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Rationality Proofs by Curve Counting

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ABSTRACT
We propose an approach for showing rationality of an algebraic variety X. We try to cover X by rational curves of certain type and count how many curves pass through a generic point. If the answer is 1, then we can sometimes reduce the question of rationality of X to the question of rationality of a closed subvariety of X. This approach is applied to the case of the so-called Ueno-Campana manifolds. Assuming certain conjectures on curve counting, we show that the previously open cases $X_{4,6}$ and $X_{5,6}$ are both rational. Our conjectures are evidenced by computer experiments. In an unexpected twist, existence of lattices D_6, E_8, and Λ_{10} turns out to be crucial.

KEYWORDS
Rationality; Ueno-Campana varieties; counting rational curves

1. Introduction

In November 2014, F. Catanese gave a talk at ICTP, Trieste about Ueno-Campana varieties. In particular, he spoke about the following open problem. Let E be the elliptic curve over $\mathbb{C}$ with complex multiplication by $\frac{1+\sqrt{-3}}{2}$ or the curve with complex multiplication by $\sqrt{-1}$. Let $\Gamma \simeq \mathbb{Z}/c\mathbb{Z}$ be the group of automorphisms of E or its subgroup with $c \geq 3$. So we have[1] $c=3$, $c=4$ or $c=6$ and c determines E uniquely. Let $X_{n,c} = E^n/\Gamma$, the quotient of E^n by the diagonal action of Γ. It is well-known that E^n/Γ is rational for $n=1$, 2. Ueno first studied these varieties in [12] and showed that E^n/Γ cannot be rational for $n \geq c = |\Gamma|$. Campana asked [1] the following question:

Problem. For which c, n is $X_{n,c}$ rational?

An introduction to the problem and the state of the art is given in [2]. In particular, unirationality of $X_{3,4}$ was proved in [4]. Then rationality of $X_{3,4}$ was proved in [3]. Rationality of $X_{3,6}$ was proved in [9]. Then [2] established unirationality of $X_{4,6}$. Rationality of $X_{4,6}$ and unirationality of $X_{5,6}$ are still open.

In this paper we give evidence towards rationality of $X_{4,6}$ and $X_{5,6}$. Below we will explain a certain curve counting problem. We could only solve this problem by a certain computer-based heuristic approach and our answer is not rigorously justified. So we formulate results of these computations as Conjectures 6.2, 6.3, 6.4.

Theorem 1.1. *If Conjecture 6.2 is true, then $X_{4,6}$ is rational.*

Theorem 1.2. *If Conjecture 6.3 is true, then $X_{5,6}$ is unirational. If moreover Conjecture 6.4 is true and $X_{4,6}$ is rational, then $X_{5,6}$ is rational.*

As a summary of all the known results we conclude:

Corollary 1.3. *Suppose* Conjectures 6.2, 6.3, 6.4 *are true. Let E be an elliptic curve over $\mathbb{C}$ and let Γ be a subgroup of the automorphism group of E. Let n be an integer such that $0 < n < |\Gamma|$. Then the quotient of E^n by the diagonal action of Γ is rational.*

It would be interesting to try to apply our methods to some other abelian varieties or other group actions.

Hopefully, the corresponding curve counting can be achieved by some clever enumerative geometry techniques. This would turn our "heuristic proofs" into real proofs.

2. The main idea

The Mori program teaches us that birational properties of varieties are very much controlled by rational curves on them. Let us try to be not too precise and make a guess, how existence of curves (or rather families of curves) would prove rationality of $X_{5,6}$ for us? It would be a good situation if some family of rational curves $\{C_s\}_{s\in S}$ existed such that the base S is rational and such that exactly one curve passes through a generic point of $X_{5,6}$. It turns out that just having the latter property is enough for establishing *unirationality* of $X_{5,6}$. To see this, consider an embedding $\iota : X_{4,6} \hookrightarrow X_{5,6}$. If through a generic point of the image of ι we have exactly one curve from our family, we are done, because then the curves can be parametrized by $X_{4,6}$, so we obtain a dominant rational map $X_{4,6} \hookrightarrow S$ and unirationality of $X_{5,6}$ as a consequence. Now notice that the union of images of all embeddings $X_{4,6} \hookrightarrow X_{5,6}$ is Zariski dense in $X_{5,6}$, so ι with the required property exists. A more careful analysis leads to the following lemma:

Lemma 2.1. *Let X be an irreducible algebraic variety of dimension n over $\mathbb{C}$, and let $\mathcal{C} = \{C_s\}_{s\in S}$ be an algebraic family of rational curves in X. Suppose for a generic point*

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[1]We include the case $c=3$ for completeness and because it helps to illustrate our techniques.

$x \in X$ there is exactly one curve from $\mathcal{C}$ containing x. Let $Z \subset X$ be an irreducible closed subvariety of dimension n – 1 such that for a generic point $x \in Z$ there is exactly one curve from $\mathcal{C}$ containing x. Suppose the curves from $\mathcal{C}$ are not contained in Z. Then the following holds:

(i) *If Z is unirational, then X is unirational.*
(ii) *If moreover Z is rational and there exists open $V \subset Z$ such that any curve from $\mathcal{C}$ intersects V in no more than one point, then X is rational.*

Proof. Denote the total space of the family of curves also by $\mathcal{C}$. It comes with maps $\pi : \mathcal{C} \to S$ and $f : \mathcal{C} \to X$. Let L be the locus of points $x \in X$ such that there is exactly one curve from $\mathcal{C}$ containing x. This is a constructible algebraic subset of X. By the assumptions, $\dim(X \setminus L) \leq n-1$. Therefore, $\dim(\overline{(X \setminus L)} \setminus (X \setminus L)) \leq n-2$. Let $U = X \setminus \overline{(X \setminus L)}$. Since $\dim(Z \cap L) = n-1$, we also have $\dim(Z \cap U) = n-1$. Let $s : U \to S$ be the algebraic map which sends a point $x \in U$ to the unique $s(x) \in S$ such that $x \in C_{s(x)}$. The pullback $s^*\mathcal{C}$ of the original family of curves to $U \cap Z$ has a natural section: for any $x \in U \cap Z$ the curve $C_{s(x)}$ contains x. Therefore over a non-empty open subset $W \subset U \cap Z$ this family is trivial. We obtain a map

$$f' : W \times \mathbb{P}^1 \subset s^*\mathcal{C} \to \mathcal{C} \to X.$$

If Z is unirational, then W is unirational. Hence $W \times \mathbb{P}^1$ is unirational. The image of f' is irreducible and contains W. Thus it is either contained in $\bar{W} = Z$, or has dimension n. The former is not possible because curves from $\mathcal{C}$ are not contained in Z. Thus the image of f' has dimension n. Therefore f' is dominant and X is unirational. The first statement has been proved.

