A$\mathbb{T}$-ALGEBRAS FROM ZERO-DIMENSIONAL DYNAMICAL SYSTEMS

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DISSERTATION ABSTRACT

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We outline a particular type of zero-dimensional system (which we call "fiberwise essentially minimal"), which, together with the condition of all points being aperiodic, guarantee that the associated crossed product C*-algebra is an AT-algebra. Since AT-algebras of real rank zero are classifiable by K-theory, this is a large step towards a classification theorem for fiberwise essentially minimal zero-dimensional systems.

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I INTRODUCTION

A C^**-algebra* is a $*$-closed norm-closed subalgebra of the bounded operators on a Hilbert space. More abstractly, these are complex Banach algebras with an involution operation that satisfies certain properties. These mathematical objects began to be studied in detail due to their use in quantum mechanics modeling observables with self-adjoint operators; C^*-algebras are the abstract objects one can use to study these operators. For a solid introduction to C^*-algebras, see [7].

An important example of a C^*-algebra (particularly in this paper) is the C^*-algebra of continuous functions from a compact Hausdorff space X to the complex numbers $\mathbb{C}$ (with pointwise operations), which we will denote by $C(X)$. There are numerous ways of constructing C^*-algebras from other C^*-algebras or from other mathematical objects. We outline the ones relevant to this paper below. Afterwards, we outline the motivation for this paper.

Given a sequence of C^*-algebras (A_n) and a sequence of maps (φ_n) from A_n to A_{n+1}, we call (A_n, φ_n) a *direct system*. To this direct system we can associate a C^*-algebra A that encodes the information of the direct system; it contains each A_n as a subalgebra and contains the connection between A_n and A_{n+1} via the map φ_n (for a more detailed account of this construction, see [10]). The important cases for this paper are AF-algebras, which are the direct limits of systems in which A_n is finite dimensional for all n, and A$\mathbb{T}$-algebras, which are the direct limits of systems in which A_n is the direct sum of matrix algebras and matrix algebras over $C(S^1)$ (where S^1 denotes the circle; note that $\mathbb{T}$ also denotes the circle, having two notations for the circle is an unfortunate but standard practice in C^*-algebras).

Let X be a compact Hausdorff space and let $h : X \to X$ be a homeomorphism of X. We call (X, h) a *dynamical system*. There is a way of associating a C^*-algebra to a dynamical system that encodes the information of the dynamical system; we denote this associated C^*-algebra by $C^*(\mathbb{Z}, X, h)$. This C^*-algebra is generated by $C(X)$ and a unitary u that satisfies $(ufu^*)(x) = f(h^{-1}(x))$. One can see that $C(X)$ encodes the information of X and u encodes the information of h. This is an example of a crossed product C^*-algebra; for more on this subject see [12].

The underlying goal of a lot of research in the field of C^*-algebras in the last few decades is K-theoretic classification, which is the goal of showing that, for various nice classes of C^*-algebras, isomorphism at the level of K-theory implies isomorphism of the C^*-algebras. This work was pioneered by George Elliott in the 70's (see [3] for his groundbreaking work classifying the simple AF-algebras introduced by Bratteli in [1]). This work was expanded by himself, Marius Dadarlat, and Guihua Gong in the 90's (see [4], [5], and [2]), where the classification result was expanded all the way up to real rank zero AH-algebras of slow dimension growth. In particular, this classification result includes A$\mathbb{T}$-algebras of real rank zero, which is precisely the type of C^*-algebra that results from the construction in this paper.

This paper proves that crossed product C^*-algebras associated to certain zero-dimensional dynamical systems are A$\mathbb{T}$-algebras with real rank zero (the real rank zero result is not in this current version, but will be included in the near future and is certainly true), and are hence classifiable by the results described in the paragraph above. This work is motivated by Putnam's work in the minimal Cantor system case (see [9]), where he proves that crossed products associated to minimal zero-dimensional systems are A$\mathbb{T}$-algebras with real rank zero. The goal of this paper is to find a more general condition for which the techinque of the proof of Theorem 2.1 of [9] can be applied (see Definition 2.6 and Definition 2.19 in the paper below).

II PRELIMINARIES

Let X be a totally disconnected compact metrizable space and let $h : X \to X$ be a homeomorphism of X. We call (X, h) a *zero-dimensional system.* Let α be the automorphism of $C(X)$ induced by h; that is, α is defined by $\alpha(f)(x) = f(h^{-1}(x))$ for all $f \in C(X)$ and all $x \in X$. Then we denote the crossed product of $C(X)$ by α by $C^*(\mathbb{Z}, X, h)$ (or, less commonly, $C^*(\mathbb{Z}, C(X), \alpha)$). We denote the "standard unitary" of $C^*(\mathbb{Z}, X, h)$ by u, so that $ufu^* = \alpha(f)$ for all $f \in C(X)$.

Definition 2.1. Let X be a totally disconnected compact metrizable space. We define a *partition* $\mathcal{P}$ of X to be a finite set of mutually disjoint compact open subsets of X whose union is X. We denote by $C(\mathcal{P})$ the subset of $C(X)$ consisting of functions that are constant on elements of $\mathcal{P}$.

For ease of notation, we will often denote a sequence $(x_n)_{n=1}^{\infty}$ just by (x_n). That such an object is a sequence will be clear from context.

Definition 2.2. We say that a sequence $(\mathcal{P}_n)$ of partitions of X is a *generating sequence of partitions of* X if $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ for all $n \in \mathbb{Z}_{>0}$, and if for every $x \in X$, there is a sequence (V_n) such that $V_n \in \mathcal{P}_n$ for all $n \in \mathbb{Z}_{>0}$ and $\bigcap_{n=1}^{\infty} V_n = \{x\}$.

The terminology "generating sequence" comes from the fact that this sequence generates the topology of X.

Notation 2.3. Let $x \in X$. We say that x is a *periodic point* if there is a nonzero integer n such that $h^n(x) = x$; otherwise, we say that x is an *aperiodic point.* We denote the orbit of x by $\mathrm{orb}(x) = \{h^n(x) \,|\, n \in \mathbb{Z}\}$. Similarly, we denote the forward and backward orbits of x by $\mathrm{orb}^+(x) = \{h^n(x) \,|\, n \in \mathbb{Z}_{\geq 0}\}$ and $\mathrm{orb}^-(x) = \{h^n(x) \,|\, n \in \mathbb{Z}_{\leq 0}\}$.

We say that a closed subset Y of X is a *minimal set* if it is h-invariant and has no nonempty h-invariant proper closed subsets. By Zorn's lemma, minimal sets exist for every zero-dimensional system. We use the following definition from [6].

Definition 2.4. We say that a dynamical system (X, h) is an *essentially minimal system* if it has a unique minimal set.

We say that (X, h) is a *minimal system* if the unique minimal set is X. Note that essentially minimal systems have no nontrivial compact open h-invariant subsets, since such a set and its complement would contain disjoint minimal sets. Also note that we do not assume that all points must be aperiodic, so $\mathbb{Z}/n\mathbb{Z}$ with the shift homemorphism is an example of a minimal zero-dimensional system.

We will use the disjoint union symbol $\bigsqcup$ to denote unions of disjoint sets. We will not always say explicitly that the sets in this union are disjoint, as this will be implied by the notation.

Notation 2.5. Let (X, h) be a zero-dimensional system and let $U \subset X$. We define the map $\lambda_U : U \to \mathbb{Z}_{>0} \cup \{\infty\}$ by $\lambda_U(x) = \inf\{n \in \mathbb{Z}_{>0} \,|\, h^n(x) \in U\}$ if this infimum exists, and $\lambda_U(x) = \infty$ otherwise. We call this map the *first return time map* of U.

Definition 2.6. Let (X, h) be a zero-dimensional system and let $\mathcal{P}$ be a partition of X. We define a *system of finite first return time maps subordinate to* $\mathcal{P}$ to be a tuple

$$\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, (K_t)_{t=1,\ldots,T}, (Y_{k,t})_{t=1,\ldots,T;k=1,\ldots,K_t}, (J_{k,t})_{t=1,\ldots,T;k=1,\ldots,K_t})$$

such that:

(a) We have $T \in \mathbb{Z}_{>0}$.

(b) For each $t \in \{1, \ldots, T\}$, X_t is a compact open subset of X. That $\mathcal{S}$ is subordinate to $\mathcal{P}$ means that, for each $t \in \{1, \ldots, T\}$, X_t is contained in an element of $\mathcal{P}$.

(c) For each $t \in \{1, \ldots, T\}$, $K_t \in \mathbb{Z}_{>0}$.

(d) For each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, $Y_{t,k}$ is a compact open subset of X_t. Moreover, for each $t \in \{1, \ldots, T\}$, $\{Y_{t,1}, \ldots, Y_{t,K_t}\}$ is a partition of X_t; that is,

$$\bigsqcup_{k=1}^{K_t} Y_{t,k} = X_t.$$

(e) For each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, $J_{t,k} \in \mathbb{Z}_{>0}$. Using the Notation 2.5, $\{J_{t,k}\} = \lambda_{X_t}(Y_{t,k})$. Moreover, for each $t \in \{1, \ldots, T\}$, $\{h^{J_{t,1}}(Y_{t,1}), \ldots, h^{J_{t,K_t}}(Y_{t,K_t})\}$ is a partition of X_t; that is,

$$\bigsqcup_{k=1}^{K_t} h^{J_{t,k}}(Y_{t,k}) = X_t.$$

(f) The set

$$\mathcal{P}_1(\mathcal{S}) = \left\{ h^j(Y_{t,k}) \,\middle|\, t \in \{1, \ldots, T\},\ k \in \{1, \ldots, K_t\}, \text{ and } j \in \{0, \ldots, J_{t,k} - 1\} \right\}$$

is a partition of X. Note that this combined with condition (e) also implies

$$\mathcal{P}_2(\mathcal{S}) = \left\{ h^j(Y_{t,k}) \,\middle|\, t \in \{1, \ldots, T\},\ k \in \{1, \ldots, K_t\}, \text{ and } j \in \{1, \ldots, J_{t,k}\} \right\}$$

is a partition of X.

Remark 2.7. We make some comments about what the objects in Definition 2.6 mean and where the name of the system comes from. Adopt the notation of Definition 2.6. Then for each $t \in \{1, \ldots, T\}$, each $k \in \{1, \ldots, K_t\}$, and each $x \in Y_{t,k}$, we have $\lambda_{X_t}(x) = J_{t,k}$. So $J_{t,k}$ is the "first return time" of each element of $Y_{t,k}$ to X_t. We enumerate some more points below.

(a) The number of "bases" (see (b) below) of $\mathcal{S}$ is T.

(b) The "bases" of $\mathcal{S}$ are $(X_t)_{t=1}^{T}$. These are the domains of the first return time maps above.

(c) For each $t \in \{1, \ldots, T\}$, K_t is the size of the partition of X_t into sets with constant first return time to X_t.

(d) For each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, $Y_{t,k}$ is a piece of X_t that has constant first return time to X_t.

(e) For each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, $J_{t,k}$ is the first return time of $Y_{t,k}$ to X_t.

Examples 2.8. If (X, h) is a minimal zero-dimensional system, then for any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$. It is shown in the proof of Theorem 2.1 of [9] that we can take $T = 1$, and X_1 can be any compact open subset of X that is contained in an element of $\mathcal{P}$.

In the comments preceding Theorem 8.3 of [6], it is implicitly stated that if (X, h) is an essentially minimal zero-dimensional system with no periodic points, then for any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$. This can be shown using the same technique of that of the proof of Theorem 2.1 of [9] by taking $T = 1$ and taking X_1 to be any compact open subset of X that intersects the unique minimal set of (X, h) and is contained in an element of $\mathcal{P}$.

Let (X, h) be a zero-dimesional system. Theorem 3.1 gives an equivalent characterization of when, given any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$. This equivalent characterization makes it easy to construct examples of such systems.

Proposition 2.9. Let (X, h) be a zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. Then:

(a) For every $t \in \{1, \ldots, T\}$, $\bigcup_{j\in\mathbb{Z}} h^j(X_t)$ is a compact open subset of X.

(b) $\bigsqcup_{t=1}^{T} \bigcup_{j\in\mathbb{Z}} h^j(X_t) = X$.

Proof. We claim that

$$\bigsqcup_{k=1}^{K_t} \bigsqcup_{j=0}^{J_{t,k}-1} h^j(Y_{t,k}) = \bigcup_{j\in\mathbb{Z}} h^j(X_t).$$

Part (a) follows from this claim since the left-hand side of the above equation is clearly compact and open, as it is the disjoint union of finitely many compact open sets. Part (b) follows from this claim since $\bigsqcup_{t=1}^{T} \bigsqcup_{k=1}^{K_t} \bigsqcup_{j=0}^{J_{t,k}-1} h^j(Y_{t,k}) = X$. To prove the claim, fix $t \in \{1, \ldots, T\}$ and set

$$W_t = \bigsqcup_{k=1}^{K_t} \bigsqcup_{j=0}^{J_{t,k}-1} h^j(Y_{t,k}).$$

Note that condition (e) of Definition 2.6 tells us that $\bigsqcup_{k=1}^{K_t} h^{J_{t,k}}(Y_{t,k}) = X_t$. Thus, W_t is an h-invariant set that contains X_t, meaning that it must contain $\bigcup_{j\in\mathbb{Z}} h^j(X_t)$. To see that W_t contains nothing more, note that $W_t \subset \bigcup_{j\in\mathbb{Z}} \bigsqcup_{k=1}^{K_t} h^j(Y_{t,k}) = \bigcup_{j\in\mathbb{Z}} h^j(X_t)$. This proves the claim, and by the argument above, proves the proposition. □

Proposition 2.10. Let (X, h) be a zero-dimensional system, let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. Then there is a system of finite first return time maps subordinate to $\mathcal{P}$, denoted by $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$, such that

(a) $T' = T$ and, for all $t \in \{1, \ldots, T\}$, $X_t = X'_t$.

(b) Using the notation of Definition 2.6, $\mathcal{P}_1(\mathcal{S}')$ and $\mathcal{P}_2(\mathcal{S}')$ are finer than $\mathcal{P}'$.

Proof. Write $\mathcal{P}' = \{U_1, \ldots, U_R\}$. Let $t \in \{1, \ldots, T\}$ and let $k \in \{1, \ldots, K_t\}$. Define the following set:

$$A_{t,k} = \big\{Y_{t,k} \cap h^{-j}(U_r) \,\big|\, r \in \{1, \ldots, R\},\ j \in \{0, \ldots, J_{t,k}\}, \text{ and } Y_{t,k} \cap h^{-j}(U_r) \neq \varnothing\big\}.$$

We claim that the union of the elements of $A_{t,k}$ is $Y_{t,k}$. First, notice that every element is a subset of $Y_{t,k}$. Next, notice that since $\mathcal{P}'$ is a partition of X, $\bigsqcup_{r=1}^{R}(Y_{t,k} \cap U_r) = Y_{t,k}$. Thus, the claim follows. We can therefore let $\mathcal{P}_{t,k}$ be a partition of $Y_{t,k}$ such that, for every $U \in \mathcal{P}_{t,k}$ and every $V \in A_{t,k}$, we either have $U \subset V$ or $U \cap V = \varnothing$. Write $\mathcal{P}_{t,k} = \{Y_{t,k}(1), \ldots, Y_{t,k}(M_{t,k})\}$.

Set $T' = T$. For each $t \in \{1, \ldots, T'\}$, define $X'_t = X_t$ and $K'_t = \sum_{k=1}^{K_t} M_{t,k}$. For each $t \in \{1, \ldots, T'\}$ and each $k' \in \{1, \ldots, K'_t\}$, let $k \in \{1, \ldots, K_t\}$ and $m \in \{1, \ldots, M_{t,k}\}$ satisfy $k' = \sum_{l=1}^{k-1} M_{t,l} + m$, and define a compact open set $Y'_{t,k'} \subset X_t$ by $Y'_{t,k'} = Y_{t,k}(m)$ and define $J_{t,k'} = J_{t,k}$.

We now show that $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by checking the conditions of Definition 2.6. It is clear that (a), (b), and (c) are satisfied. Observe that for each $t \in \{1, \ldots, T'\}$, we have

$$\begin{aligned}
\bigsqcup_{k=1}^{K'_t} Y'_{t,k} &= \bigsqcup_{k=1}^{K_t} \bigsqcup_{m=1}^{M_{t,k}} Y_{t,k}(m) \\
&= \bigsqcup_{k=1}^{K_t} Y_{t,k} \\
&= X_t \\
&= X'_t.
\end{aligned}$$

Thus, condition (d) is satisfied. Similarly, for each $t \in \{1, \ldots, T'\}$, we have

$$\begin{aligned}
\bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) &= \bigsqcup_{k=1}^{K_t} \bigsqcup_{m=1}^{M_{t,k}} h^{J'_{t,k}}(Y_{t,k}(m)) \\
&= \bigsqcup_{k=1}^{K_t} \bigsqcup_{m=1}^{M_{t,k}} h^{J_{t,k}}(Y_{t,k}(m)) \\
&= \bigsqcup_{k=1}^{K_t} h^{J_{t,k}}(Y_{t,k}) \\
&= X_t \\
&= X'_t.
\end{aligned}$$

Thus, condition (e) is satisfied. Let $x \in X$. There are precisely one $t \in \{1, \ldots, T\}$, one $k \in \{1, \ldots, K_t\}$, and one $j \in \{0, \ldots, J_{t,k} - 1\}$ such that $x \in h^j(Y_{t,k})$. Since $\mathcal{P}_{t,k}$ is a partition of $Y_{t,k}$, there is precisely one $m \in \{1, \ldots, M_{t,k}\}$ such that $h^{-j}(x) \in Y_{t,k}(m)$. Set $k' = \sum_{l=1}^{k-1} M_{l,k} + m$, so that $Y_{t,k}(m) = Y'_{t,k}$. Then since $J'_{t,k'} = J_{t,k}$, we have precisely one $t \in \{1, \ldots, T'\}$, one $k' \in \{1, \ldots, K'_t\}$, and one $j \in \{0, \ldots, J'_{t,k'} - 1\}$ such that $x \in h^j(Y'_{t,k'})$. Thus, condition (f) is met.

We now verify the conditions of the proposition. Clearly (a) is satisfied. For (b), let $t \in \{1, \ldots T'\}$, let $k' \in \{1, \ldots, K'_t\}$, and let $j \in \{0, \ldots, J'_{t,k} - 1\}$. By definition, there is some $k \in \{1, \ldots, K_t\}$ and some $m \in \{1, \ldots, M_{t,k}\}$ such that $Y'_{t,k'} = Y_{t,k}(m)$. Since $\mathcal{P}'$ is a partition of X, it is also clear that

$$\mathcal{P}'_j = \left\{ h^{-j}(U_r) \,\middle|\, r \in \{1, \ldots, R\} \right\}$$

is a partition of X. Thus, there is some $r \in \{1, \ldots, R\}$ such that $h^{-j}(U_r) \cap Y_{t,k}$ intersects $Y_{t,k}(m)$. By the definition of $\mathcal{P}_{t,k}$, this means that $Y_{t,k}(m) \subset h^{-j}(U_r) \cap Y_{t,k}$. But then $h^j(Y'_{t,k'}) = h^j(Y_{t,k}(m)) \subset U_r$. This proves that $\mathcal{P}_1(\mathcal{S}')$ and $\mathcal{P}_2(\mathcal{S}')$ are finer than $\mathcal{P}'$. This proves the proposition. □

Lemma 2.11. Let (X, h) be a zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. Let $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S})$ be as in Definition 2.6 and let $\mathcal{S}^{(1)} = (T^{(1)}, (X_t^{(1)})_{t=1,\ldots,T^{(1)}}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}_1(\mathcal{S})$. Then there is a system $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ and a system $\mathcal{S}^{(1)\prime} = (T^{(1)\prime}, (X_t^{(1)\prime})_{t=1,\ldots,T^{(1)\prime}}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}_1(\mathcal{S})$ such that:

(a) We have $T' = T$, and for all $t \in \{1, \ldots, T'\}$, we have $X_t = X'_t$.

(b) The partition $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and the partition $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

(c) For each $s \in \{1, \ldots, T^{(1)\prime}\}$, there is a $t_s \in \{1, \ldots, T\}$ and a $k_s \in \{1, \ldots, K_{t_s}\}$ such that $X_t^{(1)\prime} = Y'_{t_s,k_s}$.

Proof. For each $s \in \{1, \ldots, T^{(1)}\}$, there is some $t_s \in \{1, \ldots, T\}$, some $k_s \in \{1, \ldots, K_{t_s}\}$, and some $j_s \in \{0, \ldots, J_{t_s,k_s} - 1\}$ such that $X_s^{(1)} \subset h^{j_s}(Y_{t_s,k_s})$. Set $T^{(1)\prime} = T^{(1)}$, set $X_s^{(1)\prime} = h^{-j_s}(X_s^{(1)})$ and set $K_s^{(1)\prime} = K_s^{(1)}$ for all $s \in \{1, \ldots, T^{(1)\prime}\}$, and set $Y_{s,k}^{(1)\prime} = h^{-j_s}(Y_{s,k}^{(1)})$ and set $J_{s,k}^{(1)\prime} = J_{s,k}^{(1)}$ for all $s \in \{1, \ldots, T^{(1)\prime}\}$ and $k \in \{1, \ldots, K_t^{(1)\prime}\}$.

We now check that $\mathcal{S}^{(1)\prime} = (T^{(1)\prime}, (X_t^{(1)\prime})_{t=1,\ldots,T^{(1)\prime}}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}_1(\mathcal{S})$ by checking each of the conditions of Definition 2.6. Conditions (a) and (c) are clearly met. By construction, for each $s \in \{1, \ldots, T^{(1)\prime}\}$, there are $t_s \in \{1, \ldots, T\}$ and $k_s \in \{1, \ldots, K_{t_s}\}$ such that $X_s^{(1)\prime} \subset Y_{t_s,k_s} \in \mathcal{P}_1(\mathcal{S})$. Thus, condition (b) is met. Since for all $s \in \{1, \ldots, T^{(1)}\}$ we have $\bigsqcup_{k=1}^{K_s} Y_{s,k}^{(1)} = X_s^{(1)}$, for all $s \in \{1, \ldots, T^{(1)\prime}\}$ we have

$$\begin{aligned} \bigsqcup_{k=1}^{K_s^{(1)\prime}} Y_{s,k}^{(1)\prime} &= \bigsqcup_{k=1}^{K_s^{(1)}} h^{-j_s}(Y_{t,k}^{(1)}) \\ &= h^{-j_s}(X_s^{(1)}) \\ &= X_s^{(1)\prime}. \end{aligned}$$

Thus, condition (d) is satisfied. For condition (e), clearly $J_{s,k} \in \mathbb{Z}_{>0}$ for all $s \in \{1, \ldots, T^{(1)\prime}\}$ and all $k \in \{1, \ldots, K_s^{(1)\prime}\}$. For each $s \in \{1, \ldots, T^{(1)\prime}\}$, we also have

$$\begin{aligned}
\bigsqcup_{k=1}^{K_s^{(1)\prime}} h^{J_{s,k}^{(1)\prime}}(Y_{s,k}^{(1)\prime}) &= \bigsqcup_{k=1}^{K_s^{(1)}} h^{J_{s,k}^{(1)}-j_s}(Y_{s,k}^{(1)}) \\
&= h^{-j_s}\left(\bigsqcup_{k=1}^{K_s^{(1)}} h^{J_{s,k}^{(1)\prime}}(Y_{s,k}^{(1)})\right) \\
&= h^{-j_s}(X_s^{(1)}) \\
&= X_s^{(1)\prime}.
\end{aligned}$$

Thus, condition (e) holds. Finally, let $x \in X$. Since $\mathcal{S}^{(1)}$ is a system of finite first return time maps subordinate to $\mathcal{P}_1(\mathcal{S})$, there are precisely one $s \in \{1, \ldots, T^{(1)}\}$, one $k \in \{1, \ldots, K_s^{(1)}\}$, and one $j \in \{0, \ldots, J_{s,k}^{(1)} - 1\}$ such that $x \in h^{j-j_s}(Y_{s,k}^{(1)}) = h^j(Y_{s,k}^{(1)\prime})$. This is all that was needed to show that $\mathcal{P}_1(\mathcal{S}^{(1)\prime})$ is a partition of X. Thus, condition (f) is satisfied, proving that $\mathcal{S}^{(1)\prime}$ is a system of finite first return time maps subordinate to $\mathcal{P}_1(\mathcal{S})$.

Now, for each $t \in \{1, \ldots, T\}$, let $A_t = \{a(t,1), \ldots, a(t, M_t)\}$ denote the set of all $s \in \{1, \ldots, T^{(1)\prime}\}$ such that $t_s = t$ and $X_s^{(1)\prime} \neq Y_{t_s,k_s}$, and let $B_t = \{b(t,1), \ldots, b(t, N_t)\}$ be the set of all $k \in \{1, \ldots, K_t\}$ such that $\left(\bigsqcup_{s=1}^{T^{(1)\prime}} X_s^{(1)\prime}\right) \cap Y_{t,k} = \varnothing$ or such that $X_s^{(1)\prime} = Y_{t_s,k_s}$. Set $T' = T$, $X'_t = X_t$ and $K'_t = K_t + M_t$ for all $t \in \{1, \ldots, T'\}$, and

$$Y'_{t,k} = \begin{cases} X_s^{(1)\prime} & \text{if } s \in A_t \text{ and } k = k_s, \\ Y_{t,k_s} \setminus X_s^{(1)\prime} & \text{if } k = K_t + m \text{ for some } m \in \{1, \ldots, M_t\}, \text{ and } s = a(t,m), \\ Y_{t,k} & \text{otherwise} \end{cases}$$

and

$$J'_{t,k} = \begin{cases} J_{t,k_s} & \text{if } k = K_t + m \text{ for some } m \in \{1, \ldots, M_t\}, \text{ and } s = a(t,m), \\ J_{t,k} & \text{otherwise} \end{cases}$$

for all $t \in \{1, \ldots, T'\}$ and $k \in \{1, \ldots, K'_t\}$. We now check that $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by checking the conditions of Definition 2.6. Conditions (a), (b), and (c) are clearly met. For each $t \in \{1, \ldots, T'\}$, we have the following, where we shorten $k_{a(t,m)}$ to $k(t,m)$:

$$\begin{aligned}
\bigsqcup_{k=1}^{K'_t} Y'_{t,k} &= \left(\bigsqcup_{m=1}^{M_t} X_{a(t,m)}^{(1)\prime}\right) \sqcup \left(\bigsqcup_{m=1}^{M_t} Y_{t,k(t,m)} \setminus X_{a(t,m)}^{(1)\prime}\right) \sqcup \left(\bigsqcup_{n=1}^{N_t} Y_{t,b(t,n)}\right) \\
&= \left(\bigsqcup_{m=1}^{M_t} Y_{t,k(t,m)}\right) \sqcup \left(\bigsqcup_{n=1}^{N_t} Y_{t,b(t,n)}\right) \\
&= X_t \\
&= X'_t.
\end{aligned}$$

Thus, condition (d) is met. Similarly, for each $t \in \{1, \ldots, T'\}$, we have the following, where we again shorten $k_{a(t,m)}$ to $k(t,m)$:

$$\begin{aligned}
\bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) &= \left(\bigsqcup_{m=1}^{M_t} h^{J_{t,k(t,m)}}(X^{(1)\prime}_{a(t,m)})\right) \sqcup \left(\bigsqcup_{m=1}^{M_t} h^{J_{t,k(t,m)}}(Y_{t,k(t,m)} \setminus X^{(1)\prime}_{a(t,m)})\right) \\
&\quad \sqcup \left(\bigsqcup_{n=1}^{N_t} h^{J_{t,b(t,n)}}(Y_{t,b(t,n)})\right) \\
&= \left(\bigsqcup_{m=1}^{M_t} h^{J_{t,k(t,m)}}(Y_{t,k(t,m)})\right) \sqcup \left(\bigsqcup_{n=1}^{N_t} h^{J_{t,b(t,n)}}(Y_{t,b(t,n)})\right) \\
&= X_t \\
&= X'_t.
\end{aligned}$$

Thus, condition (e) is met. Finally, for each $x \in X$, there are precisely one $t \in \{1, \ldots, T\}$, one $k \in \{1, \ldots, K_t\}$, and one $j \in \{0, \ldots, J_{t,k} - 1\}$ such that $x \in h^j(Y_{t,k})$. If $Y_{t,k} = Y'_{t,k}$, then $J_{t,k} = J'_{t,k}$, and so $x \in h^j(Y'_{t,k})$ for precisely one $t \in \{1, \ldots, T'\}$, one $k \in \{1, \ldots, K'_t\}$ and one $j \in \{0, \ldots, J'_{t,k} - 1\}$. Otherwise, there is some $s \in A_t$ such that $X^{(1)\prime}_s \subset Y_{t,k}$. There are now two possible cases. First, if $x \in h^j(X^{(1)\prime}_s)$, then $J_{t,k} = J'_{t,k}$, and so $x \in h^j(Y'_{t,k})$ for precisely one $t \in \{1, \ldots, T'\}$, one $k \in \{1, \ldots, K'_t\}$ and one $j \in \{0, \ldots, J'_{t,k} - 1\}$. Otherwise, if $x \in h^j(Y_{t,k} \setminus X^{(1)\prime}_s)$, then $x \in h^j(Y'_{t,K_t+m})$ where m is such that $s = a(t,m)$. In this case, we also have $J_{t,k} = J'_{t,K_t+m}$, and so $x \in h^j(Y'_{t,k'})$ for precisely one $t \in \{1, \ldots, T'\}$, one $k' \in \{1, \ldots, K'_t\}$ and one $j \in \{0, \ldots, J'_{t,k'} - 1\}$ (specifically, $k' = K_t + m$) . Thus, condition (f) holds, and so $\mathcal{S}'$ is indeed a system of finite first return time maps subordinate to $\mathcal{P}$.

