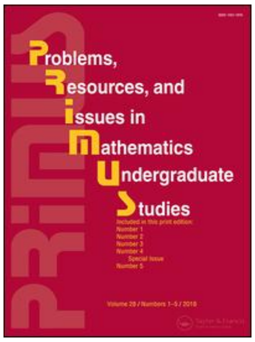

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The Extreme Apprenticeship Method

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ABSTRACT
Over the past 9 years, we have implemented, developed, and refined a novel student-centered teaching method called Extreme Apprenticeship in university mathematics context. In this paper, we report the theoretical orientation behind the method, as well as its practical implementation. We also reflect on our experiences gained from using it in our teaching.

KEYWORDS
Student-centered; novel teaching methods; inquiry-based mathematics education; instructional design; higher education; cognitive apprenticeship; programmatic flipped learning; IBL; inquiry-based program design

1. INTRODUCTION

Higher education aims to produce experts with strong professional identity and means to critically assess and develop their own skills as well as prevailing practices in their field. In this light, the objectives of a program of study in mathematics become more complex than merely transferring a set of mathematical facts into students' heads. Instead, students should learn to successfully participate in the practices of professional mathematicians.

Becoming an expert requires active participation and taking responsibility for one's own learning. In recent years, many student-centered teaching practices have been developed to guide and support students in this direction. However, it is challenging to support students' active learning in large courses. Particular difficulties include taking individual students into account in teaching, as well as arranging feedback procedures that are timely and efficient.

As a response to this challenge, we present the Extreme Apprenticeship method (XA) for teaching university mathematics students. The XA method is based on Cognitive Apprenticeship [4], and it emphasizes supporting the development of expertise in students right from the beginning of their university studies.

Earlier studies on the XA method show that students find the method satisfactory, and the passing rates have not dropped even though the workload is significantly increased and the requirement level raised [6]. Also, the XA method has enabled a shift from rote learning towards conceptual understanding [15]. In addition, the XA instructional design supports more favorable approaches to learning, higher self-efficacy and more positive experiences of the teaching–learning

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environment [10]. It has also served as a catalyst to broader, department-level change in teaching culture [16].

In this article, we describe the philosophy behind the XA method, and then focus on its practical implementation. The aim is to describe the XA method in a way that enables adaptation of its ideas in other institutions. The method has been used in courses with 10–500 students, but in this paper we mainly discuss the large course scale. Finally, we provide lessons learned and some critical advice for those who wish to apply the method in their own teaching. Our aim is to encourage other mathematics educators to integrate suitable elements of the XA method in their classes.

2. THEORETICAL ORIENTATION BEHIND EXTREME APPRENTICESHIP

Extreme Apprenticeship is a student-centered teaching method for organizing instruction in an effective and scalable manner. The method was originally created in the University of Helsinki in Finland for teaching university-level computer programming [18], and was later employed in university mathematics education [6, 16].

The XA method enables the implementation of good educational practices also in large scale courses. The label "extreme" refers originally to a software development method "Extreme programming" that "pushes ordinary development practices to the extreme" [3, p. xxi]. The same meaning can be identified with XA, but here "extreme" is also interpreted to refer to the large number of "apprentices" (i.e., students) that can be taught at the same time with the method. The XA method is not tied to teaching any particular content topic; in mathematics, it has been used in teaching many different undergraduate courses such as abstract algebra, linear algebra, logic, and probability.

In the XA method, designing the teaching structures is guided by the ideas of Lave and Wenger [13] on situated learning in a community of practice. The viewpoint of situated learning is that learners participate in a community, pursuing what is regarded as expertise in that community. In this vein, mathematics students participate in the community of mathematics learners. The participation is first peripheral, but as students progress in gaining expertise, they move in a centripetal direction inside the community. This movement is motivated by visible models of mature practice. In the process, the students acquire not only parts of the "teaching curriculum," but also the cultural and social practices that are valued and found useful by their peers, senior participants and experts of the community. Learning activities in XA are designed so that also the cultural and social practices should bring meaningful learning experiences to the students.

To instrumentalize the idea of learning in communities of practice, XA uses Cognitive Apprenticeship [4], following its idea of making cognitive processes visible. This entails both making the teacher's thinking visible to the learner and making the learner's thinking visible to the teacher. The core teaching methods of Cognitive Apprenticeship include modeling, coaching, and scaffolding.

