KAZHDAN-LUSZTIG POLYNOMIALS OF MATROIDS AND THEIR ROOTS

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KATIE R. GEDEON

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Student: Katie R. Gedeon

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Nicholas Proudfoot	Chair
Arkady Berenstein	Core Member
Benjamin Elias	Core Member
Benjamin Young	Core Member
Alisa Freedman	Institutional Representative

and

Sara D. Hodges	Interim Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Katie R. Gedeon

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The Kazhdan-Lusztig polynomial of a matroid M, denoted $P_M(t)$, was recently defined by Elias, Proudfoot, and Wakefield. These polynomials are analogous to the classical Kazhdan-Lusztig polynomials associated with Coxeter groups. For example, in both cases there is a purely combinatorial recursive definition. Furthermore, in the classical setting, if the Coxeter group is a Weyl group then the Kazhdan-Lusztig polynomial is a Poincaré polynomial for the intersection cohomology of a particular variety; in the matroid setting, if M is a realizable matroid then the Kazhdan-Lusztig polynomial is also the intersection cohomology Poincaré polynomial of a variety corresponding to M. (Though there are several analogies between the two types of polynomials, the theory is quite different.)

Here we compute the Kazhdan-Lusztig polynomials of several graphical matroids, including thagomizer graphs, the complete bipartite graph $K_{2,n}$, and (conjecturally) fan graphs. Additionally, we investigate a conjecture by the author, Proudfoot, and Young on the real-rootedness for Kazhdan-Lusztig polynomials of these matroids as well as a conjecture on the interlacing behavior of these roots.

We also show that the Kazhdan-Lusztig polynomials of uniform matroids of rank $n-1$ on n elements are real-rooted.

This dissertation includes previously published and unpublished co-authored material.

CURRICULUM VITAE

NAME OF AUTHOR: Katie R. Gedeon

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
University of California, San Diego, San Diego, CA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2018, University of Oregon
Master of Science, Mathematics, 2014, University of Oregon
Bachelor of Science, Mathematics, 2012, University of California, San Diego

AREAS OF SPECIAL INTEREST:

Matroids
Equivariant Kazhdan-Lusztig Polynomials of Matroids
Non-Equivariant Kazhdan-Lusztig Polynomials of Matroids
Real-Rootedness

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, University of Oregon, 2012-2018

GRANTS, AWARDS AND HONORS:

Johnson Fellowship for Travel and Research, University of Oregon, 2017

PUBLICATIONS:

Katie Gedeon, Nicholas Proudfoot, and Benjamin Young. Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures. *Sém. Lothar. Combin.*, 78B:Art. 80, 12, 2017.

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Soli Deo gloria

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CHAPTER I

INTRODUCTION

Kazhdan-Lusztig polynomials of matroids were first studied in [EPW] where the authors laid out the analogy between this new theory and the classical theory of Kazhdan-Lusztig polynomials of Coxeter groups. The most compelling aspect of these Kazhdan-Lusztig polynomials of matroids is that although they can be defined very simply, they exhibit (or are conjectured to exhibit) many interesting properties that suggest a deep underlying structure.

In this document, we focus on the combinatorial aspect of the theory for matroids. In particular, we have two main goals:

1. To give a closed form of the coefficients of Kazhdan-Lusztig polynomials for some families of matroids.

2. To study the behavior of the roots of these polynomials.

The closed form of the coefficients of Kazhdan-Lusztig polynomials of matroids has been a subject of interest since polynomials of this type first appeared in [EPW]. In the appendix of that paper, the authors (along with Young) explicitly computed the coefficients of the Kazhdan-Lusztig polynomials associated to some uniform and braid matroids of small rank. Proudfoot, Wakefield and Young studied uniform matroids of rank $n-1$ on n elements in [PWY] and gave a combinatorial description for the coefficients of the associated Kazhdan-Lusztig polynomials. Chapter III is dedicated to this first goal. Here, we give a closed form of the coefficients for the Kazhdan-Lusztig polynomials of the matroids associated to thagomizer graphs in Theorem 3.1, the complete bipartite graph $K_{2,n}$ in Theorem 3.5, and (conjecturally) the fan graph in Conjecture 3.8. Theorem 3.1 first appeared in [Ged] of which I am the sole author, and Theorem 3.5 first appeared in [GPY2] which was co-authored with Proudfoot and Young.

We next turn our attention to the roots of these polynomials. *A priori*, there is no reason to think that studying the roots of Kazhdan-Lusztig polynomials of matroids would bear fruit. The roots themselves have no known interpretation geometrically or algebraically, and Kazhdan-Lusztig polynomials of Coxeter groups not real-rooted in general. A conjecture on the real-rootedness of the Kazhdan-Lusztig polynomials of matroids first appeared in [GPY2], and we record this in Conjecture 4.1. We prove the real-rootedness of the Kazhdan-Lusztig polynomials associated to the uniform matroids of rank $n-1$ on n elements in Theorem 4.3, and record some computer calculations that support Conjecture 4.1 for other families of matroids later in Chapter IV. Theorem 4.3 first appeared in [GPY2] (which was co-authored with Proudfoot and Young), and records the only infinite family of matroids for which the real-rootedness of the associated Kazhdan-Lusztig polynomials is known.

Further study of the roots of Kazhdan-Lusztig polynomials of matroids revealed an even more amazing and beautiful phenomenon: interlacing roots. Conjecture 4.2 records the conditions under which we expect the roots of $P_M(t)$ and $P_{M'}(t)$ to interlace for two matroids M and M'. Conjecture 4.2 also first appeared in [GPY2] which was co-authored with Proudfoot and Young. In Chapter IV we also record some computer calculations that support Conjecture 4.2 for the families of matroids we're concerned with here.

In addition to the main results we have included above, we have other results in this document that are of a different fold. We conclude this section with a description of the structure of this document, which includes some of our other results and conjectures. Chapter II records the relevant background information that will be assumed in the later chapters.

In Chapter III, as we stated earlier, our attention is on the main goal listed above for some families of graphic matroids. The study of the coefficients of the Kazhdan-Lusztig polynomials associated to such matroids produced Conjecture 3.12, which states that these coefficients appear to be bounded by the coefficients of the Kazhdan-Lusztig polynomial of the matroid associated to the complete graph. There is no reason to believe that Kazhdan-Lusztig polynomials of matroids have bounded coefficients in general.

Section 3.1.2. explores the S_n action on the thagomizer matroid of rank $n+1$, which allows us to make a conjecture for the S_n-equivariant Kazhdan-Lusztig polynomial of this matroid (Conjecture 3.6). This categorification of Kazhdan-Lusztig coefficients was first considered for a uniform matroid of rank $n-1$ on n elements by Proudfoot, Wakefield and Young [PWY] where they were given by an irreducible representation of S_n. The equivariant Kazhdan-Lusztig polynomial for a general matroid was subsequently defined by the author, Proudfoot and Young [GPY1] where we further studied uniform matroids in this context and computed the S_n-equivariant Kazhdan-Lusztig polynomials of braid matroids of small rank. The work in this section first appeared in [Ged] of which I am the sole author.

Chapter IV is concerned with the study of the roots of the Kazhdan-Lusztig polynomials of matroids, and includes results and conjectures on the real and interlacing properties of the roots as described above. Of particular note is Section 4.3., which records the progress that was made with Mirkó Visontai towards solving Conjectures 4.1 and 4.2 for thagomizer matroids. The interesting method employed in our attempt gives a glimpse into the complex strategies and methods used to solve this sort of problem in general. (Visontai and I have no plans to publish this material.)

Finally, Appendix A records computations of the coefficients of some Kazhdan-Lusztig polynomials of thagomizer and fan matroids of small rank.

CHAPTER II

PRELIMINARIES

In this chapter, we provide necessary definitions and collect known results that will be used in the later chapters.

2.1. Matroids and Their Kazhdan-Lusztig Polynomials

A (finite) **matroid** M is a pair $(E, \mathcal{I})$ where E is a (finite) collection of objects, called the **ground set** of M, and $\mathcal{I} \subseteq 2^E$ such that

(i) $\varnothing \in \mathcal{I}$,

(ii) if $S \in \mathcal{I}$ and $S' \subseteq S$, then $S' \in \mathcal{I}$, and

(iii) if $S, T \in \mathcal{I}$ such that $|S| < |T|$, there exists $x \in T \setminus S$ such that $S \cup \{x\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the **independent sets** of M. There are other (equivalent) ways to define a matroid, which we will not explore (see [Oxl]).

For any $S \subseteq E$, the **rank** of S, denoted $\mathrm{rk}(S)$, is the size of the largest independent subset of S. We also have the **closure** operator, cl, where

$$\mathrm{cl}(S) := \{x \in E \mid \mathrm{rk}(S) = \mathrm{rk}(S \cup \{x\})\}.$$

If $S = \mathrm{cl}(S)$, we say that S is **closed**. We also refer to the closed sets as **flats** of M. These flats form a geometric lattice, which we denote by $L(M)$.

Let G be an undirected graph. The **graphic matroid** $M(G)$ has ground set $E(G)$, the edges of G, and $\mathcal{I}$ is the family of sets that form forests in G. A set of edges $S \in E(G)$ forms a flat of $M(G)$ if there does not exist $x \in E$ where $S \cup \{x\}$ creates a cycle.

We will only consider loopless graphs. If G' is a graph obtained from G by multiplying edges, then $L(M(G)) \cong L(M(G'))$. Hence from now on, we consider graphs without multiple edges.