We now check that the conditions in the lemma are satisfied. Clearly (a) is satisfied. It is clear that (b) is satisfied by construction as well, since for all $t' \in \{1, \ldots, T'\}$ and all $k' \in \{1, \ldots K'_{t'}\}$, there are $t \in \{1, \ldots, T\}$ and $k \in \{1, \ldots, K_t\}$ such that $Y'_{t',k'} \subset Y_{t,k}$. Finally, condition (c) is also clearly met by the way we defined the elements of $\mathcal{S}'$. $\square$

Lemma 2.12. Let (X, h) be a zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. Let $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S})$ be as in Definition 2.6 and let $\mathcal{S}^{(2)} = (T^{(2)}, (X^{(2)}_t)_{t=1,\ldots,T^{(2)}}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}_2(\mathcal{S})$. Then there is a system $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ and a system $\mathcal{S}^{(2)\prime} = (T^{(2)\prime}, (X^{(2)\prime}_t)_{t=1,\ldots,T^{(2)\prime}}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}_2(\mathcal{S})$ such that:

(a) We have $T' = T$, and for all $t \in \{1, \ldots, T'\}$, we have $X_t = X'_t$.

(b) The partition $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and the partition $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

(c) For each $s \in \{1, \ldots, T^{(2)\prime}\}$, there is a $t_s \in \{1, \ldots, T\}$ and a $k_s \in \{1, \ldots, K_{t_s}\}$ such that $X^{(2)\prime}_t = h^{J'_{t_s,k_s}}(Y'_{t_s,k_s})$.

Proof. For each $s \in \{1, \ldots, T^{(2)}\}$, there is some $t_s \in \{1, \ldots, T\}$, some $k_s \in \{1, \ldots, K_{t_s}\}$, and some $j_s \in \{1, \ldots, J_{t_s,k_s}\}$ such that $X^{(2)}_s \subset h^{j_s}(Y_{t_s,k_s})$. For convenience of notation, set $J_s = J_{t_s,k_s}$. Set

$T^{(2)\prime} = T^{(2)}$, set $X_s^{(2)\prime} = h^{J_s - j_s}(X_s^{(2)})$ and set $K_s^{(2)\prime} = K_s^{(2)}$ for all $s \in \{1, \dots, T^{(2)\prime}\}$, and set $Y_{s,k}^{(2)\prime} = h^{J_s - j_s}(Y_{s,k}^{(2)})$ and set $J_{s,k}^{(2)\prime} = J_{s,k}^{(2)}$ for all $s \in \{1, \dots, T^{(2)\prime}\}$ and $k \in \{1, \dots, K_t^{(2)\prime}\}$.

We now check that $\mathcal{S}^{(1)\prime} = (T^{(1)\prime}, (X_t^{(1)\prime})_{t=1,\dots,T^{(1)\prime}}, \dots)$ is a system of finite first return time maps subordinate to $\mathcal{P}_2(\mathcal{S})$ by checking each of the conditions of Definition 2.6. Conditions (a) and (c) are clearly met. By construction, for each $s \in \{1, \dots, T^{(2)\prime}\}$, there are $t_s \in \{1, \dots, T\}$ and $k_s \in \{1, \dots, K_{t_s}\}$ such that $X_s^{(2)\prime} \subset h^{J_s}(Y_{t_s,k_s}) \in \mathcal{P}_2(\mathcal{S})$. Thus, condition (b) is met. Since for all $s \in \{1, \dots, T^{(2)}\}$ we have $\bigsqcup_{k=1}^{K_s} Y_{s,k}^{(2)} = X_s^{(2)}$, for all $s \in \{1, \dots, T^{(2)\prime}\}$ we have

$$\begin{aligned}
\bigsqcup_{k=1}^{K_s^{(2)\prime}} Y_{s,k}^{(2)\prime} &= \bigsqcup_{k=1}^{K_s^{(2)}} h^{j_s}(Y_{t,k}^{(2)}) \\
&= h^{j_s}\left(\bigsqcup_{k=1}^{K_s^{(2)}} Y_{t,k}^{(2)}\right) \\
&= h^{j_s}(X_s^{(2)}) \\
&= X_s^{(2)\prime}.
\end{aligned}$$

Thus, condition (d) is satisfied. For condition (e), clearly $J_{s,k} \in \mathbb{Z}_{>0}$ for all $s \in \{1, \dots, T^{(2)\prime}\}$ and all $k \in \{1, \dots, K_s^{(2)\prime}\}$. For each $s \in \{1, \dots, T^{(2)\prime}\}$, we also have

$$\begin{aligned}
\bigsqcup_{k=1}^{K_s^{(2)\prime}} h^{J_{s,k}^{(2)\prime}}(Y_{s,k}^{(2)\prime}) &= \bigsqcup_{k=1}^{K_s^{(2)}} h^{J_{s,k}^{(2)} + J_s - j_s}(Y_{s,k}^{(2)}) \\
&= h^{J_s - j_s}\left(\bigsqcup_{k=1}^{K_s^{(2)}} h^{J_{s,k}^{(2)\prime}}(Y_{s,k}^{(2)})\right) \\
&= h^{J_s - j_s}(X_s^{(2)}) \\
&= X_s^{(2)\prime}.
\end{aligned}$$

Thus, condition (e) holds. Finally, let $x \in X$. Since $\mathcal{S}^{(2)}$ is a system of finite first return time maps subordinate to $\mathcal{P}_2(\mathcal{S})$, there is precisely one $s \in \{1, \dots, T^{(2)}\}$, one $k \in \{1, \dots, K_s^{(2)}\}$, and one $j \in \{0, \dots, J_{s,k}^{(2)} - 1\}$ such that $h^{j_s - J_s}(x) \in h^j(Y_{s,k}^{(2)})$, and hence $x \in h^{j + J_s - j_s}(Y_{s,k}^{(2)}) = h^j(Y_{s,k}^{(2)\prime})$. This is all that was needed to show that $\mathcal{P}_1(\mathcal{S}^{(2)\prime})$ is a partition of X. Thus, condition (f) is satisfied, proving that $\mathcal{S}^{(2)\prime}$ is a system of finite first return time maps subordinate to $\mathcal{P}_2(\mathcal{S})$.

Now, for each $t \in \{1, \dots, T\}$, let $A_t = \{a(t, 1), \dots, a(t, M_t)\}$ denote the set of all $s \in \{1, \dots, T^{(2)\prime}\}$ such that $t_s = t$ and $X_s^{(2)\prime} \neq h^{J_s}(Y_{t_s,k_s})$, and let $B_t = \{b(t, 1), \dots, b(t, N_t)\}$ be the set of all $k \in \{1, \dots, K_t\}$ such that $\left(\bigsqcup_{s=1}^{T^{(2)\prime}} X_s^{(2)\prime}\right) \cap Y_{t,k} = \varnothing$ or such that there is a $s \in \{1, \dots, T^{(2)\prime}\}$ such that

$X_s^{(2)\prime} = Y_{t,k}$. Set $T' = T$, set $X'_s = X_s$ and $K'_s = K_s + M_s$ for all $t \in \{1, \ldots, T'\}$, and set

$$Y'_{t,k} = \begin{cases} h^{-J_s}(X_s^{(2)\prime}) & \text{if } s \in A_t \text{ and } k = k_s, \\ Y_{t,k_s} \setminus h^{-J_s}(X_s^{(2)\prime}) & \text{if } k = K_t + m \text{ for some } m \in \{1, \ldots, M_t\}, \text{ and } s = a(t,m), \\ Y_{t,k} & \text{otherwise} \end{cases}$$

and

$$J'_{t,k} = \begin{cases} J_{t,k_s} & \text{if } k = K_t + m \text{ for some } m \in \{1, \ldots, M_t\}, \text{ and } s = a(t,m), \\ J_{t,k} & \text{otherwise} \end{cases}$$

for all $t \in \{1, \ldots, T'\}$ and $k \in \{1, \ldots, K'_t\}$. We now check that $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by checking the conditions of Definition 2.6. Conditions (a), (b), and (c) are clearly met. For each $t \in \{1, \ldots, T'\}$, we have the following, where we shorten $k_{a(t,m)}$ to $k(t,m)$.

$$\begin{aligned} \bigsqcup_{k=1}^{K'_t} Y'_{t,k} &= \left(\bigsqcup_{m=1}^{M_t} h^{-J_{a(t,m)}}(X_{a(t,m)}^{(2)\prime}) \right) \sqcup \left(\bigsqcup_{m=1}^{M_t} Y_{t,k(t,m)} \setminus h^{-J_{a(t,m)}}(X_{a(t,m)}^{(2)\prime}) \right) \sqcup \left(\bigsqcup_{n=1}^{N_t} Y_{t,b(t,n)} \right) \\ &= \left(\bigsqcup_{m=1}^{M_t} Y_{t,k(t,m)} \right) \sqcup \left(\bigsqcup_{n=1}^{N_t} Y_{t,b(t,n)} \right) \\ &= X_t \\ &= X'_t, \end{aligned}$$

Thus, condition (d) is met. Similarly, for each $t \in \{1, \ldots, T'\}$, we have the following, where we again shorten $k_{a(t,m)}$ to $k(t,m)$.

$$\begin{aligned} \bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) &= \left(\bigsqcup_{m=1}^{M_t} h^{J_{t,k(t,m)}} \left(h^{-J_{a(t,m)}}(X_{a(t,m)}^{(2)\prime}) \right) \right) \sqcup \left(\bigsqcup_{m=1}^{M_t} h^{J_{t,k(t,m)}} \left(Y_{t,k(t,m)} \setminus h^{-J_{a(t,m)}}(X_{a(t,m)}^{(2)\prime}) \right) \right) \\ &\quad \sqcup \left(\bigsqcup_{n=1}^{N_t} h^{J_{t,b(t,n)}}(Y_{t,b(t,n)}) \right) \\ &= \left(\bigsqcup_{m=1}^{M_t} h^{J_{t,k(t,m)}}(Y_{t,k(t,m)}) \right) \sqcup \left(\bigsqcup_{n=1}^{N_t} h^{J_{t,b(t,n)}}(Y_{t,b(t,n)}) \right) \\ &= X_t \\ &= X'_t, \end{aligned}$$

Thus, condition (e) is met. Finally, for each $x \in X$, there are $t \in \{1, \ldots, T\}$, $k \in \{1, \ldots, K_t\}$, and $j \in \{0, \ldots, J_{t,k} - 1\}$ such that $x \in h^j(Y_{t,k})$. If $Y_{t,k} = Y'_{t,k}$, then $J_{t,k} = J'_{t,k}$, and so $x \in h^j(Y'_{t,k})$ for precisely one $t \in \{1, \ldots, T'\}$, one $k \in \{1, \ldots, K'_t\}$ and one $j \in \{0, \ldots, J'_{t,k} - 1\}$. Otherwise, there is some $s \in A_t$ such that $h^{-J_s}(X_s^{(2)\prime}) \subset Y_{t,k}$. There are now two possible cases. First, if $x \in h^{j-J_s}(X_s^{(2)\prime})$, then $J_{t,k} = J'_{t,k}$, and so $x \in h^j(Y'_{t,k})$ for precisely one $t \in \{1, \ldots, T'\}$, one $k \in \{1, \ldots, K'_t\}$ and one

$j \in \{0, \ldots, J'_{t,k} - 1\}$. Otherwise, if $x \in h^j(Y_{t,k} \setminus h^{-J_s}(X_s^{(2)\prime}))$, then $x \in h^j(Y'_{t,K_t+m})$ where m is such that $s = a(t, m)$. In this case, we also have $J_{t,k} = J'_{t,K_t+m}$, and so $x \in h^j(Y'_{t,k'})$ for precisely one $t \in \{1, \ldots, T'\}$, one $k' \in \{1, \ldots, K'_t\}$ and one $j \in \{0, \ldots, J'_{t,k'} - 1\}$ (specifically, $k' = K_t + m$). Thus, condition (f) holds, and so $\mathcal{S}'$ is indeed a system of finite first return time maps subordinate to $\mathcal{P}$.

We now check that the conditions in the lemma are satisfied. Clearly (a) is satisfied. It is clear that (b) is satisfied by construction as well, since for all $s' \in \{1, \ldots, T'\}$ and all $k' \in \{1, \ldots K'_{s'}\}$, there are $s \in \{1, \ldots, T\}$ and $k \in \{1, \ldots, K_s\}$ such that $Y'_{s',k'} \subset Y_{s,k}$. Finally, condition (c) is also clearly met by the way we defined the elements of $\mathcal{S}'$. □

Lemma 2.13. Let (X, h) be a zero-dimensional system such that, for any partition $\mathcal{R}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{R}$. Let $\mathcal{P}$ and $\mathcal{P}'$ be partitions of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. Then there is a system $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}'$ such that, for each $t' \in \{1, \ldots, T'\}$, there is a $t \in \{1, \ldots, T\}$ such that $X'_{t'} \subset X_t$.

Proof. By Proposition 2.10, there is a system

$$\mathcal{S}^{(0)} = (T^{(0)}, (X_t^{(0)})_{t=1,\ldots,T^{(0)}}, \ldots)$$

of finite first return time maps subordinate to $\mathcal{P}$ such that $T^{(0)} = T$, $\mathcal{P}_1(\mathcal{S}^{(0)})$ is finer than $\mathcal{P}'$, and for all $t \in \{1, \ldots, T\}$, $X_t^{(0)} = X_t$.

Now, by Lemma 2.11, there is some system

$$\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$$

of finite first return time maps subordinate to $\mathcal{P}_1(\mathcal{S}^{(0)})$ such that for all $t' \in \{1, \ldots, T'\}$, there is some $t \in \{1, \ldots, T\}$ such that $X'_{t'} \subset X_t$. Since $\mathcal{S}'$ is subordinate to $\mathcal{P}_1(\mathcal{S}^{(0)})$ and since $\mathcal{P}_1(\mathcal{S}^{(0)})$ is finer than $\mathcal{P}'$, the conclusion follows. □

Lemma 2.14. Let (X, h) be a zero-dimensional system and let $(\mathcal{P}_n)$ be a generating sequence of partitions of X. Let $(\mathcal{P}'_n)$ be a sequence of partitions such that, for every $n \in \mathbb{Z}_{>0}$, $\mathcal{P}'_{n+1}$ is finer than $\mathcal{P}'_n$, and for every $n \in \mathbb{Z}_{>0}$, there is some $m_n \in \mathbb{Z}_{>0}$ such that $\mathcal{P}'_{m_n}$ is finer than $\mathcal{P}_n$. Then $(\mathcal{P}'_n)$ is a generating sequence of partitions of X.

Proof. Let $x \in X$ and let (V_n) be a sequence such that $V_n \in \mathcal{P}_n$ for all $n \in \mathbb{Z}_{>0}$ and $\bigcap_{n=1}^\infty V_n = \{x\}$. We inductively construct a sequence (U_m) such that $U_m \in \mathcal{P}'_m$ for all $m \in \mathbb{Z}_{>0}$ and $\bigcap_{m=1}^\infty U_m = \{x\}$. First, by assumption, there is an $m_1 \in \mathbb{Z}_{>0}$ such that $\mathcal{P}'_{m_1}$ is finer than $\mathcal{P}_1$. We can therefore choose $U_1, \ldots, U_{m_1}$ such that $U_1 \supset \cdots \supset U_{m_1}$, $U_{m_1} \subset V_1$, and $x \in U_m \in \mathcal{P}_m$ for all $m \in \{1, \ldots, m_1\}$. Next, there is an $m_2 \in \mathbb{Z}_{>0}$ such that $\mathcal{P}'_{m_2}$ is finer than $\mathcal{P}_2$. Since $\mathcal{P}'_{m+1}$ is finer than $\mathcal{P}'_m$ for all $m \in \mathbb{Z}_{>0}$, we are free to assume that $m_2 > m_1$. We can therefore choose $U_{m_1+1}, \ldots, U_{m_2}$ such that $U_{m_1+1} \supset \cdots \supset U_{m_2}$, $U_{m_2} \subset V_2$, and $x \in U_m \in \mathcal{P}_m$ for all $m \in \{m_1 + 1, \ldots, m_2\}$. Repeating this process yields (U_m), proving the lemma. □

Construction 2.15. Let (X,h) be a zero-dimensional system such that, for any partition $\mathcal{P}$ of X, (X,h) admits a system of finite first return time maps subordinate to $\mathcal{P}$. Let $(\mathcal{P}^{(n)})$ be a generating sequence of partitions of X. Using Proposition 2.10, we choose a system $\mathcal{S}^{(1)} = (T^{(1)}, (X_t^{(1)})_{t=1,\ldots,T^{(1)}}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}^{(1)}$ such that $\mathcal{P}_1(\mathcal{S}^{(1)})$ is finer than $\mathcal{P}^{(2)}$.

We construct a sequence of systems of finite first return time maps inductively. Let n be an integer such that $n \geq 2$ and use Lemma 2.13 with $\mathcal{S}^{(n-1)}$ in place of $\mathcal{S}$, $\mathcal{P}^{(n-1)}$ in place of $\mathcal{P}$, and $\mathcal{P}^{(n)}$ in place of $\mathcal{P}'$ to get a system $\mathcal{S}^{(n)\prime}$ of finite first return time maps subordinate to $\mathcal{P}^{(n)}$ such that, for every $t' \in \{1,\ldots,T^{(n)\prime}\}$, there is a $t \in \{1,\ldots,T^{(n-1)}\}$ such that $X_{t'}^{(n)\prime} \subset X_t^{(n)}$. Then apply Proposition 2.10 with $\mathcal{P}^{(n)}$ in place of both $\mathcal{P}$ and $\mathcal{P}'$ and with $\mathcal{S}^{(n)\prime}$ in place of $\mathcal{S}$ to get a system $\mathcal{S}^{(n)} = (T^{(n)}, (X_t^{(n)})_{t=1,\ldots,T^{(n)}}, \ldots)$ such that $T^{(n)} = T^{(n)\prime}$, $X_t^{(n)} = X_t^{(n)\prime}$ for all $t \in \{1,\ldots,T^{(n)}\}$, and $\mathcal{P}_1(\mathcal{S}^{(n)})$ is finer than $\mathcal{P}^{(n)}$. By Lemma 2.14, $(\mathcal{P}_1(\mathcal{S}^{(n)}))$ is a generating sequence of partitions, since, for all $n \in \mathbb{Z}_{>0}$, $\mathcal{P}_1(\mathcal{S}^{(n)})$ is finer than $\mathcal{P}^{(n+1)}$.

Let $x_1, x_2 \in X$. We say that $x_1 \sim x_2$ if and only if there exists a sequence (t_n) where $t_n \in \{1,\ldots,T^{(n)}\}$ for all $n \in \mathbb{Z}_{>0}$ such that $x_1, x_2 \in \bigcap_{n=1}^{\infty} \bigcup_{j\in\mathbb{Z}} h^j(X_{t_n}^{(n)})$. Define a set $Z \subset X$ by $Z = \bigcap_{n=1}^{\infty} \bigsqcup_{t=1}^{T^{(n)}} X_t^{(n)}$.

Remark 2.16. Adopt the notation of Construction 2.15. We remark that, for most choices of (t_n), the set $\bigcap_{n=1}^{\infty} \bigcup_{j\in\mathbb{Z}} h^j(X_{t_n}^{(n)})$ will be empty. In fact, it is nonempty if and only if, for every positive integer n with $n \geq 2$, we have $X_{t_n}^{(n)} \subset X_{t_{n-1}}^{(n-1)}$; if $X_{t_n}^{(n)} \not\subset X_{t_{n-1}}^{(n-1)}$, then by construction, we have $X_{t_n}^{(n)} \cap X_{t_{n-1}}^{(n-1)} = \varnothing$. Another thing to notice is that since for every $n \in \mathbb{Z}_{>0}$ the sets $X_1^{(n)}, \ldots, X_{T^{(n)}}^{(n)}$ are pairwise disjoint, the sequence (t_n) correponding to an equivalence class is unique. Finally, we can see that z is in Z if and only if there is a sequence (t_n) such that $z \in \bigcap_n X_{t_n}^{(n)}$.

Lemma 2.17. The relation $\sim$ from Construction 2.15 is an equivalence relation.

Proof. The only thing that is nonobvious about whether or not this is an equivalence relation is whether or not all elements of X have an equivalence class. But by Proposition 2.9, for every $n \in \mathbb{Z}_{>0}$, there is some $t \in \{1,\ldots,T^{(n)}\}$ such that $x \in \bigcup_{j\in\mathbb{Z}} h^j(X_t^{(n)})$. Thus, $\sim$ is indeed an equivalence relation on X. $\square$

The above equivalence relation will be referenced later, and is important for the proof of Theorem 3.1.

Lemma 2.18. The set Z in Construction 2.15 is a closed subset of X that contains exactly one element from each equivalence class of $\sim$.

Proof. It is clear that Z is a closed subset of X, as it is defined to be the intersection of closed subsets of X.

We now show that Z contains precisely one element from each equivalence class. To see this, first let (t_n) be a sequence such that, for all $n \in \mathbb{Z}_{>0}b$, we have $t_n \in \{1,\ldots,T^{(n)}\}$ and $X_{t_{n+1}}^{(n+1)} \subset X_{t_n}^{(n)}$. Then $(X_{t_n}^{(n)})$ is a decreasing sequence of nonempty compact open subsets of X, and since the union of $(\mathcal{P}^{(n)})$ generates the topology of X, $\bigcap_{n=1}^{\infty} X_{t_n}^{(n)}$ contains exactly one element, which is certainly in Z. If $x' \in X$ is another element in the same equivalence class as x, then $x' \in \bigcap_{n=1}^{\infty} \bigcup_{j\in\mathbb{Z}} h^j(X_{t_n}^{(n)})$. If we also have $x' \in Z$, then $x' \in \bigcap_{n=1}^{\infty} X_{t_n}^{(n)}$, so $x' = x$. Thus, Z indeed contains precisely one element from each equivalence class. $\square$

The above set Z will be referenced often throughout this paper and is important to the structural properties of (X, h). As with the equivalence relation defined earlier in this construction, Z is important for the proof of Theorem 3.1.

Definition 2.19. Let (X, h) be a zero-dimensional system. We say that (X, h) is *fiberwise essentially minimal* if there is a closed subset $Z \subset X$ and a continuous map $\psi : X \to Z$ such that

(a) $\psi|_Z : Z \to Z$ is the identity map.

(b) $\psi \circ h = \psi$.

(c) For each $z \in Z$, $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$ is an essentially minimal system and z is in its minimal set.

Example 2.20. We provide some examples of fiberwise essentially minimal zero-dimensional systems.

(i) Any essentially minimal zero-dimensional system is a fiberwise essentially minimal zero-dimensional system. We can take Z to be $\{z\}$ for any z in the minimal set of X and then $\psi : X \to Z$ is the map $x \mapsto z$.

(ii) Let Z be a compact metrizable totally disconnected space and let (Y, h) be an essentially minimal zero-dimensional system. Let $X = Y \times Z$ and let $h' = \mathrm{id} \times h$. Then (X, h') is an essentially minimal system, where we take $\psi : X \to Z$ to be the map $(y, z) \mapsto z$.

(iii) Let $Z = \mathbb{Z} \cup \{\infty\}$, let (Y, h) be an essentially minimal zero-dimensional system. Let $X = (Y \times \mathbb{Z})/(Y \times \{\infty\})$ and let $h' : X \to X$ be the image of $\mathrm{id} \times h$ under the quotient map. Then (X, h') is an essentially minimal system where ψ is the image of $(y, z) \mapsto (h(y), z)$ under the quotient map.

III THEOREMS

We now introduce the main theorems of the paper. These theorems will take significant work to prove, and their proofs will be located in the following sections of the paper.

Theorem 3.1. Let (X, h) be a zero-dimensional system. Then (X, h) is fiberwise essentially minimal if and only if for any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$.

The idea for Definition 2.6 came from an attempt to decipher what elements of minimality are used in the proof of Theorem 2.1 of [9], which states that the C^*-algebras associated to minimal zero-dimensional systems are A$\mathbb{T}$-algebras. As we will soon see, for a zero dimensional system (X, h), the condition that given any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$, can be used to prove that $C^*(\mathbb{Z}, X, h)$ is an A$\mathbb{T}$-algebra. Without Theorem 3.1, it is difficult to construct or visualize zero-dimensional systems that satisfy this condition. However, it is not too hard to construct examples of fiberwise essentially minimal zero-dimensional systems, as Example 2.20 illustrates.

Theorem 3.2. Let (X, h) be a fiberwise essentially minimal zero-dimensional system with no periodic points. Then $C^*(\mathbb{Z}, X, h)$ is an A$\mathbb{T}$-algebra.

Note that a later version of this paper will also prove that $C^*(\mathbb{Z}, X, h)$ has real rank zero, and is hence classifiable by [2].

IV PROOF OF THEOREM 3.1

Proposition 4.1. Let (X, h) be a zero-dimensional system, let U be a compact open subset of X, and let λ_U be as in Notation 2.5. Then λ_U is continuous.

Proof. Let $x_0 \in U$ satisfy $\lambda_U(x_0) = m$ for some $m \in \mathbb{Z}_{>0}$. We show that λ_U is continuous at x_0. We claim that since U is open, λ_U is upper semi-continuous at x_0. Since $h^m(x_0) \in U$, we have $x_0 \in h^{-m}(U)$. Then, since h^{-m} is continuous, $h^{-m}(U)$ is an open set in X, and so is $h^{-m}(U) \cap U$. For all $x \in h^{-m}(U) \cap U$, we have $h^m(x) \in U$, and so $\lambda_U(x) \leq m = \lambda_U(x_0)$. Thus, λ_U is upper semicontinuous at x_0. We claim that since U is closed, λ_U is lower semi-continuous at x_0. Suppose that λ_U is not lower semi-continuous at x_0. This means that there is a sequence (y_n) in U such that $y_n \to x_0$ and $\lambda_U(y_n) < \lambda_U(x_0)$ for all $n \in \mathbb{Z}_{>0}$. So, since $\{\lambda_U(y_n) \mid n \in \mathbb{Z}_{>0}\} \subset \{1, \ldots, \lambda_U(x_0) - 1\}$, which is a finite set, there is a subsequence (x_{n_k}) of (x_n) such that $(\lambda_U(x_{n_k}))$ is a constant sequence. Say that $\lambda_U(x_{n_k}) = m$ for all $k \in \mathbb{Z}_{>0}$. But then (x_{n_k}) is a sequence in $h^{-m}(U)$, which is closed since h^{-m} is continuous. Since $x_{n_k} \to x_0$, we conclude that $x_0 \in h^{-m}(U)$. This is a contradiction to $\lambda_U(x_n) < \lambda_U(x_0)$ for all $n \in \mathbb{Z}_{>0}$. Thus, λ_U is lower semi-continuous at x_0, and therefore continuous at x_0.