In XA, the teaching team models to the students how mathematicians work (e.g., solving problems, finding information, communicating own ideas to others). They coach the students by observing their work and giving support and feedback according to their observations. This enables a bidirectional feedback process in which students receive feedback on their work and the teaching team receives information on the progress of the students [8]. Students' work is scaffolded by tasks that have been divided into smaller and approachable goals, which are then merged together as the students start to master a topic. Teachers and tutors also scaffold students' work face-to-face by offering just enough help so that the students can discover the answers by themselves. This follows the idea of the zone of proximal development by Vygotsky [19].

The XA method can be regarded as a Flipped Learning method [5], as initial direct instruction has been moved from the whole group learning space (lectures) to the individual learning space (self-study and one-on-one instruction). The individual learning space is facilitated by carefully designed tasks and support from the teaching team. In the group space, the responsible teacher models expert thinking, and students engage creatively with the subject matter they are already familiar with. The XA method is also a version of inquiry-based mathematics education (IBME [1, 12]). Laursen and Rasmussen [12] describe how IBME unites two inquiry-themed strands of active learning and teaching: inquiry-based learning (IBL, e.g., [11, 21]) and inquiry-oriented instruction (IOI, e.g., [9, 17, 20]). Central to IBME are four pillars: students engage deeply with meaningful mathematics, student collaborate to make sense of mathematical ideas, teachers inquire into the thinking of students, and teachers foster equity in their classrooms. In the XA method, these aims are addressed by carefully designed task sequences that aim to support deep understanding, learning spaces in which students are encouraged to discuss and collaborate, pedagogical development of tutors that revolves around inquiry into the students' learning, and sets of explicitly shared norms supporting equal participation together with an open, accessible learning space.

Finally, the XA method shares features with other student-centered approaches used in tertiary education. The Moore method in mathematics [7, 14] puts emphasis on inquiry, collaboration and deep understanding, which are also some of the main driving forces behind XA (as well as IBME in general). In problem-based learning (PBL [2]), learning happens through problem solving. PBL is similar to XA in that the role of tutors as facilitators becomes crucial: they help the students by asking "questions that they should be asking themselves to better understand and manage the problems" [2, p. 5]. Tutors need to be skilled in this role, so that the students will gradually learn to ask these questions on their own.

3. CONTEXT

University studies in Finland follow the Bologna Process framework, consisting of 3 years of Bachelor's studies and 2 years of Master's studies. Students choose a specific major when they enter the university and focus on that chosen discipline from the beginning. There are no tuition fees for EU students, and the state supports Finnish students by giving them a grant to cover for their living costs.

The University of Helsinki is a research-intensive university. The Department of Mathematics and Statistics at the University of Helsinki has approximately 40 faculty members and 1000 students majoring in mathematical sciences or mathematics education. The students are selected based on their performance in the upper secondary school matriculation examination. The undergraduate courses at the Department serve mainly three groups of students: mathematics students, pre-service teachers, and students in other STEM fields. In Finnish upper secondary schools, calculus is taught up to integration, so mathematics students begin their university studies with proof-based courses beyond calculus. Typical first-year undergraduate courses are Introduction to University Mathematics (set notation and proofs), Linear Algebra and Matrices (linear algebra in finite-dimensional spaces), and Limits (real analysis with epsilon-delta definitions). The Department has a tradition of hiring undergraduate and graduate students as teaching assistants or tutors for courses.

4. IMPLEMENTING EXTREME APPRENTICESHIP

This section describes how the XA method has been implemented in teaching university mathematics courses with 100–500 students. In addition, the section illustrates how communities of practice and apprenticeship are realized, and what modeling, coaching, and scaffolding mean in the practical implementation of the method. The teaching is carried out by a teaching team that includes a responsible teacher and a number of senior students called *tutors*, who act as peer teachers in the course. The following sections elaborate on the learning cycle, teaching team, tasks, guidance, and lectures associated with the XA method.

