_n-2} d_{\pi(i)}^{1/2} e_i + a_n^{1/2} e_{k_n-1} - (\sigma_n - a_n)^{1/2} e_{k_n}. \end{aligned}$$

It is a simple calculation using (III.4.19) to check that the $\{v_i\}$ is an orthogonal set. From (III.4.16) we see that for $n \geq 2$

$$\begin{aligned}\|v_n\|^2 &= d_{\pi(k_{n-1}-1)} - a_{n-1} + d_{\pi(k_{n-1})} - \sigma_{n-1} + a_{n-1} + \sum_{i=k_{n-1}+1}^{k_n-2} d_{\pi(i)} + a_n + \sigma_n - a_n \\ &= \sum_{i=k_{n-1}-1}^{k_n-2} d_{\pi(i)} + \sigma_n - \sigma_{n-1} \\ &= \sum_{i=k_{n-1}-1}^{k_n-2} d_{\pi(i)} + \left(n - \sum_{i=1}^{k_n-2} d_{\pi(i)}\right) - \left(n - 1 - \sum_{i=1}^{k_{n-1}-2} d_{\pi(i)}\right) = 1.\end{aligned}$$

A similar calculation shows that $\|v_1\| = 1$. Define the projection P by

$$Pf = \sum \langle f, v_i \rangle v_i.$$

We see from the definition of v_n that $\langle Pe_i, e_i \rangle = d_{\pi(i)}$ for each $i \in \mathbb{N}$. Finally, we set $J = \pi(\mathbb{N})$. □

III.5. The Carpenter's Theorem part ii

In this section we will finish the proof of the Carpenter's Theorem by proving the final case where one of the quantities a and b defined in Theorem III.1.1 is infinite.

Proposition III.5.1. *If $\{d_i\}_{i\in\mathbb{N}}$ is a sequence in $[0,1]$ such that*

$$a = \sum_{d_i < 1/2} d_i = \infty \qquad \text{or} \qquad b = \sum_{d_i \geq 1/2} (1 - d_i) = \infty$$

then there is a projection P with diagonal $\{d_i\}$.

Proof. Set

$$I_0 = \{i : d_i \leq 1/2\} \quad \text{and} \quad I_1 = \{i : d_i > 1/2\}.$$

It is clear that either

$$a' = \sum_{i \in I_0} d_i = \infty$$

or $b = \infty$.

Case 1: Assume $a' = \infty$ and $|I_1| \leq 1$. If $I_1 \neq \varnothing$, then after reordering we can assume $1 \in I_1$.

We will show by induction that for each $n \in \mathbb{N}$ there is a set $L_n \subset \mathbb{N}$ and a projection P_n with the following four properties:

$$L_j \cap L_k = \varnothing \quad \text{for } j \neq k \tag{III.5.20}$$

$$P_n \text{ has diagonal } \{d_i\}_{i \in L_n} \tag{III.5.21}$$

$$\{1, 2, \ldots, n\} \subset \bigcup_{j=1}^{n} L_j \tag{III.5.22}$$

$$\sum_{i \in \mathbb{N} \setminus \bigcup_{j=1}^{n} L_j} d_i = \infty. \tag{III.5.23}$$

First, we will show that L_1 exists. Partition I into two sets J_1 and K_1 such that

$$\sum_{i \in J_1} d_i = \sum_{i \in K_1} d_i = \infty,$$

and $1 \in J_1$. Theorem III.4.3 implies that there is a subset $L_1 \subset J_1$ with $1 \in L_1$ and a projection P_1 with diagonal $\{d_i\}_{i \in L_1}$. Note that

$$\sum_{i \in \mathbb{N} \setminus L_1} d_i = \sum_{i \in J_1 \setminus L_1} d_i + \sum_{i \in K_1} d_i = \infty.$$

Next, assume we have the sets $L_1, \ldots, L_{n-1}$ and projections $P_1, P_2, \ldots, P_{n-1}$ such that (III.5.20), (III.5.21), (III.5.22) and (III.5.23) hold. Let i_0 be the smallest number in $\mathbb{N} \setminus \bigcup_{j=1}^{n-1} L_j$. From (III.5.22) we see that $i_0 \geq n$. Partition $\mathbb{N} \setminus \bigcup_{j=1}^{n-1} L_j$ into two sets J_n and K_n such that

$$\sum_{i\in J_n} d_i = \sum_{i\in K_n} d_i = \infty,$$

and $i_0 \in J_n$. Theorem III.4.3 implies that there is a subset $L_n \subset J_n$ with $i_0 \in L_n$ and a projection P_n with diagonal $\{d_i\}_{i\in L_n}$. Finally, note that

$$\sum_{i\in\mathbb{N}\setminus\bigcup_{j=1}^{n} L_j} d_i = \sum_{i\in J_n\setminus L_n} d_i + \sum_{i\in K_n} d_i = \infty.$$

This completes the induction and shows the existence of the sets $\{L_n\}_{n=1}^{\infty}$ and the projctions $\{P_n\}_{n=1}^{\infty}$ satisfying (III.5.20)–(III.5.23).

From (III.5.20) and (III.5.22) we see that $\{L_j\}_{j=1}^{\infty}$ is a partition of $\mathbb{N}$. Thus by (III.5.21)

$$P = \bigoplus_{j=1}^{\infty} P_j$$

has diagonal $\{d_i\}_{i\in\mathbb{N}}$. This completes the proof of the first case.

Case 2: Assume $a' = \infty$ and $|I_1| > 1$. We can partition I_0 into $|I_1|$ sets $\{J_j\}_{j\in I_1}$ such that

$$\sum_{i\in J_j} d_i = \infty \qquad \text{for all } j \in I_1.$$

For each $j \in I_1$ set $K_j = J_j \cup \{j\}$. Each sequence $\{d_i\}_{i\in K_j}$ has exactly one term in $[1/2, 1)$ and infinite sum. By Case 1, for each $j \in I_1$ there is a projection P_j with diagonal $\{d_i\}_{i\in K_j}$. Since $I = \bigcup_{j\in I_1} K_j$ the projection

$$P = \bigoplus_{j\in I_1} P_j$$

has the desired diagonal. This completes the proof of Case 2.

Case 3: Assume $b = \infty$. Since

$$b = \sum_{1-d_i\le 1/2} (1 - d_i).$$

by the above argument there is a projection P' with diagonal $\{1 - d_i\}$, and $I - P'$ is a projection with diagonal $\{d_i\}$. $\square$

CHAPTER IV

OPERATORS WITH THREE POINT SPECTRUM

IV.1. Statements of the main theorems

The goal of this chapter is to establish an analogue of the Schur-Horn Theorem for operators with three points in the spectrum. That is, we will give necessary and sufficient conditions for a sequence $\{d_i\}$ to be the diagonal of a self-adjoint operator with eigenvalues $\{0, A, B\}$ with specified (possibly infinite) multiplicities.

This result gives a Schur-Horn Theorem for operators with three points in the spectrum analogous to Kadison's result for orthogonal projections (Theorem III.1.1). However, we would like to emphasize two significant qualitative differences between Kadison's Theorem and our extension to operators with three point spectrum. The necessary and sufficient condition for a sequence to be the diagonal of a projection is a single trace condition, that is, an equation involving sums of diagonal terms. The requirements for a sequence to be the diagonal of an operator with a three point spectrum involve both a trace condition and a majorization inequality.

Also distinct from the case of operators with two point spectrum, it is possible for two non-unitarily equivalent operators with three point spectrum to have the same diagonal. For projections the dimension of the kernel and range (i.e. the multiplicities of 0 and 1) can be recovered from the diagonal. Indeed, if $\{d_i\}$ is the diagonal of a projection P, then

$$\dim \operatorname{ran} P = \sum d_i \quad \text{and} \quad \dim \ker P = \sum (1 - d_i).$$

However, for operators with three point spectrum the multiplicities cannot in general be determined from the diagonal, see Remark IV.4.4.

This leads to two distinct extensions of the Schur-Horn Theorem for operators with three point spectrum. In the case where the multiplicities of eigenvalues are not given we have the following general theorem characterizing diagonals of operators with three point spectrum.

Theorem IV.1.1. *Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$ with $\sum d_i = \sum (B - d_i) = \infty$. Define*

$$C = \sum_{d_i < A} d_i \quad \textit{and} \quad D = \sum_{d_i \geq A} (B - d_i).$$

There is a self-adjoint operator E with diagonal $\{d_i\}_{i \in I}$ and $\sigma(E) = \{0, A, B\}$ if and only if one of the following holds: (i) $C = \infty$, (ii) $D = \infty$, or (iii) $C, D < \infty$ and there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$C - D = NA + kB \tag{IV.1.1}$$

$$C \geq (N + k)A. \tag{IV.1.2}$$

The assumption that $\sum d_i = \sum (B - d_i) = \infty$ is not a true limitation. Indeed, the summable case $\sum d_i < \infty$ requires more restrictive conditions which can be deduced from parts (a) and (b) of Theorem IV.1.3. Theorem IV.1.3 is our second extension of the Schur-Horn Theorem which gives a complete list of characterization conditions of diagonals of operators with prescribed multiplicities. Before we state the full theorem, we need one convenient definition.

Definition IV.1.2. Let E be a bounded operator on a Hilbert space. For $\lambda \in \mathbb{C}$ define

$$m_E(\lambda) = \dim \ker(E - \lambda).$$

Theorem IV.1.3. *Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ in $[0, B]$. Define the sets*

$$I_1 = \{i \in I : d_i < A\},\ I_2 = \{i \in I : d_i \geq A\},$$

$$J_2 = \{i \in I_2 : d_i < (A+B)/2\},\ J_3 = I_2 \setminus J_2$$

and the constants (each possibly infinite)

$$C = \sum_{i \in I_1} d_i, \qquad D = \sum_{i \in I_2} (B - d_i),$$

$$C_1 = \sum_{i \in I_1} (A - d_i),\ C_2 = \sum_{i \in J_2} (d_i - A),\ C_3 = \sum_{i \in J_3} (B - d_i).$$

The following table gives the necessary and sufficient condition for $\{d_i\}$ to be the diagonal of a self-adjoint operator E with $\sigma(E) = \{0, A, B\}$ and the specified multiplicities.

	$m_E(0)$	$m_E(A)$	$m_E(B)$	Condition
(a)	Z	N	K	$\lvert I\rvert = Z+N+K$ $\sum_{i\in I} d_i = NA+KB,\ C \geq (N+K-\lvert I_2\rvert)A$
(b)	∞	N	K	$\lvert I_1\rvert = \infty,$ $\sum_{i\in I} d_i = NA+KB,\ C \geq (N+K-\lvert I_2\rvert)A$
(c)	∞	N	∞	$C+D=\infty$ *or* $C, D<\infty,\ \lvert I_1\rvert=\lvert I_2\rvert=\infty,$ $\exists\, k\in\mathbb{Z}\ C-D=NA+kB,\ C\geq A(N+k)$
(d)	Z	∞	K	$\lvert I\rvert=\infty,\ C_1\leq AZ$ $\sum_{i\in I}(d_i-A) = K(B-A)-ZA$
(e)	Z	∞	∞	$C_1\leq AZ,\ C_2+C_3=\infty$ *or* $\lvert I_1\cup J_2\rvert = \lvert J_3\rvert=\infty,\ C_1\leq AZ,\ C_2, C_3<\infty$ $\exists\, k\in\mathbb{Z},\ C_1-C_2+C_3=(Z-k)A+kB$
(f)	∞	∞	∞	$C+D=\infty$

Note that in the preceding theorem we left out the case where only B has infinite multiplicity and the case where only B has finite multiplicity. However, these two remaining cases follow easily using symmetry arguments by applying parts (b) and (e) to the operator $BI-E$ and the sequence $\{B-d_i\}$. Also, observe that case (a) corresponds to the finite dimensional case and hence it is the classical Schur-Horn Theorem (for operators with three eigenvalues), albeit written in a new form. Finally, in this chapter we only consider the case of separable Hilbert spaces, and thus the indexing set I is always taken to be a countable (possibly finite) set.

The proof of Theorem IV.1.3 breaks into 4 distinct parts. The summable cases (a) and (b) do not require many new techniques since they reduce to the study of trace class operators. In section IV.2 they are relatively easily deduced from Theorem II.2.1. The remaining 3 parts rely heavily on a technique, which was introduced in [7], of "moving" diagonal entries to more favorable configurations, where it is possible to construct required operators. In section IV.3 we deal with the case (f) involving three (or more) eigenvalues of infinite multiplicity. Much more involved combinatorial arguments are needed in section IV.4 to deal with case (c) involving two outer eigenvalues with infinite multiplicities. Finally, in section IV.5 we analyze the cases (d) and (e) where at least one of outer eigenvalues has finite multiplicity. The proofs of the necessity and the sufficiency in these last two cases require even more subtle combinatorial arguments which is partially evidenced by the complicated nature of the characterization conditions.

We finish the chapter by illustrating Theorem IV.1.3 in section IV.6. Given a sequence $\{d_i\}$ in $[0,1]$ we are interested in determining the set of inner eigenvalues A for which there exists a positive operator with spectrum $\{0,A,1\}$ and diagonal $\{d_i\}$. We show that this set is either finite or the full open interval $(0,1)$. Finally, we exhibit a few specific examples of sequences where this set has respectively 0, 1, 3, and 17 elements.

IV.2. Finite rank operators

The following is an application of Theorems II.1.1 and II.2.1, which establishes parts (a) and (b) of Theorem IV.1.3.

Theorem IV.2.1. *Let* $0 < A < B < \infty$*, let* $M \in \mathbb{N} \cup \{\infty\}$ *and let* $\{d_i\}_{i=1}^M$ *be a sequence in* $[0,B]$*. There is a self-adjoint operator* E *with diagonal* $\{d_i\}$*,* $\sigma(E) = \{0,A,B\}$*,* $m_E(A) = N < \infty$*,* $m_E(B) = K < \infty$ *and* $m_E(0) = M - N - K$ *if and only if*

$$\sum_{i=1}^{M} d_i = NA + KB \tag{IV.2.3}$$

amd

(IV.2.4) $$\sum_{d_i<A} d_i \geq (N+K-n_0)A,$$

where $n_0 = |\{i : d_i \geq A\}|$.

Proof. To prove that (IV.2.3) and (IV.2.4) are necessary, assume E is a self-adjoint operator with diagonal $\{d_i\}_{i=1}^M$, $\sigma(E) = \{0, A, B\}$, $m_E(A) = N$, $m_E(B) = K$ and $m_E(0) = M - N - K$. Since E has finite rank it has well defined trace equal to $NA + KB$; this is (IV.2.3). The eigenvalues sequence of E written in nonincreasing order is given by

$$\lambda_i = \begin{cases} B & i = 1, 2, \ldots, K \\ A & i = K+1, \ldots, K+N \\ 0 & i > K+N \end{cases} \cdot$$

Thus, using Theorem II.1.1 (or Theorem II.2.1 if $M = \infty$) we see that

$$\sum_{d_i<A} d_i = NA + KB - \sum_{d_i \geq A} d_i \geq NA + KB - (KB + (n_0 - K)A) = A(N + K - n_0),$$

which is (IV.2.4).

Next, we will show that (IV.2.3) and (IV.2.4) are sufficient for $\{d_i\}$ to be the diagonal of an operator of the specified type. Assume $\{d_i\}_{i=1}^M$ is a sequence such that (IV.2.3) and (IV.2.4) hold for some $N, K \in \mathbb{N}$.

Note that we cannot directly apply Theorems II.1.1 and II.2.1, since not every sequence $\{d_i\}$ can be written in nonincreasing order (e.g. if the sequence has an infinite number of positive terms and some zero terms). However, this does not cause real difficulty as the following argument shows. Assume that (IV.2.3) and (IV.2.4) are sufficient for positive nonincreasing sequences. Let $\{d_i'\}_{i=1}^{M'}$ be the strictly positive terms of $\{d_i\}_{i=1}^M$ in nonincreasing order. Note that $\{d_i'\}_{i=1}^{M'}$ satisfies (IV.2.3) and (IV.2.4). There is a positive operator E' with diagonal $\{d_i'\}$, $\sigma(E) \subseteq \{0, A, B\}$,

$m_{E'}(A) = N, m_{E'}(B) = K$ and $m_{E'}(A) = M' - N - K$. Let $\mathbf{0}$ be the zero operator on a separable Hilbert space with dimension $M - M'$ if $M < \infty$ and infinite dimensional if $M = \infty$. Then $E = E' \oplus \mathbf{0}$ is the desired operator.

To complete the proof, we will show (IV.2.3) and (IV.2.4) are sufficient with the assumption that $\{d_i\}$ is a positive sequence written in nonincreasing order. Define the sequence $\{\lambda_i\}_{i=1}^M$ as above. By Theorem II.1.1 or Theorem II.2.1 it is enough to show that

$$\sum_{i=1}^{m} d_i \leq \sum_{i=1}^{m} \lambda_i \tag{IV.2.5}$$

for all $m \leq M$, since the trace condition is obvious. Note that (IV.2.5) holds for $m \leq K$. For $m > K + N$ we have

$$\sum_{i=1}^{m} d_i \leq \sum_{i=1}^{M} d_i = \sum_{i=1}^{M} \lambda_i,$$

so (IV.2.5) holds for $m > K + N$.

First, we wish to show that (IV.2.5) holds for $m = n_0$. From the above we may assume $K < n_0 \leq K + N$. Using (IV.2.4) we have

$$\sum_{i=1}^{n_0} d_i = NA + KB - \sum_{d_i < A} d_i \leq NA + KB - A(N + K - n_0) = KB + (n_0 - K)A = \sum_{i=1}^{n_0} \lambda_i.$$

Now, if $K < m < n_0$ then we have

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{n_0} d_i - \sum_{i=m+1}^{n_0} d_i \leq \sum_{i=1}^{n_0} \lambda_i - (n_0 - m)A = \sum_{i=1}^{m} \lambda_i.$$

Finally, if $n_0 < m \leq K + N$ then

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{n_0} d_i + \sum_{i=n_0+1}^{m} d_i \leq \sum_{i=1}^{n_0} \lambda_i + (m - n_0)A = \sum_{i=1}^{m} \lambda_i.$$

□

IV.3. Three or more eigenvalues with infinite multiplicity

In this section we will classify the diagonals of operators with exactly three eigenvalues, each with infinite multiplicity. This will yield part (f) of Theorem IV.1.3. We will also show that a sequence with $C+D=\infty$ is the diagonal of a very general class of operators.

Theorem IV.3.1 shows $C, D < \infty$ implies that only 0 and B can have infinite multiplicity. Thus, $C+D=\infty$ is a necessary condition for a sequence to be the diagonal of a self-adjoint operator with at least three infinite multiplicities.

