SEMISIMPLICITY OF CERTAIN REPRESENTATION CATEGORIES

by

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DISSERTATION ABSTRACT

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We exhibit a correspondence between subcategories of modules over an algebra and sub-bimodules of the dual of that algebra. We then prove that the semisimplicity of certain such categories is equivalent to the existence of a Peter-Weyl decomposition of the corresponding sub-bimodule. Finally, we use this technique to establish the semisimplicity of certain finite-dimensional representations of the quantum double $D(\mathrm{U}_q(\mathfrak{sl}_2))$ for generic q.

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INTRODUCTION

Often in studying representation theory we find that certain representation categories are semisimple. This is an illuminating property, since in such categories it suffices to study the simple objects. The research presented in this paper is motivated by the desire to expand our available tools for proving semisimplicity, with a special focus on representations of the quantum double.

A familiar result in this direction is that representations of semisimple Lie algebras are semisimple [1, 28]. Pivotal to the proof of this theorem is the existence of a *Casimir* element with certain properties. The proof generalizes nicely to quantized enveloping algebras [2, 587–589]. In Section 3.3 we further generalize the proof to a Hopf algebra H, which covers both cases.

Theorem 1.1. Let H be a Hopf algebra and let $\mathcal{C}$ be an Abelian category of finite-dimensional H-modules which is closed under extension. Suppose that H has a Casimir element which acts by 0 on a simple module V if and only if V is the trivial module. If the extension of the trivial module by itself is 0, then $\mathcal{C}$ is semisimple.

In cases where the center of H is not well understood and no such Casimir element is known, we choose to pursue a different approach. Matrix coefficients of representations of Lie groups were first described by Élie Cartan, and they were used by Fritz Peter and Hermann Weyl in the 1920's to decompose representations of compact topological groups in their famous Peter-Weyl theorem. Israel Gelfand continued using matrix coefficients of representations to bring new insight to several classical problems. Their work is the inspiration for our approach.

In Section 3.1 we describe a correspondence between algebra representation categories and sub-bimodules of the dual of the algebra. We show that if A is an algebra and V is a finite-dimensional left A-module, then there is a bimodule morphism

$$\beta_V \colon V \otimes V^* \to A^* \qquad \text{given by} \qquad \beta_V(v \otimes \zeta)(a) = \zeta(a \triangleright v) \tag{1.1}$$

We thus view A^* as the best place to look for A-modules, and we examine some other properties of this correspondence.

In Section 3.2 we establish a Peter-Weyl-type theorem that makes use of this correspondence to prove semisimplicity of a category.

Theorem 1.2. Let B be a bialgebra and $\mathcal{C}$ be an Abelian category of finite-dimensional B-modules. Then $\mathcal{C}$ is semisimple if and only if the image $B^*_{\mathcal{C}}$ of $\mathcal{C}$ under the correspondence (1.1) has Peter-Weyl decomposition

$$B^*_{\mathcal{C}} = \bigoplus_V \beta_V(V \otimes V^*)$$

as an internal direct sum over all isomophism classes in $\mathcal{C}$.

Our goal is to make use of these ideas to establish semisimplicity in a new situation. It is well-known that if H is a finite-dimensional Hopf algebra and if representations of H and H^* are semisimple, then so are representations of the quantum double $D(H)$ [3, 193]. This is not necessarily true when H is infinite-dimensional; for example, not all finite-dimensional representations of $D(\mathrm{U}(\mathfrak{sl}_2))$ are semisimple. Neither are all finite-dimensional representations of $\mathrm{U}_q(\mathfrak{sl}_2)$, nor of its double $D(\mathrm{U}_q(\mathfrak{sl}_2))$, when q is specialized to a root of unity. However, thanks to our conversations with Victor Ostrik, there is a conjecture (when q is generic) which appears to be very difficult and to be open even when $\mathfrak{g} = \mathfrak{sl}_2$.

Main Conjecture 1.3. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every finite-dimensional representation of $D(\mathrm{U}_q(\mathfrak{g}))$ is semisimple.

In Section 3.4 we further conjecture what the simple $D(\mathrm{U}_q(\mathfrak{g}))$-modules are, as we now describe. Suppose that H is a Hopf algebra with invertible quasi-triangular structure R. Given a left H-module V, we can construct two left $D(H)$-modules V^+ and V^- using R and R^{-1}, respectively.

Lemma 1.4. Suppose that all left H-modules are semisimple, that $V \otimes V^*$ is semisimple for any simple H-module V, and that $V^- \ncong V^+$ when V is non-trivial. If U and V are simple H-modules, then the $D(H)$-module $U^+ \otimes V^-$ is simple.

It is unclear whether we have accounted for all simple $D(H)$-modules; although the additive span of such $U^+ \otimes V^-$ is closed under tensor multiplication, we do not know whether it is closed under extension. In our conversations with Victor Ostrik, however, we discussed this conjecture.

Conjecture 1.5. Let $\mathfrak{g}$ be a semisimple Lie algebra. The simple $D(\mathrm{U}_q(\mathfrak{g}))$-modules are, up to isomorphism, the modules $V_\lambda^+ \otimes V_\mu^- \otimes U_0$ where λ and μ are dominant integral $\mathfrak{g}$-weights and U_0 belongs to the (finite) set of one-dimensional $D(\mathrm{U}_q(\mathfrak{g}))$-modules.

Because little is known about the center of $D(\mathrm{U}_q(\mathfrak{g}))$, we have little hope of applying Theorem 1.1. Our main conjecture appears to be very difficult, so even a little progress would be quite helpful. In this paper we consider the semisimple Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ and attempt to apply Theorem 1.2.

Remark 1.6. One advantage of this strategy is that B^* is not just a left B-module, but also a right B-module. In the case of $D(H)$, the right action binds together the multiplicities of each left $D(H)$-module into a single $D(H)$-bimodule without multiplicity.

Now when H is infinite-dimensional, the structure of $D(H)^*$ is very complicated. If C is the finite dual of H, then $D(H)$ is the coalgebra $C \otimes H$ with the necessary algebra structure to make it into a Hopf algebra, as we explain in Section 2.5. It follows that the Hopf algebra $D(H)^*$ contains $H \otimes C$ as a subalgebra with the tensor algebra structure. The action of $D(H)$ on $H \otimes C$ is determined completely by the Hopf pairing of H and C. Now when H is finite-dimensional, $H \otimes C = D(H)^*$. It is surprising that $H \otimes C$ is still an interesting object in the infinite-dimensional case, as we show in Section 4.1.

Theorem 1.7. The subalgebra $H \otimes C \subset D(H)^*$ is a sub-bimodule.

In Section 3.1 we define $(H \otimes C)_f$ to be the sub-bimodule of elements of $H \otimes C$ which generate finite-dimensional $D(H)$-bimodules, and we note that this space is also an algebra. We note that $(H \otimes C)_f$ is not a new object. In fact, for any Hopf algebra H, the subalgebra $(H \otimes C)_f$ is $(H \otimes C) \cap D(H)^\circ$.

In Section 4.2 we focus our attention on the Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$. Now for finite-dimensional H, an element of $H \otimes C$ is locally-finite under the actions of $D(H)$ if and only if it is locally-finite under the actions of H. It is very helpful that the same holds for $H = \mathrm{U}_q(\mathfrak{sl}_2)$.

Proposition 1.8. An element of $\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2]$ is locally-finite under the actions of $D(\mathrm{U}_q(\mathfrak{sl}_2))$ if and only if it is locally-finite under the actions of $\mathrm{U}_q(\mathfrak{sl}_2)$.

Just as highest-weight vectors are key to the study of $\mathrm{U}_q(\mathfrak{sl}_2)$-modules, so too are highest-weight bivectors (those which are highest-weight on both the left and the right) key to achieving these results. We use Remark 1.6 extensively. We show that one particular $D(\mathrm{U}_q(\mathfrak{sl}_2))$-bimodule, which we call $H_{1,1}$, has a set of four canonical highest-weight bivectors $\{v_1, v_2, v_3, v_4\}$ which plays a very important role in the representation theory of $D(\mathrm{U}_q(\mathfrak{sl}_2))$.

Proposition 1.9. The subalgebra of highest-weight bivectors in $\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2]$ is locally-finite and is generated as an algebra by $\{v_1, v_2, v_3, v_4\}$.

This result helps us to prove our main theorem. We write $H_{\lambda,\mu}$ to denote the image of $\beta_{V_\lambda^+ \otimes V_\mu^-}$ from (1.1).

Main Theorem 1.10. As a $D(\mathrm{U}_q(\mathfrak{sl}_2))$-bimodule,

$$\big(\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2]\big)_f = \bigoplus_{\substack{\lambda,\mu \geq 0 \\ \lambda-\mu \in 2\mathbb{Z}}} H_{\lambda,\mu}$$

and this is a Peter-Weyl decomposition.

By Theorem 1.2 this proves semisimplicity of a substantial subcategory of finite-dimensional $D(\mathrm{U}_q(\mathfrak{sl}_2))$-modules.

At the end of Section 4.2 we give a presentation and basis for the algebra ${}^+(\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2])^+$. This algebra is quite surprising to us and piques our curiosity about the presentation of the entire algebra of locally-finite bivectors. We also generalize our main results to conjectures for semisimple Lie algebras $\mathfrak{g} \neq \mathfrak{sl}_2$.

CHAPTER II

BACKGROUND

2.1. Hopf algebras, modules, and comodules

The material in this section is well known, and can be found in many sources, such as Chapter 1 of [4]. In all definitions, k is a field.

Definition 2.1. An algebra A over the field k is a k-vector space with a multiplication $\mu\colon A \otimes A \to A$ satisfying $\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$ and a unit $\eta\colon k \to A$ satisfying $\mu(\eta(1), a) = \mu(a, \eta(1)) = a$, for all $a, b, c \in A$.

In more familiar notation, these conditions are written $(ab)c = a(bc)$ and $1a = a1 = a$, and associativity makes the product abc unambiguous. We will usually use this more familiar notation, but the advantage of the notation used in the definition is that the conditions can be expressed using commutative diagrams, as in Figure 2.1.

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1 \otimes \mu"] \arrow[d, "\mu \otimes 1"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[dr, "\mu"] & k \otimes A \arrow[l, "\eta \otimes 1"'] \arrow[d, "="] \\
A \otimes k \arrow[u, "1 \otimes \eta"] \arrow[r, "="] & A
\end{tikzcd}
$$

FIGURE 2.1. Conditions on the product and unit of an algebra

This makes it easy to define a dual notion, that of a coalgebra over k.

Definition 2.2. A coalgebra C over the field k is a k-vector space with a comultiplication $\Delta\colon C \to C \otimes C$ satisfying $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ and a counit $\varepsilon\colon C \to k$ satisfying $(1 \otimes \varepsilon)(\Delta c) = (\varepsilon \otimes 1)(\Delta c) = c$ for all $c \in C$.

$$\begin{tikzcd} C \otimes C \otimes C & C \otimes C \arrow[l, "1 \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes 1"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} C \otimes C \arrow[r, "\varepsilon \otimes 1"] \arrow[d, "1 \otimes \varepsilon"'] & k \otimes C \\ C \otimes k & C \arrow[l, "="'] \arrow[u, "="'] \arrow[ul, "\Delta"'] \end{tikzcd}$$

FIGURE 2.2. Conditions on the coproduct and counit of a coalgebra

We use Sweedler's notation $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$, but suppress the summation symbol, so we write $\Delta(c) = c_{(1)} \otimes c_{(2)}$. In this notation, the counit condition is $c_{(1)}\varepsilon(c_{(2)}) = \varepsilon(c_{(1)})c_{(2)} = c$, and the coassociativity condition is

$$(c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}$$

which makes the expression $\Delta^2 c = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ unambiguous.

Definition 2.3. An element c of a coalgebra is called *group-like* if $\Delta c = c \otimes c$.

Definition 2.4. A bialgebra B over the field k is a k-vector space that is both an algebra and a coalgebra, where Δ and ε are algebra homomorphisms. Equivalently, μ and η are coalgebra homomorphisms.

Definition 2.5. A Hopf algebra H is a bialgebra with an *antipode* $S \colon H \to H$ such that $S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \varepsilon(h)$.

The above condition can be expressed with the commutative diagram shown in Figure 2.3.

Definition 2.6. Let A be an algebra over k. A left A-module V is a k-vector space together with an action $\triangleright \colon A \otimes V \to V$ satisfying $1 \triangleright v = v$ and $ab \triangleright v = a \triangleright (b \triangleright v)$ for all $a, b \in A$ and $v \in V$.

$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & k \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "S \otimes 1, 1 \otimes S"] & & H \otimes H \arrow[u, "\mu"']
\end{tikzcd}$$

FIGURE 2.3. Condition on the antipode of a Hopf algebra

Definition 2.7. Let C be a coalgebra over k. A left C-comodule V is a k-vector space together with a coaction $\delta\colon V \to C \otimes V$ satisfying $(\varepsilon \otimes 1)(\delta v) = v$ and $(1 \otimes \delta) \circ \delta = (\Delta \otimes 1) \circ \delta$ for all $v \in V$.

We write $\delta(v) = v^{(-1)} \otimes v^{(0)}$, with the summation symbol suppressed. Then the first condition above can be written $\varepsilon(v^{(-1)})v^{(0)} = v$, and the second condition justifies using the notation $\delta^2 v = v^{(-2)} \otimes v^{(-1)} \otimes v^{(0)}$.

Lemma 2.8. Let A be an algebra. Then A^* is an A-bimodule with left action $(a \triangleright f)(a') = f(aa')$ and right action $(f \triangleleft a)(a') = f(a'a)$.

Lemma 2.9. The category of algebras is monoidal, with product

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

and unit $1 \otimes 1$.

Lemma 2.10. The category of coalgebras is monoidal, with coproduct

$$\delta(a \otimes b) = a^{(-1)}b^{(-1)} \otimes a^{(0)} \otimes b^{(0)}$$

and counit $\varepsilon(a \otimes b) = \varepsilon(a)\varepsilon(b)$.

Lemma 2.11. Let B be a bialgebra. Then the category of left B-modules is monoidal, when we define

$$b \triangleright (u \otimes v) = (b_{(1)} \triangleright u) \otimes (b_{(2)} \triangleright v).$$

Lemma 2.12. Let B be a bialgebra. Then the category of left B-comodules is monoidal, when we define

$$\delta(u \otimes v) = u^{(-1)} v^{(-1)} \otimes u \otimes v.$$

2.2. Examples of bialgebras and Hopf algebras

The construction we use here for $\mathrm{U}_q(\mathfrak{g})$ is detailed in [4, 92–94]. Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{t}$ be a Cartan subalgebra, and $\mathfrak{t}^*$ be its dual linear space. Let $\alpha_i \in \mathfrak{t}^*$ be a system of positive simple roots. If $(\ ,\ )$ is the symmetric bilinear form on $\mathfrak{t}^*$ derrived from the inverse of the Killing form, and $\check{\alpha}_i = 2\alpha_i/(\alpha_i, \alpha_i)$ are the coroots, then $a_{ij} = (\check{\alpha}_i, \alpha_j)$ is the Cartan matrix. Define $d_i = (\alpha_i, \alpha_i)/2$, which is always an integer. Then $\mathrm{U}_q(\mathfrak{g})$ can be defined over the field $\mathbb{C}(q)$ with generators $\{K_i^{\pm 1}, E_i, F_i\}$, with $q_i = q^{d_i}$.