Example 2.1. Consider the graph G given below.

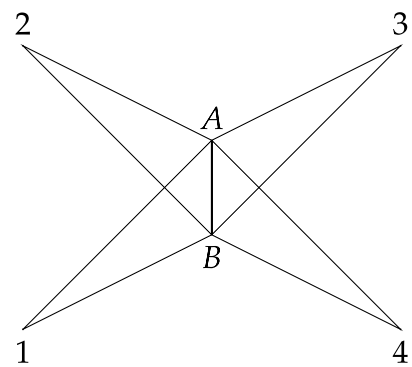

FIGURE 2.1. The thagomizer graph on 6 vertices.

Then $E(G) = \{1A, 1B, 2A, 2B, 3A, 3B, 4A, 4B, AB\}$. The independent sets of $M(G)$ include $\{AB\}$ and $\{1A, 3B, 4A\}$. The set $\{1A, 1B, AB\}$ is not an independent set, but it is a flat of rank 2. The rank of $M(G)$ is 5.

If a connected graph G has n vertices, then any spanning tree of G will have $n-1$ edges. This gives us the following lemma.

Lemma 2.2. Let M be the graphic matroid associated to a graph on n vertices. Then $\mathrm{rk}(M) = n - 1$.

We define the **characteristic polynomial** of a matroid M to be

$$\chi_M(t) := \sum_{F \in L(M)} \mu(\varnothing, F) t^{\mathrm{rk}(M) - \mathrm{rk}(F)},$$

where μ is the Möbius function on $L(M)$. If $M = M(G)$ is a graphic matroid, where G has k connected components and chromatic polynomial $\pi_G(t)$, then

$$\chi_M(t) = t^{-k}\pi_G(t).$$

There are two operations on matroids that we will consider; contraction and localization. The matroid M^F is called the **contraction** of M at F; it is the matroid on the ground set $E \setminus F$ whose lattice of flats is $L^F := \{G \setminus F \mid G \in L(M) \text{ and } G \geq F\}$. The matroid M_F is called the **localization** of M at F and is the matroid with ground set F whose lattice of flats is $L_F := \{G \in L(M) \mid G \leq F\}$.

Theorem 2.3. [EPW, Theorem 2.2] There is a unique way to assign to each matroid M a polynomial $P_M(t) \in \mathbb{Z}[t]$, called the **Kazhdan-Lusztig polynomial** of M, such that the following conditions are satisfied.

(i) If $\operatorname{rk} M = 0$, then $P_M(t) = 1$.

(ii) If $\operatorname{rk} M > 0$, then $\deg P_M(t) < \frac{1}{2} \operatorname{rk} M$.

(iii) For every M, $t^{\operatorname{rk} M} P_M(t^{-1}) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$.

We call a matroid M **non-degenerate** if $\operatorname{rk} M = 0$ or if $P_M(t)$ has degree $\lfloor \frac{\operatorname{rk} M - 1}{2} \rfloor$.

The only non-graphic matroids we will consider are uniform matroids. A **uniform matroid** of rank d on $d + m$ elements, denoted $U_{m,d}$, can be represented as linearly independent subsets of $d + m$ generic vectors in a d-dimensional vector space. If $m = 1$, then $U_{1,d}$ can be represented graphically as a cycle graph on d vertices.

We are interested in the closed form of the coefficients of Kazhdan-Lusztig polynomials, which was first calculated for $U_{1,d}$ in [PWY].

2.2. Equivariant Matroids and The Equivariant Kazhdan-Lusztig Polynomial

Let W be a finite group acting on E and preserving M. We refer to the data $\{M, E, W\}$ as an **equivariant matroid** $W \curvearrowright M$. For any $F, G \in L(M)$, let $W_F \subseteq W$ be the stabilizer of F and let $W_{FG} := W_F \cap W_G$. Note that the action of W on M induces an action of W_F on both M_F and M^F. Let $\mathrm{VRep}(W)$ be the ring of isomorphism classes of virtual representations of W and set

$$\mathrm{grVRep}(W) := \mathrm{VRep}(W) \otimes_{\mathbb{Z}} \mathbb{Z}[t].$$

Let $OS^W_{M,i} \in \mathrm{Rep}(W)$ be the degree i part of the Orlik-Solomon algebra of M. The **equivariant characteristic polynomial** of M, $H^W_M(t)$, is given by

$$H^W_M(t) := \sum_{p=0}^{\mathrm{rk}\, M} (-1)^p t^{\mathrm{rk}\, M - p} OS^W_{M,p} \in \mathrm{grVRep}(W).$$

Note that the equivariant characteristic polynomial $H^W_M(t)$ is a categorified version of the usual characteristic polynomial $\chi_M(t)$. That is, we can recover $\chi_M(t)$ from $H^W_M(t)$ by taking the graded dimension.

The **equivariant Kazhdan-Lusztig polynomial** of $W \curvearrowright M$, denoted $\mathcal{P}^W_M(t)$, is a categorified version of the Kazhdan-Lusztig polynomial and is characterized by the following properties [GPY1, Theorem 2.8].

(i) If $\mathrm{rk}\, M = 0$, $\mathcal{P}^W_M(t)$ is equal to the trivial representation in degree 0.

(ii) If $\mathrm{rk}\, M > 0$, $\deg \mathcal{P}^W_M(t) < \frac{1}{2}\,\mathrm{rk}\, M$.

(iii) For every M, $t^{\operatorname{rk} M}\mathcal{P}_M^W(t^{-1}) = \sum_{[F]\in L/W} \operatorname{Ind}_{W_F}^W \left(H_{M_F}^{W_F}(t) \otimes \mathcal{P}_{M^F}^{W_F}(t) \right).$

The polynomial $\mathcal{P}_M^W(t)$ is an element of $\operatorname{grVRep}(W)$ and we can recover $P_M(t)$ from $\mathcal{P}_M^W(t)$ by taking the graded dimension.

2.3. Lattice Paths

A **Dyck path** of semilength n is a lattice path in $\mathbb{N}^2$ beginning at $(0,0)$ and ending at $(2n,0)$ with up-steps of the form $u = (1,1)$ and down-steps of the form $d = (1,-1)$. Such a Dyck path may be expressed as a word $\alpha \in \{u,d\}^{2n}$.

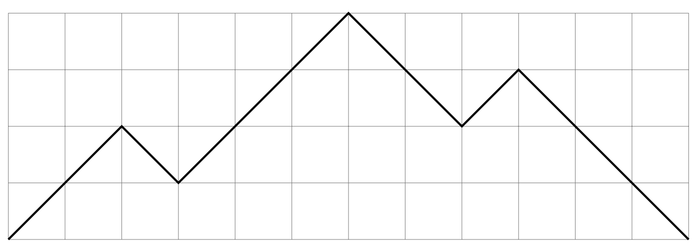

FIGURE 2.2. The Dyck path *uuduuuddduddd*.

A **long ascent** of a Dyck path is an ascent of length at least 2. Equivalently, a long ascent of a Dyck path α is a maximal subword consisting of at least two consecutive u's. The Dyck path given in Figure 2.2. has two long ascents.

Let $\mathcal{D}_n$ be the set of all Dyck paths of semilength n. We denote by $a_{n,k}$ the number of elements in $\mathcal{D}_n$ with exactly k long ascents. As noted in [STT], $a_{n,k}$ is also the number of words $\alpha \in \mathcal{D}_n$ with k occurrences of the subword uud. Additional interpretations of $a_{n,k}$ are known; see sequence A091156 in [Slo].

For each n, we now have a family of polynomials $\mathcal{F}_n(t)$ with

$$\mathcal{F}_n(t) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a_{n,k} t^k.$$

If $\mathrm{LA}(\alpha)$ is the total number of long ascents of a Dyck path α, then $\mathcal{F}_n(t)$ can also be represented as

$$\sum_{\alpha \in \mathcal{D}_n} t^{\mathrm{LA}(\alpha)}.$$

A **Motzkin path** is similar to a Dyck path, except we also allow horizontal-steps. That is, a Motzkin path of length n is lattice path in $\mathbb{N}^2$ beginning at $(0,0)$ and ending at $(n,0)$ with up-steps of the form $u = (1,1)$, down-steps of the form $d = (1,-1)$, and horizontal-steps of the form $h = (1,0)$. We denote by $\mathcal{Z}_n$ the Motzkin paths of length n and by $b_{n,k}$ the number of Motzkin paths of length n with k up-steps.

For each n, we have a family of polynomials $\mathcal{M}_n(t)$ with

$$\mathcal{M}_n(t) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} b_{n,k} t^k.$$

2.4. Real-Rootedness

A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called a **multiplier sequence** if, for any real polynomial

$$f(x) = \sum_{k=0}^{n} j_k x^k$$

with only real zeros, the polynomial

$$\Gamma[f(x)] = \sum_{k=0}^{n} \gamma_k j_k x^k$$

also has only real zeros.

Let $f(t), g(t) \in \mathbb{R}[t]$ be real-rooted polynomials with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_m$ respectively. Further assume that

$$\alpha_i \leq \alpha_{i+1} \quad \text{and} \quad \beta_i \leq \beta_{i+1}.$$

We say that $f(t)$ **interlaces** $g(t)$ whenever

$$m = n \quad \text{and} \quad \alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \beta_n$$

or

$$m = n-1 \quad \text{and} \quad \alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_{n-1} \leq \alpha_n,$$

and we write $f \prec g$. In particular, f and g can interlace only when their degrees differ by at most one. We use the phrase "f and g interlace" to mean both $f \prec g$ and $g \prec f$. We may also say "the roots of f and g interlace" to mean the same thing.