Theorem IV.3.1. *Let $0 < A < B < \infty$ and let E be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\sigma(E) = \{0, A, B\}$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ and set $d_i = \langle Ee_i, e_i\rangle$. Define*

$$C = \sum_{d_i<A} d_i, \quad D = \sum_{d_i \geq A} (B - d_i).$$

If $C, D < \infty$ then $N = m_E(A) < \infty$ and there is some $k \in \mathbb{Z}$ such that

$$C - D = NA + kB, \tag{IV.3.6}$$

$$C \geq (N + k)A. \tag{IV.3.7}$$

Proof. Define the sets $I_1 = \{i : d_i < A\}$ and $I_2 = \{i : d_i \geq A\}$. Let P be the orthogonal projection onto $\ker(E - A)$ and let Q be the projection onto $\ker(E - B)$. Define $p_i = \langle Pe_i, e_i\rangle$ and $q_i = \langle Qe_i, e_i\rangle$, so that $d_i = Ap_i + Bq_i$. Note that $p_i + q_i \leq 1$, and thus

$$(B - A)p_i = (B - A)p_i + Ap_i + Bq_i - d_i = B(p_i + q_i) - d_i \leq B - d_i.$$

Using this we obtain

$$\text{(IV.3.8)} \qquad \sum_{i\in I_2} p_i \le \frac{1}{B-A}\sum_{i\in I_2}(B-d_i) = \frac{D}{B-A} < \infty.$$

Next, we have

$$\text{(IV.3.9)} \qquad \sum_{i\in I_1} p_i = \frac{1}{A}\sum_{i\in I_1} Ap_i \le \frac{1}{A}\sum_{i\in I_1} d_i = \frac{C}{A} < \infty.$$

Together (IV.3.8) and (IV.3.9) show that P has finite trace, and thus

$$N = m_E(A) = \sum_{i\in I} p_i < \infty.$$

Define

$$a = \sum_{i\in I_1} q_i = \frac{1}{B}\sum_{i\in I_1}(d_i - Ap_i) \le \frac{1}{B}\sum_{i\in I_1} d_i = \frac{C}{B} < \infty,$$

and

$$b = \sum_{i\in I_2}(1-q_i) = \frac{1}{B}\sum_{i\in I_2}(B - d_i + Ap_i) \le \frac{D}{B} + \frac{A}{B}\sum_{i\in I_2} p_i.$$

Using (IV.3.8) we see that $b < \infty$. By Theorem III.1.1 there exists $k \in \mathbb{Z}$ such that $a - b = k$.

Now, we calculate

$$\begin{aligned} C - D &= \sum_{i\in I_1}(Ap_i + Bq_i) - \sum_{i\in I_2}(B - Ap_i - Bq_i) \\ &= \sum_{i\in I} Ap_i + B\left(\sum_{i\in I_1} q_i - \sum_{i\in I_2}(1-q_i)\right) = NA + kB, \end{aligned}$$

which shows (IV.3.6).

Finally, we calculate

$$k(B-A)+D = (a-b)(B-A) + \sum_{i\in I_2}(B - Bq_i - Ap_i) \geq bA - bB + bB - \sum_{i\in I_2} Ap_i$$
$$= bA - \sum_{i\in I_2} Ap_i = A\sum_{i\in I_2}(1-p_i-q_i).$$

Together with the fact that $p_i+q_i \leq 1$, this shows $k(B-A)+D \geq 0$, or $kB+D \geq kA$. Combining this with (IV.3.6) gives (IV.3.7). □

Next, we will show that the condition $C + D = \infty$ is sufficient to be the diagonal of any diagonalizable self-adjoint operator with the property that the largest and smallest eigenvalues have infinite multiplicity. In particular, we will prove the following theorem, which will complete the proof of part (f) of Theorem IV.1.3.

Theorem IV.3.2. *Let $\lambda \subset [0,B]$ a countable set with $0, B \in \Lambda$. Set $n_0 = n_B = \infty$, and for each $\lambda \in \Lambda \cap (0,B)$ let $n_\lambda \in \mathbb{N}\cup\{\infty\}$. If $\{d_i\}_{i\in I}$ is a sequence in $[0,B]$ such that for some (and hence all) $\alpha \in (0,B)$ we have*

$$\sum_{d_i<\alpha} d_i + \sum_{d_i\geq\alpha}(B-d_i) = \infty,$$

then there is a positive diagonalizable operator E with eigenvalues Λ and $m_E(\lambda) = n_\lambda$ for each $\lambda \in \Lambda$.

The following lemma will serve as a building block for constructing the operators in Theorem IV.3.2.

Lemma IV.3.3. *Let $0 < A < B < \infty$ and let $\{d_i\}_{i\in I}$ be a sequence in $[0,B]$. Define*

$$C = \sum_{d_i<A} d_i \quad \textit{and} \quad D = \sum_{d_i\geq B}(B-d_i).$$

If $C + D = \infty$ then there is a self-adjoint operator E with $\sigma(E) = \{0, A, B\}$, $m_E(0) = m_E(B) = \infty$, $m_E(A) = 1$, and diagonal $\{d_i\}$.

Proof. Let $I_1 = \{i \in I : d_i < A\}$. Assume $C = \infty$. There exists $i_0 \in I_1$ such that

$$\sum_{d_i \leq d_{i_0}} d_i > A.$$

This implies that

$$\sum_{\substack{d_i \leq d_{i_0} \\ i \neq i_0}} d_i > A - d_{i_0}.$$

Let K_1 be a finite subset of $\{i \in I_1 \setminus \{i_0\} : d_i \leq d_{i_0}\}$ such that

$$\sum_{i \in K_1} d_i > A - d_{i_0}.$$

Apply Lemma III.2.1 (i) to the sequences $\{d_i\}_{i \in K_1}$ and $\{d_i\}_{i=i_0}$ with $\eta_0 = A - d_{i_0}$ to obtain sequences $\{\widetilde{d}_i\}_{i \in K_1}$ and $\{\widetilde{d}_i\}_{i=i_0}$. Note that $\widetilde{d}_{i_0} = A$. Define $\widetilde{d}_i = d_i$ for $i \notin K_1 \cup \{i_0\}$. Note that

$$\sum_{i \in I_1 \setminus \{i_0\}} \widetilde{d}_i = \infty,$$

and Theorem III.1.1 implies there is a projection Q with infinite dimensional kernel and range such that BQ has diagonal $\{\widetilde{d}_i\}_{i \in I \setminus \{i_0\}}$. Let P be the identity on a one-dimensional Hilbert space. The operator $\widetilde{E} = BQ \oplus AP$ has diagonal $\{\widetilde{d}_i\}_{i \in I}$. Finally, by Lemma III.2.1 (ii) we obtain an operator E, unitarily equivalent to E, with diagonal $\{d_i\}_{i \in I}$. This completes the proof of the theorem when $C = \infty$.

Assume $D = \infty$. Define $d_i' = B - d_i$ for each $i \in I$. We have

$$\sum_{d_i' \leq B - A} d_i' = \sum_{d_i \geq A} (B - d_i) = D = \infty.$$

By the previous argument, there is a positive operator E' with diagonal $\{d_i'\}$ and $\sigma(E') = \{0, B - A, B\}$, with 0 and B having infinite multiplicity and $B - A$ having multiplicity 1. Clearly $E = B - E'$ has the desired properties. $\square$

Proof of Theorem IV.3.2. If $\Lambda = \{0, B\}$ then Theorem III.1.1 gives the desired operator. Thus we may assume $|\Lambda| \geq 3$. Set $I_1 = \{i \in I : d_i < \alpha\}$ and $I_2 = \{i : d_i \geq \alpha\}$. Partition I_1 and I_2 into sets $\{I_1^\lambda\}_{\lambda \in \Lambda}$ and $\{I_2^\lambda\}_{\lambda \in \Lambda}$ respectively, such that for each $\lambda \in \Lambda$

$$\sum_{i \in I_1^\lambda} d_i + \sum_{i \in I_2^\lambda} (B - d_i) = \infty.$$

For each $\lambda \in \Lambda \cap (0, B)$ partition I_1^λ and I_2^λ into n_λ sets $\{I_1^{\lambda,n}\}_{n=1}^{n_\lambda}$ and $\{I_2^{\lambda,n}\}_{n=1}^{n_\lambda}$ such that for each $n = 1, 2, \ldots, n_\lambda$ we have

$$\sum_{i \in I_1^{\lambda,n}} d_i + \sum_{i \in I_2^{\lambda,n}} (B - d_i) = \infty.$$

By Lemma IV.3.3, for each $\lambda \in \Lambda \cap (0, B)$ and each $n = 1, 2, \ldots, n_\lambda$ there is an self-adjoint operator $E_{\lambda,n}$ with diagonal $\{d_i\}_{i \in I_1^{\lambda,n} \cup I_2^{\lambda,n}}$ and $\sigma(E_{\lambda,n}) = \{0, \lambda, B\}$ with infinite multiplicity at 0 and B and multiplicity 1 at λ. Finally, set

$$E = \bigoplus_{\lambda \in \Lambda} \bigoplus_{n=1}^{n_\lambda} E_{\lambda,n},$$

and it is clear that E has the desired diagonal and eigenvalues. $\square$

In Theorem IV.3.2 the spectrum of E is the closure of X. To end this section we note that $C + D = \infty$ is a sufficient condition on a sequence to be the diagonal of a self-adjoint operator E with $\sigma(E) = K$ for any compact set $K \subseteq [0, B]$. Simply let Λ be a countable dense subset of K and apply Theorem IV.3.2 with any multiplicities $\{n_\lambda\}_{\lambda \in \Lambda}$. This gives us the following corollary.

Corollary IV.3.4. *Let $K \subset [0, B]$ be a compact set with $0, B \in K$. If $\{d_i\}_{i \in I}$ is a sequence in $[0, B]$ such that for some (and hence all) $\alpha \in (0, B)$ we have*

$$\sum_{d_i < \alpha} d_i + \sum_{d_i \geq \alpha} (B - d_i) = \infty$$

then there is a positive diagonalizable operator E with $\sigma(E) = K$.

The following theorem is part (c) of Theorem IV.1.3, and it is the main result of this section.

Theorem IV.4.1. *Let $0 < A < B < \infty$ and let $\{d_i\}_{i\in I}$ be a sequence in $[0,B]$. Define*

$$C = \sum_{d_i<A} d_i \quad \text{and} \quad D = \sum_{d_i\geq A}(B-d_i).$$

There is a self-adjoint operator E with $\sigma(E) = \{0,A,B\}$, $m_E(0) = m_E(B) = \infty$, $N = m_E(A) < \infty$ and diagonal $\{d_i\}_{i\in I}$ if and only if one of the following holds:

(i) $C + D = \infty$

(ii) $C, D < \infty$, $\sum d_i = \sum(B-d_i) = \infty$, *and there exists $k \in \mathbb{Z}$ such that*

$$C - D = NA + kB \tag{IV.4.10}$$

$$C \geq A(N+k). \tag{IV.4.11}$$

Proof. First, we note that the necessity direction is immediate. Indeed, if (i) fails then we have $C, D < \infty$ and we use Theorem IV.3.1 to deduce (IV.4.10) and (IV.4.11). Moreover, $\{d_i\}$ and $\{B-d_i\}$ are not summable since both E and $B-E$ are positive operators with infinite dimensional range and finite spectrum, and thus they both have infinite trace.

Next, note that Theorem IV.3.2 implies that (i) is sufficient. All that is left to prove is that (ii) is sufficient.

Define $I_1 = \{i : d_i < A\}$ and $I_2 = \{i : d_i \geq A\}$. Since $C, D < \infty$ and $\sum d_i = \sum(B-d_i) = \infty$ it must be the case that $|I_1| = |I_2| = \infty$.

First, assume B is not a limit point of $\{d_i\}_{i\in I}$. Since $D < \infty$ the set $I_2^0 = \{i \in I_2 \colon d_i < B\}$ is finite, so assume it has M elements. Let $L \subseteq I_2 \setminus I_2^0$ be a set with $|k| + 1$ elements and define $K_2 = I_2^0 \cup L$. If we consider the sequence $\{d_i\}_{i\in I_1\cup K_2}$, then we have

$$\sum_{i\in I_1\cup K_2} d_i = C + (M+|k|+1)B - \sum_{i\in K_2}(B-d_i)$$
$$= C + (M+|k|+1)B - D = NA + (M+|k|+k+1)B$$

and

$$\sum_{\substack{i\in I_1\cup K_2\\ d_i<A}} d_i = C \geq (N+k)A = (N+M+|k|+k+1-|K_2|).$$

By Theorem IV.2.1, there is a self-adjoint operator E' with diagonal $\{d_i\}_{i\in I_1\cup K_2}$, $\sigma(E') = \{0, A, B\}$, $m_{E'}(0) = \infty$, $m_{E'}(A) = N$ and $m_{E'}(B) = M+|k|+k+1$. Let I be the identity operator on an infinite dimensional Hilbert space. Then $E = E' \oplus BI$ is the desired operator.

If 0 is not a limit point, then we can use the above argument on the sequence $\{B - d_i\}$ to obtain an operator F with diagonal $\{B - d_i\}$ and eigenvalues $0, B - A$ and B which have multiplicities ∞, N and ∞, respectively. Then $B - F$ is the desired operator. Thus, for the rest of the proof we may assume that 0 and B are limit points of $\{d_i\}$.

Case 1: Assume $k \geq 0$. We have $C = NA + kB + D$, and since B is a limit point of $\{d_i\}$ we have $D > 0$ and thus $C > NA + kB$. There is a finite set $K_1 \subseteq I_1$ such that

$$C' := \sum_{i\in K_1} d_i > NA + kB.$$

Since 0 is a limit point of $\{d_i\}_{i\in I_1}$ and K_1 is finite we have $C' < C$. Define

$$\eta := C' - NA - kB < C - NA - kB = D.$$

There is a finite set $K_2 \subset I_2$ such that

$$\sum_{i\in K_2}(B - d_i) > \eta.$$

The sequences $\{d_i\}_{i\in K_1}$ and $\{d_i\}_{i\in K_2}$ are in $[0,B]$ and satisfy $\max\{d_i\}_{i\in K_1}\leq\min\{d_i\}_{i\in K_2}$, and

$$\eta\leq\max\left\{\sum_{i\in K_1}d_i,\sum_{i\in K_2}(B-d_i)\right\}.$$

Lemma III.2.1 (i) implies there are sequences $\{\widetilde{d}_i\}_{i\in K_1}$ and $\{\widetilde{d}_i\}_{i\in K_2}$ such that $\widetilde{d}_i\leq d_i$ for all $i\in K_1$ and $\widetilde{d}_i\geq d_i$ for all $i\in K_2$, and

$$\sum_{i\in K_1}\widetilde{d}_i=\left(\sum_{i\in K_1}d_i\right)-\eta=NA+kB.$$

Since $\widetilde{d}_i<A$ for all $i\in K_1$ it is clear that (IV.2.4) holds. By Theorem IV.2.1 there is a positive operator $\widetilde{E}_0$ with diagonal $\{\widetilde{d}_i\}_{i\in K_1}$, $\sigma(\widetilde{E}_0)=\{0,A,B\}$, $m_{\widetilde{E}_0}(B)=k$, $m_{\widetilde{E}_0}(A)=N$ and $m_{\widetilde{E}_0}(0)=|K_1|-k-N$. Define $\widetilde{d}_i=d_i$ for $i\notin K_1\cup K_2$, and note

$$\sum_{i\in I_1\setminus K_1}\widetilde{d}_i=\sum_{i\in I_1\setminus K_1}d_i=C-C'=D-\eta$$

and

$$\sum_{i\in I_2}(B-\widetilde{d}_i)=D-\eta.$$

By Theorem III.1.1 there is a projection Q such that BQ has diagonal $\{\widetilde{d}_i\}_{i\in(I_1\setminus K_1)\cup I_2}$. Since $|I_1\setminus K_1|=|K_2|=\infty$ we have $m_Q(1)=m_Q(0)=\infty$. Thus, the operator $\widetilde{E}=E_0\oplus BQ$ has the desired eigenvalues and multiplicities and diagonal $\{\widetilde{d}_i\}_{i\in I}$. Finally, use the second part of Lemma III.2.1 to obtain an operator E, unitarily equivalent to $\widetilde{E}$, with diagonal $\{d_i\}_{i\in I}$. This completes the proof of the first case.

Case 2: Assume $k\leq -N$. We obtain this case by applying Case 1 to the sequence $\{B-d_i\}$, to obtain the operator E_0 with $\sigma(E_0)=\{0,B-A,B\}$, $\dim\ker(E_0)=\dim\ker(B-E_0)=\infty$ and $\dim\ker((B-A)-E_0)=N$. Then $B-E_0$ is the desired operator.

Case 3: Assume $-N < k < 0$ and $C = A(N+k)$. Theorem III.1.1 implies there is a projection P with $N+k$ dimensional range, such that AP has diagonal $\{d_i\}_{i\in I_1}$. Since $|I_1| = \infty$ we also see that P has infinite dimensional kernel.

Next, note that

$$\sum_{i\in I_2}(B-d_i) = D = C - NA - kB = NA + kA - NA - kB = -k(B-A).$$

Theorem III.1.1 implies that there is a projection Q with $-k$ dimensional range, and thus $(B-A)Q$ has diagonal $\{B-d_i\}_{i\in I_2}$. Since $|I_2| = \infty$ we see that Q has infinite dimensional kernel. Finally, the operator $E = AP \oplus (B - (B-A)Q)$ has the desired diagonal and eigenvalues with the desired multiplicities.

