EQUIVARIANT DERIVED CATEGORIES OF HYPERSURFACES ASSOCIATED TO A SUM OF POTENTIALS

by

BRONSON LIM

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2017

DISSERTATION APPROVAL PAGE

Student: Bronson Lim

Title: Equivariant Derived Categories of Hypersurfaces Associated to a Sum of Potentials

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Alexander Polishchuk	Chair
Dan Dugger	Member
Victor Ostrik	Member
Vadim Vologodsky	Member
Michael Kellman	Institutional Representative

and

Scott L. Pratt	Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2017

DISSERTATION ABSTRACT

Bronson Lim

Doctor of Philosophy

Department of Mathematics

June 2017

Title: Equivariant Derived Categories of Hypersurfaces Associated to a Sum of Potentials

We construct a semi-orthogonal decomposition for the equivariant derived category of a hypersurface associated to the sum of two potentials. More specifically, if f, g are two homogeneous polynomials of degree d defining smooth Calabi-Yau or general type hypersurfaces in $\mathbb{P}^n$, we construct a semi-orthogonal decomposition of $\mathcal{D}[V(f \oplus g)/\mu_d]$. Moreover, every component of the semi-orthogonal decomposition is explicitly given by Fourier-Mukai functors.

CURRICULUM VITAE

NAME OF AUTHOR: Bronson Lim

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, Oregon
California State University, San Bernardino, California

DEGREES AWARDED:

Doctor of Philosophy, 2017, University of Oregon
Master of Arts, 2011, California State University at San Bernardino
Bachelor of Arts, 2009, California State University at San Bernardino

AREAS OF SPECIAL INTEREST:

Complex Algebraic Geometry
Matrix Factorizations
Differential Graded Categories
Finite Group Theory

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2011-2017

Teaching Associate, California State University at San Bernardino, 2010-2011

GRANTS, AWARDS AND HONORS:

Frank Anderson Teaching Award, University of Oregon, 2016

Rose Hills Scholarship, University of Oregon, 2015

Johnson Fellowship, University of Oregon, 2014

Outstanding Thesis, California State University at San Bernardino, 2011

ACKNOWLEDGEMENTS

Firstly, I thank my advisor Sasha for teaching me algebraic geometry, for introducing me to this problem and other related projects, and for his continuing patience with me. I thank my colleagues at the University of Oregon, especially Nick Howell and Ben Dyer for listening to me ramble at them. I thank Peter Gilkey for helping me find my path. I thank my girlfriend Esther, for believing in me and for nudging me along in the writing process. I thank the mathematics department at the University of Oregon for having me for the past six years. Finally, I thank my mother for making all of this possible and encouraging me to pursue what I care about.

TABLE OF CONTENTS

Chapter Page

CHAPTER I

INTRODUCTION

1.1 Semi-orthogonal decompositions in algebraic geometry

To a space X, i.e. a smooth and projective variety or more generally a smooth and proper Deligne-Mumford stack, we can associate the bounded derived category of coherent sheaves on the space, denoted $\mathcal{D}(X)$. The category $\mathcal{D}(X)$ lives in the intersection between homological algebra and algebraic geometry and has proved to be a useful tool when applied to algebro-geometric problems.

The derived category is a strong invariant of X. For example, if X has ample or anti-ample canonical bundle, a derived equivalence $\mathcal{D}(X) \simeq \mathcal{D}(Y)$ gives rise to an isomorphism $X \simeq Y$. Additionally, additive invariants such as Hochschild homology and K-theory factor through $\mathcal{D}(X)$.

Of particular interest is when $\mathcal{D}(X)$ admits a semi-orthogonal decomposition (see Section 2.2 for the definition). Roughly, a semi-orthogonal decomposition is the analogue of a group extension for triangulated categories. If $\mathcal{D}(X)$ admits a semi-orthogonal decomposition, one can hope to further understand $\mathcal{D}(X)$, or sometimes X, using the components of the decomposition.

For example, projective spaces admit the Beilinson exceptional collection:

$$\mathcal{D}(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle.$$

This can be regarded as a categorical analogue of the isomorphism $H^*(\mathbb{P}^n; \mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1})$.

An interesting example relating the derived category to birational geometry is given by cubic fourfolds. Let $X \subset \mathbb{P}^5$ be a cubic fourfold, then there is a decomposition

$$\mathcal{D}(X) = \langle \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}, \mathcal{A}_X \rangle,$$

where $\mathcal{A}_X$ is characterized as the left-orthogonal to the exceptional collection of line bundlees $\langle \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O} \rangle$ and is sometimes called the Kuznetsov subcategory (or the Kuznetsov component). It is conjectured, [17, Conjecture 1.1], that X is rational if and only if $\mathcal{A}_X$ is equivalent to $\mathcal{D}(S)$ for S a K3-surface. We will come back to this example shortly.

There are many more examples of using derived categories and semi-orthogonal decompositions to study varieties. We refer the reader to the surveys [4, Sections 4 and 5] and [7, Section 2].

1.2 Orlov's Theorem

This project was discovered when trying to understand Orlov's theorem relating the derived category of a projective hypersurface to the category of graded matrix factorizations of the defining function. We recall this theorem now to motivate the main result.

Let k be an algebraically closed field of characteristic zero and V a vector space over k of dimension n. Assume $f \in \mathrm{Sym}^d(V^\vee)$ defines a smooth projective hypersurface, say $X_f = V(f) \subset \mathbb{P}(V)$. We call f a *potential.* Let $\mathrm{HMF}^{gr}(f)$ denote the homotopy category of graded matrix factorizations of the potential f (see Section 5.1). Objects of $\mathrm{HMF}^{gr}(f)$ are $\mathbb{Z}/2$-graded, curved complexes of $\mathbb{G}_m$-equivariant vector bundles on V with curvature f. There is a natural differential on the space of morphisms between two matrix factorizations. The category $\mathrm{HMF}^{gr}(f)$ is the corresponding homotopy category.

A relationship between $\mathrm{HMF}^{gr}(f)$ and $\mathcal{D}(X_f)$ was discovered by Orlov in [20]. Orlov constructs two $\mathbb{Z}$-indexed families of exact functors $\Psi_i \colon \mathcal{D}(X_f) \to \mathrm{HMF}^{gr}(f)$ and $\Phi_i \colon \mathrm{HMF}^{gr}(f) \to \mathcal{D}(X_f)$. If X_f is Fano or Calabi-Yau, then Φ_i is a full embedding. If X_f is general type or Calabi-Yau, then Ψ_i is a full embedding. Moreover, the semi-orthogonal complement is explicitly determined.

Orlov's Theorem. *[20, Theorem 3.11] Let f be as above. For each $i \in \mathbb{Z}$, we have the following semi-orthogonal decompositions:*

$$\begin{aligned} \mathit{Fano} &: \mathcal{D}(X_f) = \langle \mathcal{O}_{X_f}(-i-n+d+1), \dots, \mathcal{O}_{X_f}(-i), \Phi_i \mathrm{HMF}^{gr}(f) \rangle; \\ \mathit{General\ Type} &: \mathrm{HMF}^{gr}(f) = \langle k^{stab}(-i), \dots, k^{stab}(-i+n-d+1), \Psi_i \mathcal{D}(X_f) \rangle; \\ \mathit{Calabi\text{-}Yau} &: \Phi_i, \Psi_i \textit{ induce mutual inverse equivalences } \mathcal{D}(X_f) \cong \mathrm{HMF}^{gr}(f). \end{aligned}$$

Here k^{stab} is a certain Koszul matrix factorization (see Example 5.1.2) associated to the residue field of $\mathrm{Sym}(V^\vee)$ *at the origin.*

1.3 Adding two potentials.

Let f, g be homogeneous polynomials of degree d defining smooth hypersurfaces $X_f \subset \mathbb{P}^{m-1}$ and $X_g \subset \mathbb{P}^{n-1}$. Let $X = V(f \oplus g) \subset \mathbb{P}^{m+n-1}$. Then X is smooth since X_f and X_g are smooth (see Proposition 3.1.1).

Suppose $d \geq \max\{m, n\}$. Then there is a $\mathbb{Z}$-indexed family of embeddings $\Psi_i : \mathcal{D}(X_f) \to \mathrm{HMF}^{gr}(f)$ and similarly $\Psi_j : \mathcal{D}(X_g) \to \mathrm{HMF}^{gr}(g)$. By tensoring, we can consider the family of embeddings:

$$\Psi_{i,j} \colon \mathcal{D}(X_f \times X_g) \cong \mathcal{D}(X_f) \otimes \mathcal{D}(X_g) \to \mathrm{HMF}^{gr}(f) \otimes \mathrm{HMF}^{gr}(g).$$

where $\Psi_{i,j} = \Psi_i \otimes \Psi_j$ and the tensor product is understood to be taken in suitable dg-enhancements[1]. We further have an identification, see [1, Corollary 5.18]

$$\mathrm{HMF}^{gr}(f) \otimes \mathrm{HMF}^{gr}(g) \cong \mathrm{HMF}^{gr,\mu_d}(f \oplus g).$$

where μ_d acts on the g variables.

If in addition $d \leq n + m$, it was noticed in [2, Example 3.10] that we can then embed $\mathrm{HMF}^{gr,\mu_d}(f \oplus g)$ into $\mathcal{D}[X/\mu_d]$ using Orlov's Theorem a second time. Fix one such embedding to get a doubly indexed family of fully-faithful functors $\Xi_{i,j} \colon \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d]$. The complement consists of mn exceptional objects, $d - m$ copies of $\mathcal{D}(X_g)$, and $d - n$ copies of $\mathcal{D}(X_f)$. Specifically, we have

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{A}, \mathcal{K}, \mathcal{D}_f, \mathcal{D}_g, \mathcal{D}_{fg} \rangle$$

where $\mathcal{A}$ consists of $(m+n-d)d$ line bundles, $\mathcal{K} \cong \langle k^{stab}(-i), \ldots, k^{stab}(-i+m-d+1)\rangle \otimes \langle k^{stab}(-j), \ldots, k^{stab}(-j+n-d+1)\rangle$, $\mathcal{D}_f = \Phi_i \mathcal{D}(X_f) \otimes \langle k^{stab}(-j), \ldots, k^{stab}(-j+n-d+1)\rangle$, $\mathcal{D}_g = \langle k^{stab}(-i), \ldots, k^{stab}(-i+m-d+1)\rangle \otimes \Phi_j \mathcal{D}(X_g)$, and $\mathcal{D}_{fg} = \Xi_{i,j}\mathcal{D}(X_f \times X_g)$.

[1] It is known that Orlov's functors lift to the dg level [10]

These functors are not easy to compute and, with the exception of $\mathcal{A}$, explicitly understanding the left and right semi-orthogonal complements to the image of $\Xi_{i,j}$ as μ_d-equivariant complexes of sheaves on X is not easy.[2]

1.4 Main result

In this dissertation, we give a more geometric definition of the functors $\Xi_{i,j}$ and show that they, miraculously, remain embeddings even if $d > n+m$. Moreover, we explicitly determine the other components in the associated semi-orthogonal decomposition.

Main Theorem. *If $m, n \geq 2$ and $d \geq \max\{m, n\}$, then there is a semi-orthogonal decomposition*

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_{g1}, \mathcal{D}_{fg}, \mathcal{D}_{g2}, \mathcal{D}_f, \mathcal{A} \rangle.$$

Here $\mathcal{D}_{g1}$ and $\mathcal{D}_{g2}$ collectively consist of $d-m$ twists of $\mathcal{D}(X_g)$ (Section 3.5), $\mathcal{D}_f$ consists of $d-n$ twists of $\mathcal{D}(X_f)$ (Section 3.4), $\mathcal{D}_{fg}$ is the image of $\Xi_{-m,-n}$ (Section 3.6), and $\mathcal{A}$ consists of an exceptional collection of line bundles (Section 3.3).

To align with the picture given by Orlov's theorem we can mutate the decomposition; however, it gets complicated quickly as we will see in Chapter V. As stated, each of the components has a simple description given by explicit Fourier-Mukai functors.

Using Orlov's theorem we can verify the main theorem in the cases that it holds. We do that, numerically, for the special case of surfaces in $\mathbb{P}^3$. In particular, our decomposition furnishes a full exceptional collection. We then show our

[2]In low dimensional cases, one can compute the image of $\mathcal{K}$ and, for an appropriate choice of functor, get twists of powers of $\Omega_{\mathbb{P}^N}|_X$.

theorem gives a full exceptional collection in the cases Orlov's theorem doesn't hold. We finish by discusssing an interesting class of cubic fourfolds.

In the case of surfaces in $\mathbb{P}^3$, Orlov's theorem holds for $d \leq 4$.

Example 1.4.1 (Surfaces with $d \leq 4$)**.** Suppose $d \leq 4$ and $m = n = 2$. Let f, g be homogeneous polynomials defining d points in $\mathbb{P}^1$. Set $S = V(f \oplus g) \subset \mathbb{P}^3$ so that S is a degree d surface in $\mathbb{P}^3$.

If $d = 2$, then Orlov's decomposition is

$$\mathcal{D}[S/\mu_2] = \langle \mathcal{O}(-1)(\chi^{0,1}), \mathcal{O}(\chi^{0,1}), \mathrm{HMF}^{gr,\mu_2}(f \oplus g) \rangle$$

and

$$\mathrm{HMF}^{gr,\mu_2}(f \oplus g) \cong \mathcal{D}(X_f \times X_g)$$

which contributes 4 exceptional objects for a total of 8.

Alternatively, $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and Beilinson's decomposition gives

$$\mathcal{D}[S/\mu_2] = \langle \mathcal{O}(-1,-1)(\chi^{0,1}), \mathcal{O}(-1,0)(\chi^{0,1}), \mathcal{O}(0,-1)(\chi^{0,1}), \mathcal{O}(\chi^{0,1}) \rangle$$

which is 8 exceptional objects.

Our decomposition gives

$$\mathcal{D}[S/\mu_2] = \langle \mathcal{D}_{fg}, \mathcal{A} \rangle$$

where $\mathcal{A}$ consists of 4 exceptional line bundles and $\mathcal{D}_{fg}$ consists of 4 exceptional objects for a total of 8.

If $d = 3$, we have Orlov's decomposition

$$\mathcal{D}[S/\mu_3] = \langle \mathcal{O}(-1)(\chi^{0,1,2}), \mathrm{HMF}^{gr,\mu_3}(f \oplus g)(\chi^{0,1,2}) \rangle.$$

The category $\mathrm{HMF}^{gr,\mu_3}(f \oplus g)$ has a semi-orthogonal decomposition

$$\mathrm{HMF}^{gr,\mu_3}(f \oplus g) = \langle k^{stab}(-i), \mathcal{D}(X_f), \mathcal{D}(X_g), \mathcal{D}(X_f \times X_g) \rangle$$

which is 16 exceptional objects for a grand total of 19 exceptional objects.

Our decomposition is

$$\mathcal{D}[S/\mu_3] = \langle \mathcal{D}_g, \mathcal{D}_{fg}, \mathcal{D}_f, \mathcal{A} \rangle$$

where $\mathcal{A}$ again has 4 exceptional objects, $\mathcal{D}_f$ has 3, $\mathcal{D}_{fg}$ has 9, and $\mathcal{D}_g$ has 3 for a grand total of 19.

If $d = 4$, we have a quartic K3 surface. Orlov's theorem gives an equivalence of categories:

$$\mathcal{D}[S/\mu_4] \cong \mathrm{HMF}^{gr,\mu_4}(f \oplus g)$$

and we have the equivalence

$$\mathrm{HMF}^{gr,\mu_4}(f \oplus g) = \mathrm{HMF}^{gr}(f) \otimes \mathrm{HMF}^{gr}(g)$$

so that

$$\mathrm{HMF}^{gr,\mu_4} = \langle \mathcal{K}, \mathcal{G}, \mathcal{F}, \mathcal{D}(X_f \times X_g) \rangle$$

where $\mathcal{G}$ consists of two copies of $\mathcal{D}(X_g)$ and similarly for $\mathcal{F}$. The category $\mathcal{K}$ consists of stabilizations of the residue field. There are 4 exceptional objects here.

The categories $\mathcal{G}, \mathcal{F}$ both have 8 exceptional objects and the category $\mathcal{D}(X_f \times X_g)$ has 16. The grand total is 36.

Our decomposition yields

$$\mathcal{D}[S/\mu_4] \cong \langle \mathcal{D}_g, \mathcal{D}_{fg}, \mathcal{D}_f, \mathcal{A} \rangle.$$

In this case $\mathcal{A}$ consists of 4 exceptional objects, $\mathcal{D}_g$ consists of 8 exceptional objects, $\mathcal{D}_{fg}$ consists of 16 exceptional objects, and $\mathcal{D}_f$ consist of 8 exceptional objects. The grand total is 36 exceptional objects.

Example 1.4.2 (Surfaces with $d > 4$)**.** If $d > 4$, Orlov's theorem no longer holds. In this case, we get an exceptional collection. Indeed, the decomposition in the main theorem gives

$$\mathcal{D}[S/\mu_d] = \langle \mathcal{D}_g, \mathcal{D}_{fg}, \mathcal{D}_f, \mathcal{A} \rangle,$$

which is an exceptional collection as in Example 1.4.1. The category $\mathcal{A}$ still has 4 exceptional objects. The categories $\mathcal{D}_g$ and $\mathcal{D}_f$ have $d(d-2)$ exceptional objects. The category $\mathcal{D}_{fg}$ has d^2 exceptional objects for a total of $3d^2 - 4d + 4$ exceptional objects.

Example 1.4.3 (Cubic fourfolds from genus 1 curves)**.** Here is an interesting example which recovers Orlov's theorem in the case $m = n = d = 3$. In this case X_f and X_g are genus 1 curves and X is a cubic fourfold. The μ_3-equivariant Orlov decomposition is

$$\mathcal{D}[X/\mu_3] = \langle \mathcal{O}(-2)(\chi^{0,1,2}), \mathcal{O}(-1)(\chi^{0,1,2}), \mathcal{O}(\chi^{0,1,2}), \mathcal{A}_X^{\mu_3} \rangle.$$

The μ_3-equivariant Kuznetsov component is equivalent to $\mathrm{HMF}^{gr,\mu_3}(f \oplus g)$ which is equivalent to $\mathcal{D}(X_f \times X_g)$.

The decomposition we give is

$$\mathcal{D}[X/\mu_3] = \langle \mathcal{D}(X_f \times X_g), \mathcal{O}(-4)(\chi), \mathcal{O}(-3)(\chi^{1,2}), \mathcal{O}(-2)(\chi^{0,1,2}), \mathcal{O}(-1)(\chi^{0,1}), \mathcal{O} \rangle.$$

It's easy to see that the decompositions differ by a sequence of mutations and possibly tensoring by a line bundle.[3]

This example was considered in [2, Example 4.7]. We conjecture that these cubic fourfolds are rational. If so, then by Kuznetsov's conjecture, $\mathcal{A}_X \cong \mathcal{D}(S)$ for some K3-surface S and there is an action of μ_3 on $\mathcal{D}(S)$ such that the μ_3-equivariant category, $\mathcal{D}(S)^{\mu_3}$ is equivalent to $\mathcal{D}(X_f \times X_g)$. It would be very interesting to verify this conjecture: explicitly determine the K3 surface S and understand the μ_3-action on $\mathcal{D}(S)$.

1.5 Further Work

We discuss four conjectures for future work related to the Main Theorem and discuss the organization of Chapters 2-5.

The first natural extension is to the case of $k \geq 2$ hypersurfaces. In particular, suppose $f_1, \dots, f_k$ defined k smooth degree d-hypersurfaces $X_i \subset \mathbb{P}(V_i)$ which are Calabi-Yau or general type. Let $X = V(f_1 \oplus \cdots \oplus f_k) \subset \mathbb{P}(\oplus_i V_i)$. Then we can use the technique discussed in Section 1.3 to see that the μ_d^{k-1}-equivariant derived category of X (with the natural action of μ_d^{k-1}) has semi-orthogonal components of

[3]Theorem 5.5.1 shows they agree on points. This implies they differ by an autoequivalence that fixes points. Since $X_f \times X_g$ is an Abelian variety, the result follows.

the form $\mathcal{D}(X_1 \times \cdots \times X_k)$, copies of subproducts of the X_i, and an exceptional collection of line bundles.

Conjecture 1.5.1. *Suppose $d \geq \max\{\dim(V_i)\}$, then there is a semi-orthogonal decomposition of $\mathcal{D}[X/\mu_d^{k-1}]$ with components:*

(i) *a copy of $\mathcal{D}(X_1 \times \cdots \times X_k)$;*

(ii) *copies of subproducts $\mathcal{D}(X_{i_1} \times \cdots \times X_{i_t})$, for a subset $\{i_1, \ldots, i_t\} \subset \{1, \ldots, k\}$, depending on d and $\dim(V_i)$;*

(iii) *an exceptional collection of line bundles.*

The decomposition in the Main Theorem should theoretically also hold in weighted projective spaces and for finitely many stacky hypersurfaces. Indeed, Orlov's theorem is still valid here and the functors involved are still well defined. We state the conjecture for two stacky hypersurfaces.

Conjecture 1.5.2. *Let $X_f = V(f) \subset \mathbb{P}(a_1, \cdots, a_m)$ and $X_g = V(g) \subset \mathbb{P}(b_1, \ldots, b_n)$ be smooth, degree d hypersurfaces (regarded as stacks) in the weighted projective stacks with weights $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$. Set $X = V(f \oplus g) \subset \mathbb{P}(a_1, \ldots, a_m, b_1, \ldots, b_n)$. Suppose $d \geq \max\{\sum a_i, \sum b_i\}$, then there is a semi-orthogonal decomposition of $\mathcal{D}[X/\mu_d]$, where the components consist of*

- *an exceptional collection of line bundles of length;*
- *a copy of $\mathcal{D}(X_f \times X_g)$;*
- *copies of $\mathcal{D}(X_g)$ and $\mathcal{D}(X_f)$ depending on d and the weights a_i, b_i.*

The Main Theorem should extend to families of hypersurfaces. More specifically, let V_1, V_2 be finite dimensional vector spaces. Consider the hypersurface

$Y \subset \mathbb{P}(S^d(V_1)^\vee \oplus S^d(V_2)^\vee) \times \mathbb{P}(V_1 \oplus V_2)$ given by $([f_1 : f_2], [x_1 : x_2])$ such that $f_1(x_1) + f_2(x_2) = 0$. The multiplicative group $\mathbb{G}_m$ acts on Y by $t \cdot ([f_1 : f_2], [x_1 : x_2]) = ([f_1 : t^{-d} f_2], [x_1 : t x_2])$. Set $\mathcal{X} = [Y/\mathbb{G}_m]$ to be the corresponding quotient stack. This quotient stack is the relative analogue of $[V(f \oplus g)/\mu_d]$ over the base $\mathbb{P}(S^d(V_1)^\vee) \times \mathbb{P}(S^d(V_2)^\vee)$.

