A CHARACTERIZATION OF ANISOTROPIC $H^1(\mathbb{R}^N)$ BY SMOOTH HOMOGENEOUS MULTIPLIERS

by

MARTIN HISEROTE

A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2019

DISSERTATION APPROVAL PAGE

Student: Martin Hiserote

Title: A Characterization of Anisotropic $H^1(\mathbb{R}^N)$ by Smooth Homogeneous Multipliers

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Dr. Marcin Bownik	Chair
Dr. Christopher Phillips	Core Member
Dr. Christopher Sinclair	Core Member
Dr. Yuan Xu	Core Member
Dr. Graham Kribs	Institutional Representative

and

Janet Woodruff-Borden	Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded June 2019

DISSERTATION ABSTRACT

Martin Hiserote

Doctor of Philosophy

Department of Mathematics

June 2019

Title: A Characterization of Anisotropic $H^1(\mathbb{R}^N)$ by Smooth Homogeneous Multipliers

We extend a well known result of Uchiyama, which gives a sufficient condition for a family of smooth homogeneous multipliers to characterize the Hardy space $H^1(\mathbb{R}^N)$, to the anisotropic setting.

CURRICULUM VITAE

NAME OF AUTHOR: Martin Hiserote

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Oregon State University, Corvallis, OR

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2019, University of Oregon
Bachelor of Science, Mathematics, 2012, Oregon State University

AREAS OF SPECIAL INTEREST:

Harmonic Analysis

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2012-2019

ACKNOWLEDGEMENTS

I thank Professor Bownik for his time and advice. Both were invaluable as I completed what follows.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

1.1. Distributions and Hardy Spaces

In this section we begin by introducing the Schwartz class and the space of tempered distributions. These will be needed to define the Hardy spaces $H^p(\mathbb{R}^n)$, whose elements are tempered distributions, as was done by Fefferman and Stein in [FS72]. We will also introduce the equivalent atomic definition for H^p with $p \leq 1$, which came later. This later characterization gives a dense subspace of H^p consisting of functions, which can be easier to work with.

The Schwartz class, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, is the space of functions in $C^\infty(\mathbb{R}^n)$ such that each of the seminorms

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)|$$

is finite for any multi-indices α, β in $\mathbb{N}_0^n$. The dual space of $\mathcal{S}$ is the space of tempered distributions, which is denoted $\mathcal{S}'$. It is well known that the Fourier transform, which we will define by

$$\mathcal{F}f(\xi) = \widehat{f} = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx,$$

is a continuous bijection on the Schwartz class. It follows from this that the Fourier transform extends by duality to a continuous bijection on $\mathcal{S}'$ given by $\widehat{f}(g) = f(\widehat{g})$ for $f \in \mathcal{S}'$ and $g \in \mathcal{S}$. In particular, this extension coincides with the usual definition of the Fourier transform for integrable functions, and more generally, for L^p functions with $1 \leq p \leq 2$. It is also possible to define convolution between a tempered distribution and a Schwartz function. If $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, then $f * \varphi$ defines an element of $\mathcal{S}'$ by $\psi \mapsto f(\varphi * \tilde{\psi})$ where $\tilde{\psi}$ is the reflection of ψ through the origin. This distribution coincides with the function $x \mapsto f(\tau_x \tilde{\varphi})$ where $\tau_x \varphi(y) = \varphi(y - x)$. There is a close relationship between bounded linear operators from L^p to L^q that commute with translations and certain tempered distributions. In particular, if $p, q \in [1, \infty)$, then we have the following well known result which can be obtained from Theorems 1.1 and 1.2 of [Hör60].

Theorem 1. *Suppose $T \colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a bounded linear operator, where $p, q \in [1, \infty)$, that commutes with translations. Then there exists a unique tempered distribution f such that the restriction of T to $\mathcal{S}$ coincides with the map $\varphi \mapsto f * \varphi$. Conversely, if f is a tempered distribution and $C > 0$ is a constant such that $\|f * \varphi\|_q \leq C \|\varphi\|_p$ holds for all φ in the Schwartz class, then $T(\varphi) = f * \varphi$ extends to a bounded linear operator from L^p to L^q that commutes with translations. Moreover, in the case where $q < p$, there are no non-trivial bounded linear operators that commute with translations.*

In the case where $p = q = 2$ it can be shown that the tempered distribution f must satisfy the condition $\widehat{f} \in L^\infty$. Additionally, by a duality argument using the Marcinkiewicz interpolation theorem, it can be shown that if convolution with f extends to a bounded linear operator on L^p for some p satisfying $1 \leq p < \infty$, then it must also be bounded on L^2. In particular, a linear operator T which commutes with translations can be bounded on an L^p space only if there exists a function $m \in L^\infty$ such that $\widehat{Tf} = m\widehat{f}$ for every function $f \in \mathcal{S}$.

We now move to the topic of Hardy spaces, beginning with their definition on the upper half plane, and then stating the more modern definition of the real Hardy spaces due to Fefferman and Stein in [FS72].

Definition 2. *Suppose F is a holomorphic function in the upper half plane $\mathbb{R}^2_+$. We define the Hardy space $\mathcal{H}^p(\mathbb{R}^2_+)$ for $p \in (0, \infty)$, by saying $F \in \mathcal{H}^p(\mathbb{R}^2_+)$ if and only if*

$$\|F\|_{\mathcal{H}^p} = \sup_{t>0} \left(\int_{\mathbb{R}} |F(x+it)|^p dx \right)^{1/p} < \infty.$$

The use of holomorphic functions can be replaced by harmonic functions, as shown in [BGS71], using a more general version of the following result.

Theorem 3. *A harmonic function u on the upper half plane is the real part of an element $f \in \mathcal{H}^p(\mathbb{R}^2_+)$ if and only if the non-tangential maximal function*

$$u^*(x) = \sup_{|x-y|<t} |u(y,t)|$$

is in $L^p(\mathbb{R})$. Furthermore, the L^p norm of u^* is equivalent to the $\mathcal{H}^p(\mathbb{R}^2_+)$ norm of f.

One generalization of Theorem 3 to higher dimensions was accomplished by Stein and Weiss using generalized Cauchy-Riemann equations which still requires reference to harmonic functions. A different generalization, which has several equivalent definitions and which removes the use of harmonicity, is due to Fefferman and Stein [FS72].

Definition 4. *Let f be a tempered distribution on $\mathbb{R}^n$ and let φ be a Schwartz function with non-zero integral. Let $\varphi_t(x) = t^{-n}\varphi(x/t)$ and define the nontangential maximal function of f with respect to φ by*

$$M_\varphi(f)(x) = \sup_{|x-y|<t} |(f * \varphi_t)(y)|$$

For $0 < p < \infty$ we say $f \in H^p(\mathbb{R}^n)$ if $\|f\|_{H^p} = \|M_\varphi f\|_p$ is finite.

It is important to note that it does not matter which Schwartz function φ is chosen as long as it has non-zero integral. If φ, ψ are two such functions, then there exists a constant $c > 0$ such that

$$(1/c) \|M_\varphi f\|_p \leq \|M_\psi f\|_p \leq c \|M_\varphi f\|_p$$

for all tempered distributions f.

An equivalent definition can also be obtained using the Poisson kernel to define a maximal function in a very similar manner. If we set

$$P(x) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}} \qquad \text{and} \qquad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}},$$

then $P_t(x) = t^{-n} P(x/t)$ defines the Poisson kernel. The definition of maximal function will make use of the convolution $f * P_t$. However, this expression does not make sense for arbitrary tempered distributions f. It will be defined if we restrict ourselves to the case where f is a bounded distribution. That is, $f * \varphi \in L^\infty$ for every $\varphi \in \mathcal{S}$. For such distributions it can be shown that $u(x,t) = f * P_t(x)$ is a well-defined harmonic function in the upper half space $\mathbb{R}^n \times \mathbb{R}_+$. This leads to the following theorem, which is also in [FS72].

Theorem 5. *Let f be a bounded distribution. Then f is in $H^p(\mathbb{R}^n)$ if and only if the non-tangential maximal function*

$$u^*(x) = \sup_{|x-y|<t} |u(y,t)|$$

is in L^p. Moreover, there exists a constant $c > 0$ such that

$$(1/c)\,\|f\|_{H^p} \leq \|u^*\|_p \leq c\,\|f\|_{H^p}\,.$$

In the case where $n = 1$, this theorem describes the same harmonic functions in the upper half space as those appearing in the result of Burkholder, Gundy, and Silverstein [BGS71]. Thus the Hardy spaces defined in Definition 4 extend Definition 2 from one dimension to n dimensions without appealing to holomorphicity or harmonicity. While not immediately obvious, the spaces $H^p(\mathbb{R}^n)$ are isomorphic to $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. As such we will focus on values of p between zero and one. We will also mention one more method of defining the space $H^p(\mathbb{R}^n)$ for $0 < p \leq 1$. We begin by giving the definition of an atom.

Definition 6. *Let $p \in (0,1]$. Let $|B|$ be the Lebesgue measure of the set B. An $H^p(\mathbb{R}^n)$ atom is a function a such that*

$$\text{The support of } a \text{ is contained in a ball } B, \tag{1.1}$$

$$|a| \leq |B|^{-1/p} \text{ almost everywhere}, \tag{1.2}$$

$$\int x^\beta a(x)dx = 0 \text{ for all } \beta \text{ such that } |\beta| \leq n(p^{-1} - 1) \tag{1.3}$$

The following theorem gives the atomic decomposition of H^p for $0 < p \leq 1$, originally due to Coifman [Coi74] in the one dimensional case.

Theorem 7. *Let $p \in (0,1]$ and let $f \in H^p(\mathbb{R}^n)$. Then there exists a sequence of H^p atoms $\{a_n\}$ and a sequence of complex numbers $\{\lambda_n\}$ such that*

$$\sum_{n=1}^{k} \lambda_n a_n \to f \text{ as } k \to \infty$$

in H^p. Such a pair of sequences is called an atomic decomposition of f. Also, there exists a constant $c > 0$ depending only on p such that

$$\sum_n |\lambda_n|^p \leq c \, \|f\|_{H^p}^p \, .$$

Conversely, there exists a constant $c > 0$ such that if $\{\lambda_n\}$ is a sequence of complex numbers such that $\sum |\lambda_n|^p$ is finite and if $\{a_n\}$ is sequence of H^p atoms, then there exists an element of $f \in H^p$ such that

$$\sum_{n=1}^{k} \lambda_n a_n \to f \;\; as \; k \to \infty$$

and

$$\|f\|_{H^p}^p \leq c \sum_n |\lambda_n|^p.$$

Moreover, defining $\|f\|_{H_{atom}}$ to be the infimum of

$$\left(\sum_n |\lambda_n|^p \right)^{1/p}$$

over all atomic decompositions of f gives a norm equivalent to the one in Definition 4.

1.2. Anisotropy on $\mathbb{R}^n$

In this section we introduce both the continuous and discrete concepts of anisotropy relevant in defining the anisotropic Hardy spaces in Section 1.3. We begin with the continuous setting. In [SW78] Stein and Wainger considered dilation structures $(\delta_t)_{t>0}$ on $\mathbb{R}^n$ of the form

$$\delta_t x = e^{\log(t)P} x$$

where P is a real matrix whose eigenvalues have positive real part. Some straightforward properties of dilations of this form are as follows:

$$\delta_t x \to 0 \text{ as } t \to 0, \tag{1.4}$$

$$|\delta_t x| \to \infty \text{ as } t \to \infty \text{ for } x \neq 0, \tag{1.5}$$

$$\delta_t^{-1} = \delta_{t^{-1}}, \tag{1.6}$$

$$\delta_{ts} = \delta_t \delta_s. \tag{1.7}$$

It follows from the assumption on P that there exist numbers $c_1, \ldots, c_4, \alpha_+, \alpha_- > 0$ such that if $|x| = 1$ then

$$c_1 t^{\alpha_-} < |\delta_t x| < c_2 t^{\alpha_+} \text{ if } t \geq 1, \tag{1.8}$$

$$c_3 t^{\alpha_+} < |\delta_t x| < c_4 t^{\alpha_-} \text{ if } t < 1 \tag{1.9}$$

It is important to note that $|\delta_t x|$ is not necessarily strictly increasing. As an example, take

$$P = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}.$$

Then we have

$$\delta_t = \begin{bmatrix} t & 10t\log(t) \\ 0 & t \end{bmatrix}$$

so, starting with $x = (0, 1)$, we have

$$|\delta_t x| = |(10t \log(t), t)| = t\sqrt{10 \log^2(t) + 1},$$

which is not monotone. In fact, the trajectory defined by $(10t \log(t), t)$ intersects the unit circle three times. Stein and Wainger construct a quasinorm ρ which is adapted to the dilation structure in the sense that $\rho(\delta_t x) = t\rho(x)$. To that end they establish the following result.

Theorem 8. *Let $\delta_t = e^{\log(t)P}$ where each eigenvalue of P has positive real part. Then the matrix*

$$B = \int_0^\infty e^{-tP^*} e^{-tP} dt$$

has the property that

$$t \mapsto \langle \delta_t x \rangle = \langle B\delta_t x, \delta_t x \rangle^{1/2}$$

is increasing as a function of t.

In particular, the function ρ can be defined by setting $\rho(x)$ to be the reciprocal of the unique $t > 0$ such that $\langle \delta_t x \rangle = 1$ for $x \neq 0$ and letting $\rho(0) = 0$. The construction also means that the set of points ω such that $\rho(\omega) = 1$ defines an ellipsoid

$$\Delta = \{\omega \in \mathbb{R}^n : \rho(\omega) = 1\} = \{\omega : \langle B\omega, \omega \rangle = 1\}.$$

With this definition Stein and Wainger show there are constants $C_q \geq 1$ and $d_1, \ldots, d_4 > 0$ such that

$$\rho(\delta_t x) = t\rho(x), \tag{1.10}$$

$$\rho(x+y) \leq C_q(\rho(x) + \rho(y)), \tag{1.11}$$

$$\rho \text{ is continuous on } \mathbb{R}^n \text{ and smooth on } \mathbb{R}^n \setminus \{0\}\,, \tag{1.12}$$

$$d_1 \, |x|^{1/\alpha_+} < \rho(x) < d_2 \, |x|^{1/\alpha_-} \text{ if } \rho(x) \geq 1, \tag{1.13}$$

$$d_3 \, |x|^{1/\alpha_-} < \rho(x) < d_4 \, |x|^{1/\alpha_+} \text{ if } \rho(x) < 1, \tag{1.14}$$

$$\begin{aligned} &dx = \rho^{\mathrm{tr}(P)-1} d\omega d\rho \text{ where } \mathrm{tr}(P) \text{ is the trace of } P, \omega \in \Delta \text{ and } \rho \in (0,\infty) \\ &\text{satisfy } x = \delta_\rho \omega, \text{ and } d\omega \text{ is a smooth measure on } \Delta. \end{aligned} \tag{1.15}$$

In Section 2 of [Bow03] Bownik established similar results for discrete dilation structures arising from dilation matrices before using them to define anisotropic Hardy spaces, which will be discussed in the next section. An expansive dilation matrix is a matrix A whose eigenvalues all have norm greater than one. For such a matrix, one can define a set of dilations by taking integer powers of A. Then there exists an ellipsoid $\mathcal{E}$ and $r > 1$ such that

$$\mathcal{E} \subset r\mathcal{E} \subset A\mathcal{E}.$$

We may also assume the Lebesgue measure of $\mathcal{E}$ is one by rescaling. It follows that the measure of $B_k = A^k\mathcal{E}$ is $b^k = |\det A|^k$. As with the continuous case, there is also a quasinorm ρ such that $\rho(Ax) = b\rho(x)$. In fact, there are many such quasinorms associated with the matrix A, though they can be shown to be equivalent. Additionally, they satisfy conditions similar to $(1.13), (1.14), (1.4)$, and

(1.5). The step quasinorm is given by

$$\rho(x) = \begin{cases} b^j & \text{if } x \in B_{j+1} \setminus B_j, \\ 0 & \text{if } x = 0. \end{cases}$$

1.3. Anisotropic Hardy Spaces

Now that the concept of anisotropy has been defined in both the continuous and discrete contexts we move to defining the anisotropic Hardy spaces. We begin by noting the contribution of Calderón and Torchinsky in [CT75] and [CT77] leading to the parabolic Hardy spaces. The dilation structures considered there were more restricted than those in [Bow03], which lead to the anisotropic Hardy spaces. We also discuss the connection between the Hardy spaces obtained by looking at the discrete and continuous cases as established in [BW].

In [CT75] and [CT77] Calderón and Torchinsky consider continuous dilations structures of the form $\delta_t = e^{\log(t)P}$ such that

$$t^\alpha |x| \leq |\delta_t x| \leq t^\beta |x| \tag{1.16}$$

for some $1 \leq \alpha \leq \beta$ and for any $t \geq 1$. Consequently, the quasinorm ρ they construct satisfies

$$\rho(x) \leq 1 \text{ if and only if } |x| \leq 1,$$

which is the equivalent of forcing the ellipsoids where $\rho(x) = 1$ in the cases described in Section 1.2 to be the unit sphere. In [CT75] Calderón and Torchinsky define a maximal function for each $a > 0$ and each complex valued function F on

$\mathbb{R}^n \times \mathbb{R}^+$ by

$$M_a(x, F) = \sup_{\rho(x-y) \le at} |F(y, t)|.$$

Definition 9. *Let P be a matrix such that $\delta_t = e^{\log(t)P}$ satisfies (1.16). Let φ be a Schwartz function with nonzero integral, and let $a > 0$. A tempered distribution f is in the parabolic Hardy space $H_P^p(\mathbb{R}^n)$ with $0 < p < \infty$ if $M_a(x, F)$ is in L^p where*

$$F(x, t) = (f * \varphi_t)(x) \quad \textit{and} \quad \varphi_t(x) = t^{-\operatorname{tr}(P)} \varphi(\delta_t^{-1} x).$$

We set $\|f\|_{H_P^p} = \|M_a(x, F)\|_p$. This definition is independent of a and φ in the sense that different choices yield equivalent norms.

For values of $p > 1$ we have, as with the isotropic Hardy spaces, that H_P^p is, up to an equivalent norm, L^p. Note also that Definition 9 extends immediately to the dilations discussed in Section 1.2.

We now turn to the anisotropic Hardy spaces found in [Bow03]. As in Section 1.2, we will use A to represent a fixed dilation matrix such that all its eigenvalues have norm greater than one, and denote the determinant of A by b. The function ρ will be the associated step quasinorm. We consider a modified version of the Schwartz class, denoted by $\mathcal{S}_A$, which turns out to be identical to the usual one. In particular, a C^∞ function φ is in $\mathcal{S}_A$ if for every multi-index α and positive integer m we have

$$\|\varphi\|_{\alpha,m} = \sup_{x \in \mathbb{R}^n} \rho(x)^m |\partial_\alpha \varphi(x)| < \infty.$$

If φ is in $\mathcal{S}_A$ and $k \in \mathbb{Z}$, then we define

$$\varphi_k(x) = b^{-k} \varphi(A^{-k} x).$$

For $N \in \mathbb{N}$ we set

$$\mathcal{S}_N = \left\{ \varphi \in \mathcal{S}_A : \|\varphi\|_{\alpha,m} \leq 1 \text{ for } |\alpha| \leq N, m \leq N \right\}.$$

With this, we can define several maximal functions.

Definition 10. *Let $\varphi \in \mathcal{S}_A$ and let $f \in \mathcal{S}'$. The nontangential maximal function of f is*

$$M_\varphi f(x) = \sup \left\{ |f * \varphi_k(y)| : x - y \in B_k, k \in \mathbb{Z} \right\}.$$

The radial maximal function of f is

$$M_\varphi^0 f(x) = \sup_{k \in Z} |f * \varphi_k(x)|.$$

If $N \in \mathbb{N}$, then the nontangential grand maximal function of f is

$$M_N f(x) = \sup_{\varphi \in \mathcal{S}_N} M_\varphi f(x).$$

The radial grand maximal function of f is

$$M_N^0 f(x) = \sup_{\varphi \in \mathcal{S}_N} M_\varphi^0 f(x).$$

In Definition 3.3 of [Bow03] Bownik gives the following definition of anisotropic Hardy spaces, where λ_- satisfies

$$1 < \lambda_- < \min \left\{ |\lambda| : \lambda \text{ is an eigenvalue of A} \right\}.$$

Definition 11. *For $0 < p < \infty$ set*

$$N_p = \begin{cases} \lfloor (1/p - 1)\log(b)/\log(\lambda_-) + 2 \rfloor & 0 < p \leq 1, \\ 2 & p > 1. \end{cases}$$

For $N \geq N_p$ define the anisotropic Hardy space by

$$H^p = H^p_A = \{f \in \mathcal{S}' : M_N f \in L^p\}$$

with $\|f\|_{H^p} = \|M_N f\|_{L^p}$.

It follows from Theorems 4.2 and 6.4 in [Bow03] that using different values of N gives spaces with equivalent norms so that we may justify the absence of N in the notation H^p. In fact, in Theorem 7.1 of [Bow03] Bownik shows that any of the four maximal functions above can be used to define the Hardy spaces in a similar fashion. As with the isotropic case, for $p > 1$ the Hardy space H^p is just the space L^p. As with the isotropic Hardy spaces, there is an atomic decomposition giving an equivalent definition for the anisotropic Hardy spaces.

Definition 12. *Let $p \in (0,1]$ and let $q \in [1,\infty]$, and let s be a positive integer. Then (p,q,s) is admissible with respect to A if $p < q$ and $s \geq \lfloor (1/p-1)\log(b)/\log(\lambda_-)\rfloor$. A (p,q,s) atom is a function a such that:*

$$\begin{aligned} &\operatorname{supp} a \subset B_j + x_0 \quad \textit{for some } j \in \mathbb{Z},\ x_0 \in \mathbb{R}^n, \\ &\|a\|_q \leq |B_j|^{1/q-1/p}, \\ &\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0 \quad \textit{for } |\alpha| \leq s. \end{aligned}$$

For an admissible triplet (p, q, s) the atomic anisotropic Hardy space $H^p_{q,s}$ is the space of all $f \in \mathcal{S}'$ such that there exist $\{\lambda_i\} \in \ell^p(\mathbb{N})$ and atoms $\{a_i\}$ such that $f = \sum_{i=1}^{\infty} \lambda_i a_i$ in $\mathcal{S}'$.

We now turn to the dual spaces of the anisotropic Hardy spaces. Define $\mathcal{B} = \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}$. Let $l \geq 0$, and $q \in [1, \infty]$, and $s \in \mathbb{N}_0$. Then we have the following definition, where $\mathcal{P}_s$ is the space of all n variable polynomials of degree at most s.

Definition 13. *The Campanto space $C^l_{q,s}$ is the space of all locally L^q functions such that, for $q < \infty$ and $q = \infty$ respectively,*

$$\|g\|_{C^l_{q,s}} = \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}_s} |B|^{-l} \left(\frac{1}{|B|} \int_B |g(x) - P(x)|^q dx \right)^{1/q} < \infty$$

$$\|g\|_{C^l_{q,s}} = \sup_{B \in \mathcal{B}} \inf_{P \in \mathcal{P}_s} |B|^{-l} \operatorname{esssup}_{x \in B} |g(x) - P(x)| < \infty.$$

The space $C^l_{q,s}/\mathcal{P}_s$ is a Banach space. Moreover, as shown in Theorem 8.3 of [Bow03], $C^{1/p-1}_{q,s}/\mathcal{P}_s$ is the dual space of H^p as long as (p, q, s) is an admissible triplet. In particular, the space $C^0_{1,0}$ is the space BMO of functions with bounded mean oscillation, well known to be the dual space of H^1. In analogy with this, we will denote the dual space of H^1_A by BMO_A or simply BMO.

We now turn to the issue of classifying the anisotropic Hardy spaces. That is, we wish to consider which dilation matrices give the same Hardy spaces. For example, it is reasonable to expect $A = 2I$ and $B = 3I$ will give the same Hardy spaces. We begin by defining the concept of equivalence up to a linear transformation.