Case 4: Assume $-N < k < 0$ and $C > A(N+k)$. Set $\eta = C - (N+k)A < C$. There is a finite set $K_1 \subset I_1$ such that

$$C' = \sum_{i\in K_1} d_i > \eta.$$

Next, note that

$$\eta = C - (N+k)A = NA + kB + D - NA - kA = D + k(B-A) < D.$$

Thus, there is a finite set $K_2 \subset I_2$ such that

$$\sum_{i\in K_2}(B-d_i) > \eta.$$

The sequences $\{d_i\}_{i\in K_1}$ and $\{d_i\}_{i\in K_2}$ are in $[0,B]$ and satisfy $\max\{d_i\}_{i\in K_1} \leq \min\{d_i\}_{i\in K_2}$, and

$$\eta \leq \max\left\{\sum_{i\in K_1} d_i, \sum_{i\in K_2}(B-d_i)\right\}.$$

Lemma III.2.1 implies there are sequences $\{\widetilde{d}_i\}_{i\in K_1}$ and $\{\widetilde{d}_i\}_{i\in K_2}$ such that $\widetilde{d}_i \leq d_i$ for all $i \in K_1$ and $\widetilde{d}_i \geq d_i$ for all $i \in K_2$,

$$\sum_{i\in K_1} \widetilde{d}_i = \sum_{i\in K_1} d_i - \eta \qquad \text{and} \qquad \sum_{i\in K_2}(B - \widetilde{d}_i) = \sum_{i\in K_2}(B - d_i) - \eta.$$

Set $\widetilde{d}_i = d_i$ for $i \in I \setminus (K_1 \cup K_2)$. Then

$$\sum_{i\in I_1} \widetilde{d}_i = \sum_{i\in I_1\setminus K_1} d_i + \sum_{i\in K_1} \widetilde{d}_i = \sum_{i\in I_1\setminus K_1} d_i + \sum_{i\in K_1} d_i - \eta = C - \eta = (N+k)A$$

and

$$\sum_{i\in I_2}(B - \widetilde{d}_i) = \sum_{i\in I_2\setminus K_2}(B - d_i) + \sum_{i\in K_2}(B - d_i) - \eta = D - \eta = -k(B - A).$$

Thus, the sequence $\{\widetilde{d}_i\}_{i\in I}$ satisfies the conditions of Case 3, so there is an operator $\widetilde{E}$ with the desired eigenvalues and multiplicities but with diagonal $\{\widetilde{d}_i\}_{i\in I}$. The second part of Lemma III.2.1 implies there is an operator E, unitarily equivalent to $\widetilde{E}$, but with diagonal $\{d_i\}_{i\in I}$. This completes the final case. $\square$

We are now in a position to prove Theorem IV.1.1. In fact we will prove the following more general theorem.

Theorem IV.4.2. *Let* $0 < A < B < \infty$ *and let* $\{d_i\}_{i\in I}$ *be a sequence in* $[0, B]$*. If there is a self-adjoint operator* E *with diagonal* $\{d_i\}_{i\in I}$ *and* $\sigma(E) = \{0, A, B\}$ *then one of the following holds:*

(i) $C = \infty$

(ii) $D = \infty$

(iii) $C, D < \infty$ *and there exist* $N \in \mathbb{N}$ *and* $k \in \mathbb{Z}$ *such that* (IV.1.1) *and* (IV.1.2) *hold.*

If (i),(ii), or (iii) holds and $\sum d_i = \sum(B - d_i) = \infty$ *then there is a self-adjoint operator* E *with diagonal* $\{d_i\}_{i\in I}$ *and* $\sigma(E) = \{0, A, B\}$*.*

Proof. First, assume that E is a self-adjoint operator with spectrum $\{0, A, B\}$ and diagonal $\{d_i\}$. If either $C = \infty$ or $D = \infty$ then we are done since this is exactly (i) or (ii). If $C, D < \infty$ then Theorem IV.4.1 shows that (IV.1.1) and (IV.1.2) hold and thus (iii) holds.

Next, assume $\{d_i\}$ is a sequence in $[0, B]$. If (i) or (ii) holds then Theorem IV.3.2 shows that there is a self-adjoint operator E with spectrum $\{0, A, B\}$ and diagonal $\{d_i\}$. Finally, if (iii) holds and $\sum d_i = \sum(B - d_i) = \infty$ then Theorem IV.4.1 shows that there is a self-adjoint operator E with spectrum $\{0, A, B\}$ and diagonal $\{d_i\}$. □

Remark IV.4.3. In Theorem IV.1.1 (and Theorem IV.4.2) the assumption that $\sum d_i = \sum(B - d_i) = \infty$ is necessary. Consider the sequence $\{A, 0, 0, \ldots\}$. This is clearly not the diagonal of any operator with spectrum $\{0, A, B\}$ since the operator would be trace class with trace A, and thus $B > A$ cannot be an eigenvalue. However, we have $C = 0$ and $D = B - A$ so that (IV.1.1) and (IV.1.2) hold with $N = 1$ and $k = -1$.

Remark IV.4.4. There exist two non-unitarily equivalent operators with three point spectrum and the same diagonal. Let $0 < A < B$ and let I_n be the identity operator of an n dimensional Hilbert space. From Theorem III.1.1, there is a projection P with infinite dimensional kernel and range such that the diagonal of BP consists of a countable infinite sequence of A's. The operator $BP \oplus AI_n$ has a diagonal consisting of a countable number of A's, however the multiplicity of the eigenvalue A is n.

IV.5. Outer eigenvalue with finite multiplicity

In the last two remaining cases ((d) and (e)) of Theorem IV.1.3 we consider operators with finite dimensional kernel. In these cases, where there is an "outer" eigenvalue with finite multiplicity, we have the following necessary condition.

Theorem IV.5.1. *Let $0 < A < B < \infty$ and let E be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with $\sigma(E) = \{0, A, B\}$ and $m_E(0) < \infty$. Let $\{e_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$ and set $d_i = \langle Ee_i, e_i\rangle$. We have*

$$\sum_{d_i<A}(A - d_i) \le Am_E(0). \tag{IV.5.12}$$

Proof. There exist mutually orthogonal projections P and Q such that $E = AP + BQ$. Note that $I - P - Q$ is a finite rank projection and thus finite trace equal to $m_E(0)$. Set $J_1 = \{i \in I : d_i < A\}$. Then we have

$$\begin{aligned}
\sum_{i\in J_1}(A - d_i) &= \sum_{i\in J_1}\big(A - A\langle Pe_i, e_i\rangle - B\langle Qe_i, e_i\rangle\big) \\
&\le \sum_{i\in J_1}\big(A - A\langle Pe_i, e_i\rangle - A\langle Qe_i, e_i\rangle\big) \\
&= A\left(\sum_{i\in J_1}\big(1 - \langle Pe_i, e_i\rangle - \langle Qe_i, e_i\rangle\big)\right) \\
&\le A\left(\sum_{i\in I}\big(1 - \langle Pe_i, e_i\rangle - \langle Qe_i, e_i\rangle\big)\right) = Am_E(0).
\end{aligned}$$

□

Next, we look at two examples which demonstrate that for operators with finite dimensional kernel the constants C and D do not capture enough information about a sequence in order to tell if it is the diagonal of an operator of the specified type.

Example IV.5.2. Consider the sequence $\{d_i\}$ consisting of $\{1 - i^{-1}\}_{i=1}^{\infty}$ and a countable infinite number of 2's. If $A = 1$ and $B = 2$ then we have $C = \infty$ and $D = 0$. By Theorem IV.5.1 this is not the diagonal of any self-adjoint operator E with $\sigma(E) = \{0, 1, 2\}$ and finite dimensional kernel, since

$$\sum_{d_i<A}(A - d_i) = \sum_{i=1}^{\infty}\frac{1}{i} = \infty.$$

Example IV.5.3. Consider the sequence $\{c_i\}$ consisting of $\{1-2^{-i}\}_{i=1}^{\infty}$ and a countable infinite number of 2's. If $A = 1$ and $B = 2$ then we have $C = \infty$ and $D = 0$. By Theorem III.1.1 there is a projection P with diagonal $\{1-2^{-i}\}_{i=1}^{\infty}$, which clearly has finite dimensional kernel. Let I be the identity operator on an infinite dimensional Hilbert space and set $E = P \oplus 2I$. This operator has diagonal $\{c_i\}$, spectrum $\{0, 1, 2\}$ and finite dimensional kernel. Note that $\{c_i\}$ and $\{d_i\}$ have the same values for C and D, but only $\{c_i\}$ is the diagonal of an operator with spectrum $\{0, 1, 2\}$ and finite dimensional kernel.

Instead of C and D we will use the following terminology from Theorem IV.1.3 in the rest of the section:

$$J_1 = \{i : d_i < A\},\ J_2 = \left\{i : d_i \in \left[A, \frac{A+B}{2}\right)\right\},\ J_3 = \left\{i : d_i \geq \frac{A+B}{2}\right\}$$

$$C_1 = \sum_{i\in J_1}(A - d_i),\ C_2 = \sum_{i\in J_2}(d_i - A),\ C_3 = \sum_{i\in J_3}(B - d_i).$$

Note that for symmetry we use the notation J_1 instead of I_1.

The next theorem shows the necessity of the conditions in part (e) of Theorem IV.1.3.

Theorem IV.5.4. *Let $0 < A < B < \infty$ and let E be a self-adjoint operator with $\sigma(E) = \{0, A, B\}$. If $m_E(0) < \infty$ and $C_2, C_3 < \infty$, then $C_1 < \infty$ and there exist $n, k \in \mathbb{Z}$ such that $n + k = m_E(0)$,*

$$C_1 - C_2 + C_3 = nA + kB \tag{IV.5.13}$$

and

$$C_1 \leq A(n + k). \tag{IV.5.14}$$

Moreover, if $m_E(A) = \infty$ *then* $|J_1 \cup J_2| = \infty$*, and if* $m_E(B) = \infty$ *then* $|J_3| = \infty$*.*

Proof. There exist mutually orthogonal projections P and Q such that $E = AP + BQ$. Let $\{e_i\}_{i\in I}$ be the orthonormal basis such that $d_i = \langle Ee_i, e_i\rangle$, and define $p_i = \langle Pe_i, e_i\rangle$ and $q_i = \langle Qe_i, e_i\rangle$ for each $i \in I$. Since $m_E(0) < \infty$, Theorem IV.5.1 implies that $C_1 < \infty$. Next, we note that

$$\sum_{i\in I}(1 - p_i - q_i) = m_{P+Q}(0) = m_E(0) < \infty. \tag{IV.5.15}$$

Using (IV.5.15) we have

$$\begin{aligned}
&\sum_{i\in J_1\cup J_2} q_i \\
&= \frac{1}{B-A}\left(\sum_{i\in J_1\cup J_2}(A - Ap_i - Aq_i) - \sum_{i\in J_1}(A - Ap_i - Bq_i) + \sum_{i\in J_2}(Bq_i + Ap_i - A)\right) \\
&= \frac{1}{B-A}\left(\sum_{i\in J_1\cup J_2}(A - Ap_i - Aq_i) - C_1 + C_2\right) \le \frac{Am_E(0) - C_1 + C_2}{B-A} < \infty.
\end{aligned}$$

Together with (IV.5.15) this also shows that $\sum_{i\in J_1\cup J_2}(1 - p_i) < \infty$. A similar calculation shows that

$$\sum_{i\in J_3}(1 - q_i), \sum_{i\in J_3} p_i < \infty.$$

By Theorem III.1.1 there exist $n, k \in \mathbb{Z}$ such that

$$\begin{aligned}
n &= \sum_{i\in J_1\cup J_2}(1 - p_i) - \sum_{i\in J_3} p_i \\
k &= \sum_{i\in J_3}(1 - q_i) - \sum_{i\in J_1\cup J_2} q_i.
\end{aligned} \tag{IV.5.16}$$

Now, we calculate

$$\begin{aligned} C_1 - C_2 + C_3 &= \sum_{i\in J_1}(A - Ap_i - Bq_i) - \sum_{i\in J_2}(Ap_i + Bq_i - A) + \sum_{i\in J_3}(B - Ap_i - Bq_i) \\ &= A\sum_{i\in J_1\cup J_2}(1-p_i) - A\sum_{i\in J_3} p_i + B\sum_{i\in J_3}(1-q_i) - B\sum_{i\in J_1\cup J_2} q_i \\ &= nA + kB, \end{aligned}$$

which shows (IV.5.13) holds.

From (IV.5.16) we have

$$n + k = \sum_{i\in I}(1 - p_i - q_i) = m_E(0),$$

and

$$C_1 = \sum_{i\in J_1}(A - Ap_i - Bq_i) \leq \sum_{i\in J_1}(A - Ap_i - Aq_i) = A\sum_{i\in I}(1 - p_i - q_i) = A(n+k).$$

This shows (IV.5.14) holds.

Finally, assume $m_E(A) = \infty$. This implies P has infinite dimensional range, and thus

$$\sum_{i\in I} p_i = m_E(A) = \infty.$$

Since $\sum_{i\in J_3} p_i < \infty$, it must be the case that $\sum_{i\in J_1\cup J_2} p_i = \infty$ and thus $|J_1 \cup J_2| = \infty$. Similarly, assuming Q has infinite dimensional range, we have $\sum_{i\in I} q_i = \infty$. Since $\sum_{i\in J_1\cup J_2} q_i < \infty$ it must be the case that $\sum_{i\in J_3} q_i = \infty$, and thus $|J_3| = \infty$. □

The next theorem shows that the conditions in part (e) of Theorem IV.1.3 are sufficient to construct the desired operator.

Theorem IV.5.5. *Let* $0 < A < B < \infty$*, let* $\{d_i\}_{i\in I}$ *be a sequence in* $[0, B]$ *with* $|J_1 \cup J_2| = \infty$ *and let* $Z \in \mathbb{N}$*. If*

$$C_1 \leq AZ \tag{IV.5.17}$$

and either of the following holds:

(i) $C_2 + C_3 = \infty$

(ii) $C_2, C_3 < \infty$ *and there exists* $n, k \in \mathbb{Z}$ *such that* $Z = n + k$ *and*

$$C_1 - C_2 + C_3 = nA + kB, \tag{IV.5.18}$$

then there is a positive operator E *with* $\sigma(E) = \{0, A, B\}$*,* $m_E(0) = Z$*,* $m_E(A) = \infty$*, and diagonal* $\{d_i\}$*. Moreover, if (i) holds then* $m_E(B) = \infty$*, and if (ii) holds then* $m_E(B) = |J_3| - k$*.*

Proof. Set

$$\eta = AZ - C_1.$$

Case 1: Assume

$$\sum_{i\in J_1} d_i, \sum_{i\in J_2\cup J_3} (B - d_i) > \eta.$$

There are finite subsets $K_1 \subset J_1$ and $K_2 \subset J_2 \cup J_3$ such that

$$\eta \leq \min\left\{\sum_{i\in K_1} d_i, \sum_{i\in K_2} (B - d_i)\right\}.$$

We can apply Lemma III.2.1 to the sequences $\{d_i\}_{i\in K_1}$ and $\{d_i\}_{i\in K_2}$, with $\eta_0 = \eta$, to obtain two sequences $\{\widetilde{d}_i\}_{i\in K_1}$ and $\{\widetilde{d}_i\}_{i\in K_2}$ such that

$$\sum_{i\in K_1} \widetilde{d}_i + \eta = \sum_{i\in K_1} d_i \quad \text{and} \quad \sum_{i\in K_2} (B - \widetilde{d}_i) + \eta = \sum_{i\in K_2} (B - d_i).$$

Setting $\widetilde{d}_i = d_i$ for $i \notin K_1 \cup K_2$ we have

$$\sum_{i\in J_1}(A-\widetilde{d}_i) = \sum_{i\in K_1}(A-\widetilde{d}_i) + \sum_{i\in J_1\setminus K_1}(A-d_i) = |K_1|A - \sum_{i\in K_1}\widetilde{d}_i + \sum_{i\in J_1\setminus K_1}(A-d_i)$$
$$= |K_1|A + \eta - \sum_{i\in K_1} d_i + \sum_{i\in J_1\setminus K_1}(A-d_i) = \eta + \sum_{i\in J_1}(A-d_i)$$
$$= \eta + C_1 = AZ.$$

Theorem III.1.1 implies there is a projection P with Z dimensional kernel such that AP has diagonal $\{\widetilde{d}_i\}_{i\in J_1}$. It is clear that if $|J_1| = \infty$ then $m_P(1) = \infty$.

If (i) holds, that is $C_2 + C_3 = \infty$, then Theorem III.1.1 implies there is a projection Q_1 such that $(B-A)Q_1$ has diagonal $\{\widetilde{d}_i - A\}_{i\in J_2\cup J_3}$. Since

$$\sum_{i\in J_2\cup J_3}(\widetilde{d}_i - A) = \sum_{i\in J_2\cup J_3}\big((B-A)-(\widetilde{d}_i - A)\big) = \infty,$$

we also see that $m_{Q_1}(0) = m_{Q_1}(1) = \infty$. Set $\widetilde{E} = AP \oplus \big((B-A)Q_1 + A\big)$. It is clear that $m_{\widetilde{E}}(0) = Z$, $\sigma(\widetilde{E}) = \{0, A, B\}$, and $m_{\widetilde{E}}(A) = m_{\widetilde{E}}(B) = \infty$. By the second part of Lemma III.2.1 there is an operator E, unitarily equivalent to $\widetilde{E}$, with diagonal $\{d_i\}_{i\in I}$.

If (ii) holds, then using (IV.5.18) we have

$$\sum_{i\in J_2}(\widetilde{d}_i - A) - \sum_{i\in J_3}(B-\widetilde{d}_i) = \eta + \sum_{i\in J_2}(d_i - A) - \sum_{i\in J_3}(B-d_i) = \eta + C_2 - C_3$$
$$= AZ - C_1 + C_1 - An - Bk = -k(B-A).$$

Theorem III.1.1 implies there is a projection Q_2 such that $(B-A)Q_2$ has diagonal $\{\widetilde{d}_i - A\}_{i\in J_2\cup J_3}$. The operator $\widetilde{E} = AP \oplus \big((B-A)Q_2 + A\big)$ has diagonal $\{\widetilde{d}_i\}_{i\in I}$, and it is clear that $m_{\widetilde{E}}(0) = Z$ and $\sigma(\widetilde{E}) = \{0, A, B\}$. Note that if $|J_2| = \infty$ then $m_{Q_2}(0) = \infty$ and we already noted that $|J_1| = \infty$ implies $m_P(1) = \infty$; in either case $m_{\widetilde{E}}(A) = \infty$. If $|J_3| = \infty$ we have $m_{Q_2}(1) = \infty$ and thus $m_{\widetilde{E}}(B) = \infty$. If $|J_3| < \infty$

then we have

$$\sum_{i\in J_2\cup J_3}(\widetilde{d}_i - A) = \sum_{i\in J_2}(\widetilde{d}_i - A) - \sum_{i\in J_3}(B-\widetilde{d}_i) + B|J_3| = B(|J_3|-k),$$

which implies $m_{Q_2}(1) = |J_3| - k$, and thus $m_{\widetilde{E}}(B) = |J_3| - k$. By the second part of Lemma III.2.1 there is an operator E, unitarily equivalent to $\widetilde{E}$, with diagonal $\{d_i\}$. This completes the proof of Case 1.