4.1. Learning Cycle

In XA, learning is built around a learning cycle consisting of three components: (1) students work on tasks, (2) they submit their coursework, and (3) attend lectures. The components are presented in Figure 1. At each step, the students interact with each other and the teaching team. The cycle starts with the students studying a new topic by solving introductory problems together with each other and with the guidance of the teaching team. Guidance is offered in a collaborative learning space in the middle of the department. Next, students hand in their solutions and receive written feedback from the teaching team. After that, students attend lectures where they get to deepen their understanding of the topics introduced in the tasks. In the lectures, the students participate in discussions facilitated by the responsible teacher. Finally, the cycle closes: the students are given more challenging tasks concerning topics that are familiar, and at the same time they start studying new topics by working on new introductory tasks. At each step of the cycle, the teaching team gives feedback to the students and also receives information on the students'

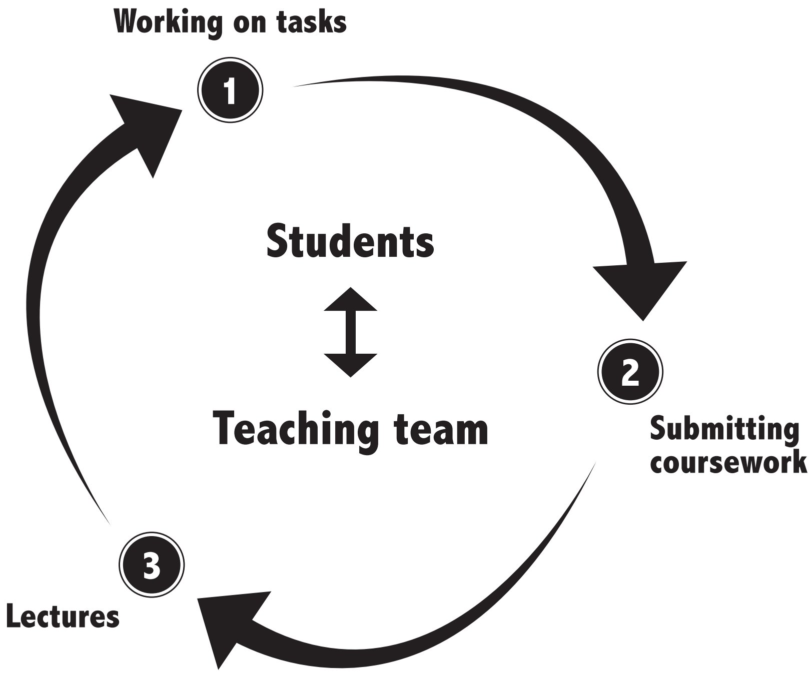

Figure 1. The Extreme Apprenticeship method is built around a learning cycle where students work on tasks, submit their coursework, and attend lectures. The students interact with each other and with the teaching team in every step of the process.

Table 1. Students' and teaching team members' activities and forms of interaction.

Step	Students . . .	Teaching team . . .	Interaction with teaching team	Peer interaction
1	work on tasks, read course material	offers guidance	inquiry-oriented discussions	students work together on tasks
2	submit coursework, resubmit based on feedback	assesses coursework	teaching team reads coursework and writes feedback	peer review exercises
3	attend lectures	responsible teacher facilitates the lecture	facilitated discussion, real-time polls	pair and share -exercises

progress, so that they can adjust their teaching accordingly. One cycle lasts for a week, and the cycles continue throughout the course.

Table 1 describes activities and forms of interaction related to each step of the cycle. Step 1 takes place in the learning space, where the students work alone or together informally on their weekly tasks. The teaching team facilitates learning by inquiring into the students' thinking. Step 2 describes the process of submitting and getting feedback on written coursework. The teaching team obtains valuable information about the students' skills from the written coursework, and students are allowed to correct their work based on the feedback. Occasionally, comments are given by other students (peer review) instead of the teaching team. In Step 3, students come to the lecture, where the responsible teacher facilitates discussion. Discussion is often based on pair-and-share, where students first turn to their neighbor for ideas, and then are asked to relate their findings to the whole group.

The following part of this section moves on to describe the learning cycle in greater detail.

4.2. Teaching Team

In an XA course, the teaching team consists of a responsible teacher and a number of tutors. Tutors are typically undergraduate and graduate mathematics or mathematics education students, and any student who has successfully completed a course is eligible to serve as a tutor. They guide students in the learning space and give written feedback to the students. The responsible teacher is usually a senior member of the teaching staff, and their role is to give the lectures and steer the teaching team. In addition, the teaching team may include developers of XA or lecturers who have taught with the method before, especially if the responsible teacher has no previous experience with XA.