Definition 14. *For two dilation matrices A_1 and A_2 we say that $H^p_{A_1}$ and $H^p_{A_2}$ are equivalent up to a linear transformation if there exists an invertible matrix P*

such that the map D_P defined by $\langle D_P f, \varphi\rangle = |\det P|^{1/p-1}\langle f, \varphi(P^{-1}\cdot)\rangle$, defines an isomorphism between $H^p_{A_1}$ and $H^p_{A_2}$. Two quasinorms ρ_1 and ρ_2 are said to be equivalent up to a linear transformation if there exists an invertible P and constant $c > 0$ such that

$$(1/c)\rho_1(x) \le \rho_2(Px) \le c\rho_1(x).$$

With this definition we have the following, which is Theorem 10.10 of [Bow03].

Theorem 15. *Let A_1 and A_2 be two dilation matrices. Then the following are equivalent if we define $\epsilon = \log|\det A_1|/\log|\det A_2|$.*

1. *The quasinorms associated to A_1 and A_2 are equivalent up to a linear transformation.*

2. *For all $r > 1$ and $m \in \mathbb{N}$ we have*

$$\sum_{|\lambda|=r^\epsilon} \dim\, \ker(A_1 - \lambda I)^m = \sum_{|\lambda|=r} \dim\, \ker(A_2 - \lambda I)^m.$$

3. *$H^p_{A_1}$ and $H^p_{A_2}$ are equivalent up to a linear transformation for all $p \in (0, 1]$.*

4. *$H^p_{A_1}$ and $H^p_{A_2}$ are equivalent up to a linear transformation for some $p \in (0, 1]$.*

Thus, we can classify anisotropic Hardy spaces up to linear transformation by classifying dilation matrices according to the second condition above.

The discrete dilation structures discussed above are more general than the continuous ones. One can simply restrict a continuous family to, say, $\{t : t = 2^k, k \in \mathbb{Z}\}$ and obtain a discrete dilation structure. Consequently, one may ask how much more is gained by studying this more general structure. We have the following two results from [BW].

Theorem 16. *Let A be a dilation matrix. Then there exists a unique one-parameter group of dilations $(\delta_t)_{t>0}$ given by $\delta_t = e^{\log(t)P}$ such that*

1. *Every eigenvalue of the generator P is positive and the trace of P is 1.*

2. *A is equivalent to δ_t for all $t > 1$.*

Theorem 17. *Let A be a dilation matrix and let P be the generator of the dilation group defined in the previous theorem. Then the Hardy spaces H^p_A, as defined in Definition 11, and H^p_P, as defined in Definition 9, coincide.*

As a consequence of Theorem 17 we see that the continuous and discrete dilation structures give the same collection of Hardy spaces.

1.4. Multipliers and Calderón-Zygmund Operators

In this section we begin by defining multiplier operators before describing certain types of Calderón-Zygmund singular integral operators. We then discuss the boundedness of the Calderón-Zygmund operators on L^p and H^p spaces. Finally, we will see a connection between certain homogeneous multipliers and singular integral operators which will be extended to a more general setting in Chapter 2.

Definition 18. *Let $m \in L^\infty(\mathbb{R}^n)$. Then the operator $T_m : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by*

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$$

is called a multiplier operator.

It follows immediately from the Plancherel Theorem that this operator is bounded on L^2. Moreover, it commutes with translations and so by Theorem 1 in Section 1.1 we know that there exists a unique tempered distribution K such that the restriction of T_m to the Schwartz class coincides with $\varphi \mapsto K * \varphi$. Moreover, the function m is the Fourier transform of the distribution K. Perhaps the simplest non-trivial example of this is the Hilbert Transform whose multiplier is $m(\xi) = -i\mathrm{sgn}(\xi)$. The corresponding distribution is given by $P.V\dfrac{1}{\pi x}$, which is the distribution defined by

$$P.V\frac{1}{\pi x}(\varphi) = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{\pi x} dx.$$

It can be shown that this operator is bounded on every L^p spaces for $1 < p < \infty$ and that it is weakly bounded on L^1. It turns out that there are fairly simple conditions on the distribution K so that these boundedness results still hold. The following result, which is essentially due to Hörmander [Hör60], can be found in the form below in Theorem 5.1 of [Duo01] along with similar results.

Theorem 19. *Suppose K is a tempered distribution which coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$. Suppose also that there is a constant $A > 0$ such that*

$$\widehat{K} \in L^\infty, \tag{1.17}$$

$$\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq A \textit{ for all } y \in \mathbb{R}^n. \tag{1.18}$$

Then convolution with K defines a bounded linear operator on L^p for $1 < p < \infty$. Moreover, it is weakly bounded on L^1.

While the operators above fail to be bounded on L^1, they are bounded from H^1 to L^1, and in fact from H^1 to itself. The following result can be found in [Ste93] as Theorem 4 of Section 3.3.

Theorem 20. *Let $\gamma > 0$ and suppose K satisfies the conditions of the previous theorem with the condition (1.18) replaced with the stricter conditions that $K \in C^{\lfloor \gamma \rfloor}(\mathbb{R}^n \setminus \{0\})$ and*

$$|\partial_\beta K(x)| \leq A|x|^{-n-\beta} \quad \text{for } |\beta| \leq \lfloor \gamma \rfloor,$$

and

$$|\partial^\beta K(x-y) - \partial^\beta K(x)| \leq A\frac{|y|^{\gamma - \lfloor \gamma \rfloor}}{|x|^{n+\gamma}}, \quad \text{for } |\beta| = \lfloor \gamma \rfloor, |x| \geq 2|y|.$$

Then, if $0 < p \leq 1$ and $\gamma > n(1/p - 1)$, the operator defined by convolution with K is bounded on $H^p(\mathbb{R}^n)$.

It is worth noting that the expression $n(1/p-1)$ also appears in the moment condition in Definition 6 where H^p atoms are defined. Similar results for more general Calderón-Zygmund operators on anisotropic Hardy spaces can be found in Section 9 of [Bow03].

The following result, which is Theorem 2 of [FR67], establishes a relationship between certain classes of homogeneous multipliers and singular kernels in the continuous anisotropic setting where the generator matrix P is diagonal.

Theorem 21. *Let P be the diagonal matrix diag$(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n > 0$. Fix a non-negative function $\chi \in C_0^\infty(-\infty, \infty)$ that is equal to 1 in a neighborhood of zero. Define $\mathscr{K}$ to be the set of functions K satisfying*

$$K(\delta_t x) = t^{-tr(P)} K(x,)$$

$$K \in C^\infty(\mathbb{R}^n \setminus \{0\}),$$

$$\int_{\{\omega : \rho(\omega) = 1\}} K(\omega) d\omega = 0.$$

Let $\mathscr{H}$ be the set of all function H satisfying

$$H(\delta_t \xi) = H(\xi),$$
$$H \in C^\infty(\mathbb{R}^n \setminus \{0\}),$$
$$\int_{\mathbb{R}^n} H(\xi)\overline{\widehat{\chi \circ \rho}}(\xi) d\xi = 0.$$

Then if $K \in \mathscr{K}$ there exists $H \in \mathscr{H}$ such that $\widehat{P.V.K} = H$, where for φ in the Schwartz class

$$P.V.K(\varphi) = \lim_{\epsilon \to 0} \int_{\rho(x) > \epsilon} K(x)\varphi(x) dx.$$

Conversely, if $H \in \mathscr{H}$ then there exists $K \in \mathscr{K}$ such that $\widehat{P.V.K} = H$.

This result will be extended to remove the assumption that P is diagonal in Chapter 2.

1.5. A Characterization of H^1 by Multipliers

The Hilbert transform defined in Section 1.4 generalizes to $\mathbb{R}^n$ as the Riesz transforms $\{R_j\}_{0 \leq j \leq n}$ where $R_0 = I$ and for $1 \leq j \leq n$ and $f \in L^2$ we have $\widehat{R_j f}(\xi) = -i \dfrac{\xi_j}{|\xi|} \widehat{f}(\xi)$. It is well known that the Riesz transforms characterize $H^1(\mathbb{R}^n)$ in the sense that there exists a constant $c > 0$ such that

$$(1/c) \|f\|_{H^1} \leq \sum_{i=0}^{n} \|R_j f\|_{L^1} \leq c \|f\|_{H^1} \tag{1.19}$$

Fefferman [Ash76] made the following conjecture related to the above.

Conjecture 22. *Let $n \in \mathbb{N}$, let $f \in L^2(\mathbb{R}^n)$, and let $K_1, \ldots, K_m$ be a collection of singular integral kernels which are homogeneous of degree $-n$, smooth away from* *the origin, and which have integral zero over the unit sphere. If the kernels are never simultaneously zero and $K_j f \in L^1$ for every $f \in L^1$ then $f \in H^1$.*

This conjecture was proven false by Garcia-Cuerva [Ash76] with singular kernels $K_1(x,y) = \dfrac{x_1^2 - x_2^2}{|x|^4}$ and $K_2(x,y) = \dfrac{-2x_1x_2}{|x|^4}$. Note that in polar coordinates these are simply $K_1 = \cos(2\theta)/r^2$ and $K_2 = \sin(2\theta)/r^2$ while the Riesz transforms can be expressed, up to a constant, as $\cos(\theta)/r^2$ and $\sin(\theta)/r^2$.

The correct version of the conjecture above, and its converse, are as follows.

Theorem 23. *Let $\theta_1, \ldots \theta_m \subset C^\infty(\mathbb{R}^n \setminus \{0\})$ be a family of functions which are homogeneous of degree zero. Then*

$$\operatorname{rank} \begin{bmatrix} \theta_1(\xi) & \cdots & \theta_m(\xi) \\ \theta_1(-\xi) & \cdots & \theta_m(-\xi) \end{bmatrix} = 2 \quad \textit{for every } \xi \in S^{n-1}$$

if and only if there exists a constant $c > 0$ such that

$$(1/c)\, \|f\|_{H^1} \leq \sum_{j=1}^{m} \left\| m_{\theta_j} f \right\|_{L^1} \leq c\, \|f\|_{H^1}$$

The necessity of the rank condition was established by Janson [Jan77] using a fairly short argument. The sufficiency of the rank condition was proved by Uchiyama [Uch82] and he gives proofs of both directions in Theorem 25.2 of [Uch01]. An analogous question can be asked in the anisotropic setting. We must change our concept of homogeneity to match the anisotropic dilation, but as we shall see in Chapter 2, the rank condition is still sufficient in this setting. It remains an open problem whether or not the condition is necessary.

CHAPTER II

SUFFICIENCY OF THE RANK CONDITION

2.1. The Main Result

In this chapter we prove Theorem 24 which generalizes the forward direction of Theorem 23 due to Uchiyama [Uch82] to the anisotropic setting. The proof below follows the argument of Uchiyama given in [Uch01]. The general structure of the proof there is modified here to fit the anisotropic case as necessary.

Throughout this chapter n is a positive integer and P is an $n \times n$ matrix with $\mathrm{tr}(P) = 1$ and whose eigenvalues have positive real part. As defined in Section 1.2, $\delta_t = e^{\log(t)P}$, ρ is the corresponding quasinorm satisfying $(1.10 - 1.15)$, and $\Delta = \{\omega : \rho(\omega) = 1\}$. We will use $\delta_t^*, \rho^*, \Delta^*$ to denote the corresponding objects coming from P^*. For simplicity, we will denote $H_P^1(\mathbb{R}^n, \mathbb{C}^k)$ by $H^1(\mathbb{R}^n, \mathbb{C}^k)$ and we will use $BMO(\mathbb{R}^n, \mathbb{C}^k)$, or simply BMO, to denote the dual of this space.

Our main result is Theorem 24 below, which is one direction of Theorem 25.2 in [Uch01].

Theorem 24. *Let $\theta_1, \dots, \theta_m \in C^\infty(\Delta^*, \mathbb{C})$ be such that*

$$\operatorname{rank} \begin{bmatrix} \theta_1(\xi) & \cdots & \theta_m(\xi) \\ \theta_1(-\xi) & \cdots & \theta_m(-\xi) \end{bmatrix} = 2 \quad \text{for every } \xi \in \Delta^*. \tag{2.1}$$

Then

$$\sup \left\{ \frac{\|f\|_{H^1(\mathbb{R}^n,\mathbb{C})}}{\sum_{j=1}^m \left\| m_{\overline{\theta_j}} f \right\|_1} : f \in H^1(\mathbb{R}^n, \mathbb{C}) \setminus \{0\} \right\} < \infty$$

and

$$BMO(\mathbb{R}^n, \mathbb{C}) = \sum_{j=1}^{m} \tilde{m}_{\theta_j} L^\infty(\mathbb{R}^n, \mathbb{C})$$

where $\widehat{m_{\theta_j} f}(\xi) = \theta_j({\delta^*}^{-1}_{\rho^*(\xi)}\xi)\widehat{f}(\xi)$ *and* $\tilde{m}_{\theta_j}$ *is defined by* $\langle \tilde{m}_{\theta_j} f_1, f_2\rangle = \langle f_1, m_{\bar{\theta}_j} f_2\rangle$

Theorem 24 follows from several results which appear in second and third sections of this chapter. We will make use of the following definition.

Definition 25. *Let S be a subspace of $H^1(\mathbb{R}^n, \mathbb{R}^k)$. Then*

$$S^\perp = \left\{ g \in BMO(\mathbb{R}^n, \mathbb{R}^k) : \langle g, f\rangle = 0 \text{ for all } f \in S \right\}$$

In Section 2 we will establish the following lemmas, which are essentially Theorem 21.2 and Corollary 21.5 in [Uch01], but adapted to the anisotropic setting.

Lemma 26. *Let $\theta_1, \ldots, \theta_m \in C^\infty(\Delta^*, \mathbb{C})$ and set*

$$S = \left\{ (m_{\bar{\theta}_j} f)_{j=1}^m : f \in H^1(\mathbb{R}^n, \mathbb{C}) \right\} \subset H^1(\mathbb{R}^n, \mathbb{C}^m) \tag{2.2}$$

so that

$$S^\perp = \left\{ g = (g_j) \in BMO(\mathbb{R}^n, \mathbb{C}^m) : \sum_{j=1}^{m} \tilde{m}_{\theta_j} g_j = 0 \in BMO(\mathbb{R}^n, \mathbb{C}) \right\}.$$

Suppose $\sum_{j=1}^m |\theta_j(\xi)| \neq 0$ *for any* $\xi \in \Delta^*$. *Then the following are equivalent:*

$$\sup\left\{ \frac{\|f\|_{H^1(\mathbb{R}^n,\mathbb{C})}}{\sum_{j=1}^m \left\| m_{\overline{\theta}_j} f \right\|_1} : f \in H^1(\mathbb{R}^n,\mathbb{C}) \setminus \{0\} \right\} < \infty, \tag{2.3}$$

$$\sup\left\{ \frac{\inf\left\{ \left\| \vec{g} - \vec{h} \right\|_\infty : h \in S^\perp \right\}}{\|\vec{g}\|_{BMO}} : \vec{g} \in BMO(\mathbb{R}^n,\mathbb{C}^m) \setminus \{0\} \right\} < \infty, \tag{2.4}$$

$$BMO(\mathbb{R}^n,\mathbb{C}) = \sum_{j=1}^m \tilde{m}_{\theta_j} L^\infty(\mathbb{R}^n,\mathbb{C}). \tag{2.5}$$

Lemma 27. *Let* k *be a positive integer and let* S *be a subspace of* $H^1(\mathbb{R}^n,\mathbb{R}^k)$. *Then there exists a constant* $C > 0$ *such that*

$$\begin{aligned} &\sup\left\{ \frac{\inf\left\{ \left\| \vec{g} - \vec{h} \right\|_\infty : \vec{h} \in S^\perp \right\}}{\|\vec{g}\|_{BMO}} : \vec{g} \in BMO(\mathbb{R}^n,\mathbb{R}^k) \setminus \{0\} \right\} \\ &\leq C \sup\left\{ \frac{\inf\left\{ \left\| \vec{g} - \vec{h} \right\|_\infty : \vec{h} \in S^\perp \right\}}{\|\vec{g}\|_{BMO}} : \vec{g} \in \mathcal{D}(\mathbb{R}^n,\mathbb{R}^k) \setminus \{0\} \right\}. \end{aligned} \tag{2.6}$$

In Section 3 we will show the following lemma, which is an anisotropic version of Theorem 22.1 of [Uch01]. and is proved using an argument adjusted to work in that setting.

Lemma 28. *Let* $\theta_1, \ldots, \theta_m \in C^\infty(\Delta^*,\mathbb{C})$ *be such that (2.1) holds, let* α_- *be as in (1.8), and let* S *be as in (2.2). Then there exists a constant* $C > 0$ *depending on* $\theta_1, \ldots, \theta_m$ *such that if* $\vec{g} \in BMO(\mathbb{R}^n,\mathbb{C}^m)$ *satisfies* $\operatorname{supp} \vec{g} \subset B(0,1)$, *then there exists* $\vec{h} \in S^\perp$ *satisfying*

$$|\vec{g}(x) - \vec{h}(x)| \leq \frac{C\,\|\vec{g}\|_{BMO}}{(1+\rho(x))^{1+\alpha_-}}, \tag{2.7}$$

With these results we can establish Theorem 24.

Proof of Theorem 24. Under the hypotheses of Theorem 24 we can use Lemma 26 to see that the conclusion of Theorem 24 is equivalent to showing

$$\sup\left\{\frac{\inf\left\{\left\|\vec{g}-\vec{h}\right\|_{\infty} : \vec{h}\in S^{\perp}\right\}}{\|\vec{g}\|_{BMO}} : \vec{g}\in BMO(\mathbb{R}^n,\mathbb{C}^m)\setminus\{0\}\right\}$$

is finite. By Lemma 27 the expression above controlled by

$$\sup\left\{\frac{\inf\left\{\left\|\vec{g}-\vec{h}\right\|_{\infty} : \vec{h}\in S^{\perp}\right\}}{\|\vec{g}\|_{BMO}} : g\in \mathcal{D}(\mathbb{R}^n,\mathbb{R}^{2m})\setminus\{0\}\right\}$$

where we have identified $\mathbb{C}^m$ with $\mathbb{R}^{2m}$. The expression above is unchanged if the restriction that the support of g is contained in the unit ball is added. This is because both the BMO and L^∞ norms are unaffected by applying the dilation operator defined by $D_t(f)(x) = f(\delta_t x)$. In particular, since $\mathcal{D}$ is contained in BMO, we can use Lemma 28 to obtain

$$\sup\left\{\frac{\inf\left\{\left\|\vec{g}-\vec{h}\right\|_{\infty} : \vec{h}\in S^{\perp}\right\}}{\|\vec{g}\|_{BMO}} : g\in \mathcal{D}(\mathbb{R}^n,\mathbb{R}^{2m})\setminus\{0\}\right\} < \infty,$$

from which the result follows. $\square$

2.2. The Proofs of Lemma 26 and Lemma 27

In this section we prove Lemma 26 and Lemma 27, which will leave only Lemma 28 to be proven. In proving Lemma 26 we will need to know the relevant multiplier operators are bounded on the space H^1. The following theorem, which generalizes Theorem 21 to the anisotropic setting, provides the means to do so in Corollary 31.

Theorem 29. *Let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi(x) = 1$ for $-1 \leq x \leq 1$. Let $\mathscr{K}$ be the set of functions K such that*

$$K \in C^\infty(\mathbb{R}^n \setminus \{0\}) \tag{2.8}$$

$$K(\delta_t x) = \frac{1}{t} K(x) \tag{2.9}$$

$$\int_\Delta K(\omega) d\omega = 0. \tag{2.10}$$

Let $\mathscr{H}$ be the set of functions H such that

$$H \in C^\infty(\mathbb{R}^n \setminus \{0\}) \tag{2.11}$$

$$H(\delta_t^* \xi) = H(\xi) \tag{2.12}$$

$$\int_{\mathbb{R}^n} H(\xi) \widehat{\chi \circ \rho}(\xi) d\xi = 0. \tag{2.13}$$

Recall that $P.V.K$ is the tempered distribution defined for φ in the Schwartz class by

$$P.V.K(\varphi) = \lim_{\epsilon \to 0} \int_{\rho(x) > \epsilon} K(x)\varphi(x) dx.$$

If $H \in \mathscr{H}$, then there exists $K \in \mathscr{K}$ such that $H = \widehat{P.V.K}$. Conversely, if $K \in \mathscr{K}$, then there exists $H \in \mathscr{H}$ such that $H = \widehat{P.V.K}$. We will refer to functions satisfying (2.11) *and* (2.12) *as smooth homogeneous multipliers.*

Proof. Let $\psi \in C_c^\infty(0,\infty)$ be such that $\int_0^\infty \frac{\psi(t)}{t}dt = 1$. Suppose $H \in \mathscr{H}$. Set $g(\xi) = H(\xi)\psi(\rho^*(\xi))$. Then we have $g \in \mathcal{S}$ and

$$\begin{aligned}
\int_0^\infty \frac{g(\delta_t^*\xi)}{t}dt &= \int_0^\infty H(\delta_t^*\xi)\frac{\psi(\rho^*(\delta_t^*\xi))}{t}dt \\
&= H(\xi)\int_0^\infty \frac{\psi(t\rho^*(\xi))}{t}dt \\
&= H(\xi)\int_0^\infty \frac{\psi(y)}{y}dy \\
&= H(\xi).
\end{aligned}$$

Since $g \in \mathcal{S}$, there exist constants $C > 0$ and $C' > 0$ such that

$$\int_0^\infty |\breve{g}(\delta_t x)|\, dt \le C + C'\int_1^\infty \frac{1}{\rho(\delta_t x)^2}dt \le C + \frac{C'}{\rho(x)^2}\int_1^\infty \frac{1}{t^2}dt$$

which is finite for all non zero x. Set $K(x) = \int_0^\infty \breve{g}(\delta_t x)dt$. Then for $s > 0$ we have

$$K(\delta_s x) = \int_0^\infty \breve{g}(\delta_{ts} x)dt = \frac{1}{s}\int_0^\infty \breve{g}(\delta_r x)dr = \frac{1}{s}K(x).$$

By the chain rule

$$\left|\frac{\partial}{\partial x_i}\left[\breve{g}\circ\delta_t\right](x)\right| = \left|\sum_{j=1}^n \breve{g}_{x_j}(\delta_t x)\left(\delta_t\right)_{ji}\right| \le \sum_{j=1}^n \left|\breve{g}_{x_j}(\delta_t x)\right| \left|\left(\delta_t\right)_{ji}\right|.$$

For $t \geq 1$ we have,

$$\left|(\delta_t)_{ji}\right| \leq \|\delta_t\| = \left\|\sum_{k=0}^{\infty} \frac{(P\log(t))^k}{k!}\right\| \leq \sum_{k=0}^{\infty} \frac{(\|P\|\log(t))^k}{k!} = t^{\|P\|}.$$

For $t \leq 1$ there exists a constant $C > 0$ such that $\left|(\delta_t)_{ji}\right| \leq C$, since the norm of δ_t goes to zero as t does. Thus, since $\breve{g}_{x_j}$ is in the Schwartz class for every natural number l there are constants C_j and $C_{j,l}$ such that for $\rho(x) \geq 1$,

$$\sum_{j=1}^{n} \left|\breve{g}_{x_j}(\delta_t x)\right| \left|(\delta_t)_{ji}\right| \leq C\chi_{[0,1)}(t) \sum_{j=1}^{n} C_j + t^{\|P\|-l}\chi_{[1,\infty)}(t) \sum_{j=1}^{n} C_{j,l}.$$

If $l > \|P\| + 1$ then the right hand side is an integrable function of t which is independent of x and which bounds

$$\left|\frac{\partial}{\partial x_i}\left[\breve{g} \circ \delta_t\right](x)\right|$$

from above. It follows from this that the first order partial derivatives of K exist for $\rho(x) > 1$ and that they are given by

$$\frac{\partial K}{\partial x_i}(x) = \int_0^{\infty} \frac{\partial}{\partial x_i}\left[\breve{g} \circ \delta_t\right](x)dt \tag{2.14}$$

Moreover, because of the homogeneity of K this holds for all nonzero x. This argument can be repeated to show the same equation holds with higher order derivatives, so $K \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$. Additionally, using (1.15) we have

$$\int_{\Delta} K(\omega)d\omega = \int_{\Delta}\int_0^{\infty} \breve{g}(\delta_t\omega)dtd\omega = \int_{\mathbb{R}^n} \breve{g}(x)dx = 0$$

with the last equality coming from $g(0)=0$. Thus, $K \in \mathscr{K}$.