Case 2: Assume

$$\sum_{i\in J_1} d_i \leq \eta.$$

This implies J_1 is a finite set and that $|J_1| \leq Z$. Since $|J_1 \cup J_2| = \infty$ this implies $|J_2| = \infty$, and thus

$$\sum_{i\in J_2}(B-d_i) = \infty.$$

Let $L_1, K_2 \subset J_2 \cup J_3$ be disjoint finite sets which satisfy three conditions:

$$\sum_{i\in K_2}(B-d_i) > BZ,$$

$|L_1| = Z - |J_1|$, and $\max\{d_i\}_{i\in L_1} \leq \min\{d_i\}_{i\in K_2}$. Set $K_1 = J_1 \cup L_1$. Apply Lemma III.2.1 to the sequences $\{d_i\}_{i\in K_1}$ and $\{d_i\}_{i\in K_2}$ with

$$\eta_0 = \sum_{i\in K_1} d_i < BZ$$

to obtain two sequences $\{\widetilde{d}_i\}_{i\in K_1}$ and $\{\widetilde{d}_i\}_{i\in K_2}$. The choice of η_0 implies that $\{\widetilde{d}_i\}_{i\in K_1}$ is a sequence of Z zeroes. Set $\widetilde{d}_i = d_i$ for $i \notin K_1 \cup K_2$.

If (i) holds, then we still have

$$\sum_{i\in J_2}(\widetilde{d}_i - A) + \sum_{i\in J_3}(B-\widetilde{d}_i) = \infty.$$

Theorem III.1.1 implies that there is a projection Q_1 such that $(B-A)Q_1$ has diagonal $\{\widetilde{d}_i - A\}_{i\in J_2\cup J_3}$. We also have $|J_2 \cup J_3| = \infty$ and $m_{Q_1}(0) = m_{Q_1}(1) = \infty$.

Let $\mathbf{0}_Z$ be the zero operator on a Z dimensional Hilbert space, and set $\widetilde{E} = \mathbf{0}_Z \oplus \big((B-A)Q_1 + A\big)$. It is clear that $\widetilde{E}$ has diagonal $\{\widetilde{d}_i\}$, $m_{\widetilde{E}}(0) = Z$, $\sigma(\widetilde{E}) = \{0, A, B\}$, and $m_{\widetilde{E}}(A) = m_{\widetilde{E}}(B) = \infty$. By the second part of Lemma III.2.1 there is an operator E, unitarily equivalent to $\widetilde{E}$, with diagonal $\{d_i\}_{i\in I}$.

If (ii) holds then by (IV.5.18) we have

$$\begin{aligned}\sum_{i\in J_2\setminus L_1}(\widetilde{d}_i - A) - \sum_{i\in J_3}(B-\widetilde{d}_i) &= \eta_0 + \sum_{i\in J_2\setminus L_1}(d_i - A) - \sum_{i\in J_3}(B-d_i)\\ &= \sum_{i\in J_1} d_i + \sum_{i\in L_1} d_i + \sum_{i\in J_2\setminus L_1}(d_i - A) - C_3\\ &= -C_1 + C_2 - C_3 + (|J_1| + |L_1|)A\\ &= -nA - kB + ZA = -k(B-A).\end{aligned}$$

Theorem III.1.1 implies there is a projection Q_2 such that $(B-A)Q_2$ has diagonal $\{\widetilde{d}_i - A\}_{i\in(J_2\cup J_3)\setminus L_1}$. Since J_2 is infinite we have $m_{Q_2}(0) = \infty$. If J_3 is infinite then we also have $m_{Q_2}(1) = \infty$. If $|J_3| < \infty$ then

$$\sum_{i\in(J_2\cup J_3)\setminus L_1}(\widetilde{d}_i - A) = \sum_{i\in J_2\setminus L_1}(\widetilde{d}_i - A) - \sum_{i\in J_3}(B-\widetilde{d}_i) + |J_3|(B-A) = (|J_3| - k)(B-A),$$

which implies $m_{Q_2}(1) = |J_3| - k$. The operator $\widetilde{E} = \mathbf{0}_Z \oplus \big((B-A)Q_2 + A\big)$ has the desired eigenvalues and multiplicities and diagonal $\{\widetilde{d}_i\}$. The second part of Lemma III.2.1 implies there is an operator E, unitarily equivalent to $\widetilde{E}$, with diagonal $\{d_i\}$. This completes the proof of this case.

Case 3: Assume

$$\sum_{i\in J_2\cup J_3}(B-d_i) \le \eta.$$

This implies J_2 is finite, since $d_i < (B+A)/2$ for all $i \in J_2$. By hypothesis $|J_1 \cup J_2| = \infty$, and thus J_1 must be infinite. Moreover, A is a limit point of $\{d_i\}_{i\in J_1}$, since

$\sum_{i\in J_1}(A-d_i)<\infty$ and $d_i<A$ for all $i\in J_1$. There is some $N_0\in\mathbb{N}$ such that

$$(B-A)N_0>\eta.$$

Choose $\alpha\in(0,A)$ such that

$$\sum_{d_i<\alpha} d_i > AN_0.$$

Set $K_1=\{i\in J_1 : d_i<\alpha\}$. Since A is a limit point of $\{d_i\}_{i\in J_1}$, we can find a set $K_2\subseteq\{i\in J_1 : d_i\geq\alpha\}$ with N_0 elements, and clearly

$$\sum_{i\in K_2}(A-d_i)<AN_0.$$

We apply Lemma III.2.1 to the sequences $\{d_i\}_{i\in K_1}$ and $\{d_i\}_{i\in K_2}$ on the interval $[0,A]$ with

$$\eta_0=\sum_{i\in K_2}(A-d_i)$$

to obtain two sequences $\{\widetilde{d}_i\}_{i\in K_1}$ and $\{\widetilde{d}_i\}_{i\in K_2}$. Using (III.2.3) we see that $\{\widetilde{d}_i\}_{i\in K_2}$ is a sequence of N_0 terms equal to A. We also have

$$\sum_{i\in K_1}(A-\widetilde{d}_i)=|K_1|A-\sum_{i\in K_1}\widetilde{d}_i=|J_1|A-\sum_{i\in K_1}d_i-\sum_{i\in K_2}(A-d_i)=\sum_{i\in K_1\cup K_2}(A-d_i).$$

Set $\widetilde{d}_i=d_i$ for $i\in I\setminus(K_1\cup K_2)$. Define the sets

$$\widetilde{J}_1=\{i:\widetilde{d}_i<A\},\ \widetilde{J}_2=\left\{i:\widetilde{d}_i\in\left[A,\frac{A+B}{2}\right)\right\},\ \widetilde{J}_3=\left\{i:\widetilde{d}_i\geq\frac{A+B}{2}\right\}.$$

We have

$$\begin{aligned}\sum_{i\in\widetilde{J}_1}(A-\widetilde{d}_i)&=\sum_{i\in J_1\setminus(K_1\cup K_2)}(A-d_i)+\sum_{i\in K_1}(A-\widetilde{d}_i)\\&=\sum_{i\in J_1\setminus(K_1\cup K_2)}(A-d_i)+\sum_{i\in K_1\cup K_2}(A-d_i)=\sum_{i\in J_1}(A-d_i)=C_1.\end{aligned}$$

Since $\widetilde{d}_i = A$ for all $i \in K_2$ we have

$$\sum_{i\in\widetilde{J}_2}(\widetilde{d}_i - A) = \sum_{i\in J_2}(d_i - A) + \sum_{i\in K_2}(\widetilde{d}_i - A) = \sum_{i\in J_2}(d_i - A) = C_2.$$

Lastly, $\widetilde{d}_i = d_i$ for all $i \in J_3$, and thus

$$\sum_{i\in\widetilde{J}_3}(B - \widetilde{d}_i) = C_3.$$

However,

$$\sum_{i\in\widetilde{J}_2\cup\widetilde{J}_3}(B - \widetilde{d}_i) = \sum_{i\in J_2\cup J_3}(B - d_i) + (B - A)N_0 > \eta.$$

This implies that $\{\widetilde{d}_i\}_{i\in I}$ satisfies the conditions of Case 1, and thus there is an operator $\widetilde{E}$ with the desired eigenvalues and multiplicities and diagonal $\{\widetilde{d}_i\}_{i\in I}$. By the second part of Lemma III.2.1, there is an operator E, unitarily equivalent to $\widetilde{E}$, with diagonal $\{d_i\}_{i\in I}$. This completes the proof of this case and the proof of the theorem. □

As a corollary of Theorems IV.5.4 and IV.5.5 we deduce part (d) of Theorem IV.1.3. This will complete the proof of Theorem IV.1.3.

Corollary IV.5.6. *Let $0 < A < B < \infty$ and let $\{d_i\}_{i\in I}$ be a sequence in $[0, B]$. Let $Z, K \in \mathbb{N}$. There exists a self-adjoint operator E with $\sigma(E) = \{0, A, B\}$, $m_E(0) = Z$, $m_E(A) = \infty$, $m_E(B) = K$ and diagonal $\{d_i\}$ if and only if $|I| = \infty$, $C_1 \leq ZA$ and*

$$\sum_{i\in I}(d_i - A) = K(B - A) - ZA. \tag{IV.5.19}$$

Proof. First, assume that $|I| = \infty$, $C_1 \leq ZA$ and (IV.5.19) holds. It is clear that $|J_3| < \infty$ and thus $|J_1 \cup J_2| = |I| - |J_3| = \infty$. We have

$$C_1 - C_2 + C_3 = -\sum_{i\in I}(d_i - A) + |J_3|(B - A) = (Z + K - |J_3|)A + (|J_3| - K)B.$$

By Theorem IV.5.5 the desired operator exists.

Next, assume the operator E exists. Note that $E - A$ is a finite rank operator, and thus it has a well defined trace. In particular

$$\sum_{i\in I}(d_i - A) = K(B - A) - AZ.$$

This implies $C_2 < \infty$ and $|J_3| < \infty$, which clearly implies $C_3 < \infty$. By Theorem IV.5.4 we have $C_1 \leq ZA$ and $|J_1 \cup J_2| = \infty$, which immediately implies $|I| = \infty$. $\square$

IV.6. Examples

To demonstrate the use of Theorem IV.1.1 we will consider the following problem: Given a sequence $\{d_i\}$ in $[0,1]$, for what values of A is there a positive operator E with $\sigma(E) = \{0, A, 1\}$ and diagonal $\{d_i\}$? First, we will prove the following general theorem.

Theorem IV.6.1. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ and set*

$$\mathcal{A} = \big\{A \in (0,1)\colon \text{there exists } E \geq 0 \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\}\big\}.$$

Either $\mathcal{A} = (0,1)$ or $\mathcal{A}$ is a finite (possibly empty) set.

Proof. For each $A \in (0,1)$ define

$$C(A) = \sum_{d_i < A} d_i \quad \text{and} \quad D(A) = \sum_{d_i \geq A} (1 - d_i).$$

Note that if $C(A) + D(A) = \infty$ for some $A \in (0,1)$ then $C(A) + D(A) = \infty$ for all $A \in (0,1)$. By Theorem IV.4.2 we have $\mathcal{A} = (0,1)$. Thus, we will assume $C(A), D(A) < \infty$ for all $A \in (0,1)$.

First, we wish to show that $\sup\mathcal{A} < 1$. Assume to the contrary that $\sup\mathcal{A} = 1$. Note that there exists $\eta \in [0,1)$ such that $\eta = C(A) - D(A) - \lfloor C(A) - D(A)\rfloor$ for all $A \in (0,1)$, where $\lfloor\cdot\rfloor$ is the greatest integer function. Thus, for each $A \in (0,1)$

there exists $m(A) \in \mathbb{Z}$ such that

$$C(A) - D(A) = m(A) + \eta.$$

By Theorem IV.4.2, for each $A \in \mathcal{A}$ there exists $N(A) \in \mathbb{N}$ and $k(A) \in \mathbb{Z}$ such that

(IV.6.20)

$$m(A) + \eta = C(A) - D(A) = N(A)A + k(A) \quad \text{and} \quad C(A) \geq (N(A) + k(A))A.$$

Using (IV.6.20) we have

$$m(A) + \eta = N(A)A + k(A) < N(A) + k(A) \leq \frac{C(A)}{A}. \tag{IV.6.21}$$

Since $\eta \geq 0$ and $m(A), N(A), k(A) \in \mathbb{Z}$, we can improve the left inequality to

$$m(A) + 1 \leq N(A) + k(A). \tag{IV.6.22}$$

Thus, for each $A \in \mathcal{A}$ we must have

$$A(m(A) + 1) \leq C(A). \tag{IV.6.23}$$

Next, note that for $A, A' \in \mathcal{A}$ with $A' > A$ we have

$$\begin{aligned} m(A') - m(A) = C(A') - C(A) + D(A) - D(A') &= \sum_{A \leq d_i < A'} d_i + \sum_{A \leq d_i < A'} (1 - d_i) \\ &= |\{i \in \mathbb{N} : A \leq d_i < A'\}|. \end{aligned}$$

Using this gives

$$C(A') - C(A) = \sum_{d_i < A'} d_i - \sum_{d_i < A} d_i = \sum_{A \leq d_i < A'} d_i < A'(m(A') - m(A)). \tag{IV.6.24}$$

Putting together (IV.6.23) and (IV.6.24) we have

$$A'(m(A') + 1) - C(A) \leq C(A') - C(A) < A'(m(A') - m(A)).$$

Rearranging this inequality gives

$$A'(m(A)+1) < C(A).$$

Since $\sup \mathcal{A} = 1$ we can let $A' \to 1$ and we have

$$m(A)+1 \le C(A).$$

Finally, since $D(A) \to 0$ as $A \to 1$, for large enough A we have $D(A) < 1-\eta$ and thus

$$C(A) < C(A) - D(A) - \eta + 1 = m(A) + 1$$

which gives a contradiction, and shows that $A_{\text{sup}} = \sup \mathcal{A} < 1$. A symmetric argument shows that $A_{\text{inf}} = \inf \mathcal{A} > 0$.

Since $C(A)$ and $m(A)$ are nondecreasing as $A \to 1$, for each $A \in \mathcal{A}$ we have $C(A_{\text{inf}}) \le C(A) \le C(A_{\text{sup}})$ and $m(A_{\text{inf}}) \le m(A) \le m(A_{\text{sup}})$. Using (IV.6.21) and (IV.6.22), for $A \in \mathcal{A}$ we have

$$m(A_{\text{inf}}) + 1 \le m(A) + 1 \le N(A) + k(A) \le \frac{C(A)}{A} \le \frac{C(A_{\text{sup}})}{A_{\text{inf}}}.$$

This shows that $\{N(A)+k(A) : A \in \mathcal{A}\}$ and $\{m(A) : A \in \mathcal{A}\}$ are finite sets of integers. Next, we note that for $A \in \mathcal{A}$ we have

$$N(A)A_{\text{sup}} \ge N(A)A = m(A) + \eta - k(A) \ge m(A_{\text{inf}}) + \eta + N(A) - \frac{C(A_{\text{sup}})}{A_{\text{inf}}}.$$

Rearranging this inequality gives

$$N(A) \le \frac{\frac{C(A_{\text{sup}})}{A_{\text{inf}}} - m(A_{\text{inf}}) - \eta}{1 - A_{\text{sup}}},$$

which implies that $\{N(A) : A \in \mathcal{A}\} \subseteq \mathbb{N}$ is finite. Since $\{N(A)+k(A) : A \in \mathcal{A}\}$ is finite, we also see that $\{k(A) : A \in \mathcal{A}\}$ is finite. Finally, we note that for $A \in \mathcal{A}$ we have

$$A = \frac{m(A) + \eta - k(A)}{N(A)},$$

which clearly implies that $\mathcal{A}$ is finite. $\square$

Next, we will find the set $\mathcal{A}$ from Theorem IV.6.1 for a few specific sequences.

Example IV.6.2. Let $\beta \in (0, 1/2)$ and define the sequence $\{d_i\}_{i \in \mathbb{Z} \setminus \{0\}}$ by

$$d_i = \begin{cases} 1 - \beta^i & i > 0 \\ \beta^{-i} & i < 0. \end{cases}$$

Define the set

$$\mathcal{A}_\beta = \big\{ A \in (0,1) : \exists\, E \geq 0 \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\} \big\}.$$

We will show that

$$\mathcal{A}_\beta = \begin{cases} \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\} & \frac{-1+\sqrt{13}}{6} \leq \beta < 1/2 \\ \{\frac{1}{2}\} & 1/3 \leq \beta < \frac{-1+\sqrt{13}}{6} \\ \varnothing & 0 < \beta < 1/3. \end{cases}$$

First, assume $A \in \mathcal{A}_\beta \cap (\beta, 1-\beta]$, and thus

$$C = \sum_{d_i < A} d_i = \sum_{i=1}^{\infty} \beta^i = \frac{\beta}{1-\beta} \quad \text{and} \quad D = \sum_{d_i \geq A} (1 - d_i) = \sum_{i=1}^{\infty} \beta^i = \frac{\beta}{1-\beta}.$$

From Theorem IV.4.2 there exists $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$0 = C - D = NA + k \tag{IV.6.25}$$

and

$$\frac{\beta}{1-\beta} = C \geq (N+k)A. \tag{IV.6.26}$$

Using (IV.6.25) and $A \leq 1-\beta$ we have

$$0 < \beta N = NA + k + \beta N \leq N(1-\beta) + k + \beta N = N + k,$$

and thus $N + k > 0$. Now, we use (IV.6.26), $\beta < A$, then $\beta < 1/2$ to see

$$N + k < (N+k)\frac{A}{\beta} \leq \beta^{-1}\frac{\beta}{1-\beta} = \frac{1}{1-\beta} < 2.$$

Since $N + k \in \mathbb{Z}$ we see that $N + k = 1$. Solving for A in (IV.6.25) we have

$$A = \frac{-k}{N} = \frac{N-1}{N} = 1 - \frac{1}{N}.$$

Since $A = 1 - N^{-1} \in (\beta, 1-\beta]$, this shows that $1 < (1-\beta)^{-1} < N \leq \beta^{-1}$. If $\beta > 1/3$, then $N \leq \beta^{-1} < 3$, and thus $N = 2$. A simple check will show that $N = 2$, $k = -1$ and $A = 1/2$ satisfy (IV.1.1) and (IV.1.2) if and only if $\beta \geq 1/3$. Thus, $\mathcal{A}_\beta \cap (\beta, 1-\beta] = \{1/2\}$ for $\beta > 1/3$ and $\mathcal{A}_\beta \cap (\beta, 1-\beta] = \varnothing$ for $\beta < 1/3$. If $\beta = 1/3$ then we see $1 < N \leq 3$, and thus $N = 2$ or $N = 3$. We have already seen that $N = 2$, $k = -1$ and $A = 1/2$ satisfy (IV.1.1) and (IV.1.2). It is simple to check that $N = 3$, $k = -2$ and $A = 2/3$ do not satisfy (IV.1.1) and (IV.1.2), and thus $\mathcal{A}_{1/3} \cap (\beta, 1-\beta] = \{1/2\}$.