Definition 2.13. Let $\mathfrak{g}$ be a complex simple Lie algebra with the structure described above. We define $\mathrm{U}_q(\mathfrak{g})$ to be the Hopf algebra generated by $\{K_i^{\pm 1}, E_i, F_i\}$ with relations $[K_i, K_j] = 0$,

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k = 0, \quad \forall i \neq j,$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k = 0, \quad \forall i \neq j.$$

and comultiplication, counit, and antipode maps

$$\Delta K_i = K_i \otimes K_i, \quad \Delta E_i = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$SK_i = K_i^{-1}, \quad SE_i = -E_i K_i^{-1}, \quad SF_i = -K_i F_i.$$

Remark 2.14. For the specific example $\mathrm{U}_q(\mathfrak{sl}_2)$, we note that there is a Casimir element

$$\Delta = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

which belongs to the center of $\mathrm{U}_q(\mathfrak{sl}_2)$.

If V is a left $\mathrm{U}_q(\mathfrak{g})$-module, we say that $v \in V$ is *highest-weight* if $E_i \triangleright v = 0$ for all i, and we denote the subspace of highest-weight vectors by ${}^+V$. We define V^+ similarly for right $\mathrm{U}_q(\mathfrak{g})$-modules. The following result is a consequence of ΔE_i and Lemma 2.11.

Lemma 2.15. If V is a $\mathrm{U}_q(\mathfrak{g})$-module, then ${}^+V$ (resp. V^+) is a subalgebra.

There is another very useful lemma about highest-weight vectors. We say that a module V is *locally finite* if every $v \in V$ generates a finite-dimensional submodule.

Lemma 2.16. If V is a locally finite $\mathrm{U}_q(\mathfrak{g})$-module, then ${}^+V$ (resp. V^+) generates V.

Another class of bialgebras is quantum matrices; the construction we use is detailed in [5, 5–6]. Let X_{ij} denote the coordinate functions on the space of $m \times n$ matrices. We consider the tensor algebra $T(\mathrm{Mat}_{m\times n})$ with generators X_{ij}, $1 \le i \le m$, $1 \le j \le n$. This tensor algebra is a bialgebra if we define comultiplication and counit on generators by

$$\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj} \quad \text{and} \quad \varepsilon(X_{ij}) = \delta_{i,j} \tag{2.1}$$

and extended these as algebra maps.

Consider the free algebra V on n generators $e_1, \dots, e_n$. This algebra is a comodule over the bialgebra $T(\mathrm{Mat}_{n\times n})$ with coaction

$$\delta(e_i) = \sum_{j=1}^{n} X_{ij} \otimes e_j \tag{2.2}$$

The bialgebra of quantum matrices $\mathbb{C}_q[\mathrm{Mat}_{n\times n}]$ is a quotient of the tensor bialgebra $T(\mathrm{Mat}_{n\times n})$ such that both the quantum symmetric algebra the quantum exterior algebra are comodules. More precisely, we define the symmetric algebra $S_q(V)$ and exterior algebra $\Lambda_q(V)$, respectively, by

$$S_q(V) = T(V)/(e_je_i - qe_ie_j \mid i < j),\ \Lambda_q(V) = T(V)/(e_i \wedge e_j + q^{-1}e_j \wedge e_i \mid i \leq j).$$

The following result is well-known; see e.g. [5].

Theorem 2.17. There exists a quadratic bi-ideal I in the bialgebra $T(\mathrm{Mat}_{n\times n})$ such that (2.2) extends to algebra homomorphisms

$$S_q(V) \to \big(T(\mathrm{Mat}_{n\times n})/I\big) \otimes S_q(V),\ \Lambda_q(V) \to \big(T(\mathrm{Mat}_{n\times n})/I\big) \otimes \Lambda_q(V) \tag{2.3}$$

We define $\mathbb{C}_q[\mathrm{Mat}_{n\times n}]$ to be the quotient of $T(\mathrm{Mat}_{n\times n})$ by the minimal such bi-ideal I.

Since the coactions (2.3) are homogeneous, we may restrict the latter coaction to the top power $\big(\Lambda_q(V)\big)^n$, which is 1-dimensional. The image of this restriction is spanned by a central, group-like element of $\mathbb{C}_q[\mathrm{Mat}_{n\times n}]$ which we call the quantum determinant $\det_q(\mathrm{Mat}_{n\times n})$ of $\mathbb{C}_q[\mathrm{Mat}_{n\times n}]$. If the quantum determinant is made to be invertible, this will produce an antipode.

Definition 2.18. The Hopf algebra $\mathbb{C}_q[\mathrm{GL}_n]$ is the localization of the quantum matrix bialgebra $\mathbb{C}_q[\mathrm{Mat}_{n\times n}]$ at the quantum determinant $\det_q(\mathrm{Mat}_{n\times n})$.

Definition 2.19. The Hopf algebra $\mathbb{C}_q[\mathrm{SL}_n]$ is the quotient of the quantum matrix bialgebra $\mathbb{C}_q[\mathrm{Mat}_{n\times n}]$ by imposing $\det_q(\mathrm{Mat}_{n\times n}) = 1$.

Example 2.20. We give a presentation of $\mathbb{C}_q[\mathrm{SL}_2]$ with the four generators $a = X_{11}$, $b = X_{12}$, $c = X_{21}$, and $d = X_{22}$. The algebra relations are

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd,$$
$$cb = bc, \quad ad - q^{-1}bc = da - qbc = 1.$$

The comultiplication and count are given in (2.1), and the antipode S is given by

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

which is read entry-wise, so $S(b) = -qb$, for example.

2.3. Hopf pairings and actions

Definition 2.21. Let H and C be Hopf algebras over a field k. A *Hopf pairing* of H and C is a map $\phi\colon C \otimes H \to k$ such that

$$\phi(c, hh') = \phi(c_{(1)}, h)\phi(c_{(2)}, h'),$$

$$\phi(cc', h) = \phi(c, h_{(1)})\phi(c', h_{(2)}),$$

and $\phi(1, h) = \epsilon(h)$, $\phi(c, 1) = \epsilon(c)$, and $\phi(Sc, h) = \phi(c, Sh)$.

We can now define left and right actions of dually paired Hopf algebras on each other.

Proposition 2.22. Let H and C be Hopf algebras over a field k, and let $\phi\colon C\otimes H \to k$ be a Hopf pairing. Then

$$h \triangleright c = c_{(1)}\phi(c_{(2)}, h) \qquad \text{and} \qquad c \triangleleft h = c_{(2)}\phi(c_{(1)}, h)$$

are left and right actions of H on C, respectively, and

$$c \triangleright h = h_{(1)}\phi(c, h_{(2)}) \qquad \text{and} \qquad h \triangleleft c = h_{(2)}\phi(c, h_{(1)})$$

are left and right actions of C on H, respectively. Furthermore, these actions and the pairing ϕ satisfy the following relations:

$$\phi(c, hh') = \phi(c \triangleleft h, h')$$

$$\phi(cc', h) = \phi(c, c' \triangleright h)$$

If we know the actions, then we can reconstruct the pairing, as we demonstrate in the following example.

Example 2.23. There is a left action of $\mathrm{U}_q(\mathfrak{gl}_n)$ on $\mathbb{C}_q[\mathrm{GL}_n]$ given by

$$E_i \triangleright X_{k\ell} = \delta_{i+1,\ell} X_{k,\ell-1}$$

$$F_i \triangleright X_{k\ell} = \delta_{i,\ell} X_{k,\ell+1}$$

$$K_i \triangleright X_{k\ell} = (q\delta_{i,\ell} + q^{-1}\delta_{i+1,\ell}) X_{k\ell}$$

This action defines a Hopf pairing

$$\phi(E_i, X_{k\ell}) = \phi(1, E_i \triangleright X_{k\ell}) = \varepsilon(\delta_{i+1,\ell} X_{ki}) = \delta_{i,k,\ell-1}$$

$$\phi(F_i, X_{k\ell}) = \phi(1, F_i \triangleright X_{k\ell}) = \varepsilon(\delta_{i,\ell} X_{k,\ell+1}) = \delta_{i,k-1,\ell}$$

$$\phi(K_i, X_{k\ell}) = \phi(1, q\delta_{i,\ell} X_{k\ell} + q^{-1}\delta_{i+1,\ell} X_{k\ell}) = q\delta_{i,k,\ell} + q^{-1}\delta_{i+1,k,\ell}$$

where $\delta_{i,j,k} = 1$ if $i = j = k$, and $\delta_{i,j,k} = 0$ otherwise. The right action is then

$$\delta_{i,k} X_{k+1,\ell} = X_{k\ell} \triangleleft E_i$$

$$\delta_{i,k-1} X_{k-1,\ell} = X_{k\ell} \triangleleft F_i$$

$$q^{-1}\delta_{i+1,k} X_{k\ell} + q\delta_{i,k} X_{k\ell} = X_{k\ell} \triangleleft K_i$$

We can also compute that

$$X_{k\ell} \triangleright E_i = (q\delta_{i,k,\ell} + q^{-1}\delta_{i-1,k,\ell}) E_i + \delta_{i,k,\ell-1}$$

$$X_{k\ell} \triangleright F_i = \delta_{k,\ell} F_i + \delta_{i,k-1,\ell} K_i^{-1}$$

$$X_{k\ell} \triangleright K_i = (q\delta_{i,k,\ell} + q^{-1}\delta_{i+1,k,\ell}) K_i$$

and

$$\delta_{i,k,\ell-1}K_i + \delta_{k,\ell}E_i = E_i \triangleleft X_{k\ell}$$

$$\delta_{i,k-1,\ell} + (q^{-1}\delta_{i,k,\ell} + q\delta_{i+1,k,\ell})F_i = F_i \triangleleft X_{k\ell}$$

$$(q\delta_{i,k,\ell} + q^{-1}\delta_{i+1,k,\ell})K_i = K_i \triangleleft X_{k\ell}$$

In Example 2.37 and Section 5.1 we will use these actions to construct $D(\mathrm{U}_q(\mathfrak{sl}_n))$ and to determine the action of $D(\mathrm{U}_q(\mathfrak{sl}_n))$ on part of its dual.

2.4. Quasi-triangular structures and braidings

As a reference for the material in this section, see Chapter 2 of [4].

Definition 2.24. Let B be a bialgebra. A quasi-triangular structure on B is an element $R \in B \otimes B$, written $R^{(1)} \otimes R^{(2)}$ with summation understood, which is invertible and satisfies

$$(\Delta \otimes 1)R = R_{13}R_{23}$$

$$(1 \otimes \Delta)R = R_{13}R_{12}$$

$$\tau(\Delta b) = R(\Delta b)R^{-1}$$

for all $b \in B$, where $\tau(\Delta b) = b_{(2)} \otimes b_{(1)}$ and where $R_{12} = R^{(1)} \otimes R^{(2)} \otimes 1$, $R_{13} = R^{(1)} \otimes 1 \otimes R^{(2)}$, and $R_{23} = 1 \otimes R^{(1)} \otimes R^{(2)}$.

Example 2.25. The Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ has quasi-triangular structure

$$R = q^{\frac{H \otimes H}{2}} \left(\sum_{n=0}^{\infty} \frac{(1-q^{-2})^n}{[n]!} (qE \otimes F)^n \right)$$

where

$$[n] = \frac{1-q^{-2n}}{1-q^{-2}} \qquad \text{and} \qquad [n]! = [n][n-1]\cdots[1]$$

and $q^{\frac{1}{2}H\otimes H}(v \otimes v') = q^{\frac{1}{2}\langle v,v'\rangle}$, as shown in [4, 83].

Proposition 2.26. Let B be a bialgebra with quasi-triangular structure R, and let V be a left B-module. Then V is a left B-comodule with either of the coactions

$$\delta_+(v) = R^{(2)} \otimes (R^{(1)} \triangleright v), \quad \delta_-(v) = (R^{-1})^{(1)} \otimes \big((R^{-1})^{(2)} \triangleright v\big).$$

Definition 2.27. Let B be a bialgebra. A dual quasi-triangular structure on B is a convolution-invertible map $\mathcal{R}\colon B \otimes B \to k$ such that

$$\begin{aligned} \mathcal{R}(ab \otimes c) &= \mathcal{R}(a \otimes c_{(1)})\mathcal{R}(b \otimes c_{(2)}) \\ \mathcal{R}(a \otimes bc) &= \mathcal{R}(a_{(1)} \otimes c)\mathcal{R}(a_{(2)} \otimes b) \\ b_{(1)}a_{(1)}\mathcal{R}(a_{(2)} \otimes b_{(2)}) &= \mathcal{R}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} \end{aligned}$$

for all $a, b, c \in B$.

Example 2.28. The bialgebra $\mathbb{C}_q[\mathrm{SL}_n]$ has dual quasi-triangular structure given by

$$\mathcal{R}(X_{ij} \otimes X_{k\ell}) = \begin{cases} q & \text{if } i = j = k = \ell, \\ 1 & \text{if } i = j \neq k = \ell, \\ (q - q^{-1}) & \text{if } i = \ell < j = k, \\ 0 & \text{otherwise,} \end{cases}$$

as shown in [4, 132].

Definition 2.29. Let $\mathcal{C}$ be a category with an associative tensor product. We say that $\mathcal{C}$ is *braided* if it is provided with functorial isomorphisms

$$\psi_{U,V} \colon U \otimes V \to V \otimes U$$

that satisfy $\psi_{U\otimes V,W} = \psi_{U,W} \circ \psi_{V,W}$ and $\psi_{U,V\otimes W} = \psi_{U,W} \circ \psi_{U,V}$ for all objects U, V, W.

Proposition 2.30. Let B be a bialgebra with dual quasi-triangular structure $\mathcal{R}$. Then every left B-comodule is also a left B-module with action

$$b \triangleright v = \mathcal{R}(b \otimes v^{(-1)})v^{(0)}.$$

Furthermore, the category of left B-comodules is braided with

$$\psi(u \otimes v) = (u^{(-1)} \triangleright v) \otimes u^{(0)}.$$

2.5. Yetter-Drinfeld modules and the quantum double

Definition 2.31. Let H be a bialgebra. Then V is a left *Yetter-Drinfeld H-module* if V is both a left H-module and a left H-comodule and if the action and coaction satisfy the relation

$$h_{(1)}v^{(-1)} \otimes (h_{(2)} \triangleright v^{(0)}) = (h_{(1)} \triangleright v)^{(-1)}h_{(2)} \otimes (h_{(1)} \triangleright v)^{(0)}.$$

If H is a Hopf algebra, then this can be written

$$\delta(h \triangleright v) = h_{(1)}v^{(-1)}S(h_{(3)}) \otimes h_{(2)} \triangleright v^{(0)}.$$

Proposition 2.32. Let H be a Hopf algebra.

1. If H is quasi-triangular, then the coaction of Proposition 2.26 makes every H-module into a Yetter-Drinfeld module.

2. If H is dual-quasi-triangular, then the action of Proposition 2.30 makes every H-comodule into a Yetter-Drinfeld module.

