ZONOTOPES AND HYPERTORIC VARIETIES

by

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A DISSERTATION

Presented to the Department of Mathematics
and the Graduate School of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

September 2015

DISSERTATION APPROVAL PAGE

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Title: Zonotopes and Hypertoric Varieties

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Degree awarded September 2015

DISSERTATION ABSTRACT

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Doctor of Philosophy

Department of Mathematics

September 2015

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Hypertoric varieties are a class of conical symplectic resolutions which are very computable. In the current literature, they are only defined constructively, using hyperplane arrangements. We provide an abstract definition of a hypertoric variety and a new construction using zonotopal tilings and relate the zonotopal construction to the hyperplane construction.

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor, Nick Proudfoot, for his guidance and support, as well as the rest of the professors in the Mathematics Department who have taught me. I would also like to thank my fellow graduate students who have spent the last seven years with me, and the office staff for all their help.

I would also like to thank all of the professors who taught me before graduate school, especially Alan Cannon, Kent Neuerberg, and Randall Wills, and the faculty mentors at the IMMERSE and Willamette Valley REU-RET programs.

Last, but certainly not least, I would like to thank my parents for being there whenever things got rough, and my family for all of their encouragement and support.

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CHAPTER I

INTRODUCTION

Hypertoric varieties were introduced by Bielawski and Dancer under the name "toric hyperkähler manifolds" in 2000 [BD00]. The definition they give is constructive: given a suitable rational hyperplane arrangement in the dual of the Lie algebra of a given torus, one can construct a manifold, and manifolds which arise this way are called toric hyperkähler manifolds. Hypertoric varieties arising from this construction have been studied in many papers, including [HS02, HH05, PW07, Pro08, Kon00]. In this paper, we provide an abstract definition for hypertoric varieties, and a new construction which includes all hypertoric varieties which come from hyperplane arrangements, as well as new hypertoric varieties that do not appear in the literature. Among our varieties, those arising from hyperplane arrangements are precisely the ones that are projective over their affinizations.

The situation may be understood by analogy to toric varieties. Given a torus T over a field k, a T-toric variety is a normal variety with an action of T that has a dense orbit. To each affine toric variety X_0, we can associate a cone σ in $\mathrm{Lie}(T)$ and its dual $\sigma^\vee \in \mathrm{Lie}(T)^*$. The cone σ contains precisely those cocharacters that act on X_0 with non-negative weights, while the cone in $\mathrm{Lie}(T)^*$ contains precisely the characters of T that extend to functions on X_0. Given a polyhedron P in $\mathrm{Lie}(T)^*$ with recession cone $\sigma^\vee$, we obtain a toric variety X_P which is projective over its affinization $\operatorname{Spec} k[X_P] = X_0$. The map $P \mapsto X_P$ is not a bijection, and the polyhedron in fact determines a presentation of X_P as a GIT quotient $\mathbb{A}^n//K$. (Equivalently, P determines a toric variety X_P along with a choice of T-equivariant ample line bundle.) In $\mathrm{Lie}(T)$, we instead consider fans Σ such that $|\Sigma| = \sigma$. Such a fan determines a toric variety X_Σ with affinization X_0, and in this case the map is proper, but not necessarily projective. In this case, toric varieties which are proper over X_0 are in bijection with fans.

By analogy, the current state of hypertoric varieties is that there is a construction which takes a central hyperplane arrangement $\mathcal{A}_0$ to an affine variety Y_0, and which takes an affinization $\mathcal{A}$ of $\mathcal{A}_0$ to a variety Y which is projective over its affinization $\operatorname{Spec} k[Y] = Y_0$. However, there is no abstract definition in the literature, and there is no equivalent to the fan construction for toric varieties. The natural combinatorial structure dual to a central arrangement is a zonotope, and the natural structure dual to an affine arrangement is a zonotopal tiling. Hence, we define a construction which takes a zonotope Z to an affine variety $Y(Z)$, and a zonotopal tiling to a variety $Y(\mathcal{T})$. Additionally, we provide an abstract definition of "hypertoric variety," and show that our varieties satisfy this definition if and only if the tiling is full-dimensional.

1.1. Preliminaries

We now give some definitions that build up to the definition of hypertoric variety. (Definition 1.4)

Fix an algebraically closed field k. We call a variety X over k **convex** if $k[X]$ is finitely generated and the natural map $\pi\colon X \to X_0 := \operatorname{Spec} k[X]$ is proper. In this case, we will also say that X is **convex over** X_0. We call X **semiprojective** if π is projective. We call a line bundle L on X **(relatively) very ample** if it is very ample with respect to the map $X \to X_0$; thus X is semiprojective if and only if it admits a very ample line bundle.

Definition 1.1 (Beauville) [Bea00] A **symplectic variety** X over k is a normal Poisson variety such that

– the Poisson structure on X is induced by a symplectic form $\omega \in \Omega^2(X^{\mathrm{reg}})$ on the regular locus

– for some (equivalently any) resolution $\pi : \tilde{X} \to X$, the form $\pi^*\omega$ extends to a 2-form on $\tilde{X}$.

We say that an action of $\mathbb{G}_m$ on X **has nonnegative weights** if the induced grading on $k[X]$ is zero in negative degrees. We say that the action **has positive weights** if, in addition, $k[X]^{\mathbb{G}_m} = k$.

Definition 1.2 A symplectic variety X over k is **conical** if there exists an action of $\mathbb{G}_m$ on X with positive weights, such that the Poisson structure is homogeneous of negative weight.

Note that the choice of $\mathbb{G}_m$ action is not part of the data of a conical symplectic variety. We call a particular $\mathbb{G}_m$ action a **weight** m **action** if the Poisson structure is homogeneous of weight $-m$.

Definition 1.3 A **partial conical symplectic resolution (PCSR)** over k is a convex conical symplectic variety X such that the map $X \to X_0$ is an isomorphism over X_0^{reg}.

We now provide our abstract definition of "hypertoric variety." Fix a torus T and $d = \dim T$.

Definition 1.4 A T**-hypertoric variety** is a PCSR X of dimension $2d$ equipped with an effective Hamiltonian action of T. We further require that the conical action of $\mathbb{G}_m$ can be chosen to commute with T.

When we have to choose a moment map μ for the action of T, we always choose the one which takes the $\mathbb{G}_m$-fixed points to the origin. The level set $\mu^{-1}(0)$ which contains these points is called the **extended core** of X. We are primarily interested in comparing the

$\mathbb{G}_m$ and T actions on the extended core, because the extended core is the only level set of μ which is preserved by $\mathbb{G}_m$.

We use $\rho(t,s)$ denote the automorphism of X induced by $(t,s) \in T \times \mathbb{G}_m$.

Definition 1.5 For a cocharacter η of T and a $T \times \mathbb{G}_m$-subvariety $X' \subset X$, we say that $\mathbb{G}_m$ **acts by** η on X' if $\rho(\eta(s),1)|_{X'} = \rho(1,s)|_{X'}$ for all $s \in \mathbb{G}_m$. We call η a **matched cocharacter** if there exists a $T \times \mathbb{G}_m$-subvariety on which $\mathbb{G}_m$ acts by η but does not act by any other cocharacter.

Definition 1.6 Given a $T \times \mathbb{G}_m$-variety X, we define the **η-twisted action** to be the action where (t,s) acts by $\rho(\eta(s)t,s)$.

In Chapter III, we define a construction $Z \mapsto Y(Z)$ which takes as input a weight 2 integral zonotope in the cocharacter lattice $N := X_*(T)$ and produces an affine Poisson $T \times \mathbb{G}_m$-variety. In Chapter IV we define $\mathcal{T} \mapsto Y(\mathcal{T})$ which takes as input a weight 2 integral zonotopal tiling in the cocharacter lattice $N := X_*(T)$ and produces a Poisson $T \times \mathbb{G}_m$-variety. It will be immediately clear from the definitions that if $\mathcal{T}$ is the trivial tiling of the zonotope Z, then $Y(Z) = Y(\mathcal{T})$ and that the $\mathbb{G}_m$ action has weight 2.

Many characteristics of $Y(\mathcal{T})$ can be determined from the combinatorics of zonotopes and zonotopal tilings, as defined in Chapter II.

- $Y(\mathcal{T})$ is a hypertoric variety if and only if $\mathcal{T}$ is a full-dimensional tiling. (Theorem 5.3).

- Given two tilings $\mathcal{T}$ and $\mathcal{T}'$, the varieties $Y(\mathcal{T})$ and $Y(\mathcal{T}')$ are isomorphic as Poisson T-varieties if and only if $\mathcal{T}$ is a translate of $\mathcal{T}'$ and are isomorphic as Poisson $T \times \mathbb{G}_m$-varieties if and only if $\mathcal{T} = \mathcal{T}'$. (Proposition 5.6)

- Every refinement of tilings $\mathcal{T} \leq \mathcal{T}'$ induces a map $Y(\mathcal{T}') \to Y(\mathcal{T})$. The affinization of $Y(\mathcal{T})$ is the map $Y(\mathcal{T}) \to Y(|\mathcal{T}|)$. In particular, $Y(\mathcal{T})$ is affine if and only if $\mathcal{T}$ is the trivial tiling of some zonotope. (Corollary 4.1)

- Strictly convex support functions on $\mathcal{T}$ are in bijection with T-equivarant ample line bundles on $Y(\mathcal{T})$. In particular, $Y(\mathcal{T})$ is projective if and only if $\mathcal{T}$ is regular. (Proposition 5.9)

- There is one extended core component $E(\mathcal{T})_v$ for each vertex v of $\mathcal{T}$, a toric variety with associated fan Σ_v. The intersection of two components $E(\mathcal{T})_v \cap E(\mathcal{T})_{v'}$ is a toric variety with fan Σ_Z, where Z is the smallest zonotope containing v and v', and is empty if $\mathcal{T}$ contains no such zonotope. (Proposition 5.5)

- The inclusion of a face of a zonotope $F \subset Z$ corresponds to the inclusion of a subvariety $Y(F) \subset Y(Z)$. (Note that if a zonotope has positive codimension, then the associated variety is not a hypertoric variety.) (Corollary 3.12) The variety $Y(\mathcal{T})$ is the colimit of the directed system $\{Y(Z) | Z \in \mathcal{T}\}$, where morphisms are given by face inclusions. (Corollary 4.3)

- For a cocharacter η, the η-twisted $\mathbb{G}_m$ action has nonnegative weights if and only if $\eta \in |\mathcal{T}|$ and it has positive weights if and only if η is in the interior of $|\mathcal{T}|$. (Proposition 5.7)

We also make the conjecture that we have described all hypertoric varieties.