Next, assume $A \in \mathcal{A}_\beta \cap (1-\beta^m, 1-\beta^{m+1}]$ for some $m \in \mathbb{N}$. We have

$$C = \frac{\beta}{1-\beta} + \sum_{i=1}^{m}(1-\beta^i) = m + \frac{\beta^{m+1}}{1-\beta} \quad \text{and} \quad D = \sum_{i=m+1}^{\infty}\beta^i = \frac{\beta^{m+1}}{1-\beta}$$

By Theorem IV.4.2 there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$m = C - D = NA + k \tag{IV.6.27}$$

and

$$m + \frac{\beta^{m+1}}{1-\beta} = C \geq (N+k)A. \tag{IV.6.28}$$

Using (IV.6.27) and $A \leq 1 - \beta^{m+1}$ we have

(IV.6.29) $m < m + N\beta^{m+1} \leq NA + k + N\beta^{m+1} \leq N(1-\beta^{m+1}) + k + N\beta^{m+1} = N + k.$

Using (IV.6.28) and $A > 1 - \beta^m$ we have

$$m + \frac{\beta^{m+1}}{1-\beta} \geq (N+k)A > (N+k)(1-\beta^m).$$

Rearranging, and using $\beta < 1/2$ we have

(IV.6.30) $$N + k < \left(m + \frac{\beta^{m+1}}{1-\beta}\right)\frac{1}{1-\beta^m} < \left(m + \frac{1}{2^m}\right)\frac{2^m}{2^m-1} = m + \frac{1+m}{2^m-1}.$$

A simple calculation shows that $\frac{1+m}{2^m-1} \leq 1$ for all $m \geq 2$. Combining this with (IV.6.29) shows that $m < N + k < m + 1$ for $m \geq 2$. Since $N + k \in \mathbb{Z}$ this shows that $\mathcal{A}_\beta \cap (1 - \beta^2, 1) = \varnothing$.

Now, restrict to $A \in (1 - \beta, 1 - \beta^2]$. In this case (IV.6.29) and (IV.6.30) imply $1 < N+k < 3$, which implies $N+k = 2$. Solving (IV.6.27) for A and using $N+k = 2$ we have

$$A = \frac{1-k}{N} = 1 - \frac{1}{N}.$$

Since $A > 1 - \beta > 1/2$ this implies $N > 1/\beta > 2$. From (IV.6.28) we see

$$1 + \frac{\beta^2}{1-\beta} \geq 2A = 2 - \frac{2}{N}.$$

Rearranging this we have

$$N \leq \frac{2-2\beta}{1-\beta-\beta^2}.$$

For $\beta < \frac{-1+\sqrt{13}}{6}$ we have $\frac{2-2\beta}{1-\beta-\beta^2} < 3$ and thus $N < 3$. Combined with the fact that $N > 2$, we see $\mathcal{A} \cap (1 - \beta, 1 - \beta^2] = \varnothing$ for $\beta < \frac{-1+\sqrt{13}}{6}$. Finally, assume $\frac{-1+\sqrt{13}}{6} \leq \beta < 1/2$. Then $\frac{2-2\beta}{1-\beta-\beta^2} < 4$ and we must have $N = 3$, $A = \frac{2}{3}$ and $k = -1$.

It is clear that (IV.1.1) holds. For (IV.1.2), we use the fact that $\beta \geq \frac{-1+\sqrt{13}}{6}$ to see

$$C = 1 + \frac{\beta^2}{1-\beta} \geq \frac{4}{3} = (N+k)A.$$

Thus, by Theorem IV.4.2, for $\beta \geq \frac{-1+\sqrt{13}}{6}$ we have $2/3 \in \mathcal{A}_\beta$.

Finally, since $\{d_i\}$ is symmetric about $1/2$, if $A \in \mathcal{A}_\beta$ then $1 - A \in \mathcal{A}_\beta$. Thus $\mathcal{A}_\beta \cap (0, \beta] = \{1/3\}$ for $\frac{-1+\sqrt{13}}{6} \leq \beta < 1/2$ and the set is empty for $\beta < \frac{-1+\sqrt{13}}{6}$. $\square$

In the above example, note that for any choice of β, we have $C - D \in \mathbb{Z}$ for any choice of $A \in (0,1)$. Thus, Theorem III.1.1 implies that there is a projection with diagonal $\{d_i\}$. However, if $\beta < 1/3$ then there is no $A \in (0,1)$ so that $\{d_i\}$ is the diagonal of a self-adjoint operator E with $\sigma(E) = \{0, A, 1\}$. The next example is not the diagonal of any projection, but we will show that it is the diagonal of many different operators with three point spectrum.

Example IV.6.3. Let $\{d_i\}_{i\in\mathbb{Z}}$ be given by

$$d_i = \begin{cases} 2^{i-1} & i \leq 0 \\ 1 - 2^{-i-1} & i > 0. \end{cases}$$

Let

$$\mathcal{A} = \big\{A \in (0,1) \colon \text{there exists } E \geq 0 \text{ with } \sigma(E) = \{0, A, 1\} \text{ and diagonal } \{d_i\}\big\}.$$

We claim that

$$\mathcal{A} = \left\{\frac{1}{2n} : n = 1, 2, \ldots, 8\right\} \cup \left\{\frac{2n-1}{2n} : n = 1, 2, \ldots, 8\right\}.$$

The sequence $\{d_i\}$ is symmetric about $1/2$, and thus $A \in \mathcal{A}$ implies $1 - A \in \mathcal{A}$. Hence, it is enough to show that

$$\mathcal{A} \cap \left[\tfrac{1}{2}, 1\right) = \left\{\frac{2n-1}{2n} : n = 1, 2, \ldots, 8\right\}.$$

Assume $A \in \mathcal{A} \cap (1-2^{-m}, 1-2^{-m-1}]$ for some $m \geq 1$. We have

$$C = m - \frac{1}{2} + \frac{1}{2^m} \quad \text{and} \quad D = \frac{1}{2^m}.$$

Since $A \in \mathcal{A}$, Theorem IV.4.2 implies that there exist $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that

$$C - D = m - \frac{1}{2} = NA + k \tag{IV.6.31}$$

$$C = m - \frac{1}{2} + 2^{-m} \geq (N+k)A. \tag{IV.6.32}$$

Using (IV.6.31) and $A \leq 1 - 2^{-m-1}$ we have

(IV.6.33)

$$m-1 < m-\frac{1}{2}+N2^{-m-1} = NA+k+N2^{-m-1} \leq N(1-2^{-m-1})+k+N2^{-m-1} = N+k.$$

From (IV.6.32) and $A > 1 - 2^{-m}$ we have

$$m - \frac{1}{2} + 2^{-m} \geq (N+k)A > (N+k)(1-2^{-m}).$$

Rearranging gives

$$N + k < \left(m - \frac{1}{2} + 2^{-m}\right) \frac{2^m}{2^m - 1} = m + \frac{m - 2^{m-1} + 1}{2^m - 1}. \tag{IV.6.34}$$

For $m \geq 4$, a simple calculation shows $\frac{m-2^{m-1}+1}{2^m-1} < 0$ and thus $N+k < m$. However, from (IV.6.33) we have $N + k > m - 1$. Since $N + k \in \mathbb{Z}$ this is a contradiction and shows that $\mathcal{A} \cap (1-2^{-m}, 1-2^{-m-1}] = \varnothing$ for $m \geq 4$.

One can easily check that $A = 1/2$ satisfies (IV.1.1) and (IV.1.2) with $N = 1$ and $k = -1$ (or $N = 3$ and $k = -2$). All that is left is to find $\mathcal{A} \cap (1-2^{-m}, 1-2^{-m-1}]$ for $m = 1, 2$ and 3. The calculation for each m is similar, so the cases of $m = 2$ and 3 will be left to the reader.

Assume $A \in \mathcal{A} \cap (1/2, 3/4]$. In this case we have $C = 1$ and $D = 1/2$. From (IV.6.33) and (IV.6.34) we have $0 < N + k < 2$ and thus $N + k = 1$. Using this and solving (IV.6.31) for A we have

$$A = \frac{\frac{1}{2} - k}{N} = \frac{N - \frac{1}{2}}{N} = 1 - \frac{1}{2N}.$$

From the inequalities $1/2 < A = 1 - 1/(2N) \leq 3/4$ we obtain $1 < N \leq 2$. Thus $N = 2, A = 3/4$ and $k = -1$. One can easily check that (IV.1.1) and (IV.1.2) are satisfied for these values of A, N and k. □

CHAPTER V

THE SCHUR-HORN PROBLEM IN FRAME THEORY

V.1. Frames

In this chapter we exhibit a connection between diagonals of self-adjoint operators and frames. The material in this chapter is contained in a paper co-authored with Marcin Bownik [7] which has been accepted for publication in *Journal für die reine und angewandte Mathematik*.

First, we need to introduce some basic notions from frame theory.

Definition V.1.1. A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum |\langle f, f_i\rangle|^2 \le B\|f\|^2 \qquad \text{for all } f \in \mathcal{H}. \tag{V.1.1}$$

The numbers A and B are called the *frame bounds*. The supremum over all A and infimum over all B which satisfy (V.1.1) are called the *optimal frame bounds*. If $A = B$, then $\{f_i\}$ is said to be a *tight frame*. In addition, if $A = B = 1$, then $\{f_i\}$ is called a *Parseval frame*.

The basic connection between frame theory and operator theory is via the following operators.

Definition V.1.2. If $\{f_i\}_{i\in I}$ is a frame we call the operator $T : \mathcal{H} \to \ell^2(I)$, given by

$$Tf = \{\langle f, f_i\rangle\}_{i\in I}, \tag{V.1.2}$$

the *analysis operator*. The adjoint $T^* : \ell^2(I) \to \mathcal{H}$ given by

$$T^*\big(\{a_i\}_{i\in I}\big) = \sum_{i\in I} a_i f_i \tag{V.1.3}$$

is called the *synthesis operator*. The operator $S = T^*T$ given by

$$Sf = \sum_{i\in I} \langle f, f_i\rangle f_i \tag{V.1.4}$$

is called the *frame operator*.

Many standard facts about frames can be found in [11]. We will find use for the following proposition in the next section.

Proposition V.1.3. *If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$ and S is the frame operator, then $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame for $\mathcal{H}$.*

V.2. Frame norms and diagonals

In this section we reformulate the problem of characterizing norms of frames to an equivalent problem of characterizing diagonals of positive operators with prescribed lower and upper bounds. We start with the following basic fact.

Proposition V.2.1. *Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_i\}_{i\in I}$ and let $0 < A \le B < \infty$. If E is a positive operator with $\sigma(E) \subseteq \{0\} \cup [A, B]$, then $\{Ee_i\}$ is a frame for the Hilbert space $E(\mathcal{H})$ with frame bounds A^2 and B^2.*

Proof. Let $f \in E(\mathcal{H})$. We have

$$\sum_{i\in I} |\langle f, Ee_i\rangle|^2 = \sum_{i\in I} |\langle Ef, e_i\rangle|^2 = \|Ef\|^2.$$

This clearly implies that B^2 is an upper frame bound. Since $f \in E(\mathcal{H})$ we have $\|Ef\| \geq A\|f\|$, which shows that A^2 is a lower frame bound. $\square$

Our goal is to establish the converse statement. That is, any frame in $\mathcal{H}$ is an image of an orthonormal basis of a larger Hilbert space $\mathcal{K} \supset \mathcal{H}$ under a positive operator. This generalizes the classical dilation theorem for Parseval frames due to Han and Larson [14, Proposition 1.1], which says that Parseval frames are images of orthonormal bases under orthogonal projections. Proposition V.2.2 is essentially contained in the work of Antezana, Massey, Ruiz, and Stojanoff [1, Proposition 4.5]. In particular, the authors of [1] established the relationship of our problem with the Schur-Horn Theorem of majorization theory which we state in a convenient form in Theorem V.2.3.

Proposition V.2.2. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with optimal frame bounds A^2 and B^2. Then there exists an isometry $\Phi : \mathcal{H} \to \ell^2(I)$ and a positive operator $E : \ell^2(I) \to \Phi(\mathcal{H})$ such that $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ and $Ee_i = \Phi f_i$, where $\{e_i\}_{i\in I}$ is the coordinate basis of $\ell^2(I)$. If S is the frame operator of $\{f_i\}_{i\in I}$ and $\mathbf{0_e}$ is the zero operator on $\Phi(\mathcal{H})^\perp$, then E^2 is unitarily equivalent to $S \oplus \mathbf{0_e}$.*

Proof. Let S be the frame operator of $\{f_i\}$. By Proposition V.1.3, $\{S^{-1/2}f_i\}$ is a Parseval frame. Set $p_i = S^{-1/2}f_i$, and let Φ be the analysis operator of $\{p_i\}$. Since $\{p_i\}$ is a Parseval frame, Φ is an isometry. Let P be the orthogonal projection onto $\Phi(\mathcal{H})$. As a consequence of the Han-Larson dilation theorem for Parseval frames [14, Proposition 1.1] we have $Pe_i = \Phi p_i$ for all $i \in I$. Hence, we also have $\Phi^* e_i = p_i$. Define the operator $E = \Phi S^{1/2}\Phi^*$. Clearly, E is a self-adjoint operator on $\ell^2(I)$.

Observe that

$$Ee_i = \Phi S^{1/2}\Phi^* e_i = \Phi S^{1/2} p_i = \Phi S^{1/2} S^{-1/2} f_i = \Phi f_i.$$

Thus,

$$\|Ef\|^2 = \sum_{i\in I} |\langle Ef, e_i\rangle|^2 = \sum_{i\in I} |\langle f, Ee_i\rangle|^2 = \sum_{i\in I} |\langle f, \Phi f_i\rangle|^2.$$

Since Φ is an isometry, $\{\Phi f_i\}_{i\in I}$ is a frame for $\Phi(\mathcal{H})$ with optimal frame bounds A^2 and B^2. The frame property now implies $A^2\|f\|^2 \leq \|Ef\|^2 \leq B^2\|f\|^2$, which in turn implies that $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$.

Finally, define $U : \mathcal{H} \oplus \Phi(\mathcal{H})^\perp \to \ell^2(I)$ by

$$Uf = \begin{cases} \Phi f & f \in \mathcal{H}, \\ f & f \in \Phi(\mathcal{H})^\perp. \end{cases}$$

It is clear that U is unitary, since $\Phi : \mathcal{H} \to \Phi(\mathcal{H})$ is an isometric isomorphism. Note that $\Phi^*\Phi$ is the identity on $\mathcal{H}$, thus

$$E^2 = \Phi S^{1/2}\Phi^*\Phi S^{1/2}\Phi^* = \Phi S \Phi^*.$$

Finally, for $f \in \mathcal{H}$,

$$E^2 U f = E^2 \Phi f = \Phi S f = USf,$$

and for $f \in \Phi(\mathcal{H})^\perp$,

$$E^2 U f = E^2 f = \Phi S \Phi^* f = 0 = U\mathbf{0}_{\mathbf{e}} f.$$

This proves the last part of Proposition V.2.2. □

One should remark that Han and Larson [14] gave a different extension of their frame dilation result than Proposition V.2.2. In [14, Proposition 1.6] it is shown that any frame is an image of a Riesz basis under an orthogonal projection, and the frame and Riesz bounds are the same.

Next, we show that the problem of characterizing the sequence of norms of a frame can be reformulated into the problem of characterizing the diagonals of a certain set of self-adjoint operators. The characterization of this set of operators is the content of the next chapter. This reformulation is due to Antezana, Massey, Ruiz, and Stojanoff [1], who established the relationship of the frame norm problem with the Schur-Horn Theorem. Consequently, a characterization of norms of finite frames follows from the Schur-Horn Theorem. The special tight case $A = B$ is a celebrated theorem of Kadison [16, 17], which gives a complete characterization of diagonals of projections.

Theorem V.2.3. *Suppose* $0 < A \leq B < \infty$, $\mathcal{H}$ *is a Hilbert space, and* $\{e_i\}_{i\in I}$ *is the coordinate basis of* $\ell^2(I)$. *The following sets are equal:*

$$\mathcal{N} = \left\{ \{\|f_i\|^2\}_{i\in I} \;\middle|\; \{f_i\}_{i\in I} \text{ is a frame for } \mathcal{H} \text{ with optimal bounds } A \text{ and } B \right\},$$

$$\mathcal{D} = \left\{ \{\langle E e_i, e_i\rangle\}_{i\in I} \;\middle|\; E \text{ is self-adjoint on } \ell^2(I) \text{ with rank} = \dim \mathcal{H} \right.$$
$$\left. \text{and } \{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B] \right\}.$$

Proof. First we show $\mathcal{D} \subseteq \mathcal{N}$. Let $\{d_i\}_{i\in I} \in \mathcal{D}$ be the diagonal of E. Since $E \geq 0$, it has a positive square root $E^{1/2}$ with $\{\sqrt{A}, \sqrt{B}\} \subseteq \sigma(E^{1/2}) \subseteq \{0\} \cup [\sqrt{A}, \sqrt{B}]$. By Proposition V.2.1 the sequence $\{E^{1/2}e_i\}_{i\in I}$ is a frame for the Hilbert space $E^{1/2}(\ell^2(I))$ with frame bounds A and B. Since $\{\sqrt{A}, \sqrt{B}\} \subseteq \sigma(E^{1/2})$ it is clear that the bounds A and B are optimal. Since

$$\|E^{1/2}e_i\|^2 = \langle E^{1/2}e_i, E^{1/2}e_i\rangle = \langle E e_i, e_i\rangle = d_i,$$

this shows that $\{d_i\} \in \mathcal{N}$.

Next, we will show that $\mathcal{N} \subseteq \mathcal{D}$. Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with optimal frame bounds A and B. By Proposition V.2.2 there is a isometry $\Phi : \mathcal{H} \to \ell^2(I)$ and a positive operator $E : \ell^2(I) \to \Phi(\mathcal{H})$ with $\{\sqrt{A}, \sqrt{B}\} \subseteq \sigma(E) \subseteq \{0\} \cup [\sqrt{A}, \sqrt{B}]$ such that $Ee_i = \Phi f_i$. Since $\{A, B\} \subseteq \sigma(E^2) \subseteq \{0\} \cup [A, B]$, and

$$\langle E^2 e_i, e_i\rangle = \langle Ee_i, Ee_i\rangle = \|Ee_i\|^2 = \|\Phi f_i\|^2 = \|f_i\|^2,$$

this shows that $\{\|f_i\|^2\}_{i\in I} \in \mathcal{D}$. □

A similar results holds for Riesz bases.