We will now show $\widehat{P.V.K} = H$. Let $\varphi \in \mathcal{S}$. Then

$$P.V.K(\varphi) = \lim_{\epsilon \to 0} \int\limits_{\rho(x)>\epsilon} K(x)\varphi(x)dx = \lim_{\epsilon \to 0} \int\limits_{\rho(x)>\epsilon} K(x)\left[\varphi(x) - \varphi(0)\chi(\rho(x))\right]dx$$

where the last equality holds since for $\epsilon > 0$ we have

$$\int\limits_{\rho(x)>\epsilon} K(x)\chi(\rho(x))dx = \int\limits_{\epsilon}^{\infty}\int\limits_{\Delta} K(\delta_t\omega)\chi(\rho(\delta_t\omega))d\omega dt = \int\limits_{\epsilon}^{\infty}\frac{\chi(t)}{t}\int\limits_{\Delta} K(\omega)d\omega dt = 0.$$

Also, from the definition of K and the fact that $\check{g} \in \mathcal{S}$, if $\gamma \in (1, 1+\alpha_-)$ then there exist constants $C > 0$ and $C' > 0$ such that

$$\begin{aligned}
&\int\limits_{\rho(x)>\epsilon} \left|K(x)\left[\varphi(x) - \varphi(0)\chi(\rho(x))\right]\right| dx \le \int\limits_{\mathbb{R}^n}\int\limits_{0}^{\infty} \left|\check{g}(\delta_t x)\left[\varphi(x) - \varphi(0)\chi(\rho(x))\right]\right| dtdx \\
&\le \int\limits_{\mathbb{R}^n}\int\limits_{0}^{1} C|\varphi(x) - \varphi(0)\chi(\rho(x))|dtdx + \int\limits_{\mathbb{R}^n}\int\limits_{1}^{\infty} \frac{C'|\varphi(x) - \varphi(0)\chi(\rho(x))|}{(t\rho(x))^{\gamma}}dtdx \\
&\le C\int\limits_{\mathbb{R}^n} |\varphi(x) - \varphi(0)\chi(\rho(x))|dx + C'\int\limits_{\mathbb{R}^n} \frac{|\varphi(x) - \varphi(0)\chi(\rho(x))|}{\rho(x)^{\gamma}}dx.
\end{aligned}$$

Because φ and $\chi \circ \rho$ are in the Schwartz class,

$$C\int\limits_{\mathbb{R}^n} |\varphi(x) - \varphi(0)\chi(\rho(x))|dx + C'\int\limits_{\rho(x)\ge 1} \frac{|\varphi(x) - \varphi(0)\chi(\rho(x))|}{\rho(x)^{\gamma}}dx$$

is finite. Since $\chi(x) = 1$ for $\rho(x) \le 1$, we have, for some constant $C'' > 0$,

$$C'\int\limits_{\rho(x)\le 1} \frac{|\varphi(x) - \varphi(0)\chi(\rho(x))|}{\rho(x)^{\gamma}}dx = C'\int\limits_{\rho(x)\le 1} \frac{|\varphi(x) - \varphi(0)|}{\rho(x)^{\gamma}}dx \le \int\limits_{\Delta}\int\limits_{0}^{1} \frac{C''}{\rho^{\gamma-\alpha_-}}d\rho d\omega,$$

which is finite. It follows from this that

$$P.V.K(\varphi) = \int_0^\infty \int_{\mathbb{R}^n} \breve{g}(\delta_t x)\,[\varphi(x) - \varphi(0)\chi(\rho(x))]\,dxdt.$$

For fixed t the functions $\breve{g} \circ \delta_t$ and $\chi \circ \rho$ are in L^2 and so

$$\int_{\mathbb{R}^n} \breve{g}(\delta_t x)\chi(\rho(x))dx = \int_{\mathbb{R}^n} t^{-1} g(\delta_t^* \xi)\widehat{\chi \circ \rho}(\xi)d\xi.$$

There exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_0^\infty t^{-1}|g(\delta_t^*\xi)||\widehat{\chi \circ \rho}(\xi)|dtd\xi \\
&\quad \leq C \int_{\mathbb{R}^n} \int_0^{\rho^*(\xi)^{-1}} t^{-1}|\delta_t^*\xi||\widehat{\chi \circ \rho}(\xi)|dtd\xi + C \int_{\mathbb{R}^n} \int_{\rho^*(\xi)^{-1}}^\infty \frac{|\widehat{\chi \circ \rho}(\xi)|}{t^2\rho^*(\xi)}dtd\xi.
\end{aligned}$$

The two integrals on the right side above are finite because for $0 \leq t \leq \rho^*(\xi)^{-1}$ we have $|\delta_t^*\xi| \leq C(t\rho^*(\xi))^{1/\alpha_+}$ and because $\widehat{\chi \circ \rho} \in \mathcal{S}$. So,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} \breve{g}(\delta_t x)\chi \circ \rho(x)dxdt &= \int_0^\infty \int_{\mathbb{R}^n} t^{-1}g(\delta_t^*\xi)\widehat{\chi \circ \rho}(\xi)d\xi dt \\
&= \int_{\mathbb{R}^n} \int_0^\infty t^{-1}g(\delta_t^*\xi)\widehat{\chi \circ \rho}(\xi)dtd\xi \\
&= \int_{\mathbb{R}^n} \widehat{\chi \circ \rho}(\xi)H(\xi)d\xi \\
&= 0,
\end{aligned}$$

with the last equality holding since $H \in \mathscr{H}$. Thus,

$$P.V.K(\varphi) = \int_0^\infty \int_{\mathbb{R}^n} \breve{g}(\delta_t x)\varphi(x)dxdt.$$

By the same argument just given, this integral is the same as

$$\int_{\mathbb{R}^n} \breve{\varphi}(\xi)H(\xi)d\xi = H(\breve{\varphi}).$$

For the other direction, suppose $K \in \mathscr{K}$ and set $g(x) = K(x)\psi(\rho(x))$. Then

$$\int_0^\infty g(\delta_t x)dt = K(x)\int_0^\infty t^{-1}\psi(t\rho(x))dt = K(x).$$

Note that since $g \in \mathcal{S}$ and $\widehat{g}(0) = 0$ we have, for some constant $C > 0$,

$$\int_0^\infty t^{-1}|\widehat{g}(\delta_t^*\xi)|dt \leq \int_0^1 Ct^{-1}|\delta_t^*\xi|dt + \int_1^\infty \frac{C}{t^2\rho^*(\xi)}dt,$$

which is finite. Set

$$H(\xi) = \int_0^\infty t^{-1}\widehat{g}(\delta_t^*\xi)dt.$$

Then

$$H(\delta_s\xi) = \int_0^\infty t^{-1}\widehat{g}(\delta_{ts}^*\xi)dt = H(\xi)$$

and $H \in C^\infty(\mathbb{R}^n \setminus \{0\})$ by the same argument used with K above.

Now consider

$$\int_{\mathbb{R}^n} H(\xi)\widehat{\chi \circ \rho}(\xi)d\xi = \int_{\mathbb{R}^n}\int_0^\infty \widehat{\chi \circ \rho}(\xi)t^{-1}\widehat{g}(\delta_t^*\xi)dtd\xi.$$

By the same argument given at the end of the last part, we can change the order of integration. Applying Plancherel's theorem and expanding the definition of g then gives

$$\begin{aligned}\int\limits_{\mathbb{R}^n} H(\xi)\widehat{\chi\circ\rho}(\xi)d\xi &= \int\limits_0^\infty\int\limits_{\mathbb{R}^n} \widehat{\chi\circ\rho}(\xi)t^{-1}\widehat{g}(\delta_t^*\xi)d\xi dt \\ &= \int\limits_0^\infty\int\limits_{\mathbb{R}^n} \chi(\rho(x))K(\delta_t x)\psi(\rho(\delta_t x))dxdt.\end{aligned}$$

If the support of ψ is contained in, say, $\gamma_1 \leq \rho(x) \leq \gamma_2$, then the inner integral on the right is

$$\int\limits_{\gamma_1/t\leq\rho(x)\leq\gamma_2/t} \frac{\chi(\rho(x))K(x)\psi(t\rho(x))}{t}dx = \int\limits_{\gamma_1/t}^{\gamma_2/t} \frac{\chi(\rho)\psi(t\rho)}{t\rho}\int\limits_{\Delta} K(\omega)d\omega d\rho = 0.$$

So, $H \in \mathscr{H}$. Using the same argument as before, it can be shown that $H = \widehat{P.V.K}$.

$\square$

Corollary 30. *Let χ be as in Theorem 29. Let $H \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfy $H(\delta_t^*\xi) = H(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and for all $t > 0$. Then there exist $a \in \mathbb{C}$ and $K \in \mathscr{K}$ such that for $f \in \mathcal{S}$,*

$$T_H(f)(x) = af(x) + (P.V.K * f)(x),$$

where T_H is the operator defined by $\widehat{T_H(f)}(\xi) = H(\xi)\widehat{f}(\xi)$.

Proof. It follows from the assumptions on H and χ that the integrals

$$\int_{\mathbb{R}^n} H(\xi)\widehat{\chi\circ\rho}(\xi)d\xi$$

and

$$\int_{\mathbb{R}^n} \widehat{\chi \circ \rho}(\xi) d\xi$$

are both finite. By subtracting an appropriate constant a from H, we obtain a function H_1 in $\mathscr{H}$. By Theorem 29, there exists a $K \in \mathscr{K}$ such that

$$\widehat{T_H(f)}(\xi) = H(\xi)\widehat{f}(\xi) = H_1(\xi)\widehat{f}(\xi) + a\widehat{f}(\xi) = (P.\widehat{V.K * f})(\xi) + a\widehat{f}(\xi).$$

The result follows from taking the inverse Fourier transform. $\square$

Corollary 31. *Let $H \in C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfy $H(\delta_t^* \xi) = H(\xi)$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$ and all $t > 0$. Then T_H is bounded on $H^p(\mathbb{R}^n)$.*

Proof. By [BW] Theorem 2.12 it is enough to establish that T_H is bounded on the discrete anisotropic Hardy spaces corresponding to, say, $\{\delta_{2^k}\}_{k \in \mathbb{Z}}$. The boundedness of T_H on these spaces follows from Theorem 9.8 of [Bow03], and Corollary 30, and Theorem 29 above. $\square$

Corollary 32. *Let $p \in (0, 1]$. Let $\theta_1, \ldots, \theta_m \in C^\infty(\Delta^*, \mathbb{C})$ be such that*

$$\inf_{\xi \in \Delta} \sum_{j=1}^{m} |\theta_j(\xi)| > 0.$$

Then there exists a constant $C > 0$ such that for all $f \in L^2(\mathbb{R}^n, \mathbb{C})$ we have

$$\|f\|_{H^p} \leq C \sum_{j=1}^{m} \left\| m_{\theta_j} f \right\|_{H^p}.$$

Proof. Set

$$\theta_j^*(\xi) = \frac{\overline{\theta_j(\xi)}}{\sum_{k=1}^{m} |\theta_k(\xi)|^2}.$$

Then the sum of all $\theta_j^*\theta_j$ is 1 so

$$\|f\|_{H^p} = \left\| \sum_{j=1}^{m} m_{\theta_j^*} m_{\theta_j} f \right\|_{H^p} \leq \sum_{j=1}^{m} \left\| m_{\theta_j^*} m_{\theta_j} f \right\|_{H^p}.$$

Because $m_{\theta_j^*}$ is bounded on H^p for each j by Corollary 31, we have for some constant $C > 0$

$$\|f\|_{H^p} \leq C \sum_{j=1}^{m} \left\| m_{\theta_j} f \right\|_{H^p}.$$

□

Before moving to the proofs of Lemma 26 and Lemma 27 we will need one more theorem and a corollary.

Theorem 33. *Let S be a subspace of $H^1(\mathbb{R}^n, \mathbb{R}^k)$. Then*

$$\sup \left\{ \frac{\left\|\vec{f}\right\|_{H^1}}{\left\|\vec{f}\right\|_1} : f \in S \setminus \{0\} \right\} \tag{2.15}$$

$$= \sup \left\{ \frac{\inf \left\{ \left\| \vec{g} - \vec{h} \right\|_\infty : \vec{h} \in S^\perp \right\}}{\|\vec{g}\|_{BMO}} : \vec{g} \in BMO(\mathbb{R}^n, \mathbb{R}^k) \setminus \{0\} \right\} \tag{2.16}$$

Proof. Let $T^\perp = \left\{ \vec{g} \in L^\infty(\mathbb{R}^n, \mathbb{R}^k) : \langle g, f \rangle = 0 \text{ for all } f \in S \right\}$. Then, considering S as a subspace of H^1, its dual space is $S'_{H^1} = BMO/S^\perp$. Alternatively, considering S a subspace of L^1, its dual space is $S'_{L^1} = L^\infty/T^\perp$. The latter is isomorphic, through the inclusion of L^∞ into BMO, to $\left(L^\infty + S^\perp\right)/S^\perp$ with the norm given by $\|\vec{g}\| = \inf \left\{ \left\| \vec{g} - \vec{h} \right\|_\infty : \vec{h} \in S^\perp \right\}$.

Thus, the linear mapping

$$M\colon S \subset L^1 \to S \subset H^1,\ f \mapsto f$$

is bounded if and only if its adjoint,

$$BMO/S^{\perp} \to \left(L^{\infty} + S^{\perp}\right)/S^{\perp},\ \vec{g} + S^{\perp} \mapsto \vec{g} + S^{\perp}$$

defines a bounded linear operator, and the operator norms agree. Note that the adjoint map only makes sense if $BMO = L^{\infty} + S^{\perp}$ and that if this is not the case then the expressions in both (2.15) and (2.16) are infinite. Otherwise, note that the operator norm of the first map is, by definition, (2.15) while the second map has norm given by

$$\sup\left\{ \frac{\inf\left\{\left\|\vec{g} - \vec{h}\right\|_{\infty} : \vec{h} \in S^{\perp}\right\}}{\inf\left\{\left\|\vec{g} - \vec{h}\right\|_{BMO} : \vec{h} \in S^{\perp}\right\}} : g \in BMO(\mathbb{R}^n, \mathbb{R}^k) \setminus S^{\perp} \right\}. \tag{2.17}$$

To see that this coincides with (2.16), note that (2.16) can be rewritten as

$$\sup\left\{ \frac{\inf\left\{\left\|\vec{g} - \vec{h}\right\|_{\infty} : \vec{h} \in S^{\perp}\right\}}{\left\|\vec{g} - \vec{h'}\right\|_{BMO}} : \vec{g} \in BMO(\mathbb{R}^n, \mathbb{R}^k) \setminus \left\{S^{\perp}\right\} \text{ and } \vec{h'} \in S^{\perp} \right\},$$

which is equivalent to (2.17). $\square$

Corollary 34. *If S is a subspace of $H^1(\mathbb{R}^n, \mathbb{R}^k)$ then the following are equivalent:*

$$\sup \left\{ \frac{\left\|\vec{f}\right\|_{H^1}}{\left\|\vec{f}\right\|_1} : f \in S \setminus \{0\} \right\} < \infty, \tag{2.18}$$

$$\sup \left\{ \frac{\inf \left\{ \left\|\vec{g} - \vec{h}\right\|_\infty : h \in S^\perp \right\}}{\|\vec{g}\|_{BMO}} : \vec{g} \in BMO(\mathbb{R}^n, \mathbb{R}^k) \setminus \{0\} \right\} < \infty, \tag{2.19}$$

$$BMO = L^\infty + S^\perp. \tag{2.20}$$

Proof. From Theorem 33 and its proof, (2.18) and (2.19) are equivalent to each other and (2.19) implies $BMO = L^\infty + S^\perp$.

Conversely, if $BMO = L^\infty + S^\perp$ then the map

$$\left(L^\infty + S^\perp\right)/S^\perp \to BMO/S^\perp,\ \vec{g} + S^\perp \mapsto \vec{g} + S^\perp$$

must be bijective. We already know this map defines a bounded linear operator since it is the adjoint, up to the isomorphism of S'_{H^1} and $\left(L^\infty + S^\perp\right)/S^\perp$, of the inclusion of $S \subset H^1$ into $S \subset L^1$.

Thus, by the inverse mapping theorem,

$$BMO/S^\perp \to \left(L^\infty + S^\perp\right)/S^\perp, \vec{g} + S^\perp \mapsto \vec{g} + S^\perp$$

defines a bounded linear operator so (2.19) holds. $\square$

We can now prove Lemma 26 as a corollary of the above.

Proof of Lemma 26. By Corollaries 31 and 32 we know that

$$\|f\|_{H^1} \approx \sum_{j=1}^{m} \left\| m_{\overline{\theta}_j} f \right\|_{H^1}.$$

We also have

$$\sum_{j=1}^{m} \left\| m_{\overline{\theta}_j} f \right\|_{H^1} \approx \left\| \vec{f} \right\|_{H^1}$$

where $\vec{f} = (m_{\overline{\theta}_1} f, \ldots, m_{\overline{\theta}_m} f)$, so the equivalence of (2.3) and (2.4) follows from Corollary 34. Moreover, they are equivalent to $BMO = L^\infty + S^\perp$ which we will show coincides with (2.5). Viewing elements of $BMO(\mathbb{R}^n, \mathbb{R}^{2m})$ as being of the form $\vec{g} = (\mathrm{Re}(g_1), \mathrm{Im}(g_1), \ldots, \mathrm{Re}(g_m), \mathrm{Im}(g_m))$, with

$g_1, \ldots, g_m \in BMO(\mathbb{R}^n, \mathbb{C})$, we can consider the map

$$BMO(\mathbb{R}^n, \mathbb{R}^{2m}) \to \sum_{j=1}^{m} \tilde{m}_{\theta_j} BMO(\mathbb{R}^n, \mathbb{C}),\ \vec{g} \mapsto \sum_{j=1}^{m} \tilde{m}_{\theta_j} g_j.$$

Since the kernel of this map is exactly $S^\perp$ it follows that $BMO = L^\infty + S^\perp$ is equivalent to

$$\sum_{j=1}^{m} \tilde{m}_{\theta_j} BMO(\mathbb{R}^n, \mathbb{C}) = \sum_{j=1}^{m} \tilde{m}_{\theta_j} L^\infty(\mathbb{R}^n, \mathbb{C}).$$

To see that this is equivalent to (2.5), we need to show

$$BMO(\mathbb{R}^n, \mathbb{C}) = \sum_{j=1}^{m} \tilde{m}_{\theta_j} BMO(\mathbb{R}^n, \mathbb{C}).$$

To see this, set $\theta_j^* = \dfrac{\overline{\theta_j}}{\sum_{k=1}^m |\theta_k|^2}$. Then we have

$$BMO(\mathbb{R}^n,\mathbb{C}) \supset \sum_{j=1}^m \tilde{m}_{\theta_j} BMO(\mathbb{R}^n,\mathbb{C}) \supset \sum_{j=1}^m \tilde{m}_{\theta_j}\tilde{m}_{\theta_j^*} BMO(\mathbb{R}^n,\mathbb{C}) = BMO(\mathbb{R}^n,\mathbb{C}),$$

from which the result follows. $\square$

We now prove Lemma 27, which also makes use of Theorem 33.

Proof of Lemma 27. If $\vec{g} \in BMO(\mathbb{R}^n,\mathbb{R}^k)$ and $r > 0$ then we define the truncation of $\vec{g}$ at height r by

$$\operatorname{tr}(\vec{g},r)(x) = \frac{\vec{g}(x)}{\max\{1, |\vec{g}(x)|/r\}}.$$

It is clear that $|\operatorname{tr}(\vec{g},r)(x)| \le r$ and so $\operatorname{tr}(\vec{g},r) \in L^\infty$. Moreover, because for any $h \in BMO(\mathbb{R}^n,\mathbb{R})$ and $c \in \mathbb{R}$ we have the inequality

$$|\operatorname{tr}(h,r)(x) - \operatorname{tr}(c,r)(x)| \le |h(x) - c|,$$

we have

$$\|\operatorname{tr}(\vec{g},r)(x)\|_{BMO} \le \|\vec{g}\|_{BMO}$$

Also, for any $\vec{f} \in H^1(\mathbb{R}^n,\mathbb{R}^k)$ and $\vec{g} \in BMO(\mathbb{R}^n,\mathbb{R}^k)$ we have

$$\lim_{r\to\infty} \int \vec{f}(x)\cdot \operatorname{tr}(\vec{g},r)(x)dx = \langle \vec{g}, \vec{f}\rangle_{BMO},$$

which follows by taking an L^∞ atomic decomposition of f. With this in hand we have the following, first making use of Theorem 33,

$$
\begin{aligned}
&\sup\left\{\frac{\inf\left\{\left\|\vec{g}-\vec{h}\right\|_\infty : \vec{h}\in S^\perp\right\}}{\|\vec{g}\|_{BMO}} : \vec{g}\in BMO(\mathbb{R}^n,\mathbb{R}^k)\setminus\{0\}\right\}\\
&\quad=\sup\left\{\frac{|\langle\vec{g},\vec{f}\rangle_{BMO}|}{\|\vec{g}\|_{BMO}\left\|\vec{f}\right\|_1} : \vec{f}\in S\setminus\{0\}, \vec{g}\in BMO(\mathbb{R}^n,\mathbb{R}^k)\setminus\{0\}\right\}\\
&\quad=\sup\left\{\frac{|\langle\vec{g},\vec{f}\rangle_{BMO}|}{\|\vec{g}\|_{BMO}\left\|\vec{f}\right\|_1} : \vec{f}\in S\setminus\{0\}, \vec{g}\in L^\infty(\mathbb{R}^n,\mathbb{R}^k)\setminus\{0\}\right\}.
\end{aligned}
$$

We now wish to restrict to $\vec{g}$ with compact support. To that end, we let $\vec{g}\in L^\infty$, and $f\in H^1$, and take $\varphi\in\mathcal{D}(\mathbb{R}^n,\mathbb{R})$ such that $\varphi(x)=1$ for $|x|\le 1$. Let $B(0,r)=\delta_r\Delta$ and $\vec{g}_{B(0,r)}=\frac{1}{|B(0,r)|}\int_{B(0,r)}\vec{g}(x)dx$. Then for $r>0$, using the fact that $\vec{f}$ has zero average,

$$
\begin{aligned}
&\int \vec{f}\cdot\left\{\varphi(\delta_r^{-1}x)(\vec{g}(x)-\vec{g}_{B(0,r)})\right\}dx\\
&\quad=\int\varphi(\delta_r^{-1}x)\vec{f}(x)\cdot\vec{g}(x)dx+\vec{g}_{B(0,r)}\cdot\int(1-\varphi(\delta_r^{-1}x))\vec{f}(x)dx,
\end{aligned}
$$

As $r\to\infty$ this tends to

$$\int\vec{f}(x)\cdot\vec{g}(x)dx$$

Additionally, for some constant $C(\varphi)>0$

$$\left\|\varphi(\delta_r^{-1}\cdot)(\vec{g}(\cdot)-\vec{g}_{B(0,r)})\right\|_{BMO}\le C(\varphi)\,\|\vec{g}\|_{BMO}.$$

Consequently,

$$\sup\left\{\frac{|\langle\vec{g},\vec{f}\rangle_{BMO}|}{\|\vec{g}\|_{BMO}\left\|\vec{f}\right\|_1}:\vec{f}\in S\setminus\{0\},\vec{g}\in L^\infty(\mathbb{R}^n,\mathbb{R}^k)\setminus\{0\}\right\}$$
$$=C\sup\left\{\frac{|\langle\vec{g},\vec{f}\rangle_{BMO}|}{\|\vec{g}\|_{BMO}\left\|\vec{f}\right\|_1}:\vec{f}\in S\setminus\{0\},\vec{g}\in L^\infty(\mathbb{R}^n,\mathbb{R}^k)\setminus\{0\}\operatorname{supp}\vec{g}\text{ is compact}\right\}.$$

To obtain the desired result it remains to reduce to the case where $\vec{g}$ is smooth. This can be done by mollifying $\vec{g}$. In fact we can reduce to g supported in $B(0,1)$.