Conjecture 1.7 *For every T-hypertoric variety Y, there exists a full-dimensional weight-two zonotopal tiling $\mathcal{T}$ in N (unique up to translation by Theorem 5.3) such that Y is isomorphic to $Y(\mathcal{T})$ as a Poisson T-variety.*

We end this section by giving examples of results on toric varieties and analogous results on hypertoric varieties. (Table 1.1)

Toric result	Hypertoric result
A cone σ in N determines an affine toric variety $X(\sigma)$.	A zonotope Z in N determines an affine hypertoric variety $Y(Z)$.
A cone $\sigma^\vee$ in $M := N^*$ determines an affine toric variety $X(\sigma)$.	A hyperplane arrangement $\mathcal{A}_0$ in M determines an affine hypertoric variety $Y(\mathcal{A})$.
A polytope P whose recession cone is $\sigma^\vee$ determines projective toric partial resolution $X(P)$ of $X(\sigma)$.	An affine hyperplane arrangement $\mathcal{A}$ with associated central arrangement $\mathcal{A}_0$ determines a projective hypertoric resolution $Y(\mathcal{A})$ of $Y(\mathcal{A}_0)$.
A refinement Σ of σ determines a toric partial resolution $X(\Sigma)$ of $X(\sigma)$	A tiling $\mathcal{T}$ of Z determines a hypertoric partial resolution $Y(\mathcal{T})$ of $Y(Z)$.
A support function ϕ on Σ determines a T-equivariant line bundle $L(\phi)$ on $X(\Sigma)$, which is ample if and only if ϕ is strictly convex.	A support function ϕ on $\mathcal{T}$ determines a T-equivariant line bundle $L(\phi)$ on $Y(\mathcal{T})$, which is ample if and only if ϕ is strictly convex.

TABLE 1.1. A comparison of results on toric and hypertoric varieties

CHAPTER II

COMBINATORICS

We now present standard material on oriented matroids, zonotopes, and hyperplane arrangements. See for example [BLVS$^+$99] or [Zie95].

Throughout this chapter, $\mathbf{a} = (a_1, \ldots, a_n)$ is any configuration of vectors in a lattice N which spans $N_{\mathbb{R}}$[1], and ν is an additional integral vector in N. Given such a configuration, we describe three related constructions: the oriented matroid $\mathcal{M}(\mathbf{a})$, the linear hyperplane arrangement $\mathcal{A}(\mathbf{a})$, and the zonotope $Z(\mathbf{a})$.

2.1. Sign Vectors and Oriented Matroids

We refer to an element of the set $\{+1, -1, 0\}$, which we abbreviate $\{+, -, 0\}$, as a **sign**, and for any index set E, an element of $\{+, -, 0\}^E$ as a **sign vector**.[2] Given a real number λ, we use $\operatorname{sign}(\lambda)$ to mean $+$, $-$, or 0 if λ is positive, negative, or zero, and given a vector (λ_i) of real numbers, we define the sign vector $\operatorname{sign}(\lambda)$ componentwise. Finally, we define a partial order on signs by $0 < +$ and $0 < -$, and define a partial order on sign vectors componentwise.

Given a sign vector u, we define the support of u to be the set $\{i \in E | u_i \neq 0\}$. Given two sign vectors u and v, we define the separation set $S(u, v)$ to be the subset $\{i | u_i = -v_i \neq 0\}$ of their mutual support on which they disagree. We define uv as the componenent wise product $(uv)_i = u_i v_i$, and we say $u \perp v$ if uv consists of only 0s, or has at least one $+$ and at least one $-$. Finally, we define $u \circ v$ by $(u \circ v)_i = u_i$ if $u_i \neq 0$, and v_i otherwise.

[1] All of the combinatorial definitions are valid for any configuration, spanning or not. The difference is only important in Remark 2.6.

[2] We will almost always use either $[n] := \{1, \ldots, n\}$ or $[n] \cup 0$ as index sets.

Given a vector configuration $\mathbf{a}$, we can define two sets of sign vectors $\mathcal{V} = \{\text{sign}(\lambda) | \sum \lambda_i a_i = 0\}$ and $\mathcal{V}^* = \{\text{sign}(\lambda(\mathbf{a})) | \lambda \in N^*\}$. These are our primary examples of oriented matroids:

Definition 2.1 An oriented matroid $\mathcal{M}$ is a set E and a collection $\mathcal{F}$ of sign vectors indexed by E satisfying:

- $000 \cdots 0 \in \mathcal{F}$
- If $u \in \mathcal{F}$, then $-u \in \mathcal{F}$
- If $u, v \in \mathcal{F}$, then $u \circ v \in \mathcal{F}$
- If $u, v \in \mathcal{F}$ and $i \in S(u, v)$, then there is $w \in \mathcal{F}$ such that $w_i = 0$ and $w_j = (u \circ v)_j$ for all $j \notin S(u, v)$.

We have the following from [Zie95]:

Proposition 2.2 *Let $\mathbf{a} \in N^n$. Then $\mathcal{M}(\mathbf{a}) = ([n], \mathcal{V})$ and $\mathcal{M}^*(\mathbf{a}) = ([n], \mathcal{V}^*)$ are oriented matroids.*

Proposition 2.3 *Let $(E, \mathcal{F})$ be an oriented matroid. Then $(E, \mathcal{F}^\perp)$ is also an oriented matroid, where $\mathcal{F}^\perp := \{u \colon u \perp v \text{ for all } v \in \mathcal{F}\}$.*

The entire set $\mathcal{F}$ can be reconstructed from its maximal elements $\mathcal{F}_{\max}$ or its minimal nonzero elements $\mathcal{F}_{\min}$ with respect to the partial order $<$.

Proposition 2.4 *Let $\mathbf{a} \in N^n$. Then $\mathcal{V} = (\mathcal{V}^*)^\perp = (\mathcal{V}^*_{min})^\perp = (\mathcal{V}^*_{max})^\perp$ and $\mathcal{V}^* = \mathcal{V}^\perp = \mathcal{V}^\perp_{min} = \mathcal{V}^\perp_{max}$.*

For any sign vector u on an index set E and set $E' \subseteq E$, we define $u|_{E'}$ to be the sign vector obtained by deleting all entries indexed by $E \setminus E'$. If $\mathcal{M} = (E, \mathcal{F})$ is a matroid and $E' \subseteq E$, we define $\mathcal{M}|_{E'}$ to be $(E', \{u|_{E'} \colon u \in \mathcal{F}\})$.

If u and u' are sign vectors on E and E', we use $u|u'$ to represent the sign vector on $E \sqcup E'$ obtained by concatenating their entries (and likewise for $\mathbf{a}|\mathbf{a}'$).

2.2. Zonotopes and Linear Hyperplane Arrangements

Given a configuration $\mathbf{a}$ in N, we can make two geometric constructions:

Definition 2.5 A (weight 2) zonotope in $N_{\mathbb{R}}$ is a polytope which is Minkowski sum of integral line segments of even length. We define $Z(\mathbf{a}) := \sum_{i=1}^{n}[-1,1] \cdot a_i$ and more generally, for any sign vector u, $Z(\mathbf{a}, u) := \sum_{i=1}^{n} u_i a_i + \sum_{u_i=0} \cdot a_i$.

Remark 2.6 Zonotopes of the form $Z(\mathbf{a})$ are precisely the full-dimensional zonotopes centered at 0. We will always use such a zonotope, and all other zonotopes will be subsets of the form $Z(\mathbf{a}, u)$.

The term "weight 2" is a reference to the length of $[-1, 1]$. We assume all zonotopes are weight 2 until Section 5.3.

Alternately, we may view $Z(\mathbf{a})$ as the convex hull of the points $\sum_{i=1}^{n} \pm a_i$. It is clear that the zonotope is unchanged if we permute the a_i or replace a_i with $-a_i$; we call such an action a **relabeling**. It is also unchanged if we replace ℓ copies of a_i with a single vector ℓa_i or vice versa; we call this a **multiplicity operation**. Conversely, we may recover $\mathbf{a}$ up to relabeling and multiplicity: each edge is a translate of $[-m, m] \cdot a_i$ for some a_i; the vectors a_i and $-a_i$ appear a total of m times if both length and multiple appearances are counted.

Figure 2.1 gives an example of a zonotope and a relabeling.

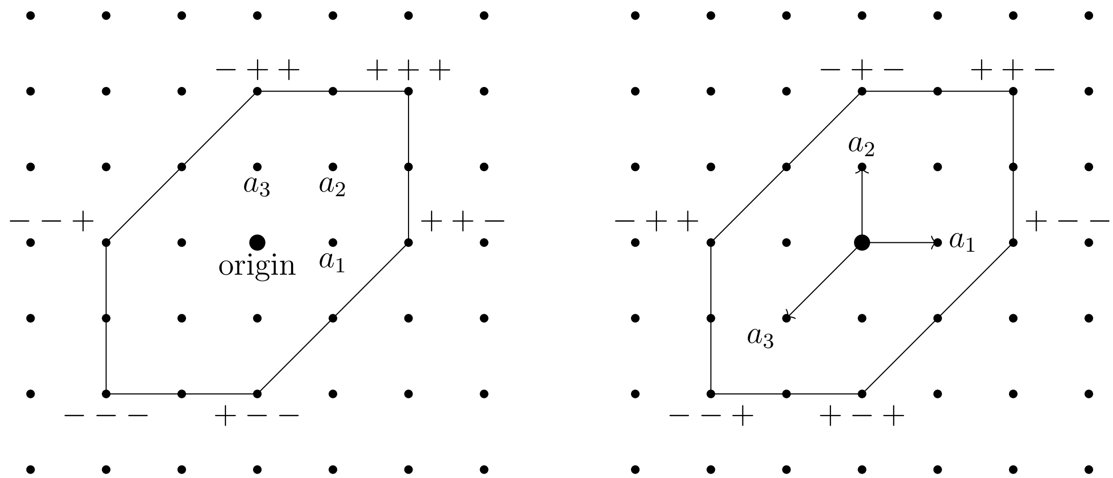


FIGURE 2.1. A zonotope with vertices labeled, and a relabeling of the same zonotope. To avoid confusion, no axes are drawn, and vectors are based at the origin.

Definition 2.7 We call a zonotope Z a **parallelotope** if there is an injective affine map $\phi\colon \mathbb{R}^n \to N_{\mathbb{R}}$ such that $\phi(\mathbb{Z}^n) \subset N$ and $\phi([-1,1]^n) = Z$. We call a parallelotope a **cube** if $\phi(\mathbb{Z}^n) = N$.

