GROUP ACTIONS ON HYPERPLANE ARRANGEMENTS

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DISSERTATION ABSTRACT

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In this dissertation, we will look at two families of algebras with connections to hyperplane arrangements that admit actions of finite groups. One of the fundamental questions to ask is how these decompose into irreducible representations. For the first family of algebras, we will use equivariant cohomology techniques to reduce the computation to an easier one. For the second family, we will use two decompositions over the intersection lattice of the hyperplane arrangement to aid us in computation.

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To my wife, Elizabeth.

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CHAPTER I

INTRODUCTION

In the representation theory of a finite group, one of the fundamental questions about any representation is how it decomposes into irreducible representations. One very rich class of examples comes from decomposing cohomology algebras of spaces that admit an action of a finite group. Many have studied examples of this kind. In [Pro1] and [Leh2], the authors study the action of a Coxeter group on a toric variety and its corresponding cohomology algebra. Many such as Lehrer [Leh1] have studied how Coxeter groups act on the Orlik-Solomon algebra of a Coxeter hyperplane arrangement.

One of the examples that motivated this dissertation and is the example of the cohomology of a configuration space of n ordered points in $\mathbb{R}^k$. The example in the case that k is even is studied in [Leh1] and [CF] under the guise of the Orlik-Solomon algebra and the cohomology of the pure braid group respectively. In the case that $k = 3$, or more generally k is odd, the cohomology of the configuration space is isomorphic as an S_n-representation to the regular representation. One proof method that will be employed in this dissertation is to equip the space with an S^1-action and use equivariant cohomology to show that the cohomology algebra is isomorphic to the associated graded algebra of the cohomology of the fixed point set of the S^1-action. If the space also admits a finite group action that commutes with the action of S^1, then these two representations are isomorphic.

Another motivating example is that of the coinvariant algebra. The coinvariant algebra of a Weyl group W is constructed by taking a polynomial ring and taking the quotient by the polynomials in positive degree invariant under the action of

W. In the case that $W = S_n$, this corresponds to modding out by the ideal of symmetric polynomials in positive degree. We may again ask the question of how this decomposes as an S_n-representation. This ring is isomorphic to the cohomology ring of the flag variety, and the action of S_n is induced by an action on the space. Using S^1-equivariant cohomology and the above S_n-action, we find that the coinvariant algebra is isomorphic to the associated graded of the cohomology of a finite set. This finite set has $n!$ points, a free and transitive S_n-action, and is isomorphic to the regular representation of S_n.

In this dissertation, we will look at a two classes of examples of this kind. The first class of example is very similar to the Orlik-Solomon algebra, and its data too comes from a hyperplane arrangement. The Orlik-Solomon algebra is constructed by complexifying the arrangement, and taking the cohomology of the complement. Similarly, we will construct a space $M_3(\mathcal{A})$, which in the case of a central hyperplane arrangement is obtained by tensoring the ambient vector space and hyperplanes by $\mathbb{R}^3$ and taking the complement of the resulting subspaces. If $\mathcal{A}$ has the action of a finite group, then so does $M_3(\mathcal{A})$ as does the cohomology of $M_3(\mathcal{A})$.

In Chapter II we will prove, using techniques in equivariant cohomology, that the cohomology algebra of $M_3(\mathcal{A})$ is the associated graded algebra of the cohomology of the complement $M_1(\mathcal{A})$ of the original arrangement $\mathcal{A}$ with respect to a filtration defined by Varchencko and Gelfand (Corollary 2.28). We will also give presentations of the S^1-equivariant cohomology and ordinary cohomology of the space $M_3(\mathcal{A})$ (Theorem 2.23, Corollary 2.25). A hyperplane arrangement with a choice of coorientation for each hyperplane determines an oriented matroid, which in turn determines the equivariant cohomology ring of the complement. We will extend some of our results to the setting of arbitrary (not necessarily representable) oriented matroids.

As an application of these results we will reduce the question about the decomposition of $H^*(M_3(\mathcal{A}))$ into irreducible representations to a question of how $H^*(M_1(\mathcal{A}))$ decomposes (Proposition 2.40). In the case that $\mathcal{A}$ is a Coxeter arrangement with corresponding reflection group W, this will yield the regular representation.

The second class of example we will look at is called the Orlik-Terao algebra. We will consider it in a special case corresponding to a sequence of hyperplane arrangements $\mathcal{B}_n$ with the action of corresponding symmetric groups S_n. The algebra is generated by the reciprocals of defining linear forms $\left\{\frac{1}{x_i - x_j} \mid 1 \leq i < j \leq n\right\}$ of the hyperplanes of $\mathcal{B}_n$.

One way we will address this in Chapter III is via a new concept called representation stability introduced by Church and Farb in [CF]. Representation stability in the symmetric group setting describes a property of a sequence $\{V_n\}$ of symmetric group representations where V_i is acted on by S_i. A sequence is representation stable if the maps between the representations are nice with respect to the actions of the symmetric groups, and if eventually the decomposition into irreducibles follows a constant pattern. While this won't answer the question of how the algebra decomposes, it lets us know if the decomposition will eventually become constant. In this chapter we will prove that the graded pieces of the Orlik-Terao algebra corresponding to $\mathcal{B}_n$ will form a representation stable sequence (Theorem 3.10).

Finally, in Chapter IV, we take a more geometric approach to this problem. As in the first class of examples, we interpret the Orlik-Terao algebra as a cohomology ring.

The space is called a hypertoric variety, and its combinatorial data come from a vector or hyperplane arrangement. In [BP] it is shown that the T-equivariant intersection cohomology isomorphic as a graded vector space to the Orlik-Terao algebra. Using the data of a stratification of the hypertoric variety and a spectral sequence defined in [BGS], we give a recursive way to calculate how the algebra decomposes (Proposition 4.5). While we won't fully answer the question of how this algebra decomposes into irreducible representations, this gives us valuable insight into possible strategies for proceeding in the future.

CHAPTER II

SUBSPACE ARRANGEMENTS

Definition 2.1. Let V be a finite dimensional real vector space, and let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in V given by $H_i = \omega_i^{-1}(0)$ for some non-constant affine linear form $\omega_i : V \to \mathbb{R}$. Let $\overline{\omega_i}$ be the associated linear map. Define affine linear maps $\omega_{i,k} : V^k \to \mathbb{R}^k$ by

$$\omega_{i,k}(v_1, \ldots, v_k) = (\omega_i(v_1), \overline{\omega_i}(v_2), \ldots, \overline{\omega_i}(v_k)).$$

The space $M_k(\mathcal{A})$ is defined to be the complement of the union of the affine subspaces

$$\omega_{i,k}^{-1}(0, 0, \ldots, 0).$$

When $k = 1$, this is just the complement of the arrangement. When $k = 2$, this is isomorphic to the complement of the complexified arrangement. When $\mathcal{A}$ is the braid arrangement $\mathcal{B}_n$, $M_k(\mathcal{A})$ is the configuration space of n ordered points in $\mathbb{R}^k$.

Consider the ring[1] $H^0(M_1(\mathcal{A}))$ of locally constant functions on $M_1(\mathcal{A})$. Varchenko and Gelfand defined a filtration of this ring via Heaviside functions. Let

$$H_i^+ = \{v \in V \mid \omega_i(v) > 0\},$$

and let

$$H_i^- = \{v \in V \mid \omega_i(v) < 0\}.$$

[1] All cohomology rings in this dissertation will be taken with coefficients in $\mathbb{Q}$.

Define the Heaviside function $x_i \in H^0(M_1(\mathcal{A}))$ by putting

$$x_i(v) = \begin{cases} 1 & v \in H_i^+ \\ 0 & v \in H_i^-. \end{cases}$$

Proposition 2.2. [VG, Thm 4.5] Consider the map $\psi : \mathbb{Q}[e_1, \ldots, e_n] \to H^0(M_1(\mathcal{A}))$ taking e_i to x_i, and let $\mathcal{I}_1$ be the kernel. This map is surjective, and $\mathcal{I}_1$ is generated by the following relations:

$$(1) \quad e_i^2 - e_i$$

$$(2) \quad \prod_{i \in S^+} e_i \prod_{j \in S^-} (e_j - 1) \text{ if } \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing.$$

$$(3) \quad \prod_{i \in S^+} e_i \prod_{j \in S^-} (e_j - 1) - \prod_{i \in S^+} (e_i - 1) \prod_{j \in S^-} e_j$$

$$\text{if } \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing \text{ and } \bigcap_{i \in (S^+ \cup S^-)} H_i \neq \varnothing.$$

Remark 2.3. Note that only families (1) and (2) are necessary to generate $\mathcal{I}_1$. However, in the case that $\mathcal{A}$ is central, only families (1) and (3) are necessary to generate $\mathcal{I}_1$ [VG, Thm 4.5]. We include family (3) as this is the presentation we will get when we set the equivariant parameter $u = 1$ in our presentation of the equivariant cohomology ring.

Let $P^k \subseteq H^0(M_1(\mathcal{A}))$ be the space of functions representable by polynomials in $\{x_i\}_{i=1}^n$ of degree less than or equal to k. Varchenko and Gelfand show[2] that the

[2] They show this only for central arrangements, but the proof extends easily to the affine case.

associated graded algebra of $H^0(M_1(\mathcal{A}))$ is isomorphic as a graded vector space to $H^*(M_2(\mathcal{A}))$ [VG, Cor 2.2].

However, as $H^*(M_2(\mathcal{A}))$ is non-commutative, it cannot be isomorphic as a ring. Proudfoot shows that one obtains an isomorphism of rings if one works with coefficients in $\mathbb{F}_2$ rather than $\mathbb{Q}$, and he gives an equivariant cohomology interpretation of this fact in the discussion after Theorem 3.1 [Pro2]. This is not so satisfying, however, because the signs in the presentation of $H^*(M_2(\mathcal{A}))$ are subtle, and all of this structure is lost by working over $\mathbb{F}_2$. In this chapter, we'll show that the right space to look at is $M_3(\mathcal{A})$ rather than $M_2(\mathcal{A})$. More precisely, we prove the following theorem.

Theorem 2.4. Let $\mathcal{A}$ be a real hyperplane arrangement.

(a) The associated graded algebra of $H^0(M_1(\mathcal{A}))$ with respect to the VG filtration is isomorphic as a graded ring to $H^*(M_3(\mathcal{A}))$, with degrees halved. (That is, $P^k/P^{k-1} \cong H^{2k}(M_3(\mathcal{A}))$.)

(b) If W is a finite group acting on $\mathcal{A}$, then $H^0(M_1(\mathcal{A}))$ and $H^*(M_3(\mathcal{A}))$ are isomorphic as W-representations.

Remark 2.5. A special case of Theorem 2.4(b) is the action of S_n on the braid arrangement $\mathcal{B}_n$. In this case we recover the fact that the cohomology of the configuration space of n ordered points in $\mathbb{R}^3$ is isomorphic to the regular representation of S_n [Ati].

Remark 2.6. One might wonder why we are only looking at $k = 1, 2$, or 3. In [dLS, 5.6] it was shown (at least for central arrangements) that $H^*(M_k(\mathcal{A})) \cong H^*(M_{k-2}(\mathcal{A}))$ for all $k > 3$.

Last, we extend Theorem 2.4(a) to the setting of oriented matroids. Here, $H^*(M_3(\mathcal{A}))$ is replaced by an algebra defined by Cordovil in [Cor] and $H^0(M_1(\mathcal{A}))$ is replaced by an analogous algebra for oriented matroids as defined in [GR].

2.1. Equivariant Cohomology

Let G be a connected Lie group acting on a space X. Choose a contractible space EG on which G acts freely. Define

$$X_G := EG \times_G X$$

to be the quotient of $EG \times X$ by the relation

$$(g \cdot e, x) \sim (e, g \cdot x).$$

When X is a point, the resulting space $X_G = EG/G$ is called the classifying space, and is usually denoted BG. For arbitrary X, X_G is a fiber bundle over BG with fiber X. This fiber bundle is trivial if and only if the action of G on X is trivial.

Definition 2.7. The G-equivariant cohomology of X is defined to be the ordinary cohomology of X_G, i.e.

$$H^*_G(X) := H^*(X_G).$$

Remark 2.8. Note that in the definition of equivariant cohomology, we have made a choice of a space EG. However, if we choose a different contractible space with free G-action, we get an isomorphic cohomology ring.

Suppose that X is an m-dimensional oriented G-manifold with $Y \subseteq X$ a closed, oriented, codimension k G-submanifold. We denote by $[Y]_G \in H^k_G(X)$ the class of the codimension k submanifold $Y_G \subseteq X_G$.