□

2.3. Smooth Atoms

The following lemma, which is a generalization of Lemma 22.3 of [Uch01], shows multipliers map smooth atoms into smooth molecules. Recall that Δ is the ellipsoid corresponding to the dilation structure arising from P.

Lemma 35. *Let $\theta\in C^\infty(\Delta,\mathbb{C})$. Let $I=\delta_l\left([0,1)^n+k\right)$ where $k\in\mathbb{Z}^n$, and $l>0$, and let $x_I=\delta_l k$. Let $b\in C^2(\mathbb{R}^n,\mathbb{R})$ satisfy*

$$\operatorname{supp} b\subset I, \tag{2.21}$$

$$\left|\nabla^2\left[b\circ\delta_l\right]\right|_{L^\infty}\leq 1, \tag{2.22}$$

$$\int_{\mathbb{R}^n} b(x)dx=0. \tag{2.23}$$

Recall α_- and α_+ from equations (1.8) *and* (1.9). *Set $p = m_\theta b$. Then, for some constant $C(\theta) > 0$ depending only on θ, we have*

$$p \in C^1(\mathbb{R}^n, \mathbb{C}), \tag{2.24}$$

$$|p(x)| + |\nabla [p \circ \delta_l] (\delta_{l^{-1}} x)| \leq \frac{C(\theta)}{(1 + \rho(\delta_{l^{-1}}(x - x_I)))^{1+\alpha_-}}, \tag{2.25}$$

$$\int_{\mathbb{R}^n} p(x) dx = 0. \tag{2.26}$$

Proof. In this proof we will use C, C' to denote positive constants. Different instances of C and C' do not necessarily denote the same constant. We first assume $l = 1$ and $k = 0$, so I is the unit cube. Let $\mathcal{S}_0$ be the set of functions in the Schwartz class whose Fourier transform is compactly supported away from the origin. Let $\psi \in \mathcal{S}_0$ satisfy

$$\int_0^\infty \widehat{\psi}(\delta_t^* \xi) \frac{dt}{t} = 1 \quad \text{ for } \xi \neq 0$$

and define η by $\widehat{\eta}(\xi) = \widehat{\psi}(\xi) \theta \left(\delta^*_{\rho^{-1}(\xi)} \xi \right)$. Note that $\widehat{\eta} \in \mathcal{S}$ so $\eta \in \mathcal{S}$ as well. We will now show

$$\widehat{p} \int_\epsilon^{1/\epsilon} \frac{\widehat{\psi}(\delta_t^* \cdot) dt}{t} \to \widehat{p}$$

in L^2 as $\epsilon \to 0$. Fix $\delta > 0$ such that supp $\widehat{\psi} \subseteq \{\xi \in \mathbb{R}^n \mid 1/\delta \leq \rho^*(\xi) \leq \delta\}$ and let ϵ satisfy $\delta > \epsilon > 0$. If $R = \delta / \epsilon$ and ξ satisfies $1/R \leq \rho^*(\xi) \leq R$ then we have

$$\int_\epsilon^{1/\epsilon} \frac{\widehat{\psi}(\delta_t^* \xi) dt}{t} = 1.$$

It follows that

$$\widehat{p(\xi)} - \widehat{p(\xi)} \int_{\epsilon}^{1/\epsilon} \frac{\widehat{\psi}(\delta_t \xi) dt}{t}$$

is supported on the set $S_R = \{\xi \in \mathbb{R}^n | \rho^*(\xi) \leq 1/R \text{ or } \rho^*(\xi) \geq R\}$. Moreover, since there exists a constant $C > 0$ with

$$\int_0^{\infty} \frac{|\widehat{\psi}(\delta_t \xi)| dt}{t} \leq C$$

we can conclude the $L^2(\mathbb{R}^n)$ norm of

$$\widehat{p}(\xi) - \widehat{p}(\xi) \int_{\epsilon}^{1/\epsilon} \frac{\widehat{\psi}(\delta_t \xi) dt}{t}$$

is bounded above, independently of R, by a multiple of the $L^2(S_R)$ norm of $\widehat{p}$. Since $\widehat{p}$ is in $L^2(\mathbb{R}^n)$ the $L^2(S_R)$ norm will converge to zero as $R \to \infty$ and so since $R \to \infty$ as $\epsilon \to 0$ we conclude

$$\widehat{p}(\xi) \int_{\epsilon}^{1/\epsilon} \frac{\widehat{\psi}(\delta_t \xi) dt}{t}$$

converges to $\widehat{p}$ in L^2 as $\epsilon \to 0$.

We next establish

$$\left| \int_{\epsilon}^{1/\epsilon} \eta_{\delta_t} * b(x) \frac{dt}{t} \right| \leq \frac{C(\theta)}{(1 + \rho(x - x_I))^{1+\alpha_-}}$$

where $\eta_{\delta_t}(x) = t^{-1} \eta(\delta_t^{-1} x)$. Let ϵ be such that $1 > \epsilon > 0$ and x be such that $\rho(x - z) \geq 1$ for all $z \in I$. Then by using the support, (2.21), and mean, (2.23), conditions on b we have

$$\left|\int_{\epsilon}^{1/\epsilon} \eta_{\delta_t} * b(x) \frac{dt}{t}\right| = \left|\int_{\epsilon}^{1/\epsilon} \frac{dt}{t} \int_I \left[\eta_{\delta_t}(x-y) - \eta_{\delta_t}(x-x_I)\right] b(y)dy\right|$$

The integral in t can be split up into $t < 1$ and $t > 1$. Applying the mean value theorem, the chain rule, and the fact that $\eta \in \mathcal{S}$ gives the existence of $z \in I$, and a constant $C > 0$

$$\begin{aligned} t^{-1} \left|\eta_{\delta_t}(x-y) - \eta_{\delta_t}(x-x_I)\right| &\leq t^{-2} |\nabla \eta(\delta_{t^{-1}}(x-z))| \, \|\delta_{t^{-1}}\| \\ &\leq t^{-(2+\alpha_+)} \frac{C}{(1+\rho(\delta_{t^{-1}}(x-z)))^{2+\alpha_+}} \\ &\leq \frac{C}{(t+\rho((x-z)))^{2+\alpha_+}}. \end{aligned}$$

Using the quasi triangle inequality for ρ we have, recalling that C_q is the constant appearing in (1.11),

$$\begin{aligned} t + \rho(x-x_I) &\leq t + c_q(\rho(x-z) + \rho(z-x_I)) \\ &\leq t + C' + c_q\rho(x-z) \\ &\leq C'(t+\rho(x-z)) \end{aligned}$$

so

$$\frac{1}{(t+\rho(x-z))^{2+\alpha_+}} \leq \frac{C'}{(t+\rho(x-x_I))^{2+\alpha_+}}.$$

Thus,

$$\begin{aligned}
\left|\int_\epsilon^1 \frac{dt}{t}\int_I [\eta_{\delta_t}(x-y)-\eta_{\delta_t}(x-x_I)]\, b(y)dy\right| &\le \int_\epsilon^1 dt \int_I \frac{C}{(t+\rho(x-x_I))^{2+\alpha_+}}|b(y)|dy \\
&\le C\int_0^1 \frac{dt}{(t+\rho(x-x_I))^{2+\alpha_+}} \\
&\le \frac{C}{\rho(x-x_I)^{1+\alpha_+}} \\
&\le \frac{C}{\rho(x-x_I)^{1+\alpha_-}} \\
&\le \frac{C}{(1+\rho(x-x_I))^{1+\alpha_-}}.
\end{aligned}$$

For $t \geq 1$ we use the same bounds as above except using α_- in place of α_+. In particular, we make use of the bound

$$t^{-1}\left|\eta_{\delta_t}(x-y)-\eta_{\delta_t}(x-x_I)\right| \le \frac{C}{(t+\rho(x-z))^{2+\alpha_-}} \le \frac{C}{(t+\rho(x-x_I))^{2+\alpha_-}}.$$

We then obtain

$$\begin{aligned}
&\left|\int_1^{1/\epsilon} \frac{dt}{t}\int_I [\eta_{\delta_t}(x-y)-\eta_{\delta_t}(x-x_I)]\, b(y)dy\right| \\
&\le \int_1^{1/\epsilon} dt \int_I \frac{C}{(t+\rho(x-x_I))^{2+\alpha_-}}|b(y)|dy \\
&\le C\int_1^{\infty} \frac{dt}{(t+\rho(x-x_I))^{2+\alpha_-}} \\
&\le \frac{C}{(1+\rho(x-x_I))^{1+\alpha_-}}.
\end{aligned}$$

Thus the claim holds for x with $\rho(x-x_I) \geq 1$.

It remains to look at the case where $\rho(x-x_I) \leq 1$ where we no longer need to establish any decay estimates, but merely boundedness. As above we will split the integral in t into pieces with $t < 1$ and $t > 1$. We have, making use of the fact that η has zero mean along with the fact that $\eta \in \mathcal{S}$, and applying the mean value theorem to b,

$$\begin{aligned}\left|\int_\epsilon^1 \eta_{\delta_t} * b(x)\frac{dt}{t}\right| &\leq \int_\epsilon^1 \int_I |\eta_{\delta_t}(x-y)|\,|b(y)-b(x)|\,dy\frac{dt}{t} \\ &\leq \int_\epsilon^1 \int_I \frac{C\,|x-y|\,dtdy}{t^2\left(1+\rho(\delta_{t^{-1}}(x-y))\right)^2}.\end{aligned}$$

Substituting $z = \delta_{t^{-1}}(y-x)$ we can get an upper bound on the last integral of

$$\int_\epsilon^1 \int_{\mathbb{R}^n} \frac{C\,|\delta_t z|\,dzdt}{t\left(1+\rho(z)\right)^2} \leq \int_0^1 \int_{\mathbb{R}^n} \frac{C|z|dzdt}{t^{1-\alpha_+}\left(1+\rho(z)\right)^2} \leq C.$$

For $t \geq 1$, we simply have

$$\begin{aligned}\left|\int_1^{1/\epsilon} \eta_{\delta_t} * b(x)\frac{dt}{t}\right| &\leq \int_1^{1/\epsilon} \|\eta_{\delta_t}\|_\infty \|b\|_1 \frac{dt}{t} \\ &\leq \int_1^\infty \frac{Cdt}{t^2} \\ &\leq C.\end{aligned}$$

We have now established

$$\left|\int_\epsilon^{1/\epsilon} \eta_{\delta_t} * b(x)\frac{dt}{t}\right| \leq \frac{C}{(1+\rho(x-x_I))^{1+\alpha_-}}.$$

Next, we establish

$$\left|\nabla\int_\epsilon^{1/\epsilon} \eta_{\delta_t} * b(x)\frac{dt}{t}\right| \leq \frac{C}{(1+\rho(x-x_I))^{1+\alpha_-}}.$$

Let ϵ satisfy $1 > \epsilon > 0$. We will again look first at x such that $\rho(x - x_I) \geq 1$ and split the integral into pieces where $t \leq 1$ and $t \geq 1$.

For $t \leq 1$ we have, by the chain rule and $\eta \in \mathcal{S}$,

$$t^{-1} \left|\nabla \eta_{\delta_t}(x-y)\right| \leq \frac{C}{(t+\rho(x-y))^{2+\alpha_+}}$$

so

$$\begin{aligned}
\left|\nabla \int_\epsilon^1 \eta_{\delta_t} * b(x) \frac{dt}{t}\right| &\leq \int_\epsilon^1 \frac{dt}{t} \int_I \left|\nabla \eta_{\delta_t}(x-y)\right| |b(y)| \, dy \\
&\leq \int_\epsilon^1 \int_I \frac{C|b(y)|}{\left(t+\rho(x-y)\right)^{2+\alpha_+}} dy dt \\
&\leq \int_\epsilon^1 \int_I \frac{C|b(y)|}{\left(t+\rho(x-x_I)\right)^{2+\alpha_+}} dy dt \\
&\leq \int_\epsilon^1 \frac{C}{\left(t+\rho(x-x_I)\right)^{2+\alpha_+}} dt \\
&\leq \frac{C}{\rho(x-x_I)^{1+\alpha_+}} \\
&\leq \frac{C}{\left(1+\rho(x-x_I)\right)^{1+\alpha_-}},
\end{aligned}$$

where the last inequality makes use of $\rho(x - x_I) \geq 1$.

For $t \geq 1$ we obtain, by replacing α_+ with α_- in the bound for $\nabla\eta$,

$$
\begin{aligned}
\left|\nabla \int_1^{1/\epsilon} \eta_{\delta_t} * b(x) \frac{dt}{t}\right| &\leq \int_1^{1/\epsilon} \frac{dt}{t} \int_I |\nabla \eta_{\delta_t}(x-y)|\,|b(y)|\,dy \\
&\leq \int_1^{1/\epsilon} \int_I \frac{C|b(y)|}{(t+\rho(x-y))^{2+\alpha_-}} dydt \\
&\leq \int_1^{1/\epsilon} \int_I \frac{C|b(y)|}{(t+\rho(x-x_I))^{2+\alpha_-}} dydt \\
&\leq \int_1^{1/\epsilon} \frac{C}{(t+\rho(x-x_I))^{2+\alpha_-}} dt \\
&\leq \frac{C}{(1+\rho(x-x_I))^{1+\alpha_-}}.
\end{aligned}
$$

We now turn to the case where $\rho(x-x_I) \leq 1$. Once more, we split the relevant integral into two parts depending on the size of t. For $t \leq 1$ we have,

$$
\begin{aligned}
\left|\nabla \int_\epsilon^1 \eta_{\delta_t} * b(x) \frac{dt}{t}\right| &\leq \int_\epsilon^1 \int_I |\eta_{\delta_t}(x-y)|\,|\nabla b(y) - \nabla b(x)|\,dy \frac{dt}{t} \\
&\leq \int_\epsilon^1 \int_I \frac{C\,|x-y|\,dtdy}{t^2\,(1+\rho(\delta_{t^{-1}}(x-y)))^2}.
\end{aligned}
$$

Substituting $z = \delta_{t^{-1}}(y-x)$, we can get an upper bound on the last integral of

$$
\int_\epsilon^1 \int_{\mathbb{R}^n} \frac{C\,|\delta_t z|\,dzdt}{t\,(1+\rho(z))^2} \leq \int_0^1 \int_{\mathbb{R}^n} \frac{C|z|dzdt}{t^{1-\alpha_+}\,(1+\rho(z))^2} \leq C.
$$

For $t \geq 1$ we have,

$$
\begin{aligned}
\left|\nabla \int_1^{1/\epsilon} \eta_{\delta_t} * b(x) \frac{dt}{t}\right| &\leq \int_1^{1/\epsilon} \|\nabla \eta_{\delta_t}\|_\infty \|b\|_1 \frac{dt}{t} \\
&\leq \int_1^\infty \frac{Cdt}{t^{2+\alpha_-}} \\
&\leq C.
\end{aligned}
$$

We have now established

$$\left|\nabla \int_{\epsilon}^{1/\epsilon} \eta_{\delta_t} * b(x) \frac{dt}{t}\right| \leq \frac{C}{(1+\rho(x-x_I))^{1+\alpha_-}}.$$

Furthermore, by examining the integrals involved in the estimates above we see that the convergence is uniform in x as $\epsilon \to 0$.

Set

$$F_\epsilon(x) = \int_{\epsilon}^{1/\epsilon} \eta_{\delta_t} * b(x) \frac{dt}{t}.$$

Then we have shown F_ϵ satisfies

$$|F_\epsilon(x)| + |\nabla F_\epsilon(x)| \leq \frac{C}{(1+\rho((x-x_I)))^{1+\alpha_-}}.$$

We now show p has properties (2.24)–(2.26). Taking the Fourier transform of F_ϵ we obtain

$$\widehat{F_\epsilon}(\xi) = \int_{\epsilon}^{1/\epsilon} \widehat{\eta}(\delta_t^* \xi)\widehat{b}(\xi) \frac{dt}{t} = \widehat{p}(\xi) \int_{\epsilon}^{1/\epsilon} \frac{\psi(\delta^* \xi) dt}{t},$$

which we have already shown converges to $\widehat{p}$ in L^2. Because of the uniform convergence of $\{F_\epsilon\}_{1>\epsilon.0}$ and $\{\nabla F_\epsilon\}_{1>\epsilon>0}$, for any compact set K in $\mathbb{R}^n$ there exists a C^1 function, g, satisfying (2.24) and (2.25), to which $\{F_\epsilon\}_{1>\epsilon>0}$ converges uniformly and also in L^2 norm. Since we also know the family converges to p in L^2 norm it follows that $p = g$ in $L^2(K)$. It follows that (2.24) and (2.25) hold for p. For (2.26) note that $\widehat{p}(0) = 0$ since $\widehat{b}(0) = 0$.

It remains to show the result for general dilated cubes of the form $I = \delta_l\left([0,1]^n + k\right)$. To that end, suppose b satisfies (2.21)–(2.23) for the cube I and consider the function B defined by

$$B(x) = b(\delta_l(x + k)).$$

Then B is supported in the unit cube, it has mean zero, and it satisfies the condition $|\nabla^2 B(x)| \leq 1$. So if $q = m_\theta B$, then P satisfies (2.24)–(2.26) with $l = 1$ and $k = 0$. Taking the Fourier transform of p we find

$$\widehat{p}(\xi) = l e^{-2\pi i k \cdot \delta_l^* \xi} \widehat{q}(\delta_l^* \xi)$$

from which it follows that

$$p(x) = q(\delta_{l^{-1}} x - k).$$

Now (2.24)–(2.26) can be seen to hold for p using the fact that they hold for q. $\square$

2.4. Smooth Molecules

This section deals with molecules of the form resulting from the conclusion of Lemma 35. In particular, Lemma 37 will establish control on certain norms corresponding to families of such molecules indexed by dilated cubes. We will construct such a family later from a corresponding family of functions satisfying the hypotheses of Lemma 35. The following lemma will be useful in proving that result. The results in this section are generalizations of Lemmas 22.4 and 22.5 in [Uch01].

Lemma 36. *Suppose $I = \delta_{2^l}([0,1)^n + k)$, and $J = \delta_{2^p}([0,1)^n + q)$, and $|I| \geq |J|$. Suppose also that $p_I, p_J \in C^1(\mathbb{R}^n, \mathbb{C})$ and let $x_I = \delta_{2^l} k$ and $x_J = \delta_{2^p} q$.*

$$|p_I(x)| + |\nabla [p_I \circ \delta_{2^l}] (\delta_{2^{-l}} x)| \leq \frac{1}{(1 + \rho(\delta_{2^{-l}}(x - x_I)))^{1+\alpha-}}, \tag{2.27}$$

$$|p_J(x)| + |\nabla [p_J \circ \delta_{2^p}] (\delta_{2^{-p}} x)| \leq \frac{1}{(1 + \rho(\delta_{2^{-l}}(x - x_J)))^{1+\alpha-}}, \tag{2.28}$$

$$\int p_I(x) dx = \int p_J(x) dx = 0, . \tag{2.29}$$

Then there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{R}^n} p_I(x) p_J(x) dx \right| \leq \frac{C|I||J|^{1+\alpha-/2}}{(|I| + \rho(x_I - x_J))^{1+\alpha-/2}}. \tag{2.30}$$

Proof. This follows from Lemma 5.1 of [BH06]. $\square$

Lemma 37. *Let $\{\lambda_I\}_I$ be an indexed set of non-negative real numbers and let $\{p_I\}_I$ be an indexed set of $C^1(\mathbb{R}^n, \mathbb{C})$ functions where the index I runs over all dilated cubes of the form $I = \delta_{2^l}([0,1)^n + k)$ with $l \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$. Suppose*

$$\sum_I \lambda_I^2 |I| \leq \infty, \tag{2.31}$$

and that for all I

$$|p_I(x)| + |\nabla [p_I \circ \delta_{2^l}] (\delta_{2^{-l}} x)| \leq \frac{1}{(1 + \rho(\delta_{2^{-l}}(x - x_I)))^{1+\alpha-}}, \tag{2.32}$$

$$\int p_I(x) dx = 0. \tag{2.33}$$

Then there exists a constant $C > 0$ such that

$$\left\|\sum_I \lambda_I p_I\right\|_2^2 \leq C \sum_I \lambda_I^2 |I| \tag{2.34}$$

$$\left\|\sum_I \lambda_I p_I\right\|_{BMO}^2 \leq C \left\|\Lambda\right\|_{\mathcal{C}} = \sup_{B \in \mathcal{B}} \frac{\Lambda(B \times (0, |B|))}{|B|} \tag{2.35}$$

where $\mathcal{B}$ is the set of all shifted dilates of the unit ball. $\Lambda = \sum_I \lambda_I^2 |I| \delta_{(x_I, |I|)}$ and $\delta_{(x_I, |I|)}$ is the Dirac measure at the point $(x_I, |I|)$ in $\mathbb{R}^n \times \mathbb{R}_+$.

Proof. In this proof C is used to denote a positive constant. Separate instances of C do not necessarily denote the same constant. We start by establishing (2.34). By (2.32) and (2.33) we can apply Lemma 36 which, in conjunction with the Cauchy-Schwartz inequality, gives

$$\begin{aligned}
\left\|\sum_I \lambda_I p_I\right\|_2^4 &\leq \left(2 \sum_I \sum_{J : |J| \leq |I|} \lambda_I \lambda_J \left| \int p_I(x) p_J(x) dx \right| \right)^2 \\
&\leq C \left(\sum \sum \frac{\lambda_I \lambda_J |I| |J|^{1+\alpha_-/2}}{(|I| + \rho(x_I - x_J))^{1+\alpha_-/2}} \right)^2 \\
&\leq C \left(\sum \sum \frac{\lambda_I^2 |I| |J|^{1+\alpha_-/2}}{(|I| + \rho(x_I - x_J))^{1+\alpha_-/2}} \right) \\
&\quad \cdot \left(\sum \sum \frac{\lambda_J^2 |I| |J|^{1+\alpha_-/2}}{(|I| + \rho(x_I - x_J))^{1+\alpha_-/2}} \right).
\end{aligned}$$

Since $|J| \leq |I|$ we have, using the quasi triangle inequality for ρ,

$$\int_J \frac{dy}{(|I| + \rho(x_I - y))^{1+\alpha_-/2}} \geq \frac{C|J|}{(|I| + \rho(x_I - x_J))^{1+\alpha_-/2}}$$

so that for a dilated cube I we have

$$\begin{aligned}
\sum_{J:|J|\le|I|} \frac{|J|^{1+\alpha_-/2}}{(|I|+\rho(x_I-x_J))^{1+\alpha_-/2}} &\le C \sum_{l:2^l\le|I|} \sum_{J:|J|=2^l} \frac{|J|^{1+\alpha_-/2}}{(|I|+\rho(x_I-x_J))^{1+\alpha_-/2}} \\
&\le C \sum_{l:2^l\le|I|} 2^{l\alpha_-/2} \int_{\mathbb{R}^n} \frac{dy}{(|I|+\rho(x_I-y))^{1+\alpha_-/2}} \\
&= C \frac{(2|I|)^{\alpha_-/2}}{2^{\alpha_-/2}-1} \int_{\mathbb{R}^n} \frac{|I|^{-\alpha_-/2}dy}{(1+\rho(y))^{1+\alpha_-/2}} \\
&= C.
\end{aligned}$$

Similarly,

$$\int_I \frac{dy}{(|I|+\rho(y-x_J))^{1+\alpha_-/2}} \ge \frac{C|I|}{(|I|+\rho(x_I-x_J))^{1+\alpha_-/2}}$$

so that for a dilated cube J we have

$$\begin{aligned}
\sum_{I:|I|\ge|J|} \frac{|I||J|^{\alpha_-/2}}{(|I|+\rho(x_I-x_J))^{1+\alpha_-/2}} &\le C \sum_{l:2^l\ge|J|} |J|^{\alpha_-/2} \int_{\mathbb{R}^n} \frac{dy}{(2^l+\rho(y-x_J))^{1+\alpha_-/2}} \\
&\le C \sum_{l:2^l\ge|J|} |J|^{\alpha_-/2} 2^{-l\alpha_-/2} \int_{\mathbb{R}^n} \frac{dy}{(1+\rho(y))^{1+\alpha_-/2}} \\
&\le C.
\end{aligned}$$

Consequently we obtain

$$\begin{aligned}
\left\|\sum \lambda_I p_I\right\|_2^4 \le C &\left(\sum\sum \frac{\lambda_I^2|I||J|^{1+\alpha_-/2}}{(|I|+\rho(x_I-x_J))^{1+\alpha_-/2}}\right) \\
&\cdot \left(\sum\sum \frac{\lambda_J^2|I||J|^{1+\alpha_-/2}}{(|I|+\rho(x_I-x_J))^{1+\alpha_-/2}}\right) \\
\le C &\left(\sum \lambda_I^2|I|\right)\left(\sum \lambda_J^2|J|\right)
\end{aligned}$$

as desired.