Definition V.2.4. A sequence $\{f_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is called a *Riesz basis* if it is complete and there exist $0 < A \leq B < \infty$ such that

$$A \sum |a_i|^2 \leq \left\| \sum a_i f_i \right\|^2 \leq B \sum |a_i|^2 \tag{V.2.5}$$

for all finitely supported sequences $\{a_i\}_{i\in I}$. The numbers A and B are called the *Riesz bounds*. The supremum over all A and infimum over all B which satisfy (V.2.5) are called the *optimal Riesz bounds*.

Equivalently, a Riesz basis is a frame such that its synthesis operator T^*, and thus its analysis operator T, is an isomorphism. Moreover, optimal Riesz and frame bounds are the same. Therefore, an analogue of Proposition V.2.2 for Riesz bases involves operators E without zero in the spectrum. Consequently, we have the following analogue of Theorem V.2.3.

Theorem V.2.5. *Suppose* $0 < A \leq B < \infty$, $\mathcal{H}$ *is a Hilbert space, and* $\{e_i\}_{i\in I}$ *is the coordinate basis of* $\ell^2(I)$. *The following sets are equal:*

$$\mathcal{N} = \Big\{ \{\|f_i\|^2\}_{i\in I} \;\Big|\; \{f_i\}_{i\in I} \text{ is a Riesz basis for } \mathcal{H} \text{ with optimal bounds } A \text{ and } B \Big\},$$

$$\mathcal{D} = \Big\{ \{\langle Ee_i, e_i\rangle\}_{i\in I} \;\Big|\; E \text{ is self-adjoint on } \ell^2(I) \text{ and } \{A, B\} \subseteq \sigma(E) \subseteq [A, B] \Big\}.$$

CHAPTER VI

LOCALLY INVERTIBLE OPERATORS

VI.1. Statement of the main theorem

The goal of this chapter is to characterize the diagonals of self-adjoint operators E with spectrum $\sigma(E)$ such that $A, B \in \sigma(E) \subseteq \{0\} \cup [A, B]$. Using Theorem V.2.3 this gives a characterization of all possible sequences of norms of a frame with prescribed optimal bounds A and B. The material in this chapter is contained in a paper co-authored with Marcin Bownik [7] which has been accepted for publication in *Journal für die reine und angewandte Mathematik*.

The problem of characterizing norms of frames with prescribed frame operator has attracted a significant number of researchers. Casazza and Leon [8, 9] gave explicit and algorithmic construction of finite tight frames with prescribed norms. Moreover, Casazza, Fickus, Kovačević, Leon, and Tremain [10] characterized norms of finite tight frames in terms of their "fundamental frame inequality" using frame potential methods of Benedetto and Fickus [6]. An alternative approach using projection decomposition was undertaken by Kornelson and Larson [12, 19], which yields some necessary and some sufficient conditions for infinite dimensional Hilbert spaces. Antezana, Massey, Ruiz, and Stojanoff [1] established the connection of this problem with the infinite dimensional Schur-Horn problem and gave refined necessary conditions and sufficient conditions. Finally, Kadison [16, 17] gave the complete answer for Parseval frames, which easily extends to tight frames by scaling.

Our main result can be thought as infinite Schur-Horn Theorem for positive locally invertible operators. Note that the assumption of $\{d_i\}$ being nonsummable in Theorem VI.1.1 is not a true limitation. Indeed, the summable case requires more restrictive conditions reflected in Theorem VI.2.1.

Theorem VI.1.1. *Let $0 < A < B < \infty$ and $\{d_i\}_{i\in I}$ be a nonsummable sequence in $[0, B]$. Define*

$$C = \sum_{d_i<A} d_i \quad \text{and} \quad D = \sum_{d_i\geq A} \big(B - d_i\big). \tag{VI.1.1}$$

Then there is a positive operator E on a Hilbert space $\mathcal{H}$ with $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ and diagonal $\{d_i\}$ if and only if one of the following holds:

(i) $C = \infty$

(ii) $D = \infty$

(iii) $C, D < \infty$ *and*

$$\text{there exists } n \in \mathbb{N} \cup \{0\} \quad nA \leq C \leq A + B(n - 1) + D. \tag{VI.1.2}$$

As a corollary of Theorem VI.1.1 and Theorem V.2.3 we obtain the characterization of sequences of frame norms.

Corollary VI.1.2. *Let $0 < A < B < \infty$ and $\{d_i\}$ be a nonsummable sequence in $[0, B]$. There exists a frame $\{f_i\}$ for some Hilbert space with optimal frame bounds A and B and $d_i = ||f_i||^2$ if and only if (i), (ii), or (iii) from Theorem VI.1.1 hold.*

We would like to emphasize that the non-tight case is not a mere generalization of the tight case $A = B$ established by Kadison [16, 17], see Theorem III.1.1. Indeed, the non-tight case is qualitatively different from the tight case, since by setting $A = B$ in Theorem VI.1.1 we do not get the correct necessary and sufficient condition (III.1.1) previously discovered by Kadison, see Remark VI.3.4. Furthermore, the non-tight summable and nonsummable cases require different characterization conditions. This is again unlike the tight case, where the same condition (III.1.1) works in either case.

The proof of Theorem VI.1.1 breaks into 3 distinctive parts. The summable case does not require many new techniques since it reduces to the study of trace class operators, and thus it can be deduced from the work of Arveson-Kadison [5] and Kaftal-Weiss [18]. However, the nonsummable case is much more involved. The sufficiency part of Theorem VI.1.1 requires special techniques of "moving" diagonal entries to more favorable configurations, where it is possible to construct required operators. This is done in Section 4 by considering a variety of cases, some of which are tight in the sense that the required operator has a three point spectrum. It is worth adding that our construction is quite explicit and algorithmic, always leading to diagonalizable operators. Finally, Section 5 contains the necessity proof of Theorem VI.1.1. This part is shown using arguments involving trace class operators and Kadison's Theorem III.1.1.

Theorem VI.1.1 has an analogue for operators without zero in the spectrum, see Theorem VI.5.3. This result is much easier to prove and it leads to a characterization of norms of Riesz bases with prescribed bounds. Finally, in the last section we illustrate how our main theorem can be applied to determine the set $\mathcal{A}$ of possible lower bounds of positive operators with fixed diagonal $\{d_i\}$. While we show that it is always closed, $\mathcal{A}$ can take distinct configurations depending on the choice of diagonal.

VI.2. The summable case

The goal of this section is to establish the summable case of our main Theorem VI.1.1. This special case can be deduced from a finite rank version of the Schur-Horn Theorem (Theorem II.2.3).

Theorem VI.2.1. *Suppose* $0 < A \leq B < \infty$ *and* $M \in \mathbb{N} \cup \{\infty\}$. *Let* $\{d_i\}_{i=1}^M$ *be a summable sequence in* $[0, B]$. *There is a positive, rank* $N+1$ *operator* E *on a Hilbert space* $\mathcal{H}$ *with diagonal* $\{d_i\}$ *and* $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ *if and only if*

$$\sum_{i=1}^{M} d_i \in [AN + B, A + BN] \tag{VI.2.3}$$

and

$$\sum_{d_i < A} d_i \geq A(N - m_0 + 1), \quad \text{with } m_0 = |\{i : d_i \geq A\}|. \tag{VI.2.4}$$

Proof. Assume an operator E is as in Theorem VI.2.1. Because each of the $N+1$ nonzero eigenvalues of E is at most B, and A is an eigenvalue, we have $\sum d_i = \operatorname{tr}(E) \leq A + BN$. Similarly, since each of the $N+1$ nonzero eigenvalues of E is at least A, and B is an eigenvalue, we have $\sum d_i = \operatorname{tr}(E) \geq AN + B$. After rearranging $\{d_i\}$ in non-increasing order, Theorem II.1.3 yields

$$\sum_{d_i < A} d_i = \sum_{i=m_0+1}^{M} d_i \geq \sum_{i=m_0+1}^{M} \lambda_i = \sum_{i=m_0+1}^{N+1} \lambda_i \geq A(N - m_0 + 1),$$

where $\{\lambda_i\}$ are the eigenvalues of E in non-increasing order (with multiplicity). This shows that (VI.2.3) and (VI.2.4) are necessary.

Conversely, assume we have a sequence $\{d_i\}_{i=1}^M$ which satisfies (VI.2.3) and (VI.2.4). If $\sum d_i < A + BN$ then there exists unique $n_0 \in \{1, 2, \ldots, N\}$ and $x \in [A, B)$ such that

$$\sum_{i=1}^{M} d_i = A(N-n_0) + x + Bn_0. \tag{VI.2.5}$$

We set

$$\lambda_i = \begin{cases} B & i \in \{1, \ldots, n_0\} \\ x & i = n_0 + 1 \\ A & i \in \{n_0+2, \ldots, N+1\}. \end{cases}$$

If $\sum d_i = A + BN$, simply let $n_0 = N-1$ and $x = B$. By Theorem II.2.3, we need only check that the majoriztion property (II.2.5) holds for $\{d_i\}$ and $\{\lambda_i\}$.

Combining (VI.2.4) and (VI.2.5), we have

$$\sum_{i=1}^{m_0} d_i \leq Bn_0 + x + A(m_0 - n_0 - 1). \tag{VI.2.6}$$

For $m \leq m_0$, we have

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{m_0} d_i - \sum_{i=m+1}^{m_0} d_i \leq \sum_{i=1}^{m_0} d_i + A(m - m_0).$$

For $m_0 < m \leq N+1$, we have

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{m_0} d_i + \sum_{i=m_0+1}^{m} d_i \leq \sum_{i=1}^{m_0} d_i + A(m - m_0).$$

In either case, combining these with (VI.2.6) yields

$$\sum_{i=1}^{m} d_i \leq Bn_0 + x + A(m - n_0 - 1) \leq \sum_{i=1}^{m} \lambda_i \qquad \text{for } n_0 + 1 \leq m \leq N+1.$$

Finally, for $m > N+1$ and $m < n_0 + 1$ the majorization property is trivial. □

As a corollary of Theorems V.2.3 and VI.2.1 we have the following.

Corollary VI.2.2. *Suppose* $0 < A \leq B < \infty$ *and* $M \in \mathbb{N} \cup \{\infty\}$. *Let* $\{d_i\}_{i=1}^{M}$ *be a summable sequence in* $[0, B]$. *There exists a frame* $\{f_i\}$ *for an* $(N+1)$*-dimensional* *space with optimal frame bounds A and B and $d_i = ||f_i||^2$ if and only if* (VI.2.3) *and* (VI.2.4) *hold.*

In the nonsummable case the condition (VI.2.3) makes no sense. However, we can give an alternate set of conditions which will generalize.

Theorem VI.2.3. *Suppose $0 < A \leq B < \infty$ and $M \in \mathbb{N} \cup \{\infty\}$. Let $\{d_i\}_{i=1}^M$ be a summable sequence in $[0, B]$. Define the numbers*

$$C = \sum_{d_i < A} d_i \quad \textit{and} \quad D = \sum_{d_i \geq A} \big(B - d_i\big).$$

There is a positive, rank $N+1$ operator E on a Hilbert space $\mathcal{H}$ with diagonal $\{d_i\}$ and $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ if and only if

$$\text{(VI.2.7)} \quad C \in [A(N - m_0 + 1), A + B(N - m_0) + D], \quad \textit{with } m_0 = |\{i : d_i \geq A\}|$$

and

$$\text{(VI.2.8)} \qquad \sum_{i=1}^M d_i \geq AN + B.$$

Proof. Assuming (VI.2.3) and (VI.2.4) we have

$$C - D = \sum_{d_i < A} d_i - \sum_{d_i \geq A} (B - d_i) = \sum_{i=1}^M d_i - m_0 B \leq A + BN - m_0 B,$$

which shows $C \leq A + B(N - m_0) + D$. The other parts of (VI.2.7) and (VI.2.8) are obvious. Similarly, assuming (VI.2.7) and (VI.2.8) we see

$$\sum_{i=1}^M d_i = C - D + m_0 B \leq A + B(N - m_0) + D - D + m_0 B = A + BN,$$

and the other parts of (VI.2.3) and (VI.2.4) are obvious. □

Note that if $\{d_i\}$ is not summable, then (VI.2.8) is trivially satisfied. Thus it is a reasonable and correct guess that a variant of (VI.2.7) is the necessary and sufficient condition.

VI.3. The nonsummable case of the Carpenter's Theorem

The goal of this section is to prove the sufficiency part of our main theorem. In the terminology of Kadison [16, 17], this is a non-tight version of the Carpenter's Theorem.

Theorem VI.3.1. *Suppose* $0 < A < B < \infty$. *Let* $\{d_i\}_{i\in I}$ *be a nonsummable sequence in* $[0, B]$ *and*

$$C = \sum_{d_i < A} d_i, \qquad D = \sum_{d_i \geq A} (B - d_i).$$

If

$$C \in \bigcup_{n=0}^{\infty} [An, A + B(n-1) + D] \cup \{\infty\}, \tag{VI.3.9}$$

then there is a positive diagonalizable operator E *on a Hilbert space* $\mathcal{H}$ *with* $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ *and diagonal* $\{d_i\}_{i\in I}$.

Remark VI.3.2. In Theorem VI.3.1, the index set I may or may not be countable and $\mathcal{H}$ may or may not be separable. The case of $\mathcal{H}$ being non-separable can be reduced to the separable case. We will use the convention that a "sequence" $\{d_i\}_{i\in I}$ can have an indexing set of any cardinality. Note that, if $D = \infty$, then the first interval in the union is $[0, \infty]$ so (VI.3.9) is always satisfied. Similarly, if $C = \infty$, then (VI.3.9) is always satisfied. Moreover, if $A - B + D < 0$, then we interpret the interval $[0, A - B + D]$ to be $\varnothing$. Thus, if $D < B - A$ then $C = 0$ does not satisfy (VI.3.9). Finally, note that the set in (VI.3.9) reduces to a finite union of intervals since it always contains an infinite interval $[(n+1)A, \infty]$, where $n = \lceil A/(B - A) \rceil$.

In the tight case $A = B$, the condition (VI.3.9) is necessary but not sufficient. The correct condition was discovered by Kadison [16, 17], see Theorem III.1.1. We state it in a rescaled form that is convenient for our purposes in this chapter.

Theorem VI.3.3 (Kadison)**.** *Let $\{d_i\}_{i\in I}$ be a sequence in $[0,B]$. For $\alpha \in (0,B)$ define*

$$a = \sum_{d_i<\alpha} d_i, \qquad b = \sum_{d_i\geq\alpha} (B - d_i).$$

Then, there is an orthogonal projection P such that BP has a diagonal $\{d_i\}_{i\in I}$ if and only if

$$a - b \in B\mathbb{Z} \cup \{\pm\infty\} \tag{VI.3.10}$$

with the convention that $\infty - \infty = 0$.

Remark VI.3.4. Note that the condition (VI.3.10) is independent of the choice of $\alpha \in (0,B)$. That is, if (III.1.1) holds for some α, then it must hold for all $\alpha \in (0,B)$. To see that conditions (VI.3.9) and (VI.3.10) are different in the tight case, consider the sequence $\{d_i\}$ which contains the terms $\{n^{-2}\}_{n=2}^{\infty}$ and $\{1-2^{-n}\}_{n=1}^{\infty}$. For $B = 1$ and $\alpha = 1/2$ we have $a - b = \frac{\pi^2-6}{6} - 1 \notin \mathbb{Z}$. By Theorem VI.3.3 there is no projection with diagonal $\{d_i\}$, although (VI.3.9) is satisfied since $C = \infty$.

We are ready to give the proof of Theorem VI.3.1, which breaks into several cases.

Proof of Theorem VI.3.1. Throughout this proof let $\{a_i\}$ and $\{b_i\}$ be the subsequences of $\{d_i\}$ in $[0,A)$ and $[A,B]$, respectively.

Case 1. Assume $C = \infty$.

Partition $\{a_i\}$ into a countable number of sequences $\{a_i^{(k)}\}$ for each $k \in \mathbb{N}$, each with infinite sum. For each $k \in \mathbb{N}$ we apply Theorem VI.3.3 on $[0, A + \frac{B-A}{k}]$ with

$\alpha = A$. Since

$$\sum_{a_i^{(k)} < \alpha} a_i^{(k)} = \infty,$$

there is a projection $P_k \neq I$ on a Hilbert space $\mathcal{H}_k$ such that the diagonal of $(A + \frac{B-A}{k})P_k$ is $\{a_i^{(k)}\}$. Let S be the diagonal operator with the diagonal $\{b_i\}$ on a Hilbert space $\mathcal{H}_0$. Then, the operator

$$E = S \oplus \bigoplus_{k=1}^{\infty} \left(A + \frac{B-A}{k}\right) P_k$$

on the Hilbert space $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ has diagonal $\{d_i\}$. By construction $\sigma(E)$ is the closure of $\{0\} \cup \{A + \frac{B-A}{k} : k \in \mathbb{N}\} \cup \{b_i\}$. This implies $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$.

Case 2. Assume $D = \infty$.

First, suppose that A is not an accumulation point of $\{b_i\}$. Partition $\{b_i\}$ into two sequences $\{b_i^{(1)}\}$ and $\{b_i^{(2)}\}$ such that

(VI.3.11) $$\sum_{i=1}^{\infty} (B - b_i^{(k)}) = \infty \quad \text{for} \quad k = 1, 2.$$

Let $\{c_i\}$ be the sequence consisting of $\{a_i\}$ and $\{b_i^{(1)}\}$. By Theorem VI.3.3 on $[0, B]$ with $\alpha = A$ and

$$\sum_{c_i \geq \alpha} (B - c_i) = \infty,$$

there is a projection P_1 on a Hilbert space $\mathcal{H}_1$ such that BP_1 has diagonal $\{c_i\}$.

Define

$$k_i = \frac{b_i^{(2)} - A}{B - A}.$$

The sequence $\{k_i\}$ is in $[0, 1]$ and 0 is not an accumulation point. Thus, there exists $\alpha \in (0, 1)$ such that

$$\sum_{k_i \geq \alpha} (1 - k_i) = \infty.$$

By Theorem VI.3.3 there is a projection P_2 on a Hilbert space $\mathcal{H}_2$ with diagonal $\{k_i\}$. The operator $S = (B-A)P_2 + AI$ is diagonalizable with eigenvalues A and B, and diagonal $\{b_i^{(2)}\}$. Thus, the operator $E = BP_1 \oplus S$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$ has diagonal $\{d_i\}$ and $\sigma(E) = \{0, A, B\}$.