Now let $x_0 \in U$ satisfy $\lambda_U(x_0) = \infty$. We now show that λ_U is continuous at x_0. Suppose not; that is, suppose there is a sequence (x_n) in U converging to x_0 such that that $\lim_{n\to\infty} \lambda_U(x_n) \neq \infty$. By passing to a subsequence, we may assume that $(\lambda_U(x_n))$ is bounded, and then by passing to another subsequence, we may assume that $(\lambda_U(x_n))$ is constant, say equal to k. This means that $h^k(x_n) \in U$ for all $n \in \mathbb{Z}_{>0}$. But then since h^k is continuous, $\lim_{n\to\infty} h^k(x_n) = h^k(x_0)$. Since U is closed, $\lim_{n\to\infty} h^k(x_n) \in U$. Thus, $h^k(x_0) \in U$, a contradiction to $\lambda_U(x_\infty) = \infty$. Thus, λ_U is continuous at x_0.

Altogether, we see that λ_U is continuous. □

The following proposition is contained in Theorem 1.1 of [6].

Proposition 4.2. Let (X, h) be an essentially minimal zero-dimensional system, let Y be its unique minimal set, let $y \in Y$, let U be a compact open neighborhood of y, and let λ_U be as in Notation 2.5. Then we have the following:

(a) $\bigcup_{j\in\mathbb{Z}} h^j(U) = X$.

(b) $\operatorname{ran}(\lambda_U)$ is a finite subset of $\mathbb{Z}_{>0}$.

(c) $U = \{h^{\lambda_U(x)}(x) \,\big|\, x \in U\}$.

(d) $\bigcup_{j\in\mathbb{Z}_{>0}} h^j(U) = \bigcup_{j\in\mathbb{Z}_{<0}} h^j(U) = X$.

Proof. Part (a) is contained in Theorem 1.1 of [6], but we include the proof for the convenience of the reader. First, note that $\bigcup_{j\in\mathbb{Z}} h^j(U)$ is an h-invariant open set in X, which means $X \setminus \bigcup_{j\in\mathbb{Z}} h^j(U)$ is an h-invariant closed set. Since this h-invariant closed set must contain a minimal set, and y is not in this set, we conclude that (X, h) would not have a unique minimal set unless $X \setminus \bigcup_{j\in\mathbb{Z}} h^j(U)$ was empty.

For part (b), let $x \in U$. Theorem 1.1 of [6] implies that there is some $j \in \mathbb{Z}_{>0}$ such that $h^j(x) \in U$. Thus, the range of λ_U is contained in $\mathbb{Z}_{>0}$. By Proposition 4.1, λ_U is continuous, and since U is compact, λ_U therefore has finite range.

For part (c), define a map $\widetilde{\lambda}_U : U \to \mathbb{Z}_{<0} \cap \{-\infty\}$ by $\widetilde{\lambda}_U(x) = \sup\{n \in \mathbb{Z}_{<0} \,\big|\, h^n(x) \in U\}$ if this supremum exists, and $\widetilde{\lambda_U}(x) = -\infty$ otherwise. By Theorem 1.1 of [6], for each $x \in U$, there is some $j \in \mathbb{Z}_{<0}$ such that $h^j(x) \in U$. Thus, the range of $\widetilde{\lambda}_U$ is contained in $\mathbb{Z}_{<0}$. To see that $\widetilde{\lambda}_U$ is continuous, we apply Proposition 4.1 with h^{-1} in place of h. Thus, $\widetilde{\lambda}_U$ has finite range. Write $\mathrm{ran}(\widetilde{\lambda_U}) = \{J_1, \ldots, J_K\}$ and let $k \in \{1, \ldots, K\}$. If $x \in \widetilde{\lambda}_U^{-1}(J_k)$, then it is easy to see that we must have $\lambda_U(x) = -J_k$. Thus, $\mathrm{ran}(\lambda_U) = \{-J_1, \ldots, -J_K\}$, and so we get

$$\{h^{\lambda_U}(x) \,\big|\, x \in U\} = \bigcup_{k=1}^{K} h^{-J_k}(\lambda_U^{-1}(-J_k)) = \bigcup_{k=1}^{K} \widetilde{\lambda}_U^{-1}(J_k) = U.$$

Part (d) is an immediate consequence of (a) and (c). $\square$

Proof of Theorem 3.1. $(\Rightarrow)$. Let Z and ψ be as in Definition 2.19. Let $\mathcal{P}$ be a partition of X. Let $X_1', \ldots, X_T'$ be the elements of $\mathcal{P}$ with nontrivial intersection with Z. For each $t \in \{1, \ldots, T\}$, set $X_t = \psi^{-1}(X_t' \cap Z) \cap X_t'$. Since $X_1', \ldots, X_T'$ are pairwise disjoint, it follows that $X_1, \ldots, X_T$ are also pairwise disjoint.

Fix $t \in \{1, \ldots, T\}$. Let $\lambda_{X_t} : X_t \to \mathbb{Z}_{>0}$ be as in Notation 2.5 and set $Z_t = X_t \cap Z$. Let $z \in Z_t$ and set $V_z = X_t \cap \psi^{-1}(z)$. Note that V_z is a compact open neighborhood of z in $\psi^{-1}(z)$. Since $h|_{\psi^{-1}(z)}$ is essentially minimal and V_z contains an element from the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$, $\lambda_{X_t}|_{V_z}$ is a finite subset of $\mathbb{Z}_{>0}$ by Proposition 4.2(b). Since this holds for all $z \in Z_t$, we see that $\mathrm{ran}(\lambda_{X_t})$ is a subset of $\mathbb{Z}_{>0}$. Since X_t is a compact open subset of X and since λ_{X_t} is continuous by Proposition 4.1, it follows that $\mathrm{ran}(\lambda_{X_t})$ is a finite set; thus, we can write $\mathrm{ran}(\lambda_{X_t}) = \{J_{t,1}, \ldots, J_{t,K_t}\}$. For each $k \in \{1, \ldots, K_t\}$, define $Y_{t,k} = \lambda_{X_t}^{-1}(J_{t,K_t})$.

We now check that what was defined above satisfies the conditions of Definition 2.6. Conditions (a) and (c) are clearly met. For each $t \in \{1, \ldots, T\}$, note that cince $X_t' \cap Z$ is compact and open in Z and since ψ is continuous, $\psi^{-1}(X_t' \cap Z)$ is compact and open, and hence X_t is compact and open. Furthermore, since $X_t \subset X_t'$, and X_t' is an element of $\mathcal{P}$, it follows that X_t is contained in an element of $\mathcal{P}$. Thus, (b) holds. For each $t \in \{1, \ldots, T\}$, since $\mathrm{ran}(\lambda_{X_t}) = \{J_{t,1}, \ldots, J_{t,K_t}\}$, we also have

$$\begin{aligned} \bigsqcup_{k=1}^{K_t} Y_{t,k} &= \bigsqcup_{k=1}^{K_t} \lambda_{X_t}^{-1}(J_{t,k}) \\ &= X_t. \end{aligned}$$

Thus, (d) holds. Recall that for each $z \in Z$, $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$ is an essentially minimal system and z is in its minimal set, and so since V_z is a compact open neighborhood of z, Proposition 4.2(c) tells us that $\{h^{\lambda_{V_z}}(x) \mid x \in V_z\} = V_z$. Thus, for each $t \in \{1, \ldots, T\}$, we have

$$\begin{aligned}\bigsqcup_{k=1}^{K_t} h^{J_{t,k}}(Y_{t,k}) &= \{h^{\lambda_{X_t}(x)}(x) \mid x \in X_t\} \\ &= \bigsqcup_{z \in Z_t} \{h^{\lambda_{V_z}(x)}(x) \mid x \in V_z\}.\end{aligned}$$

Since $\bigsqcup_{z \in Z_t} V_z = X_t$, this proves that (e) indeed holds. Next, let $x \in X$. There is precisely one $z \in Z$ such that $x \in \psi^{-1}(z)$, and precisely one $t \in \{1, \ldots, T\}$ such that $z \in Z_t$. Let $j \in \mathbb{Z}_{\geq 0}$ be the smallest nonnegative integer such that $h^{-j}(x) \in V_z \subset X_t$. This integer exists by applying Proposition 4.2(a), which applies since$(\psi^{-1}(z), h|_{\psi^{-1}(z)})$ is an essentially minimal system, z is in its minimal set, and V_z is a compact open neighborhood of z. It is clear that there is precisely one $k \in \{1, \ldots, K_t\}$ such that $h^{-j}(x) \in Y_{t,k}$. Then note that $j \in \{0, \ldots, J_{t,k} - 1\}$ since either $j = 0$ or $h^k(x) \notin X_t$ for all $k \in \{-j, \ldots, -1\}$. Thus, this proves that (f) holds as well. Altogether, we see that (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$.

($\Leftarrow$). Let $(\mathcal{P}^{(n)})$ be a generating sequence of partitions of X. Use Construction 2.15 to construct a sequence $(\mathcal{S}^{(n)})$ of finite first return time maps and adopt the notation of the construction. Define a map $\psi : X \to Z$ by $\psi(x) = z$ if $z \in Z$ and $x \sim z$. Recall that by Lemma 2.17, $\sim$ is an equivalence relation, and by Lemma 2.18, ψ is a well-defined map.

We claim that ψ and Z satisfy the conditions of Definition 2.19. To see that ψ is continuous, let $x \in X$ and let V be an open neighborhood of $\psi(x)$ in Z. Since $x \sim \psi(x)$, there is a sequence (t_n) such that $t_n \in \mathbb{Z}_{>0}$ for all $n \in \mathbb{Z}_{>0}$ and $x, \psi(x) \in \bigcap_{n=1}^{\infty} \bigcup_{j \in \mathbb{Z}} h^j(X_{t_n}^{(n)})$. Since $\psi(x) \in Z$, we have $\psi(x) \in \bigcap_{n=1}^{\infty} X_{t_n}^{(n)}$; because of this and because $(\mathcal{P}^{(n)})$ is a generating sequence of partitions, there is some $n \in \mathbb{Z}_{>0}$ such that $X_{t_n}^{(n)} \subset \psi^{-1}(V)$, and so $\psi(X_{t_n}^{(n)}) \subset V$. Set $U = \bigcup_{j \in \mathbb{Z}} h^j(X_{t_n}^{(n)})$, which contains x. Notice that since the equivalence classes of elements in U are the same as the equivalence classes of elements in $X_{t_n}^{(n)}$, we have $\psi(U) = \psi(X_{t_n}^{(n)}) \subset V$. Since $x \in U$, U is a neighborhood of x such that $\psi(U) \subset V$, which proves that ψ is continuous.

It is obvious that $\psi|_Z$ is the identity. To see that $\psi \circ h = \psi$, let $x \in X$ and let (t_n) be a sequence such that $t_n \in \mathbb{Z}_{>0}$ and $x \in \bigcup_{j \in \mathbb{Z}} h^j(X_{t_n}^{(n)})$ for all $n \in \mathbb{Z}_{>0}$. Then clearly $h(x) \in \bigcup_{j \in \mathbb{Z}} h^j(X_{t_n}^{(n)})$ for all $n \in \mathbb{Z}_{>0}$, so $x \sim h(x)$. Thus, $\psi \circ h = \psi$.

Let $z \in Z$. It is left to show that $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$ is an essentially minimal system and z is in its minimal set. By Theorem 1.1 of [6], it suffices to show that, for every neighborhood V of z in $\psi^{-1}(z)$, we have $\bigcup_{j \in \mathbb{Z}} h^j(V) = \psi^{-1}(z)$. So let V be a neighborhood of z in $\psi^{-1}(z)$, let V' be a neighborhood of z in X such that $V' \cap \psi^{-1}(z) = V$, and let (t_n) be a sequence such that $t_n \in \mathbb{Z}_{>0}$ for all $n \in \mathbb{Z}_{>0}$ and $z \in \bigcap_{n=1}^{\infty} X_{t_n}^{(n)}$. Since $(\mathcal{P}^{(n)})$ is a generating sequence of partitions, there is some $n \in \mathbb{Z}_{>0}$ such that $X_{t_n}^{(n)} \subset V'$. Let $x \in \psi^{-1}(z)$, so $x \sim z$. This means that in particular, we have $x \in X_{t_n}^{(n)}$. This tells us that $\bigcup_{j \in \mathbb{Z}} h^j(X_{t_n}^{(n)} \cap \psi^{-1}(z)) = \psi^{-1}(z)$, and since $X_{t_n}^{(n)} \cap \psi^{-1}(z) \subset V' \cap \psi^{-1}(z) = V$, this shows us that $\bigcup_{j \in \mathbb{Z}} h^j(V) = \psi^{-1}(z)$, as desired. $\square$

V PROOF OF THEOREM 3.2

The following two propositions are well-known (see [8]).

Proposition 5.1. Let (X, h) be a zero-dimensional system. Then there is an isomorphism φ : $K_0(C(X)) \to C(X, \mathbb{Z})$ that sends $[\chi_E]$ (where E is a compact open subset of X) to $\chi_E \in C(X, \mathbb{Z})$.

A particular consequence of the above proposition is that if E_1 and E_2 are compact open subset of X such that $E_1 \neq E_2$, then $[\chi_{E_1}] \neq [\chi_{E_2}]$.

Let $\mathcal{T}$ denote the Toeplitz algebra, the universal C^*-algebra generated by a single isometry s. Let A be a unital C^*-algebra and let α be an automorphism of A, and let u be the standard unitary of $C^*(\mathbb{Z}, A, \alpha)$. We denote by $\mathcal{T}(A, \alpha)$ the Toeplitz extension of A by α, which is the subalgebra of $C^*(\mathbb{Z}, A, \alpha) \otimes \mathcal{T}$ generated by $A \otimes 1$ and $u \otimes s$. The ideal generated by $A \otimes (1 - ss^*)$ is isomorphic to $A \otimes \mathcal{K}$, and the quotient by this ideal is isomorphic to $C^*(\mathbb{Z}, A, \alpha)$.

Proposition 5.2. Let (X, h) be a zero-dimensional system. Let α be the automorphism of $C(X)$ induced by h; that is, α is defined by $\alpha(f)(x) = f(h^{-1}(x))$ for all $f \in C(X)$ and all $x \in X$. Let δ be the connecting map obtained from the exact sequence

$$0 \longrightarrow C(X) \otimes \mathcal{K} \xrightarrow{\ \iota\ } \mathcal{T}(C(X), \alpha) \xrightarrow{\ \pi\ } C^*(\mathbb{Z}, A, \alpha) \longrightarrow 0,$$

where $K_0(C(X) \otimes \mathcal{K})$ is identified with $K_0(C(X))$ in the standard way. Let $i : C(X) \to C^*(\mathbb{Z}, X, h)$ be the natural inclusion. Then there is an exact sequence

$$0 \longrightarrow K_1(C^*(\mathbb{Z}, X, h)) \xrightarrow{\ \delta\ } K_0(C(X)) \xrightarrow{\mathrm{id}-\alpha_*} K_0(C(X)) \xrightarrow{\ i_*\ } K_0(C^*(\mathbb{Z}, X, h)) \longrightarrow 0.$$

Proof. Since $K_1(C(X)) = 0$, this follows immediately from Theorem 2.4 of [8]. $\square$

Lemma 5.3. Let (X, h) be a zero-dimensional system and let $E \subset X$ be a compact open h-invariant subset of X. Set $p = \chi_E$. Then $\delta([pup+(1-p)]) = [p]$. Moreover, if E is nonempty, then $[pup+(1-p)] \neq 0$.

Proof. We use the exact sequence in Proposition 5.2 and the definition of the connecting map as in Definition 8.1.1 of [11]. Let p_1 be the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and let

$$w = \begin{pmatrix} up \otimes s + (1-p) \otimes 1 & p \otimes (1 - ss^*) \\ 0 & pu^* \otimes s^* + (1-p) \otimes 1 \end{pmatrix} \in M_2(\mathcal{T}(A, \alpha)).$$

It is straightfoward to check that

$$\pi(w) = \begin{pmatrix} up + (1-p) & 0 \\ 0 & (up + (1-p))^* \end{pmatrix}.$$

We also have

$$w^* p_1 w = \begin{pmatrix} 1 & 0 \\ 0 & p \otimes (1 - ss^*) \end{pmatrix}.$$

Thus, $\delta([up + (1-p)]) = [p]$ as desired.

If E is nonempty, by Proposition 5.1, $[p] \neq 0$. Since δ is injective, this means that $[up+(1-p)] \neq 0$. ☐

Lemma 5.4. Let (X, h) be a zero-dimensional system and let E be an h-invariant compact open subset of X. Set $p = \chi_E$. Then $K_1(pC^*(\mathbb{Z}, X, h)p)$ is torsion-free.

Proof. Set $A = C^*(\mathbb{Z}, X, h)$ for convenience of notation. By Proposition 5.1, $K_0(C(X))$ is torsion-free. By Proposition 5.2, since $K_1(A)$ embeds into $K_0(C(X))$, $K_1(A)$ must be torsion-free as well. Since p is a central projection, we have $A \cong pAp \oplus (1-p)A(1-p)$. This means that we have $K_1(A) \cong K_1(pAp) \oplus K_1((1-p)A(1-p))$. Since $K_1(pAp)$ is a direct summand in a torsion-free group, it itself is torsion-free. ☐

Lemma 5.5. Let A be a unital C^*-algebra, let p be a projection in A, and let v be a unitary in pAp. Then if $[v+(1-p)] \neq 0$ in $K_1(A)$, we have $[v] \neq 0$ in $K_1(pAp)$.

Proof. Suppose that $[v] = 0$ in $K_1(pAp)$. This means that there is an $n \in \mathbb{Z}_{>0}$ such that $v \oplus \underbrace{p \oplus \cdots \oplus p}_{n-1 \text{ times}}$ is homotopic to $\underbrace{p \oplus \cdots \oplus p}_{n \text{ times}}$ in the unitary group of $M_n(pAp)$. Let $(x_t)_{t\in[0,1]}$ be this homotopy. Define a homotopy $(y_t)_{t\in[0,1]}$ in $M_n(A)$ by $y_t = x_t + \underbrace{(1-p) \oplus \cdots \oplus (1-p)}_{n \text{ times}}$ for all $t \in [0,1]$. Then $y_0 = (v+(1-p)) \oplus \underbrace{1 \oplus \cdots \oplus 1}_{n-1 \text{ times}}$ and $y_1 = \underbrace{1 \oplus \cdots \oplus 1}_{n \text{ times}}$, which shows that $[v+(1-p)] = 0$ in $K_1(A)$. ☐

Proposition 5.6. Let (X, h) be a zero-dimensional system such that, for any partition $\mathcal{P}$ of X, (X, h) admits a system of finite first return time maps subordinate to $\mathcal{P}$. Then (X, h) has no periodic points if and only if for every partition $\mathcal{P}$ and every $N \in \mathbb{Z}_{>0}$, there is a system $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ such that $J_{t,k} \geq N$ for all $t \in \{1, \ldots, T\}$ and all $k \in \{1, \ldots, K_t\}$.

Proof. ($\Rightarrow$). Let $N \in \mathbb{Z}_{>0}$ and let $\mathcal{P}$ be a partition of X. Since all points are aperiodic, for each $x \in X$, there is a compact open neighborhood U_x such that $U_x, h(U_x), \ldots, h^{N-1}(U_x)$ are pairwise disjoint. Then $(U_x)_{x\in X}$ is a compact open cover of X, and hence has a finite compact open refinement. By taking appropriate intersections, this refinement can be taken to be a partition $\mathcal{P}'$ of X. Let $\mathcal{P}''$ be a partition of X finer than both $\mathcal{P}$ and $\mathcal{P}'$ and let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}''$. Since $\mathcal{P}''$ is finer than $\mathcal{P}$, this $\mathcal{S}$ is also subordinate to $\mathcal{P}$. Since the first $N-1$ iterates of any element of this partition are pairwise disjoint, we must have $J_{t,k} \geq N$ for all $t \in \{1, \ldots, T\}$ and all $k \in \{1, \ldots, K_t\}$.

($\Leftarrow$). Suppose that $x \in X$ is a periodic point of (X, h), let $M \in \mathbb{Z}_{>0}$ satisfy $h^M(x) = x$, let $N \in \mathbb{Z}_{>0}$ be larger than M, and let $\mathcal{P}$ be a partition of X. Let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$ and let $\mathcal{P}_1(\mathcal{S})$ be as in Definition 2.6. Since $\mathcal{P}_1(\mathcal{S})$ is a partition of X, there are $t \in \{1, \ldots, T\}$, $k \in \{1, \ldots, K_t\}$, and $j \in \{0, \ldots, J_{t,k}-1\}$ such that $x \in h^j(Y_{t,k})$. It is clear that $h^{-j}(x) \in Y_{t,k}$ and $h^{-j}(x) \in h^M(Y_{t,k})$, and so $J_{t,k} \leq M < N$. ☐

Lemma 5.7. Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ be a partition of X, let Z and ψ be as in Definition 2.19, and let $X_1, \ldots, X_T$ be compact open subsets of X, each of which is contained in an element of $\mathcal{P}$. Then there is a system $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ (where $X_1, \ldots, X_T$ are as in the first sentence) if and only if for all $z \in Z$, there is precisely one $t \in \{1, \ldots, T\}$ such that X_t intersects $\psi^{-1}(z)$, and this intersection intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$.

Proof. ($\Rightarrow$). Suppose there is some $z \in Z$ such that, for all $t \in \{1, \ldots, T\}$, X_t does not intersect the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$. Since $\bigcup_{n\in\mathbb{Z}} h^n(X_t)$ is an h-invariant open set that doesn't contain z, it hence doesn't contain $\overline{\mathrm{orb}(z)}$. Since this is true for all $t \in \{1, \ldots, T\}$, this contradicts Proposition 2.9(b).

($\Leftarrow$). Let $z \in Z$ and let $t \in \{1, \ldots, T\}$ satisfy $X_t \cap \psi^{-1}(z) \neq \varnothing$. By our assumptions, $X_t \cap \psi^{-1}(z)$ is a compact open subset of $\psi^{-1}(z)$ intersecting the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$. By Proposition 4.2, $\lambda_{X_t\cap\psi^{-1}(z)}(x) < \infty$ for all $x \in X_t \cap \psi^{-1}(z)$. Since this holds for all $z \in Z \cap X_t$, it follows that $\lambda_{X_t}(x) < \infty$ for all $x \in X_t$. By Proposition 4.1, λ_{X_t} is continuous, and so $\mathrm{ran}(\lambda_{X_t})$ is a finite subset of $\mathbb{Z}_{>0}$. Write $\mathrm{ran}(\lambda_{X_t}) = \{J_{t,1}, \ldots J_{t,K_t}\}$ and, for each $k \in \{1, \ldots, K_t\}$, define $Y_{t,k} = \lambda_{X_t}^{-1}(J_{t,k})$.

We now claim that $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by checking the conditions of Definition 2.6. That (a), (b), and (c) are satisfied is clear. Condition (d) is satisfied due to the continuity of λ_{X_t} for each $t \in \{1, \ldots, T\}$. Condition (e) is satisfied due to Proposition 4.2(c). Now, let $x \in X$. By assumption, there is precisely one $t \in \{1, \ldots T\}$ such that $X_t \cap \psi^{-1}(\psi(x)) \neq \varnothing$. By Proposition 4.2(d), $x \in \bigcup_{n\in\mathbb{Z}_{>0}} h^n(X_t \cap \psi^{-1}(\psi(x)))$. Let $j \in \mathbb{Z}_{>0}$ be the smallest nonzero positive integer such that $x \in h^j(X_t)$. Let $k \in \{1, \ldots, K_t\}$ satisfy $x \in h^j(Y_{t,k})$. Then since j was chosen to be minimal, $x \notin h^l(X_t)$ for all $l \in \{0, \ldots, j-1\}$, and so we must have $j \in \{0, \ldots, J_{t,k} - 1\}$. Suppose $k' \in \{1, \ldots, K_t\}$ and $j' \in \{0, \ldots, J_{t,k'} - 1\}$ are such that $x \in h^{j'}(Y_{t,k'})$. We have $h^{-j}(x) \in X_t$, $h^{J_{t,k}-j}(x) \in X_t$, and $h^{-j+l}(x) \notin X_t$ for all $l \in \{1, \ldots, J_{t,k} - 1\}$, and so this means $j' = j$ and hence $k' = k$. Thus, condition (f) is satisfied. This proves the claim and consequently proves the lemma. $\square$

Lemma 5.8. Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $N \in \mathbb{Z}_{>0}$, let $\mathcal{P}$ be a partition of X, and let Z and ψ be as in Definition 2.19. Then there is a system $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ such that:

(a) We have $Z \cap \bigsqcup_{t=1}^{T} X_t = Z$.

(b) For all $t \in \{1, \ldots, T\}$, $X_t \cap Z \neq \varnothing$.

(c) For all $t \in \{1, \ldots, T\}$ and for all $n \in \{0, \ldots, N-1\}$, $h^n(X_t)$ is contained in an element of $\mathcal{P}$.

Proof. Let $X_1', \ldots, X_{T'}'$ be the elements of $\mathcal{P}$ that have nonempty intersection with Z. Let $t \in \{1, \ldots, T'\}$ and write $\mathcal{P} = \{U_1, \ldots, U_R\}$. We claim that

$$\widetilde{\mathcal{P}}_t = \left\{ \bigcap_{n=1}^{N-1} (X_t' \cap h^{-n}(U_{r_n})) \;\middle|\; r_n \in \{1, \ldots, R\} \text{ for } n \in \{1, \ldots, N-1\}; \bigcap_{n=1}^{N-1} (X_t' \cap h^{-n}(U_{r_n})) \neq \varnothing \right\}$$

is a partition of X'_t. Clearly $\widetilde{\mathcal{P}}_t$ is a finite set and all elements of $\widetilde{\mathcal{P}}_t$ are compact open subsets of X. Each element of $\widetilde{\mathcal{P}}_t$ is also contained in X'_t since $X'_t \cap h^{-n}(U_r) \subset X'_t$ for all $n \in \{1, \dots, N-1\}$ and all $r \in \{1, \dots, R\}$.

What is left to show is that each element of X'_t is in an element of $\widetilde{\mathcal{P}}_t$ and that the elements of $\widetilde{\mathcal{P}}_t$ are pairwise disjoint. Let $x \in X'_t$. For each $n \in \{1, \dots, N-1\}$, choose $r_n \in \{1, \dots, R\}$ such that $h^n(x) \in U_{r_n}$. Then

$$x \in \bigcap_{n=1}^{N-1} (X_t \cap h^{-n}(U_{r_n})).$$

So X_t is the union of all elements of $\widetilde{\mathcal{P}}_t$. Now, for each $n \in \{1, \dots, N-1\}$, choose $r'_n \in \{1, \dots, R\}$. If

$$x \in \bigcap_{n=1}^{N-1} (X_t \cap h^{-n}(U_{r'_n})),$$

then it must be the case that $h(x) \in U_{r_1}$ and $h(x) \in U_{r'_1}$, but since $\mathcal{P}$ is a partition of X, this must mean that $r_1 = r'_1$. We can repeat this process for $h^2(x), \dots, h^{N-1}(x)$, showing that $r_n = r'_n$ for all $n \in \{1, \dots, N-1\}$. Thus, elements of $\widetilde{\mathcal{P}}_t$ are pairwise disjoint, so $\widetilde{\mathcal{P}}_t$ is indeed a partition of X_t.