We now show

$$\left\| \sum_I \lambda_I p_I \right\|_{BMO}^2 \leq C \left\| \Lambda \right\|_{\mathcal{C}},$$

which will complete the proof. By rescaling we may assume

$$\left\| \Lambda \right\|_{\mathcal{C}} = 1. \tag{2.36}$$

Define $z = \sup_{x \in [0,1]^n} \rho(x)$. Let $B = B(x_B, r_B)$ and write

$$\sum_I \lambda_I p_I = \sum_{\substack{I : x_I \in \delta_{2zC_q} B, \\ |I| \leq r_B}} \lambda_I p_I + \sum_{\substack{I : x_I \notin \delta_{2zC_q} B, \\ |I| \leq r_B}} \lambda_I p_I + \sum_{I : |I| > r_B} \lambda_I p_I = q_1 + q_2 + q_3.$$

Note that because

$$\frac{1}{|B|} \int_B \left| \sum_I \lambda_I p_I(x) - q_3(x_B) \right| dx \leq \frac{\|q_1\|_{L^1(B)}}{|B|} + \frac{\|q_2\|_{L^1(B)}}{|B|} + \frac{\|q_3 - q_3(x_B)\|_{L^1(B)}}{|B|}$$

it is enough to show that each of $\dfrac{\|q_1\|_{L^1(B)}}{|B|}$, $\dfrac{\|q_2\|_{L^1(B)}}{|B|}$, and $\dfrac{\|q_3 - q_3(x_B)\|_{L^1(B)}}{|B|}$ is bounded above by a constant.

Starting with the q_1 term we have, making use of (2.36) for the last inequality,

$$\begin{aligned}
\frac{\|q_1\|_{L^1(B)}}{|B|} &\leq \frac{\|q_1\|_{L^2(\mathbb{R}^n)}}{|B|^{1/2}} \\
&\leq \frac{C\left(\displaystyle\sum_{\substack{x_I \in \delta_{2zC_q}B, \\ |I| \leq r_B}} \lambda_I^2 |I|\right)^{1/2}}{|B|^{1/2}} \\
&\leq C.
\end{aligned}$$

Similarly, for the q_2 term we have,

$$\begin{aligned}
\frac{\|q_2\|_{L^1(B)}}{|B|} &\leq C \sum_{l=-\infty}^{\log_2(r_B)} \sum_{\substack{I: x_I \notin \delta_{2zC_q}B \\ |I| = 2^l}} \int_B \frac{|p_I(x)|\,dx}{|B|} \\
&\leq C \sum \sum \frac{1}{(1+\rho(x_B - x_I)/2^l)^{1+\alpha_-}} \\
&\leq C \sum_{l=-\infty}^{\log_2(r_B)} 2^{-l} \int_{B(x_B, c' r_B)^c} \frac{dy}{(1+\rho(x_B - y)/2^l)^{1+\alpha_-}} \\
&\leq C \sum 2^{l\alpha_-} \int_{B(0, c' r_B)} \frac{dy}{\rho(y)^{1+\alpha_-}} \\
&\leq C.
\end{aligned}$$

Lastly, we have

$$\begin{aligned}
\|q_3(x) - q_3(x_B)\|_{L^1(B)} &\leq \sum_{l=\log_2(r_B)+1}^{\infty} \sum_{I:|I|=2^l} \lambda_I \int_B |p_I(x) - p_I(x_B)|\, dx \\
&\leq C \sum_l \sum_I \int_B \left| [p_I \circ \delta_{2^l}] \left(\delta_{2^{-l}} x\right) - [p_I \circ \delta_{2^l}] \left(\delta_{2^{-l}} x_B\right) \right| dx \\
&\leq C|B| \sum_l \sum_I \sup_{x \in B} \left| \nabla \left[p_I \circ \delta_{2^l}\right] \left(\delta_{2^{-l}} x\right) \right| \sup_{x \in B(0,r_B)} |\delta_{2^{-l}} x| \\
&\leq \sum_l \sum_I (r_B 2^{-l})^{\alpha_+} \frac{C|B|}{\left(1 + \rho(x_B - x_I)/2^l\right)^{1+\alpha_-}} \\
&\leq C|B| \sum_l 2^{-l} (r_B 2^{-l})^{\alpha_+} \int_{\mathbb{R}^n} \frac{dy}{\left(1 + \rho(x_B - y)/2^l\right)^{1+\alpha_-}} \\
&\leq C|B| \sum_l (r_B 2^{-l})^{\alpha_+} \\
&\leq C|B|,
\end{aligned}$$

from which the result follows. □

2.5. Multipliers and BMO

In this section we begin with a family of multipliers satisfying the hypotheses of Theorem 24 and construct a second family in Lemma 38. This second family is used in Lemma 39 to construct, starting from certain BMO functions, an element of $S^\perp$, where $S^\perp$ is as in Lemma 26. This will be our means of constructing the desired function $\vec{h} \in S^\perp$ for Lemma 28. Recall that Δ^* is the ellipsoid corresponding to the dilation structure arising from P^*. The results in this sections are generalizations of 24.2′ and 24.3′ in [Uch01].

Lemma 38. *Let $\theta_1, \ldots, \theta_m \in C^\infty(\Delta^*, \mathbb{C})$ be such that*

$$\operatorname{rank}\begin{bmatrix} \theta_1(\xi) & \cdots & \theta_m(\xi) \\ \theta_1(-\xi) & \cdots & \theta_m(-\xi) \end{bmatrix} = 2.$$

Let $\vec{v} \in S^{2m-1}$. Then there exist $\Theta_1, \ldots, \Theta_m \in C^\infty(\Delta^*, \mathbb{C})$ and a constant $C > 0$ such that

$$\sum_{j=1}^{m} \theta_j(\xi)\Theta_j(\xi) = 1, \tag{2.37}$$

$$\sum_{j=1}^{m} \{v_{2j-1} \operatorname{Re}(\Theta_j(\xi) + \Theta_j(-\xi)) + v_{2j} \operatorname{Im}(\Theta_j(\xi) + \Theta_j(-\xi))\} = 0, \tag{2.38}$$

$$\sum_{j=1}^{m} \{v_{2j-1} \operatorname{Im}(\Theta_j(\xi) - \Theta_j(-\xi)) - v_{2j} \operatorname{Re}(\Theta_j(\xi) - \Theta_j(-\xi))\} = 0, \tag{2.39}$$

$$|\nabla \Theta_j| \leq C(\theta_1, \ldots, \theta_m, P). \tag{2.40}$$

Proof. Because Δ^* is compact and because the function θ_j is smooth on Δ^* for all j we can construct, for $1 \leq j, k \leq m$, smooth functions $\psi_{j,k} \in C^\infty(\Delta^*, \mathbb{C})$ such that

$$\psi_{j,k} = \psi_{k,j},$$

$$\psi_{j,k}(\xi) = \psi_{j,k}(-\xi),$$

$$\operatorname{rank}\begin{bmatrix} \theta_j(\xi) & \theta_k(\xi) \\ \theta_j(-\xi) & \theta_k(-\xi) \end{bmatrix} = 2 \text{ if } \xi \in \operatorname{supp}(\psi_{j,k}),$$

$$\sum \psi_{j,k}(\xi) = 1 \text{ for all } \xi \in \Delta^*.$$

For $z \in \mathbb{C}$ define

$$R(z) = \begin{bmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{bmatrix} \text{ and } S(z) = \begin{bmatrix} \mathrm{Re}(z) & \mathrm{Im}(z) \\ \mathrm{Im}(z) & -\mathrm{Re}(z) \end{bmatrix}.$$

Suppose $1 \leq j < k \leq m$ and suppose $\xi \in \Delta^*$ and $\mu = (\mu_1, \ldots, \mu_4) \in S^3$.

Define

$$B_{j,k}(\xi, \mu) = \begin{bmatrix} R(\theta_j(\xi)) & R(\theta_k(\xi)) & R(0) & R(0) \\ R(0) & R(0) & R(\theta_j(-\xi)) & R(\theta_k(-\xi)) \\ R(\mu_1 - i\mu_2) & R(\mu_3 - i\mu_4) & S(\mu_1 + i\mu_2) & S(\mu_3 + i\mu_4) \end{bmatrix}.$$

Then $B_{j,k}(\xi, \mu)$ has full rank as long as $\xi \in \mathrm{supp}\psi_{j,k}$. Fix $(\xi, \mu) \in \Delta^* \times S^3$ in the support of $\psi_{j,k}$. Then since $B_{j,k}(\xi, \mu)$ has rank 6 we can find 6 of its columns which are linearly independent. Call the other two columns the ith and lth columns. Define $e_i = (0, \ldots, 1, \ldots, 0)$ and $e_l = (0, \ldots, 1, \ldots, 0)$ where the 1 appears in the ith and jth position respectively. Define $C_{j,k}(\xi, \mu)$ to be the 8×8 matrix which consists of the matrix $B_{j,k}(\xi, \mu)$ with the rows e_i and e_j appended. Then $C_{j,k}(\xi, \mu)$ is invertible and so

$$C_{j,k}(\xi, \mu)^{-1}\psi_{j,k}(\xi) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a well defined element of $\mathbb{R}^8$. Moreover, there exists an open set $U \subset \Delta^* \times S^3$ consisting of points (ξ, μ) such that the same 6 columns are linearly independent for each $(\zeta, \nu) \in U$ and such that U does not intersect

$$U^* = \left\{(\zeta, \nu) \in \Delta^* \times S^3 : (-\zeta, \nu) \in U\right\}.$$

So, we can define a smooth function $G\colon U \to \mathbb{R}^8$ such that for $(\zeta, \nu) \in U$ we have

$$B_{j,k}(\zeta, \nu) G(\zeta, \nu) = \psi_{j,k}(\zeta) \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.41}$$

Note that for $(\xi, \mu) \notin \operatorname{supp} \psi_{j,k}$ we may simply take U small enough so that $G \equiv \vec{0}$ satisfies the equation above. We now define complex valued functions $\Theta_{j,k,U}, \Theta_{k,j,U} \in C^\infty(\Delta^* \times S^3)$ by

$$\begin{bmatrix} \operatorname{Re} \Theta_{j,k,U}(\xi, \mu) \\ \operatorname{Im} \Theta_{j,k,U}(\xi, \mu) \\ \operatorname{Re} \Theta_{k,j,U}(\xi, \mu) \\ \operatorname{Im} \Theta_{k,j,U}(\xi, \mu) \\ \operatorname{Re} \Theta_{j,k,U}(-\xi, \mu) \\ \operatorname{Im} \Theta_{j,k,U}(-\xi, \mu) \\ \operatorname{Re} \Theta_{k,j,U}(-\xi, \mu) \\ \operatorname{Im} \Theta_{k,j,U}(-\xi, \mu) \end{bmatrix} = G(\xi, \mu).$$

That this gives smooth and well-defined functions follows from the facts that G is smooth and that U and U^* do not intersect. Combining this definition with (2.41) leads to the following system of equations for each $(\xi, \mu) \in U \cup U^*$.

$$\theta_j(\xi)\Theta_{j,k,U}(\xi,\mu) + \theta_k(\xi,\mu)\Theta_{k,j,U}(\xi,\mu) = \psi_{j,k}(\xi), \tag{2.42}$$

$$\begin{aligned} &\mu_1 \operatorname{Re}(\Theta_{j,k,U}(\xi,\mu) + \Theta_{j,k,U}(-\xi,\mu)) \\ &\quad + \mu_2 \operatorname{Im}(\Theta_{j,k,U}(\xi,\mu) + \Theta_{j,k,U}(-\xi,\mu)) \\ &\quad + \mu_3 \operatorname{Re}(\Theta_{k,j,U}(\xi,\mu) + \Theta_{k,j,U}(-\xi,\mu)) \\ &\quad + \mu_4 \operatorname{Im}(\Theta_{k,j,U}(\xi,\mu) + \Theta_{k,j,U}(-\xi,\mu)) = 0, \end{aligned} \tag{2.43}$$

$$\begin{aligned} &\mu_1 \operatorname{Im}(\Theta_{j,k,U}(\xi,\mu) - \Theta_{j,k,U}(-\xi,\mu)) \\ &\quad - \mu_2 \operatorname{Re}(\Theta_{j,k,U}(\xi,\mu) - \Theta_{j,k,U}(-\xi,\mu)) \\ &\quad + \mu_3 \operatorname{Im}(\Theta_{k,j,U}(\xi,\mu) - \Theta_{k,j,U}(-\xi,\mu)) \\ &\quad - \mu_4 \operatorname{Re}(\Theta_{k,j,U}(\xi,\mu) - \Theta_{k,j,U}(-\xi,\mu)) = 0. \end{aligned} \tag{2.44}$$

Since we have a neighborhood, U, as above for each point $(\xi, \mu) \in \Delta^* \times S^3$ and since $\Delta^* \times S^3$ is compact, we know there exist a natural number N and points $\zeta_1, \dots, \zeta_N \in \Delta^* \times S^3$ such that the corresponding sets $U_1, \dots, U_N$ cover $\Delta^* \times S^3$. Since $\Delta^* \times S^3$ is compact, and since $U_i \cap U_i^*$ is empty for each i there exist smooth real valued functions $\psi_1, \dots, \psi_N$ defined on $\Delta^* \times S^3$ satisfying

$$\sum_{l=1}^{N} \psi_l(\xi,\mu) = 1 \tag{2.45}$$

and such that for $i \in \{1, \ldots, N\}$ we have

$$\operatorname{supp} \psi_i \subset U_i \cup U_i^*, \tag{2.46}$$

$$\psi_i(\xi, \mu) = \psi_i(-\xi, \mu). \tag{2.47}$$

Fix $(\xi, \mu) \in \Delta^* \times S^3$. Define $\Theta_{j,k}$ and $\Theta_{k,j}$ by

$$\Theta_{j,k}(\xi, \mu) = \sum_{l=1}^{N} \psi_l(\xi, \mu)\Theta_{j,k,U_l}(\xi, \mu), \qquad \Theta_{k,j}(\xi, \mu) = \sum_{l=1}^{N} \psi_l(\xi, \mu)\Theta_{k,j,U_l}(\xi, \mu)$$

with $\psi_l(\xi, \mu)\Theta_{j,k,U_l}(\xi, \mu)$ taken to be zero if $(\xi, \mu) \notin U_l$. From this definition and (2.42)–(2.44) we obtain

$$\theta_j(\xi)\Theta_{j,k}(\xi, \mu) + \theta_k(\xi, \mu)\Theta_{k,j}(\xi, \mu) = \psi_{j,k}(\xi), \tag{2.48}$$

$$\begin{aligned} &\mu_1 \operatorname{Re}(\Theta_{j,k}(\xi, \mu) + \Theta_{j,k}(-\xi, \mu)) \\ &\quad + \mu_2 \operatorname{Im}(\Theta_{j,k}(\xi, \mu) + \Theta_{j,k}(-\xi, \mu)) \\ &\quad + \mu_3 \operatorname{Re}(\Theta_{k,j}(\xi, \mu) + \Theta_{k,j}(-\xi, \mu)) \\ &\quad + \mu_4 \operatorname{Im}(\Theta_{k,j}(\xi, \mu) + \Theta_{k,j}(-\xi, \mu)) = 0, \end{aligned} \tag{2.49}$$

$$\begin{aligned} &\mu_1 \operatorname{Im}(\Theta_{j,k}(\xi, \mu) - \Theta_{j,k}(-\xi, \mu)) \\ &\quad - \mu_2 \operatorname{Re}(\Theta_{j,k}(\xi, \mu) - \Theta_{j,k}(-\xi, \mu)) \\ &\quad + \mu_3 \operatorname{Im}(\Theta_{k,j}(\xi, \mu) - \Theta_{k,j}(-\xi, \mu)) \\ &\quad - \mu_4 \operatorname{Re}(\Theta_{k,j}(\xi, \mu) - \Theta_{k,j}(-\xi, \mu)) = 0. \end{aligned} \tag{2.50}$$

If $r > 0$, define $\Theta_{j,k}(\xi, r\mu) = \Theta_{j,k}(\xi, \mu)$ and $\Theta_{k,j}(\xi, r\mu) = \Theta_{k,j}(\xi, \mu)$. Additionally, define

$$\Theta_{j,k}(\xi, (0,0,0,0)) = \begin{cases} \dfrac{\overline{\theta_j(\xi)}\psi_{j,k}(\xi)}{|\theta_j(\xi)|^2 + |\theta_k(\xi)|^2} & \xi \in \operatorname{supp} \psi_{j,k}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Theta_{k,j}(\xi, (0,0,0,0)) = \begin{cases} \dfrac{\overline{\theta_k(\xi)}\psi_{j,k}(\xi)}{|\theta_j(\xi)|^2 + |\theta_k(\xi)|^2} & \xi \in \operatorname{supp} \psi_{j,k}, \\ 0 & \text{otherwise.} \end{cases}$$

This extends the definition of $\Theta_{j,k}$ and $\Theta_{k,j}$ to $\Delta^* \times \mathbb{R}^4$ such that the corresponding versions of (2.48)–(2.50) still hold.

We can now define our desired functions Θ_j. Set $v_{j,k} = (v_{2j-1}, v_{2j}, v_{2k-1}, v_{2k})$. Let $j \in \{1, \dots, m\}$ and let $\xi \in \Delta^*$ and let $\vec{v} \in S^{2m-1}$. Then define

$$\Theta_j(\xi) = \sum_{l=1}^{j-1} \Theta_{j,l}(\xi, v_{l,j}) + \sum_{l=j+1}^{m} \Theta_{j,l}(\xi, v_{j,l}).$$

We first check that (2.37) holds. We have,

$$\begin{aligned} \sum_{j=1}^{m} \theta_j(\xi)\Theta_j(\xi) &= \sum_{j=1}^{m}\sum_{k=1}^{j-1} \theta_j(\xi)\Theta_{j,k}(\xi, v_{k,j}) \\ &\quad + \sum_{j=1}^{m}\sum_{k=j+1}^{m} \theta_j(\xi)\Theta_{j,k}(\xi, v_{j,k}) \\ &= \sum_{j=1}^{m}\sum_{k=j+1}^{m} \left(\theta_k(\xi)\Theta_{k,j}(\xi, v_{j,k}) + \theta_j(\xi)\Theta_{j,k}(\xi, v_{j,k})\right) \\ &= \sum\sum \psi_{j,k}(\xi) \\ &= 1. \end{aligned}$$

This is (2.37).

Next we prove (2.38). We have

$$
\begin{aligned}
&\sum_{j=1}^{m} v_{2j-1}\operatorname{Re}(\Theta_j(\xi)+\Theta_j(-\xi)) + v_{2j}\operatorname{Im}(\Theta_j(\xi)+\Theta_j(-\xi))\\
&=\sum_{j=1}^{m}\sum_{k=1}^{j-1} v_{2j-1}\operatorname{Re}(\Theta_{j,k}(\xi,v_{k,j})+\Theta_{j,k}(-\xi,v_{k,j}))\\
&\qquad\qquad + v_{2j}\operatorname{Im}(\Theta_{j,k}(\xi,v_{k,j})+\Theta_{j,k}(-\xi,v_{k,j}))\\
&+\sum_{j=1}^{m}\sum_{k=j+1}^{m} v_{2j-1}\operatorname{Re}(\Theta_{j,k}(\xi,v_{j,k})+\Theta_{j,k}(-\xi,v_{j,k}))\\
&\qquad\qquad + v_{2j}\operatorname{Im}(\Theta_{j,k}(\xi,v_{j,k})+\Theta_{j,k}(-\xi,v_{j,k}))\\
&=\sum_{j=1}^{m}\sum_{k=j+1}^{m} v_{2k-1}\operatorname{Re}(\Theta_{k,j}(\xi,v_{j,k})+\Theta_{k,j}(-\xi,v_{j,k}))\\
&\qquad\qquad + v_{2k}\operatorname{Im}(\Theta_{k,j}(\xi,v_{j,k})+\Theta_{k,j}(-\xi,v_{j,k}))\\
&\qquad\qquad + v_{2j-1}\operatorname{Re}(\Theta_{j,k}(\xi,v_{j,k})+\Theta_{j,k}(\xi,v_{j,k}))\\
&\qquad\qquad + v_{2j}\operatorname{Im}(\Theta_{j,k}(\xi,v_{j,k})+\Theta_{j,k}(-\xi,v_{j,k}))\\
&=\sum\sum 0\\
&=0.
\end{aligned}
$$

The proof of (2.39) is similar to the proof of (2.38) and is omitted.

It remains to show (2.40). Note that by construction,

$$
\begin{aligned}
&\sup\left\{|\nabla_\xi\Theta_{j,k}(\xi,\mu)| : \xi\in\Delta^*, \mu\in\mathbb{R}^4\right\}\\
&=\sup\left\{|\nabla_\xi\Theta_{j,k}(\xi,\mu)| : \xi\in\Delta^*, \mu\in S^3\cup\{0,0,0,0\}\right\}.
\end{aligned}
$$

Since Δ is compact, and since $\Theta_{j,k}(\cdot,(0,0,0,0)) \in C^\infty(\Delta^*)$ and since $\Theta_{j,k}(\cdot,\cdot) \in C^\infty(\Delta^* \times S^3)$ it follows that

$$\sup\left\{|\nabla_\xi \Theta_{j,k}(\xi,\mu)| : \xi \in \Delta^*, \mu \in S^3 \cup \{0,0,0,0\}\right\} < \infty.$$

This completes the proof of Lemma 38. $\square$

Lemma 39. *Let $\vec{v} \in S^{2m-1}$. Let $\theta_1, \ldots, \theta_m$, and $\Theta_1, \ldots, \Theta_m$ be as in Lemma 38. Let $S^\perp$ be as in Lemma 26. Let $b \in L^2(\mathbb{R}^n, \mathbb{R}) \cap BMO(\mathbb{R}^n, \mathbb{R})$. Set*

$$\vec{v}' = (-v_2, v_1, \cdots, -v_{2m}, v_{2m-1})$$

and let $\Theta'_1, \ldots, \Theta'_m$ be the family of functions constructed in Lemma 38 using $\vec{v}'$ in place of $\vec{v}$. Let

$$b' = \sum_{j=1}^m m_{\theta_j}((v_{2j-1} + iv_{2j}b))$$

and

$$\vec{p} = \begin{bmatrix} (v_1 + iv_2)b - m_{\Theta_1}(\operatorname{Re} b') - im_{\Theta'_1}(\operatorname{Im} b') \\ \vdots \\ (v_{2m-1} + iv_{2m})b - m_{\Theta_m}(\operatorname{Re} b') - im_{\Theta'_m}(\operatorname{Im} b') \end{bmatrix}.$$

Then

$$\vec{p} \in S^\perp \tag{2.51}$$

$$\vec{p}(x) \cdot \vec{v} = b(x). \tag{2.52}$$

Proof. We first establish (2.51). Note that by (2.37) we have

$$\sum_{j=1}^{m} m_{\theta_j}\Big((v_{2j-1}+iv_{2j})b - m_{\Theta_j}(\operatorname{Re} b') - im_{\Theta'_j}(\operatorname{Im} b')\Big)$$

$$= b' - \operatorname{Re} b' - i\operatorname{Im} b'$$

$$= 0.$$

For (2.52) we note that if $f \in L^2(\mathbb{R}^n, \mathbb{R})$ then we have

$$\begin{aligned}
\operatorname{Re}(m_{\Theta_j} f) &= \frac{m_{\Theta_j} f + \overline{m_{\Theta_j} f}}{2} \\
&= \frac{m_{\Theta_j} f + m_{\overline{\widetilde{\Theta_j}}} f}{2} \\
&= \frac{\mathcal{F}^{-1}\left[\Big(\operatorname{Re}(\Theta_j + \widetilde{\Theta_j}) + i\operatorname{Im}(\Theta_j - \widetilde{\Theta_j})\Big)\widehat{f}\right]}{2}.
\end{aligned}$$

Similarly,

$$\operatorname{Im}(m_{\Theta_j} f) = \frac{\mathcal{F}^{-1}\left[\Big(\operatorname{Im}(\Theta_j + \widetilde{\Theta_j}) - i\operatorname{Re}(\Theta_j - \widetilde{\Theta_j})\Big)\widehat{f}\right]}{2}.$$

Consequently,

$$(m_{\Theta_1} f, \ldots, m_{\Theta_m} f) \cdot \vec{v} = \sum_{j=1}^{m} v_{2j-1}\operatorname{Re}(m_{\Theta_j} f) + v_{2j}\operatorname{Im}(m_{\Theta_j} f) = 0,$$

where the last equality follows from (2.38) and (2.39). Then

$$(im_{\Theta'_1} f, \ldots, im_{\Theta'_m} f) \cdot \vec{v} = -(m_{\Theta'_1} f, \ldots, m_{\Theta'_m} f) \cdot \vec{v}'.$$

It then follows from the definition of $\vec{p}$ that

$$\vec{p}\cdot\vec{v} = \sum_{j=1}^{m}(v_{2j-1}^2 + v_{2j}^2)b = b,$$

as desired. $\square$

2.6. Averaging Operators

In this section we define several averaging operators. They will serve as auxiliary functions allowing us to bridge the gap between inequalities later on. The result of this section is a generalization of Lemma 22.7 of [Uch01].

