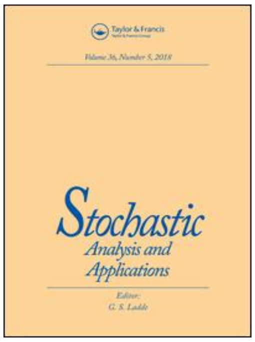

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Time-dependent weak rate of convergence for functions of generalized bounded variation

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ABSTRACT

Let W denote the Brownian motion. For any exponentially bounded Borel function g the function u defined by $u(t,x) = \mathbb{E}[g(x + \sigma W_{T-t})]$ is the stochastic solution of the backward heat equation with terminal condition g. Let $u^n(t,x)$ denote the corresponding approximation generated by a simple symmetric random walk with time steps $2T/n$ and space steps $\pm\sigma\sqrt{T/n}$ where $\sigma > 0$. For a class of terminal functions g having bounded variation on compact intervals, the rate of convergence of $u^n(t,x)$ to $u(t,\ x)$ is considered, and also the behavior of the error $u^n(t,x) - u(t,x)$ as t tends to T.

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1. Introduction

The objective of this article is to study the rate of convergence of a finite-difference approximation scheme for the backward heat equation. The error analysis is carried out for a large class of exponentially bounded terminal condition functions having bounded variation on compact intervals.

During the past decades, convergence rates of finite-difference schemes for parabolic boundary value problems have been studied with varying assumptions on the regularity of the initial/terminal condition, the domain of the solution, properties of the possible boundary data, etc. (see, e.g., [1–5]). In order to study the convergence, several techniques have been applied. Our approach is probabilistic: The solution of the PDE is represented in terms of Brownian motion, and the approximation scheme is realized using an appropriately scaled sequence of simple symmetric random walks in the same probability space, in the spirit of Donsker's theorem. The possible discontinuities of the terminal function produce error bounds which are not uniform over the time-nets under consideration, and hence the time dependence of the error is of particular interest here.

To explain our setting in more detail, fix a finite time horizon $T > 0$, a constant $\sigma > 0$, and consider the backward heat equation

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$$\frac{\partial}{\partial t}u+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}u=0,\quad (t,x)\in\big[0,T)\times\mathbb{R},\quad u(T,x)=\ g(x),\quad x\in\mathbb{R}. \tag{1}$$

The terminal condition $g:\mathbb{R}\to\mathbb{R}$ is assumed to belong to the class $GBV_{\exp}$ consisting of exponentially bounded functions that have bounded variation on compact intervals (the precise description is given in Definition 2.3). The stochastic solution to the problem (1) is given by

$$u(t,x):=\mathbb{E}\big[g(\sigma W_T)|\sigma W_t=x\big]=\mathbb{E}\big[g(x+\sigma W_{T-t})\big],\quad (t,x)\in[0,T]\times\mathbb{R}, \tag{2}$$

where $(W_t)_{t\geq 0}$ denotes the standard Brownian motion. To approximate the solution (2), we proceed as follows. Given an even integer $n\in 2\mathbb{N}$, a level $z_0\in\mathbb{R}$, and time and space step sizes $\delta>0$ and $h>0$, respectively, define

$$\mathcal{T}^n:=\left\{t_k^n:=2k\delta\ \Big|\ 0\leq k\leq\frac{n}{2},k\in\mathbb{Z}\right\},\quad \mathcal{S}_{z_0}^n:=\{z_0+2mh\ |\ m\in\mathbb{Z}\}. \tag{3}$$

The finite-difference scheme we will consider is given by the following system of equations defined on grids $\mathcal{G}_{z_0}^n:=\mathcal{T}^n\times\mathcal{S}_{z_0}^n\subset[0,T]\times\mathbb{R}$,

$$\begin{cases}\dfrac{v^n(t_k^n,x)-v^n(t_{k-1}^n,x)}{t_k^n-t_{k-1}^n}+\dfrac{\sigma^2}{2}\dfrac{v^n(t_k^n,x+2h)-2v^n(t_k^n,x)+v^n(t_k^n,x-2h)}{(2h)^2}=0,\\ v^n(T,\ \cdot\ )=g.\end{cases} \tag{4}$$

Letting $\delta:=\frac{T}{n}$ and $h:=\sigma\sqrt{\frac{T}{n}}$, the system (4) can be rewritten in an equivalent form as

$$\begin{cases}v^n(t_{k-1}^n,x)=\dfrac{1}{4}\big[v^n(t_k^n,x+2h)+2v^n(t_k^n,x)+v^n(t_k^n,x-2h)\big],\\ v^n(T,\ \cdot\ )=g.\end{cases} \tag{5}$$

This scheme is explicit: Given the set of terminal values $\{g(x)\ |\ x\in\mathcal{S}_{z_0}^n\}$, the solution v^n of (5) is uniquely determined by a backward recursion. We extend the function v^n in continuous time by letting

$$v^n(t,x):=v^n(t_k^n,x)\qquad\text{for}\qquad t\in[t_k^n,t_{k+1}^n),\ 0\leq k<\frac{n}{2}, \tag{6}$$

and study the error of approximation $\varepsilon_n(t,x)$ on $(t,x)\in[0,T)\times\mathcal{S}_{z_0}^n$, where

$$\varepsilon_n(t,x):=v^n(t,x)-u(t,x). \tag{7}$$

Theorem 2.5, the main result of this article, states that for a constant $C>0$ depending only on g,

$$|\varepsilon_n(t,x)|\leq\frac{C\psi(x)}{\sqrt{n(T-t)}}\ \mathbb{1}_{\{t\neq t_k^n\}}+\frac{C\psi(x)}{\sqrt{n(T-t_k^n)}},\quad (t,x)\in[t_k^n,t_{k+1}^n)\times\mathcal{S}_{z_0}^n,\ 0\leq k<\tfrac{n}{2}, \tag{8}$$

where the function $\psi(x)=\psi(|x|,g,\sigma,T)>0$ is given explicitly in Section 2.

In the 1950s, Juncosa and Young [2] considered a finite difference approximation of the (forward) heat equation on a semi-infinite strip $[0,\infty)\times[0,1]$, where the initial condition was assumed to have bounded variation. Using Fourier methods, they proved

[2, Theorem 7.1] that the error is $O(n^{-\frac{1}{2}})$ uniformly on $[t, \infty) \times [0, 1]$ for any fixed $t > 0$. However, they did not study the order of the blow-up of the error as $t \downarrow 0$ (which translates to $t \uparrow T$ in the case of the backward equation (1)). Indeed, the bound (8) suggests that the convergence is not uniform in (t, x). Nevertheless, one obtains the rate $n^{-\frac{1}{2}}$ on any compact subset of $[0, T) \times \mathbb{R}$, and this rate is also sharp for the class $GBV_{\exp}$. The blow-up in (8) vanishes if g has more regularity: For α-Hölder continuous g with $\alpha \in (0, 1]$ and $\sigma = 1$, it was shown in [6, Corollary 4] that $|\varepsilon_n(t, x)| \le Cn^{-\frac{\alpha}{2}}$ holds uniformly in (t, x), where $C = C(T) > 0$ is a constant.

The main result is derived using the following probabilistic approach. Let $(\xi_i)_{i=1,2,\dots}$ be a sequence of i.i.d. Rademacher random variables, and define

$$u^n(t, x) := \mathbb{E}\left[g(x + \sigma W^n_{T-t})\right], \quad (t, x) \in [0, T] \times \mathbb{R}, \tag{9}$$

where $(W^n_t)_{t\in[0,T]}$ is the random walk given by

$$W^n_t := \sqrt{\frac{T}{n}} \sum_{i=1}^{2\left\lceil \frac{t}{2T/n} \right\rceil} \xi_i, \quad t \in [0, T] \tag{10}$$

($\lceil\cdot\rceil$ denotes the ceiling function). The key observation is that the function u^n, when restricted to $\mathcal{G}^n_{z_0}$, is the unique solution of (5) for every $z_0 \in \mathbb{R}$. Relation (6) also holds for u^n by definition. Moreover, since the random walk $(W^n_t)_{t\in[0,T]}$ affects the value of u^n only through its distribution, we may consider a special setting where the Rademacher variables $\xi_1, \xi_2, \dots$ are chosen in a suitable way. Defining these variables as the values of the Brownian motion $(W_t)_{t\ge0}$ sampled at certain first stopping times (see Section 2.1) enables us to apply techniques from stochastic analysis for the estimation of the error (7) where $v^n = u^n$.

The above procedure was applied in Walsh [7] (cf. Rogers and Stapleton [8]) in relation to a problem arising in mathematical finance. More precisely, the weak rate of convergence of European option prices given by the binomial tree scheme (Cox–Ross–Rubinstein model) to prices implied by the Black–Scholes model is analyzed in [7] (cf. Heston and Zhou [9]). A detailed error expansion is presented in [7, Theorem 4.3] for terminal conditions belonging to a certain class of piecewise C^2 functions.

Using similar ideas, we complement this result by considering a large class of functions containing the class considered in [7]. Moreover, instead of studying the error at time $t = 0$ only, we derive a time-dependent error bound. Finally, a gap is closed in the proof done in [7]. It concerns the estimate [7, Proposition 11.2(iv)] for which a detailed proof is given in Section 5.2.

It is argued in [7, Sections 7 and 12] that the rate remains unaffected if the geometric Brownian motion is replaced with a Brownian motion, and the binomial tree is replaced with a random walk. It seems plausible that also our time-dependent results in the Brownian setting can be transferred into the geometric setting with essentially the same upper bounds.

The article is organized as follows. In Section 2, we introduce the notation, recall the construction of a simple random walk using first hitting times of the Brownian motion, and formulate the main result Theorem 2.5. Using this sequence of first hitting times, the error (7) will be split into three parts. Estimates for the *adjustment error* and the *local error* are derived in Section 3, and the *global error* is treated in Section 4. Section 5 contains the result for the sharpness of the rate and the key moment estimates applied in Section 4. The remaining auxiliary results and estimates can be found in the appendix, where also the construction of the terminal function class and its properties are briefly discussed.

2. The setting and the main result

2.1. Notation related to the random walk

Consider a standard Brownian motion $(W_t)_{t\geq 0}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$, where $(\mathcal{F}_t)_{t\geq 0}$ stands for the natural filtration of $(W_t)_{t\geq 0}$. Let $(X_t)_{t\geq 0} := (\sigma W_t)_{t\geq 0}$, where $\sigma > 0$ is a given constant. By $\tau_{(-h,h)}$ we denote the first exit time of the process $(X_t)_{t\geq 0}$ from the open interval $(-h, h)$,

$$\tau_{(-h,h)} := \inf\{t \geq 0 : |X_t| = h\} = \inf\{t \geq 0 : |W_t| = h/\sigma\}, \quad h > 0.$$

In order to represent the error (7), we construct a random walk on the space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. Following [7], we define

$$\tau_0 := 0 \quad \text{and} \quad \tau_k = \tau_k(h) := \inf\{t \geq \tau_{k-1} : |X_t - X_{\tau_{k-1}}| = h\} \tag{11}$$

recursively for $k = 1, 2, \ldots$. Then τ_k is a $\mathbb{P}$-a.s. finite $(\mathcal{F}_t)_{t\geq 0}$-stopping time for all $k \geq 0$, and the process $(X_{\tau_k})_{k=0,1,\ldots}$ is a symmetric simple random walk on $\mathbb{Z}^h := \{mh : m \in \mathbb{Z}\}$. For every integer $k \geq 1$, we also let

$$\Delta\tau_k := \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta X_{\tau_k} := X_{\tau_k} - X_{\tau_{k-1}}.$$

The strong Markov property of $(X_t)_{t\geq 0}$ implies that $(\Delta\tau_k, \Delta X_{\tau_k})_{k=1,2,\ldots}$ is an i.i.d. process such that for each $k \geq 1$, we have $\mathbb{P}(\Delta X_{\tau_k} = \pm h) = 1/2$,

$$(\Delta\tau_k, \Delta X_{\tau_k}) \stackrel{d}{=} (\tau_{(-h,h)}, X_{\tau_{(-h,h)}}), \quad \text{and} \quad (\Delta\tau_k, \Delta X_{\tau_k}) \text{ is independent of } \mathcal{F}_{\tau_{k-1}+}.$$

Moreover, as shown in [8, Proposition 1], the increments ΔX_{τ_1} and $\Delta\tau_1$ are independent. Consequently, the *processes* $(\Delta\tau_k)_{k=1,2,\ldots}$ and $(\Delta X_{\tau_k})_{k=1,2,\ldots}$ are independent (see also [7, Proposition 11.1] and [10, Proposition 2.4]).

We deduce, in particular, that for all $N \geq 1$ the random variable X_{τ_N} is distributed as $h\sum_{k=1}^{N} \xi_k$, where $(\xi_k)_{k=1,2,\ldots}$ is an i.i.d. sequence of Rademacher random variables. Therefore, for W^n_{T-t} defined in (10), we have the equality in law

$$X_{\tau_N} \stackrel{d}{=} \sigma W^n_{T-t} \quad \text{provided that} \quad (h, N) = \left(\sigma\sqrt{\frac{T}{n}}, 2\left\lceil \frac{T-t}{2T/n} \right\rceil\right).$$

Note that in this case the sequence of stopping times $(\tau_k)_{k=0,1,\ldots}$ (11) depends on n via $h = h(n)$.

The error (7) will be split into three parts, where each of these parts will take into account different properties of the given function g. For this purpose, let us introduce some more notation. Let θ_n denote the smallest multiple of $2T/n$ greater than or equal to $T - t$. That means, for given $n \in 2\mathbb{N}$ and $t \in [0, T)$,

$$\theta_n := \frac{n_\theta T}{n}, \quad \text{where} \quad n_\theta := 2\left\lceil \frac{T-t}{2T/n} \right\rceil \in \{2, 4, ..., n\}. \tag{12}$$

It is clear that

$$0 \le \theta_n - (T-t) \le \frac{2T}{n} \quad \text{and} \quad \theta_n \downarrow T-t \quad \text{as} \quad n \to \infty.$$

Note also that the connection between lattice points $t_k^n = 2kT/n \in \mathcal{T}^n$ introduced in (3) and the time instant $\theta_n \in (0, T]$ is explained as follows:

$$t \in [t_k^n, t_{k+1}^n) \quad \text{if and only if} \quad \theta_n = T - t_k^n, \quad 0 \le k \le \tfrac{n}{2} - 1. \tag{13}$$

2.2. The class of terminal functions

The approximation error will be estimated for functions g belonging to the class $GBV_{\exp}$ introduced below. This class is contained in the class of exponentially bounded Borel functions.

Definition 2.1 (The class $\mathcal{B}_{\exp}$). A function $g : \mathbb{R} \to \mathbb{R}$ is said to be exponentially bounded if there exist constants $A, b \ge 0$ such that

$$|g(x)| \le Ae^{b|x|} \quad \text{for all } x \in \mathbb{R}. \tag{14}$$

The class of all Borel functions with the above property will be denoted by $\mathcal{B}_{\exp}$.

The function class $GBV_{\exp}$ generalizes functions of bounded variation (which are bounded) by allowing exponential growth. While these functions have bounded variation on each compact interval, their total variation may be unbounded (or undefined) over unbounded intervals. See [11] and Appendix A.1 for more information on this topic.

Definition 2.2 ([11, Definition 3.2]). Denote by $\mathcal{M}$ the class of all set functions

$$\mu : \{G \in \mathcal{B}(\mathbb{R}) : G \text{ is bounded}\} \to \mathbb{R}$$

that can be written as a difference of two measures $\mu^1, \mu^2 : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that $\mu^1(K) < \infty$ nad $\mu^2(K) < \infty$ for all compact sets $K \in \mathcal{B}(\mathbb{R})$.

Below it is understood that $[a, b) = \emptyset$ whenever $a \ge b$.