Definition 2.8 Fix an edge E of a zonotope Z. We define a **zone** to be all faces of Z which contain an edge parallel to E.

We note that for any face F of $Z(\mathbf{a})$, there is a unique sign vector u so that $F = Z(\mathbf{a}, u)$.[3] Given a zonotope Z, we may construct sign vectors without explicit reference to $\mathbf{a}$: we number the zones $\mathcal{Z}_1, \ldots, \mathcal{Z}_n$ of Z. For each zone $\mathcal{Z}_i$, the set $\bigcup_{F \notin \mathcal{Z}_i} F$ contains two connected components; we arbitrarily label one the positive side and one the negative side. Then to each face F, we give it the sign vector whose i^{th} coordinate is $+$ or $-$ if F is on the positive or negative side of $\mathcal{Z}_i$, and 0 if $F \in \mathcal{Z}_i$.

[3]In the case of a non-face subzonotope, neither existence nor uniqueness of the sign vector is guaranteed.

Definition 2.9 Let F be a face of a zonotope Z; then we define the cone associated to F, $C_{F,Z} := \mathbb{R}_+(Z - F) = \{r(n_1 - n_2)\colon r \in \mathbb{R}, n_1 \in Z, n_2 \in F\}$. If there is no ambiguity, we write $C_{F,Z}$ as C_F.

Definition 2.10 Given $\mathbf{a} \in N^n$, for each nonzero a_i, we define $H_i = a_i^\perp \subset M := N^*$, and we define $\mathcal{A}(\mathbf{a}) = \{H_i\}_{1 \leq i \leq n}$. We also define H_i^+ and H_i^- to be the closed half spaces which a_i maps to $[0, \infty)$ and $(-\infty, 0]$, respectively.

Given a sign vector u, we define $M_u = \bigcap_{u_i \neq 0} H_i^{u_i} \cap \bigcap_{u_i = 0} H_i$. We have that H_u is not empty if and only if $u \in \mathcal{V}^*$. For each $\chi \in M$, we can associate a unique smallest u so that $\chi \in M_u$.

There is a bijection between the faces of $Z(\mathbf{a})$ and of $\mathcal{A}(\mathbf{a})$ given by sign vectors. We may describe this bijection more directly by using the fact that elements of M are linear functionals on N (and thus on Z). Given M_u, Z_u consists of the subset of Z that maximizes all elements of M_u (it suffices to choose an interior element of M_u). Conversely, given Z_u, M_u consists of the elements of M that are maximized at every point of Z_u.

2.3. Zonotopal Tilings

All of the structures in this section are informally "one dimension up" from those in the previous section. More precisely, an affine oriented matroid is an oriented matroid on the ground set $E \cup \{0\}$, an affine hyperplane arrangement in $M_\mathbb{R}$ is equivalent to a hyperplane arrangement in $M_\mathbb{R} \oplus \mathbb{R}$, and many (though crucially, not all) zonotopal tilings of a zonotope $Z \subset N_\mathbb{R}$ are equivalent to a zonotope $\tilde{Z} \subset N_\mathbb{R} \oplus \mathbb{R}$.

An affine oriented matroid is an oriented matroid with a distinguished element. More precisely,

Definition 2.11 An **affine oriented matroid** $(E, \mathcal{F}, g)$ is a set E, and element $g \in E$, and a set $\mathcal{F}$ of sign vectors on E such that $(E, \mathcal{F})$ is an oriented matroid and there is at least one sign vector $u \in \mathcal{F}$ with $u_g \neq 0$. (In matroid terminology, g is not a loop.) The **positive covectors** $\mathcal{F}_+$ are the set of sign vectors $\{u \in \mathcal{F} \colon u_g = +\}$. A positive covector u is called a **bounded covector** if there is no sign vector v in $\mathcal{F}$ with $u < v$ and $v_g = 0$.

In practice, we will always use 0 as the distinguished element, and conversely if $0 \in E$ then we are viewing the associated matroid as an affine matroid. We say that $(E \cup \{0\}, F)$ is an affine matroid over the oriented matroid $(E, \{u|_E \colon u \in \mathcal{F} \text{ and } u_0 = 0\})$.

Note that $\mathcal{F} = ((\mathcal{F}_+)^\perp)^\perp$, so that any affine oriented matroid is identified by its positive covectors.

Definition 2.12 A zonotopal tiling $\mathcal{T}$ is a collection of zonotopes such that

- $|\mathcal{T}| := \bigcup_{Z \in \mathcal{T}} Z$ is a zonotope.

- If F is a face of $Z \in \mathcal{T}$, then $F \in \mathcal{T}$.

- If $Z, Z' \in \mathcal{T}$, then the intersection $Z \cap Z'$ is a face of both Z and Z'.

In this case we says that $\mathcal{T}$ is a zonotopal tiling of $|\mathcal{T}|$.

Figure 2.2 gives two examples of zonotopal tilings.

Definition 2.13 Let $\mathbf{a} \in N^n$ be a vector configuration, and $\mathcal{M}_+$ be the positive covectors of some oriented matroid over the matroid $\mathcal{M}(\mathbf{a})$. Then $\mathcal{T}(\mathbf{a}, \mathcal{M}_+) := \{Z(\mathbf{a}, u) | u \in \mathcal{M}_+\}$.

Tilings correspond to oriented matroid extensions. [RGZ94, Theorem 1.7]

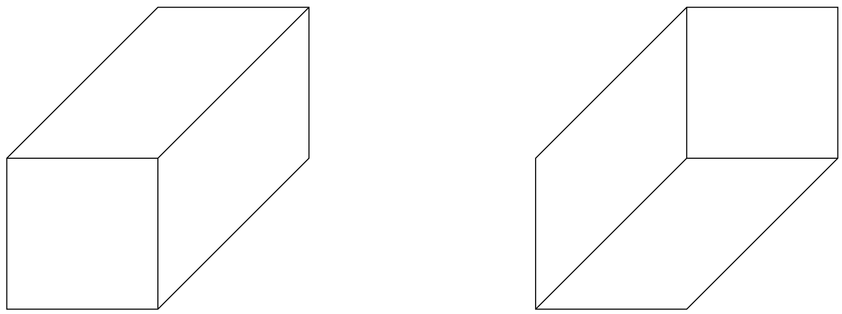


FIGURE 2.2. Two different tilings of the same zonotope.

Theorem 2.14 (Bohne-Dress) *Let* $\mathbf{a} \in N^n$ *be a configuration. Then* $\mathcal{T}(\mathbf{a}, \mathcal{M}_+)$ *is a zonotopal tiling of* $Z(\mathbf{a})$*. Furthermore, all zonotopal tilings of* $Z(\mathbf{a})$ *arise this way.*

To describe the inverse map, we generalize the notion of zones to tilings.

Definition 2.15 Divide the edges of $\mathcal{T}$ into equivalence classes using the relation generated by $E \sim E'$ if E and E' are opposite edges of some zonotope $Z \in \mathcal{T}$. Fix one such equivalence class; the collection of all zonotopes which contain an edge from this class is called a **zone** of the tiling. (See Figure 2.3.)

Remark 2.16 If all edges of $|\mathcal{T}|$ are edges of $\mathcal{T}$, then $E \sim E'$ if and only if E is parallel to E'.

If $\mathcal{T}$ is a tiling, then we obtain a sign vector for each zonotope as before: number the zones and designate one side as "positive" and one as "negative." From here we obtain both a configuration $\mathbf{a}$ and a set $\mathcal{M}_+$ of sign vectors. By the Bohne-Dress Theorem, these are the sign vectors of an orientation. If we wish to use a designated representation $Z = Z(\mathbf{a}')$, then we may transform $\mathbf{a}$ to $\mathbf{a}'$ by relabeling and multiplicity operations, provided we apply the same transformations to $\mathcal{M}_+$.

Definition 2.17 Let Z be a zonotope in $\mathcal{T}$. Then the fan associated to Z is

$$\Sigma_Z := \{C_{Z,Z'} | Z \text{ is a face of } Z' \in \mathcal{T}\}.$$

We note that a zonotopal tiling can be recovered from the set of vertices and their associated fans.

Definition 2.18 Given two tilings $\mathcal{T}$ and $\mathcal{T}'$ with $|\mathcal{T}| = |\mathcal{T}'|$, we say that $\mathcal{T}'$ refines $\mathcal{T}$ if, for every $Z' \in \mathcal{T}'$, there is a $Z \in \mathcal{T}$ such that $Z' \subseteq Z$. In this case, we write $\mathcal{T}' \leq \mathcal{T}$.

If $\mathcal{T} = \mathcal{T}(\mathbf{a}, \mathcal{M}_+)$ and $\mathcal{T}' = \mathcal{T}(\mathbf{a}, \mathcal{M}'_+)$, we say that $\mathcal{M}_+$ and $\mathcal{M}'_+$ are compatible if, for $u \in \mathcal{M}_+$ and $u' \in \mathcal{M}'_+$, $Z(\mathbf{a}, u) \subset Z(\mathbf{a}, u')$ implies $u \geq u'$.

Definition 2.19 We define a **support function** ϕ for $\mathcal{T}$ to be a piecewise linear function $|\mathcal{T}| \to \mathbb{R}$ such that

- $\phi(N \cap |\mathcal{T}|) \subset \mathbb{Z}$
- ϕ is affine-linear on each zonotope of $\mathcal{T}$, and
- if v' and v are opposite vertices of $|\mathcal{T}|$, then $\phi(v') = -\phi(v)$.

A support function is called **strictly convex** if it is convex and the maximal domains of linearity are the maximal zonotopes of $\mathcal{T}$. If there exists a strictly convex support function for $\mathcal{T}$, then $\mathcal{T}$ is called a **regular** zonotope.

Note that in the case of a zonotope centered at 0, the third condition becomes $\phi(-v) = -\phi(v)$. If m and $m + a_i$ are both in the i^{th} zone, then $\phi(m + a_i) - \phi(m)$ is independent of m; we call this integer the slope along a_i and refer to it as r_i. The points (a_i, r_i) and $(-a_i, -r_i)$ in $N_{\mathbb{R}} \times \mathbb{R}$ are then the vertices of a zonotope $Z(\phi)$ (centered at zero by the third condition), and for any $a \in N_{\mathbb{R}}$, $\phi(a)$ is the maximum value of r such that $(a, r) \in Z(\phi)$.