Let $\widetilde{\mathcal{P}}$ be a partition of X that contains all elements of $\widetilde{\mathcal{P}}_t$ for all $t \in \{1, \dots, T\}$ and is finer than $\mathcal{P}$. Let $X''_1, \dots, X''_T$ be the elements of $\widetilde{\mathcal{P}}$ that have nonempty intersection with Z. For each $t \in \{1, \dots, T\}$, define $X_t = X''_t \cap \psi^{-1}(X''_t \cap Z)$. Then $X_1, \dots, X_T$ satisfy conclusions (a) and (b) of this lemma, and by construction satisfy the hypotheses of Lemma 5.7. Thus, there is a system $\mathcal{S} = (T, (X_t)_{t=1,\dots,T}, \dots)$ of finite first return time maps subordinate to $\mathcal{P}$. To see that conclusion (c) of the lemma is satisfied, for each $t \in \{1, \dots, T\}$, there exists $s \in \{1, \dots, T'\}$ such that X_t is contained in an element of $\widetilde{\mathcal{P}}_s$, and therefore for every $n \in \{0, \dots, N\}$, $h^n(X_t)$ is contained in an element of $\mathcal{P}$. $\square$

Lemma 5.9. Let A be a unital C^*-algebra, let $N \in \mathbb{Z}_{>0}$ and let $v \in A$ be a unitary with finite spectrum. Then there is a unitary $w \in C^*(v)$, such that $\|w - 1\| \leq \pi/N$ and $w^N = v$.

Proof. Write $\mathrm{sp}(v) = \{\lambda_1, \dots, \lambda_K\} \subset S^1$. Since $\mathrm{sp}(v) \neq S^1$, by functional calculus there is a self-adjoint element $b \in A$ such that $\exp(b) = v$ and such that $\mathrm{sp}(b) \subset [-\pi, \pi]$. Setting $c = (1/N)b$, we have $\mathrm{sp}(c) \subset [-\pi/N, \pi/N]$. Set $w = \exp(c)$, a unitary in A. Clearly $w^N = \exp(Nc) = \exp(b) = v$. We compute

$$\begin{aligned} \|w - 1\| &= \|\exp(c) - 1\| \\ &\leq \max_{\lambda \in \mathrm{sp}(c)} |\exp(\lambda) - 1| \\ &\leq \max_{\lambda \in \mathrm{sp}(c)} |\lambda - 0| \\ &\leq \pi/N, \end{aligned}$$

finishing the proof. $\square$

Lemma 5.10. Let A be a C^*-algebra, let $L \in \mathbb{Z}_{>0}$, and let $a, a_1, \dots, a_m$ be positive elements in A such that $a = \sum_{m=1}^{M} a_m$ and $a_m \perp a_{m'}$ for $m, m' \in \{1, \dots, M\}$ with $m \neq m'$. Then $\|a\| = \max_{1 \leq l \leq M} \|a_m\|$.

Proof. Let $\mathcal{H}$ be a Hilbert space and let $\pi : A \to B(\mathcal{H})$ be a faithful representation. Then from the operator norm on $B(\mathcal{H})$, we know $\|\pi(a)\| = \max_{1\leq l\leq L} \|\pi(a_l)\|$. Since the representation is faithful, the conclusion follows. $\square$

Lemma 5.11. Let A be a unital C^*-algebra, let $a \in A$, let $\varepsilon > 0$, let $M \in \mathbb{Z}_{>0}$, and let $p_1, \ldots, p_M$ and $q_1, \ldots, q_M$ be projections in A such that $\sum_{m=1}^{M} p_m = \sum_{m=1}^{M} q_m = 1$. Then $p_m a q_n = 0$ for all $m, n \in \{1, \ldots, M\}$ with $m \neq n$ implies $\|a\| \leq \max_m \|p_m a q_m\|$.

Proof. Set $\varepsilon = \max_m \|p_m a q_m\|$. The hypotheses imply that

$$a = \left(\sum_{m=1}^{M} p_m\right) a \left(\sum_{m=1}^{M} q_m\right) = \sum_{m=1}^{M} p_m a q_m.$$

Now consider

$$\begin{aligned} a^* a &= \left(\sum_{m=1}^{M} q_m a^* p_m\right)\left(\sum_{m=1}^{M} p_m a q_m\right) \\ &= \sum_{m=1}^{M} q_m a^* p_m a q_m \\ &= \sum_{m=1}^{M} (p_m a q_m)^* (p_m a q_m). \end{aligned}$$

We can apply Lemma 5.10 with a replaced by $a^* a$ and a_m replaced by $(p_m a q_m)^*(p_m a q_m)$ for all $m \in \{1, \ldots, M\}$. To check the hypotheses of the lemma, note for all $m, m' \in \{1, \ldots, M\}$ with $m \neq m'$, we have $q_m \perp q_{m'}$, and so $(p_m a q_m)^*(p_m a q_m) \perp (p_{m'} a q_{m'})^*(p_{m'} a q_{m'})$. Now, for all $m \in \{1, \ldots, M\}$, we have $\|p_m a q_m\| \leq \varepsilon$, and so $\|(p_m a q_m)^*(p_m a q_m)\| = \|p_m a q_m\|^* \leq \varepsilon^2$. Thus, Lemma 5.10 tells us that $\|a^* a\| \leq \varepsilon^*$, and hence $\|a\| \leq \varepsilon$ as desired. $\square$

Lemma 5.12. Let A be a unital C^*-algebra, let $n \in \mathbb{Z}_{>0}$, let $(e_{i,j})_{1\leq i,j\leq n}$ be matrix units for a unital copy of M_n inside A (call this B_0), and let $u \in A$ be a unitary. Let B be the C^*-subalgebra of A generated by B_0 and u. Suppose that u commutes with $e_{i,j}$ for all $i, j \in \{1, \ldots, n\}$ and that $\mathrm{sp}(u) = S^1$. Then $B \cong C(S^1) \otimes M_n$.

Proof. Recall that $C(S^1) \otimes M_n$ is the universal C^*-algebra generated by $(f_{i,j})_{1\leq i,j\leq n}$ and v satisfying the relations

(a) $f_{i,j} f_{i',j'} = \delta_{j,i'} f_{i,j'}$ for all $i, j, i', j' \in \{1, \ldots, n\}$,

(b) $\sum_{i=1}^{n} f_{i,i} = 1$,

(c) $vv^* = v^* v = 1$,

(d) $f_{i,j} v = v f_{i,j}$ for all $i, j \in \{1, \ldots, n\}$.

Identify v with $z \otimes 1$, where $z \in C(S^1)$ is the identity map. Let $(g_{i,j})_{1\leq i,j\leq n}$ be the standard matrix units for M_n and identify $f_{i,j}$ with $1 \otimes g_{i,j}$ for all $i, j \in \{1, \ldots, n\}$. Since $(e_{i,j})_{1\leq i,j\leq n}$ and u

satisfy the relations (a)–(d) as well, by the universal property there is a surjective $*$-homomorphism $\varphi : C(S^1) \otimes M_n \to B$ such that $\varphi(z \otimes 1) = u$ and $\varphi(1 \otimes g_{i,j}) = e_{i,j}$ for all $i, j \in \{1, \dots, n\}$.

For each $i, j \in \{1, \dots, n\}$, let $N_{i,j}$ and $M_{i,j}$ be integers such that $M_{i,j} \leq N_{i,j}$. For each $i, j \in \{1, \dots, n\}$ and each $k \in \{M_{i,j}, \dots, N_{i,j}\}$, let $\lambda_{i,j,k}$ be a complex number. Then define

$$x = \sum_{i,j=1}^{n} \left(\sum_{k=M_{i,j}}^{N_{i,j}} \lambda_{i,j,k} z^k \right) \otimes g_{i,j} \in C(S^1) \otimes M_n.$$

Note that elements of the above form are dense in $C(S^1) \otimes M_n$. It is clear that

$$\varphi(x) = \sum_{i,j=1}^{n} \left(\sum_{k=M_{i,j}}^{N_{i,j}} \lambda_{i,j,k} u^k \right) e_{i,j}.$$

For each $i, j \in \{1, \dots, n\}$, since $\mathrm{sp}(u) = S^1$, we have $C^*(u) \cong C(S^1)$, and so $\sum_{k=M_{i,j}}^{N_{i,j}} \lambda_{i,j,k} u^k = 0$ if and only if $\lambda_{i,j,k} = 0$ for all $k \in \{N_{i,j}, \dots, M_{i,j}\}$. But this means that $\varphi(x) = 0$ implies that $x = 0$, meaning that $\ker(\varphi)$ is trivial. Thus, φ is an isomorphism. $\square$

Lemma 5.13. Let (X, h) be a zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\dots,T}, \dots)$ and $\mathcal{S}' = (T', (X'_t)_{t=1,\dots,T'}, \dots)$ be systems of finite first return time maps subordinate to $\mathcal{P}$ such that $\bigsqcup_{t=1}^{T} X_t = \bigsqcup_{t=1}^{T'} X'_t$. Then $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ if and only if $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

Proof. ($\Rightarrow$). Set $\widehat{X} = \bigsqcup_{t=1}^{T} X_t$ and set $\widehat{X}' = \bigsqcup_{t=1}^{T'} X'_t$ (note that $\widehat{X} = \widehat{X}'$, but the distinction will be important in our reasoning later). Let $t \in \{1, \dots, T\}$ and let $k \in \{1, \dots, K_t\}$. By assumption, there are $s(1), \dots, s(M) \in \{1, \dots, T'\}$ and there are $l(m,1), \dots, l(m, N_m) \in \{1, \dots, K'_{s(m)}\}$ for each $m \in \{1, \dots, M\}$ such that $\bigsqcup_{m=1}^{N} \bigsqcup_{n=1}^{N_m} Y'_{s(m),l(m,n)} = Y_{t,k}$. Then clearly $\bigsqcup_{m=1}^{N} \bigsqcup_{n=1}^{N_m} h^{J_{t,k}}(Y'_{s(m),l(m,n)}) = h^{J_{t,k}}(Y_{t,k})$. Now, for each $x \in Y_{t,k}$, we have $\lambda_{\widehat{X}}(x) = \lambda_{X_t}(x) = J_{t,k}$. But then for all $m \in \{1, \dots, M\}$, we have $\lambda_{X'_{s(m)}}(x) = \lambda_{\widehat{X}'}(x) = \lambda_{\widehat{X}}(x) = J_{t,k}$. Thus, for each $m \in \{1, \dots, M\}$ and each $n \in \{1, \dots, N_m\}$, we have $J_{t,k} = J'_{s(m),l(m,n)}$, and so $h^{J_{t,k}}(Y_{s(m),l(m,n)})$ is an element of $\mathcal{P}_2(\mathcal{S}')$. Thus, $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$. The proof of ($\Leftarrow$) is analogous to this. $\square$

Lemma 5.14. Let (X, h) be a zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let $\mathcal{S} = (T, (X_t)_{t=1,\dots,T}, \dots)$ and $\mathcal{S}' = (T', (X'_t)_{t=1,\dots,T'}, \dots)$ be systems of finite first return time maps subordinate to $\mathcal{P}$ such that $\bigsqcup_{t=1}^{T} X_t = \bigsqcup_{t=1}^{T'} X'_t$. Then $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ if and only if for each $s \in \{1, \dots, T'\}$ and each $l \in \{1, \dots, K'_s\}$, there is a $t \in \{1, \dots, T\}$ and a $k \in \{1, \dots, K_t\}$ such that $Y'_{s,l} \subset Y_{t,k}$.

Proof. ($\Rightarrow$). Let $s \in \{1, \dots, T'\}$ and let $l \in \{1, \dots, K'_s\}$. Since $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$, there is a $t \in \{1, \dots, T\}$, a $k \in \{1, \dots, K_t\}$, and a $j \in \{0, \dots, J_{t,k} - 1\}$ such that $Y'_{s,l} \subset h^j(Y_{t,k})$. But by assumption, there is $t' \in \{1, \dots, T\}$ such that $Y'_{s,l} \subset X_{t'}$. Then since $\mathcal{P}_1(\mathcal{S})$ is a partition of X, we must have $t' = t$, and hence we must have $j = 0$.

($\Leftarrow$). Let $t \in \{1, \dots, T\}$, let $k \in \{1, \dots, K_t\}$, and let $j \in \{0, \dots, J_{t,k} - 1\}$. Let $s \in \{1, \dots, T'\}$, $l \in \{1, \dots, K'_s\}$, and $i \in \{0, \dots, J_{s,l} - 1\}$ satisfy $h^i(Y'_{s,l}) \cap h^j(Y_{t,k}) \neq \varnothing$. Now, note that, for all

$x \in h^i(Y'_{s,l}) \cap h^j(Y_{t,k})$, we have $h^{-1}(x), \dots, h^{-j+1}(x) \notin \bigsqcup_{t=1}^{T} X_t$ and $h^{-j}(x) \in \bigsqcup_{t=1}^{T} X_t$. Similarly, we have $h^{-1}(x), \dots, h^{-i+1}(x) \notin \bigsqcup_{t=1}^{T} X_t$ and $h^{-i}(x) \in \bigsqcup_{t=1}^{T} X'_t$. Since $\bigsqcup_{t=1}^{T} X'_t = \bigsqcup_{t=1}^{T} X_t$, this means that $i = j$, so $Y'_{s,l} \cap Y_{t,k} \neq \varnothing$. But by assumption this means we must have $Y'_{s,l} \subset Y_{t,k}$, and therefore $h^i(Y'_{s,l}) \subset h^j(Y_{t,k})$. This proves that $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$. □

Lemma 5.15. Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let Z be as in Definition 2.19. Then there is a system $\mathcal{S} = (T, (X_t)_{t=1,\dots,T}, \dots)$ of finite first return time maps subordinate to $\mathcal{P}$ such that:

(a) We have $\bigsqcup_{t=1}^{T} X_t \cap Z = Z$.

(b) For each $t \in \{1, \dots, T\}$, $X_t \cap Z \neq \varnothing$.

Proof. Apply the construction in the proof of $(\Rightarrow)$ of Theorem 3.1. □

Lemma 5.16. Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ be a partition of X, and let Z and ψ be as in Definition 2.19. Apply Lemma 5.15 to get a system $\mathcal{S} = (T, (X_t)_{t=1,\dots,T}, \dots)$ satisfying the conclusions of the lemma. Then there is a system $\mathcal{S}' = (T', (X'_t)_{t=1,\dots,T'}, \dots)$ such that:

(a) We have $\bigsqcup_{t=1}^{T'} X'_t = \bigsqcup_{t=1}^{T} X_t$.

(b) The partition $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

(c) For each $t \in \{1, \dots T'\}$, $Y'_{t,1} \cap Z \neq \varnothing$ and $Y'_{t,k} \cap Z = \varnothing$ for all $k \in \{2, \dots, K'_t\}$.

Proof. For each $t \in \{1, \dots, T\}$, let $A_t = \{a(t, 1), \dots, a(t, N_t)\}$ be the set of all $k \in \{1, \dots, K_t\}$ such that $Y_{t,k} \cap Z \neq \varnothing$. Set $T' = \sum_{t=1}^{T} N_t$.

Let $s \in \{1, \dots, T'\}$. There is some $t \in \{1, \dots, T\}$ and some $n \in \{1, \dots, N_t\}$ such that $s = \sum_{r=1}^{t-1} N_r + n$. Define $X'_s = \psi^{-1}(Y_{t,a(t,n)} \cap Z) \cap X_t$. Let $B_s = \{b(s, 1), \dots, b(s, K'_s)\}$ be the set of all $k \in \{1, \dots, K_t\}$ such that $Y_{t,k} \cap X'_s \neq \varnothing$, taking $b(s, 1) = a(t, n)$. For each $k \in \{1, \dots, K'_s\}$, set $Y'_{s,k} = Y_{t,b(s,k)} \cap X'_s$ and set $J'_{s,k} = J_{t,b(s,k)}$.

We now show that $\mathcal{S}' = (T', (X'_t)_{t=1,\dots,T'}, \dots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by verifying the conditions of Definition 2.6. It is clear that conditions (a) and (c) are satisfied. Let $s \in \{1, \dots, T'\}$ and let $t \in \{1, \dots, T\}$ and $n \in \{1, \dots, N_t\}$ satisfy $s = \sum_{r=1}^{t-1} N_r + n$. Notice that $Y_{t,a(t,n)} \cap Z$ is compact and open in Z, and so by the continuity of ψ, $\psi^{-1}(Y_{t,a(t,n)} \cap Z)$ is compact and open in X. Thus, X'_s is compact and open. Since X_t is contained in an element of $\mathcal{P}$, so is X'_s. Thus, condition (b) is satisfied. For each $k \in \{1, \dots, K'_s\}$, it is clear that $Y'_{s,k}$ is a compact open subset of X'_s, since it is the intersection of two compact open subsets of X. Moreover, it is nonempty by construction. Now, notice that if $k \in \{1, \dots, K_t\} \setminus B_s$, we have $Y_{t,k} \cap X'_s = \varnothing$, and using this fact at the second step below, we have

$$\bigsqcup_{k=1}^{K'_s} Y'_{s,k} = \bigsqcup_{k=1}^{K'_s} Y_{t,b(s,k)} \cap X'_s$$

$$
\begin{aligned}
&= \bigsqcup_{k=1}^{K_t} Y_{t,k} \cap X'_s \\
&= X_t \cap X'_s \\
&= X'_s.
\end{aligned}
$$

Thus, condition (d) holds. For all $k \in \{1, \ldots, K_t\} \setminus B_s$, since $\psi^{-1}(Y_{t,a(t,n)} \cap Z)$ is h-invariant and $Y_{t,k} \cap \psi^{-1}(Y_{t,a(t,n)} \cap Z) = \varnothing$, we have $h^{J_{t,k}}(Y_{t,k}) \cap X'_s = \varnothing$. Thus, using this fact at the second step below, we have

$$
\begin{aligned}
\bigsqcup_{k=1}^{K'_s} h^{J'_{s,k}}(Y'_{s,k}) &= \bigsqcup_{k=1}^{K'_s} h^{J_{t,b(s,k)}}(Y_{t,b(s,k)}) \cap X'_s \\
&= \bigsqcup_{k=1}^{K_t} h^{J_{t,k}}(Y_{t,k}) \cap X'_s \\
&= X_t \cap X'_s \\
&= X'_s.
\end{aligned}
$$

Thus, condition (e) holds.

It still remains to verify that condition (f) holds. Let $x \in X$. There is precisely one $t \in \{1, \ldots, T\}$, one $k \in \{1, \ldots, K_t\}$, and one $j \in \{0, \ldots, J_{t,k} - 1\}$ such that $x \in h^j(Y_{t,k})$. Observe that

$$
\bigsqcup_{r=1}^{T} \bigsqcup_{m=1}^{N_r} Y_{r,a(r,m)} \cap Z = Z. \tag{1}
$$

Thus, there is exactly one $n \in \{1, \ldots, N_t\}$ such that $x \in \psi^{-1}(Y_{t,a(t,n)} \cap Z)$. Let $s = \sum_{r=1}^{t-1} N_r + n$. We can now see that there is exactly one $k \in B_s$ such that $h^{-j}(x) \in Y'_{s,k}$. Since $J'_{s,k} = J_{t,a(t,n)}$, we have $j \in \{0, \ldots, J'_{s,k} - 1\}$. Thus, condition (f) holds.

We now show that $\mathcal{S}'$ satisfies the conclusions of the lemma. It is clear that $\bigsqcup_{t=1}^{T'} X'_t \subset \bigsqcup_{t=1}^{T} X_t$. Since $\bigsqcup_{t=1}^{T} \bigsqcup_{n=1}^{N_t} Y_{t,a(t,n)} \cap Z = Z$, we have $\bigsqcup_{t=1}^{T} \bigsqcup_{n=1}^{N_t} \psi^{-1}(Y_{t,a(t,n)} \cap Z) = X$. Hence, $\bigsqcup_{t=1}^{T'} X'_t = \bigsqcup_{t=1}^{T} X_t$. Thus, conclusion (a) of the lemma is satisfied. Now, note that by construction, for every $s \in \{1, \ldots, T'\}$ and every $l \in \{1, \ldots, K'_s\}$, there is a $t \in \{1, \ldots, T\}$ and a $k \in \{1, \ldots, K_t\}$ such that $Y'_{s,l} \subset Y_{t,k}$. Thus, by Lemma 5.14, we see $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$, and by Lemma 5.14, we see that $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$. Thus, conclusion (b) of the lemma is satisfied. Now, let $s \in \{1, \ldots, T'\}$ and let $t \in \{1, \ldots, T\}$ and $n \in \{1, \ldots, N_t\}$ satisfy $s = \sum_{r=1}^{t-1} N_r + n$. Since $Y'_{s,1} = Y_{t,a(t,n)} \cap X'_s$, it follows that $Y'_{s,1} \cap Z \neq \varnothing$. Let $k \in \{2, \ldots, K'_s\}$. Then there is no $n \in \{1, \ldots, N_t\}$ such that $Y_{t,b(s,k)} = Y_{t,a(t,n)}$. Thus, by (1), we have $Y'_{s,k} \cap Z = \varnothing$. Thus, condition (c) of the lemma is satisfied. □

Lemma 5.17. Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ be a partition of X, let $N \in \mathbb{Z}_{>0}$ and let Z and ψ be as in Definition 2.19. Apply Lemma 5.15 to get a system $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ satisfying the conclusions of the lemma. Then there is a system $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ such that:

(a) The partition $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

(b) For all $t \in \{1, \ldots, T'\}$ and all $k \in \{2, \ldots, K'_s\}$, we have $J_{t,k} > N$.

(c) For each $t \in \{1, \ldots, T'\}$, $Y_{t,1} \cap Z \neq \varnothing$ and $Y_{t,k} \cap Z = \varnothing$ for all $k \in \{2, \ldots, K'_t\}$.

(d) We have $\bigsqcup_{t=1}^{T} X_t \cap Z = Z$.

Proof. We now prove this lemma by induction, with the base case $N = 0$ proved by Lemma 5.16. So suppose that $\mathcal{S}$ satisfies the conclusions of the lemma with $N-1$ in place of N. Let $t \in \{1, \ldots, T\}$ and let B_t be the set of all $k \in \{2, \ldots, K_t\}$ such that $J_{t,k} = N$. Define $X''_t = X_t \setminus \bigsqcup_{k \in B_t} Y_{t,k}$.

Let $z \in Z$. Since $\mathcal{S}$ satisfies conclusion (c) of this lemma, there is a $t \in \{1, \ldots, T\}$ such that $z \in Y_{t,1}$. Thus, $z \in X''_t$. Let $s \in \{1, \ldots, T\}$ satisfy $s \neq t$. Then $\psi^{-1}(z) \cap X_s = \varnothing$, and so since $X''_s \subset X_s$, we conclude that $\psi^{-1}(z) \cap X''_s = \varnothing$. Thus, by Lemma 5.7, setting $T = T''$, there is a system $\mathcal{S}'' = (T'', (X''_t)_{t=1,\ldots,T''}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$.

By applying Proposition 2.10 with $\mathcal{S}''$ in place of $\mathcal{S}$ and where $\mathcal{P}'$ is a partition finer than both $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S})$, we may assume that $\mathcal{P}_1(\mathcal{S}'')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S}'')$ is finer than $\mathcal{P}_2(\mathcal{S})$. By applying Lemma 5.16 with $\mathcal{S}''$ in place of $\mathcal{S}$, we may additionally assume that $\bigsqcup_{t=1}^{T''} Y''_{t,1} \cap Z = Z$.

For each $t \in \{1, \ldots, T''\}$, let C_t be the set of all $k \in \{1, \ldots, K''_t\}$ such that $J''_{t,k} = N$. Let $D_1 = \{a(1), \ldots, a(L_1)\}$ be the set of all $t \in \{1, \ldots, T''\}$ such that $C_t = \varnothing$. Let $D_2 = \{a(L_1 + 1), \ldots, a(L_2)\}$ be the set of all $t \in \{1, \ldots, T''\}$ such that $1 \in C_t$. Let $D_3 = \{a(L_2 + 1), \ldots, a(L_3)\}$ be the set of all $t \in \{1, \ldots, T''\}$ such that $C_t \neq \varnothing$ but $1 \notin C_t$.

Let $t \in \{1, \ldots, L_1\}$. Set $X'_t = X''_{a(t)}$ and $K'_t = K''_{a(t)}$, and for each $k \in \{1, \ldots, K'_t\}$, we set

$$Y'_{t,k} = Y''_{a(t),k} \tag{2}$$

and $J'_{t,k} = J''_{a(t),k}$. It is clear that for all $k \in \{1, \ldots, K'_t\}$, we have $J_{t,k} > N$.

Let $t \in \{L_1 + 1, \ldots, L_2\}$. Set $X'_t = X''_{a(t)}$. Define

$$Y'_{t,1} = \bigsqcup_{k \in C_{a(t)}} Y''_{a(t),k} \tag{3}$$

and set $J'_{t,1} = J''_{a(t),1}$. Write $\{1, \ldots, K''_{a(t)}\} \setminus C_{a(t)} = \{b(t, 2), \ldots, b(t, K'_t)\} = C^c_{a(t)}$. For each $k \in \{2, \ldots, K'_t\}$, define

$$Y'_{t,k} = Y''_{a(t),b(t,k)} \tag{4}$$

and $J'_{t,k} = J''_{a(t),b(t,k)}$. By construction, it is clear that for all $k \in \{2, \ldots, K'_t\}$, we have $J_{t,k} > N$.

Let $t \in \{L_2 + 1, \ldots, L_3\}$. Then set $\widetilde{X}'_t = X''_{a(t)} \setminus \bigsqcup_{k \in C_{a(t)}} Y''_{a(t),k}$. Now, apply Lemma 5.7 with $(\psi^{-1}(X''_{a(t)} \cap Z), h|_{\psi^{-1}(X''_{a(t)} \cap Z)})$ in place of (X, h), $T = 1$, $\widetilde{X}'_t$ in place of X_1, and

$$\mathcal{P}^{(t)} = \{U \cap \psi^{-1}(X''_{a(t)} \cap Z) \,\big|\, U \in \mathcal{P} \text{ and } U \cap \psi^{-1}(X''_{a(t)} \cap Z) \neq \varnothing\}$$

in place of $\mathcal{P}$. We then get a system $\mathcal{S}^{(t)} = \{T^{(t)}, (X^{(t)}_s)_{s=1,\ldots,T^{(t)}}, \ldots\}$ of finite first return time maps subordinate to $\mathcal{P}^{(t)}$, and by using Proposition 2.10, we are free to assume that $\mathcal{P}_1(\mathcal{S}^{(t)})$ is finer than

$$\{U \cap \psi^{-1}(X''_{a(t)} \cap Z) \,\big|\, U \in \mathcal{P}_1(\mathcal{S}'') \text{ and } U \cap \psi^{-1}(X''_{a(t)} \cap Z) \neq \varnothing\} \tag{5}$$

and $\mathcal{P}_2(\mathcal{S}^{(t)})$ is finer than

$$\{U \cap \psi^{-1}(X''_{a(t)} \cap Z) \,\big|\, U \in \mathcal{P}_2(\mathcal{S}'') \text{ and } U \cap \psi^{-1}(X''_{a(t)} \cap Z) \neq \varnothing\}.$$

Using Lemma 5.16, we may assume that $\mathcal{S}^{(t)}$ satisfies the conclusions of the lemma with $Z \cap X_t$ in place of Z. Let $s \in \{1, \dots, T^{(t)}\}$. Set $u = L_2 + s + \sum_{l=L_2+1}^{t-1} T^{(l)}$, set $X'_u = X^{(t)}_s$, set $K'_u = K^{(t)}_s$, and for each $k \in \{1, \dots, K'_u\}$, set $Y'_{u,k} = Y^{(t)}_{s,k}$ and set $J'_{u,k} = J^{(t)}_{s,k}$. Let $k \in \{1, \dots, K'_t\}$. By (5), there is some $l \in (\{1, \dots, K''_{a(t)}\} \setminus C_{a(t)})$ such that $Y'_{u,k} \subset Y''_{a(t),l}$. But since $J''_{a(t),l} > N$ and since $X'_u \subset X''_{a(t)}$, we have $J'_{u,k} > N$.