Definition 2.3 (The class $GBV_{\exp}$). Denote by $GBV_{\exp}$ the class of functions $g : \mathbb{R} \to \mathbb{R}$ which can be represented as

$$g(x) = c + \mu([0, x)) - \mu([x, 0)) + \sum_{i=1}^{\infty} \alpha_i \, \mathbb{1}_{\{x_i\}}(x), \quad x \in \mathbb{R}, \tag{15}$$

where $c \in \mathbb{R}$ is a constant, $\mu \in \mathcal{M}$, and $\mathcal{J} = (\alpha_i, x_i)_{i=1,2,...} \subset \mathbb{R}^2$ is a countable set such that $x_i \ne x_j$ whenever $i \ne j$. In addition, we require that for some constant $\beta \ge 0$,

$$\int_{\mathbb{R}} e^{-\beta|x|} d|\mu|(x) + \sum_{i=1}^{\infty} |\alpha_i| e^{-\beta|x_i|} < \infty. \tag{16}$$

To give some examples of classes of functions contained in $GBV_{\exp}$, we have the remark below. See Appendix A.1 for the proof.

Remark 2.4 (Examples of functions belonging to the class $GBV_{\exp}$).

(i) Every polynomial belongs to the class $GBV_{\exp}$.
(ii) Each increasing (resp. decreasing) function $g \in \mathcal{B}_{\exp}$ belongs to $GBV_{\exp}$.
(iii) Each convex (resp. concave) function $g \in \mathcal{B}_{\exp}$ belongs to $GBV_{\exp}$.
(iv) $\mathcal{K}_{\exp} \subset GBV_{\exp}$, where $\mathcal{K}_{\exp}$ is the class of functions $g : \mathbb{R} \to \mathbb{R}$ considered in Walsh [7] (pp. 340, 345–346, and 348), i.e. they satisfy the below criteria:
(v) g, g', and g'' belong to $\mathcal{B}_{\exp}$
(vi) g, g', and g'' have at most finitely many discontinuities and no oscillatory discontinuities
(vii) $g(x) = \frac{1}{2}(g(x+) + g(x-))$ at each point $x \in \mathbb{R}$.

2.3. The main result

The main result of this article, Theorem 2.5, describes the approximation error between the solution of the backward heat equation (1) and its finite-difference approximation (9) for terminal functions belonging to the class $GBV_{\exp}$.

Theorem 2.5. *Let $n \in 2\mathbb{N}$, and let u and u^n be the functions introduced in (2) and (9). Suppose that $g \in GBV_{\exp}$ is a function given by (15) and that $\beta \geq 0$ is as in (16). Then, for all $(t, x) \in [0, T) \times \mathbb{R}$,*

$$(i) \quad |u^n(t,x) - u(t,x)| \leq \frac{C_{\beta,\sigma,T}}{\sqrt{n(T-t)}} e^{\beta|x|}, \qquad t \neq t_k^n, \quad 0 \leq k < \frac{n}{2},$$

$$(ii) \quad \left|u^n(t_k^n,x) - u(t_k^n,x)\right| \leq \frac{C_{\beta,\sigma,T}}{\sqrt{n(T-t_k^n)}} e^{\beta|x|}, \quad 0 \leq k < \frac{n}{2},$$

where $C_{\beta,\sigma,T} := C\sqrt{T} e^{3\beta^2\sigma^2 T}$ and $C > 0$ is a constant depending only on g.

Remark 2.6. The error bounds in Theorem 2.5 grow exponentially as functions of the variable x. If the terminal condition g satisfies (16) with $\beta = 0$ one obtains bounds which are uniform in x.

Proof of Theorem 2.5. The function $u^n(t,x)$ is constant in t on intervals of length $2T/n$, while $t \mapsto u(t,x)$ is continuous. Therefore, for $t \in [t_k^n, t_{k+1}^n)$, we will split the error and write

$$u^n(t,x) - u(t,x) = (u^n(t_k^n,x) - u(t_k^n,x)) + (u(t_k^n,x) - u(t,x)),$$

where

$$\varepsilon_n^{\mathrm{adj}}(t,x) := u(t_k^n,x) - u(t,x) = \mathbb{E}\left[g(x + X_{\theta_n}) - g(x + X_{T-t})\right] \tag{17}$$

will be called the *adjustment error*. Next, we exploit the construction of the random walk $(X_{\tau_k})_{k\geq 0}$ by Skorokhod embedding from the process $(X_t)_{t\geq 0}$ and let

$$J_n(\omega) := \inf\{2m \in 2\mathbb{N} : \tau_{2m}(\omega) > \theta_n\}, \quad \omega \in \Omega, \tag{18}$$

which is the index of the first stopping time $\tau_0, \tau_2, \tau_4, \ldots$ exceeding the value θ_n. Consequently, by construction, $X_{\tau_{J_n}}$ will be 'rather close' to X_{θ_n} for large n. Therefore, we will write

$$u^n(t_k^n, x) - u(t_k^n, x) = \mathbb{E}\big[g(x + X_{\tau_{n_\theta}}) - g(x + X_{\theta_n})\big] = \varepsilon_n^{\text{glob}}(t,x) + \varepsilon_n^{\text{loc}}(t,x),$$

where the first term on the right-hand side

$$\varepsilon_n^{\text{glob}}(t,x) := \mathbb{E}\big[g(x + X_{\tau_{n_\theta}}) - g(x + X_{\tau_{J_n}})\big] \tag{19}$$

is referred to as the *global error*, and second term

$$\varepsilon_n^{\text{loc}}(t,x) := \mathbb{E}\big[g(x + X_{\tau_{J_n}}) - g(x + X_{\theta_n})\big] \tag{20}$$

denotes the *local error*. The local error is influenced by the smoothness properties of the terminal condition g, while for the global error only integrability properties of g are needed.

Since $g \in GBV_{\exp}$, there exists a constant $A = A(\beta) \geq 0$ such that $|g(x)| \leq Ae^{\beta|x|}$ for all $x \in \mathbb{R}$. Indeed, relation (14) is satisfied for a function g given by (15) by letting $b := \beta$ and A to be equal to the sum of $|c|$ and the left-hand side of (16).

Consequently, by Theorems 3.1, 3.8, and 4.3 (the bounds for the error terms $\varepsilon_n^{\text{adj}}, \varepsilon_n^{\text{loc}}$, and $\varepsilon_n^{\text{glob}}$, respectively), there exists a constant $C > 0$ such that

$$|u^n(t,x) - u(t,x)| \leq Ce^{\beta|x| + 3\beta^2\sigma^2 T}\left(\frac{\sqrt{T}}{\sqrt{n(T-t)}}\mathbb{1}_{\{t \neq t_k^n\}} + \frac{\sqrt{T}}{\sqrt{n(T-t_k^n)}} + \frac{T}{n(T-t_k^n)}\right).$$

It remains to observe that since $\sqrt{n(T-t_k^n)} \geq \sqrt{2T}$ for all integers $0 \leq k < n/2$, it holds

$$\frac{T}{n(T-t_k^n)} \leq \frac{\sqrt{T}}{\sqrt{n(T-t_k^n)}} \leq \frac{\sqrt{T}}{\sqrt{n(T-t)}}.$$

□

3. The adjustment error and the local error

3.1. The adjustment error

The purpose of this subsection is to derive an upper bound for the modulus of the adjustment error defined in (17) for a terminal function g belonging to $GBV_{\exp}$.

Theorem 3.1. *Let $n \in 2\mathbb{N}$, and suppose that $g \in GBV_{\exp}$ and $\beta \geq 0$ is as in (16). Then, for all $(t,x) \in [0,T) \times \mathbb{R}$,*

$$|\varepsilon_n^{\text{adj}}(t,x)| \leq \frac{A_\beta\sqrt{T}}{\sqrt{n(T-t)}}e^{\beta|x| + \beta^2\sigma^2 T}\ \mathbb{1}_{\{t \neq t_k^n \forall 0 < k < \frac{n}{2}\}},$$

where $A_\beta = 2\int_{\mathbb{R}} e^{-\beta|y|}d|\mu|(y)$.

Proof. Denote by p_s the density of $X_s = \sigma W_s$ for $s > 0$, and consider the function

$$u(s,x) = \mathbb{E}\big[g(x + X_{T-s})\big] = \int_{\mathbb{R}} g(x+z)p_{T-s}(z)dz, \quad 0 \le s < T.$$

Fix $n \in 2\mathbb{N}$ and suppose that $t_k^n = 2kT/n$ is the lattice point such that $t \in \big[t_k^n, t_{k+1}^n\big)$. If $t = t_k^n$, (13) implies that $\theta_n = T - t$, and thus $\varepsilon_n^{\text{adj}}(t,x) = 0$ by (17). Suppose then $t \in \big(t_k^n, t_{k+1}^n\big)$ and use the representation (29) for the function $z \mapsto g(x+z)$ in order to rewrite

$$\begin{aligned} u(t_k^n, x) - u(t,x) &= \int_{\mathbb{R}} \Big[g\Big(x + z\sqrt{T - t_k^n}\Big) - g\Big(x + z\sqrt{T-t}\Big)\Big] p_1(z)dz \\ &= \int_{\mathbb{R}}\int_{[0,\infty)} \Big[\mathbb{1}_{(y-x,\infty)}\Big(z\sqrt{T-t_k^n}\Big) - \mathbb{1}_{(y-x,\infty)}\Big(z\sqrt{T-t}\Big)\Big] d\mu(y)p_1(z)dz \\ &\quad - \int_{\mathbb{R}}\int_{(-\infty,0)} \Big[\mathbb{1}_{(-\infty,y-x]}\Big(z\sqrt{T-t_k^n}\Big) - \mathbb{1}_{(-\infty,y-x]}\Big(z\sqrt{T-t}\Big)\Big] d\mu(y)p_1(z)dz \\ &=: I_1 - I_2. \end{aligned} \tag{21}$$

Since g is exponentially bounded, one may apply Fubini's theorem to rewrite

$$I_1 = \int_{\mathbb{R}}\int_{[0,\infty)} \Big[\mathbb{1}_{\left(\frac{y-x}{\sqrt{T-t_k^n}},\infty\right)}(z) - \mathbb{1}_{\left(\frac{y-x}{\sqrt{T-t}},\infty\right)}(z)\Big] d\mu(y)p_1(z)dz = \int_{[0,\infty)}\int_{\frac{y-x}{\sqrt{T-t_k^n}}}^{\frac{y-x}{\sqrt{T-t}}} p_1(z)dzd\mu(y). \tag{22}$$

The mean value theorem and the fact $\sqrt{T-t} < \sqrt{T-t_k^n}$ imply for arbitrary $y \in \mathbb{R}$ that

$$\begin{aligned} e^{\beta|y|}\left|\int_{\frac{y-x}{\sqrt{T-t_k^n}}}^{\frac{y-x}{\sqrt{T-t}}} p_1(z)dz\right| &\le e^{\beta|x| + \beta|y-x|} p_1\left(\frac{|y-x|}{\sqrt{T-t_k^n}}\right)\frac{|y-x|}{\sqrt{T-t_k^n}}\,\frac{\sqrt{T-t_k^n} - \sqrt{T-t}}{\sqrt{T-t}} \\ &\le \frac{e^{\beta|x|}}{\sqrt{2\pi}}\left(\sup_{z\in(0,\infty)} z e^{z\beta\sigma\sqrt{T} - z^2/2}\right)\frac{\sqrt{T-t_k^n} - \sqrt{T-t}}{\sqrt{T-t}} \\ &\le \frac{e^{1+\beta|x|+\beta^2\sigma^2 T}}{\sqrt{\pi}}\,\frac{\sqrt{T}}{\sqrt{n(T-t)}} \end{aligned}$$

where the estimates $\sqrt{T-t_k^n} - \sqrt{T-t} \le \sqrt{t - t_k^n} \le \sqrt{2T/n}$ and

$$\sup_{z\in(0,\infty)} z e^{z\beta\sigma\sqrt{T} - z^2/2} \le \sup_{z\in(0,\infty)} e^{z\left(1+\beta\sigma\sqrt{T}\right) - z^2/2} \le e^{\left(1+\beta\sigma\sqrt{T}\right)^2/2} \le e^{1+\beta^2\sigma^2 T}$$

were applied. Consequently, it follows by (22) that

$$|I_1| \le \frac{e^{1+\beta|x|+\beta^2\sigma^2 T}}{\sqrt{\pi}}\left(\int_{[0,\infty)} e^{-\beta|y|} d|\mu|(y)\right)\frac{\sqrt{T}}{\sqrt{n(T-t)}}, \tag{23}$$

and an analogous computation for the integral I_2 yields

$$|I_2| \le \frac{e^{1+\beta|x|+\beta^2\sigma^2 T}}{\sqrt{\pi}}\left(\int_{(0,\infty)} e^{-\beta|y|} d|\mu|(y)\right)\frac{\sqrt{T}}{\sqrt{n(T-t)}}. \tag{24}$$

Since $\varepsilon_n^{\text{adj}}(t,x) = u(t_k^n, x) - u(t,x)$, the relations (21, 23), and (24) imply the claim. □

3.2. The local error

Suppose that $(h,\theta)\in(0,\infty)\times(0,T]$. The aim of this subsection to derive an upper bound for the absolute value of the error

$$\varepsilon^{\mathrm{loc}}_{h,\theta}(g) := \mathbb{E}\left[g(X_{\tau_J}) - g(X_\theta)\right] \tag{25}$$

as a function of (h,θ), where $g\in GBV_{\exp}$. The random variable J is given by

$$J = J(h,\theta) = \inf\{2m : \tau_{2m} > \theta\},$$

where τ_k was defined in (11). Afterward, upper bounds for the error (25) are derived in the dynamical setting, where the step size h and the level θ will depend on n. Observe that $J = J_n$ holds for J_n defined in (18) when one substitutes $(h,\theta) = \left(\sigma\sqrt{T/n},\ \frac{2T}{n}\left\lceil\frac{T-t}{2T/n}\right\rceil\right)$.

Let us start by introducing the following notation:

$$\mathbb{Z}^h_o := \left\{(2k+1)h : k\in\mathbb{Z}\right\},\quad \mathbb{Z}^h_e := \{2kh : k\in\mathbb{Z}\}$$

(o refers to "odd" and e refers to "even"); then $\mathbb{Z}^h = \mathbb{Z}^h_o\cup\mathbb{Z}^h_e$. In addition, we will abbreviate

$$d_o(x) := \operatorname{dist}\left(x,\mathbb{Z}^h_o\right),\quad d_e(x) := \operatorname{dist}\left(x,\mathbb{Z}^h_e\right) = h - d_o(x),\quad x\in\mathbb{R}. \tag{26}$$

As in [7], we project functions onto piecewise linear functions in order to compute the conditional expectation $\mathbb{E}\left[g(X_{\tau_J})|\mathcal{F}_\theta\right]$.

Definition 3.2. Define operators Π_o and Π_e acting on functions $u:\mathbb{R}\to\mathbb{R}$ by

- $\Pi_e u(x) := u(x)$ if $x\in\mathbb{Z}^h_e$ and $x\mapsto\Pi_e u(x)$ linear in $[2kh,(2k+2)h]\ \forall k\in\mathbb{Z}$,
- $\Pi_o u(x) := u(x)$ if $x\in\mathbb{Z}^h_o$ and $x\mapsto\Pi_o u(x)$ linear in $\left[(2k-1)h,(2k+1)h\right]\ \forall k\in\mathbb{Z}$.

The key ingredient in the estimation of the error $\varepsilon^{\mathrm{loc}}_{h,\theta}(g)$ is the following result, which was proposed in [7, Section 9]. For the convenience of the reader, a sketch of the proof is given below. Recall Definition 2.1 for the class $\mathcal{B}_{\exp}$, and denote by $\mathbb{N}_0 := \{0,1,2,\ldots\}$ the set of non-negative integers.