Example 2.9. The group $T = S^1$ acts freely on the contractible space $ET = S^\infty$ with quotient $BT = \mathbb{C}P^\infty$. Hence, $H^*_T(pt)$ is an polynomial ring generated in degree 2.

Consider the inclusion of a point into $\mathbb{C}$ at the origin. If we let T act on $\mathbb{C}$ by multiplication, then the inclusion is equivariant, and the restriction map

$$H^*_T(\mathbb{C}) \xrightarrow{\sim} H^*_T(pt)$$

is an isomorphism. The origin in $\mathbb{C}$ is a closed, oriented, codimension 2 T-submanifold, and we denote the image of $[0]_T$ in $H^*_T(pt)$ by u. This class is nonzero, and thus $H^*_T(pt) \cong \mathbb{Q}[u]$.

The ordinary cohomology of a connected space X has the structure of a $\mathbb{Q}$-algebra coming from the induced map of the map $X \to pt$. For equivariant cohomology, we get the structure of an $H^*_G(pt)$-algebra from the map $X_G \to BG$. We can compute $H^*_G(X)$ as a module over $H^*_G(pt)$ using the Serre spectral sequence associated to this map.

We say that X is **equivariantly formal** if this spectral sequence collapses at the E_2 page $H^*(BG) \otimes H^*(X)$. Equivalently, X is equivariantly formal if $H^*_G(X)$ is isomorphic to $H^*_G(pt) \otimes H^*(X)$ as an $H^*_G(pt)$-module and the map to $H^*(X)$ induced by the inclusion $X \hookrightarrow X_G$ is obtained by setting all positive degree elements of $H^*_G(pt)$ to zero. If X is equivariantly formal, then any lift of any $\mathbb{Q}$-basis of $H^*(X)$ to $H^*_G(X)$ is an $H^*_G(pt)$-basis.

Example 2.10. Let $T = S^1$ act on $\mathbb{R}^3 \setminus \{0\}$ by rotation about the x-axis. Since $\mathbb{R}^3 \setminus \{0\}$ is homotopy equivalent to S^2, its cohomology ring is concentrated in even degree. Since the same is true for BT, the Serre spectral sequence degenerates at the E_2 page and $\mathbb{R}^3 \setminus \{0\}$ is equivariantly formal.

Denote by Z^+ and Z^- the positive and negative x-axes, respectively. Choose an orientation of $\mathbb{R}^3 \setminus \{0\}$, and orient Z^+ and Z^- outward. The classes $[Z^+]_T$ and $[Z^-]_T$ represent classes in $H^2_T(\mathbb{R}^3 \setminus \{0\})$ as they are codimension 2 T-submanifolds of the 3-manifold $\mathbb{R}^3 \setminus \{0\}$.

Consider the projection π of $\mathbb{R}^3 \setminus \{0\}$ onto the second and third coordinates. This maps is equivariant, and induces the map

$$H^*_T(pt) \cong H^*_T(\mathbb{R}^2) \xrightarrow{\pi^*} H^*_T(\mathbb{R}^3 \setminus \{0\}).$$

Note that $\pi^{-1}(0) = -Z^- \sqcup Z^+$, where $-Z^-$ denotes Z^- with the opposite orientation. Thus the image of u is $[Z^+]_T - [Z^-]_T$. Since these two subvarieties don't intersect, we get a map

$$\mathbb{Q}[x, y]/\langle xy \rangle \to H^*_T(\mathbb{R}^3 \setminus \{0\})$$

sending x to $[Z^-]_T$ and y to $[Z^+]_T$. We claim that this is in fact an isomorphism.

Note that the forgetful map $H^*_T(\mathbb{R}^3 \setminus \{0\}) \to H^*(\mathbb{R}^3 \setminus \{0\}) \cong H^*(S^2)$ sends the classes of $[Z^-]_T$ and $[Z^+]_T$ to the same class, and this class generates $H^*(S^2)$. Hence, the ring generated by $[Z^-]_T$ and $[Z^+]_T$ contains the image of $H^*_T(pt)$ and surjects onto $H^*(\mathbb{R}^3 \setminus \{0\})$. Since $\mathbb{R}^3 \setminus \{0\}$ is equivariantly formal, this implies that the map $\mathbb{Q}[x, y]/\langle xy \rangle \to H^*_T(\mathbb{R}^3 \setminus \{0\})$ is surjective.

This map is also injective by a graded dimension count. Since $\mathbb{R}^3 \setminus \{0\}$ is equivariantly formal, we have $H_T^*(\mathbb{R}^3 \setminus \{0\}) \cong H^*(\mathbb{R}^3 \setminus \{0\}) \otimes \mathbb{Q}[u]$ as a graded $\mathbb{Q}[u]$-module, and we're done.

Proposition 2.11. Let X be an equivariantly formal T-space, and let $F = X^T$. Then

$$H_T^*(X)/\langle u \rangle \cong H^*(X) \text{ and}$$
$$H_T^*(X)/\langle u - 1 \rangle \cong H^*(F).$$

Proof. By [GKM, (6.2)(2)], the restriction map on equivariant cohomology induces an isomorphism

$$H_T^*(X)[u^{-1}] \to H_T^*(F)[u^{-1}] \cong H^*(F)[u, u^{-1}]$$

and hence

$$H^*(F) \cong H_T^*(X)/\langle u - 1 \rangle$$

This first isomorphism follows from formality. □

Corollary 2.12. The ring $H^*(F)$ has a natural filtration, its Rees algebra is isomorphic to $H_T^*(X)$, and its associated graded algebra is isomorphic to $H^*(X)$.

Example 2.13. In Example 2.10, we computed the equivariant cohomology of $\mathbb{R}^3 \setminus \{0\}$. The presentation that $H^*(Z^- \sqcup Z^+)$ inherits from $H_T^*(\mathbb{R}^3 \setminus \{0\})$ is the same presentation provided in [VG]

$$H^*(Z^- \sqcup Z^+) = \mathbb{Q}[y]/\left\langle y^2 - y \right\rangle.$$

If we filter this ring by degree, then its corresponding associated graded algebra is

$$\mathrm{gr}_{\deg} H^*(Z^- \sqcup Z^+) \cong \mathbb{Q}[y]/\left\langle y^2 \right\rangle$$

which is isomorphic to $H^*(\mathbb{R}^3 \setminus \{0\})$.

Suppose that W is a group that acts on X. If the action of W commutes with the action of T, then W also acts on $F := X^T$.

Proposition 2.14. If the action of W commutes with the action of T, then W acts on $H_T^*(X)$. If X is equivariantly formal, then the isomorphisms of Proposition 2.11 are W-equivariant.

Proof. If we let W act on ET trivially, then it is clear that W acts on $ET \times X$. To show that the action respects the relation $(y, tx) \sim (ty, x)$ we use the facts that the actions commute and hence

$$w \cdot (y, tx) = (y, wtx) = (y, twx) \sim (ty, wx) = w \cdot (ty, x).$$

Since W acts on X_T and the projection to BT is W-invariant, we obtain an action of W on $H_T^*(X)$ that fixes u. If X is formal, then $H_T^*(X)$ is a flat family of W-representations and the result follows. □

Remark 2.15. If W is a finite group, the category of W-representations is semisimple and hence W-representations are completely reducible. Hence, $H^*(X)$ and $H^*(F)$ are isomorphic as W-representations as $\mathrm{gr}\, H^*(F) \cong H^*(X)$ and complementary subrepresentations are isomorphic to quotients.

2.2. Equivariant Formality

In this section we will establish that $M_3(\mathcal{A})$ is equivariantly formal. To do this we will appeal to the reasoning in Example 2.10 and show that $H^*(M_3(\mathcal{A}))$ is concentrated in even degree.

Suppose $H_1 \in \mathcal{A}$, let $\mathcal{A}' = \mathcal{A} \setminus H_1$, and let

$$\mathcal{A}'' = \mathcal{A}^{H_1} = \{H \cap H_1 \mid H \in \mathcal{A}', H \cap H_1 \neq \varnothing\}.$$

Note that $M_3(\mathcal{A}') = M_3(\mathcal{A}) \cup M_3(\mathcal{A}'')$. The submanifold $M_3(\mathcal{A}'')$ has codimension 3 in $M_3(\mathcal{A}')$.

Proposition 2.16. $H^*(M_3(\mathcal{A}))$ is concentrated in even degrees.

The proof of this proposition is strongly influenced by the proof of [OT, 5.80,5.81] and the discussion on p. 213.

Proof. When $\mathcal{A}$ consists of one hyperplane, we may reduce this to the case of a point in a line. In this case, $M_3(\mathcal{A})$ is homotopic to S^2, and thus its cohomology is concentrated in even degree.

Suppose this result is true for $|\mathcal{A}| \leq n$. Suppose that $\mathcal{A}$ has $n+1$ hyperplanes. Inside $M_3(\mathcal{A}')$, $M_3(\mathcal{A}'')$ has a tubular neighborhood E which is the disk bundle of the trivial bundle over $M_3(\mathcal{A}'')$. Let $E_0 = E \cap M_3(\mathcal{A})$ be the punctured disk bundle. The Thom isomorphism theorem gives us that

$$H^{k+1}(E, E_0) \cong H^{k-2}(E)$$

as the Thom space for the trivial bundle is the suspension of $E \sqcup \{pt\}$ iterated 3 times.

By excision $H^{k+1}(M_3(\mathcal{A}'), M_3(\mathcal{A})) \cong H^{k+1}(E, E_0) \cong H^{k-2}(E)$. Since E is homotopy equivalent to $M_3(\mathcal{A}'')$, we get that $H^{k-2}(E) \cong H^{k-2}(M_3(\mathcal{A}''))$. The long exact sequence in cohomology gives

$$\cdots \to H^k(M_3(\mathcal{A}')) \to H^k(M_3(\mathcal{A})) \to H^{k+1}(M_3(\mathcal{A}'), M_3(\mathcal{A})) \to \cdots .$$

Replacing $H^{k+1}(M_3(\mathcal{A}'), M_3(\mathcal{A}))$ with $H^{k-2}(M_3(\mathcal{A}''))$ and using the inductive hypothesis we get that the terms to the left and right of $H^k(M_3(\mathcal{A}))$ in the long exact sequence are nontrivial only when k is even. □

Corollary 2.17. The sequence

$$0 \to H^k(M_3(\mathcal{A}')) \to H^k(M_3(\mathcal{A})) \to H^{k-2}(M_3(\mathcal{A}'')) \to 0$$

is exact.

Proof. Starting with the long exact sequence in the previous proof, we use that the odd degree cohomology vanishes and the result follows. □

From this short exact sequence we can deduce the following recursion which is also exhibited by the Orlik-Solomon algebra.

Corollary 2.18. The Poincaré polynomial of $H^*(M_3(\mathcal{A}))$ follows the recursion

$$\mathrm{Poin}(M_3(\mathcal{A})) = \mathrm{Poin}(M_3(\mathcal{A}')) + t^2\mathrm{Poin}(M_3(\mathcal{A}'')).$$

Note that the t^2 comes from the fact that the map from $H^k(M_3(\mathcal{A}))$ to $H^{k-2}(M_3(\mathcal{A}''))$ is a degree -2 map.

2.3. Main Results

We first define an action of $T = S^1$ on $M_3(\mathcal{A})$. In Example 2.10, we let T act by rotation about the x-axis, or equivalently, we thought of $\mathbb{R}^3$ as $\mathbb{R} \oplus \mathbb{C}$ and let T act by multiplication on the complex coordinate, leaving the real coordinate fixed. We may extend this action to $V^3 \cong V \otimes (\mathbb{R} \oplus \mathbb{C})$. Note that $\omega_{i,3}$ is T-equivariant and $0 \in \mathbb{R}^3$ is T-fixed, so T acts on

$$M_3(\mathcal{A}) = \bigcap_{i=1}^{n} \omega_{i,3}^{-1}(\mathbb{R}^3 \setminus \{0\}).$$

The fixed point set of this action is $M_1(\mathcal{A})$.

As $M_3(\mathcal{A})$ is equivariantly formal, Corollary 2.12 therefore yields the following result.

Proposition 2.19. With respect to the filtration coming from the equivariant cohomology of $M_3(\mathcal{A})$ via Corollary 2.12,

$$\operatorname{gr} H^0(M_1(\mathcal{A})) \cong H^*(M_3(\mathcal{A})).$$

The remainder of this section will be devoted to computing presentations of $H^*(M_3(\mathcal{A}))$ and $H^*_T(M_3(\mathcal{A}))$, which we will use to show that the filtration of $H^0(M_1(\mathcal{A}))$ coming from equivariant cohomology is the same as the one coming from Proposition 2.2. Our calculation of $H^*_T(M_3(\mathcal{A}))$ will make use of a generating set of the ordinary cohomology of $M_3(\mathcal{A})$.