Let $h(t) \in \mathbb{R}[t]$ be an n-degree polynomial of the form

$$h(t) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k t^k (1+t)^{n-2k}.$$

Then $\{\gamma_k\}_k$ is called the **gamma vector** of h. The generating function for the gamma vector is called the **gamma polynomial** of $h(t)$ and is denoted $\gamma(h;t)$.

Recall that an n-degree polynomial

$$f(x) = \sum_{k=0}^{n} j_k x^k$$

is called **palindromic** if $j_i = j_{n-i}$. The following theorem is well known (e.g. see [Pet, Observation 4.2]).

Theorem 2.4. If h has palindromic coefficients, then $h(t)$ is real-rooted if and only if $\gamma(h;t)$ is real-rooted.

Let $N_{n,k}$ be the number of paths in $\mathcal{D}_n$ with exactly k occurrences of the subword ud. Then the set $N_{n,k}$ is known as the set of **Narayana numbers**. We set

$$\mathcal{N}_n(t) := \sum_{k=0}^{n-1} N_{n,k} t^k.$$

Then $\mathcal{N}_n(t)$ is called the n-th **Narayana polynomial**. By [Cok, Equation 4.4], we have

$$\mathcal{N}_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} b_{n-1,k} t^k (1+t)^{n-2k};$$

hence the n-th Motzkin polynomial is equal to $\gamma(\mathcal{N}_{n+1};t)$. In particular, since the Narayana polynomials are known to be real-rooted, this tells us that $\mathcal{M}_n(t)$ is real-rooted.

CHAPTER III

GRAPHIC MATROIDS

The results stated in Theorem 3.5 are from a joint project with Nicholas Proudfoot and Benjamin Young, though I was the primary contributor. This work originally appeared in [GPY2]. The results stated in Theorem 3.1, Lemma 3.2, and Conjecture 3.6 were published in [Ged], of which I was the sole author.

In this chapter, we study the Kazhdan-Lusztig polynomials and equivariant Kazhdan-Lusztig polynomials of some families of graphic matroids.

3.1. Thagomizer and Complete Bipartite Graphs

Let τ_n be the matroid associated with the graph obtained from the bipartite graph $K_{2,n}$ by adding an edge between the two distinguished vertices. We also let κ_n be the matroid associated to $K_{2,n}$.

We call τ_n a **thagomizer matroid**. The ground set of τ_n has size $2n+1$ and the rank of τ_n is $n+1$. Note that the underlying graph of τ_4 is given in Example 2.1.

3.1.1. The Kazhdan-Lusztig polynomials

Let $P_{\tau_n}(t)$ be the **Kazhdan-Lusztig polynomial** of τ_n and set

$$\Phi_\tau(t,u) := \sum_{n=0}^{\infty} P_{\tau_n}(t)u^{n+1}.$$

Let $c_{n,k}$ be the k-th coefficient of $P_{\tau_n}(t)$ and note that the degree of $P_{\tau_n}(t)$ is at most $\lfloor \frac{n}{2} \rfloor$. The following theorem is our first main result.

Theorem 3.1. The following (equivalent) statements hold.

(1) For all n and k, $c_{n,k}$ is the number of Dyck paths of semilength n with k long ascents.

(2) The generating function $\Phi_\tau(t,u)$ is equal to $\dfrac{1-\sqrt{1-4u(1-u+tu)}}{2(1-u+tu)}$.

We begin this section with a description of the flats $F \in L(\tau_n)$ given by the underlying graph. Let AB be the distinguished edge. For any $j \in \{1,\dots,n\}$, we call the subgraph with edges Aj and Bj a **spike**.

If $\operatorname{rk} F = i$, then either

1. F contains exactly one edge from i distinct spikes, or

2. F is the union of $i-1$ spikes and AB.

For example, when $n=4$, a rank 2 flat of the first type is given by $\{A1, B3\}$ and a rank 2 flat of the second type is given by $\{AB, A4, B4\}$ (see Figure 2.1.).

In the first case, the localization $(\tau_n)_F$ yields a Boolean matroid of rank i, and the contraction τ_n^F gives a matroid whose lattice of flats is isomorphic to that of τ_{n-i}. In the second case, the localization $(\tau_n)_F$ yields τ_{i-1}, and the contraction τ_n^F gives a matroid whose lattice of flats is isomorphic to that of a Boolean matroid of rank $n-i+1$.

The characteristic polynomial of a rank i Boolean matroid is equal to $(t-1)^i$. For thagomizer matroids, it is clear that $\chi_{\tau_i}(t) = (t-1)(t-2)^i$ by a simple deletion/contraction argument.

If F is of the first type and $\operatorname{rk} F = n-i$, there are $\binom{n}{n-i}$ ways to choose the spikes and 2^{n-i} choices of edges. If F is of the second type and $\operatorname{rk} F = i$, there are only $\binom{n}{i-1}$ choices.

We first turn our attention towards proving the following lemma.

Lemma 3.2. We have the following (equivalent) equations.

1. For all n, $t^{n+1}P_{\tau_n}(t^{-1}) = (t-1)^{n+1} + \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i}P_{\tau_i}(t)$.

2. $\Phi_\tau(t^{-1}, tu) = \dfrac{ut-u}{1+u-tu} + \Phi_\tau\left(t, \dfrac{u}{1+2u-2tu}\right)$.

Proof. There are $\binom{n}{i} \cdot 2^{n-i}$ flats of the first type of rank $n-i$ and $\binom{n}{i}$ flats of the second type of rank $i+1$. Note that for any Boolean matroid M, $P_M(t) = 1$ [EPW, Corollary 2.10]. Then we have

$$t^{n+1}P_{\tau_n}(t^{-1}) = \sum_{i=0}^{n} \binom{n}{i} \left(2^{n-i}(t-1)^{n-i}P_{\tau_i}(t) + (t-1)(t-2)^i\right) \tag{3.1}$$

$$= (t-1)^{n+1} + \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i}P_{\tau_i}(t) \tag{3.2}$$

which is the formula given in Lemma 3.2(1). Now our defining recursion tells us that

$$\begin{aligned}
\Phi_\tau(t^{-1}, tu) &= \sum_{n=0}^{\infty} P_{\tau_n}(t^{-1})t^{n+1}u^{n+1} \\
&= \sum_{n=0}^{\infty}(t-1)^{n+1}u^{n+1} + \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}2^{n-i}(t-1)^{n-i}P_{\tau_i}(t)u^{n+1}.
\end{aligned}$$

We let $m = n-i$ which allows us to write the second summand as

$$\sum_{i=0}^{\infty} P_{\tau_i}(t)u^{i+1} \sum_{m=0}^{\infty} 2^m \binom{m+i}{i}(t-1)^m u^m.$$

Recall the identity

$$\sum_{\ell=0}^{\infty} \binom{r+\ell}{r} x^\ell = \frac{1}{(1-x)^{r+1}}$$

and set $\ell = m$ and $x = 2u(t-1)$. This gives

$$\begin{aligned}\Phi_\tau(t^{-1}, tu) &= u(t-1)\sum_{n=0}^{\infty}(t-1)^n u^n + \sum_{i=0}^{\infty}\frac{P_{\tau_i}(t)u^{i+1}}{(1-2u(t-1))^{i+1}} \\ &= \frac{u(t-1)}{1-u(t-1)} + \sum_{i=0}^{\infty} P_{\tau_i}(t)\left(\frac{u}{1-2u(t-1)}\right)^{i+1} \\ &= \frac{ut-u}{1+u-tu} + \Phi_\tau\left(t, \frac{u}{1+2u-2tu}\right).\end{aligned}$$

This completes the proof of Lemma 3.2. □

Now we are ready to prove Theorem 3.1. Let $a_{n,k}$ be as in Section 2.3., and set

$$G(t,u) := \sum_{n,k\geq 0} a_{n,k} t^k u^n.$$

It was shown in [STT, Section 1] that $G(t,u)$ satisfies

$$u(1-u+tu)\cdot(G(t,u))^2 - G(t,u) + 1 = 0$$

which gives

$$G(t,u) = \frac{1-\sqrt{1-4u(1-u+tu)}}{2u(1-u+tu)}.$$

A priori, this formula should have a $\pm$ sign. However, a plus sign would not result in a formal power series with positive coefficients. Hence we conclude that the formula for $G(t,u)$ includes a negative sign instead.

Let $g(t,u) := u \cdot G(t,u)$. Since we'd like to show that $\Phi_\tau(t,u) = u\cdot G(t,u)$, we first check that $g(t,u)$ satisfies the functional equation in Lemma 3.2(2).

We have

$$g(t,u) = \frac{1-\sqrt{1-4u(1-u+tu)}}{2(1-u+tu)}$$

and hence

$$
\begin{aligned}
g(t^{-1}, tu) &= \frac{1-\sqrt{1-4tu(1-tu+u)}}{2(1-tu+u)} \\
&= \frac{ut-u}{1-tu+u} + \frac{1-2ut+2u-\sqrt{1-4tu(1-tu+u)}}{2(1-tu+u)} \\
&= \frac{ut-u}{1-tu+u} + \frac{1-\frac{1}{1+2u-2tu}\sqrt{1-4tu(1-tu+u)}}{\frac{2(1+u-tu)}{1+2u-2tu}} \\
&= \frac{ut-u}{1-tu+u} + \frac{1-\sqrt{1-\frac{4u(1+2u-2tu-u+tu)}{(1+2u-2tu)^2}}}{\frac{2(1+u-tu)}{1+2u-2tu}} \\
&= \frac{ut-u}{1-tu+u} + g\left(t, \frac{u}{1+2u-2tu}\right).
\end{aligned}
$$

Lastly, we note that both $c_{n,k}$ and $a_{n,k}$ are zero if $n > 2k$ and that $g(t,0) = \Phi_\tau(t,0) = 1$. Then $g(t,u) = \Phi_\tau(t,u)$ which equivalently tells us that $c_{n,k} = a_{n,k}$. This completes the proof of Theorem 3.1. □

Since we are interested in the closed form of the coefficients of $P_{\tau_n}(t)$, we record the following corollary to Theorem 3.1.