Proposition 2.33. Let H be a bialgebra with a braided category $\mathcal{C}$ as in Proposition 2.30. If the objects of $\mathcal{C}$ are Yetter-Drinfeld modules, then $(1 \otimes \psi) \circ \delta = \delta \circ \psi$.

Definition 2.34. Let $\mathcal{C}$ be a monoidal category. The *Drinfeld center* of $\mathcal{C}$ is the monoidal category whose objects are objects X of $\mathcal{C}$ together with a natural isomorphism $\psi_X \colon X \otimes Y \to Y \otimes X$, where Y is any other object, such that $\psi_{X \otimes Y} = (\mathrm{id} \otimes \psi_Y) \circ (\psi_X \otimes \mathrm{id})$ for all X, Y.

Proposition 2.35. Let H be a bialgebra and let $\mathcal{C}$ be the category of left H-modules. Then an object of $\mathcal{C}$ is a Yetter-Drinfeld module if and only if it belongs to the Drinfeld center of $\mathcal{C}$.

To prove this, we already saw in Proposition 2.30 how the coaction can be used to produce a twisting. On the other hand, since H is a left H-module where the action is left multiplication, we can define $\delta(v) = \psi_V(v \otimes 1)$ where $1 \in H$.

If H is a Hopf algebra and $\mathcal{C}$ is the category of left H-modules, then the Drinfeld center of $\mathcal{C}$ is also the category of left modules over a Hopf algebra related to H, called the Drinfeld double, or quantum double, of H, which we now define.

Definition 2.36. Let H and C be Hopf algebras over a field k, and let $\phi \colon C \otimes H \to k$ be a Hopf pairing. We define a Hopf algebra called the *quantum double* $D(H)$ as follows. As a coalgebra, $D(H) = C \otimes H$ with the tensor coalgebra structure of

Lemma 2.10, and thus both C and H are sub-coalgebras of $D(H)$. As an algebra, C^{op} and H are subalgebras. Specifically, the multiplication $\cdot$ of $D(H)$ is given by $c \cdot c' = c'c$ for all $c, c' \in C$, by $h \cdot h' = hh'$ for all $h, h' \in H$, and by the cross-relation

$$h \cdot c = c_{(2)} \cdot h_{(2)} \phi(c_{(1)}, Sh_{(1)}) \phi(c_{(3)}, h_{(3)}).$$

We note that according to Proposition 2.22, the above is equivalent to

$$h \cdot c = (h_{(3)} \triangleright c \triangleleft Sh_{(1)}) \cdot h_{(2)}.$$

Example 2.37. The Hopf algebras $\mathrm{U}_q(\mathfrak{gl}_n)$ and $\mathbb{C}_q[\mathrm{GL}_n]$ are dually paired as shown in Example 2.23. Then as a coalgebra, $D(\mathrm{U}_q(\mathfrak{gl}_n)) = \mathbb{C}_q[\mathrm{GL}_n] \otimes \mathrm{U}_q(\mathfrak{gl}_n)$ as defined in Lemma 2.10. Also, $D(\mathrm{U}_q(\mathfrak{gl}_n))$ has subalgebras $\mathrm{U}_q(\mathfrak{gl}_n)$, as presented in Definition 2.13, and $\mathbb{C}_q[\mathrm{GL}_n]^{\mathrm{op}}$, as given in Definition 2.18 but with opposite multiplication. The cross-relations are

$$\begin{aligned}
E_i X_{k\ell} &= (q\delta_{i,\ell} + q^{-1}\delta_{i+1,\ell}) X_{k\ell} E_i - (q^2 \delta_{i,\ell} + \delta_{i+1,\ell}) \delta_{i,k} X_{k+1,\ell} K_i + \delta_{i+1,\ell} X_{k,\ell-1} \\
F_i X_{k\ell} &= (q^{-1}\delta_{i+1,k} + q\delta_{i,k}) X_{k\ell} F_i + (q^{-1}\delta_{i+1,k} + q\delta_{i,k}) \delta_{i,\ell} X_{k,\ell+1} K_i^{-1} - q^{-1} \delta_{i+1,k} X_{k-1,\ell} \\
K_i X_{k\ell} &= (q^2 \delta_{i+1,k,\ell+1} + \delta_{i,k,\ell} + \delta_{i+1,k,\ell} + q^{-2} \delta_{i+1,k+1,\ell}) X_{k\ell} K_i
\end{aligned}$$

Here $\delta_{i,j,k} = 1$ if $i = k = j$, and $\delta_{i,j,k} = 0$ otherwise.

We note that the quantum determinant of $\mathbb{C}_q[\mathrm{GL}_n]$ is central and group-like here, and the action by $\mathrm{U}_q(\mathfrak{gl}_n)$ on it is by the counit.

Example 2.38. In particular, the algebra $D(\mathrm{U}_q(\mathfrak{sl}_2))$ has cross-relations

$$
\begin{aligned}
&Ea = qaE - q^2cK && Ec = qcE \\
&Eb = q^{-1}bE - dK + a && Ed = q^{-1}dE + c \\
&Fa = qaF + qbK^{-1} && Fc = q^{-1}cF + q^{-1}dK^{-1} - q^{-1}a \\
&Fb = qbF && Fd = q^{-1}dF - q^{-1}b \\
&Ka = aK && Kc = q^2cK \\
&Kb = q^{-2}bK && Kd = dK
\end{aligned}
$$

It may be helpful in an investigation of highest-weight vectors to observe that c quasi-commutes with E.

We will also be interested in the dual $D(H)^*$ of the quantum double. If H is infinite-dimensional, then this Hopf algebra is very complicated. As we will see, even knowing the finite dual $D(H)^\circ$ is as complicated as knowing the entire category of finite-dimensional $D(H)$-modules. However, there is a Hopf subalgebra of $D(H)^*$ which is equal to $D(H)^*$ if H is finite-dimensional but is much easier to describe when H is infinite-dimensional.

Proposition 2.39. Let H and C be Hopf algebras over a field k, and let $\phi\colon C\otimes H \to k$ be a Hopf pairing. There is a subalgebra $H\otimes C \subset D(H)^*$ which has the tensor algebra structure of Lemma 2.9. (See [4, 345].)

We will suppress the tensor symbol when writing elements of $H \otimes C$.

Proposition 2.40. Let H be a Hopf algebra with quasi-triangular structure R. There is an embedding $\Phi_R\colon H\text{-mod} \hookrightarrow D(H)\text{-mod}$ which gives each H-module the structure of a $D(H)$-module.

To prove the above proposition we use the coaction $\delta(v) = R^{(2)} \otimes (R^{(1)} \triangleright v)$ as in Proposition 2.26 and then define the action of elements of c by

$$c \triangleright v = \phi(c, v^{(-1)})v^{(0)}$$

where ϕ is the pairing between C and H. In the same way we could show using Proposition 2.30 that if H has a dual quasi-triangular structure then there is an embedding of H-comod into $D(H)$-mod.

Because the Hopf algebra $\mathrm{U}_q(\mathfrak{sl}_2)$ is quasi-triangular, every $\mathrm{U}_q(\mathfrak{sl}_2)$-module is a $D(\mathrm{U}_q(\mathfrak{sl}_2))$-module. Because the quasi-triangular structure R is invertible, we can use either R or R^{-1} to construct a $D(\mathrm{U}_q(\mathfrak{sl}_2))$-module from an H-module.

Proposition 2.41. There are exactly n one-dimensional $D(\mathrm{U}_q(\mathfrak{sl}_n))$-modules.

Proof. A one-dimensional left $D(\mathrm{U}_q(\mathfrak{sl}_n))$-module is an algebra homomorphism $\phi\colon D(\mathrm{U}_q(\mathfrak{sl}_n)) \to k$. We know that the only one-dimensional $\mathrm{U}_q(\mathfrak{sl}_n)$-module is the trivial module, where

$$\phi(K_i) = \phi(K_i^{-1}) = 1, \quad \phi(E_i) = \phi(F_i) = 0.$$

We refer to the cross-relations in Example 2.37. By commuting E_i past $X_{k,i+1}$, we find that $\phi(X_{k,i}) = 0$ if $i \neq k$ and $\phi(X_{i+1,i+1}) = \phi(X_{i,i})$. By commuting F_i past $X_{i+1,n}$ we find that $\phi(X_{i,n}) = 0$. Thus

$$\phi(X_{k,k}) = \phi(X_{\ell,\ell}), \quad \phi(X_{k,\ell}) = 0 \quad \forall k \neq \ell.$$

However, the quantum determinant implies that $\prod_{k=1}^{n} \phi(X_{k,k}) = 1$. Thus if $\phi(X_{1,1})$ is any nth root of unity, then this determines the homomorphism. $\square$

CHAPTER III

SOME SEMISIMPLICITY RESULTS

3.1. A correspondence of subcategories and sub-bimodules

If B is an algebra and V is a left B-module, then V^* is a right B-module, and we can define a map $\beta_V \colon V \otimes V^* \to B^*$ so that $\beta_V(v \otimes f)$ is a linear function on B given by

$$\beta_V(v \otimes f)(b) = f(b \triangleright v) = (f \triangleleft b)(v) \tag{3.1}$$

for all $v \in V$, $f \in V^*$. We refer to $\beta_V(v \otimes f)$ a *matrix coefficient*.

Lemma 3.1. If U and V are isomorphic B-modules, then $\beta_U(U \otimes U^*) = \beta_V(V \otimes V^*)$.

Proof. Let $\phi \colon V \to U$ be an isomorphism of B-modules. Then

$$\begin{aligned} \beta_U(\phi v \otimes \phi^* f)(b) &= \phi^* f(b \triangleright \phi v) \\ &= \phi^* f(\phi(b \triangleright v)) \\ &= f(b \triangleright v) \\ &= \beta_V(v \otimes f) \end{aligned}$$

for any $v \in V$ and $f \in V^*$. □

Lemma 3.2. The maps $\{\beta_V \mid V \in B\text{-mod}\}$ in (3.1) are morphisms of B-bimodules, and $\beta_{U \oplus V} = \beta_U \oplus \beta_V$.

Proof. Let $b, b' \in B$, $v \in V$, and $f \in V^*$. We see that

$$\begin{aligned} \beta_V((b' \triangleright v) \otimes f)(b) &= f(b \triangleright (b' \triangleright v)) \\ &= f(bb' \triangleright v) \\ &= (f \triangleleft (bb'))(v) \\ &= \beta_V(v \otimes f)(bb') \\ &= (b' \triangleright \beta_V(v \otimes f))(b) \end{aligned}$$

and

$$\begin{aligned} \beta_V(v \otimes (f \triangleleft b'))(b) &= (f \triangleleft b')(b \triangleright v) \\ &= f(b'b \triangleright v) \\ &= (f \triangleleft b'b)(v) \\ &= \beta_V(v \otimes f)(b'b) \\ &= (\beta_V(v \otimes f) \triangleleft b')(b). \end{aligned}$$

Let U and V be B-modules, and let $u \in U$, $v \in V$, $f \in U^*$, and $g \in V^*$. Then

$$\begin{aligned} \beta_{U \oplus V}\big((u \oplus v) \otimes (f \oplus g)\big)(b) &= (f \oplus g)\big(b \triangleright (u \oplus v)\big) \\ &= f(b \triangleright u) + g(b \triangleright v) \\ &= \beta_U(u \otimes f)(b) + \beta_V(v \otimes g)(b) \end{aligned}$$

so $\beta_{U \oplus V} = \beta_U \oplus \beta_V$. □

This means that β's effectively ignore multiplicity:

Corollary 3.3. The image of $\beta_{V \oplus V}$ is equal to the image of β_V.

Lemma 3.4. If V is a simple B-module, then β_V is injective.

Proof. We have $\beta_V(v \otimes f)(B) = 0$ if and only if $f(B \triangleright v) = 0$. This is true if and only if $v = 0$ or $f = 0$, i.e. if and only if $v \otimes f = 0$. $\square$

We now define a correspondence between additive subcategories of B-mod and sub-bimodules of B^*. For any additive category $\mathcal{C}$ of B-modules, we denote by $B^*_{\mathcal{C}}$ the span of the images of $\{\beta_V \mid V \in \mathcal{C}\}$. On the other hand, given a sub-bimodule D of B^*, we define $\mathcal{C}(D)$ to be the full subcategory of objects $V \in \mathcal{C}$ such that $\beta_V(V \otimes V^*) \subset D$.

Remark 3.5. Lemma 3.2 shows why when defining our correspondence we assume that D is a bimodule and $\mathcal{C}$ is additive. However, it is still not clear whether the containments $\mathcal{C} \subseteq \mathcal{C}(B^*_{\mathcal{C}})$ and $B^*_{\mathcal{C}(D)} \subseteq D$ are equalities.

Proposition 3.6. If B is a bialgebra and $\mathcal{C}$ is monoidal, then $B^*_{\mathcal{C}}$ is a subalgebra of B^*. On the other hand, D is a subalgebra of B^* if and only if $\mathcal{C}(D)$ is monoidal.

Proof. Let U and V be objects of $\mathcal{C}$, and let $u \in U$, $v \in V$, $f \in U^*$, and $g \in V^*$. Then

$$\begin{aligned}\beta_{U\otimes V}\big((u \otimes v) \otimes (g \otimes f)\big)(b) &= (f \otimes g)\big(b \triangleright (u \otimes v)\big) \\ &= f(b_{(1)} \triangleright u) \cdot g(b_{(2)} \triangleright v) \\ &= \beta_U(u \otimes f)(b_{(1)}) \cdot \beta_V(v \otimes g)(b_{(2)})\end{aligned}$$

so $\beta_{U\otimes V} = \beta_U \beta_V$. $\square$

Definition 3.7. An element of B^* is *locally finite* if it generates a finite-dimensional bimodule. If D is a sub-bimodule of B^*, the *locally finite* part D_f is the sub-bimodule of locally finite elements of D.

Now the product of two locally finite elements belongs to the tensor product of their respective finite-dimensional submodules, which proves the following lemma.

Lemma 3.8. If a sub-bimodule D of B^* is a subalgebra, then the sub-bimodule D_f of locally finite elements is also a subalgebra.

3.2. A Peter-Weyl-type theorem

In this section we present a Peter-Weyl-type theorem relating semisimplicity of $\mathcal{C}$ with a Peter-Weyl decomposition of $B^*_{\mathcal{C}}$. We fail to find a complete reference for this theorem in the literature, although one direction of the implication is well known and for this part we appreciated the proof given in a lecture by David Jordan.

Definition 3.9. Let $D \subset B^*$ be a sub-bimodule. We say that D has a Peter-Weyl decomposition if

$$D = \bigoplus \beta_V(V \otimes V^*)$$

as an internal direct sum over all isomorphism classes of simple objects $V \in \mathcal{C}(D)$. This is well defined by Lemma 3.1.

Theorem 3.10. Let B be an algebra and let $\mathcal{C}$ be an Abelian category of finite-dimensional B-modules. Then $\mathcal{C}$ is semisimple if and only if $B^*_{\mathcal{C}}$ has a Peter-Weyl decomposition.

Before we prove the theorem, we note the following well known result which is proved by induction on the length of a Krull-Schmidt decomposition.

Lemma 3.11. Let B be an algebra and let $\mathcal{C}$ be an Abelian category of finite-dimensional B-modules. Then $\mathcal{C}$ is semisimple if and only if $\mathrm{Ext}^1(U, V) = 0$ for all simple B-modules U and V.