Lemma 2.20. Consider the map $\psi : \mathbb{Q}[e_1, \ldots, e_n] \to H^*(M_3(\mathcal{A}))$ taking e_i to $\omega_{i,3}^*([Z^+]) \in H^2(M_3(\mathcal{A}))$. This map is surjective.

Remark 2.21. In the case that $\mathcal{A}$ is central, this is established in [dLS] as they provide a presentation of $H^*(M_3(\mathcal{A}))$.

Before we prove the result in the case that $\mathcal{A}$ is affine, we describe a construction called coning. Given an affine arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in a vector space V, the cone of $\mathcal{A}$ denoted $c\mathcal{A}$ consists of hyperplanes $\{cH_1, \ldots, cH_n\}$ corresponding to the hyperplanes of $\mathcal{A}$ along with a new hyperplane cH_0 in $V \oplus \mathbb{R}$ whose defining linear form is $c\omega_0(v, r) = -r$. While any scalar multiple of this linear form would define the same hyperplane, the reasoning behind the choice of $-r$ will be apparent when we start the computation. If $H_i = \omega_i^{-1}(0) = \overline{\omega_i}^{-1}(a_i)$, then $cH_i := \{(v, r) \mid \overline{\omega_i}(v) = ra_i\}$. Let $c\omega_i : V \times \mathbb{R} \to \mathbb{R}$ be defined by

$$c\omega_i(v, r) = \overline{\omega_i}(v) - ra_i.$$

Thus $cH_i = c\omega_i^{-1}(0)$. The resulting arrangement $c\mathcal{A}$ is central.

Note that there is a natural inclusion of $M_1(\mathcal{A})$ into $M_1(c\mathcal{A})$ by $v \mapsto (v, 1)$. Similarly, we get an inclusion of $M_3(\mathcal{A})$ into $M_3(c\mathcal{A})$ given by

$$i : (v_1, v_2, v_3) \mapsto (v_1, v_2, v_3, (1, 0, 0)).$$

Note that $c\omega_{0,3}^{-1}(Z^+)$ doesn't intersect the embedding of $M_3(\mathcal{A})$ into $M_3(c\mathcal{A})$ as $c\omega_{0,3}^{-1}(Z^+)$ contains everything of the form $(v_1, v_2, v_3, -r, 0, 0)$ in $M_3(c\mathcal{A})$ such that $r > 0$.

Proof. To prove Lemma 2.20, we will show the following. The map

$$i^* : H^*(M_3(c\mathcal{A})) \to H^*(M_3(\mathcal{A}))$$

induced by the inclusion $i : M_3(\mathcal{A}) \to M_3(c\mathcal{A})$ is surjective and the class $c\omega_{0,3}^*([Z^+])$ is in the kernel.

First, note that $c(\mathcal{A}') = (c\mathcal{A})'$ where the right hand side is constructed by removing the coned version cH_1 of the distinguished hyperplane H_1 in $\mathcal{A}$. Also, if $\mathcal{A}'' = \mathcal{A}^{H_1}$, then $c(\mathcal{A}'') = (c\mathcal{A})'' := (c\mathcal{A})^{cH_1}$ (up to differences in multiplicities). Thus we get a short exact sequence

$$0 \to H^*(M_3(c(\mathcal{A}'))) \to H^*(M_3(c\mathcal{A})) \to H^*(M_3(c(\mathcal{A}''))) \to 0.$$

Suppose that $\mathcal{A}$ consists of one hyperplane. Then $c\mathcal{A}$ consists of two intersecting hyperplanes. In this scenario,

$$H^*(M_3(\mathcal{A})) \cong \mathbb{Q}[\omega_{1,3}^*([Z^+])]/\langle \omega_{1,3}^*([Z^+])^2 \rangle$$

and

$$H^*(M_3(c\mathcal{A})) \cong \mathbb{Q}[c\omega_{1,3}^*([Z^+]), c\omega_{0,3}^*([Z^+])]/\langle c\omega_{1,3}^*([Z^+])^2, c\omega_{0,3}^*([Z^+])^2 \rangle.$$

Since $c\omega_{0,3}^{-1}(Z^+)$ doesn't intersect the embedding of $M_3(\mathcal{A})$ into $M_3(c\mathcal{A})$ there is a surjective map i^* given by

$$i^*(c\omega_{1,3}^*([Z^+])) = \omega_{1,3}^*([Z^+]),\ i^*(c\omega_{0,3}^*([Z^+])) = 0$$

as

$$i^{-1}(c\omega_{1,3}^{-1}(Z^+)) = \omega_{1,3}^{-1}(Z^+).$$

Suppose that this is true for arrangements with $k \leq n$ hyperplanes. There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(M_3((c\mathcal{A})')) & \longrightarrow & H^*(M_3(c\mathcal{A})) & \longrightarrow & H^{*-2}(M_3((c\mathcal{A})'')) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow & & \\ 0 & \longrightarrow & H^*(M_3(\mathcal{A}')) & \longrightarrow & H^*(M_3(\mathcal{A})) & \longrightarrow & H^{*-2}(M_3(\mathcal{A}'')) & \longrightarrow & 0 \end{array} .$$

The left square commutes because it is induced from inclusions of spaces. The right square commutes by naturality of the Thom isomorphism.

By our inductive hypothesis, the downward maps on the left and right are surjective. By the short five lemma, the map $i^* : H^*(M_3(c\mathcal{A})) \to H^*(M_3(\mathcal{A}))$ is also surjective. The statement about the kernel follows from the inductive hypothesis and the commutativity of the diagram.

This tells us that, not only is there a map $\psi : \mathbb{Q}[e_1, \ldots, e_n] \to H^*(M_3(\mathcal{A}))$ taking e_i to $\omega_{i,3}^*([Z^+]) \in H^2(M_3(\mathcal{A}))$, but also that this map is surjective. □

In the following result, we give a proof that an algebra which we will call $B(\mathcal{A})$ satisfies the same recursion on Poincaré polynomials that $H^*(M_3(\mathcal{A}))$ satisfies. This recursion will be used in the proof of Theorem 2.23 to establish that $B(\mathcal{A}) \cong H^*(M_3(\mathcal{A}))$.

Lemma 2.22. Let $B(\mathcal{A}) := \mathbb{Q}[e_{H_1}, \ldots, e_{H_n}]/\mathcal{I}_0$ with $\deg e_{H_i} = 2$ where $\mathcal{I}_0$ is generated by the following families of relations:

1) $e_{H_i}^2$ for $i \in \{1, \ldots, n\}$

2) $\prod_{i \in S} e_{H_i}$ if $\bigcap_{i \in S} H_i = \varnothing$

3) $$\sum_{k \in S^-} \left(\prod_{i \in S^+} e_{H_i} \times \prod_{k \neq j \in S^-} e_{H_j} \right) - \sum_{k \in S^+} \left(\prod_{k \neq i \in S^+} e_{H_i} \times \prod_{j \in S^-} e_{H_j} \right)$$

if $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$ and $\bigcap_{i \in (S^+ \cup S^-)} H_i \neq \varnothing$.

The Poincaré polynomial of the algebra $B(\mathcal{A})$ satisfies the relation

$$\mathrm{Poin}(B(\mathcal{A})) = \mathrm{Poin}(B(\mathcal{A}')) + t^2 \mathrm{Poin}(B(\mathcal{A}'')).$$

Proof. This is shown in [Cor, 2.7] in the case that $\mathcal{A}$ is a central arrangement. Additionally, it is shown that $B(\mathcal{A})$ has a "no broken circuit" basis.

First we will construct a complex

$$0 \to B(\mathcal{A}) \xrightarrow{t} B(c\mathcal{A}) \xrightarrow{\varphi} B(\mathcal{A}) \to 0.$$

Eventually, we will show that in fact this is a short exact sequence. First we need to define the maps. Let S be a collection[3] of hyperplanes in $\mathcal{A}$, let $e_S = \prod_{H \in \Phi} e_H$, let

[3] If S is an empty collection, then let $e_S = 1$.

$cS = \{cH\}_{H \in S}$, and let

$$\Phi : \mathbb{Q}[e_{cH_0}, e_{cH_1}, \ldots, e_{cH_n}]/\langle e_{cH_i}^2 \rangle \to \mathbb{Q}[e_{H_1}, \ldots, e_{H_n}]/\langle e_{H_i}^2 \rangle$$

be given by

$$\Phi(e_{cS}) = e_S,\ \Phi(e_{cH_0} e_{cS}) = 0.$$

To show that this descends to a map $\varphi : B(c\mathcal{A}) \to B(\mathcal{A})$, we need to show that generators of families 2 and 3 in $\mathcal{I}_0(c\mathcal{A})$ map to $\mathcal{I}_0(\mathcal{A})$. Note that since $c\mathcal{A}$ is central, families 1 and 3 generate $\mathcal{I}_0(c\mathcal{A})$. Family 3 is addressed in [OT, 3.47, 3.49]. The map t is also similarly addressed by [OT, 3.47, 3.50] and sends e_S to $e_{cH_0} e_{cS}$.

Next, we will show that as vector spaces $B(\mathcal{A}) \cong BC(\mathcal{A})$, the broken circuit module as defined in [OT]. To define $BC(\mathcal{A})$, we first need to impose an ordering on $\mathcal{A}$. We call a collection of hyperplanes dependent if $\bigcap_{i \in S} H_i \neq \varnothing$ and if there exists a decomposition of S into $S^+ \sqcup S^-$ so that

$$\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing.$$

This condition is equivalent to the defining linear forms being linearly dependent. We call a collection of hyperplanes a circuit if it is minimally dependent. A collection S is called a broken circuit if adding on a hyperplane H greater than all the hyperplanes in S, with respect to the ordering imposed above, results in a circuit. The algebra $BC(\mathcal{A})$ is defined to the the free $\mathbb{Q}$-module generated by 1 and the set $\{e_S\}$ where the collections S intersect nontrivially and don't contain any broken circuits. This is known as a "no broken circuit" or NBC basis.

We already know that the natural map from $BC(c\mathcal{A})$ to $B(c\mathcal{A})$ induces an isomorphism of graded vector spaces in the case when $\mathcal{A}$ is central [Cor, 2.8]. To prove this in the affine case, we construct a commutative diagram [OT, 3.55]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & BC(\mathcal{A}) & \longrightarrow & BC(c\mathcal{A}) & \longrightarrow & BC(\mathcal{A}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B(\mathcal{A}) & \longrightarrow & B(c\mathcal{A}) & \longrightarrow & B(\mathcal{A}) & \longrightarrow & 0 \end{array}$$

with the top row known to be exact. Following the same reasoning as [OT, 3.55], since the middle vertical map $BC(c\mathcal{A}) \to B(c\mathcal{A})$ is known to be an isomorphism of vector spaces, then by a diagram chase, $BC(\mathcal{A}) \cong B(\mathcal{A})$.

Finally, we conclude that since

$$\mathrm{Poin}(B(c\mathcal{A})) = (1+t^2)\mathrm{Poin}(B(\mathcal{A}))$$

and

$$\mathrm{Poin}(B(\mathcal{H})) = \mathrm{Poin}(B(\mathcal{H}')) + t^2\mathrm{Poin}(B(\mathcal{H}''))$$

where $\mathcal{H}$ is a central arrangement, then

$$\mathrm{Poin}(B(\mathcal{A})) = \mathrm{Poin}(B(\mathcal{A}')) + t^2\mathrm{Poin}(B(\mathcal{A}'')).$$

□

In the next result we will compute a presentation of the T-equivariant cohomology of $M_3(\mathcal{A})$ and along the way, we will prove that $B(\mathcal{A}) \cong H^*(M_3(\mathcal{A}))$.

Theorem 2.23. Consider the map $\psi : \mathbb{Q}[e_1, \dots, e_n, u] \to H^*_T(M_3(\mathcal{A}))$ taking e_i to $\omega^*_{i,3}([Z^+]_T) \in H^2_T(M_3(\mathcal{A}))$, and let $\mathcal{I}$ be the kernel. This map is surjective, and $\mathcal{I}$ is generated by the following families of relations:

1) $e_i(e_i - u)$ for $i \in \{1, \dots, n\}$

2) $\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (e_j - u)$ if $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$

3) $u^{-1} \left(\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (e_j - u) - \prod_{i \in S^+} (e_i - u) \times \prod_{j \in S^-} e_j \right)$

if $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$ and $\bigcap_{i \in (S^+ \cup S^-)} H_i \neq \varnothing$.

Remark 2.24. Note that the expression inside the parentheses in family (3) is a multiple of u, so the whole thing is a polynomial.