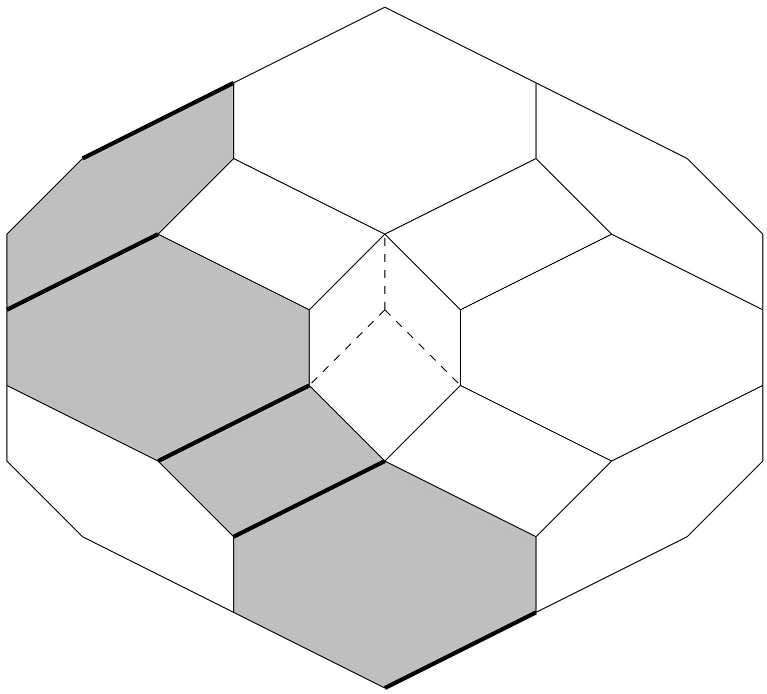


FIGURE 2.3. The bold line segments are an equivalence class. Together with the gray two-dimensional zonotopes, they form a zone of the tiling. The tiling which includes the dashed edges and the three parallelotopes which contain them is called the non-Pappus tiling, and is well-known to be non-regular.

Such a tuple $(r_1, \ldots, r_n)$ can also be used to define an affine arrangement by translating H_i by r_i; more precisely, we define $\tilde{H}_i := \{a_i + r_i = 0\}$. Then each hyperplane still has a positive and a negative side as before, and we may again define chambers M_u. There is of course a bijection between affine arrangements and support functions given by using the same r_i. However, this has a more geometric meaning.

Given a configuration $\mathbf{a}$ and a tuple $(r_1, \ldots, r_n)$, define $\tilde{a}_i := (a_i, r_i) \in N \oplus \mathbb{R}$. Then the lower faces of $Z(\tilde{\mathbf{a}})$ determine a convex support function on a tiling of $Z(\mathbf{a})$, and the arrangement $\{\tilde{H}_i\}$ is the the intersection of the arrangement $\mathcal{A}(\tilde{\mathbf{a}})$ with $N_{\mathbb{R}} \times \{1\}$.

CHAPTER III

VARIETIES FROM ZONOTOPES

Rather than an abstract lattice N, we now assume that T is a torus over a field k, and $\mathbf{a}$ is a vector configuration in $N := X_*(T)$ which is compatible with k in the sense that no nonzero a_i has a length which is a multiple of the characteristic of k.

A configuration $\mathbf{a} \subset N$ determines a map from the coordinate torus $\mathbb{G}_m^n$ to the torus T. In general, we only have an exact sequence

$$1 \to K \to \mathbb{G}_m^n \to T \to T' \to 1$$

where the connected component K° of K is a torus, and K/K° is a finite group, and T and T' are tori. If the configuration spans $N_\mathbb{R}$, then T' is trivial, and if the sublattice generated by $\mathbf{a}$ is saturated, then $K = K^\circ$. Unless otherwise noted, we assume for the rest of this chapter that $\mathbf{a}$ spans $N_\mathbb{R}$. (We do not assume that the sublattice they generate is saturated.) If N' is a lattice, then we define $T_{N'} := N' \otimes \mathbb{G}_m$ to be the torus with cocharacter lattice N'. If $N' \subset N$, then $T_{N'}$ is a finite quotient of a subtorus of $T = T_N$. This allows us to write a short exact sequence of tori

$$1 \to K^\circ \to \mathbb{G}_m^n \to T_{\mathbb{Z}\mathbf{a}} \to 1$$

which is easier to work with. In particular, if we choose isomorphisms $K^\circ \cong \mathbb{G}_m^k$ and $T \cong \mathbb{G}_m^d$, then the map from $\mathrm{Lie}(\mathbb{G}_m^n)$ to $\mathrm{Lie}(T)$ is given by an $n \times d$ matrix $A = (a_1|\cdots|a_n)$. Then a cocharacter β of $\mathbb{G}_m^n$ is a cocharacter of K° if $\sum_{i=1}^n \beta_i a_i = 0$. The dependences β are a vector space of dimension k; we choose a basis $\{b_j\}$ of these to be the rows of a matrix B. Then the map $\mathrm{Lie}(K) \to \mathrm{Lie}(\mathbb{G}_m^n)$ is given by B^T. By choosing this basis carefully, we can make computations easier. Among other things, throughout this paper it may be assumed that $\mathrm{sign}(b_j)$ is minimal.

3.1. Varieties from Arrangements and Signed Arrangements

In this section we define the variety $Y(Z)$ for any zonotope $Y(Z)$. First, given a spanning configuration $\mathbf{a}$, we define $Y(\mathbf{a})$ as the categorical quotient by K of the level set $\mu_K^{-1}(0)$. We then define the variety $Y(\mathbf{a}, u, N)$ by removing coordinate hyperplanes from $T^*\mathbb{A}^n$ before taking the quotient. Finally, we show that the variety obtained by either construction depends only on the zonotope $Z(\mathbf{a})$ or $Z(\mathbf{a}, u)$.

We coordinatize $T^*\mathbb{A}^n = \operatorname{Spec} k[x_{1+}, \ldots, x_{n+}, x_{1-}, \ldots, x_{n-}]$. If necessary for clarity, we may write $x_{i,+}$ or $x_{i,-}$ instead. For convenience, we often write $x_{i\pm}$ to mean both x_{i+} and x_{i-}; thus we could instead write $k[x_{i\pm} \colon 1 \le i \le n]$. We use $x_+ = (x_{1+}, \ldots, x_{n+})$ and likewise for x_-. Finally, we define $x_i = x_{i+}x_{i-}$.

Given a configuration $\mathbf{a} \in N^n$ of length n, we define [1]

$$U(\mathbf{a}, N) := \operatorname{Spec} k[x_{1+}, \ldots, x_{n+}, x_{1-}, \ldots, x_{n-}] \cong T^*\mathbb{A}^n$$

with the following structures:

- The Poisson structure is given by the symplectic form $\sum_{i=1}^n dx_{i+} \wedge dx_{i-}$.
- The $\mathbb{G}_m^n$ action is given by $t \cdot (x_+, x_-) = (tx_+, t^{-1}x_-)$.
- This action is Hamiltonian with moment map $\mu_n(x_+, x_-) = (x_1, \ldots, x_n) \in \mathbb{A}^n \cong \mathrm{Lie}(\mathbb{G}_m^n)^*$
- The $\mathbb{G}_m$ action has weight one on all variables $s \times (x_+, x_-) = (sx_+, sx_-)$.

[1] We do not need all the data of $(\mathbf{a}, N)$ to define $U(\mathbf{a}, N)$, but this keeps $U(\mathbf{a}, N)$ consistent with other notation. In particular, we use the length of $\mathbf{a}$.

We will define the associated affine hypertoric variety $Y(\mathbf{a}, N)$ as the algebraic symplectic quotient $U(\mathbf{a}, N)////K$, which will require both taking a level set of the moment map μ_K and taking a GIT quotient by K. This may be done in either order, and it is useful to have notation for the result of either operation done alone.

Definition 3.1 Let $\mathbf{a}$ be a configuration of vectors in N that spans $N_{\mathbb{R}}$. Then we define $L(\mathbf{a}, N) = V(\mu_K) \subseteq U(\mathbf{a}, N)$ and $X(\mathbf{a}, N) := U(\mathbf{a}, N)//K$, and $Y(\mathbf{a}, N) = L(\mathbf{a}, N)//K \cong V(\mu_K) \subset X(\mathbf{a}, N)$.

Proposition 3.2 *Let $\mathbf{a}'$ and $\mathbf{a}''$ be vector configurations in lattices N' and N'', respectively. Then $Y(\mathbf{a}'|\mathbf{a}'', N' \oplus N'') \cong Y(\mathbf{a}', N') \times Y(\mathbf{a}'', N'')$ as $T_{N'\oplus N''} \cong T_{N'} \times T_{N''}$-Poisson varieties.*

Proof: We may choose B to be in block diagonal form. Then we have

$$\begin{aligned} k[Y(\mathbf{a}'|\mathbf{a}'', N' \oplus N'')] &= k[T^*\mathbb{A}^{n'} \times T^*\mathbb{A}^{n''}]^{K'\oplus K''}/(\mu_{K'}, \mu_{K''}) \\ &\cong k[T^*\mathbb{A}^{n'}]^{K'}/(\mu_{K'}) \otimes k[T^*\mathbb{A}^{n''}]^{K''}/(\mu_{K''}) \\ &\cong k[Y(\mathbf{a}', N')] \otimes k[Y(\mathbf{a}'', N'')] \end{aligned}$$

$\square$

In particular, we may add or delete 0 entries of $\mathbf{a}$ without affecting the variety, since $Y((0), 0)$ is a point.