Finally, set $T' = L_2 + \sum_{l=L_2+1}^{L_3} T^{(l)}$. We now show that $\mathcal{S}' = (T', (X'_t)_{t=1,\dots,T'}, \dots)$ is indeed a system of finite first return time maps subordinate to $\mathcal{P}$ by verifying the conditions of Definition 2.6. That conditions (a) and (c) hold is clear. For condition (b), let $t \in \{1, \dots, T'\}$. It is clear that X'_t is a compact open subset of X. For $t \in \{1, \dots, L_2\}$, we have $X'_t \subset X''_{a(t)}$; for $t \in \{L_2 + 1, \dots, T'\}$, let $L \in \{L_2 + 1, \dots, L_3\}$ and let $s \in \{1, \dots, T^{(L)}\}$ satisfy $t = L_2 + \sum_{l=L_2+1}^{L-1} T^{(l)} + s$, and then we have $X'_t \subset X''_{a(L)}$. Thus, in both cases, X'_t is contained in an element of $\mathcal{P}$. This verifies condition (b). For conditions (d) and (e), let $t \in \{1, \dots, L_1\}$. We then have

$$\begin{aligned} X'_t &= X''_{a(t)} \\ &= \bigsqcup_{k=1}^{K''_{a(t)}} Y''_{a(t),k} \\ &= \bigsqcup_{k=1}^{K'_t} Y'_{t,k} \end{aligned}$$

and

$$\begin{aligned} X'_t &= X''_{a(t)} \\ &= \bigsqcup_{k=1}^{K''_{a(t)}} h^{J''_{a(t),k}}(Y''_{a(t),k}) \\ &= \bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}). \end{aligned}$$

Now, let $t \in \{L_1 + 1, \dots, L_2\}$. We then have

$$X'_t = X''_{a(t)}$$

$$
\begin{aligned}
&= \bigsqcup_{k=1}^{K''_{a(t)}} Y''_{a(t),k} \\
&= \left(\bigsqcup_{k \in C_{a(t)}} Y''_{a(t),k} \right) \sqcup \left(\bigsqcup_{k=2}^{K'_t} Y''_{a(t),b(t,k)} \right) \\
&= Y'_{t,1} \sqcup \left(\bigsqcup_{k=2}^{K'_t} Y'_{t,k} \right) \\
&= \bigsqcup_{k=1}^{K'_t} Y'_{t,k}
\end{aligned}
$$

and

$$
\begin{aligned}
X'_t &= X''_{a(t)} \\
&= \bigsqcup_{k=1}^{K''_{a(t)}} h^{J''_{a(t),k}}(Y''_{a(t),k}) \\
&= \left(\bigsqcup_{k \in C_{a(t)}} h^{J''_{a(t),k}}(Y''_{a(t),k}) \right) \sqcup \left(\bigsqcup_{k=2}^{K'_t} h^{J''_{a(t),k}}(Y''_{a(t),b(t,k)}) \right) \\
&= h^{J'_{t,1}}(Y'_{t,1}) \sqcup \left(\bigsqcup_{k=2}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) \right) \\
&= \bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}).
\end{aligned}
$$

Finally, let $u \in \{L_2+1, \ldots, T'\}$. Let $t \in \{L_2+1, \ldots, L_3\}$ and $s \in \{1, \ldots, T^{(t)}\}$ such that $u = L_2 + s + \sum_{l=L_2}^{t-1} T^{(l)}$. Then since $\mathcal{S}^{(t)}$ is a system of finite first return time maps, we get

$$
\begin{aligned}
X'_u &= X^{(t)}_s \\
&= \bigsqcup_{k=1}^{K^{(t)}_s} Y^{(t)}_{s,k} \\
&= \bigsqcup_{k=1}^{K'_u} Y'_{u,k}
\end{aligned}
$$

and

$$
\begin{aligned}
X'_u &= X^{(t)}_s \\
&= \bigsqcup_{k=1}^{K^{(t)}_s} h^{J^{(t)}_{s,k}}(Y^{(t)}_{s,k})
\end{aligned}
$$

$$= \bigsqcup_{k=1}^{K'_u} h^{J'_{u,k}}(Y'_{u,k}).$$

Thus, conditions (d) and (e) are verified. For condition (f), we have

$$\bigsqcup_{t\in D_1} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k}) = \bigsqcup_{t=1}^{L_1} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k}),$$

$$\begin{aligned}
\bigsqcup_{t\in D_2} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J''_{t,1}-1} h^j(Y''_{t,k}) &= \left(\bigsqcup_{t\in D_2} \bigsqcup_{k\in C_t} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k})\right) \sqcup \left(\bigsqcup_{t\in D_2} \bigsqcup_{k\in C_t^c} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k})\right) \\
&= \left(\bigsqcup_{t=L_1+1}^{L_2} \bigsqcup_{j=0}^{J'_{t,1}-1} h^j(Y'_{t,1})\right) \sqcup \left(\bigsqcup_{t=L_1+1}^{L_2} \bigsqcup_{k=2}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k})\right) \\
&= \bigsqcup_{t=L_1+1}^{L_2} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k}),
\end{aligned}$$

and

$$\begin{aligned}
\bigsqcup_{t\in D_3} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k}) &= \bigsqcup_{t\in D_3} \psi^{-1}(\psi(X''_t)) \\
&= \bigsqcup_{t\in D_3} \bigsqcup_{s=1}^{T^{(t)}} \bigsqcup_{k=1}^{K_t^{(t)}} \bigsqcup_{j=0}^{J_{s,k}^{(t)}-1} h^j(Y_{s,k}^{(t)}) \\
&= \bigsqcup_{t=L_2+1}^{T'} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\bigsqcup_{t=1}^{T'} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k}) &= \left(\bigsqcup_{t=1}^{L_1} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k})\right) \sqcup \left(\bigsqcup_{t=L_1+1}^{L_2} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k})\right) \\
&\quad \sqcup \left(\bigsqcup_{t=L_2+1}^{L_3} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k})\right) \\
&= \left(\bigsqcup_{t\in D_1} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k})\right) \sqcup \left(\bigsqcup_{t\in D_2} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k})\right) \\
&\quad \sqcup \left(\bigsqcup_{t\in D_3} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k})\right)
\end{aligned}$$

$$= \bigsqcup_{t=1}^{T''} \bigsqcup_{k=1}^{K''_t} \bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y''_{t,k})$$
$$= X.$$

Thus, condition (f) is satisfied.

We now show that $\mathcal{S}'$ satisfies the conclusions of the lemma. The fact that $\mathcal{S}'$ satisfies conclusion (a) of the lemma follows from (2), (3), (4), and (5), along with the fact that $\mathcal{P}_1(\mathcal{S}'')$ is finer than $\mathcal{P}_1(\mathcal{S})$. Our construction also showed that conclusions (b), (c), and (d) are satisfied. This proves the lemma. ☐

Lemma 5.18. Let (X, h) be a fiberwise essentially minimal zero-dimensional system, let $\mathcal{P}$ be a partition of X, let $N \in \mathbb{Z}_{>0}$, and let $\mathcal{S} = (T, (X_t)_{t=1,\dots,T}, \dots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. Then there is a system $\mathcal{S}' = (T', (X'_t)_{t=1,\dots,T'}, \dots)$ of finite first return time maps subordinate to $\mathcal{P}$ such that, setting $\widehat{X}_t = X_t \setminus (Y'_{t,1} \cap h^{J'_{t,1}}(Y'_{t,1}))$ for each $t \in \{1, \dots, T'\}$, we have:

(a) The partition $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S}')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

(b) For each $t \in \{1, \dots, T'\}$, there is an $s \in \{1, \dots, T\}$ such that $X'_t \subset X_s$.

(c) For each $t \in \{1, \dots, T'\}$ and each $z \in \psi(X_t)$, $Y'_{t,1}$ intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$.

(d) For all $t \in \{1, \dots, T'\}$ and all $k \in \{2, \dots, K'_s\}$, we have $J_{t,k} > N$.

(e) The sets $\widehat{X}_t, h(\widehat{X}_t), \dots, h^N(\widehat{X}_t)$ are pairwise disjoint.

Proof. By applying Lemma 5.17, we may assume that $\mathcal{S}$ satisfies the conclusions of the lemma. Let $t \in \{1, \dots, T\}$. Define
$$B_t = h^{J_{t,1}}(Y_{t,1}) \cap \left(\bigsqcup_{k=2}^{K_t} Y_{t,k} \right)$$
and for each $n \in \{1, \dots, N\}$ define
$$C_{t,n} = B_t \cap \left(\bigcup_{k=2}^{K_t} h^n(Y_{t,1} \cap h^{J_{t,k}}(Y_{t,k})) \right).$$
Set $T' = T$ and define
$$X''_t = X_t \setminus \left(\bigcup_{n=1}^{N} \bigcup_{m=1}^{n} h^{-m}(C_{t,n}) \cap X_t \right).$$

Let $z \in Z$. Since $\mathcal{S}$ satisfies the conclusions of Lemma 5.17, there is precisely one $t \in \{1, \dots, T\}$ such that $z \in X_t$. If $z \in X''_t$, then clearly X_t intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$. Suppose $z \notin X''_t$. Then there is some $n \in \{1, \dots, N\}$ such that $z \in \bigcup_{m=1}^{n} h^{-m}(C_{t,n}) \cap X_t$, which means that there is some $m \in \{1, \dots, n\}$ such that
$$h^m(z) \in X''_t. \tag{6}$$

Since $h^m(z)$ is in the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$, we see X''_t intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$. By Lemma 5.7, there is a system $\mathcal{S}'' = (T'', (X''_t)_{t=1,\ldots,T''}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$. By using Proposition 2.10, we are free to assume that $\mathcal{P}_1(\mathcal{S}'')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and that $\mathcal{P}_2(\mathcal{S}'')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

Let $t \in \{1, \ldots, T\}$, suppose that $J_{t,1} \leq N$, and suppose that $\{x \in X''_t \,|\, \lambda_{X''_t}(x) = J_{t,1}\}$ is nonempty. We claim that

$$\{x \in X''_t \,|\, \lambda_{X''_t}(x) = J_{t,1}\} = Y_{t,1} \cap X''_t.$$

Let $x \in X''_t$ and suppose $\lambda_{X''_t}(x) = J_{t,1}$. Then since $X''_t \subset X_t$, we have $\lambda_{X_t}(x) \leq J_{t,k}$. Thus, by (b) of Lemma 5.17, we have $x \in Y_{t,1}$. Thus, $\{x \in X''_t \,|\, \lambda_{X''_t}(x) = J_{t,1}\} \subset Y_{t,1} \cap X''_t$. So suppose $x \in X''_t \cap Y_{t,1}$ and suppose $h^{J_{t,1}}(x) \notin X''_t$. This means that there is some $n \in \{1, \ldots, N\}$ and some $m \in \{1, \ldots, n\}$ such that $h^{J_{t,1}}(x) \in h^{-m}(C_{t,n})$. But then notice that $h^j(h^{J_{t,1}}(x)) \notin X''_t$ for all $j \in \{0, \ldots, n-m\}$, and so since $x \in X''_t$, this must mean that $n - m \leq J_{t,1}$. But since $h^{J_{t,1}+m}(x) \in C_{t,n}$, we have $h^{J_{t,1}+m-n}(x) \in h^{-n}(C_{t,n}) \subset X_t$, a contradiction to $\lambda_{X_t}(x) = J_{t,1}$. Thus, $\{x \in X''_t \,|\, \lambda_{X''_t}(x) = J_{t,1}\} = Y_{t,1} \cap X''_t$, and so we are free to assume that

$$\{x \in X''_t \,|\, \lambda_{X''_t}(x) = J_{t,1}\} = Y''_{t,1},$$

by combining all $Y''_{t,k}$ with $J''_{t,k} = J_{t,1}$, since this assumption does not contradict the fact that $\mathcal{P}_1(\mathcal{S}'')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and the fact that $\mathcal{P}_2(\mathcal{S}'')$ is finer than $\mathcal{P}_2(\mathcal{S})$, since we have $J''_{t,1} = J_{t,1}$ and $h^j(Y''_{t,1}) \subset h^j(Y_{t,1})$ for all $j \in \{0, \ldots, J_{t,1}\}$. Moreover, for all $k \in \{2, \ldots, K''_t\}$, we have $Y''_{t,k} \subset \bigsqcup_{l=2}^{K_t} Y_{t,k}$, and so $J''_{t,k} > N$.

Now, note that by (6), we see that

$$Z \subset \bigcup_{j=-N}^{0} \bigsqcup_{t=1}^{T''} h^j(X''_t). \tag{7}$$

Also, observe that if $t \in \{1, \ldots, T''\}$, $k \in \{1, \ldots, K''_t\}$, and $z \in Z \cap \bigcup_{j=-N}^{0} h^j(Y''_{t,k})$, then $Y''_{t,k}$ intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$.

Let $\{a(1), \ldots, a(L)\}$ be the set of all $t \in \{1, \ldots, T''\}$ such that $Z \cap \bigcup_{j=-N}^{0} h^j(Y''_{t,1}) \neq \varnothing$. For each $l \in \{1, \ldots, L\}$, define

$$X'_l = X''_{a(l)} \cap \psi^{-1}\left(Z \cap \bigcup_{j=-N}^{0} h^j(Y''_{a(l),1})\right).$$

Let $\{b(l,1), \ldots, b(l, K'_l)\}$ be the set of all $k \in \{1, \ldots, K''_{a(t)}\}$ such that $Y''_{a(l),k} \cap X'_l \neq \varnothing$, making the choice $b(l,1) = 1$. For each $k \in \{1, \ldots, K'_l\}$, define $Y'_{l,k} = Y''_{a(l),b(l,k)} \cap X'_l$ and define $J'_{l,k} = J''_{a(l),b(l,k)}$.

Set

$$\widetilde{X} = X \setminus \left(\bigsqcup_{l=1}^{L} \bigsqcup_{k=1}^{K'_l} \bigsqcup_{j=0}^{J'_{t,l}-1} h^j(Y'_{l,k})\right). \tag{8}$$

If $\widetilde{X}$ is empty, set $T' = L$. Otherwise, set $\widetilde{Z} = Z \cap \widetilde{X}$ and set $\widetilde{\psi} = \psi|_{\widetilde{X}}$. Notice that if $x \in \widetilde{X}$, then $\psi(x) \notin \bigsqcup_{l=1}^{L} \bigcup_{j=-N}^{0} h^j(Y''_{a(l),1})$, and therefore $\psi(x) \in \widetilde{Z}$. Thus, $(\widetilde{X}, h|_{\widetilde{X}})$ is an essentially minimal zero-dimensional system where $\widetilde{\psi} : \widetilde{X} \to \widetilde{Z}$ satisfies the requirements of Definition 2.19. Notice that for each $z \in \widetilde{Z}$, there is a $t \in \{1, \ldots, T''\}$ and a $k \in \{2, \ldots, K_t''\}$ such that

$$z \in \left(\widetilde{X} \cap \bigcup_{j=-N}^{0} h^j(Y''_{t,k}) \right). \tag{9}$$

Let $\{a'(1), \ldots, a'(\widetilde{T})\}$ be the set of all $t \in \{1, \ldots, T''\}$ such that $\widetilde{Z} \cap \bigcup_{j=-N}^{0} h^j(X''_t) \neq \varnothing$. For each $t \in \{1, \ldots, \widetilde{T}\}$, set $\widetilde{X}_t = \widetilde{X} \cap \left(X''_{a'(t)} \setminus Y''_{a'(t),1} \right)$. Then from (9), we see that for every $z \in \widetilde{Z}$, there is a $t \in \{1, \ldots, \widetilde{T}\}$ such that $\widetilde{X}_t$ intersects the minimal set of $(\widetilde{\psi}^{-1}(z), h|_{\widetilde{\psi}^{-1}(z)})$. Thus, by Lemma 5.7, there is a system

$$\widetilde{\mathcal{S}} = (\widetilde{T}, (\widetilde{X}_t)_{t=1,\ldots,\widetilde{T}} \ldots)$$

of finite first return time map subordinate to

$$\widetilde{\mathcal{P}} = \left\{ U \cap \widetilde{X} \,\middle|\, U \in \mathcal{P} \text{ and } U \cap \widetilde{X} \neq \varnothing \right\}.$$

For each $t \in \{1, \ldots, \widetilde{T}\}$, let $\{c(t, 1), \ldots, c(t, N_t)\}$ be the set of all $k \in \{1, \ldots, \widetilde{K}_t\}$ such that $\widetilde{Z} \cap \bigcup_{j=-N}^{0} h^j(\widetilde{Y}_{t,k}) \neq \varnothing$.

Let $t \in \{1, \ldots, \widetilde{T}\}$, let $n \in \{1, \ldots, N_t\}$ and set $s = L + n + \sum_{r=1}^{t-1} N_r$. Define

$$X'_s = \widetilde{X}_t \cap \psi^{-1} \left(\widetilde{Z} \cap \bigcup_{j=-N}^{0} h^j(\widetilde{Y}_{t,c(t,n)}) \right).$$

Let $\{d(s, 1), \ldots, d(s, K'_s)\}$ be the set of all $k \in \{1, \ldots, \widetilde{K}_t\}$ such that $\widetilde{Y}_{t,k} \cap X'_s \neq \varnothing$, where we make the choice $d(s, 1) = c(t, n)$. For each $l \in \{1, \ldots, K'_s\}$, set $Y'_{s,l} = X'_s \cap \widetilde{Y}_{t,d(s,l)}$ and set $J'_{s,l} = \widetilde{J}_{t,d(s,l)}$.

Set $T' = L + \sum_{r=1}^{\widetilde{T}} N_r$. We now check that, for (X, h), $\mathcal{S}' = (T', (X'_t)_{t=1,\ldots,T'}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by verifying the conditions of Definition 2.6. That conditions (a) and (c) hold is clear. Let $t \in \{1, \ldots, L\}$. Then $\bigcup_{j=-N}^{0} h^j(Y''_{t,1})$ is compact and open in X, and so $Z \cap \bigcup_{j=-N}^{0} h^j(Y''_{a(t),1})$ is compact and open in Z, and by the continuity of ψ, $\psi^{-1}(Z \cap \bigcup_{j=-N}^{0} h^j(Y''_{a(t),1}))$ is compact and open in X, and so X'_t is therefore compact and open in X. It is also clear by construction that X'_t is nonempty. Moreover, since $X'_t \subset X''_{a(t)}$, and $X''_{a(t)}$ is contained in an element of $\mathcal{P}$, so is X'_t. Now, let $t \in \{L + 1, \ldots, T'\}$. By the exact same reasoning, X'_t is a nonempty compact open subset of X. Since $\widetilde{\mathcal{S}}$ is subordinate to $\widetilde{\mathcal{P}}$, and since every element of $\widetilde{\mathcal{P}}$ is contained in an element of $\mathcal{P}$, we see that X'_t is contained in an element of $\mathcal{P}$. Thus, condition (b) holds. Now, let $t \in \{1, \ldots, L\}$. Observe that

$$\begin{aligned} \bigsqcup_{k=1}^{K'_t} Y'_{t,k} &= \bigsqcup_{k=1}^{K'_t} Y''_{a(t),b(t,k)} \cap X'_t \\ &= \bigsqcup_{k=1}^{K''_t} Y''_{a(t),k} \cap X'_t \end{aligned}$$

$$
\begin{aligned}
&= X''_{a(t)} \cap X'_t \\
&= X'_t
\end{aligned}
$$

and

$$
\begin{aligned}
\bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) &= \bigsqcup_{k=1}^{K'_t} h^{J'_{a(t),b(t,k)}}(Y''_{a(t),b(t,k)}) \cap X'_t \\
&= \bigsqcup_{k=1}^{K''_t} h^{J_{a(t),b(t,k)}}(Y''_{a(t),k}) \cap X'_t \\
&= X''_{a(t)} \cap X'_t \\
&= X'_t.
\end{aligned}
$$

Now, let $t \in \{L_1, \ldots, T'\}$, and let $s \in \{1, \ldots, T''\}$ and $n \in \{1, \ldots, N_s\}$ satisfy $t = L + n + \sum_{r=1}^{s-1} N_r$. Then we have

$$
\begin{aligned}
\bigsqcup_{k=1}^{K'_t} Y'_{t,k} &= \bigsqcup_{k=1}^{K'_t} \widetilde{Y}_{s,d(t,k)} \cap X'_s \\
&= \bigsqcup_{k=1}^{\widetilde{K}_s} \widetilde{Y}_{s,k} \cap X'_s \\
&= \widetilde{X}_s \cap X'_t \\
&= X'_t
\end{aligned}
$$

and

$$
\begin{aligned}
\bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) &= \bigsqcup_{k=1}^{K'_t} h^{\widetilde{J}_{s,d(t,k)}}(\widetilde{Y}_{s,d(t,k)}) \cap X'_s \\
&= \bigsqcup_{k=1}^{\widetilde{K}_s} h^{\widetilde{J}_{s,d(t,k)}}(\widetilde{Y}_{s,k}) \cap X'_s \\
&= \widetilde{X}_s \cap X'_t \\
&= X'_t.
\end{aligned}
$$

Thus, conditions (d) and (e) hold. For condition (f), for each $t \in \{1, \ldots, \widetilde{T}\}$, set $N'_t = L + \sum_{r=1}^{t} N_r$ and observe that

$$
\begin{aligned}
\widetilde{X}_t &= \widetilde{X}_t \cap \bigsqcup_{n=1}^{N_t} {}^{-1}\left(\widetilde{Z} \cap \bigcup_{j=-N}^{0} h^j(\widetilde{Y}_{t,c(t,n)})\right) \\
&= \bigsqcup_{n=1}^{N_t} \left(\widetilde{X}_t \cap {}^{-1}\left(\widetilde{Z} \cap \bigcup_{j=-N}^{0} h^j(\widetilde{Y}_{t,c(t,n)})\right)\right)
\end{aligned}
$$

$$= \bigsqcup_{s=N'_{t-1}+1}^{N'_t} \bigsqcup_{k=1}^{K'_s} \left(\widetilde{X}_t \cap \quad {}^{-1} \left(\widetilde{Z} \cap \bigcup_{j=-N}^{0} h^j(\widetilde{Y}_{t,d(s,k)}) \right) \right)$$

$$= \bigsqcup_{s=N'_{t-1}+1}^{N'_t} X'_s$$

In particular, we see that for each $k \in \{1, \ldots, \widetilde{K}_t\}$, we have $\widetilde{Y}_{t,k} \subset \bigsqcup_{s=N_{t-1}+1}^{N_t} X'_s$. With this in mind, we have

$$\begin{aligned}
\widetilde{X} &= \bigsqcup_{t=1}^{\widetilde{T}} \bigsqcup_{k=1}^{\widetilde{K}_t} \bigsqcup_{j=0}^{J_{t,k}-1} h^j(\widetilde{Y}_{t,k}) \\
&= \bigsqcup_{t=1}^{\widetilde{T}} \bigsqcup_{k=1}^{\widetilde{K}_t} \bigsqcup_{j=0}^{J_{t,k}-1} h^j \left(\bigsqcup_{s=N'_{t-1}+1}^{N'_t} X_s \cap \widetilde{Y}_{t,k} \right) \\
&= \bigsqcup_{t=1}^{\widetilde{T}} \bigsqcup_{s=N'_{t-1}}^{N'_t} \bigsqcup_{k=1}^{K'_s} \bigsqcup_{j=0}^{\widetilde{J}_{t,d(s,k)}-1} h^j \left(X_s \cap \widetilde{Y}_{t,d(s,k)} \right) \\
&= \bigsqcup_{t=L+1}^{T'} \bigsqcup_{k=1}^{K'_s} \bigsqcup_{j=0}^{J_{t,k}-1} h^j(Y'_{s,k}).
\end{aligned}$$

Thus, this combined with (8) gives us

$$\bigsqcup_{t=1}^{T'} \bigsqcup_{k=1}^{K'_s} \bigsqcup_{j=0}^{J_{t,k}-1} h^j(Y_{t,k}) = X.$$

This proves that condition (f) holds.

We now show that $\mathcal{S}'$ satisfies the conclusions of the lemma. That (a) is satisfied follows from the fact that all elements of $\mathcal{P}_1(\mathcal{S}'')$ and all elements of $\mathcal{P}_1(\widetilde{\mathcal{S}})$ are contained in elements of $\mathcal{P}_1(\mathcal{S})$, and all elements of $\mathcal{P}_2(\mathcal{S}'')$ and all elements of $\mathcal{P}_2(\widetilde{\mathcal{S}})$ are contained in elements of $\mathcal{P}_2(\mathcal{S})$. From the choices we made in our construction, (b) and (c) are satisfied. To see that (d) is satisfied, first note that for all $t \in \{1, \ldots, L\}$ and for all $k \in \{2, \ldots, K'_t\}$, we earlier showed that $J_{t,k} > N$. Next notice that for all $t \in \{1, \ldots, T''\}$ and all $x \in \bigsqcup_{k=2}^{K''_t} Y''_{t,k}$, we have $\lambda_{X''_t}(x) > N$. Thus, for all $t \in \{1, \ldots, \widetilde{T}\}$ and all $x \in \widetilde{X}_t$, we have $\lambda_{\widetilde{X}_t}(x) > N$. Thus, for all $t \in \{L+1, \ldots, T'\}$ and all $k \in \{1, \ldots, K'_t\}$, we have $J'_{t,k} > N$. For (e), notice that if $t \in \{L+1, \ldots, T'\}$, then $Y'_{t,k} > N$ for all $k \in \{1, \ldots, K'_t\}$, and so $\widehat{X}_t, h(\widehat{X}_t), \ldots, h^N(\widehat{X}_t)$ are clearly pairwise disjoint. So let $t \in \{1, \ldots, L\}$, let $x \in \widehat{X}_t$, and suppose there is some $n \in \{1, \ldots, N\}$ such that $h^n(x) \in \widehat{X}_t$. This means that $\lambda_{X'_t}(x) \leq N$, and, from our work above, we know that means $x \in Y_{a(t),1}$. By definition of $\widehat{X}_t$, this means that there is some $k \in \{2, \ldots, K'_{a(t)}\}$ such that $x \in Y_{a(t),1} \cap h^{J_{a(t),k}}(Y_{a(t),k})$. We now have two cases. First, suppose that $h^n(x) \in B_{a(t)}$. But since $x \in Y_{a(t),1} \cap h^{J_{a(t),k}}(Y_{a(t),k})$, this would mean that $h^n(x) \in C_{t,n}$, which means that $x \notin X''_t$, a contradiction. The second possibility is that $h^n(x) \in Y_{a(t),1} \cap h^{J_{a(t),l}}(Y_{a(t),l})$ for some $l \in \{2, \ldots, K_{a(t)}\}$. But then since $J_{a(t),l} > n$, we would have $x = h^{-n}(h^n(x)) \notin X_{a(t)}$, a contradiction. Thus, $\widehat{X}_t \cap h^n(\widehat{X}_t) = \varnothing$. This proves that $\mathcal{S}'$ satisfies the conclusions of the lemma and therefore proves the lemma. $\square$

Lemma 5.19. Let (X, h) be a fiberwise essentially minimal system, let $\mathcal{P}$ be a partition of X, let $N \in \mathbb{Z}_{>0}$, let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$, and let $\mathcal{S}' = (T', (X_t')_{t=1,\ldots,T'}, \ldots)$ satisfy the conclusions of Lemma 5.18. Let D be the set of all $t \in \{1, \ldots, T'\}$ such that $J_{t,1} < N$. Make the following definitions.

(a) Set $T'' = T'$.

(b) For each $t \in D$, set $X_t'' = X_t' \setminus \bigsqcup_{n=0}^{N} h^n \left(Y_{t,1}' \cap \left(\bigsqcup_{k=2}^{K_t'} h^{J_{t,k}'}(Y_{t,k}')\right)\right)$.