Corollary 3.3. For any n,

$$
c_{n,k} = \frac{1}{n+1}\binom{n+1}{k}\sum_{j=2k}^{n}\binom{j-k-1}{k-1}\binom{n+1-k}{n-j}.
$$

To see where this is recorded as the closed form for $a_{n,k}$, view [STT] and sequence A091156 in [Slo].

Remark 3.4. The total number of Dyck paths of semilength n is equal to the n-th Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. Thus Theorem 3.1 implies that $P_{\tau_n}(1) = \mathcal{C}_n$ and Corollary 3.3 implies that the leading coefficient of $P_{\tau_{2n}}(t)$ is $\mathcal{C}_n$. Interestingly, $\mathcal{C}_n$ also appears as the leading coefficient of the Kazhdan-Lusztig polynomial of the uniform matroid of rank $2n-1$ on $2n$ elements (see [EPW] Appendix A and [PWY]).

Next we turn our attention to the complete bipartite graph $K_{2,n}$ and its associated matroid κ_n. The ground set of κ_n has size $2n$ and the rank of κ_n is $n+1$.

Theorem 3.5. If $n \geq 2$,

$$P_{\kappa_n}(t) = P_{\tau_n}(t) + t.$$

Proof. Like with the thagomizer matroid, there are two types of flats for κ_n. Let $\{A, B, 1, 2, \cdots, n\}$ be the vertices of $K_{2,n}$ labelled in the obvious way. We also refer to the subgraph $K_{2,1}$ with vertices $\{A, B, i\}$, $i \in \{1, \ldots, n\}$ as a **spike**.

If $F \in L(\kappa_n)$ and $\operatorname{rk} F = j$, then either

1. F contains exactly one edge from j distinct spikes, or

2. F is the union of $j-1$ spikes.

In the first case, the localization $(\kappa_n)_F$ yields a Boolean matroid of rank i, and the contraction κ_n^F gives a matroid whose lattice of flats is isomorphic to that of τ_{n-j} when $j > 0$. If $j = 0$, the contraction instead yields a matroid with a lattice of flats isomorphic to that of κ_n.

In the second case, the localization $(\kappa_n)_F$ gives a matroid whose lattice of flats is isomorphic to that of κ_j, and the contraction κ_n^F is a Boolean matroid of rank $n-j$.

It is well-known that $\chi_{\kappa_j}(t) = (t-1)^j + (t-1)(t-2)^j$.

There are $\binom{n}{i} \cdot 2^{n-i}$ flats of the first type of rank $n-i$ and $\binom{n}{j}$ flats of the second type of rank $j+1$. Then the recursion says

$$t^{n+1}P_{\kappa_n}(t^{-1}) = P_{\kappa_n}(t) + \sum_{i=0}^{n-1} \binom{n}{i} 2^{n-i}(t-1)^{n-i}P_{\tau_i}(t) + \sum_{j=1}^{n} \binom{n}{j} \left((t-1)^j + (t-1)(t-2)^j\right).$$

Recall Equation 3.2, which gives

$$t^{n+1}P_{\tau_n}(t^{-1}) = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i}P_{\tau_i}(t) + \sum_{i=0}^{n} \binom{n}{i} (t-1)(t-2)^i.$$

Then we have

$$\begin{aligned} t^{n+1}P_{\kappa_n}(t^{-1}) - t^{n+1}P_{\tau_n}(t^{-1}) &= P_{\kappa_n}(t) - P_{\tau_n}(t) + \sum_{j=1}^{n} \binom{n}{j} (t-1)^j - (t-1) \\ &= P_{\kappa_n}(t) - P_{\tau_n}(t) + t^n - t, \end{aligned}$$

hence

$$t^{n+1}P_{\kappa_n}(t^{-1}) - P_{\kappa_n}(t) = t^{n+1}P_{\tau_n}(t^{-1}) - P_{\tau_n}(t) + t^n - t.$$

Since both $P_{\kappa_n}(t)$ and $P_{\tau_n}(t)$ have degree strictly less than $(n+1)/2$, this completes the proof. $\square$

3.1.2. The equivariant Kazhdan-Lusztig polynomials

Now we turn our attention back to the thagomizer matroid τ_n. Though the full automorphism group of τ_n is the signed permutation group on n elements, here we only consider the action of the symmetric group S_n. Let

$$\mathcal{P}_n(t) := \mathcal{P}^{S_n}_{\tau_n}(t) \quad \text{and} \quad \phi(t,u) := \sum_{u=0}^{\infty} \mathcal{P}_n(t)u^{n+1}.$$

Let Y_n be all partitions of n of the form $[a, n-a-2i-\eta, 2^i, \eta]$ where $\eta \in \{0,1\}$, $i \geq 0$ and $1 < a < n$. For any partition λ of n, we let V_λ be the irreducible representation of S_n indexed by λ.

For any partition λ, we set

$$\delta(\lambda) = \begin{cases} \lambda_1 - \lambda_2 + 1 & \lambda \neq [n-1,1] \\ \lambda_1 - 1 & \text{otherwise} \end{cases}$$

and

$$\omega(\lambda) = \begin{cases} 1 & \lambda_{\ell(\lambda)} \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 3.6. For all $n > 0$, we have

$$\mathcal{P}_n(t) = \sum_{\lambda \in Y_n} \delta(\lambda) V_\lambda t^{\ell(\lambda)-1}(t+1)^{\omega(\lambda)} + V_{[n]}((n-1)t+1).$$

We have checked this conjecture for thagomizer matroids of rank at most 20 using SageMath [Dev]. For our calculations, we worked in the symmetric function setting (see Proposition 3.8). Furthermore, we know the coefficients of $\mathcal{P}_n(t)$ will be honest representations by [GPY1, Corollary 2.12] since τ_n is S_n-equivariantly realizable.

Remark 3.7. Unlike the analogous statements for uniform matroids, Conjecture 3.6 is less enlightening than Theorem 3.1(1) (see [GPY1], Theorem 3.1 and Remark 3.4). That is, the coefficients of the Kazhdan-Lusztig polynomial of a uniform matroid are more cleanly expressed when given as the dimension of a certain representation of the symmetric group. This is not the case for thagomizer matroids.

The remainder of this section is devoted to understanding the results that allow us to derive the recursive formula and functional equation for the Frobenius characteristic of $\mathcal{P}_n(t)$. Let

$$W(t) := (t-1)\mathbb{C} \quad \text{and} \quad V(t) := (t-2)\mathbb{C}$$

as virtual graded vector spaces. Then $W(t)^{\otimes r}$ is the equivariant characteristic polynomial of a rank r Boolean matroid and $W(t) \otimes V(t)^{\otimes r}$ is the equivariant characteristic polynomial of M_r. Both $W(t)^{\otimes r}$ and $W(t) \otimes V(t)^{\otimes r}$ are virtual graded representations of S_r, where S_r acts by permuting the factors of the graded tensor product. Note that the equivariant Kazhdan-Lusztig polynomial of a Boolean matroid is the trivial representation in degree zero.

We'd like to categorify the recursive formula given in Lemma 3.2(1). Recall Equation 3.1:

$$t^{n+1}P_{\tau_n}(t^{-1}) = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i}P_{\tau_i}(t) + \sum_{i=0}^{n} \binom{n}{i}(t-1)(t-2)^i.$$

The first sum is over flats of rank $n-i$ of the first type mentioned above, i.e. where F contains exactly one edge from i distinct spikes. For flats of this type, summing over $[F] \in L(\tau_n)/S_n$ gives

$$\sum_{m+j+i=n} \mathrm{Ind}_{S_m \times S_j \times S_i}^{S_n} \left(W(t)^{\otimes m} \otimes W(t)^{\otimes j} \otimes \mathcal{P}_i(t) \right) \quad \in \mathrm{grVRep}(S_n) \tag{3.5}$$

where S_m permutes the vertices that are connected to A by an edge in F, S_j permutes the vertices that are connected to B by an edge in F, and S_i permutes the vertices that are not adjacent to any edge in F. Similarly, summing over flats of the second type, where F is the union of $i-1$ spikes, gives

$$\sum_{i=0}^{n} \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n} \left(W(t) \otimes V(t)^{\otimes i} \right) \quad \in \mathrm{grVRep}(S_n) \tag{3.6}$$

where S_{n-i} is acting trivially.

As in [GPY1, Section 3.1], we now translate to symmetric functions. We consider the **Frobenius characteristic**

$$\mathrm{ch} : \mathrm{grVRep}(S_n) \xrightarrow{\sim} \Lambda_n[t]$$

where Λ_n is the space of symmetric functions of degree n in infinitely many formal variables $\{x_i \mid i \in \mathbb{N}\}$.

