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Discovering and Proving Infinite Pochhammer Sum Identities

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ABSTRACT
We consider nested sums involving the Pochhammer symbol at infinity and rewrite them in terms of a small set of constants, such as powers of π, $\log(2)$, or zeta values. In order to perform these simplifications, we view the series as specializations of generating series. For these generating series, we derive integral representations in terms of root-valued iterated integrals or directly in terms of cyclotomic harmonic polylogarithms. Using substitutions, we express the root-valued iterated integrals as cyclotomic harmonic polylogarithms. Finally, by applying known relations among the cyclotomic harmonic polylogarithms, we derive expressions in terms of several constants. We provide an algorithimic machinery to prove identities which so far could only be proved using classical hypergeometric approaches. These methods are implemented in the computer algebra package `HarmonicSums`.

KEYWORDS
binomial sums; Pochhammer symbol; holonomic functions; multiple zeta values

AMS SUBJECT CLASSIFICATIONS
05A10; 68W30; 11M32

1. Infinite nested Pochhammer sums

The goal of this article is to find and prove identities of the following form:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{1}{j^2}}{i}}{(n+1)!}=3\zeta_3,$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{1}{j}}{i^3}}{(n+1)!}=\frac{2l_2^4}{3}-4\zeta_3l_2+\frac{2}{3}\pi^2l_2^2+16p_4-\frac{13\pi^4}{180},$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{4}\right)_n\sum_{i=1}^{n}\frac{1}{i^2}}{(n+1)!}=\frac{7\pi^2}{18}-\frac{16C}{3}-6l_2^2+2\pi l_2,$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n\left(\sum_{i=1}^{n}\frac{1}{3i+1}-\sum_{i=1}^{n}\frac{1}{3i+2}\right)}{(n+1)!}=\frac{\pi}{\sqrt{3}}-\frac{3}{4}-\frac{\sqrt{3\pi}\Gamma\left(\frac{5}{6}\right)}{\sqrt{23}\Gamma\left(\frac{1}{3}\right)},$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{1}{2j+1}}{2i+1}}{(2n+1)^2n!}=\frac{1}{96}\pi\left(4\pi^2l_2-72l_2^2+56l_2^3-9\zeta_3\right),$$

$$\sum_{i=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n\sum_{i=1}^{n}\frac{\sum_{j=1}^{i}\frac{\sum_{k=1}^{j}\frac{\sum_{l=1}^{k}\frac{\sum_{m=1}^{l}\frac{1}{m}}{l}}{k}}{j}}{i}}{(n+1)!}=180\zeta_5-\frac{\pi^5}{\sqrt{3}},$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\sum_{i=1}^{n}\frac{1}{i^9}}{(n+1)!}=&-\frac{2339\pi^8l_2}{453600}-\frac{79\pi^6l_2^3}{5670}-\frac{1}{75}\pi^4l_2^5-\frac{8}{945}\pi^2l_2^7\\&+\frac{8l_2^9}{2835}-\frac{79\pi^6\zeta_3}{3780}-\frac{1}{5}\pi^4l_2^2\zeta_3-\frac{4}{9}\pi^2l_2^4\zeta_3+\frac{16}{45}l_2^6\zeta_3-\frac{4}{3}\pi^2l_2\zeta_3^2\\&+\frac{16}{3}l_2^3\zeta_3^2+\frac{8\zeta_3^3}{3}-\frac{3\pi^4\zeta_5}{10}-4\pi^2l_2^2\zeta_5+8l_2^4\zeta_5+48l_2\zeta_3\zeta_5\\&-6\pi^2\zeta_7+72l_2^2\zeta_7+\frac{340\zeta_9}{3},\end{aligned}$$

where $(x)_n$ denotes the Pochhammer symbol, $l_k:=\log(k), \zeta_k:=\sum_{n=1}^{\infty}\frac{1}{n^k}$, and C denotes Catalan's constant.

Note that similar identities were given in [Ablinger 17, Liu and Wang 19]. Such identities are of interest in physics: In particular, such sums have been studied in order to perform calculations of higher-order corrections to scattering processes in particle physics [Ablinger et al. 14, Davydychev and Kalmykov 01, 04, Fleischer et al. 99, Jegerlehner et al. 03, Kalmykov and Veretin 00, Kalmykov et al. 07, Ogreid and Osland 98, Weinzierl 04]. Moreover, similar identities were also considered [Borwein et al. 01, Borwein and Lisoněk 00, Lehmer 85, Zhi-Wei 11, Zucker 85], and there is a connection to Apéry's proof of the irrationality of $\zeta(3)$ (see [Borwein and Borwein 87]) [Weinzierl 04, Zhi-Wei 11, Zucker 85].

While [Ablinger 17] basically deals with sums of the form

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}f(n) \text{ and } \sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{n!}f(n),$$

we are going to consider a much wider class of sums in the frame of this paper. In addition, we will state a general computer algebra method to evaluate a large class of sums in terms of nested integrals. Moreover, we will be able to prove a structural theorem, about

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when such a sum can be expressed in terms of the so-called *cyclotomic polylogarithms*.

The main purpose of this article is to present methods which can be automated; hence, not all identities presented in this paper are new identities. To make more precise which class of sums we are considering, some definitions are in place. Let $r \in \mathbb{N}$ and let $a_i, c_i \in \mathbb{N}$ and $b_i \in \mathbb{N}_0$ for $1 \leqslant i \leqslant r$ then we call $\mathrm{S}_{(a_1,b_1,c_1),\ldots,(a_r,b_r,c_r)}(n)$ defined as

$$\mathrm{S}_{(a_1,b_1,c_1),\ldots,(a_r,b_r,c_r)}(n) := \sum_{i_1=1}^{n} \frac{1}{(a_1 i_1 + b_1)^{c_1}} \sum_{i_2=1}^{i_1} \frac{1}{(a_2 i_2 + b_2)^{c_2}} \cdots \sum_{i_r=1}^{i_{r-1}} \frac{1}{(a_r i_r + b_r)^{c_r}} \tag{1–1}$$

a *cyclotomic harmonic sum* (compare [Ablinger and Blümlein 13, Ablinger et al. 11, 14, Ablinger 13]) of depth r. Note that if $a_i = 1$ and $b_i = 0$ for $1 \leqslant i \leqslant r$ we write

$$\mathrm{S}_{c_1,c_2,\ldots,c_r}(n) := \mathrm{S}_{(1,0,c_1),(1,0,c_2),\ldots,(1,0,c_r)}(n), \tag{1–2}$$

and we call $\mathrm{S}_{c_1,c_2,\ldots,c_r}(n)$ a *multiple harmonic sum* (see, e.g., [Ablinger et al. 13, Ablinger 13, Blümlein and Kurth 99, Blümlein 00, Vermaseren 99]).

The sums we are considering take the form

$$\sum_{n=1}^{\infty} \frac{(p)_n}{(an+b)^c (n+d)!} f(n) \tag{1–3}$$

where $a, b, c, d \in \mathbb{N}_0, p \in \mathbb{R}$, and $f(n)$ is a cyclotomic harmonic sum. We will refer to (1–3) as *Pochhammer sum*.

We are going to find representations of these Pochhammer sums in terms of special classes of integrals (that are similar to the iterated integrals in [Ablinger et al. 14] and correspond to the iterated integrals in [Ablinger 17]). These classes of integrals are iterated integrals over *hyperexponential* functions. More precisely a function $f(x)$ is called *hyperexponential* if

$$\frac{f'(x)}{f(x)} = q(x)$$

where $q(x)$ is a rational function in x.