(c) For each $t \in \{1, \ldots, T'\} \setminus D$, set $X_t'' = X_t'$.

(d) Let $t \in D$. Let $E_t = \{a(1), \ldots, a(L_t)\}$ be the set of all $k \in \{1, \ldots, K_t'\}$ such that

$$h^{J_{t,k}'}(Y_{t,k}' \cap X_t'') \cap Y_{t,1}' \neq \varnothing.$$

Let $E_t' = \{a(L_t + 1), \ldots, a(K_t'')\}$ be the set of all $k \in \{1, \ldots, K_t'\}$ such that

$$h^{J_{t,k}'}(Y_{t,k}' \cap X_t'') \cap \left(\bigsqcup_{l=2}^{K_l'} Y_{t,l}'\right) \neq \varnothing.$$

For all $k \in \{1, \ldots, L_t\}$ such that $a(k) \neq 1$, set $Y_{t,k}'' = Y_{t,a(k)}' \cap h^{-J_{t,a(k)}'}(Y_{t,1}') \cap X_t''$ and set $J_{t,k}'' = J_{t,a(k)}' + m_t J_{t,1}'$ where m_t is the integer such that $N = m_t J_{t,l}' - m_t'$ for $0 < m_t' \leq J_{t,1}'$. If $a(k) = 1$, then set $Y_{t,k}'' = Y_{t,1}' \cap h^{-J_{t,1}'}(Y_{t,1}') \cap X_t''$ and set $J_{t,k}'' = J_{t,1}'$. For all $k \in \{L_t + 1, \ldots, K_t''\}$, set $Y_{t,k}'' = Y_{t,a(k)}' \cap h^{-J_{t,a(k)}} \left(\bigsqcup_{l=2}^{K_t'} Y_{t,l}'\right) \cap X_t''$ and set $J_{t,k}'' = J_{t,a(k)}'$.

(e) Let $t \in \{1, \ldots, T'\} \setminus D$. Set $K_t'' = K_t'$. For each $k \in \{1, \ldots, K_t'\}$, set $Y_{t,k}'' = Y_{t,k}'$ and set $J_{t,k}'' = J_{t,k}'$.

Then $\mathcal{S}'' = (T'', (X_t'')_{t=1,\ldots,T''}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$. Moreover, the partition $\mathcal{P}_1(\mathcal{S}'')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and the partition $\mathcal{P}_2(\mathcal{S}'')$ is finer than $\mathcal{P}_2(\mathcal{S})$.

Proof. We check that $\mathcal{S}'' = (T'', (X_t'')_{t=1,\ldots,T''}, \ldots)$ is a system of finite first return time maps subordinate to $\mathcal{P}$ by verifying the conditions of Definition 2.6. It is clear that (a) holds. Moreover, for $t \in \{1, \ldots, T''\} \setminus D$, it is clear that conditions (b) – (e) hold, so we check conditions (b) – (e) for $t \in D$.

For each $t \in D$, we have $X_t'' \subset X_t'$, so since X_t' is contained in an element of $\mathcal{P}$, so is X_t''. It is also clear that X_t'' is a compact open subset of X. Thus, condition (b) holds.

For each $t \in D$ and each $k \in \{1, \ldots, K_t'\}$, we must have $h^{J_{t,k}'}(Y_{t,k}') \cap Y_{t,l}'$ for some $l \in \{1, \ldots, K_t'\}$, and so either $k \in E_t$ or $k \in E_t'$ (or both). In particular, at least one of E_t and E_t' is nonempty, and so $K_t'' > 0$. Thus, condition (c) holds.

Let $t \in D$. For each $k \in \{1, \dots, K''_t\}$, it is clear that $Y''_{t,k}$ is a compact open subset of X. Now, for each $k \in \{1, \dots, K'_t\}$, notice that

$$\left(h^{J'_{t,k}}(Y'_{t,k}) \cap Y'_{t,1}\right) \sqcup \left(h^{J'_{t,k}}(Y'_{t,k}) \cap \left(\bigsqcup_{l=2}^{K'_t} Y'_{t,l}\right)\right) = h^{J'_{t,k}}(Y'_{t,k}).$$

Thus,

$$\begin{aligned} X'_t &= \bigsqcup_{k=1}^{K'_t} h^{J'_{t,k}}(Y'_{t,k}) \\ &= \bigsqcup_{k=1}^{K'_t} \left(\left(h^{J'_{t,k}}(Y'_{t,k}) \cap Y'_{t,1}\right) \sqcup \left(h^{J'_{t,k}}(Y'_{t,k}) \cap \left(\bigsqcup_{l=2}^{K'_t} Y'_{t,l}\right)\right)\right). \end{aligned}$$

And so we have

$$\begin{aligned} X''_t &= X'_t \cap X''_t \\ &= X''_t \cap \bigsqcup_{k=1}^{K'_t} \left(\left(h^{J'_{t,k}}(Y'_{t,k}) \cap Y'_{t,1}\right) \sqcup \left(h^{J'_{t,k}}(Y'_{t,k}) \cap \left(\bigsqcup_{l=2}^{K'_t} Y'_{t,l}\right)\right)\right) \\ &= \bigsqcup_{k=1}^{K'_t} \left(\left(h^{J'_{t,k}}(Y'_{t,k}) \cap Y'_{t,1} \cap X''_t\right) \sqcup \left(h^{J'_{t,k}}(Y'_{t,k}) \cap \left(\bigsqcup_{l=2}^{K'_t} Y'_{t,l}\right) \cap X''_t\right)\right) \\ &= \left(\bigsqcup_{k=1}^{L_t} Y''_{t,k}\right) \sqcup \left(\bigsqcup_{k=L_t+1}^{K''_t} Y''_{t,k}\right) \\ &= \bigsqcup_{k=1}^{K''_t} Y_{t,k}. \end{aligned}$$

Thus, condition (d) holds.

Let $t \in D$. Let $\widehat{X}_t$ be as in the statement of Lemma 5.18. We claim that for all $k \in \{1, \dots, K''_t\}$, we have $h^{J''_{t,k}}(Y''_{t,k}) \subset X''_t$. First, suppose $k \in \{1, \dots, L_t\}$ and $a(k) \neq 1$. Then

$$h^{J'_{t,a(k)}}(Y''_{t,k}) \subset Y'_{t,1} \cap h^{J'_{t,a(k)}}(Y'_{t,a(k)}) \subset \widehat{X}_t.$$

We claim that

$$\left(\bigsqcup_{n=0}^{N} h^n(Y'_{t,1} \cap h^{J'_{t,a(k)}}(Y'_{t,a(k)}))\right) \cap X'_t \subset Y'_{t,1} \tag{10}$$

Otherwise, there is some integer p such that $0 \leq pJ'_{t,1} \leq N$ and

$$h^{mJ'_{t,1}}(Y'_{t,1} \cap h^{J'_{t,a(k)}}(Y'_{t,a(k)})) \subset Y'_{t,l} \cap h^{J'_{t,1}}(Y'_{t,1})$$

But then $h^{mJ'_{t,1}}(\widehat{X}_t) \cap \widehat{X}_t \neq \varnothing$, a contradiction to the fact that $\mathcal{S}'$ satisfies (e) of Lemma 5.18. Thus,

(10) holds. So since $(m_t - 1)J'_{t,1} \leq N$, we have

$$h^{(m_t-1)J'_{t,1}}(Y'_{t,1} \cap h^{J'_{t,a(k)}}(Y'_{t,a(k)})) \subset Y'_{t,1}$$

and so therefore

$$\begin{aligned} h^{J''_{t,k}}(Y''_{t,k}) &= h^{J'_{t,a(k)}+m_tJ'_{t,1}}(Y''_{t,k}) \\ &= h^{m_tJ'_{t,1}}(Y'_{t,1} \cap h^{J'_{t,a(k)}}(Y'_{t,a(k)})) \\ &\subset h^{J'_{t,1}}(Y'_{t,1}) \\ &\subset X'_t \end{aligned}$$

We claim that

$$h^{J''_{t,k}}(Y''_{t,k}) \cap \left(\bigsqcup_{n=0}^{N} h^n \left(Y'_{t,1} \cap \left(\bigsqcup_{l=2}^{K_t} h^{J'_{t,l}}(Y'_{t,l}) \right) \right) \right) \neq \varnothing. \tag{11}$$

Suppose not. Then there is some $n \in \{0, \ldots, N\}$ and some $l \in \{2, \ldots, K'_t\}$ such that

$$h^{J''_{t,k}-n}(Y''_{t,k}) \cap Y'_{t,1} \cap h^{J'_{t,l}}(Y'_{t,l}) \neq \varnothing.$$

But notice that

$$h^{J''_{t,k}-m_tJ'_{t,1}}(Y_{t,k}) = J_{t,a(k)}(Y''_{t,k}) \cap Y'_{t,1}$$

and so since $m_tJ'_{t,1} - n < J'_{t,1} + J'_{t,l}$, it must be the case that

$$h^{J''_{t,k}-n}(Y''_{t,k}) = h^{m_tJ'_{t,1}-n}(h^{J_{t,a(k)}}(Y_{t,a(k)}) \cap Y'_{t,1})$$

is disjoint from $h^{J_{t,l}}(Y_{t,l})$, which is a contradiction. Thus, (11) holds. Putting (10) and (11) together, we see $h^{J''_{t,k}}(Y''_{t,k}) \subset X''_t$. Next, suppose $k \in \{1, \ldots, L_t\}$ and $a(k) = 1$.

Now, let $k \in \{L_t + 1, \ldots, K''_t\}$. Then $h^{J''_{t,k}}(Y''_{t,k}) = h^{J'_{t,a(k)}}(Y'_{t,a(k)}) \cap \bigsqcup_{l=2}^{K'_t}(Y_{t,l})$, and so by (10), we have $h^{J''_{t,k}}(Y''_{t,k}) \subset X''_t$. Altogether, we see that for all $k \in \{1, \ldots, K''_t\}$, we have

$$h^{J''_{t,k}}(Y''_{t,k}) \subset X''_t. \tag{12}$$

Next, we claim that

$$X''_t \subset \bigsqcup_{k=1}^{K''_t} h^{J''_{t,k}}(Y''_{t,k}) \tag{13}$$

Let $x \in X''_t$. Since $X''_t \subset X'_t$, there is some $l \in \{1, \ldots, K_t\}$ such that $x \in h^{J'_{t,l}}(Y'_{t,l})$. Suppose first that $l \neq 1$. Notice that by definition of X''_t, we have $x \notin Y'_{t,1}$. By (10), we have $h^{-J'_{t,l}}(x) \in X''_t$. This means that $h^{-J'_{t,l}}(x) \in Y'_{t,l} \cap h^{-J_{t,l}}\left(\bigsqcup_{m=2}^{K_t} Y'_{t,m}\right) \cap X'_t$, and so $h^{-J'_{t,l}}(x) \in Y''_{t,k}$ where $k \in \{L_t+1, \ldots, K''_t\}$ and $a(k) = l$. Since we have $J''_{t,k} = J'_{t,l}$, it follows that $x \in h^{J''_{t,k}}(Y''_{t,k})$. Now suppose that $x \in h^{J'_{t,1}}(Y'_{t,1})$ and $h^{-J'_{t,1}}(x) \in X''_t$. Then since $\lambda_{X''_t}(x) \geq \lambda_{X'_t}(x)$, we must have $\lambda_{X''_t}(h^{-J'_{t,1}}(x)) = J'_{t,1}$, and so we must have $x \in h^{J''_{t,k}}(Y''_{t,k})$ where $k \in \{1, \ldots, L_t\}$ is such that $a(k) = 1$.

Finally, suppose that $x \in h^{J'_{t,1}}(Y'_{t,1})$ and $h^{J'_{t,1}}(x) \notin X''_t$. Then there is some $l \in \{1, \ldots, K'_t\}$ such that $h^{-m_t J'_{t,1}}(x) \in h^{J'_{t,l}}(Y'_{t,l}) \cap Y'_{t,1}$, and so it is clear by definition that $x \in h^{J''_{t,k}}(Y''_{t,k})$ where $k \in \{L_1 + 1, \ldots, K''_t\}$ is such that $a(k) = l$. Finally, notice that for $k, l \in \{1, \ldots, K''_t\}$ with $k \neq l$, we have

$$h^{J'_{t,a(k)}}(Y_{t,a(k)}) \cap h^{J_{t,a(l)}}(Y_{t,a(l)}) = \varnothing,$$

so

$$h^{J'_{t,a(k)}+m_t J'_{t,1}}(Y_{t,a(k)}) \cap h^{J_{t,a(l)}+m_t J'_{t,1}}(Y_{t,a(l)}) = \varnothing,$$

and therefore

$$h^{J''_{t,k}}(Y''_{t,k}) \cap h^{J''_{t,l}}(Y''_{t,l}) = \varnothing. \tag{14}$$

From (12), (13), and (14), we see that condition (e) holds.

Notice that since $T' = T''$ and since $X''_t \subset X'_t$ for all $t \in \{1, \ldots, T'\}$, it is clear that

$$\bigcup_{i=0}^{\infty} h^i(X'_t) = \bigcup_{i=0}^{\infty} h^i(X''_t) \tag{15}$$

for all $t \in \{1, \ldots, T'\}$. Now, let $t \in D$ and let $k \in \{1, \ldots, L_t\}$. Then by (10), we have

$$\bigsqcup_{j=0}^{J''_{t,k}-1} h^j(Y''_{t,k}) \subset \left(\bigsqcup_{j=0}^{J_{t,a(k)}-1} h^j(Y'_{t,a(k)}) \right) \cup \left(\bigsqcup_{j=0}^{J'_{t,1}-1} h^j(Y'_{t,1}) \right). \tag{16}$$

Let $x \in X$. Then there is exactly one $t \in \{1, \ldots, T'\}$, exactly one $k \in \{1, \ldots, K'_t\}$, and exactly one $j \in \{0, \ldots, J'_{t,k} - 1\}$ such that $x \in h^j(Y'_{t,k})$. We keep x, t, k, and j until we finish verifying condition (f).

Suppose $t \notin D$. By (15) and by the fact that $h^i(Y''_{t,l}) = h^i(Y'_{t,l})$ for all $l \in \{1, \ldots, K'_t\}$ and all $i \in \{0, \ldots, J'_{t,l} - 1\}$, it follows that t is the unique element of $\{1, \ldots, T''\}$, k is the unique element of $\{1, \ldots, K''_t\}$, and j is the unique element of $\{0, \ldots, J''_{t,k} - 1\}$ such that $x \in h^j(Y''_{t,k})$.

Now, suppose $t \in D$, suppose $k > 1$, and suppose $h^{J'_{t,k}-j}(x) \in Y'_{t,1}$. Then by definition, $h^{-j}(x) \in Y''_{t,l}$ where $l \in \{1, \ldots, L_t\}$ is such that $a(l) = k$. Then this, combined with (15), (16), and the fact that $J''_{t,l} > J'_{t,k}$ tells us that t is the unique element of $\{1, \ldots, T''\}$, l is the unique element of $\{1, \ldots, K''_t\}$, and j is the unique element of $\{0, \ldots, J''_{t,l} - 1\}$ such that $x \in h^j(Y''_{t,l})$.

Next, suppose $t \in D$, suppose $k > 1$, and suppose $h^{J'_{t,k}-j}(x) \notin Y'_{t,1}$. Then by definition, $h^{-j}(x) \in Y''_{t,l}$ where $l \in \{L_t + 1, \ldots, K''_t\}$ and $a(l) = k$. Then this, combined with (15), (16), and the fact that $J''_{t,l} = J'_{t,k}$ tells us that t is the unique element of $\{1, \ldots, T''\}$, l is the unique element of $\{1, \ldots, K''_t\}$, and j is the unique element of $\{0, \ldots, J''_{t,l} - 1\}$ such that $x \in h^j(Y''_{t,l})$.

Suppose $t \in D$, suppose $k = 1$, and suppose $h^{-j}(x) \in \bigsqcup_{n=0}^{N} h^n(\widehat{X}_t)$. There there is exactly one $n \in \{0, \ldots, N + J'_{t,1} - 1\}$ and exactly one $l \in \{2, \ldots, K'_t\}$ such that $h^{-n}(x) \in h^{J'_{t,l}}(Y'_{t,l}) \cap Y'_{t,1}$. Set $i = J'_{t,l} + n$ let m be the unique element of $\{1, \ldots, L_1\}$ such that $a(m) = l$. Then this, combined with (15), (16), and the fact that $J''_{t,l} = J'_{t,k}$ tells us that t is the unique element of $\{1, \ldots, T''\}$, m is the unique element of $\{1, \ldots, K''_t\}$, and i is the unique element of $\{0, \ldots, J''_{t,m} - 1\}$ such that $x \in h^i(Y''_{t,m})$.

Finally, suppose $t \in D$, suppose $k = 1$, and suppose $h^{-j}(x) \notin \bigsqcup_{n=0}^{N} h^n(\widehat{X}_t)$. Then $h^{-j}(x) \in X_t''$, and so by $\square$

Lemma 5.20. Let (X, h) be a fiberwise essentially minimal zero-dimensional system with no periodic points and let $N \in \mathbb{Z}_{>0}$. Then there is a partition $\mathcal{P}$ of X such that, for every system $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$, we have $J_{t,k} > N$ for all $t \in \{1, \ldots, T\}$ and all $k \in \{1, \ldots, K_t\}$.

Proof. For each $x \in X$, since x is aperiodic, there is a compact open neighborhood U_x such that $U_x, h(U_x), \ldots, h^N(U_x)$ are pairwise disjoint. Then $(U_x)_{x \in X}$ is a compact open cover of X, and hence has a finite compact open refinement. By taking appropriate intersections, this refinement can be taken to be a partition $\mathcal{P}$ of X.

Now let $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ be a system of finite first return time maps subordinate to $\mathcal{P}$. By the above, for each $t \in \{1, \ldots, T\}$, we have $X_t, h(X_t), \ldots, h^N(X_t)$ are pairwise disjoint. Thus, it is clear that for each $t \in \{1, \ldots, T\}$ and each $k \in \{1, \ldots, K_t\}$, we have $J_{t,k} > N$. $\square$

Proof of Theorem 3.2. Let $\varepsilon > 0$, let $N \in \mathbb{Z}_{>0}$ satisfy $\pi/N < \varepsilon$, and let $\mathcal{P}$ be a partition of X. By possibly passing to a finer partition, we may assume that $\mathcal{P}$ satisfies the conclusion of Lemma 5.20. Following the proof of Theorem 2.1 of [9], we will show that there is a C^*-subalgebra A of $C^*(\mathbb{Z}, X, h)$ which is isomorphic to a direct sum of matrix algebras and matrix algebras over $C(S^1)$ such that $C(\mathcal{P}) \subset A$ and such that A contains a unitary u' such that $\|u' - u\| < \varepsilon$. By using the semiprojectivity of circle algebras to construct a direct system, this will imply that $C^*(\mathbb{Z}, X, h)$ is an A$\mathbb{T}$-algebra.

Let Z and ψ be as in Definition 2.19. Use Lemma 5.8 to find a system $\mathcal{S}'' = (T'', (X_t'')_{t=1,\ldots,T''}, \ldots)$ of finite first return time maps satisfying the conclusions of the lemma. By applying Proposition 2.10, we are free to assume that both $\mathcal{P}_1(\mathcal{S}'')$ and $\mathcal{P}_2(\mathcal{S}'')$ are finer than $\mathcal{P}$. By applying Lemma 5.18 with $\mathcal{S}''$ in place of $\mathcal{S}$, we get a system $\mathcal{S} = (T, (X_t)_{t=1,\ldots,T}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ satisfying the conclusions of the lemma. By conclusion (b) of Lemma 5.18, $\mathcal{S}''$ still satisfies conclusion (c) of Lemma 5.8. Now, notice the conclusion (c) of Lemma 5.18 says that for each $t \in \{1, \ldots, T\}$ and each $z \in \psi(X_t)$, $Y_{t,1}'$ intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$, and hence, $h^{J_{t,1}}(Y_{t,1})$ intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$. Since by Proposition 4.2(b) we have $\bigsqcup_{t=1}^{T} \bigcup_{j \in \mathbb{Z}} h^j(X_t) = X$, it follows that for every $z \in Z$, there is a $t \in \{1, \ldots, T\}$ such that $h^{J_{t,1}}(Y_{t,1})$ intersects the minimal set of $(\psi^{-1}(z), h|_{\psi^{-1}(z)})$. Thus, we can apply Lemma 5.7 with $h^{J_{1,1}}(Y_{1,1}), \ldots, h^{J_{T,1}}(Y_{T,1})$ in place of $X_1, \ldots, X_T$ to get a system $\mathcal{S}' = (T', (X_t')_{t=1,\ldots,T'}, \ldots)$ of finite first return time maps subordinate to $\mathcal{P}$ where $T' = T$ and $X_t' = h^{J_{t,1}}(Y_{t,1})$ for all $t \in \{1, \ldots, T'\}$. By applying Proposition 2.10, we may assume that $\mathcal{P}_1(\mathcal{S}')$ is finer than both $\mathcal{P}_1(\mathcal{S})$ and $\mathcal{P}_2(\mathcal{S})$.

For each $t \in \{1, \ldots, T\}$, each $k \in \{1, \ldots, K_t\}$, and each $i, j \in \{0, \ldots, J_{t,k} - 1\}$, define

$$e_{i,j}^{(t,k)} = \chi_{h^i(Y_{t,k})} u^{i-j} \chi_{h^j(Y_{t,k})}.$$

We claim that these elements are matrix units for a finite dimensional C^*-subalgebra of $C^*(\mathbb{Z}, X, h)$ (which we will denote by A_1) isomorphic to $\bigoplus_{t=1}^{T} \bigoplus_{k=1}^{K_t} M_{J_{t,k}}$. To see this, let $t, t' \in \{1, \ldots, T\}$,

$k \in \{1, \ldots, K_t\}$, $k' \in \{1, \ldots, K_{t'}\}$, $i, j \in \{0, \ldots, J_{t,k} - 1\}$, and $i', j' \in \{0, \ldots, J_{t',k'} - 1\}$, and observe the following:

$$e_{i,j}^{(t,k)} e_{i',j'}^{(t',k')} = \chi_{h^i(Y_{t,k})} u^{i-j} \chi_{h^j(Y_{t,k})} \chi_{h^{i'}(Y_{t',k'})} u^{i'-j'} \chi_{h^{j'}(Y_{t',k'})}.$$

Note that $\chi_{h^j(Y_{t,k})} \chi_{h^{i'}(Y_{t',k'})}$, and hence $e_{i,j}^{(t,k)} e_{i',j'}^{(t',k')}$, is 0 unless $t = t'$, $k = k'$, and $j = i'$, as $\mathcal{P}_1(\mathcal{S})$ is a partition of X. In the case where we do have these equalities, we have

$$\begin{aligned} e_{i,j}^{(t,k)} e_{j,j'}^{(t,k)} &= \chi_{h^i(Y_{t,k})} u^{i-j} \chi_{h^j(Y_{t,k})} \chi_{h^j(Y_{t,k})} u^{j-j'} \chi_{h^{j'}(Y_{t,k})} \\ &= \chi_{h^i(Y_{t,k})} u^{i-j} \chi_{h^j(Y_{t,k})} u^{j-j'} \chi_{h^{j'}(Y_{t,k})} \\ &= \chi_{h^i(Y_{t,k})} u^{i-j'} \chi_{h^{j'}(Y_{t,k})} \\ &= e_{i,j'}^{(t,k)}. \end{aligned}$$

Thus, we indeed do have a system of matrix units for a C^*-subalgebra of $C^*(\mathbb{Z}, X, h)$, which we will call A_1, that is isomorphic to $\bigoplus_{t=1}^{T} \bigoplus_{k=1}^{K_t} M_{J_{t,k}}$. Notice that $C(\mathcal{P}_1(\mathcal{S}))$ is equal to the set of diagonal matrices in A_1. Since by $\mathcal{P}_1(\mathcal{S})$ is finer than $\mathcal{P}$, we have $C(\mathcal{P}) \subset C(\mathcal{P}_1(\mathcal{S}))$, and so it follows that

$$C(\mathcal{P}) \subset A_1. \tag{17}$$

Define an element $v_1 \in A_1$ by

$$v_1 = \sum_{t=1}^{T} \sum_{k=1}^{K_t} \left(\chi_{Y_{t,k}} u^{1-J_{t,k}} \chi_{h^{J_{t,k}-1}(Y_{t,k})} + \sum_{j=0}^{J_{t,k}-2} \chi_{h^{j+1}(Y_{t,k})} u \chi_{h^j(Y_{t,k})} \right).$$

To see what v_1 does, let $t \in \{1, \ldots, T\}$ and let $k \in \{1, \ldots, K_t\}$, and observe that for $j \in \{0, \ldots, J_{t,k}-2\}$ we have

$$\begin{aligned} v_1 \chi_{h^j(Y_{t,k})} v_1^* &= \chi_{h^{j+1}(Y_{t,k})} u \chi_{h^j(Y_{t,k})} u^* \chi_{h^{j+1}(Y_{t,k})} \\ &= \chi_{h^{j+1}(Y_{t,k})}. \end{aligned} \tag{18}$$

We also have

$$\begin{aligned} v_1 \chi_{h^{J_{t,k}-1}(Y_{t,k})} v_1^* &= \chi_{Y_{t,k}} u^{1-J_{t,k}} \chi_{h^{J_{t,k}-1}(Y_{t,k})} u^{J_{t,k}-1} \chi_{Y_{t,k}} \\ &= \chi_{Y_{t,k}}. \end{aligned} \tag{19}$$

Define $u_1 = v_1^* u$. To see what u_1 does, let $t \in \{1, \ldots, T\}$ and let $k \in \{1, \ldots, K_t\}$, and observe that for $j \in \{0, \ldots, J_{t,k} - 2\}$ we have

$$\begin{aligned} u_1 \chi_{h^j(Y_{t,k})} u_1^* &= v_1^* u \chi_{h^j(Y_{t,k})} u^* v_1 \\ &= v_1^* \chi_{h^{j+1}(Y_{t,k})} v_1 \\ &= \chi_{h^j(Y_{t,k})}, \end{aligned} \tag{20}$$

where the step line is justified by (18). We also have

$$
\begin{aligned}
u_1\chi_{h^{-1}(Y_{t,k})}u_1^* &= v_1^*u\chi_{h^{-1}(Y_{t,k})}u^*v_1 \\
&= v_1^*\chi_{Y_{t,k}}v_1 \\
&= \chi_{h^{J_{t,k}-1}(Y_{t,k})}, \qquad (21)
\end{aligned}
$$

where the step line is justified by (19).

Using $\mathcal{S}'$, we can similarly construct A_2, v_2, and u_2 in analogy with the above. To be specific, A_2 is the finite dimensional C^*-algebra generated by the matrix units

$$
e_{i,j}^{(t,k)\prime} = \chi_{h^i(Y'_{t,k})}u^{i-j}\chi_{h^j(Y'_{t,k})}
$$

for $t \in \{1, \ldots, T'\}$, $k \in \{1, \ldots, K'_t\}$, and $j \in \{0, \ldots, J'_{t,k}\}$. Moreover, we define

$$
v_2 = \sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{Y'_{t,k}}u^{1-J'_{t,k}}\chi_{h^{J'_{t,k}-1}(Y'_{t,k})} + \sum_{j=0}^{J'_{t,k}-2}\chi_{h^{j+1}(Y'_{t,k})}u\chi_{h^j(Y'_{t,k})}\right),
$$

and we define $u_2 = v_2^*u$.