To prove the second statement, we assume without loss of generality that $W \subset V$. If Z is rational, then W, and hence also $W \times \mathbb{P}^1$ is rational. So it is enough to show that a generic point of X has not more than one preimage under f'. Suppose $x \in U$ has at least two preimages. This means there are $(v_1, t_1), (v_2, t_2) \in W \times \mathbb{P}^1$ that go to x. Since there is exactly one curve from $\mathcal{C}$ passing through x, and that curve can intersect W in at most one point, we obtain $v_1 = v_2$. On the other hand, for each $v \in W$ there is at most finitely many values of t such that there exist t' such that $f'(v,t) = f'(v,t')$. So the dimension of such pairs (v, t) is at most n – 1, and therefore the dimension of the space of such x is also at most n – 1. So a generic x has no more than one preimage. □

Although the proof of Lemma 2.1 is essentially trivial, we see that proving unirationality/rationality of X is reduced to unirationality/rationality of Z, and a purely curve counting question.

A similar idea appeared in [10], where the authors show that existence of a unique *quasi-line* passing through *two* general points implies rationality.

2.1. Counting curves on a computer

The families of curves we will be dealing with are such that one can write down explicitly a system of equations whose solutions correspond to curves passing through a given point. So we can implement the following strategy. Pick a big prime number, for instance $p = 1{,}000{,}003$ or $p = 1{,}000{,}033$. We will work over $F = \mathrm{GF}(p)$. Generate a random point $x \in X(F)$. Compute the number of curves passing through x by counting solutions over $\bar{F}$ of the corresponding system of equations by the standard Gröbner basis techniques.[2] If this number is k, p is large and x is "sufficiently random," then we expect x to behave like a generic point, so the number of curves for a generic point over the complex numbers should also be k. More precisely, by the Weil conjectures the probability of hitting the bad locus where the statement is not true is roughly c/p where c is the number of geometric components of the bad locus defined over F. In our 10,000 trials for the Conjectures 6.3 and 6.4 we witnessed 1–2 failures, which gives an estimate on the number of components of the bad locus at the order of ≈ 100. The bad locus at least has to contain the divisors D_v for vectors v of H-norm 12 whose number $336/6 = 56$ is of similar order, see Section 6.3.[3]

3. Rational curves

There are exactly three pairs E, Γ where E is an elliptic curve over $\mathbb{C}$ and Γ is a subgroup of the group of automorphisms of E with $|\Gamma| > 2$. Consider an elliptic curve E of the form $x^2 - y^3 = z^6$ in $\mathbb{P}(3,2,1)$ or $x^2 - y^4 = z^4$ in $\mathbb{P}(2,1,1)$ or $x^3 - y^3 = z^3$ in $\mathbb{P}(1,1,1) = \mathbb{P}^2$. The equation of the curve in all cases is $x^a - y^b = z^c$ in $\mathbb{P}(\frac{c}{a}, \frac{c}{b}, 1)$. We choose (1, 1, 0) as the zero point on E. There are $\gcd(a,b)$ points with $z = 0$, which we call "points at infinity". Let ζ be a primitive root of unity of order c. The group Γ of the roots of unity of order c acts on E by

$$\zeta(x,y,z) = (x,y,\zeta z).$$

We construct rational curves in E^n/Γ as follows. Let $k \geq 1$ be an integer, and let $R(t, u)$ be a homogeneous polynomial of degree ck. For each $i = 1, 2, \ldots, n$ let $P_i(u,v), Q_i(u,v)$ be relatively prime homogeneous polynomials of degrees $\frac{kc}{a}, \frac{kc}{b}$ respectively satisfying

$$P_i^a - Q_i^b = R. \tag{1}$$

Let $\widetilde{C}$ be the curve given by equation $R(u,v) = w^c$ in $\mathbb{P}(1,1,k)$. The group Γ acts on $\widetilde{C}$ by $\zeta(u,v,c) = (u,v,\zeta c)$ and $\widetilde{C}/\Gamma = \mathbb{P}^1$. For each i we have a Γ-equivariant map $f_i : \widetilde{C} \to E$ by

$$(u,v,w) \to (P_i(u,v), Q_i(u,v), w).$$

Quotienting out by Γ we obtain a commutative diagram

$$\begin{tikzcd} \widetilde{C} \arrow[r, "f=(f_1,\ldots,f_n)"] \arrow[d] & E^n \arrow[d] \\ C \cong \mathbb{P}^1 \arrow[r] & E^n/\Gamma. \end{tikzcd}$$

3.1. Discrete invariants

To every such curve we associate *discrete invariants* as follows. For each $i \neq j$ we have

[2]In our computations we used SAGE [11], which delegates Gröbner basis computations to Singular [6] and certain lattice algorithms to GAP [8].
[3]We thank the anonymous referee for this observation.

$$P_i^a - Q_i^b = P_j^a - Q_j^b.$$

Thus we have

$$\prod_{l=0}^{a-1}\left(P_i - \zeta^{\frac{lc}{a}}P_j\right) = \prod_{r=0}^{b-1}\left(Q_i - \zeta^{\frac{rc}{b}}Q_j\right).$$

Denote

$$G_{i,j}^{l,r} = \gcd\left(P_i - \zeta^{\frac{lc}{a}}P_j, Q_i - \zeta^{\frac{rc}{b}}Q_j\right),$$
$$M_{i,j}^{l,r} = \deg G_{i,j}^{l,r} \quad (0 \le l < a,\ 0 \le r < b).$$

Using the assumption that P_i and Q_i are relatively prime and considering contribution of an arbitrary linear form in u, v to various $M_{i,j}^{l,r}$, we establish the following:

$$\frac{kc}{a} = \deg\left(P_i - \zeta^{\frac{lc}{a}}P_j\right) = \sum_{r=0}^{b-1} M_{i,j}^{l,r},$$
$$\frac{kc}{b} = \deg\left(Q_i - \zeta^{\frac{rc}{a}}Q_j\right) = \sum_{l=0}^{a-1} M_{i,j}^{l,r}.$$

Note that $\gcd(G_{i,j}^{l,r}, G_{i,j}^{l',r'}) = 1$ whenever $l \neq l'$ and $r \neq r'$ because otherwise all the four polynomials P_i, P_j, Q_i, Q_j have a common divisor.

3.2. Cohomology classes

It is useful to match the discrete invariants M to the homology classes of the strict pullbacks of our curves in $H_2(\widetilde{E^n/\Gamma}, \mathbb{Z})$, where $\widetilde{E^n/\Gamma}$ is the blowup of E^n/Γ in the fixed points of Γ. It is possible to describe this homology group explicitly, but we will not do this. Instead we will think of the homology class of a rational curve as above consisting of two pieces of data:

(i) The homology class of $\widetilde{C}$ in $H_2(E^n, \mathbb{Z})$.
(ii) For each Γ-fixed point $x \in E^n$ the intersection number of the strict pullback of $\widetilde{C}$ to the blowup of E^n in x with the exceptional divisor. This, roughly speaking, counts how may points on $\widetilde{C}$ go to x.

Furthermore, the homology class of $\widetilde{C}$ in $H_2(E^n, \mathbb{Z})$ can be specified by the following data.