Then, an *iterated integral* over the hyperexponential functions $f_1(x), f_2(x), \ldots, f_k(x)$ is defined recursively by

$$\mathrm{G}(f_1(\tau), f_2(\tau), \ldots, f_k(\tau), x) = \int_0^x f_1(\tau_1) \mathrm{G}(f_2(\tau), \ldots, f_k(\tau), \tau_1) d\tau_1,$$

with the special case $\mathrm{G}(x) = 1$. Since some letters might have a non-integrable singularity at the base point $x = 0$ we consistently define

$$\mathrm{G}(f(\tau), x) := \int_0^x \left(f(t) - \frac{c}{t}\right) dt + c \log(x),$$

where c takes the unique value such that the integrand on the right hand side is integrable at $t = 0$. It is important to note that this definition preserves the derivative $\frac{d}{dx}\mathrm{G}(f(\tau), x) = f(x)$. In general, we set

$$\begin{aligned}
&\mathrm{G}(f_1(\tau), \ldots, f_j(\tau), x) \\
&:= \int_0^x \left(f_1(t) \mathrm{G}(f_2(\tau), \ldots, f_j(\tau), t) - \sum_{i=0}^{k} c_i \frac{\log(t)^i}{t} \right) dt \\
&\quad + \sum_{i=0}^{k} \frac{c_i}{i+1} \log(x)^{i+1},
\end{aligned}$$

where k and $c_0, \ldots, c_k$ are chosen to remove any non-integrable singularity. Again the result is unique and retains

$$\frac{d}{dx} \mathrm{G}(f_1(\tau), \ldots, f_j(\tau), x) = f_1(x) \mathrm{G}(f_2(\tau), \ldots, f_j(\tau), x).$$

In the following, we will define a subclass of iterated integrals (compare [Ablinger et al. 11]). For $a \in \mathbb{N}$ and $b \in \mathbb{N}$, $b < \varphi(a)$, where φ denotes Euler's totient function, we define

$$f_a^b : (0,1) \mapsto \mathbb{R}$$
$$f_a^b(x) = \begin{cases} \dfrac{1}{x}, & \text{if } a = b = 0 \\ \dfrac{x^b}{\Phi_a(x)}, & \text{otherwise,} \end{cases}$$

where $\Phi_a(x)$ denotes the ath cyclotomic polynomial, e.g., the first cyclotomic polynomials are given by

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \quad \text{etc.}
\end{aligned}$$

Now, let $m_i = (a_i, b_i) \in \mathbb{N}^2$, $b_i < \varphi(a_i)$; for $x \in (0,1)$, we define *cyclotomic polylogarithms* recursively as follows (compare, e.g., [Ablinger et al. 11]):

$$\mathrm{H}(x) = 1,$$
$$\mathrm{H}_{m_1,\ldots,m_k}(x) = \begin{cases} \dfrac{1}{k!}(\log x)^k, & \text{if } m_i = (0,0) \\ \int_0^x f_{a_1}^{b_1}(y) \mathrm{H}_{m_2,\ldots,m_k}(y) dy, & \text{otherwise.} \end{cases}$$

We call k the weight of a cyclotomic polylogarithm and in case the limit exists we extend the definition to $x = 1$ and write

$$\mathrm{H}_{m_1,\ldots,m_k}(1) := \lim_{x \to 1} \mathrm{H}_{m_1,\ldots,m_k}(x).$$

Note that restricting the alphabet to the letters $(0,0), (1,0)$, and $(2,0)$ leads to *harmonic polylogarithms* [Remiddi and Vermaseren 00].

The proposed strategy to prove and find Pochhammer sum identities reads as follows and follows the method proposed in [Ablinger 17]):

Step 1: Rewrite the sums in terms of nested integrals.
Step 2: Rewrite the integrals in terms of cyclotomic polylogarithms (see [Ablinger 17, Section 4]).
Step 3: Provide a sufficiently strong database to eliminate relations among these cyclotomic polylogarithms and find reduced expressions (see Section 4).

This article focuses on Step 1, and we will present three different possibilities to find integral representations of Pochhammer sums. In order to accomplish this task, we view infinite sums as specializations of generating functions [Ablinger 17, Ablinger et al. 14]. Namely, if we are given an integral representation of the generating function of a sequence, then we can obtain an integral representation for the infinite sum over that sequence if the limit $x \to 1$ can be carried out. This approach to infinite sums can be summarized by the following formula:

$$\sum_{i=1}^{\infty} f(i) = \lim_{x\to 1} \sum_{i=1}^{\infty} x^i f(i).$$

For details on Step 2 (implemented in the command `SpecialGLToH` in `HarmonicSums`) and on Step 3 we refer to [Ablinger 17]. It has to be mentioned that we computed and used relation tables of harmonic polylogarithms at one up to weight 12, for cyclotomic polylogarithms of cyclotomy 4 and 6 we computed and used relation tables of cyclotomic polylogarithms at 1 up to weight 6. The size of these tables amounts to several gigabytes. Note that the full strategy has been implemented in the Mathematica package `HarmonicSums`[1] [Ablinger 14].

To complete this introduction we define a number of constants that will appear throughout this article:

$$\begin{array}{lll}
l_2 := \log(2) & l_3 := \log(3) & \zeta_3 := S_3(\infty); \\
\zeta_5 := S_5(\infty); & \zeta_7 := S_7(\infty); & \zeta_9 := S_9(\infty); \\
\zeta_{11} := S_{11}(\infty); & C := \text{Catalan}; & p_4 := \text{Li}_4(\tfrac{1}{2}); \\
p_5 := \text{Li}_5(\tfrac{1}{2}); & p_6 := \text{Li}_6(\tfrac{1}{2}); & p_7 := \text{Li}_7(\tfrac{1}{2}); \\
p_8 := \text{Li}_8(\tfrac{1}{2}); & p_9 := \text{Li}_9(\tfrac{1}{2}); & s_1 := S_{-5,-1}(\infty); \\
s_2 := S_{5,-1,-1}(\infty); & s_3 := S_{-5,1,1}(\infty); & s_4 := S_{5,3}(\infty); \\
s_5 := S_{-7,-1}(\infty); & s_6 := S_{-5,-1,-1,-1}(\infty); & s_7 := S_{-5,-1,1,1}(\infty); \\
h_1 := H_{(3,0),(0,0)}(1); & h_2 := H_{(3,0),(0,0),(1,0)}(1); & h_3 := H_{(3,0),(0,0),(0,0),(0,0)}(1); \\
h_4 := H_{(3,0),(0,0),(1,0),(1,0)}(1); & h_6 := H_{(5,1)}(1); & h_6 := H_{(5,3)}(1); \\
h_7 := H_{(5,1),(0,0)}(1); & h_8 := H_{(5,2),(0,0)}(1); &
\end{array}$$

Here, we extend the definition (1–2) to negative indices by

$$S_{c_1,c_2,\ldots,c_r}(n) := \sum_{i_1=1}^{n} \frac{\text{sign}(c_1)^{i_1}}{|i_1|^{c_1}} \sum_{i_2=1}^{i_1} \frac{\text{sign}(c_2)^{i_2}}{|i_2|^{c_2}} \cdots \sum_{i_r=1}^{i_{r-1}} \frac{\text{sign}(c_r 1)^{i_r}}{|i_r|^{c_r}}.$$

Note that these constants do not possess any further relations induced by the algebraic properties given in Ablinger [Ablinger 17, Section 4], namely shuffle, stuffle, multiple argument, and duality relations, but it is presently not known, whether these constants obey further algebraic equations or not. In few of this question corresponding to logarithms of integers, we refer to Baker's theorem [Baker 66]. Note that for the subclass of multiple zeta values and Euler sums, we use the slightly different, but equivalent, set of constants compared to [Blümlein et al. 10]. For details on how the constants of [Blümlein et al. 10] can be rewritten in terms of the constants given above, we refer to the accompanying Mathematica notebook.