Let $t \in \{1, \ldots, T'\}$ and let $k \in \{1, \ldots, K'_t\}$. To see what v_2 does, observe that for $j \in \{0, \ldots, J'_{t,k} - 2\}$ we have

$$
\begin{aligned}
v_2\chi_{h^j(Y'_{t,k})}v_2^* &= \chi_{h^{j+1}(Y'_{t,k})}u\chi_{h^j(Y'_{t,k})}u^*\chi_{h^{j+1}(Y'_{t,k})} \\
&= \chi_{h^{j+1}(Y'_{t,k})}. \qquad (22)
\end{aligned}
$$

We also have

$$
\begin{aligned}
v_2\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}v_2^* &= \chi_{Y'_{t,k}}u^{1-J'_{t,k}}\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}u^{J'_{t,k}-1}\chi_{Y'_{t,k}} \\
&= \chi_{Y'_{t,k}}. \qquad (23)
\end{aligned}
$$

To see what u_2 does, for $j \in \{0, \ldots, J'_{t,k} - 2\}$, we have

$$
\begin{aligned}
u_2\chi_{h^j(Y'_{t,k})}u_2^* &= v_2^*u\chi_{h^j(Y'_{t,k})}u^*v_2 \\
&= v_2^*\chi_{h^{j+1}(Y'_{t,k})}v_2 \\
&= \chi_{h^j(Y'_{t,k})}, \qquad (24)
\end{aligned}
$$

where the last step is justified by (22). We also have

$$
\begin{aligned}
u_2\chi_{h^{-1}(Y'_{t,k})}u_2^* &= v_2^*u\chi_{h^{-1}(Y'_{t,k})}u^*v_2 \\
&= v_2^*\chi_{Y'_{t,k}}v_2
\end{aligned}
$$

$$= \chi_{h^{J'_{t,k}-1}(Y'_{t,k})}, \tag{25}$$

where the last step is justified by (23).

Since $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and since $X'_t \subset X_t$ for all $t \in \{1, \ldots, T'\}$, we have $A_1 \subset A_2$. Now consider the unitary $v_2 v_1^*$, which is in A_2. Before our computations, first note that if $U \subset X$ is a compact open set with $U \cap \bigsqcup_{t=1}^{T'} h^{-1}(X'_t) = \varnothing$, then

$$v_2 \chi_U v_2^* = \chi_{h(U)}. \tag{26}$$

Let $t \in \{1, \ldots, T\}$. We have

$$\begin{aligned}
v_2 v_1^* \chi_{Y_{t,1}} v_1 v_2^* &= v_2 \chi_{h^{J_{t,1}-1}(Y_{t,1})} v_2^* && \text{by (19)} \\
&= v_2 \chi_{h^{-1}(X'_t)} v_2^* \\
&= \sum_{k=1}^{K'_t} \left(v_2 \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} v_2^* \right) \\
&= \sum_{k=1}^{K'_t} \chi_{Y'_{t,k}} && (27) \\
&= \chi_{X'_t} \\
&= \chi_{h^{J_{t,1}}(Y_{t,1})}. && (28)
\end{aligned}$$

Now let $k \in \{2, \ldots, K'_t\}$. Since $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$ and since $h^{J_{t,1}}(Y_{t,1}) = X'_t = \bigsqcup_{l=1}^{K'_t} Y'_{t,l}$, there is a set

$$F_{t,k} \subset \left\{ (s, l, j) \,\middle|\, s \in \{1, \ldots, T'\},\ l \in \{1, \ldots, K'_s\}, \text{ and } j \in \{1, \ldots, J'_{s,l} - 1\} \right\}$$

such that

$$h^{J_{t,k}}(Y_{t,k}) = \bigsqcup_{(s,l,j) \in F_{t,k}} h^j(Y'_{s,l}).$$

$$\begin{aligned}
v_2 v_1^* \chi_{Y_{t,k}} v_1 v_2^* &= v_2 \chi_{h^{J_{t,k}-1}(Y_{t,k})} v_2^* \\
&= \sum_{(s,l,j) \in F_{t,k}} v_2 \chi_{h^{j-1}(Y'_{s,l})} v_2^* \\
&= \sum_{(s,l,j) \in F_{t,k}} \chi_{h^j(Y'_{s,l})} && \text{by (26)} \\
&= \chi_{h^{J_{t,k}}(Y_{t,k})}. && (29)
\end{aligned}$$

In particular, by (28) and (29), we see that

$$v_2 v_1^* \chi_{X_t} v_1 v_2^* = \chi_{X_t}. \tag{30}$$

Set

$$Y = \bigsqcup_{t=1}^{T} X_t$$

Recall that $A_1 \subset A_2$, and so χ_Y and v_1 are elements of A_2. Thus, (30) tells us that $\chi_Y v_2 v_1^* \chi_Y$ is a unitary in $\chi_Y A_2 \chi_Y$.

Set $v = \chi_Y v_2 v_1^* \chi_Y$. Since $\chi_Y A_2 \chi_Y$ is a finite dimensional C^*-algebra, v has finite spectrum. By Lemma 5.9, there is a unitary w in $\chi_Y A \chi_Y$ with $w^N = v$ and $\|w - \chi_Y\| \leq \pi/N < \varepsilon$.

Define

$$z = \sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j w^{N-j} u^{-j} \chi_{h^j(Y)} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)}.$$

It is easy to see that z is a unitary, since $z = \sum_{j=0}^{N} z_j$ for unitaries $z_j \in \chi_{h^j(Y)} C^*(\mathbb{Z}, X, h) \chi_{h^j(Y)}$ for $j \in \{0, \ldots, N-1\}$ and a unitary $z_N = \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)}$. We claim that A, the C^*-algebra generated by zA_1z^* and u_2, has the desired properties. Specifically, we claim that

$$A \cong \bigoplus_{t=1}^{T} \left(\left(C(S^1) \otimes M_{J_{t,1}} \right) \oplus \left(\bigoplus_{k=2}^{K_t} M_{J_{t,k}} \right) \right),$$

A contains $C(\mathcal{P})$, and A contains a unitary u' such that $\|u' - u\| < \varepsilon$.

First, we want $C(\mathcal{P}) \subset A$. Because $\mathcal{P}_1(\mathcal{S})$ is finer than $\mathcal{P}$, we have $C(\mathcal{P}) \subset A_1$, so all that is left to show is that z commutes with $C(\mathcal{P})$. To see this, let $U \in \mathcal{P}$. Since for all $t \in \{1, \ldots, T\}$ and for all $n \in \{0, \ldots, N-1\}$, $h^n(X_t)$ is contained in an element of $\mathcal{P}$, we can write $U = \bigsqcup_{r=0}^{R} U_r$ where $U_0 \subset X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)$ and for all $r \in \{1, \ldots, R\}$, there are $q_r \in \{1, \ldots, T\}$ and $m_r \in \{0, \ldots, N-1\}$ such that $U_r = h^{m_r}(X_{q_r})$. By (30) and (28), we know that v commutes with $\chi_{h^{-m_r}(U_r)}$ for every $r \in \{1, \ldots, R\}$. So by Lemma 5.9, w commutes with $\chi_{h^{-m_r}(U_r)}$ for all $r \in \{1, \ldots, R\}$ as well. We now have:

$$\begin{aligned}
\chi_U z &= \chi_U \left(\sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j w^{N-j} u^{-j} \chi_{h^j(Y)} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right) \\
&= \chi_U \left(\sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j w^{N-j} \chi_Y u^{-j} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right) \\
&= \chi_U \left(\sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j \chi_Y w^{N-j} u^{-j} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right) \\
&= \chi_U \left(\sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j w^{N-j} u^{-j} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right) \\
&= \left(\sum_{r=0}^{R} \chi_{U_r} \right) \left(\sum_{j=0}^{N-1} \sum_{t=1}^{T} \chi_{h^j(Y)} u^j w^{N-j} u^{-j} \chi_{h^j(Y)} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r=1}^{R} \chi_{U_r} u^{m_r} w^{N-m_r} u^{-m_r} \chi_{h^{m_r}(Y)} + \chi_{U_0} \\
&= \sum_{r=1}^{R} \chi_{U_r} u^{m_r} \chi_{h^{-m_r}(U_r)} w^{N-m_r} u^{-m_r} \chi_{h^{m_r}(Y)} + \chi_{U_0} \\
&= \sum_{r=1}^{R} \chi_{U_r} u^{m_r} w^{N-m_r} \chi_{h^{-m_r}(U_r)} u^{-m_r} \chi_{h^{m_r}(Y)} + \chi_{U_0} \\
&= \sum_{r=1}^{R} \chi_{U_r} u^{m_r} w^{N-m_r} u^{-m_r} \chi_{U_r} + \chi_{U_0}.
\end{aligned}
$$

A similar computation yields the same thing for $z\chi_U$. Thus, z commutes with χ_U for all $U \in \mathcal{P}$, which shows that z commutes with $C(\mathcal{P})$.

Now, we define $u' = zv_1z^*u_2$, a unitary in A. We still must show that $\|u' - u\| < \varepsilon$. We have

$$
\begin{aligned}
\|u' - u\| &= \|zv_1z^*u_2 - u\| \\
&= \|zv_1z^*u_2 - v_2u_2\| \\
&= \|zv_1z^* - v_2\| \\
&= \|zv_1z^* - v_2zz^*\| \\
&= \|zv_1 - v_2z\|. \qquad (31)
\end{aligned}
$$

We will now show that $\|zv_1 - v_2z\| < \varepsilon$.

Now, notice that for each $t \in \{1, \dots, T\}$ and each $k \in \{1, \dots, K_t\}$, we have $n < J_{t,k} - 1$ by (**??**). Thus, we have

$$
\begin{aligned}
v_1\chi_{h^n(Y)} &= \left(\sum_{t=1}^{T} \sum_{k=1}^{K_t} \left(\chi_{Y_{t,k}} u^{1-J_{t,k}} \chi_{h^{J_{t,k}-1}(Y_{t,k})} + \sum_{j=0}^{J_{t,k}-2} \chi_{h^{j+1}(Y_{t,k})} u \chi_{h^j(Y_{t,k})} \right) \right) \left(\sum_{t=1}^{T} \sum_{k=1}^{K_t} \chi_{h^n(Y_{t,k})} \right) \\
&= \left(\sum_{t=1}^{T} \sum_{k=1}^{K_t} \sum_{j=0}^{J_{t,k}-2} \chi_{h^{j+1}(Y_{t,k})} u \chi_{h^j(Y_{t,k})} \right) \left(\sum_{t=1}^{T} \sum_{k=1}^{K_t} \chi_{h^n(Y_{t,k})} \right) \\
&= \sum_{t=1}^{T} \sum_{k=1}^{K_t} \chi_{h^{n+1}(Y_{t,k})} u \chi_{h^n(Y_{t,k})} \\
&= \sum_{t=1}^{T} \sum_{k=1}^{K_t} u \chi_{h^n(Y_{t,k})} \\
&= u\chi_{h^n(Y)}. \qquad (32)
\end{aligned}
$$

Now, since $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$, so $h^n(Y)$ is the union of some of the members of $\mathcal{P}_1(\mathcal{S}')$. Also, if $t \in \{1, \dots, T'\}$, $k \in \{1, \dots, K'_t\}$, and $j \in \{0, \dots, J'_{t,k} - 1\}$ are such that $h^j(Y'_{t,k}) \subset h^n(Y)$, then actually we have $j \neq J'_{t,k} - 1$, since

$$
h^{J'_{t,k}-1}(Y'_{t,k}) \subset h^{-1}(X'_t)
$$

$$
\begin{aligned}
&= h^{J_{t_s,k_s}-1}(Y_{t_s,k_s}) \\
&\subset h^{-1}(X_{t_s}),
\end{aligned}
$$

and $h^{-1}(X_t)$ trivially intersects $h^n(X_t)$ since $J_{t,k} > N$ for all $k \in \{1, \dots, K_t\}$. This means that there is a set

$$
F_n \subset \big\{(t,k,j) \,\big|\, t \in \{1,\dots,T'\},\ k \in \{1,\dots,K'_t\}, \text{ and } j \in \{0,\dots,J'_{t,k}-2\}\big\},
$$

such that

$$
h^n(Y) = \bigsqcup_{(t,k,j)\in F_n} h^j(Y'_{t,k}).
$$

Thus, we have the following:

$$
\begin{aligned}
v_2 \chi_{h^n(Y)} &= \left(\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{Y'_{t,k}} u^{1-J'_{t,k}} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^{j+1}(Y'_{t,k})} u \chi_{h^j(Y'_{t,k})}\right)\right)\left(\sum_{(t,k,j)\in F_n} \chi_{h^j(Y'_{t,k})}\right) \\
&= \sum_{(t,k,j)\in F_n} \chi_{h^{j+1}(Y'_{t,k})} u \chi_{h^j(Y'_{t,k})} \\
&= \sum_{(t,k,j)\in F_n} u \chi_{h^j(Y'_{t,k})} \\
&= u\chi_{h^n(Y)}. \qquad (33)
\end{aligned}
$$

If $n \in \{0,\dots,N-2\}$, we have

$$
\begin{aligned}
\chi_{h^{n+1}(Y)}(zv_1 - v_2 z)\chi_{h^n(Y)} &= \chi_{h^{n+1}(Y)}(zv_1\chi_{h^n(Y)} - v_2 z \chi_{h^n(Y)}) \\
&= \chi_{h^{n+1}(Y)}(zu\chi_{h^n(Y)} - v_2 z \chi_{h^n(Y)}) && \text{by (32)} \\
&= \chi_{h^{n+1}(Y)}(zu\chi_{h^n(Y)} - v_2 \chi_{h^n(Y)} z) \\
&= \chi_{h^{n+1}(Y)}(zu\chi_{h^n(Y)} - u \chi_{h^n(Y)} z) && \text{by (33)} \\
&= \chi_{h^{n+1}(Y)} u^{n+1} w^{N-(n+1)} (\chi_Y - w) u^{-n} \chi_{h^n(Y)}. && (34)
\end{aligned}
$$

We also have

$$
\begin{aligned}
\chi_{h^N(Y)}(zv_1 - v_2 z)\chi_{h^{N-1}(Y)} &= \chi_{h^N(Y)}(zv_1\chi_{h^{N-1}(Y)} - v_2 z \chi_{h^{N-1}(Y)}) \\
&= \chi_{h^N(Y)}(zu\chi_{h^{N-1}Y)} - v_2 z \chi_{h^{N-1}(Y)}) && \text{by (32)} \\
&= \chi_{h^N(Y)}(zu\chi_{h^{N-1}(Y)} - v_2 \chi_{h^{N-1}(Y)} z) \\
&= \chi_{h^N(Y)}(zu\chi_{h^{N-1}(Y)} - u \chi_{h^{N-1}(Y)} z) && \text{by (33)} \\
&= \chi_{h^N(Y)} u \chi_{h^{N-1}(Y)} - \chi_{h^N(Y)} u \chi_{h^{N-1}(Y)} z \\
&= \chi_{h^N(Y)} u \chi_{h^{N-1}(Y)} - \chi_{h^N(Y)} u \chi_{h^{N-1}(Y)} u^{N-1} w u^{-(N-1)} \chi_{h^{N-1}(Y)} \\
&= \chi_{h^N(Y)} u \chi_{h^{N-1}(Y)} - \chi_{h^N(Y)} u^N w u^{-(N-1)} \chi_{h^{N-1}(Y)} \\
&= \chi_{h^N(Y)} u^N \chi_Y u^{-(N-1)} \chi_{h^{N-1}(Y)} - \chi_{h^N(Y)} u^N w u^{-(N-1)} \chi_{h^{N-1}(Y)}
\end{aligned}
$$

$$= \chi_{h^N(Y)} u^N (\chi_Y - w) u^{-(N-1)} \chi_{h^{N-1}(Y)} \tag{35}$$

So since $\|w - \chi_Y\| < \varepsilon$, (34) and (35) give us

$$\|\chi_{h^{n+1}(Y)}(zv_1 - v_2 z)\chi_{h^n(Y)}\| < \varepsilon. \tag{36}$$

Now, let p be any projection orthogonal to $\chi_{h^{n+1}(Y)}$. We have

$$\begin{aligned} pzv_1\chi_{h^n(Y)} &= pzu\chi_{h^n(Y)} && \text{by (32)} \\ &= p\chi_{h^{n+1}(Y)}zu \\ &= 0. && (37) \end{aligned}$$

Similarly, we have

$$\begin{aligned} pv_2z\chi_{h^n(Y)} &= pv_2\chi_{h^n(Y)}z \\ &= pu\chi_{h^n(Y)}z && \text{by (33)} \\ &= p\chi_{h^{n+1}(Y)}u \\ &= 0. && (38) \end{aligned}$$

Thus, (37) and (38) yield

$$p(zv_1 - v_2z)\chi_{h^n(Y)} = 0. \tag{39}$$

Let $E \subset X$ be any compact open set such that $E \subset X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)$. We then have

$$\begin{aligned} z\chi_E &= \left(\sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j w^{N-j} u^{-j} \chi_{h^j(Y)} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right) \chi_E \\ &= \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \chi_E \\ &= \chi_E. && (40) \end{aligned}$$

Similarly,

$$\chi_E z = \chi_E. \tag{41}$$

Now, let $t \in \{1, \ldots, T\}$ and let $k \in \{1, \ldots, K_t\}$. Note that $h^{J_{t,k}-1}(Y_{t,k}) \subset X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)$ by condition (a) above. Also, since $h^{J_{t,k}}(Y_{t,k}) \subset Y$, we have

$$\begin{aligned} z\chi_{h^{J_{t,k}}(Y_{t,k})} &= \left(\sum_{j=0}^{N-1} \chi_{h^j(Y)} u^j w^{N-j} u^{-j} \chi_{h^j(Y)} + \chi_{X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)} \right) \chi_{h^{J_{t,k}}(Y_{t,k})} \\ &= \chi_Y w^N \chi_{h^{J_{t,k}}(Y_{t,k})} \\ &= \chi_Y v_2 v_1^* \chi_{h^{J_{t,k}}(Y_{t,k})} \end{aligned}$$

$$
\begin{aligned}
&= v_2 \chi_{h^{-1}(Y)} v_1^* \chi_{h^{J_{t,k}}(Y_{t,k})} \\
&= v_2 v_1^* \chi_{h(Y)} \chi_{h^{J_{t,k}}(Y_{t,k})} \\
&= v_2 v_1^* \chi_{h^{J_{t,k}}(Y_{t,k})}. && (42)
\end{aligned}
$$

We therefore have

$$
\begin{aligned}
(zv_1 - v_2 z)\chi_{h^{J_{t,k}-1}(Y_{t,k})} &= zv_1 \chi_{h^{J_{t,k}-1}(Y_{t,k})} - v_2 z \chi_{h^{J_{t,k}-1}(Y_{t,k})} \\
&= z\chi_{Y_{t,k}} v_1 - v_2 z \chi_{h^{J_{t,k}-1}(Y_{t,k})} \\
&= z\chi_{Y_{t,k}} v_1 - v_2 \chi_{h^{J_{t,k}-1}(Y_{t,k})} && \text{by (40)} \\
&= v_2 v_1^* \chi_{Y_{t,k}} v_1 - v_2 \chi_{h^{J_{t,k}-1}(Y_{t,k})} && \text{by (42)} \\
&= v_2 \chi_{h^{J_{t,k}-1}(Y_{t,k})} - v_2 \chi_{h^{J_{t,k}-1}(Y_{t,k})} \\
&= 0. && (43)
\end{aligned}
$$

Thus,

$$
(zv_1 - v_2 z)\chi_{h^{-1}(Y)} = 0. \tag{44}
$$

Set

$$
\widehat{Y} = X \setminus \bigsqcup_{j=-1}^{N} h^j(Y).
$$

We have $z\chi_{\widehat{Y}} = \chi_{\widehat{Y}}$ by (40) since $\widehat{Y} \subset X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y)$. Since $\mathcal{P}_1(\mathcal{S}')$ is finer than $\mathcal{P}_1(\mathcal{S})$, there is a set

$$
G \subset \{(t,k,j) \,\big|\, t \in \{1,\ldots,T'\}; k \in \{1,\ldots,K'_t\}; j \in \{1,\ldots,J'_{t,k}-2\}\}
$$

such that

$$
\widehat{Y} = \bigsqcup_{(t,k,j)\in G} h^j(Y'_{t,k}).
$$

Note that if $(t,k,j) \in G$, then $j \neq J'_{t,k} - 1$, since $h^{J'_{t,k}-1}(Y'_{t,k}) \subset h^{-1}(Y)$, which is disjoint from $\widehat{Y}$. Using (40) at the first step, we therefore have

$$
\begin{aligned}
v_2 z \chi_{\widehat{Y}} &= v_2 \chi_{\widehat{Y}} \\
&= \left(\sum_{t=1}^{T'} \sum_{k=1}^{K'_t} \left(\chi_{Y'_{t,k}} u^{1-J'_{t,k}} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^{j+1}(Y'_{t,k})} u \chi_{h^j(Y'_{t,k})} \right) \right) \left(\sum_{(t,k,j)\in G} h^j(Y'_{t,k}) \right) \\
&= \left(\sum_{t=1}^{T'} \sum_{k=1}^{K'_t} \sum_{j=0}^{J'_{t,k}-2} \chi_{h^{j+1}(Y'_{t,k})} u \chi_{h^j(Y'_{t,k})} \right) \left(\sum_{(t,k,j)\in G} h^j(Y'_{t,k}) \right) \\
&= \sum_{(t,k,j)\in G} \chi_{h^{j+1}(Y'_{t,k})} u \chi_{h^j(Y'_{t,k})} \\
&= u \chi_{\widehat{Y}}. && (45)
\end{aligned}
$$

We also have

$$
\begin{aligned}
zv_1\chi_{\widehat{Y}} &= z\left(\sum_{t=1}^{T}\sum_{k=1}^{K_t}\left(\chi_{Y_{t,k}}u^{1-J_{t,k}}\chi_{h^{J_{t,k}-1}(Y_{t,k})} + \sum_{j=0}^{J_{t,k}-2}\chi_{h^{j+1}(Y_{t,k})}u\chi_{h^j(Y_{t,k})}\right)\right)\chi_{\widehat{Y}} \\
&= z\left(\sum_{t=1}^{T}\sum_{k=1}^{K_t}\sum_{j=N}^{J_{t,k}-2}\chi_{h^{j+1}(Y_{t,k})}u\chi_{h^j(Y_{t,k})}\right) \\
&= \left(\sum_{j=0}^{N-1}\chi_{h^j(Y)}u^j w^{N-j}u^{-j}\chi_{h^j(Y)} + \chi_{X\setminus\bigsqcup_{j=0}^{N-1}h^j(Y)}\right)\left(\sum_{t=1}^{T}\sum_{k=1}^{K_t}\sum_{j=N}^{J_{t,k}-2}\chi_{h^{j+1}(Y_{t,k})}u\chi_{h^j(Y_{t,k})}\right) \\
&= \left(\chi_{X\setminus\bigsqcup_{j=0}^{N-1}h^j(Y)}\right)\left(\sum_{t=1}^{T}\sum_{k=1}^{K_t}\sum_{j=N}^{J_{t,k}-2}\chi_{h^{j+1}(Y_{t,k})}u\chi_{h^j(Y_{t,k})}\right) \\
&= \left(\sum_{t=1}^{T}\sum_{k=1}^{K_t}\sum_{j=N}^{J_{t,k}-2}\chi_{h^{j+1}(Y_{t,k})}u\chi_{h^j(Y_{t,k})}\right) \\
&= u\chi_{\widehat{Y}}. \qquad (46)
\end{aligned}
$$

Putting our recent work together, we get

$$
\begin{aligned}
(zv_1 - v_2z)\chi_{\widehat{Y}} &= zv_1\chi_{\widehat{Y}} - v_2z\chi_{\widehat{Y}} \\
&= zv_1\chi_{\widehat{Y}} - u\chi_{\widehat{Y}} && \text{by (45)} \\
&= u\chi_{\widehat{Y}} - u\chi_{\widehat{Y}} && \text{by (46)} \\
&= 0. && (47)
\end{aligned}
$$

Precisely the same argument shows that

$$
(zv_1 - v_2z)\chi_{h^N(Y)} = 0. \qquad (48)
$$

We now apply Lemma 5.11 with $M = N+3$, $a = zv_1 - v_2z$, $p_n = \chi_{h^n(Y)}$ for all $n \in \{1, \ldots, N\}$, $q_n = \chi_{h^{n-1}(Y)}$ for all $n \in \{1, \ldots, N+1\}$, $p_{N+1} = \chi_Y$, $p_{N+2} = q_{N+2} = \chi_{h^{-1}(Y)}$, and $p_{N+3} = q_{N+3} = \chi_{\widehat{Y}_1}$. By (36), we have $\|p_naq_n\| < \varepsilon$ for all $n \in \{1, \ldots, N\}$. By (39), for $n \in \{1, \ldots, N\}$ with $n \neq m$, we have $q_map_n = 0$ for all $m \in \{1, \ldots, M\}$ such that $m \neq n$. By (48), we have $p_maq_{N+1} = 0$ for all $m \in \{1, \ldots, M\}$. By (44), we have $p_maq_{N+2} = 0$ for all $m \in \{1, \ldots, M\}$. By (47), we have $p_maq_{N+3} = 0$ for all $m \in \{1, \ldots, M\}$. Thus, Lemma 5.11 applies to give us $\|zv_1 - v_2z\| < \varepsilon$. Thus, by (31), we have $\|u' - u\| < \varepsilon$.

We will now show that

$$
A \cong \bigoplus_{t=1}^{T}\left(\left(C(S^1)\otimes M_{J_{t,1}}\right) \oplus \left(\bigoplus_{k=2}^{K_t} M_{J_{t,k}}\right)\right),
$$

To do this, we first claim that u_2 and z commute. First note our formula for u_2:

$$\begin{aligned}
u_2 &= v_2^* u \\
&= \left(\sum_{t=1}^{T'} \sum_{k=1}^{K'_t} \left(\chi_{Y'_{t,k}} u^{1-J'_{t,k}} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^{j+1}(Y'_{t,k})} u \chi_{h^j(Y'_{t,k})} \right) \right)^* u \\
&= \left(\sum_{t=1}^{T'} \sum_{k=1}^{K'_t} \left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}-1} \chi_{Y'_{t,k}} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^j(Y'_{t,k})} u^* \chi_{h^{j+1}(Y'_{t,k})} \right) \right) u \\
&= \left(\sum_{t=1}^{T'} \sum_{k=1}^{K'_t} \left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}-1} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^j(Y'_{t,k})} u^* \right) \right) u \\
&= \sum_{t=1}^{T'} \sum_{k=1}^{K'_t} \left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^j(Y'_{t,k})} \right). \qquad (49)
\end{aligned}$$

Let $t \in \{1, \ldots, T'\}$ and let $k \in \{1, \ldots, K'_t\}$. Since

$$h^{-1}(Y'_{t,k}) \subset h^{-1}(Y) \subset X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y),$$

by (40), we have

$$z\chi_{h^{-1}(Y'_{t,k})} = \chi_{h^{-1}(Y'_{t,k})}.$$

By (41), we have

$$\chi_{h^{-1}(Y'_{t,k})} z = \chi_{h^{-1}(Y'_{t,k})}.$$

Thus, we have

$$z\chi_{h^{-1}(Y'_{t,k})} = \chi_{h^{-1}(Y'_{t,k})} z. \qquad (50)$$

Since

$$h^{J'_{t,k}-1}(Y'_{t,k}) \subset h^{-1}(Y) \subset X \setminus \bigsqcup_{j=0}^{N-1} h^j(Y),$$

by a process similar to the above, we have

$$z\chi_{h^{J'_{t,k}-1}(Y'_{t,k})} = \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} z. \qquad (51)$$

Since

$$\bigsqcup_{j=0}^{N-1} h^j(Y) \subset \bigsqcup_{t=1}^{T'} \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{k,t}-2} h^j(Y'_{k,t}),$$

we have

$$\left(\sum_{j=0}^{N-1}\chi_{h^j(Y)}u^jw^{N-j}u^{-j}\chi_{h^j(Y)}\right)\left(\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\sum_{j=0}^{J'_{k,t}-2}\chi_{h^j(Y'_{k,t})}\right)=\sum_{j=0}^{N-1}\chi_{h^j(Y)}u^jw^{N-j}u^{-j}\chi_{h^j(Y)} \tag{52}$$

and

$$\left(\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\sum_{j=0}^{J'_{k,t}-2}\chi_{h^j(Y'_{k,t})}\right)\left(\sum_{j=0}^{N-1}\chi_{h^j(Y)}u^jw^{N-j}u^{-j}\chi_{h^j(Y)}\right)=\sum_{j=0}^{N-1}\chi_{h^j(Y)}u^jw^{N-j}u^{-j}\chi_{h^j(Y)}. \tag{53}$$

It is also clear that if E is any subset of $X\setminus\bigsqcup_{j=0}^{N-1}h^j(Y)$, then

$$z\chi_E=\chi_E z \tag{54}$$

Again, since w is obtained via functional calculus at $v_2v_1^*$, w^{N-j} commutes with u as well. Therefore, we have

$$\begin{aligned}uz&=u\chi_{h^j(Y)}u^jw^{N-j}u^{-j}\chi_{h^j(Y)}\\&=\chi_{h^{j+1}(Y)}u^juw^{N-j}u^{-j}\chi_{h^j(Y)}\\&=\chi_{h^{j+1}(Y)}u^jw^{N-j}uu^{-j}\chi_{h^j(Y)}\\&=\chi_{h^{j+1}(Y)}u^jw^{N-j}u^{-j}\chi_{h^{j+1}(Y)}u\\&=zu.\end{aligned} \tag{55}$$

We have

$$\begin{aligned}zu_2&=z\left(\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}u^{J'_{t,k}}\chi_{h^{-1}(Y'_{t,k})}+\sum_{j=0}^{J'_{t,k}-2}\chi_{h^j(Y'_{t,k})}\right)\right)\\&=\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(z\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}u^{J'_{t,k}}\chi_{h^{-1}(Y'_{t,k})}+\sum_{j=0}^{J'_{t,k}-2}z\chi_{h^j(Y'_{t,k})}\right)\\&=\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}zu^{J'_{t,k}}\chi_{h^{-1}(Y'_{t,k})}+\sum_{j=0}^{J'_{t,k}-2}z\chi_{h^j(Y'_{t,k})}\right)\\&=\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}u^{J'_{t,k}}z\chi_{h^{-1}(Y'_{t,k})}+\sum_{j=0}^{J'_{t,k}-2}z\chi_{h^j(Y'_{t,k})}\right)&&\text{by (55)}\\&=\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}u^{J'_{t,k}}\chi_{h^{-1}(Y'_{t,k})}z+\sum_{j=0}^{J'_{t,k}-2}z\chi_{h^j(Y'_{t,k})}\right)\\&=\sum_{t=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})}u^{J'_{t,k}}\chi_{h^{-1}(Y'_{t,k})}z+\sum_{j=0}^{J'_{t,k}-2}\chi_{h^j(Y'_{t,k})}z\right)&&\text{by (54)}\end{aligned}$$

$$= u_2 z.$$

Thus, we see z and u_2 indeed commute.