Let $\mathcal{H}_i \subset \mathbb{P}(S^d(V_i)^\vee) \times \mathbb{P}(V_i)$ be the universal hypersurfaces. The analogue of $X_f \times X_g$ is unfortunately not $\mathcal{H}_1 \times \mathcal{H}_2$ and is slightly more complicated. Consider the affine version of the universal hypersurfaces:

$$H_1 \times H_2 \subset S^d(V_1)^\vee) \times S^d(V_2)^\vee \times \mathbb{P}(V_1) \times \mathbb{P}(V_2).$$

Define an action of $\mathbb{G}_m^2$ on $H_1 \times H_2$ by

$$(t_1, t_2) \cdot (f_1, f_2, [v_1, v_2]) = (t_1 f_1, t_1 t_2^{-d} f_2, [v_1, v_2]).$$

Let $\mathcal{B} = [H_1 \times H_2/\mathbb{G}_m^2]$ be the corresponding quotient stack. This is the analogue of $X_{f_1} \times X_{f_2}$ in the sense that the coarse moduli of $\mathcal{B}$ is $\mathcal{H}_1 \times \mathcal{H}_2$ and the fiber over $([f_1], [f_2])$ under the projection $\mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{P}(S^d(V_1)^\vee) \times \mathbb{P}(S^d(V_2)^\vee)$ is $X_{f_1} \times X_{f_2}$.

The action of $\mathbb{G}_m^2$ is not effective with kernel μ_d and so $\mathcal{B}$ is a μ_d-gerbe over $\mathcal{H}_1 \times \mathcal{H}_2$. Over $\mathcal{B}$ there is a $\mathbb{P}^1$-bundle given by descending the bundle $\mathbb{P}_{\mathcal{H}_1 \times \mathcal{H}_2}(\mathcal{O}_{H_1}(-1) \boxplus \mathcal{O}_{X_2}(-1))$ to $\mathcal{B}$, i.e. the $\mathbb{G}_m^2$ action on $H_1 \times H_2$ lifts to this bundle via

$$(t_1, t_2) \cdot (f_1, f_2, [v_1 : v_2]) = (t_1 f_1, t_1 t_2^{-d} f_2, [v_1 : t_2 v_2]).$$

Let $\mathcal{P}$ be the corresponding μ_d-gerbe. We can now use $\mathcal{O}_{\mathcal{P}}$ to define a Fourier-Mukai functor $\Phi \colon \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{X})$.

Conjecture 1.5.3. *The functor Φ is fully-faithful under appropriate conditions on d and $\dim(V_i)$ with explicit semi-orthogonal complement consisting of copies of $\mathcal{D}(\mathcal{B})$, $\mathcal{D}(H_1)$, $\mathcal{D}(H_2)$, and an exceptional collection of line bundles.*

By specializing to a specific fiber, we could hope to extend the Main Theorem to possibly singular hypersurfaces, which is very interesting.

The Main Theorem may also generalize to the case where X_f and X_g are replaced with smooth complete intersections. However, it appears that the number of hypersurfaces we intersect to get X_f and X_g must be equal. Otherwise it is unclear how to add them and get something smooth. Here is a toy example indicating how it might be setup.

Example 1.5.1. Suppose f_1, f_2 are degree d polynomials defining a smooth complete intersection X_f in $\mathbb{P}(V_1)$ and g_1, g_2 are degree d defining a smooth complete intersection X_g in $\mathbb{P}(V_2)$. Then $X = V(f_1 \oplus g_1, f_2 \oplus g_2)$ define a smooth complete intersection in $\mathbb{P}(V_1 \oplus V_2)$. There is again an action of μ_d which scales the g_i variables.

If $d >> 0$, then it's easy to see the natural inclusions $X_f, X_g \hookrightarrow X$ induce fully-faithful functors $\mathcal{D}(X_f), \mathcal{D}(X_g) \hookrightarrow \mathcal{D}[X/\mu_d]$. The functors $\Xi_{-m,-n}$ still make sense and the proof of fully-faithfulness in the Main Theorem should carry over to this case. We conjecture that it is still fully-faithful and that there is a semi-orthogonal decomposition of $\mathcal{D}[X/\mu_d]$ analogous to the Main Theorem.

Example 1.5.2. If say $X_f = V(f_1, f_2) \subset \mathbb{P}(V_1)$ and $X_g = V(g) \subset \mathbb{P}(V_2)$, then the only possible choice for X is $X = V(f_1 \oplus g, f_2 \oplus g) \subset \mathbb{P}(V_1 \oplus V_2)$. This is the same as $V(f_1 - f_2, g)$ which is not always smooth even if X_f and X_g are smooth.

In Chapter II, we recall preliminary facts about triangulated and equivariant triangulated categories.In Chapter III we define all of the terms in the above decomposition and show they are embeddings. In Chapter IV we prove the Main Theorem and also discuss the special case $m = 1$, which is not covered by the Main Theorem. In Chapter V we compare our functors to the ones described in Section 1.3 on the structure sheaf of points.

CHAPTER II

PRELIMINARIES

2.1 Triangulated Categories

Throughout k is an algebraically closed field of characteristic zero. For an overview of triangulated categories in algebraic geometry see [14].

Definition 2.1.1. A **triangulated category** $\mathcal{T}$ is a k-linear category together with an autoequivalence $[1] : \mathcal{T} \to \mathcal{T}$ and a class of exact triangles

$$t \to u \to v \to t[1]$$

certain axioms, see [13, Section 1.1]. The autoequivalence [1] is sometimes called a *shift functor*.

Example 2.1.1. Most (but not all) examples come from Abelian categories. If $\mathcal{A}$ is an Abelian category, then the *derived category* of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) := \mathrm{Com}(\mathcal{A})[S^{-1}].$$

That is, we take the category of chain complexes[1] of objects in $\mathcal{A}$. Then we localize at the class of quasi-isomorphisms. The shift functor is given by shifting the complex *to the left*:

$$\mathcal{K}^{\cdot}[1] = \mathcal{K}^{\cdot+1}.$$

[1] We only consider bounded complexes in this dissertation. However, the use of unbounded complexes still has many uses.

Example 2.1.2. If $(X, \mathcal{O}_X)$ is a locally ringed space (for us X will be either a scheme or a stack), then the category of $\mathcal{O}_X$-modules is an Abelian category. By Example 2.1.1, we can consider the derived category of $\mathcal{O}_X$-modules, denoted $\mathcal{D}(\mathcal{O}_X - \mathrm{Mod})$.

If X is a locally Noetherian scheme (or stack), then we can consider the subcategory, $\mathfrak{Coh}(X)$, of coherent $\mathcal{O}_X$-modules. This is still Abelian and we define the derived category of X to be the derived category of the Abelian category $\mathfrak{Coh}(X)$:

$$\mathcal{D}(X) := \mathcal{D}(\mathfrak{Coh}(X)).$$

As mentioned in the introduction, the derived category is a strong invariant of X and is our primary object of study.

Another subcategory of $\mathcal{O}_X$-modules which is interesting is the subcategory of perfect complexes, denote $\mathfrak{Perf}(X)$. A complex $\mathcal{F}^{\cdot}$ is *perfect* if it is locally quasi-isomorphic to a bounded complex of locally free sheaves. In the case X is Noetherian, this is the same as being quasi-isomorphic to a complex of vector bundles. If X is quasi-projective and smooth, then $\mathfrak{Perf}(X) = \mathcal{D}(X)$. Thus the subcategory of perfect complexes is a categorical measure of smoothness (see Definition 5.2.1).

2.2 Semi-orthogonal decompositions

Semi-orthogonal decompositions allow us to decompose our triangulated category into simpler pieces. The first example was found by Beilinson in [3]. Roughly, the idea is the following: Suppose we have a triangulated subcategory $\mathcal{K}$ of a triangulated category $\mathcal{T}$. We can form the Drinfeld-Verdier localization of $\mathcal{T}$

by $\mathcal{K}$. Recall, the localization $\mathcal{C}$ is

$$\mathcal{C} = \mathcal{T}[\Sigma^{-1}]$$

where Σ is the class of morphisms, σ, such that $\mathrm{Cone}(\sigma) \in \mathcal{K}$. Consider the sequence of triangulated categories

$$\mathcal{K} \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} \mathcal{C}.$$

If $\mathcal{K}$ is nice enough (see 2.3), then the embedding functor ($\iota\colon \mathcal{K} \to \mathcal{T}$) will have left and right adjoints, say $\iota_{L,R}$. Take an object $t \in \mathcal{T}$, then there is an exact triangle in $\mathcal{T}$ of the form:

$$\iota(\iota_L(t)) \to t \to \mathrm{Cone} \to$$

where Cone is the cone of the map $\iota(\iota_L(t)) \to t$. If $\mathcal{C}$ is also nice enough, then Cone is in the image of the right adjoint to π. It's not hard to see that $\iota(\iota_L(t))$ and Cone are unique up to isomorphism and $\mathrm{Hom}_{\mathcal{T}}(\iota(\iota_L(t), \mathrm{Cone})) = 0$. This is made precise with the next definition.

Definition 2.2.1. Let $\mathcal{T}$ be a triangulated category. A **semi-orthogonal decomposition** of $\mathcal{T}$, written

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$$

is a sequence of full triangulated subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $\mathcal{T}$ such that:

(i) $\mathrm{Hom}_{\mathcal{T}}(a_i, a_j) = 0$ for $a_i \in \mathcal{A}_i$, $a_j \in \mathcal{A}_j$, and $i > j$;

(ii) For any $t \in \mathcal{T}$, there is a sequence of morphisms

$$0 = a_n \to a_{n-1} \to \cdots \to a_1 \to a_0 = t$$

where $\mathrm{Cone}(a_i \to a_{i-1}) \in \mathcal{A}_i$.

Condition (i) is called the *semi-orthogonality* condition and condition (ii) is called the *fullness* or *generation* condition.

Example 2.2.1. The most common examples of semi-orthogonal decompositions occur when $\mathcal{T} = \mathcal{D}(X)$ for some smooth projective scheme over k. In this case, there is often a vector bundle $\mathcal{E}$ such that $\mathrm{Ext}^*_X(\mathcal{E}, \mathcal{E}) \cong k[0]$. Such an object in $\mathcal{D}(X)$ is called **exceptional**. The subcategory generated by $\mathcal{E}$ is also abusively denoted by $\mathcal{E}$ and there are two semi-orthogonal decompositions (that it exists follows from Example 2.3.1):

$$\mathcal{D}(X) = \langle \mathcal{E}^\perp, \mathcal{E} \rangle = \langle \mathcal{E}, {}^\perp\mathcal{E} \rangle$$

where $\mathcal{E}^\perp = \{\mathcal{F}^\cdot \in \mathcal{D}(X) \mid \mathrm{Ext}^*_X(\mathcal{E}, \mathcal{F}^\cdot) = 0\}$ and ${}^\perp\mathcal{E}$ is defined similarly.

Example 2.2.2. Beilinson's example, [3], is given by

$$\mathcal{D}(\mathbb{P}^n) = \langle \mathcal{O}(-n), \ldots, \mathcal{O} \rangle.$$

Semi-orthogonality is Serre's theorem and fullness is given by the Beilinson spectral sequence (or equivalently Beilinson's resolution of the diagonal). In other words, $\mathcal{D}(\mathbb{P}^n)$ is made of iterated extensions of $\mathcal{D}(\mathrm{Spec}(k))$. As mentioned before, this is, in a very precise way, a categorical analogue of the fact

$$H^*(\mathbb{P}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1}).$$

Projective spaces are an important example of a scheme with the (rare) Diagonal Resolution Property[2]: If X is any smooth projective variety such that the structure sheaf of the diagonal $\mathcal{O}_\Delta$ in $X \times X$ has a resolution by sheaves of the form $\pi_1^*\mathcal{E}_i \otimes \pi_2^*\mathcal{F}_i$, then it has the Diagonal Resolution Property. An important consequence is that the full triangulated subcategory of $\mathcal{D}(X)$ generated by either the $\mathcal{E}_i$ or the $\mathcal{F}_i$ is $\mathcal{D}(X)$ itself. If in addition we have semi-orthogonality between them, then this furnishes a semi-orthogonal decomposition of $\mathcal{D}(X)$.

Example 2.2.3. It is not always the case that the derived category of projective scheme admits a semi-orthogonal decomposition. Indeed, let C be a smooth and proper curve of genus $g > 1$. Then Okawa proves, see [18], that $\mathcal{D}(C)$ does not admit a semi-orthogonal decomposition. For $g = 1$, there is no semi-orthogonal decomposition because the Serre functor is the shift functor. For $g = 0$, we have Beilinson's exceptional collection of Example 2.2.2. In a sense, this result can be viewed as saying that curves provide one dimensional building blocks for triangulated categories.

2.3 Admissible and saturated triangulated subcategories

As mentioned in Section 2.2, triangulated subcategories (or quotients) possessing adjoints lead the way to semi-orthogonal decompositions. In this section,

[2]There is a weaker diagonal resolution property, where we require that the diagonal is cut out by the zero locus of a section of a vector bundle on $X \times X$. See [22] for results in this direction.

we introduce admissibility and saturatedness that characterize those subcategories which possess adjoints.

Definition 2.3.1. Let $\mathcal{A} \subset \mathcal{T}$ be a full triangulated subcategory of a triangulated category. We say $\mathcal{A}$ is **admissible** if the embedding functor $\iota : \mathcal{A} \to \mathcal{T}$ has a left and right adjoint[3].

If $\mathcal{A}$ is admissible, then it follows formally that $\mathcal{T}$ admits two semi-orthogonal decompositions

$$\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle.$$

where

$$\mathcal{A}^{\perp} := \{t \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(a[i], t) = 0 \text{ for all } a \in \mathcal{A}, i \in \mathbb{Z}\};$$
$${}^{\perp}\mathcal{A} := \{t \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(t, a[i]) = 0 \text{ for all } a \in \mathcal{A}, i \in \mathbb{Z}\}.$$

We refer to $\mathcal{A}^{\perp}$ as the *right orthogonal* and ${}^{\perp}\mathcal{A}$ as the *left orthogonal* to $\mathcal{A}$ in $\mathcal{T}$.

Indeed, semi-orthogonality is by definition. Then suppose $\iota \colon \mathcal{A} \to \mathcal{T}$ is the embedding functor and $\iota_{L,R}$ are the left and right adjoints. It follows that

$$\iota(\iota_L(t)) \to t \to \mathrm{Cone} \to,$$

where Cone is the cone of the obvious mapping, gives $\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$. To see Cone $\in {}^{\perp}\mathcal{A}$ we apply $\mathrm{Hom}_{\mathcal{T}}(-, \iota(a))$ for some $a \in \mathcal{A}$:

$$\mathrm{Hom}_{\mathcal{T}}(\mathrm{Cone}, \iota(a)) \to \mathrm{Hom}_{\mathcal{T}}(t, \iota(a)) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{T}}(\iota(\iota_L(t)), \iota(a)) \to .$$

[3]Sometimes, we define left (or right) admissible to mean the existence of a left (or right) adjoint. We won't need these more general notions in this dissertation.

The last term is

$$\mathrm{Hom}_{\mathcal{T}}(\iota(\iota_L(t)), \iota(a)) \cong \mathrm{Hom}_{\mathcal{A}}(\iota_L(t), a) \cong \mathrm{Hom}_{\mathcal{T}}(t, \iota(a))$$

and ϵ is the obvious isomorphism.

Definition 2.3.2. A triangulated category $\mathcal{T}$ is called **saturated** if every cohomological functor (contravariant or covariant) $H\colon \mathcal{T} \to \mathrm{Vect}_k$ of finite type is representable.

We have the following important proposition regarding saturated subcategories, see [5, Proposition 2.6].

Proposition 2.3.1. *Let $\mathcal{A}$ be a saturated triangulated category and $\iota\colon \mathcal{A} \to \mathcal{T}$ is a full embedding. Then $\mathcal{A}$ is an admissible subcategory of $\mathcal{T}$.*

Example 2.3.1. The derived category of coherent sheaves on a smooth projective variety, X, is saturated, [5, Theorem 2.14]. If $\mathcal{E}$ is an exceptional object of $\mathcal{D}(X)$, then there is a full embedding $\iota_{\mathcal{E}}\colon \mathcal{D}(\mathrm{Spec}(k)) \to \mathcal{D}(X)$ given by $\iota_{\mathcal{E}}(V) = \mathcal{E} \otimes V$. This justifies Example 2.2.1.

Example 2.3.2. Let X be smooth projective and let $\mathcal{A}$ be the full triangulated subcategory of $\mathcal{D}(X)$ generated by the structure sheaves of closed points, $\mathcal{O}_p$ for $p \in X$. Then $\mathcal{A}$ is not saturated. If it were, then the functor of cohomology $\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{O}_X, -)$ would be representable. This is not possible as the only objects in $\mathcal{A}$ are torsion sheaves (with zero dimensional support) and there are always nonzero extensions between a torsion sheaf and itself, whereas cohomology of a sheaf with zero dimensional support is just global sections.

We will use the following proposition in conjuction with Proposition 2.6.1 in Section 4.5.

Proposition 2.3.2. *[5, Theorem 2.10] If $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ are full, saturated, triangulated subcategories such that $\langle \mathcal{A}, \mathcal{B} \rangle$ is semi-orthogonal, then $\langle \mathcal{A}, \mathcal{B} \rangle$ is saturated.*

2.4 Equivariant triangulated categories

The main triangulated categories we will concern ourselves with are derived categories of quotient stacks. The notion of a G-equivariant objects is central here.

Definition 2.4.1. Suppose G is a finite group. An action of G on a triangulated category $\mathcal{T}$ is the following data, [16, §3.1]:

- For every $g \in G$, an exact autoequivalence $g^* : \mathcal{T} \to \mathcal{T}$;
- For every $g, h \in G$, an isomorphism of functors $\varepsilon_{g,h} : (gh)^* \xrightarrow{\sim} h^* \circ g^*$ satisfying the usual associativity conditions.

Definition 2.4.2. A G-equivariant object of $\mathcal{T}$ is a pair (t, θ), where $t \in \mathcal{T}$ and θ is a collection of isomorphisms $\theta_g : t \xrightarrow{\sim} g^* t$ for all $g \in G$ satifying the usual associativity diagram. An object of $t \in \mathcal{T}$ together with an equivariant structure θ is called a **linearization** of t.

For any action of G on a triangulated category $\mathcal{T}$, we can form the category of equivariant objects of $\mathcal{T}$ denoted $\mathcal{T}^G$.[4]

Example 2.4.1. Suppose a finite group G acts on a scheme X, then there is an exact equivalence $\mathcal{D}[X/G] \cong \mathcal{D}(X)^G$, see [25, Section 3.8]. So in this case there is a

[4] It is not true that $\mathcal{T}^G$ is always triangulated, i.e. that the triangulated structure on $\mathcal{T}$ descends to $\mathcal{T}^G$, see [11, Example 8.4]. In fact, this example shows that even if the G-action lifts to the dg level, it may not be true that the G-equivariant objects of a pretriangulated dg category is pretriangulated.

natural triangulated structure on $\mathcal{D}(X)^G$. A good reference for equivariant derived categories of coherent sheaves is [8, Section 4].

Example 2.4.2. Let $(\mathcal{F}, \theta)$ be an equivariant object in $\mathcal{D}(X)$. Further, let $\chi : G \to \mathbb{G}_m$ be a multiplicative character of G. Define a new equivariant object $(\mathcal{F}(\chi), \theta \cdot \chi)$ where $\mathcal{F}(\chi) = \mathcal{F}$ as an object of $\mathcal{D}(X)$ but the maps $\theta_g \cdot \chi : \mathcal{F} \to g^*\mathcal{F}$ are twisted by χ.

If $\mathcal{F}$ is a vector bundle, then an equivariant structure is equivalent to a compatible fiberwise G-representation. Tensoring with a character corresponds to tensoring the fiberwise G-respresentation by the same character. Thus if $\mathcal{F}$ admits one linearization it can admit several distinct linearizations. We denote these adjusted linearizations by $\mathcal{F}(\chi)$ and say $\mathcal{F}(\chi)$ is the twist of $\mathcal{F}$ by χ.

More generally, if V is a representation of G, and $(\mathcal{F}, \theta)$ is a G-equivariant sheaf, we can tensor with V to get a new G-equivariant sheaf $(\mathcal{F} \otimes V, \theta \otimes \rho_V)$, where $\rho_V \colon G \to \mathrm{GL}(V)$ is the representation and $\mathcal{F} \otimes V$ is $\mathcal{F}^{\oplus \dim(V)}$. We will denote this new G-equivariant sheaf by $\mathcal{F} \otimes V$.

This can be made very precise as follows. Let $\pi \colon X \to \star$ be projection to a point. Giving $\star$ the trivial G-action makes π equivariant and so we have a G-equivariant pullback functor

$$\pi^* \colon \mathcal{D}[\star/G] \to \mathcal{D}[X/G].$$

The category $\mathcal{D}[\star/G]$ is the derived category of the category of G-representations. So by $\mathcal{F} \otimes V$, we mean the object $\mathcal{F} \otimes \pi^* V$.