In the following sections, we will use different methods to compute integral representations of the generating function. In Section 2, we will use holonomic closure properties, while in sections 3 and 4 we will use rewrite rules. In Section 4, we will consider a subclass of Pochhammer sums, for which we can directly find representations in terms of cyclotomic polylogarithms, i.e., we do not have to deal with Step 2 of the proposed strategy.

2. Using closure properties of holonomic functions to derive generating functions

In the following, we repeat important definitions and properties (compare [Ablinger 16, Ablinger et al. 14, Kauers and Paule 11]). Let $\mathbb{K}$ be a field of characteristic 0. A function $f = f(x)$ is called *holonomic* (or *D-finite*) if there exist polynomials $p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x]$ (not all p_i being 0) such that the following holonomic differential equation holds:

[1]The package `HarmonicSums` (Version 1.0 16/05/19) together with a Mathematica notebook containing a list of illustrative examples can be downloaded at http://www.risc.jku.at/research/combinat/software/HarmonicSums.

$$p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0. \tag{2–1}$$

We emphasize that the class of holonomic functions is rather large due to its closure properties. Namely, if we are given two such differential equations that contain holonomic functions $f(x)$ and $g(x)$ as solutions, one can compute holonomic differential equations that contain $f(x)+g(x)$, $f(x)g(x)$, or $\int_0^x f(y)dy$ as solutions. In other words, any composition of these operations over known holonomic functions $f(x)$ and $g(x)$ is again a holonomic function $h(x)$. In particular, if for the inner building blocks $f(x)$ and $g(x)$, the holonomic differential equations are given, also the holonomic differential equation of $h(x)$ can be computed.

Of special importance is the connection to recurrences. A sequence $(f_n)_{n\geqslant 0}$ with $f_n \in \mathbb{K}$ is called holonomic (or *P-finite*) if there exist polynomials $p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n]$ (not all p_i being 0) such that the holonomic recurrence

$$p_d(n)f_{n+d} + \cdots + p_1(n)f_{n+1} + p_0(n)f_n = 0 \tag{2–2}$$

holds for all $n \in \mathbb{N}$ (from a certain point on). In the following, we utilize the fact that holonomic functions are precisely the generating functions of holonomic sequences: if $f(x)$ is holonomic, then the coefficients f_n of the formal power series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n$$

form a holonomic sequence. Conversely, for a given holonomic sequence $(f_n)_{n\geqslant 0}$, the function defined by the above sum (i.e., its generating function) is holonomic (this is true in the sense of formal power series, even if the sum has a zero radius of convergence). Note that given a holonomic differential equation for a holonomic function $f(x)$, it is straightforward to construct a holonomic recurrence for the coefficients of its power series expansion. For a recent overview of this holonomic machinery and further literature, we refer to [Kauers and Paule 11].

Since cyclotomic sums are holonomic sequences with respect to n and the iterated integrals we consider are holonomic functions with respect to x, we can use holonomic closure properties to derive integral representations of Pochhammer sums: Given a Pochhammer sum

$$\sum_{n=1}^{\infty} \frac{(p)_n}{(an+b)^c(n+d)!} g(n),$$

where $g(n)$ is a cyclotomic sum. We proceed as proposed in on page 3: define

$$f_n := \frac{(p)_n}{(an+b)^c(n+d)!} g(n)$$

and try to find an iterated integral representation of

$$f(x) := \sum_{n=1}^{\infty} x^n f_n$$

using the following steps:

1. Compute a holonomic recurrence equation for $(f_n)_{n\geqslant 0}$.
2. Compute a holonomic differential equation for $f(x)$.
3. Compute initial values for the differential equation.
4. Solve the differential equation to get a closed form representation for $f(x)$.

This procedure is implemented in the packages HarmonicSums and can be called by

ComputeGeneratingFunction

$$\left[\frac{\text{Pochhammer}[p,n]}{(an+b)^c(n+d)!} g[n], x, \{n, 1, \infty\}\right].$$

We will succeed in finding a closed form representation for $f(x)$ in terms of iterated integrals, if we can find a full solution set of the derived differential equation. The command `ComputeGeneratingFunction` internally uses the differential solver implemented in HarmonicSums, which finds all solutions of holonomic differential equations that can be expressed in terms of iterated integrals over hyperexponential alphabets [Ablinger 16, Ablinger et al. 14, Bronstein 92, Petkovšek 92, Hendriks and Singer 99]; these solutions are called d'Alembertian solutions [Abramov and Petkovšek 94], in addition for differential equations of order two it finds all solutions that are Liouvillian [Ablinger 17a, Kovacic 86, Hendriks and Singer 99].

If we succeed in finding a closed form representation for $f(x)$ in terms of iterated integrals, we proceed with Step 2 and Step 3 of the proposed strategy. Hence, we send $x \to 1$ and try to transform these iterated integrals to expression in terms of cyclotomic polylogarithms and finally we use relations between cyclotomic polylogarithms at one to derive an expression in terms of known constants.

The Pochhammer sum

$$\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2(n-1)!} \tag{2–3}$$

will deal as a representative example to illustrate all three different methods that are presented in this article. First, we work out the sum using the method presented above.

Example 1. We consider the sum (2–3) and start to derive a recurrence for

$$f_n := \frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2 (n-1)!};$$

we find:

$$(2+n)(4+n)^2(1+2n)(3+2n)f_n - 2(1+n)$$
$$(5+n)^2(3+2n)(5+2n)f_{n+1}$$
$$+ 4(1+n)(2+n)(3+n)(6+n)^2 f_{n+2} = 0.$$

Using the closure properties of holonomic functions, we find the following differential equation

$$96f(x) + 3(-250+343x)f'(x) + 3(144-590x+481x^2)f''(x)$$
$$+ x(352-942x+599x^2)f^{(3)}(x) + 8x^2(9-20x+11x^2)f^{(4)}(x)$$
$$+ 4(-1+x)^2 x^3 f^{(5)}(x) = 0,$$

satisfied by

$$\sum_{n=1}^{\infty} x^n \frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2 (n-1)!}.$$

We can solve this differential equation for example using the differential equation solver implemented in `HarmonicSums`:

$$\mathbf{SolveDE}\big[96f[x] + 3(-250+343x)f'[x]$$
$$+ 3(144-590x+481x^2)f''[x]$$
$$+ x(352-942x+599x^2)f^{(3)}[x]$$
$$+ 8x^2(9-20x+11x^2)f^{(4)}[x]$$
$$+ 4(-1+x)^2x^3f^{(5)}[x] == 0, f[x], x\big].$$