The tutors are chosen based on their academic performance and an interview that examines their motivation to teach and capability to interact with students. There are two kinds of tutors: senior tutors and junior tutors. Senior tutors have worked as a tutor before, and they offer support for the less experienced junior tutors. In addition, senior tutors design the tasks together with the responsible teacher. The tutors are paid for their work. As senior tutors have more responsibilities, they also have a higher salary.

In XA, it is important that the members of a teaching team share all information on the progress of the students, as well as their experiences on the tutor practices. For these reasons, the teaching team meets every week. Half of the meeting is spent reflecting on general pedagogical topics, such as how to interact with students. During the other half of the meeting, the team discusses the current week's tasks and possible challenges students might face when solving them.

Working as a tutor can be demanding, especially for the undergraduate tutors. That is why a special effort is made to build a community of tutors that supports its members. The community is built by having the tutors discuss with each other in small groups in weekly meetings, and having them organize an informal meeting for themselves outside the campus at the beginning of term.

In fact, the teaching team forms another community of practice that is centered around teaching. In this community, the tutors become learners who learn student-centered pedagogy and teamwork skills from the responsible teacher as well as more senior tutors. Hence, the tutors have a dual role: they function as "masters" when guiding students, and as "apprentices" when discussing course pedagogy with the responsible teacher.

4.3. Tasks

Each week, students are given a set of problems designed by the teaching team. The students are encouraged to work together with their peers in the learning space where the teaching team offers guidance. Some of the problems are introductory

First task concerning linear independence:

Start reading chapter 5 in the course material. It concerns linear independence. Denote $w_1 = (1,1,1)$, $w_2 = (1,2,3)$ and $w_3 = (1,-1,2)$. We wish to determine whether w_1, w_2 and w_3 are linearly independent.

1. What kind of an equation should we consider? What do we want to show concerning the solutions?
2. What kind of a system of linear equations does the equation give?
3. Are the vectors linearly independent? (Justify your answer carefully. You can use a computer in solving the system of linear equations if you wish.)

Second task concerning linear independence:

Determine whether the vectors $w_1 = (3,3,13)$, $w_2 = (0,0,-2)$ and $w_3 = (1,1,3)$ are linearly independent.

A further task concerning linear independence:

Assume that v and w are vectors of $\mathbb{R}^n$. Are vectors v, w and 0 linearly independent?

Figure 2. A sequence of tasks that introduces linear independence to students.

tasks that are approachable and introduce new topics or ideas. They scaffold the students' work. Others are more challenging tasks that are related to topics that the students are already familiar with. The introductory tasks create a flipped classroom component to the course, as they concern topics that have not been discussed in the lectures yet. Since the topics are new to the students, they need to read the course material in order to solve the tasks. This steers students towards practicing their reading skills.

The first task in Figure 2 is an example of an introductory task in a linear algebra course. The second and third tasks would be introduced later when the students start to master the topic. The second task is used to verify that core content of the course (in this case checking linear independence of given vectors) is reached. This kind of task would be asked to be written up and submitted for feedback. The third task is more conceptual, and it pushes the students' thinking further.

Every week, students hand in solutions to the problems, and the teaching team selects one or two tasks for inspection and feedback. The chosen tasks cover an important topic or teach an essential skill. The students are encouraged to improve their solutions when necessary: they receive points for their solution only when the solution is good enough. On the other hand, by reading the students' answers, the teaching team receives information on students' performance and can take that into account when planning the next tasks and lectures. The feedback given to the