Proposition 3.1. *For each curve $\widetilde{C} \subset E^n$ there exists a unique $n \times n$ Hermitian matrix $H(\widetilde{C})$ with entries in $\mathbb{Q}[\zeta]$ such that for any vector $v \in \mathbb{Z}[\zeta]^n$ we have*

$$v^* H(\widetilde{C}) v = D_v \cdot \widetilde{C},$$

where D_v is the divisor class given by the pullback of $0 \in E$ to E^n via the map $\pi_v : E^n \to E$ given by $(x_1, x_2, \ldots, x_n) \to \sum_i v_i x_i$, and v^* denotes the conjugate transpose of v.

Proof. It is well-known that the function $v \to D_v \cdot \widetilde{C}$ is quadratic in v. Thus there exists a unique symmetric $\mathbb{Q}$-bilinear form $B : \mathbb{Q}[\zeta]^n \times \mathbb{Q}[\zeta]^n \to \mathbb{Q}$ such that $B(v, v) = D_v \cdot \widetilde{C}$ for all v. But we have $D_{\zeta v} = D_v$. This implies $B(\zeta v, \zeta v) = B(v, v)$, hence $B(\zeta v, \zeta v') = B(v, v')$ for any pair of vectors v, v'. Let $H(\widetilde{C}) : \mathbb{Q}[\zeta]^n \to \mathbb{Q}[\zeta]^n$ be the unique $\mathbb{Q}$-linear map such that

$$B(v, v') = \mathrm{Re}(v^* H(\widetilde{C}) v') \quad \text{for all pairs } v, v' \in \mathbb{Q}[\zeta]^n.$$

We have

$$\mathrm{Re}(v^* H(\widetilde{C}) \zeta v') = B(v, \zeta v') = B(\bar{\zeta} v, v') = \mathrm{Re}(v^* \zeta H(\widetilde{C}) v').$$

Since this holds for all v, v' the map $H(\widetilde{C})$ must be $\mathbb{Q}[\zeta]$-linear. So it can be represented by a matrix with entries in $\mathbb{Q}[\zeta]$, and that matrix must be Hermitian because the form B was symmetric. □

It is clear that the diagonal entries of $H(\widetilde{C})$ are simply the degrees of the components f_i of f, $f_i : \widetilde{C} \to E$. Let us calculate the degree of these components for our construction. Consider the function

$$\frac{x}{z^{\frac{c}{a}}}.$$

This is a rational function of degree b on E because for a generic $t \in \mathbb{C}$ there are exactly b solutions to $\frac{x}{z^{\frac{c}{a}}} = t$ corresponding to the bth roots of $t^a - 1$. Its pullback to $\widetilde{C}$ is the function

$$\frac{P_i(u, v)}{w^{c/a}}.$$

Now the equation $\frac{P_i(u,v)}{w^{c/a}} = t$ has $ck \cdot \frac{c}{a}$ solutions: ck values of u/v obtained by solving $P_i(u, v)^a = tR(u, v)$, and c/a values of w/v^k for each of these. Thus the degree of f_i is

$$\frac{ck \cdot \frac{c}{a}}{b} = \frac{kc^2}{ab}.$$

A recipe to calculate the off-diagonal entries from the matrices $M_{i,j}$ will be given in the next section on a case-by-case basis.

3.3. Calculating k

Finally, we calculate the value of k as a function of n for which we expect to have finite number of our curves passing through a generic point of E^n. The first coefficient of $R(u, v)$ can be normalized to 1, and we have kc remaining coefficients. A generic point is given by pairs x_i, y_i satisfying $x_i^a - y_i^b = 1$, and we can parametrize our curve so that the point $(u, v, w) = (1, 0, 1)$ goes to $(x_i, y_i, 1)$. This fixes the first coefficient of P_i and Q_i. Then the condition for a polynomial R to be of the form $P^a - Q^b$ is of codimension $k\left(c - \frac{c}{a} - \frac{c}{b}\right)$. Thus the expected dimension of the space of solutions is $kc - nk\left(c - \frac{c}{a} - \frac{c}{b}\right)$. We want this number to be equal to 2 because there is a two-dimensional group of translations and rotations acting on solutions that needs to be gauged out. Thus we have

$$2 = kc - nk\left(c - \frac{c}{a} - \frac{c}{b}\right).$$

Note that we have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ in all the three cases, so we obtain

$$k = \frac{2}{c - n}.$$

3.4. Summary of the approach

We summarize our strategy for proving rationality of varieties of the form E^n/Γ corresponding to triples $(a,b,c)=(3,3,3), (a,b,c)=(2,4,4), (a,b,c)=(2,3,6)$ and $n<c$.

- Calculate $k=\frac{2}{c-n}$. Suppose it is an integer.[4]
- List possible $a\times b$ matrices M and figure out which matrices correspond to which off-diagonal values of H. Obtain a list of possible off-diagonal entries $\mathbf{h}=\{h_1,h_2,\dots,h_m\}$.
- List possible $n\times n$ matrices H up to integral change of basis which are positive-definite, have $\frac{kc^2}{ab}$ on the diagonal, and have only off-diagonal entries from the list $h_1,h_2,\dots,h_m$.
- For each $n\times n$ matrix H list the degrees $M_{i,j}^{l,r}$.
- For a point $p=(p_1,p_2,\dots,p_n)\in E^n, p_i=(x_i,y_i,1)$ try to compute how many curves with discrete invariants $M_{i,j}^{l,r}$ pass through p. A curve is determined by a sequence of homogeneous polynomials $G_{i,j}^{l,r}(u,v)$ with first coefficient 1 of degrees $M_{i,j}^{l,r}$. These polynomials must satisfy $\gcd(G_{i,j}^{l,r},G_{i,j}^{l',r'})=1$ whenever $l\neq l'$ and $r\neq r'$, and the equations obtained by elimination of $P_1,\dots,P_n$ and $Q_1,\dots,Q_n$ from the following $(i,j=1,\dots,n, l=0,\dots,a-1, r=0,\dots,b-1)$ system of *main equations*:

$$
\begin{aligned}
P_i-\zeta^{\frac{lc}{a}}P_j&=\left(x_i-\zeta^{\frac{lc}{a}}x_j\right)\prod_{r=0}^{b-1}G_{i,j}^{l,r},\\
Q_i-\zeta^{\frac{rc}{b}}Q_j&=\left(y_i-\zeta^{\frac{rc}{b}}y_j\right)\prod_{l=0}^{a-1}G_{i,j}^{l,r}.
\end{aligned}
\tag{2}
$$

- If we are lucky and the answer to the previous step is 1 for a generic point p, then try to construct a vector $v\in\mathbb{Z}[\zeta]^n$ such that for a generic point $p\in D_v$ the number of curves is also one, and the number of intersection points of $D_v/\Gamma\cap C$ outside the set of fixed points of Γ is at most 1.

4. Example for $(a,b,c)=(3,3,3)$

In this case the group Γ has order $c=3$, so we have only one case $n=2, k=2$. The discrete invariant has the form of a matrix

$$M=\begin{pmatrix} M^{0,0} & M^{0,1} & M^{0,2}\\ M^{1,0} & M^{1,1} & M^{1,2}\\ M^{2,0} & M^{2,1} & M^{2,2}\end{pmatrix}$$

of non-negative integers with all the row and column sums equal 2. To calculate the 2×2 matrix $H(\widetilde{C})$ we already know that the diagonal entries are 2. Let

$$H=\begin{pmatrix} 2 & h_{1,2}\\ \bar{h}_{1,2} & 2\end{pmatrix}$$

One can relate $h_{1,2}$ to M by the following. Let $\Delta\subset E\times E$ be the diagonal. Then we have

[4]The only cases with $n>1$ when this number is not an integer are $(a,b,c)=(2,3,6)$ with $n=2,3$. In these cases the method can still be applied. The curve $\widetilde{C}$ should pass through Γ-fixed points of orders different from 6, which implies a slightly different general shape of the Equation (1). We do not include these situations here because it would complicate the notations, and because these cases are already known to be rational anyway.