Proposition 3.3 *Let $\mathbf{a}$ be a configuration of vectors in N that spans $N_{\mathbb{R}}$. Then $Y(\mathbf{a}, N) = Y(\mathbf{a}, \mathbb{Z}\mathbf{a})/(K/K^\circ)$.*

Proof: We have that $L(\mathbf{a}) := L(\mathbf{a}, N) = L(\mathbf{a}, \mathbb{Z}\mathbf{a})$, since it does not depend upon the ambient lattice. Then $k[Y(\mathbf{a}, N)] = k[L(\mathbf{a})]^K = (k[L(\mathbf{a})]^{K^\circ})^{K/K^\circ} = k[Y(\mathbf{a}, \mathbb{Z}\mathbf{a})]^{K/K^\circ} = \operatorname{Spec} k[Y(\mathbf{a}, \mathbb{Z}\mathbf{a})]//(K/K^\circ)$, and the quotient is geometric since K/K° is finite. □

Proposition 3.4 *If* $\mathbf{a}$ *and* $\mathbf{a}'$ *are configurations of vectors in* N*, compatible with* k*, with* $Z(\mathbf{a}) = Z(\mathbf{a}')$*, then* $Y(\mathbf{a}, N) \cong Y(\mathbf{a}', N)$*.*

Proof: It suffices to consider the case where $\mathbf{a}$ and $\mathbf{a}'$ differ by a permuation, a sign change, or a multiplicity operation. In the first two cases, the isomorphisms are obvious. In the third case, we assume that the operation takes place at the end, so that $a_i = a'_i$ for $i < n$, and $a_n = \ell a'_i$ for $i \geq n$. We choose B to have the form

$$B = \left(\begin{array}{c|c} B'' & 0 \\ \hline \ell C & 1 \end{array}\right)$$

and

$$B' = \left(\begin{array}{c|ccc} B'' & 0 & 0 & 0 \\ \hline C & 1 & -1 & 0 \\ 0 & & \ddots & \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Define a map $k[Y(\mathbf{a}')] \to k[Y(\mathbf{a})]$ by $x'_{i\pm} \mapsto x_{i\pm}$ for $i < n$, and $x'_{n\pm} \cdots x'_{n'\pm} \mapsto x_{n\pm}/\ell$ for $i \geq n$, and $x'_i \mapsto x_n/\ell$. □

Note that the compatibility with k is essential; if ℓ divides $char k$, then no such map exists. (There is a map $Y(\mathbf{a}) \to Y(\mathbf{a}')$ which is an isomorphism of varieties, but not of Poisson varieties.)

We also may define varieties using sign vectors; if $u_i \neq 0$, then we remove the locus $x_{i,u_i} = 0$ for each nonzero u_i. We use $x_u := \prod x_{i,u_i}$, where $x_{i,0} = 1$. We begin with $U(\mathbf{a}, u, N) := \operatorname{Spec} k[x_{i\pm}]_{x_u} \subset U(\mathbf{a}, N)$, and define the other three analogously: $L(\mathbf{a}, u, N) := U(\mathbf{a}, u, N) \cap L(\mathbf{a}, N)$, $X(\mathbf{a}, u, N) := U(\mathbf{a}, u, N)//K$, and $Y(\mathbf{a}, u, N) := L(\mathbf{a}, u, N)//K \cong V(\mu_K) \subset X(\mathbf{a}, u, N)$. Note that $U(\mathbf{a}, N) = U(\mathbf{a}, 0, N)$, and thus likewise for $L(\mathbf{a}, N)$, $X(\mathbf{a}, N)$, and $Y(\mathbf{a}, N)$.

Lemma 3.5 *Let $(\mathbf{a}, u)$ be a signed configuration and $(\mathbf{a}', u')$ be the signed configuration obtained by appending $\eta \in N$ to $\mathbf{a}$ and $+$ to u. Then $Y(\mathbf{a}, u) \cong Y(\mathbf{a}', u')$ as T-varieties, with $\mathbb{G}_m$ actions twisted by η.*

Proof: Write B' by appending a column of 0s to B, and then appending a single row expressing $a'_{n'}$ as a linear combination of a_i. Then $Y(\mathbf{a}', u', N) = (Y(\mathbf{a}, u, N) \times k[x_{n',\pm}, x_{n'+}^{-1}])^{K'/K}/(f)$, where

$$f = x_{i-} - \frac{1}{b_{n',d'} x_{i+}} \sum_{i=1}^{n} b_{i,d'} x_i$$

Hence $Y(\mathbf{a}', u', N) = (Y(\mathbf{a}, u, N) \times k[x_{n',+}^{\pm 1}])^{K'/K}$, and there is a bijection between K-invariant monomials in the first n variables, and K'-invariant monomials in all n' variables given by the last row of B. $\square$

In particular, we may append to $\mathbf{a}$ any configuration of vectors which sums to 0 and to u the corresponding number of +s without changing $Y(\mathbf{a}, u, N)$.

Proposition 3.6 *Let $(\mathbf{a}, u)$ and $(\mathbf{a}, u')$ be signed configurations in N. If $Z(\mathbf{a}, u) = Z(\mathbf{a}', u')$, then $Y(\mathbf{a}, u, N) \cong Y(\mathbf{a}', u'N)$ as Poisson $T \times \mathbb{G}_m$-varieties. If $Z(\mathbf{a}, u) =$*

$Z(\mathbf{a}', u') + \eta$ for some cocharacter η, then they are isomorphic as Poisson T-varieties, but the $\mathbb{G}_m$ action on $Y(\mathbf{a}, u, N)$ is the η-twisted action on $Y(\mathbf{a}', u', N)$.

The converse of both statements is true; this is proved in the next chapter. In light of Proposition 3.6, we make the following definition:

Definition 3.7 Let $Z = Z(\mathbf{a}, u)$ be a zonotope in N such that $\mathbf{a}$ spans $N_{\mathbb{R}}$. Then $Y(Z, N) := Y(\mathbf{a}, u, N)$. If N is understood, we write $Y(Z)$ for $Y(Z, N)$.

Proposition 3.8 *There is a $T_{N\cap\mathbb{R}\mathbf{a}} \times \mathbb{G}_m$-equivariant isomorphism $Y(\mathbf{a}, u, N \cap \mathbb{R}\mathbf{a}) \times T^*(T_N/T_{\mathbb{Z}\mathbf{a}}) \cong Y(\mathbf{a}, u, N)$ that is T-equivariant on the second factor.*

Proof: Let $\mathbf{e} = (e_1, \ldots, e_\ell)$ be vectors in N that descend to a basis of $N/\mathbb{Z}\mathbf{a}$. Then $(U(\mathbf{a}, u, N) \times T \times T)////\mathbb{G}_m^n = U(\mathbf{a}|\mathbf{e}|\mathbf{e}, u| + \cdots + | - \cdots -, N)////K'$, where K' is the kernel of the map given by the configuration $\mathbf{a}|\mathbf{e}|\mathbf{e}$. $\square$

Remark 3.9 All zonotopes are expressible in the form $Z(\mathbf{a}, u)$, with many nonzero signs in the case of full-dimensional zonotopes. We can also define $Y(\mathbf{a}, u, N)$ where $\mathbf{a}$ does not span N. Since we wish Proposition 3.6 to hold under this new definition, we should be able to append pairs of opposite vectors to $\mathbf{a}$, with corresponding +s appended to u, without changing Y.

Let $\mathbf{a}$ be any configuration of n vectors in N. Then we define $Y(\mathbf{a}, N) = (U(\mathbf{a}, u, N) \times T \times T)////\mathbb{G}_m^n$, where $\mathbb{G}_m$ acts on $U(\mathbf{a}, u, N)$ as normal, and on $T \times T$ via $(A(t), A(t)^{-1})$, where A is the map $\mathbb{G}_m^n \to T$.

3.2. Zonotopes and the Extended Core

Before talking about zonotopes, we need a more concrete description of the action of $\mathbb{G}_m^n$ (particularly K) on $U(\mathbf{a}, N)$. Let π_i be the projection $U(\mathbf{a}, N) \to T^*\mathbb{A}^1 = \operatorname{Spec} k[x_{i\pm}]$, $\beta \in \mathbb{Z}^n$ be a character of $\mathbb{G}_m^n$, and $p \in T^*\mathbb{A}$. Then there are three possibilities for lifting the map $\phi_{\beta,i,p} : \mathbb{G}_m \to T^*\mathbb{A}^1$ given by $t \mapsto t^\beta \cdot p$ to a map $\mathbb{A}^1 \to T^*\mathbb{A}^1$.

- If the action is nontrivial and p is off the coordinate axes then the orbit is closed, and $\phi_{\beta,i}$ may not be extended.
- If the action is nontrivial and p is on one coordinate axis, then the origin is in the closure of $t^\beta \cdot p$, and exactly one of $\phi_{\beta,i}$ or $\phi_{-\beta,i}$ may be extended.
- If p is a fixed point, then either p is the origin or $\beta_i = 0$, and both $\phi_{\beta,i}$ and $\phi_{-\beta,i}$ may be extended.

We now consider a point $p \in U(\mathbf{a}, N)$ and the map $\phi_{\beta,p} : \mathbb{G}_m \to U(\mathbf{a}, N)$ given by $t \mapsto t^\beta \cdot p$. We define a sign vector indexed by the coordinates of $\mu_n(p)$ which are zero, where $u_{p,i} = +$ if and only if $x_{i+} \neq 0$, $u_{p,i} = -$ if and only if $x_{i-} \neq 0$. The map $\phi_{\beta,p}$ may be extended if and only if it may be extended coordinatewise.

We decompose the set $\mu_n^{-1}(0)$, where at least one of x_{i+} or x_{i-} is 0 for every i, as the union of $\mathbb{G}_m$-orbits O'_u where $x_{i,u_i} \neq 0$, and $x_{i,j} = 0$ otherwise. Clearly, O'_u is in the closure of O'_v if and only if $u \leq v$, but we can say more about how this closure interacts with the action of $\mathbb{G}_m^n$.

Proposition 3.10 *Let $\beta \in \mathbb{Z}^n$ be a cocharacter of $\mathbb{G}_m^n$, let $p \in O'_u$. Then $t^\beta \cdot p$ is closed if and only if $u \perp \operatorname{sign}(\beta)$. The map ϕ_β may be extended if and only if $(u \operatorname{sign}(\beta))$ has no $-s$.*

The interesting case is where the orbit is not closed and the map may be extended: in this case the image of 0 under the extension is in the closure of O'_u but not in O'_u.

Proposition 3.11 *The K-orbit of a point $p \in O'_u$ is closed if and only if $u \in \mathcal{M}(\mathbf{a})$. If $u \notin \mathcal{M}(\mathbf{a})$, then the unique closed K-orbit in the closure of $K \cdot p$ is O'_v, where u covers v.*

Proof: If $u \in \mathcal{M}(\mathbf{a})$, then for every cocharacter β of K, $\mathrm{sign}(\beta) \in \mathcal{M}^*(\mathbf{a})$, so that $\beta(\mathbb{G}_m) \cdot p$ is closed. If $u \notin \mathcal{M}(\mathbf{a})$, then there is some cocharacter β such that $\mathrm{sign}(\beta)$ is not perpendicular to u. Since $\mathrm{sign}(\beta) \perp v$, it must be that $\mathrm{sign}(\beta)$ and v have disjoint support. Aassume that wherever $\mathrm{sign}(\beta)$ and u agree, they both have $+$. Then we have that $\lim \beta(t) \in O'_w$, where $w_i = 0$ if $u_i = \mathrm{sign}(\beta_i) = +$ and $w_i = u_i$ otherwise. If $w = v$, then we are done; otherwise we may proceed by induction, since $u > w \geq v$. □

Since $u \in \mathcal{M}(\mathbf{a})$ if and only if $Z(\mathbf{a}, u)$ is a face of $Z(\mathbf{a})$, this implies:

Corollary 3.12 *Let Z be a zonotope and F be a face of Z. Then there is a canonical inclusion $Y(F) \to Y(Z)$.*

3.3. The Extended Core

Definition 3.13 We define the extended core $E(\mathbf{a}, u, N)$ of $Y(\mathbf{a}, u, N)$ to be the subvariety[2] $\mu^{-1}(0)$, where μ is the moment map for the T action.

