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Extended formulations of lower-truncated transversal polymatroids

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ABSTRACT
Extended formulations of (k, l)-sparsity matroids defined on graphs with n vertices and m edges are investigated by Iwata et al. [*Extended formulations for sparsity matroids*, Math. Program. 158 (2016), pp. 565–574]. This note interprets results on (k, l)-lower-truncated transversal polymatroids by the first author in 1983, from the viewpoint of extended formulations, and shows the same $O(mn)$ bound when $k \geq l$ and a better bound $O(m^2)$ when $k < l$. A unified polymatroidal approach is given to derive more general understanding.

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1. Introduction

A polytope P may be expressed as the projection of a polytope Q with less facets in higher-dimensional space. The extension complexity $\mathrm{xc}(P)$ of P is the minimum number of facets of such polytopes. For polymatroids, any linear optimization over them can be solved efficiently by a greedy algorithm, while a certain matroid has exponential extension complexity, as shown by Rothvoß [11] (see also Rothvoß [12] for graph matchings). This poses a problem of investigating a nice class of matroids with polynomial extension complexity.

Martin [9] firstly shows a class of matroids with polynomial extended formulations by reformulating problems with new auxiliary variables as follows. The base polytope of a graphic matroid, for a graph $G' = (V, E)$ with vertex set V and edge set E, is shown to have an extended formulation of size $O(|V|^3)$ ($O(|V||E|)$ as pointed out in [2]). The base polytope of a transversal matroids on U, over a bipartite graph $G = (U, W; A)$ with left vertex set U, right vertex set W and edge set $A \subseteq U \times W$, is shown to have an extended formulation of size $O(|U||W|)$. Iwata et al. [8] show the base polytope of a (k, l)-sparsity matroid on G' has extension complexity of $O(|V||E|)$ when $k \geq l$, and $O(|V|^2|E|)$ when $k < l$ by devising randomized communication protocols as an extension of the protocol in Faenza et al. [2]. For bipartite matchings, the Birkhoff polytope on perfect matchings gives a polynomial-size extended formulation directly [3], which directly implies the above result of transversal matroids.

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This note discusses a general framework to regard the sparsity matroid results as special cases from the viewpoint of lower-truncated polymatroids and their derivatives, including (k, l)-lower-truncated transversal matroids, which are defined and algorithmically investigated by Imai [6] (see also [7] and also [10] for polymatroidal treatments on bipartite graphs). First we bound the extension complexity of lower truncation of a general polymatroid. For a (k, l)-lower-truncated transversal polymatroid over U on bipartite graph $G = (U, W; A)$ with integer parameters k, l, we show that the extension complexity of its base polytope is $O(|U||A|)$ in general and $O(|W||A|)$ when $k \geq l$. When applied to a (k, l)-sparsity matroid, the bounds are the same for $k \geq l$, while our bound is better than [8] when $k < l$. Moreover, our approach explicitly describes extended formulations of these bounds, which may be directly used for linear optimization. These bounds are given in a technical report [5], and this note focuses on the direct descriptions of extended formulations in this general framework.

2. Extension complexity of polymatroids

For a polytope P, another polytope Q in the same or higher dimensional space is called an extension of P if P is derived as a linear projection of Q. The extended complexity $\mathrm{xc}(P)$ of P is the minimum number of facets of any extension of P.

Edmonds introduced a polymatroid as a polytope in his seminal paper [1] by using the lower truncation from its beginning, and we use his terminology below in this section to pay respect to the paper. A set function $\rho\colon 2^E \to \mathbb{R}$ is a β_0-function if it satisfies the following: (1) $\rho(X) \geq 0$ for $\emptyset \neq X \subseteq E$, (2) $\rho(Y) \leq \rho(X)$ for $Y \subseteq X \subseteq E$ (monotonicity), (3) $\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$ for $X, Y \subseteq E$ (submodularity). Then, a polytope

$$P(\rho) = \{x \in \mathbb{R}^E \mid x \geq 0,\ x(X) \leq \rho(X)\ (\emptyset \neq X \subseteq E)\}$$

is a polymatroid, where $x(X) = \sum_{e \in X} x(e)$.

For this polymatroid $P(\rho)$, consider l satisfying $0 \leq l \leq \min\{\rho(\{e\}) \mid e \in E\}$. Then, $\rho'\colon 2^E \to \mathbb{R}$ defined by $\rho'(X) = \rho(X) - l\,(X \subseteq E)$, (E, ρ') is a β_0-function, which defines a polymatroid $P(\rho')$. This polymatroid is called an l-lower-truncated polymatroid obtained from (E, ρ), simply a lower-truncated polymatroid. The membership problem for $P(\rho')$ can be characterized by using $P(\rho)$ as follows, where $\chi_e \in \mathbb{R}^E$ for $e \in E$ is a unit vector on the underlying set (E here) with $\chi_e(e) = 1$ and others 0.