By checking initial values we find

$$\frac{1}{7350x^3}\Bigg(-1776+808x-319x^2-888x^3-600x^4+6656\left[\mathrm{G}\left(\frac{\sqrt{1-\tau}}{\tau};x\right)-\mathrm{G}\left(\frac{1}{\tau};x\right)\right]$$
$$+3360\left[\mathrm{G}\left(\frac{1}{\tau},\frac{1}{\tau};x\right)-\mathrm{G}\left(\frac{1}{\tau},\frac{\sqrt{1-\tau}}{\tau};x\right)+\mathrm{G}\left(\frac{\sqrt{1-\tau}}{\tau},\frac{1}{1-\tau};x\right)+\mathrm{G}\left(\frac{\sqrt{1-\tau}}{\tau},\frac{1}{\tau};x\right)\right.$$
$$\left.-\mathrm{G}\left(\frac{\sqrt{1-\tau}}{\tau},\frac{\sqrt{1-\tau}}{\tau};x\right)\right]+4\sqrt{1-x}\Bigg((404-218x-111x^2-75x^3)\left[-\mathrm{G}\left(\frac{1}{1-\tau};x\right)\right.$$
$$\left.-\mathrm{G}\left(\frac{1}{\tau};x\right)+\mathrm{G}\left(\frac{\sqrt{1-\tau}}{\tau};x\right)\right]+222(2-x-x^2)\Bigg)\Bigg). \tag{2–4}$$

At this point we send $x \to 1$ and use the command `SpecialGLToH` in `HarmonicSums` to derive an expression in terms of cyclotomic polylogarithms (compare [Ablinger 17, Section 3]). This leads to

$$-\frac{9367}{7350}-\frac{3328H_{(0,0)}(1)}{3675}+\frac{8}{35}H_{(0,0)}(1)^2-\frac{64H_{(2,0)}(1)}{3675}$$
$$-\frac{32}{35}H_{(2,0)}(1)^2-\frac{16}{35}H_{(0,0),(0,0)}(1)-\frac{32}{35}H_{(0,0),(1,0)}(1)$$
$$-\frac{32}{35}H_{(2,0),(0,0)}(1)+\frac{64}{35}H_{(2,0),(1,0)}(1)+\frac{64}{35}H_{(2,0),(2,0)}(1).$$

Finally, we can use relations between cyclotomic polylogarithms at one (compare [Ablinger 17, Section 4]) to derive

$$\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2(n-1)!} = \frac{-9367+560\pi^2+6720l_2^2-128l_2}{7350}. \tag{2–5}$$

Note that in the last step of this example we are actually only dealing with harmonic polylogarithms (see [Remiddi and Vermaseren 00]).

Let us now list several identities that could be computed using this method:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n S_{1,1,1}(n)}{(n+1)!} = 18\zeta_3 - \frac{\pi^3}{\sqrt{3}},$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n S_2(n)}{(n+1)!} = \frac{5\pi^2}{16}+\frac{27h_1}{8}+\frac{3}{8}\sqrt{3}\pi l_3 - \frac{27l_3^2}{16},$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{4}\right)_n S_2(n)}{(n+1)!} = -\frac{16C}{3}+\frac{7\pi^2}{18}-6l_2^2+2\pi l_2,$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{(2,1,1)}(n)}{(2n+1)^2 n!} = \frac{1}{4}\pi l_2(3l_2-2),$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{(2,1,1),(2,1,1)}(n)}{(2n+1)^2 n!}$$
$$= \frac{1}{96}\pi\left(4\pi^2 l_2 - 72l_2^2 + 56l_2^3 - 9\zeta_3\right).$$

Several formulas that can be found in [Liu and Wang 19] can be also discovered and proved using the described method. Here, we are going to list some of them:

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(S_1(n)^2-S_2(n)\right)}{(n+1)!}=8l_2^2+\frac{2\pi^2}{3},$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(S_1(n)^3-3S_1(n)S_2(n)+2S_3(n)\right)}{(n+1)!}=24\zeta_3+16l_2^3+4\pi^2l_2,$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(S_1(n)^3-3S_1(n)S_2(n)+2S_3(n)\right)}{(n+1)!}=-96Cl_2+16\pi C+72\zeta_3+36l_2^3-18\pi l_2^2+13\pi^2l_2-\frac{9\pi^3}{2},$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(S_1(n)^2-S_2(n)\right)}{(n+1)!}=288Cl_2+48\pi C+216\zeta_3+108l_2^3+54\pi l_2^2+39\pi^2l_2+\frac{27\pi^3}{2},$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}\right)_n\left(S_1(n)^2-S_2(n)\right)}{(n+1)!}=-\frac{32C}{3}+12l_2^2-4\pi l_2+\frac{13\pi^2}{9},$$

$$\sum_{n=0}^{\infty}\frac{\left(\frac{3}{4}\right)_n\left(S_1(n)^2-S_2(n)\right)}{(n+1)!}=32C+36l_2^2+12\pi l_2+\frac{13\pi^2}{3},$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_{n-1}\left(S_1(n)^2-2S_1(n)+S_2(n)\right)}{n!}=8.$$

Note that this method can also be used to compute integral representations of sums of the form

$$\sum_{n=1}^{\infty}\frac{x^n(3)_nS_3(n)}{n^2n!}.$$

Here, we find

$$\sum_{n=1}^{\infty}\frac{x^n(3)_nS_3(n)}{n^2n!}=H_{(0,0),(1,0),(0,0),(0,0),(1,0)}(x)-\frac{3\text{Li}_2(x)^2}{4}-\frac{\text{Li}_3(x)}{2(-1+x)}-\frac{3}{2}\log(1-x)\text{Li}_3(x)+\frac{3\text{Li}_4(x)}{2}+\text{Li}_5(x).$$

and sending for instance $x\to\frac{1}{2}$ we get:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n(3)_nS_3(n)}{n^2n!}.=-\frac{\pi^4}{192}-\frac{\pi^2l_2}{12}+\frac{\pi^4l_2}{288}-\frac{1}{16}\pi^2l_2^2+\frac{l_2^3}{6}-\frac{5}{72}\pi^2l_2^3+\frac{l_2^4}{16}+\frac{11l_2^5}{120}+\frac{3p_4}{2}+3l_2p_4+4p_5+\frac{7\zeta_3}{8}-\frac{7\pi^2\zeta_3}{48}+\frac{21l_2\zeta_3}{16}+\frac{7}{8}l_2^2\zeta_3-\frac{81\zeta_5}{64}.$$

Finally, we consider

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_nS_{11}(n)}{(n+1)!}, \tag{2–6}$$

proceeding as proposed, we find a differential equation of order 16:

$$
\begin{aligned}
&430080f(x) + 210(-4096 + 1592275x)f'(x) \\
&+ 42(-33554432 - 6356812x + 407269601x^2)f''(x) \\
&+ (671088640 - 33746963856x - 8037305736x^2 + 192200072453x^3)f^{(3)}(x) \\
&+ x(11047661360 - 204994450032x - 61653276602x^2 + 771941124781x^3)f^{(4)}(x) \\
&+ 13x^2(3812823056 - 38280317036x - 13991732902x^2 + 109483643797x^3)f^{(5)}(x) \\
&+ 26x^3(3555308396 - 22952549314x - 9866689087x^2 + 53652573053x^3)f^{(6)}(x) \\
&+ 572x^4(152216474 - 697858881x - 344085550x^2 + 1394066246x^3)f^{(7)}(x) \\
&+ 143x^5(323583896 - 1123610312x - 623388464x^2 + 1975831409x^3)f^{(8)}(x) \\
&+ 143x^6(103854560 - 285705072x - 175728306x^2 + 451597351x^3)f^{(9)}(x) \\
&+ 286x^7(10492016 - 23636810x - 15928139x^2 + 34105982x^3)f^{(10)}(x) \\
&+ 3x^8(130094536 - 246156812x - 180008400x^2 + 328091581x^3)f^{(11)}(x) \\
&+ x^9(32842216 - 53242200x - 41920782x^2 + 66163633x^3)f^{(12)}(x) \\
&+ x^{10}(1762640 - 2487876x - 2095294x^2 + 2904173x^3)f^{(13)}(x) \\
&+ 2x^{11}(28900 - 35986x - 32239x^2 + 39703x^3)f^{(14)}(x) \\
&+ 4x^{12}(262 - 291x - 276x^2 + 305x^3)f^{(15)}(x) \\
&+ 8(-1 + x)^2x^{13}(1 + x)f^{(16)}(x) = 0.
\end{aligned}
$$

Solving this differential equation is possible but takes quite some time, so this indicates that we might look for more feasible methods to find generating function representations for Pochhammer sums of that kind. In the following sections, we will introduce rewrite rules, which will allow to compute generating function representations of Pochhammer sums without having to solve differential equations.