Proposition 3.3. *Let $(h,\theta)\in(0,\infty)\times(0,T]$ and define a random variable*

$$L = \ L(h,\theta) := \sup\{m\in\mathbb{N}_0 : \tau_m < \theta\} \tag{27}$$

(τ_L is equal to the largest of the stopping times $\tau_0,\tau_1,\ldots$ less than θ). Then, for a function $g\in\mathcal{B}_{\exp}$,

$$\varepsilon^{\mathrm{loc}}_{h,\theta}(g) = \mathbb{E}\left[\Pi_e g(X_\theta) - g(X_\theta)\right] + \mathbb{E}\left[\left(\Pi_o\Pi_e g(X_\theta) - \Pi_e g(X_\theta)\right)\mathbb{P}(L\ \mathit{even}|X_\theta)\right]. \tag{28}$$

Proof. If $g\in\mathcal{B}_{\exp}$, then also $\Pi_e g\in\mathcal{B}_{\exp}$ and $\Pi_o\ \Pi_e g\in\mathcal{B}_{\exp}$. The expectations on the right-hand side of (28) thus exist and are finite. Using the Markov property of the process $(X_t)_{t\geq 0}$, it can be shown that

$$\mathbb{E}\left[g(X_{\tau_J})|\mathcal{F}_\theta\right] = \Pi_e g(X_\theta)\qquad \mathbb{P}\text{-a.s. on } \{L \text{ odd}\},$$

$$\mathbb{E}\big[g(X_{\tau_J})|\mathcal{F}_\theta\big] = \Pi_o\Pi_e g(X_\theta) \quad \mathbb{P}\text{-a.s. on } \{L \text{ even}\},$$

see [7, Section 9]. Consequently, since $\mathbb{1}_{\{L \text{ odd}\}} + \mathbb{1}_{\{L \text{ even}\}} = 1$ $\mathbb{P}$-a.s.,

$$\begin{aligned}
\mathbb{E}[g(X_{\tau_J})] &= \mathbb{E}\big[\mathbb{E}\big[g(X_{\tau_J})|\mathcal{F}_\theta\big]\ \mathbb{1}_{\{L \text{ odd}\}}\big] + \mathbb{E}\big[\mathbb{E}\big[g(X_{\tau_J})|\mathcal{F}_\theta\big]\ \mathbb{1}_{\{L \text{ even}\}}\big] \\
&= \mathbb{E}\big[\Pi_e g(X_\theta)\mathbb{P}(L \text{ odd}|X_\theta)\big] + \mathbb{E}\big[\ \Pi_o\Pi_e g(X_\theta)\mathbb{P}(L \text{ even}|X_\theta)\big] \\
&= \mathbb{E}\big[\Pi_e g(X_\theta)\big] + \mathbb{E}\big[\big(\Pi_o\Pi_e g(X_\theta) - \Pi_e g(X_\theta)\big)\mathbb{P}(L \text{ even}|X_\theta)\big].
\end{aligned}$$

□

If $g \in GBV_{\exp}$ is a function given by (15) and $g^x := g(x + \cdot\,)$ for given $x \in \mathbb{R}$, then

$$g^x(z) = c + \int_{[0,\infty)} \mathbb{1}_{(y-x,\infty)}(z)d\mu(y) - \int_{(-\infty,0)} \mathbb{1}_{(-\infty,y-x]}(z)d\mu(y) + \sum_{i=1}^{\infty} \alpha_i \mathbb{1}_{\{x_i-x\}}(z). \tag{29}$$

Using the representation (29) and linearity, the estimation of the error $\varepsilon^{\text{loc}}_{h,\theta}(g^x)$ essentially reduces to the estimation of integrals whose integrands consist of indicator functions or their linear approximations given by the operators Π_e and Π_o (introduced in Definition 3.2). The following lemma enables us to interchange the order of integration or summation with the application of these operators.

Lemma 3.4. *Suppose that $(h,\theta) \in (0,\infty) \times (0,T]$ and that $g \in GBV_{\exp}$ admits the representation (15). Then, for all $x \in \mathbb{R}$,*

(i)
$$\begin{aligned}
\Pi_e g^x(z) = c &+ \int_{[0,\infty)} \Pi_e \mathbb{1}_{(y-x,\infty)}(z)d\mu(y) - \int_{(-\infty,0)} \Pi_e \mathbb{1}_{(-\infty,y-x]}(z)d\mu(y) \\
&+ \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}^h_e} \alpha_i \Pi_e \mathbb{1}_{\{x_i-x\}}(z), \quad z \in \mathbb{R},
\end{aligned}$$

(ii)
$$\begin{aligned}
\Pi_o\Pi_e g^x(z) = c &+ \int_{[0,\infty)} \Pi_o\Pi_e \mathbb{1}_{(y-x,\infty)}(z)d\mu(y) - \int_{(-\infty,0)} \Pi_o\Pi_e \mathbb{1}_{(-\infty,y-x]}(z)d\mu(y) \\
&+ \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}^h_e} \alpha_i \Pi_o\Pi_e \mathbb{1}_{\{x_i-x\}}(z), \quad z \in \mathbb{R}.
\end{aligned}$$

Idea of the proof. The representations in (i) and (ii) follow by using the representation (29), linearity of the operations $f \mapsto \Pi_e f$, $f \mapsto \Pi_o f$, and $f \mapsto \int f d|\mu|$, and relation (83). □

Proposition 3.5. *Let $(h,\theta) \in (0,\infty) \times (0,T]$. Suppose that $g \in GBV_{\exp}$ admits the representation (15) and that $\beta \geq 0$ is as in (16). Then, for all $x \in \mathbb{R}$,*

$$\big|\mathbb{E}\big[g^x(X_{\tau_J}) - g^x(X_\theta)\big]\big| \leq \frac{7}{\sqrt{2\pi}}\frac{h}{\sigma\sqrt{\theta}} e^{3\beta h+\beta|x|+\beta^2\sigma^2T/2}\left(\int_{\mathbb{R}} e^{-\beta|y|}d|\mu|(y) + \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}^h_e} |\alpha_i|e^{-\beta|x_i|}\right). \tag{30}$$

Proof. Let us denote by $p = p(\,\cdot\,, \theta)$ the density of X_θ:

$$p(z,\theta) = \frac{1}{\sqrt{2\pi\sigma^2\theta}} \exp\left(-\frac{z^2}{2\sigma^2\theta}\right), \quad z \in \mathbb{R}.$$

By Lemma 3.4, we may decompose the expectation on the left-hand side of (30) as follows:

$$\begin{aligned}
\mathbb{E}\left[g^x(X_{\tau_J}) - g^x(X_\theta)\right] &= \int_{\mathbb{R}} \int_{[0,\infty)} \left[\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \mathbb{1}_{(y-x,\infty)}(z)\right] d\mu(y) p(z) dz \\
&\quad - \int_{\mathbb{R}} \int_{(-\infty,0)} \left[\Pi_e \mathbb{1}_{(-\infty,y-x]}(z) - \mathbb{1}_{(-\infty,y-x]}(z)\right] d\mu(y) p(z) dz \\
&\quad + \int_{\mathbb{R}} \int_{[0,\infty)} \left[\Pi_o \Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \Pi_e \mathbb{1}_{(y-x,\infty)}(z)\right] d\mu(y) q(z) p(z) dz \\
&\quad - \int_{\mathbb{R}} \int_{(-\infty,0)} \left[\Pi_o \Pi_e \mathbb{1}_{(-\infty,y-x]}(z) - \Pi_e \mathbb{1}_{(-\infty,y-x]}(z)\right] d\mu(y) q(z) p(z) dz \\
&\quad + \int_{\mathbb{R}} \sum_{i \in \mathbb{N}: x_i - x \in \mathbb{Z}_e^h} \alpha_i \left[\Pi_e \mathbb{1}_{\{x_i - x\}}(z) - \mathbb{1}_{\{x_i - x\}}(z)\right] p(z) dz \\
&\quad + \int_{\mathbb{R}} \sum_{i \in \mathbb{N}: x_i - x \in \mathbb{Z}_e^h} \alpha_i \left[\Pi_o \Pi_e \mathbb{1}_{\{x_i - x\}}(z) - \Pi_e \mathbb{1}_{\{x_i - x\}}(z)\right] q(z) p(z) dz \\
&=: E^{(1)} - E^{(2)} + E^{(3)} - E^{(4)} + E^{(5)} + E^{(6)},
\end{aligned} \tag{31}$$

where $q = q(z,\theta)$ is the function introduced in (57) which satisfies $q(z) = \mathbb{P}(L \text{ even} | X_\theta = z)$ (Leb-a.e.). To show (30), we derive upper estimates for the quantities $|E^{(i)}|, 1 \le i \le 6,$ in the following steps.

Step 1: $E^{(1)}$ and $E^{(2)}$. Suppose that $y - x \in [2kh, (2k+2)h)$ for some $k \in \mathbb{Z}$. Then

$$|\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \mathbb{1}_{(y-x,\infty)}(z)| \le \mathbb{1}_{[2kh,(2k+2)h)}(z),$$

and since for each $z \in [2kh, (2k+2)h)$ it holds that $|y| \le 2h + |x| + |z|,$ we have

$$\begin{aligned}
&e^{\beta|y|} \int_{\mathbb{R}} |\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \mathbb{1}_{(y-x,\infty)}(z)| p(z) dz \\
&\le e^{2\beta h + \beta|x|} \int_{\mathbb{R}} e^{\beta|z|} |\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \mathbb{1}_{(y-x,\infty)}(z)| p(z) dz \\
&\le e^{2\beta h + \beta|x|} \int_{2kh}^{(2k+2)h} e^{\beta|z|} p(z) dz \\
&\le \frac{2}{\sqrt{2\pi}} e^{2\beta h + \beta|x| + \beta^2\sigma^2 T/2} \frac{h}{\sigma\sqrt{\theta}}.
\end{aligned}$$

Consequently, by Fubini's theorem,

$$
\begin{aligned}
|E^{(1)}| &\le \int\limits_{[0,\infty)} e^{-\beta|y|}\left(e^{\beta|y|}\int\limits_{\mathbb{R}} |\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \mathbb{1}_{(y-x,\infty)}(z)|p(z)dz\right) d|\mu|(y) \\
&\le \frac{h}{\sigma\sqrt{\theta}}\frac{2}{\sqrt{2\pi}} e^{2\beta h+\beta|x|+\beta^2\sigma^2 T/2}\int_{[0,\infty)} e^{-\beta|y|}d|\mu|(y).
\end{aligned} \tag{32}
$$

In fact, it also holds that

$$
|E^{(2)}| \le \frac{h}{\sigma\sqrt{\theta}}\frac{2}{\sqrt{2\pi}} e^{2\beta h+\beta|x|+\beta^2\sigma^2 T/2}\int_{(-\infty,0)} e^{-\beta|y|}d|\mu|(y) \tag{33}
$$

since $|\Pi_e \mathbb{1}_{(-\infty,y-x]}(z) - \mathbb{1}_{(-\infty,y-x]}(z)| = |\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \mathbb{1}_{(y-x,\infty)}(z)|$ for all $z \in \mathbb{R}$, which is a direct consequence of the relation

$$
\Pi_e \mathbb{1}_{(-\infty,r]} = 1 - \Pi_e \mathbb{1}_{(r,\infty)}, \quad r \in \mathbb{R}. \tag{34}
$$

Step 2: $E^{(3)}$ and $E^{(4)}$. Suppose $y - x \in [2kh, (2k+2)h)$ for some $k \in \mathbb{Z}$. Then $|y| \le 3h + |x| + |z|$ holds for all $z \in \big[(2k-1)h, (2k+3)h\big)$, and by (84) we may estimate

$$
\begin{aligned}
&e^{\beta|y|}\int_{\mathbb{R}} |\Pi_o\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \Pi_e \mathbb{1}_{(y-x,\infty)}(z)|q(z)p(z)dz \\
&\le e^{3\beta h+\beta|x|}\int_{\mathbb{R}} e^{\beta|z|}|\Pi_o\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \Pi_e \mathbb{1}_{(y-x,\infty)}(z)|q(z)p(z)dz \\
&\le e^{3\beta h+\beta|x|}\int_{(2k-1)h}^{(2k+3)h} e^{\beta|z|}\frac{d_o(z)}{4h}q(z)p(z)dz \\
&\le \frac{1}{4}e^{3\beta h+\beta|x|}\int_{(2k-1)h}^{(2k+3)h} e^{\beta|z|}p(z)dz \\
&\le \frac{1}{\sqrt{2\pi}}e^{3\beta h+\beta|x|+\beta^2\sigma^2 T/2}\frac{h}{\sigma\sqrt{\theta}},
\end{aligned}
$$

where we used the fact that $d_o(z) \le h$ and $q(z) \le 1$ for all $z \in \mathbb{R}$. Hence, by Fubini's theorem,

$$
\begin{aligned}
|E^{(3)}| &\le \int\limits_{[0,\infty)} e^{-\beta|y|}\left(e^{\beta|y|}\int\limits_{\mathbb{R}} |\Pi_o\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \Pi_e \mathbb{1}_{(y-x,\infty)}(z)|q(z)p(z)dz\right) d|\mu|(y) \\
&\le \frac{h}{\sigma\sqrt{\theta}}\frac{1}{\sqrt{2\pi}} e^{3\beta h+\beta|x|+\beta^2\sigma^2 T/2}\int\limits_{[0,\infty)} e^{-\beta|y|}d|\mu|(y).
\end{aligned} \tag{35}
$$

Moreover, by (34) and by the linearity of Π_o, we obtain

$$
|E^{(4)}| \le \frac{h}{\sigma\sqrt{\theta}}\frac{1}{\sqrt{2\pi}} e^{3\beta h+\beta|x|+\beta^2\sigma^2 T/2}\int\limits_{(-\infty,0)} e^{-\beta|y|}d|\mu|(y), \tag{36}
$$

since $|\Pi_o\Pi_e \mathbb{1}_{(-\infty,y-x]}(z) - \Pi_e \mathbb{1}_{(-\infty,y-x]}(z)| = |\Pi_o\Pi_e \mathbb{1}_{(y-x,\infty)}(z) - \Pi_e \mathbb{1}_{(y-x,\infty)}(z)|$, $z \in \mathbb{R}$.

Step 3: $E^{(5)}$. Notice that $\int_{\mathbb{R}} \varphi(z)p(z)dz = 0$ for the function $\varphi(z) := \sum_{i\in\mathbb{N}: x_i - x \in \mathbb{Z}_e^h} \mathbb{1}_{\{x_i - x\}}(z)$ since $\varphi = 0$ a.e. Notice also that for $\xi \in \mathbb{Z}_e^h$, it holds that $\Pi_e \, \mathbb{1}_{\{\xi\}} \le$

$\mathbb{1}_{[\xi-2h,\,\xi+2h]}$. Therefore, since it also holds (for each x_i) that $|x_i| \le 2h + |x| + |z|$ whenever $|z-(x_i-x)| \le 2h$,

$$\begin{aligned}
|E^{(5)}| &\le \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} \left|\alpha_i \int_{\mathbb{R}} \Pi_e \mathbb{1}_{\{x_i-x\}}(z)p(z)dz\right| \\
&\le \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i| e^{-\beta|x_i|} \int_{\mathbb{R}} e^{\beta|x_i|} p(z) \mathbb{1}_{[(x_i-x)-2h,\,(x_i-x)+2h]}(z)dz \\
&\le \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i| e^{-\beta|x_i|} \int_{(x_i-x)-2h}^{(x_i-x)+2h} e^{\beta|x_i|} p(z)dz \\
&\le \frac{h}{\sigma\sqrt{\theta}} \frac{4}{\sqrt{2\pi}} e^{2\beta h+\beta|x|+\beta^2\sigma^2 T/2} \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i| e^{-\beta|x_i|}.
\end{aligned} \tag{37}$$

Step 4: $E^{(6)}$. If $\xi \in \mathbb{Z}_e^h$, relations (83), (88), and the linearity of Π_o imply that

$$\begin{aligned}
&\Pi_o\Pi_e\mathbb{1}_{\{\xi\}}(z) - \Pi_e\mathbb{1}_{\{\xi\}}(z) \\
&= \frac{1}{4h}\left(\Pi_o\,|\cdot\; -(\xi-2h)|(z) - |z-(\xi-2h)|\right) - \frac{1}{2h}\left(\Pi_o\,|\cdot\; -\xi|(z) - |z-\xi|\right) \\
&\quad + \frac{1}{4h}\left(\Pi_o\,|\cdot\; -(\xi+2h)|(z) - |z-(\xi+2h)|\right) \\
&= \frac{d_o(z)}{4h}\left(\mathbb{1}_{[\xi-3h,\,\xi-h)}(z) - 2\mathbb{1}_{[\xi-h,\,\xi+h)}(z) + \mathbb{1}_{[\xi+h,\,\xi+3h)}(z)\right), \quad z\in\mathbb{R}.
\end{aligned}$$

Therefore, since $|x_i| \le 3h + |x| + |z|$ whenever $|z-(x_i-x)| \le 3h$, we get

$$\begin{aligned}
|E^{(6)}| &\le \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i| \int_{\mathbb{R}} |\Pi_o\Pi_e\mathbb{1}_{\{x_i-x\}}(z) - \Pi_e\mathbb{1}_{\{x_i-x\}}(z)| q(z)p(z)dz \\
&\le \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i| e^{-\beta|x_i|} \int_{(x_i-x)-3h}^{(x_i-x)+3h} e^{\beta|x_i|} \frac{d_o(z)}{2h} q(z)p(z)dz \\
&\le \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} \frac{|\alpha_i|}{2} e^{-\beta|x_i|+3\beta h+\beta|x|} \int_{(x_i-x)-3h}^{(x_i-x)+3h} e^{\beta|z|} p(z)dz \\
&\le \frac{h}{\sigma\sqrt{\theta}} \frac{3}{\sqrt{2\pi}} e^{3\beta h+\beta|x|+\beta^2\sigma^2 T/2} \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i| e^{-\beta|x_i|}.
\end{aligned} \tag{38}$$

The proof is completed by combining relation (31) with the bounds (32)–(33) and (35)–(38). □

Before presenting the main result of this subsection, Theorem 3.8, we provide an auxiliary convention regarding the notation. It enables us to distinguish between the general setting (h,θ) and the specific n-dependent setting (h_n,θ_n) also in the later sections.