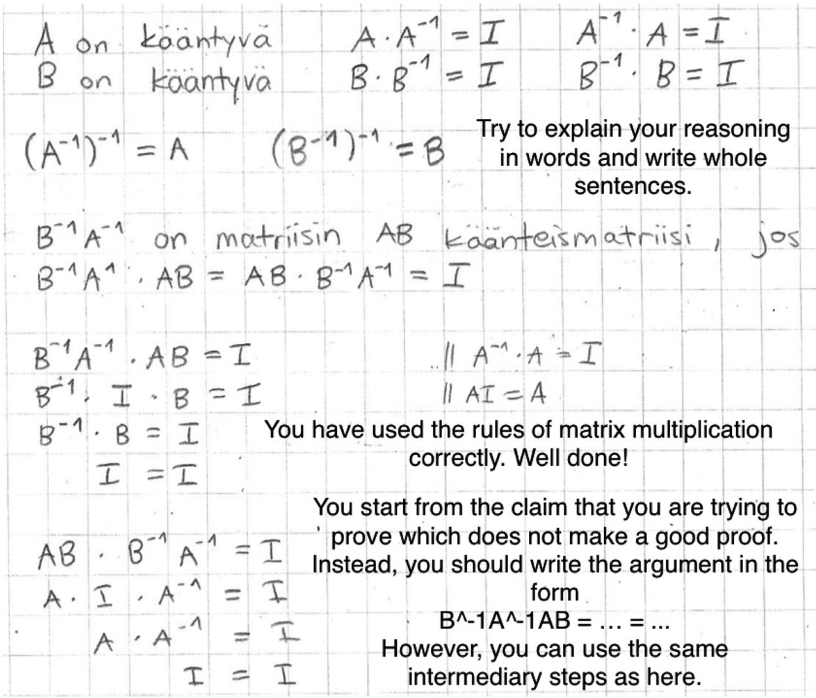

Figure 3. Teaching team's comments to a student's solution illustrate what is required from the students regarding mathematical writing style. (The solution is written in Finnish and the comments are translated into English.) The task was "Assume that A and B are invertible matrices. Show that also AB is invertible and its inverse is $B^{-1}A^{-1}$." First the student states in lines 1–2 what it means for A and B to be invertible. Then the student explains in lines 4–5 what it means that $B^{-1}A^{-1}$ is the inverse of AB. Finally, starting from line 6, the student presents her proof for $B^{-1}A^{-1}$ being the inverse of AB.

students concerns both reasoning and the language and readability of the solution. It is not enough to have the calculations correct. The students also need to communicate their thinking well and strive for good mathematical style. This illustrates one aspect of what the word "extreme" in the name of the method means: good writing practices are pushed to the extreme.

To show how much it is possible to demand from the students, we present a typical student solution alongside the teaching team's feedback (Figure 3). The solution could be taken as logically valid, but would not pass the inspection. The teaching team coaches the student in the writing process by giving feedback that leads the student towards good mathematical style.

4.4. Guidance

The teaching team guides students in drop-in sessions in an open learning space where students can come to get help with their mathematical problems. The members of the teaching team model to the students how mathematicians work. They coach the students and scaffold their work by leading them subtly towards the discovery of a solution through a process of questioning and listening.

In an iterative process over several years, the teaching team has put together 11 guidelines for instruction:

1. *Listen.* Encourage the student to talk, and listen to what they say. Let the student's needs lead your instruction.
2. *Guide individually.* Students are different. Some may need help with the basics and require very concrete advice. Others are just asking for a small hint. Try to find out what the student needs and address that need.
3. *Let the student do and discover.* The aim is that the student works towards a solution with the support of the teaching team. Guide in such a way that the student can experience aha moments.
4. *Be encouraging.* Students may be very insecure in what comes to mathematics and feel that they are not doing well enough. Be encouraging and try to find something good in the student's work.
5. *Be active.* Circulate among students on your own initiative and say hi to them. It is easier for the students to ask questions if the teacher has opened the conversation.
6. *Divide your attention.* Do not let one student take too much of your time. Note also that sometimes it is good to let the student think about the problem on their own.
7. *Help the student in reading the course material.* Reading mathematics is difficult for the students, and they may try to use the teacher as a data bank. The teacher should encourage the students to read the course material and advise them how to do this.
8. *The teacher does not need to know everything.* They can investigate the topic together with the student. This way, the student sees how a more experienced mathematics student works.
9. *Teach study skills.* The aim of guidance is not only to help the student to solve a task, but to show how mathematicians tackle problems they face.
10. *Do not take emotional outbursts personally.* React to students' problems and outbursts with compassion and empathy, but do not let them worry you too much. The reason behind outbursts can be, for example, the insecurity of the student.
11. *Encourage cooperation.* Students should learn to discuss mathematics. Encourage students to collaborate, especially if many of them are working alone on the same problem.

To ensure that the students get support according to their needs, they can spend as much time as they want in the learning space, and come and go when they wish. In total, there are 10–20 hours of guidance available per week. It is worth noting that this guidance is not a "help center," providing additional support to the students, but actually forms the main part of teaching in the XA method. Members of the teaching team wear colorful vests, so that the students know who the teachers are and can easily approach them. The vests also help the tutors, who are students themselves, to change their role from student to teacher. Apart from the drop-in sessions and lectures, there is no other form of contact teaching, so the drop-in sessions are the only place where students are in direct contact with the tutors.