This is also the subvariety of $X(\mathbf{a}, u)$, defined by the ideal (μ_n). The ideal (μ_n) is not prime, but it has an obvious decomposition as $(\mu_n) = \bigcap_u ((x_{1,u_1}), \ldots, x_{n,u_n})$.

We may now describe the coordinate ring $k[Y(Z)]$ more explicitly:

[2] That is, we give the reduced scheme structure to the subscheme $\mu^{-1}(0)$.

By Definition 3.1, $Y(Z) \cong \operatorname{Spec} k[x_{i\pm}]^K/(\mu_K)$ is a subset of $X(Z) := \operatorname{Spec} k[x_{i\pm}]^K$. The variety $X(Z)$ is an affine toric variety, called the **Lawrence toric variety** associated to Z. It is clear that the components $x_i = x_{i+}x_{i-}$ of the map μ_n are primitive K-invariant monomials, and therefore all other primitive K-invariant monomials contain at most one of x_{i+} or x_{i-} for each i. We call any monomial in $k[x_{i\pm}]$ which is not in the ideal generated by μ_n an **extended core monomial**, or an **EC monomial** for short. Thus, the coordinate ring $k[X(Z)] = k[x_{i\pm}]^K$ is generated by n moment map components $x_{i+}x_{i-}$, and some number of invariant EC monomials.

Note that only *invariant* EC monomials (monomials in $k[x_{i\pm}]^K$ which are not in the ideal (μ)) are functions on the extended core, but all EC monomials are sections of line bundles (and thus determine subvarieties).

Note that the extended core is a $T \times \mathbb{G}_m$-variety, whose components are T-toric varieties. More specifically:

Definition 3.14 For any sign vector u, we define V_u to be the subvariety defined by $x_{i,-u_i} = 0$, and we define O_u to be the open subset of V_u where $x_{i,u_i} \neq 0$.

Proposition 3.15 *The components of the extended core are the V_u for maximal covectors u. The associated cone is the cone over $u\mathbf{a} := (u_i a_i)_{1 \le i \le n}$.*

Proof: By relabeling $\mathbf{a}$, it suffices to prove that the statement when u is the sign vector of all +s, or the subvariety where $x_{i-} = 0$ for all i. However, this is the T-toric variety corresponding to the cone over the a_i. This toric variety has the same dimension as T if and only if the cone over the a_i is pointed, which occurs exactly when u is a maximal covector. □

It is also clear the O_u is the dense T-orbit of V_u.

We now use matched cocharacters and positive weight cocharacters to recover Z from $Y(Z)$.

Proposition 3.16 *If $\eta \in Z(\mathbf{a}, u)$ for some $u \in \mathcal{M}(\mathbf{a})$, then $\mathbb{G}_m$ acts by η on V_u. In particular, the matched cocharacters of $Y(Z)$ are the vertices of Z.*

Proof: The action of $\mathbb{G}_m$ on V_u is given by the cocharacter $\sum a_i u_i$. Furthermore, the action of $\mathbb{G}_m$ by a_i is trivial if $u_i = 0$ since $x_{i+} = x_{i-} = 0$. But every $\eta \in Z_u$ is of the form $\eta = \sum u_i a_i + \sum_{u_i=0} c_i a_i$ for some integers c_i. $\square$

Proposition 3.17 *Let $\eta \in N$. Then the action of $\mathbb{G}_m$ on $Y(Z)$ by $s \times_\eta p := s\eta(s)p$ has nonnegative weights if and only if $\eta \in Z$. It has positive weights if and only if η is in the interior of Z.*

Proof: Moment map components always have weight 2, and a cocharacter has positive weights on $Y(Z)$ if and only if it has positive weights on $E(Z)$, if and only if it has positive weights on each component of $E(Z)$ where it is nonzero, and similarly for non-negative weights. Finally, it has positive weight on a component corresponding to a vertex v if it is in the interior of the cone Σ_v, and non-negative weights if it lies in the cone at all. $\square$

3.4. Smoothness

We now give a criterion for $Y(Z)$ to be smooth:

Proposition 3.18 *The following statements are equivalent:*

1. *Z is a cube*

2. *The variety $Y(Z)$ is smooth*

3. *The variety $Y(Z)$ is smooth at the origin*

Proof: If Z is a cube, then we may write it as $Z = (e_1, \ldots, e_n)$ for some basis $e_1, \ldots, e_n$ of N. Hence, $Y(Z) = T^*\mathbb{A}^n$. If Z is not a cube, note that $Y(Z)$ is a codimension k subvariety of $X(Z)$, and $T_0^*Y(Z)$ is a codimension k subspace of $T_0^*X(Z)$. Since $X(Z)$ is singular at the origin, so is $Y(Z)$. $\square$

If $Y(Z)$ is not smooth, then its regular locus is what might combinatorially be expected.

Proposition 3.19 *The regular locus $Y(Z)^{reg} = \bigcup_{F \text{ is a cube}} Y(F)$.*

Proof: If $p \in Y(F)$ where F is a cube, then p is a regular point since $Y(F)$ is smooth. For the converse, let F be the smallest face such that $p \in Y(F)$ and suppose that F is not a cube. Then p maps to $(0, p') \in Y(F) \times T^*(T_{N/N'})$ under the isomorphism of 3.8, and hence is a singular point. $\square$

Proposition 3.20 *If Z is a parallelotope, then $Y(Z)$ is an orbifold.*

Proof: In this case $Z = Z(\mathbf{a})$ for some basis of $N_{\mathbb{R}}$. $\square$

CHAPTER IV

VARIETIES FROM TILINGS

We now construct varieties of the form $Y(\mathcal{T})$ for zonotopal tilings $\mathcal{T}$. We are now removing a set of codimension at least 2 from $T^*\mathbb{A}^n$, although the definition does not take this form; instead of describing $U(\mathbf{a}, \mathcal{M}_+, N)$ as the complement of a set of codimension 2, we describe it as the union of $U(\mathbf{a}, u, N)$ for $u \in \mathcal{M}^+$.

Let $\mathcal{M}_+$ be the positive sign vectors of an affine oriented matroid over $\mathcal{M}(\mathbf{a})$. Then we define $U(\mathbf{a}, \mathcal{M}_+, N) = \bigcup_{u \in \mathcal{M}_+} U(\mathbf{a}, u, N) \subseteq U(\mathbf{a}, N)$. As before, we define $X(\mathbf{a}, \mathcal{M}_+, N) := U(\mathbf{a}, \mathcal{M}_+, N)//K$, $L(\mathbf{a}, \mathcal{M}_+, N) = U(\mathbf{a}, \mathcal{M}_+, N) \cap L(\mathbf{a}, N)$, and $Y(\mathbf{a}, \mathcal{M}_+, N) = L(\mathbf{a}, \mathcal{M}_+, N)$.

The variety $X(\mathbf{a}, \mathcal{M}_+, N)$ is called a Lawrence toric variety, with affinization $X(\mathbf{a}, N)$. Its fan is necessarily a subdivision of the cone associated to $X(\mathbf{a}, N)$. We denote the associated cone Σ_Z, and refer to the rays of Σ_Z as ρ_{i+} if they are associated to x_{i+} or ρ_{i-} if they are associated to x_{i-}. To a sign vector u, we associate the cone $\sigma_u := \mathbb{R}_+(\{\rho_{i,j} | j \leq u_i\})$. For any tiling $\mathcal{T}$ of Z, the map $\phi : Z_u \mapsto \sigma_u$ then associates a collection of cones $\{\phi(Z)\}_{Z \in \mathcal{T}}$ to $\mathcal{T}$. This collection determines a fan[1], and $U(\mathcal{T})$ is the open set associated to it; hence $X(\mathcal{T}) := U(\mathcal{T})//K$ is a partial toric resolution of $X(|\mathcal{T}|)$.

Because $L(\mathbf{a}, \mathcal{M}_+, N) \subset L(\mathbf{a}, N)$ with complement at least codimension 2, we have that their coordinate rings are equal, and thus so are K-invariant functions. The induced map $Y(\mathbf{a}, \mathcal{M}_+, N) \to Y(\mathbf{a}, N)$ is thus the affinization map. On the level of tilings, we have the following proposition.

[1]The collection is not itself a fan, but it does determine one; it includes all maximal cones of the fan.

Proposition 4.1 *Let $\mathcal{T}$ be any zonotopal tiling. Then $Y(\mathcal{T})_0 \cong Y(|\mathcal{T}|)$.*

Note that if $\mathcal{T}$ consists of a zonotope Z and its faces, then $U(Z) = U(\mathcal{T})$, so that we we may consider Z as a tiling.

4.1. Refinements and Partial Resolutions

For any refinement of tilings $\mathcal{T} \leq \mathcal{T}'$ of Z, we have an inclusion $U(\mathcal{T}') \subset U(\mathcal{T})$, which induces a T-equivariant Poisson map $Y(\mathcal{T}') \to Y(\mathcal{T})$. For each zonotope $Z = Z(\mathbf{a}, u) \in \mathcal{T}$, the K-orbits in $U(\mathbf{a}, u, N)$ which are in $U(\mathbf{a}, \mathcal{T}', N)$, are precisely the K-orbits indexed by $v \geq u$.

The next lemma says that these maps are all partial resolutions.