2.5 Fourier-Mukai functors

Let X and Y be smooth projective varieties and denote the two projections

$$\pi_X\colon X \times Y \to X \text{ and } \pi_Y\colon X \times Y \to Y.$$

Definition 2.5.1. Let $\mathcal{K} \in \mathcal{D}(X \times Y)$. The induced **Fourier-Mukai transform** or **Fourier-Mukai functor** is the functor

$$\Phi_{\mathcal{K}}\colon \mathcal{D}(X) \to \mathcal{D}(Y)$$

given by

$$\mathcal{E}^{\cdot} \mapsto \pi_{Y*}(\pi_X^*(\mathcal{E}^{\cdot}) \otimes \mathcal{K}).$$

The object $\mathcal{K}$ is called a *Fourier-Mukai kernel*. It can be used to define a functor $\mathcal{D}(Y) \to \mathcal{D}(X)$ in an analogous way. In this dissertation, we will explicitly state which way the functor goes to avoid awful notation like $\Phi_{\mathcal{K}}^{X\to Y}$.

Remark 2.5.1. If G is a finite group acting on Y. We can define *equivariant Fourier-Mukai functors*. See [8, Section 4.1] for a short discussion and [1, Section 2] for a long discussion.

In this case, where X, Y are smooth projective, all triangulated functors $\mathcal{D}(X) \to \mathcal{D}(Y)$ are of Fourier-Mukai type. In fact, this is still true after in the quasi-compact separated world, [24, Theorem 8.9]

One important aspect of having a Fourier-Mukai functor is the existence of adjoints. Indeed, suppose $\mathcal{K}$ is a Fourier-Mukai kernel defining an exact functor

$\Phi_{\mathcal{K}} \colon \mathcal{D}(X) \to \mathcal{D}(Y)$. Define

$$\mathcal{K}_L = \mathcal{K}^\vee \otimes \pi_Y^* \omega_Y[\dim(Y)]$$

and

$$\mathcal{K}_R = \mathcal{K}^\vee \otimes \pi_X^* \omega_X[\dim(Y)].$$

Proposition 2.5.1 ([14, Proposition 5.9]). *Let $\Phi_{\mathcal{K}} \colon \mathcal{D}(X) \to \mathcal{D}(Y)$ be the Fourier-Mukai transform with Fourier-Mukai kernel $\mathcal{K}$. Then $\Phi_{\mathcal{K}_L}$ is the left adjoint and $\Phi_{\mathcal{K}_R}$ is the right adjoint to $\Phi_{\mathcal{K}}$.*

We shall also need the well known fully-faithfulness criterion of Bondal and Orlov. A consequence of the proof, which doesn't seem to be widely used, is that the target category need not be $\mathcal{D}(Y)$ for some smooth stack Y. The functor just needs to have a right adjoint. We will use $\mathcal{O}_p$ to denote the structure sheaf of a closed point $p \in X$.

Theorem 2.5.1 ([14, Proposition 7.1]). *Let X be a smooth projective variety over k and $\mathcal{T}$ be a triangulated category. Suppose $F : \mathcal{D}(X) \to \mathcal{T}$ is an exact functor with a right adjoint G. Then F is fully-faithful if and only if for any two closed points $x, y \in X$ we have*

$$\operatorname{Hom}_{\mathcal{T}}(F(\mathcal{O}_x), F(\mathcal{O}_y)[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ and } i \notin [0, \dim(X)]. \end{cases}$$

Proof. The proof in [14, Proposition 7.1] only requires that the functor F has a right adjoint. $\square$

Remark 2.5.2. We will use Theorem 2.5.1 when $\mathcal{T}$ is the derived category of a smooth quotient stack over k. In this case, the existence of a (left and) right adjoint is given by Proposition 2.5.1 and the existence of dualizing sheaves in this case.

Remark 2.5.3. It would be interesting to know if X can be replaced with a DM stack or, more specifically, a global quotient stack $[X/G]$ with G a finite group. That is, suppose $\Psi\colon \mathcal{D}[X/G] \to \mathcal{T}$ is an exact functor, then Ψ is fully-faithful if and only if for all stacky points $x, y \in [X/G]$ we have

$$\operatorname{Hom}_{\mathcal{T}}(F(\mathcal{O}_x), F(\mathcal{O}_y)[i]) = \begin{cases} k & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ and } i \notin [0, \dim(X)]. \end{cases}.$$

Note, since G is finite $\dim[X/G] = \dim(X)$. The current proof relies on using the Hilbert scheme; however, the G-Hilbert scheme is not as well understood, if it exists.

2.6 Spanning classes

The easy part to prove in a semi-orthogonal decomposition is vanishing of extension groups. The harder part is proving fullness. The notion of a spanning class, introduced by Bridgeland in [6], gives a natural way of proving fullness.

Definition 2.6.1. Let $\mathcal{T}$ be a triangulated category. A subclass of objects $\Omega \subset \mathcal{T}$ is called a **spanning class** if for every $t \in \mathcal{T}$ the following two conditions hold:

- $\operatorname{Hom}_{\mathcal{T}}(t, \omega[i]) = 0$ for all $\omega \in \Omega$ and all $i \in \mathbb{Z}$ implies $t = 0$;
- $\operatorname{Hom}_{\mathcal{T}}(\omega[i], t) = 0$ for all $\omega \in \Omega$ and all $i \in \mathbb{Z}$ implies $t = 0$.

In the presence of a Serre functor, it suffices to require only one of the two conditions.

Example 2.6.1. If X is a smooth projective variety over k, then a spanning class is furnished by the structure sheaves of closed points:

$$\Omega = \{\mathcal{O}_x \mid x \in X \text{ is a closed point}\}.$$

This is a derived analogue of the fact that a sheaf on a variety is zero if and only if the stalks at each closed point are zero.

More generally, if $\mathcal{X}$ is a smooth and proper Deligne-Mumford stack over k, then we can consider the coarse moduli space $\pi : \mathcal{X} \to \mathfrak{X}$ and take the following collection as a spanning class:

$$\Omega = \{\mathcal{O}_{\mathcal{Z}} \mid \mathcal{Z} \text{ is a closed substack of } \mathcal{X} \text{ and } \pi(\mathcal{Z}) \text{ is a closed point of } \mathfrak{X}\}.$$

In the cases we will consider, $\mathcal{X}$ is a global quotient stack $\mathcal{X} = [X/G]$ with G a finite group. The class Ω is provided by (twists of) orbits of points under the G-action. On one extreme, the orbit could be free: if $p \in X$ and $|G \cdot p| = |G|$, then set $Z = G \cdot p$ to be the corresponding G-cluster. Then $\mathcal{O}_Z \in \Omega$. On the other extreme, the orbit could be a single point: if $G \cdot p = p$, then for each irreducible representation of V, we have $\mathcal{O}_p \otimes V \in \Omega$. Our actions will either be one of these extremes so we do not discuss the intermediate cases.

The next proposition shows how useful spanning classes can be.

Proposition 2.6.1. *Suppose Ω is a spanning class for $\mathcal{T}$ and $\mathcal{A}$ is a full, admissible, triangulated subcategory containing Ω, then $\mathcal{A} = \mathcal{T}$.*

Proof. Since $\mathcal{A}$ is admissible, there is a semi-orthogonal decomposition of $\mathcal{T}$ of the form:

$$\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$$

The condition that $\mathcal{A}$ contains a spanning class implies that $\mathcal{A}^{\perp}$ must be trivial.

$\square$

Example 2.6.2. The condition that $\mathcal{A}$ is admissible cannot be removed. Indeed, let X be a smooth projective scheme. Let $\mathcal{A}$ be the full subcategory of $\mathcal{D}(X)$ from Example 2.3.2. Then, by definition, $\mathcal{A}$ has a spanning class; however, the only objects in $\mathcal{A}$ are torsion (with zero dimensional support) and so they cannot generate.

It will be convenient to use more than one embedding in Section 4.4. The following theorem tells us which other functors we can use.

Theorem 2.6.1. *Suppose $F : \mathcal{D}(X) \to \mathcal{T}$ is a full embedding where X is smooth and projective over k. Further suppose there exists a saturated subcategory $\mathcal{A}$ containing $F(\Omega)$, where Ω is a spanning class for $\mathcal{D}(X)$. Then $F(\mathcal{D}(X)) \subset \mathcal{A}$.*

Proof. It is sufficient to prove that the right adjoint $G : \mathcal{T} \to \mathcal{D}(X)$ is zero on $\mathcal{A}^{\perp}$. Suppose $b \in \mathcal{A}^{\perp}$. For every $\omega \in \Omega$ and $i \in \mathbb{Z}$ we have

$$\mathrm{Ext}^i_X(\omega, G(b)) \cong \mathrm{Hom}_{\mathcal{A}}(F(\omega), b[i]) = 0.$$

Since Ω is a spanning class, we conclude $G(b) = 0$ for all $b \in \mathcal{A}^{\perp}$. $\square$

2.7 Derived Category of $[\mathbb{P}^{m+n-1}/\mu_d]$

This section serves to illustrate how to work with equivariant triangulated categories of quotient stacks as well as provide a useful semi-orthogonal decomposition for $\mathcal{D}[\mathbb{P}^1/\mu_d]$ and more generally a semi-orthogonal decomposition for cyclic quotients with fixed locus a smooth divisor (see Theorem 2.7.2).

For the rest of this dissertation, we set $\chi\colon \mu_d \to \mathbb{G}_m$ denote the standard primitive character $\chi(\lambda) = \lambda$.

There is an action of μ_d on $\mathbb{P}^{m+n-1}$, where $\mathbb{P}^{m+n-1}$ has coordinates $[x_1 : \ldots : x_m : y_1 : \ldots : y_n]$ and μ_d acts by scaling the variables $y_1, \ldots, y_n$:

$$\lambda \cdot [x_1 : \cdots : x_m : y_0 : \cdots : y_n] = [x_1 : \cdots : x_m : \lambda y_1 : \cdots : \lambda y_n].$$

In terms of the homogeneous coordinate algebra $k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, the variables y_i have weight χ^{-1} and the variables x_i have trivial weight.

Let $H_y = V(x_1, \ldots, x_m)$ and $H_x = V(y_1, \ldots, y_n)$. The fixed locus of the μ_d action is $(\mathbb{P}^{m+n-1})^{\mu_d} = H_x \sqcup H_y$. Therefore the sheaves $\mathcal{O}_{H_x}$ and $\mathcal{O}_{H_y}$ have a natural equivariant structure given by the identity morphism. As in Example 2.4.2, we can form the equivariant sheaves $\mathcal{O}_{H_x}(\chi^i)$ and $\mathcal{O}_{H_y}(\chi^i)$ for $i = 0, \ldots, d-1$.

We equip $\mathcal{O}(-1)$ with the μ_d-linearization $\theta_\lambda\colon \mathcal{O}(-1) \to \lambda^*\mathcal{O}(-1)$ given by fiberwise multiplication by λ and consider $\mathcal{O}(i)$ with the induced μ_d-linearizations. We can also twist these sheaves by characters to get the equivariant line bundles $\mathcal{O}_{\mathbb{P}^{m+n-1}}(i)(\chi^j)$ for $i \in \mathbb{Z}$ and $j = 0, \ldots, d-1$.

The canonical bundle on $\mathbb{P}^{m+n-1}$ is $\mathcal{O}(-m-n)$. It is locally trivial as a μ_d-equivariant bundle; however, the identification $\omega_{\mathbb{P}^{m+n-1}} \cong \mathcal{O}(-m-n)$ may involve twisting by a character. To determine the twist, we recall the Euler exact sequence on $\mathbb{P}^{m+n-1}$

$$0 \to \Omega^1 \to \mathcal{O}(-1)^{\oplus m+n} \xrightarrow{\alpha} \mathcal{O} \to 0$$

where $\alpha = (x_1, \ldots, x_m, y_1, \ldots, y_n)$. Since the sections y_i have weight -1, the above Euler exact sequence admits the following μ_d-linearization:

$$0 \to \Omega^1 \to (\oplus_{i=1}^m \mathcal{O}(-1)) \oplus (\oplus_{j=1}^n \mathcal{O}(-1)(\chi^{-1})) \xrightarrow{\alpha} \mathcal{O} \to 0$$

Now taking determinants yields $\omega_{[\mathbb{P}^{m+n-1}/\mu_d]} \cong \mathcal{O}_{\mathbb{P}^{m+n-1}}(-m-n)(\chi^{-n})$ as μ_d-equivariant sheaves. Serre duality therefore takes the following form:

Proposition 2.7.1 (Serre Duality)**.** *For any $\mathcal{F}, \mathcal{G} \in \mathcal{D}[\mathbb{P}^{m+n-1}/\mu_d]$ there is a natural isomorphism*

$$\mathrm{Ext}^*_{[\mathbb{P}^{m+n-1}/\mu_d]}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Ext}^{m+n-1-*}_{[\mathbb{P}^{m+n-1}/\mu_d]}(\mathcal{G}, \mathcal{F}(-m-n)(\chi^{-n})).$$

Let us consider the case $m = n = 1$. In this case, we can describe a useful semi-orthogonal decomposition of $[\mathbb{P}^1/\mu_d]$.

The projective line is a coarse moduli space for $[\mathbb{P}^1/\mu_d]$ and the mapping $\pi\colon [\mathbb{P}^1/\mu_d] \to \mathbb{P}^1$ is defined by the μ_d-equivariant morphism $\tilde{\pi}\colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $[x : y] \mapsto [x^d : y^d]$. Since $\tilde{\pi}$ can be described as as the d-uple embedding $\mathbb{P}^1 \to \mathbb{P}^{\frac{1}{2}d(d+1)}$ followed by the linear projection onto the x^d, y^d variables, we have $\pi^*\mathcal{O}_{\mathbb{P}^1}(-1) \cong \mathcal{O}_{[\mathbb{P}^1/\mu_d]}(-d)$.

The fixed orbit consists of two points $\{p = [1 : 0], q = [0 : 1]\}$. In the notation before, we have $H_x = \{p\}$ and $H_y = \{q\}$. The following semi-orthogonal decomposition is used in Section 4.5.

Theorem 2.7.1. *For each $i \in \mathbb{Z}$ there is a semi-orthogonal decomposition*

$$\begin{aligned}\mathcal{D}([\mathbb{P}^1/\mu_d]) &= \langle \mathcal{O}_p(\chi^{d-1}), \ldots, \mathcal{O}_p(\chi), \mathcal{O}_q(\chi^{-(d-1)}), \ldots, \mathcal{O}_q(\chi^{-1}), \pi^*\mathcal{D}(\mathbb{P}^1)\rangle \\ &= \langle \mathcal{O}_p(\chi^{d-1}), \ldots, \mathcal{O}_p(\chi), \mathcal{O}_q(\chi^{-(d-1)}), \ldots, \mathcal{O}_q(\chi^{-1}), \mathcal{O}(-di), \mathcal{O}(-d(i-1))\rangle.\end{aligned}$$

We prove something slightly more general than Theorem 2.7.1 regarding μ_d-actions. See also [15] for more on the derived categories of cyclic quotients.

Theorem 2.7.2. *Let μ_d act on a smooth projective variety X of dimension n. Suppose the geometric quotient $\pi \colon X \to X/\mu_d$ is smooth and the fixed locus $X^{\mu_d} = Z$ is a smooth divisor such that μ_d acts freely on $X \backslash Z$ such that $\mathcal{N}_{Z/X} \cong \mathcal{L}(\chi^{-1})$ for some fixed line bundle $\mathcal{L}$ on Z, where $\mathcal{N}_{Z/X}$ is the normal bundle. Let $\iota \colon Z \hookrightarrow X$ denote the inclusion. Then there is a semi-orthogonal decomposition of $\mathcal{D}[X/\mu_d]$:*

$$\mathcal{D}[X/\mu_d] = \langle \iota_*(\mathcal{D}(Z))(\chi), \ldots, \iota_*(\mathcal{D}(Z))(\chi^{d-1}), \pi^*\mathcal{D}(X/\mu_d)\rangle.$$

Proof. We first show $\iota_* : \mathcal{D}(Z) \to \mathcal{D}[X/\mu_d]$ is fully-faithful using Theorem 2.5.1. Pick $z \in Z$. Since $\mathcal{N}_{Z/X,z} \cong \chi^{-1}$, we have an isomorphism of μ_d-representations:

$$\operatorname{Ext}^*_X(\mathcal{O}_z, \mathcal{O}_z) \cong \Lambda^*(T_z X) \cong \Lambda^*(\mathbf{1}^{\oplus n-1} \oplus \chi^{-1}).$$

It follows that ι_* is fully-faithful. Semi-orthogonality follows from this identification as well.

To see fullness, take an object $\mathcal{F} \in \mathcal{D}[X/\mu_d]$ such that $\mathcal{F}$ is left orthogonal to

$$\langle \iota_*(\mathcal{D}(Z))(\chi), \ldots, \iota_*(\mathcal{D}(Z))(\chi^{d-1})\rangle.$$

As the action of μ_d on $X \setminus Z$ is free, we can apply [23, Theorem 2.4] to see $\mathcal{F} \in \pi^*\mathcal{D}(X/\mu_d)$. □

Remark 2.7.1. Of course the theorem can be adapted to the case where $\mathcal{N}_{Z/X}$ has different weights. However, the components $\iota_*\mathcal{D}(Z)(\chi^i)$ may need to be reordered to ensure semi-orthogonality. The correct reordering for $[\mathbb{P}^1/\mu_d]$ is provided in the statement and so Theorem 2.7.2 proves Theorem 2.7.1.

Remark 2.7.2. The sheaves $\mathcal{O}(n)$ have a natural μ_d-equivariant structure and so we could equally as well have considered an equivariantized Beilinson's exceptional collection:

$$\mathcal{D}[\mathbb{P}^1/\mu_d] = \langle \mathcal{O}(-1), \mathcal{O}(-1)(\chi) \ldots, \mathcal{O}(-1)(\chi^{d-1}), \mathcal{O}, \mathcal{O}(\chi), \ldots, \mathcal{O}(\chi^{d-1})\rangle.$$

We could then tediously argue that the decomposition of Theorem 2.7.1 is a mutation of Beilinson's collection. The above argument is more pleasant and Theorem 2.7.2 will be needed in Section 4.6.

The classical Grothendieck splitting theorem decomposes any vector bundle on $\mathbb{P}^1$ as a sum of line bundles. We have the following equivariant version of this result which is used in Section 3.6 in the case $G = \mu_d$.

Theorem 2.7.3 (Equivariant Grothendieck Splitting). *Let $\mathcal{E}$ be a rank r vector bundle on $[\mathbb{P}^1/G]$, where G is a finite Abelian group. Then there exists $n_i \in \mathbb{Z}$,*

$\chi_i \in \hat{G}$ *for* $i = 1, \ldots, r$ *such that*

$$\mathcal{E} \cong \oplus_{i=1}^{r} \mathcal{O}(n_i)(\chi_i)$$

Proof. The proof is almost identical to the classical proof, see [19, Section 2.1]. The Abelian condition ensures the irreducible representations are one dimensional and the twist by characters χ_i show up when looking for an equivariant global section of $\mathcal{E}(n_i)$ with $n_i >> 0$. $\square$

Remark 2.7.3. In the case $G = \mu_d$, the character group is generated by χ. If $\mathcal{E}$ is a μ_d-equivariant vector bundle of rank r on $\mathbb{P}^1$, then we have

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}(n_i)(\chi^{s_i})$$

for $0 \leq s_i < d - 1$.

CHAPTER III

SETUP AND EMBEDDINGS

3.1 Setup

Let $X_f \subset \mathbb{P}^{m-1}$ and $X_g \subset \mathbb{P}^{n-1}$ be smooth degree d hypersurfaces. Let $X = V(f \oplus g) \subset \mathbb{P}^{m+n-1}$ be the hypersurface associated to the Thom-Sebastiani sum of potentials. We impose the conditions $d \geq n \geq m \geq 2$, i.e. the hypersurfaces involved are Calabi-Yau or general type and are non-empty and $\dim(X_g) \geq \dim(X_f)$. The latter condition is for purely computational purposes.

Proposition 3.1.1. *The hypersurface X is smooth.*

Proof. The gradient of $f \oplus g$ is

$$\nabla(f \oplus g) = [\nabla f \ \nabla g].$$

Suppose $\nabla(f \oplus g)([p : q]) = 0$. Then for each i and j, we have $\partial_{x_i}(f)(p) = 0$ and $\partial_{y_j}(g)(q) = 0$. By the Euler formula, we have

$$f(p) = \frac{1}{d} \sum_i \partial_{x_i} f(p) = 0$$

and

$$g(q) = \frac{1}{d} \sum_j \partial_{y_j} g(q) = 0.$$

Since f, g both define smooth hypersurfaces, it must be that $p = [0 : \cdots : 0]$ and $q = [0 : \cdots : 0]$ which is impossible. We conclude X is smooth. $\square$

The action of μ_d on $\mathbb{P}^{m+n-1}$ from Section 2.7 descends to X and we consider the quotient stack $[X/\mu_d]$. The fixed loci are given by the intersections with $(\mathbb{P}^{m+n-1})^{\mu_d} = H_x \sqcup H_y$:

$$X^{\mu_d} = X \cap (H_x \sqcup H_y) \cong X_f \sqcup X_g.$$

3.2 Equivariant geometry of X

Line bundles associated to hyperplane sections $\mathcal{O}_X(iH)$ have d distinct equivariant structures. These equivariant line bundles are of the form $\mathcal{O}_X(iH)(\chi^j)$.

Proposition 3.2.1 (Serre Duality). *The triangulated category $\mathcal{D}[X/\mu_d]$ has the Serre functor $(-) \otimes \mathcal{O}_X(d-m-n)(\chi^{-n})[m+n-2]$.*

Proof. Since $[X/\mu_d]$ is a smooth substack of $[\mathbb{P}^{m+n-1}/\mu_d]$ with normal bundle isomorphic to $\mathcal{O}_X(d)$, we can use the adjunction formula

$$\omega_{[X/\mu_d]} \cong \omega_{[\mathbb{P}^{m+n-1}/\mu_d]} \otimes \mathcal{O}_{[X/\mu_d]}(d) \cong \mathcal{O}_X(d-m-n)(\chi^{-n}).$$

□

For Fano hypersurfaces it is easy to see that line bundles are exceptional. With this extra μ_d action, all line bundles on $[X/\mu_d]$ are exceptional.