Since z and u_2 commute, the C^*-algebra $\widehat{A}$ generated by A_1 and u_2 is unitarily equivalent to A (via z). Thus, $A \cong \widehat{A}$, and we will therefore work with $\widehat{A}$ for the remainder of the proof. Recall that for each $t \in \{1, \dots, T'\}$, we have $X'_t = h^{J_{t,1}}(Y_{t,1})$. For convenience of notation during the rest of the proof, set

$$\widehat{p} = \chi_{X \setminus \bigsqcup_{t=1}^{T'} \bigsqcup_{j=0}^{J_{t,1}-1} h^j(Y_{t,1})}.$$

Let

$$\widehat{u} = \sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} u_2 e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p}.$$

We claim $\widehat{u}$ is a unitary in $\widehat{A}$. First, for $s \in \{1, \dots, T'\}$, and $j \in \{0, \dots, J_{t,1} - 1\}$, observe that

$$\begin{aligned}
u_2 e^{(t,1)}_{J_{t,1}-1,j} e^{(t,1)}_{j,J_{t,1}-1} u_2^* &= u_2 e^{(t,1)}_{J_{t,1}-1,J_{t,1}-1} u_2^* \\
&= u_2 \chi_{h^{J_{t,1}-1}(Y_{t,1})} u_2^* \\
&= u_2 \chi_{h^{-1}(X'_t)} u_2^* \\
&= v_2^* u \chi_{h^{-1}(X'_s)} u^* v_2 \\
&= v_2^* \chi_{X'_t} v_2 \\
&= \sum_{k=1}^{K'_t} v_2^* \chi_{Y'_{t,k}} v_2 \\
&= \sum_{k=1}^{K'_t} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} \\
&= \chi_{h^{-1}(X'_t)} \\
&= \chi_{h^{J_{t,1}-1}(Y_{t,1})}.
\end{aligned} \tag{56}$$

Now, observe that

$$\begin{aligned}
\widehat{u}\widehat{u}^* &= \left(\sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} u_2 e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p} \right) \left(\sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} u_2^* e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p} \right) \\
&= \sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} \left(u_2 e^{(t,1)}_{J_{t,1}-1,j} e^{(t,1)}_{j,J_{t,1}-1} u_2^* \right) e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p} \\
&= \sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} \chi_{h^{J_{t,1}-1}(Y_{t,1})} e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p} && \text{by (56)} \\
&= \sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} e^{(t,1)}_{J_{t,1}-1,J_{t,1}-1} e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} e_{j,j}^{(t,1)} + \widehat{p} \\
&= \sum_{t=1}^{T'} \sum_{j=0}^{J_{t,1}-1} \chi_{h^j(Y_{t,1})} + \widehat{p} \\
&= 1.
\end{aligned}
$$

A similar computation shows $\widehat{u}^*\widehat{u} = 1$. Thus, $\widehat{u}$ is a unitary.

We claim that A_1 and $\widehat{u}$ commute. To see this, it is clear that we only need to check commutativity with matrix units of the form $e_{i,j}^{(t,1)}$ for $s \in \{1, \ldots, T'\}$. But with this in mind, we have

$$
\begin{aligned}
\widehat{u}e_{i,j}^{(t,1)}\widehat{u}^* &= e_{i,J_{t,1}-1}^{(t,1)} u_2 e_{J_{t,1}-1,i}^{(t,1)} e_{i,j}^{(t,1)} e_{j,J_{t,1}-1} u_2^* e_{J_{t,1}-1,j}^{(t,1)} \\
&= e_{i,J_{t,1}-1}^{(t,1)} u_2 e_{J_{t,1}-1,J_{t,1}-1}^{(t,1)} u_2^* e_{J_{t,1}-1,j}^{(t,1)} \\
&= e_{i,J_{t,1}-1}^{(t,1)} e_{J_{t,1}-1,J_{t,1}-1}^{(t,1)} e_{J_{t,1}-1,j}^{(t,1)} && \text{by (56)} \\
&= e_{i,j}^{(t,1)}.
\end{aligned}
$$

Thus, A_1 and $\widehat{u}$ commute.

We claim that A_1 and $\widehat{u}$ generate $\widehat{A}$. To see this, notice that

$$
\left(\sum_{t=1}^{T'} e_{J_{t,1}-1,0}^{(t,1)}\right) \widehat{u} \left(\sum_{t=1}^{T'} e_{0,J_{t,1}-1}^{(t,1)}\right) = \sum_{t=1}^{T'} e_{J_{t,1}-1,J_{t,1}-1}^{(t,1)} u_2 e_{J_{t,1}-1,J_{t,1}-1}^{(t,1)},
$$

which, when added to $\chi_{X \setminus \bigsqcup_{t=1}^{T'} h^{J_{t,1}-1}(Y_{t,1})} = \chi_{X \setminus \bigsqcup_{t=1}^{T'} h^{-1}(X_t')}$, yields u_2.

Let $t \in \{1, \ldots, T\}$, let $k \in \{2, \ldots, K_t\}$, and let $i, j \in \{0, \ldots, J_{t,k} - 1\}$. We have

$$
\begin{aligned}
e_{i,j}^{(t,k)}\widehat{u} &= \chi_{h^i(Y_{t,k})} u^{j-i} \chi_{h^j(Y_{t,k})} \widehat{p} \\
&= \chi_{h^i(Y_{t,k})} u^{j-i} \chi_{h^j(Y_{t,k})} \\
&= e_{i,j}^{(t,k)}
\end{aligned}
$$

and similarly $\widehat{u}e_{i,j}^{(t,k)} = e_{i,j}^{(t,k)}$. Thus, setting $p_{t,k} = \sum_{i=0}^{J_{t,k}-1} e_{i,i}^{(t,k)}$, we have

$$
p_{t,k}\widehat{A}p_{t,k} \cong M_{J_{t,k}}. \tag{57}
$$

Fix $t \in \{1, \ldots, T'\}$ and set $p_t = \sum_{j=0}^{J_{t,1}-1} \chi_{h^j(Y_{t,1})} = \sum_{j=0}^{J_{t,1}-1} e_{j,j}^{(t,1)}$. We now claim that $p_t u_2 p_t$ and $p_t \widehat{u} p_t$ are unitaries in $p_t C^*(\mathbb{Z}, X, h) p_t$. To show this, we show that p_t commutes with u_2 and $\widehat{u}$.

It is obvious that p_t commutes with $\widehat{u}$, since $\widehat{u}$ commutes with A_1. So to show that p_t commutes with u_2, we first claim that, for each $j \in \{-J_{t,1}, \ldots, -2\}$, we have

$$
h^j(X_t') \subset \bigsqcup_{k=1}^{K_t'} \bigsqcup_{j'=0}^{J_{t,k}'-2} h^{j'}(Y_{t,k}'). \tag{58}
$$

To see this, note that

$$\bigsqcup_{k=1}^{K'_s}\bigsqcup_{j'=0}^{J'_{t,k}-2} h^{j'}(Y'_{t,k}) = \bigsqcup_{k=1}^{K'_s}\bigsqcup_{j'=0}^{J'_{t,k}-1} h^{j'}(Y'_{t,k}) \setminus h^{-1}(X'_t) = \bigcup_{j'\in\mathbb{Z}} h^j(X'_t) \setminus h^{-1}(X'_t),$$

and then note that $Y_{t,1} = h^{-J_{t,1}}(X'_t)$, and so since $Y_{t,1}, h(Y_{t,1}), \ldots, h^{J_{t,1}-1}(Y_{t,1})$ are pairwise disjoint, it follows that $h^j(X'_t) \cap h^{-1}(X'_t) = \varnothing$ for all $j \in \{-J_{t,1}, \ldots, -2\}$. Thus, the claim follows. Now,

$$\begin{aligned}
u_2 p_t &= \left(\sum_{s=1}^{T'}\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})} + \sum_{j=0}^{J'_{s,k}-2} \chi_{h^j(Y'_{s,k})}\right)\right)\left(\sum_{j=0}^{J_{t,1}-1} \chi_{h^j(Y_{t,1})}\right) \\
&= \left(\sum_{s=1}^{T'}\sum_{k=1}^{K'_s}\left(\chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})} + \sum_{j=0}^{J'_{s,k}-2} \chi_{h^j(Y'_{s,k})}\right)\right)\left(\sum_{j=-J_{t,1}}^{-1} \chi_{h^j(X'_t)}\right) \\
&= \left(\sum_{k=1}^{K'_t} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})}\right) \chi_{h^{-1}(X'_s)} + \sum_{j=-J_{t,1}}^{-2} \chi_{h^j(X'_t)} && \text{by (58)} \\
&= \left(\sum_{k=1}^{K'_t} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})}\right) \sum_{k=1}^{K'_t} \chi_{h^{-1}(Y'_{t,k})} + \sum_{j=-J_{t,1}}^{-2} \chi_{h^j(X'_t)} \\
&= \sum_{k=1}^{K'_t} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} + \sum_{j=-J_{t,1}}^{-2} \chi_{h^j(X'_t)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
p_t u_2 &= \left(\sum_{j=0}^{J_{t,1}-1} \chi_{h^j(Y_{t,1})}\right)\left(\sum_{s=1}^{T'}\sum_{k=1}^{K'_s}\left(\chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})} + \sum_{j=0}^{J'_{s,k}-2} \chi_{h^j(Y'_{s,k})}\right)\right) \\
&= \left(\sum_{j=-J_{t,1}}^{-1} \chi_{h^j(X'_t)}\right)\left(\sum_{s=1}^{T'}\sum_{k=1}^{K'_s}\left(\chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})} + \sum_{j=0}^{J'_{s,k}-2} \chi_{h^j(Y'_{s,k})}\right)\right) \\
&= \chi_{h^{-1}(X'_t)}\left(\sum_{k=1}^{K'_t} \chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})}\right) + \sum_{j=-J_{t,1}}^{-2} \chi_{h^j(X'_s)} && \text{by (58)} \\
&= \sum_{k=1}^{K'_s} \chi_{h^{J'_{s,k}-1}(Y'_{s,k})}\left(\sum_{k=1}^{K'_s} \chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})}\right) + \sum_{j=-J_{t,1}}^{-2} \chi_{h^j(X'_s)} \\
&= \sum_{k=1}^{K'_s} \chi_{h^{J'_{s,k}-1}(Y'_{s,k})} u^{J'_{s,k}} \chi_{h^{-1}(Y'_{s,k})} + \sum_{j=-J_{t,1}}^{-2} \chi_{h^j(X'_s)}.
\end{aligned}$$

Thus, p_t commutes with u_t, and so $p_t u_2 p_t$ is a unitary in $p_t C^*(\mathbb{Z}, X, h) p_t$.

Set $\widehat{X}'_t = \bigcup_{j\in\mathbb{Z}} h^j(X'_t) = \bigsqcup_{k=1}^{K'_t}\bigsqcup_{j=0}^{J'_{t,k}-1} h^j(Y'_{t,k})$, an h-invariant compact open subset of X. Set $r_t = \chi_{\widehat{X}'_t}$, a projection that therefore commutes with u, which means that it is central in $C^*(\mathbb{Z}, X, h)$. We claim that $[r_t \widehat{u} r_t] = J_{t,1}[r_t u_2 r_t]$ in $K_1(r_t C^*(\mathbb{Z}, X, h) r_t)$. For each $j \in \{0, \ldots, J_{t,1} - 1\}$, set $D_j =$

$\{0,\dots,J_{t,1}-1\}\setminus\{j\}$ and set

$$w_j = e^{(t,1)}_{j,J_{t,1}-1} u_2 e^{(t,1)}_{J_{t,1}-1,j} + \sum_{i\in D_j} e_{i,i} + (r_t - p_t).$$

Note that $w_{J_{t,1}-1} = r_t u_2 r_t$. We have

$$\begin{aligned}
\prod_{j=0}^{J_{t,1}-1} w_j &= \prod_{j=0}^{J_{t,1}-1}\left(e^{(t,1)}_{j,J_{t,1}-1} u_2 e^{(t,1)}_{J_{t,1}-1,j} + \sum_{i\in D_j} e^{(t,1)}_{i,i} + (r_s - p_s)\right) \\
&= \sum_{j=0}^{J_{t,1}-1} e^{(t,1)}_{j,J_{t,1}-1} u_2 e^{(t,1)}_{J_{t,1}-1,j} + (r_t - p_t) \\
&= r_t \widehat{u} r_t. \qquad (59)
\end{aligned}$$

Let $j \in \{0,\dots,J_{t,1}-2\}$. Define $D'_j = \{0,\dots,J_{t,1}-2\}\setminus\{j\}$,

$$\widehat{p}_j = \sum_{i\in D'_j} e^{(t,1)}_{i,i} + (r_t - p_t),$$

and

$$w'_j = e^{(t,1)}_{j,J_{t,1}-1} + e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p}_j.$$

We have

$$w'_j r_t u_2 r_t = \left(e^{(t,1)}_{j,J_{t,1}-1} + e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p}_j\right)\left(\sum_{k=1}^{K'_t}\left(\chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} + \sum_{j=0}^{J'_{t,k}-2} \chi_{h^j(Y'_{t,k})}\right)\right).$$

We break the right hand side of this computation into steps. First, recall that $h^{-1}(X'_t) = h^{J_{t,1}-1}(Y_{t,1})$. With this in mind, we have

$$\begin{aligned}
e^{(t,1)}_{j,J_{t,1}-1} r_t u_2 r_t &= \left(\chi_{h^j(Y_{t,1})} u^{j-(J_{t,1}-1)} \chi_{h^{J_{t,1}-1}(Y_{t,1})}\right)\left(\sum_{k=1}^{K'_t} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})}\right) \\
&= \sum_{k=1}^{K'_t} \chi_{h^j(Y_{t,1})} u^{j-(J_{t,1}-1)} \chi_{h^{-1}(X'_t)} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} \\
&= \sum_{k=1}^{K'_t} \chi_{h^j(Y_{t,1})} u^{j-(J_{t,1}-1)} \chi_{h^{J'_{t,k}-1}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} \\
&= \sum_{k=1}^{K'_t} \chi_{h^j(Y_{t,1})} u^{j-(J_{t,1}-1)+J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} \\
&= \sum_{k=1}^{K'_t} \chi_{h^{j-J_{t,1}}(X'_t)} u^{j-(J_{t,1}-1)+J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})}
\end{aligned}$$

$$= \sum_{k=1}^{K'_t} \chi_{h^{j-J_{t,1}+J'_{t,k}}(Y'_{t,k})} u^{j-(J_{t,1}-1)+J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})}. \tag{60}$$

Since

$$h^j(Y_{t,1}) \subset \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j'=0}^{J'_{t,k}-2} h^{j'}(Y'_{t,k}),$$

we have

$$\begin{aligned} e^{(t,1)}_{J_{t,1}-1,j} r_t u_2 r_t &= \chi_{h^{J_{t,1}-1}(Y_{t,1})} u^{J_{t,1}-1-j} \chi_{h^j(Y_{t,1})} \left(\sum_{k=1}^{K'_t} \sum_{j=0}^{J'_{t,k}-2} \chi_{h^j(Y'_{t,k})} \right) \\ &= \chi_{h^{J_{t,1}-1}(Y_{t,1})} u^{J_{t,1}-1-j} \chi_{h^j(Y_{t,1})} \\ &= e^{(t,1)}_{J_{t,1}-1,j}. \end{aligned} \tag{61}$$

Note that $\widehat{p}_j = \chi_{\widehat{E}_j}$ where

$$\begin{aligned} \widehat{E}_j &= \left(\bigsqcup_{i \in D'_j} h^i(Y_{t,1}) \right) \sqcup \left(\left(\bigsqcup_{k=1}^{K'_t} \bigsqcup_{i=0}^{J'_{t,k}-1} h^i(Y'_{t,k}) \right) \setminus \left(\bigsqcup_{j=0}^{J_{t,1}-1} h^j(Y_{t,1}) \right) \right) \\ &= \left(\bigsqcup_{k=1}^{K'_t} \bigsqcup_{i=0}^{J'_{t,k}-1} h^i(Y'_{t,k}) \right) \setminus \left(h^{J_{t,1}-1}(Y_{t,1}) \sqcup h^j(Y_{t,1}) \right) \\ &= \left(\bigsqcup_{k=1}^{K'_t} \bigsqcup_{i=0}^{J'_{t,k}-1} h^i(Y'_{t,k}) \right) \setminus \left(h^{-1}(X'_t) \sqcup h^j(Y_{t,1}) \right) \\ &= \left(\bigsqcup_{k=1}^{K'_t} \bigsqcup_{i=0}^{J'_{t,k}-2} h^i(Y'_{t,k}) \right) \setminus h^j(Y_{t,1}). \end{aligned} \tag{62}$$

Thus, $\widehat{E}_j \subset \bigsqcup_{k=1}^{K'_t} \bigsqcup_{j=0}^{J'_{t,k}-2} h^j(Y'_{t,k})$, and so we have

$$\begin{aligned} \widehat{p}_j r_t u_2 r_t &= \widehat{p}_j \left(\sum_{k=1}^{K'_t} \sum_{j=0}^{J'_{t,k}-2} \chi_{h^j(Y'_{t,k})} \right) \\ &= \widehat{p}_j. \end{aligned} \tag{63}$$

From (60), (61), and (63), we have

$$w'_j r_t u_2 r_t = \sum_{k=1}^{K'_t} \chi_{h^{j-J_{t,1}+J'_{t,k}}(Y'_{t,k})} u^{j-(J_{t,1}-1)+J'_{t,k}} \chi_{h^{-1}(Y'_{t,k})} + e^{(t,1)}_{J_{t,1}-1,j} + \widehat{p}_j.$$

Now, simplify $w_j' r_t u_2 r_t w_j'$, which rewritten is

$$\left(\sum_{k=1}^{K_t'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{j-(J_{t,1}-1)+J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} + e_{J_{t,1}-1,j}^{(t,1)} + \widehat{p}_j\right)\left(e_{j,J_{t,1}-1}^{(t,1)} + e_{J_{t,1}-1,j}^{(t,1)} + \widehat{p}_j\right)$$

From (62), it is clear that

$$\widehat{p}_j e_{j,J_{t,1}-1}^{(t,1)} = 0.$$

Since $j \neq J_{t,1}-1$, we have $h^j(Y_{t,1}) \cap h^{-1}(X_t') = \varnothing$, so

$$\sum_{k=1}^{K_t'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{j-(J_{t,1}-1)+J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} e_{j,J_{t,1}-1}^{(t,1)} = 0.$$

Thus, we have

$$\begin{aligned} w_j' r_t u_2 r_t e_{j,J_{t,1}-1}^{(t,1)} &= e_{J_{t,1}-1,j}^{(t,1)} e_{j,J_{t,1}-1}^{(t,1)} \\ &= e_{j,j}. \end{aligned} \tag{64}$$

Since $J_{t,1}-1 \notin D_j$, and from (62), we can easily see that

$$\left(e_{J_{t,1}-1,j}^{(t,1)} + \widehat{p}_j\right) e_{J_{t,1}-1,j}^{(t,1)} = 0.$$

Thus, we have

$$\begin{aligned} w_j' r_t u_2 r_t e_{J_{t,1}-1,j}^{(t,1)} &= \sum_{k=1}^{K_s'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{j-(J_{t,1}-1)+J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} \chi_{h^{J_{t,1}-1}(Y_{t,1})} u^{J_{t,1}-1-j} \chi_{h^j(Y_{t,1})} \\ &= \sum_{k=1}^{K_s'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{j-(J_{t,1}-1)+J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} \chi_{h^{-1}(X_s')} u^{J_{t,1}-1-j} \chi_{h^j(Y_{t,1})} \\ &= \sum_{k=1}^{K_s'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{j-(J_{t,1}-1)+J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} u^{J_{t,1}-1-j} \chi_{h^j(Y_{t,1})} \\ &= \sum_{k=1}^{K_s'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{j-(J_{t,1}-1)+J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} u^{J_{t,1}-1-j} \chi_{h^{j-J_{t,1}}(Y_{t,k}')} \\ &= \sum_{k=1}^{K_s'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{J_{t,k}'} \chi_{h^{j-J_{t,1}}(Y_{t,k}')}. \end{aligned}$$

But then notice that

$$\begin{aligned} e_{j,J_{t,1}-1}^{(t,1)} u_2 e_{J_{t,1}-1,j}^{(t,1)} &= \chi_{h^j(Y_{t,1})} u^{J_{t,1}-1-j} \chi_{h^{J_{t,1}-1}(Y_{t,1})} u_2 \chi_{h^{J_{t,1}-1}(Y_{t,1})} u^{j-(J_{t,1}-1)} \chi_{h^j(Y_{t,1})} \\ &= \sum_{k=1}^{K_s'} \chi_{h^{j-J_{t,1}+J_{t,k}'}(Y_{t,k}')} u^{J_{t,1}-1-j} \chi_{h^{J_{t,k}'-1}(Y_{t,k}')} u^{J_{t,k}'} \chi_{h^{-1}(Y_{t,k}')} u^{j-(J_{t,1}-1)} \chi_{h^{j-J_{t,1}}(Y_{t,k}')} \end{aligned}$$

$$= \sum_{k=1}^{K'_s} \chi_{h^{j-J_{t,1}+J'_{t,k}}(Y'_{t,k})} u^{j-(J_{t,1}-1)+J'_{t,k}} \chi_{h^{j-J_{t,1}}(Y'_{t,k})}.$$

Thus,

$$w'_j r_t u_2 r_t e^{(t,1)}_{J_{t,1}-1,j} = e^{(t,1)}_{j,J_{t,1}-1} u_2 e^{(t,1)}_{J_{t,1}-1,j}. \tag{65}$$

Finally, it is immediately clear that

$$\begin{aligned} w'_j r_t u_2 r_t \widehat{p}_j &= \widehat{p}_j \widehat{p}_j \\ &= \widehat{p}_j. \end{aligned} \tag{66}$$

So by (64), (65), and (66), we see

$$\begin{aligned} w'_j r_t u_2 r_t w'_j &= e^{(t,1)}_{j,j} + \sum_{k=1}^{K'_s} \chi_{h^{j-J_{t,1}+J'_{t,k}}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{j-J_{t,1}}(Y'_{t,k})} + \widehat{p}_j \\ &= \sum_{k=1}^{K'_s} \chi_{h^{j-J_{t,1}+J'_{t,k}}(Y'_{t,k})} u^{J'_{t,k}} \chi_{h^{j-J_{t,1}}(Y'_{t,k})} + \sum_{i \in D_j} e^{(t,1)}_{i,i} + (r_t - p_t) \\ &= e_{j,J_{t,1}-1} u_2 e_{J_{t,1}-1,j} + \sum_{i \in D_j} e^{(t,1)}_{i,i} + (r_t - p_t) \\ &= w_j. \end{aligned}$$

Now note that w'_j is a unitary in $r_t A_2 r_t$, and since $r_t A_2 r_t$ is a finite-dimensional C^*-subalgebra of $r_t C^*(\mathbb{Z}, X, h) r_t$, w'_j has trivial K_1-class. Thus, in $K_1(r_t C^*(\mathbb{Z}, X, h) r_t)$, we have $[w_j] = [r_t u_2 r_t]$, so by (59), we have

$$[r_t \widehat{u} r_t] = J_{t,1} [r_t u_2 r_t]. \tag{67}$$

We now show that $[r_t u_2 r_t] \neq 0$. First note that $r_t v_2 r_t \in r_t A_2 r_t$, and since $r_t A_2 r_t$ is finite-dimensional, we have $[r_t v_2 r_2] = 0$. From Lemma 5.3, we have $[r_t u r_t] \neq 0$. Thus,

$$\begin{aligned} [r_t u_2 r_t] &= [r_t v_2^* u r_t] \\ &= -[r_t v_2 r_t] + [r_t u r_t] \\ &\neq 0. \end{aligned} \tag{68}$$

By Lemma 5.4, $r_t C^*(\mathbb{Z}, X, h) r_t$ has torsion-free K_1, so (67) and (68) give us

$$[r_t \widehat{u} r_t] \neq 0. \tag{69}$$

A very straightforward computation shows

$$r_t \widehat{u} r_t = p_t \widehat{u} p_t + (r_t - p_t)$$

This fact combined with (69) and Lemma 5.5 gives us $[p_t \widehat{u} p_t] \neq 0$ in $K_1(p_t C^*(\mathbb{Z}, X, h) p_t)$. Thus, $\mathrm{sp}(p_t \widehat{u} p_t) = S^1$. So because of this, because $p_t \widehat{u} p_t$ commutes with $e_{i,j}^{(t,1)}$ for all $i, j \in \{0, \ldots, J_{t,1} - 1\}$, and because $p_t \widehat{u} p_t$ and $(e_{i,j}^{(t,1)})_{0 \leq i,j \leq J_{t,1}-1}$ generate $p_t \widehat{A} p_t$, by Lemma 5.12, we have

$$p_t \widehat{A} p_t \cong C(S^1) \otimes M_{J_{t,1}}. \tag{70}$$

Altogether, from (57) and (70), we get

$$\widehat{A} \cong \bigoplus_{t=1}^{T} \left(\big(C(S^1) \otimes M_{J_{t,1}}\big) \oplus \left(\bigoplus_{k=2}^{K_t} M_{J_{t,k}} \right) \right),$$

finishing the proof of the theorem. $\square$

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