3. Using rewrite rules to derive generating functions

In this section, we are going to state rewrite rules which will allow us to find integral representations of the generating functions of Pochhammer sums without having to solve differential equations. We will summarize these rewrite rules in the following lemmas. We start with the base cases where there is no inner sum present:

Lemma 2. *Let $\mathbb{K}$ be a field of characteristic 0. Then, the following identities hold in the ring $\mathbb{K}[[x]]$ of formal power series with $a, c \in \mathbb{N}$ and $b, d \in \mathbb{Z}$:*

$$\sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+d)!} = \frac{(1-x)^{d-p}}{x^d}(p)_{-d}, d<0, \tag{3–1}$$

$$\sum_{n=1}^{\infty} x^n \frac{(p)_n}{n} = (1-x)^{-p}-1, \tag{3–2}$$

$$\sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+d)!} = (1-x)^{d-p}\frac{p}{d!x^d}\int_0^x (1-t)^{p-d-1}t^d dt, d>0, \tag{3–3}$$

$$\sum_{n=1}^{\infty} x^n \frac{(p)_n}{(a\ n+b)^c(n+d)!} = \frac{x^{-\frac{b}{a}}}{a}\int_0^x t^{\frac{b}{a}-1}\sum_{n=1}^{\infty} t^n \frac{(p)_n}{(a\ n+b)^{c-1}(n+d)!}dt. \tag{3–4}$$

In case an inner sum is present, we will make use of the following three lemmas.

Lemma 3. *Let $\mathbb{K}$ be a field of characteristic 0 and let $f : \mathbb{N} \to \mathbb{K}$. Then the following identity holds in the ring $\mathbb{K}[[x]]$ of formal power series with $d<0$:*

$$\sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+d)!}\sum_{i=1}^{n} f(i) = \frac{(1-x)^{d-p}}{x^d}\left((p)_{-d}\sum_{i=1}^{-d} f(i) + \int_0^x \frac{(1-t)^{p-d-1}}{t^{1-d}} \sum_{n=1}^{\infty} t^n \frac{(p)_n}{(n+d-1)!}f(n)dt\right). \tag{3–5}$$

Proof. Both sides satisfy the following initial value problem for $y(x)$, which has a unique solution near $x=0$:

$$y'(x) - \frac{px-d}{(1-x)x}y(x) = \frac{1}{(1-x)x}\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+d-1)!}f(n),$$
$$y(0)=0.$$

□

Lemma 4. *Let $\mathbb{K}$ be a field of characteristic 0 and let $f:\mathbb{N}\to\mathbb{K}$. Then the following identity holds in the ring $\mathbb{K}[[x]]$ of formal power series with $d\geqslant 0$:*

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+d)!}\sum_{i=1}^{n}f(i)$$
$$=\frac{(1-x)^{d-p}}{x^d}\int_0^x t^{d-1}(1-t)^{p-d-1}\sum_{n=1}^{\infty}t^n\frac{(p)_n}{(n+d-1)!}f(n)dt. \tag{3–6}$$

Proof. Both sides satisfy the following initial value problem for $y(x)$, which has a unique solution near $x=0$:

$$y'(x) - \frac{p\ x-d}{(1-x)x}y(x) = \frac{1}{(1-x)x}\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+d-1)!}f(n),$$
$$y(0)=0.$$

□

Lemma 5. *Let $\mathbb{K}$ be a field of characteristic 0 and let $f:\mathbb{N}\to\mathbb{K}$. Then the following identity holds in the ring $\mathbb{K}[[x]]$ of formal power series with $a,c\in\mathbb{N}$ and $b\in\mathbb{Z}$:*

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(a\ n+b)^c(n+d)!}\sum_{i=1}^{n}f(i)$$
$$=\frac{x^{-\frac{b}{a}}}{a}\int_0^x t^{\frac{b}{a}-1}\sum_{n=1}^{\infty}t^n\frac{(p)_n}{(a\ n+b)^{c-1}(n+d)!}\sum_{i=1}^{n}f(i)dt. \tag{3–7}$$

Proof. Both sides satisfy the following initial value problem for $y(x)$, which has a unique solution near $x=0$:

$$y'(x) - \frac{b}{a\ x}y(x) = \frac{1}{a\ x}\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(a\ n+b)^{c-1}(n+d)!}\sum_{i=1}^{n}f(i),$$
$$y(0)=0.$$

□

Note that formulas related to the previous lemmas concerning binomial sums can be found in [Ablinger et al. 14].

Let us now, for the second time, consider (2–3) and illustrate how the previous lemmas can be used as rewrite rules to find integral representations of Pochhammer sums.

Example 6. We again look for a closed form representation in terms of iterated integrals of

$$\sum_{n=1}^{\infty}x^n\frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2(n-1)!}.$$

We start by using Lemma 5 twice:

$$\sum_{n=1}^{\infty}x^n\frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2(n-1)!}$$
$$=x^{-3}\int_0^x t^2\sum_{n=1}^{\infty}t^n\frac{S_1(n)\left(-\frac{1}{2}\right)_n}{(3+n)(n-1)!}dt$$
$$=x^{-3}\int_0^x t^{-1}\int_0^t u^2\sum_{n=1}^{\infty}u^n\frac{S_1(n)\left(-\frac{1}{2}\right)_n}{(n-1)!}dudt.$$

Now we apply Lemma 3 followed by applying (3–4) and (3–1)

$$\sum_{n=1}^{\infty}x^n\frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2(n-1)!}$$
$$=x^{-3}\int_0^x t^{-1}\int_0^t\frac{u^3}{\sqrt{1-u}}\left(\int_0^u\frac{1}{v^2\sqrt{1-v}}\sum_{n=1}^{\infty}v^n\frac{\left(-\frac{1}{2}\right)_n}{n(n-2)!}dv-\frac{1}{2}\right)dudt$$
$$=x^{-3}\int_0^x t^{-1}\int_0^t\frac{u^3}{\sqrt{1-u}}\left(\int_0^u\frac{1}{v^2\sqrt{1-v}}\int_0^v\frac{1}{w}\sum_{n=1}^{\infty}w^n\frac{\left(-\frac{1}{2}\right)_n}{(n-2)!}dwdv-\frac{1}{2}\right)dudt$$
$$=x^{-3}\int_0^x t^{-1}\int_0^t\frac{u^3}{\sqrt{1-u}}\left(\int_0^u\frac{1}{v^2\sqrt{1-v}}\int_0^v\frac{-w}{4(1-w)^{\frac{3}{2}}}dwdv-\frac{1}{2}\right)dudt.$$

At this point, we rewrite the expression in terms of iterated integrals (this can be done by hand or by using the command `GLIntegrate` of `HarmonicSums`) and arrive again at (2–4) and hence we can proceed as in Example 1 to arrive at

$$\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_n S_1(n)}{(3+n)^2(n-1)!}=\frac{-9367+560\pi^2+6720l_2^2-128l_2}{7350}.$$

Note that this method is implemented in the package `HarmonicSums` using the command `PochhammerSumToGL`. Calling

$$\mathbf{PochhammerSumToGL}\left[\frac{\text{Pochhammer}\left[-\frac{1}{2},n\right]\text{S}[1,n]}{(3+n)^2(n-1)!},x,\{n,1,\infty\}\right]$$

will immediately (after regrouping) give (2–4).