Proof of Theorem 3.10. Suppose that $B^*_{\mathcal{C}}$ has a Peter-Weyl decomposition, and suppose by way of contradiction that $\mathcal{C}$ has an indecomposable object V and a short exact sequence

$$0 \to V_1 \to V \to V_2 \to 0$$

where V_1 and V_2 are both simple. Now V must be cyclic; if not, then any $v \in V \setminus V_1$ would generate a complement to V_1, and we have assumed that V is indecomposable.

Now the dual short exact sequence $0 \to V_2^* \to V^* \to V_1^* \to 0$ has the same properties. Choose a cyclic vector $f \in V^*$. We define $\iota_f \colon V \to V \otimes V^*$ by $\iota_f(v) = v \otimes f$. We claim that $\beta_V \circ \iota_f$ is an injective morphism of B-modules. Indeed, $\beta_V(v \otimes f)$ is the zero map if and only if $(f \triangleleft b)(v) = 0$ for all $b \in B$, and since f generates V^* this implies $v = 0$. Thus $\beta_V \circ \iota_f$ embeds V into $B_{\mathcal{C}}^*$, which is semisimple, so V is semisimple, contradicting our assumption that V was indecomposable. Therefore $\mathcal{C}$ is semisimple by Lemma 3.11.

Suppose now that $\mathcal{C}$ is semisimple. Let $V \cong \bigoplus_{i=1}^{n} \left(\bigoplus_{j=1}^{n_i} V_i \right)$ be a decomposition of a module $V \in \mathcal{C}$ as a sum of simple modules V_i. By Lemma 3.2 and Corollary 3.3, the image of β_V is equal to $\sum \beta_{V_i}(V_i \otimes V_i^*)$. By Lemma 3.4, the sum is direct. □

3.3. Semisimplicity via a Casimir element

In this section we present a theorem proving the semisimplicity of certain representations of Hopf algebras when a Casimir element is available. The proof is a straightforward generalization of proofs given in other sources. For example, [1, 28] for semisimple Lie algebras and [2, 587–589] for $\mathrm{U}_q(\mathfrak{sl}_2)$.

Let H be a Hopf algebra. Recall that if V is an irreducible left H-module and $c \in H$ belongs to the center of H, then c acts on V as multiplication by some scalar.

Theorem 3.12. Let H be a Hopf algebra, let $\mathcal{C}$ be an Abelian category of finite-dimensional left H-modules which is closed under extension, and let $\mathbf{1}$ denote the trivial one-dimensional H-module. Suppose that there exists an element c from the center of H with the following property: For any simple V in $\mathcal{C}$, $c \triangleright V = 0$ if and only if $V \cong \mathbf{1}$. Suppose furthermore that $\mathrm{Ext}^1(\mathbf{1},\mathbf{1}) = 0$. Then every H-module in $\mathcal{C}$ is semisimple.

We call the element c a *Casimir* element of H. Given a finite-dimensional left H-module V, the strategy for the proof is to show that for any submodule $W \subset V$ there is another submodule $W' \subset V$ such that $V = W \oplus W'$. We first consider a couple of special cases.

Lemma 3.13. Let H be a Hopf algebra as described in Theorem 3.12, and let $V \in \mathcal{C}$ be a finite-dimensional left H-module. If $W \subset V$ is an irreducible submodule with $V/W \cong \mathbf{1}$, then there exists another submodule $W' \subset V$ such that $V = W \oplus W'$.

Proof. The Casimir element c satisfies $c \triangleright \bar{v} = 0$ for all $\bar{v} \in V/W$. If $W = \mathbf{1}$, then $V \cong \mathbf{1} \oplus \mathbf{1}$ since $\mathrm{Ext}^1(\mathbf{1},\mathbf{1}) = 0$. If $W \neq \mathbf{1}$, then we know that $c \triangleright W \neq 0$ by the hypothesis of Theorem 3.12. Therefore the submodule $\ker(c)$ of V satisfies $\ker(c) \cap W = 0$, so $V = W \oplus \ker(c)$. □

Lemma 3.14. Let H be a Hopf algebra as described in Theorem 3.12, and let $V \in \mathcal{C}$ be a finite-dimensional left H-module. If $W \subset V$ is a submodule with $V/W \cong \mathbf{1}$, then there exists another submodule $W' \subset V$ such that $V = W \oplus W'$.

Proof. If W is irreducible, then this follows from Lemma 3.13. So, suppose that W has a proper nonzero submodule $U \subset W$. Then we may write the short exact sequence of H-modules

$$0 \to W/U \to V/U \to V/W \to 0.$$

Now $\dim(W/U) < \dim(W)$. We use induction on the dimension of W, noting that in the base case W is irreducible. So by hypothesis the short exact sequence splits and there is a submodule $U' \subset V$ such that $V/U = W/U \oplus U'/U$. Note that $U'/U \cong \mathbf{1}$ since $V/W \cong \mathbf{1}$. We now write the short exact sequence of H-modules

$$0 \to U \to U' \to U'/U \to 0.$$

Now $\dim(U) < \dim(W)$, so by hypothesis the short exact sequence splits and there is a submodule $W' \subset U'$ such that $U' = U \oplus W'$. It follows that $V/U = W/U \oplus W'$. Thus $W \cap W' = 0$, and we conclude that $V = W \oplus W'$. $\square$

We are now ready to prove Theorem 3.12.

Proof of Theorem 3.12. Let $V \in \mathcal{C}$ be a finite-dimensional left H-module, and suppose $W \subset V$ is a proper non-zero submodule. We know that $\mathrm{Hom}_k(V, W)$ is a left H-module with action given by

$$(h \triangleright \phi)(v) = h_{(1)} \triangleright \phi(Sh_{(2)} \triangleright v).$$

We define two subspaces L and L' of $\mathrm{Hom}_k(V, W)$ as follows:

$$L = \{\phi \colon \exists\, f(\phi) \in k \text{ such that } \phi(w) = f(\phi)w \text{ for all } w \in W\},$$
$$L' = \{\phi \colon \phi(w) = 0 \text{ for all } w \in W\}.$$

We wish to show that L and L' are submodules of $\mathrm{Hom}_k(V, W)$. Let $h \in H$, $\phi \in L$, and $w \in W$. Then

$$\begin{aligned}(h \triangleright \phi)(w) &= h_{(1)} \triangleright \phi(Sh_{(2)} \triangleright w) \\ &= h_{(1)} \triangleright f(\phi)(Sh_{(2)} \triangleright w) \\ &= f(\phi)(h_{(1)}Sh_{(2)} \triangleright w) \\ &= f(\phi)\varepsilon(h)w.\end{aligned}$$

Thus $(h \triangleright \phi) \in L$, so L is an H-module. Similarly L' is an H-module. We note that $L/L' \cong \mathbf{1}$, so by Lemma 3.14 there is an H-module L'' such that $L = L' \oplus L''$. Let us choose some nonzero $\phi \in L''$, scaled as necessary so that $f(\phi) = 1$. Since L'' is an H-module we have $\big[\big(h - \varepsilon(h)\big) \triangleright \phi\big] \in L''$ for all $h \in H$. But our calculation above shows that $\big[\big(h - \varepsilon(h)\big) \triangleright \phi\big](w) = 0$ for all $w \in W$, so $\big[\big(h - \varepsilon(h)\big) \triangleright \phi\big] \in L'$. Since L' and L'' have trivial intersection, $\big[\big(h - \varepsilon(h)\big) \triangleright \phi\big] = 0$. That is,

$$(h \triangleright \phi)(v) = \varepsilon(h)\phi(v)$$

for all $v \in V$. Thus ϕ is not merely k-linear, but is a homomorphism of H-modules. It is surjective since it belongs to L. Therefore $V = W \oplus \ker(\phi)$. $\square$

3.4. Some semisimple categories of $D(\mathrm{U}_q(\mathfrak{g}))$-modules

In this section we demonstrate a method of constructing simple $D(H)$-modules, where H is a bialgebra, which we learned from Victor Ostrik. We stated in Proposition 2.26 that if H has quasi-triangular structure R, then we can construct coactions

$$\delta_+(v) = R^{(2)} \otimes (R^{(1)} \triangleright v), \quad \delta_-(v) = (R^{-1})^{(1)} \otimes \big((R^{-1})^{(2)} \triangleright v\big)$$

for any H-module. We refer to the resulting $D(H)$-modules (See Proposition 2.40) as V^+ and V^-, respectively.

Lemma 3.15. Let H be a bialgebra and let U and V be non-isomorphic simple H-modules. Then $U^- \ncong V^+$, and furthermore $V^- \cong V^+$ if and only if $\delta_- = \delta_+$.

Proof. That $U^- \ncong V^+$ is obvious since U^- and V^+ retain the H-module structures of U and V, respectively, and H is a subalgebra of $D(H)$.

Now assume that $f\colon V \to V$ is an isomorphism such that

$$\delta_- = (\mathrm{id}_H \otimes f)^{-1} \circ \delta_+ \circ f.$$

Since V is simple, Schur's Lemma implies that $f(v) = cv$ for all $v \in V$, where c is a non-zero constant. Therefore, $\delta_- = \delta_+$. $\square$

Lemma 3.16. Let H be a quasi-triangular Hopf algebra. Suppose that $V \otimes V^*$ is semisimple for any simple H-module V, that the category of finite-dimensional left H-modules is semisimple, and that $\delta_- \neq \delta_+$ except for the trivial H-module. Let U and V be simple H-modules. Then the $D(H)$-module $U^+ \otimes V^-$ is simple. Furthermore, the tensor category generated by all such $U^+ \otimes V^-$ is semisimple.

Proof. Let U and V be simple H-modules. We know from Lemma 3.15 that $\mathrm{Hom}_{D(H)}(V^+, V^-)$ is nonzero if and only if V is the trivial module. We have

$$\begin{aligned}
\mathrm{End}_{D(H)}(U^+ \otimes V^-) &= \mathrm{Hom}_{D(H)}(U^+ \otimes V^-, U^+ \otimes V^-) \\
&= \mathrm{Hom}_{D(H)}\left(U^+ \otimes (U^+)^*, V^+ \otimes (V^-)^*\right) \\
&= \mathrm{Hom}_{D(H)}\left((U \otimes U^*)^+, (V \otimes V^*)^-\right)
\end{aligned}$$

which is $\mathbb{C}(q)$ since the only contribution is from the trivial submodules of $(U \otimes U^*)^+$ and $(V \otimes V^*)^-$. Thus $U^+ \otimes V^-$ is simple.

Now let U, V, W, and Y be left H-modules. We have

$$\left(U^+ \otimes V^-\right) \otimes \left(W^+ \otimes Y^-\right) \cong U^+ \otimes W^+ \otimes V^- \otimes Y^- \cong (U \otimes W)^+ \otimes (V \otimes Y)^-$$

so the lemma is proved. □

Definition 3.17. Recall that $\mathrm{U}_q(\mathfrak{g})$ is quasi-triangular with simple modules V_λ. We define $V_{\lambda,\mu} = V_\lambda^+ \otimes V_\mu^-$.

Corollary 3.18. The $D(\mathrm{U}_q(\mathfrak{sl}_2))$-modules $V_{\lambda,\mu}$ are simple, and the category they generate is semisimple.

Proof. Let V be a simple, non-trivial $\mathrm{U}_q(\mathfrak{sl}_2)$-module. Let $v \in V$ be a highest-weight vector. Since V is non-trivial, we have $F \triangleright v \neq 0$. Given the quasi-triangular structure from Example 2.25, we find that v generates a 1-dimensional H-comodule under δ_+, but not under δ_-. Thus $\delta_- \neq \delta_+$, so the hypotheses of Lemma 3.16 are satisfied. □

The following conjectures are due to our conversations with Victor Ostrik.

Conjecture 3.19. The category of finite-dimensional left $D(\mathrm{U}_q(\mathfrak{g}))$-modules is semisimple.

Conjecture 3.20. The simple $D(\mathrm{U}_q(\mathfrak{g}))$-modules are, up to isomorphism, of the form $V_{\lambda,\mu} \otimes U_0$ where λ and μ are dominant integral $\mathfrak{g}$-weights and U_0 belongs to the (finite) set of one-dimensional $D(\mathrm{U}_q(\mathfrak{g}))$-modules.

These conjectures are extremely difficult even for $\mathfrak{g} = \mathfrak{sl}_2$. However, we find that they are consistent with our main results in Chapter IV.

CHAPTER IV

MAIN RESULTS

Proofs of results presented in this chapter are found in Chapter VI.

4.1. The actions of $D(H)$ on $H \otimes C$

It follows from Lemma 2.8 that $D(H)^*$ is a $D(H)$-bimodule, but the actions are very complicated when H is infinite-dimensional. However, there is a sub-bimodule that we are able to describe.

Theorem 4.1. The subalgebra $H \otimes C \subset D(H)^*$ of Proposition 2.39 is closed under both the left and right actions of $D(H)$ described in Section 2.3, and thus $H \otimes C$ is a bimodule algebra over $D(H)$. The left and right actions on generators are given by

$$
\begin{aligned}
c \blacktriangleright \bar{h} &= (\bar{h} \triangleleft c_{(2)}) \cdot Sc_{(1)}c_{(3)} & \qquad \bar{h} \blacktriangleleft c &= c \triangleright \bar{h} \\
c \blacktriangleright \bar{c} &= Sc_{(1)}\bar{c}c_{(2)} & \bar{c} \blacktriangleleft c &= \varepsilon(c)\bar{c} \\
h \blacktriangleright \bar{h} &= \varepsilon(h)\bar{h} & \bar{h} \blacktriangleleft h &= Sh_{(1)}\bar{h}h_{(2)} \\
h \blacktriangleright \bar{c} &= h \triangleright \bar{c} & \bar{c} \blacktriangleleft h &= Sh_{(1)}h_{(3)} \cdot (\bar{c} \triangleleft h_{(2)})
\end{aligned}
$$

We use the solid triangles $\blacktriangleright$ and $\blacktriangleleft$ to distinguish these actions from the actions of C and H on each other, for which we use $\triangleright$ and $\triangleleft$. We note that $\bar{h} \blacktriangleleft c = c \triangleright \bar{h}$ makes sense because the subalgebra $C^{\mathrm{op}} \subset D(H)$ has multiplication opposite to that of C.

Since in Theorem 4.1 we have calculated explicit formulas for the actions of $D(H)$ on $H \otimes C$, we will seek to describe its locally finite part $(H \otimes C)_f$ and thus to describe all objects of the category $\mathcal{C}\big((H \otimes C)_f\big)$. Ideally we would be able to describe the finite dual $D(H)^\circ$ and thus all finite-dimensional $D(H)$-modules, but that is much more difficult.

For examples of these actions, see Sections 5.1 and 5.2.

4.2. Semisimplicity of certain $D(\mathrm{U}_q(\mathfrak{sl}_2))$-modules

If $H = \mathrm{U}_q(\mathfrak{g})$, we recall $V_{\lambda,\mu} = V_\lambda^+ \otimes V_\mu^-$ from Definition 3.17. We define $H_{\lambda,\mu}$ to be $\beta_{V_{\lambda,\mu}}(V_{\lambda,\mu} \otimes V_{\lambda,\mu}^*)$ which is a sub-bimodule of $D(H)_f^*$ as shown in Section 3.2.