Definition 40. *Suppose $\{\lambda_I\}_I$ is an indexed set of non-negative real numbers where I is taken over all dilated cubes in $\mathbb{R}^n$. For each $j \in \mathbb{Z}$ define*

$$\eta_j^{(1)}(x) = \sum_{I:|I|=2^j} \frac{\lambda_I}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-/2}},$$

$$\eta_j^{(2)}(x) = \sum_{k=j}^{\infty}\left(\frac{99}{100}\right)^{(k-j)\alpha_-}\eta_k^{(1)}(x),$$

$$\eta_j^{(3)}(x) = \left(\sum_{I:|I|=2^j} \frac{\lambda_I^2}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-/2}}\right)^{1/2},$$

$$\eta_j^{(4)}(x) = \left(\sum_{k=j}^{\infty}\left(\frac{99}{100}\right)^{(k-j)\alpha_-}\eta_k^{(3)}(x)^2\right)^{1/2},$$

$$\eta_j^{(0)}(x,y) = \begin{cases} \eta_j^{(2)}(x)\,|\delta_{2^{-j}}(x-y)| & \text{if } \rho(x-y) < 2^j, \\ \displaystyle\sum_{k=j}^{\log_2\rho(x-y)}\left(\eta_k^{(2)}(x)+\eta_k^{(2)}(y)\right) & \text{if } \rho(x-y) \geq 2^j. \end{cases}$$

Lemma 41. *Let $\{\lambda_I\}_I$ be as in Definition 40. Let $f_j\colon \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}_+$ be defined by $f_j(x) = (x, 2^j)$. Then the measure $\delta_{t=2^j}$ is defined by setting $\delta_{t=2^j}(S)$ to be the*

Lebesgue measure of $f_j^{-1}(S)$. There exists a constant $C > 0$ such that

$$\eta_j^{(1)}(x) \leq C\eta_j^{(1)}(y) \text{ if } \rho(x-y) \leq 2^j, \tag{2.53}$$

$$\eta_j^{(4)}(x) \leq C\eta_j^{(4)}(y) \text{ if } \rho(x-y) \leq 2^j, \tag{2.54}$$

$$\sum_{k=j}^{\infty} \left(\frac{9}{10}\right)^{(k-j)\alpha_-} \eta_k^{(4)}(x)^2 \leq C\eta_j^{(4)}(x)^2, \tag{2.55}$$

$$2^{-j} \int \frac{\eta_j^{(4)}(y)^2 dy}{(1+\rho(x-y)/2^j)^{1+\alpha_-/2}} \leq C\eta_j^{(4)}(x)^2, \tag{2.56}$$

$$\eta_j^{(1)}(x) \leq \eta_j^{(2)}(x) \leq C\eta_j^{(4)}(x) \leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2}, \tag{2.57}$$

$$\eta_j^{(0)}(x,y) \leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} \log_2(2+\rho(x-y)/2^j), \tag{2.58}$$

$$\sum_{I:|I|=2^j} \frac{\lambda_I \eta_{j+1}^{(0)}(x,x_I)}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-}} \leq C \min\left\{\eta_j^{(4)}(x)^2, \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} \eta_j^{(1)}(x)\right\} \tag{2.59}$$

$$\left\| \sum_{j=-\infty}^{\infty} \eta_j^{(4)}(x)^2 \delta_{t=2^j} \right\|_{\mathcal{C}} \leq C \left\|\Lambda\right\|_{\mathcal{C}}. \tag{2.60}$$

Proof. In this proof C denotes a positive constant. Different instances of C do not necessarily denote the same constant. For (2.53) we have

$$\begin{aligned}
\eta_j^{(1)}(y) &= \sum_{I:|I|=2^j} \frac{\lambda_I}{(1+\rho(y-x_I)/2^j)^{1+\alpha_-/2}} \\
&\geq \sum_{I:|I|=2^j} \frac{\lambda_I}{(1+C_q\rho(x-y)/2^j + C_q\rho(x-x_I)/2^j)^{1+\alpha_-/2}} \\
&\geq \sum_{I:|I|=2^j} \frac{\lambda_I}{((1+C_q)(1+\rho(x-x_I)/2^j))^{1+\alpha_-/2}} \\
&= C\eta_j^{(1)}(x).
\end{aligned}$$

Similarly, for (2.54) we have

$$\begin{aligned}\eta_j^{(4)}(y)^2 &= \sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^j} \frac{\lambda_I^2}{(1+\rho(y-x_I)/2^k)^{1+\alpha_-/2}} \\ &\geq C \sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^j} \frac{\lambda_I^2}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}} \\ &= C\eta_j^{(4)}(x)^2.\end{aligned}$$

For (2.55),

$$\begin{aligned}&\sum_{k=j}^{\infty} \left(\frac{9}{10}\right)^{(k-j)\alpha_-} \eta_k^{(4)}(x)^2 \\ &= \sum_{k=j}^{\infty} \left(\frac{9}{10}\right)^{(k-j)\alpha_-} \sum_{l=k}^{\infty} \left(\frac{99}{100}\right)^{(l-k)\alpha_-} \sum_{I:|I|=2^j} \frac{\lambda_I^2}{(1+\rho(x-x_I)/2^l)^{1+\alpha_-/2}} \\ &\leq \sum_{k=j}^{\infty} \left(\frac{9}{10}\right)^{(k-j)\alpha_-} \left(\frac{99}{100}\right)^{(j-k)\alpha_-} \sum_{l=j}^{\infty} \left(\frac{99}{100}\right)^{(l-j)\alpha_-} \sum_{I:|I|=2^j} \frac{\lambda_I^2}{(1+\rho(x-x_I)/2^l)^{1+\alpha_-/2}} \\ &= \eta_j^{(4)}(x)^2 \sum_{k=j}^{\infty} \left(\frac{10}{11}\right)^{(k-j)\alpha_-} \\ &= C\eta_j^{(4)}(x)^2.\end{aligned}$$

In order to prove (2.56) we need the following inequality, which holds for $k \geq j$.

$$\begin{aligned}2^{-j} \int & \frac{dy}{(1+\rho(z-y)/2^k)^{1+\alpha_-/2}(1+\rho(x-y)/2^j)^{1+\alpha_-/2}} \\ &\leq \frac{C}{(1+\rho(z-x)/2^k)^{1+\alpha_-/2}}.\end{aligned} \tag{2.61}$$

To see this, we split the integral up into two pieces defined by the ball $B = B(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) \le r\}$ and its complement where

$$r = 2^{k-1} / \left[C_q(1 + \rho(z-x)/2^k)\right]$$

and obtain

$$\begin{aligned}
&2^{-j} \int \frac{dy}{\left(1+\rho(z-y)/2^k\right)^{1+\alpha_-/2}\left(1+\rho(x-y)/2^j\right)^{1+\alpha_-/2}} \\
&\le 2^{-j} \sup_{y\in B} \frac{1}{\left(1+\rho(z-y)/2^k\right)^{1+\alpha_-/2}} \left\| \frac{1}{\left(1+\rho(x-\cdot)/2^j\right)^{1+\alpha_-/2}} \right\|_1 \\
&\quad + 2^{-j} \left\| \frac{1}{\left(1+\rho(z-\cdot)/2^k\right)^{1+\alpha_-/2}} \right\|_1 \sup_{y\in B^c} \frac{1}{\left(1+\rho(x-y)/2^j\right)^{1+\alpha_-/2}} \\
&\le C \left(\sup_{y\in B} \frac{1}{\left(1+\rho(z-y)/2^k\right)^{1+\alpha_-/2}} + 2^{k-j} \sup_{y\in B^c} \frac{1}{\left(1+\rho(x-y)/2^j\right)^{1+\alpha_-/2}} \right).
\end{aligned}$$

For $y \in B$ we have

$$\begin{aligned}
1 + \rho(z-x)/2^k &\le 1 + C_q\rho(z-y)/2^k + C_q\rho(x-y)/2^k \\
&\le C_q(1+\rho(z-y)/2^k) + 2^{-1}(1+\rho(z-x)/2^k),
\end{aligned}$$

so

$$1 + \rho(z-x)/2^k \le 2C_q(1+\rho(z-y)/2^k)$$

and

$$\sup_{y\in B} \frac{1}{\left(1+\rho(z-y)/2^k\right)^{1+\alpha_-/2}} \le \frac{C}{\left(1+\rho(z-x)/2^k\right)^{1+\alpha_-/2}}.$$

For $y \in B^c$ we have

$$1 + \rho(x-y)/2^j \ge 1 + C_q^{-1} 2^{k-j-1}(1+\rho(z-x)/2^k)$$

so

$$C_q 2^{j+1-k}(1+\rho(x-y)/2^j) \geq C_q 2^{j+1-k} + 1 + \rho(z-x)/2^k$$
$$\geq (1+2c_q)(1+\rho(z-x)/2^k),$$

from which we obtain

$$\frac{1}{(1+\rho(x-y)/2^j)^{1+\alpha_-/2}} \leq \frac{C2^{j-k}}{(1+\rho(z-x)/2^k)^{1+\alpha_-/2}}$$

from which (2.61) follows. Returning to proving (2.56) we have

$$\begin{aligned}
&2^{-j}\int \frac{\eta_j^{(4)}(y)dy}{(1+\rho(x-y)/2^j)^{1+\alpha_-/2}} \\
&= 2^{-j}\int \sum_{k=j}^{\infty}\left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{\lambda_I^2 dy}{(1+\rho(y-x_I)/2^k)^{1+\alpha_-/2}(1+\rho(x-y)/2^j)^{1+\alpha_-/2}} \\
&\leq \sum_{k=j}^{\infty}\left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{C\lambda_I^2}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}} \\
&= C\eta_j^{(4)}(x)^2.
\end{aligned}$$

The first inequality of (2.57) is immediate since the first term of the expression for $\eta_j^{(2)}(x)$ is exactly $\eta_j^{(1)}(x)$. For the second inequality in (2.57) we have, using the

Cauchy-Schwarz inequality twice,

$$\begin{aligned}
\eta_j^{(2)}(x) &= \sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{\lambda_I}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}} \\
&\leq \sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \left(\sum_{I:|I|=2^k} \frac{\lambda_I^2}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}\right)^{1/2} \\
&\quad \cdot \left(\sum_{I:|I|=2^k} \frac{1}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}\right)^{1/2} \\
&\leq \left(\sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{\lambda_I^2}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}\right)^{1/2} \\
&\quad \cdot \left(\sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{1}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}\right)^{1/2} \\
&= \eta_j^{(4)}(x) \left(\sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{1}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}\right)^{1/2}.
\end{aligned}$$

Thus, we need to show

$$\left(\sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{1}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}\right)^{1/2} \leq C.$$

Note that

$$\begin{aligned}
\int_I \frac{dy}{(1+\rho(x-y)/2^k)^{1+\alpha_-/2}} &\geq \int_I \frac{dy}{(1+C_q\rho(x-y)/2^k + C_q\rho(x_I-y)/2^k)^{1+\alpha_-/2}} \\
&\geq \int_I \frac{C dy}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}} \\
&= \frac{C2^k}{(1+\rho(x-x_I)/2^k)^{1+\alpha_-/2}}
\end{aligned}$$

so

$$\begin{aligned}
\sum_{I:|I|=2^k} \frac{1}{\left(1+\rho(x-x_I)/2^k\right)^{1+\alpha_-/2}} &\leq C \sum_{I:|I|=2^k} 2^{-k} \int_I \frac{dy}{\left(1+\rho(x-y)/2^k\right)^{1+\alpha_-/2}} \\
&= C2^{-k} \int_{\mathbb{R}^n} \frac{dy}{\left(1+\rho(x-y)/2^k\right)^{1+\alpha_-/2}} \\
&= C.
\end{aligned}$$

Thus, we have

$$\left(\sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{1}{\left(1+\rho(x-x_I)/2^k\right)^{1+\alpha_-/2}} \right)^{1/2} \leq C.$$

For the third inequality in (2.57),

$$\eta_j^{(4)}(x) \leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2},$$

we need only recall that

$$\lambda_I \leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2}$$

so that

$$\begin{aligned}
\eta_j^{(4)}(x)^2 &= \sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{\lambda_I^2}{\left(1+\rho(x-x_I)/2^k\right)^{1+\alpha_-/2}} \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}} \sum_{k=j}^{\infty} \left(\frac{99}{100}\right)^{(k-j)\alpha_-} \sum_{I:|I|=2^k} \frac{1}{\left(1+\rho(x-x_I)/2^k\right)^{1+\alpha_-/2}} \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}.
\end{aligned}$$

This completes the proof of (2.57).

The proof of (2.58) is straightforward from (2.57). First, if $\rho(x-y) < 2^j$ then

$$\begin{aligned}
\eta_j^{(0)}(x,y) =& \eta_j^{(2)}(x)|A_{2^{-j}}(x-y)| \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}} \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} \log_2(2+\rho(x-y)/2^j).
\end{aligned}$$

For $\rho(x-y) \geq 2^j$, we have

$$\begin{aligned}
\eta_j^{(0)}(x,y) = & \sum_{k=j}^{\log_2 \rho(x-y)} \left(\eta_j^{(2)}(x) + \eta_j^{(2)}(y)\right) \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} (\log_2 \rho(x-y) - j + 1) \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} log_2(2+\rho(x-y)/2^j).
\end{aligned}$$

We move on to establishing (2.59). Using (2.58) we have

$$\begin{aligned}
&\sum_{I:|I|=2^j} \frac{\lambda_I \eta_{j+1}^{(0)}(x,x_I)}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-}} \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} \sum_{I:|I|=2^j} \frac{\lambda_I \log_2(2+\rho(x-x_I)/2^{j+1}))}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-}} \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} \sum_{I:|I|=2^j} \frac{\lambda_I}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-/2}} \frac{\log_2(2+\rho(x-x_I)/2^{j+1}))}{(1+\rho(x-x_I)/2^j)^{\alpha_-/2}} \\
&\leq C \left\|\Lambda\right\|_{\mathcal{C}}^{1/2} \eta_j^{(1)}(x).
\end{aligned}$$

Also,

$$
\begin{aligned}
&\sum_{I:|I|=2^j} \frac{\lambda_I \eta_{j+1}^{(0)}(x, x_I)}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}} \\
&\quad \leq C \sum_{I:|I|=2^j} \frac{\lambda_I}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}} \sum_{k=j+1}^{\log_2(2^{j+1}+\rho(x-x_I))} \left(\eta_k^{(2)}(x)+\eta_k^{(2)}(x_I)\right) \\
&\quad \leq C \sum_{k>j} \sum_{\substack{I:|I|=2^j \\ x_I \notin B(x, 2^k-2^{j+1})}} \frac{\lambda_I \left(\eta_k^{(2)}(x)+\eta_k^{(2)}(x_I)\right)}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}} \\
&\quad \leq C \left(\sum\sum \frac{\lambda_I^2}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}}\right)^{1/2} \\
&\qquad \cdot \left(\sum\sum \frac{\left(\eta_k^{(2)}(x)+\eta_k^{(2)}(x_I)\right)^2}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}}\right)^{1/2},
\end{aligned}
$$

where the last inequality follows from Cauchy-Schwarz. Since $x_I \notin B(x, 2^k - 2^{j+1})$ we have

$$
\frac{1}{1+\rho(x-x_I)/2^j} \leq \frac{1}{1+2^{k-j}-2} \leq 2^{j-k}
$$

so that

$$
\frac{1}{\left(1+\rho(x-x_I)/2^j\right)^{\alpha_-/2}} \leq 2^{(j-k)\alpha_-/2}.
$$

Additionally, using that $\rho(x-x_I) \geq 2^k - 2^{j+1}$, we have

$$
\frac{1}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}} \leq \frac{C2^{(j-k)(1+\alpha_-)}}{\left(1+\rho(x-x_I)/2^k\right)^{1+\alpha_-}}
$$

so that

$$\left(\sum\sum \frac{\lambda_I^2}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}}\right)^{1/2} \left(\sum\sum \frac{\left(\eta_k^{(2)}(x)+\eta_k^{(2)}(x_I)\right)^2}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}}\right)^{1/2}$$

$$\leq C\left(\sum\sum \frac{2^{(j-k)\alpha_-/2}\lambda_I^2}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-/2}}\right)^{1/2}$$

$$\cdot\left(\sum\sum \frac{\left(\eta_k^{(2)}(x)^2+\eta_k^{(2)}(x_I)^2\right)2^{(j-k)(1+\alpha_-)}}{\left(1+\rho(x-x_I)/2^k\right)^{1+\alpha_-}}\right)^{1/2}.$$

From here we make use of (2.54) and (2.57) followed by (2.56) and then finally (2.55) to obtain

$$\left(\sum\sum \frac{2^{(j-k)\alpha_-/2}\lambda_I^2}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-/2}}\right)^{1/2}$$

$$\cdot\left(\sum\sum \frac{\left(\eta_k^{(2)}(x)^2+\eta_k^{(2)}(x_I)^2\right)2^{(j-k)(1+\alpha_-)}}{\left(1+\rho(x-x_I)/2^j\right)^{1+\alpha_-}}\right)^{1/2}$$

$$\leq C\left(\sum_{k>j} 2^{(j-k)\alpha_-/2}\eta_j^{(3)}(x)^2\right)^{1/2}\left(\sum_{k>j} 2^{(j-k)\alpha_-}\int_{\mathbb{R}^n} \frac{\eta_k^{(4)}(x)^2+\eta_k^{(4)}(y)^2}{2^k\left(1+\rho(x-y)/2^k\right)^{1+\alpha_-}}dy\right)^{1/2}$$

$$\leq C\eta_j^{(3)}(x)\left(\sum_{k>j} 2^{(j-k)\alpha_-}\eta_k^{(4)}(x)^2\right)^{1/2}$$

$$\leq C\eta_j^{(4)}(x),$$

and the result follows.

Finally, we prove (2.60). Let $B = B(x_B, r_B)$. Then we have

$$\int_B \sum_{j=-\infty}^{\log_2(r_B)} \eta_j^{(3)}(x)^2 dx = \sum_{j=-\infty}^{\log_2(r_B)} \sum_{I:|I|=2^j} \lambda_I^2 \int_B \frac{dx}{(1+\rho(x-x_I)/2^j)^{1+\alpha_-/2}}$$
$$\leq C \sum \sum \lambda_I^2 \min\left\{|I|, |B| \left(\operatorname{dist}(x_I, B)/|I|\right)^{1+\alpha_-/2}\right\}.$$

If $|I| \leq |B| \left(\operatorname{dist}(x_I, B)/|I|\right)^{1+\alpha_-/2}$ then by rearranging we obtain

$$\operatorname{dist}(x_I, B) \leq (|B||I|^{\alpha_-/2})^{1/(1+\alpha_-/2)} \leq C r_B.$$

Thus,

$$1 + \frac{\rho(x_I - x_B)}{r_B} \leq 1 + C_q \frac{\operatorname{dist}(x_B, B) + r_B}{r_B} \leq C,$$

from which it follows that

$$|I| \leq \frac{C|I|}{(1+\rho(x_I - x_B))^{1+\alpha_-/2}}.$$

If $|B| \left(\operatorname{dist}(x_I, B)/|I|\right)^{1+\alpha_-/2} \leq |I|$ then by rearranging we obtain

$$\left(\operatorname{dist}(x_I, B)/|I|\right)^{1+\alpha_-/2} \geq |B||I|^{\alpha_-/2}.$$

This gives

$$\frac{1+\rho(x_I - x_B)/r_B}{\operatorname{dist}(x_I, B)} \leq \frac{1+C_q}{\operatorname{dist}(x_I, B)} + \frac{C_q}{r_B} \leq \frac{C}{|B|^{1/(1+\alpha_-/2)}|I|^{\alpha_-/(2(1+\alpha_-/2))}},$$

from which it follows that

$$|B| \left(\operatorname{dist}(x_I, B)/|I|\right)^{1+\alpha_-/2} \leq \frac{C|I|}{(1+\rho(x_I - x_B)/r_B)^{1+\alpha_-/2}}.$$

Thus, we have

$$\int_B \sum_{j=-\infty}^{\log_2(r_B)} \eta_j^{(3)}(x)^2 dx \leq C \sum \sum \frac{\lambda_I^2 |I|}{(1+\rho(x_I - x_B)/r_B)^{1+\alpha_-/2}}$$
$$\leq C \|\Lambda\|_{\mathcal{C}} |B|.$$

Hence, for any $m \in \mathbb{N}$ we have

$$\begin{aligned}
&\int \int_{Q(B)} \sum_{j=-\infty}^{\infty} \eta_{j+m}^{(3)}(x)^2 \delta_{t=2^j} \\
&= \sum_{j=-\infty}^{\log_2(r_B)+m} \int_B \eta_j^{(3)}(x)^2 dx \\
&= \sum_{j=-\infty}^{\log_2(r_B)} \int_B \eta_j^{(3)}(x)^2 dx + \sum_{j=\log_2(r_B)+1}^{\log_2(r_B)+m} \int_B \eta_j^{(3)}(x)^2 dx \\
&\leq C(1+m) \|\Lambda\|_{\mathcal{C}} |B|.
\end{aligned}$$

So

$$\left\| \sum_{j=-\infty}^{\infty} \eta_{j+m}^{(3)}(x)^2 \delta_{t=2^j} \right\|_C \leq C(1+m) \|\Lambda\|_{\mathcal{C}}$$

and

$$\begin{aligned}
\left\| \sum_{j=-\infty}^{\infty} \eta_j^{(4)}(x)^2 \delta_{t=2^j} \right\|_C &\leq \sum_{m=0}^{\infty} .99^{m\alpha_-} \left\| \sum_{j=-\infty}^{\infty} \eta_{j+m}^{(3)}(x)^2 \delta_{t=2^j} \right\|_C \\
&\leq C \sum_{m=0}^{\infty} .99^{m\alpha_-} (1+m) \|\Lambda\|_{\mathcal{C}} \\
&\leq C \|\Lambda\|_{\mathcal{C}}.
\end{aligned}$$

□

2.7. Proof of Lemma 28

Before getting to the proof of Lemma 28 we need a few more lemmas. The first will give an approximate decomposition of elements $\vec{g}$ in BMO with support in $B(0,1)$ in terms of smooth atoms. Note that each component of these atoms satisfies the relevant conditions of Lemma 35. Moreover, each component satisfies the relevant condition of Lemma 39. The following lemma is a generalization of Lemma 22.6 of [Uch01].