Assumption 3.6. For given $t \in [0,T)$ and $n \in 2\mathbb{N}$, we substitute $(h,\theta) = (h_n,\theta_n)$, where

$$h_n = \sigma\sqrt{\frac{T}{n}}, \quad \theta_n = \frac{n_\theta T}{n} \quad \text{and} \quad n_\theta = 2\left\lceil \frac{T-t}{2T/n} \right\rceil$$

as in (12). For notational convenience, we will drop the subscript n from h_n.

Remark 3.7. The special choice $(h,\theta)=(h_n,\theta_n)$ in Assumption 3.6 affects the objects below used throughout this text:

$$\begin{aligned}
\tau_k &= \inf\{s>\tau_{k-1} : |X_s - X_{\tau_{k-1}}| = h\}, \quad (X_{\tau_k})_{k=0,1,\dots}, \quad (\mathcal{F}_{\tau_k})_{k=0,1,\dots},\\
J_n &= J = \inf\{2m\in 2\mathbb{N} : \tau_{2m} > \theta_n\}, \quad L_n = L = \sup\{m\in\mathbb{N}_0 : \tau_m < \theta_n\},\\
\mathbb{Z}_e^h &= \{2kh : k\in\mathbb{Z}\}, \quad \mathbb{Z}_o^h = \{(2k+1)h : k\in\mathbb{Z}\}, \quad \mathbb{Z}^h = \mathbb{Z}_o^h\cup\mathbb{Z}_e^h,\\
d_o(y) &= \mathrm{dist}\left(y,\mathbb{Z}_o^h\right), \quad d_e(y) = \mathrm{dist}\left(y,\mathbb{Z}_e^h\right), \text{ and } \quad p(y) = \mathbb{P}(X_{\theta_n}\in dy)/dy.
\end{aligned}$$

This choice also affects the function $q = q(\,\cdot\,,h,\theta)$ defined below in (5.3). In particular, Proposition 5.3 implies that $q(y) = \mathbb{P}\left(L_n \text{ even}|X_{\theta_n} = y\right)$ for $y\notin\mathbb{Z}^h$.

Recall that $\varepsilon_n^{\mathrm{loc}}(t,x) = \mathbb{E}\left[g(x+X_{\tau_{J_n}}) - g(x+X_{\theta_n})\right]$ as defined in (20).

Theorem 3.8. *Let $n\in 2\mathbb{N}$. Suppose that the function $g\in GBV_{\exp}$ admits the representation (15) and that $\beta\geq 0$ is as in (16). Then, under Assumption 3.6, there exists a constant $C>0$ such that for all $(t,x)\in[0,T)\times\mathbb{R}$,*

$$|\varepsilon_n^{\mathrm{loc}}(t,x)| \leq \frac{C\sqrt{T}}{\sqrt{n(T-t_k^n)}} e^{\beta|x|+3\beta^2\sigma^2 T}, \quad t\in\left[t_k^n, t_{k+1}^n\right), \quad 0\leq k<\tfrac{n}{2}.$$

Proof. Proposition 3.5 with $(h,\theta)=(h,\theta_n)$ as stated in Assumption 3.6 combined with the relation $h(\sigma^2\theta_n)^{-1/2} = n_\theta^{-1/2}$ yield

$$|\varepsilon_n^{\mathrm{loc}}(t,x)| \leq \frac{C_{\beta,\sigma,T}e^{\beta|x|}}{\sqrt{n_\theta}}\left(\int_{\mathbb{R}} e^{-\beta|y|}d|\mu|(y) + \sum_{i\in\mathbb{N}:x_i-x\in\mathbb{Z}_e^h} |\alpha_i|e^{-\beta|x_i|}\right),$$

where the coefficient $C_{\beta,\sigma,T}>0$ implied by (30) can be estimated as follows:

$$C_{\beta,\sigma,T} = \frac{7}{\sqrt{2\pi}}e^{3\beta h+\beta^2\sigma^2T/2} \leq \frac{7}{\sqrt{2\pi}}e^{\frac{5}{2}\beta\sigma\sqrt{T}+\beta^2\sigma^2T/2} \leq Ce^{3\beta^2\sigma^2T}$$

for a constant $C>0$. Since $n_\theta T = n(T-t_k^n)$ for $t\in\left[t_k^n,t_{k+1}^n\right)$ by (13), we obtain the desired result. □

4. The global error

Our aim is to derive an upper bound for the modulus of the global error defined in (19), that is

$$\varepsilon_n^{\mathrm{glob}}(t,x) = \mathbb{E}\left[g\left(x+X_{\tau_{n_\theta}}\right) - g(x+X_{\tau_{J_n}})\right],$$

where the function g is an exponentially bounded Borel function and $(X_{\tau_k})_{k=0,1,\dots}$ is the random walk considered in Section 2.1. For this purpose, we need a collection of estimates related to the behavior of the random walk (X_{τ_k}) and the random variable J_n. A part of these estimates are given in this section, while the more involved ones are presented later in Section 5.2 and Appendix A.2.

Note: Assumption 3.6 is taken as a standing assumption throughout Section 4.

Recall the definitions of n_θ and θ_n given in (12), and that $J_n(\omega) = \inf\{2m \in 2\mathbb{N} : \tau_{2m}(\omega) > \theta_n\}$ as was defined in (18). A result similar to the lemma below was proved in [7, Corollary 11.4].

Lemma 4.1. *For any $b \geq 0$, it holds that*

$$\text{(i)} \quad \mathbb{E}\left[e^{b|X_{\tau_{n_\theta}}|}\right] \leq 2e^{b^2\sigma^2 T/2}, \tag{39}$$

$$\text{(ii)} \quad \mathbb{E}\left[e^{b|X_{\tau_{J_n}}|}\right] \leq 2e^{b\sigma\sqrt{2T}+b^2\sigma^2 T/2}. \tag{40}$$

Proof. (i) Since $X_{\tau_{n_\theta}} = \sum_{k=1}^{n_\theta} \Delta X_{\tau_k}$, where $(\Delta X_{\tau_k})_{k=1,2,\dots}$ is a sequence of i.i.d. random variables with $\mathbb{P}(\Delta X_{\tau_k} = \pm h) = 1/2$ for $h = \sigma\sqrt{T/n}$ (see Section 2.1),

$$\mathbb{E}\left[e^{b|X_{\tau_{n_\theta}}|}\right] \leq 2\mathbb{E}\left[e^{bX_{\tau_{n_\theta}}}\right] = 2\left(\mathbb{E}\left[e^{b\Delta X_{\tau_1}}\right]\right)^{n_\theta} = 2(\cosh(bh))^{n_\theta} \leq 2e^{b^2h^2 n_\theta/2} \leq 2e^{b^2\sigma^2 T/2},$$

since $\cosh(y) \leq e^{y^2/2}$ holds for any $y \in \mathbb{R}$.

(ii) Firstly, observe that by the definition of J_n we have $|X_{\tau_{J_n}} - X_{\theta_n}| \leq 2h$. Secondly, since for a standard normal Z random variable it holds that $\mathbb{E}[e^{u|Z|}] \leq 2e^{u^2/2}$ for all $u \in \mathbb{R}$,

$$\mathbb{E}\left[e^{b|X_{\tau_{J_n}}|}\right] \leq \mathbb{E}\left[e^{b|X_{\tau_{J_n}}-X_{\theta_n}|+b|X_{\theta_n}|}\right] \leq e^{2bh}\mathbb{E}\left[e^{b\sigma\sqrt{\theta_n}|Z|}\right] \leq 2e^{b\sigma\sqrt{2T}+b^2\sigma^2 T/2}.$$

□

The following upper bounds are later needed for the estimation of the global error.

Proposition 4.2.

(i) *Suppose that $p \geq 0$, $g \in \mathcal{B}_{\exp}$, and that $b \geq 0$ is as in (14). Then there exists a constant $C_p > 0$ such that for all $x \in \mathbb{R}$,*

$$\sup_{(n,t)\in 2\mathbb{N}\times[0,T)} \left| \mathbb{E}\left[\left(\frac{|X_{\tau_{n_\theta}}|}{\sqrt{\sigma^2\theta_n}}\right)^p g\left(x + X_{\tau_{n_\theta}}\right)\right]\right| \leq C_p e^{b|x|+b^2\sigma^2 T}. \tag{41}$$

Moreover, for every $p > 0$ there exists a constant $C_p > 0$ such that

$$\text{(ii)} \quad \sup_{(n,t)\in 2\mathbb{N}\times[0,T)} n_\theta^p \mathbb{P}\left(|X_{\tau_{n_\theta}}/h| > n_\theta^{3/5}\right) \leq C_p, \tag{42}$$

$$\text{(iii)} \quad \sup_{(n,t)\in 2\mathbb{N}\times[0,T)} n_\theta^p \mathbb{P}\left(|J_n - n_\theta| > n_\theta^{3/5}\right) \leq C_p. \tag{43}$$

Proof. (i) Observe that

$$S_{n_\theta} := \frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}} = \frac{1}{\sqrt{\sigma^2\theta_n}} \sum_{k=1}^{n_\theta} \Delta X_{\tau_k} \stackrel{d}{=} \frac{1}{\sqrt{n_\theta}} \sum_{k=1}^{n_\theta} \xi_i,$$

where $(\xi_i)_{i=1,2,\dots}$ is an i.i.d. Rademacher sequence (see Section 2.1). Hence,

$$\mathbb{E}\left[e^{rS_{n_\theta}}\right] = \left(\cosh\left(\frac{r}{\sqrt{n_\theta}}\right)\right)^{n_\theta} \leq \left(e^{r^2/(2n_\theta)}\right)^{n_\theta} = e^{r^2/2}, \quad r \in \mathbb{R}.$$

Consequently, by the symmetry of S_{n_θ} and Markov's inequality,

$$\mathbb{P}\big(|S_{n_\theta}| > r\big) = 2\mathbb{P}\big(e^{rS_{n_\theta}} > e^{r^2}\big) \le 2e^{-r^2}\mathbb{E}[e^{rS_{n_\theta}}] \le 2e^{-r^2/2}, \quad r > 0,$$

and thus, uniformly in (n, t), for $p > 0$,

$$\mathbb{E}|S_{n_\theta}|^p = p\int_0^\infty r^{p-1}\mathbb{P}\big(|S_{n_\theta}| > r\big)dr \le 2p\int_0^\infty r^{p-1}e^{-r^2/2}dr := \tilde{C}_p < \infty. \tag{44}$$

Hölder's inequality, (44), and (39) then imply that

$$|\mathbb{E}\Big[|S_{n_\theta}|^p g\big(x + X_{\tau_{n_\theta}}\big)\Big]| \le Ae^{b|x|}\big(\mathbb{E}|S_{n_\theta}|^{2p}\big)^{1/2}\big(\mathbb{E}\big[e^{2b|X_{\tau_{n_\theta}}|}\big]\big)^{1/2} \le 2A\tilde{C}_{2p}^{1/2}e^{b|x|+b^2\sigma^2 T}.$$

This proves (41) for $p > 0$, and the case $p = 0$ can be seen from the last line as well.

(ii) Since $h\sqrt{n_\theta} = \sqrt{\sigma^2\theta_n}$, by Markov's inequality and (44) we obtain

$$\mathbb{P}\Big(|X_{\tau_{n_\theta}}/h| > n_\theta^{3/5}\Big) = \mathbb{P}\Big(|S_{n_\theta}| > n_\theta^{1/10}\Big) \le \mathbb{E}|S_{n_\theta}|^q n_\theta^{-q/10} \le C_q n_\theta^{-q/10} \tag{45}$$

for all $q > 0$. Choose $q \ge 10p$ and multiply both sides of (45) by n_θ^p to obtain (42).

(ii) For every $K > 0$, Markov's inequality and Proposition 5.9 imply that

$$\mathbb{P}\Big(|J_n - n_\theta| > n_\theta^{3/5}\Big) \le \mathbb{E}|J_n - n_\theta|^K n_\theta^{-3K/5} \le C_K n_\theta^{-K/10} \tag{46}$$

for some constant $C_K > 0$. For given $p > 0$, it remains to choose $K \ge 10p$ and multiply both sides of (46) by n_θ^p. □

The proof of the main result of this section follows closely the proof of [7, Theorem 8.1].