In the remainder of this section, we let $H = \mathrm{U}_q(\mathfrak{sl}_2)$, $C = \mathbb{C}_q[SL_2]$, and $\mathcal{H} = (\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2])_f$. In this case, λ and μ are nonnegative integers. We claim that $\mathcal{H}$ has a Peter-Weyl decomposition, namely the following.

Main Theorem 4.2. As a $D(H)$-bimodule,

$$\mathcal{H} = \bigoplus_{\substack{\lambda,\mu \geq 0 \\ \lambda - \mu \in 2\mathbb{Z}}} H_{\lambda,\mu} \tag{4.1}$$

and this is a Peter-Weyl decomposition of $\mathcal{H}$.

We recall the notation $\mathcal{C}(D)$ from Section 3.1. Then Theorems 3.10 and 4.2 have the following corollary:

Corollary 4.3. The category $\mathcal{C}(\mathcal{H})$ is semisimple.

We now highlight some of the results leading to Main Theorem 4.2.

Definition 4.4. If $v \in \mathcal{H}$, we say that v is bihomogeneous of biweight (ω_1, ω_2) if there exist scalars ω_1 and ω_2 such that $K \blacktriangleright v = q^{\omega_1} v$ and $v \blacktriangleleft K^{-1} = q^{\omega_2} v$.

In Section 5.3 we will compute $H_{1,1}$ explicitly and prove the following proposition about the subalgebra ${}^+{H_{1,1}}^+ = \{v \in H_{1,1} \mid E \blacktriangleright v = v \blacktriangleleft E = 0\}$ of highest-weight bivectors (Recall Lemma 2.15).

Proposition 4.5. The algebra ${}^{+}H_{1,1}{}^{+}$ is the linear span of four vectors v_1, v_2, v_3, and v_4 of biweight $(2,2)$, $(2,0)$, $(0,2)$, and $(0,0)$, respectively.

The following result provides an upper bound for $\mathcal{H}$ in $H \otimes C$.

Theorem 4.6. As an algebra, ${}^{+}(H \otimes C)^{+}$ is generated by $\{v_1, v_2, v_3, v_4\}$.

In fact, in the proof we will give a basis of ${}^{+}(H \otimes C)^{+}$. Now Lemma 2.15, Lemma 3.8, and Theorem 4.6 have the following corollary since $v_1, v_2, v_3, v_4 \in H_{1,1}$.

Corollary 4.7. The algebra ${}^{+}(H \otimes C)^{+}$ is locally finite.

Corollary 4.8. ${}^{+}\mathcal{H}^{+} = {}^{+}(H \otimes C)^{+}$.

As a consequence of Theorem 4.6, Corollary 4.8, and Lemma 2.16, we have the following:

Corollary 4.9. As a $D(H)$-bimodule, $\mathcal{H}$ is generated by the algebra $\langle v_1, v_2, v_3, v_4 \rangle$.

The above results are all we need to prove the main theorem. However, the algebra ${}^{+}\mathcal{H}^{+} = \langle v_1, v_2, v_3, v_4 \rangle$ is very interesting in its own right. The vectors v_1 and v_4 are central in this algebra, but v_2 and v_3 have homogeneous relations in degree 4. Namely,

$$\begin{aligned} v_2^3 v_3 - (q^2+1+q^{-2}) v_2^2 v_3 v_2 + (q^2+1+q^{-2}) v_2 v_3 v_2^2 - v_3 v_2^3 &= 0 \\ v_2 v_3^3 - (q^2+1+q^{-2}) v_3 v_2 v_3^2 + (q^2+1+q^{-2}) v_3^2 v_2 v_3 - v_3^3 v_2 &= 0 \\ v_3 v_2^2 v_3 - v_2 v_3^2 v_2 &= 0 \end{aligned}$$

In fact $\langle v_1, v_2, v_3, v_4 \rangle$ appears to have polynomial growth with Hilbert series

$$h(t) = \frac{1}{(1-t^2)^2(1-t)^4} - \frac{t^2}{(1-t^2)(1-t)^2}.$$

Surprisingly, ${}^{+}\mathcal{H}^{+}$ is isomorphic to the quotient by $(v_3v_2^2v_3 - v_2v_3^2v_2)$ of the positive part of $\mathrm{U}_q(\mathfrak{g})$, where $\mathfrak{g}$ is the Kac-Moody algebra with Cartan matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Moreover, in Section 6.2 we prove that ${}^{+}\mathcal{H}^{+}$ has basis

$$\{v_3^{\ell}v_1^{m}v_5^{p}v_2^{s}\} \cup \{v_3^{\ell}v_1^{m}v_6^{r}v_2^{s}\} \cup \{v_3^{\ell}v_4^{n}v_5^{p}v_2^{s}\} \cup \{v_3^{\ell}v_4^{n}v_6^{r}v_2^{s}\}$$

where v_5 and v_6 are the highest-weight bivectors of $H_{2,0}$ and $H_{0,2}$. This algebra appeared in another context in [6], and it would be very interesting to continue this line of research.

Problem 4.10. Find a presentation for the algebra $\mathcal{H}$.

Problem 4.11. Find a presentation for the algebras ${}^{+}(\mathrm{U}_q(\mathfrak{sl}_n) \otimes \mathbb{C}_q[\mathrm{SL}_n])^{+}$ and $(\mathrm{U}_q(\mathfrak{sl}_n) \otimes \mathbb{C}_q[\mathrm{SL}_n])_f$.

We conclude this section with some conjectures for other semisimple Lie algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra and G be the corresponding simply-connected algebraic group.

Conjecture 4.12. The highest-weight bivectors in the $D(\mathrm{U}_q(\mathfrak{g}))$-bimodule $\mathrm{U}_q(\mathfrak{g}) \otimes \mathbb{C}_q[G]$ are locally-finite.

Conjecture 4.13. As a $D(\mathrm{U}_q(\mathfrak{g}))$-bimodule, there is a Peter-Weyl decomposition

$$\big(\mathrm{U}_q(\mathfrak{g}) \otimes \mathbb{C}_q[G]\big)_f \cong \bigoplus H_{\lambda,\mu} \tag{4.2}$$

where the sum is over all dominant weights λ and μ such that $\lambda - \mu$ belongs to the root lattice of $\mathfrak{g}$.

Conjecture 4.14. The sum $\bigoplus H_{\omega_i,\omega_i}$ over all fundamental weights ω_i generates $(\mathrm{U}_q(\mathfrak{g}) \otimes \mathbb{C}_q[G])_f$ as an algebra.

CHAPTER V

EXAMPLES

5.1. The actions of $D(\mathrm{U}_q(\mathfrak{sl}_n))$ on $\mathrm{U}_q(\mathfrak{sl}_n) \otimes \mathbb{C}_q[\mathrm{SL}_n]$

Using Example 2.23 and Theorem 4.1 we calculate the left and right actions of $D(\mathrm{U}_q(\mathfrak{sl}_n))$ on the generators of $\mathrm{U}_q(\mathfrak{sl}_n) \otimes \mathbb{C}_q[\mathrm{SL}_n]$.

From $h \blacktriangleright \bar{h} = \varepsilon(h)\bar{h}$, we have

$$E_i \blacktriangleright \bar{h} = 0, \quad F_i \blacktriangleright \bar{h} = 0, \quad \text{and} \quad K_i \blacktriangleright \bar{h} = 1.$$

From $h \blacktriangleright \bar{c} = h \triangleright \bar{c}$, we have

$$E_i \blacktriangleright X_{k\ell} = \delta_{i+1,\ell} X_{k,\ell-1}, \quad F_i \blacktriangleright X_{k\ell} = \delta_{i,\ell} X_{k,\ell+1},$$
$$\text{and} \quad K_i \blacktriangleright X_{k\ell} = (q\delta_{i,\ell} + q^{-1}\delta_{i+1,\ell}) X_{k\ell}.$$

Let us denote $SX_{k\ell} = Y_{k\ell}$. From $c \blacktriangleright \bar{h} = (\bar{h} \triangleleft c_{(2)}) \cdot Sc_{(1)}c_{(3)}$, we have

$$X_{k\ell} \blacktriangleright E_i = K_i Y_{ki} X_{i+1,\ell} + \textstyle\sum_j E_i Y_{kj} X_{j\ell},$$
$$X_{k\ell} \blacktriangleright F_i = Y_{k,i+1} X_{i\ell} + q^{-1} F_i Y_{ki} X_{i\ell} + q F_i Y_{k,i+1} X_{i+1,\ell},$$
$$\text{and} \quad X_{k\ell} \blacktriangleright K_i = q K_i Y_{ki} X_{i\ell} + q^{-1} K_i Y_{k,i+1} X_{i+1,\ell}.$$

From $c \blacktriangleright \bar{c} = Sc_{(1)}\bar{c}c_{(2)}$, we have

$$X_{k\ell} \blacktriangleright X_{mn} = \textstyle\sum_j Y_{kj} X_{mn} X_{j\ell}.$$

From $\bar{h} \blacktriangleleft h = Sh_{(1)}\bar{h}h_{(2)}$, we have

$$\bar{h} \blacktriangleleft E_i = -E_i K_i^{-1}\bar{h}K_i + \bar{h}E_i, \quad \bar{h} \blacktriangleleft F_i = -K_i F_i \bar{h} + K_i \bar{h} F_i,$$
$$\text{and} \quad \bar{h} \blacktriangleleft K_i = K_i^{-1}\bar{h}K_i.$$

From $\bar{c} \blacktriangleleft h = Sh_{(1)}h_{(3)} \cdot (\bar{c} \triangleleft h_{(2)})$, we have

$$X_{k\ell} \blacktriangleright E_i = -(q^{-1}\delta_{i+1,k} + q\delta_{i,k})E_i X_{k\ell} + \delta_{i,k}K_i X_{k+1,\ell} + E_i X_{k\ell},$$
$$X_{k\ell} \blacktriangleright F_i = -K_i F_i X_{k\ell} + \delta_{i,k-1}K_i X_{k-1,\ell} + (q\delta_{i+1,k} + q^{-1}\delta_{i,k})K_i F_i X_{k\ell},$$
$$\text{and} \quad X_{k\ell} \blacktriangleleft K_i = (q^{-1}\delta_{i+1,k} + q\delta_{i,k})X_{k\ell}.$$

From $\bar{h} \blacktriangleleft c = c \triangleright \bar{h}$, we have

$$E_i \blacktriangleleft X_{k\ell} = (q\delta_{i,k,\ell} + q^{-1}\delta_{i-1,k,\ell})E_i + \delta_{i,k,\ell-1},$$
$$F_i \blacktriangleleft X_{k\ell} = \delta_{k,\ell}F_i + \delta_{i,k-1,\ell}K_i^{-1},$$
$$\text{and} \quad K_i \blacktriangleleft X_{k\ell} = (q\delta_{i,k,\ell} + q^{-1}\delta_{i+1,k,\ell})K_i.$$

From $\bar{c} \blacktriangleleft c = \varepsilon(c)\bar{c}$, we have

$$X_{mn} \blacktriangleleft X_{k\ell} = \delta_{k,\ell}X_{mn}.$$

In the next section we specialize to the case $H = \mathrm{U}_q(\mathfrak{sl}_2)$.

5.2. The actions of $D(\mathrm{U}_q(\mathfrak{sl}_2))$ on $\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2]$

Here we specialize our example from Section 5.1 to $D(\mathrm{U}_q(\mathfrak{sl}_2))$ acting on the algebra $\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2]$. From $h \blacktriangleright \bar{h} = \varepsilon(h)\bar{h}$, we have

$$
\begin{array}{cccc}
E \blacktriangleright E = 0 & E \blacktriangleright F = 0 & E \blacktriangleright K = 0 & E \blacktriangleright K^{-1} = 0 \\
F \blacktriangleright E = 0 & F \blacktriangleright F = 0 & F \blacktriangleright K = 0 & F \blacktriangleright K^{-1} = 0 \\
K \blacktriangleright E = E & K \blacktriangleright F = F & K \blacktriangleright K = K & K \blacktriangleright K^{-1} = K^{-1} \\
K^{-1} \blacktriangleright E = E & K^{-1} \blacktriangleright F = F & K^{-1} \blacktriangleright K = K & K^{-1} \blacktriangleright K^{-1} = K^{-1}
\end{array}
$$

From $h \blacktriangleright \bar{c} = h \triangleright \bar{c}$, we have

$$
\begin{array}{cccc}
E \blacktriangleright a = 0 & E \blacktriangleright b = a & E \blacktriangleright c = 0 & E \blacktriangleright d = c \\
F \blacktriangleright a = b & F \blacktriangleright b = 0 & F \blacktriangleright c = d & F \blacktriangleright d = 0 \\
K \blacktriangleright a = qa & K \blacktriangleright b = q^{-1}b & K \blacktriangleright c = qc & K \blacktriangleright d = q^{-1}d \\
K^{-1} \blacktriangleright a = q^{-1}a & K^{-1} \blacktriangleright b = qb & K^{-1} \blacktriangleright c = q^{-1}c & K^{-1} \blacktriangleright d = qd
\end{array}
$$

From $c \blacktriangleright \bar{h} = (\bar{h} \triangleleft c_{(2)}) \cdot Sc_{(1)}c_{(3)}$, we have

$$
\begin{array}{llll}
a \blacktriangleright E &= E + qKcd & a \blacktriangleright F &= q^{-1}F - qba + (1-q^2)Fbc \\
a \blacktriangleright K &= qK + (q^2-1)Kbc & a \blacktriangleright K^{-1} &= q^{-1}K^{-1} + (1-q^2)K^{-1}bc \\
b \blacktriangleright E &= Kd^2 & b \blacktriangleright F &= -qb^2 + (1-q^2)Fbd \\
b \blacktriangleright K &= (q^2-1)Kbd & b \blacktriangleright K^{-1} &= (1-q^2)K^{-1}bd \\
c \blacktriangleright E &= -q^{-1}Kc^c & c \blacktriangleright F &= a^2 + (q-q^{-1})Fac \\
c \blacktriangleright K &= (q^{-1}-q)Kac & c \blacktriangleright K^{-1} &= (q-q^{-1})K^{-1}ac \\
d \blacktriangleright E &= E - q^{-1}Kcd & d \blacktriangleright F &= qF + ab + (1-q^{-2})Fbc \\
d \blacktriangleright K &= q^{-1}K + (q^{-2}-1)Kbc & d \blacktriangleright K^{-1} &= qK^{-1} + (1-q^{-2})K^{-1}bc
\end{array}
$$

From $c \blacktriangleright \bar{c} = Sc_{(1)}\bar{c}c_{(2)}$, we have

$$
\begin{array}{ll}
a \blacktriangleright a = a + (q-1)bca & a \blacktriangleright b = qb + (q^2-q)b^2c \\
a \blacktriangleright c = qc + (q^2-q)bc^2 & a \blacktriangleright d = d + (q-1)dbc \\
b \blacktriangleright a = (1-q)b + (q-1)b^2c & b \blacktriangleright b = (1-q^{-1})db^2 \\
b \blacktriangleright c = (1-q^{-1})dcb & b \blacktriangleright d = (1-q^{-1})d^2b \\
c \blacktriangleright a = (1-q)a^2c & c \blacktriangleright b = (1-q)abc \\
c \blacktriangleright c = (1-q)ac^2 & c \blacktriangleright d = (1-q^{-1})c + (q^{-1}-1)bc^2 \\
d \blacktriangleright a = a + (q^{-1}-1)abc & d \blacktriangleright b = q^{-1}b + (q^{-2}-q^{-1})b^2c \\
d \blacktriangleright c = q^{-1}c + (q^{-2}-q^{-1})bc^2 & d \blacktriangleright d = d + (q^{-1}-1)bcd
\end{array}
$$