Let $s[\lambda] := \mathrm{ch}\, V_\lambda$ be the Schur function corresponding to λ and set

$$p_n(t) := \mathrm{ch}\, \mathcal{P}_n(t), \quad w_n(t) := \mathrm{ch}\, W(t)^{\otimes n} \quad \text{and} \quad v_n(t) := \mathrm{ch}\, V(t)^{\otimes n}.$$

Applying Frobenius characteristic to Equations 3.5 and 3.6, we obtain

$$t^{n+1} p_n(t^{-1}) = (t-1) \sum_{\ell=0}^{n} v_\ell(t) s[n-\ell] + \sum_{i+j+m=n} p_i(t) w_j(t) w_m(t).$$

Finally, we pass to generating functions, working in the ring $\Lambda[[t,u]]$ of formal power series in the variables $\{t, u, x_1, x_2, \ldots\}$ that are symmetric in the x variables.We let

$$s(u) := \sum_n s[n] u^n, \quad w(t,u) := \sum_n w_n(t) u^n$$

and

$$v(t,u) := \sum_n v_n(t) u^n.$$

Note that

$$w(t,u) = \frac{s(tu)}{s(u)}$$

by [GPY1, Proposition 3.9]. The results of this section can be summarized in the following proposition.

Proposition 3.8. We have the following (equivalent) equations.

1. For $n > 0$, $t^{n+1} p_n(t^{-1}) = (t-1) \sum_{\ell=0}^{n} v_\ell(t) s[n-\ell] + \sum_{i+j+m=n} p_i(t) w_j(t) w_m(t)$.

2. $\phi(t^{-1}, tu) = (t-1) u s(u) v(t,u) + w(t,u)^2 \phi(t,u)$.

In [GPY1], we were able to compute the equivariant Kazhdan-Lusztig polynomial for uniform matroids by showing that our "guess" satisfied a recursion analogous to the one found in Proposition 3.8(2). That case was much simpler; we only had to consider singular applications of the Pieri rule. In this case, $w(t,u)^2$ requires multiple applications of the Pieri rule while $v_n(t) = s[n]\big[v_1(t)\big]$ involves a plethysm. This makes proving Conjecture 3.6 much more difficult.

3.2. Fan Graphs

Let F_n be the graphical join of the path graph on $n+1$ vertices with the empty graph on 1 vertex; then F_n is the **fan graph** on $n+2$ vertices with n blades. Denote by Δ_n the matroid associated to F_n, which we call the **fan matroid**. Note that $\operatorname{rk} \Delta_n = n+1$ and the underlying ground set of Δ_n has size $2n+1$.

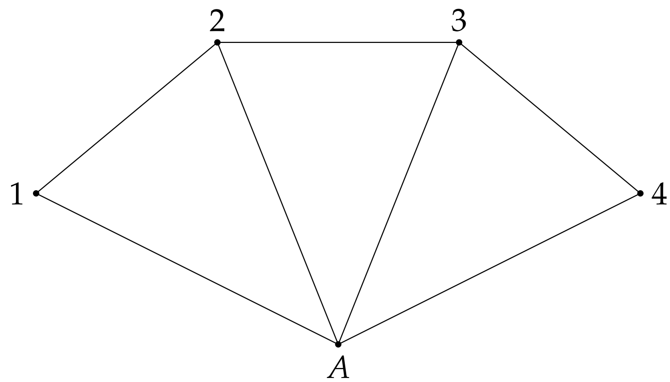

FIGURE 3.1. The fan graph F_3.

We will only consider the non-equivariant Kazhdan-Lusztig polynomial, $P_{\Delta_n}(t)$, associated to Δ_n. Set

$$P_{\Delta_n}(t) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} d_{n,k} t^k \quad \text{and} \quad \Phi_\Delta(t,u) := \sum_{n=0}^{\infty} P_{\Delta_n}(t) u^{n+1}.$$

Just like with thagomizer matroids, we would like to give a description of the coefficients $d_{n,k}$ and of $\Phi_\Delta(t,u)$. However, we have only managed to do this conjecturally, hence we have the following conjecture analogous to Theorem 3.1.

Conjecture 3.8. The following (equivalent) statements hold.

1. For all n and k, $d_{n,k}$ is the number of Motzkin paths of length n with k up steps.
2. The generating function $\Phi_\Delta(t,u)$ is equal to $\dfrac{u\left(1-u-\sqrt{(u-1)^2-4tu^2}\right)}{2tu^2}$.

This conjecture has been checked for fan matroids of rank at most 20.

We again begin with a description of the flats of Δ_n. Let A be the distinguished vertex of F_n and label the other vertices $1, 2, \ldots, n+1$. We refer to the subgraph $\{A, j, j+1\}$ as a **blade**. For convenience, we refer to any edge of the form Aj as a **blue edge** and any edge of the form $\{j, j+1\}$ as a **red edge**.

We describe the subgraphs of F_n that form the flats in $L(\Delta_n)$. Let $U := \{v = (v_0, \ldots, v_\ell)\}$ be the set of compositions of n with v_0 and v_ℓ allowed to be zero. For each $v = (v_0, \ldots, v_\ell) \in U$, v_i gives the number of blue edges in between the red edges included in the subgraph. Because these v will give flats, we include the corresponding blue edge whenever $v_i = 1$. For example in the case of F_3, $v = (0, 1, 2)$ gives the subgraph $\{A1, A2, 12\}$. Note that this is different from $v = (0, 1, 2, 0)$ which gives the subgraph $\{A1, A2, 12, A4\}$.

Consider again $v = (0, 1, 2)$ for the graph F_3. This gives two choices of flats in $L(\Delta_3)$ since we have a choice as to whether or not we should include the edge 34. The two corresponding subgraphs are $\{A1, A2, 12, A4\}$ and $\{A1, A2, 12, A4, 34\}$.

When exactly will we have such a choice? If $v_0 = 0$ or 1, there are no choices since $v_0 = 0$ gives no blue edge to chose, and $v_0 = 1$ forces us to include the edge 12. The analogous statement is true of v_ℓ. However, if v_0 or v_ℓ is greater than or equal to 2, we may make some choices on which blue edges to include in our flat.

For any of the "middle" v_i, we again have no choices if $v_i = 1$, since we are forced to include a blue edge. Additionally, we have no choices if $v_i = 2$; including either of these edges would force us to include an additional red edge if we want this subgraph to define a flat. To see why this is true, consider $v = (0, 2, 1, 0)$ in F_3, which gives the subgraph $\{A1, A2, 12, A4\}$. In order to include the edge 23, we would be forced to include the edge $A3$ (and subsequently, the edge 34) in order for our subgraph to define a flat.

We pause to consider an additional example; let $v = (0, 3, 0)$ in F_3. This corresponds to the subgraph $\{A1, A4\}$. In this case, we cannot choose either of the red edges 12 or 34 if we want our subgraph to be a flat. This shows us that if $v_i \geq 2$ for a "middle" i value, we can choose at most $v_i - 2$ red edges from this section of F_n.

Let $J_v = \{j = (j_0, \dots, j_\ell)\}$ be subsets of v such that

$$0 \leq j_0 \leq v_0 - 1, \quad 0 \leq j_\ell \leq v_\ell - 1,$$

and

$$0 \leq j_i \leq v_i - 2 \quad \text{for} \quad 0 < i < \ell.$$

Then j_i tells us how many of the v_i blue edges we are allowed to include in the subgraph in order for this subgraph to define a flat in $L(\Delta_n)$. Hence the sum of all $j_i \in j \in J_v$ as v ranges over the set U gives the total number of flats in $L(\Delta_n)$.

Fix $j \in J_v$ for some $v \in U$. The next question to consider is how many choices of flats we have for each $j_i \in j$. Denote this value by $\mathcal{C}_i(v, j)$. Then

$$\mathcal{C}_i(v, j) = \binom{\zeta_i(v, j) + j_i}{j_i}$$

where

$$\zeta_i(v, j) = \begin{cases} v_i - j_i & \text{if } \ell = 0 \\ \max(v_i - 1 - j_i, 0) & \text{if } i = 0 \text{ or } \ell, \text{ and } \ell \neq 0 \\ \max(v_i - 2 - j_i, 0) & \text{else.} \end{cases}$$

This matches our earlier discussion of choices of red edges to include in our subgraph. Additionally, if $\ell = 0$, i.e. if $v = (n)$, our subgraph includes no blue edges, and there are $\binom{v_0}{j_0}$ choices of red edges available.

Finally, we turn our attention to the localization and contraction of Δ_n at each type of flat. We note that in general, we expect both the localization and contraction to have lattices of flats isomorphic to the matroid associated to a product of fan graphs, where by product we mean glued together at the distinguished vertex A. Let $F_{v,j}$ be the flat associated to a choice of $v \in U$ and $j \in J_v$.

Contracting at a flat $F_{v,j}$ gives a product of fan graphs and path graphs as described above. We note that the Kazhdan-Lusztig polynomial of a matroid associated to this type of product of graphs, where two graphs are glued together at a single vertex, is equal to the product of the Kazhdan-Lusztig polynomials of the matroids associated to each individual graph. We now see that contracting at $F_{v,j}$ results in a matroid whose Kazhdan-Lusztig polynomial is given by

$$\prod_{i=0}^{\ell} P_{\Delta_{\zeta_i(v,j)}}(t).$$

We turn to the localization, and note that for a fixed v, the flat will be determined by which blue edges we include in our subgraph. Recall that the possible values for j tell us that we are not allowed to include any blue edges which share a vertex with chosen red edges. So if the subgraph corresponding to our flat includes any smaller fans, each of these blue edges must have come from a $v_i = 1$. Hence we know that the total number of blades in our subgraph is given by

$$\alpha(v) = \#\{0 < i < \ell \mid v_i = 1\}.$$

We already know that the total number of blue edges that do not share a vertex with chosen red edges is equal to $\sum_{m=0}^{\ell} j_m$. We also note that the number of red edge vertices (excluding the distinguished vertex) which are not included in any smaller fan in our subgraph is equal to $\ell - \alpha(v)$.