Theorem 4.3. *Let $n \in 2\mathbb{N}$. Suppose that $g \in \mathcal{B}_{\exp}$ and that $b \ge 0$ is as in (14). Then there exists a constant $C > 0$ such that for all $(t, x) \in [0, T) \times \mathbb{R}$,*

$$|\varepsilon_n^{\text{glob}}(t, x)| \le \frac{CT}{n(T - t_k^n)}e^{b|x|+3b^2\sigma^2 T}, \quad t \in \big[t_k^n, t_{k+1}^n\big), \quad 0 \le k < \frac{n}{2}.$$

Proof. The rough idea behind the estimation of the global error is to decompose it into a sum of a part, which corresponds to certain moments of the random variables $X_{\tau_{n_\theta}}$ and $J_n - n_\theta$, and to a part, which can be bounded by a term which is "of the order" n_θ^{-p} for some $p > 1$. Define a set

$$\Gamma_{n_\theta} := \{|X_{\tau_{n_\theta}}/h| \vee |J_{n_\theta} - n_\theta| \le n_\theta^{3/5}\} \tag{47}$$

and decompose the error $\varepsilon_n^{\text{glob}}(t, x)$ into the sum of expectations $E^{(1)}$ and $E^{(2)}$, where

$$E^{(1)} := \mathbb{E}\big[g\big(x + X_{\tau_{n_\theta}}\big) - g\big(x + X_{\tau_{J_n}}\big); \Gamma_{n_\theta}\big],$$
$$E^{(2)} := \mathbb{E}\Big[g\big(x + X_{\tau_{n_\theta}}\big) - g\big(x + X_{\tau_{J_n}}\big); \Gamma_{n_\theta}^{\complement}\Big].$$

Using the estimates of Lemma 4.1 and Proposition 4.2, it can be shown that

$$|E^{(2)}| \le \tilde{C}_0 n_\theta^{-3/2}e^{b|x|+b^2\sigma^2 T+b\sigma\sqrt{2T}} \tag{48}$$

for some constant $\tilde{C}_0 > 0$. This is done in Lemma A.2(i). Estimation of $|E^{(1)}|$ requires more subtlety. Denote the probability mass functions of $X_{\tau_{n_\theta+k}}/h$ and $J_n - n_\theta$ by

$$P_{n_\theta+k}(y) := \mathbb{P}\big(X_{\tau_{n_\theta+k}} = yh\big) \quad \text{and} \quad P^J_{n_\theta}(y) := \mathbb{P}(J_n - n_\theta = y), \quad y \in \mathbb{Z}. \tag{49}$$

By Lemma A.2(ii), there exists a constant $\tilde{C}_1 > 0$ such that

$$\begin{aligned} |E^{(1)}| \le & \left| \sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh) P^J_{n_\theta}(k) P_{n_\theta}(y) \left(\frac{k}{2n_\theta} - \frac{3k^2+4ky^2}{8n_\theta^2} + \frac{3k^2y^2}{4n_\theta^3} - \frac{k^2y^4}{8n_\theta^4} \right) \right| \\ & + \tilde{C}_1 n_\theta^{-3/2} e^{b|x|+b^2\sigma^2 T}. \end{aligned} \tag{50}$$

Next, we use relation (78) in order to rewrite the double sum on the right-hand side of (50) as

$$\begin{aligned} E^{(3)} :=& \sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh) P^J_{n_\theta}(k) P_{n_\theta}(y) \left(\frac{k}{2n_\theta} - \frac{3k^2+4ky^2}{8n_\theta^2} + \frac{3k^2y^2}{4n_\theta^3} - \frac{k^2y^4}{8n_\theta^4} \right) \\ =& \frac{1}{n_\theta} \Bigg\{ \frac{1}{2}\mathbb{E}\big[g\big(x+X_{\tau_{n_\theta}}\big)\big]\mathbb{E}[J_n - n_\theta] - \frac{3}{8}\mathbb{E}\big[g\big(x+X_{\tau_{n_\theta}}\big)\big]\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 \\ & - \frac{1}{2}\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^2 g\big(x+X_{\tau_{n_\theta}}\big)\right]\mathbb{E}[J_n-n_\theta] + \frac{3}{4}\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^2 g\big(x+X_{\tau_{n_\theta}}\big)\right]\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 \\ & - \frac{1}{8}\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^4 g\big(x+X_{\tau_{n_\theta}}\big)\right]\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 \Bigg\} \\ =& \frac{1}{n_\theta} \Bigg\{ \mathbb{E}\big[g\big(x+X_{\tau_{n_\theta}}\big)\big]\left(\frac{1}{2}\mathbb{E}[J_n-n_\theta] - \frac{3}{8}\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2\right) \\ & + \mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^2 g\big(x+X_{\tau_{n_\theta}}\big)\right]\left(\frac{3}{4}\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 - \frac{1}{2}\mathbb{E}[J_n-n_\theta]\right) \\ & - \frac{1}{8}\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^4 g\big(x+X_{\tau_{n_\theta}}\big)\right]\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 \Bigg\}. \end{aligned} \tag{51}$$

By Proposition 5.7, there exist constants $c_1, c_2 > 0$ such that $|\mathbb{E}[J_n - n_\theta] - \frac{4}{3}| \le c_1 n_\theta^{-\frac{1}{2}}$ and $\mathbb{E}\left[\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 - \frac{2}{3}\right] \le c_2 n_\theta^{-\frac{1}{2}}$. In particular, the following inequalities hold:

$$\begin{aligned} \left| \frac{1}{2}\mathbb{E}[J_n-n_\theta] - \frac{3}{8}\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 - \frac{5}{12} \right| &\le \left(\frac{c_1}{2} + \frac{3c_2}{8}\right)\frac{1}{\sqrt{n_\theta}}, \\ \left| \frac{3}{4}\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 - \frac{1}{2}\mathbb{E}[J_n-n_\theta] + \frac{1}{6} \right| &\le \left(\frac{c_1}{2} + \frac{3c_2}{4}\right)\frac{1}{\sqrt{n_\theta}}, \\ \left| \frac{1}{8}\mathbb{E}\left(\frac{J_n-n_\theta}{\sqrt{n_\theta}}\right)^2 - \frac{1}{12} \right| &\le \frac{c_2}{8\sqrt{n_\theta}}. \end{aligned}$$

Consequently, by (51) and (41), there exist constants $\tilde{C}_2, \tilde{C}_3 > 0$ such that

$$\begin{aligned}|E^{(3)}| &\leq \frac{5}{12n_\theta}\left|\mathbb{E}\left[g\left(x+X_{\tau_{n_\theta}}\right)\right]\right| + \frac{1}{6n_\theta}\left|\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^2 g\left(x+X_{\tau_{n_\theta}}\right)\right]\right| \\ &\quad + \frac{1}{12n_\theta}\left|\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^4 g\left(x+X_{\tau_{n_\theta}}\right)\right]\right| + \frac{\tilde{C}_2 e^{b|x|+b^2\sigma^2 T}}{n_\theta^{3/2}} \\ &\leq \frac{\tilde{C}_3}{n_\theta} e^{b|x|+b^2\sigma^2 T} + \frac{\tilde{C}_2 e^{b|x|+b^2\sigma^2 T}}{n_\theta^{3/2}}.\end{aligned} \tag{52}$$

To complete the proof, it remains to observe that $n_\theta^{-\frac{3}{2}} \leq n_\theta^{-1}\left(1/\sqrt{2}\right)$, to combine (48), (50), and (52), and to recall that $n_\theta T = n(T - t_k^n)$ for $t \in \left[t_k^n, t_{k+1}^n\right)$. □

5. Technical results

5.1. First exit times of Brownian bridges and the sharpness of the rate

Let $(h, \theta) \in (0, \infty) \times (0, T]$ and recall $L = \sup\{m \in \mathbb{N}_0 : \tau_m < \theta\}$ as was defined in (27). In this subsection we derive a representation for the function

$$y \mapsto \mathbb{P}\left(L \text{ even} | X_\theta = y\right) \tag{53}$$

based on first exit time probabilities of a Brownian bridge. This representation (60) together with the associated estimates derived in [12] is applied in order to prove Proposition 5.5, the main result of this subsection.

Definition 5.1 (Brownian bridge). Let $x, y \in \mathbb{R}$ and $l > 0$. A Gaussian process $(B_t^{x,l,y})_{t\in[0,l]}$ with mean and covariance functions given by

$$\mathbb{E}\left[B_t^{x,l,y}\right] = x + \frac{t}{l}(y - x), \quad 0 \leq t \leq l,$$
$$\mathrm{Cov}\left(B_s^{x,l,y}, B_t^{x,l,y}\right) = s\left(1 - \frac{t}{l}\right), \quad 0 \leq s \leq t \leq l,$$

is called a (generalized) Brownian bridge from x to y of length l.

Remark 5.2. By comparing mean and covariance functions, it is easy to verify that a Brownian bridge $(B_t^{x,l,y})_{t\in[0,l]}$ is equal in law with the transformed processes below:

$$\left(B_{l-t}^{y,l,x}\right)_{t\in[0,l]} \qquad \text{('time reversal')} \tag{54}$$

$$\left(x + B_t^{0,l,y-x}\right)_{t\in[0,l]} \qquad \text{('translation')} \tag{55}$$

$$\left(-B^{-x,l,-y}\right)_{t\in[0,l]} \qquad \text{('reflection around the } x\text{-axis')}. \tag{56}$$

A continuous version of a Brownian bridge $(B_t^{x,\theta,y})_{t\in[0,\theta]}$ can be thought as a random function on the canonical space $\left(C[0,\theta], \mathcal{B}(C[0,\theta]), \mathbb{P}_{x,\theta,y}\right)$, where $\mathbb{P}_{x,\theta,y}$ denotes the associated probability measure. In the following proposition we give different characterizations for the function (53) in terms of hitting times. For all $c \in \mathbb{R}$, $a < b$, and $\omega \in C[0,\theta]$, we let

$$H_c(\omega) := \inf\{t \in [0,\theta] : \omega_t = c\}, \quad H_{(a,b)}(\omega) := \inf\{t \in [0,\theta] : \omega_t \notin (a,b)\},$$
$$\hat{H}_c(\omega) := \sup\{t \in [0,\theta] : \omega_t = c\}, \quad \hat{H}_{(a,b)}(\omega) := \sup\{t \in [0,\theta] : \omega_t \notin (a,b)\}.$$

Proposition 5.3. *Let $(h,\theta) \in (0,\infty) \times (0,T]$. Suppose that $\left(B_t^{y/\sigma,\theta,0}\right)_{t\in[0,\theta]}$ is a Brownian bridge on a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$, and define*

$$q(y) = q(y,h,\theta) := \tilde{\mathbb{P}}\left(\left(B_t^{y/\sigma,\theta,0}\right)_{t\in[0,\theta]} \text{ hits } \mathbb{Z}_e^{h/\sigma} \text{ before hitting } \mathbb{Z}_o^{h/\sigma}\right), \quad y \in \mathbb{R}. \tag{57}$$

Then, for all $k \in \mathbb{Z}$,

$$\text{(i)} \quad q(y) = \mathbb{P}(L \text{ even} | X_\theta = y), \quad y \notin \mathbb{Z}^h, \tag{58}$$

$$\text{(ii)} \quad q(y) = \begin{cases} \mathbb{P}_{y/\sigma,\theta,0}\left(H_{2kh/\sigma} < H_{(2k+1)h/\sigma}\right), & y \in (2kh,(2k+1)h), \\ \mathbb{P}_{y/\sigma,\theta,0}\left(H_{2kh/\sigma} < H_{(2k-1)h/\sigma}\right), & y \in ((2k-1)h,2kh), \end{cases} \tag{59}$$

$$\text{(iii)} \quad q(y) = \begin{cases} \dfrac{d_o(y)}{h} + \dfrac{\sigma}{h}\mathbb{E}_{\tilde{\mathbb{P}}}\left[B^{0,\theta,y/\sigma}_{\tilde{H}_{(-((2k+1)h-y)/\sigma,\,(y-2kh)/\sigma)}}\right], & y \in (2kh,(2k+1)h), \\ \dfrac{d_o(y)}{h} - \dfrac{\sigma}{h}\mathbb{E}_{\tilde{\mathbb{P}}}\left[B^{0,\theta,y/\sigma}_{\tilde{H}_{(-(2kh-y)/\sigma,\,(y-(2k-1)h)/\sigma)}}\right], & y \in ((2k-1)h,2kh). \end{cases} \tag{60}$$

Here $\tilde{H}_{(a,b)} = \inf\{t \in [0,\theta] : B_t^{0,\theta,y/\sigma} \notin (a,b)\}$, and $\mathbb{P}$ refers to the probability measure on the space $(\Omega,\mathcal{F},\mathbb{P})$ considered in Section 2.

Proof. Item (ii) is clear. To show (i), observe that if $X_\theta(\omega) \in (2kh,(2k+1)h)$ and $L(\omega)$ is even, the path $t \mapsto X_t(\omega)$ does hit $2kh$ at $\tau_L(\omega)$ and afterwards, i.e. on $[\tau_L(\omega),\theta)$, it does not hit any other mh $(m \neq 2k)$ and hence stays inside $((2k-1)h,(2k+1)h)$. Therefore, the last entry of this path into $(2kh,(2k+1)h)$ occurs via $2kh$, and thus

$$\begin{aligned} \mathbb{P}(L \text{ even}, X_\theta \in (2kh,(2k+1)h)) &= \mathbb{P}_0\left(\sigma\omega_{\hat{H}_{(2kh,(2k+1)h)}(\omega)} = 2kh, \ \sigma\omega_\theta \in (2kh,(2k+1)h)\right) \\ &= \mathbb{P}_0\left(\omega_{\hat{H}_{(2kh/\sigma,(2k+1)h/\sigma)}(\omega)} = \frac{2kh}{\sigma}, \ \omega_\theta \in \left(\frac{2kh}{\sigma},\frac{(2k+1)h}{\sigma}\right)\right) \\ &= \mathbb{P}_0\left(\hat{H}_{2kh/\sigma} > \hat{H}_{(2k+1)h/\sigma}\right), \end{aligned}$$

where $\mathbb{P}_0$ denotes the Wiener measure on $(C[0,\theta],\mathcal{B}(C[0,\theta]))$. Thus, for $y \in (2kh,(2k+1)h)$,

$$\mathbb{P}(L \text{ even}|X_\theta = y) = \mathbb{P}_{0,\theta,y/\sigma}\left(\hat{H}_{2kh/\sigma} > \hat{H}_{(2k+1)h/\sigma}\right) = \mathbb{P}_{y/\sigma,\theta,0}\left(H_{2kh/\sigma} < H_{(2k+1)h/\sigma}\right) = q(y),$$

where we used relations (54), (59), and the fact that $\mathbb{P}(\,\cdot\,|X_\theta = y) = \mathbb{P}_{0,\theta,y/\sigma}$ on $(C[0,\theta],\mathcal{B}(C[0,\theta]))$ (see e.g. [13, Chapter 1]). The case $y \in ((2k-1)h,2kh)$ is similar.

For (iii), assume $y \in ((2k-1)h, 2kh)$; the case $y \in (2kh, (2k+1)h)$ is similar. It is clear that whenever $z \notin (a,b)$, $a < 0 < b$, and $\tilde{H}_{(a,b)} = \inf\{t \in [0,\theta] : B_t^{0,\theta,z} \notin (a,b)\}$,

$$\mathbb{P}_{0,\theta,z}(H_a < H_b) = \frac{b}{b-a} - \frac{1}{b-a}\mathbb{E}_{\tilde{\mathbb{P}}}\left[B^{0,\theta,z}_{\tilde{H}_{(a,b)}}\right]. \tag{61}$$

In addition, from (59) we deduce that

$$\begin{aligned} q(y) &= \mathbb{P}_{y/\sigma,\theta,0}\big(H_{2kh/\sigma} < H_{(2k-1)h/\sigma}\big) \\ &= \mathbb{P}_{0,\theta,-y/\sigma}\Big(H_{(2kh-y)/\sigma} < H_{((2k-1)h-y)/\sigma}\Big) \\ &= \mathbb{P}_{0,\theta,y/\sigma}\Big(H_{(y-2kh)/\sigma} < H_{(y-(2k-1)h)/\sigma}\Big) \end{aligned} \tag{62}$$

by (55) and (56). Substitute $z = y/\sigma$, $a = \frac{y-2kh}{\sigma}$, and $b = \frac{y-(2k-1)h}{\sigma}$. Then $z \notin (a,b)$, $a < 0 < b$, $b - a = h/\sigma$, and hence by (61), (62), and $d_o(y) = y - (2k-1)h$,

$$q(y) = \frac{d_o(h)}{h} - \frac{\sigma}{h}\mathbb{E}_{\tilde{\mathbb{P}}}\left[B^{0,\theta,y/\sigma}_{\tilde{H}_{((y-2kh)/\sigma,\,(y-(2k-1)h)/\sigma)}}\right].$$

□

Before we proceed to prove the sharpness result for the class $GBV_{\exp}$, Proposition 5.5, we list the assertions of [12] which are needed for the proof.