Proposition 4.1.

$$\Delta\cdot\widetilde{C}=2+M^{0,0}+M^{1,2}+M^{2,1}.$$

Proof. Consider curves $E_{l,r}\subset E\times E$ defined by equations on $(x_i,y_i,z_i)\in E$ $(i=1,2)$:

$$\frac{x_1}{z_1}=\zeta^l\frac{x_2}{z_2},\quad \frac{y_1}{z_1}=\zeta^r\frac{y_2}{z_2}.$$

It turns out, that $E_{0,0},E_{1,2},E_{2,1}$ do not intersect. Therefore they have the same homology class. So, by counting the intersection points

$$
\begin{aligned}
E_{0,0}\cdot\widetilde{C}&=\frac{1}{3}(E_{0,0}+E_{1,2}+E_{2,1})\cdot\\
\widetilde{C}&\geq M^{0,0}+M^{1,2}+M^{2,1}+2.
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
E_{0,1}\cdot\widetilde{C}&\geq M^{0,1}+M^{1,0}+M^{2,2}+2,\\
E_{0,2}\cdot\widetilde{C}&\geq M^{0,2}+M^{2,0}+M^{1,1}+2.
\end{aligned}
$$

The divisor $[E_{0,0}]+[E_{0,1}]+[E_{0,2}]$ is linearly equivalent to $[E\times D]+[D\times E]$ where D is the divisor at infinity of E, which has degree 3. Thus we obtain

$$(E_{0,0}+E_{0,1}+E_{0,2})\cdot\widetilde{C}=12.$$

Hence the inequalities are equalities. □

The diagonal corresponds to the vector $(1,-1)$. This gives us

$$4-h_{1,2}-\bar{h}_{1,2}=2+M^{0,0}+M^{1,2}+M^{2,1}.$$

Similarly we obtain the evaluation for the vector $(\zeta,-1)$, which corresponds to the curve $E_{1,1}$:

$$4-\bar{\zeta}h_{1,2}-\zeta\bar{h}_{1,2}=2+M^{1,1}+M^{0,2}+M^{2,0},$$

which allows to calculate $h_{1,2}$:

$$h_{1,2}=2+2\zeta+\frac{\zeta^2(M^{1,1}+M^{0,2}+M^{2,0})-M^{0,0}-M^{1,2}-M^{2,1}}{1-\zeta}.$$

Going over the set of possible M we find the set of possible values of $h_{1,2}$:

$$h_{1,2}\in\{0,1,\zeta,\zeta^2,-1,-\zeta,-\zeta^2\}.$$

Up to a integral change of basis (a matrix $g\in\mathrm{GL}_2(\mathbb{Z}[\zeta])$ sends H to g^*Hg) we have two possible matrices, with determinants 3 and 4:

$$H_3=\begin{pmatrix}2&-1\\-1&2\end{pmatrix},\quad H_4=\begin{pmatrix}2&0\\0&2\end{pmatrix}.$$

For H_3 we have three possible matrices M:

$$M=\begin{pmatrix}2&0&0\\0&1&1\\0&1&1\end{pmatrix},\quad M=\begin{pmatrix}1&0&1\\1&0&1\\0&2&0\end{pmatrix},\quad M=\begin{pmatrix}1&1&0\\0&0&2\\1&1&0\end{pmatrix}.$$

For H_4 we have six possible matrices M:

$$\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix},\begin{pmatrix}0&0&2\\2&0&0\\0&2&0\end{pmatrix},\begin{pmatrix}0&2&0\\0&0&2\\2&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix},\begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix},\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

Some matrices do not produce any curves passing through generic points, for instance the first three matrices corresponding to H_4. To illustrate our method we give here an explicit parametrization of the curves corresponding to H_3:

4.1. H_3 curves

It is enough to consider only the first matrix, because the other 2 can be obtained from it by automorphisms:

$$M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We want to determine how many curves pass through a given point. Take $p_i = (x_i, y_i, 1)$ for $i = 1, 2$ points on E. If a curve $\widetilde{C}$ passes through $p = (p_1, p_2)$ then we can choose the coordinates u, v such that p is at $v = 0$. We can still apply affine transformations $(u, v) \to (au + bv, v)$. Such a curve is then completely determined by the homogeneous polynomials $G_{l,r}(u, v)$ of degrees $M^{l,r}$ with first coefficient 1 satisfying the following equations, which follow from $\sum_{l=0}^{2} \zeta^l (P_1 - \zeta^l P_2) = 0$ and a similar equation for Q:

$$\sum_{l=0}^{2} \zeta^l (x_1 - \zeta^l y_1) \prod_{r=0}^{2} G_{l,r} = 0, \quad \sum_{r=0}^{2} \zeta^l (x_2 - \zeta^l y_2) \prod_{l=0}^{2} G_{l,r} = 0.$$

In our situation, we have one polynomial of degree 2 and four polynomials of degree 1:

$$(x_1 - y_1)G_{0,0} + (\zeta x_1 - \zeta^2 y_1)G_{1,1}G_{1,2} + (\zeta^2 x_1 - \zeta y_1)G_{2,1}G_{2,2},$$
$$(x_2 - y_2)G_{0,0} + (\zeta x_2 - \zeta^2 y_2)G_{1,1}G_{2,1} + (\zeta^2 x_2 - \zeta y_2)G_{1,2}G_{2,2}.$$

The polynomial of degree 2 is $G_{0,0}$. Note that $x_1 - y_1 \neq 0$ and $x_2 - y_2 \neq 0$. So we can eliminate $G_{0,0}$ from the equations:

$$G_{1,1}\Big((x_2 - y_2)(\zeta x_1 - \zeta^2 y_1)G_{1,2} - (x_1 - y_1)(\zeta x_2 - \zeta^2 y_2)G_{2,1}\Big),$$
$$= G_{2,2}\Big((x_1 - y_1)(\zeta^2 x_2 - \zeta y_2)G_{1,2} - (x_2 - y_2)(\zeta^2 x_1 - \zeta y_1)G_{2,1}\Big).$$

The polynomials $G_{1,1}, G_{2,2}$ must be relatively prime, for otherwise P_1, Q_1, P_2, Q_2 would all share a factor. This implies

$$\begin{aligned}
&(2\zeta + 1)(x_2 y_1 - x_1 y_2)G_{1,1} \\
&= \Big((x_1 - y_1)(\zeta^2 x_2 - \zeta y_2)G_{1,2} - (x_2 - y_2)(\zeta^2 x_1 - \zeta y_1)G_{2,1}\Big), \\
&(2\zeta + 1)(x_2 y_1 - x_1 y_2)G_{2,2} \\
&= \Big((x_2 - y_2)(\zeta x_1 - \zeta^2 y_1)G_{1,2} - (x_1 - y_1)(\zeta x_2 - \zeta^2 y_2)G_{2,1}\Big).
\end{aligned}$$