Lemma 2.1: *$x \in P(\rho')$ iff $x + l\chi_e \in P(\rho)$ for each $e \in E$.*

Proof: Suppose $x \in P(\rho')$. For each nonempty $X \subseteq E$, we have $x(X) \leq \rho'(X) = \rho(X) - l$, and hence $(x + l\chi_e)(X) \leq x(X) + l \leq \rho(X)$, which implies $x + l\chi_e \in P(\rho)$.

Conversely, if $x + l\chi_e \in P(\rho)$ for $e \in E$, then, for X with $e \in X \subseteq E$, we have $x(X) + l \leq \rho(X)$. This holds for each $e \in E$, and the lemma follows. ■

This is an extension of Theorem 2.1 in [6], whose variant is Lemma 3.4 in this note.

Theorem 2.2: $\mathrm{xc}(P(\rho')) \leq |E| \cdot \mathrm{xc}(P(\rho))$.

Proof: Suppose that $P(\rho)$ can be represented as $\{x \mid Fx + Gy = h,\ y \geq 0\}$, where F, G are some matrices, h is some vector, $x \in \mathbb{R}^E$, $y \in \mathbb{R}^d$ with $d = \mathrm{xc}(P(\rho))$. Then, $P(\rho')$ can be expressed by $\{x \mid Fx + Gy^{(e)} = h - F \cdot l\chi_e,\ y^{(e)} \geq 0\ (e \in E)\}$, with introducing independent $y^{(e)}$ for each $e \in E$. In the expression, the total number of inequalities is $d|E|$. ■

Deletion/contraction of an element and truncation with respect to some vector $a \in \mathbb{R}^d$ for polymatroid $P(\rho)$ are simpler operations than lower truncations, and yield polymatroids whose extention complexity is $O(\mathrm{xc}(P(\rho)) + |E|)$.

3. Network flow and lower-truncated transversal polymatroids

Let $\widetilde{N} = (\widetilde{V}, \widetilde{A}; S, t; \widetilde{c})$ be a network with a vertex set $\widetilde{V}$, a directed edge set $\widetilde{A}$, a set of sources $S \subset \widetilde{V}$, a unique sink $t \in \widetilde{V} - S$, and a capacity $\widetilde{c} \in \mathbb{R}^{\widetilde{A}}$ where $\widetilde{c}(a)(\geq 0)$ is a capacity of $a \in \widetilde{A}$. For $f \in \mathbb{R}^{\widetilde{A}}$, we define $\partial^+ f \in \mathbb{R}^{\widetilde{V}}$ by $\partial^+ f(v) = -\sum_{e=(u,v)\in\widetilde{A}} f(e) + \sum_{e=(v,w)\in\widetilde{A}} f(e)$ for $v \in \widetilde{V}$. The restriction of $\partial^+ f$ to a subdomain $X(\subseteq \widetilde{V})$ is denoted by $\partial^+ f|_X$. $f \in \mathbb{R}^{\widetilde{A}}$ is a flow if $0 \leq f(a) \leq \widetilde{c}(a)$ $(a \in \widetilde{A})$, $\partial^+ f|_{V-(S\cup\{t\})} = 0$ and $\partial^+ f|_S \geq 0$.

A cut function $\widetilde{c}$: $2^{\widetilde{V}-\{t\}} \to \mathbb{R}$ is defined by $\widetilde{c}(Y) = \sum_{e=(u,w)\in\widetilde{A},\ u\in Y,\ w\in\widetilde{V}-Y} \widetilde{c}(e)$ for $Y \subseteq \widetilde{V} - \{t\}$. $\gamma_{\widetilde{N}}$: $2^S \to \mathbb{R}$ is defined to be $\gamma_{\widetilde{N}}(X) = \min\{\widetilde{c}(Y) \mid Y \subseteq \widetilde{V} - \{t\},\ Y \cap S = X\}$ for $X \subseteq S$. $(S, \gamma_{\widetilde{N}})$ is a polymatroid and the following is well known.

Lemma 3.1: *For $x \in \mathbb{R}^S$, $x \in P(\gamma_{\widetilde{N}})$ iff there is a flow f with $x = \partial^+ f|_S$. Hence* $\mathrm{xc}(P(\gamma_{\widetilde{N}})) \leq 2|\widetilde{A}| + |S|$.

Let $G = (U, W; A)$ be a bipartite graph with left vertex set U, right vertex set W and directed edge set $A \subseteq U \times W$ where edges are directed from U to W. By adding a new vertex t with new directed edges $\{(w,t) \mid w \in W\}$ and setting a capacity c on the new directed edge set A', we derive a network $N = (U \cup W \cup \{t\}, A'; U, t; c)$ and we denote the integer-valued set function over 2^U for network N by γ_N, as $\gamma_{\widetilde{N}}$ is defined for $\widetilde{N}$.