Reconsidering (2–6) we find

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n S_3(n)}{(n+1)!}=-\frac{5\pi^3}{32\sqrt{3}}+\frac{9}{16}\sqrt{3}\pi h_1-\frac{15\pi^2 l_3}{32}-\frac{81h_1 l_3}{16}-\frac{9}{32}\sqrt{3}\pi l_3^2+\frac{27l_3^3}{32}+6\zeta_3, \tag{3–8}$$

$$\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{1,1,1,1,1}(n)}{(n+1)!}=60\zeta_5, \tag{3–9}$$

$$\sum_{i=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n S_{1,1,1,1,1}(n)}{(n+1)!}=180\zeta_5-\frac{\pi^5}{\sqrt{3}}, \tag{3–10}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{11}(n)}{(n+1)!}&=-4\mathrm{G}\left(\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{\sqrt{1-\tau}-1}{\tau};1\right)\\
&\quad+2\mathrm{G}\left(\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{1}{\tau},\frac{\sqrt{1-\tau}-1}{\tau};1\right)\\
&=-\frac{677\pi^{10}l_2}{475200}-\frac{2339\pi^8 l_2^3}{680400}-\frac{79\pi^6 l_2^5}{28350}-\frac{2\pi^4 l_2^7}{1575}-\frac{4\pi^2 l_2^9}{8505}+\frac{16l_2^{11}}{155925}\\
&\quad-\frac{2339\pi^8\zeta_3}{453600}-\frac{79\pi^6 l_2^2\zeta_3}{1890}-\frac{1}{15}\pi^4 l_2^4\zeta_3-\frac{8}{135}\pi^2 l_2^6\zeta_3+\frac{8}{315}l_2^8\zeta_3\\
&\quad-\frac{1}{5}\pi^4 l_2\zeta_3^2-\frac{8}{9}\pi^2 l_2^3\zeta_3^2+\frac{16}{15}l_2^5\zeta_3^2-\frac{4\pi^2\zeta_3^3}{9}+\frac{16}{3}l_2^2\zeta_3^3-\frac{79\pi^6\zeta_5}{1260}\\
&\quad-\frac{3}{5}\pi^4 l_2^2\zeta_5-\frac{4}{3}\pi^2 l_2^4\zeta_5+\frac{16}{15}l_2^6\zeta_5-8\pi^2 l_2\zeta_3\zeta_5+32l_2^3\zeta_3\zeta_5+24\zeta_3^2\zeta_5\\
&\quad+72l_2\zeta_5^2-\frac{9\pi^4\zeta_7}{10}-12\pi^2 l_2^2\zeta_7+24l_2^4\zeta_7+144l_2\zeta_3\zeta_7-\frac{170\pi^2\zeta_9}{9}\\
&\quad+\frac{680}{3}l_2^2\zeta_9+372\zeta_{11}.\end{aligned}$$

Note that all the identities listed in Section 2 can also be computed using rewrite rules. But using these rewrite rules turns out to be much more efficient. We are now going to list several additional identities that could be computed with the help of this command:

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_4(n)}{n(n+1)!}=-\frac{\pi^4}{20}-\frac{2}{3}\pi^2 l_2^2-\frac{4}{9}\pi^2 l_2^3+\frac{4l_2^4}{3}+\frac{8l_2^5}{15}+16l_2 p_4+16p_5-\frac{7\pi^2\zeta_3}{12}+8l_2\zeta_3+7l_2^2\zeta_3-\frac{63\zeta_5}{8}, \tag{3–11}$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{3,1}(n)}{(n+1)!n^3}=\frac{13\pi^4}{180}+\frac{\pi^6}{189}+\frac{199\pi^6 l_2}{7560}-\frac{2}{3}\pi^2 l_2^2-\frac{4}{27}\pi^4 l_2^3-\frac{2l_2^4}{3}+\frac{8}{45}\pi^2 l_2^5$$
$$-16p_4+\frac{16}{3}\pi^2 l_2 p_4+\frac{16\pi^2 p_5}{3}+24s_1+\frac{200 l_2 s_1}{7}+\frac{136 s_2}{7}$$
$$-\frac{200 s_3}{7}-\frac{3\pi^2\zeta_3}{2}+\frac{29\pi^4\zeta_3}{168}+4l_2\zeta_3-3\pi^2 l_2\zeta_3+\frac{5}{6}\pi^2 l_2^2\zeta_3 \qquad (3–12)$$
$$-\frac{3}{2}l_2^4\zeta_3-36p_4\zeta_3+\frac{9\zeta_3^2}{8}-\frac{243}{7}l_2\zeta_3^2+\frac{75\zeta_5}{4}-\frac{2935\pi^2\zeta_5}{168}$$
$$-9l_2\zeta_5-\frac{111}{2}l_2^2\zeta_5+\frac{12685\zeta_7}{112},$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_3(n)}{(n+1)!n^5}=\frac{37\pi^4}{360}+\frac{89\pi^6}{5040}-\frac{63031\pi^8}{3024000}+\frac{2}{3}\pi^2 l_2+\frac{37}{180}\pi^4 l_2+\frac{463\pi^6 l_2}{7560}$$
$$+\frac{1}{3}\pi^2 l_2^2+\frac{47}{180}\pi^4 l_2^2+\frac{1079\pi^6 l_2^2}{7560}-\frac{8l_2^3}{3}+\frac{2}{9}\pi^2 l_2^3+\frac{47}{270}\pi^4 l_2^3-\frac{l_2^4}{3}$$
$$-\frac{5}{18}\pi^2 l_2^4+\frac{19}{180}\pi^4 l_2^4-\frac{2l_2^5}{15}-\frac{1}{9}\pi^2 l_2^5+\frac{2l_2^6}{5}-\frac{1}{27}\pi^2 l_2^6+\frac{4l_2^7}{35}+\frac{l_2^8}{35}$$
$$-8p_4-\frac{4}{3}\pi^2 p_4+\frac{4}{9}\pi^4 p_4+16l_2^2 p_4+16p_5+\frac{8}{3}\pi^2 p_5+32l_2 p_5$$
$$-32l_2^2 p_5-\frac{16}{3}\pi^2 p_6-128l_2 p_6+64l_2^2 p_6-128p_7+384l_2 p_7$$
$$+1100s_5-\frac{134}{3}\pi^2 s_1-32l_2 s_1-104l_2^2 s_1+\frac{6939}{40}s_4-16s_3 \qquad (3–13)$$
$$+32s_2-64l_2 s_2+128s_6 80s_7-4\zeta_3+\frac{7\pi^2\zeta_3}{12}-\frac{13\pi^4\zeta_3}{60}-7l_2\zeta_3$$
$$-\frac{271}{90}\pi^4 l_2\zeta_3-7l_2^2\zeta_3-\frac{20}{9}\pi^2 l_2^3\zeta_3+\frac{4}{3}l_2^5\zeta_3-160p_5\zeta_3-\frac{43\pi^2\zeta_3^2}{2}$$
$$-9l_2\zeta_3^2+32l_2^2\zeta_3^2-\frac{203\zeta_5}{8}-\frac{249\pi^2\zeta_5}{16}-\frac{203}{4}l_2\zeta_5-\frac{361}{12}\pi^2 l_2\zeta_5$$
$$-\frac{203}{4}l_2^2\zeta_5+\frac{201}{2}l_2^3\zeta_5+\frac{393\zeta_3\zeta_5}{8}+\frac{3955\zeta_7}{16}-\frac{11533}{16}l_2\zeta_7$$
$$+640p_8+48l_2 s_3.$$