Assume $x_2 y_1 - x_1 y_2 \neq 0$. Then we uniquely reconstruct $G_{1,1}, G_{2,2}$ from $G_{1,2}, G_{2,1}$. Again $G_{1,2}, G_{2,1}$ are relatively prime, and by applying affine transformations we can move them to an arbitrary pair of distinct linear polynomials with first coefficient 1, for instance u, $u - v$. So under our assumptions there is at most one curve passing through p. Vice versa, to show that the curve exist we just need to make sure that in our construction the pairs $(G_{0,0}, G_{1,2})$, $(G_{0,0}, G_{2,1})$, $(G_{0,0}, G_{1,1})$, $(G_{0,0}, G_{2,2})$, $(G_{1,1}, G_{2,2})$, $(G_{1,2}, G_{2,1})$ are relatively prime. This requires another condition: $x_1 x_2 - y_1 y_2 \neq 0$.

So we have shown that the curve is unique provided

$$z_1 \neq 0, \quad z_2 \neq 0, \quad x_2 y_1 - x_1 y_2 \neq 0, \quad x_1 x_2 - y_1 y_2 \neq 0.$$

This means we have to remove the divisors given by vectors $(1, \pm\zeta^i)$, (0, 1), (1, 0). Taking any other divisor class we will satisfy conditions for part (i) of Lemma 2.1. To show rationality we need to satisfy the assumptions of part (ii). So we need a divisor with small intersection number with $\widetilde{C}$, that is a vector not of the form $(\pm\zeta^i, 0), (0, \pm\zeta^i), (\pm\zeta^i, \pm\zeta^j)$ whose length is small with respect to the form H. Take $v = (1, 2 + \zeta)$, which corresponds to the divisor D_v consisting of $(p_1, p_2) \in E^2$ such that $p_1 + 2p_2 + \zeta p_2 = 0$. We have $v^* H v = 5$. So there is at most 5 points of intersection in $D_v \cap \widetilde{C}$. Going down to E^2/Γ we obtain at most $\lfloor \frac{5}{3} \rfloor = 1$ of points of intersection $(D_v/\Gamma) \cap C$ satisfying $z_i \neq 0$. Clearly, D_v/Γ is rational. So the conditions of Lemma 2.1 are satisfied.

4.2. H_4 curves

In this case computer experiments showed that there are three curves passing through a generic point for each of the last three matrices M. However these curves can be distinguished by their incidence information with the Γ-fixed points, so probably it is possible to use these curves for an alternative rationality proof.

4.3. Total curve count

In total we obtain three curves for H_3 and nine curves for H_4. However, these curves can be distinguished by our discrete invariants and by their intersections with $z_1 = z_2 = 0$.

5. Examples for $(a, b, c) = (2, 4, 4)$

If $(a, b, c) = (2, 4, 4)$, we can have $n = 2$ or $n = 3$. Here $\zeta = \sqrt{-1}$. For $n = 2$ we obtain $k = 1$. For $n = 3$ we obtain $k = 2$. The matrices $M_{i,j}$ are 2×4 with column sums k and row sums $2k$. The matrices H have $2k$ on the diagonal.

Proposition 5.1. *The intersection number of the diagonal $\Delta \subset E \times E$ and $\widetilde{C}$ is given by*

$$\Delta \cdot \widetilde{C} = 2k + 2M^{0,0} + 2M^{1,0}.$$

Proof. We have curves $E_{l,r} \subset E \times E$ given by equations ($l = 0$, 1, $r = 0, 1, 2, 3$)

$$\frac{x_1}{z_1^2} = (-1)^l \frac{x_2}{z_2^2}, \quad \frac{y_1}{z_1} = \zeta^r \frac{y_2}{z_2}.$$

The pairs representing the same homology class are listed as follows $(E_{0,0}, E_{1,2}), (E_{0,1}, E_{1,3}), (E_{0,2}, E_{1,0}), (E_{0,3}, E_{1,1})$. So

$$E_{0,0} \cdot \widetilde{C} = \frac{1}{2}(E_{0,0} + E_{1,2}) \cdot \widetilde{C} \geq 2M^{0,0} + 2M^{1,2} + 2k.$$

This is because each root of $\gcd(P_1 - P_2, Q_1 - Q_2)$ has multiplicity 4 in $E_{0,0} \cdot \widetilde{C}$, and there are further $kc = 4k$

points with $w=0$ on $\widetilde{C}$ which map to the points with $z_1 = z_2 = 0$. Producing similar inequality for $E_{1,0}$ and adding to the one above we obtain

$$(E_{1,0}+E_{0,0})\cdot\widetilde{C}\geq 8k.$$

On the other hand, $E_{1,0}+E_{0,0}$ is equivalent to $E\times D + D\times E$, where D is the divisor at infinity consisting of two points. So the intersection equals $8k$. Therefore our inequalities must be equalities. □

This allows us to compute $h_{i,j}$ as a function of the entries of $M_{i,j}$. The diagonal corresponds to the vector $e_i - e_j$, so we have

$$4k - h_{i,j} - \bar{h}_{i,j} = 2k + 2M^{0,0}_{i,j} + 2M^{1,2}_{i,j}.$$

Hence $\operatorname{Re} h_{i,j} = k - M^{0,0}_{i,j} - M^{1,2}_{i,j}$. The vector $e_i - \zeta e_j$ corresponds to the curve $E_{1,3}$, so the corresponding intersection number is

$$4k - \zeta h_{i,j} - \bar{\zeta}\bar{h}_{i,j} = 2k + 2M^{1,3}_{i,j} + 2M^{0,1}_{i,j}.$$

We obtain $\operatorname{Im} h_{i,j} = -k + M^{0,1}_{i,j} + M^{1,3}_{i,j}$. Thus

$$h_{i,j} = k(1-\zeta) - M^{0,0}_{i,j} - M^{1,2}_{i,j} + \zeta M^{0,1}_{i,j} + \zeta M^{1,3}_{i,j}.$$

5.1. The case k = 1, n = 2

There are two H-matrices (up to automorphisms) for $n=2$, $k=1$, of determinants 2 and 4:

$$H_2 = \begin{pmatrix} 2 & \zeta-1 \\ -\zeta-1 & 2\end{pmatrix},\quad H_4 = \begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix}.$$

For H_2 there is only one matrix M:

$$M = \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}.$$

For H_4 there are two matrices:

$$M = \begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix},\quad M = \begin{pmatrix} 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{pmatrix}.$$

It is not so difficult to check that each of the three matrices M leads to a good family of curves.