Lemma 5.4 ([12, Lemmas 4.1 and 4.2(i)]). *Let $(h,\theta) \in (0,\infty) \times (0,\infty)$ and suppose that $a < 0 < b$ and $y \notin (a,b)$. Then*

$$\left|\mathbb{E}_{\tilde{\mathbb{P}}}\left[B^{0,\theta,y}_{\tilde{H}_{(a,b)}}\right]\right| \le \frac{\mathbb{E}_{0,\theta,y}\left[H_{(a,b)}\right]}{\theta}\Big(|y| + 2(|a| \vee b) + 3\sqrt{2\theta}\Big), \tag{63}$$

$$\mathbb{E}_{0,\theta,y}\left[H_{(a,b)}\right] \le \begin{cases} b(2|a|+y) \wedge \theta, & y \ge b, \\ |a|(2b+|y|) \wedge \theta, & y \le a. \end{cases} \tag{64}$$

Proposition 5.5. *Under Assumption 3.6, there exists a function $g \in GBV_{\exp}$ such that*

$$0 < \liminf_{n\to\infty} n^{\frac{1}{2}}\varepsilon_n(0,0) \le \limsup_{n\to\infty} n^{\frac{1}{2}}\varepsilon_n(0,0) < \infty. \tag{65}$$

Proof. For simplicity, let $T = \sigma = 1$ and $g := \mathbb{1}_{[0,\infty)}$. Then $h = n^{-\frac{1}{2}}$, $g \in GBV_{\exp}$, and the location of the jump of g belongs to the set $\mathbb{Z}_e^h$ for all $n \in \mathbb{N}$. Observe that then $\varepsilon_n^{\mathrm{adj}}(0,0) = 0$ by Theorem 3.1 and $|\varepsilon_n^{\mathrm{glob}}(0,0)| \le Cn^{-1}$ by Theorem 4.3, where $C > 0$ is some constant. Consequently, it suffices to show that (65) is valid for the local error $\varepsilon_n^{\mathrm{loc}}(0,0)$.

The expression $n^{\frac{1}{2}}\varepsilon_n^{\mathrm{loc}}(0,0)$ is bounded from above by Theorem 3.8. For the lower bound, we note that by Definition 3.2,

$$\Pi_e \mathbb{1}_{[0,\infty)}(x) = \left(1 \wedge \frac{x+2h}{2h}\right)\mathbb{1}_{[-2h,\infty)}(x), \quad \Pi_o\Pi_e \mathbb{1}_{[0,\infty)}(x) = \left(1 \wedge \frac{x+3h}{4h}\right)\mathbb{1}_{[-3h,\infty)}(x), \quad x \in \mathbb{R}.$$

Consequently, for $(h,\theta) = \left(n^{-\frac{1}{2}}, 1\right)$, Proposition 3.3 and relation (58) yield

$$\begin{aligned}
\varepsilon_n^{\text{loc}}(0,0) &= \mathbb{E}\left[\Pi_e \mathbb{1}_{[0,\infty)}(W_1) - \mathbb{1}_{[0,\infty)}(W_1)\right] + \mathbb{E}\left[\left(\Pi_o \Pi_e \mathbb{1}_{[0,\infty)}(W_1) - \Pi_e \mathbb{1}_{[0,\infty)}(W_1)\right) q(W_1)\right] \\
&= \int_{-2h}^{0} \frac{x+2h}{2h} p(x)dx + \int_{-3h}^{h} \left[\frac{x+3h}{4h} - \left(\frac{x+2h}{2h} \mathbb{1}_{[-2h,0)}(x) + \mathbb{1}_{[0,\infty)}(x)\right)\right] q(x)p(x)dx \\
&= \int_{-2h}^{0} \frac{x+2h}{2h}(1-q(x))p(x)dx + \int_{-3h}^{0} \frac{x+3h}{4h} q(x)p(x)dx + \int_{0}^{h} \frac{x-h}{4h} q(x)p(x)dx \\
&= \int_{-2h}^{0} \frac{x+2h}{2h}(1-q(x))p(x)dx + \int_{-3h}^{-h} \frac{x+3h}{4h} q(x)p(x)dx + \frac{1}{2}\int_{0}^{h} q(x)p(x)dx \\
&\geq p(h)\int_{-h}^{0} \frac{x+2h}{2h}(1-q(x))dx
\end{aligned}$$

by the symmetry of the functions $p > 0$ and $q \in [0,1]$ and substitution $x \mapsto -x$. Next, notice that for $k = 0$ and $\sigma = \theta = 1$, relations (60), (63), and (64) imply that whenever $x \in (-h, 0)$,

$$\begin{aligned}
\left| q(x) - \frac{d_o(x)}{h} \right| = \frac{1}{h}\left| \mathbb{E}_{\tilde{\mathbb{P}}}\left[B^{0,1,x}_{\tilde{H}_{(x,h+x)}} \right] \right| &\leq \frac{1}{h}\left(|x| + 2\big(|x| \vee (x+h)\big) + 3\sqrt{2} \right) \mathbb{E}_{0,1,x}\left[H_{(x,h+x)} \right] \\
&\leq \left(3h + 3\sqrt{2}\right) \frac{|x|}{h} (2h - |x|) \\
&\leq 3h\left(h + \sqrt{2}\right).
\end{aligned}$$

Consequently, for $x \in (-h, 0)$,

$$1 - q(x) = 1 - \frac{d_o(x)}{h} - \left| q(x) - \frac{d_o(x)}{h} \right| \geq \frac{d_e(x)}{h} - 3h\left(h+\sqrt{2}\right) = \frac{|x|}{h} - 3h\left(h+\sqrt{2}\right),$$

and thus there exist constants $C_1, C_2 > 0$ (not depending on h) such that

$$\begin{aligned}
\varepsilon_n^{\text{loc}}(0,0) &\geq p(h)\int_{-h}^{0} \frac{x+2h}{2h}\frac{|x|}{h} dx - 3h\left(h+\sqrt{2}\right)p(h)\int_{-h}^{0} \frac{x+2h}{2h} dx \\
&\geq \left[C_1 h - C_2 h^2 \left(h + \sqrt{2}\right) \right] p(h).
\end{aligned}$$

The relation $h = n^{-\frac{1}{2}}$ then implies that $\liminf_{n\to\infty} n^{\frac{1}{2}} \varepsilon_n^{\text{loc}}(0,0) \geq C_1 p(0) > 0$. □

Remark 5.6. In [7, Proposition 9.8] it is stated that the rate for the local error is h (i.e. $n^{-\frac{1}{2}}$) instead of h^2 (i.e. n^{-1}) whenever the terminal condition g has a discontinuity at a non-lattice point $x \notin \mathbb{Z}^h$. By contrast, Proposition 3.5 implies that only the jumps that occur at even lattice points contribute to the error. This discrepancy is a result of the choice of different step functions: In [7], only step functions of the type $\tilde{\mathbb{1}}_{[a,\infty)} := \mathbb{1}_{\{a\}}/2 + \mathbb{1}_{(a,\infty)}$ are considered.

5.2. Moment estimates for the random variable J_n

In this subsection, moment estimates are presented for the random variable

$$J_n = \inf\{2m \in 2\mathbb{N} : \tau_{2m} > \theta_n\},$$

(which was introduced in (18)), which are applied in Section 4. We begin with a proposition which generalizes [7, Proposition 11.2(ii)–(iii)] to the time-dependent setting $(t \neq 0)$. For the proof, the reader is referred to [14, Section 6.1].

Proposition 5.7. *Suppose that Assumption 3.6 holds. Then there exists a constant $C>0$ such that for all $(n,t) \in 2\mathbb{N} \times [0,T)$,*

$$\text{(i)} \quad \left|\mathbb{E}[J_n] - n_\theta - \frac{4}{3}\right| \leq \frac{C}{\sqrt{n_\theta}}, \quad \text{(ii)} \quad \left|\mathbb{E}\left(\frac{J_n - n_\theta}{\sqrt{n_\theta}}\right)^2 - \frac{2}{3}\right| \leq \frac{C}{\sqrt{n_\theta}}.$$

To derive an estimate for $\mathbb{E}|J_n - n_\theta|^K$ for arbitrary $K>0$, we recall (see, e.g. [15, Theorem 14.12]) a version of the Azuma–Hoeffding inequality.

Proposition 5.8 (Azuma–Hoeffding inequality). *Suppose that $(M_j)_{j=0,1,\ldots}$ is a martingale started for which $M_0 = 0$ holds. In addition, assume that for all $i \geq 1$ there exists a constant $\alpha_i > 0$ such that $|M_i - M_{i-1}| \leq \alpha_i$ a.s. Then, for all $k \in \mathbb{N}$ and every $s>0$,*

$$\mathbb{P}(M_k \geq s) \leq \exp\left(-\frac{s^2}{2\sum_{j=1}^k \alpha_i^2}\right).$$

The following result, proved below in the time-dependent setting, can be found in [7, Proposition 11.2(iv)] for $t=0$. The original proof, however, does not cover the case corresponding to the inequality (67) for the set A_3.

Proposition 5.9. *Suppose that Assumption 3.6 holds, and let $K>0$. Then there exists a constant $C_K > 0$ depending at most on K such that*

$$\mathbb{E}|J_n - n_\theta|^K \leq C_K n_\theta^{K/2} \quad \textit{for all} \quad (n,t) \in 2\mathbb{N} \times [0,T). \tag{66}$$

Proof. It suffices to prove the claim for $K \geq 2$, since the case $K \in (0,2)$ then follows by Jensen's inequality. Since $|J_n - n_\theta|$ is a non-negative random variable,

$$\frac{1}{K}\mathbb{E}|J_n - n_\theta|^K = \int_0^\infty z^{K-1}\mathbb{P}(|J_n - n_\theta| > z)dz.$$

We show that there exist constants $C_K^{(1)}, C_K^{(2)}, C_K^{(3)} > 0$ corresponding to the sets $A_1 = (0,2], A_2 = (2, n_\theta]$ and $A_3 = (n_\theta, \infty)$ such that

$$I_k(n_\theta) := \int_{A_k} z^{K-1}\mathbb{P}(|J_n - n_\theta| > z)dz \leq C_K^{(k)} n_\theta^{K/2} \quad \text{for all } n_\theta. \tag{67}$$

Step 1: Since $K \geq 2$ and $n_\theta \geq 2$, we have that

$$I_1(n_\theta) = \int_0^2 z^{K-1}\mathbb{P}(|J_n - n_\theta| > z)dz \leq \int_0^2 z^{K-1}dz \leq 2^K/K \leq C_K^{(1)} n_\theta^{K/2}.$$

Step 2: Suppose that $n_\theta > 2$ and define $\delta_{n_\theta}(u) := \frac{2}{n_\theta}\lfloor \frac{n_\theta u}{2} \rfloor$. Then

$$\begin{aligned} I_2(n_\theta) &= \int_2^{n_\theta} z^{K-1}\mathbb{P}(|J_n - n_\theta| > z)dz = n_\theta^K \int_{2/n_\theta}^1 u^{K-1}\mathbb{P}(|J_n - n_\theta| > n_\theta u)du \\ &\leq n_\theta^K \int_{2/n_\theta}^1 u^{K-1}\mathbb{P}\big(|J_n - n_\theta| > \delta_{n_\theta}(u) n_\theta\big)du. \end{aligned} \tag{68}$$

The idea here is to estimate the tail probability inside the integral on the last line of (68) with the help of Lemma A.6. To proceed, fix a constant $a \in (0, 1]$ small enough such that for every $m \in \mathbb{N}$,

$$\delta_m(u) < \frac{\pi^2}{12 + \pi^2} \quad \text{and} \quad H\left(\sqrt{\frac{3\delta_m(u)}{1+\delta_m(u)}}\right) \wedge H\left(\sqrt{\frac{3\delta_m(u)}{1-\delta_m(u)}}\right)$$
$$> 1/4 \quad \text{hold for all} \quad u \le a, \tag{69}$$

where the function H is defined below in (91). Depending on the value of n_θ, we split the right-hand side of (68) into the sum of the integrals

$$I_{2,1}(n_\theta) := n_\theta^K \int_{2/n_\theta}^{a} u^{K-1} \mathbb{P}\big(|J_n - n_\theta| > \delta_{n_\theta}(u) n_\theta\big) du \quad (\text{for } a > 2/n_\theta, \text{ otherwise } 0),$$
$$I_{2,2}(n_\theta) := n_\theta^K \int_{a \vee (2/n_\theta)}^{1} u^{K-1} \mathbb{P}\big(|J_n - n_\theta| > \delta_{n_\theta}(u) n_\theta\big) du.$$

If $a \in (2/n_\theta, 1)$, by (69) and the fact that $n_\theta(1 + \delta_{n_\theta}(u))$ and $n_\theta(1 - \delta_{n_\theta}(u))$ are even integers, we may apply Lemma A.6 and estimate

$$\begin{aligned} I_{2,1}(n_\theta) &= n_\theta^K \int_{2/n_\theta}^{a} u^{K-1} [\mathbb{P}(J_n > n_\theta(1 + \delta_{n_\theta}(u))) + \mathbb{P}(J_n < n_\theta(1 - \delta_{n_\theta}(u)))] du \\ &\le n_\theta^K \int_{2/n_\theta}^{a} u^{K-1} \left[\exp\left(-\frac{3}{8}\frac{n_\theta \delta_{n_\theta}^2(u)}{1 + \delta_{n_\theta}(u)}\right) + \exp\left(-\frac{3}{8}\frac{n_\theta \delta_{n_\theta}^2(u)}{1 - \delta_{n_\theta}(u)}\right)\right] du \\ &\le 2n_\theta^K \int_{2/n_\theta}^{a} u^{K-1} \exp\left(-\frac{3}{8}\frac{n_\theta \delta_{n_\theta}^2(u)}{1 + \delta_{n_\theta}(u)}\right) du. \end{aligned} \tag{70}$$

By the properties of the floor function, for $u \in (0, 1]$ it holds that

$$\frac{\delta_{n_\theta}^2(u)}{1 + \delta_{n_\theta}(u)} = \frac{\left(\frac{2}{n_\theta}\left\lfloor\frac{n_\theta u}{2}\right\rfloor\right)^2}{1 + \frac{2}{n_\theta}\left\lfloor\frac{n_\theta u}{2}\right\rfloor} \ge \frac{\left(\frac{2}{n_\theta}\left(\frac{n_\theta u}{2} - 1\right)\right)^2}{1 + u} > \frac{\left(u - \frac{2}{n_\theta}\right)^2}{2}, \tag{71}$$

and thus the right-hand side of (70) can be bounded from above by

$$\begin{aligned} 2n_\theta^K \int_{2/n_\theta}^{a} u^{K-1} e^{-\frac{3n_\theta(u-2/n_\theta)^2}{16}} du &\le 2n_\theta^K \int_0^{1-2/n_\theta} \left(u + \frac{2}{n_\theta}\right)^{K-1} e^{-\frac{3n_\theta u^2}{16}} du \\ &\le 2^{K-1} n_\theta^K \int_0^{1-2/n_\theta} \left(u^{K-1} + \left(\frac{2}{n_\theta}\right)^{K-1}\right) e^{-\frac{3n_\theta u^2}{16}} du. \end{aligned} \tag{72}$$

By substituting $x = u\sqrt{n_\theta}$ and identifying the right-hand side of (72) as an integral with respect to a Gaussian measure, it can be verified that this integral multiplied by $n_\theta^{K/2}$ is bounded by some constant $\tilde{C}_K^{(2,1)} > 0$. Hence, $I_{2,1}(n_\theta) \le C_K^{(2,1)} n_\theta^{K/2}$ for all n_θ, where $C_K^{(2,1)} > 0$ depends only on K.

Let us then consider the integral $I_{2,2}(n_\theta)$. If $a/3 \le 2/n_\theta$, then $n_\theta \le 6/a$, $a \le a \vee (2/n_\theta)$, and thus $I_{2,2}(n_\theta) \le 6^K (Ka^K)^{-1}$. On the other hand, if $a/3 \in (2/n_\theta, 1)$, by Lemma A.6,

$$\begin{aligned}
I_{2,2}(n_\theta) &= n_\theta^K \int_a^1 u^{K-1}[\mathbb{P}(J_n > n_\theta(1+\delta_{n_\theta}(u))) + \mathbb{P}(J_n < n_\theta(1-\delta_{n_\theta}(u)))]du \\
&\le n_\theta^K \int_a^1 u^{K-1}\big[\mathbb{P}\big(J_n > n_\theta\big(1+\delta_{n_\theta}(a/3)\big)\big) + \mathbb{P}\big(J_n < n_\theta\big(1-\delta_{n_\theta}(a/3)\big)\big)\big]du \\
&\le n_\theta^K \left[\exp\left(-\frac{3}{8}\frac{n_\theta \delta_{n_\theta}(a/3)^2}{1+\delta_{n_\theta}(a/3)}\right) + \exp\left(-\frac{3}{8}\frac{n_\theta \delta_{n_\theta}(a/3)^2}{1-\delta_{n_\theta}(a/3)}\right)\right]\int_a^1 u^{K-1}du \\
&\le 2n_\theta^K \exp\left(-\frac{3}{8}\frac{n_\theta \delta_{n_\theta}(a/3)^2}{1+\delta_{n_\theta}(a/3)}\right)\int_a^1 u^{K-1}du \\
&\le 2n_\theta^K \exp\left(-\frac{3n_\theta(a/3-2/n_\theta)^2}{16}\right).
\end{aligned}$$

Here we used the fact that $u \mapsto \delta_{n_\theta}(u)$ is nondecreasing, that $n_\theta\big(1+\delta_{n_\theta}(a/3)\big)$ and $n_\theta\big(1-\delta_{n_\theta}(a/3)\big)$ are (even) integers, condition (69), and inequality (71). Notice that the right-hand side converges to zero as $n_\theta \to \infty$. Consequently, there exists a constant $C_K^{(2,2)} > 0$ such that $I_{2,2}(n_\theta) \le C_K^{(2,2)}$ for all n_θ, and (67) for $k=2$ follows.