Proposition 3.2.2. *Line bundles are exceptional objects of $\mathcal{D}[X/\mu_d]$.*

Proof. It is sufficient to prove $H^*(\mathcal{O}_X)^{\mu_d} \cong k$. We have an equivariant exact sequence on $\mathbb{P}^{m+n-1}$:

$$0 \to \mathcal{O}_{\mathbb{P}^{m+n-1}}(-d) \xrightarrow{f \oplus g} \mathcal{O}_{\mathbb{P}^{m+n-1}} \to \mathcal{O}_X \to 0.$$

We therefore only need to show the vanishing of $H^{m+n-2}(\mathcal{O}_X)^{\mu_d}$. We have an isomorphism

$$H^{m+n-2}(\mathcal{O}_X)^{\mu_d} \cong H^{m+n-1}(\mathcal{O}_{\mathbb{P}^{m+n-1}}(-d))^{\mu_d}.$$

If $d < m+n$, then the latter group is zero and we are finished. Suppose $d \geq m+n$. Then

$$H^{m+n-1}(\mathcal{O}_{\mathbb{P}^{m+n-1}}(-d))^{\mu_d} \cong H^0(\mathcal{O}_{\mathbb{P}^{m+n-1}}(d-m-n)(\chi^{-n}))^{\mu_d}.$$

The latter has a basis of monomials of the form $x^I y^J$ where $I = (i_1, \ldots, i_m)$, $J = (j_1, \ldots, j_n)$ and $x^I = x_1^{i_1} \cdots x_m^{i_m}$, $y^J = y_1^{j_1} \ldots y_n^{j_n}$ such that $|I| + |J| = d-m-n$ and $|J| + n$ is a positive multiple of d. It follows that $|J| = d-n$ is the only possible option. Hence, $d-m-n = |I|+|J| = |I|+d-n$ from which we conclude $|I| = -m$, impossible. $\square$

Proposition 3.2.3. *For* $0 < i-j < m$, $H^*(\mathcal{O}_X(\chi^{j-i})) = 0$.

Proof. Clearly equivariant global sections are zero. By Proposition 3.2.2, there is an isomorphism

$$H^{m+n-2}(\mathcal{O}_X(\chi^{j-i}))^{\mu_d} \cong H^{m+n-1}(\mathcal{O}_{\mathbb{P}}^{m+n-1}(d-m-n)(\chi^{i-j-n}))^{\mu_d}.$$

As in the preceeding proof, we must have a monomial $x^I y^J$ where $|I| + |J| = d - m - n$ and $|J| + n + j - i$ is a positive multiple of d. Thus $|J| = d-n-j+i$ is the only possible obtion. Hence, $d-m-n = |I|+|J| = |I|+d-n-j+i$. Therefore $|I| = i-j-m < m-m = 0$, which is impossible. $\square$

3.3 Subcategory of exceptional line bundles.

Define subcategories $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ of $\mathcal{D}[X/\mu_d]$ as follows.

$$\begin{aligned}
\mathcal{A}_1 = \langle & \mathcal{O}_X(-(n-1)-(m-1))(\chi^{-(n-1)}), \\
& \mathcal{O}_X(-(n-1)-(m-1)+1)(\chi^{-(n-2),-(n-1)}), \\
& \dots, \mathcal{O}_X(-(n-1)-1)(\chi^{-(n-m)-1,\dots,-(n-1)})\rangle; \\
\mathcal{A}_2 = \langle & \mathcal{O}_X(-(n-1))(\chi^{-(n-m),\dots,-(n-1)}), \\
& \mathcal{O}_X(-(n-1)+1)(\chi^{-(n-m)+1,\dots,-(n-1)+1}), \\
& \dots, \mathcal{O}_X(-(m-1)-1)(\chi^{-1,\dots,-(m-1)})\rangle; \\
\mathcal{A}_3 = \langle & \mathcal{O}_X(-(m-1))(\chi^{0,\dots,-(m-1)}), \\
& \mathcal{O}_X(-(m-2))(\chi^{0,\dots,-(m-2)}), \dots, \mathcal{O}_X\rangle.
\end{aligned}$$

It is understood that if $m = n$, then $\mathcal{A}_2$ is zero. Further, the notation $\mathcal{O}_X(i)(\chi^{j_1,\dots,j_k})$ means the subcategory generated by the exceptional objects $\mathcal{O}_X(i)(\chi^{j_1}), \dots, \mathcal{O}_X(i)(\chi^{j_k})$. By Proposition 3.2.3, there is a semi-orthogonal decomposition:

$$\mathcal{O}_X(i)(\chi^{j_1,\dots,j_k}) = \langle \mathcal{O}_X(i)(\chi^{j_k}), \dots, \mathcal{O}_X(i)(\chi^{j_1})\rangle,$$

where $j_1 > j_2 > \dots > j_k$.

Proposition 3.3.1. *The decomposition of the subcategories $\mathcal{A}_i$ for $i = 1, 2, 3$ is semi-orthogonal. Moreover, $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\rangle$ is semi-orthogonal.*

Proof. We only show the semi-orthogonality of the decomposition for $\mathcal{A}_3$. The semi-orthogonality of the decomposition for $\mathcal{A}_1$ and $\mathcal{A}_2$ is similar.

The sheaves in $\mathcal{A}_3$ are of the form $\mathcal{O}(-i_1)(\chi^{-j_1})$ for $0 \le i \le m-1$ and $0 \le i_1 \le i_2$. Pick two such sheaves with $i_1 \le i_2$ and $j_1 \le j_2$. By Serre Duality and the closed substack exact sequence:

$$H^{m+n-2}(\mathcal{O}_X(i_1 - i_2)(\chi^{j_1-j_2})) \cong H^0(\mathcal{O}_{\mathbb{P}^{m+n-1}}(d + i_2 - i_1 - n - m)(\chi^{j_2-j_1-n})).$$

We have the following inequalities:

$$m - 1 \ge i_2 - i_1 \ge 0,$$
$$m - 1 \ge j_2 - j_1 \ge 0.$$

An equivariant global section is of the form $x^I y^J$ where $|I| + |J| = d + i_2 - i_1 - n - m$ such that $|J| = j_2 - j_1 - n + d$. But

$$d + i_2 - i_1 - n - m = |I| + |J| = |I| + d + j_2 - j_1 - n.$$

Hence, $|I| = i_2 - i_1 - (j_2 - j_1) - m \le i_2 - i_1 - m \le m - 1 - m = -1$, which is impossible. Thus $H^*(\mathcal{O}_X(i_1 - i_2)(\chi^{j_1-j_2})) = 0$ and the semi-orthogonal decomposition for $\mathcal{A}_3$ is verified.

We now check $\langle \mathcal{A}_2, \mathcal{A}_3 \rangle$, the semi-orthogonality computations for $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ and $\langle \mathcal{A}_1, \mathcal{A}_3 \rangle$ are similar. Recall, for $\mathcal{A}_2$ to exist we require $n > m$. The relevant group is

$$H^{m+n-2}(\mathcal{O}_X(i_1 - (m-1) - i_2)(\chi^{j_1-j_2}))^{\mu_d}$$
$$\cong H^0(\mathcal{O}_{\mathbb{P}^{m+n-1}}(d + i_2 - i_1 - n - 1)(\chi^{j_2-j_1-n}))^{\mu_d}$$

for $i_1 = 0, \ldots, m-1$, $j_1 = 0, \ldots, i_1$, $i_2 = 1, \ldots, n-m$, $j_2 = i_2, \ldots, (m-1)+i_2$.

If $d+i_2-i_1-n-1 < 0$, there is nothing to prove. Assume $d+i_2-i_1-n-1 \geq 0$. Let $x^I y^J$ be an equivariant global section. Since $j_2 - j_1 - n < 0$ we require $|J| = d + j_2 - j_1 - n$. Then

$$d + i_2 - i_1 - n - 1 = |I| + |J| = |I| + j_2 - j_1 - n$$

forces $|I| = i_2 - j_2 + j_1 - i_1 - 1$. However, $i_2 - j_2 \leq 0$ and $j_1 - i_1 \leq 0$ so $|I| \leq -1$ which is impossible. This finishes the proof. □

3.4 Embedding $\mathcal{D}(X_f)$

Let $\iota_f \colon X_f \to X$ be given by $\iota_f([x_1 : \ldots : x_m]) = [x_1 : \ldots : x_m : 0 : \ldots : 0]$. Clearly ι_f is μ_d-equivariant as it coincides with a component of the fixed locus. Let ι_{f*} denote the corresponding equivariant pushforward functor $\iota_{f*} \colon \mathcal{D}(X_f) \to \mathcal{D}[X/\mu_d]$. This means first include $\mathcal{D}(X_f)$ into the trivial component of

$$\mathcal{D}[X_f/\mu_d] = \bigoplus_{i=0}^{d-1} \mathcal{D}(X_f)(\chi^i),$$

then use the equivariant pushforward.

Proposition 3.4.1. *If $d > n$, then ι_{f*} is fully-faithful.*

Proof. We use Theorem 2.5.1. Let $p \in X_f$ be a closed point and identify p with $\iota_f(p)$. It is sufficient to show vanishing of $\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_p, \mathcal{O}_p) \cong (\Lambda^* T_p X)^{\mu_d}$ for $* > m-2$. From the normal bundle exact sequence

$$0 \to TX \to T\mathbb{P}^{m+n-1}|_X \to \mathcal{O}_X(d) \to 0$$

and the identification $T_p\mathbb{P}^{m+n-1} \cong \mathbf{1}^{\oplus m-1} \oplus \chi^{\oplus n}$ coming from the μ_d-linearized Euler exact sequence, we see

$$T_pX \cong \mathbf{1}^{\oplus m-2} \oplus \chi^{\oplus n}$$

If $d > n$, then $(\Lambda^* T_pX)^{\mu_d} = 0$ for $* > m-2$. $\square$

Define the following subcategories of $\mathcal{D}(X)^{\mu_d}$:

$$\mathcal{D}_f^i = \iota_{f*}(\mathcal{D}(X_f))(\chi^i).$$

Orlov's theorem predicts that for some i, these subcategories appear in the semi-orthogonal decomposition. The next proposition tells us which ones are needed.

Proposition 3.4.2. *For $0 < i_1 - i_2 < d - n$ we have the semi-orthogonality*

$$\langle \mathcal{D}_f^{i_1}, \mathcal{D}_f^{i_2} \rangle.$$

Proof. Pick a closed point $p \in X_f$, then we have the isomorphism:

$$\mathrm{Ext}^*_X(\mathcal{O}_p(\chi^{i_2}), \mathcal{O}_p(\chi^{i_1}))^{\mu_d} \cong \Lambda^*(\mathbf{1}^{\oplus m-2} \oplus \chi^{\oplus n})(\chi^{i_1-i_2}).$$

Provided $0 < i_1 - i_2 < d-n$, we will have a nontrivial weight on the extension group for every $*$. $\square$

Definition 3.4.1. Let $\mathcal{D}_f$ be the strictly full subcategory of $\mathcal{D}[X/\mu_d]$ generated by $\mathcal{D}_f^1, \ldots, \mathcal{D}_f^{d-n}$.

The follwing corollary is immediate.

Corollary 3.4.1. *For $d > n$, we have a semi-orthogonal decomposition*

$$\mathcal{D}_f = \langle \mathcal{D}_f^{d-n}, \mathcal{D}_f^{d-n-1}, \ldots, \mathcal{D}_f^1 \rangle.$$

3.5 Embedding $\mathcal{D}(X_g)$

Similarly to $\mathcal{D}_f$, we have a closed embedding $\iota_g : X_g \to X$ given by $\iota_g([y_1 : \ldots : y_n]) = [0 : \ldots : 0 : y_1 : \ldots : y_n]$, which is the inclusion of the other component of the fixed locus and so is μ_d-equivariant. Let $\iota_{g*} \colon \mathcal{D}(X_g) \to \mathcal{D}[X/\mu_d]$ be the associated equivariant pushforward. The following results are analogous to Propositions 3.4.1, 3.4.2 and Corollary 3.4.1. We include the proofs for completeness

Proposition 3.5.1. *If $d > m$, then ι_{g*} is fully-faithful.*

Proof. As in Proposition 3.4.1, for each closed point $q \in X_g$, we have an identification

$$T_pX \cong \mathbf{1}^{\oplus n-2} \oplus (\chi^{-1})^{\oplus m}.$$

If $d > m$, then $(\Lambda^* T_q X)^{\mu_d} = 0$ for $* > n - 2$. □

Define the subcategories

$$\mathcal{D}_g^i = \iota_{g*}(\mathcal{D}(X_g))(\chi^i)$$

of $\mathcal{D}([X/\mu_d])$.

Proposition 3.5.2. *For $m - d < i_1 - i_2 < 0$ we have*

$$\langle \mathcal{D}_g^{i_1}, \mathcal{D}_g^{i_2} \rangle.$$

Proof. Pick a closed point $q \in X_g$, then have the isomorphism

$$\mathrm{Ext}_X^*(\mathcal{O}_q(\chi^{i_2}, \mathcal{O}_q(\chi^{i_1}))) \cong \Lambda^*(\mathbf{1}^{\oplus n-2} \oplus (\chi^{-1})^{\oplus m}) \otimes \chi^{i_1 - i_2}.$$

Provided $0 > i_1 - i_2 > m - d$, the extension group will always have a nontrivial weight and so the it must vanish. □

Corollary 3.5.1. *For $d > m$ we have a semi-orthogonal decomposition*

$$\mathcal{D}_g = \langle \mathcal{D}_g^{m-d}, \mathcal{D}_g^{m-d+1}, \ldots, \mathcal{D}_g^{-1} \rangle.$$

In the case $n > m$, it will be necessary to split $\mathcal{D}_g$ into two subcategories.

Definition 3.5.1. Define subcategories of $\mathcal{D}[X/\mu_d]$:

$$\mathcal{D}_{g1} = \langle \mathcal{D}_g^{m-d}, \mathcal{D}_g^{m-d+1}, \ldots, \mathcal{D}_g^{m-n-1} \rangle.$$

and

$$\mathcal{D}_{g2} = \langle \mathcal{D}_g^{m-n}, \mathcal{D}_g^{m-n+1}, \ldots, \mathcal{D}_g^{-1} \rangle.$$

We have $\mathcal{D}_g = \langle \mathcal{D}_{g1}, \mathcal{D}_{g2} \rangle$, where it is understood that if $m = n$, then $\mathcal{D}_g = \mathcal{D}_{g1}$.

3.6 Embedding $\mathcal{D}(X_f \times X_g)$

Let $Y = \mathbb{P}(\mathcal{O}_{X_f}(-1) \boxplus \mathcal{O}_{X_g}(-1))$, i.e. the projectivization of the rank 2 vector bundle $\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)$ over $X_f \times X_g$. Let $\pi \colon Y \to X_f \times X_g$ be the projection. Consider the commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[d, "\pi"] \arrow[dr, "\sigma"] & X_f \times X_g \times X \arrow[d, "\pi_X"] \\ X_f \times X_g & X \end{tikzcd}$$

The cyclic group μ_d acts on Y by scaling the second coordinate of the fiber. We endow $X_f \times X_g$ with the trivial action rendering the diagram μ_d-equivariant.

Define a family of (equivariant) *Fourier-Mukai functors*

$$\Xi_{i,j} \colon \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d]$$

using the kernel $\iota_* \mathcal{O}_Y \otimes \pi_X^* \mathcal{O}_X(iH)(\chi^j)$, i.e.

$$\Xi_{i,j}(\mathcal{F}^{\cdot}) = \mathbf{R}\pi_{X*}(\pi^*_{X_f \times X_g}(\mathcal{F}^{\cdot}) \otimes \iota_* \mathcal{O}_Y \otimes \mathcal{O}_X(iH)(\chi^j)),$$

where it is understood that before applying $\Xi_{i,j}$ we precompose with the embedding

$$\mathcal{D}(X_f \times X_g) \hookrightarrow \mathcal{D}[X_f \times X_g/\mu_d] = \bigoplus_{i=0}^{d-1} \mathcal{D}(X_f \times X_g)(\chi^i)$$

into the trivial component. Then the derived push and pull functors are taken equivariantly.

Proposition 3.6.1. *Let $(p, q) \in X_f \times X_g$ be a closed point. Then*

$$\Xi_{i,j}(\mathcal{O}_{p,q}) = \mathcal{O}_{l(p,q)}(i)(\chi^j)$$

where $l_{p,q}$ is the line joining $\iota_f(p)$ to $\iota_g(q)$ inside X.

Proof. Since the kernel is flat over $X_f \times X_g$, we just have to take the restriction of the kernel to $\{(p, q)\} \times X$. This is precisely the structure sheaf of the line $l(p, q)$ with a twist. □

We show $\Xi_{i,j}$ is an embedding using Theorem 2.5.1.

Lemma 3.6.1. *Let $\mathcal{N}$ denote the normal bundle to $l(p, q)$ inside X. Then*

$$\mathcal{N} \cong \left(\bigoplus_{i=1}^{m-2} \mathcal{O}_{l(p,q)}(1)\right) \oplus \left(\bigoplus_{j=1}^{n-2} \mathcal{O}_{l(p,q)}(1)(\chi)\right) \oplus \mathcal{O}_{l(p,q)}(2-d)(\chi).$$

Proof. By the μ_d-equivariant Grothendieck splitting theorem (Theorem 2.7.3), we have an isomorphism

$$\mathcal{N} \cong \bigoplus_{i=1}^{m+n-3} \mathcal{O}_{l(p,q)}(n_i)(\chi^{j_i})$$

for some $n_i \in \mathbb{Z}$ and weights j_i.

As X is a degree d hypersurface in $\mathbb{P}^{m+n-1}$ and $l(p, q)$ is a linear subvariety of $\mathbb{P}^{m+n-1}$, the normal bundle $\mathcal{N}$ fits into the following equivariant exact sequence:

$$0 \to \mathcal{N} \to \left(\bigoplus_{i=1}^{m-1} \mathcal{O}_{l(p,q)}(1)\right) \oplus \left(\bigoplus_{j=1}^{n-1} \mathcal{O}_{l(p,q)}(1)(\chi)\right) \to \mathcal{O}_{l(p,q)}(d) \to 0$$

on $l(p,q)$. The weights come from the description of the morphism

$$\mathcal{O}_{l(p,q)}(1)^{\oplus m+n-2} \to \mathcal{O}_{l(p,q)}(d).$$

It is given by multiplication by

$$(\partial_{u_1} f|_{l(p,q)}, \ldots, \partial_{u_{m-1}} f|_{l(p,q)}, \partial_{v_1} g|_{l(p,q)}, \ldots, \partial_{v_{n-1}} g|_{l(p,q)}),$$

where $u_1, \ldots, u_{m-1}$ are linear sections cutting out $p \in \mathbb{P}^{m-1}$ and $v_1, \ldots, v_{n-1}$ are linear sections cutting out $q \in \mathbb{P}^{m-1}$.

Up to a linear change of coordinates, we can assume this mapping is

$$(u_1^{d-1}, 0, \ldots, 0, v_1^{d-1}, 0, \ldots, 0).$$

Hence,

$$\mathcal{N} \cong \left(\bigoplus_{i=1}^{m-2} \mathcal{O}_{l(p,q)}(1)\right) \oplus \left(\bigoplus_{j=1}^{n-2} \mathcal{O}_{l(p,q)}(1)(\chi)\right) \oplus \mathcal{O}(i)(\chi^j)$$

Since $\deg(\mathcal{N}) = m+n-2-d$ we must have $i = 2-d$. By checking the fibers of the normal bundle exact sequence at $p = [1:0]$, we have

$$0 \to \mathcal{N}_p \to \mathbf{1}^{\oplus m-1} \oplus \chi^{\oplus n-1} \to \mathbf{1} \to 0$$

as μ_d-representations. Therefore $\mathcal{N}_p \cong \mathbf{1}^{\oplus m-2} \oplus \chi^{\oplus n-1}$ and it follows that $j = 1$. $\square$

Lemma 3.6.2. *For $(p,q),(p',q') \in X_f \times X_g$. If $p \neq p'$ or $q \neq q'$, then*

$$\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p',q')}) = 0.$$

Proof. If $p \neq p'$ and $q \neq q'$, then the subvarieties $l(p,q)$ and $l(p',q')$ are disjoint. The vanishing follows. Without loss of generality, suppose $p = p'$. We must compute

$$\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q')}) \cong \mathrm{Ext}^*_{\mathcal{O}_{X,p}}(\mathcal{O}_{l(p,q),p}, \mathcal{O}_{l(p,q'),p})^{\mu_d}$$

Let $R = \widehat{\mathcal{O}_{X,p}}$. Then by the Cohen structure theorem, we have $R \cong k[[x_1, \ldots, x_{m-2}, y_1, \ldots, y_n]]$. The action of μ_d on $\mathrm{Spec}(R)$ endows $x_1, \ldots, x_{m-2}$ with trivial weight and $y_1, \ldots, y_n$ with weight -1. The completions of $\mathcal{O}_{l(p,q),p}$ and $\mathcal{O}_{l(p,q'),p}$ are isomorphic to the modules

$$M_q = R/(x_1, \ldots, x_{m-2}, y_2, \ldots, y_n) \cong k[[y_1]]$$

and

$$M_{q'} = R/(x_1, \ldots, x_{m-2}, y_1, y_3, \ldots, y_n) \cong k[[y_2]],$$

respectively.

Since M_q is cut out by the regular sequence $x_1, \ldots, x_{m-2}, y_2, \ldots, y_n$, we have the following equivariant Koszul resolution

$$(\otimes_{i=1}^{m-2} Re_{x_i} \xrightarrow{x_i} R) \otimes (\otimes_{j=2}^{n} Re_{y_j}(\chi^{-1}) \xrightarrow{y_j} R)$$

of M_q. We apply $\mathrm{Hom}_R(-, M_{q'})$:

$$(\otimes_{i=1}^{m-2} M_{q'} e_{x_i}^\vee \xrightarrow{0} M_{q'}) \otimes (\otimes_{j=3}^{n} M_{q'} e_{y_j}^\vee \xrightarrow{0} M_{q'}(\chi)) \otimes (M_{q'} \xrightarrow{y_2} M_{q'}(\chi))$$
$$\cong (\otimes_{i=1}^{m-2} M_{q'} e_{x_i}^\vee \xrightarrow{0} M_{q'}) \otimes (\otimes_{j=3}^{n} M_{q'} e_{y_j}^\vee \xrightarrow{0} M_{q'}\chi) \otimes k(\chi)$$

Since $d \geq n$, the terms appearing in $\mathrm{Ext}_R^*(M_q, M_{q'})$ will all have nontrivial weight. Indeed, the weights will be between 1 and $n-1$. Hence, $\mathrm{Ext}_R^*(M_q, M_{q'})^{\mu_d} = 0$. Since completion is faithful, we have the desired vanishing. □

We can now prove $\Xi_{i,j}$ is fully-faithful.