Finally, suppose that A is an accumulation point of $\{b_i\}$. Partition $\{b_i\}$ into two infinite sequences $\{b_i^{(1)}\}$ and $\{b_i^{(2)}\}$ each with infimum A. Then (VI.3.11) holds. Let P_1 be a projection on $\mathcal{H}_1$ as before. Let S be the diagonal operator on a Hilbert space $\mathcal{H}_2$ with $\{b_i^{(2)}\}$ on the diagonal. The operator $E = BP_1 \oplus S$ has diagonal $\{d_i\}$. Clearly, $\sigma(S) \subseteq [A, B]$, and since $\inf\{b_i^{(2)}\} = A$ we have $A \in \sigma(S)$. We also have $\{0, B\} = \sigma(BP_1)$. This implies $\{A, B\} \subseteq \sigma(E) = \sigma(BP_1) \cup \sigma(S) \subseteq \{0\} \cup [A, B]$ as desired.

Case 3. Assume $C, D < \infty$ and $C \in [An, A + B(n-1) + D]$ for some $n \in \mathbb{N}$.

We claim that it is enough to prove Case 3 when $\{d_i\}$ is countable. The fact that $C, D < \infty$ implies that the sequence $\{d_i\}$ contains at most countably many terms in $(0, B)$. Assume that there exists an operator E with the desired spectrum and diagonal consisting of only the terms of $\{d_i\}$ in $(0, B)$. Let I be the identity operator on a Hilbert space of dimension $|\{i : d_i = B\}|$, and let $\mathbf{0}$ be the zero operator on a Hilbert space of dimension $|\{i : d_i = 0\}|$. The operator $E \oplus BI \oplus \mathbf{0}$ has the same spectrum as E and diagonal $\{d_i\}$. However, it may happen that the sequence of terms contained in $(0, B)$ is summable. This would imply that $\{d_i\}$ must contain infinitely many terms equal to B (since $\{d_i\}$ is assumed to be nonsummable). In this case we consider the sequence of terms in $(0, B)$ together with a countable infinite sequence with each term equal to B. If we can find an operator E with this diagonal sequence and the desired spectrum, then $E \oplus BI \oplus \mathbf{0}$ is again the desired operator. This proves our claim.

Let $n \in \mathbb{N}$ be the largest integer such that $C \in [An, A + B(n-1) + D]$. Since $\{d_i\}$ is not summable, $\{b_i\}$ is an infinite sequence. First, assume $C = An$. By

Theorem VI.3.3 on $[0, A]$ there is a projection P on a Hilbert space $\mathcal{H}_1$ such that AP has diagonal $\{a_i\}$. Let $\mathcal{H}_2$ be an infinite dimensional Hilbert space, and let S be a diagonal operator with $\{b_i\}$ on the diagonal. Since $\{b_i\}$ is an infinite sequence in $[A, B]$ and $D < \infty$ we clearly have $B \in \sigma(S)$ and thus $E = AP \oplus S$ is the desired operator.

Next, assume $C \in (An, A + B(n-1)]$ and set $x = C - An$. Since $\sup\{b_i\} = B$, there is some $i_0 \in \mathbb{N}$ such that $b_{i_0} + x \geq B$. Define the sequence $\{\widetilde{a}_i\}$ to be the sequence consisting of $\{a_i\}$ and b_{i_0}. This sequence is summable and

$$\sum \widetilde{a}_i = C + b_{i_0} = An + x + b_{i_0} \geq An + B,$$

$$\sum \widetilde{a}_i = C + b_{i_0} \leq A + B(n-1) + b_{i_0} \leq A + Bn.$$

Since

$$\sum_{\widetilde{a}_i < A} \widetilde{a}_i = C \geq nA,$$

and there is exactly one term in $\{\widetilde{a}_i\}$ in $[A, \infty)$, the sequence meets the conditions of Theorem VI.2.1. Thus there is an operator S_1 with A and B as eigenvalues, $\sigma(S_1) \subseteq \{0\} \cup [A, B]$ and diagonal $\{\widetilde{a}_i\}$. Define $\{\widetilde{b}_i\}$ to be the sequence $\{b_i\}_{i \neq i_0}$. Let S_2 be the diagonal operator with $\{\widetilde{b}_i\}$ on the diagonal. The operator $E = S_1 \oplus S_2$ is the desired operator.

Next, assume $C \in (A + B(n-1), A + B(n-1) + D)$ and set $x = C - A - B(n-1)$. Since $x < D$ and $x < C$, there are $N, M \in \mathbb{N}$ such that

$$\sum_{i=1}^{N} a_i \geq x \quad \text{and} \quad \sum_{i=1}^{M} (B - b_i) \geq x.$$

Apply Lemma III.2.1 part (i) to the sequences $\{a_i\}_{i=1}^{N}$ and $\{b_i\}_{i=1}^{M}$ with $\eta_0 = x$ to get new sequences $\{\widetilde{a}_i\}_{i=1}^{N}$ and $\{\widetilde{b}_i\}_{i=1}^{M}$ satisfying (III.2.2) and (III.2.3). Let $\{\widetilde{b}_i\}_{i=1}^{\infty}$ be the sequence consisting of $\{\widetilde{b}_i\}_{i=1}^{N}$ and $\{b_i\}_{i=N+1}^{\infty}$ and similarly define $\{\widetilde{a}_i\}$. We purposely omit indexing for $\{\widetilde{a}_i\}$ since the original sequence $\{a_i\}$ might be either finite or infinite. Set

$$\widetilde{C} = \sum \widetilde{a}_i \quad \text{and} \quad \widetilde{D} = \sum (B - \widetilde{b}_i). \tag{VI.3.12}$$

We have $\widetilde{C} = A + B(n-1)$ and we can apply the previous case to get an operator $\widetilde{E}$ with $\{A, B\} \subseteq \sigma(\widetilde{E}) \subseteq \{0\} \cup [A, B]$ with diagonal consisting of $\{\widetilde{a}_i\}$ and $\{\widetilde{b}_i\}$. Then, Lemma III.2.1 part (ii) yields an operator E with the same spectrum as $\widetilde{E}$ and diagonal $\{a_i\} \cup \{b_i\}$.

Finally, assume $C = A + B(n-1) + D$. First, we look at the case where $C = A$. This implies $n = 1$ and $D = 0$. Thus, $\{b_i\}$ is an infinite sequence with each term equal to B. By Theorem VI.3.3 there is a projection P such that AP has diagonal $\{a_i\}$. Let F be the diagonal operator with $\{b_i\}$ on the diagonal. Then $AP \oplus F$ has the desired spectrum and diagonal. Now, we may assume $C > A$.

Arrange the sequence $\{a_i\}$ in nonincreasing order and define

$$M_0 = \max\left\{m : \sum_{i=1}^{m} a_i \leq A\right\} \quad \text{and} \quad x = A - \sum_{i=1}^{M_0} a_i.$$

Observe that $M_0 \geq 1$ and there is $N \geq M_0 + 1$ such that

$$\sum_{i=M_0+1}^{N} a_i \geq x.$$

It is also clear that

$$\sum_{i=1}^{M_0} (A - a_i) \geq x.$$

Apply Lemma III.2.1 part (i) to the sequences $\{a_i\}_{i=M_0+1}^{N}$ and $\{a_i\}_{i=1}^{M_0}$ on the interval $[0, A]$ with $\eta_0 = x$ to get new sequences $\{\widetilde{a}_i\}_{i=M_0+1}^{N}$ and $\{\widetilde{a}_i\}_{i=1}^{M_0}$ satisfying (III.2.2) and (III.2.3). Let $\{\widetilde{a}_i\}$ be the sequence consisting of $\{\widetilde{a}_i\}_{i=1}^{N}$ and $\{a_i\}_{i \geq N+1}$. By

(III.2.3) observe that

$$\sum_{i=1}^{M_0}\widetilde{a}_i = \sum_{i=1}^{M_0} a_i + x = A, \quad \text{and} \quad \sum_{i\geq M_0+1}\widetilde{a}_i = \sum_{i\geq M_0+1} a_i - x = C - A.$$

Thus, by Theorem VI.3.3 we can construct a rank one projection P such that the operator AP has diagonal $\{\widetilde{a}_i\}_{i=1}^{M_0}$. Define

$$a = \sum_{i\geq M_0+1}\widetilde{a}_i \quad \text{and} \quad b = \sum_{i=1}^{\infty}(B - b_i),$$

and note that $a - b = C - A - D = (n-1)B$. Thus, by Theorem VI.3.3 there is a projection Q such that BQ has diagonal consisting of $\{\widetilde{a}_i\}_{i\geq M_0+1}$ and $\{b_i\}$. Now, $\widetilde{E} = AP \oplus BQ$ has diagonal consisting of $\{\widetilde{a}_i\}$ and $\{b_i\}$ and $\sigma(\widetilde{E}) = \{0, A, B\}$. Then, Lemma III.2.1 part (ii) yields an operator E with the same spectrum as $\widetilde{E}$ and diagonal $\{a_i\} \cup \{b_i\}$.

Case 4. Assume $C \in [0, A - B + D]$.

Using the same argument as in Case 3, it suffices to consider only countable sequences $\{d_i\}$. Note that it is implicitly assumed that $D \geq B - A > 0$. First, assume $D = B - A$. This implies $C = 0$ and $a_i = 0$ for all i. Since $\sum(B - b_i) = B - A$, by Theorem VI.3.3, there exists a projection P such that $(B - A)P$ has diagonal $\{B - b_i\}$. Thus, $E = BI - (B - A)P$ has the desired spectrum and diagonal $\{b_i\}$. For the rest of Case 4 we may assume $D > B - A$.

Now, assume $C = 0$. Reorder $\{b_i\}$ so that $b_1 = \min\{b_i\}$ and set $\eta_0 = b_1 - A$. We have

$$\sum_{i=2}^{\infty}(B - b_i) = D - (B - b_1) > B - A - B + b_1 = \eta_0.$$

So there is some N such that

$$\sum_{i=2}^{N}(B - b_i) \geq \eta_0.$$

Apply Lemma III.2.1 part (i) to $\{b_1\}$ and $\{b_i\}_{i=2}^N$ on the interval $[0, B]$ to obtain new sequences $\{\widetilde{b}_1\}$ and $\{\widetilde{b}_i\}_{i=2}^N$. Let $\{\widetilde{b}_i\}_{i=1}^\infty$ be the sequence consisting of $\{\widetilde{b}_i\}_{i=1}^N$ and $\{b_i\}_{i=N+1}^\infty$. Let $\widetilde{E}$ be the operator with $\{\widetilde{b}_i\}$ on the diagonal (recall all of the a_i are 0). Clearly, $\{A, B\} \subseteq \sigma(\widetilde{E}) \subseteq [A, B]$. Using Lemma III.2.1 part (ii) there exists an operator E with the desired diagonal and spectrum.

Finally, we assume $C > 0$. Again, assume $b_1 = \min\{b_i\}$. Fix any $\varepsilon > 0$ with $\varepsilon < \min(A, C)$. Since

$$\varepsilon + B - A < C + B - A \leq D = \sum_{i=2}^{\infty}(B - b_i) + (B - b_1),$$

by subtracting $(B - b_1)$ from both sides we have

$$\varepsilon + b_1 - A < \sum_{i=2}^{\infty}(B - b_i).$$

Thus, there exists $M \geq 2$ such that

$$\sum_{i=2}^{M}(B - b_i) > \varepsilon + b_1 - A.$$

Apply Lemma III.2.1 part (i) to the sequences $\{b_1\}$ and $\{b_i\}_{i=2}^M$ on the interval $[0, B]$, with $\eta_0 = \varepsilon + b_1 - A$, to obtain sequences $\{\widetilde{b}_1\}$ and $\{\widetilde{b}_i\}_{i=2}^M$. By Lemma III.2.1 part (i) we have

$$\widetilde{b}_1 = b_1 - (\varepsilon + b_1 - A) = A - \varepsilon,$$

and

$$\sum_{i=2}^{M}(B - \widetilde{b}_i) = \sum_{i=2}^{M}(B - b_i) - (\varepsilon + b_1 - A).$$

Let $\{\widetilde{d}_i\}$ be the sequence consisting of $\{a_i\}$, $\{\widetilde{b}_i\}_{i=1}^M$, and $\{b_i\}_{i=M+1}^\infty$. Set

$$\widetilde{C} = \sum_{\widetilde{d}_i < A} \widetilde{d}_i = C + A - \varepsilon,$$

and

$$\widetilde{D} = \sum_{\widetilde{d}_i \geq A} (B - \widetilde{d}_i) = \sum_{i=2}^{M} (B - \widetilde{b}_i) + \sum_{i=M+1}^{\infty} (B - b_i)$$

$$= D - (B - b_1) - (\varepsilon + b_1 - A) = D - B + A - \varepsilon.$$

Observe that $\widetilde{C} > A$. We also have

$$\widetilde{C} = A + \varepsilon = \widetilde{D} - D + B + C \leq \widetilde{D} - D + B + A - B + D = \widetilde{D} + A,$$

so that $\widetilde{C} \in [A, A + \widetilde{D}]$. By the argument in Case 3, there is an operator $\widetilde{E}$ with diagonal $\{\widetilde{d}_i\}$ and the desired spectrum. By Lemma III.2.1 part (ii) there is an operator E unitarily equivalent to E with diagonal $\{d_i\}$. This completes the proof of Theorem VI.3.1. □

VI.4. The nonsummable case of the Pythagorean theorem

The goal of this section is to prove the necessity part of our main theorem. The summable case was already shown in Section VI.2. The nonsummable case requires special arguments involving trace-class operators and Kadison's Theorem VI.3.3. In the terminology of Kadison [16, 17], this is a non-tight version of the Pythagorean theorem.

Theorem VI.4.1. *Suppose* $0 < A < B < \infty$. *Let* E *be a positive operator with* $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$. *Let* $\{e_i\}_{i \in I}$ *be an orthonormal basis for* $\mathcal{H}$ *and let* $d_i = \langle E e_i, e_i \rangle$. *If*

$$C = \sum_{d_i < A} d_i < \infty \quad \text{and} \quad D = \sum_{d_i \geq A} (B - d_i) < \infty, \tag{VI.4.13}$$

then

$$C \in \bigcup_{n=0}^{\infty} [nA, A + B(n-1) + D].$$

Furthermore, $K = B(I - P) - E$ *is a positive trace class operator on* $\mathcal{H}$*, where* P *is the orthogonal projection onto* $\ker(E) \subseteq \ker(K)$*.*

Observe that Theorem VI.4.1 does not require the assumption that $\{d_i\}$ is non-summable. However, if $\{d_i\}$ is summable, Theorem VI.4.1 gives only a necessary, but not sufficient, condition, see Theorem VI.2.3.

Proof of Theorem VI.4.1. We claim that it is sufficient to consider the case where $\{d_i\}$ is at most countable. The condition (VI.4.13) implies that the sequence $\{d_i\}$ contains at most countably many terms in $(0, B)$. Thus, we need only consider sequences $\{d_i\}$ which contain an uncountable number of terms equal to 0 or B. Let $\{e_i\}_{i \in I}$ be the orthonormal basis with respect to which E has diagonal $\{d_i\}_{i \in I}$. Let $J = \{i : d_i = 0\} \cup \{i : d_i = B\}$. Since E is a positive operator with $\|E\| = B$, for each $i \in J$, e_i is an eigenvector of E. Let E' be E acting on $\overline{\text{span}}\{e_i\}_{i \in I \setminus J}$. Note that E acting on $\overline{\text{span}}\{e_i\}_{i \in J}$ is B times some projection Q. Thus, we have the orthogonal decomposition $E = E' \oplus BQ$. The operator E' has countable (possibly finite) diagonal consisting of the terms of $\{d_i\}$ contained in $(0, B)$. Thus, E' has the same values of C and D as E. If the conclusions of the theorem hold for E', then because $E = E' \oplus BQ$, they also hold for E.

By the above, we can take indexing set to be $I = \mathbb{Z} \setminus \{0\}$. For convenience, we reorder the basis so that $d_i \in [A, B]$ for $i > 0$ and $d_i \in [0, A)$ for $i < 0$. The case when there are only finitely many $d_i \in [A, B]$, or $d_i \in [0, A)$, does not cause any extra difficulties, and it is left to the reader.

Let $k_i = \langle K e_i, e_i \rangle$ and $n_i = \langle P e_i, e_i \rangle$ be the diagonal entries of K and P, respectively. Observe that K is a positive operator and thus

$$B(1 - n_i) - d_i = k_i \geq 0 \qquad \text{for all } i \in \mathbb{Z} \setminus \{0\}. \tag{VI.4.14}$$

Since $Bn_i \leq B - d_i$, we have

$$\sum_{i=1}^{\infty} Bn_i \leq \sum_{i=1}^{\infty}(B - d_i) \leq D < \infty.$$

Hence,

$$\sum_{i=1}^{\infty} k_i = \sum_{i=1}^{\infty}(B - d_i) - B\sum_{i=1}^{\infty} n_i \leq D < \infty. \tag{VI.4.15}$$

Since $\sigma(E) \subseteq \{0\} \cup [A, B]$, we have $A(I - P) \leq E$. Thus, $B - Bn_i \leq \frac{B}{A}d_i$, which immediately shows

$$\sum_{i=1}^{\infty}(B - Bn_{-i}) \leq \frac{B}{A}\sum_{i=1}^{\infty} d_{-i} = \frac{BC}{A} < \infty.$$

Using (VI.4.14),

$$\sum_{i=1}^{\infty} k_{-i} = \sum_{i=1}^{\infty}(B(1 - n_{-i}) - d_{-i}) \leq \frac{BC}{A} - C < \infty. \tag{VI.4.16}$$

Since K is a positive operator, (VI.4.15) and (VI.4.16) show that K is trace class. Observe that the diagonal entries of P satisfy

$$a = \sum_{i=1}^{\infty} n_i < \infty \quad \text{and} \quad b = \sum_{i=1}^{\infty}(1 - n_{-i}) < \infty.$$

Despite the fact that the above splitting of $\{n_i\}$ may not be the same as in Theorem VI.3.3, for any $\alpha \in (0, 1)$ it differs only by a finite number of terms from the standard splitting such that $n_i < \alpha$ for $i < 0$ and $n_i \geq \alpha$ for $i > 0$. And this change does not affect the property of $a - b$ being an integer. Thus, by Theorem VI.3.3 applied to the projection P we have $n_0 := b - a \in \mathbb{Z}$. Using (VI.4.15), and (VI.4.16) again we have

$$\operatorname{tr}(K) = \sum_{i \in \mathbb{Z} \setminus \{0\}} k_i = D - C + Bn_0 \leq D + \frac{BC}{A} - C. \tag{VI.4.17}$$

This immediately yields the lower bound for C:

$$An_0 \le C. \tag{VI.4.18}$$

Since $A \in \sigma(E)$ we know that $B - A$ is an eigenvalue of K and thus $B - A \le \mathrm{tr}(K)$. Again using (VI.4.17) we see that

$$B - A \le D - C + Bn_0.$$

This yields the upper bound

$$C \le A + B(n_0 - 1) + D. \tag{VI.4.19}$$

If $n_0 \ge 0$ then (VI.4.18) and (VI.4.19) show that $C \in [n_0 A, A + B(n_0 - 1) + D]$ as desired. If $n_0 \le -1$ then $B(n_0 - 1) \le -B$ and thus (VI.4.19) and the fact that $C \ge 0$ shows $C \in [0, A - B + D]$ as desired. This completes the proof of Theorem VI.4.1. □

As a corollary of Theorems V.2.3, VI.3.1, and VI.4.1 we obtain the following result.