To conclude this section we consider the sum

$$\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_n S_{(3,1,1)}(n)}{(n-1)!n}.$$

We find that it equals

$$\frac{1-2G\left(\frac{\sqrt{1-\tau}}{1-\tau^{1/3}};1\right)+2G\left(\frac{\sqrt{1-\tau}}{1+\tau^{1/3}+\tau^{2/3}};1\right)-33G\left(\sqrt{1-\tau}\tau^{1/3};1\right)-2G\left(\frac{\sqrt{1-\tau}\tau^{1/3}}{1+\tau^{1/3}+\tau^{2/3}};1\right)}{45}. \qquad (3–14)$$

Here, we fail to transform the iterated integrals in terms of cyclotomic polylogarithms; however, since the integrals are simple enough, we are able to perform the integrals in (3–14) for example by using Mathematica and find the result

$$\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{2}\right)_n S_{(3,1,1)}(n)}{(n-1)!n}=-\frac{4\pi^{3/2}}{27\sqrt{3}\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{11}{6}\right)}.$$

Another example where we fail to transform the iterated integrals in terms of cyclotomic polylogarithms but still can do the integrals is

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n\left(S_{(3,1,1)}(n)-S_{(3,2,1)}(n)\right)}{(n+1)!}=-\frac{3}{4}+\frac{\pi}{\sqrt{3}}-\frac{\sqrt{3\pi}\Gamma\left(\frac{5}{6}\right)}{\sqrt{23}\Gamma\left(\frac{1}{3}\right)}.$$

In the following section, we will consider a subclass of Pochhammer sums, for which we will always be able to derive a representation in terms of cyclotomic polylogarithms.

4. Using rewrite rules to directly derive generating functions in terms of cyclotomic polylogarithms

In this section, we will deal with a subclass of the Pochhammer sums, namely we restrict the inner sum to be a multiple harmonic sum and we set $p=1/q$ with $q\in\mathbb{Z}\setminus\{0\}$ and $a=1$ in (1–3), i.e., we are considering sums of the form

$$\sum_{n=1}^{\infty}\frac{(p)_n}{(n+b)^c(n+d)!}\mathrm{S}_{c_1,c_2,\ldots,c_r}(n),\tag{4–1}$$

where $c,c_i\in\mathbb{N},b,d\in\mathbb{Z}$ and $p=\frac{1}{q}$ with $q\in\mathbb{Z}\setminus\{0\}$. Considering a Pochhammer sum in this subclass we could again use the rewrite rules presented in Section 3 to find an integral representation, however we can also use the following lemmas. These new rewrite rules will directly lead to cyclotomic polylogarithms. We again start with the base cases where no inner sum is present (compare Lemma 2):

Lemma 7. *Let $\mathbb{K}$ be a field of characteristic 0. Then, the following identities hold in the ring $\mathbb{K}[[x]]$ of formal power series with $c\in\mathbb{N}$ and $b,d\in\mathbb{Z}$:*

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+d)!}=-\frac{(1-x)^{d-p}x^{-d}}{|p|\ d!}\int_1^{(1-x)^{|p|}}\frac{\left(1-t^{\frac{1}{|p|}}\right)^d}{t^{1-\mathrm{sign}(p)}t^{\frac{d}{|p|}}}dt,\ d>0,\tag{4–2}$$

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+b)^c(n+d)!}=\frac{-1}{|p|\ x^b}\int_1^{(1-x)^{|p|}}\frac{\left(1-t^{\frac{1}{|p|}}\right)^{b-1}}{t^{1-\frac{1}{|p|}}}\sum_{n=1}^{\infty}\frac{(p)_n\left(1-t^{\frac{1}{|p|}}\right)^n}{(n+b)^{c-1}(n+d)!}dt.\tag{4–3}$$

In the cases where there is an inner multiple harmonic sum present we can refine the Lemmas 3, 4, and 5 and get the following result.

Lemma 8. *Let $\mathbb{K}$ be a field of characteristic 0 and let $f:\mathbb{N}\to\mathbb{K}$. Then the following identities hold in the ring $\mathbb{K}[[x]]$ of formal power series with $c\in\mathbb{N},b,d\in\mathbb{Z}$ and $\mathrm{S}_{m_1,\ldots,m_r}(n)$ a multiple harmonic sum:*

$c=0,d<0:$

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+d)!}\mathrm{S}_{m_1,\ldots,m_r}(n)=\frac{(1-x)^{d-p}}{x^d}\left((p)_{-d}s(-d)-\frac{1}{|p|}\int_1^{(1-x)^{|p|}}\frac{\left(1-t^{\frac{1}{|p|}}\right)^{d-1}}{t^{1-\mathrm{sign}(p)}t^{\frac{d}{|p|}}}\sum_{n=1}^{\infty}\frac{\left(1-t^{\frac{1}{|p|}}\right)^n(p)_n}{(n+d-1)!n^{m_1}}\bar{s}(n)dt\right);\tag{4–4}$$

$c=0,d\geqslant 0:$

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+d)!}\mathrm{S}_{m_1,\ldots,m_r}(n)=-\frac{(1-x)^{d-p}x^{-d}}{|p|}\int_1^{(1-x)^{|p|}}\frac{\left(1-t^{\frac{1}{|p|}}\right)^{d-1}}{t^{1-\mathrm{sign}(p)}t^{\frac{d}{|p|}}}\sum_{n=1}^{\infty}\frac{\left(1-t^{\frac{1}{|p|}}\right)^n(p)_n}{(n+d-1)!n^{m_1}}\bar{s}(n)dt;\tag{4–5}$$

$c>0:$

$$\sum_{n=1}^{\infty}x^n\frac{(p)_n}{(n+b)^c(n+d)!}\mathrm{S}_{m_1,\ldots,m_r}(n)=-\frac{x^{-b}}{|p|}\int_1^{(1-x)^{|p|}}t^{\frac{1}{|p|}-1}\left(1-t^{\frac{1}{|p|}}\right)^{b-1}\sum_{n=1}^{\infty}\left(1-t^{\frac{1}{|p|}}\right)^n\frac{(p)_n}{(n+b)^{c-1}(n+d)!}s(n)dt.\tag{4–6}$$

Here, we use the abbreviations $s(n) := \mathrm{S}_{m_1,\dots,m_r}(n)$ and $\bar{s}(n) := \mathrm{S}_{m_2,\dots,m_r}(n)$.

Proof. For all these equalities, it is possible to find an initial value problem, which has a unique solution near $x = 0$ and is satisfied by both sides of the respective equation. □

Note that the polynomials arising in the left hand sides of the equations in Lemmas 7 and 8 are of the form t^i or $(1-t^i)^k$ for $i,k \in \mathbb{Z}$, hence integrating over these integrands will lead to cyclotomic polylogarithms. Therefore, the Pochhammer sums of the form (4–1) will be expressible in terms of cyclotomic polylogarithms, and we can state the following structural theorem.

Theorem 9. *Any sum of the form*

$$\sum_{n=1}^{\infty} \frac{\left(\frac{1}{q}\right)_n}{(n+b)^c (n+d)!} \mathrm{S}_{c_1,c_2,\dots,c_r}(n), \tag{4–7}$$

where $c, c_i \in \mathbb{N}, b, d \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$, can be expressed in terms of cyclotomic polylogarithms at one.