Theorem 3.6.1. *The functors $\Xi_{i,j}$ are fully-faithful for all i, j.*

Proof. Using Theorem 2.5.1 and Lemma 3.6.2 we only need to show

$$\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q)}) = \begin{cases} k & * = 0 \\ 0 & * \notin [0, m+n-4] \end{cases}.$$

That $\mathrm{Hom}_{[X/\mu_d]}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q)}) \cong k$ is clear.

For vanishing, we use the local-to-global spectral sequence. Since $l(p,q)$ and X are smooth, this reduces to:

$$\mathrm{H}^r(\Lambda^s \mathcal{N}) \Rightarrow \mathrm{Ext}^{r+s}_{[X/\mu_d]}(\mathcal{O}_{l(p,q)}, \mathcal{O}_{l(p,q)}).$$

here $\mathcal{N}$ is the normal bundle from Lemma 3.6.1. So we must compute $\mathrm{H}^r(\Lambda^s\mathcal{N})$ for $(r,s) = (0, m+n-3), (1, m+n-4)$. We will compute separately.

For the case $(r,s) = (0, m+n-3)$, we have

$$\Lambda^{m+n-3}\mathcal{N} \cong \mathcal{O}_{l(p,q)}(m+n-4+2-d)(\chi^{n-1}) \cong \mathcal{O}(m+n-d-2)(\chi^{n-1}).$$

Suppose $m+n-d-2 \geq 0$ (otherwise there is nothing to check), then we would require a monomial of the form $x^a y^{n-1}$ with $a \geq 0$ and $a+n-1 = m+n-d-2$. Solving for a, we have $a = m-d-1 \leq -1$, but $d \geq m$, which is impossible.

Now for the case $(r,s) = (1, m+n-4)$. Since $l(p,q) \cong \mathbb{P}^1$, the only way $H^1(\Lambda^{m+n-4}\mathcal{N})$ can be nonvanishing is if $\mathcal{O}(2-d)(\chi)$ is involved in the product. In which case, the isotypical summands of $\Lambda^{m+n-4}\mathcal{N}$ involving $\mathcal{O}(2-d)(\chi)$ are:

$$\mathcal{O}(m+n-3-d)(\chi^{n-2}), \mathcal{O}(m+n-3-d)(\chi^{n-1}).$$

If $m+n-3-d \geq -1$, then the first cohomology group is zero without equivariance. Assume $m+n-3-d \leq -2$. By Serre duality, we have an isomorphism

$$H^1(\mathcal{O}_{l(p,q)}(m+n-3-d))^{\mu_d} \cong H^0(\mathcal{O}_{l(p,q)}(d+1-m-n)(\chi^{-n,1-n}))^{\mu_d}$$

We remark that $d > n$ here; otherwise, if $d = n$, then $m-3 \leq -2$ forces $m = 1$ and we assume $m \geq 2$. In particular, the weights $\chi^{-n,1-n}$ are nontrivial above. We must find a monomial of the form $x^a y^{d-n}$ where $a \geq 0$ and $a+d-n = d+1-m-n$. This forces $a = 1-m$, which is absurd. Similarly, we would need a monomial of the form $x^a y^{d-n+1}$ with $a \geq 0$ and $a+d-n+1 = d+1-m-n$ and hence $a = -m$, which is still absurd. Thus there are no equivariant global sections and the group vanishes.

We conclude $\Xi_{0,0}$ is fully-faithful and so $\Xi_{i,j}$ is fully-faithful for all $i,j \in \mathbb{Z}$ as it differs from $\Xi_{0,0}$ by an autoequivalence. □

Definition 3.6.1. Let $\mathcal{D}_{fg} = \Xi_{-m,-n}\mathcal{D}(X_f \times X_g)$.

By Lemma 3.6.1 we have $\Xi_{-m,-n}$ is a full embedding. The main result can now be stated.

Main Theorem. *In the above notation, we have a semi-orthogonal decomposition*

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_{g1}, \mathcal{D}_{fg}, \mathcal{D}_{g2}, \mathcal{D}_f, \mathcal{A} \rangle.$$

The proof of this theorem will occupy Chapter IV. In Sections 4.1 and 4.2 we finish proving that the decomposition is semi-orthogonal. In Sections 4.3 and 4.4, we analyze other sheaves that we can construct from the components present. In Section 4.5 we complete the proof of fullness.

It is worth noting that in the cases $(m, n) = (2, 2), (2, 3), (3, 3)$, there is an easier proof of this result. The idea of the proof is what is used in the subsequent sections and so we believe it does no harm in proving these special cases now.

Pf. in the special cases. We will only do the case $(m, n) = (2, 2)$ with the understanding that the other two are similar. The subcategories are as follows:

$$\begin{aligned}
\mathcal{D}_{g1} &= \mathcal{D}_g = \langle \mathcal{D}_g^{2-d}, \dots, \mathcal{D}_g^{-1} \rangle, \\
\mathcal{D}_{fg} &= \Xi_{-2,-2}(\mathcal{D}(X_f \times X_g)), \\
\mathcal{D}_f &= \langle \mathcal{D}_f^{d-2}, \dots, \mathcal{D}_f^{1} \rangle, \\
\mathcal{A} &= \langle \mathcal{O}_X(-2)(\chi^{-1}), \mathcal{O}_X(-1)\chi^{0,-1}, \mathcal{O}_X \rangle,
\end{aligned}$$

Define $\mathcal{T} = \langle \mathcal{D}_g, \mathcal{D}_{fg}, \mathcal{D}_f, \mathcal{A} \rangle$. We will prove orthogonality in §IV, the difficult part is fullness. To do this we show $\mathcal{T}$ has a spanning class. This will be sufficient to conclude $\mathcal{D}[X/\mu_d] = \mathcal{T}$.

Using Example 2.6.1, we see that the collection of objects consisting of free orbits, say $\mathcal{O}_Z$ where $Z = \{\lambda \cdot z\}_{\lambda \in \mu_d}$ and $\lambda \cdot z \neq z\}$, as well as the sheaves $\mathcal{O}_{\iota_f(p)}(\chi^i)$ and $\mathcal{O}_{\iota_g(q)}(\chi^i)$ for $i = 1, \ldots, d$ form a spanning class.

Let $J = J(X_f, X_g)$ inside of X denote the join of X_f and X_g. A free orbit $Z \subset X \setminus J$ is a complete intersection with respect to two sections $s_x \in \Gamma(\mathcal{O}_X(1))$ and $s_y \in \Gamma(\mathcal{O}_X(1)(\chi))$. It follows that the corresponding resolution of $\mathcal{O}_Z$ given by

$$0 \to \mathcal{O}_X(-2)(\chi^{-1}) \to \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1)(\chi^{-1}) \to \mathcal{O}_X$$

is in the subcategory $\mathcal{A}$, hence the free orbits $\mathcal{O}_Z \subset \mathcal{T}$ provided we are away from J.

To see we have the remaining objects of the spanning class, we will use Theorem 2.7.1. Let $l(p, q)$ denote the line joining $\iota_f(p)$ to $\iota_g(q)$. We will see in §4.3 that the objects $\mathcal{O}_{l(p,q)}(-d)$ and $\mathcal{O}_{l(p,q)}$ are in $\mathcal{T}$ for all p, q. It remains to see that the twists of the fixed orbits are in $\mathcal{T}$.

Using $\mathcal{D}_g$ and $\mathcal{D}_f$ we only need one additional twist, say $\mathcal{O}_p$, $\mathcal{O}_q$ (or in the case (m,n) = (2,3),(3,3) we will also need $\mathcal{O}_p(\chi^{-1}), \mathcal{O}_q(\chi))$. To do that, we notice that $\mathrm{Cone}(\mathcal{O}_X(-1) \to \mathcal{O}_X) \cong \mathcal{O}_{J(p,X_g)} \in \mathcal{A}$ by cutting out p with a section of $\mathcal{O}_X(1)$. We then have the exact sequence

$$0 \to \mathcal{O}_{J(p,X_g)} \to \bigoplus_{q \in X_g} \mathcal{O}_{l(p,q)} \to \mathcal{O}_p^{\oplus d-1} \to 0.$$

Since $\mathcal{T}$ is saturated, it follows that $\mathcal{O}_p \in \mathcal{T}$. In the case $(m, n) = (2, 3), (3, 3)$ we can look at a similar sequence using $\mathcal{O}_{J(p,X_g)}(-1)(\chi^{-1}) \in \mathcal{A}$. A similar argument shows $\mathcal{O}_q \in \mathcal{T}$ and Theorem 2.7.1 finishes the proof. □

CHAPTER IV

PROOF OF MAIN THEOREM

In this chapter we finish proving semi-orthogonality (Section 4.1) and prove fullness (Sections 4.3)

We finish the chapter by studying the case $m = 1$. This is the case of a cyclic cover.

4.1 Semi-orthogonality between D_{fg} and $\mathcal{A}$

As before, we let $\mathcal{D}_{fg}$ be the image of the fully-faithful functor $\Xi_{-m,-n}$. Let us compute the semi-orthogonality $\langle \mathcal{D}_{fg}, \mathcal{A} \rangle$. We have the formula

$$\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_X(-i)(\chi^{-j}), \mathcal{O}_{l(p,q)}(-m)(\chi^{-n})) \cong H^*(\mathbb{P}^1; \mathcal{O}(i-m)(\chi^{j-n}))^{\mu_d}.$$

Lemma 4.1.1. *There is a semi-orthogonal decomposition $\langle \mathcal{D}_{fg}, \mathcal{A} \rangle$.*

Proof. We only check the semi-orthogonality $\langle \mathcal{D}_{fg}, \mathcal{A}_3 \rangle$ as the other computations are similar. The objects in $\mathcal{A}_3$ are of the form $\mathcal{O}(-i)(\chi^{-j})$, where $0 \leq i \leq m-1$ and $0 \leq j \leq i$. In this case we have $\mathcal{O}(i-m)(\chi^{j-n})$ is a negative line bundle so

$$H^1(\mathcal{O}(i-m)(\chi^{j-n})) \cong H^0(\mathcal{O}(m-i-2)(\chi^{n-j-1}))^\vee$$

Since $i \geq j \geq 0$ we have

$$n-1 \geq n-j-1 \geq n-i-1$$

and so we need a monomial of the form $x^a y^{n-j-1}$ where $a \geq 0$ and $a+n-j-1 = m-i-2$. This is impossible because

$$a+n-j-1 \geq n-i-1 \geq m-i-1 > m-i-2.$$

□

4.2 Semi-orthogonality between $\mathcal{D}_g, \mathcal{D}_f, \mathcal{A}$.

For $\mathcal{D}_f$ to be present in the semi-orthogonal decomposition, we need $d > n$ and for $\mathcal{D}_g$ to be present, we require $d > m$.

Lemma 4.2.1. *We have the semi-orthogonality $\langle \mathcal{D}_g, \mathcal{D}_f, \mathcal{A} \rangle$.*

Proof. That $\mathcal{D}_g$ and $\mathcal{D}_f$ are semi-orthogonal is clear. We only show $\mathcal{D}_f$ is right orthogonal to $\mathcal{A}$. The claim that $\mathcal{D}_g$ is also right orthogonal is analgous.

Let $p \in X_f$ and consider the sheaves $\mathcal{O}_p(\chi^{-j})$. We compute

$$\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}(-i_1)(\chi^{-i_2}), \mathcal{O}_p(\chi^{-j})) \cong \Gamma(\mathcal{O}_p(\chi^{i_2-j}))^{\mu_d}$$

which is nonzero if and only if $i_2 - j = 0$. Since $0 \leq i_2 \leq n-1$, if we choose $n \leq j \leq d-1$, then

$$1-d \leq i_2 - j \leq -1.$$

These are precisely the weights in $\mathcal{D}_f$. This shows the semi-orthogonality $\langle \mathcal{D}_f, \mathcal{A} \rangle$.

□

Lemma 4.2.2. *We have the semi-orthogonality $\langle \mathcal{D}_{g1}, \mathcal{D}_{fg}, \mathcal{D}_{g2}, \mathcal{D}_f \rangle$.*

Proof. Again, we only prove the semi-orthogonality $\langle \mathcal{D}_{fg}, \mathcal{D}_f \rangle$ the other claims are analogous. The only possible nonzero extension group in the standard spanning class for $\mathcal{D}_{fg}$ is

$$\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_p(\chi^{-j}), \mathcal{O}_l(-m)(\chi^{-n})) \cong (\mathrm{Ext}^*_X(\mathcal{O}_p, \mathcal{O}_l)(\chi^{j-n}))^{\mu_d},$$

where l is the line between $p \in X_f$ and and any point $q \in X_g$. Set $R = \widehat{\mathcal{O}_{X,p}}$, then $R \cong k[[x_1, x_2, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}]]$. Here the variables x_i have weight 0 and the variables y_j have weight -1. The sheaf $\mathcal{O}_p$ corresponds to the graded module

$$k_p = R/(x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1})$$

The sheaf $\mathcal{O}_l$ corresponds to the graded module

$$M_q = R/(x_1, \ldots, x_{m-1}, y_2, \ldots, y_{n-1}).$$

We can take the Koszul resolution of k_p:

$$\left(\otimes_{i=1}^{m-1} Re_{x_i} \xrightarrow{x_i} R\right) \otimes \left(\otimes_{j=1}^{n-1} Re_{y_j}(\chi^{-1}) \xrightarrow{y_j} R\right) \to k_p$$

and apply $\mathrm{Hom}(-, M_q)$. This will kill all of the maps except y_1. The resulting complex has general term $M_q e^\vee_{x_I} \wedge e^\vee_{y_J}$, where $0 \leq |I| \leq m-1$ and $0 \leq |J| \leq n-1$. It's easy to see that the cohomology of this complex has general term $ke^\vee_{x_I} \wedge e^\vee_{y_J}$ where $0 \leq |I| \leq m-1$ and $1 \leq |J| \leq n-1$. The weights therefore vary between 1 and $n-1$. Thus the summands of the extension group are of the form:

$$k(\chi), \ldots, k(\chi^{n-1}).$$

Thus the general term of $\mathrm{Ext}^*_{[X/\mu_d]}(\mathcal{O}_p(\chi^{-j}), \mathcal{O}_l(-m)(\chi^{-n}))$ is of the form

$$k(\chi^{1+j-n}), \ldots, k(\chi^{n-1+j-n}) = k(\chi^{j-1}).$$

Since $n \leq j \leq d-1$, these terms all have nonzero weight and so the equivariant extension group vanishes. $\square$

This completes the semi-orthogonal claim in the Main Theorem. It remains to see fullness.

Definition 4.2.1. Define a full-subcategory $\mathcal{T}$ of $\mathcal{D}[X/\mu_d]$ by

$$\mathcal{T} = \langle \mathcal{D}_{g1}, \mathcal{D}, \mathcal{D}_{g2}, \mathcal{D}_f, \mathcal{A} \rangle.$$

4.3 Koszul complexes and Joins

By Proposition 2.3.2, the subcategory $\mathcal{T}$ is saturated and hence admissible. Using Proposition 2.6.1, it suffices to check that $\mathcal{T}$ has a spanning class. This is done in Section 4.5. To do so, we need to construct more sheaves in $\mathcal{T}$ using the subcategories present. Essential to these constructions is the Koszul complex of a regular section of a vector bundle.

Let E be a μ_d-equivariant locally free sheaf of rank r over X and $s \in \Gamma(E)^{\mu_d}$ be an equivariant global section. Then we have the corresponding Koszul complex:

$$0 \to \Lambda^r E^\vee \to \Lambda^{r-1} E^\vee \to \cdots \to E^\vee \xrightarrow{s^\vee} \mathcal{O}_X.$$

Denote this complex by $\mathcal{K}(E, s)$.

If the zero locus of s is codimension r, then the Koszul complex is exact and is a locally free resolution of $\mathcal{O}_{Z(s)}$, where $Z(s)$ is the vanishing locus of s. Even if the Koszul complex is not exact we can still learn information from its cohomology sheaves, see Lemma 4.3.2.

Let $Z \subset X$ be a free orbit of the μ_d action away from the join of X_f and X_g inside X, i.e. pick $p \notin X \setminus J(X_f, X_g)$ and let $Z = \{\lambda p \mid \lambda \in \mu_d\}$. These free orbits are also called μ_d-clusters.

Lemma 4.3.1. *For each μ_d-cluster $Z \subset X$, we have $\mathcal{O}_Z \in \mathcal{T}$.*

Proof. In this case, we notice that Z is the intersection of X with $m + n - 2$ hyperplanes. Moreover, we can pick sections $s_i \in \Gamma(\mathcal{O}_X(1))^{\mu_d}$ and section $t_j \in \Gamma(\mathcal{O}_X(1)(\chi))^{\mu_d}$ where $i = 1, \ldots, m-1$ and $j = 1, \ldots, n-1$ such that Z is the vanishing locus of $s = (s_1, \ldots, s_{m-1}, t_1, \ldots, t_{n-1}) \in \Gamma(E)$, where $E = \left(\bigoplus_{i=1}^{m-1} \mathcal{O}_X(1)\right) \oplus \left(\bigoplus_{j=1}^{n-1} \mathcal{O}_X(1)(\chi)\right)$.

The summands of $\mathcal{K}(E, s)$ are precisely the sheaves that occur in $\mathcal{A}$. We conclude for any free orbit Z in $X \setminus J(X_f, X_g)$, we know $\mathcal{O}_Z \in \mathcal{A}$. $\square$

Let $p \in X_f$ and $q \in X_g$. As before, denote by $l(p,q) \cong \mathbb{P}^1$ the line joining $\iota_f(p)$ to $\iota_g(q)$. Contrary to the case of free orbits outside of $J(X_f, X_g)$, the structure sheaves of both fixed orbits and free orbits in the join are not complete intersections. Moreover, the structure sheaf $\mathcal{O}_{l(p,q)}$ is not a complete intersection subvariety. We can still take the corresponding Koszul complex cutting it out. Indeed, there exists a section $s = (s_1, \ldots, s_{m-1}, t_1, \ldots, t_{n-1}) \in \Gamma(E)$, where E is as before, such that $V(s) = l(p,q)$.

Lemma 4.3.2. *As above, let $\mathcal{K}(E,s)$ be the Koszul complex cutting out $l(p,q)$. Then*

$$\mathcal{H}^*(\mathcal{K}(E,s)) = \begin{cases} \mathcal{O}_{l(p,q)} & * = 0 \\ \mathcal{O}_{l(p,q)}(-d) & * = -1 \\ 0 & * \neq 0, -1 \end{cases}$$

Proof. By Bezout's theorem, we can assume the intersection $X_f \cap V(s_1, \ldots, s_{m-2})$ consists of d points, say $p_1, \ldots, p_d$. The intersection $X_g \cap V(t_1, \ldots, t_{n-1})$ is $\{q\}$. Let J denote the join in X of $\{p_1, \ldots, p_d\}$ with $\{q\}$.

The Koszul complex associated to the sections $(s_1, \ldots, s_{m-2}, t_1, \ldots, t_{n-1})$ is quasi-isomorphic to the following complex

$$0 \to \mathcal{O}_J(-1) \to \mathcal{O}_J \to 0.$$

To each point $\{p_1, \ldots, p_d\} \in X_f \subset \mathbb{P}^1$ there exists a linear section s_i such that $V(s_i) = p_i$. We have the exact sequence

$$0 \to \mathcal{O}_{l(p,q)}(-d) \xrightarrow{s_2 \cdots s_d} \mathcal{O}_J(-1) \xrightarrow{s_1} \mathcal{O}_J(-1) \to \mathcal{O}_{l(p_1,q)} \to 0.$$

The claim follows. $\square$

Lemma 4.3.3. *The subvarieties X_g and X_f are complete intersection subvarieties.*

Proof. We show how to get X_g, X_f is analogous. The zero locus of the section $s_{X_g} = (x_1, \ldots, x_m)$ of the vector bundle $E_{X_g} = \mathcal{O}_X(1)^{\oplus m}$ is X_g. $\square$

The summands of the Koszul resolution, $\mathcal{K}(E_{X_g}, s_{X_g})$ are of the form $\mathcal{O}_X(-m+i)$ for $i = 0, \ldots, m$.

Lemma 4.3.4. *For $i = 1, \ldots, d - m$. The components of $\mathcal{K}(E_{X_g}, s_{X_g})(-(n-1) + i + t)(\chi^{-(n-1)+t})$ are in $\mathcal{T}$ for $t = 0, \ldots, n-1$.*

Remark 4.3.1. The restriction of the equivariant structure on a hyperplane divisor of X to X_g is not the trivial structure. In particular, we have isomorphisms:

$$\mathcal{O}_X(iH)|_{X_g} \cong \mathcal{O}_{X_g}(ih)(\chi^{-i}).$$

Proof. We check the base case $i = 1$. In this case, we have explicitly:

$$\begin{aligned} \mathcal{K}(E_{X_g}, s_{X_g})(1) &\to \mathcal{O}_{X_g}(1)(\chi^{-1}) \\ \mathcal{K}(E_{X_g}, s_{X_g})(\chi^{-1}) &\to \mathcal{O}_{X_g}(\chi^{-1}) \\ \vdots &\to \vdots \\ \mathcal{K}(E_{X_g}, s_{X_g})(-(n-1)+1)(\chi^{-(n-1)}) &\to \mathcal{O}_{X_g}(-(n-1)+1)(\chi^{-1}) \end{aligned}$$

All line bundles appearing in the resolution are already in $\mathcal{T}$ except the line bundle appearing in degree zero. Since $\mathcal{O}_{X_g}(j)(\chi^{-1}) \in \mathcal{T}$ for all j, we know that the line bundles in degree zero are also in $\mathcal{T}$.