5.2. The case k = 2, n = 3

With $n=3$ the set of possibilities is much bigger. We have 19 possible matrices M. They produce the following list of 13 possible off-diagonal entries for H:

$$0, \pm 2, \pm 2\zeta, \pm 2\zeta\pm 2, \pm\zeta\pm 1.$$

To construct a curve we need to choose three of them to get $M_{i,j}$ with $(i,j)=(1,2),(1,3),(2,3)$. So there are $19^3 = 6859$ possibilities. Classifying all the possible positive definite 3×3 matrices H up to $\mathrm{GL}_3(\mathbb{Z}[\zeta])$ action produces 14 cases with determinants

$$8, 16, 16, 24, 32, 32, 32, 36, 40, 44, 48, 48, 56, 64.$$

Counting curves on a computer produces Table 1.[5]

[5]The values in the table are conjectural, they were obtained by testing on random points over the finite field $GF_{1,000,033}$, see Section 2.1. We used exactly the same computer program as for testing Conjectures 6.2, 6.3, 6.4 below.

Table 1. Curve counts for $(a,b,c)=(2,4,4)$, $n=3$, $k=2$.

det(H)	8	16	24	32	36	40	44	48	56	64
#C	0	0,1	4	8,10,18	8	20	24	48,60	80	212

The H-matrices with 0 curves are the following matrices with determinants 8 resp. 16:

$$\begin{pmatrix} 4 & -2\zeta-2 & -2\\ 2\zeta-2 & 4 & -\zeta-1\\ -2 & \zeta-1 & 4\end{pmatrix},\quad \begin{pmatrix} 4 & -2 & -2\\ -2 & 4 & -\zeta-1\\ -2 & \zeta-1 & 4\end{pmatrix}.$$

It turns out that nonexistence of these curves is explained by the fact that the matrices can be conjugated to

$$\begin{pmatrix} 4 & 2 & 0\\ 2 & 4 & -\zeta+3\\ 0 & \zeta+3 & 4\end{pmatrix},\quad \begin{pmatrix} 4 & 2 & \zeta+3\\ 2 & 4 & \zeta+1\\ -\zeta+3 & -\zeta+1 & 4\end{pmatrix},$$

which contain forbidden off-diagonal entries $\zeta+3$.

Note that for each matrix H there are several triples of matrices $M_{i,j}$. The table was obtained by adding the point counts for all triples. In some situations the total number of curves can be greater than 1, but for some individual triples $M_{i,j}$ the number is 1. We will work with the matrix of determinant 16 which gives one curve. The matrix is

$$H_{16} = \begin{pmatrix} 4 & 2 & 2\zeta\\ 2 & 4 & 2\\ -2\zeta & 2 & 4\end{pmatrix}.$$

The matrices $M_{i,j}$ are as follows:

$$M_{1,2} = \begin{pmatrix} 0 & 1 & 2 & 1\\ 2 & 1 & 0 & 1\end{pmatrix},\quad M_{1,3} = \begin{pmatrix} 1 & 2 & 1 & 0\\ 1 & 0 & 1 & 2\end{pmatrix},$$

$$M_{2,3} = \begin{pmatrix} 0 & 1 & 2 & 1\\ 2 & 1 & 0 & 1\end{pmatrix}.$$

Computer experiments show that exactly one curve passes through a generic point of E^3. To apply Lemma 2.1 in full generality we need to choose a divisor. So we look for a vector v whose H-norm v^*Hv is small, but not too small. All vectors of norm 4 do not produce good divisors: through a generic point of such divisor there are no curves of our type. There are no vectors of norms between 4 and 8. There are 252 vectors of norm 8. Let $\mathrm{Aut}(H)$ be the group of matrices $g\in\mathrm{GL}_3(\mathbb{Z}[\zeta])$ such that $g^*Hg = H$. The vectors of norm 8 form 3 $\mathrm{Aut}(H)$-orbits. Some of these vectors are also such that through a generic point of the corresponding divisor there are no curves. In the orbit of $v=(1,-2,1)$, which consists of 192 vectors, for 168 vectors[6] the curve count is 1 and for the remaining 24 it is 0. This vector produces a divisor D_v/Γ satisfying the conditions of Lemma 2.1. We have $D_v\cdot\widetilde{C}=8$, so if we show that at least one intersection point is at infinity, we obtain that the number

[6]Elements of $\mathrm{Aut}(H)$ acting on E^3 do not change H, but they still permute the 8 points at infinity. So the true symmetry group of the system is not $\mathrm{Aut}(H)$, but a certain congruence subgroup. This explains why we obtain different curve counts for vectors of the same $\mathrm{Aut}(H)$-orbit.

of finite intersection points of D_v/Γ with C is at most $\lfloor \frac{7}{4} \rfloor = 1$. The points at infinity of D_v are four points out of the total $2^3 = 8$ points at infinity on E^3. These are the points (p_1, p_2, p_3) satisfying $p_1 - 2p_2 + p_3 = 0$. The points at infinity are of order 2, so this condition is equivalent to $p_1 = p_3$. Let C' be the projection of $\widetilde{C} \subset E \times E \times E$ to $E \times E$ using coordinates 1, 3. So it is enough to show that C' intersects Δ at infinity. The intersection number is 8, but there are only four finite intersection points because $M_{1,3}^{0,0} = 1$. Thus there must be intersections at infinity.

6. Examples for $(a, b, c) = (2, 3, 6)$

Finally, we turn to the most interesting example, which includes open cases. We have $(a, b, c) = (2, 3, 6)$ and $n = 4$ or $n = 5$. Here $\zeta = e^{\frac{2\pi i}{6}}$. For $n = 4$ we obtain $k = 1$. For $n = 5$ we obtain $k = 2$. The matrices $M_{i,j}$ are 2×3 with column sums $2k$ and row sums $3k$. The matrices H have $6k$ on the diagonal. Some things are simpler because there is only one point at infinity, and the correspondence between M-matrices and the off-diagonal entries of the H-matrix are bijective.

Proposition 6.1. *The intersection number of the diagonal $\Delta \subset E \times E$ and $\widetilde{C}$ is given by*

$$\Delta \cdot \widetilde{C} = 6k + 6M^{0,0}.$$

Proof. We have curves $E_{l,r} \subset E \times E$ given by equations ($l = 0,\ 1,\ r = 0,\ 1,\ 2$)

$$\frac{x_1}{z_1^3} = (-1)^l \frac{x_2}{z_2^3}, \quad \frac{y_1}{z_1^2} = \zeta^{2r} \frac{y_2}{z_2^2}.$$

We have

$$E_{0,0} \cdot \widetilde{C} \geq 6M^{0,0} + 6k.$$

This is because each root of $\gcd(P_1 - P_2, Q_1 - Q_2)$ has multiplicity 6 in $E_{0,0} \cdot \widetilde{C}$, and there are further $kc = 6k$ points with $w = 0$ on $\widetilde{C}$ which map to the points with $z_1 = z_2 = 0$. Producing similar inequality for $E_{1,0}$ and adding to the one above we obtain

$$(E_{1,0} + E_{0,0}) \cdot \widetilde{C} \geq 24k.$$

On the other hand, the divisor of the rational function $\frac{y_1}{z_1^2} - \frac{y_2}{z_2^2}$ is

$$E_{1,0} + E_{0,0} - 2(E \times D + D \times E),$$

where D is the point at infinity. Therefore

$$(E_{1,0} + E_{0,0}) \cdot \widetilde{C} = 2(E \times D + D \times E) \cdot \widetilde{C} = 24k.$$