In a bipartite graph G, define $\Gamma(X)$ to be $\{w \mid (u,w) \in A,\ u \in X\}$ $(X \subseteq U)$. Hall's theorem states that $X(\subseteq U)$ is covered by a matching iff $|Y| \leq |\Gamma(Y)|$ for all $Y \subseteq X$. In the network N, we set the capacity c by $c(e) = +\infty$ $(e \in A)$ and $c(e) = 1$ $(e = (w,t),\ w \in W)$, then $|\Gamma(X)| = \gamma_N(X)$ holds. (U, γ_N) is a transversal polymatroid, and its restriction by 1_U is a transversal matroid over U in G. By applying Lemma 3.1 with eliminating some redundant capacity constraints from the network structure, we have the following.

Lemma 3.2: $\mathrm{xc}(P(\gamma_N)) = |A| + |W|$. *For the transversal matroid over U of bipartite graph G, its independence polytope has extension complexity of $|A| + |U| + |W|$.*

For $G = (U, W; A)$, let k, l be positive integers with $d'k - l > 0$, where d' is the minimum degree of a vertex in U. Define $\gamma_{k,l}$: $2^U \to \mathbb{R}$ by $\gamma_{k,l}(X) = k\gamma_N(X) - l$ $(X \subseteq U)$. $(U, \gamma_{k.l})$ is a polymatroid, called a (k,l)-lower-truncated transversal polymatroid. This is an integral polymatroid, and its truncation with respect to 1_U is called a lower-truncated transversal matroid [6]. We denote this matroid by $M(G; k, l)$ (denoted simply by $M(k,l)$ in [6]). Note that $M(G; 1, 0)$ is a transversal matroid on U, as investigated above. Note that a class of transversal matroids includes both uniform matroids and partition matroids, and results below apply to those fundamental matroids.

Combining Theorem 2.2 and Lemma 3.2, we have the following.

Theorem 3.3: $\mathrm{xc}(P(\gamma_{k,l})) \leq |U|(|A| + |W|)$.

This theorem can also be obtained more or less directly by using Theorem 2.1 in [6], which was used to solve greedy-type optimization problem for the lower-truncated transversal polymatroids. When $k \geq l$, this can be modified as in Theorem 2.5 in [6] by using W instead of U in its bipartite structure. For the network $N = (U \cup W \cup \{t\}, A'; U, t; c)$, we replace c with a new capacity c_w for $w \in W$ defined by $c_w(e) = +\infty$ $(e \in A)$, $c_w(e) = k$ $(e = (w', t), w' \in W - \{w\})$, and $c_w(e) = k - l$ $(e = (w, t))$.

Lemma 3.4 (Theorem 2.5 in [6]): *For $x \in \mathbb{R}^U$, $x \in P(\gamma_{k,l})$ iff there is a flow f with $x = \partial^+ f|_S$ in network $N_w = (U \cup W \cup \{t\}, A'; U, t; c_w)$ for each $w \in W$.*

Using this lemma in this case, we may have a better upper bound.

Theorem 3.5: *When $k \geq l$, $\mathrm{xc}(P(\gamma_{k,l})) \leq |W|(|A| + |W|)$.*

Summarizing these results for the lower-truncated transversal matroid, we have only to add $|U|$ inequalities of $x \leq 1_U$ to the extended formulations.

Theorem 3.6: *For lower-truncated transversal matroid $M(G; k, l)$, its independence polytope has extended complexity (1) $|U|(|A| + |W| + 1)$ in general, and (2) $|U| + |W|(|A| + |W|)$ when $k \geq l$.*

4. Sparsity matroids

Consider a case of $|\Gamma(\{u\})| = 2$ $(u \in U)$ in the bipartite graph G. Let $G' = (V, E)$ be an undirected graph with vertex set $V = W$ and edge set $E = U$. G' can be regarded as a graph obtained from G by subdividing each edge e of G' by a vertex e of G. The set $V(X)$ of vertices incident to edges in $X \subseteq E$ in G' is equal to $\Gamma(X)$ in G. For G' and positive integers k, l with $2k - l > 0$, $\{X \mid |Y| \leq k|V(X)| - l \ (\emptyset \neq Y \subseteq X)\}$ is the set of independent sets of a matroid, called (k, l)-sparsity matroid [13] or count matroid [4], which is isomorphic to $M(G; k, l)$ for G. $M(G; 1, 1)$ is a graphic matroid of G', $M(G; k, k)$ is the union of k identical graphic matroids of G', and $M(G; 2, 3)$ is the rigidity matroid of G'. Restating Theorem 3.6, we have the following.

Theorem 4.1: *For the independence polytope of a (k, l)-sparsity matroid of a graph $G' = (V, E)$, there is an extended formulation of size $|E|(2|E| + |V| + 1)$ in general and that of size $|E| + |V|(2|E| + |V|)$ when $k \geq l$.*

Applying this theorem, we see that $M(G; k, k)$ has extension complexity of $O(|V||E|)$ and $M(G; 2, 3)$ has extension complexity of $O(|E|^2)$, which improves $O(|V|^2|E|)$ bound in [8].

5. Concluding remarks

In this note we expand a class of matroids whose polytope has a polynomial-size extension formulation, basically utilizing bipartite and network structures underlying the class. This extends to polymatroids and its lower-truncated ones defined by network flow. It would be an interesting problem to investigate the extension complexity for linear matroids.

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