Suppose true for $1, \ldots, i$ we show true for $i + 1 \leq d - m$. In which case we have the following twists of the above diagram:

$$\begin{aligned} \mathcal{K}(E_{X_g}, s_{X_g})(i+1) &\to \mathcal{O}_{X_g}(i+1)(\chi^{-i-1}) \\ \mathcal{K}(E_{X_g}, s_{X_g})(i)(\chi^{-1}) &\to \mathcal{O}_{X_g}(i)(\chi^{-i-1}) \\ \vdots &\to \vdots \\ \mathcal{K}(E_{X_g}, s_{X_g})(-(n-1)+i+1)(\chi^{-(n-1)}) &\to \mathcal{O}_{X_g}(-(n-1)+i+1)(\chi^{-i-1}). \end{aligned}$$

Again, all of the sheaves except those in the extreme right of the resolution are already in $\mathcal{T}$ by induction. That the line bundles in degree zero are in $\mathcal{T}$ follows since $\mathcal{O}_{X_g}(j)(\chi^{-i-1}) \in \mathcal{T}$ as $i+1 \leq d-m$. $\square$

Lemma 4.3.5. *Let $J(X_f, q)$ denote the join of X_f and q inside X. Then*

$$\mathcal{O}_{J(X_f,q)}(i) \in \mathcal{T}$$

for $i = 0, \ldots, d-m$.

Proof. The subvariety $J(X_f, q)$ is a complete interesection:

$$(\otimes_{i=1}^{n-1} \mathcal{O}_X(-1)(\chi^{-1}) \to \mathcal{O}_X) \to \mathcal{O}_{J(X_f,q)}.$$

Twisting by $\mathcal{O}_X(i)$ for $i = 0, \ldots, d-m$ gives a general component of the Koszul resolution as

$$\mathcal{O}_X(-(n-1)+i+t)(\chi^{-(n-1)+t}).$$

The statement now follows from Lemma 4.3.4. $\square$

As X_f is also a complete intersection subvariety, we have the following similar statement for $\mathcal{O}_{J(p,X_g)}$ with $p \in X_f$.

Lemma 4.3.6. *Let $\mathcal{O}_{J(p,X_g)}$ denote the join of $p \in X_f$ with X_g. Then*

$$\mathcal{O}_{J(p,X_g)}(i)(\chi^i) \in \mathcal{T}$$

for $i = 0, \ldots, d-n$.

Proof. The presence of twists comes from the fact that X_f is cut out be the section $s = (y_1, \ldots, y_n) \in \Gamma(\mathcal{O}_X(1)^{\oplus n}(\chi))$. $\square$

Our goal is to show $\mathcal{T}$ has a spanning class. Recall, Example 2.6.1, if $\mathcal{X}$ is a smooth DM stack with coarse moduli space $\pi\colon \mathcal{X} \to \mathfrak{X}$, the sheaves

$$\Omega = \{\mathcal{Z} \subset \mathcal{X} \mid \mathcal{Z} \text{ is a closed substack of } \mathcal{X} \text{ and } \pi(\mathcal{Z}) \text{ is a closed point of } \mathfrak{X}\}$$

form a spanning class.

For $\mathcal{X} = [X/\mu_d]$, these sheaves are the structure sheaves of the free orbits and twists of the structure sheaves of fixed orbits by all characters. In Lemma 4.3.1 we saw that the structure sheaves of the free orbits away from $J(X_f, X_g)$ have Koszul resolutions using the sheaves in $\mathcal{A}$. We will get the remaining sheaves by showing for all $p \in X_f$ and $q \in X_g$ we have $\mathcal{D}[l(p,q)/\mu_d] \subset \mathcal{D}[X/\mu_d]$. For that we use Theorem 2.7.1 and is carried out in the next two sections.

4.4 Other kernels.

It will be convenient to use the images of other Fourier-Mukai kernels from $\Xi_{-m,-n}$ to $\Xi_{d-m,0}$ and $\Xi_{d-n,d-n}$. We justify their use in this subsection.

Using Theorem 2.6.1 we must verify that the image of the spanning class $\{\mathcal{O}_{(p,q)}\}$ under $\Xi_{d-m,0}$ and $\Xi_{d-n,d-n}$ factors through $\mathcal{T}$. We will need to start by verifying the line bundles in Theorem 2.7.1 are in $\mathcal{T}$.

Lemma 4.4.1. *For all $p \in X_f$ and $q \in X_g$ we have $\mathcal{O}_{l(p,q)}(-d)$ and $\mathcal{O}_{l(p,q)}$ in $\mathcal{T}$.*

Proof. By Lemma 4.3.2, it suffices to show $\mathcal{O}_{l(p,q)}(-d) \in \mathcal{T}$. We have the exact sequences

$$0 \to \mathcal{O}_{l(p,q)}(-m-i-1)(\chi^{-n}) \to \mathcal{O}_{l(p,q)}(-m-i)(\chi^{-n}) \to \mathcal{O}_q(\chi^{-(n-m)+i}) \to 0.$$

Since $\mathcal{O}_{l(p,q)}(-m)(\chi^{-n}), \mathcal{O}_q(\chi^{-(n-m)}), \ldots, \mathcal{O}_q(\chi^{-1}) \in \mathcal{T}$, we have $\mathcal{O}_{l(p,q)}(-n)(\chi^{-n}) \in \mathcal{T}$ by induction.

Now consider the sequences

$$0 \to \mathcal{O}_{l(p,q)}(-n-i-1)(\chi^{-n-i-1}) \to \mathcal{O}_{l(p,q)}(-n-i)(\chi^{-n-i}) \to \mathcal{O}_p(\chi^{-n-i}) \to 0.$$

Since $\mathcal{O}_{l(p,q)}(-n)(\chi^{-n}), \mathcal{O}_p(\chi^1), \ldots, \mathcal{O}_p(\chi^{d-n}) \in \mathcal{T}$, we have $\mathcal{O}_{l(p,q)}(-d) \in \mathcal{T}$ by induction and this completes the proof. $\square$

Lemma 4.4.2. *For all $p \in X_f$ and $q \in X_g$ we have $\mathcal{O}_{l(p,q)}(d-m), \mathcal{O}_{l(p,q)}(d-n)(\chi^{d-n}) \in \mathcal{T}$.*

Proof. Use the exact sequences in the proof of Lemma 4.4.1. $\square$

Proposition 4.4.1. *The functors $\Xi_{d-m,0}$ and $\Xi_{d-n,d-n}$ factor through $\mathcal{T}$.*

Proof. By Proposition 2.3.2, it follows from Lemma 4.4.2 and Theorem 2.6.1. $\square$

4.5 Proof of Fullness

We now compute $\Xi_{d-m,0}(\mathcal{O}_{X_f\times\{q\}}(-i))$ and $\Xi_{d-n,d-n}(\mathcal{O}_{\{p\}\times X_g}(-j))$ for various $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$ to show that this gives us the missing sheaves: $\mathcal{O}_q(\chi^{0,1,\ldots,m-1}), \mathcal{O}_p(\chi^{0,-1,\ldots,-(n-1)})$. In particular, we show there exists triangles

$$\mathcal{O}_{J(X_f,q)}(d-m-i) \to \Xi_{d-m,0}(\mathcal{O}_{X_f\times\{q\}}(-i)) \to T_q \to$$

and

$$\mathcal{O}_{J(p,X_g)}(d-n-j)(\chi^{d-n}) \to \Xi_{d-n,d-n}(\mathcal{O}_{\{p\}\times X_g}) \to T_p \to$$

for every $p \in X_f$ and $q \in X_g$, where T_q and T_p are certain torsion sheaves supported at q and p, respectively. Since the first two objects are in $\mathcal{T}$ we have $T_q, T_p \in \mathcal{T}$.

We then build a filtration of T_q, T_p and argue then that $\mathcal{T}$ has the remaining elements of the spanning class. To perform these computations, we need to add an auxiliary blowup.

Let $q \in X_g$ and let $\mathbb{P}^m$ be the linear subspace spanned by $x_1, \dots, x_m, q$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{P}(\mathcal{O}_{X_f}(-1) \boxplus \mathcal{O}_{X_g}(-1)) \arrow[d] & \mathbb{P}(\mathcal{O}_{X_f}(-1) \oplus \mathcal{O}_{X_f}) = Z \arrow[l] \arrow[d, "\pi"] \arrow[r, "\iota_q"] \arrow[dr, "\sigma"] & \mathrm{Bl} = \mathrm{Bl}_q \mathbb{P}^m \arrow[d, "\beta"] \\
X_f \times X_g & X_f \arrow[l, "j"] & \mathbb{P}^m
\end{tikzcd}$$

where j includes X_f via $j(x) = (x, q)$, Z is the fibered product, and σ also denotes the restriction of $\sigma\colon Y \to \mathbb{P}^{m+n-1}$ which factors through $\mathbb{P}^m$. The mapping ι_q includes Y as the divisor $dH - dE$ in Bl. Here $\mathbb{P}^m$ has coordinates $[x_1 : \cdots : x_m : y]$ and μ_d acts by scaling the y coordinate. The μ_d-action lifts to Bl fixing the exceptional divisor pointwise and thus rendering the enter diagram μ_d-equivariant.

Recall, the canonical bundle of $[\mathbb{P}^m/\mu_d]$ is isomorphic to $\mathcal{O}_{\mathbb{P}^m}(-m-1)\chi^{-1}$. The usual formula for the canonical bundle of a blowup yields $\omega_{\mathrm{Bl}} \cong \beta^*\mathcal{O}_{\mathbb{P}^m} \otimes \mathcal{O}((m-1)E)$ which admits a μ_d-linearization since the divisors involved are invariant under the μ_d-action. It remains to determine if there is a twist by a character. Restricting to $\mathrm{Bl} \setminus E$ gives the isomorphism

$$\omega_{\mathrm{Bl}}|_{\mathrm{Bl}\setminus E} \cong \beta^*\omega_{\mathbb{P}^m}|_{\mathrm{Bl}\setminus E}$$

and so it follows $\omega_{\mathrm{Bl}} \cong \beta^*\mathcal{O}_{\mathbb{P}^m}(-m-1) \otimes \mathcal{O}_{\mathrm{Bl}}((m-1)E)(\chi^{-1})$.

Let H_1 be a hyperplane section of X_f and H be a hyperplane section of $\mathbb{P}^m$ which restricts to H_1 under the inclusion $X_f \to \mathbb{P}^m$ where $[x] \mapsto [x : 0]$. Then $H - E|_Y \cong \pi^* H_1$.

Theorem 4.5.1 (Equivariant Grothendieck Duality). *There is a natural isomorphism:*

$$\mathbf{R}\beta_*\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(D) \cong \mathbf{R}\mathcal{H}om_{[\mathbb{P}^m/\mu_d]}(\mathbf{R}\beta_*(\omega_{[\mathrm{Bl}/\mu_d]}(-D)), \omega_{[\mathbb{P}^m/\mu_d]})$$

for any μ_d-invariant divisor D on Bl.

Proof. This follows since β is μ_d-equivariant and the usual Grothendieck duality, [14, Theorem 3.34], is natural, hence commutes with automorphisms. □

The divisors on Bl are, up to equivalence, well known to be of the form $aH + bE$ for $a, b \in \mathbb{Z}$. Using the projection formula, we have

$$\mathbf{R}\beta_*\omega_{\mathrm{Bl}}(-(aH + bE) \cong (R\beta_*\mathcal{O}_{[\mathrm{Bl}/\mu_d]}((m-1-b)E)) \otimes \omega_{[\mathbb{P}^m/\mu_d]}(-aH)$$

and by Grothendieck duality

$$\mathbf{R}\beta_*(\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(aH + bE)) \cong \mathbf{R}\mathcal{H}om_{[\mathbb{P}^m/\mu_d]}(\mathbf{R}\beta_*(\mathcal{O}_{[\mathrm{Bl}/\mu_d]}((m-1-b)E), \mathcal{O}_{\mathbb{P}^m})(-aH).$$

Remark 4.5.1. Since $\{q\}$ is codimension m, there is a canonical isomorphism

$$\mathbf{R}\beta_*\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(kE) \cong \mathcal{O}_{[X/\mu_d]}$$

for $k = 0, \ldots, m-1$.

If $k > 0$, then $R\beta_*\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-kE) \cong \mathcal{I}_q^k$, where $\mathcal{I}_q$ is the ideal sheaf for the closed subscheme $\{q\}$ in $\mathbb{P}^m$. Moreover, $\mathbf{R}\beta_*^i\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(kE) = 0$ unless $i = 0, m-1$.

For $i = 0, \ldots, m-1$, we consider $\Xi_{d-m,0}(\mathcal{O}_{X_f}(-iH_1))$. On Bl we have the divisor exact sequence

$$0 \to \mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-dH + dE) \to \mathcal{O}_{[\mathrm{Bl}/\mu_d]} \to \iota_{q*}\mathcal{O}_Y \to 0.$$

Since $\iota_{q*}\pi^*\mathcal{O}_{X_f}(-iH_1) = \iota_{q*}\mathcal{O}_Y(-iH_1) \cong \iota_{q*}\mathcal{O}_Y \otimes \mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-i(H-E))$, we can consider the twist of the divisor exact sequence

$$0 \to \mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-dH + dE - iH + iE) \to \mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-iH + iE) \to \mathcal{O}_Y(-iH_1) \to 0.$$

Using the long exact sequence of cohomology sheaves for $\mathbf{R}\beta_*$, we see there is an isomorphism $\mathbf{R}\sigma_*^k\mathcal{O}_Y(-iH_1) \cong \mathbf{R}\beta_*^{k+1}\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-(d+i)H + (d+i)E)$. It follows from Remark 4.5.1, that the only possible nonzero higher direct image is $\mathbf{R}\sigma_*^{m-2}\mathcal{O}_Y(-iH_1)$.

Lemma 4.5.1. *If $m > 2$, then for $i = 0, \ldots, m-1$, we have a distinguished triangle*

$$\mathcal{O}_{J(X_f,q)}(d-m-i) \to \Xi_{d-m,0}(\mathcal{O}_{X_f}(-iH_1)) \to \mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+1+i})^\vee)(-m-i)[2-m] \to$$

where $\mathcal{I}_q$ is the ideal sheaf of $\{q\}$ in $\mathbb{P}^m$, and $(\mathcal{I}_q^{d-m+1+i})^\vee$ is the derived dual. In particular,

$$\mathcal{H}^*(\Xi_{d-m,0}(\mathcal{O}_{X_f}(-iH_1))) \cong \begin{cases} \mathcal{O}_{J(X_f,q)}(d-m-i) & * = 0 \\ \mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+1+i})^\vee)(-m-i) & * = m-2 \\ 0 & * \neq 0, m-2 \end{cases}.$$

If $m = 2$, then for $i = 0, 1$ there is an exact sequence

$$0 \to \mathcal{O}_{J(X_f,q)}(d-2-i) \to \Xi_{d-2,0}(\mathcal{O}_{X_f}(-iH_1)) \to \mathcal{O}_q^{\oplus d-1}(\chi^{2+i-d}) \to 0.$$

Proof. The case $m = 2$ is easy to see directly and the vanishing statements for $m > 2$ follow from the preceeding discussion. It remains to show the isomorphisms. Since $d + i > 0$, the only possible higher direct image is in degree $m - 1$. We have an exact sequence

$$0 \to \mathcal{O}_{[\mathbb{P}^m/\mu_d]}(-d-i) \to \mathcal{O}_{[\mathbb{P}^m/\mu_d]}(-i) \to \mathcal{O}_{J(X_f,q)}(-i) \to 0.$$

Now $\mathcal{H}^0(\Xi_{d-m,0}(\mathcal{O}_{X_f}(-iH_1))) \cong \mathcal{H}^0(\Xi_{0,0}(\mathcal{O}_{X_f}(-iH_1)))(d-m) \cong \mathcal{O}_{J(X_f,q)}(d-m-i)$. This gives us the first arrow for $m > 2$. If $m = 2$, the first arrow is defined similarly but it is not surjective onto $\mathcal{H}^0(\Xi_{d-m,0}(\mathcal{O}_{X_f}(-iH_1)))$.

For the second we need to compute $\mathbf{R}\beta^{m-1}\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-(d+i)H + (d+i)E)$. By Grothendieck duality and the derived functor spectral sequence, we have an isomorphism

$$\begin{aligned}
&\mathbf{R}\beta_*^{m-1}\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(-(d+i)H + (d+i)E) \\
&\cong \mathbf{R}\mathcal{H}om_{[\mathbb{P}^n/\mu_d]}(\mathbf{R}\beta_*(\mathcal{O}_{[\mathrm{Bl}/\mu_d]}(m-1-d-i)E), \mathcal{O}_{[\mathbb{P}^n/\mu_d]})(-d-i) \\
&\cong \mathbf{R}\mathcal{H}om_{[\mathbb{P}^n/\mu_d]}(\mathcal{I}_q^{d-m+i+1}, \mathcal{O}_{[\mathbb{P}^n/\mu_d]})(-d-i).
\end{aligned}$$

and the second isomorphism follows by twisting by $(d-m)$. $\square$

Corollary 4.5.1. *For $i = 0, \ldots, d-m$, we have $\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+i+1})^\vee)(-m-i) \in \mathcal{T}$.*

Proof. This now follows by Lemmas 4.3.5 and 4.5.1. $\square$

It remains to compute $\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+i+1})^\vee)(-m-i)$ for $i = 0, \ldots, m-1$.

We seek to understand the sheaves in Corollary 4.5.1. To that end, there is an exact sequence of sheaves on $\mathbb{P}^m$, where we have identified the conormal bundle of $\{q\}$ with $\Omega_{\mathbb{P}^m,q}$:

$$0 \to \mathcal{I}_q^{r+1} \to \mathcal{I}_q^r \to S^r(\Omega_{\mathbb{P}^m,q}) \cong \mathcal{O}_q^{\oplus N(r)}(\chi^r) \to 0,$$

where $N(r) = \binom{m+r-1}{r}$. Since $\mathcal{O}_q$ is a smooth closed subscheme of codimension m, we know

$$\mathcal{H}^m((\mathcal{O}_q(\chi^r))^\vee) \cong \mathcal{O}_q \otimes \omega_{\mathbb{P}^m}^\vee(\chi^{-r}).$$

Since $\omega_{[\mathbb{P}^m/\mu_d]} \cong \mathcal{O}_{\mathbb{P}^m}(-m-1)(\chi^{-1})$, we see

$$\mathcal{H}^m((\mathcal{O}_q(\chi^r))^\vee) \cong \mathcal{O}_q(\chi^{-r-m}).$$

Taking the derived dual of the above sequence now yields the short exact sequence

$$0 \to \mathcal{H}^{m-1}((\mathcal{I}_q^r)^\vee) \to \mathcal{H}^{m-1}((\mathcal{I}_q^{r+1})^\vee) \to \mathcal{O}_q^{\oplus N(r)}(\chi^{-r-m}) \to 0.$$

Lemma 4.5.2. *For $r > 0$, the sheaves $\mathcal{H}^{m-1}((\mathcal{I}_q^r)^\vee)$ have a filtration by sheaves $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r+1} = \mathcal{H}^{m-1}((\mathcal{I}_q^r)^\vee)$ such that*

$$\mathcal{F}_i/\mathcal{F}_{i+1} \cong \mathcal{O}_q^{\oplus N(r)}(\chi^{-i-m}).$$

In particular, $\mathcal{H}^{m-1}((\mathcal{I}_q^{r+1})^\vee) \in \langle \mathcal{O}_q(\chi^{-m}), \mathcal{O}_q(\chi^{-1-m}), \ldots, \mathcal{O}_q(\chi^{-r-m})\rangle$.

Proof. Immediate from the previous discussion and the observation $\mathcal{H}^{m-1}(\mathcal{I}_q^\vee) \cong \mathcal{O}_q(\chi^{-m})$. $\square$

Using Lemma 4.5.2, we have the following filtration of $\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+1+i})^\vee)$:

$$\begin{array}{ccc}
\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+1+i})^\vee)(d-m-i) & \longrightarrow & \mathcal{O}_q^{\oplus N(d-m+i)}(\chi^{-(d-m)}) \\
\uparrow & & \\
\mathcal{H}^{m-1}((\mathcal{I}_q^{d-m+i})^\vee)(d-m-i) & \longrightarrow & \mathcal{O}_q^{\oplus N(d-m-1+i)}(\chi^{-(d-m)+1}) \\
\uparrow & & \\
\vdots & & \vdots \\
\uparrow & & \\
\mathcal{H}^{m-1}((\mathcal{I}_q^{2})^\vee)(d-m-i) & \longrightarrow & \mathcal{O}_q^{\oplus N(1)}(\chi^{-1+i}) \\
\uparrow & & \\
\mathcal{H}^{m-1}((\mathcal{I}_q)^\vee)(d-m-i) & \longrightarrow & \mathcal{O}_q(\chi^{i})
\end{array}$$

Lemma 4.5.3. *For all $i = 0, \ldots, d-1$ we have $\mathcal{O}_q(\chi^i) \in \mathcal{T}$.*

Proof. We proceed by induction on i. When $i = 0$ we use the filtration given by Lemma 4.5.2 and the fact that $\mathcal{O}_q(\chi^{-j}) \in \mathcal{T}$ for $j = 1, \ldots, d-m$ to see $\mathcal{O}_q \in \mathcal{T}$. Suppose we have $\mathcal{O}_q, \ldots, \mathcal{O}_q(\chi^i)$, then we use the filtration again with $i+1$ to see $\mathcal{O}_q(\chi^{i+1}) \in \mathcal{T}$. $\square$

Lemma 4.5.4. *For all $i = 0, \ldots, d-1$ we have $\mathcal{O}_p(\chi^i) \in \mathcal{T}$.*

Proof. This is analogous to the results proved in this section using the images $\Xi_{d-n,d-n}(\mathcal{O}_{X_g}(-jH_2))$. One requires the extra twist because the hyperplane section that restricts to H_2 has nontrivial equivariant structure. $\square$

Proof of Main Result. By Lemmas 4.4.1, 4.5.3, and 4.5.4 we have shown for all $p \in X_f$ and $q \in X_g$ we have $\mathcal{D}[l(p,q)/\mu_d] \subset \mathcal{T}$. We have also shown in Lemma

4.3.1 that the structure sheaves of free orbits not in the join are in $\mathcal{T}$. Thus $\mathcal{T}$ has a spanning class. Since $\mathcal{T}$ is admissible, by Proposition 2.6.1 we conclude $\mathcal{T} = \mathcal{D}[X/\mu_d]$. $\square$

4.6 The Case $m = 1$

For completeness, we devote this section to understanding $[X/\mu_d]$ when $m = 1$. We will independently study $n = 1$ and $n > 1$. In the case $n > 1$, the hypersurface X is called a *cyclic hypersurface*. We also compare the decompositions to the work in [16]

Let $m, n = 1$, then $\mathcal{D}(X_f)$ and $\mathcal{D}(X_g)$ will not appear. The associated quotient stack is, up to a change of coordinates, $X = V(x^d + y^d) \subset \mathbb{P}^1$. In particular, $|X| = d$ and the μ_d action permutes the linear factors. It follows that $[X/\mu_d]$ is a scheme and is represented by $\mathrm{Spec}(k)$. There is a single exceptional object given by $\mathcal{O}_X$ and so

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{O}_X \rangle.$$

Let $n > 1$. Then $f(x) = x^d$ and $g(y_1, \ldots, y_n)$ be a degree d polynomial defining a smooth hypersurface in $\mathbb{P}^{n-1}$. Let $\pi\colon X \to \mathbb{P}^{n-1}$ be the linear projection onto the y variables. This is well defined since $[1 : 0 : \ldots : 0] \notin X$. The map π is a degree d mapping ramified along the divisor $\iota_g\colon X_g \hookrightarrow X$. In particular, we have the following commutative diagram.

$$\begin{tikzcd}
 & X \arrow[d, "\pi"] \\
X_g \arrow[ur, "\iota_g"] \arrow[r] & \mathbb{P}^{n-1}
\end{tikzcd}$$

Endowing $\mathbb{P}^{n-1}$ with the trivial μ_d action renders the diagram commutative. Moreover, it is not hard to see that π exhibits $\mathbb{P}^{n-1}$ as a coarse moduli space.