Step 3: To estimate $I_3(n_\theta)$, we apply the Azuma–Hoeffding inequality to the tail distribution of the random variable $J_n = \inf\{2m \in 2\mathbb{N} : \tau_{2m} > \theta_n\}$. Recall that $\tau_i - \tau_{i-1}, i = 1, 2, \ldots,$ are i.i.d. (see Section 2.1) and that $\frac{n}{T}\theta_n = n_\theta$ according to (4). Let $\zeta_i := \frac{n}{T}(\tau_i - \tau_{i-1}),\ i \ge 1$. Then, for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}(J_n \ge 2m) = \mathbb{P}(\tau_{2m-2} \le \theta_n) &= \mathbb{P}\left(\sum_{i=1}^{2m-2}\zeta_i \le \frac{n}{T}\theta_n\right) \le \mathbb{P}\left(\sum_{i=1}^{2m-2}\zeta_i \wedge N \le n_\theta\right) \\
&= \mathbb{P}\left(\sum_{i=1}^{2m-2}(c_N - \zeta_i \wedge N) \ge (2m-2)c_N - n_\theta\right),
\end{aligned} \tag{73}$$

where $N \in \mathbb{N}$ is chosen such that $3/4 < c_N := \mathbb{E}[\zeta_i \wedge N] < \mathbb{E}[\zeta_i] = 1$. Then $|\mathbb{E}[\zeta_i \wedge N] - \zeta_i \wedge N| \le N$ for all $i \ge 1$, and by (73) and the Azuma–Hoeffding inequality (Proposition 5.8),

$$\mathbb{P}(J_n \ge 2m) \le \exp\left(-\frac{((2m-2)c_N - n_\theta)^2}{2(2m-2)N^2}\right), \quad m \in \mathbb{N}.$$

Since $J_n > 0$, we have

$$\begin{aligned}
I_3(n_\theta) &= \int_{n_\theta}^{\infty} z^{K-1}\mathbb{P}(J_n - n_\theta \ge z)dz \\
&= \int_{n_\theta}^{2n_\theta+2} z^{K-1}\mathbb{P}(J_n \ge z + n_\theta)dz + \sum_{m=3}^{\infty}\int_{(m-1)n_\theta+2}^{mn_\theta+2} z^{K-1}\mathbb{P}(J_n \ge z + n_\theta)dz \\
&\le \int_{n_\theta}^{2n_\theta+2} z^{K-1}\mathbb{P}(J_n \ge 2n_\theta)dz + \sum_{m=3}^{\infty}\int_{(m-1)n_\theta+2}^{mn_\theta+2} z^{K-1}\mathbb{P}(J_n \ge mn_\theta + 2)dz \\
&\le \int_{n_\theta}^{2n_\theta+2} z^{K-1}e^{-\frac{n_\theta}{2N^2}\frac{((2-2/n_\theta)c_N-1)^2}{(2-2/n_\theta)}}dz + \sum_{m=3}^{\infty}\int_{(m-1)n_\theta+2}^{mn_\theta+2} z^{K-1}e^{-\frac{n_\theta}{2N^2}\frac{(mc_n-1)^2}{m}}dz \\
&=: I_{3,1}(n_\theta) + I_{3,2}(n_\theta).
\end{aligned}$$

Since $c_N \in (3/4, 1)$, there exist constants $c, c' > 0$ such that

$$I_{3,1}(n_\theta) = \int_{n_\theta}^{2n_\theta+2} z^{K-1} e^{-\frac{n_\theta}{2N^2}\frac{((2-2/n_\theta)c_N-1)^2}{(2-2/n_\theta)}} dz \le 2(2n_\theta+2)^{K-1} e^{-\frac{n_\theta c}{N^2}} \le C_K^{(3,1)},$$

$$I_{3,2}(n_\theta) = \sum_{m=3}^{\infty} \int_{(m-1)n_\theta+2}^{mn_\theta+2} z^{K-1} e^{-\frac{n_\theta}{2N^2}\frac{(mc_N-1)^2}{m}} dz \le \sum_{m=3}^{\infty} (mn_\theta+2)^K e^{-\frac{n_\theta mc'}{N^2}} \le C_K^{(3,2)},$$

where $C_K^{(3,1)}, C_K^{(3,2)} > 0$ depend at most on K. This proves (67) for $k=3$. □

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Appendix A

A.1. The class GBV_{exp}

The present subsection gives some insight to the properties of the terminal function class $GBV_{\exp}$. For a function $g : \mathbb{R} \to \mathbb{R}$, we let

$$T_g(x) := \sup\left\{ \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})|, \ N \in \mathbb{N}, \ -\infty < x_0 < x_1 < \cdots < x_N = x \right\}.$$

If $\lim_{x\to\infty} T_g(x) < \infty$, the function g is said to be of bounded variation. The class of functions with this property is denoted by BV.

In this article, an extension of BV instead of the class itself is chosen as the class of terminal functions. The reason for this is the fact that, by definition, each function in BV is bounded. Consequently, e.g. the class of polynomials is not contained in BV even though a polynomial has bounded variation on every compact interval. To find a large class of functions which may be unbounded but also have the latter property, we will follow the presentation given in [11].

Recall the class $\mathcal{M}$ given by Definition 2.2, which consists of set functions μ (acting on bounded Borel sets on $\mathbb{R}$) that can be written as a difference of two measures $\mu^1, \mu^2 : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ such that $\mu^1(K)$ and $\mu^2(K)$ are finite for all compact sets $K \in \mathcal{B}(\mathbb{R})$. In [11, Theorem 3.3] it is proved that such a decomposition can be chosen to be orthogonal and minimal: There exists a unique pair of measures μ^+, μ^- on $\mathcal{B}(\mathbb{R})$ such that μ^+ and μ^- are mutually singular, and $\mu^+ \le \mu^1$ and $\mu^- \le \mu^2$ hold for all the other decompositions $\mu = \mu^1 - \mu^2$. Even though $\mu \in \mathcal{M}$ is not itself a signed measure (it is undefined on unbounded sets), the aforementioned result, based on the Hahn decomposition theorem, allows us to define the total variation measure associated to μ by setting

$$|\mu| : \mathcal{B}(\mathbb{R}) \to [0, \infty], \quad |\mu| := \mu^+ + \mu^-.$$

Consequently, the integral in (16) appearing in Definition 2.3 of the class $GBV_{\exp}$ is defined.

As a special case of the result [11, Theorem 4.3], one may verify that $BV \subset GBV_{\exp}$ holds true. To close this subsection, we sketch the proof of the item (i) and (iv) of Remark 2.4. The proofs for the items (ii) and (iii) are left to the reader.

Proof of Remark 2.4. (i) To show that every polynomial $f(x) = \sum_{k=0}^{N} a_k x^k$, $a_k \in \mathbb{R}, N \in \mathbb{N}$ belongs to the class $GBV_{\exp}$, let

$$c = a_0, \quad d\mu = \sum_{k=1}^{N} k a_k x^{k-1} dx, \quad \text{and} \quad \mathcal{J} = \emptyset$$

to be the parameters appearing in the representation (5) for the function f. It remains to observe that this μ satisfies the condition (16), since for every $\beta > 0$,

$$\int_{\mathbb{R}} e^{-\beta|x|} d|\mu|(x) = \int_{\mathbb{R}} e^{-\beta|x|} \left| \sum_{k=1}^{N} k a_k x^{k-1} \right| dx < \infty.$$

(iv) Denote the points of discontinuity of the function $g \in \mathcal{K}_{\exp}$ by $x_1 < x_2 < \cdots < x_N$, $N \ge 0$. Then, the function $h : \mathbb{R} \to \mathbb{R}$ defined as

$$h(x) := g(x) - \sum_{i=1}^{N} \Delta g(x_i) \left(\frac{1}{2} \mathbb{1}_{\{x_i\}}(x) + \mathbb{1}_{(x_i, \infty)}(x) \right), \quad \Delta g(x_i) := g(x_i+) - g(x_i-),$$

is continuous. In addition, by the fundamental theorem of calculus for piecewise continuous functions, it holds that

$$h(y) - h(x) = \int_x^y g'(s) ds, \quad x, y \in \mathbb{R}.$$

One then checks that the function g satisfies (5) provided that the set function μ is given by

$$\mu(A) := \int_A g'(s)ds + \sum_{i=1}^{N} \Delta g(x_i) \ \mathbb{1}_{\{x_i \in A\}}, \quad A \in \mathcal{B}(\mathbb{R}) \quad \text{is a bounded set,}$$

$\mathcal{J} := \left(\frac{1}{2}\Delta g(x_i), x_i\right)_{i=1}^{N}$, and $c := g(0) - \sum_{i=1}^{N} \frac{1}{2}\Delta g(x_i) \ \mathbb{1}_{\{x_i\}}(0)$. It remains to verify that

$$\int_{\mathbb{R}} e^{-\beta|x|} d|\mu|(x) + \sum_{i=1}^{N} \frac{|\Delta g(x_i)|}{2} e^{-\beta|x_i|} < \infty$$

holds for a sufficiently large $\beta \geq 0$ by the exponential boundedness of g'. □

A.2. Auxiliary results for the proof of Theorem 4.3

Under Assumption 3.6, let us recall from (49) the notation $P_{n_\theta+k}(y) = \mathbb{P}\big(X_{\tau_{n_\theta+k}} = hy\big)$ and $P^J_{n_\theta}(y) = \mathbb{P}(J_n - n_\theta = y)$, $y \in \mathbb{Z}$. Notice also that for all $k \in 2\mathbb{N}$,

$$P_k(y) = \binom{k}{\frac{k+y}{2}} 2^{-k}, \quad y \in 2\mathbb{Z}, \quad |y| \leq k.$$

As in [7], we define the "effective order" of a monomial $\frac{k^p y^q}{n^r}$ with $p, q, r \in \mathbb{N}_0$ to be

$$\breve{O}\left(\frac{k^p y^q}{n^r}\right) := \frac{p+q}{2} - r.$$

We will use the following result from [7] in the proof of Lemma A.2.

Proposition A.1 ([7, Proposition 11.5]). *Let*

$$R : \mathcal{D}(R) \to \mathbb{R}, \quad R(n,k,y) := \frac{P_{n+k}(y)}{P_n(y)}; \tag{74}$$

$$R^{(1)} : 2\mathbb{N} \times (2\mathbb{Z})^n \to \mathbb{R}, \quad R^{(1)}(n,k,y) := \frac{k}{2n} - \frac{3k^2 + 4ky^2}{8n^2} + \frac{3k^2y^2}{4n^3} - \frac{k^2y^4}{8n^4}, \tag{75}$$

where

$$\mathcal{D}(R) := \left\{(n,k,y) \in 2\mathbb{N} \times (2\mathbb{Z})^2 : |k| \vee |y| \leq n^{3/5}\right\}.$$

Then there exists a constant $C_0 > 0$, an integer n_0, and a finite sum $R^{(2)}$ of monomials of effective order at most $-3/2$ such that for all $(n,k,y) \in \mathcal{D}(R)$ with $n > n_0$,

$$|R(n,k,y) - \left[1 - R^{(1)}(n,k,y) + R^{(2)}(n,k,y)\right]| \leq C_0 n^{-3/2}. \tag{76}$$

The lemma below presents upper estimates which are applied in the proof of Theorem 4.3.

Lemma A.2. *Suppose that $g \in \mathcal{B}_{\exp}$ and that $b \geq 0$ is as in (14). Suppose also that $R^{(1)}$ is as in (75) and that Γ_{n_θ} is given by (47). Then there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$ and $n_\theta \in 2\mathbb{N}$,*

$$(i) \quad \left|\mathbb{E}\left[g\left(x + X_{\tau_{n_\theta}}\right) - g\left(x + X_{\tau_{J_n}}\right); \Gamma^{\complement}_{n_\theta}\right]\right| \leq C n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T + b\sigma\sqrt{2T}},$$

$$(ii) \quad \Big|\mathbb{E}\left[g\left(x + X_{\tau_{n_\theta}}\right) - g\left(x + X_{\tau_J}\right); \Gamma_{n_\theta}\right] - \sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x + yh) P^J_{n_\theta}(k) P_{n_\theta}(y) R^{(1)}(n_\theta, k, y)\Big| \leq C n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T}.$$

Proof. (i) Since $\Gamma_{n_\theta}^{\complement} \subset \{|X_{\tau_{n_\theta}}/h| > n_\theta^{3/5}\} \cup \left\{|J_n - n_\theta| > n_\theta^{3/5}\right\}$, we may use Hölder's inequality, (42), and (43) to show that there exists a constant $C' > 0$ such that

$$|\mathbb{E}\left[g(x + X_{\tau_{n_\theta}}) - g(x + X_{\tau_{J_n}}); \Gamma_{n_\theta}^{\complement}\right]| \le C' n_\theta^{-3/2}\left(\mathbb{E}|g(x + X_{\tau_{n_\theta}}) - g(x + X_{\tau_{J_n}})|^2\right)^{1/2}.$$

The claim follows, since by the triangle inequality, (39) and (40), and the fact that $g \in \mathcal{B}_{\exp}$, there exists another constant $\tilde{C} > 0$ such that

$$\left(\mathbb{E}|g(x + X_{\tau_{n_\theta}}) - g(x + X_{\tau_{J_n}})|^2\right)^{1/2} \le \tilde{C} e^{b|x| + b^2\sigma^2 T + b\sigma\sqrt{2T}}.$$

(ii) The proof of item (ii) is done in several intermediate steps and only sketched here for the sake of brevity. More details can be found in [14, Lemma A.3].