Finally, we recall that we are interested in the characteristic polynomial of the localization $\chi_{M_F}(t)$. The characteristic polynomial of Δ_n is given by $\chi(t) = (t-1)(t-2)^n$ and the characteristic polynomial of the matroid associated to the path graph on n vertices is given by $\chi(t) = (t-1)^n$. Hence, for any flat $F_{v,j}$, we have

$$\chi_{M_{F_{v,j}}}(t) = (t-1)^{\beta(v,j)}(t-2)^{\alpha(v)}$$

where

$$\beta(v,j) = \ell - \alpha(v) + \sum_{m=0}^{\ell} j_m.$$

Collecting this information together, we have the following lemma.

Lemma 3.9. For $n > 0$, we have

$$t^{n+1}P_{\Delta_n}(t^{-1}) = \sum_{v \in U}\sum_{j \in J_v}\prod_{i=0}^{\ell}\left(\mathcal{C}_i(v,j)P_{\Delta_{\zeta_i(v,j)}}(t)\right)(t-1)^{\beta(v,j)}(t-2)^{\alpha(v)}.$$

Note that the flats of Δ_n are more complex than those of τ_n or κ_n, which suggests that proving Conjecture 3.8 will be more difficult than it was to prove Theorems 3.1 and 3.5. Indeed, we would hope to simplify the equation given in Lemma 3.9 as we did to obtain the equation in Lemma 3.2(1). But we have not been able to do so thus far.

We turn our attention to the two variable generating function $\Phi_\Delta(t,u)$. Based on the equation given in Lemma 3.9, we could produce a two-variable equation involving $\Phi_\Delta(t,u)$ as well. However, this formula is not enlightening, so we do not produce it here.

Let $b_{n,k}$ be the number of Motzkin paths of length n with k up-steps, as in Section 2.3., and set

$$M(t,u) := \sum_{n,k\geq 0} b_{n,k} t^k u^n.$$

It is known (see Sequence A055151 in [Slo]) that $M(t,u)$ satisfies

$$tu^2\big(M(t,u)\big)^2 + (u-1)M(t,u) + 1 = 0$$

which gives

$$M(t,u) = \frac{1-u-\sqrt{(u-1)^2-4tu^2}}{2tu^2}.$$

As with the function $G(t,u)$ in Section 3.1., we would *a priori* conclude that this formula should include a $\pm$ sign. But again, a plus sign would not result in giving a formal power series with positive coefficients.

Now we see that in order to prove Conjecture 3.8, we would need to show that $\Phi_\Delta(t,u) = u \cdot M(t,u)$. If we were to replicate the argument given in Section 3.1. for thagomizer matroids, we would want to show that $\Phi_\Delta(t,u)$ satisfies an equation analogous to the one given in Lemma 3.2(2). Considering instead $M(t,u)$, we obtain the following.

Lemma 3.10. $M(u,t)$ satisfies the following equation:

$$M(t^{-1},tu) = \frac{1}{1-tu+u} \cdot M\Big(t, \frac{u}{1-tu+u}\Big).$$

Proof. We have

$$(1-tu+u)\cdot M(t^{-1},tu) = \frac{1}{\frac{1}{1-tu+u}}\cdot\frac{1-tu-\sqrt{(tu-1)^2-4tu^2}}{2tu^2}$$

$$\begin{aligned}
&= \frac{\frac{1}{1-tu+u}}{\left(\frac{1}{1-tu+u}\right)^2}\cdot\frac{1-tu+u-u-\sqrt{t^2u^2-2tu+1-4tu^2+2(u^2-u+tu^2)-2(u^2-u+tu^2)}}{2tu^2}\\
&= \frac{1-\frac{u}{1-tu+u}-\sqrt{\frac{u^2-2u(1-tu+u)+(1-tu+u)^2-4tu^2}{(1-tu+u)^2}}}{2t\left(\frac{u}{1-tu+u}\right)^2}\\
&= \frac{1-\frac{u}{1-tu+u}-\sqrt{\left(\frac{u}{1-tu+u}-1\right)^2-4t\left(\frac{u}{1-tu+u}\right)^2}}{2t\left(\frac{u}{1-tu+u}\right)^2}\\
&= M\Big(t,\frac{u}{1-tu+u}\Big)
\end{aligned}$$

□

This gives us the following conjecture.

Conjecture 3.11.

$$\Phi(t^{-1},tu) = t\cdot\Phi\Big(t,\frac{u}{1-tu+u}\Big).$$

Note that proving this conjecture would almost immediately give Conjecture 3.8, based on our work above.

3.3. Complete Graphs

In general, little is known about the equivariant and non-equivariant Kazhdan-Lusztig polynomials of the matroid associated to the complete graph on n vertices. However, a study of the coefficients of the Kazhdan-Lusztig polynomials associated to many different graphical matroids leads us to an interesting conjecture. Let B_n be the matroid associated to this graph.

Conjecture 3.12. Let G be a 2-connected graph with $n \neq 4$ vertices, and let $P_M(t)$ be the Kazhdan-Lusztig polynomial of the matroid M associated to G. Then the k-th coefficient of $P_M(t)$ is less than or equal to the k-th coefficient of $P_{B_n}(t)$.

This has been checked on all 2-connected graph with at most 9 vertices. Note that we only consider 2-connected graphs for the following reason: the matroid associated to a 1-connected graph is the direct sum of the matroids associated to the two (or more) subgraphs that were glued together at a vertex make the larger graph. We recall [EPW, Proposition 2.7] which states that for any two matroids M_1 and M_2,

$$P_{M_1 \oplus M_2}(t) = P_{M_1}(t) P_{M_2}(t).$$

Hence the Kazhdan-Lusztig polynomial of the matroid associated to a 1-connected graph may not even have the expected degree to be able to make such a conjecture.

Conjecture 3.12 says that largest coefficients of a Kazhdan-Lusztig polynomial are obtained when a (simple) graph has the maximum number of edges. When $n = 4$, $P_{\kappa_2}(t) = 2t + 1$ gives the only known exception.

The example of $P_{\kappa_2}(t)$ given above shows exactly why Conjecture 3.12 is surprising; if one deletes an edge from T_n, one obtains $K_{2,n}$ and the linear term of the associated Kazhdan-Lusztig polynomial *increases*. In other words, for a generic graph G, and it's associated matroid $M(G)$, we do not expect the coefficients of $P_{M(G)}(t)$ to dominate the coefficients of $P_{M(G \setminus e)}(t)$, for any edge e of G.

We end this section with the following example, which shows that the Petersen graph satisfies Conjecture 3.12, since this graph often appears as a counterexample to stated conjectures.

Example 3.13. Let $\mathcal{G}$ be the Petersen graph on 10 vertices. Then $P_{M(\mathcal{G})}(t) = 456t^4 + 3585t^3 + 2185t^2 + 176t + 1$ and the k-th coefficient is less than or equal to the k-th coefficient of $P_{B_{10}}(t) = 76545t^4 + 204400t^3 + 147466t^2 + 968t + 1$.

CHAPTER IV

ROOTS OF KAZHDAN-LUSZTIG POLYNOMIALS OF MATROIDS

Conjecture 4.2 and Theorem 4.3 are results of joint work with Nicholas Proudfoot and Benjamin Young, though I was the primary contributor to ideas and proofs. This work originally appeared in [GPY2]. Section 4.3. includes joint work with Mirkó Visontai who originally noticed the pattern that appears in Figure 4.1. All included work was written entirely by me, and I was the sole contributor to all proofs in this section.

In this chapter, we study the roots of the Kazhdan-Lusztig polynomials of some matroids. We give some conjectures based on the observed behavior of these roots and prove one of these conjectures for a family of uniform matroids.

4.1. Conjectures

We first recall a conjecture given on the roots of Kazhdan-Lusztig polynomials [GPY2, Conjecture 3.2].

Conjecture 4.1. For every matroid M, all roots of $P_M(t)$ lie on the negative real axis.

Further analysis on the behavior of the roots of Kazhdan-Lusztig polynomials of uniform matroids led us to believe that, in some cases, the roots of two Kazhdan-Lusztig polynomials interlace. We consider the following example which gives the Kazhdan-Lusztig polynomials of some uniform matroids and their roots.

Kazhdan-Lusztig polynomial	Roots
$P_{U_{1,8}}(t) = 84t^3 + 120t^2 + 27t + 1$	$-1.16, -0.222, -0.046$
$P_{U_{1,9}}(t) = 42t^4 + 300t^3 + 225t^2 + 35t + 1$	$-6.315, -0.628, -0.163, -0.037$
$P_{U_{1,10}}(t) = 330t^4 + 825t^3 + 385t^2 + 44t + 1$	$-1.931, -0.413, -0.126, -0.03$

TABLE 4.1. Kazhdan-Lusztig polynomials of some uniform matroids and their roots.