Theorem 4.6.1. *There is a semi-orthogonal decomposition*

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_g^1, \dots, \mathcal{D}_g^{d-1}, \pi^* \mathcal{D}(\mathbb{P}^{n-1}) \rangle,$$

where $\mathcal{D}_g^i = \iota_{g*}\mathcal{D}(X_g)(\chi^i)$.

Proof. The action of μ_d on $X \setminus X_g$ is easily seen to be free. Hence Theorem 2.7.2 applies. □

The case of a cyclic cover of a variety was investigated in [16, §8.3]. In particular, they discuss the equivariant derived category of cyclic hypersurfaces where $d \leq n$, here $X \subset \mathbb{P}^n$ and $X_g \subset \mathbb{P}^{n-1}$. For completeness, we recall their result. Since $d \leq n$, we have the standard semi-orthogonal decomposition of a hypersurface

$$\mathcal{D}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \dots, \mathcal{O}_X(d-n) \rangle,$$

where $\mathcal{A}_X$ is characterized as the right orthogonal to $\langle \mathcal{O}_X, \dots, \mathcal{O}_X(d-n) \rangle$. The category $\mathcal{A}_X$ is also quasi-equivalent to the homotopy category of graded matrix factorizations of the potential f, where f is the defining equation for X.

Theorem 4.6.2. *In the above notation, if* $d \leq n$*, then there is a decomposition*

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{A}_X^{\mu_d}, \mathcal{O}_X(\chi^{0,\dots,d-1}), \dots \mathcal{O}_X(d-n)(\chi^{0,\dots,d-1}) \rangle$$

where

$$\mathcal{A}_X^{\mu_d} = \langle \mathcal{A}_{X_g}, \mathcal{A}_{X_g}(\chi), \dots, \mathcal{A}_{X_g}(\chi^{n-2}) \rangle.$$

where $\mathcal{A}_{X_g}$ is viewed as a subcategory of $\mathcal{D}[X/\mu_d]$ via ι_{g*}.

Remark 4.6.1. Note, their results do not apply when $d > n$ because $\pi\colon X \to \mathbb{P}^{n-1}$ is not a cyclic cover in the sense of [16]. Indeed, suppose X is the relative spectrum associted to the the line bundle $\mathcal{O}(i)$ with a section $s \in \Gamma(\mathcal{O}(di))$ over $\mathbb{P}^{n-1}$. That is, we consider the sheaf of algebras

$$\mathcal{A} = \mathcal{O}(-di) \oplus \cdots \oplus \mathcal{O}$$

on $\mathbb{P}^{n-1}$ and $X \cong \operatorname{Spec}(\mathcal{A})$. Then $\mathbf{R}\pi_*\mathcal{O}_X \cong \mathcal{A}$ as sheaves on $\mathbb{P}^{n-1}$. Therefore we would have an isomorphism

$$H^{n-1}(\mathcal{O}_X) \cong \bigoplus_{i=1}^{d} H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(-di)).$$

Using the divisor exact sequence for X we see $H^{n-1}(\mathcal{O}_X) \cong H^n(\mathcal{O}_{\mathbb{P}^n}(-d)) \cong H^0(\mathcal{O}_{\mathcal{P}^n}(d-n-1))$ which is of dimension $\binom{d-1}{d-n-1}$. For the right hand side we have

$$\bigoplus_{i=0}^{d} H^{n-1}(\mathcal{O}_{\mathbb{P}^{n-1}}(-di)) \cong \bigoplus_{i=0}^{d} H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(di-n)),$$

which is of dimension

$$\sum_{i=1}^{d} \binom{di}{di-n} > \binom{d}{d-n} = \binom{d}{n} > \binom{d-1}{n} = \binom{d-1}{d-1-n}.$$

It follows that π cannot be a cyclic cover.

When $d = n$ the subcategory $\mathcal{A}_{X_g}$ is all of $\mathcal{D}(X_g)$. Using the notation $\mathcal{D}_g^i = \iota_{g*}(\mathcal{D}(X_g))(\chi^i)$, the decomposition of Theorem 4.6.2 is

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_g^0, \mathcal{D}_g^1, \ldots, \mathcal{D}_g^{n-2}, \mathcal{O}_X(\chi^{0,\ldots,n-1}) \rangle.$$

The decomposition of Theorem 4.6.1 is

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{D}_g^1, \ldots, \mathcal{D}_g^{n-1}, \pi^*(\mathcal{D}(\mathbb{P}^{n-1})) \rangle.$$

It follows that our decomposition agrees with theirs up to a twist by a character.

CHAPTER V

COMPARISON WITH ORLOV'S FUNCTORS ON POINTS

In this chapter, we show in the case of two Calabi-Yau hypersurfaces, i.e. $m = n = d$, that the functor Ξ agrees with Orlov's on points. To do so we remind the reader of graded matrix factorizations and singularity categories in Section 5.1. In Section 5.3, we discuss Orlov's theorem and how to compute Orlov's functors. In Section 5.4, we do some computations on special objects. In Section 5.5 we show that our functor agree, up to a mutation with Orlov's functor on points.

5.1 Graded Matrix Factorizations

Let $R = \mathrm{Spec}(\mathrm{Sym}(V^\vee))$ be functions on a finite dimensional vector space V. Then R has a natural $\mathbb{Z}$-grading so that

$$R_d = \mathrm{Sym}^d(V^\vee),$$

which corresponds to the diagonal action of $\mathbb{G}_m$ on V with weight 1.

By a graded R-module, we mean an R-module M together with a $\mathbb{Z}$-grading such that

$$R_m \cdot M_n \subset M_{m+n}.$$

We will refer to the $\mathbb{Z}$-grading on M as the *internal grading*. Morphisms between graded R-modules are R-module morphisms that preserve the grading. Denote the category of $\mathbb{Z}$-graded R-modules by $Gr - R$. This is an Abelian category.

If $M \in Gr - R$, then there is a internal grading shift functor, denoted by (1), where we define the graded R-Module $M(1)$ by $(M(1))_m = M_{m+1}$. Denote by (q)

the qth power of (1). If $M, N \in Gr - R$, then there is a natural grading on the (ungraded) R-module $\mathrm{Hom}_R(M, N)$, where

$$\mathrm{Hom}_R(M, N)_m = \bigoplus_m \mathrm{Hom}_{Gr-R}(M, N(m)).$$

Call this, now graded, R-module $\underline{\mathrm{Hom}}_R(M, N)$. As usual, it is the internal hom in $Gr - R$.

Let $f \in R_d$ be a degree d polynomial. Then f defines a hypersurface $X = V(f) \subset \mathbb{P}(V)$. We assume X is smooth or equivalently the affine cone $C(X) \subset V$ has an isolated singularity at the origin.

Definition 5.1.1. A graded matrix factorization is a pair of morphism

$$\delta_0 \colon P_{-1} \to P_0, \ \delta_{-1} \colon P_0 \to P_{-1}(d)$$

between graded projective R-modules such that

$$\delta_{-1}\delta_0 = f = \delta_0\delta_{-1}.$$

When no confusion arises, we denote a matrix factorization by $\mathcal{P}$.

Equivalently, a graded matrix factorization is a curved complex of $\mathbb{G}_m$-equivariant vector bundles on V with curvature f.

Example 5.1.1. Consider the case $\dim(V) = 2$. Then $R \cong k[x, y]$ with its usual $\mathbb{Z}$-grading. Consider $f = xy$. Then there are two (inequivalent) matrix factorizations

$$l_x \colon R \xrightarrow{x} R(1) \xrightarrow{y} R(2)$$

and

$$l_y \colon R \xrightarrow{y} R(1) \xrightarrow{x} R(2).$$

It will follow from Orlov's Theorem that these matrix factorizations are inequivalent.

Example 5.1.2 (Koszul matrix factorizations)**.** Let $s_1, \ldots, s_r$ be a homogeneous regular sequence in R such that $f \in (s_1, \ldots, s_r)$. Pick homogeneous elements $t_1, \ldots, t_r$ such that

$$f = t_1 s_1 + \cdots + t_r s_r.$$

Consider the Koszul complex associated to s_i:

$$K^{\cdot} \colon \otimes_{i=1}^{r} (R(|-s_i|)e_i^{\vee} \xrightarrow{s_i} R),$$

where the $e_i^{\vee}$ is just a place-holder. Objects of this complex are of the form $R(-|s_{i_1}| - \cdots - |s_{i_k}|)e_{i_1}^{\vee} \wedge \cdots \wedge e_{i_k}^{\vee}$ for a subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, r\}$. The differential is given by contraction with

$$s = s_1 e_1 + \cdots + s_r e_r.$$

Call this morphism ι_s. Define the morphism

$$d_s = (t_1 e_1^{\vee} + \cdots + t_r e_r^{\vee}) \wedge (-).$$

Define the associated *Koszul matrix factorization* as follows. Set

$$P_{-1} = \bigoplus_{i \text{ odd}} K^i(d(i-1)), \;\; P_0 = \bigoplus_{i \text{ even}} K^i(di)$$

with differential given by ι_s and d_s. Clearly $d_s\iota_s = \iota_s d_s = f$.

An important example of a Koszul matrix factorization is furnished by $R = k[x_1, \ldots, x_n]$ and $s_i = x_i$ for $i = 1, \ldots, n$. Then we can take, up to scale, $t_i = \partial_{x_i} f$. This matrix factorization is often called the *stabilization of the residue field* (see Proposition 5.2.1).

Morphisms between two graded matrix factorizations are morphisms of underlying R-modules making the relevant diagram commute. Suppose we have two matrix factorizations $\mathcal{P}, \mathcal{P}'$. Then the set of morphisms between them can be enriched to a $\mathbb{Z}$-graded vector space. For $n = 2l$, define

$$\mathrm{Hom}_{MF^{gr}(f)}(\mathcal{P}, \mathcal{P}')_n = \mathrm{Hom}_{Gr-R}(P_{-1}, P'_{-1}(dl)) \oplus \mathrm{Hom}_{Gr-R}(P_0, P'_0(dl))$$

and for $n = 2l + 1$, define

$$\mathrm{Hom}_{MF^{gr}(f)}(\mathcal{P}, \mathcal{P}')_n = \mathrm{Hom}_{Gr-R}(P_0, P'_{-1}(d(l+1))) \oplus \mathrm{Hom}_{Gr-R}(P_{-1}, P'_0(dl)).$$

This gives a category enriched in graded vector spaces. We can further enhance it to a dg category, see [1, Definition 3.1] for a more precise statement, by defining

$$d\varphi = \delta_{\mathcal{P}'} \circ \varphi \pm \varphi \circ \delta_{\mathcal{P}}.$$

There is also a natural triangulated structure on the category $\mathrm{MF}^{gr}(f)$ making it a pretriangulated dg category. We set $\mathrm{HMF}^{gr}(f)$ to be the homotopy category.

Matrix factorizations show up, in different guises, in several places in algebraic geometry, commutative algebra, and physics. One interesting example connections matrix factorization to commutative algebra through maximal Cohen-Macaulay modules over the *hypersurface algebra* $A = R/(f)$.

Recall, a graded maximal Cohen-Macaulay A-module is a graded A-module such that

$$\operatorname{depth}_A(M) = \operatorname{depth}(A) = n - 1.$$

If we have a graded maximal Cohen-Macaulay A-module, then by the Auslander-Buchsbaum formula, M has a projective resolution as an R-module of length 2:

$$F^{\cdot} : F_{-1} \xrightarrow{\delta_{-1}} F_0 \to M \to 0.$$

Multiplication by f induces a map of chain complexes $\cdot f \colon F^{\cdot} \to F^{\cdot}(d)$. Since f acts as zero on M, this map is null-homotopic. Define $\delta_0 \colon F_0 \to F_{-1}(d)$ to be this nullhomotopy. Then by definition $\delta_0\delta_{-1} = \delta_{-1}\delta_0 = f$.

Conversely, given a matrix factorization $(F_{\cdot}, \delta_{\cdot})$, we can construct a maximal Cohen-Macaulay module $\operatorname{cok}(\delta_{-1})$ by taking the cokernel of δ_{-1}:

$$F_{-1} \xrightarrow{\delta_{-1}} F_0 \to \operatorname{cok}(\delta_{-1}) \to 0.$$

Since $\delta_0\delta_{-1} = f$, which is injective, we know δ_{-1} is injective. So $\operatorname{pd}_R(M) = 1$. Moreover, M is an A-module. Indeed, pick $m \in M$ and $\bar{m} \in F_0$ a preimage under the projection. Then $fm = \pi(f\bar{m}) = \pi(\delta_{-1}\delta_0(\bar{m})) = 0$ since $\pi \circ \delta_{-1} = 0$. Finally, M is maximal Cohen-Macaulay since $\operatorname{depth}_A(M) = n - 1$.

There is also a natural triangulated structure on $\mathrm{MCM}^{gr}(A)$ and we have proved part of the following theorem well known triangulated equivalence.

Theorem 5.1.1. *The functor*

$$\mathrm{cok}\colon \mathrm{HMF}^{gr}(f) \xrightarrow{\sim} \mathrm{MCM}^{gr}(A)$$

is a triangulated equivalence between the category of graded Maximal Cohen-Macualay modules over A and the cagtegory of matrix factorizations of f.

5.2 Singularity Categories

Let's define singularity categories in full generality first. We then specialize to the affine cone over $X = V(f)$. Let X be a scheme, not necessarily smooth, and let $\mathfrak{Perf}(X)$ denote the full subcategory of $\mathcal{D}(X)$ consisting of perfect complexes from Example 2.1.2.

Definition 5.2.1. The **singularity category** of X is the Drinfeld-Verdier localization

$$\mathcal{D}_{Sg}(X) = \mathcal{D}(X)/\mathfrak{Perf}(X).$$

It is immediate from these definitions that if X is quasi-projective and smooth, then the singularity category vanishes.

If in addition X has an action of an algebraic group G, we can take G-equivariant objects in Definition 5.2.1 to define the G-equivariant (or G-graded) singularity category

$$\mathcal{D}_{Sg,G}(X) = \mathcal{D}(X)^G/\mathfrak{Perf}(X)^G.$$

We will be primarily interested in the following example.

Example 5.2.1. Let $X = V(f) \subset \mathbb{P}(V)$ be a smooth degree d hypersurface. Then $C(X) = V(f) \subset V$, the affine cone over X, has an isolated singularity at the origin. There is a natural $\mathbb{G}_m$-action by scaling the variables. We consider the graded singularity category denoted

$$\mathcal{D}_{Sg,gr}(C(X)) = \mathcal{D}_{Sg,\mathbb{G}_m}(C(X)).$$

In the case where $X = V(f \oplus g)$, where f, g are degree d polynomials in different variables, then there is an additional μ_d-action on the g-variables (or the f-variables). We can then also consider the $\mathbb{G}_m \times \mu_d$-graded singularity category

$$\mathcal{D}_{Sg,gr,\mu_d} = \mathcal{D}_{Sg,\mathbb{G}_m \times \mu_d}(C(X)).$$

Example 5.2.2. Objects in the singularity category $\mathcal{D}_{Sg,gr}(C(X))$ can be thought of as objects with support at the origin. Indeed, the primary example of such an object is $k(i)$, for $i \in \mathbb{Z}$. Here k is the graded A-module given by $A/(x_1, \ldots, x_n)$. Since A is not smooth at the origin, k does not possess a finite free resolution and so these objects are nontrivial.

Theorem 5.2.1. *Let $X = V(f) \subset \mathbb{P}(V)$. If $C(X)$ has an isolated singularity at the origin, then there is an equivalence of categories*

$$\mathrm{HMF}^{gr}(f) \simeq \mathcal{D}_{Sg,gr}(C(X))$$

Proof. We only describe the functor. Let $\text{cok}\colon \text{HMF}^{gr}(f) \to \mathcal{D}(\text{gr} - A)$ denote the cokernel functor, which takes a graded matrix factorization $(F_\cdot, \delta_\cdot)$ to the cokernel of δ_{-1}. $\square$

Remark 5.2.1. By Theorem 5.1.1, we also have a triangulated equivalence $\text{MCM}^{gr}(A) \cong \mathcal{D}_{Sg,gr}(C(X))$.

Remark 5.2.2. There are natural extensions of Theorem 5.2.1 and Proposition 5.2.1 to the case of a $\mathbb{G}_m \times \mu_d$-action (as in Example 5.2.1).

Definition 5.2.2. In view of Theorem 5.2.1, we define the *stabilization* functor, $\text{stab}\colon \mathcal{D}_{Sg,gr}(C(X)) \to \text{HMF}^{gr}(f)$ to be the quasi-inverse to cok.

We end with a proposition which tells us how to compute stabilizations for complete intersection subschemes.

Proposition 5.2.1 ([21, Lemma 1.6.2])**.** *Let $K(s,t)$ be a Koszul matrix factorization cutting out a subscheme of Z of $C(X)$, then* $\text{stab}(\mathcal{O}_Z) \cong K(s,t)$.

5.3 Orlov's Theorem

For the rest of this chapter, we work with the hypersurface algebra A from Example 5.2.1. To get an understanding of Orlov's functors we need to understand various semi-orthogonal decompositions that define them.

Let $gr - A$ denote the full subcategory of $Gr - A$ generated by coherent modules. For each $i \in \mathbb{Z}$, we have truncation functors $\text{tr}_{\geq i}$ on $\text{Gr} - A$, defined by

$$\text{tr}_{\geq i}(M)_n = \begin{cases} M_n & n \geq i \\ 0 & n < i \end{cases}.$$

Define the full triangulated subcategory $gr - A_{\geq i}$ to be the image of $\mathrm{tr}_{\geq i}$ on the subcategory $gr - A$.

Define $S_{<i}$ to be the full subcategory of finite dimensional graded A-modules generated by $k(e)$ for $e > -i$. In other words, these are finite dimensional A-modules concentrated in degrees less than i. The kernel of $\mathrm{tr}_{\geq i}$ is clearly $S_{<i}$. Simlarly define $S_{\geq i}$.

Let $P_{<i}$ be the full subcategory generated by projective A-modules $A(e)$ for $e > -i$. Similarly define $P_{\geq i}$. Let $tors - A$ denote the full subcategory of $gr - A$ generated by finite dimensional A-modules and $grproj - A$ the full subcategory of graded projective A-modules.

Lemma 5.3.1 ([20, Lemma 2.3])**.** *The subcategories $S_{<i}$ and $P_{<i}$ are left and right admissible for any $i \in \mathbb{Z}$. Moreover, there are semi-orthogonal decompositions*

$$\mathcal{D}(gr - A) = \langle S_{<i}, \mathcal{D}(gr - A_{\geq i})\rangle;$$

$$\mathcal{D}(gr - A) = \langle \mathcal{D}(gr - A_{\geq i}), P_{<i}\rangle;$$

$$\mathcal{D}(tors - A) = \langle S_{<i}, S_{\geq i}\rangle;$$

$$\mathcal{D}(grproj - A) = \langle P_{\geq i}, P_{<i}\rangle.$$

For each $i \in \mathbb{Z}$, there is a natural projection

$$\pi_i \colon \mathcal{D}(gr - A_{\geq i}) \to \mathcal{D}_{Sg,gr}(C(X))$$

which evidently kills $P_{\geq i}$ and factors through the quotient to induce an equivalence, which we abusively call π_i:

$$\pi_i\colon \mathcal{D}(gr - A_{\geq i})/P_{\geq i} \xrightarrow{\sim} D_{Sg,gr}(C(X)).$$

For each $i \in \mathbb{Z}$, define functors $\omega_i\colon \mathcal{D}(X) \to \mathcal{D}(gr - A_{\geq i})$ by the formula

$$\omega_i(\mathcal{F}^{\cdot}) = \bigoplus_{n=i}^{\infty} \mathbf{R}\Gamma(\mathcal{F}^{\cdot}(n)).$$

If we set $\mathbf{R}\Gamma_*$ to be the derived graded global sections functor, then we have

$$\omega_i = \mathrm{tr}_{\geq i} \circ \mathbf{R}\Gamma_*$$

Lemma 5.3.2 ([20, Lemma 2.4])**.** *The subcategories $S_{\geq i}$ and $P_{\geq i}$ are right and left admissible. Moreover, for any $i \in \mathbb{Z}$, there are semi-orthogonal decompositions*

$$\mathcal{D}(gr - A_{\geq i}) = \langle D_i, S_{\geq i}\rangle$$
$$\mathcal{D}(gr - A_{\geq i}) = \langle P_{\geq i}, T_i\rangle$$

where $D_i = \omega_i(\mathcal{D}(X))$ and T_i is equivalent to $D_{Sg,gr}(C(X))$.