Proof. By Proposition 2.16, $H^*(M_3(\mathcal{A}))$ is concentrated in even degrees, so $M_3(\mathcal{A})$ is formal. By formality, any lift of any generating set for $H^*(M_3(\mathcal{A}))$ is a generating set for $H_T^*(M_3(\mathcal{A}))$ over $H_T^*(pt)$. Hence by Lemma 2.20, $H_T^*(M_3(\mathcal{A}))$ is generated by $e_1, \dots, e_n$ and u.

The class $\omega_{i,3}^*([Z^+]_T)$ is represented by the oriented T-submanifold

$$Y_i^+ = \omega_{i,3}^{-1}(Z^+).$$

Let $u \in H_T^2(M_3(\mathcal{A}))$ be the image of the generator of $H_T^2(pt)$. By functoriality, we have $u = \omega_{i,3}^*(u)$ for all i.

Recall from Example 2.10 that $[Z^-]_T = y - u \in H_T^*(\mathbb{R}^3 \setminus 0)$. Hence,

$$e_i - u = \omega_{i,3}^*(y - u) \in H_T^*(M_3(\mathcal{A}))$$

is represented by the oriented T-submanifold

$$Y_i^- = \omega_{i,3}^{-1}(Z^-).$$

The next step is to check that the three families belong to $\mathcal{I}$. For all i, $\psi(e_i(e_i - u)) = 0$ because $\psi(e_i(e_i - u)) = \omega_{i,3}^*(y(y - u)) = \omega_{i,3}^*(0) = 0$.

For the second family of relations, we need to show that if

$$\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$$

then

$$\bigcap_{i \in S^+} Y_i^+ \cap \bigcap_{j \in S^-} Y_j^- = \varnothing.$$

Let $\pi_1 : V^3 \longrightarrow V$ be the projection onto the first coordinates. That is, it should restrict the 3-arrangement complement to the real arrangement complement. If $p \in \bigcap_{i \in S^+} Y_i^+ \cap \bigcap_{j \in S^-} Y_j^-$, then $\omega_{i,3}(p) \in Z^+$ for all $i \in S^+$ and $\omega_{j,3}(p) \in Z^-$ for all $j \in S^-$. Then

$$\pi_1(p) \in \bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^-$$

and hence the intersection is not empty.

For the third family, since $H_T^*(M_3(\mathcal{A}))$ is free over $H_T^*(pt) \cong \mathbb{Q}[u]$, we only need to show that

$$\left(\prod_{i \in S^+} e_i \times \prod_{j \in S^-} (e_j - u) - \prod_{i \in S^+} (e_i - u) \times \prod_{j \in S^-} e_j \right) = 0 \qquad (*)$$

if $\bigcap_{i \in S^+} H_i^+ \cap \bigcap_{j \in S^-} H_j^- = \varnothing$ and $\bigcap_{i \in S} H_i \neq \varnothing$. Choose a point $p \in \bigcap_{i \in S} H_i$. The involution of V given by reflection through p takes H_i^+ to H_i^- and vice versa for all $i \in S$, hence we also have

$$\bigcap_{i \in S^+} H_i^- \cap \bigcap_{j \in S^-} H_j^+ = \varnothing.$$

Taking the difference of the corresponding relations from family 2) yields the relation $(*)$. We have now shown that there is a surjective map

$$\mathbb{C}[e_1, \ldots, e_n, u]/\mathcal{I} \to H_T^*(M_3(\mathcal{A})).$$

Setting u to 0 in $\mathcal{I}$ gives us the ideal $\mathcal{I}_0$ from Lemma 2.22. Thus $\mathcal{I}_0$ is contained in the kernel of the map $\mathbb{Q}[e_1, \ldots, e_n] \to H^*(M_3(\mathcal{A}))$. Therefore, there is a map, which is surjective by Lemma 2.20, from the algebra $B(\mathcal{A})$ to $H^*(M_3(\mathcal{A}))$. By Corollary 2.18 and Lemma 2.22, these algebras have the same Poincaré polynomials and must be isomorphic. Hence we have the following commutative diagram with exact rows and surjective columns.

$$\begin{array}{ccccccccc}
 & & \mathcal{I} & \longrightarrow & \mathbb{Q}[e_1, \ldots, e_n, u] & \xrightarrow{\psi} & H_T^*(M_3(\mathcal{A})) & \longrightarrow & 0 \\
 & & \phi \downarrow u=0 & & \phi \downarrow u=0 & & u=0 \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_0 & \longrightarrow & \mathbb{Q}[e_1, \ldots, e_n] & \longrightarrow & H^*(M_3(\mathcal{A})) & \longrightarrow & 0
\end{array}$$

We would like to prove that $\mathcal{I} = \ker(\psi)$. Assume not, and let $a \in \ker\psi \setminus \mathcal{I}$ be a homogeneous class of minimal degree. Since $\psi(a) = 0$, $\phi(a) \in \mathcal{I}_0$. Hence, there is a $b \in \mathcal{I}$ so that $\phi(a-b) = 0$. Hence, $a - b = cu$ for some $c \in \mathbb{Q}[e_1, \ldots, e_n, u]$. By formality, since $cu \in \ker(\psi)$, then $c \in \ker(\psi)$. Since $a - b = cu$, then $cu \notin \mathcal{I}$, and hence $c \notin \mathcal{I}$. This contradicts the assumption that a is of minimal degree in the set $\ker(\psi) \setminus \mathcal{I}$. □

As a byproduct of this proof, we obtain a presentation for the ordinary cohomology of $M_3(\mathcal{A})$.

Corollary 2.25. Let $B(\mathcal{A})$ be the algebra defined in Lemma 2.22. Then

$$B(\mathcal{A}) \cong H^*(M_3(\mathcal{A})).$$

Remark 2.26. If $\mathcal{A}$ is central, then families (1) and (3) in Lemma 2.22 are sufficient to generate $\mathcal{I}_0$. This was proven for central arrangements in [dLS, 5.5][4]. Note the similarity to the presentation of $H^*(M_2(\mathcal{A}))$, which appears in [OS, 2 & 5.2] for central arrangements and in [OT, 3.45 & 5.90] for affine arrangements.

Remark 2.27. By Proposition 2.11, if we set $u = 1$ we get a presentation for $H^0(M_1(\mathcal{A}))$. This is precisely the presentation given in Proposition 2.2, and the degree filtration is precisely the filtration defined using Heaviside functions. In particular, this demonstrates that the filtration introduced in [VG] is very natural from the point of view of equivariant cohomology. Combining this observation with Proposition 2.19, we obtain our main result.

Corollary 2.28. The associated graded algebra of $H^*(M_1(\mathcal{A}))$ with respect to the VG filtration is isomorphic to $H^*(M_3(\mathcal{A}))$.

[4]Family 1) was omitted in error in [dLS, 5.5]

2.4. Oriented Matroids

As a central hyperplane arrangement gives rise to a matroid, one can ask if these results generalize to matroids. More precisely, a real central hyperplane arrangement with choices of linear forms determines an oriented matroid.

Definition 2.29. Let E be a set. A signed subset X of E is a set $\underline{X} \subset E$ together with a partition (X^+, X^-) of $\underline{X}$ into two distinguished subsets. The set $\underline{X} = X^+ \cup X^-$ is the support of X. The opposite of a signed set X, denoted by $-X$, is the signed set with $(-X)^+ = X^-$ and $(-X)^- = X^+$.

Alternatively, one could consider a signed subset of E to be a map of sets $\Phi_X : E \to \{-1, 0, 1\}$ with $X^+ = \Phi_X^{-1}(1)$, and $X^- = \Phi_X^{-1}(-1)$.

Definition 2.30. A collection $\mathcal{C}$ of signed subsets of a set E is the set of signed circuits of a loop-free oriented matroid on E if and only if it satisfies the following axioms:

1. for all $X \in \mathcal{C}$, $|\underline{X}| > 1$
2. $\mathcal{C} = -\mathcal{C}$
3. for all $X, Y \in \mathcal{C}$, if $\underline{X} \subseteq \underline{Y}$, then $X = Y$ or $X = -Y$
4. for all $X, Y \in \mathcal{C}$, $X \neq -Y$, and $e \in X^+ \cap Y^-$ there is a $Z \in \mathcal{C}$ such that

$$Z^+ \subseteq (X^+ \cup Y^+) \setminus \{e\} \text{ and } Z^- \subseteq (X^- \cup Y^-) \setminus \{e\}.$$

Example 2.31. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central hyperplane arrangement determined by linear forms $\{\omega_1, \dots, \omega_n\}$, and let $E = \{1, \dots, n\}$. Let

$$H_i^+ = \{v \mid \omega_i(v) > 0\}$$

and

$$H_i^- = \{v \mid \omega_i(v) < 0\}.$$

A signed circuit is a minimal signed subset X such that

$$\left(\bigcap_{i \in X^+} H_i^+\right) \bigcap \left(\bigcap_{j \in X^-} H_j^-\right) = \varnothing.$$

We denote by $\mathcal{C}(\mathcal{A})$ the collection of signed circuits of the oriented matroid determined by a hyperplane arrangement $\mathcal{A}$.

Definition 2.32. [GR] Let $(E, \mathcal{C})$ be an oriented matroid. The algebra $P(\mathcal{C})$ is defined to be the quotient of the polynomial ring $\mathbb{Q}[x_a]_{a \in E}$ by the ideal generated by the following elements:

1. $x_a^2 - x_a$ for all $a \in E$.

2. $\left(\prod_{a \in X^+} x_a\right)\left(\prod_{a \in X^-}(1 - x_a)\right)$ for all $X \in \mathcal{C}$.

3. $\left(\prod_{a \in X^+} x_a\right)\left(\prod_{a \in X^-}(1 - x_a)\right) - \left(\prod_{a \in X^-} x_a\right)\left(\prod_{a \in X^+}(1 - x_a)\right)$
   for all $X \in \mathcal{C}$.

Remark 2.33. We see that in fact only families (1) and (2) are required to generate $P(\mathcal{C})$ from axiom (2) of the definition of oriented matroid. However, as in the case of Proposition 2.2, we may replace family (2) with family (3).

Example 2.34. By Proposition 2.2,

$$P(\mathcal{C}(\mathcal{A})) \cong H^0(M_1(\mathcal{A})).$$

Filtering $P(\mathcal{C})$ by degree generalizes the VG filtration.

Analogous to the above construction, Cordovil defines a commutative algebra associated to an oriented matroid that is isomorphic to $H^*(M_3(\mathcal{A}))$ in the case that the oriented matroid is represented by a hyperplane arrangement.

Definition 2.35. [Cor] Consider the map

$$\widetilde{\partial} : \mathcal{C} \to \mathbb{Z}[E] \qquad X \mapsto \sum_{a \in \underline{X}} \mathbf{\Phi_X}(a) \prod_{\substack{b \in \underline{X} \\ b \neq a}} x_b$$

where $\underline{X} = \{i_1, \dots, i_m\}, i_1 < \cdots < i_m$, and $X \in \mathcal{C}$ is the signed circuit supported by $\underline{X}$ so that $\Phi_X(i_1) = 1$. The algebra $\mathbb{A}(\mathcal{C})$ is defined to be the quotient of the polynomial ring $\mathbb{Q}[x_a]_{a \in E}$ by the ideal generated by the following elements:

1. $x_a^2 = 0$
2. $\widetilde{\partial}(X) = 0$ for all $X \in \mathcal{C}$.

Example 2.36. By Corollary 2.25 and Remark 2.26,

$$\mathbb{A}(\mathcal{C}(\mathcal{A})) \cong H^*(M_3(\mathcal{A})).$$

Theorem 2.37. With respect to the filtration of $P(\mathcal{C})$ by polynomial degree in the generators, the associated graded algebra of $P(\mathcal{C})$ is isomorphic to $\mathbb{A}(\mathcal{C})$.

Proof. Given $X \in \mathcal{C}$, family (3) of $P(\mathcal{C})$ becomes

$$\left(\prod_{a\in X^+} x_a\right)\left(\prod_{a\in X^-}(1-x_a)\right) - \left(\prod_{a\in X^-} x_a\right)\left(\prod_{a\in X^+}(1-x_a)\right)$$

$$= \widetilde{\partial}(X) + \text{ lower degree terms.}$$

This tells us that there is a natural surjection of rings $\mathbb{A}(\mathcal{C}) \to \operatorname{gr} P(\mathcal{C})$. By [Cor, 2.8], the dimensions of the two algebras are equal. Hence this map is an isomorphism. □

Note that if $\mathcal{A}$ of a central hyperplane arrangement and $\mathcal{C} = \mathcal{C}(\mathcal{A})$, this agrees with our earlier result, Corollary 2.28. There are several advantages of the approach taken in Section 3. Proposition 2.19 shows *a priori* that the cohomology ring of $M_1(\mathcal{A}) = M_3(\mathcal{A})^T$ admits a filtration whose associated graded algebra is isomorphic to the cohomology ring of $M_3(\mathcal{A})$ without having to work directly with any presentations. Furthermore, Remark 2.27 motivates the appearance of Heaviside functions in the work of Varchenko and Gelfand.