From $\bar{h} \blacktriangleleft h = Sh_{(1)}\bar{h}h_{(2)}$, we have

$$
\begin{aligned}
(1-q^{-2})E^2 &= E \blacktriangleleft E & (1-q^2)EF - \tfrac{K-K^{-1}}{q-q^{-1}} &= F \blacktriangleleft E \\
(q^2-1)EK &= K \blacktriangleleft E & (q^{-2}-1)EK^{-1} &= K^{-1} \blacktriangleleft E \\
K \cdot \tfrac{K-K^{-1}}{q-q^{-1}} &= E \blacktriangleleft F & 0 &= F \blacktriangleleft F \\
(1-q^2)K^2F &= K \blacktriangleleft F & (1-q^{-2})F &= K^{-1} \blacktriangleleft F \\
q^{-2}E &= E \blacktriangleleft K & q^2F &= F \blacktriangleleft K \\
K &= K \blacktriangleleft K & K^{-1} &= K^{-1} \blacktriangleleft K \\
q^2E &= E \blacktriangleleft K^{-1} & q^{-2}F &= F \blacktriangleleft K^{-1} \\
K &= K \blacktriangleleft K^{-1} & K^{-1} &= K^{-1} \blacktriangleleft K^{-1}
\end{aligned}
$$

From $\bar{c} \blacktriangleleft h = Sh_{(1)}h_{(3)} \cdot (\bar{c} \triangleleft h_{(2)})$, we have

$$
\begin{aligned}
(1-q)Ea + Kc &= a \blacktriangleleft E & (1-q)Eb + Kd &= b \blacktriangleleft E \\
(1-q^{-1})Ec &= c \blacktriangleleft E & (1-q^{-1})Ed &= c \blacktriangleleft E \\
(q^{-1}-1)KFa &= a \blacktriangleleft F & (q^{-1}-1)KFb &= b \blacktriangleleft F \\
(q-1)KFc + Ka &= c \blacktriangleleft F & (q-1)KFd + Kb &= c \blacktriangleleft F \\
qa &= a \blacktriangleleft K & qb &= b \blacktriangleleft K \\
q^{-1}c &= c \blacktriangleleft K & q^{-1}d &= c \blacktriangleleft K \\
q^{-1}a &= a \blacktriangleleft K^{-1} & q^{-1}b &= b \blacktriangleleft K^{-1} \\
qc &= c \blacktriangleleft K^{-1} & qd &= c \blacktriangleleft K^{-1}
\end{aligned}
$$

From $\bar{h} \blacktriangleleft c = c \triangleright \bar{h}$, we have

$$\begin{aligned} qE &= E \blacktriangleleft a & F &= F \blacktriangleleft a \\ qK &= K \blacktriangleleft a & q^{-1}K^{-1} &= K^{-1} \blacktriangleleft a \\ 1 &= E \blacktriangleleft b & 0 &= F \blacktriangleleft b \\ 0 &= K \blacktriangleleft b & 0 &= K^{-1} \blacktriangleleft b \\ 0 &= E \blacktriangleleft c & K^{-1} &= F \blacktriangleleft c \\ 0 &= K \blacktriangleleft c & 0 &= K^{-1} \blacktriangleleft c \\ q^{-1}E &= E \blacktriangleleft d & F &= F \blacktriangleleft d \\ q^{-1}K &= K \blacktriangleleft d & qK^{-1} &= K^{-1} \blacktriangleleft d \end{aligned}$$

From $\bar{c} \blacktriangleleft c = \varepsilon(c)\bar{c}$, we have

$$\begin{aligned} a &= a \blacktriangleleft a & b &= b \blacktriangleleft a & c &= c \blacktriangleleft a & d &= d \blacktriangleleft a \\ 0 &= a \blacktriangleleft b & 0 &= b \blacktriangleleft b & 0 &= c \blacktriangleleft b & 0 &= d \blacktriangleleft b \\ 0 &= a \blacktriangleleft c & 0 &= b \blacktriangleleft c & 0 &= c \blacktriangleleft c & 0 &= d \blacktriangleleft c \\ a &= a \blacktriangleleft d & b &= b \blacktriangleleft d & c &= c \blacktriangleleft d & c &= d \blacktriangleleft d \end{aligned}$$

5.3. Some simple sub-bimodules of $(\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2])_f$

In this section we exhibit three simple sub-bimodules of $(\mathrm{U}_q(\mathfrak{sl}_2) \otimes \mathbb{C}_q[\mathrm{SL}_2])_f$. In each example, we examine the subspace of highest-weight bivectors, i.e. those vectors annihilated by both the left and right actions of E.

The first example, $H_{1,1}$, is 16-dimensional and has a 4-dimensional subspace of highest-weight bivectors. The tensor square of $H_{1,1}$ is 100-dimensional and is the internal direct sum of four non-isomorphic simple sub-bimodules of dimensions 81, 9, 9, and 1, respectively:

$$H_{1,1} \otimes H_{1,1} \cong H_{2,2} \oplus H_{2,0} \oplus H_{0,2} \oplus H_{0,0}$$

The other two examples we include here are $H_{2,0}$ and $H_{0,2}$, each of which has a one-dimensional subspace of highest-weight bivectors.

Example 5.1. Let $H = \mathrm{U}_q(\mathfrak{sl}_2)$ and $C = \mathbb{C}_q[\mathrm{SL}_2]$. Then $H_{1,1}$ is the 16-dimensional $D(H)$-sub-bimodule of $(H \otimes C)_f$ with basis

$$v_{11} = EK^{-1}$$

$$v_{12} = K^{-1}$$

$$v_{13} = F$$

$$v_{14} = \Delta$$

$$v_{21} = (q - q^{-1})EK^{-1}ac - c^2$$

$$v_{22} = (q - q^{-1})K^{-1}ac$$

$$v_{23} = (q - q^{-1})Fac + a^2$$

$$v_{24} = (q - q^{-1})\Delta ac - \tfrac{q+q^{-1}}{q-q^{-1}}Kac - q^{-2}FKc^2 + Ea^2$$

$$v_{31} = (q^{-1} - q)EK^{-1}db + qd^2$$

$$v_{32} = (q^{-1} - q)K^{-1}db$$

$$v_{33} = (q^{-1} - q)Fdb - qb^2$$

$$v_{34} = (q^{-1} - q)\Delta db + \tfrac{q+q^{-1}}{q-q^{-1}}Kdb + q^{-1}FKd^2 - qEb^2$$

$$v_{41} = (q - q^{-1})EK^{-1}bc - dc$$

$$v_{42} = (q - q^{-1})K^{-1}bc$$

$$v_{43} = (q - q^{-1})Fbc + qab$$

$$v_{44} = (q - q^{-1})\Delta bc - \tfrac{q+q^{-1}}{q-q^{-1}}Kbc - q^{-2}FKdc + qEab - \tfrac{1}{q-q^{-1}}K$$

As a left $D(H)$-module, $H_{1,1} = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, where each V_i is the left $D(H)$-module with basis $\{v_{1i}, v_{2i}, v_{3i}, v_{4i}\}$. In this basis, the left action of $x \in D(H)$

on $v \in V_i$ is given by $x \blacktriangleright v = \phi(x)v$, where

$$\phi(E) = \begin{bmatrix} 0 & 0 & q^{-1}-q & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -q^{-1}-q & 0 \end{bmatrix} \qquad \phi(F) = \begin{bmatrix} 0 & 1-q^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1+q^{-2} & 0 & 0 \end{bmatrix}$$

$$\phi(K) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \phi(K^{-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi(a) = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q & 0 & 0 & q \end{bmatrix} \qquad \phi(b) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & q^{-1}-q & 0 & 0 \end{bmatrix}$$

$$\phi(c) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1}-q & 0 \end{bmatrix} \qquad \phi(d) = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q^{-1} & 0 & 0 & q^{-1} \end{bmatrix}$$

As a right $D(H)$-module, $H_{1,1} = V_1' \oplus V_2' \oplus V_3' \oplus V_4'$, where each V_i' is the right $D(H)$-module with basis $\{v_{i1}, v_{i2}, v_{i3}, v_{i4}\}$. In this basis, the right action of $x \in D(H)$

on $v \in V_i'$ is given by $v \blacktriangleleft x = \phi'(x)v$, where

$$\phi'(E) = \begin{bmatrix} 0 & q^{-2}-1 & 0 & 0 \\ 0 & 0 & \frac{q^2+1}{q-q^{-1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q^2 & 0 \end{bmatrix} \qquad \phi'(F) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{q^2+1}{q^{-1}-q} & 0 & 0 & 0 \\ 0 & 1-q^{-2} & 0 & 0 \\ q^2-1 & 0 & 0 & 0 \end{bmatrix}$$

$$\phi'(K) = \begin{bmatrix} q^{-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \phi'(K^{-1}) = \begin{bmatrix} q^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi'(a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & \frac{q^{-1}}{q^{-1}-q} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix} \qquad \phi'(b) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\phi'(c) = \begin{bmatrix} 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \phi'(d) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 0 & \frac{q}{q-q^{-1}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix}$$

We see that $\phi(E)$ is a rank-2 matrix with null space spanned by $\{v_{1i}, v_{2i}\}$ and $\phi'(E)$ is a rank-2 matrix with null space spanned by $\{v_{i1}, v_{i4}\}$. It follows that the subspace $\{v \in H_{1,1} \mid E \blacktriangleright v = v \blacktriangleleft E = 0\}$ of highest-weight bivectors is spanned by $\{v_{11}, v_{14}, v_{21}, v_{24}\}$. We note that

$$K \blacktriangleright v_{21} = q^2 v_{21}, \quad K \blacktriangleright v_{24} = q^2 v_{24}, \quad K \blacktriangleright v_{11} = v_{11}, \quad K \blacktriangleright v_{14} = v_{14}$$

and

$$v_{21} \blacktriangleleft K = q^{-2} v_{21}, \quad v_{24} \blacktriangleleft K = v_{24}, \quad v \blacktriangleleft v_{11} = q^{-2} v_{11}, \quad v_{14} \blacktriangleleft K = v_{14}.$$

Thus these vectors are canonical up to scalar multiple, since they are distinguished by their biweights. Elsewhere in this paper, we refer to these four highest-weight bivectors as

$$\begin{aligned}
v_1 &= (q - q^{-1}) E K^{-1} ac - c^2 \\
v_2 &= (q - q^{-1}) \Delta ac - \tfrac{q+q^{-1}}{q-q^{-1}} K ac - q^{-2} F K c^2 + E a^2 \\
v_3 &= E K^{-1} \\
v_4 &= \Delta
\end{aligned} \tag{5.1}$$

Example 5.2. Let $H = \mathrm{U}_q(\mathfrak{sl}_2)$ and $C = \mathbb{C}_q[\mathrm{SL}_2]$. Then $H_{2,0} \subset (H_{1,1})^2$ is the 9-dimensional $D(H)$-sub-bimodule of $(H \otimes C)_f$ with basis

$$v_{11} = K^{-1}c^2$$

$$v_{12} = (q - q^{-1})Fc^2 + qac$$

$$v_{13} = q^{-3}(q - q^{-1})^2 F^2 K c^2 + q^{-1}(q^2 - q^{-2})FKac + Ka^2$$

$$v_{21} = q^{-1}K^{-1}dc$$

$$v_{22} = q^{-1}(q - q^{-1})Fdc + bc + \tfrac{1}{q+q^{-1}}$$

$$v_{23} = q^{-4}(q - q^{-1})^2 F^2 K dc + q^{-2}(q^2 - q^{-2})FKbc + q^{-2}(q - q^{-1})FK + Kab$$

$$v_{31} = K^{-1}d^2$$

$$v_{32} = (q - q^{-1})Fd^2 + db$$

$$v_{33} = q^{-3}(q - q^{-1})^2 F^2 K d^2 + q^{-2}(q^2 - q^{-2})FKdb + Kb^2$$

As a left $D(H)$-module, $H_{2,0} = V_1 \oplus V_2 \oplus V_3$, where each V_i is the left $D(H)$-module with basis $\{v_{1i}, v_{2i}, v_{3i}\}$. In this basis, the left action of $x \in D(H)$ on $v \in V_i$

is given by $x \blacktriangleright v = \phi_i(x)v$, where

$$\phi_i(E) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & q+q^{-1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i(F) = \begin{bmatrix} 0 & 0 & 0 \\ q+q^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\phi_i(K) = \begin{bmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{bmatrix} \qquad \phi_i(K^{-1}) = \begin{bmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{bmatrix}$$

$$\phi_i(a) = \begin{bmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{bmatrix} \qquad \phi_i(b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\phi_i(c) = \begin{bmatrix} 0 & 1-q^{-2} & 0 \\ 0 & 0 & q^2-q^{-2} \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i(d) = \begin{bmatrix} q^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{bmatrix}$$

As a right $D(H)$-module, $H_{2,0} = V_1' \oplus V_2' \oplus V_3'$, where each V_i' is the right $D(H)$-module with basis $\{v_{i1}, v_{i2}, v_{i3}\}$. In this basis, the right action of $x \in D(H)$ on $v \in V_i'$

is given by $v \blacktriangleleft x = \phi_i'(x)v$, where

$$\phi_i'(E) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & q+q^{-1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i'(F) = \begin{bmatrix} 0 & 0 & 0 \\ q+q^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\phi_i'(K) = \begin{bmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{bmatrix} \qquad \phi_i'(K^{-1}) = \begin{bmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{bmatrix}$$

$$\phi_i'(a) = \begin{bmatrix} q^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{bmatrix} \qquad \phi_i'(b) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\phi_i'(c) = \begin{bmatrix} 0 & q-q^{-1} & 0 \\ 0 & 0 & q-q^{-3} \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i'(d) = \begin{bmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{bmatrix}$$

We see that $\phi_i(E)$ is a rank-2 matrix with null space spanned by $\{v_{1i}\}$ and $\phi_i'(E)$ is a rank-2 matrix with null space spanned by $\{v_{i1}\}$. It follows that the subspace $\{v \in H_{2,0} \mid E \blacktriangleright v = v \blacktriangleleft E = 0\}$ of highest-weight bivectors is spanned by v_{11}. Elsewhere in this paper, we refer to this bivector as

$$v_5 = K^{-1}c^2 \tag{5.2}$$

We note that $\dfrac{q(q^2+1)}{q^2-1} v_5 = (1-q^2)v_1v_4 - [v_2, v_3]_{q^2}$.