The learning space is designed to encourage the students to collaborate with each other and interact with the teaching team. There are blackboards on the walls, and the tables are arranged into groups with a white surface on which one can draw with a whiteboard marker. This helps the students to work together and share their thoughts with each other and with the members of the teaching team. The learning space is open and accessible. It is located close to areas students use frequently such as classrooms and the students' social spaces. This means that it is easy for the students to come to the learning space and start working on their mathematics whenever they feel like it.

4.5. Lectures

As the students have worked on weekly problems and read the course material prior to the lectures, it is not necessary to use the lecture time to deliver content. Instead, the lectures are used as a common discussion arena, where students' ideas and misconceptions can be shared and reflected upon. They also offer the teacher an opportunity to model expert's thinking. The focus of the conversation may be on how the topics of the course are linked, what the meanings and consequences of definitions and theorems are, how mathematicians approach and discuss problems, or how they read and write formal mathematics.

In analogy with classical apprenticeship, lectures resemble a joint meeting with the whole workshop, in which the master discusses examples and makes the stages of the crafting process visible to the apprentices.

To promote discussion, various pair and small group activities are used, enhanced by online polls. This way the students get to practice cooperation and communication with one another. The lecture activities are based not only on the course syllabus but also on the information the teaching team has acquired when guiding students in the learning space and reading and commenting on their coursework. The lectures are usually not fully structured in advance, so that the common discussions can be let to direct the course of the lecture to some extent.

5. REFLECTIONS AND ADVICE FOR PRACTITIONERS

In this section, we go back to our experiences of using the XA method and offer some advice for other practitioners. First, we draw into focus what we see as essential features that should be present in any implementation of the method. Then, we make remarks on possible challenges and provide some lessons we have learned during the process.

5.1. Essential Features of the Method

Central to the XA method is the idea of participation in a community of practice of learners in a meaningful and supported way. The teachers and tutors model mathematical thinking and practices to the students, and there is coaching and scaffolding available from the teaching team at the time students need it. However, there is a lot of flexibility in how all this is implemented. For example, when the XA method was introduced to our department, it was modified to suit mathematics and the teaching culture in the department and was heavily influenced by existing good teaching practices.

There are some features of XA that we consider crucial in implementing the method successfully. One of them is the *community of tutors*. When the tutors are almost peers to the students, they are easy to approach, and they know very well what kind of problems the students are facing. In this way, the tutor role fills in a gap in the chain connecting first-year students with the faculty. The tutors need support in their work, offered by the responsible teacher and more experienced tutors. Regular meetings and pedagogical discussions enhance the professional development of the tutors, but they also create a teamwork atmosphere, where all parties trust and learn from each other.

The tutors also enable the effective *bidirectional feedback mechanism* in large courses. While interacting with the students or with their work, they bring their observations to the teaching team as well as imparting the team's expertise to the students. This way, the cognitive processes of both the teacher and the students are made visible, which is a key principle of Cognitive Apprenticeship.

By the principles of situated learning, students will gain expertise not only in the intended course content but also in whatever strategies are necessary for them to succeed. Therefore, in the XA method, these tendencies should be guided towards *meaningful learning products*. For example, in our implementation we wanted students to interact with the topics prior to the lectures. Most of them would not do this by merely recommending it to them, so we have set actual course tasks to be completed before lectures. This way it becomes a new norm to study before the lecture. Moreover, there are many ways to provide students with the necessary information, one example being to give them videos prepared by the lecturer. However, this way the students would soon become proficient in learning from videos, and while this could also be valuable, we specifically wanted them to learn to read mathematical texts instead. For this reason, we give them the information in the form of written course material.

However, the students cannot be expected to have the necessary study skills from the start. This brings us to the last essential feature in the XA method, which is *support in adopting study skills*. Using the previous example, we have carefully chosen the written course material to be easily approachable, and we have designed the first tasks so that they scaffold the adoption of new concepts. Furthermore, the tutors are advised to discuss reading strategies with the students while guiding them, and the teacher regularly addresses the same questions in the lectures, as well.

In this article, we are focusing on implementing XA in large basic level university courses. The method has also been applied to small advanced courses and in K–12 studies. The main difference in these contexts is that there are fewer tutors available (or none), and therefore the community of pedagogically oriented teaching team is missing. Then the role of the responsible teacher shifts more toward the role of a tutor: the lectures are replaced by group sessions, in which the teacher divides the time between guiding the students in their work and facilitating discussions among the whole group. In other aspects, the principles of XA remain unchanged.