Remark 2.38. In a future paper we will generalize Theorem 2.37 to pointed oriented matroids, which are the combinatorial analogues of non-central arrangements.

Question 2.39. In [GR] and [Pro2], when extending the Orlik-Solomon algebra to the setting of oriented matroids, the Salvetti complex is used to give this a geometric realization. In particular, when the oriented matroid is representable by a hyperplane arrangement, the Salvetti complex is a deformation retract of the complexified hyperplane complement. Is there such a complex that one can construct for the corresponding 3-arrangement?

2.5. Finite Group Actions

Proposition 2.19 has applications for the representation theory of finite groups. Let $\mathcal{A}$ be a real hyperplane arrangement.

Proposition 2.40. Suppose that a finite group W acts on $\mathcal{A}$. Then W acts on both $M_1(\mathcal{A})$ and $M_3(\mathcal{A})$, and $H^*(M_3(\mathcal{A})) \cong H^*(M_1(\mathcal{A}))$ as W-representations.

Proof. We will first show that the actions of W and T on $M_3(\mathcal{A})$ commute. The group T acts trivially on the first coordinate of $M_3(\mathcal{A}) \subset V^3$, so we only need to show that the actions commute on the last two coordinates. If we treat the last two coordinates as one complex coordinate, we see that T acts by scalar multiplication. Hence, these two actions commute.

By Proposition 2.14, $H^*(M_3(\mathcal{A})) \cong \operatorname{gr} H^*(M_1(\mathcal{A}))$ as W-representations. The result now follows from Remark 2.15. □

The most interesting class of examples we have are Weyl groups acting on Coxeter arrangements.

Example 2.41. Let $\mathcal{A}$ be a Coxeter arrangement with Weyl group W. In this case W acts simply transitively on the chambers of $\mathcal{A}$, so $H^*(M_3(\mathcal{A})) \cong_W H^0(M_1(\mathcal{A}))$ is the regular representation.

Remark 2.42. When $\mathcal{A} = \mathcal{B}_n$ is the braid arrangement and $M_3(\mathcal{A})$ is the configuration space of n points in $\mathbb{R}^3$, this is already known [Ati].

Example 2.43. Consider the arrangement $\mathcal{B}_4$ whose Weyl group is the symmetric group S_4. Computing each graded component as an S_4-representation we get

$$\begin{aligned}
H^0(M_3(\mathcal{B}_4)) &\cong \tau \\
H^2(M_3(\mathcal{B}_4)) &\cong \rho + \Lambda^2(\rho) \\
H^4(M_3(\mathcal{B}_4)) &\cong \rho + \Lambda^2(\rho) + 2\omega + \sigma \\
H^6(M_3(\mathcal{B}_4)) &\cong \rho + \Lambda^2(\rho)
\end{aligned}$$

where τ corresponds to the partition (4), ρ corresponds to the partition $(3,1)$, $\Lambda^2(\rho)$ corresponds to the partition $(2,1,1)$, ω corresponds to the partition $(2,2)$, and σ corresponds to the partition $(1,1,1,1)$. Adding up the representations of all of the graded components yields the regular representation of S_4.

Example 2.44. The semi-order arrangement $\mathcal{S}_3$ is defined by the affine linear forms

$$\omega_{ij} = x_i - x_j - 1, \quad 1 \le i, j \le 3, H_{ij} = \omega_{ij}^{-1}(0).$$

Looking at the action of the symmetric group S_3 on $H^0(M_1(\mathcal{S}_3))$ we get

$$H^0(M_1(\mathcal{S}_3)) \cong_{S_3} 5\tau + 2\sigma + 6\rho.$$

where τ corresponds to the partition (3), ρ corresponds to the partition $(2,1)$, and σ corresponds to the partition $(1,1,1)$.

As is illustrated in Figure 2.1., the bounded region containing the origin corresponds to a τ, the remaining bounded regions correspond to $\tau + \sigma + 2\rho$, and the unbounded regions correspond to $3\tau + \sigma + 3\rho$.

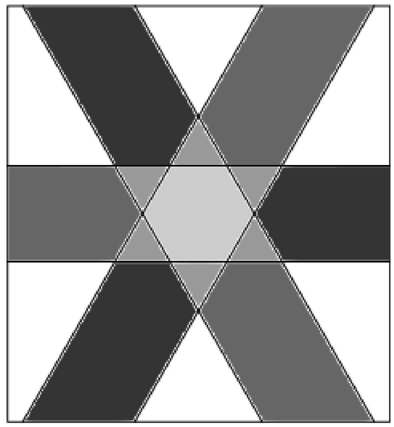

FIGURE 2.1. The semi-order arrangement $\mathcal{S}_3$ with shaded orbits.

The algebra $H^*(M_3(\mathcal{S}_3))$ decomposes as

$$H^0(M_3(\mathcal{S}_3)) \cong_{S_3} \tau$$

$$H^2(M_3(\mathcal{S}_3)) \cong_{S_3} \tau + \sigma + 2\rho$$

$$H^4(M_3(\mathcal{S}_3)) \cong_{S_3} 3\tau + \sigma + 4\rho$$

as S_3-representations.

CHAPTER III

REPRESENTATION STABILITY

In [Ter], Terao studies the algebra generated by the reciprocals of defining linear forms of a hyperplane arrangement. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a central hyperplane arrangement in a k-dimensional vector space V with $H_i = \omega_i^{-1}(0)$ for some linear form $\omega_i \in V^*$. We define the algebra $C(\mathcal{A})$ to be the subalgebra of the fraction field of $\mathbb{Q}[V]$ generated by the set

$$\left\{\frac{1}{\omega_i}\right\}_{i=1}^{n}.$$

Remark 3.1. Note that we made a choice of linear form ω_i for each hyperplane H_i. This algebra is independent of this choice as any other defining linear form is a scalar multiple of it. This algebra is also not sensitive to multiplicities of hyperplanes.

As with the Orlik-Solomon algebra, this algebra decomposes over its intersection lattice, the lattice of all possible subspaces formed as intersections of hyperplanes in the arrangement.

Theorem 3.2. [Ter] If $L(\mathcal{A})$ is the intersection lattice of the arrangement $\mathcal{A}$, then

$$C(\mathcal{A}) = \bigoplus_{X \in L(\mathcal{A})} C_X(\mathcal{A})$$

where $C_X(\mathcal{A})$ is the subalgebra generated as a vector space by reciprocals of products $\omega_{i_1} \ldots \omega_{i_l}$ with $\cap_{j=1}^{l} \omega_i^{-1}(0) = X$.

In this chapter, we are interested in studying this algebra in the case where $\mathcal{A} = B_n$ the braid arrangement. It is easy to see that

$$C(\mathcal{B}_n) = \mathbb{Q}\left[\frac{1}{x_i - x_j}\right]_{1\leq i\leq j\leq n}$$

where the linear form $x_i - x_j$ corresponds to the hyperplane H_{ij}.

Letting $z_{ij} = \frac{1}{x_i - x_j}$ we have that $z_{ji} = -z_{ij}$, and $C(\mathcal{B}_n)$ admits an S_n action given by permuting the indices. An interesting question that one can ask about this algebra is how it decomposes into irreducible S_n-representations. While we won't answer this question explicitly, we will show that each graded piece of $C(\mathcal{B}_n)$ exhibits a property known as representation stability and in Chapter IV we will describe a program to compute this recursively.

3.1. Representation Stability

Representation stability is an idea introduced in 2010 by Church and Farb in [CF] as an analogue to homological stability. A sequence of spaces $\{X_n\}$ equipped with maps $X_n \to X_{n+1}$ exhibits homological stability if the induced maps on homology $H_i(X_n) \to H_i(X_{n+1})$ are isomorphisms for sufficiently large values of n. A sequence of representations of a family of groups exhibits representation stability if eventually each term in the sequence decomposes into irreducible representations in a predictable way.

Here we will focus on symmetric group representation stability where irreducible representations of S_n correspond to partitions of n. However, given a partition $\mu = (\mu_1, \ldots, \mu_k)$ of m, we will represent by $V(\mu)$ the irreducible S_n representation corresponding to the partition $(n - m, \mu_1, \ldots, \mu_k)$ where $n \geq m + \mu_1$.

The reason to use this notation is to establish a consistent vocabulary between representations of different symmetric groups. For example, $V(0)$ corresponds to the trivial representation of S_n for all $n \geq 1$ and $V(1,1)$ corresponds to $\Lambda^2\rho$ for all $n \geq 3$ where ρ is the standard $(n-1)$-dimensional representation of S_n.

Let $\{V_n\}$ be a sequence of vector spaces so that V_i is an S_i-representation along with maps $\varphi_i : V_i \to V_{i+1}$ with the property that the map is S_i-equivariant. In this case, S_i acts on V_{i+1} via the standard inclusion $S_i \hookrightarrow S_{i+1}$.

Definition 3.3. [CF] A sequence $\{V_n\}$ as described above is representation stable if the following three conditions hold.

1. The maps $\varphi_n : V_n \to V_{n+1}$ are eventually injective.
2. For large enough n, the S_{n+1}-span of $\varphi_i(V_n)$ is V_{n+1}.
3. The multiplicities of the irreducible representations eventually stabilize. That is, for large enough n,
$$V_n = \bigoplus_{\mu} V(\mu)^{\oplus c_\mu}$$
where c_μ does not depend on n and the sum is over partitions μ of positive integers m with the property that $n - m \geq \mu_1$.

Example 3.4. Let M be a connected orientable manifold. In [Chu], Church proves that for the sequence of configuration spaces $\{C_n(M)\}_{n=1}^{\infty}$, the corresponding sequence $H^i(C_n(M))$ is S_n-representation stable. Here the maps $H^i(C_n(M)) \to H^i(C_{n+1}(M))$ are induced by the maps $C_{n+1}(M) \to C_n(M)$ given by forgetting the last coordinate. As a special case of this, we see that for a fixed i, $\{H^i(M_3(\mathcal{B}_n))\}$ forms a representation stable sequence as $M_3(\mathcal{B}_n) = C_n(\mathbb{R}^3)$.

Example 3.5. While the sequence of regular representations $\mathbb{C}[S_n]$ with maps induced by the standard inclusion $S_n \hookrightarrow S_{n+1}$ satisfy conditions 1 and 2 of the definition, condition 3 is not satisfied as new irreducible representations arise with each iteration of n.

3.2. Algebra of Reciprocals of Linear Forms

Let $C^i(\mathcal{B}_n)$ be the i^{th} graded piece of $C(\mathcal{B}_n)$ where $z_{ij} = \frac{1}{x_i - x_j}$ is in degree 1, and let $C^i_X(\mathcal{B}_n) = C_X(\mathcal{B}_n) \cap C^i(\mathcal{B}_n)$. For the sequence of algebras $C^i(\mathcal{B}_n)$, it is clear that the first two conditions of representation stability are satisfied where the map $C^i(\mathcal{B}_n) \to C^i(\mathcal{B}_{n+1})$ is given by $z_{ij} \mapsto z_{ij}$. However, the multiplicity stable condition is not immediately clear. The rest of this section will be devoted to proving that $\{C^i(\mathcal{B}_n)\}$ is multiplicity stable. This argument will be strongly influenced by the proof that the cohomology of the pure braid group is multiplicity stable shown in [CF].