Lemma 42. *Let* $\vec{g} \in BMO(\mathbb{R}^n, \mathbb{R}^d)$ *and let* $\operatorname{supp} \vec{g} \subset B(0,1)$. *Define* $z = \sup_{x\in[0,1]^n} \rho(x)$. *Then there exists a constant* $C > 0$, *independent of* $\vec{g}$, *such that there exist indexed sets* $\{\lambda_I\}_I$ *and* $\{\vec{b}_I\}_I$ *of non-negative real numbers and functions in* $\mathcal{D}(\mathbb{R}^n, \mathbb{R}^d)$ *respectively, where the index runs over all* I *of the form* $I = \delta_{2^l}\left([0,1)^n + k\right)$ *for* $l \in \mathbb{Z}$ *and* $k \in \mathbb{Z}^n$, *such that*

$$\begin{aligned} &\lambda_I = 0 \text{ and } \vec{b}_I \equiv 0 \text{ unless } I \subset B(0, C_q^2 + 1), \\ &\text{and } l \le -\log_2(C_q^2(1+z)) = j' \end{aligned} \tag{2.62}$$

$$\|\Lambda\|_{\mathcal{C}} \le C \|\vec{g}\|_{BMO}^2 \tag{2.63}$$

$$\operatorname{supp} \vec{b}_I \subset B(0, |I|) + I \tag{2.64}$$

$$\int \vec{b}_I dx = \vec{0} \tag{2.65}$$

$$\left|\nabla^2 \left[\vec{b}_I \circ \delta_{2^l}\right]\right| \le 1 \tag{2.66}$$

$$\operatorname{supp}\left(\vec{g} - \sum_I \lambda_I \vec{b}_I\right) \subset B(0, C_q^3 + 2C_q) \tag{2.67}$$

$$\left\|\vec{g} - \sum_I \lambda_I \vec{b}_I\right\|_{L^\infty} \le C \|\vec{g}\|_{BMO} \tag{2.68}$$

where $\nabla^2 \vec{f}$ is the vector consisting of all second partial derivatives of $\vec{f}$.

Proof. In this proof C is used to denote a positive constant. Different instances of C do not necessarily denote the same constant. Let $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{R})$ satisfy $\operatorname{supp} \varphi \subset B(0,1)$ and $\int_{\mathbb{R}^n} \varphi = 0$, and

$$\int_0^\infty \widehat{\varphi}(\delta_t^* \xi)^2 \frac{dt}{t} = 1 \quad \text{ if } \xi \neq 0.$$

For a cube $I_{l,k} = I$ let $T(I) = I \times (|I|/2, |I|]$ and define

$$\lambda'_{I_{l,k}} = \begin{cases} \left(|I|^{-1/2} \int \int_{T(I)} |\vec{g} * \varphi_{\delta_t}(y)|^2 \, \frac{dydt}{t} \right)^{1/2} & l \leq -\log_2(C_q^2(1+z)), \\ 0 & \text{otherwise} \end{cases}$$

$$\vec{b}'_{I_{l,k}}(x) = \begin{cases} \int \int_{T(I)} \vec{g} * \varphi_{\delta_t}(y) \varphi_{\delta_t}(x-y) \frac{dydt}{t\lambda'_{I_{l,k}}} & \lambda'_{I_{j,k}} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

First we show $\left| \nabla^2 \left[\vec{b}'_{I_{l,k}} \circ \delta_{2^l} \right] (x) \right| \leq C$. If $\lambda'_{I_{l,k}} = 0$, then this is clear. If not, then since $1 \leq 2^l/t \leq 2$ holds for $2^{l-1} \leq t \leq 2^l$, we have for $1 \leq i, j \leq n$.

$$\begin{aligned} \left|\partial_i \partial_j \left[\varphi \circ \delta_{2^l t^{-1}}\right](x - \delta_{2^{-l}} y)\right| \leq & C \left| \sup_{1 \leq a,b \leq n} \partial_a \partial_b \varphi(\delta_{2^l t^{-1}} x - \delta_{t^{-1}} y) \right| \|\delta_{2^l t^{-1}}\|^2 \\ \leq & C'. \end{aligned}$$

It then follows from the above and Hölder's inequality that for $1 \leq r \leq d$,

$$\begin{aligned}
&\left|\partial_i\partial_j\left[\vec{b}'_{l,k}\circ\delta_{2^l}\right]_r(x)\right| \\
&\quad = \left|\int\int_{T(I)}\vec{g}_r * \varphi_{\delta_t}(y)\partial_i\partial_j\left[\varphi\circ\delta_{2^l t^{-1}}\right](x-\delta_{2^{-l}}y)\frac{dydt}{t^2\lambda'_{I_{l,k}}}\right| \\
&\quad \leq \frac{C'}{\lambda'_{l,k}}\left(\int\int_{T(I)}\left|\vec{g}_r * \varphi_{\delta_t}(y)\right|^2\frac{dydt}{t}\right)^{1/2}\cdot\left(\int\int_{T(I)}\frac{dydt}{t^3}\right)^{1/2} \\
&\quad = |I|^{1/2}\cdot\frac{C''(\varphi)|I|^{1/2}}{|I|} \\
&\quad = C''(\varphi),
\end{aligned}$$

from which the result follows. Set $\vec{b}_I = \vec{b}'_I/(n^2 d C''(\varphi))$ and set $\lambda_I = n^2 d C''(\varphi)\lambda'_I$. Then (2.66) holds.

For (2.65) we note that $\int_{\mathbb{R}^n}\varphi_{\delta_t}(x-y)dx = 0$ for all $(y,t)\in T(I)$.

Turning to (2.64), since the support of φ is $B(0,1)$, we know that $\varphi_{\delta_t}(x-y)$ is nonzero only if $x-y\in B(0,t)$. In particular, if $(y,t)\in T(I)$, then $\varphi_{\delta_t}(x-y)$ is nonzero only if $x\in B(0,|I|)+I$.

For (2.62) note that because φ is supported in $B(0,1)$, the function φ_{A_t} is supported in $B(0,|I|)$ for $t\leq|I|$. It follows that the support of $\vec{g}*\varphi_{\delta_t}$ is contained in $B(0,1)+B(0,|I|)$ for that range of t. If $x\in\operatorname{supp}(\vec{g}*\varphi_{\delta_t})\cap I$ and $y\in I$ then we have

$$\begin{aligned}
\rho(y) &\leq C_q\left(\rho(x-y)+\rho(x)\right) \\
&\leq C_q\left(|I|z + C_q\left(1+|I|\right)\right) \\
&= C_q^2 + |I|\left(C_q z + C_q^2\right).
\end{aligned}$$

Thus, if $l \leq -\log_2(C_q^2(1+z))$ then I only intersects with the support of $\vec{g} * \varphi_{\delta_{|I|}}$ if $I \subset B(0, C_q^2 + 1)$.

We now show (2.63). By the definition of λ_I we have

$$\lambda_I^2 |I| = C \int_{T(I)} \frac{|\vec{g} * \varphi_{\delta_t}(y)|^2 dy dt}{t}$$

If $B = B(x_B, r)$ and S_B is the union over all $T(I)$ for dyadic cubes I such that $x_I \in B$ and $|I| \leq r$ then we have

$$\begin{aligned} \int_{B \times [0,r]} \Lambda &= \int_{S_B} \frac{|\vec{g} * \varphi_{\delta_t}(y)|^2 dy dt}{t} \\ &\leq \int_{T(B')} \frac{|\vec{g} * \varphi_{\delta_t}(y)|^2 dy dt}{t}, \end{aligned}$$

where $B' = B(x_B, rC_q(1+z))$, and $T(B') = B \times [0, rC_q(1+z)]$. Since we are integrating over $T(B')$ the support of φ_{δ_t} is contained in $B(0, rC_q(1+z))$ so that if $x \in B'$ and $\varphi_{\delta_t}(x-y)$ is nonzero then

$$\begin{aligned} \rho(y - x_B) &\leq C_q(\rho(x - x_B) + \rho(x - y)) \\ &\leq 2C_q^2 r(1+z) \\ &= r''. \end{aligned}$$

So, letting $B'' = B(x_B, r'')$ we then have

$$\begin{aligned}
\int_{T(B')} \frac{|(\vec{g} * \varphi_{\delta_t})(y)|^2 dydt}{t} &= \sum_{i=1}^{d} \int_{T(B')} \frac{|(g_i - (g_i)_{B''})\chi_{B''} * \varphi_{\delta_t}(y)|^2 dydt}{t} \\
&\leq \sum \int_{R_+^{n+1}} \frac{|(g_i - (g_i)_{B''})\chi_{B''} * \varphi_{\delta_t}(y)|^2 dydt}{t} \\
&\leq C \sum \int_{B''} |g_i(y) - (g_i)_{B''}|^2 dy \\
&\leq C \sum |B''| \, \|g_i\|_{BMO}^2 ,
\end{aligned}$$

where the last inequality is a consequence of Corollary 6.3 in [Hyt10].

It then follows that

$$\frac{1}{|B|} \int_{B \times [0,r]} \Lambda \leq C \sum_{i=1}^{d} \|g_i\|_{BMO}^2$$

which gives (2.63).

We now consider the support of $\vec{g} - \sum_I \lambda_I \vec{b}_I$ to establish (2.67). We know from (2.62) that we only need to consider those cubes I for which $|I| \leq 1/(C_q^2(1+z))$ and $I \subset B(0, C_q^2 + 1)$. Combining this with (2.64), each function b_I is supported in

$$\begin{aligned}
B(0, |I|) + I \subset & B\left(0, \frac{1}{C_q^2(1+z)}\right) + B(0, C_q^2 + 1) \\
& \subset B\left(0, C_q\left(\frac{1}{C_q^2(1+z)} + C_q^2 + 1\right)\right) \\
& \subset B\left(0, C_q^3 + 2C_q\right).
\end{aligned}$$

Since this is a larger set than the support of $\vec{g}$ we obtain

$$\operatorname{supp}\left(\vec{g} - \sum_I \lambda_I \vec{b}_I\right) \subset B(0, C_q^3 + 2C_q)$$

which is (2.67).

It remains to show (2.68). In fact, this is a straightforward consequence of

$$\vec{g}(x) - \sum \lambda_I \vec{b}_I(x) = \int_{\mathbb{R}^n \times \left(2^{j'}, \infty\right)} \vec{g} * \varphi_{\delta_t}(y) \varphi_{\delta_t}(x-y) dy dt/t$$

and

$$|\vec{g} * \varphi_{\delta_t}(y)| \leq \frac{C \|\vec{g}\|_{L^1}}{t} \leq \frac{C \|\vec{g}\|_{BMO}}{t}.$$

□

The following lemma will put together most of the results in the last several sections. Its conclusion yields the pieces that will enable us to prove Lemma 28.

Lemma 43. *Let $A > 0$ be a sufficiently large constant. Let $\vec{g} \in BMO$ satisfy* $\operatorname{supp} \vec{g} \subset B(0,1)$ *and $\|\vec{g}\|_{BMO} \leq A^{-100}$. Then there exist $\{\vec{b}_I\}_I$, and $\{\lambda_I\}_I$, as in Lemma 42 and there exist $\{\vec{p}_I\}_{I:|I|\leq 2^{j'}}$ and $\{\vec{\varphi}_j\}_{j\leq j'}$ such that the following hold:*

$$|\vec{p}_I(x)| + \left|\nabla \left[\vec{p}_I \circ \delta_{|I|}\right] (\delta_{|I|^{-1}} x)\right| \leq \frac{A}{(1+\rho(x-x_I)/|I|)^{1+\alpha_-}}, \tag{2.69}$$

$$\int \vec{p}_I(x) dx = 0, \tag{2.70}$$

$$\vec{p}_I \in S^\perp, \tag{2.71}$$

$$|\vec{\kappa}_j(x)| \leq 1 \textit{ where } \vec{\kappa}_j = \sum_{I:|I|\geq 2^j} \lambda_I (\vec{b}_I + \vec{p}_I) - \sum_{k\geq j} \vec{\varphi}_k, \tag{2.72}$$

$$|\vec{\varphi}_j(x)| \leq A^{10} \min\left\{\eta_j^{(4)}(x)^2, \|\vec{g}\|_{BMO} \eta_j^{(1)}(x)\right\}, \tag{2.73}$$

$$|\nabla [\vec{\varphi}_j \circ A_{2^j}] (x)| \leq A^{10} \|\vec{g}\|_{BMO} \eta_j^{(1)}(x), \tag{2.74}$$

$$supp \vec{\varphi}_j \subset \left\{ x \in \mathbb{R}^n : \sum_{I:|I|>2^j} \lambda_I |\vec{b}_I(x) + \vec{p}_I(x)| \geq \frac{9}{10} \right\}. \tag{2.75}$$

Proof. In this proof C is used to denote a positive constant. Different instances of C do not necessarily denote the same constant. For every dilated cube I with $|I| > 2^{j'}$ define $\vec{p}_I = 0$. For every $j > j'$ define $\vec{\varphi}_j = 0$. Note that for $j \geq j' + 1$ each of (2.69)–(2.75) holds since each function involved is zero. We will construct the desired functions for other values of j inductively. To that end, suppose $j \leq j'$ and suppose that we have constructed functions $\{\vec{p}_I\}_{|I| \geq j}$ and $\{\vec{\varphi}_k\}_{k \geq j}$ such that (2.69)–(2.75) all hold.

Let $\vec{U}(y) = y/\|y\|$. If I is a dilated cube with $|I| = 2^{j-1}$ then by applying Lemma 39 with $b(x) = -\vec{b}_I(x) \cdot \vec{U}(\vec{\kappa}_j(x_I))$ and $\vec{\nu} = \vec{U}(\vec{\kappa}_j(x_I))$ gives a C^1 function $\vec{p}_I$ in $S^{\perp}$ such that

$$\left(\vec{p}_I(x) + \vec{b}_I(x)\right) \cdot U(\vec{\kappa}_j(x_I)) = 0$$

It follows from Lemma 35 that (2.69)–(2.71) all hold.

Before constructing $\vec{\varphi}_{j-1}$ we first establish

$$|\vec{\kappa}_j(x) - \vec{\kappa}_j(y)| \leq A^2 \eta_j^{(0)}(x, y), \tag{2.76}$$

$$|\nabla \left[\vec{\kappa}_j \circ \delta_{2^j}\right] (\delta_{2^{-j}} x)| \leq A^2 \eta_j^{(2)}(x). \tag{2.77}$$

We first prove (2.76). Suppose $\rho(x - y) \leq 2^k$. Then

$$\begin{aligned}
&\left| \sum_{I:|I|=2^k} \lambda_I \left(\vec{b}_I(x) + \vec{p}_I\right) - \sum_{I:|I|=2^k} \lambda_I \left(\vec{b}_I(x) + \vec{p}_I(x)\right) \right| \\
&\quad \leq \sum \lambda_I \left(|\vec{b}_I(x) - \vec{b}_I(y)| + |\vec{p}_I(x) - \vec{p}_I(y)|\right) \\
&\quad \leq \sum \frac{CA\lambda_I |\delta_{2^{-k}}(x - y)|}{(1 + \rho(x - x_I)/|I|)^{1+\alpha_-}} \\
&\quad \leq A^2 \eta_k^{(1)}(x) |\delta_{2^{-k}}(x - y)|.
\end{aligned}$$

Also we have

$$|\vec{\varphi}_k(x) - \vec{\varphi}_k(y)| \le CA\,\|\vec{g}\|_{BMO}\,\eta_k^{(1)}(x)|\delta_{2^{-k}}(x-y)|.$$

Thus, if $\rho(x-y) \le 2^j$ then, by summing over $k \ge j$, we obtain

$$\begin{aligned}|\vec{\kappa}_j(x) - \vec{\kappa}_j(y)| &\le 2A^2 \sum_{k=j}^{\infty} |\delta_{2^{-k}}(x-y)|\eta_k^{(1)}(x)\\ &\le A^3|\delta_{2^{-j}}(x-y)|\eta_k^{(2)}(x)\end{aligned}$$

Similarly, if $\rho(x-y) \ge 2^k$ we have

$$\begin{aligned}&\left|\sum_{I:|I|=2^k} \lambda_I\left(\vec{b}_I(x) + \vec{p}_I\right) - \sum_{I:|I|=2^k} \lambda_I\left(\vec{b}_I(x) + \vec{p}_I(x)\right)\right|\\ &\le \sum \lambda_I\left(|\vec{b}_I(x) + \vec{p}_I(x)| + |\vec{b}_I(y) + \vec{p}_I(y)|\right)\\ &\le CA\left(\eta_k^{(1)}(x) + \eta_k^{(1)}(y)\right)\end{aligned}$$

and

$$|\vec{\varphi}_k(x) - \vec{\varphi}_k(y)| \le \eta_k^{(1)}(x) + \eta_k^{(1)}(y).$$

Summing over $k \ge j$ we obtain

$$\begin{aligned}|\vec{\kappa_j}(x) - \vec{\kappa_j}(y)| &\le \sum_{k=j}^{\log_2(\rho(x-y))} CA\left(\eta_k^{(1)}(x) + \eta_k^{(1)}(y)\right) + \sum_{k=\log_2(\rho(x-y))+1}^{\infty} CA\eta_k^{(1)}(x)\\ &\le A^2 \sum_{k=j}^{\log_2(\rho(x-y))} \left(\eta_k^{(2)}(x) + \eta_k^{(1)}(y)\right)\end{aligned}$$

This establishes (2.76).

Note that by (2.76) we have, for $\rho(x - \delta_{2^j} y) < 2^j$,

$$\begin{aligned}
\frac{|\left[\vec{\kappa}_j \circ \delta_{2^j}\right](\delta_{2^{-j}} x) - \left[\vec{\kappa}_j \circ \delta_{2^j}\right](y)|}{|\delta_{2^{-j}} x - y|} &\leq \frac{A^2 \eta_j^{(0)}(x, \delta_{2^j y})}{|\delta_{2^{-j}}(x - \delta_{2^j} y)|} \\
&\leq \frac{A^2 \eta_j^{(2)}(x) |\delta_{2^{-j}}(x - \delta_{2^j} y)|}{|\delta_{2^{-j}}(x - \delta_{2^j} y)|} \\
&= A^2 \eta_j^{(2)}(x),
\end{aligned}$$

from which (2.77) follows.

We next construct an auxiliary $C^1(\mathbb{R}^n, \mathbb{R})$ function ψ_j such that $|\psi_j(x)| \leq 1$ and

$$\psi_j(x) = \begin{cases} 1 & \text{if } |\vec{\kappa}_j(x)| \geq \dfrac{99}{100} \\ 0 & \text{if } |\vec{\kappa}_j(x)| \leq \dfrac{9}{10} \end{cases} \tag{2.78}$$

$$|\nabla \left[\psi_j \circ \delta_{2^j}\right](\delta_{2^{-k}} x)| \leq 1. \tag{2.79}$$

Note that by (2.77) we know that the function $\vec{\kappa}_j \circ \delta_{2^j}$ has gradient bounded above by $A^2 \left\| \eta_j^{(2)} \right\|_{L^\infty} \leq A^{-97}$. Consequently, if $\vec{\kappa}_j \circ \delta_{2^j}(x) \geq \dfrac{99}{100}$ and $\vec{\kappa}_j \circ \delta_{2^j}(y) \leq \dfrac{9}{10}$ then we have by the mean value theorem

$$|x - y| \geq \frac{9}{100 A^{-97}} \geq A > 1$$

In particular, there exists a C^1 function Ψ_j such that

$$\Psi_j(x) = \begin{cases} 1 & \text{if } |\vec{\kappa}_j \circ \delta_{2^j}(x)| \geq \dfrac{99}{100} \\ 0 & \text{if } |\vec{\kappa}_j \circ \delta_{2^j}(x)| \leq \dfrac{9}{10} \end{cases}$$

$$|\nabla \Psi_j(x)| \leq 1.$$

Setting $\psi_j(x) = \Psi_j(\delta_{2^{-j}} x)$ gives our desired function.

Define

$$\vec{\rho}(x) = \sum_{I:|I|=2^{j-1}} \lambda_I \left(\vec{b}_I(x) + \vec{p}_I(x)\right), \tag{2.80}$$

$$\vec{\tau}(x) = \vec{\kappa}_j(x) + \vec{\rho}(x), \tag{2.81}$$

$$\vec{\varphi}_{j-1}(x) = \psi_j(x)\left(|\vec{\tau}(x)| - |\vec{\kappa}_j(x)|\right) U(\vec{\tau}(x)), \tag{2.82}$$

and

$$\vec{\kappa}_{j-1}(x) = \vec{\tau}(x) - \vec{\varphi}_{j-1}(x).$$

We must show (2.72)–(2.75) hold for $\vec{\kappa}_{j-1}$ and $\vec{\varphi}_{j-1}$.

We first prove (2.72). Note that if $\vec{\kappa}_j(x) \geq \dfrac{99}{100}$ then we have

$$|\vec{\kappa}_{j-1}(x)| = |\vec{\tau}(x)| - |\vec{\tau}(x)| + |\vec{\kappa}_j(x)| = \vec{\kappa}_j(x) \leq 1.$$

If $\vec{\kappa}_j(x) \le \dfrac{99}{100}$ then we have

$$\begin{aligned}|\vec{\kappa}_{j-1}(x)| &= |(1-\psi_j(x))|\vec{\tau}(x)| + \psi_j(x)|\vec{\kappa}_j(x)| \\ &\le |\vec{\kappa}_j(x)| + |\vec{\rho}(x)| \\ &\le .99 + A^2\eta^{(1)}_{j-1}(x) \\ &\le 1.\end{aligned}$$

We next prove (2.75). Note that by construction we have

$$\begin{aligned}\operatorname{supp}\vec{\varphi}_{j-1} \subset \operatorname{supp}\vec{\psi}_j & \\ &\subset \left\{x \in \mathbb{R}^n : |\vec{\kappa}_j(x)| \ge \frac{9}{10}\right\} \\ &\subset \left\{x \in \mathbb{R}^n : \sum \lambda_I|\vec{b}_I(x) + \vec{p}_I(x)| + \sum |\vec{\varphi}_k(x)| \ge \frac{9}{10}\right\}.\end{aligned}$$

By the induction hypothesis we know the support of $\vec{\varphi}_j$ is contained in the region where $\sum \lambda_I|\vec{b}_I(x) + \vec{p}_I(x)| \ge \dfrac{9}{10}$ so that the last set above is contained in

$$\left\{x \in \mathbb{R}^n : \sum \lambda_I|\vec{b}_I(x) + \vec{p}_I(x)| \ge .9\right\}.$$

so that (2.75) holds for $\vec{\varphi}_{j-1}$.