Corollary VI.4.2. *Let* $0 < A < B < \infty$ *and let* $\{d_i\}_{i \in I}$ *be a nonsummable sequence in* $[0, B]$. *The following are equivalent:*

(i) $\{d_i\}_{i \in I}$ *satisfies* (VI.3.9),

(ii) *there is a positive operator* E *on a Hilbert space* $\ell^2(I)$ *with* $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ *and diagonal* $\{d_i\}_{i \in I}$,

(iii) *there exists a frame* $\{f_i\}_{i \in I}$ *for some infinite dimensional Hilbert space* $\mathcal{H}$ *with optimal frame bounds* A *and* B *and* $d_i = ||f_i||^2$.

Proof. The equivalence (i) $\iff$ (ii) follows directly from Theorems VI.3.1 and VI.4.1. Assume (ii). By Theorem V.2.3, there exists a frame $\{f_i\}_{i \in I}$ with optimal frame bounds A and B and $d_i = ||f_i||^2$. This frame lives on a Hilbert space $\mathcal{H}$ with $\dim \mathcal{H}$ equal to rank of E. Since E is positive with infinite trace, $\mathcal{H}$ is infinite dimensional, which shows (iii). The implication (iii) $\implies$ (ii) similarly follows from Theorem V.2.3. $\square$

VI.5. Without zero in the spectrum

The goal of this section is to establish an analogue of Theorem VI.1.1 for positive operators without zero in the spectrum. This result turns out to be less involved than our main theorem. As a consequence, we obtain a characterization of norms of Riesz bases with optimal bounds A and B. In the finite case, we obtain this result immediately from the Schur-Horn Theorem.

Theorem VI.5.1. *Let $0 < A \leq B < \infty$. Let $\{d_i\}_{i=1}^{N+1}$ be a sequence in $[A, B]$. There is a positive operator $E : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ with $\{A, B\} \subseteq \sigma(E) \subseteq [A, B]$ with diagonal $\{d_i\}$ if an only if*

$$\sum_{i=1}^{N+1} d_i \in [AN + B, A + BN]. \tag{VI.5.20}$$

Without zero in the spectrum the diagonal must be in $[A, B]$, and thus there is no summable infinite dimensional case. We can reformulate the condition (VI.5.20) to something that generalizes to the infinite dimensional case.

Corollary VI.5.2. *Let $0 < A \leq B < \infty$. Let $\{d_i\}_{i=1}^{N+1}$ be a sequence in $[A, B]$. Define the numbers*

$$C = \sum_{i=1}^{N+1}(d_i - A), \quad D = \sum_{i=1}^{N+1}(B - d_i). \tag{VI.5.21}$$

There is a positive operator $E : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ *with* $\{A, B\} \subseteq \sigma(E) \subseteq [A, B]$ *with diagonal* $\{d_i\}$ *if and only if*

(VI.5.22) $$C, D \geq B - A.$$

Proof. The condition (VI.5.20) implies

$$C = \sum_{i=1}^{N+1} d_i - (N+1)A \geq AN + B - NA - A = B - A$$

and

$$D = (N+1)B - \sum_{i=1}^{N+1} d_i \geq NB + B - A - NB = B - A.$$

Conversely, it is also clear that these inequalities imply (VI.5.20). □

We can now state the infinite dimensional case.

Theorem VI.5.3. *Let* $0 < A \leq B < \infty$*. Let* $\{d_i\}_{i\in I}$ *be a sequence in* $[A, B]$*. Define*

(VI.5.23) $$C = \sum_{i\in I}(d_i - A), \quad D = \sum_{i\in I}(B - d_i).$$

There is a positive operator E *with* $\{A, B\} \subseteq \sigma(E) \subseteq [A, B]$ *with diagonal* $\{d_i\}$ *if an only if*

(VI.5.24) $$C, D \geq B - A.$$

Proof. We can assume that I is countable, since the non-separable case follows from simple modifications as in the proof of Theorem VI.3.1. Suppose that E is a positive operator as in Theorem VI.5.3. First, we assume $D < \infty$. The operator $BI - E$ is a positive trace class operator with trace D. This implies that $D = \sum(B - \lambda)$, where the sum runs over all eigenvalues λ of E, repeated according to multiplicity. We also see that each $x \in \sigma(E) \setminus \{B\}$ is an eigenvalue of E. Thus, A is an eigenvalue of E and $D \geq B - A$. Next, we assume $C < \infty$. The operator $E - AI$ is trace class with trace C. Since B is in the spectrum of E, it is an eigenvalue of E, and thus $C \geq B - A$. Finally, if $C = D = \infty$, then (VI.5.24) trivially holds.

Conversely, suppose that $\{d_i\}$ is a sequence in $[A, B]$ satisfying (VI.5.24). If we assume $C, D > B - A$, then we can find some $N \in \mathbb{N}$ such that both

$$\sum_{i=1}^{N+1}(B - d_i) \geq B - A \quad \text{and} \quad \sum_{i=1}^{N+1}(d_i - A) \geq B - A.$$

By Corollary VI.5.2, there is an operator E_1 on an $N+1$-dimensional Hilbert space $\mathcal{H}_{N+1}$ such that $\{A, B\} \subseteq \sigma(E_1) \subseteq [A, B]$ and diagonal $\{d_i\}_{i=1}^{N+1}$. Let E_2 be the diagonal operator on the infinite dimensional Hilbert space $\mathcal{H}_\infty$ with $\{d_i\}_{i=N+2}^{\infty}$ on the diagonal. Now, $E = E_1 \oplus E_2$ on $\mathcal{H}_{N+1} \oplus \mathcal{H}_\infty$ is the desired operator. Next, we assume $D = B - A$. By Theorem VI.3.3 there is a rank 1 operator K with eigenvalue $B - A$ and diagonal $\{B - d_i\}_{i=1}^{\infty}$. Then $E = BI - K$ is the desired operator. Finally, assume $C = B - A$. By Theorem VI.3.3 there is a rank 1 operator K with eigenvalue $B - A$ and diagonal $\{d_i - A\}_{i=1}^{\infty}$. Then $E = K + AI$ is the desired operator. $\square$

As a corollary of Theorem V.2.5 we have the following result.

Corollary VI.5.4. *Let* $0 < A \leq B < \infty$ *and let* $\{d_i\}$ *be a sequence in* $[A, B]$. *There exists a Riesz basis* $\{f_i\}$ *with optimal bounds* A *and* B *and* $d_i = ||f_i||^2$ *if and only if* (VI.5.24) *holds.*

VI.6. Examples

The goal of this section is to illustrate our main theorem. We start with the definition of the set of possible lower bounds of positive operators with a fixed diagonal.

Definition VI.6.1. Let $\{d_i\}_{i \in \mathbb{N}}$ be a given nonsummable sequence in $[0, 1]$. Define

$$\mathcal{A} = \left\{ A \in (0,1] \colon \begin{array}{l} \text{there exists } E \geq 0 \text{ with diagonal } \{d_i\}_{i\in\mathbb{N}} \\ \text{and } A \in \sigma(E) \subseteq \{0\} \cup [A, 1] \end{array} \right\}.$$

Without loss of generality we can assume that $\sup d_i = 1$. Indeed, if $\sup d_i < 1$, then by Theorem VI.1.1 there exists a positive operator E with diagonal $\{d_i\}$ and $\{A, 1\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, 1]$ for any $0 < A \leq 1$. This fact can also be deduced from a result of Kornelson and Larson [19, Theorem 6]. Thus, we have always $\mathcal{A} = (0, 1]$ and this case is not interesting.

Example VI.6.2. Take any $0 < \beta < 1$ and define $d_i = 1 - \beta^i$ for $i \in \mathbb{N}$. First, we determine the set $\mathcal{A}$ near 0. We claim that

(VI.6.25)
$$\begin{aligned} (0, 1-\beta] &\subseteq \mathcal{A} \qquad && \text{for } 1/2 \leq \beta < 1, \\ \mathcal{A} \cap (0, 1-\beta] &= [(1-2\beta)/(1-\beta), 1-\beta] \qquad && \text{for } 0 < \beta < 1/2. \end{aligned}$$

Indeed, if $A \in (0, 1-\beta]$, we have $C = 0$ and $D = \sum_{i=1}^{\infty} \beta^i = \beta/(1-\beta)$. The condition (VI.3.9) holds if and only if $A - 1 + D \geq 0$ and thus $A \geq (1-2\beta)/(1-\beta)$. This shows the first claim. Next, we claim there exists

(VI.6.26)
$$\delta = \delta(\beta) > 0 \qquad (1-\delta, 1) \cap \mathcal{A} = \varnothing.$$

Moreover, $1 \in \mathcal{A}$ if and only if β is of the form $\beta = N/(N+1)$ for some $N \in \mathbb{N}$ by a simple application of Theorem VI.3.3.

Indeed, assume that $A \in (1-\beta^i, 1-\beta^{i+1}]$ for some $i \in \mathbb{N}$. Then

$$C = i + \frac{\beta^{i+1} - \beta}{1-\beta}, \qquad D = \frac{\beta^{i+1}}{1-\beta}.$$

Suppose that $C \in [nA, A + n - 1 + D]$ for some $n \in \mathbb{N}$. Then

$$(1-\beta^i)n \leq An \leq C \leq A + n - 1 + D \leq n + \frac{\beta^{i+2}}{1-\beta}.$$

The upper bound on C yields $i \leq n + \frac{\beta}{1-\beta} - \beta^{i+1}$ and thus $i \leq n + \lfloor \frac{\beta}{1-\beta} \rfloor$. On the other hand, the lower bound $(1-\beta^i)(i - \lfloor \frac{\beta}{1-\beta} \rfloor) \leq C$ yields

$$i \geq \left\{ \frac{\beta}{1-\beta} \right\} (\beta^{-i} - 1), \tag{VI.6.27}$$

where $\{\cdot\}$ is the fractional part. Obviously, (VI.6.27) must fail for sufficiently large i provided that $\beta \neq N/(N+1)$ for some $N \in \mathbb{N}$. In the special case of $\beta = N/(N+1)$, the upper bound on C actually yields $i \leq n + \frac{\beta}{1-\beta} - 1$. A similar argument as before shows that the lower bound for C must fail for sufficiently large i (depending on β). Therefore, in either case we have (VI.6.26).

Finally, we claim that

$$\mathcal{A} = [(1-2\beta)/(1-\beta), 1-\beta] \qquad \text{for } 0 < \beta < 1/2. \tag{VI.6.28}$$

By (VI.6.25), it suffices to consider $A > 1-\beta$. Since (VI.6.27) fails for $0 < \beta < 1/2$ and $i \geq 2$, we have $(1-\beta^2, 1) \cap \mathcal{A} = \varnothing$. Moreover, $1 \notin \mathcal{A}$ by Theorem VI.3.3. Finally, if $A \in (1-\beta, 1-\beta^2]$ then $C = 1-\beta$ and $D = \frac{\beta^2}{1-\beta}$. It is easy to see that $A - 1 + D < C < A$. Thus, $\mathcal{A} \cap (1-\beta, 1-\beta^2] = \varnothing$, which shows (VI.6.28).

Example VI.6.3. Let $\beta \approx 0.57$ be the real root of $\beta^3 - (1-\beta)^2 = 0$, and take $d_i = 1 - \beta^i$ for $i \in \mathbb{N}$. We will show that

$$\mathcal{A} = (0, 1-\beta] \cup \left[1-\beta^2, \frac{1}{3}(2 + 2\beta - \beta^2)\right]. \tag{VI.6.29}$$

By previous consideration we have $(0, 1-\beta] \subseteq \mathcal{A}$. Moreover, a simple numerical calculation shows that the inequality (VI.6.27) fails for $i \geq 5$. Thus, $(1-\beta^5, 1] \cap \mathcal{A} = \varnothing$.

Assume that $A \in (1-\beta, 1-\beta^2)$. We have $C = 1-\beta$ and $D = \frac{\beta^2}{1-\beta}$. Note that $C < A$, but

$$A - 1 + D < \frac{\beta^2}{1-\beta} - \beta^2 = \frac{\beta^3}{1-\beta} = 1 - \beta = C$$

and thus $\mathcal{A} \cap (1-\beta, 1-\beta^2) = \varnothing$. But, if $A = 1-\beta^2$ then we have $A - 1 + D = C$, so that $1 - \beta^2 \in \mathcal{A}$.

Next, assume that $A \in (1-\beta^2, 1-\beta^3]$. We have $C = 2 - \beta - \beta^2$ and $D = \frac{\beta^3}{1-\beta} = 1 - \beta$. Since $\beta < \frac{3}{5}$ we see that $2\beta < 2 - \beta$ and

$$A \leq 1 - \beta^3 = 2\beta - \beta^2 < 2 - \beta - \beta^2 = C.$$

Now,

$$A + D \geq 1 - \beta^2 + 1 - \beta = C$$

so that $C \in [A, A + D]$ and $(1-\beta^2, 1-\beta^3] \subseteq \mathcal{A}$. A similar calculation shows that $(1-\beta^3, 1-\beta^4] \subseteq \mathcal{A}$.

Now, assume that $A \in (1-\beta^4, \frac{1}{3}(2+2\beta-\beta^2)]$, we have $C = 2 + 2\beta - \beta^2$, so that $3A \leq C$. We have $D = 2\beta - 1$, and using the fact that $\beta > \frac{1}{2}$ we easily see that

$$A + 2 + D \geq 1 - \beta^4 + 2 + 2\beta - 1 = 1 + 3\beta + \beta^2 \geq 2 + 2\beta - \beta^2 = C.$$

Thus $C \in [3A, A + 2B + D]$ and $(1-\beta^4, \frac{1}{3}(2+2\beta-\beta^2)] \subseteq \mathcal{A}$. Finally, assume $A \in (\frac{1}{3}(2+2\beta-\beta^2), 1-\beta^5]$. Again, we have $C = 2 + 2\beta - \beta^2$, so that $3A > C$. Using the numerical estimates $\beta \in (\frac{1}{2}, \frac{3}{5})$ we easily obtain $2A \leq C$. However,

$$A + 1 + D \leq 1 - \beta^5 + 1 + 2\beta - 1 = 2 - \beta + 2\beta^2 < 2 + 2\beta - \beta^2 = C,$$

which shows that $C \in (A + 1 + D, 3A)$ and thus $(\frac{1}{3}(2+2\beta-\beta^2), 1-\beta^5] \cap \mathcal{A} = \varnothing$. This shows (VI.6.29).

In general, determining the set $\mathcal{A}$ for sequences satisfying (VI.4.13) is not an easy task since it boils down to checking condition (VI.3.9) for all possible values of $0 < A < 1$. This often leads to computing countably many infinite series (VI.1.1) and verifying whether (VI.3.9) holds or not. In the above examples involving geometric series this task actually reduces to checking a finite number of conditions using properties (VI.6.25) and (VI.6.26). Nevertheless, we have the following general fact about $\mathcal{A}$.

Theorem VI.6.4. *Let* $\{d_i\}_{i\in\mathbb{N}}$ *be a sequence in* $[0,1]$ *with* $\sup d_i = 1$. *The set* $\mathcal{A}\cup\{0,1\}$ *is closed.*

Proof. For any $A \in (0,1]$ define the numbers

$$C(A) = \sum_{d_i<A} d_i \quad \text{and} \quad D(A) = \sum_{d_i\geq A}(1-d_i).$$

By Theorem VI.1.1 $A \notin \mathcal{A}$ if and only if $C(A), D(A) < \infty$ and there exists $n \in \mathbb{N}$

$$A + n - 2 + D(A) < C(A) < An. \tag{VI.6.30}$$

Let $A_0 \in (0,1)\setminus\mathcal{A}$. First, assume $A_0 \neq d_i$ for all $i \in \mathbb{N}$. This implies there is some $\varepsilon > 0$ such that for all $A \in (A_0-\varepsilon, A_0+\varepsilon)$ we have $C(A) = C(A_0)$ and $D(A) = D(A_0)$. By continuity, there exists $\delta > 0$ such that (VI.6.30) holds for $|A - A_0| < \delta$. Thus, $(A_0-\delta, A_0+\delta)\cap\mathcal{A} = \varnothing$.

Now, assume $A_0 = d_i$ for some $i \in \mathbb{N}$, and let $k \in \mathbb{N}$ be the number of terms in the sequence $\{d_i\}$ equal to A_0. There is some $\varepsilon > 0$ such that $(A_0-\varepsilon, A_0+\varepsilon)$ contains no $d_i \neq A_0$. Note that for $A \in (A-\varepsilon, A_0]$ we have $C(A) = C(A_0)$ and $D(A) = D(A_0)$. The same argument as above shows that there is some $\delta > 0$ such that $(A_0-\delta, A_0]\cap\mathcal{A} = \varnothing$. Finally, for each $A \in (A_0, A_0+\varepsilon)$ we have $C(A) = C(A_0) + kA_0$ and $D(A) = D(A_0) - k + kA_0$, and (VI.6.30) is equivalent to the existence of $n \in \mathbb{N}$ such that

$$A + n - k - 2 + D(A_0) < C(A_0) < A(n-k) + (A - A_0)k.$$

Since (VI.6.30) holds for $A = A_0$ with $n = n_0$, the above holds with $n = n_0 + k$ and $A \in (A_0, A_0+\delta)$ for some $\delta > 0$. This shows that $(A_0, A_0+\delta)\cap\mathcal{A} = \varnothing$. $\square$

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