Therefore our inequalities must be equalities. □

This allows us to compute $h_{i,j}$ as a function of the entries of $M_{i,j}$. The diagonal corresponds to the vector $e_i - e_j$, so we have

$$12k - h_{i,j} - \bar{h}_{i,j} = 6k + 6M_{i,j}^{0,0}.$$

The vector $e_i - \zeta e_j$ corresponds to the curve $E_{1,2}$, so the corresponding intersection number is

Table 2. Curve counts for $(a, b, c) = (2, 3, 6)$, $n = 4$, $k = 1$.

det(H)	144	432	576	864	1296
#C	1	6	12	0	72

$$12k - \zeta h_{i,j} - \bar{\zeta}\bar{h}_{i,j} = 6k + 6M_{i,j}^{1,2}.$$

So we can recover $h_{i,j}$:

$$h_{i,j} = (4 - 2\zeta)k - (2 + 2\zeta)M_{i,j}^{0,0} + (4\zeta - 2)M_{i,j}^{1,2}.$$

6.1. The case $k = 1$, $n = 4$

The diagonal entries of H are 6 and the possible off-diagonal entries are in the set

$$\mathbf{h} = \{0, 2\zeta - 4, 2\zeta + 2, 4\zeta - 2, -4\zeta + 2, -2\zeta - 2, -2\zeta + 4\}.$$

We classified all matrices H up to $GL_4(\mathbb{Z}[\zeta])$-equivalence satisfying the following conditions:

(i) H is positive definite.
(ii) $H_{i,i} = 6$ for $i = 1, 2, 3, 4$.
(iii) There is no vector $v \in \mathbb{Z}[\zeta]^4$ such that $v^* H v < 6$.
(iv) For any $v_1, v_2 \in \mathbb{Z}[\zeta]^4$ such that $v_i^* H v_i = 6$ we have $v_1 H v_2 \in \mathbf{h}$.

It turns out there are five matrices with determinants $144, 432, 576, 864, 1296$:

$$H_{144} = \begin{pmatrix} 6 & 2\zeta - 4 & 0 & 0 \\ -2\zeta - 2 & 6 & 2\zeta - 4 & 0 \\ 0 & -2\zeta - 2 & 6 & 2\zeta - 4 \\ 0 & 0 & -2\zeta - 2 & 6 \end{pmatrix},$$

$$H_{432} = \begin{pmatrix} 6 & 2\zeta - 4 & 0 & 0 \\ -2\zeta - 2 & 6 & 2\zeta - 4 & 0 \\ 0 & -2\zeta - 2 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix},$$

$$H_{576} = \begin{pmatrix} 6 & 2\zeta - 4 & 0 & 0 \\ -2\zeta - 2 & 6 & 0 & 0 \\ 0 & 0 & 6 & 2\zeta - 4 \\ 0 & 0 & -2\zeta - 2 & 6 \end{pmatrix},$$

$$H_{864} = \begin{pmatrix} 6 & 2\zeta - 4 & 0 & 0 \\ -2\zeta - 2 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix},$$

$$H_{1296} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix},$$

Note that the off-diagonal values 0 resp. $2\zeta - 4$ correspond to $M_{i,j} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, M_{i,j} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. The curve counts are given in Table 2.[7] It is not clear why curves corresponding to H_{864} do not pass through generic points.

We turn our attention to the matrix $H = H_{144}$, which already implies unirationality of $X_{4,6}$ and will also imply rationality if we find a "good" divisor class. The group

$$\mathrm{Aut}(H) = \{g \in \mathrm{GL}_4(\mathbb{Z}(\zeta)) \mid g^*Hg = H\}$$

has order 155,520 and acts transitively on the 240 vectors of H-norm 6 and on the 2160 vectors of H-norm 12. Vectors of norm 6 intersect C only at infinity, so we pick a vector of norm 12. Some of the vectors of norm 12 correspond to the "diagonals," for instance $v = (1, 0, 1, 0)$. For this vector we obtained 0 curves. However picking $v = (0, 1, 2, 1)$, and any other vector not of the form $(0, 0, \zeta^i, \zeta^j)$ for some i, j or a permutation of such, we obtain 1 curve.

Note that for any v of norm 12 and any curve C of our kind the number of intersection points $\#(C \cap D_v/\Gamma)$ outside of the Γ-fixed points is at most 1. This is true because $\widetilde{C} \cdot D_v = 12$, and the intersection $D_v \cap C$ contains at least one point at infinity.

So we make the following Conjecture, which by Lemma 2.1 implies rationality of $X_{4,6}$:

Conjecture 6.2. For $p \in E^4$ denote by $\#(p)$ the number of curves $\widetilde{C}$ of our type corresponding to the matrix H_{144} and containing p. Then for a generic point $p \in E^4$ we have $\#(p) = 1$. Moreover, for a generic point $p \in D_{0,1,2,1}$ we have $\#(p) = 1$, where

$$D_{0,1,2,1} = \{(p_1, p_2, p_3, p_4) \in E^4 \mid p_2 + 2p_3 + p_4 = 0\}.$$

We verified this conjecture by testing the statement on 10,000 random points on $D_{0,1,2,1}$ and 10,000 random points on E^4 over the field $\mathrm{GF}_{1,000,003}$. Only 1 point got "unlucky" and the number of curves was 0. For every other point the number was 1. Counting the curves took ≈ 0.05 s per point on an ordinary laptop.

6.2. The case $k = 2$, $n = 5$

Finally we turn to the most interesting case. The diagonal entries of H are 12 and the possible off-diagonal entries are in the set

$$\begin{aligned}\mathbf{h} = \{&0,\ -4\zeta+8,\ 2\zeta-4,\ 6,\ 4\zeta-8,\ -2\zeta+4,\ 2\zeta+2,\\ &-4\zeta-4,\ -4\zeta+2,\ 4\zeta-2,\\ &-6\zeta,\ 6\zeta-6,\ 4\zeta+4,\ -6\zeta+6,\ -2\zeta-2,\ 6\zeta,\\ &-8\zeta+4,\ -6,\ 8\zeta-4\}.\end{aligned}$$

We could not classify all such matrices H up to $\mathrm{GL}_5(\mathbb{Z}[\zeta])$-equivalence because the set of possibilities is too big. However the following matrix seems to be the only matrix up to $\mathrm{GL}_5(\mathbb{Z}[\zeta])$-equivalence of the smallest possible determinant $24^3 = 13824$.

$$H_{13824} = \begin{pmatrix} 12 & 4\zeta+4 & 4\zeta+4 & 4\zeta+4 & 4\zeta+4 \\ -4\zeta+8 & 12 & 4\zeta+4 & 4\zeta+4 & 4\zeta+4 \\ -4\zeta+8 & -4\zeta+8 & 12 & 4\zeta+4 & 6 \\ -4\zeta+8 & -4\zeta+8 & -4\zeta+8 & 12 & 6 \\ -4\zeta+8 & -4\zeta+8 & 6 & 6 & 12 \end{pmatrix}$$