5.2. Lessons Learned

There have been and still are many challenges in implementing the XA method. Looking back to the beginning, running a pilot course with the method required financial support from the department, so it was crucial that the atmosphere in the department was very positive towards teaching experiments. After the pilot course, modifications were made to bring down the expenses.

From the students' point of view, importing a new teaching method will unavoidably cause confusion. In the beginning of XA, student feedback on the method was mixed. On one hand, students appreciated the plethora of guidance offered to them. On the other hand, some of them stated that the teaching method was strange and they wanted to have more traditional lectures and tasks. However, over the years students started to receive the XA method better and better. This was partly due to the development of the teaching method and communicating its idea better to the students, and also largely due to a change in the learning culture. It has become a new norm to the students that some of the courses were always taught with the XA method. Now the senior students, who had already experienced XA, say positive things about the method and share their learning strategies with new students. However, motivating regularly why student-centered methods are used and what their benefits are, is still necessary.

Another way to reduce resistance among students is to introduce the novel teaching method in the first courses that the students take when they arrive to university, since at that point they have few presumptions regarding university teaching. The method can then be used in other courses as well, as it is already familiar as one of many teaching methods that are in use in the department.

A difficult issue in departments with a strong culture of voluntary participation is that the students in XA courses may not come to the lectures or the learning space at all. In our department, students have traditionally flocked to the first lectures of any course to see if they can benefit from them and if they do not find them valuable, never appear again. However, with the XA method the purpose of lectures is not direct information transfer, but rather to work as a common discussion arena, and it may well be that students will not appreciate this from only visiting a couple of lectures in the beginning of the course. A similar problem lies with the guidance offered in the learning space. Students who think they can complete the weekly tasks on their own may decide never to come to work in the learning space. Even though they might still benefit from the discussions with other students and tutors, they may not understand this in advance. In the future, it would be important to find new ways to engage different kinds of students and help them understand the value of face-to-face learning.

Regarding members of staff, a recurring challenge is to find the responsible teachers for XA courses, as many faculty members are new to student-centered teaching methods. In addition, many structures built over the years to support traditional teaching do not necessarily suit a new teaching method. We have noticed that a faculty member who takes an XA course needs plenty of support especially in the beginning. In our case, the support comes from people who are experienced with the XA method. They can, for example, help new teachers in designing tasks and in running the weekly meetings of the teaching team. Also the experienced senior tutors in the teaching team can offer a lot of support to the new responsible teacher. However, all this takes resources, so there has to be a critical mass of XA proficient teachers and tutors to be able to introduce the method to new courses and new staff members.

6. CONCLUSION

Traditional teaching in large mathematics courses in a university typically consists of mass lectures and weekly problem classes. In this kind of setting, it is difficult to develop and maintain student-centered pedagogies. Individual student's needs are easily unnoticed in the large population, as lecturers gear their teaching toward the average student or sometimes the "ideal" one. Feedback mechanisms are typically slow and inefficient: students get feedback for their work 1 or 2 weeks too late for them, and teacher gets feedback for the teaching after the end of the course. Even the kindest and most friendly lecturer remains distant to the individual student, who is afraid to approach them when surrounded by hundreds of peers.

In this article, we have presented the theoretical orientation and practical implementation of a student-centered teaching method for university mathematics that partially solves the challenge of large course context. The method, called Extreme Apprenticeship, is based on principles of situated learning and Cognitive Apprenticeship. As an inquiry-oriented teaching method, with a drop-in workspace roamed by pedagogically instructed peer tutors, the method can provide timely and scaffolded support for individual students. Through the functionally structured teaching team, it enables continuous bidirectional feedback between students and the responsible teacher.

Nine years of experience with the Extreme Apprenticeship method have taught us a great deal about the method and about student thinking and student-centered teaching methods in general. We feel that the XA method has not only benefited both students and teachers in our department but also helped to improve the general atmosphere around teaching and learning of mathematics. It has shown that student-centered pedagogies can be implemented in large course settings, and that it is possible to bridge the gap between lecturer and student with carefully appointed peer tutors. We hope that this article will inspire other mathematics educators to try out the XA method either as it is, or to modify and apply it to their own ends.

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