Example 5.3. Let $H = \mathrm{U}_q(\mathfrak{sl}_2)$ and $C = \mathbb{C}_q[\mathrm{SL}_2]$. Then $H_{0,2} \subset (H_{1,1})^2$ is the $D(H)$-sub-bimodule of $(H \otimes C)_f$ with basis

$$v_{1,1} = q^{-1}(q - q^{-1})^2 E^2 K^{-1} a^2 - q(q^2 - q^{-2}) Eac + Kc^2$$

$$v_{1,2} = (q^{-1} - q) EK^{-1} a^2 + qac$$

$$v_{1,3} = K^{-1} a^2$$

$$v_{2,1} = q^{-1}(q - q^{-1})^2 E^2 K^{-1} ab - (q^2 - q^{-2}) Ebc - (q - q^{-1}) E + q^{-1} K dc$$

$$v_{2,2} = (q^{-1} - q) EK^{-1} ab + bc + \tfrac{1}{q+q^{-1}}$$

$$v_{2,3} = K^{-1} ab$$

$$v_{3,1} = q^{-1}(q - q^{-1})^2 E^2 K^{-1} b^2 - (q^2 - q^{-2}) Edb + Kd^2$$

$$v_{3,2} = (q^{-1} - q) EK^{-1} b^2 + db$$

$$v_{3,3} = K^{-1} b^2$$

As a left $D(H)$-module, $H_{0,2} = V_1 \oplus V_2 \oplus V_3$, where each V_i is the left $D(H)$-module with basis $\{v_{1i}, v_{2i}, v_{3i}\}$. In this basis, the left action of $x \in D(H)$ on $v \in V_i$

is given by $x \blacktriangleright v = \phi_i(x)v$, where

$$\phi_i(E) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & q+q^{-1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i(F) = \begin{bmatrix} 0 & 0 & 0 \\ q+q^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\phi_i(K) = \begin{bmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{bmatrix} \qquad \phi_i(K^{-1}) = \begin{bmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{bmatrix}$$

$$\phi_i(a) = \begin{bmatrix} q^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{bmatrix} \qquad \phi_i(b) = \begin{bmatrix} 0 & 0 & 0 \\ q^{-1}-q^3 & 0 & 0 \\ 0 & q^{-1}-q & 0 \end{bmatrix}$$

$$\phi_i(c) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i(d) = \begin{bmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{bmatrix}$$

As a right $D(H)$-module, $H_{0,2} = V_1' \oplus V_2' \oplus V_3'$, where each V_i' is the right $D(H)$-module with basis $\{v_{i1}, v_{i2}, v_{i3}\}$. In this basis, the right action of $x \in D(H)$ on $v \in V_i'$

is given by $v \blacktriangleleft x = \phi_i'(x)v$, where

$$\phi_i'(E) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & q+q^{-1} \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i'(F) = \begin{bmatrix} 0 & 0 & 0 \\ q+q^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\phi_i'(K) = \begin{bmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{bmatrix} \qquad \phi_i'(K^{-1}) = \begin{bmatrix} q^{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{bmatrix}$$

$$\phi_i'(a) = \begin{bmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{bmatrix} \qquad \phi_i'(b) = \begin{bmatrix} 0 & 0 & 0 \\ q^{-2}-q^2 & 0 & 0 \\ 0 & 1-q^2 & 0 \end{bmatrix}$$

$$\phi_i'(c) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \phi_i'(d) = \begin{bmatrix} q^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{bmatrix}$$

We see that $\phi_i(E)$ is a rank-2 matrix with null space spanned by $\{v_{1i}\}$ and $\phi_i'(E)$ is a rank-2 matrix with null space spanned by $\{v_{i1}\}$. It follows that the subspace $\{v \in H_{0,2} \mid E \blacktriangleright v = v \blacktriangleleft E = 0\}$ of highest-weight bivectors is spanned by v_{11}. Elsewhere in this paper, we refer to this bivector as

$$v_6 = q^{-1}(q-q^{-1})^2E^2K^{-1}a^2 - q(q^2-q^{-2})Eac + Kc^2 \tag{5.3}$$

We note that $\dfrac{q^3}{q^4-1}\, v_6 = (1-q^2)v_1v_4 - [v_3, v_2]_{q^2}$.

Remark 5.4. We see that $H_{2,0}$ is not isomorphic to $H_{0,2}$ because b annihilates $H_{2,0}$ but does not annihilate $H_{0,2}$.

Conjecture 5.5. Based on these examples, we conjecture that $H_{\lambda,\mu}$ is generated as a $D(H)$-bimodule by $h_{\lambda,\mu}$ where

$$h_{\lambda,\mu} = \begin{cases} K^{-(\lambda+\mu)/2}c^{\lambda-\mu} & \text{if } \lambda \geq \mu, \\ K^{-(\lambda+\mu)/2}b^{\mu-\lambda} & \text{if } \lambda < \mu. \end{cases}$$

CHAPTER VI

PROOFS

6.1. Proof of Theorem 4.1

We define a pairing $\langle\ ,\ \rangle\colon D(H)\otimes(H\otimes C)\to k$ by

$$\langle c\cdot h,\bar{h}\cdot\bar{c}\rangle=\phi(c,\bar{h})\phi(\bar{c},h) \tag{6.1}$$

As described in Section 2.3, we can use this pairing to define left and right actions of $D(H)$ on $H\otimes C$. The rest of the proof is a direct calculation.

We define $\blacktriangleleft\colon(C^*\otimes H^*)\otimes D(H)\to C^*\otimes H^*$ so that

$$\begin{aligned}
&\langle c'\cdot h',(c^*\cdot h^*)\blacktriangleleft(c\cdot h)\rangle\\
&\qquad=\langle(c\cdot h)\cdot(c'\cdot h'),c^*\cdot h^*\rangle\\
&\qquad=\langle c\cdot c'_{(2)}\cdot h_{(2)}\cdot h',c^*\cdot h^*\rangle\phi(c'_{(1)},Sh_{(1)})\phi(c'_{(3)},h_{(3)})\\
&\qquad=\phi(c\cdot c'_{(2)},c^*)\phi(h^*,h_{(2)}\cdot h')\phi(c'_{(1)},Sh_{(1)})\phi(c'_{(3)},h_{(3)})\\
&\qquad=\phi(c'_{(2)}c,c^*)\phi(h^*,h_{(2)}h')\phi(c'_{(1)},Sh_{(1)})\phi(c'_{(3)},h_{(3)})\\
&\qquad=\phi(c'_{(2)},c^*_{(1)})\phi(c,c^*_{(2)})\phi(h^*_{(1)},h_{(2)})\phi(h^*_{(2)},h')\phi(c'_{(1)},Sh_{(1)})\phi(c'_{(3)},h_{(3)})\\
&\qquad=\phi(c',Sh_{(1)}c^*_{(1)}h_{(3)})\phi(c,c^*_{(2)})\phi(h^*_{(1)},h_{(2)})\phi(h^*_{(2)},h')\\
&\qquad=\langle c'\cdot h',Sh_{(1)}c^*_{(1)}h_{(3)}\cdot h^*_{(2)}\rangle\phi(c,c^*_{(2)})\phi(h^*_{(1)},h_{(2)})\\
&\qquad=\langle c'\cdot h',Sh_{(1)}(c\rhd c^*)h_{(3)}\cdot(h^*\lhd h_{(2)})\rangle
\end{aligned}$$

and thus

$$(c^* \cdot h^*) \blacktriangleleft c = (c \triangleright c^*) \cdot h^*$$

$$(c^* \cdot h^*) \blacktriangleleft h = Sh_{(1)} c^* h_{(3)} \cdot (h^* \triangleleft h_{(2)})$$

We define $\blacktriangleright \colon D(H) \otimes (C^* \otimes H^*) \to C^* \otimes H^*$ so that

$$\begin{aligned}
\langle c' \cdot h', h \blacktriangleright (c^* \cdot h^*) \rangle &= \langle (c' \cdot h')(h), c^* \cdot h^* \rangle \\
&= \langle c' \cdot h' h, c^* \cdot h^* \rangle \\
&= \phi(c', c^*) \phi(h^*, h' h) \\
&= \phi(c', c^*) \phi(h^*_{(1)}, h') \phi(h^*_{(2)}, h) \\
&= \langle c' \cdot h', c^* \cdot h^*_{(1)} \rangle \phi(h^*_{(2)}, h) \\
&= \langle c' \cdot h', c^* \cdot (h \triangleright h^*) \rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle c' \cdot h', c \blacktriangleright (c^* \cdot h^*) \rangle &= \langle (c' \cdot h')(c), c^* \cdot h^* \rangle \\
&= \langle c' \cdot c_{(2)} \cdot h'_{(2)}, c^* \cdot h^* \rangle \phi(c_{(1)}, Sh'_{(1)}) \phi(c_{(3)}, h'_{(3)}) \\
&= \phi(c_{(2)} c', c^*) \phi(h^*, h'_{(2)}) \phi(c_{(1)}, Sh'_{(1)}) \phi(c_{(3)}, h'_{(3)}) \\
&= \phi(c_{(2)}, c^*_{(1)}) \phi(c', c^*_{(2)}) \phi(h^*, h'_{(2)}) \phi(c_{(1)}, Sh'_{(1)}) \phi(c_{(3)}, h'_{(3)}) \\
&= \phi(c_{(2)}, c^*_{(1)}) \phi(c', c^*_{(2)}) \phi(h^*, h'_{(2)}) \phi(Sc_{(1)}, h'_{(1)}) \phi(c_{(3)}, h'_{(3)}) \\
&= \phi(c_{(2)}, c^*_{(1)}) \phi(c', c^*_{(2)}) \phi(Sc_{(1)} h^* c_{(3)}, h') \\
&= \phi(c_{(2)}, c^*_{(1)}) \langle (c' \cdot h', c^*_{(2)} \cdot (Sc_{(1)} h^* c_{(3)}) \rangle \\
&= \langle (c' \cdot h', (c^* \triangleleft c_{(2)}) \cdot Sc_{(1)} h^* c_{(3)} \rangle
\end{aligned}$$

Thus

$$c \blacktriangleright (c^* \cdot h^*) = (c^* \triangleleft c_{(2)}) \cdot Sc_{(1)} h^* c_{(3)}$$

$$h \blacktriangleright (c^* \cdot h^*) = c^* \cdot (h \triangleright h^*)$$

The theorem is proved. $\square$

6.2. Proof of Theorem 4.6

We recall the vectors from (5.1), (5.2), and (5.3).

$$v_1 = (q - q^{-1}) E K^{-1} ac - c^2$$

$$v_2 = (q - q^{-1}) \Delta ac - \tfrac{q+q^{-1}}{q-q^{-1}} Kac - q^{-2} FKc^2 + Ea^2$$

$$v_3 = EK^{-1}$$

$$v_4 = \Delta$$

$$v_5 = K^{-1} c^2$$

$$v_6 = q^{-1} (q - q^{-1})^2 E^2 K^{-1} a^2 - q(q^2 - q^{-2}) Eac + Kc^2$$

We will show that ${}^+(H \otimes C)^+$ has basis

$$\{v_3^\ell v_1^m v_5^p v_2^s\} \cup \{v_3^\ell v_1^m v_6^r v_2^s\} \cup \{v_3^\ell v_4^n v_5^p v_2^s\} \cup \{v_3^\ell v_4^n v_6^r v_2^s\}.$$

In order to do this computation, we will first embed ${}^+(H \otimes C)^+ \hookrightarrow {}^+\mathcal{L}_1{}^+ \hookrightarrow {}^+\mathcal{L}_2{}^+$, where ${}^+\mathcal{L}_1{}^+$ and ${}^+\mathcal{L}_2{}^+$ are localized algebras. Specifically, let A be the algebra with basis

$$\left\{ v_3^m v_4^n v_5^p K^k a^\ell c^\epsilon \mid m, n, p, \ell \in \mathbb{Z}_{\geq 0},\ k \in \mathbb{Z},\ \epsilon \in \{0, 1\} \right\}$$

The relations of $H \otimes C$ imply the following; see Proposition 2.39.

Lemma 6.1. The vector v_4 is central in $H \otimes C$, and

$$0 = [v_3, v_5]_{q^2} = [v_3, K]_{q^{-2}} = [v_3, a] = [v_3, c] = [v_5, K] = [v_5, a]_{q^2} = [v_5, c].$$

Corollary 6.2. Both v_3 and c are Ore elements in A.

Let $\mathcal{L}_1 = A[v_3^{-1}]$ and $\mathcal{L}_2 = A[v_3^{-1}, c^{-1}]$. Clearly ${}^+\mathcal{L}_1{}^+ \hookrightarrow {}^+\mathcal{L}_2{}^+$. We wish to show that ${}^+(H \otimes C)^+ \hookrightarrow {}^+\mathcal{L}_1{}^+$.

Lemma 6.3. The subspace ${}^+C \subset C$ of vectors satisfying $E \blacktriangleright c = 0$ is spanned by $\{a^i c^j \mid i, j \geq 0\}$. Furthermore, if $v \in \mathcal{H}$, then $E \blacktriangleright v = 0$ if and only if $v \in H \otimes {}^+C$.

Proof. Recall that $H \otimes C$ is spanned by $\{E^i F^j \Delta^k K^{\pm\ell} a^p c^r b^m d^n \mid ij = pn = 0\}$. Because

$$E \blacktriangleright E^i F^j \Delta^k K^\ell a^p c^r b^m d^n = E^i F^j \Delta^k K^\ell a^p c^r (E \blacktriangleright b^m d^n)$$

we consider $E \blacktriangleright b^m d^n$.

Now

$$E \blacktriangleright b^m = \frac{1 - q^{2m}}{1 - q^2} q^{1-m} a b^{m-1}, \qquad E \blacktriangleright d^n = \frac{1 - q^{2n}}{1 - q^2} q^{1-n} c d^{n-1},$$

$$E \blacktriangleright b^m d^n = \frac{q^{-2m} - 1}{1 - q^2} q^{2-n} b^{m-1} d^{n-1} + \frac{2 - q^{2n} - q^{-2m}}{1 - q^2} q^{1-n} c b^m d^{n-1}.$$

For $x \in H \otimes C$, consider the terms in $E \blacktriangleright x$ with maximum $n - p$. Of those, consider the terms with maximum $m - r$. Of those, consider the terms with maximum m. The coefficient of each of those terms is a nonzero multiple of the coefficient of exactly one term in x. Thus the only way to get $E \blacktriangleright x = 0$ is for $x \in H \otimes {}^+C$. $\square$

Lemma 6.4. $H \otimes {}^+C$ is a subalgebra of $\mathcal{L}_1$.

Proof. We write $E = v_3 K$,

$$\begin{aligned} F &= (FE)(v_3K)^{-1} \\ &= \left(v_4 - \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}\right) q^2 v_3^{-1} K^{-1} \\ &= q^2 v_3^{-1} v_4 K^{-1} - \frac{q}{(q-q^{-1})^2} v_3^{-1} - \frac{q^3}{(q-q^{-1})^2} v_3^{-1} K^{-2} \end{aligned}$$

and $c^{2p+\epsilon} = v_5^p K^p c^\epsilon$. □

We thus have established an embedding of algebras ${}^+(H \otimes C)^+ \hookrightarrow {}^+\mathcal{L}_1{}^+ \hookrightarrow {}^+\mathcal{L}_2{}^+$.

Proposition 6.5. The set $\{v_1^s v_3^m v_4^n v_5^p \mid s, n \in \mathbb{Z}_{\geq 0},\ m, p \in \mathbb{Z}\}$ is a basis of ${}^+\mathcal{L}_2{}^+$.