Proposition 4.2 *The map $Y(\mathcal{T}') \to Y(\mathcal{T})$ is proper and an isomorphism over the regular locus of $Y(\mathcal{T})$.*

Proof: Choose compatible representations $\mathcal{T}' = \mathcal{T}(\mathbf{a}, \mathcal{M}'_+)$ and $\mathcal{T} = \mathcal{T}(\mathbf{a}, \mathcal{M}_+)$. On the level of Lawrence toric varieties, $X(\mathbf{a}, \mathcal{M}'_+, N) \to X(\mathbf{a}, \mathcal{M}_+, N)$ is proper; this remains when taking a closed subvariety. The map is an isomorphism over the regular locus because cubes cannot be subdivided. $\square$

The following description of $Y(\mathcal{T})$ is now evident:

Proposition 4.3 *The variety $Y(\mathcal{T})$ is the colimit of the directed system $\{Y(Z) | Z \in \mathcal{T}\}$ with morphisms given by the inclusions of faces as in Corollary 3.12.*

CHAPTER V

THE MAIN THEOREM

We now prove that $Y(\mathcal{T})$ is a hypertoric variety for any full-dimensional tiling $\mathcal{T}$. It is clear that there is a positive weight $\mathbb{G}_m$ action, that the action of T is effective. By Proposition 4.2 we have that $Y(\mathcal{T})$ is convex and a partial resolution. In the case that Z is a parallelotope, we have symplecticness by the following result of Beauville:

Lemma 5.1 *If Z is a full-dimensional parallelotope, then $Y(Z)$ is a hypertoric variety.*

Proof: In this case K is finite, so $Y(Z) = \mathbb{A}^{2n}/K$, which is symplectic by [Bea00, 2.4]. □

We prove that $Y(\mathcal{T})$ is symplectic in general by using the maps $\phi_{\mathcal{T}',\mathcal{T}}$ and the following lemma:

Lemma 5.2 *Suppose that $X \to Y$ is a Poisson map of normal varieties that is an isomorphism over Y^{reg}. Then X is symplectic if and only if Y is.*

Proof: Choose any resolution $\widetilde{X} \to X$. If either X or Y is symplectic, then so is $\widetilde{X}$ since it is a resolution of both. And if $\widetilde{X}$ is symplectic, then both X and Y are, since the map to each is an isomorphism over the regular locus. □

Theorem 5.3 *Let $\mathcal{T}$ be any full-dimensional zonotopal tiling. Then $Y(\mathcal{T})$ is a hypertoric variety.*

Proof: Choose a refinement $\mathcal{T}'$ of $\mathcal{T}$ by parallelotopes. To prove normality, let $Z \in \mathcal{T}$. Then $Y(\mathcal{T}'|_Z)$ is normal, hence $Y(Z) = Y(\mathcal{T}'|_Z)_0$ is too. Since the $Y(Z)$ cover $Y(\mathcal{T})$,

$Y(\mathcal{T})$ is normal. To prove symplecticness, we note that $Y(\mathcal{T}')$ is symplectic, since by Proposition 5.1 it has a cover by symplectic varieties; hence by Lemma 5.2 $Y(\mathcal{T})$ is symplectic too. $\square$

5.1. The Extended Core

We now state the analogues of Propositions 3.16 and 3.17 for tilings.

Proposition 5.4 *If $\eta \in Z(\mathbf{a}, u)$, then $\mathbb{G}_m$ acts by η on V_u.*

Proof: The proof of 3.16 applies without modification. $\square$

We also note that $V_u \subset V_v$ if and only if $u \geq v$. Hence we can describe the extended core more completely:

Proposition 5.5 *There is one extended core component $E(\mathcal{T})_v$ for each vertex v of $\mathcal{T}$, a toric variety with associated fan Σ_v. The intersection of two components $E(\mathcal{T})_v \cap E(\mathcal{T})_{v'}$ is a toric variety with fan Σ_Z, where Z is the smallest zonotope containing v and v', and is empty if $\mathcal{T}$ contains no such zonotope.*

Proposition 5.6 *Given two tilings $\mathcal{T}$ and $\mathcal{T}'$, $Y(\mathcal{T})$ and $Y(\mathcal{T}')$ are isomorphic as Poisson T-varieties if and only if $\mathcal{T} = \mathcal{T}' + \eta$. In this case, the $\times$ action on $Y(\mathcal{T})$ is the $\times_\eta$ action on $Y(\mathcal{T}')$.*

Proof: If $Y(\mathcal{T})$ and $Y(\mathcal{T}')$ are isomorphic as Poisson T-varieties, then they have the same number of extended core components, and hence $\mathcal{T}$ and $\mathcal{T}'$ have the same number of vertices. Furthermore, the fans associated to corresponding vertices are the same. This means that there are bijections between the zones of $\mathcal{T}$ and $\mathcal{T}'$ as well. If $\mathcal{T}$ and

$\mathcal{T}'$ are not translation-equivalent, then at least one zone is of different widths on the two tilings. But then the two varieties do not have the same resolutions, and hence are not isomorphic. $\square$

Because of the map $\phi_{\mathcal{T},\mathcal{T}'}$, we can extend Proposition 3.17 to all varieties of the form $Y(\mathcal{T})$.

Corollary 5.7 *Let $\eta \in N$. Then the action of $\mathbb{G}_m$ on $Y(\mathcal{T})$ by $s \times_\eta p := s\eta(s)p$ has nonnegative weights if and only if $\eta \in |\mathcal{T}|$. It has positive weights if and only if η is in the interior of $|\mathcal{T}|$.*

5.2. Line Bundles

For line bundles, we confine our discussion to the smooth case. Suppose that we have a zonotopal tiling $\mathcal{T}$ consisting of cubes, and a support function ϕ. This support function determines n integers r_i, which we may interpret as a character $(r_1, \ldots, r_n)$ on T^n, which descends to a character α on K. Conversely, a character of K may be lifted to a character of T^n, which provides the r_i to determine a support function.

Proposition 5.8 *Let $\mathcal{T}$ be a tiling of cubes. Then strictly convex support functions on $\mathcal{T}$ are in bijection with T-equivariant ample line bundles on $Y(\mathcal{T})$.*

Proof: We have a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, equal] \arrow[d, two heads] & H^2_{T^n}(T^*\mathbb{A}^n) \arrow[r, "\approx"] \arrow[d, two heads] & H^2_T(Y) \arrow[d, two heads] \\
X^*(K) \arrow[r, equal] & H^2_K(T^*\mathbb{A}^n) \arrow[r, "\approx"] & H^2(Y)
\end{tikzcd}$$

of Kirwan maps. Given $r \in \mathbb{Z}^n$, we have a T-equivariant line bundle in $H^2_T(Y)$. The cone of strictly convex support functions is the GIT cone under this bijection; by [BPW, 2.22], the cone of strictly convex support functions in $\mathbb{Z}^n$ corresponds to the ample cone in $H^2(Y)$. $\square$

Proposition 5.9 *If ϕ is a strictly convex support function for $\mathcal{T}$, then the set $U(\mathcal{T})$ is precisely the semistable set for the character α.*

Proof: We may lift ϕ to a support function on the fan of $X(\mathcal{T})$; by the theory of toric varieties (for example, Section 14.2 of [CLS11]), $X(\mathcal{T}) = T^*\mathbb{A}^n//_\alpha K$. $\square$

Proposition 5.10 *Let $\mathcal{T}$ be a zonotopal tiling. Then $Y(\mathcal{T})$ is a projective hypertoric variety if and only if $\mathcal{T}$ is regular.*

Proof: Let $\mathcal{T}'$ be a regular tiling of $|\mathcal{T}|$. Then by the ample cone of any conical symplectic resolution of $Y(|\mathcal{T}|)$ is a projective GIT quotient at some character of $\mathbb{G}_m^n$. But we already know these quotients; they are the hypertoric varieties associated to regular tilings. $\square$

5.3. Alternate Weights

We have chosen to focus on weight 2 actions of $\mathbb{G}_m$. However, we can produce the same results with any weight.

Definition 5.11 A **weight** m **zonotope** is a set of the form $Z_m(\mathbf{a}) := \sum_{i=0}^n [0, m] \cdot a_i$ or a translate $Z_m(\mathbf{a}, \nu) := \nu + Z(\mathbf{a})$.

Note that although $Z(\mathbf{a}) \neq Z_2(\mathbf{a})$, the two are translates of each other; hence this definition of "weight 2 zonotope" agrees with our earlier one.

We define $Y_m(\mathbf{a})$, $Y_m(Z)$, and $Y_m(\mathcal{T})$ to be the same as before, except that we begin by equipping $T^*\mathbb{A}^n$ with the action $s \times (z, w) = (s^m z, w)$. All of our proofs follow, except the proof of conicality; we must translate $Z_m(\mathbf{a})$ so that the origin is an interior point; this may not be possible in the case $m = 1$.

CHAPTER VI

RECOVERING THE TILING

We have two methods for recovering the tiling $\mathcal{T}$ from the variety $Y(\mathcal{T})$. Under conjecture 1.7, this would provide classification of all hypertoric varieties.

First, we can focus on the extended core. In this case, we decompose the core into its components, each of which is a T-toric variety. To each component, we thus may associate a fan, and we may determine by which cocharacter $\mathbb{G}_m$ acts. In this case, Conjecture 1.7 is then a statement that these fans are compatible, in the sense that they do come from a zonotopal tiling.

Conjecture 6.1 *Let Y be any T-hypertoric variety. Then:*

1. *For any matched cocharacter η, there is a unique orbit O_η on which it acts.*

2. *There is a zonotopal tiling $\mathcal{T}$ in N with vertex set equal to the matched cocharacters of Y, and with Σ_η equal to the fan of the toric variety V_η.*

3. *Y is isomorphic to $Y(\mathcal{T})$ as a Poisson $T \times \mathbb{G}_m$-variety.*

Second, given a T-hypertoric variety Y, we can find a cover by affine hypertoric varieties. For each of these affine patches, we can recover the associated zonotope by finding the cocharacters which act with positive weights. In this case, Conjecture 1.7 implies that each affine patch has an associated zonotope, and also that these zonotopes and their faces will then form a zonotopal tiling.

Conjecture 6.2 1. *Let Y be an affine T-hypertoric variety. Then there is a zonotope $Z(Y)$ such that a cocharacter η has positive weights on Y if and only if $\eta \in Z(Y)$.*

2. *Let Y be any hypertoric variety. Then Y is covered by affine hypertoric varieties $Y_1, \ldots, Y_\ell$, and $Z(Y_1), \ldots, Z(Y_\ell)$ are the maximial zonotopes of a zonotopal tiling $\mathcal{T}$, and $Y \cong Y(\mathcal{T})$ as a Poisson $T \times \mathbb{G}_m$-variety.*

Proposition 6.3 *Conjecture 1.7 implies Conjectures 6.1 and 6.2*

We note that both Proposition 6.1 and 6.2 are true in the case of varieties of the form $Y(\mathcal{T})$.