Let us now, for the third time, consider (2–3) and illustrate how the previous lemmas can be used as rewrite rules to directly find a representation in terms of cyclotomic polylogarithms.

Example 10. We seek a closed form representation in terms of cyclotomic polylogarithms of

$$\sum_{n=1}^{\infty} x^n \frac{\left(-\frac{1}{2}\right)_n \mathrm{S}_1(n)}{(3+n)^2 (n-1)!},$$

so we use (4–6) twice:

$$\sum_{n=1}^{\infty} x^n \frac{\left(-\frac{1}{2}\right)_n \mathrm{S}_1(n)}{(3+n)^2 (n-1)!}$$
$$= -\frac{2}{x^3}\int_1^{\sqrt{1-x}} t(1-t^2)^2 \sum_{n=1}^{\infty} \frac{(1-t^2)^n \left(-\frac{1}{2}\right)_n \mathrm{S}_1(n)}{(3+n)(n-1)!}\, dt$$
$$= \frac{4}{x^3}\int_1^{\sqrt{1-x}} \frac{t}{1-t^2} \int_1^t u(1-u^2)^2 \sum_{n=1}^{\infty} \frac{(1-u^2)^n \left(-\frac{1}{2}\right)_n \mathrm{S}_1(n)}{(n-1)!}\, du\; dt.$$

Now, we apply (4–4) followed by applying (4–3) and (3–1):

$$\sum_{n=1}^{\infty} x^n \frac{\left(-\frac{1}{2}\right)_n \mathrm{S}_1(n)}{(3+n)^2 (n-1)!}$$
$$= -\frac{4}{x^3}\int_1^{\sqrt{1-x}} \frac{t}{1-t^2} \int_1^t (1-u^2)^3 \left(2\int_1^u \frac{\sum_{n=1}^{\infty} \frac{(1-v^2)^n \left(-\frac{1}{2}\right)_n}{n(n-2)!}}{(1-v^2)^2}\, dv + \frac{1}{2}\right) du\; dt$$
$$= \frac{4}{x^3}\int_1^{\sqrt{1-x}} \frac{t}{1-t^2} \int_1^t (1-u^2)^3 \left(\int_1^u \frac{\int_1^v \frac{4w \sum_{n=1}^{\infty} \frac{(1-w^2)^n \left(-\frac{1}{2}\right)_n}{(n-2)!}}{1-w^2}\, dw}{(1-v^2)^2}\, dv - \frac{1}{2}\right) du\; dt$$
$$= \frac{4}{x^3}\int_1^{\sqrt{1-x}} \frac{t}{1-t^2} \int_1^t (1-u^2)^3 \left(\int_1^u \frac{\int_1^v \frac{w^2-1}{w^2}\, dw}{(1-v^2)^2}\, dv - \frac{1}{2}\right) du\; dt.$$

Now, we can send $x \to 1$ and rewrite this expression directly in terms of cyclotomic harmonic polylogarithms (again this can be done by hand or by using the command `GLIntegrate` of `HarmonicSums`) and arrive again at

$$-\frac{9367}{7350} - \frac{64 H_{(2,0)}(1)}{3675} - \frac{32}{35} H_{(0,0),(1,0)}(1) - \frac{32}{35} H_{(2,0),(0,0)}(1) + \frac{64}{35} H_{(2,0),(1,0)}(1). \tag{4–8}$$

Finally, we can again use relations between cyclotomic polylogarithms at one to derive (2–5).

Note that this is implemented in the command `PochhammerSumToH`, so calling

$$\textbf{PochhammerSumToH}\left[\frac{\mathrm{Pochhammer}\left[-\frac{1}{2}, n\right] \mathrm{S}[1,n]}{(n+3)^2 (n-1)!}, x, \{n, 1, \infty\}\right]$$

will immediately give (after pretty printing)

$$\frac{-11856+2552x+537x^2-600x^3}{7350x^3}+\frac{4\sqrt{1-x}(1482+107x+75x^3)}{3675x^3}$$
$$+\left(\frac{64}{3675x^3}+\frac{4\sqrt{1-x}(1276+218x+111x^2+75x^3)}{3675x^3}\right)H_{-1}\left(\sqrt{1-x}\right)$$
$$-\frac{4\sqrt{1-x}(1276+218x+111x^2+75x^3)H_0\left(\sqrt{1-x}\right)}{3675x^3}$$
$$+H_{-1}(1)\left(-\frac{64}{3675x^3}-\frac{4\sqrt{1-x}(1276+218x+111x^2+75x^3)}{3675x^3}+\frac{64H_1\left(\sqrt{1-x}\right)}{35x^3}\right)$$
$$-\frac{96H_{-1,0}(1)}{35x^3}+\frac{32H_{-1,0}\left(\sqrt{1-x}\right)}{35x^3}-\frac{64H_{-1,1}(1)}{35x^3}-\frac{64H_{0,-1}(1)}{35x^3}+\frac{32H_{0,1}(1)}{35x^3}$$
$$-\frac{64H_{1,-1}\left(\sqrt{1-x}\right)}{35x^3}+\frac{32H_{1,0}\left(\sqrt{1-x}\right)}{35x^3}.$$

Sending $s \to 1$ will give (4–8).

To conclude we are going to list several identities that could be computed with the help of this command (note that these identities could have also be computed using the methods presented in the previous sections):

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{5}\right)_n S_1(n)}{(n+1)!}=\frac{25h_6}{4}, \tag{4–9}$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{5}\right)_n S_2(n)}{(n+1)!}=\frac{875h_6^2}{48}+\frac{125}{12}\sqrt{5}h_6^2+\frac{125h_7}{16}+\frac{125h_8}{8}, \tag{4–10}$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{2,2,2}(n)}{(n+1)!}=\frac{2\pi^6}{189}-\frac{9\zeta_3^2}{4}-\frac{15l_2\zeta_5}{2}, \tag{4–11}$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{2,2}(n)}{(n+2)!}=\frac{2\pi^2}{9}+\frac{\pi^4}{45}-\frac{8l_2^2}{3}-\zeta_3-2l_2\zeta_3, \tag{4–12}$$

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n S_{2,2}(n)}{n(n+2)!}=-\frac{\pi^2}{9}-\frac{2\pi^4}{45}+\frac{4l_2^2}{3}+\frac{\zeta_3}{2}+4l_2\zeta_3+\frac{15\zeta_5}{16}, \tag{4–13}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{3}\right)_n S_{2,2}(n)}{(n+3)!}=&\frac{9}{320}+\frac{3\sqrt{3}\pi}{320}+\frac{51\pi^2}{512}+\frac{11\pi^3}{128\sqrt{3}}+\frac{91\pi^4}{5760}+\frac{1377h_1}{1280}\\&-\frac{27}{128}\sqrt{3}\pi h_1-\frac{135\pi^2h_1}{128}+\frac{243h_2}{64}+\frac{27}{64}\sqrt{3}\pi h_2+\frac{1539h_3}{128}\\&-\frac{243h_4}{16}-\frac{27l_3}{320}+\frac{153\sqrt{3}\pi l_3}{1280}+\frac{11\pi^3l_3}{128\sqrt{3}}-\frac{27}{128}\sqrt{3}\pi h_1l_3\\&+\frac{243h_2l_3}{64}-\frac{1377l_3^2}{2560}-\frac{81\zeta_3}{64}+\frac{39}{64}\sqrt{3}\pi\zeta_3-\frac{81l_3\zeta_3}{64}.\end{aligned} \tag{4–14}$$

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