Let $\mathcal{S}$ be a partition of $\{1, \dots, n\}$, and suppose $\mathcal{S}$ decomposes as $\mathcal{S} = \{\sqcup_{l=1}^m \mathcal{S}_l\} \sqcup \{\alpha_1\} \sqcup \cdots \sqcup \{\alpha_p\}$ where each $\mathcal{S}_l$ contains more than one element. Denote by $\overline{\mathcal{S}}$ the corresponding partition of n. If $\overline{\mathcal{S}} = \mu = (\mu_1, \dots, \mu_k)$, then let $\varphi(\mu) = \sum(\mu_l - 1) = n - k$. Define

$$H_A = \bigcap_{\substack{(\alpha,\beta)\in A\times A \\ \alpha\neq\beta}} H_{\alpha\beta}$$

for a subset A of $\{1, \dots, n\}$. Let

$$X_{\mathcal{S}} = \bigcap_{l=1}^{m} H_{\mathcal{S}_l}$$

be the corresponding vector space in the intersection lattice. Note that S_n acts on the set of partitions $\mathcal{S}$ of $\{1, \dots, n\}$. Let $\mathrm{Stab}(\mathcal{S}) \subset S_n$ be the stabilizer of $\mathcal{S}$. The group

$\mathrm{Stab}(\mathcal{S}) = G_{\mathcal{S}} \times S_{p(\mathcal{S})}$, where $G_{\mathcal{S}}$ stabilizes $\sqcup_{l=1}^m \mathcal{S}_l$ and $S_{p(\mathcal{S})}$ permutes the singletons. Note that $G_{\mathcal{S}}$ permutes the $\mathcal{S}_l$ that are of the same size and elements within them. In particular, if $\beta \in \mathcal{S}_l$, then either $\sigma(\beta) \in \mathcal{S}_l$ or $\sigma(\beta) \in \mathcal{S}_r$ and $\sigma(\mathcal{S}_l) = \mathcal{S}_r$.

While the following result is not used explicitly in the proof that the sequence $\{C^i(\mathcal{B}_n)\}$ is representation stable, it tells us the structure of the stabilizers of the $C^i_{X_{\mathcal{S}}}(\mathcal{B}_n)$ in $C^i(\mathcal{B}_n)$.

Lemma 3.6. The stabilizer of the subspace $C_{X_{\mathcal{S}}}(\mathcal{B}_n)$ is equal to $\mathrm{Stab}(\mathcal{S})$.

Proof. Suppose $\sigma \in \mathrm{Stab}(\mathcal{S})$ and $z_{i_1j_1}\cdots z_{i_kj_k} \in C_{X_{\mathcal{S}}}$. By the correspondence of partitions and the intersection lattice, all the pairs $\{i_p, j_p\}$ are subsets of some $\mathcal{S}_l$ for some l. We would like to show that $\sigma \cdot z_{i_1j_1}\cdots z_{i_kj_k} \in C_{X_{\mathcal{S}}}$. Note that

$$\sigma \cdot z_{i_1j_1}\cdots z_{i_kj_k} = z_{\sigma(i_1)\sigma(j_1)}\cdots z_{\sigma(i_k)\sigma(j_k)}.$$

Suppose that $x \in X_{\mathcal{S}}$. Then x has the property that

$$x_{i_p} - x_{j_p} = 0 \text{ for all } 1 \le p \le k.$$

Since all the pairs $\{i_k, j_k\}$ are subsets of some $\mathcal{S}_l$ for some l, then $\{\sigma(i_p), \sigma(j_p)\} \subset \mathcal{S}_{l^*}$ for some l^*. Hence,

$$x_{\sigma(i_p)} - x_{\sigma(j_p)} = 0 \text{ for all } 1 \le p \le k.$$

Thus $x \in \sigma \cdot X_{\mathcal{S}} = \cap_{l=1}^k \ker(x_{\sigma(i_l)} - x_{\sigma(j_l)})$. The permutation action of the symmetric group on a subspace of $\mathbb{R}^{n-1}$ preserves dimension, so $X_{\mathcal{S}} = \sigma \cdot X_{\mathcal{S}}$. Thus $\sigma \cdot z_{i_1j_1}\cdots z_{i_kj_k} \in C_{X_{\mathcal{S}}}$.

If $\sigma \notin \mathrm{Stab}(\mathcal{S})$, then there exists an $\mathcal{S}_l$ so that $\sigma(\mathcal{S}_l) \neq \mathcal{S}_{l^*}$ for any $1 \le l^* \le k$. Hence, there exist $\alpha, \beta \in \sigma(\mathcal{S}_l)$ so that $(z_{\alpha\beta})^{-1}(x) = x_\alpha - x_\beta \neq 0$ for some $x \in X_{\mathcal{S}}$.

□

Note that all partitions $\mathcal{S}$ of $\{1,\dots,n\}$ with the property that $\overline{\mathcal{S}} = \mu$ for a given μ have isomorphic stabilizers. Thus for each partition $\mu \vdash n$, choose a representative partition $\mathcal{S}_\mu$ of $\{1,\dots,n\}$. Note that the action of S_n on the set of partitions of $\{1,\dots,n\}$ preserves partition type. It is easy to see that for a given $\mathcal{S}$, the S_n-span of $C^i_{X_\mathcal{S}}(\mathcal{B}_n)$ is the sum over all partitions of type $\mu = \overline{\mathcal{S}}$ and that as an S_n-representation, this is

$$\operatorname{Ind}^{S_n}_{\operatorname{Stab}(\mathcal{S}_\mu)} C^i_{X_{\mathcal{S}_\mu}}(\mathcal{B}_n).$$

For a fixed i, for which partitions $\mathcal{S}$ is $C^i_{X_\mathcal{S}}$ nontrivial? If $\varphi(\mathcal{S}) \leq i$, then $C^i_{X_\mathcal{S}}$ is nontrivial. In particular, $\varphi(\mathcal{S})$ is equal to the minimum number of pairs (α,β) needed to make $X_\mathcal{S} = \bigcap_{(\alpha,\beta)} \ker(x_\alpha - x_\beta)$.

By the Orlik-Terao decomposition in Theorem 3.2,

$$C^i(\mathcal{B}_n) = \bigoplus_{\mathcal{S}\vdash\{1,\dots,n\},\varphi(\overline{\mathcal{S}})\leq i} C^i_{X_\mathcal{S}}(\mathcal{B}_n)$$

$$\cong \bigoplus_{\mu\vdash n,\varphi(\mu)\leq i} \operatorname{Ind}^{S_n}_{\operatorname{Stab}(\mathcal{S}_\mu)} C^i_{X_{\mathcal{S}_\mu}}(\mathcal{B}_n) \cong \bigoplus_{\mu\vdash n,\varphi(\mu)\leq i} \operatorname{Ind}^{S_n}_{G_{\mathcal{S}_\mu}\times S_{p(\mathcal{S}_\mu)}} C^i_{X_{\mathcal{S}_\mu}}(\mathcal{B}_n).$$

To establish that the sequence of representations becomes multiplicity stable, we need to establish a relationship between $C^i(\mathcal{B}_n)$ and $C^i(\mathcal{B}_m)$ where $m \geq n$. Let $\mathcal{S}$ be a partition of $\{1,\dots,n\}$ and let

$$\mathcal{S}[m] := \mathcal{S} \sqcup \{n+1\} \sqcup \cdots \sqcup \{m\}.$$

Lemma 3.7. Let $\mathcal{S}$ be a partition of $\{1, \ldots, n\}$. Then

$$\mathrm{Stab}(\mathcal{S}[m]) \cong G_{\mathcal{S}} \times S_{m-n+p(\mathcal{S})}.$$

and

$$C^i_{X_{\mathcal{S}}}(\mathcal{B}_n) \cong C^i_{X_{\mathcal{S}[m]}}(\mathcal{B}_m)$$

as $G_{\mathcal{S}}$-representations.

Proof. To justify the first assertion, we appeal to the definition of $G_{\mathcal{S}}$ and note that it only depends on the non-singleton subsets. To justify the second assertion we notice that the subspace $X_{\mathcal{S}[m]}$ is defined by the vanishing of the same linear forms as $X_{\mathcal{S}}$. □

This says that if we can show that a sequence of induced representations is representation stable and that new types of partitions eventually stop showing up, then $C^i(\mathcal{B}_n)$ will be multiplicity stable. To address the first part, we use a result of Hemmer.

Theorem 3.8. [Hem] (Hemmer) Fix $k \geq 1$, a subgroup $H < S_k$, and a representation V of H.

1. The sequence of S_n-representations $\left\{\mathrm{Ind}^{S_n}_{H \times S_{n-k}} V\right\}$ is representation stable. Here $H \times S_{n-k}$ acts on V by letting S_{n-k} act trivially.

2. The decomposition of this sequence stabilizes once $n \geq 2k$.

Remark 3.9. Despite the fact that this says that the sequence stabilizes for $n \geq 2k$, it is possible for it to stabilize sooner.

With this result, we may prove our main theorem.

Theorem 3.10. The sequence $\{C^i(\mathcal{B}_n)\}$ is a representation stable.

Proof. For a fixed i, as n increases, the partitions $\mu = (\mu_1, \ldots, \mu_k)$ with $\varphi(\mu) \leq i$ have the property that eventually (once $n > 2i$) some of the μ_a must equal 1. This is because if we want $\varphi(\mu) \leq i$, then the length of the partition k must eventually be more than half of n. Hence, the new partitions that arise beyond $2i$ subject to the condition that $\varphi(\mu) \leq i$ are ones that come from partitions of smaller numbers by adding on singletons.

Hence, for $n \geq 2i$

$$C^i(\mathcal{B}_n) \cong \bigoplus_{\mu \vdash 2i, \varphi(\mu) \leq i} \mathrm{Ind}^{S_n}_{G_{\mathcal{S}_\mu} \times S_{p(\mathcal{S}_\mu)+n-2i}} C^i_{X_{\mathcal{S}_\mu[n]}}(\mathcal{B}_n)$$

$$\cong \bigoplus_{\mu \vdash 2i, \varphi(\mu) \leq i} \mathrm{Ind}^{S_n}_{G_{\mathcal{S}_\mu} \times S_{p(\mathcal{S}_\mu)+n-2i}} C^i_{X_{\mathcal{S}_\mu}}(\mathcal{B}_{2i}).$$

The second isomorphism is by Lemma 3.7. We notice that the summation no longer depends on n and the representation we are inducing isn't changing as a $G_{\mathcal{S}_\mu}$-representation. The only terms that change as we increase n are $S_{p(\mathcal{S}_\mu)+n-2i}$, which acts trivially on $C^i_{X_{\mathcal{S}_\mu}}(\mathcal{B}_{2i})$, and S_n. Thus this sequence then stabilizes by Theorem 3.8. □

As a corollary to this proof and Hemmer's result, we get the following.

Corollary 3.11. The sequence $\{C^i(\mathcal{B}_n)\}$ stabilizes for $n \geq 4i$.

Proof. For a fixed i, we achieve all partitions μ of n with $\varphi(\mu) \leq i$ once $n \geq 2i$. Theorem 3.8 says that the decomposition of the induced representations stabilizes once n is twice this. □

3.3. Examples

In the previous sections we've proven that $\{C^i(\mathcal{B}_n)\}$ is S_n-representation stable. However, we haven't commented on how they decompose for a fixed i. Here we provide the examples of $\{C^1(\mathcal{B}_n)\}$ and $\{C^2(\mathcal{B}_n)\}$.

Example 3.12. The sequence $\{C^1(\mathcal{B}_n)\}$ stabilizes for $n \geq 3$ as

$$C^1(\mathcal{B}_n) \cong_{S_n} V(1) + V(1,1).$$

This is the standard representation of S_n plus its exterior square. Notice that the representation $V(1,1)$ only makes sense when $n \geq 3$ as it corresponds to the partition $(n-2,1,1)$ of n.

The sequence $\{C^2(\mathcal{B}_n)\}$ stabilizes for $n \geq 6$ as

$$C^2(\mathcal{B}_n) \cong_{S_n} V(0)+V(1)^{\oplus 2}+V(2)^{\oplus 3}+V(1,1)+V(1,1,1)+V(2,1)^{\oplus 2}+V(2,2)+V(1,1,1,1).$$

Notice that the representation $V(2,2)$ only exists when $n \geq 6$ as it correspond to the partition $(n-4,2,2)$ of n.

To perform this computation, we notice three types of monic monomials $z_{ij}z_{kl}$ in $C^2(\mathcal{B}_n)$. The first type consists of those with only two distinct indices. These are in fact squares of elements of $C^1(\mathcal{B}_n)$ and thus there are $\binom{n}{2}$ such monomials. These monomials correspond to a partition $\mathcal{S}$ of type $(2,1,\ldots,1)$. There are no relations between these monomials, so they form a basis of $C^2_{X_\mathcal{S}}(\mathcal{B}_n)$. This partition type first occurs in $C^2(\mathcal{B}_2)$ and the corresponding monomial is z_{12}^2. If we look at the vector space generated by z_{12}^2 in $C^2(\mathcal{B}_n)$ for large enough n, we get that its S_n span is

$$\mathrm{Ind}_{S_2\times S_{n-2}}^{S_n}\mathbb{C}[z_{12}^2]$$

as it is stabilized by $S_2 \times S_{n-2}$.