We consider $H = H_{13824}$. Note that the off-diagonal value $H_{i,j} = 4\zeta + 4$ resp. $H_{i,j} = 6$ corresponds to $M_{i,j} = \begin{pmatrix} 0 & 4 & 2 \\ 4 & 0 & 2 \end{pmatrix}$ resp. $M_{i,j} = \begin{pmatrix} 0 & 3 & 3 \\ 4 & 1 & 1 \end{pmatrix}$. There are exactly 336 vectors of H-norm 12. Since every curve $\widetilde{C}$ has 12 points at infinity, these curves cannot pass through generic points of divisors corresponding to these vectors. Just for reference we mention that the size of the group $\mathrm{Aut}(H) = \{g \in \mathrm{GL}_5(\mathbb{Z}[\zeta]) \mid g^*Hg = H\}$ is 6912. The vectors of H-norm 12 form three orbits of sizes 48, 192, 96, represented by the basis vectors e_1, e_3, e_5. The next possible H-norm is 18, and there are 768 vectors of norm 18 forming a single $\mathrm{Aut}(H)$-orbit. For such a vector v we have $D_v \cdot \widetilde{C} = 18$, and at least 12 points of intersection are at infinity. Therefore $|D_v/\Gamma \cap C| \le 1$. Some vectors represent "generalized diagonals," for instance $(0, 0, 1, 0, -\zeta)$. We found that our curves do not pass through generic points on the corresponding divisors. Taking any vector different from those do seem to produce good divisors, for instance we take $v = (1, 0, \zeta, 0, -1)$.

The following conjecture implies unirationality of $X_{5,6}$ by part (i), Lemma 2.1:

Conjecture 6.3. For $p \in E^5$ denote by $\#(p)$ the number of curves $\widetilde{C}$ of our type corresponding to the matrix H_{13824} and containing p. Then for a generic point $p \in E^5$ we have $\#(p) = 1$.

The following conjecture together with rationality of $X_{4,6}$ implies rationality of $X_{5,6}$ by Lemma 2.1:

Conjecture 6.4. With $\#(p)$ defined in Conjecture 6.3, for a generic point $p \in D_{1,0,\zeta,0,-1}$ we have $\#(p) = 1$, where

$$D_{1,0,\zeta,0,-1} = \{(p_1, p_2, p_3, p_4, p_5) \in E^5 \mid p_1 + \zeta p_3 = p_5\}.$$

6.3. Computations for $n = 5$

The computations in these cases take much more time than in the $n = 4$ case. It can probably be explained by the fact that the set of divisors where the number of curves is not 1 is huge: for instance, it must contain all divisors D_v corresponding to the 336 vectors v of H-norm 12. Another issue is that when we create the ideal parametrizing our curves, we have besides equations also inequalities of the form

$$\text{resultant }(P_i, Q_i) \neq 0 \quad (1 \le i \le 5). \tag{3}$$

Each inequality is imposed by adding an extra variable J_i and an extra equation

$$J_i \text{ resultant }(P_i, Q_i) = 1 \quad (1 \le i \le 5). \tag{4}$$

Note that the degrees of P_i and Q_i are 6 and 4 respectively, so the resultant has degree 24 and these extra

[7] Similarly to Table 1, the values in Table 2 are conjectural, they were obtained by testing on random points over the finite field $\mathrm{GF}_{1,000,003}$, see Section 2.1. The same computer program was used.

equations are very long. On the other hand, when we tried to keep only the equations without the inequalities the length of the scheme of solutions grew up to 99. The scheme turned out to contain a single isolated point and several very fat points failing the conditions $\gcd(P_i, Q_i) = 1$.

The computation with the inequalities (3) takes $\approx$ 1 hour 15 min on an ordinary laptop (for each random point on E^5). It turns out, it is better to extend the set of inequalities that translate to Equation (4) by a much larger set of 32 inequalities

$$\text{resultant}\ (G_{i,j}^{l,r}, G_{i,j}^{l',r'}) \neq 0 \quad \begin{pmatrix} 1 \le i < j \le 5, 0 \le l < l' \le 1, 0 \le r, r' \le 2: \\ r \neq r', M_{i,j}^{l,r} \neq 0, M_{i,j}^{l',r'} \neq 0 \end{pmatrix}. \tag{5}$$

For each inequality we have to create a new variable and a new equation as in (4). These inequalities formally follow from (3) as explained in Section 3.1, but their degrees are much smaller. On the other hand, inequalities 5 do not seem to imply 3. Thus we must additionally test that every solution we find satisfies 3.

It turns out, that it is faster to build the ideal step-by-step. On each *step* we add some new equations and recompute the Gröbner basis. In the very beginning we choose a cell of the cell decomposition of the weighted projective space we do computations in. The total number of variables is 120 (we have 10 pairs $1 \le i < j \le 5$ and for each pair i, j we have six polynomials $G_{i,j}^{l,r}$ whose degrees are given by the entries of $M_{i,j}$). Among these variables 42 have weight 1, 38 have weight 2, 22 have weight 3, and 18 have weight 4. We order the variables by weight, and if the weights agree by the degree of the polynomial they are coefficients of. Because we should consider the solutions up to translation, we set the very first variable to 0. The choice of a cell in the weighted projective space means we set the first r variables to 0, the $r+1$st variable to 1. We need to do this for every r, $1 \le r \le 119$. Then we have three steps (for each r):

i. Add equations coming from elimination of P_i, Q_i from the main Equation (2).
ii. Add variables and equations representing Equation (5).
iii. Add variables and equations representing Equation (3).

Then we compute the dimension over the base field of the quotient ring with respect to the ideal obtained in the final *step*. This number divided by the weight of the variable we made equal to 1 is the number of points in the given cell. If after some step we obtain the ideal Equation (1), this means there are no solutions in a given cell, so we abort and pass to the next cell, that is next value of r. In all situations we encountered, all the solutions belonged to the biggest cell.

Complete computation for each point $p \in E^5(\mathrm{GF}_{1,000,003})$ takes $\approx$ 3 min on an ordinary laptop. Initially, we made 10 trials for each of the Conjectures 6.3, 6.4, and obtained exactly 1 curve in all cases. The referee suggested that a more extensive testing would provide more evidence for the conjectures, so we ran 10,000 trials for each of the Conjectures 6.3, 6.4 on a cluster (200 cores, $\approx$ 5 h). We found two failures of Conjecture 6.3 and one failure of Conjecture 6.4, which we believe is a convincing evidence, see Section 2.1. The source code and the output logs are available online at https://mellit.xyz/post/rationality/.

Remark 6.1. The quadratic form induced by $H_{13,824}$ on the rank 10 lattice $\mathbb{Z}[\zeta]^5$ is proportional to the so-called laminated lattice Λ_{10}, see [5]. We discovered this fact with the help of OEIS ([7], sequence A006909) by searching for the sequence of numbers of vectors of given norm, which begins as follows: $1, 0, 336, 768$. In fact, the matrix H_{144} from Section 6.1 in a similar way corresponds to the lattice E_8. The matrix H_{16} from Section 5.2 corresponds to the lattice D_6.

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Declaration of Interest

No potential conflict of interest was reported by the author.

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