Step 1: Let us first show that there exists a constant $C > 0$ such that for all $x \in \mathbb{R}$ and n_θ,

$$\left|\sum_{k=2-n_\theta}^{\infty}\sum_{y=-n_\theta}^{n_\theta} g(x + yh) P_{n_\theta}^J(k) P_{n_\theta}(y) R^{(2)}(n_\theta, k, y)\ \mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}}\right| \le C n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T}, \tag{77}$$

where $R^{(2)}$ is as in Proposition A.1. Using the relations $h = \sigma\sqrt{T/n}$, $\theta_n = n_\theta T/n$ and (49), it can be shown that for given integers $p, q, r \in \mathbb{N}_0$ and subsets $\Lambda_1, \Lambda_2 \subset \mathbb{Z}$,

$$\begin{aligned}&\sum_{k=2-n_\theta}^{\infty}\sum_{y=-n_\theta}^{n_\theta} g(x + yh) P_{n_\theta}^J(k) P_{n_\theta}(y) \frac{k^p y^q}{n_\theta{}^r}\mathbb{1}_{\{y \in \Lambda_1, k \in \Lambda_2\}} \\ &= n_\theta^{(p+q)/2 - r}\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^q g(x + X_{\tau_{n_\theta}}); X_{\tau_{n_\theta}}/h \in \Lambda_1\right]\mathbb{E}\left[\left(\frac{J_n - n_\theta}{\sqrt{n_\theta}}\right)^p; J_n - n_\theta \in \Lambda_2\right].\end{aligned} \tag{78}$$

By the definition of $R^{(2)}$, there exists an integer $N \in \mathbb{N}$, a vector $(a_i)_{i=1}^N \subset \mathbb{R}$, and vectors $(p_i)_{i=1}^N, (q_i)_{i=1}^N, (r_i)_{i=1}^N \in \mathbb{N}_0^N$ such that $(p_i + q_i)/2 - r_i \le -3/2$ for all $1 \le i \le N$, and

$$R^{(2)}(n_\theta, k, y) = \sum_{i=1}^N a_i \frac{k^{p_i} y^{q_i}}{n_\theta^{r_i}} \quad \text{for} \quad (n_\theta, k, y) \in \mathcal{D}(R).$$

Therefore, by the relation (78), the left-hand side of (77) can be rewritten and estimated by

$$\begin{aligned}&\left|\sum_{i=1}^N a_i n_\theta^{\frac{p_i+q_i}{2} - r_i}\mathbb{E}\left[\left(\frac{X_{\tau_{n_\theta}}}{\sqrt{\sigma^2\theta_n}}\right)^{q_i} g(x + X_{\tau_{n_\theta}}); |X_{\tau_{n_\theta}}/h| \le n_\theta^{3/5}\right]\mathbb{E}\left[\left(\frac{J - n_\theta}{\sqrt{n_\theta}}\right)^{p_i}; |J_n - n_\theta| \le n_\theta^{3/5}\right]\right| \\ &\le n_\theta^{-3/2}\sum_{i=1}^N |a_i|\mathbb{E}\left[\left(\frac{|X_{\tau_{n_\theta}}|}{\sqrt{\sigma^2\theta_n}}\right)^{q_i}|g(x + X_{\tau_{n_\theta}})|\right]\mathbb{E}\left(\frac{|J_n - n_\theta|}{\sqrt{n_\theta}}\right)^{p_i} \\ &\le \tilde{C} n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T},\end{aligned}$$

where $\tilde{C} > 0$ is some constant implied by (41) and (66). This proves (77).

Step 2: Let us show that for some constant $C > 0$ and for all $x \in \mathbb{R}$ and $n_\theta \in 2\mathbb{N}$,

$$\left|\sum_{k=2-n_\theta}^{\infty}\sum_{y=-n_\theta}^{n_\theta} g(x + yh) P_{n_\theta}^J(k) P_{n_\theta}(y) R^{(1)}(n_\theta, k, y)\ \mathbb{1}_{\left\{|y| \vee |k| > n_\theta^{3/5}\right\}}\right| \le C n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T}. \tag{79}$$

By (75) and (78), it is sufficient to prove that for given $p, q, r \in \mathbb{N}_0$ there exists a constant $C_{p,q,r} > 0$ such that for all $x \in \mathbb{R}$,

$$\left|\sum_{k=2-n_\theta}^{\infty}\sum_{y=-n_\theta}^{n_\theta} g(x + yh) P_{n_\theta}^J(k) P_{n_\theta}(y) \frac{k^p y^q}{n_\theta{}^r}\ \mathbb{1}_{\left\{|y| \vee |k| > n_\theta^{3/5}\right\}}\right| \le C_{p,q,r} n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T}. \tag{80}$$

Relation (80) can be verified by writing $\left\{|y| \vee |k| > n_\theta^{3/5}\right\} = \left\{|y| > n_\theta^{3/5}\right\} \cup \left\{|k| > n_\theta^{3/5},\ |y| \le n_\theta^{3/5}\right\}$ and considering the corresponding sums separately using relation (78) and similar calculations as in Step 1. Indeed, the case $\left\{|y| > n_\theta^{3/5}\right\}$ can be shown using Hölder's inequality and relations (39), (42), (44), and (66). The case $\left\{|k| > n_\theta^{3/5}, |y| \le n_\theta^{3/5}\right\}$ follows from Hölder's inequality, (41), (43), and (66).

Step 3: Since the processes $(\Delta\tau_k)_{k=1,2,\dots}$ and $(\Delta X_{\tau_k})_{k=1,2,\dots}$ are independent (see Section 2.1), the random variable J_n and the process $(X_{\tau_k})_{k=0,1,\dots}$ are also independent. Taking also into account that $\operatorname{supp}P_{n_\theta+k} = \{m \in 2\mathbb{Z} : |m| \le n_\theta + k\}$ (for each $k \in 2\mathbb{N}$) and $\operatorname{supp}P^J_{n_\theta} = \{m - n_\theta : m \in 2\mathbb{N}\}$, it can be shown that

$$\begin{aligned}
&\mathbb{E}\left[g\left(x + X_{\tau_{n_\theta}}\right) - g\left(x + X_{\tau_{J_n}}\right); \Gamma_{n_\theta}\right] \\
&= \sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh)P^J_{n_\theta}(k)P_{n_\theta}(y)\left(1 - \frac{P_{n_\theta+k}(y)}{P_{n_\theta}(y)}\right) \mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}}.
\end{aligned}$$

Thus, by (74)–(78), there exist constants $C_0, C_1 > 0$ and $n_0 \in 2\mathbb{N}$ such that whenever $n_\theta > n_0$,

$$\begin{aligned}
&\Bigg|\sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh)P^J_{n_\theta}(k)P_{n_\theta}(y)R^{(1)}(n_\theta,k,y)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}} \\
&\quad - \mathbb{E}\left[g\left(x + X_{\tau_{n_\theta}}\right) - g\left(x + X_{\tau_J}\right); \Gamma_{n_\theta}\right]\Bigg| \\
&= \left|\sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh)P^J_{n_\theta}(k)P_{n_\theta}(y)\left(R^{(1)}(n_\theta,k,y) - [1 - R(n_\theta,k,y)]\right)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}}\right| \\
&\le C_0 n_\theta^{-3/2} \sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} |g(x+yh)|P^J_{n_\theta}(k)P_{n_\theta}(y)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}} \\
&\quad + \left|\sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh)P^J_{n_\theta}(k)P_{n_\theta}(y)R^{(2)}(n_\theta,k,y)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}}\right| \\
&\le C_0 n_\theta^{-3/2}\mathbb{E}\left[\left|g\left(x + X_{\tau_{n_\theta}}\right)\right|; |X_{\tau_{n_\theta}}/h| \le n_\theta^{3/5}\right] + C_1 n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T} \\
&\le C_2 n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T}
\end{aligned} \tag{81}$$

for some constant $C_2 > 0$ implied by (41). Consequently, we get the claim for all $n_\theta > n_0$ by the triangle inequality, (79), and (81). By letting

$$M := \sup_{(n,k,y): n \le n_0} \left|\left(R^{(1)}(n,k,y) - [1 - R(n,k,y)]\right)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}}\right| < \infty,$$

for $n_\theta \le n_0$ we find another constant $C_3 = C_3(n_0) > 0$ such that

$$\begin{aligned}
&\left|\sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} g(x+yh)P^J_{n_\theta}(k)P_{n_\theta}(y)\left(R^{(1)}(n_\theta,k,y) - [1 - R(n_\theta,k,y)]\right)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}}\right| \\
&\le M \sum_{k=2-n_\theta}^{\infty} \sum_{y=-n_\theta}^{n_\theta} |g(x+yh)|P^J_{n_\theta}(k)P_{n_\theta}(y)\mathbb{1}_{\left\{|y| \vee |k| \le n_\theta^{3/5}\right\}} \\
&\le M\mathbb{E}\left[\left|g\left(x + X_{\tau_{n_\theta}}\right)\right|; |X_{\tau_{n_\theta}}/h| \le n_\theta^{3/5}\right]\mathbb{P}\left(|J_{n_\theta} - n_\theta| \le n_\theta^{3/5}\right) \\
&\le C_3 n_\theta^{-3/2} e^{b|x| + b^2\sigma^2 T}
\end{aligned} \tag{82}$$

by (41). Combine (79), (81), and (82) to complete the proof. □

A.3. Auxiliary results for Sections 3 and 5

The following identities are applied in proof of Lemma 3.4.

Lemma A.3. *Let $h > 0$ and recall the operators Π_e and Π_o given by Definition 3.2.*

(i) *For all $\xi \in \mathbb{R}$, it holds that*

$$\Pi_e \mathbb{1}_{\{\xi\}}(x) = \frac{\mathbb{1}_{\{\xi \in \mathbb{Z}_e^h\}}}{4h} \big(|x - (\xi - 2h)| + |x - (\xi + 2h)| - 2|x - \xi|\big), \quad x \in \mathbb{R}. \tag{83}$$

(ii) *If $y \in [2kh, (2k+2)h)$ for $k \in \mathbb{Z}$, then in terms of d_o defined in (26),*

$$|\Pi_o \Pi_e \mathbb{1}_{(y,\infty)}(x) - \Pi_e \mathbb{1}_{(y,\infty)}(x)| = \frac{d_o(x)}{4h} \mathbb{1}_{[(2k-1)h,\,(2k+3)h)}(x), \quad x \in \mathbb{R}. \tag{84}$$

Proof. (i) It is obvious by the definition of Π_e that $\Pi_e \mathbb{1}_{\{\xi\}} \equiv 0$ for $\xi \notin \mathbb{Z}_e^h$. If $\xi \in \mathbb{Z}_e^h$, then

$$\Pi_e \mathbb{1}_{\{\xi\}}(x) = \begin{cases} \frac{x-(\xi-2h)}{2h}, & (\xi - 2h) \le x < \xi, \\ \frac{(\xi+2h)-x}{2h}, & \xi \le x < (\xi + 2h), \end{cases} \tag{85}$$

and zero elsewhere, so it suffices to verify that (85) agrees with the representation given in (83).

(ii) Suppose that $y \in [2kh, (2k+2)h)$ for some $k \in \mathbb{Z}$. One checks that

$$\Pi_e \mathbb{1}_{(y,\infty)}(x) = \frac{1}{2} + \frac{1}{4h}|x - 2kh| - \frac{1}{4h}|x - (2k+2)h|, x \in \mathbb{R}. \tag{86}$$

Then, by the linearity of Π_o and by (86), we have for every $x \in \mathbb{R}$ that

$$\begin{aligned} &\Pi_o \Pi_e \mathbb{1}_{(y,\infty)}(x) - \Pi_e \mathbb{1}_{(y,\infty)}(x) \\ &= \frac{1}{4h}\big(\Pi_o |\cdot \; -2kh|(x) - |x - 2kh|\big) - \frac{1}{4h}\big(\Pi_o |\cdot \; -(2k+2)h|(x) - |x - (2k+2)h|\big) \\ &= \frac{d_o(x)}{4h}\Big(\mathbb{1}_{[(2k-1)h,\,(2k+1)h)}(x) - \mathbb{1}_{[(2k+1)h,\,(2k+3)h)}(x)\Big), \end{aligned} \tag{87}$$

since it holds for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$ that

$$\Pi_o |\cdot \; -2mh|(x) - |x - 2mh| = d_o(x)\mathbb{1}_{[(2m-1)h,\,(2m+1)h)}(x). \tag{88}$$

Taking the absolute values of both sides of (87) then completes the proof. □

The proof of the next lemma, which is based on the Laplace transform of the stopping time τ_1 defined in (11), follows the approach of [7, Proposition 11.3] and is given in [14, Section 6.2].

Lemma A.4. *Under Assumption 3.6, suppose that $n_\theta \in 2\mathbb{N}$ and a constant $\xi > 0$ are such that $n_\theta \xi \in \mathbb{N}$. Then for every $\rho \in \left(0, \frac{\pi^2}{12}\xi\theta_n\sqrt{n_\theta}\right)$ it holds that*

$$(i) \quad \mathbb{P}\big(\sqrt{n_\theta}(\tau_{n_\theta \xi} - \xi\theta_n) > \rho\big) \le \exp\left(-\frac{3}{2}\frac{\rho^2}{\xi\theta_n^2} H\left(\sqrt{\frac{3\rho}{\xi\theta_n\sqrt{n_\theta}}}\right)\right), \tag{89}$$

$$(ii) \quad \mathbb{P}\big(\sqrt{n_\theta}(\tau_{n_\theta \xi} - \xi\theta_n) < -\rho\big) \le \exp\left(-\frac{3}{2}\frac{\rho^2}{\xi\theta_n^2} H\left(\sqrt{\frac{3\rho}{\xi\theta_n\sqrt{n_\theta}}}\right)\right), \tag{90}$$

where the function $H : (0, \pi/2) \to \mathbb{R}$ is given by

$$H(x) := 1 + \frac{6}{x^4}\left(\frac{x^2}{2} + \log\cos x\right). \tag{91}$$

Remark A.5. The above estimates are non-trivial only whenever H is positive. Since $H(0+) = 1/2$, it holds that $H(x) > 0$ for small enough x. Notice that the condition $\rho \in \left(0, \frac{\pi^2}{12}\xi\theta_n\sqrt{n_\theta}\right)$ ensures that $\sqrt{\frac{3\rho}{\xi\theta_n\sqrt{n_\theta}}} \in (0, \pi/2)$, which is the domain of H.

The purpose of Lemma A.4 and its immediate application is the following result related to the tail probabilities of J_n—an essential tool in proof of Proposition 5.9. While it resembles inequality (42) in [7], the time-dependent setting causes some changes.

Lemma A.6. *Under Assumption 3.6, suppose that* $n_\theta \in 2\mathbb{N}$, $\delta \in \left(0, \frac{\pi^2}{12+\pi^2}\right)$, *and let H be as in (91). Then*

$$(i) \quad \mathbb{P}(J_n > n_\theta(1+\delta)) \leq \exp\left(-\frac{3}{2}\frac{n_\theta\delta^2}{1+\delta}H\left(\sqrt{\frac{3\delta}{1+\delta}}\right)\right) \quad \text{if} \quad n_\theta(1+\delta) \in 2\mathbb{N},$$

$$(ii) \quad \mathbb{P}\big(J_n < n_\theta(1-\delta)\big) \leq \exp\left(-\frac{3}{2}\frac{n_\theta\delta^2}{1-\delta}H\left(\sqrt{\frac{3\delta}{1-\delta}}\right)\right) \quad \text{if} \quad n_\theta(1-\delta) \in 2\mathbb{N}.$$

Proof. Fix $n_\theta \in 2\mathbb{N}$, $\delta \in \left(0, \frac{\pi^2}{12+\pi^2}\right)$, and let $\rho := \delta\theta_n\sqrt{n_\theta}$. For (i), let $\xi := 1+\delta$ and suppose that $n_\theta(1+\delta) = n_\theta\xi \in 2\mathbb{N}$. Then (the first equality follows from the definition of J_n in (18))

$$\mathbb{P}(J_n > n_\theta\xi) = \mathbb{P}(\tau_{n_\theta\xi} < \theta_n) = \mathbb{P}\big(\sqrt{n_\theta}(\tau_{n_\theta\xi} - \xi\theta_n) < -\rho\big) \leq \exp\left(-\frac{3}{2}\frac{\rho^2}{\xi\theta_n^2}H\left(\sqrt{\frac{3\rho}{\xi\theta_n\sqrt{n_\theta}}}\right)\right)$$

by (90), since the choice of δ ensures that the pair (ξ, ρ) satisfies the assumptions of Lemma A.4. To show (ii), let now $\xi := 1-\delta$ and suppose that $n_\theta(1-\delta) = n_\theta\xi \in 2\mathbb{N}$. Then, by (89),

$$\mathbb{P}(J_n < n_\theta\xi) = \mathbb{P}(\tau_{n_\theta\xi} > \theta_n) = \mathbb{P}\big(\sqrt{n_\theta}(\tau_{n_\theta\xi} - \xi\theta_n) > \rho\big) \leq \exp\left(-\frac{3}{2}\frac{\rho^2}{\xi\theta_n^2}H\left(\sqrt{\frac{3\rho}{\xi\theta_n\sqrt{n_\theta}}}\right)\right)$$

since the pair (ξ, ρ) satisfies the assumptions of Lemma A.4 due to the choice of δ. □