The second type of monomial corresponds to those with three distinct indices. For a given partition $\mathcal{S}$ of type $(3, 1, \ldots)$, we get 3 different types of monomials. However, there is a quadratic relation $z_{ij}z_{jk} = z_{ik}z_{jk} + z_{ij}z_{ik}$, so we choose two for a basis of $C^2_{X_\mathcal{S}}$. This partition type first occurs in $C^2(\mathcal{B}_3)$ and the stabilizer of this vector space is $S_3 \times S_{n-3}$. Its S_n span is

$$\mathrm{Ind}_{S_3\times S_{n-3}}^{S_n}\mathbb{C}[z_{ik}z_{jk}, z_{ij}z_{ik}]$$

The final type of monomial corresponds to those with four distinct indices. For a given partition $\mathcal{S}$ of type $(2, 2, 1, \ldots, 1)$, we get only one monomial. This partition type first occurs in $C^2(\mathcal{B}_4)$ and the stabilizer of this vector space is $H \times S_{n-4}$ where H is a subgroup of S_4 of order 8. To compute this, instead of using the character table of $H \times S_{n-4}$, we first induce up to $S_4 \times S_{n-4}$ by looking at the $S_4 \times S_{n-4}$-span of the monomial. Then we induce up to S_n.

$$\mathrm{Ind}_{H\times S_{n-4}}^{S_n}\mathbb{C}[z_{ij}z_{kl}] \cong \mathrm{Ind}_{S_4\times S_{n-4}}^{S_n}\mathbb{C}[z_{ij}z_{kl}, z_{ik}z_{jl}, z_{il}z_{jk}].$$

Adding all of these together, we get $C^2(\mathcal{B}_n)$ as an S_n-representation. Notice that the Hemmer result says that this should stabilize for $n \geq 8$, but in fact it stabilizes at $n = 6$, so in this case the lower bound is not strict.

In Chapter IV we will compute the example of $C^2(\mathcal{B}_6)$ in a different way using a more geometric decomposition.

CHAPTER IV

HYPERTORIC VARIETIES

The algebra $C(\mathcal{A})$ introduced in Chapter III was studied in [PW] and [BP] as the equivariant intersection cohomology of a hypertoric variety. Since the algebra $C(\mathcal{A})$ gets its data from a hyperplane arrangement, it isn't a stretch to believe that the topology of a hypertoric variety can be encoded in a vector arrangement. In this chapter we will define hypertoric varieties and use the geometry to assist us in decomposing $C(\mathcal{A})$ into irreducible S_n-representations.

4.1. Hypertoric Varieties

In this section we will define the hypertoric variety corresponding to a vector arrangement, equip it with a stratification, and provide a presentation of the equivariant intersection cohomology of the hypertoric variety.

Let T be an algebraic torus, and let $T^n = (\mathbb{C}^\times)^n$ be the n-dimensional coordinate torus. Given a spanning set of cocharacters

$$\mathcal{V} = \{a_1, \ldots, a_n\} \in \mathrm{Hom}(\mathbb{C}^\times, T),$$

we can define a surjective map $\pi : T^n \to T$ given by

$$\pi(z_1, \ldots, z_n) = \prod_{i=1}^{n} a_i(z_i).$$

Let K be the kernel of the map π. Let T^n act on $T^*\mathbb{A}^n$ by the standard action. The restriction of the action of T^n to K is hamiltonian with moment map

$$\mu(z,w) = \iota^* \sum_{i=1}^{n} (z_i w_i) e_i$$

where $\iota : \mathfrak{k} \to \mathfrak{t}^n$ is induced by the inclusion $K \to T^n$.

The resulting quotient

$$\mathfrak{M}(\mathcal{V}) = \mu^{-1}(0)//K = \operatorname{Spec}\left(\mathbb{C}[\mu^{-1}(0)]^K\right)$$

is the hypertoric variety corresponding to the vector arrangement $\mathcal{V}$.

Remark 4.1. If we identify $\operatorname{Hom}(\mathbb{C}^\times, T)$ with $\mathfrak{t}_\mathbb{Z}$, the integer lattice of the Lie algebra of T, we can define a central hyperplane arrangement $\mathcal{A}$ with hyperplanes $\{H_1, \ldots, H_n\}$ given by

$$H_i = \left\{x \in (\mathfrak{t}^d)^*_\mathbb{R} \mid x \cdot a_i = 0\right\}$$

with normal vector $a_i \in \mathfrak{t}_\mathbb{Z}$. Note that different vector arrangements may yield the same hyperplane arrangement.

Example 4.2. Let T be the maximal torus inside of $SL(n,\mathbb{C})$ with Weyl group $W \cong S_n$. Let $\mathcal{A}_{n-1}$ be the vector arrangement in $\mathfrak{t}_\mathbb{R}$ consisting of the coroots of $\mathfrak{sl}_n$. The reflecting hyperplanes of this root system form the braid arrangement $\mathcal{B}_n$. We will think about $\mathfrak{t}_\mathbb{R}$ as the subspace of $\mathbb{R}^n$ determined by the vanishing of $x_1+\cdots+x_n$. Note that W acts faithfully and transitively on $\mathcal{A}_{n-1}$ by construction. Let $\mathfrak{M}_n = \mathfrak{M}(\mathcal{A}_{n-1})$ be the hypertoric variety corresponding to this vector arrangement.

After taking the quotient of $\mu^{-1}(0)$ by K, we are left with an action of T. The resulting space is singular, so instead of computing the T-equivariant cohomology, we compute its T-equivariant intersection cohomology.

Proposition 4.3. [BP, 4.5] If $\mathcal{V}$ is a vector arrangement with corresponding hyperplane arrangement $\mathcal{A}$, then

$$IH^*_T(\mathfrak{M}(\mathcal{V}); \mathbb{C}) \cong C(\mathcal{A}).$$

As stated in Chapter III, $C(\mathcal{A})$ is not sensitive to altering vectors by scalar multiplication or multiplicities. This says that the equivariant intersection cohomology doesn't depend on the coorientations or multiplicities of the hyperplane arrangement.

Example 4.4. In the example of $\mathcal{A}_{n-1}$ and $\mathfrak{M}_n$, we get that

$$IH^*_T(\mathfrak{M}_n) \cong C(\mathcal{B}_n),$$

as the reflecting hyperplanes of $\mathcal{A}_{n-1}$ form the braid arrangement $\mathcal{B}_n$. Since $\mathcal{A}_{n-1}$ admits an action of the group $W \cong S_n$, so do $\mathfrak{M}_n$ and $IH^*_T(\mathfrak{M}_n)$. To see that these S_n actions are the same, we just need to show that they are the same in degree 2. To do this we appeal to [PW, p.19-20] to say that

$$IH^2_T(\mathfrak{M}_n) \cong_{S_n} H^2(M_3(\mathcal{B}_n)) \cong_{S_n} C^2(\mathcal{B}_n).$$

The hypertoric variety $\mathfrak{M}_n$ admits a stratification indexed by partitions of $\{1, \ldots, n\}$. Let H_{ij} be the hyperplane defined by the vanishing of $x_i - x_j$ with normal vector in the direction where $x_i > x_j$. Given a partition $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_k\}$

with $\mathcal{S}_1, \ldots, \mathcal{S}_k$ representing the subsets of size greater than or equal to 2, define the subspace

$$H_{\mathcal{S}} = \left\{ x \in \mathbb{R}^{n-1} \mid x_j = x_l \text{ if } j, l \in \mathcal{S}_i \text{ for some } i \right\}.$$

For a partition $\mathcal{S}$, define the restriction of the arrangement B_n to be

$$\mathcal{B}_n^{\mathcal{S}} := \{ H_{jl} \cap H_{\mathcal{S}} \mid j, l \notin \mathcal{S}_i \text{ for any } i \}$$

and the localization to be

$$(\mathcal{B}_n)_{\mathcal{S}} := \{ H_{jl}/H_{\mathcal{S}} \mid j, l \in \mathcal{S}_i \text{ for some } i \}.$$

Note that for the empty partition E, $\mathcal{B}_n^E = \mathcal{B}_n$ and $(\mathcal{B}_n)_E$ is the empty arrangement in a point.

The stratification of $\mathfrak{M}_n$ is given by

$$\mathfrak{M}_n = \bigsqcup_{\mathcal{S} \vdash \{1, \ldots, n\}} \mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$$

where

$$\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}}) = \{ [z, w] \in \mathfrak{M}(\mathcal{B}_n) \mid z_{ij} = w_{ij} = 0 \text{ if and only if } i, j \in \mathcal{S}_k \}.$$

By $[z, w]$, we mean the equivalence class of (z, w) in $\mu^{-1}(0)$.

To show that this is indeed a stratification, [PW, 2.4] show that there exists a space $N_{\mathcal{S}}$ with a basepoint s for each stratum $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$ satisfying the following property. There is an analytic neighborhood of $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$ in $\mathfrak{M}_n$ which is locally biholomorphic to a neighborhood of $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}}) \times \{s\}$ in $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}}) \times N_{\mathcal{S}}$. The space $N_{\mathcal{S}}$ is called a normal slice to $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$ in $\mathfrak{M}_n$ and $\mathfrak{M}((\mathcal{B}_n)_{\mathcal{S}})$ is shown to be a normal slice to $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$ in [PW, 2.4].

The tori that act on the strata are naturally quotients of T and hence the strata admit an action of T. Collecting the strata that correspond to all of the partitions of type μ, we see that all the normal slices are isomorphic, we get an action of $T \rtimes W$, and this stratification is $T \rtimes W$-equivariant.

4.2. Program for Computation

To decompose $C(\mathcal{B}_n)$ as an S_n-representation, we may use the geometric interpretation of this ring as the equivariant intersection cohomology of a hypertoric variety along with its T-equivariant stratification. From the stratification, we get a spectral sequence described below that degenerates at its E_1-page and the following result.

Proposition 4.5. As an S_n-representation, the algebra $C(\mathcal{B}_n)$, thought of as the equivariant intersection cohomology of the hypertoric variety $\mathfrak{M}(\mathcal{B}_n)$, decomposes over the stratification by partitions as follows:

$$C(\mathcal{B}_n) \cong \bigoplus_{\mathcal{S} \vdash \{1,\ldots,n\}} IH_T^*(\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})) \otimes IH_c^*(\mathfrak{M}((\mathcal{B}_n)_{\mathcal{S}}))$$

$$\cong \bigoplus_{\mathcal{S} \vdash \{1,\ldots,n\}} H^*(M_3(B_n^{\mathcal{S}})) \otimes \mathrm{sym}(\mathfrak{t}^*/H_{\mathcal{S}}) \otimes IH_c^*(\mathfrak{M}((\mathcal{B}_n)_{\mathcal{S}})).$$

This follows from a spectral sequence outlined in [BGS, 3.4] and a lemma from [BP, 5.5]. Filter $\mathfrak{M}_n$ by $\mathfrak{M}(\mathcal{B}_n) = X_0 \supset X_1 \supset \cdots \supset X_{n-1} = \varnothing$ where X_i consists of all the strata of codimension greater than or equal to $2i$, and let $i_k : X_k \setminus X_{k+1} \to \mathfrak{M}(\mathcal{B}_n)$ be the inclusion. To compute $IH_T^*(\mathfrak{M}_n)$, we take the limit of the spectral sequence with E_1-page given by

$$E_1^{p,q} = H^{p+q}(i_p^! IC_{\mathfrak{M}_n,T}) \Rightarrow \mathbb{H}^{p+q}(IC_{\mathfrak{M}_n,T}) \cong IH_T^{p+q}(\mathfrak{M}_n)$$

by [BGS]. By $E_1^{p,q} \Rightarrow IH_T^{p+q}(\mathfrak{M}_n)$ we mean that the spectral sequence converges to the associated graded algebra of $IH_T^m(\mathfrak{M}_n)$ with respect to the filtration coming from the decomposition

$$IH_T^m(\mathfrak{M}_n) \cong \bigoplus_{p+q=m} E_\infty^{p,q}.$$

The lemma from [BP] tells us that

$$H^*(i_p^! IC_{\mathfrak{M}_n,T}) \cong \bigoplus_{\varphi(\mathcal{S})=p} H_T^*(\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})) \otimes IH_c^*((\mathfrak{M}(\mathcal{B}_n)_{\mathcal{S}}))$$

as the codimension $2p$ strata correspond to the partitions with $\varphi(\mathcal{S}) = p$. Since the terms of the spectral sequence are concentrated in even degree, it degenerates at the E_1-page and the first isomorphism of our proposition follows.

The second isomorphism comes from the fact that there is a subtorus of T that acts trivially on $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$, its Lie algebra is $H_{\mathcal{S}}^{\perp}$, the quotient torus acts freely, and the quotient of $\mathring{\mathfrak{M}}(\mathcal{B}_n^{\mathcal{S}})$ by this torus is $M_3(\mathcal{B}_n^{\mathcal{S}})$ [PW, §6].