This analysis leads us to strengthen the statement of Conjecture 4.1 to include this interlacing phenomenon. In particular, we consider the Kazhdan-Lusztig polynomials associated to a matroid M and the contraction of that matroid at some element e in the ground set, M/e. We would like to say that the roots of these polynomials interlace, however the polynomial with the smallest root depends on the rank of M, as we see in Table 4.1..

Assume that both M and M/e are non-degenerate and connected matroids with positive rank. If the rank of M is odd and greater than 1, the maximum possible degree of $P_M(t)$ is one greater than that of $P_{M/e}(t)$, as long as e is not a loop. In this case, we would like to say that $P_M(t)$ interlaces $P_{M/e}(t)$. Otherwise, we would want to say that $P_{M/e}(t)$ interlaces $P_M(t)$. To make this precise, we let $Q_M(t) := t^{\operatorname{rk} M-1} P_M(-t^{-2})$.

Conjecture 4.2. Let M be a non-degenerate connected matroid and let e be an element of the ground set of M such that e is not a loop and M/e is also non-degenerate. Then $Q_M(t)$ interlaces $Q_{M/e}(t)$.

Note that Conjecture 4.2 actually says that $tP_{M/e}(t)$ interlaces $P_M(t)$. Conjecture 4.2 is the most precise statement we could make about the observed interlacing of roots in the Kazhdan-Lusztig polynomials we have studied.

However, it does not capture all interlacing behavior that we have noticed. For example, notice that contracting $K_{2,6}$ at any edge gives a graph whose matroid has a lattice of flats isomorphic to that of τ_5. Hence Conjecture 4.2 concerns the matroids κ_6 and τ_5 when $M = \kappa_6$. But we have also noticed that the roots of $P_{\kappa_6}(t)$ interlace those of $P_{\kappa_5}(t)$. We will explore this more in the next section.

4.2. Examples

We give some examples and state conjectures in all cases where the coefficients of $P_M(t)$ are known. We reserve the study of $P_{\tau_n}(t)$ until the next section.

We first consider the family of uniform matroids $U_{1,d}$ for $d > 0$. By [PWY, Theorem 1.2(1)], we have

$$P_{U_{1,d}}(t) = \sum_{i \geq 0} \frac{1}{i+1} \binom{d-i-1}{i} \binom{d+1}{i} t^i \tag{4.1}$$

for all $d > 0$.

Theorem 4.3. All of the roots of $P_{U_{1,d}}(t)$ lie on the negative real axis.

Proof. For any fixed positive integer d, the sequence

$$\Gamma(d) := \left\{ \frac{1}{(i+1)!(d+1-i)!} \right\}$$

is a multiplier sequence [Zha, Lemma 2.5]. Consider the polynomial

$$h_d(t) := \sum_{i \geq 0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{d-i-1}{i} t^i.$$

This polynomial is real-rooted [Zha, Lemma 3.2], hence

$$\Gamma(d)[h_d(t)] = \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{1}{(i+1)!(d+1-i)!} \binom{d-i-1}{i} t^i$$

is also real-rooted. Then since

$$P_{U_{1,d}}(t) = (d+1)! \cdot \Gamma(d)[h_d(t)],$$

we conclude that $P_{U_{1,d}}(t)$ is real-rooted. Since the coefficients of $P_{U_{1,d}}(t)$ are positive (including the constant coefficient), it cannot have any non-negative real roots, therefore all of the roots lie on the negative real axis. □

Theorem 4.3 gives the only case in which we have been able to prove Conjecture 4.1 for an infinite family of non-degenerate matroids.

The contraction of $U_{1,d}$ at any e in the ground set gives a matroid whose lattice of flats is isomorphic to that of $U_{1,d-1}$. Hence Conjecture 4.2 says that we expect $Q_{U_{1,d}}(t)$ to interlace $Q_{U_{1,d-1}}(t)$. This has been checked on a computer for all $2 \leq d \leq 50$.

We also study the roots of Kazhdan-Lusztig polynomials for other uniform matroids. Conjectures 4.1 and 4.2 have been checked with a computer for $M = U_{m,d}$ with $1 \leq d \leq 10$ and $1 \leq m \leq 50$. However, we obtain an interesting conjecture when we consider the family of matroids $U_{m,d}$ with fixed d instead of fixed m.

Conjecture 4.4. For all $m, m' > 0$, the polynomials $P_{U_{m,d}}(t)$ and $P_{U_{m',d}}(t)$ interlace.

This conjecture has been checked for all $1 \leq d \leq 10$ and $1 \leq m \leq 50$. Note that when d is fixed, the degree of the Kazhdan-Lusztig polynomials is fixed as well [GPY1, Theorem 3.1], hence our conjecture concerns $P_{U_{m,d}}(t)$ instead of $Q_{U_{m,d}}(t)$. When we investigate the roots of $U_{m,d}$ for fixed d and increasing m, we observe additional interesting behavior. An example of this behavior is recorded in Table 4.2.

Matroid M	Roots of $P_M(t)$
$U_{1,7}$	$-3.60973, -0.33038, -0.05989$
$U_{10,7}$	$-2.16755, -0.15237, -0.00008$
$U_{100,7}$	$-1.73704, -0.12211, -5.53328 \times 10^{-10}$
$U_{1000,7}$	$-1.68671, -0.11929, -7.00869 \times 10^{-16}$
$U_{1,5}$	$-1.68102, -0.11898$

TABLE 4.2. Roots of $P_M(t)$ for some uniform matroids.

This leads us to strengthen the statement of Conjecture 4.4 to the following.

Conjecture 4.5. Let $r_{k,m,d}$ be the k-th (largest) root of $P_{U_{m,d}}(t)$. Then

$$r_{k,m,d} < r_{k,m+1,d}$$

and

$$\lim_{m\to\infty} \left(r_{k,m,d}\right) = \begin{cases} 0 & \text{if } k = 1 \\ r_{k-1,1,d-2} & \text{else.} \end{cases}$$

We now turn our attention to other matroids. Recall the matroid B_n associated to the complete graph. Both Conjectures 4.1 and 4.2 have been checked for B_n for all values of $n \leq 30$. These conjectures have also been checked for all values of $n \leq 30$ for the matroid κ_n associated to $K_{2,n}$.

Finally we consider fan graphs. Recall Conjecture 3.8 which tells us that $P_{\Delta_n}(t) = \mathcal{M}_n(t)$ where $\mathcal{M}_n(t)$ is the n-th Motzkin polynomial as defined in Chapter II. If this conjecture is true, then by Theorem 2.4, since the n-th Narayana polynomial $\mathcal{N}_n(t)$ is real-rooted, we have that $P_{\Delta_n}(t)$ is as well.

Here we note that Conjecture 4.2 doesn't apply unless e is one of the outer edges of F_n, i.e. e must be a red edge or a blue edge of the form $\{A, n+1\}$. That is, Conjecture 4.2 states that we expect the roots of $P_{F_n}(t)$ and $P_{F_{n+1}}(t)$ to interlace. Both Conjectures 4.1 and 4.2 have been checked for F_n whenever $n \leq 30$.

4.3. Thagomizer Matroids

We turn our attention toward Conjectures 4.1 and 4.2 for thagomizer matroids. Visontai noticed a very interesting way to obtain these Kazhdan-Lusztig polynomials, which we will now describe.

Starting with 1 at the origin, we build a cone of polynomials to the right. We call the (i, j) entry of this cone $\zeta_{i,j}(x)$ and set

$$\zeta_{i,j}(x) = \begin{cases} 0 & \text{if } j > i, \\ (-1)^i & \text{if } i = j, \\ x\zeta_{i-1,0}(x) + \zeta_{i-1,1}(x) & \text{if } j = 0, \\ 2x\zeta_{i-1,j}(x) + \zeta_{i-1,j+1}(x) - \zeta_{i-1,j-1}(x) & \text{else.} \end{cases}$$

This gives the cone in Figure 4.1., which can be seen below.

Notice that the $\zeta_{i,0}(x)$ appear to be $Q_{\tau_i}(x)$ (see Table A.1). Our hope was that by describing the thagomizer Kazhdan-Lusztig polynomials in this way, we might be able to use the recursive formula for obtaining the column polynomials

$$
\begin{array}{lllllll}
 & & & & & -1 & \\
 & & & & 1 & 9x & \\
 & & & -1 & -7x & -31x^2+4 & \\
 & & 1 & 5x & 17x^2-3 & 49x^3-26x & \cdots \\
 & -1 & -3x & -7x^2+2 & -15x^3+13x & -31x^4+54x^2-5 & \\
1 & x & x^2-1 & x^3-4x & x^4-11x^2+2 & x^5-26x^3+15x &
\end{array}
$$

FIGURE 4.1. The cone of $\zeta_{i,j}(x)$ polynomials.

to prove that each column of polynomials interlaces the previous column. This is a common strategy to prove interlacing of families of polynomials, see for example [Brä] and [LW]. Ultimately, Visontai and I have not been successful, though both Conjectures 4.1 and 4.2 have been checked for τ_n whenever $n \leq 50$. The rest of this section is dedicated to proving the statement

$$\zeta_{i,0}(x) = Q_{\tau_i}(x).$$

4.3.1. Cone path bijection

We think of the method for obtaining $\zeta_{n,0}(x)$ as a collection of paths in the cone from $\zeta_{0,0}(x)$. Each path can be described as a word in S, U, and D where S represents a sideways step to the right, U represents an up-diagonal step, and D represents a down-diagonal step. Let Π_n be the total collection of such paths; Π_n is identical as a set to the set of Motzkin paths $\mathcal{Z}_n$, but the context here is different and hence we will use the notation Π_n. Then, for example, we have

$$\Pi_3 = \{UDS, USD, SUD, SSS\}.$$

Notice that based on our definition of $\zeta_{i,j}(x)$, S represents a map that multiplies by either x or $2x$, while U and D are maps that multiply by ± 1. Therefore, if the number of S steps in a cone word α is k, then α contributes to the x^k term of $\zeta_{n,0}(x)$.