For (2.73) and (2.74) we first rewrite $\vec{\varphi}_{j-1}$ in a more useful form. For $|\vec{\kappa}_j(x)| > 0$ we have

$$\begin{aligned}|\vec{\tau}(x)| - |\vec{\kappa}_j(x)| &= \sqrt{|\vec{\kappa}_j(x)|^2 + 2(\langle\vec{\kappa}_j(x), \vec{\rho}(x)\rangle) + |\vec{\rho}(x)|^2} - |\vec{\kappa}_j(x)| \\ &= |\vec{\kappa}_j(x)|\left(\sqrt{1 + 2|\vec{\kappa}_j(x)|^{-1}(\vec{\rho}(x)\cdot U(\vec{\kappa}_j(x))) + |\vec{\kappa}_j(x)|^{-2}|\vec{\rho}(x)|^2} - 1\right).\end{aligned}$$

For simplicity we set

$$\begin{aligned}
&v(x) = \sqrt{1+x} - 1 \\
&\mu(x) = 2|\vec{\kappa}_j(x)|^{-1}(\vec{\rho}(x) \cdot U(\vec{\kappa}_j(x))) + |\vec{\kappa}_j(x)|^{-2}|\vec{\rho}(x)|^2.
\end{aligned}$$

So, if $|\vec{\kappa}_j(x)| > 0$ then

$$|\vec{\tau}(x)| - |\vec{\kappa}_j(x)| = |\vec{\kappa}_j(x)| v(\mu(x))$$

and

$$|\vec{\varphi}_{j-1}(x)| = \begin{cases} \psi_j(x)|\vec{\kappa}_j(x)| v(\mu(x)) U(\vec{\tau}(x)) & |\vec{\kappa}_j(x)| \geq \dfrac{9}{10}, \\ 0 & \text{otherwise.} \end{cases}$$

We next establish that if $|\vec{\kappa}_j(x)| \geq \dfrac{9}{10}$ then

$$|\vec{\rho}(x) \cdot U(\vec{\kappa}_j(x))| \leq A^4 \min\left\{\eta_{j-1}^{(4)}(x)^2, \|\vec{g}\|_{BMO}^2\right\} \tag{2.83}$$

$$|\nabla\left[\vec{\rho} \circ \delta_{2^{j-1}} \cdot U(\vec{\kappa}_j \circ \delta_{2^{j-1}})\right](\delta_{2^{-(j-1)}} x)| \leq A^4 \|\vec{g}\|_{BMO}^2 \tag{2.84}$$

We first prove (2.83). We have

$$\begin{aligned}
|\vec{\rho}(x) \cdot U(\vec{\kappa}_j(x))| &= \left| \sum_{I:|I|=2^{j-1}} \lambda_I \left(\vec{b}_I(x) + \vec{p}_I(x)\right) \cdot U(\vec{\kappa}_j(x)) \right| \\
&= \left| \sum_{I:|I|=2^{j-1}} \lambda_I \left(\vec{b}_I(x) + \vec{p}_I(x)\right) \cdot \left(U(\vec{\kappa}_j(x)) - U(\vec{\kappa}_j(x_I))\right) \right| \\
&\leq \sum_{I:|I|=2^{j-1}} \lambda_I \left|\vec{b}_I(x) + \vec{p}_I(x)\right| \left|U(\vec{\kappa}_j(x)) - U(\vec{\kappa}_j(x_I))\right|.
\end{aligned}$$

Note that

$$\begin{aligned}|U(x)-U(y)| &= |U(x)(1-|x|/|y|)+(x-y)/|y|| \\ &\leq |x-y|/|y| + |1-|x|/|y|| \\ &= |x-y|/|y| + ||y|-|x||/|y| \\ &\leq 2|x-y|/|y|.\end{aligned}$$

With our assumption that $|\vec{\kappa}_j(x)| \geq \dfrac{9}{10}$ we obtain

$$\begin{aligned}&\sum_{I:|I|=2^{j-1}} \lambda_I \left|\vec{b}_I(x)+\vec{p}_I(x)\right| |U(\vec{\kappa}_j(x)) - U(\vec{\kappa}_j(x_I))| \\ &\quad\leq \sum_{I:|I|=2^{j-1}} \lambda_I \left|\vec{b}_I(x)+\vec{p}_I(x)\right| 3\,|\vec{\kappa}_j(x)-\vec{\kappa}_j(x_I)| \\ &\quad\leq \sum_{I:|I|=2^{j-1}} \frac{6A^3\lambda_I\eta_j^{(0)}(x,x_I)}{(1+\rho(x-x_I)/2^{j-1})^{1+\alpha_-}}\end{aligned}$$

from which (2.83) follows.

The proof of (2.84) is similar:

$$\begin{aligned}&|\nabla\left[\vec{\rho}\circ\delta_{2^{j-1}}\cdot U(\vec{\kappa}_j\circ\delta_{2^{j-1}})\right](\delta_{2^{-(j-1)}}x)| \\ &\quad\leq \left|\sum_{I:|I|=2^{j-1}} \lambda_I\left(U(\vec{\kappa}_j(x))-U(\vec{\kappa}_j(x_I)\right)\nabla\left(\vec{b}_I\circ\delta_{2^{j-1}}+\vec{p}_I\circ\delta_{2^{j-1}}\right)(\delta_{2^{-(j-1)}}x)\right| \\ &\qquad+\left|\sum_{I:|I|=2^{j-1}} \lambda_I\left(\vec{b}_I(x)+\vec{p}_I(x)\right)\nabla(U(\vec{\kappa}_j\circ\delta_{2^{j-1}}))(\delta_{2^{-(j-1)}}x)\right| \\ &\quad\leq \sum_{I:|I|=2^{j-1}} \frac{6A^3\lambda_I\eta_j^{(0)}(x,x_I)}{(1+\rho(x-x_I)/2^{j-1})^{1+\alpha_-}} + \sum_{I:|I|=2^{j-1}} \frac{A^4\lambda_I\eta_j^{(2)}(x)}{(1+\rho(x-x_I)/2^{j-1})^{1+\alpha_-}} \\ &\quad\leq A^5\left\|\vec{g}\right\|_{BMO}^2.\end{aligned}$$

We next show that if $|\vec{\kappa}_j(x)| \geq \dfrac{9}{10}$ then

$$|\mu(x)| \leq A^5 \min\left\{\eta_{j-1}^{(4)}(x)^2, \|\vec{g}\|_{BMO}^2\right\} \tag{2.85}$$

$$|\nabla\left[\mu \circ \delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)| \leq A^7 \|\vec{g}\|_{BMO}^2, \tag{2.86}$$

which will allow us to complete the proof of Lemma 43.

For (2.85) we apply the definition of $\vec{\rho}$ as well as our condition on $|\vec{\kappa}_j(x)|$ and (2.83).

For (2.86), we have

$$\begin{aligned}
&|\nabla\left[\mu \circ \delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)| \\
&\quad \leq 2|\vec{\kappa}_j(x)|^{-2}|\nabla\left[\vec{\kappa}_j \circ \delta_{2^{(j-1)}}\right](\delta_{2^{-(j-1)}}x)||\vec{\rho}(x)| \\
&\qquad + 2|\vec{\kappa}_j(x)|^{-1}|\nabla\left[\vec{\rho} \circ \delta_{2^{j-1}} \cdot U(\vec{\kappa}_j \circ \delta_{2^{j-1}})\right](\delta_{2^{-(j-1)}}x)| \\
&\qquad + 2|\vec{\kappa}_j(x)|^{-3}|\nabla\left[\vec{\kappa}_j \circ \delta_{2^{(j-1)}}\right](\delta_{2^{-(j-1)}}x)||\vec{\rho}(x)|^2 \\
&\qquad + 2|\vec{\kappa}_j(x)|^{-2}||\vec{\rho}(x)||\nabla\left[\vec{\rho} \circ \delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)| \\
&\quad \leq 3A^4\eta_j^{(2)}(x)\eta_{j-1}^{(1)}(x) + 3A^4\|\vec{g}\|_{BMO}^2 + 3A^4\eta_j^{(2)}(x)\eta_{j-1}^{(1)}(x) + 3A^4\eta_{j-1}^{(1)}(x)^2 \\
&\quad \leq A^7\|\vec{g}\|_{BMO}^2.
\end{aligned}$$

We prove (2.73). Note that if $|\vec{\kappa}_j(x)| < \dfrac{9}{10}$ then $\vec{\varphi}_{j-1}(x) = 0$, and that if $|\vec{\kappa}_j(x)| \geq \dfrac{9}{10}$ then we have

$$|\vec{\varphi}_{j-1}(x)| \leq |v(\mu(x))| \leq |\mu(x)| \leq A^{10} \min\left\{\eta_j^{(4)}(x)^2, \|\vec{g}\|_{BMO}\,\eta_j^{(1)}(x)\right\},$$

as desired.

Finally, we prove (2.74). We have

$$
\begin{aligned}
&|\nabla\left[\vec{\varphi}_{j-1}\circ\delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)| \\
&\quad\leq |\nabla\left[\vec{\psi}_j\circ\delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)|\vec{\kappa}_j(x)|v(\mu(x))|U(\vec{\tau}(x))| \\
&\qquad + |\psi_j(x)||\nabla\left[\vec{\kappa}_j\circ\delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)|v(\mu(x))|U(\vec{\tau}(x))| \\
&\qquad + |\psi_j(x)||\vec{\kappa}_j(x)|v'(\mu(x))\nabla\left[\mu\circ\delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}}x)|U(\vec{\tau}(x))| \\
&\qquad + |\psi_j(x)||\vec{\kappa}_j(x)|v(\mu(x))|\vec{\tau}(x)|^{-1}|\nabla\left[\vec{\tau}\circ\delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}})| \\
&\quad\leq |\mu(x)| + A^2\eta_j^{(2)}(x)|\mu(x)| + .5A^7\left\|\vec{g}\right\|_{BMO}^2 + |\mu(x)||\vec{\tau}(x)|^{-1}|\nabla\left[\vec{\tau}\circ\delta_{2^{j-1}}\right](\delta_{2^{-(j-1)}})| \\
&\quad\leq A^7\left\|\vec{g}\right\|_{BMO}^2 + 2A^5\left\|\vec{g}\right\|_{BMO}^2 \\
&\quad\leq A^8\left\|\vec{g}\right\|_{BMO}^2,
\end{aligned}
$$

as desired. This completes the proof of Lemma 43. $\square$

We now prove our final lemma. Once it is proven we will establish Lemma 28 through an iterative process.

Lemma 44. *There exists a positive constants C', C'', C''' such that if $\vec{g}$ and $S^\perp$ are as in Lemma 28, then there exist $\vec{h}\in S^\perp$ and $\vec{v}\in BMO$ satisfying*

$$\left\|\vec{h}\right\|_{BMO}\leq C'\left\|\vec{g}\right\|_{BMO}, \tag{2.87}$$

$$\left\|\vec{v}\right\|_{BMO}\leq C'\left\|\vec{g}\right\|_{BMO}^2, \tag{2.88}$$

$$\operatorname{supp}\vec{v}\subset B(0,C''), \tag{2.89}$$

$$\left|\vec{g}(x)-\vec{h}(x)-\vec{v}(x)\right|\leq 2\chi_{B(0,C''')}(x)+\chi_{B(0,C''')^c}(x)\rho(x)^{-(1+\alpha_-)}. \tag{2.90}$$

Proof. In the following proof, C denotes a positive constant. Different instances of C do not necessarily denote the same constant. Note that if we assume $\left\|\vec{g}\right\|_{BMO}>$

A^{-100} then the result is straightforward. Take $\vec{h} = 0$ and $\vec{v} = \vec{g}$ and $C' \geq A^{100}$. If $\|\vec{g}\|_{BMO}$, then we can use Lemma (43). If (2.69)–(2.75) hold the we can define

$$\vec{h} = -\sum_I \lambda_I \vec{p}_I \quad \text{and} \quad \vec{v} = \sum_{k=-\infty}^{\infty} \vec{\varphi}_k.$$

We first check that $\vec{h}$ satisfies (2.87) and is in $S^\perp$. By (2.62) and (2.63) we have

$$\sum_I \lambda_I^2 |I| = \frac{|B(0,C''')|}{|B(0,C''')|} \int_{B(0,C''')} \Lambda \leq C \|\Lambda\|_{\mathcal{C}} \leq C \|\vec{g}\|_{BMO}^2 < \infty.$$

So, since (2.69) and (2.70) also hold we can apply Lemma 37 to establish that the sum defining $\vec{h}$ converges in L^2 and BMO and that (2.34) and (2.35) hold. Combining (2.35) with (2.63) gives (2.87). The fact that $\vec{h}$ is in $S^\perp$ follows from the fact that each p_I is.

We next look at $\vec{v}$ and show (2.88) and (2.89) hold. First, because of (2.73) and because

$$\sum \left\| \eta_k^{(4)} \right\|_2^2 \leq C \sum_I \lambda_I^2 |I|$$

we know that the sum defining $\vec{v}$ converges in L^1. In particular, it is finite almost everywhere.

Since supp $\sum_I \lambda_I \vec{b}_I \subset B(0,C)$ we have, for $x \notin B(0,C'')$,

$$\begin{aligned}
\sum_I \lambda_I \left(|\vec{b}_I(x)| + |\vec{p}_I(x)| \right) &= \sum_I \lambda_I |\vec{p}_I(x)| \\
&\leq A \|\vec{g}\|_{BMO}^2 \cdot \frac{A}{(1+\rho(x-x_I)/|I|)^{1+\alpha_-}} \\
&\leq \frac{A^{-97}}{\rho(x)^{1+\alpha_-}} \\
&< \frac{9}{10},
\end{aligned}$$

so that by (2.75) we can conclude that $\vec{v}$ is supported in $B(0, C'')$ giving (2.89).

Let $B = B(x_B, r_B)$. Then we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left| \vec{v}(x) - \sum_{k=\log_2(r_B)+1}^{\infty} \vec{\varphi}_k(x_B) \right| dx \\
&\quad \leq \frac{1}{|B|}\int_B \sum_{k=-\infty}^{\log_2(r_B)} |\vec{\varphi}_k(x)|\, dx + \frac{1}{|B|}\int_B \sum_{k=\log_2(r_B)+1}^{\infty} |\vec{\varphi}_k(x) - \vec{\varphi}_k(x_B)|\, dx.
\end{aligned}$$

For the first term we have

$$\begin{aligned}
\frac{1}{|B|}\int_B \sum_{k=-\infty}^{\log_2(r_B)} |\vec{\varphi}_k(x)|\, dx &\leq \frac{1}{|B|}\int_B \sum_{k=-\infty}^{\log_2(r_B)} A^{10}\eta_k^{(4)}(x)^2 dx \\
&\leq A^{10} \left\| \sum \eta_k^{(4)}(x)^2 \delta_{t=2^k} \right\|_C \\
&\leq A^{11} \|\vec{g}\|_{BMO}^2 .
\end{aligned}$$

For the second term we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B \sum_{k=\log_2(r_B)+1}^{\infty} |\vec{\varphi}_k(x) - \vec{\varphi}_k(x_B)|\, dx \\
&\quad \leq \frac{1}{|B|}\int_B \sum_{k=\log_2(r_B)+1}^{\infty} \sup_{x\in B} |\nabla[\vec{\varphi}_k \circ \delta_{2^k}](\delta_{2^{-k}}x)| \sup_{x\in B} |\delta_{2^{-k}}(x - x_B)|\, dx \\
&\quad \leq \sum_{\log_2(r_B)+1}^{\infty} A^{10} \|\vec{g}\|_{BMO} \left\| \eta_k^{(1)} \right\|_{\infty} (2^{-k} r_B)^{\alpha-} \\
&\quad \leq C A^{10} \|\vec{g}\|_{BMO}^2 .
\end{aligned}$$

Combining the above gives (2.88).

We now prove that (2.90) holds. Note that by (2.72) for each j we have

$$\left|\sum_I \lambda_I \vec{b}_I(x) - \vec{h}(x) - \vec{v}(x)\right| = \left|\sum_{I:|I|<2^j} \lambda_I(\vec{b}_I(x) + \vec{p}_I(x)) - \sum_{k<j} \vec{\varphi}_k(x) + \vec{\kappa}(x)\right|$$
$$\leq 1 + \left|\sum_{I:|I|<2^j} \lambda_I(\vec{b}_I(x) + \vec{p}_I(x)) - \sum_{k<j} \vec{\varphi}_k(x)\right|.$$

Letting $j \to \infty$ we obtain

$$\left|\sum_I \lambda_I \vec{b}_I(x) - \vec{h}(x) - \vec{v}(x)\right| \leq 1.$$

For $x \notin B(0, C'')$ we have

$$\left|\sum_I \lambda_I \vec{b}_I(x) - \vec{h}(x) - \vec{v}(x)\right| = \left|\sum_I \lambda_I \vec{b}_I(x) - \vec{h}(x)\right| \leq \frac{A^{-97}}{(1 + \rho(x - x_I)/|I|)^{1+\alpha_-}}.$$

Combining these gives

$$\left|\sum \lambda_I \vec{b}_I(x) - \vec{h}(x) - \vec{v}(x)\right| \leq \chi_{B_A(0,C''')}(x) + \chi_{B_A(0,C''')^c}(x)\rho(x)^{-(1+\alpha_-)}.$$

Using (2.67) and (2.68) we then obtain (2.90). $\square$

We can now give a proof of Lemma 28, which completes the proof of Theorem 24.

Proof of Lemma 28. Set $C_g = (1 + C''')^{-2(1+\alpha_-)}$. We consider only $\|\vec{g}\|_{BMO} = C_g/C'$ since Lemma 28 holds under scaling. Now, we know there exist $\vec{h}_1 \in S^\perp$ and $\vec{v}_1 \in$

BMO satisfying

$$\begin{aligned}
&\left\|\vec{h}_1\right\|_{BMO} \leq C_g,\\
&\|\vec{v}_1\|_{BMO} \leq C_g^2/C',\\
&\mathrm{supp}\vec{v}_1 \subset B(0,C''),\\
&|\vec{g}(x) - \vec{h}_1(x) - \vec{v}_1(x)| \leq 2\chi_{B(0,C''')}(x) + \chi_{B(0,C''')^c}(x)\rho(x)^{-(1+\alpha_-)}.
\end{aligned}$$

We next iterate the above by replacing $\vec{g}$ with $C_g^{-1}\vec{v}(\delta_{C''}\cdot)$ to obtain functions $\vec{h}_2' \in S^{\perp}$ and $\vec{v}_2' \in BMO$ such that the above hold with the proper replacements.

In particular, if we set $\vec{h}_2(x) = C_g\vec{h'}_2(\delta_{C''}^{-1}x)$ and $\vec{v}_2(x) = C_g\vec{v'}(\delta_{C''}^{-1}x)$ then we have

$$\begin{aligned}
&\left\|\vec{h}_2\right\|_{BMO} \leq C_g^2,\\
&\|\vec{v}_2\|_{BMO} \leq C_g^3/C',\\
&\mathrm{supp}\vec{v}_2 \subset B(0,C''^2),\\
&|\vec{v}_1(x) - \vec{h}_2(x) - \vec{v}_2(x)| \leq 2C_g\left(\chi_{B(0,C'''^2)}(x) + \rho(\delta_{C'''}^{-1}x)^{-(1+\alpha_-)}\chi_{B(0,C'''^2)_c}(x)\right).
\end{aligned}$$

Repeating this process by rescaling $\vec{v}_j$ at each step so that it is contained in $B(0,1)$ and has BMO norm bounded above by C_g/C', we obtain collections

$$\left\{\vec{h}_j\right\} \text{ in } S^{\perp} \quad \text{and} \quad \{\vec{v}_j\} \text{ in } BMO$$

such that for $j \geq 1$, and with $\vec{v}_0 = \vec{g}$,

$$\left\|\vec{h}_j\right\|_{BMO} \leq C_g^j, \tag{2.91}$$

$$\|\vec{v}_j\|_{BMO} \leq C_g^{j+1}/C', \tag{2.92}$$

$$\text{supp } \vec{v}_j \subset B(0, C''^j), \tag{2.93}$$

$$\left|\vec{v}_{j-1}(x) - \vec{h}_j(x) - \vec{v}_j(x)\right| \leq 2C_g^{j-1}\left(\chi_{B(0,C'''^j)}(x) + \rho(\delta_{C'''}^{-(j-1)}x)^{-(1+\alpha_-)}\chi_{B(0,C'''^j)^c}(x)\right). \tag{2.94}$$

From (2.94) we have

$$\left|\vec{v}_{j-1}(x) - \vec{h}_j(x) - \vec{v}_j(x)\right| \leq CC_g^{j-1}(1 + C''')^{(j-1)(1+\alpha_-)}(1 + \rho(x))^{-(1+\alpha_-)}.$$

Thus, by making repeated use of the triangle inequality and recalling that $C_g = (1 + C''')^{-2(1+\alpha_-)}$, we obtain

$$\begin{aligned}\left|\vec{g}(x) - \sum_{k=1}^{j}\vec{h}_k(x) - \vec{v}_j(x)\right| &\leq \sum_{k=0}^{j-1} C(1 + C''')^{-(j-1)(1+\alpha_-)}(1 + \rho(x))^{-(1+\alpha_-)} \\ &\leq C(1 + \rho(x))^{-(1+\alpha_-)}.\end{aligned}$$

This gives us

$$|\vec{g}(x) - \sum_{k=1}^{j}\vec{h}_k(x)| \leq C(1 + \rho(x))^{-(1+\alpha_-)} + |\vec{v}_j(x)|.$$

By (2.92) and (2.93) we have

$$\|\vec{v}_j\|_1 \leq CC'''^jC_g^j \leq C(1 + C''')^{-j(1+\alpha_-)}$$

which goes to zero as $j \to \infty$.

Similarly, we have

$$\begin{aligned}\left\|\vec{h}_j\right\|_1 &\leq \left\|\vec{h}_j \chi_{B(0,C'''j)}\right\|_{BMO} + \left\|\vec{h}_j \chi_{B(0,C'''j)^c}\right\|_1 \\ &\leq 2CC_g^j C'''^j + \left\|\vec{h}_j \chi_{B(0,C'''j)^c}\right\|_1 .\end{aligned}$$

From (2.93) and (2.94) we have

$$|\vec{h}_j(x)| \leq 2C_g^{j-1} \rho(\delta_{C'''}^{-(j-1)} x)^{-(1+\alpha_-)}.$$

It follows from this that

$$\left\|\vec{h}_j \chi_{B(0,C'''j)^c}\right\|_1 \leq CC_g^{j-1} C'''^{-(j-1)(1+\alpha_-)} \leq C(1+C''')^{-j(1+\alpha_-)}$$

so that

$$\left\|\vec{h}_j\right\|_1 \leq C(1+C''')^{-j(1+\alpha_-)}.$$

Note that the right hand side above is summable over $j \geq 1$. In particular, this means that the partial sums

$$\sum_{k=1}^{j} \vec{h}_k$$

converge in L^1 to some $\vec{h}$. It follows that there is a sequence $\{l_n\}$ such that $\vec{g}(x) - \sum_{k=0}^{l_n} \vec{h}_k(x)$ converges almost everywhere to $\vec{g}(x) - \vec{h}(x)$. Combining this with the fact that $\|\vec{v}_j\|_1 \to 0$ and

$$|\vec{g}(x) - \sum_{k=1}^{j} \vec{h}_k(x)| \leq C(1+\rho(x))^{-(1+\alpha_-)} + |\vec{v}_j(x)|$$

we conclude

$$|\vec{g}(x) - \vec{h}(x)| \leq C(1+\rho(x))^{-(1+\alpha_-)}.$$

$\square$

REFERENCES CITED

[Ash76] J. M. Ash, *Studies in harmonic analysis*, 1976, MAA Studies in Mathematics. Vol. 13, pp. xv+319. MR 0442565

[BGS71] D. L. Burkholder, R. F. Gundy, and M. L. Silverstein, *A maximal function characterization of the class H^p*, Trans. Amer. Math. Soc. **157** (1971), 137–153. MR 0274767

[BH06] Marcin Bownik and Kwok-Pun Ho, *Atomic and molecular decompositions of anisotropic Triebel-Lizorkin spaces*, Trans. Amer. Math. Soc. **358** (2006), no. 4, 1469–1510. MR 2186983

[Bow03] Marcin Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. **164** (2003), no. 781, vi+122. MR 1982689

[BW] Marcin Bownik and Daniel Wang, *A PDE characterization of anisotropic Hardy spaces*, unpublished.

[Coi74] Ronald R. Coifman, *A real variable characterization of H^p*, Studia Math. **51** (1974), 269–274. MR 0358318

[CT75] A.-P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1–64. MR 0417687

[CT77] ______, *Parabolic maximal functions associated with a distribution. II*, Advances in Math. **24** (1977), no. 2, 101–171. MR 0450888

[Duo01] Javier Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001, Translated and revised from the 1995 Spanish original by David Cruz-Uribe. MR 1800316

[FR67] E. B. Fabes and N. M. Rivière, *Symbolic calculus of kernels with mixed homogeneity*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 106–127. MR 0236773

[FS72] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), no. 3-4, 137–193. MR 0447953

[Hör60] Lars Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140. MR 0121655

[Hyt10] Tuomas Hytönen, *A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa*, Publ. Mat. **54** (2010), no. 2, 485–504. MR 2675934

[Jan77] Svante Janson, *Characterizations of H^1 by singular integral transforms on martingales and* $\mathbf{R}^n$, Math. Scand. **41** (1977), no. 1, 140–152. MR 0463789

[Ste93] Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993, With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. MR 1232192

[SW78] Elias M. Stein and Stephen Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), no. 6, 1239–1295. MR 508453

[Uch82] Akihito Uchiyama, *A constructive proof of the Fefferman-Stein decomposition of BMO* ($\mathbf{R}^n$), Acta Math. **148** (1982), 215–241. MR 666111

[Uch01] ———, *Hardy spaces on the Euclidean space*, Springer Monographs in Mathematics, Springer-Verlag, Tokyo, 2001, With a foreword by Nobuhiko Fujii, Akihiko Miyachi and Kôzô Yabuta and a personal recollection of Uchiyama by Peter W. Jones. MR 1845883