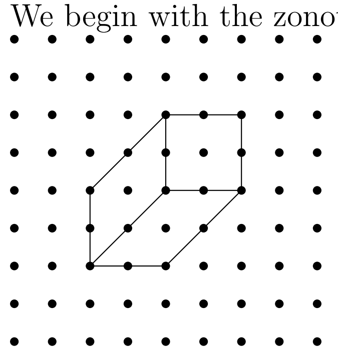
CHAPTER VII

A WORKED EXAMPLE

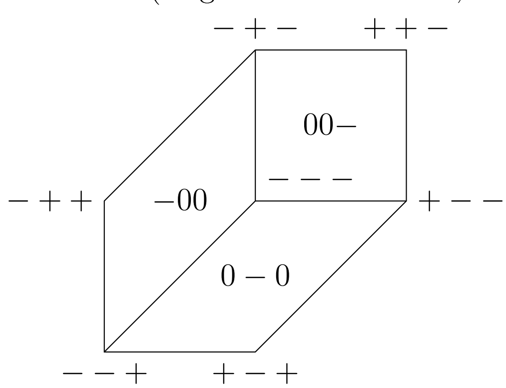
In this chapter we work through an extended example by constructing $T^*\mathbb{C}P^2$ from a zonotopal tiling, then recovering the tiling from the variety.

7.1. Constructing $T^*\mathbb{C}P^2$

We begin with the zonotopal tiling



which we center at the origin. First, we write the underlying zonotope as $|\mathcal{T}| = Z(\mathbf{a})$, where $a_1 = (0, 1)$, $a_2 = (1, 0)$, and $a_3 = (-1, -1)$, which determines a sign vector for each face of $\mathcal{T}$. (Edges are not labeled, but their labels can be inferred.)



We also choose the matrix $B = (111)$ so that $B^T A = 0$.

We are now ready to begin the geometric construction; we start with the affine space $T^*\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[z_1, z_2, z_3, w_1, w_2, w_3]$ and define an open set for each maximal zonotope $Z \in \mathcal{T}$. The open set $U(\mathcal{T}, Z_{+00}) = \operatorname{Spec}\mathbb{C}[z_1, \ldots, w_3]_{z_1}$, and we have a similar description for each of the other two. The union is $U(\mathcal{T}) = T^*\mathbb{C}^3 \setminus V(z_1, z_2, z_3)$. We then have that $Z(\mathbf{a})$ is the familiar description of $T^*\mathbb{C}P^2$: all 6-tuples of numbers such that $\mu_K(z, w) := z_1w_1 + z_2w_2 + z_3w_3 = 0$, and z_i are not all 0, modulo the torus K.

7.2. Recovering the Tiling

We now describe how to recover the tiling, using only intrinsic properties of $T^*\mathbb{C}P^2$.

The orbit O_{---} is where $z_i \neq 0 = w_i$ for all i. Its closure is $V_{---} = (\mathbb{C}^3 \setminus (0,0,0))//K$, which we recognize as $\mathbb{C}P^2$. Coordinates for O_{---} are z_1/z_3 and z_2/z_3, on which $\mathbb{G}_m$ acts with weight 0, so the matched cocharacter for O_{---} is $(0,0)$. Finally, we have the fan which is associated to V_{---} as a toric variety in Figure 7.1 (a).

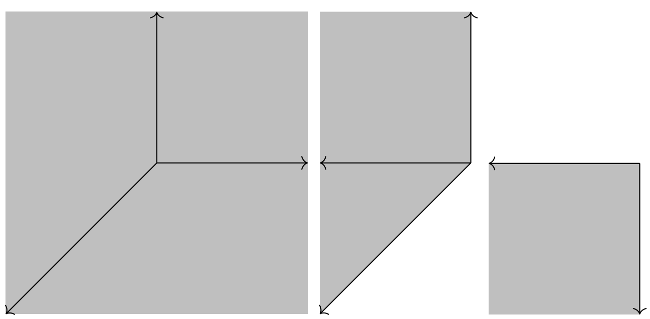
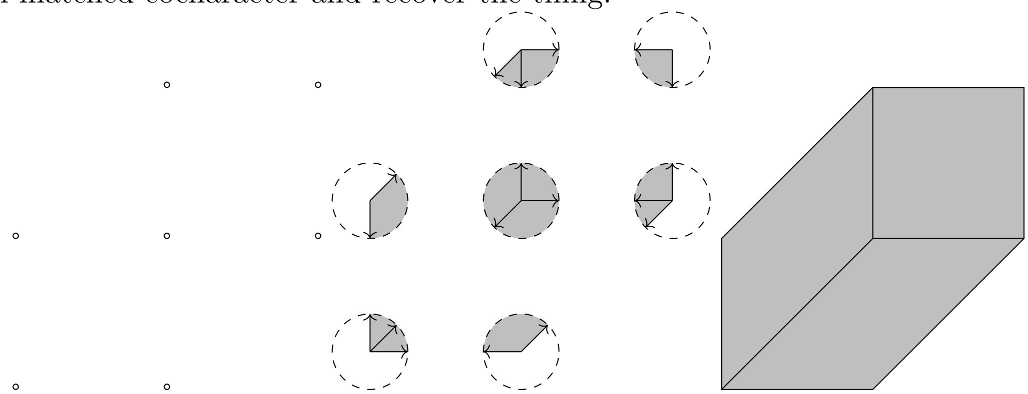


FIGURE 7.1. The fans for the toric varieties V_{---}, V_{+--}, and V_{++-}.

The oribit O_{+--} is where $z_1 = w_2 = w_3 = 0$ and $w_1, z_2, z_3 \neq 0$. Its closure is $V_{+--} = (\mathbb{C}^3 \setminus \mathbb{C} \times (0,0))//K$, which is the blowup of $\mathbb{A}^2$ at a point. Coordinates for O_{+--} are z_2w_1 and z_3w_1. On the one hand, we have $(t_1, t_2) \cdot z_2w_1 = t_1^{-1}t_2z_2w_1$ and $(t_1, t_2) \cdot z_3w_1 = t_1^{-1}z_3w_1$; on the other hand, we have $s \times z_2w_1 = s^2z_2w_1$ and $s \times z_3w_1 = s^2z_3w_1$. Thus $s^2 = t_1^{-1}t_2$ and $s^2 = t_1^{-1}$, so $t_1 = s^{-2}$ and $t_2 = 1$, and the matched cocharacter is $(-2, 0)$. Finally, the fan for V_{+--} is shown in Figure 7.1 (b).

The last type of orbit is given by O_{++-}, which has closure $V_{++-} = (\mathbb{C}^3 \setminus \mathbb{C} \times \mathbb{C} \times 0)//K = \mathbb{A}^2$. Coordinates on O_{++-} are $z_1 w_3$ and $z_2 w_3$. Using the same method as before, we find that the matched cocharacter is $(-2, -2)$, and the fan is Figure 7.1 (c).

After performing a similar analysis on the other sign vectors, we can place a fan at each matched cocharacter and recover the tiling:



We finish by considering some twisted $\mathbb{G}_m$ actions. To illustrate all the possibilities, we choose an interior point of Z, a point in the relative interior of an edge, a vertex, and a point not in Z.

$$\mathbb{G}_m \times_{(1,1)} (z, w) = (s^2 z_1, s^2 z_2, s z_3, w_1, w_2, s w_3)$$

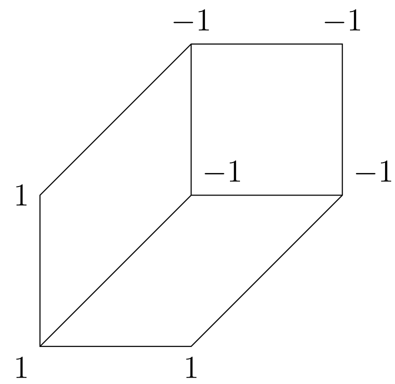
$$\mathbb{G}_m \times_{(1,2)} (z, w) = (s^2 z_1, s^3 z_2, s z_3, w_1, s^{-1} w_2, s w_3)$$

$$\mathbb{G}_m \times_{(1,3)} (z, w) = (s^2 z_1, s^4 z_2, s z_3, w_1, s^{-2} w_2, s w_3)$$

$$\mathbb{G}_m \times_{(2,2)} (z, w) = (s^3 z_1, s^3 z_2, s z_3, s^{-1} w_1, s^{-1} w_2, s w_3)$$

Since the invariant monomials are precisely the $z_i w_j$, it is easy to see that the minimum weight monomials have weights 1, 0, -1, and 0. We also note that $(2, 2)$, which lies on a codimension 2 face, has 2 weight zero monomials, while $(2, 1)$, which lies on a codimension 1 face, has 1 weight zero monomial.

Finally, we choose the support function which takes the following values on vertices:



The slopes along a_1, a_2, and a_3 are 0, 0, and 1, so this corresponds to the character of T^n which takes (t_1, t_2, t_3) to t_3. But this restricts to the usual character α of K to write $T^*\mathbb{C}P^2$ as a projective GIT quotient $T^*\mathbb{C}^3//_\alpha K$.

REFERENCES CITED

[BD00] Roger Bielawski and Andrew S. Dancer. The geometry and topology of toric hyperkähler manifolds. *Comm. Anal. Geom.*, 8(4):727–760, 2000.

[Bea00] Arnaud Beauville. Symplectic singularities. *Invent. Math.*, 139(3):541–549, 2000.

[BLVS$^+$99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter M. Ziegler. *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[BPW] Tom Braden, Nicholas Proudfoot, and Ben Webster. Quantizations of conical symplectic resolutions I: local and global structure. arXiv:1208.3863.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[HH05] Megumi Harada and Tara S. Holm. The equivariant cohomology of hypertoric varieties and their real loci. *Comm. Anal. Geom.*, 13(3):527–559, 2005.

[HS02] Tamás Hausel and Bernd Sturmfels. Toric hyperKähler varieties. *Doc. Math.*, 7:495–534 (electronic), 2002.

[Kon00] Hiroshi Konno. Cohomology rings of toric hyperkähler manifolds. *Internat. J. Math.*, 11(8):1001–1026, 2000.

[Pro08] Nicholas Proudfoot. A survey of hypertoric geometry and topology. In *Toric Topology*, volume 460 of *Contemp. Math.*, pages 323–338. Amer. Math. Soc., Providence, RI, 2008.

[PW07] Nicholas Proudfoot and Ben Webster. Intersection cohomology of hypertoric varieties. *J. Algebraic Geom.*, 16(1):39–63, 2007.

[RGZ94] Jürgen Richter-Gebert and Günter M. Ziegler. Zonotopal tilings and the Bohne-Dress theorem. In *Jerusalem combinatorics '93*, volume 178 of *Contemp. Math.*, pages 211–232. Amer. Math. Soc., Providence, RI, 1994.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.