Proof. We can write $(q - q^{-1})a = v_1 v_3^{-1} c^{-1} + v_3^{-1} v_5 K c^{-1}$, so a basis of $\mathcal{L}_2$ is

$$\left\{ v_1^s v_3^m v_4^n v_5^p K^k c^\epsilon \mid s, n \in \mathbb{Z}_{\geq 0},\ m, p, k \in \mathbb{Z},\ \epsilon \in \{0, 1\} \right\}.$$

We already know that $E \blacktriangleright v = 0$ for all $v \in \mathcal{L}_2$. Our goal, therefore, is to show that the solutions to $v \blacktriangleleft E = 0$ are those vectors v belonging to the subspace spanned by $\{v_1^s v_3^m v_4^n v_5^p \mid s, n \in \mathbb{Z}_{\geq 0},\ m, p \in \mathbb{Z}\}$. So supposing that

$$0 = \left(\sum v_1^s v_3^m v_4^n v_5^p K^k \big(\alpha_{s,m,n,p,k} + c\beta_{s,m,n,p,k}\big) \right) \blacktriangleleft E$$

we will show that $\beta_{s,m,n,p,k} = 0$ for all indices, and $\alpha_{s,m,n,p,k} = 0$ unless $k = 0$. Now

$$K^k \blacktriangleleft E = (q^{2k} - 1) E K^k = (q^{2k} - 1) v_3 K^{k+1}$$

and

$$
\begin{aligned}
K^k c \blacktriangleleft E &= (K^k \blacktriangleleft E)(c \blacktriangleleft K) + K^k(c \blacktriangleleft E) \\
&= (q^{2k}-1)EK^k q^{-1}c + K^k(1-q^{-1})Ec \\
&= (q^{2k-1}-q^{-1})v_3K^{k+1}c + (1-q^{-1})K^k v_3 Kc \\
&= (q^{2k-1}-q^{-1})v_3K^{k+1}c + (q^{2k}-q^{2k-1})v_3K^{k+1}c \\
&= (q^{2k}-q^{-1})v_3K^{k+1}c
\end{aligned}
$$

so

$$
\begin{aligned}
0 &= \Big(\sum v_1^s v_3^m v_4^n v_5^p K^k\big(\alpha_{s,m,n,p,k} + c\beta_{s,m,n,p,k}\big)\Big) \blacktriangleleft E \\
&= \sum v_1^s v_3^m v_4^n v_5^p \Big(K^k\big(\alpha_{s,m,n,p,k} + c\beta_{s,m,n,p,k}\big) \blacktriangleleft E\Big) \\
&= \sum v_1^s v_3^m v_4^n v_5^p v_3 K^{k+1}\Big((q^{2k}-1)\alpha_{s,m,n,p,k} + c(q^{2k}-q^{-1})\beta_{s,m,n,p,k}\Big)
\end{aligned}
$$

Thus if $\alpha_{s,m,n,p,k} \neq 0$ then $q^{2k}=1$, so $k=0$ since q is not a root of unity. Also if $\beta_{s,m,n,p,k} \neq 0$ then $2k=-1$ which is impossible for $k \in \mathbb{Z}$. $\square$

We just gave a basis of ${}^+\mathcal{L}_2{}^+$. Our next task is step back to ${}^+\mathcal{L}_1{}^+$ by determining which highest-weight bivectors have only nonnegative powers of c.

Proposition 6.6. The algebra ${}^+\mathcal{L}_1{}^+$ is generated by $\{v_1, v_3^{\pm 1}, v_4, v_5, v_6\}$.

Proof. We wish to find $\big\langle v_1, v_3^{\pm1}, v_4, v_5^{\pm1}\big\rangle \cap A[v_3^{-1}]$. We notice that c does not occur in v_3 or v_4, but only in v_1 and v_5. Therefore, the only way for $v_1^s v_3^m v_4^n v_5^{-1}$ to be in $A[v_3^{-1}]$ is for $s > 0$. In fact, we must have $s \geq 2$ so that c^2 is a factor in $v_1^s v_3^m v_4^n$. But $v_1^2 v_5^{-1} = v_6$. $\square$

Our final task is to step back to ${}^+(H \otimes C)^+$ by reintroducing F and determining which highest-weight bivectors can be written with nonnegative powers of E. This will be much more challenging than Proposition 6.6.

We recall that a basis of $H \otimes {}^+C$ is $\{E^m F^n K^p a^\ell c^k \mid m, n, \ell, k \in \mathbb{Z}_{\geq 0},\, p \in \mathbb{Z}\}$. We define

$$\deg_E(x) = \max\{m \in \mathbb{Z} \mid x = E^m y \text{ for some } y \in H \otimes {}^+C\}.$$

Definition 6.7. Let $\lambda\colon H \otimes {}^+C \to H \otimes {}^+C$ be the map given on the basis by

$$\lambda\big(E^m F^n K^p a^\ell c^k\big) = \begin{cases} 0 & \text{if } m > 0, \\ F^n K^p a^\ell c^k & \text{if } m = 0. \end{cases}$$

Remark 6.8. If B is the algebra generated by $\{F, K, a, c\}$, then in fact $\lambda\colon H \otimes {}^+C \to B$ is a morphism of right B-modules, and is the quotient by the right ideal generated by E.

In particular,

$$\begin{aligned} \lambda(v_1) &= -c^2 \\ \lambda(v_3) &= 0 \\ \lambda(v_4) &= \frac{q^{-1}}{(q-q^{-1})^2} K + \frac{q}{(q-q^{-1})^2} K^{-1} \\ \lambda(v_5) &= K^{-1} c^2 \\ \lambda(v_6) &= K c^2 \end{aligned} \tag{6.2}$$

Proposition 6.9. Let R be the algebra generated by $\{v_1, v_3, v_4, v_5, v_6\}$. Let M be the left R-module with R-basis $\{v_2^n\}$ where the action is $r \triangleright r' v_2^n = r r' v_2^n$. We note that because R and M are subspaces of $H \otimes {}^+C$, we may give $\lambda(M)$ the structure of a left R-module with action $r \triangleright x = \lambda(r)x$. Then λ is an R-module homomorphism from M to $\lambda(M)$.

Proof. Now $K^{\pm 1}$, a, and c each quasi-commute with E, so it follows that $\lambda(v_i v_j) = \lambda(v_i)\lambda(v_j)$ for $v_i, v_j \in \{v_1, v_3, v_4, v_5, v_6\}$ since F does not appear in any of those vectors. Thus λ, when restricted to R, is a ring homomorphism. By the same argument, $\lambda(v_i v_2^n) = \lambda(v_i)\lambda(v_2^n)$ when $i \neq 2$. $\square$

Note that $\lambda(v_2^2) \neq \lambda(v_2)\lambda(v_2)$ because of the complicated relations between E and F.

Lemma 6.10. If we filter $H \otimes C$ by F-degree, where Δ has degree 0, then the monomial in $\lambda(v_2^s)$ of highest F-degree is

$$(-q^{-2}FKc^2)^s = (-1)^s q^{-s-s^2} F^s K^s c^{2s}.$$

Lemma 6.11. Let $x \in \langle v_1, v_2, v_4, v_5, v_6 \rangle \subset H \otimes^+ C$. Then x can be written as a finite sum $x = \sum v_3^i x_i$ where each x_i belongs to the subspace spanned by

$$S = \{v_1^m v_5^p v_2^s\} \cup \{v_1^m v_6^r v_2^s\} \cup \{v_4^n v_5^p v_2^s\} \cup \{v_4^n v_6^p v_2^s\}.$$

We will show in Proposition 6.12 that S is linearly independent. Here we are simply removing $v_5 v_6$ and $v_1 v_4$ using the identities

$$v_5 v_6 = v_1^2 \qquad \text{and} \qquad v_1 v_4 = v_3 v_2 - \frac{q}{(q-q^{-1})^2} v_5 - \frac{q^{-1}}{(q-q^{-1})^2} v_6.$$

This introduces v_3's, but in such a way that they appear to the left of the v_2's. We use the fact that v_2 and v_3 both quasi-commute with $\{v_1, v_4, v_5, v_6\}$, which commute with each other.

Proposition 6.12. The set S from Lemma 6.11 is linearly independent, and the restriction of λ to the span of S is injective.

Proof. Let $x = \sum_{i \in I} \alpha_i x_i$ be a linear combination of elements of S and suppose that $\lambda(x) = 0$. We will show that $\alpha_i = 0$ for all $i \in I$, which will prove both parts of the proposition.

We filter $H \otimes {}^+C$ by F-degree, where Δ has degree 0. Now let us fix $s = \max_{i \in I}(\deg(\lambda(x_i)))$. By Proposition 6.9 and Lemma 6.10, the monomials in $\lambda(x)$ of degree s are linear combinations of the degree-s parts of $\{\lambda(x_i) \mid x_i \text{ has } v_2^s\}$. So we define $I' = \{i \in I \mid x_i \text{ has } v_2^s\}$ and we consider $x_s = \sum_{i \in I'} \alpha_i x_i' v_2^s$ where $x_i = x_i' v_2^s$ and x_i' belongs to

$$S' = \{v_1^m v_5^p\} \cup \{v_1^m v_6^r\} \cup \{v_4^n v_5^p \mid n > 0\} \cup \{v_4^n v_6^r \mid n > 0\}$$

Since $\lambda(v_2^s) \neq 0$, it follows from Proposition 6.9 that $\lambda(x) = 0$ if and only if $\lambda(\sum \alpha_i x_i') = 0$.

Now from (6.2) we have

$$\lambda(v_1^m v_5^p) = (-1)^m K^{-p} c^{2(m+p)}$$

$$\lambda(v_1^m v_6^r) = (-1)^m K^r c^{2(m+r)}$$

$$\lambda(v_4^n v_5^p) = \sum_{j=0}^{n} \binom{n}{j} \frac{q^{n-2j}}{(q-q^{-1})^{2n}} K^{-(n-2j)-p} c^{2p}$$

$$\lambda(v_4^n v_6^r) = \sum_{j=0}^{n} \binom{n}{j} \frac{q^{n-2j}}{(q-q^{-1})^{2n}} K^{-(n-2j)+r} c^{2r}$$

If $\{x'_i\}_{i\in I'}$ and $\{v_4^n v_5^p \mid n > 0\} \cup \{v_4^n v_6^r \mid n > 0\}$ have nonempty intersection, we may fix n to be the maximum $n > 0$ that occurs.

Suppose that for some $i \in I'$, we have $x'_i \in \{v_4^n v_5^p\}$. When $j = 0$, the term $K^{-(n+p)}c^{2p}$ appears in $\lambda(x'_i)$. Since the power of c is strictly less than twice the absolute value of the power of K, this term does not appear in $\lambda(v_1^m v_5^{p'})$ or $\lambda(v_1^m v_6^r)$. This term would appear in $\lambda(v_4^{n'} v_6^r)$ only if $r = p$ and $n' = n + 2i + 2p$ for some $i \geq 0$. But if $p = r > 0$, then this implies $n' > n$, which contradicts the maximality of n.

Suppose that for some $i \in I'$, we have $x'_i \in \{v_4^n v_6^r\}$. When $j = n$, the term $K^{n+r}c^{2r}$ appears in $\lambda(x'_i)$. Since the power of c is strictly less than twice the absolute value of the power of K, this term does not appear in $\lambda(v_1^m v_5^p)$ or $\lambda(v_1^m v_6^{r'})$. This term would appear in $\lambda(v_4^{n'} v_5^p)$ only if $p = r$ and $2i - n' = n + 2r$ for some $i \leq n'$. But if $r = p > 0$, then this implies $n' > n$, which contradicts the maximality of n.

Thus $\{x'_i\}_{i\in I'}$ and $\{v_4^n v_5^p \mid n > 0\} \cup \{v_4^n v_6^r \mid n > 0\}$ have empty intersection, so $x_s = \sum_{i\in I'} \alpha_i x'_i v_2^s$ where x'_i belongs to

$$S'' = \{v_1^m v_5^p\} \cup \{v_1^m v_6^r \mid r > 0\}$$

But $v(v_1^m v_5^p)$ and $v(v_1^{m'} v_6^r)$ are linearly independent when $r > 0$ due to the power of K. We conclude that $\alpha_i = 0$ for all $i \in I'$. Since s was maximal, it must be that $s = 0$, so in fact $\alpha_i = 0$ for all $i \in I$. $\square$

By Theorem 6.6 and the quasi-commutativity of v_1, v_3, v_4, v_5, and v_6, we may write any highest-weight bivector $w \in {}^+A[v_3^{-1}]^+ \cap (H \otimes C)$ as

$$w = y_0 + \sum_{i=1}^{m} v_3^{-i} y_i, \quad y_0 \in \langle v_1, v_3, v_4, v_5, v_6 \rangle, \quad y_i \in \langle v_1, v_4, v_5, v_6 \rangle.$$

Now we apply Lemma 6.11 to each y_i. We get a new sum

$$w = x_0 + \sum_{i=1}^{n} v_3^{-i} x_i, \quad x_0 \in \langle v_1, v_2, v_3, v_4 \rangle, \quad x_i \in \mathrm{span}(S).$$

Now $v_3^{n-1} w = z + v_3^{-1} x_n$ where $z \in \langle v_1, v_2, v_3, v_4 \rangle$, so $x_n = v_3^n w - v_3 z$. Then $\lambda(x_n) = \lambda(v_3^n w - v_3 z) = 0$. Since $x_n \in \mathrm{span}(S)$, it follows from Proposition 6.12 that $x_n = 0$. Similarly $x_i = 0$ for $1 \le i < n$. Thus $w = x_0$, so $w \in \langle v_1, v_2, v_3, v_4 \rangle$.

The theorem is proved. $\square$

6.3. Proof of Main Theorem 4.2

Lemma 6.13. The algebra ${}^+(H \otimes C)^+$ is contained in $\bigoplus_{\substack{\lambda,\mu \ge 0 \\ \lambda-\mu \in 2\mathbb{Z}}} H_{\lambda,\mu}$.

Proof. We have defined $H_{\lambda,\mu}$ to be the image of $\beta_{V_{\lambda,\mu}}$, where $V_{\lambda,\mu}$ is $V_\lambda^+ \otimes V_\mu^-$. Now we know that ${}^+(H \otimes C)^+ \subset \sum_{n=0}^{\infty} \left(\beta_{V_{1,1}}(V_{1,1} \otimes V_{1,1}^*)\right)^n$ by Theorem 4.6. But products $V_{\lambda_1,\mu_1} \otimes V_{\lambda_2,\mu_2}$ obey the Pierri Rule, implying that tensor powers of $V_{1,1}$ span the subalgebra $\bigoplus V_{\lambda,\mu}$ where $\lambda - \mu$ is even. Therefore powers of $H_{1,1}$ decompose as sums of $H_{\lambda,\mu}$, where $\lambda - \mu \in 2\mathbb{Z}$, since these are the images of $\beta_{V_{\lambda,\mu}}$. $\square$

By Corollary 4.9, each vector in $\mathcal{H}$ belongs to a sub-bimodule generated by vectors in ${}^+(H \otimes C)^+$. By Lemma 6.13, each of those vectors belongs to some $H_{\lambda,\mu}$ with $\lambda - \mu \in 2\mathbb{Z}$. The theorem is proved. $\square$

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