To do the actual computation we use the decomposition above to compute to the highest degree in which $IH^*(\mathfrak{M}_n)$ is nonzero. We do this as $IH_T^*(\mathfrak{M}) \cong IH^*(\mathfrak{M}) \otimes H_T^*(pt)$. This means many of the terms corresponding to particular types of partitions can be eliminated from the computation. The following result shows that we only need to consider the components of the decomposition corresponding to partitions with $\varphi(\mathcal{S}) < n - 2$.

Lemma 4.6. As the top non-zero degree of $IH^*(\mathfrak{M}_n)$ is $2n-4$, the only partitions $\mathcal{S}$ of $\{1,\ldots,n\}$ whose corresponding components in the above decomposition that contribute in degree less than or equal to $2n-4$ have the property that $\varphi(\mathcal{S}) < n-2$.

Proof. The value $\varphi(\mathcal{S})$ is the dimension of the ambient vector space $H_{\mathcal{S}}$ of the restricted arrangement $(\mathcal{B}_n)_{\mathcal{S}}$. If the ambient vector space of a hyperplane arrangement $\mathcal{A}$ has dimension d, then every collection of $d+1$ normal vectors is linearly dependent. Hence, every collection of d normal vectors is a broken circuit. Hence, $IH^*(\mathfrak{M}(\mathcal{A}))$ has top degree $2d-2$ by [PW, 4.3] and thus $IH^*_c(\mathfrak{M}(\mathcal{A}))$ has bottom degree greater than or equal to $2d+2$.

Hence, if $\varphi(\mathcal{S}) \geq n-2$, then the minimal degree of $IH^*_c(\mathfrak{M}((B_n)_{\mathcal{S}}))$ is greater than or equal to $2n-2$. □

4.3. Examples

In this section, we will decompose $IH^*(\mathfrak{M}_4)$ as a representation of S_4 and $IH^2_T(\mathfrak{M}_6)$ as a representation of S_6. Recall, we computed the second example previously in Chapter III.

Example 4.7. To exhibit this program for computation, we provide the example of the braid arrangement $\mathcal{B}_4$. According to the above lemma, we only need to compute this for partitions of type $(1,1,1,1)$ and $(2,1,1)$ as $\varphi(\mu) < 2$ in each of these cases. Figure 4.1. illustrates the intersection lattice and its correspondence with partitions $\mathcal{S}$ of $\{1,2,3,4\}$. It also indicates the value of $\varphi(\mathcal{S})$ of each partition.

In this example, as gradings are important, we will represent the result as a polynomial in q where q is in degree 2 and the coefficients are in the representation ring of S_4. We will use the correspondence $V(0)=\tau$, $V(1)=\rho$, $V(2)=\chi$, $V(1,1)=\Lambda^2\rho$, and $V(1,1,1)=\sigma$.

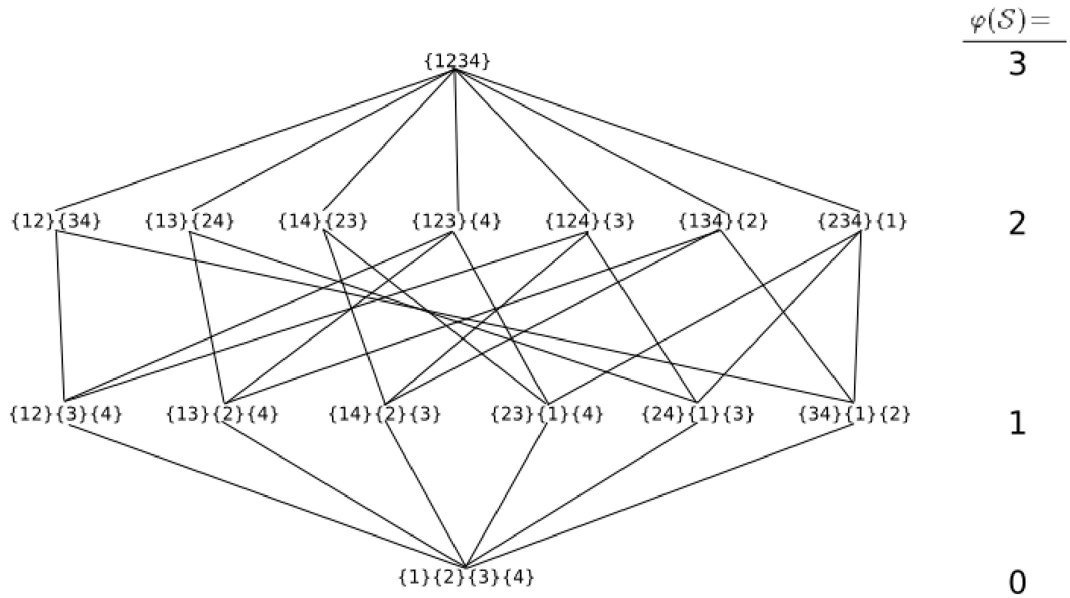

FIGURE 4.1. The intersection lattice of $\mathcal{B}_4$.

For the partitions of $\{1\}, \{2\}, \{3\}, \{4\}$ of type $\mu = (1,1,1,1)$ there is only one representative, and that is $\mathcal{S} = \{1\}\{2\}\{3\}\{4\}$. For this partition, we get that $\mathcal{B}_4^{\mathcal{S}}$ is the entire arrangement B_4, $H_{\mathcal{S}} = \{pt\}$, and $IH_c^*(\mathfrak{M}((\mathcal{B}_4)_{\mathcal{S}}))$ is isomorphic to $\mathbb{C}$ in degree 0. Hence, we just need to calculate the equivariant Poincaré polynomial for $H^*(M_3(\mathcal{B}_4))$.

$$P(H^*(M_3(\mathcal{B}_4)), q) = \tau + (\rho + \wedge^2\rho)q + (\rho + \sigma + 2\chi + \wedge^2\rho)q^2 + (\rho + \Lambda^2\rho)q^3$$

For the partitions of the type $\mu = (2,1,1)$, we choose representative $\mathcal{S} = \{12\}\{3\}\{4\}$. In this case, $(\mathcal{B}_4)^{\mathcal{S}} \cong \mathcal{B}_3$ and $(\mathcal{B}_4)_{\mathcal{S}}$ consists of a point in a line. The three components corresponding to this partition type are

$$H^*(M_3(\mathcal{B}_4^{\mathcal{S}})) \cong \mathbb{C}[x_{13}, x_{14}, x_{34}]/\langle x_{ij}^2, x_{13}x_{14} - x_{13}x_{34} + x_{14}x_{34}\rangle,$$

the polynomial algebra

$$\mathrm{sym}(H_{\mathcal{S}}) = \mathrm{sym}(\mathfrak{t}/H_{12}) = \mathbb{C}[x_{12}],$$

and

$$IH_c^*(\mathfrak{M}((\mathcal{B}_4)_{\mathcal{S}})) \cong \mathbb{C}$$

which is acted on trivially.

The stabilizer of the action of S_n on this partition is

$$W_{\mathcal{S}} = \langle (12), (34) \rangle.$$

This group has character table given by

	e	(12)	(34)	(12)(34)
χ_{12}	1	-1	1	-1
χ_{34}	1	1	-1	-1
$\chi_{12} \otimes \chi_{34}$	1	-1	-1	1

Calculating the equivariant Poincaré polynomial (with coeffs in the representation ring) of these components yields:

$$P(B(\mathcal{B}_4^{\mathcal{S}}), q) = \tau + (2\chi_{34} + \tau)q + (\tau + \chi_{34})q^2$$

$$P(\mathrm{sym}(H_{\mathcal{S}}), q) = \tau + \chi_{12}q + \tau q^2 + \chi_{12}q^3 + \cdots = \frac{1}{1 - \chi_{12}q}$$

$$P(IH_c^*(\mathfrak{M}((\mathcal{B}_n)_{\mathcal{S}})), q) = \tau q^2.$$

Tensoring these components yields a Poincaré polynomial whose given by

$$P(B(\mathcal{B}_4^{\mathcal{S}}) \otimes \mathrm{sym}(H_{\mathcal{S}}) \otimes IH_c^*(\mathfrak{M}((\mathcal{B}_4)_{\mathcal{S}}), q) = \tau q^2 + \cdots .$$

To get all of the partitions of this type, we induce from $W_{\mathcal{S}}$ up to W using the Weyl character formula. This decomposes as

$$\mathrm{Ind}_{W_{\mathcal{S}}}^{W}(\tau) = \rho + \chi + \tau$$

and thus

$$P\left(\bigoplus_{\overline{\mathcal{S}}=(2,1,1)} B(\mathcal{B}_4^{\mathcal{S}}) \otimes \mathrm{sym}(V_{\mathcal{S}}) \otimes IH_c^*(\mathfrak{M}((\mathcal{B}_4)_{\mathcal{S}}), q\right) = (\rho + \chi + \tau)q^2 + \cdots .$$

Hence,

$$P(IH_{T^d}^*(\mathfrak{M}(\mathcal{B}_4); \mathbb{C}), q) = \tau + (\rho + \wedge^2\rho)q + (2\rho + \sigma + 3\chi + \wedge^2\rho + \tau)q^2 + \cdots .$$

By equivariant formality, since $IH_{T^d}^*(\mathfrak{M}(\mathcal{B}_4); \mathbb{C}) \cong H^*(\mathfrak{M}(\mathcal{B}_4); \mathbb{C}) \otimes \mathrm{sym}(\rho)$, then

$$IH^0(\mathfrak{M}(\mathcal{B}_4); \mathbb{C}) \cong \tau$$

as $\mathrm{sym}^0(\rho) \cong \tau$.

For the degree 2 portion,

$$IH^2_{T^d}(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \cong IH^0(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \otimes \mathrm{sym}^1(\rho) \oplus IH^2(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \otimes \mathrm{sym}^0(\rho)$$

$$\cong \rho \oplus IH^2(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}).$$

Hence,

$$IH^2(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \cong \wedge^2\rho.$$

Finally, for the piece in degree 4,

$$IH^4_{T^d}(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \cong$$

$$IH^0(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \otimes \mathrm{sym}^2(\rho) \oplus IH^2(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \otimes \mathrm{sym}^1(\rho) \oplus IH^4(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \otimes \mathrm{sym}^0(\rho).$$

Thus,

$$IH^4(\mathfrak{M}(\mathcal{B}_4);\mathbb{C}) \cong \chi.$$

Therefore the Poincaré polynomial for $IH^*_T(\mathfrak{M}_4;\mathbb{C})$ is

$$P(IH^*_T(\mathfrak{M}_4;\mathbb{C}), q) = (\tau q + \wedge^2\rho q + \chi q^2)\left(\sum_{i=1}^{\infty} \mathrm{sym}^i(\rho) q^i\right).$$

In the next example, we decompose $C^2(\mathcal{B}_6) \cong IH^4_T(\mathfrak{M}_6)$ as was computed in Chapter III.

Example 4.8. Following the proof of Lemma 4.6, we only need to consider the pieces of the decomposition corresponding to the partitions $(1,1,1,1,1,1)$ and $(2,1,1,1,1)$.

This gives us

$$IH_T^*(\mathfrak{M}_6) \cong H^*(M_3(\mathcal{B}_6)) \oplus \mathrm{Ind}_{S_2\times S_4}^{S_6}\left(H^*(M_3(\mathcal{B}_5)) \otimes \mathbb{C}[x_{12}] \otimes IH^*((\mathfrak{M}_n)_{\mathcal{S}})\right).$$

The right-hand summand we have already computed mostly in the previous example. As we saw, $\mathbb{C}[x_{12}] \otimes IH^*((\mathfrak{M}_n)_{\mathcal{S}})$ has lowest nonzero degree 4, so we only need to compute $H^0(M_3(\mathcal{B}_5))$. This is trivial as an $S_2 \times S_4$ representation, so the right-hand summand in degree 4 is

$$\mathrm{Ind}_{S_2\times S_4}^{S_6}\left(H^0(M_3(\mathcal{B}_5)) \otimes \mathbb{C} \otimes IH^4((\mathfrak{M}_n)_{\mathcal{S}})\right) \cong_{S_6} \mathrm{Ind}_{S_2\times S_4} V(0) \cong_{S_6} V(0)+V(1)+V(2).$$

The degree 4 part of the left-hand summand $H^*(M_3(\mathcal{B}_6))$ decomposes as

$$H^4(M_3(\mathcal{B}_6)) \cong_{S_6} V(1)+2V(2)+V(1,1)+V(1,1,1)+2V(2,1)+V(2,2)+V(1,1,1,1).$$

This matches our result from Chapter III.

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