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The matching number and Hamiltonicity of graphs

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ABSTRACT
The matching number of a graph G is the size of a maximum matching in the graph. In this note, we present a sufficient condition involving the matching number for the Hamiltonicity of graphs.

KEYWORDS
Matching number; Hamiltonicity

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05C70; 05C45

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let $G = (V(G),\ E(G))$ be a graph. A matching M in G is a set of pairwise nonadjacent edges. A maximum matching is a matching that contains the largest possible number of edges. The matching number, denoted $m(G)$, of a graph G is the size of a maximum matching. For a vertex u and a vertex subset U in G, we use $N_U(u)$ to denote all the neighbors of u in U. We use $G_1 \vee G_2$ to denote the join of two disjoint graphs G_1 and G_2. We define $\mathcal{F} := \{G : K_{p,q} \subseteq G \subseteq K_p \vee (qK_1)$, where $q \geq p + 1 \geq 3\}$. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle.

The purpose of this note is to present a sufficient condition based on the matching number for the Hamiltonicity of graphs. The main result is as follows:

Theorem 1. *Let G be a graph of order $n \geq 3$ with matching number m and connectivity κ ($\kappa \geq 2$). If $m \leq \kappa$, then G is Hamiltonian or $G \in \mathcal{F}$.*

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose G is not Hamiltonian. Since $\kappa \geq 2$, G contains a cycle. Choose a longest cycle C in G and give an orientation on C. For a vertex u on C, we use u^+ to denote the successor of u along the direction of C. u^{+2} is defined as the successor of u^+ along the direction of C. Since G is not Hamiltonian, there exists a vertex, $x_0 \in V(G)\backslash V(C)$. By Menger's theorem, we can find s ($s \geq \kappa$) pairwise disjoint (except for x_0) paths $P_1, P_2, \ldots, P_s$ between x_0 and $V(C)$. Let u_i be the end vertex of P_i on C, where $1 \leq i \leq s$. Then, a standard proof in Hamiltonian graph theory yields that $S := \{x_0, u_1^+, u_2^+, \ldots, u_s^+\}$ is independent (otherwise G would have cycles which are longer than C). Obviously, the edges of $u_1u_1^+, u_2u_2^+, \ldots,$ and $u_su_s^+$ form a matching in G. Thus, $\kappa \leq s \leq m \leq \kappa$. Therefore, $\kappa = s = m$. The remainder of proofs consists of five claims and their proofs.

Claim 1. Let H be the component in $V(G)\backslash V(C)$ that contains x_0. Then, H consists of the singleton x_0.

Proof of Claim 1. Suppose, to the contrary, that Claim 1 is not true. Then, we can find an edge, say e, in H. Then, the edges of e, $u_1u_1^+, u_2u_2^+, \ldots,$ and $u_su_s^+$ form a matching in G, giving a contradiction of $\kappa + 1 = s + 1 \leq m = \kappa$. □

Claim 2. $u_{i+1} = u_i^{+2}$ for each i with $1 \leq i \leq s$, where u_{s+1} is regarded as u_1, namely $C = u_1u_1^+u_2u_2^+ \ldots u_su_s^+u_1$.

Proof of Claim 2. Suppose, to the contrary, that there exists one i with $1 \leq i \leq s$ such that $u_{i+1} \neq u_i^{+2}$. Without loss of generality, we assume that $u_2 \neq u_1^{+2}$. Then, the edges of $x_0u_1, u_1^+u_1^{+2}, u_2u_2^+, u_3u_3^+, \ldots,$ and $u_su_s^+$ form a matching in G, giving a contradiction of $\kappa + 1 = s + 1 \leq m = \kappa$. □

Claim 3. If $V(G)\backslash(V(C) \cup \{\ x_0\ \})$ is not empty, then $V(G)\backslash(V(C) \cup \{\ x_0\ \})$ is an independent set.

Proof of Claim 3. Using the similar arguments as the ones in the proofs of Claim 1, we can prove that Claim 3 is true. □

Claim 4. If the independent set $V(G)\backslash(V(C) \cup \{x_0\} := \{x_1, x_2, \ldots, x_r\}$ is nonempty, then $N_C(x_i) = \{u_1, u_2, \ldots, u_s\}$ for each i with $1 \leq i \leq r$.

Proof of Claim 4. Suppose, to the contrary, that there exists one i with $1 \leq i \leq s$ such that $N_C(x_i) \neq \{u_1, u_2, \ldots, u_s\}$. Without loss of generality, we assume that $N_C(x_1) \neq \{u_1, u_2, \ldots, u_s\}$. Using the similar arguments as the ones in

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the proofs of Claim 2, we can prove that $C = z_1 z_1^+ z_2 z_2^+ \ldots z_s z_s^+ z_1$, where $N_C(x_1) = \{z_1, z_2, \ldots, z_s\}$. Since $N_C(x_1) \neq \{u_1, u_2, \ldots, u_s\}$, we must have that $N_C(x_1) = \{z_1, z_2, \ldots, z_s\} = \{u_1^+, u_2^+, \ldots, u_s^+\}$. In this case, we can easily find a cycle in G which is longer than C, giving a contradiction. □

Claim 5. $u_i^+ u_j \in E$ for each i with $1 \leq i \leq s$, where u_{s+1} is regarded as u_1.

Proof of Claim 5. If $i = 1$, it is obvious that $u_1^+ u_1 \in E$ and $u_1^+ u_2 \in E$. Suppose, to the contrary, that there exists one j with $3 \leq j \leq s$ such that $u_1^+ u_j \notin E$. Then, $G[V(G) \setminus \{u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_s\}]$ is disconnected, contradiction to the assumption that the connectivity of G is κ. Similarly, we can prove that $u_i^+ u_j \in E$ for each i with $2 \leq i \leq s$, where u_{s+1} is regarded as u_1. □

Claims 1–5 imply that $G \in \mathcal{F}$. So we complete the proof of Theorem 1.

Reference

[1] Bondy, J. A., Murty, U. S. R. (1976). *Graph Theory with Applications*. London: Macmillan.