We can now state the main theorem from [20].

Theorem 5.3.1. *Set $a = n - d$. The triangulated categories $\mathcal{D}(X)$ and $\mathcal{D}_{Sg,gr}(C(X))$ are related as follows. For each $i \in \mathbb{Z}$ there are functors $\Phi_i\colon \mathcal{D}_{Sg,gr}(C(X)) \to \mathcal{D}(X)$ and $\Psi_i\colon \mathcal{D}(X) \to \mathcal{D}_{Sg,gr}(C(X))$ such that* *(i) if $a > 0$, Φ is fully-faithful and there is a semi-orthogonal decomposition*

$$\mathcal{D}(X) = \langle \mathcal{O}_X(-i-a-1), \ldots, \mathcal{O}_X(-i), \Phi_i \mathcal{D}_{Sg,gr}(C(X)) \rangle;$$

(ii) if $a < 0$, Ψ is fully-faithful and there is a semi-orthogonal decomposition

$$\mathcal{D}_{Sg,gr}(C(X)) = \langle k^{stab}(-i), \ldots, k^{stab}(-i+a+1), \Psi_i \mathcal{D}(X) \rangle;$$

(iii) if $a = 0$, the functors Ψ_i and Φ_i induce mutually inverse equivalences of categories $\mathcal{D}(X) \simeq \mathcal{D}_{Sg,gr}(C(X))$.

Remark 5.3.1. In view of Theorem 5.2.1, we will can replace all instances of $\mathcal{D}_{Sg,gr}(C(X))$ with $\mathrm{cok}(\mathrm{HMF}^{gr}(f))$.

Remark 5.3.2. Theorem 5.3.1 holds equally as well in the case of the $\mathbb{G}_m \times \mu_d$ action of Example 5.2.1.

We will need an understanding of both the functors Φ_i and Ψ_i. Let's start with Φ_i. This functor is defined in the proof of the theorem as the composite

$$\mathcal{D}_{Sg,gr}(A) \xrightarrow{\sim} T_i \hookrightarrow D(gr-A) \xrightarrow{sh} \mathcal{D}(X).$$

The most difficult part of this computation is the first step, where we need to compute the image of an object under the quasi-inverse to π_i. Using Lemmas 5.3.1 and 5.3.2, we have a semi-orthogonal decomposition for each $i \in \mathbb{Z}$:

$$\mathcal{D}(gr-A) = \langle S_{<i}, P_{\geq i}, T_i \rangle.$$

With that in mind, we have the following recipe for computing $\Phi_i(N)$.

1. find a graded A-module projecting onto N;

2. truncate it with $\mathrm{tr}_{\geq i}$ to make it left orthogonal to $S_{<i}$;

3. mutate it through the necessary objects in $P_{\geq i}$ so that it is left orthogonal to both $P_{\geq i}$;

4. sheafify it to get a complex of coherent sheaves on X.

Orlov defines Ψ_i to be the composite

$$\mathcal{D}(X) \xrightarrow{\omega_{i-a}} \mathcal{D}_{i-a} \hookrightarrow \mathcal{D}(gr - A) \xrightarrow{q} \mathcal{D}_{Sg,gr}(C(X))$$

where q is the natural quotient map. This is straightforward except we will want specific representatives in the singularity category so that we can compute their stabilizations using Proposition 5.2.1.

5.4 Computations in the Calabi-Yau case

We will now specialize to the case where $\mathrm{Spec}(A)$ is the cone over a Calabi-Yau hypersurface. Let $p = [p_0 : \cdots : p_n] \in X$ be such that $p_i \neq 0$. Let l_p denote the graded A-module given by $\omega_0(\mathcal{O}_p)$. Then there is an isomorphism

$$l_p \cong A/(X_0 - p_0 X_i, \ldots, X_n - p_n X_i).$$

Of course, this is the structure sheaf of the line and as such is just a polynomial algebra in one variable. It follows that for each $i \in \mathbb{Z}$, we have an isomorphism $\omega_i(\mathcal{O}_p) \cong l_p(-i)$. Hence, there is an exact sequence of graded A-modules

$$0 \to \omega_{i-1}(\mathcal{O}_p) \xrightarrow{x_i} \omega_i(\mathcal{O}_p) \to k(-i+1) \to 0.$$

Lemma 5.4.1 ([12, Lemma 1.3])**.** *There is an isomorphism* $\Psi_i(\mathcal{O}_p) \cong l_p(-i)$

Proof. We only need to check $l_p(-i) \in {}^{\perp}P_{\geq i}$. Let $e \leq -i$, then we need to compute $\mathrm{Ext}^*_{gr-A}(l_p(-i), A(e))$. Using the exact sequence above, we have an exact triangle

$$\mathrm{Ext}^*_{gr-A}(k(-i+1), A(e)) \to \mathrm{Ext}^*_{gr-A}(l_p(-i), A(e)) \to \mathrm{Ext}^*_{gr-A}(l_p(-i+1), A(e)) \to .$$

Since A is Gorenstein with Gorenstein parameter 0, we have

$$\mathrm{Ext}^*_{gr-A}(k(-i+1), A(e)) \cong (k(e+i-1))_0[n]$$

Since $e \leq -i$, we know $e+i-1 \leq -1$ and so the degree zero part of $k(e+i-1)$ is zero. This completes the proof. □

We now turn to line bundles. The following computation is also stated in [1, Remark 6.14].

Lemma 5.4.2 ([12, Lemma 1.2])**.** *There is an isomorphism*

$$\Psi_1(\mathcal{O}_X) \cong k[-1]$$

Proof. In this case, ω_1 has no higher cohomology so it is simply graded global sections. In which case, $\omega_1(\mathcal{O}_X) \cong A_{\geq 1}$ and there is an exact triangle in $gr - A$:

$$0 \to A_{\geq 1} \to A \to k \to 0.$$

Fix $e \leq -1$, then we have

$$\mathrm{Ext}^*_{gr-A}(k, A(e)) \to \mathrm{Ext}^*_{gr-A}(A, A(e)) \to \mathrm{Ext}^*_{gr-A}(A_{\geq 1}, A(e)) \to .$$

Which gives an exact sequence of graded A-modules:

$$k(e)[-n] \to A(e) \to \mathrm{Ext}^*_{gr-A}(A_{\geq 1}, A(e))$$

Passing to degree zero pieces and noting $e \leq -1$, this gives

$$0[-n] \to 0 \to \mathrm{Ext}^*_{gr-A}(A_{\geq 1}, A(e))_0$$

and so $\Psi_1(\mathcal{O}_X) \cong A_{\geq 1}$. In this singularity category, this is isomorphic to $k[-1]$ using the same exact sequence above. □

Corollary 5.4.1. *For $i \in \mathbb{Z}$, there is an isomorphism in the singularity category*

$$\Psi_{-i+1}(\mathcal{O}_X(i)) \cong k(i-1)[-1]$$

Proof. This is similar to the proof of Lemma 5.4.2. We have

$$\begin{aligned} \omega_{-i+1}(\mathcal{O}_X(i)) &= \bigoplus_{n=-i+1}^{\infty} \mathbf{R}\Gamma(\mathcal{O}_X(i+n)) \\ &= \left(\bigoplus_{j=1}^{\infty} \Gamma(\mathcal{O}_X(1)) \right) (i-1) \end{aligned}$$

The rest is the same. □

For computations besides those done in Corollary 5.4.1, the picture is much more complicated. Indeed, we will work out the image of $\mathcal{O}_X(1)$ under Ψ_1, to illustrate this. There is a short exact sequence

$$0 \to A(1)_{\geq 1} \to A(1) \to t \to 0$$

where t is a torsion module with

$$t_i = \begin{cases} k^{\oplus n+1} & i = 0 \\ k & i = -1 \\ 0 & i \neq 0, -1 \end{cases} .$$

So that t fits in the exact sequence

$$0 \to k^{\oplus n+1} \to t \to k(1) \to 0.$$

We claim $\Psi_1(\mathcal{O}_X(1)) \cong t[-1]$. Clearly $\omega_1(\mathcal{O}_X(1)) \cong A(1)_{\geq 1}$, we just need to check orthogonality. We have, for $e \leq -1$,

$$\mathrm{Ext}^*_{gr-A}(t, A(e))_0 \to A(e-1)_0 \to \mathrm{Ext}^*_{gr-A}(A(1)_{\geq 1}, A(e))_0.$$

We compute

$$A(e-1)_0 \to \mathrm{Ext}^*_{gr-A}(t, A(e))_0 \to k^{\oplus n+1}(e)_0$$

which shows $\mathrm{Ext}^*_{gr-A}(A(1)_{\geq 1}, A(e))_0 = 0$ since $e \leq -1$. It follows that there is an isomorphism in the singularity category

$$\Psi_1(\mathcal{O}_X(1)) \cong t[-1].$$

In general, the objects $\Psi_1(\mathcal{O}_X(j))$ for $j > 0$ will be iterated extensions of shifts of k.

5.5 Comparison of Functors

In this section, we compare our functors to Orlov's on points.

Let X_f and X_g define Calabi-Yau hypersurface in $\mathbb{P}(V)$, where $\dim(V) = n+1$. Let A_f and A_g denote the corresponding graded hypersurface algebras. Let $R = \mathrm{SpecSym}(V^\vee \oplus V^\vee)$ and $A = R/f \oplus g$. Then A is $\mathbb{Z} \times \mu_d$-graded. If we set $x_0, \dots, x_n$ to be the f-variables and $y_0, \dots, y_n$ to be the y-variables, then the μ_{n+1}-weight of the coordinate functions x_i is 0, while for y_i it is -1.

Denote Orlov's functors for X_f by Ψ_i^f and for X_g by Ψ_j^g. Define

$$\Psi_{i,j} = \Psi_i^f \otimes \Psi_j^g \colon \mathcal{D}(X_f \times X_g) \to \mathrm{HMf}^{gr,\mu_d}(f \oplus g)$$

to be the tensor product of Ψ_i^f and Ψ_j^g (in a dg enhancement) using the equivalences:

$$\mathcal{D}(X_f) \otimes \mathcal{D}(X_g) \cong \mathcal{D}(X_f \otimes X_g)$$

via $(\mathcal{F}^\cdot, \mathcal{G}^\cdot) \mapsto \pi^*_{X_f}\mathcal{F}^\cdot \otimes \pi^*_{X_g}\mathcal{G}^\cdot$ and

$$\mathrm{HMF}^{gr}(f) \otimes \mathrm{HMF}^{gr}(g) \cong \mathrm{HMF}^{gr,\mu_d}(f \oplus g)$$

described in [2, Proposition 2.39].

Proposition 5.5.1. *Let $p \in X_f$ and $q \in X_g$. Let $S_{p,q}$ denote the $\mathbb{Z} \times \mu_d$-graded A-module corresponding to the structure sheaf of the plane containing l_p and l_q. Then*

$$\Psi_{i,j}(\mathcal{O}_{p,q}) \cong S_{p,q}^{stab}(-i-j)(\chi^j).$$

Proof. Using Lemma 5.4.1, we know $\Psi_i^f(\mathcal{O}_p) \cong l_p(-i)$ and $\Psi_j^g(\mathcal{O}_p) \cong l_q(-j)$. Both of these have Koszul resolutions cutting them out and so the stabilization is given by Koszul matrix factorizations. Thus $l_p(-i)^{stab} \boxtimes l_q(-j)^{stab}$ is also given by a Koszul resolution, where we take all of the sections cutting out l_p and l_q. The zero locus of this Koszul resolution is precisely the plane containing l_p and l_q in $V \times V$.

The last step is to check the grading shift and the weights involved. The $\mathbb{Z}$-grading is evidently $-i-j$. However, the μ_d-grading is not. Indeed, twisting by $(-j)$ in the y-variables corresponds to a μ_{n+1}-twist by j. This completes the proof. $\square$

Let us recall our decomposition for this case

$$\mathcal{D}[X/\mu_{n+1}] = \langle \mathcal{D}(X_f \times X_g), \mathcal{A} \rangle$$

where $\mathcal{A}$ is the subcategory

$$\mathcal{A} = \langle \mathcal{O}_X(-2n)(\chi^{-n}), \mathcal{O}_X(-2n-1)(\chi^{-n,-n+1}), \ldots, \mathcal{O}_X(-1)(\chi^{0,-1}), \mathcal{O}_X \rangle.$$

Using the inverse Serre functor $(-) \otimes \mathcal{O}_X(n+1)$, we have

$$\mathcal{D}[X/\mu_d] = \langle \mathcal{A}, \mathcal{D}(X_f \times X_g)(n+1) \rangle.$$

Let $\mathcal{A}_{\geq -n}$ be the subcategory of $\mathcal{A}$ given by

$$\mathcal{B} = \langle \mathcal{O}(-n)(\chi^{0,\ldots,-n}), \ldots, \mathcal{O}_X(-1)(\chi^{0,-1}), \mathcal{O}_X \rangle$$

and $\mathcal{A}_{<-n}$ be the subcategory generated by the remaining exceptional objects so that

$$\mathcal{A} = \langle \mathcal{A}_{<-n}, \mathcal{A}_{\geq -n} \rangle.$$

We now match Orlov's decomposition (at least in spirit):

$$\begin{aligned}
\mathcal{D}[X/\mu_{n+1}] &= \langle \mathcal{A}_{<-n}, \mathcal{A}_{\geq -n}, \mathcal{D} \rangle \\
&= \langle \mathcal{A}_{\geq -n}, \mathcal{D}, \mathcal{A}_{<-n}(n+1) \rangle && \text{use Serre functor} \\
&= \langle \mathcal{A}_{\geq -n}, \mathcal{A}_{<-n}(n+1), R_{\mathcal{A}_{<-n}(n+1)}\mathcal{D} \rangle && \text{mutate to the right}
\end{aligned}$$

By definition of $\mathcal{A}$, we have

$$\langle \mathcal{A}_{\geq -n}, \mathcal{A}_{<-n}(n+1) \rangle = \langle \mathcal{O}_X(-n)(\chi^{all}), \ldots, \mathcal{O}_X(\chi^{all}) \rangle,$$

where χ^{all} means take all weights.

We are left to compare Orlov's composite functor

$$\Phi_0 \circ \mathrm{cok} \circ \Psi_{0,0} \colon \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d]$$

to the composite given by the Main Theorem followed by a right mutation:

$$\mathbf{R}_{\mathcal{A}_{<-n}(n+1)} \circ \Xi_{0,0} \colon \mathcal{D}(X_f \times X_g) \to \mathcal{D}[X/\mu_d].$$

To being comparing, we will need the following lemma.

Lemma 5.5.1 ([9, Lemma 3.16]). *Let M be a finitely generated graded A-module, then there is an isomorphism of graded A-modules*

$$\mathrm{Ext}^*_{gr-A}(M, A)) \cong \mathrm{Ext}^{*+1}_{gr-R}(M, R(-d))).$$

Proof. The proof given in [9, Lemma 3.16] extends to the category of graded A-modules. □

Lemma 5.5.2. *We have vanishing*

$$\mathrm{Ext}^*_A(S_{p,q}, A(e)(\chi^i)) = 0$$

for $-n \geq e \geq 0$ and $e \leq i \leq 0$.

Proof. Using Lemma 5.5.1, we have

$$\mathrm{Ext}^*_A(S_{p,q}, A(e)(\chi^i)) = S_{p,q}(n-1)(\chi^i)[1-2n].$$

The statement follows by taking degree zero pieces. □

Lemma 5.5.3. *There is an isomorphism*

$$\mathrm{Ext}^*_A(S_{p,q}, A(e)(\chi^i)) \cong \mathrm{Ext}^*_X(\mathcal{O}_{l(p,q)}, \mathcal{O}_X(e)(\chi^i))$$

for $-n \leq e \leq 0$ and $-n-1 \leq i < e$.

Proof. In the proof of Lemma 5.5.2, we saw

$$\mathrm{Ext}^*_A(S_{p,q}, A(e)(\chi^i)) = (S_{p,q}(n-1)(\chi^i))_0^{\mu_{n+1}}[1-2n]$$

To finish the claim, we compute the other extension group. We have

$$\begin{aligned}\mathrm{Ext}^*_X(\mathcal{O}_{l(p,q)}, \mathcal{O}_X(e)(\chi^i)) &\cong \mathrm{Ext}^{2n-*}_X(\mathcal{O}_X(e)(\chi^i), \mathcal{O}_{l(p,q)}(-n-1)) \\ &\cong H^{2n-*}(\mathcal{O}_{l(p,q)}(-e-n-1)(\chi^{-i})) \\ &\cong H^{*+1-2n}(\mathcal{O}_{l(p,q)}(n+e-1)(\chi^{i-1})) \\ &\cong (S_{p,q}(n+e-1)(\chi^{i-1}))_0^{\mu_{n+1}}[1-2n]\end{aligned}$$

□

Theorem 5.5.1. *For a point $(p,q) \in X_f \times X_g$, we have*

$$(\Phi_0 \circ \mathrm{cok} \circ \Psi)(\mathcal{O}_{p,q}) = (\mathbf{R}_{\mathcal{A}_{<-n}(n+1)} \circ \Xi_{0,0})(\mathcal{O}_{p,q}).$$

Proof. Follows from Lemmas 5.5.2 and 5.5.3 and the discussion on how to compute Φ_0 in Section 5.3. □

It follows from Theorem 5.5.1, that Orlov's functors and ours (after a mutation) are the same up to a twist by a line bundle. That is, there exists a line bundle $\mathcal{L}$ on $X_f \times X_g$ such that

$$(\Phi_0 \circ \mathrm{cok} \circ \Psi) \circ (\mathcal{L} \otimes (?)) \cong (\mathbf{R}_{\mathcal{A}_{<-n}(n+1)} \circ \Xi_{0,0})$$

Conjecture 5.5.1. *Orlov's functors and our functors agree, after an appropriate choice, and up to a mutation. That is,* $\mathcal{L} \cong \mathcal{O}$*.*

It would also be interesting to do these computations in the case d, m, n arbitrary (but $d \leq m + n$). An argument similar to the one carried out should still hold for points. For now we just leave it as a conjecture.

Conjecture 5.5.2. *There exists appropriate choices of* i, j *such that Conjecture 5.5.1 extends to arbitrary Calabi-Yau or general type hypersurfaces provided* $d \leq n + m$*.*

REFERENCES CITED

[1] Matthew Ballard, David Favero, and Ludmil Katzarkov. A category of kernels for equivariant factorizations and its implications for Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 120:1–111, 2014.

[2] Matthew Ballard, David Favero, and Ludmil Katzarkov. A category of kernels for equivariant factorizations, II: further implications. *J. Math. Pures Appl. (9)*, 102(4):702–757, 2014.

[3] A. A. Beĭlinson. Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. *Funktsional. Anal. i Prilozhen.*, 12(3):68–69, 1978.

[4] A. Bondal and D. Orlov. Derived categories of coherent sheaves. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 47–56. Higher Ed. Press, Beijing, 2002.

[5] A. I. Bondal and M. M. Kapranov. Representable functors, Serre functors, and reconstructions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(6):1183–1205, 1337, 1989.

[6] Tom Bridgeland. Equivalences of triangulated categories and Fourier-Mukai transforms. *Bull. London Math. Soc.*, 31(1):25–34, 1999.

[7] Tom Bridgeland. Derived categories of coherent sheaves. In *International Congress of Mathematicians. Vol. II*, pages 563–582. Eur. Math. Soc., Zürich, 2006.

[8] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.

[9] W. Bruns and H.J. Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[10] Andrei Căldăraru and Junwu Tu. Curved A_∞ algebras and Landau-Ginzburg models. *New York J. Math.*, 19:305–342, 2013.

[11] A. Elagin. On equivariant triangulated categories. http://adsabs.harvard.edu/abs/2014arXiv1403.7027E, March 2014.

[12] Lennart Galinat. Orlov's equivalence and maximal Cohen-Macaulay modules over the cone of an elliptic curve. *Math. Nachr.*, 287(13):1438–1455, 2014.

[13] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[14] D. Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[15] A. Krug, D. Ploog, and P. Sosna. Derived categories of resolutions of cyclic quotient singularities. *ArXiv e-prints*, January 2017.

[16] A. Kuznetsov and A. Perry. Derived categories of cyclic covers and their branch divisors. http://adsabs.harvard.edu/abs/2014arXiv1411.1799K, November 2014.

[17] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 219–243. Birkhäuser Boston, Inc., Boston, MA, 2010.

[18] Shinnosuke Okawa. Semi-orthogonal decomposability of the derived category of a curve. *Adv. Math.*, 228(5):2869–2873, 2011.

[19] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2011. Corrected reprint of the 1988 edition, With an appendix by S. I. Gelfand.

[20] Dmitri Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II*, volume 270 of *Progr. Math.*, pages 503–531. Birkhäuser Boston, Inc., Boston, MA, 2009.

[21] Alexander Polishchuk and Arkady Vaintrob. Matrix factorizations and cohomological field theories. *J. Reine Angew. Math.*, 714:1–122, 2016.

[22] Piotr Pragacz, Vasudevan Srinivas, and Vishwambhar Pati. Diagonal subschemes and vector bundles. *Pure Appl. Math. Q.*, 4(4, Special Issue: In honor of Jean-Pierre Serre. Part 1):1233–1278, 2008.

[23] Sophie Térouanne. Sur la catégorie $\mathcal{D}^{b,G}(X)$ pour G réductif fini. *C. R. Math. Acad. Sci. Paris*, 336(6):483–486, 2003.

[24] Bertrand Toën. The homotopy theory of *dg*-categories and derived Morita theory. *Invent. Math.*, 167(3):615–667, 2007.

[25] Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In *Fundamental algebraic geometry*, volume 123 of *Math. Surveys Monogr.*, pages 1–104. Amer. Math. Soc., Providence, RI, 2005.