We weight the cone words in the following way; beginning with a weight of 1, if there is an S step above the bottom axis (i.e. if S occurs between a U and a D), multiply the weight by 2. For example, the words

$$UDUD, \quad USDS, \quad USSD, \quad \text{and} \quad SSSS$$

contribute terms

$$1, \quad -2x^2, \quad -4x^2, \quad \text{and} \quad x^4$$

respectively, and hence have weights of

$$1, \quad 2, \quad 4, \quad \text{and} \quad 1.$$

Let $\omega(\alpha) = 2^\ell$ be the weight of a cone word α with ℓ total S steps above the x-axis, and let $s(\alpha)$ be the total number of S steps in α.. Note that there are $\frac{n-s(\alpha)}{2}$ total D steps for any path α of length n. Then we can write $\zeta_{n,0}(x)$ as

$$\zeta_{n,0}(x) = \sum_{\alpha \in \Pi_n} (-1)^{(n-s(\alpha))/2} \omega(\alpha) x^{s(\alpha)}.$$

Recall that $\mathcal{F}_n(t)$ can be given by

$$\mathcal{F}_n(t) = \sum_{\beta \in \mathcal{D}_n} t^{\mathrm{LA}(\beta)}$$

where $\mathcal{D}_n$ is the set of Dyck paths of semilength n and $\mathrm{LA}(\beta)$ is the total number of long ascents of β. Then we have

$$Q_{\tau_n}(t) = t^n \sum_{\beta \in \mathcal{D}_n} \left(-t^{-2}\right)^{\mathrm{LA}(\beta)}$$

hence

$$Q_{\tau_n}(t) = \sum_{\beta \in \mathcal{D}_n} (-1)^{\mathrm{LA}(\beta)} t^{n-2\,\mathrm{LA}(\beta)}.$$

We would like to prove the existence of a map Ξ_n where

$$\Xi_n : \{\beta \in \mathcal{D}_n \mid \mathrm{LA}(\beta) = k\} \to \{\alpha \in \Pi_n \mid s(\alpha) = n - 2k\},$$

and $|\Xi_n^{-1}(\alpha)| = \omega(\alpha)$. Doing so would enable us to conclude that $\zeta_{n,0}(x) = Q_{\tau_n}(x)$.

To do this, we will consider the decomposition of Dyck paths into 2-step subpaths. This is a process that takes a Dyck path

$$\beta = uuududduddd$$

and returns the decomposition

$$uu|uu|du|dd|ud|dd.$$

The key observation here is that there is a relationship between the number of occurrences of $|uu|$ in 2-step subpath decompositions and the number of long ascents of Dyck paths. Hence we will prove the existence of the map Ξ_n by proving the following lemma.

Lemma 4.6. Let $w(\beta)$ be the number of times $|uu|$ occurs in the 2-step subpath decomposition of $\beta \in \mathcal{D}_n$.

(1) The sets $\{\beta \in \mathcal{D}_n \mid \mathrm{LA}(\beta) = k\}$ and $\{\gamma \in \mathcal{D}_n \mid w(\gamma) = k\}$ have the same cardinality.

(2) There exists a map

$$\bar{\Xi}_n : \{\gamma \in \mathcal{D}_n \mid w(\gamma) = k\} \to \{\alpha \in \Pi_n \mid s(\alpha) = n - 2k\}$$

with $|\bar{\Xi}_n^{-1}(\alpha)| = \omega(\alpha)$.

We first prove part (2) of Lemma 4.6. We note that the statement in Lemma 4.6(1) is known (e.g. see [Slo] sequence A091156), but we could not find a proof in the literature, hence we prove it at the end of this section.

Let $\gamma \in \mathcal{D}_n$ with $w(\gamma) = k$. We translate the path γ into a cone word α by replacing $|uu|$ with a U, $|dd|$ with a D, and both $|ud|$ and $|du|$ with an S. Hence the Dyck paths

$$uu|du|dd|ud \quad \text{and} \quad uu|uu|dd|dd$$

are translated into the cone paths

$$USDS \quad \text{and} \quad UUDD$$

respectively. Since the number of u's in γ must equal the number of d's in γ, there must also be k occurrences of $|dd|$ in the decomposition of γ. Hence the number of S's that show up in the resulting cone path must be equal to $n - 2k$.

Under this correspondence, there are obviously $|\omega(\alpha)|$ Dyck paths that get sent to the cone word α: each S on the bottom axis must have been translated from a ud, and any other S could have come from either ud or du. That is to say, both of the Dyck paths

$$uu|du|dd|ud \quad \text{and} \quad uu|ud|dd|ud$$

are sent to the cone word $USDS$, and no other Dyck paths can be sent to this word. This is another way of saying that the preimage of an S that occurs between a U and a D has cardinality 2, while every other S has a preimage of cardinality 1. Then the preimage of any cone word α under this map is exactly $\omega(\alpha)$. This completes the proof of Lemma 4.6(2).

Finally, we turn our attention towards proving the statement in Lemma 4.6(1). The result is clear for $n = 2, 3$. Now let $\alpha \in \mathcal{D}_n$ and suppose the statement holds for smaller values of n.

Recall that a **return** of a Dyck path is a down step ending on the x-axis. An **irreducible** Dyck path is a Dyck path with exactly one return.

If β is reducible, say $\beta = \oplus_{i=1}^{j} \hat{\beta}_i$ with each $\hat{\beta}_i$ irreducible of semilength strictly less than n, note that

$$\mathrm{LA}(\beta) = \sum_{i=1}^{j} \mathrm{LA}(\hat{\beta}_i)$$

and

$$w(\beta) = \sum_{i=1}^{j} w(\hat{\beta}_i).$$

Hence

$$\#\{\beta \in \mathcal{D}_n \mid \beta \text{ reducible and } \operatorname{LA}(\beta) = k\}$$

is equal to

$$\#\{\gamma \in \mathcal{D}_n \mid \gamma \text{ reducible and } w(\gamma) = k\}.$$

It remains to show the analogous statement for irreducible Dyck paths.

If β is irreducible, then it necessarily begins with a uu, and we can write

$$\beta = u\beta' d$$

with β' a Dyck path of semilength $n-1$. Consider the 2-step subpath decomposition of β'; it looks like a shift of the 2-step subpath decomposition of β. Since we are interested in the number of occurrences of $|uu|$ in the 2-step subpath decomposition, we will study the effect of such a shift on this decomposition.

There are only three possible subwords of β of length 3 that contain uu:

$$uuu, \quad uud, \quad \text{and} \quad duu.$$

In the first case, a shift in the 2-step decomposition preserves the number of $|uu|'s$, that is

$$|uu|u \quad \rightsquigarrow \quad u|uu|.$$

In the second case, the number of $|uu|$'s decreases by one and similarly in the third case, the number increases by one. Hence it suffices to count the occurrences of each of these subwords.

Notice that a long ascent is always preceded by a d unless it occurs at the beginning of a Dyck path. Then if a Dyck path does not begin with a long ascent, the number of occurrences of uud and duu are the same in that path. Otherwise, there is one more occurrence of uud.

Then if β' begins with a long-ascent, then the number of long-ascents in β and β' is the same. The number of occurrences of $|uu|$ is also the same since shifting destroys an occurrence of uuu at the beginning of β, but there is one more occurrence of uud than there is of duu.

Otherwise, there is one more long-ascent in β than in β'. Then β' has one fewer occurrence of uuu than β, but the same number of uud's and duu's. Hence, after shifting we see that β' has one fewer occurrence of $|uu|$ than β as well.

We conclude that if $\mathrm{LA}(\beta) = \mathrm{LA}(\beta')$, it must be the case that $w(\beta) = w(\beta')$, and otherwise $\mathrm{LA}(\beta) = \mathrm{LA}(\beta') + 1$ occurs only when $w(\beta) = w(\beta') + 1$. Since β' is a Dyck path of semilength less than n, this completes the proof. $\square$

APPENDIX

COMPUTATIONS

We include computer generated computations of Kazhdan-Lusztig polynomials $P_M(t)$ of some matroids mentioned in this document. For computations of some uniform matroids and braid matroids, see [EPW, Appendix].

A.1. Thagomizer Matroids

TABLE A.1 Kazhdan-Lusztig polynomials for the thagomizer matroid τ_n

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1
t		1	4	11	26	57	120	247	502	1013	2036	4083	8178
t^2				2	15	69	252	804	2349	6455	16962	43086	106587
t^3						5	56	364	1800	7515	27940	95458	305812
t^4								14	210	1770	11055	57035	257257
t^5										42	792	8217	62062
t^6												132	3003

A.2. Fan Matroids

TABLE A.2 Kazhdan-Lusztig polynomials for the fan matroid Δ_n

$n =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
t		1	3	6	10	15	21	28	36	45	55	66	78	91	105
t^2				2	10	30	70	140	252	420	660	990	1430	2002	2730
t^3						5	35	140	420	1050	2310	4620	8580	15015	25025
t^4								14	126	630	2310	6930	18018	42042	90090
t^5										42	462	2772	12012	42042	126126
t^6												132	1716	12012	60060
t^7														429	6435

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